ISSUES REGARDING SYNCHRONIZATION PROBLEMS FOR NETWORKS AND INTERNAL STABILITY OF LINEAR SYSTEMS WITH CONSTRAINTS

By

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ISSUES REGARDING SYNCHRONIZATION PROBLEMS FOR NETWORKS AND INTERNAL STABILITY OF LINEAR SYSTEMS WITH CONSTRAINTS

Abstract

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My Ph.D. thesis research accomplishments span two different disciplines. The first one is synchronization in multi-agent systems and the other one is internal stabilization of linear systems with constraints. These two disciplines are studied respectively in Part I and Part II in this thesis.

In Part I, I solve the synchronization problems toward generality of network structure, from *homogeneous networks* (i.e., the agent models in the network are identical) to *heterogeneous networks* (i.e., the agent models in the network are non-identical). For homogeneous networks, I propose different design methodologies for solving the *state synchronization* problems for both *full-state coupling* (i.e., each agent measures its own state relative to that of neighboring agents) and *partial-state coupling* (i.e., each agent measures its own output relative to that of neighboring agents). For heterogeneous networks, I consider the *output synchronization* problem and the *output regulation* problem for two scenarios based on the information available for each agent: *introspective agents* and *non-introspective agents*. While in both cases, each agent collects information of its own output relative to that of neighboring agents, an introspective agent also acquires some sort of self-knowledge. I also consider the *semi-global regulation of output synchronization* for heterogeneous networks of introspective, invertible linear agents subject to actuator saturation. Finally,

I consider the case that the network communications are subject to unknown uniform constant communication delay.

In Part II, I study the issues regarding the internal stabilization of linear systems subject to actuator saturation. I design saturated globally stabilizing linear static state feedback control laws for continuous-time linear systems mixed with single integrators, double integrators, and neutrally stable dynamics. I also completely characterize the dynamic behavior of the discrete-time double integrator with a saturated locally stabilizing linear state feedback law. These are the first step toward my further goal: to completely characterize under what conditions one can utilize a linear static/dynamic state feedback control laws to globally asymptotically stabilize linear systems subject to actuator saturation.

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To my dear parents and all my friends

Chapter 1

Introduction

1.1 Synchronization in Multi-agent Systems

The problem of achieving *synchronization* among agents in a network—that is, asymptotic agreement on the agents' state or output trajectories—has received substantial attention in recent years (see $[2, 44, 55, 100]$ and references therein). The essential difficulty of the synchronization problem is the lack of a central authority with the ability to control the network as a whole. Instead, each agent must implement a controller based on limited information about itself and its surroundings—typically in the form of measurements of its own state or output relative to that of neighboring agents in the network.

The research on synchronization can be generally divided into two categories: one studies *homogeneous* networks (i.e., networks where the agent models are identical) and the other one studies *heterogeneous* networks (i.e., networks where the agent models are non-identical).

1.1.1 Homogeneous Networks

Much of the attention has been directed toward *state synchronization* in *homogeneous* networks. Depending on the information the agents collect from the network, the study on the state synchronization for homogeneous network can be bifurcated into two categories. The case where each agent receives information about its own state relative to that of neighboring agents, which is referred to as *full-state coupling*, have been considered in [44–46, 52, 54, 56, 61, 83]. Roy, Saberi, and Herlugson [62], Tuna [83], and Yang, Roy, Wan, and Saberi [103] considered this type of problem for more general observation topologies and more complex identical agent models than previously considered. Others have studied the case where the agents receive relative information about their own partial-state output, which is referred to as *partial-state coupling*, see for example, [32, 48, 49, 84]. A key idea in the work of [32], which was expanded upon by [107], is the development of a distributed observer. This observer makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. Many of the results on the synchronization problem are rooted in the seminal work of [98, 99].

1.1.2 Heterogeneous Networks

A limited amount of research has also been conducted on *heterogeneous* networks. The synchronization literature for heterogeneous networks can be further divided into two categories. One studies the *state synchronization* while the other one studies the *output synchronization*.

1.1.2.1 State Synchronization

In [50], the authors presented a robust state-synchronization design for networks of nonlinear systems with relative degree one, where each agent implements a sufficiently strong feedback based on the difference between its own state and that of a common reference model. In the work of [101], it is assumed that a common Lyapunov function candidate is available, which is used to analyze stability with respect to a common equilibrium point. Depending on the system, some agents may also implement feedbacks to ensure stability, based on the difference between those agents' states and the equilibrium point. [114] analyzed state synchronization in a network of nonlinear agents based on the network topology and the existence of certain time-varying matrices. Controllers can be designed based on this analysis, to the extent that the available information and actuation allows for the necessary manipulation of the network topology.

1.1.2.2 Output Synchronization

The above-cited works focus on synchronizing the agents' internal states. In heterogeneous networks, however, the physical interpretation of one agent's state may be different from that of another agent. Indeed, the agents may be governed by models of different dimensions. In this case, comparing the agents' internal states is not meaningful, and it is more natural to aim for *output synchronization*—that is, agreement on some partial-state output from each agent. The study on output synchronization for heterogeneous networks can be further classified into two categories depending on the information the agents collect, that is, *introspective* agents and *non-introspective* agents. The agent is *introspective* if it possesses some sort of self-knowledge, while the agent is *non-introspective* if it have no knowledge of their own state or output separate from what is received via the network. This distinction is significant because introspective agents have much greater freedom to manipulate their internal dynamics (e.g., through the use of pre-feedbacks) and thus change the way that they present themselves to the rest of the network. The notion of a non-introspective agent is also practically relevant; for example, two vehicles in close proximity may be able to measure their relative distance without either of them having knowledge of their absolute position.

Introspective Agents Chopra and Spong [10] focused on output synchronization for weakly minimum-phase systems of relative degree one, using a pre-feedback within each agent to create a single-integrator system with decoupled zero dynamics. Pre-feedbacks were also used by [2] to facilitate passivity-based designs. The authors have previously considered output synchronization for right-invertible agents, using pre-compensators and an observer-based pre-feedback within each agent to yield a network of asymptotically identical agents [104]. Kim, Shim, and Seo [25] studied output synchronization for uncertain single-input single-output, minimum-phase systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output. A similar approach was taken by [97], which showed that a necessary condition for output synchronization in heterogeneous networks is the existence of a *virtual exosystem* that produces a trajectory to which all the agents asymptotically converge. If one knows the model of an observable virtual exosystem without exponentially unstable modes, which each agent is capable of tracking, then it can be implemented within each agent and synchronized via the network. The agent can then be made to track the model with the help of a local observer estimating the agent's states.

Non-introspective Agents In this aspect, the results are very limited. In [113], the authors solve the output synchronization problem for a well-defined class of heterogeneous networks of non-introspective agents is by [113]. In their work, the only information available to each agent is a linear combination of outputs received over the network. However, the agents are assumed to be passive—a strict requirement that, among other things, requires the agents to be weakly minimum-phase and of relative degree one. Grip et al. [17] study the output synchronization problem for heterogeneous networks of non-introspective linear agents. In [17], the author assume, in the spirit of [32], that the agents can exchange information about their internal estimates using the network's communication infrastructure besides a linear combination of their own output relative to that of neighboring agents.

1.1.3 Organization

Part I is composed of eleven chapters and can be divided into several subparts:

- Chapter 2 consists of the article Yang, Roy, Wan, and Saberi [103] while Chpater 3 consists of the article Yang, Stoorvogel, and Saberi [107]. Chapter 2 and 3 consider the synchronization problem for homogeneous networks with full-state coupling and partial-state coupling respectively.

- Chapter 4 consists of the article Yang, Saberi, Stoorvogel, and Grip [104] while Chpater 5 consists of the article Wang, Saberi and Yang [95]. Chapter 4 and 5 study the synchronization problems for heterogeneous networks of introspective right-invertible linear agents in continuous-time setting and discrete-time setting respectively.

- Chapter 6 consists of the article Grip, Yang, Saberi, and Stoorvogel [17]. Chapter 6 studies the output synchronization problem and the *regulation of output synchronization* problem for heterogeneous networks of non-introspective right-invertible linear time-invariant agents.

- Chapter 7 consists of the article Yang, Stoorvogel, Grip, and Saberi [105]. This chapter consider the *semi-global regulation of output synchronization* problem for heterogeneous networks of non-introspective, invertible linear time-invariant agents subject to actuator saturation.

- Chapter 8 consists of the article Wang, Saberi, Stoorvogel, Grip, and Yang [92]. This chapter studies the state synchronization problem for homogeneous networks with uniform constant communication delay.

- Chapter 9 consists of the article Wang, Saberi, Stoorvogel, Grip, and Yang [93]. This chapter studies the output synchronization problem and the output regulation problem for heterogeneous networks of introspective right-invertible linear agents with uniform constant communication delay.

Some notations are different among chapters in Part I. I apologize for any inconvenience they may cause.

1.2 Internal Stabilization of Linear Systems subject to Actuator Saturation

Constraints on inputs and other variables of a dynamic system are ubiquitous. Often they occur in the form of magnitude as well as rate saturation of a variable. Clearly, the capacity of every device is capped. Valves can only be operated between fully open and fully closed states, pumps and compressors have a finite throughput capacity, and tanks can only hold a certain volume. Force, torque, thrust, stroke, voltage, current, flow rate, and so on, are limited in their activation range in all physical systems. Servers can serve only so many consumers. In circuits, transistors and amplifiers are saturating components. Every physically conceivable actuator, sensor, or transducer has bounds on the magnitude as well as on the rate of change of its output. Thus, the *saturation* of a device presents a hard constraint.

Part II of the thesis is concerned with the case where there is magnitude constraint on the linear system's actuator/input. Linear systems subject to actuator saturation have been the subject of extensive study. See for instance two special issues [4, 69], and references therein. One of the fundamental goal is to design a feedback such that the closed-loop system is globally asymptotically stable. Internal stabilization for this class of systems has a long history. The negative result given by Fuller [14] established that a chain of integrators with order greater or equal to three cannot be globally stabilized by any saturating linear static state feedback control law with only one input channel. Sontag and Sussmann [78] and Yang, Sontag and Sussmann [109] established that, global stabilization of linear systems subject to actuator saturation can be achieved if and only if the linear system in the absence of actuator saturation is stabilizable, and has all its open-loop poles in the closed left-half plane for continuous-time linear systems and in the closed unit disc for discrete-time linear systems (equivalently, *asymptotically null controllable with bounded control*). In general, this requires nonlinear feedback control laws. We have only very limited insight into which linear controller yields global stability and which one does not. For certain cases, global stabilization can be achieved by linear static state feedback control laws. For example, in both continuous-time and discrete-time settings, it is well-known that there exist linear static state feedback control laws which globally stabilize neutrally stable linear systems subject to actuator saturation, see for instance [76]. The extension of the above result in continuous-time setting has been established in [87]. More precisely, the paper [87] shown that systems which are asymptotically null controllable with bounded inputs can be globally stabilized by linear static state feedback control laws if all non-zero eigenvalues on the imaginary axis are semi-simple (geometric and algebraic multiplicities are equal) while zero is allowed to be an eigenvalue whose Jordan blocks can be at most of size 2×2 (which are associated with double integrators).

One of the goal of Part II is to investigate under what condition the linear static state feedback controllers can be designed to globally asymptotically stabilize the linear system subject to actuator saturation.

Also in continuous-time setting, it is well-known that a linear static state feedback law which locally stabilizes the double integrator subject to actuator saturation[1] also globally stabilizes the system in the presence of actuator saturation, see for instance, [75, 87]. However, similar result has not yet been obtained for the discrete-time case. The goal of Part II is to investigate whether the equivalent of the double integrator subject to actuator saturation in discrete-time is globally asymptotically stable when a locally stabilizing linear state feedback law is used. The answer turns out to be no.

1.2.1 Organization

Part II is composed of two chapters that are organized as follows:

- Chapter 10 consists of the article [106]. This chapter reexamines the classical issue whether linear or nonlinear static state feedback control laws are needed for globally stabilizing linear systems subject to actuator saturation.

- Chapter 11 consists of the article [108]. This chapter completely characterizes the dynamic behavior of the discrete-time double integrator with a saturated locally stabilizing linear state feedback law.

[1]Note that a linear state feedback law with arbitrary negative feedback gains locally stabilizes the double integrator.

Part I

Synchronization in Multi-agent Systems

Chapter 2

Constructing Consensus Controllers for Networks with Identical General Linear Agents

2.1 Introduction

A multitude of networks in nature automatically *synchronize*, that is, states of individual network components or agents dynamically evolve toward a common value or trajectory. In complement, control-theorists have recently sought to develop a decentralized protocol that brings a network's components into *consensus*, that is, to deliberately drive the states of network components to a common value *that depends on the initial component states in a prescribed manner*.

Thus far, most studies on consensus control have been limited to the case that the agents' open-loop internal dynamics are described by an integrator chain (e.g., single- or double-integrator models [44–46, 52, 54, 56, 61]). Very recently, a few researchers have begun to consider consensus control among agents with general identical linear internal dynamics, see [83–85].

In this chapter, we address consensus control for networks whose agents have identical but arbitrary multi-input LTI open-loop dynamics, and a quite-general observation topology. Specifically, we exploit a high-gain decentralized control scheme to obtain consensus for this general agent model, and for a broad class of network topologies.

Let us briefly overview the literature on consensus. We note that the consensus problem actually has a long history in the computer science community [37]. The control-theoretic approach to consensus - i.e., the use of a feedback methodology to synchronize agents' local states in a network to a prescribed function of their initial states - is relatively new, but has been extensively studied in the control community during the last five years and has yielded some advances in e.g., sensor networking ([44–46, 60]) and autonomous vehicle control applications ([52, 54, 56, 61]). Although this literature is extensive, however, much of it fundamentally derives from a classical work of Chua ([98, 99]) that gives conditions on a network's topology and agent dynamics for synchronization. Pogromsky ([48, 49]) has given a control-theoretic interpretation of the classical synchronization result, that captures the essence of the consensus problem. Recently, Li ([32]) and Tuna ([84, 85]) gave conditions for consensus, for networks with agents having identical but general linear internal dynamics and with topology described by directed Laplacian matrices. Also, Tuna ([83]) has considered the consensus control problem in the case that agents have general internal dynamics, using an optimal-control approach. Other than these recent results, explicit design of controllers for consensus (i.e., design of controllers such that the closed-loop meets Chua's condition) has been achieved only for very simple agent models (integrator chains). Also, the efforts on consensus control have focused on network interactions described by a Laplacian matrix [44–46, 52, 54, 56], see our studies [60–62] for analysis of a more general network model. We also note that consensus in networks with time-varying topologies has been studied extensively; we refer the reader to Blondel's summary [6], which shows that general results in the time-varying case can be extracted from an early result of Tsitsiklis [82]. Yet another focus of the consensus literature has been on prescribing the dependence of the agreed-upon value on the initial conditions, or agreement-law design, see [61]. Finally, cursory studies of consensus under delay ([29, 44–46]) and actuator saturation [51] are available.

Noting that the ongoing research on consensus is progressing toward models of increasing generality (from first- [44–46, 54, 62] to second-order and integrator-chain internal dynamics [52, 56, 61], and recently to general agents' internal dynamics [32, 83–85]), we view the problem of constructing consensus controllers for a network with a general network-topology model along with general linear models for the agents' dynamics as a key open problem. In pursuing this aim, here we develop decentralized controllers for consensus in a network of identical agents which have general multi-input LTI internal dynamics and rather general interaction topology. Importantly, our design extends existing efforts in that in our development it permits consensus for a very broad class of sensing/communication topologies (not only ones specified by Laplacian matrices). Also, we show how the agreement law can be assigned while achieving consensus. To solve the consensus problem for general multi-input LTI agents, we apply a high-gain controller design methodology. This methodology provides a general approach to solving the consensus problem, and so in essence shows how the simultaneous-stabilization condition of Chua can be met when feedback control for the agents is permitted.

The remainder of the chapter is organized as follows. In section 2.2, we model in detail the agent internal dynamics, sensing/communication topology, controller architecture, and the consensus task considered in our development. In doing so, we also describe the sense in which our model generalizes and encompasses those in the literature. In section 2.3, we give network and agent theoretic conditions for completion of the consensus tasks using the described controller architecture for time invariant topology. In doing so, we draw extensively on the classical time-scale-based design of control systems, which permits us to study consensus in the broad class of network models introduced here. In section 2.4, we give network and agent theoretic conditions for completion of the consensus task problem with varying topologies.

2.2 Problem Formulation

In this section, we introduce a general model for networked autonomous agents (Section 2.2.1), for which we seek consensus control. We then comprehensively introduce the consensus control problem, and present a controller architecture for achieving consensus (Section 2.2.2).

2.2.1 A Model for Networked Autonomous Agents

We study a network of identical agents with general linear time-invariant (LTI) internal dynamics, that interact through an arbitrary linear observation topology. The autonomous agent network model that we introduce encompasses and generalizes many of the models considered in the consensus literature and more generally the autonomous-agent control literature (with respect to both the agents' internal dynamics and their interactions). Of particular interest, it encompasses models for both distributed computational processes in networks (such as those used in sensor networking applications, see e.g. [44–46, 60]) and networks with mechanical or electromechanical hardware (such as autonomous-vehicle teams [51, 54, 56, 61]).

Here, let us describe the agents' internal dynamics, their networked observations, and the framework for control in the model. Subsequently, for convenience, we also assemble the agents' dynamics into a single state-space representation, and introduce some terminologies regarding the network model's dynamics.

2.2.1.1 The Agent Model

We consider a network of N identical multiple input linear time-invariant (LTI) agents of the form

$$\dot{\hat{x}}_i = \hat{A}\hat{x}_i + \hat{B}\hat{u}_i,$$

for $i \in \{1, \ldots, N\}$, where $x_i \in \mathbb{R}^n$ is agent i's *local state*, and $\hat{u}_i \in \mathbb{R}^m$ is agent i's *local input.* For ease of presentation, let us assume that the matrix $\hat{B}$ has full column rank, and the pair $(\hat{A}, \hat{B})$ is controllable.

2.2.1.2 Network Interactions

In many application areas, the fundamental challenge in achieving consensus among autonomous agents stems from the decentralization of the agents' observations, that is, from the fact that each agent only has partial and complex information about the local states in the network. To permit consensus control for a broad family of applications, we thus consider a quite-general model for the observations made by the agents.

In particular, we consider the rather general case that each agent observes a linear combination of multiple agents' local states. That is, we assume that each agent i makes the observation

$$\hat{y}_i = \sum_{j=1}^{N} g_{ij}\hat{x}_j, \tag{2.1}$$

where we term $\hat{y}_i \in \mathbb{R}^n$ as the agent i's *observation* and term the scalars g_{ij} as *observation weights.* Noting that the observation weight g_{ij} represents the influence (through sensing or networked communication) of each agent j's state on agent i's observation, we find it natural to assemble the weights into an $N \times N$ *topology matrix* $G = [g_{ij}]$. We note that the topology matrix G entirely describes the observation model of the agents. In the literature, the topology matrix is often assumed to be a Laplacian matrix, which has properties that signs of nonzero off-diagonal entries are identical and the sum of the entries on each row is equal to zero. In this case, each observation can be interpreted as a positive combination of relative differences between the local agent's state and each neighbor's state. Here, we allow observations that are arbitrary combinations of multiple agents' states, and so can capture measurement capabilities (including e.g. of absolute and relative states, averages of multiple agents' states, differences between relative-state measurements, etc). We will consider both the general case where the topology matrix can have arbitrary row sums, and the special case that the topology matrix has zero row sums. Even in the zero-row-sum case, we note that the sign pattern of the topology matrix's entries may be arbitrary, in distinction with the directed-Laplacian case. Using this more general formulation, we will clarify that consensus can be achieved for a wide family of observation capabilities, including those captured by certain broad classes of matrices such as D-stable ones.

Variations in network's observation topology are ubiquitous in a range of autonomous agent applications, because of the harsh environments in which the agents operate or because of limitations in the agents' sensing/communication capabilities, among other causes. Numerous articles have studied autonomous agent control and/or synchronization under topological variation, using both deterministic and stochastic models for the variation. Here, we also study consensus control under topological variation, using a classical deterministic model for the variation. In particular, we consider the case where each agent i makes the observation

$$\hat{y}_i = \sum_{j=1}^{N} g_{ij}(t)\hat{x}_j$$

at time t, where the *time-t topology matrix* $G(t) = [g_{ij}(t)]$ is selected from the a finite set of $N \times N$

matrices $G_1, \ldots, G_z$ (i.e., $G(t) \in G_1, \ldots, G_z$ at all times t). For convenience, we also impose the technical condition that $G(t)$ is right-continuous. Thus, we notice that there exists a (either finite or infinite) sequence of times such that, between any two subsequent times (and including the earlier one), the time-t topology matrix is a constant matrix G_i, $i \in \{1, \ldots, z\}$.

2.2.1.3 Framework for Control

A decentralized feedback control paradigm is required, that is, agent i only has the observation $\hat{y}_i$ available and can only set the input $\hat{u}_i$. In the broadest sense, we assume that agent i determines its input $\hat{u}_i(t)$ at time t from concurrent and past observations: $\hat{y}_i(\tau)$, $0 \leq \tau \leq t$. That is, the agent i's controller constitutes a functional mapping from the signal $\hat{y}_i(\tau)$, $\tau \in [0, t]$, to the vector $\hat{u}_i(t)$.

In achieving consensus, we will consider the family of static (memoryless) linear controllers. We will describe this specific controller architecture once we have introduced the agreement problem.

2.2.1.4 Assembled Dynamics and Terminology

We find it convenient to assemble the agents' individual dynamics and observations into a single state-space equation. To this end, we define the *full state vector* as $\hat{x} = [\hat{x}_1^{\mathrm{T}}, \ldots, \hat{x}_N^{\mathrm{T}}]^{\mathrm{T}}$, the *full input vector* as $\hat{u} = [\hat{u}_1^{\mathrm{T}}, \ldots, \hat{u}_N^{\mathrm{T}}]^{\mathrm{T}}$, and the *full observation vector* as $\hat{y} = [\hat{y}_1^{\mathrm{T}}, \ldots, \hat{y}_N^{\mathrm{T}}]^{\mathrm{T}}$. In terms of these quantities, we obtain the following representation of the dynamics when the sensing topology is fixed:

$$\dot{\hat{x}} = (I_N \otimes \hat{A})\hat{x} + (I_N \otimes \hat{B})\hat{u}, \tag{2.2a}$$

$$\hat{y} = (G \otimes I_n)\hat{x}, \tag{2.2b}$$

where the notation '$\otimes$' represents the Kronecker product. We refer to whole model—the dynamics (2.2) together with the decentralized feedback control paradigm—as a *sensing-agent network*, or

SAN.

In the case where the topology may vary, the dynamics of the networks are as follows:

$$\dot{\hat{x}} = (I_N \otimes \hat{A})\hat{x} + (I_N \otimes \hat{B})\hat{u}, \tag{2.3a}$$

$$\hat{y} = (G(t) \otimes I_n)\hat{x}, \tag{2.3b}$$

where the characteristics of the evolving topology matrix $G(t)$ were described above. We refer to the model in this as a *sensing-agent network with topological variation*, or SAN-VT.

2.2.2 The Consensus Control Problem and Static LTI Feedback Architecture

2.2.2.1 The Consensus Control Problem

At its essence, consensus control has to do with feedback design to achieve synchronization among networked agents. That is, we seek controllers for the SAN that make the manifold in which all the agents' states are identical asymptotically stable. Beyond this fundamental goal, consensus control applications sometimes require more refined shaping of dynamics (for instance, designing the trajectory on the asymptotically stable manifold). Here, let us introduce the core consensus task, and then discuss the design of the trajectory on the asymptotically stable manifold.

Since consensus has to do with asymptotic stability of the manifold where all the agents' states are identical, it is convenient for us to define *relative state vectors* that are nil when the agents' states are identical. Formally, let us define the *relative state vectors* as $\hat{q}_i = \hat{x}_i - \hat{x}_N$, for $i \in \{1, \ldots, N-1\}$ (where we have chosen to measure the states relative to $\hat{x}_N$ for notational simplicity). Now that we have defined the relative state vectors, we are ready to formally define the consensus task:

Definition 2.1. *An SAN is said to achieve consensus, if its feedback controller has been designed so that the manifold* $\hat{q}_1 = \ldots = \hat{q}_{N-1} = 0$ *is asymptotically stable.*

Let us make several comments on the definition for consensus:

1) Consensus controls are needed in a variety of application areas, ranging from satellite antenna alignment to vehicle-group formation and sensor fusion, see [44, 61, 102] for just some of the relevant literature.

2) Conceptually, an SAN essentially achieves consensus if the local states of the agents reach the same value or the same trajectory, or in other words agree. Formally, however, we note that consensus is a stronger condition in that we require not only attractivity to the manifold where the local states are identical, but stability in the sense of Lyapunov of this manifold; this stronger definition is natural in feedback controller design, and matches with the existing literature on consensus. We kindly ask the reader to see the broad literature on nonlinear control for a careful deconstruction of the difference between attractivity and stability. For the linear dynamics that we study here, the notions are identical.

3) Asymptotic stability of the state $\hat{x}(t)$ (with the origin as the equilibrium point) is sufficient for consensus. However, consensus is possible even when stability is not: only equalization of the various agents' states is needed. In fact, our definition does not enforce any condition on the dynamics on the manifold where the states are equal; the dynamics on the manifold may depend on the initial conditions in an arbitrary way, and may be time-varying. Thus, our definition encompasses both the concepts of consensus and tracking-consensus introduced in the literature [44–46, 51, 52, 54, 56, 60].

In contrast to the traditional studies of synchronization, we explicitly allow for controller design in seeking consensus in SANs. This design freedom can potentially allow not only for stabilization of the manifold of interest, but shaping of the trajectory on the manifold. Motivated by numerous applications (in particular, computational applications such as sensor fusion ones), we are especially interested in shaping the dependence of the asymptotic dynamics on its initial conditions. This task of shaping the dependence of the asymptotic dynamics on the initial conditions has been termed *agreement law design*, see the initial work of Olfati-Saber and co-workers [44–46] as well as the systematic treatment in our earlier work [60]. Here, let us formalize the notion of an agreement law (and of agreement law design) for SANs.

Definition 2.2. *Consider an SAN that achieves consensus upon use of a particular feedback controller. Now consider a functional mapping from the initial states of the agents and time to an n-component vector, say $f(\hat{x}_1(0), \ldots, \hat{x}_N(0), t)$. This function is said to be the agreement law of the SAN (upon use of the particular controller), if*

$$\lim_{t\to\infty} \left(\hat{x}_i(t) - f(\hat{x}_1(0), \ldots, \hat{x}_N(0), t)\right) = 0, \quad \forall i \in \{1, \ldots, N\}.$$

We note that, when an SAN achieves consensus using a particular controller, it has a unique agreement law. We will be interested in characterizing and designing the agreement laws of SANs that achieve consensus.

Finally, let us discuss the controller architecture that we propose for achieving consensus.

2.2.2.2 Static LTI Control Architecture

Our goal is to design a controller for an SAN, so as to achieve consensus and (additionally) set the agreement law. Classical research on state feedback controller design, together with our recent efforts on stabilization through decentralized control [89–91], suggest that a linear static controller design should permit consensus under broad conditions on the network topology. Thus, we focus in this chapter on a linear static (memoryless) feedback control architecture. We note that observer-type dynamical controller and non-linear control architecture have also been considered, see [3, 32]. In particular, we consider applying the controller $\hat{u}_i = \hat{K}_i \hat{y}_i$, where $\hat{K}_i \in \mathbb{R}^{m \times n}$, for each agent $i \in \{1, \dots, N\}$. We will study how the *gain matrices* $\hat{K}_i$ can be designed, to achieve consensus and shape the agreement law.

We find it convenient to assemble the control laws for each agent into a single relation. Doing so, we find that $\hat{u} = \hat{K}\hat{y}$, where $\hat{K} = \text{blkdiag}(\hat{K}_1, \dots, \hat{K}_N)$.

We notice that the control architecture that we consider is fundamentally a decentralized architecture, in that each agent can only use its own observation and govern local actuator's input.

2.3 Constructing Controllers for Consensus

In this section, we develop broad conditions under which the SAN achieves consensus, in the process explicitly specifying the static decentralized controllers that can achieve consensus. Our efforts here significantly enhance existing research on consensus control, in that 1) consensus is achieved for general agent internal dynamics, 2) a systematic high-gain methodology for designing consensus control is obtained, and 3) connections to ongoing research on synchronization and dynamical-network control/design are made explicitly.

Let us first present a key implicit condition on the network topology under which consensus can be achieved for an SAN, where the proof allows construction of the high-gain consensus controller. After doing so, we will show that this condition encompasses a very broad range of network topologies, including not only the Laplacian topology matrices commonly considered in the literature but a wide family of asymmetric topology matrices.

Theorem 2.1. *Consider an SAN with topology matrix G. A static LTI decentralized controller can be designed for the SAN to achieve consensus, if there exists a diagonal matrix D such that either 1) all the eigenvalues of DG are in the open left-half complex plane; or 2) all the eigenvalues* *of DG are in the closed left-half complex plane, and the only eigenvalue on the jω-axis is at the origin, with the corresponding right eigenvector* $\mathbf{1}$.

Proof. We prove this theorem by first transforming the (identical) agent's open-loop system dynamics into a special form, which facilitates the design of a high-gain controller and the use of time-scale ideas to achieve consensus. For case 1) of the theorem, we use a time-scale design technique [26,67] to show that we can place the closed-loop eigenvalues in the open left-half complex plane, thus proving that consensus is achieved. For case 2) of the theorem, we consider the dynamics of the relative state, and then the time-scale analysis tells us that we can place the eigenvalues of the relative state's closed-loop system matrix in the open left-half complex plane. Thus, we prove that consensus is also achieved.

From [64], we know that for any controllable pair $(\hat{A}, \hat{B})$, there exist non-singular state transformation T_s and input transformation T_i, such that $x_i = T_s\hat{x}_i$, $u_i = T_i\hat{u}_i$, $x_i = [x_{i,1}^{\mathrm{T}}, \ldots, x_{i,m}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^n$, $x_{i,j} = [x_{i,j,1}, x_{i,j,2}, \ldots, x_{i,j,l_j}]^{\mathrm{T}} \in \mathbb{R}^{l_j}$, and $u_i = [u_{i,1}, u_{i,2}, \ldots, u_{i,m}]^{\mathrm{T}} \in \mathbb{R}^m$ satisfy

$$\dot{x}_{i,j} = A_j x_{i,j} + B_j \left(u_{i,j} + \sum_{l=1, l\neq j}^{m} E_{j,l} x_{i,l} \right), \quad j = 1, 2, \ldots, m,$$

where matrices $A_j \in \mathbb{R}^{l_j \times l_j}$, $B_j \in \mathbb{R}^{l_j \times 1}$ have the following special structures:

$$A_j = \begin{bmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & \ddots & 1 & 0 \\ 0 & \ldots & \ldots & \ldots & 0 & 1 \\ E_{j,j,1} & E_{j,j,2} & \ldots & \ldots & E_{j,j,l_j-1} & E_{j,j,l_j} \end{bmatrix}, \quad B_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and

$$E_{j,l} = \begin{bmatrix} E_{j,l,1} & E_{j,l,2} & \dots & E_{j,l,l_l} \end{bmatrix}.$$

We note that the state transformation T_s and input transformation T_i transform each agent's model into m integrator chains, with the length of the j-th chain being l_j. The triple subindex x_{i,j,l_j} denotes the state variable of the i-th agent, j-th chain and l_j-th level. The chains for each agent are coupled only at the bottom layer, and the input signal $u_{i,j}$ is injected into the bottom layer of each integrator chain.

Now let us consider the design of feedback controller architecture. The controller in the new coordinates for agent i can be expressed as

$$u_i = K_i \sum_{j=1}^{N} g_{i,j} x_j,$$

where $K_i = T_i \hat{K}_i T_s^{-1} \in \mathbb{R}^{m \times n}$. Here we design a high-gain controller K_i of the following form:

$$K_i = \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \dots & \frac{\beta_{1,l_1}}{\epsilon_1} & & & \\ & & & \ddots & & \\ & & & & \frac{\beta_{m,1}}{\epsilon_m} & \dots & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix} d_i,$$

where ϵ_j is sufficiently small and $d_i \in \mathbb{R}$ is a scalar. We limit ourselves to the case where K_i is block diagonal so that the scalar input $u_{i,j}$ of j-th chain of the agent i, only feeds back the state information of its local chain. Also, the gain matrix K_i for different agents only differs by a scalar factor d_i. We find it convenient to assemble the agents' individual states and inputs into a single-space equation. By defining the *full state vector* as $x = [x_1^{\mathrm{T}}, \dots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, the *full input vector* as $u = [u_1^{\mathrm{T}}, \dots, u_N^{\mathrm{T}}]^{\mathrm{T}}$, and introducing a diagonal matrix $D = \mathrm{diag}(d_1, \dots, d_N)$, we express the feedback law for the SAN as

$$u = \left(DG \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \dots & \frac{\beta_{1,l_1}}{\epsilon_1} & & & \\ & & & \ddots & & \\ & & & & \frac{\beta_{m,1}}{\epsilon_m} & \dots & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix} \right) x.$$

Now, let us consider case 1), where all the eigenvalues of DG are in the open left-half complex plane. We will use the time-scale technique to show that the high-gain controller can stabilize the SAN and hence consensus is achieved. First, let us assemble the last state variable of each chain of all agents' into a vector $\eta \in \mathbb{R}^{Nm}$,

$$\eta = \begin{bmatrix} x_{1,1,l_1}, & x_{1,2,l_2}, & \dots, & x_{1,m,l_m}, & \dots, & \dots, & x_{N,1,l_1}, & x_{N,2,l_2}, & \dots, & x_{N,m,l_m} \end{bmatrix}^{\mathrm{T}},$$

and the rest of state variables into another vector $\zeta \in \mathbb{R}^{N(n-m)}$,

$$\zeta = \begin{bmatrix} x_{1,1,1}, & \dots, & x_{1,1,l_1-1}, & \dots, & x_{1,m,1}, & \dots, & x_{1,m,l_m-1}, & \dots, & x_{N,1,1}, & \dots, x_{N,m,l_m-1} \end{bmatrix}^{\mathrm{T}}.$$

With some algebra, we express the closed-loop system dynamics separated in the slow and fast time scales as

$$\begin{aligned}
\dot{\zeta} &= (I_N \otimes R)\zeta + (I_N \otimes S)\eta, \\
\dot{\eta} &= \left(DG \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \dots & \frac{\beta_{1,l_1-1}}{\epsilon_1} & & & \\ & & & \ddots & & \\ & & & & \frac{\beta_{m,1}}{\epsilon_m} & \dots & \frac{\beta_{m,l_m-1}}{\epsilon_m} \end{bmatrix} + I_N \otimes P \right) \zeta \\
&\quad + \left(DG \otimes \begin{bmatrix} \frac{\beta_{1,l_1}}{\epsilon_1} & & \\ & \ddots & \\ & & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix} + I_N \otimes Q \right) \eta,
\end{aligned}$$

where

$$R = \mathrm{blkdiag}(R_1, \dots, R_m) \in \mathbb{R}^{(n-m)\times(n-m)}, \quad S = \mathrm{blkdiag}(S_1, \dots, S_m) \in \mathbb{R}^{(n-m)\times m},$$

$$R_j = \begin{bmatrix} 0_{(l_j-2)\times 1} & I_{l_j-2} \\ 0 & 0_{1\times(l_j-2)} \end{bmatrix} \in \mathbb{R}^{(l_j-1)\times(l_j-1)}, \quad S_j = \begin{bmatrix} 0_{(l_j-2)\times 1} \\ 1 \end{bmatrix} \in \mathbb{R}^{(l_j-1)\times 1},$$

$$P = \begin{bmatrix} E_{1,1,1} & \dots & E_{1,1,l_1-1} & \dots & E_{1,m,1} & \dots & E_{1,m,l_m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ E_{m,1,1} & \dots & E_{m,1,l_1-1} & \dots & E_{m,m,1} & \dots & E_{m,m,l_m-1} \end{bmatrix}, \quad Q = \begin{bmatrix} E_{1,1,l_1} & \dots & E_{1,m,l_m} \\ \vdots & \vdots & \vdots \\ E_{m,1,l_1} & \dots & E_{m,m,l_m} \end{bmatrix}.$$

Since ϵ_j for $j = 1, \dots, m$ are sufficiently small and DG is nonsingular, the time-scale methodology [26] shows that the Nm fast eigenvalues can be divided into m groups, and for each group $j = 1, \dots, m$, the eigenvalues are located at

$$\lambda_{f_j} = \frac{\beta_{j,l_j}}{\epsilon_j} \lambda(DG) + \mathcal{O}(1).$$

Since the eigenvalues of DG are in the open left-half complex plane, we can choose all the parameters $\beta_{1,l_1}, \dots, \beta_{m,l_m}$ to be positive to ensure that the fast eigenvalues are in the open left-half complex plane.

Now, let's consider the $N(n-m)$ slow eigenvalues. From [26], since ϵ_j are sufficiently small, we know that the slow eigenvalues are the eigenvalues of matrix A_0 shown below plus some small perturbation.

$$
\begin{aligned}
A_0 &= I_N \otimes R - (I_N \otimes S) \left(DG \otimes \begin{bmatrix} \frac{\beta_{1,l_1}}{\epsilon_1} & & \\ & \ddots & \\ & & \frac{\beta_{m,l_m}}{\epsilon_m} \end{bmatrix} \right)^{-1} \\
&\quad \times \left(DG \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \dots & \frac{\beta_{1,l_1-1}}{\epsilon_1} & & & \\ & & & \ddots & & \\ & & & \frac{\beta_{m,1}}{\epsilon_m} & \dots & \frac{\beta_{m,l_m-1}}{\epsilon_m} \end{bmatrix} \right) \\
&= I_N \otimes R - (I_N \otimes S) \left((DG)^{-1} \otimes \begin{bmatrix} \frac{\epsilon_1}{\beta_{1,l_1}} & & \\ & \ddots & \\ & & \frac{\epsilon_m}{\beta_{m,l_m}} \end{bmatrix} \right) \\
&\quad \times \left(DG \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\epsilon_1} & \dots & \frac{\beta_{1,l_1-1}}{\epsilon_1} & & & \\ & & & \ddots & & \\ & & & \frac{\beta_{m,1}}{\epsilon_m} & \dots & \frac{\beta_{m,l_m-1}}{\epsilon_m} \end{bmatrix} \right) \\
&= I_N \otimes R - (I_N \otimes S) \left(I_N \otimes \begin{bmatrix} \frac{\beta_{1,1}}{\beta_{1,l_1}} & \dots & \frac{\beta_{1,l_1-1}}{\beta_{1,l_1}} & & & \\ & & & \ddots & & \\ & & & \frac{\beta_{m,1}}{\beta_{m,l_m}} & \dots & \frac{\beta_{m,l_m-1}}{\beta_{m,l_m}} \end{bmatrix} \right) \\
&= I_N \otimes \begin{bmatrix} A_{0,1} & & \\ & \ddots & \\ & & A_{0,m} \end{bmatrix},
\end{aligned}
$$

where the matrix $A_{0,j} \in \mathbb{R}^{(l_j-1)\times(l_j-1)}$ for $j = 1, \ldots, m$ has the following special structure:

$$A_{0,j} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & 1 \\ -\frac{\beta_{j,1}}{\beta_{j,l_j}} & \ldots & \ldots & -\frac{\beta_{j,l_j-2}}{\beta_{j,l_j}} & -\frac{\beta_{j,l_j-1}}{\beta_{j,l_j}} \end{bmatrix}.$$

Therefore, the slow eigenvalues can be placed arbitrarily close to $n - m$ locations, and there are N eigenvalues at each location. For each $j = 1, \ldots, m$, we can choose $\beta_{j,1}, \ldots, \beta_{j,l_j-1}$ such that the slow eigenvalues

$$\lambda_{s_j} = \lambda(A_{0,j}) + \mathcal{O}(\epsilon_j)$$

are in the open left-half complex plane. Since all the eigenvalues of the closed-loop system are in the open left-half complex plane. We have proved that the consensus is achieved for the SAN.

Now, let us consider case 2) of the theorem. To begin, we find it convenient to give the closed-loop dynamics:

$$\dot{x} = \left(I_N \otimes A + DG \otimes \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_m \end{bmatrix} \right) x, \tag{2.4}$$

where matrix $A \in \mathbb{R}^{n\times n}$ is the system matrix of the (identical) local agent

$$A = \begin{bmatrix} A_1 & B_1E_{1,2} & \ldots & B_1E_{1,m} \\ B_2E_{2,1} & A_2 & \ldots & B_2E_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_mE_{m,1} & B_mE_{m,2} & \ldots & A_m \end{bmatrix},$$

and

$$F_j = \begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \frac{\beta_{j,1}}{\epsilon_j} & \dots & \frac{\beta_{j,l_j}}{\epsilon_j} \end{bmatrix} \in \mathbb{R}^{l_j \times l_j}.$$

Next, let us study the dynamics of relative state vectors for the system. Specifically, let us define a transformed relative state vector $q_i = x_i - x_N$, for $i = 1, \dots, N-1$ (where, incidentally, q_i can be obtained from $\hat{q}_i$, see Definition 2.1 from Section 2.1, through the same transformation T_s). To do so, we find it convenient to assemble all the transformed relative state vectors into a single global relative vector $q = [q_1^{\mathrm{T}}, \dots, q_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$. Through a state transformation of (2.4), we find that the global relative vector satisfies an autonomous differential equation (see e.g. [46, 98] for similar analysis). In particular, with some algebra similar to that of [12, 98, 100], we can express the dynamics of the global relative vectors as

$$\dot{q} = \left(I_{N-1} \otimes A + \overline{DG} \otimes \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_m \end{bmatrix} \right) q, \tag{2.5}$$

where $\overline{DG}$ is formed by removing the last row and column from $DG - D_N \mathbf{1} g_N^{\mathrm{T}}$, and g_N^{T} is the last row of G. Equivalently, $\overline{DG}$ can be viewed as being formed by subtracting the last row of DG from all other rows, and then removing the last row and column. It is easy to see that $\overline{DG}$ has $N-1$ eigenvalues, which are the non-zero eigenvalues of DG. Thus all the eigenvalues of $\overline{DG}$ are in the open left-half complex plane, and we automatically see that the problem of stabilizing the relative state dynamics is identical to the stabilization of the state dynamics achieved for case 1). Specifically using a high-gain controller, we can place all $(N-1)n$ eigenvalues of the state matrix of dynamics (2.5) in the open left-half complex plane. Hence, the manifold where the agents' states are identical is made asymptotically stable. We have proved the consensus is achieved for the SAN. $\square$

Let us make a couple remarks about the implicit condition for consensus:

- We note that the first case of Theorem 2.1 is a condition for stabilization of the dynamics, since in this case the invariant manifold comprises only the origin.
- The proof of Theorem 2.1 provides a high-gain methodology for achieving consensus, under the condition that a diagonal D can be found to place the eigenvalues of DG either in the open left-half complex plane or in the closed left-half complex plane with a single zero eigenvalue on the $j\omega$-axis and corresponding eigenvector $\mathbf{1}$. We stress that this time-scale assignment (or high gain) approach is in analogy with the classical designs used for centralized plants, and is needed for agents with general models (rather than only integrator-chain dynamics); this need for a time-scale assignment design becomes clear from certain classical representations of linear systems, such as the Brunovsky canonical form and the special coordinate basis (e.g., [67]). We also stress that the time-scale assignment design (like all controller designs) must be tuned/refined with several performance metrics in mind, including disturbance- and noise- response metrics and robustness measures. In this first effort, we focus solely on shaping the network's internal dynamics, and leave further refinement of the design to future work.
- We note that our design methodology for case 1) of Theorem 2.1, allows us to place $N(n-m)$ eigenvalues arbitrarily close to $n-m$ locations in the complex plane, in groups of N. Meanwhile, the remaining Nm eigenvalues are within $\mathcal{O}(1)$ of the eigenvalues of $\frac{\beta_{j,l_j}}{\epsilon_j}DG$.
- A simple eigen-analysis for case 2) of Theorem 2.1, where DG has one zero eigenvalue, shows that the state matrix of the closed-loop system (2.4) has n eigenvalues that are exactly the

same as those of local agents' open-loop system matrix A. This remark will be useful for agreement law design.

- The consensus control problem has been considered in [32, 83, 85] for identical but general open-loop dynamics, for the case where the poles of the open-loop system are all in the closed left-half complex plane. In our development, since we allow arbitrary dynamics on the consensus manifold (including possibly unstable dynamics), we do not constrain the location of the open-loop poles.

- With just a little effort, Theorem 2.1 can be extended to the case that the pair $(\hat{A}, \hat{B})$ is stabilizable (rather than controllable). We exclude the details.

The condition and controller construction given in Theorem 2.1 makes it clear that consensus can be achieved, whenever a diagonal scaling D can be found to place the eigenvalues of DG in a single half plane. The problem of finding a diagonal D to shape the spectrum of DG has been explored in both the classical numerical-methods and control literature [13,61], as well as in recent works on dynamical network control [59,88]. Let us therefore recall a useful result from Fisher and Fuller's paper [13].

Lemma 2.1. *There exists a diagonal matrix D such that the eigenvalues of DG are all in the open left-half complex plane (or, alternatively, in the open right-half complex plane) if there exists a permutation matrix P_1 such that all the leading principal minors of $P_1GP_1^{-1}$ are nonzero.*

Based on Lemma 2.1 given in [13], we can obtain a broad explicit condition on the network topology matrix G for an SAN to achieve consensus. This condition encompasses those given in the literature. Here is the result:

Theorem 2.2. *Consider an SAN with topology matrix G. A static decentralized controller can be designed for the SAN to achieve consensus, if either 1) G has a nested sequence of N principal minors (of dimensions $1 \times 1, 2 \times 2, \ldots, N \times N$) all of full rank or 2) G has a nested sequence of $N-1$ principal minors (of dimensions $1 \times 1, 2 \times 2, \ldots, N-1 \times N-1$) of full rank and further the vector $\mathbf{1}$ is in the null space of G.*

Proof. Let us consider case 1) of the theorem. In the case that G has a nested sequence of N principal minors all of full rank, the papers [13] and [61] give a systematic method for constructing a diagonal matrix D, such that all the eigenvalues of DG are in the open left-half complex plane. Hence, the result follows from case 1) of Theorem 2.1.

For case 2) of the theorem, we can design a diagonal matrix D such that $N-1$ eigenvalues of DG are in the open left-half complex plane. Also the vector $\mathbf{1}$ is in the null space of DG, since the vector $\mathbf{1}$ is in the null space of G. Thus another eigenvalue of matrix DG is zero and the corresponding right eigenvector is $\mathbf{1}$. Hence, the result follows from case 2) of Theorem 2.1. $\square$

Notice that the first condition of Theorem 2.2, that is, the sequential-full-rank condition, is in fact satisfied for a broad class of matrices, including grounded Laplacian ones and more generally diagonally-dominant matrices. The second condition of Theorem 2.2 encompasses a broad class of matrices, including Laplacian matrices of connected undirected graph, or a digraph which contains a directed spanning tree, as well as a class of matrices known as *D-semistable matrices* with additional property that the matrix has no eigenvalues on the $j\omega$-axis other than the single eigenvalue at the origin with the corresponding right eigenvector $\mathbf{1}$. For the definition of D-semistability, please see [19, 61]. Since this notion is not very widely used, let us recall that D-semistability is defined as follows: a matrix G is said to be D-semistable if the eigenvalues of the matrix DG are in the closed left half plane and the eigenvalues of DG on the $j\omega$-axis are simple, for all positive diagonal matrix

D. This broad class includes a wide family of matrices with more general entry sign pattern than the Laplacian matrix, and hence admits consensus control for a wider set of observation capabilities.

We have constructed a high-gain controller for identical but general agents' internal dynamics to achieve consensus. In many cases, the agreement law – the dependence of the consensus value or trajectory on the initial conditions – is of importance. Let us characterize the agreement law, when a particular controller is used under the conditions of Theorem 2.2. We will do so by characterizing the dynamics on the consensus manifold through eigen-analysis.

In case 1) of Theorem 2.2, we see automatically that agreement law is $f(x_1(0), \ldots, x_N(0), t) = 0$, that is, the final state is nil for all initial conditions.

Now, let us consider case 2) of Theorem 2.2. Without loss of generality, assume that the local agent's system matrix A has $k \leq n$ distinct eigenvalues, $\lambda_1, \ldots, \lambda_k$, each with algebraic multiplicity p_i, where $i = 1, \ldots, k$. Using the Jordan form representation, we find that

$$A = VJV^{-1},$$

where the Jordan canonical form of A is $J = \text{blkdiag}(J_1, J_2, \ldots, J_k)$, and each Jordan Block J_i can be subpartitioned as

$$J_i = \text{blkdiag}(J_{i,1}, J_{i,2}, \ldots, J_{i,p_i}),$$

where each $n_{ij} \times n_{ij}$ subblock $J_{i,j}$ is of the form below:

$$J_{i,j} = \begin{bmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_i & 1 \\ 0 & \ldots & \ldots & 0 & \lambda_i \end{bmatrix}.$$

$V = [V_1, V_2, \ldots, V_k]$ is the matrix whose columns are right eigenvectors and generalized right eigenvectors of A, the rows of V^{-1} are left eigenvectors and generalized left eigenvectors of A, the partitioning of V and V^{-1} matches that of J, blkdiag( ) represents a block diagonal matrix with entries specifying the blocks.

Let ω^{T} be the normalized left eigenvector of G associated with the zero eigenvalue. From the proof of Theorem 2.2, we know that DG has all except a single zero eigenvalue in the open left-half complex plane, the right eigenvector of DG associated with zero eigenvalue is $\mathbf{1}$, and the normalized left eigenvector associated with the zero eigenvalue is $\omega_0^{\mathrm{T}} = (1/\omega^{\mathrm{T}} D^{-1}\mathbf{1})\omega^{\mathrm{T}} D^{-1}$, where we have assumed that an invertible D is being used. For $i = 1, \ldots, k$ and $j = 1, \ldots, p_i$, let us denote the j-th right eigenvector of A associated with λ_i as $v_{i,j,1}$, and the associated generalized right eigenvectors as $v_{i,j,2}, \ldots, v_{i,j,n_{ij}}$. Similarly, let us denote the j-th left eigenvector of A associated with λ_i as $\omega_{i,j,1}^{\mathrm{T}}$, and the associated generalized left eigenvectors as $\omega_{i,j,2}^{\mathrm{T}}, \ldots, \omega_{i,j,n_{ij}}^{\mathrm{T}}$.

With just a little algebra, we find that the closed-loop system matrix (3.7) has an eigenvalue λ_i, $i = 1, \ldots, k$ with algebraic multiplicity p_i. Further, the j-th left eigenvector associated with λ_i is $\omega_0^{\mathrm{T}} \otimes \omega_{i,j,1}^{\mathrm{T}}$, and the corresponding generalized left eigenvectors are $\omega_0^{\mathrm{T}} \otimes \omega_{i,j,2}^{\mathrm{T}}, \ldots, \omega_0^{\mathrm{T}} \otimes \omega_{i,j,n_{ij}}^{\mathrm{T}}$. Similarly, the j-th right eigenvector associated with λ_i is $\mathbf{1} \otimes v_{i,j,1}$, and the corresponding generalized right eigenvectors are $\mathbf{1} \otimes v_{i,j,1}, \ldots, \mathbf{1} \otimes v_{i,j,n_{ij}}$.

Without loss of generality, let us assume that the first $k_1 \leq k \leq n$ eigenvalues of the local agent system matrix A, $\lambda_1, \ldots, \lambda_{k_1}$ are in the closed right-half complex plane. From the analysis in the proof of Theorem 2.1, we immediately find that

$$\lim_{t \to \infty} \left(x(t) - \sum_{i=1}^{k_1} (\mathbf{1} \otimes V_i) e^{J_i t} (\omega_0^{\mathrm{T}} \otimes W_i^{\mathrm{T}}) x(0) \right) = 0,$$

where for $i = 1, \ldots, k_1$ and $j = 1, \ldots, p_i$,

$$e^{J_i t} = \mathrm{blkdiag}(e^{J_{i,1} t}, e^{J_{i,2} t}, \ldots, e^{J_{i,p_i} t}),$$

$$e^{J_{i,j}t} = e^{\lambda_i t} \begin{bmatrix} 1 & t & \dots & \dots & \frac{t^{n_{ij}-1}}{(n_{ij}-1)!} \\ 0 & 1 & t & \ddots & \frac{t^{n_{ij}-2}}{(n_{ij}-2)!} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \dots & \ddots & 1 & t \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix},$$

$$V_i = \begin{bmatrix} V_{i,1} & \dots & V_{i,p_i} \end{bmatrix}, \quad V_{i,j} = \begin{bmatrix} v_{i,j,1} & \dots & v_{i,j,n_{ij}} \end{bmatrix},$$

$$W_i^{\mathrm{T}} = \begin{bmatrix} W_{i,1}^{\mathrm{T}} \\ \vdots \\ W_{i,p_i}^{\mathrm{T}} \end{bmatrix}, \quad \text{and} \quad W_{i,j}^{\mathrm{T}} = \begin{bmatrix} \omega_{i,j,n_{ij}}^{\mathrm{T}} \\ \omega_{i,j,n_{ij-1}}^{\mathrm{T}} \\ \vdots \\ \omega_{i,j,1}^{\mathrm{T}} \end{bmatrix}.$$

Hence, for $\ell = 1, \dots, N$:

$$\lim_{t \to \infty} \left(x_\ell(t) - \sum_{i=1}^{k_1} V_i e^{J_i t} (\omega_0^{\mathrm{T}} \otimes W_i^{\mathrm{T}}) x(0) \right) = 0.$$

Hence, the agreement law of the SAN (upon use of a particular high-gain controller with matrix D) is

$$f(x_1(0), \dots, x_N(0), t) = \frac{1}{\omega^{\mathrm{T}} D^{-1} \mathbf{1}} \sum_{i=1}^{k_1} V_i e^{J_i t} \left((\omega^{\mathrm{T}} D^{-1}) \otimes W_i^{\mathrm{T}} \right) x(0).$$

When DG has a zero eigenvalue, we thus see that the asymptotic trajectory depends on the initial conditions of all the agents. Notice that the agreement law - the dependence of the asymptotic dynamics on initial conditions - in general is a time varying function, which depends on the closed right-half complex plane modes of the agent's internal dynamics; thus, tracking in consensus is also possible.

By selecting D, we see that the agreement law can be designed. In [60], Roy and co-workers studied selection of D so that a desired agreement law is achieved while the eigenvalues of DG

are left in the closed left-half complex plane. We can apply these results to achieve agreement assignment in the general case studied here.

2.4 Consensus Controller Design under Topological Variation

In this section, we will consider controller design for consensus in the SAN-VT, that is, for a sensing-agent network model that is subject to variations in the observation topology. Such controller design for consensus under topological variation is relevant in several application domains, including for control of autonomous vehicle teams and sensor networks (which both tend to operate in harsh environments with limited actuation/power, and so maybe routinely subject to sensing failures and other topological variations). Our work on controller design under topological variation is complementary to numerous studies on modeling and analyzing synchronization/consensus under topological variation, see e.g. Blondel and co-workers' recent article for a succinct overview [6]. We also note the connection of our work to several recent works on design of network controllers under arbitrary and stochastic topological variation [58]; in comparison, the results presented here permit design for much more general agent models and a broad class of network topologies.

In our efforts to design controllers for the SAN-VT, we distinguish between two paradigms regarding information dissemination on the topology changes. The first case is that the controller can detect when the network topology changes, and so (formally) the controller has available the index of the topology at the current time; in this case, (switching) gain parameters that depend on the index of the current topology can be used. The second is that current network topology is unknown to the controller, and so a single set of gain parameters is used.

Here, we develop conditions under which the SAN-VT achieves consensus, for both information paradigms. As in our earlier development, we separately consider the case where the stable manifold is only the origin (i.e., all agents' states converge to the origin) and the case of a more general consensus manifold.

Let us consider the paradigm that the current topology is known to the controller, and present two conditions under which consensus can be achieved. Here is the first:

Theorem 2.3. *Consider an SAN-VT, and assume that the controller has available the index of the current topology at each instant. A linear decentralized controller that switches with the network topology can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:*

(1) Assumption 1: At least one of the possible topologies G_i, $i = 1, \ldots, z$ has a nested sequence of N principal minors that all have full rank.

(2) Assumption 2: Time epochs during which no topology satisfies the premise of Assumption 1 *are upper bounded in duration (say by T_1), while time epochs during which any particular possible topology G_i that satisfies Assumption* 1 *is in force are lower bounded by T_2.*

To prove Theorem 2.3, we first find it convenient to develop the following lemma regarding controller design during a time interval when a particular topology (that is amenable to control) is in force.

Lemma 2.2. *Consider a particular topology G_i of an SAN-VT such that Assumption* 1 *of Theorem 2.3 is in force, i.e., G_i has a sequence of n nested principal minors all of full rank. Say that there is an epoch $\mathcal{T} = [t_A, t_B)$ of duration greater than T_2 such that $G(t) = G_i$ for all $t \in \mathcal{T}$. Then, for any $\gamma > 0$, a controller can be designed for the SAN-VT so that $||x(t)|| \leq \frac{1}{\gamma}||x(t_A)||$ for all $t \in [t_A + T_2, t_B)$. Furthermore, for each such design, there exists γ_1 and $\lambda_1 > 0$ such that $||x(t)|| \leq \gamma_1 e^{-\lambda_1(t-t_A)}||x(t_A)||$ for all $t \in \mathcal{T}$.*

Let us first prove the lemma:

Proof of Lemma 2.2. Consider application of a high-gain stabilizing controller for consensus, as developed in Theorem 2.1, during the epoch $[t_A, t_B)$. Notice that, since G_i has a sequence of leading principal minors all of full rank, we will design the controller based on the first assumption in Theorem 2.1. For any such controller, classical results on high-gain state feedback control clarify that, for any sufficiently high gain, $||x(t)||$ can be made less than $\frac{1}{\gamma}||x(t_A)||$ for any γ and after any fixed interval of time (while the model remains in force). Thus, we immediately recover that a controller can be designed to achieve $||x(t)|| \leq \frac{1}{\gamma}||x(t_A)||$ for all $t \in [t_A + T_2, t_B)$. Whatever asymptotically-stabilizing high gain controller is used, the state $x(t)$ is bounded in the interim and the state approaches the origin exponentially (from properties of linear systems), and so the theorem is proved. $\square$

We notice that reduction of the state's norm to an arbitrary level within an interval is possible for any stabilizing controller developed through Theorem 2.1, and is achieved for any sufficiently high gain.

Let us now apply the lemma to prove Theorem 2.3.

Proof of Theorem 2.3. Let us label the sequence of switching times for the observation topology as $t_0, t_1, \ldots$. We consider applying a feedback control of the following form: during the intervals $[t_i, t_{i+1}]$ such that the corresponding topology matrix G_j satisfies the sequential full rank condition (which we call the "good" intervals), we apply a stabilizing linear high-gain controller as per Theorem 2.1. During the remaining intervals (which we call "bad" intervals), we set the feedback control to nil. If the gains during the good intervals are chosen sufficiently large, we claim that asymptotic stability and hence consensus is achieved.

To formalize this, let us first consider $||x(t)||$ at the end of each good interval; for convenience, we label these times as $\widehat{t}_1, \widehat{t}_2, \ldots$, and also label the initial time as $\widehat{t}_0 = t_0$. We will bound $||x(\widehat{t}_{i+1})||$ with respect to $||x(\widehat{t}_i)||$, for $i = 0, 1, 2, \ldots$. To do so, we note that, during the epoch $[\widehat{t}_i, \widehat{t}_{i+1})$ the concluding good interval of interest (which has duration of at least T_2) may be preceded by several bad intervals with total duration of at most T_1. Using exponential bounds on the transition matrix norm during each bad interval and noting the bound on the total duration, we immediately can bound $||x(t)||$ before the beginning of the good interval (say t_i^*) as follows: $||x(t)|| \leq \mu ||x(\widehat{t}_i)||$ for $\widehat{t}_i \leq t \leq t_i^*$, for some $\mu > 0$. Next, from Lemma 2.2, we see that the high-gain controller during the concluding good interval can be selected so that $||x(\widehat{t}_{i+1})|| \leq \frac{1}{\gamma}||x(t_i^*)||$ for any $\gamma > 0$. Choosing the controller to achieve $\gamma = 2\mu$, we immediately recover that $||x(\widehat{t}_{i+1})|| \leq \frac{1}{2}||x(\widehat{t}_i)||$. Thus, we see that $||x(\widehat{t}_i)|| \leq (\frac{1}{2})^i ||x(t_0)||$.

Now let us consider $||x(t)||$ for $t \in [\widehat{t}_i, \widehat{t}_{i+1})$. Noting the bound on the state during the bad intervals and noting the exponential bound during the good intervals (from Lemma 2.2), we recover that $||x(t)|| \leq \mu\gamma_1 ||x(\widehat{t}_i)||$ for some fixed $\gamma_1 > 0$ (which for convenience we can take to be the largest among those given by Lemma 2.2 for the topologies satisfying Assumption 1), for $\widehat{t}_i \leq t < \widehat{t}_{i+1}$. Thus, we automatically find that $||x(t)|| \leq (\frac{1}{2})^i \mu\gamma_1 ||x(t_0)||$ for $\widehat{t}_i \leq t < \widehat{t}_{i+1}$.

Now consider two cases. The first case is that there is an infinite sequence of topologies, in which case we obtain asymptotic stability directly from the expression $||x(t)|| \leq (\frac{1}{2})^i \mu\gamma_1 ||x(t_0)||$. Alternately, if a (good) interval persists beyond a particular time $\tilde{t}$, we can directly invoke the exponentially-decaying bound on the response upon stabilizing control together with boundedness in the earlier time period to verify asymptotic stability.[1] □

[1]Our proof here is for asymptotic stability rather than uniform asymptotic stability, as per the definition of consensus. However, uniform asymptotic stability can also be proved here with a little more effort, by exploiting the exponential decay of $||x(t)||$ in long-duration "good" intervals.

We note that Theorem 2.3 holds whether or not the open-loop agent plant has closed left-half complex plane eigenvalues; in the case where it has open right-half complex plane or unstable eigenvalues, stabilization is still possible because the state can be driven to the consensus manifold at a faster rate during the good intervals than it escapes during the bad ones. In practice, various constraints may limit that capability to rapidly drive the state to the consensus manifold in short periods, and so the result is most apt for the (typical) case of open-loop poles in the closed left-half complex plane.

Theorem 2.3 is concerned with the case that the consensus manifold is only the origin. We also seek to verify consensus for the more general case, i.e., in the case of a general consensus manifold. Here is the result:

Theorem 2.4. *Consider an SAN-VT, and assume that the controller has available the index of the current topology at each instant. A linear decentralized controller that switches with the network topology can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:*

(1) Assumption 1: At least one of the possible topologies G_i, $i = 1, \dots, z$, has a nested sequence of $N - 1$ principal minors of full rank and further the vector $\mathbf{1}$ is in the null space of that topology matrix.

(2) Assumption 2: Time epochs during which no topology satisfies the premise of Assumption 1 are upper bounded in duration (say by T_1), while time epochs during which any particular possible topology G_i that satisfies Assumption 1 is in force are lower bounded by T_2.

Proof. Let us label the sequence of switching times for the observation topology as $t_0, t_1, \dots$. We consider applying a feedback control of the following form: during the intervals $[t_i, t_{i+1}]$ such that the corresponding topology matrix G_j satisfies the sequential full rank condition (which we call the "good" intervals), we apply a linear high-gain controller that achieves consensus as per Theorem 2.1.

To show that such a controller achieves consensus, we progress as follows. We consider each interval such that the network topology is fixed. In the proof of Theorem 2.1, we have shown that the global relative vector $q = [q_1^{\mathrm{T}}, \dots, q_{N-1}^{\mathrm{T}}]^{\mathrm{T}}$ has the following dynamics:

$$\dot{q} = \left(I_{N-1} \otimes A + \overline{DG} \otimes \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_m \end{bmatrix} \right) q, \tag{2.6}$$

where $\overline{DG}$ can be viewed as being formed by subtracting the last row of DG from all other rows and then removing the last row and column, and G is the particular G_i in force during that interval. We see that $\overline{DG}$ has $N-1$ eigenvalues, which are the non-zero eigenvalues of DG. We thus automatically see that the stabilization of the relative state dynamics for the time varying topology is identical to the stabilization of the state dynamics for time varying topology achieved in Theorem 2.3. That is, during the good intervals, we apply a linear high-gain controller as per Theorem 2.1. During the remaining intervals (which we call "bad" intervals), we set the feedback control to nil. Then, following the proof of Theorem 2.3, we see that the relative state is asymptotically stable and hence the consensus manifold where all the agents' states are identical is made asymptotically stable for the time varying topology. $\square$

Now, let us consider the paradigm that current network topology is unknown to the controller, and so a single set of gain parameters is used. To achieve stabilization/consensus in this case, we seek for a single set of gain parameters that causes the state during each constant-topology interval to either be exponentially decrescent at a fast rate or to be only slowly growing. We argue that such gains can be found if all the topology matrices G_i either fall in the broad class of *D-stable* matrices or are nil. This model for the observation topology is a broadly applicable one, for instance in the case that the network has one or more designed modes of operation and also may be subject to global network failures.

Theorem 2.5. *Consider an SAN-VT, and assume that the current network topology is unknown to the controller. A linear time-invariant decentralized controller can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:*

(1) Assumption 1: At least one of the possible topology matrices G_i, $i = 1, \ldots, z$, is D-stable, and all topology matrices are either D-stable or the zero matrix.

(2) Assumption 2: Time epochs during which the topology remains the zero matrix are upper bounded in duration (say by T_1), while time epochs during which any particular possible topology G_i that is D-stable is in force are lower bounded by T_2.

Proof. Let us label the sequence of switching times for the observation topology as $t_0, t_1, \ldots$. We consider applying a time-invariant high-gain feedback control, and then show than the SAN-VT can achieve consensus with this controller.

Let us consider the dynamics during the intervals $[t_i, t_{i+1}]$ such that the corresponding topology matrix G_j is D-stable (which we call the "good" intervals). For any particular good interval, all the eigenvalues of DG_i for an arbitrary positive definite matrix D, are in the open left-half complex plane. Therefore, we can choose a single diagonal matrix D for all good intervals so as to place the eigenvalues of the DG_i in the open left-half complex plane. From Lemma 2.2, we notice that reduction of the state's norm to an arbitrary level within an interval is possible for any stabilizing controller developed through Theorem 2.1, and is achieved for any sufficiently high gain. It is thus clear that an LTI controller can be designed that reduces that state by any desired fraction during each good interval. Let us consider applying this controller.

During the remaining "bad" intervals (for which the topology matrices are zero matrices), the closed-loop dynamics are entirely independent of the control used. Thus, we immediately recover that the norm of the state at the ends of these intervals (which are also upper-bounded in duration) can be bounded as a fixed multiple of the norm at the beginning. The remainder of the proof thus follows as in Theorem 2.3. □

Theorem 2.5 only develops the case that the consensus manifold is the origin. Next, we consider the case of a more general consensus manifold. We find that the result is related to the notion of D-semistablity [60].

Now, we are ready to present the result:

Theorem 2.6. *Consider an SAN-VT, and assume that the current network topology is unknown to the controller. A linear time-invariant decentralized controller can be designed for the SAN-VT to achieve consensus, if the following two assumptions hold:*

(1) Assumption 1: At least one of the possible topology matrices G_i, $i = 1, \ldots, z$ is D-semistable, and all topology matrices are either D-semistable or the zero matrix. Furthermore, there exists a single positive diagonal matrix D such that, for each G_i that is D-semistable, DG_i has no eigenvalue on the $j\omega$-axis other than the single eigenvalue at the origin, and the corresponding right eigenvector is $\mathbf{1}$.

(2) Assumption 2: Time epochs during which the topology remains the zero matrix are upper bounded in duration (say by T_1), while time epochs during which any particular possible topology G_i that is D-semistable is in force are lower bounded by T_2 .

Proof. The proof closely follows the proofs of Theorem 2.4 and Theorem 2.5, and so we omit the details. Notice here Assumption 1 means that there must exists a diagonal matrix D for all possible topologies G_i that satisfies D-semistable condition such that the condition is satisfied. $\square$

Let us make a couple remarks about our results:

- In general, it is hard to test whether a given matrix is D-stable or D-semistable. However, there are several important classes of matrices that are known to be D-stable or D-semistable, and are representative of many common network interactions: these include Laplacian, grounded Laplacian, diagonally-dominant, and symmetric positive-definite matrices, among others. We strongly refer the reader to [60] for details.

- Regarding Theorem 2.4 and Theorem 2.6, we note that the network dynamics on the consensus manifold may be quite complex, and may be persistently dependent on the particular sequence of the underlying network topologies. We leave it to future work to pursue design of the trajectory on the consensus manifold in this case.

- We note that Theorem 2.5 and Theorem 2.6 encompass the case without network failure, i.e. the case that none of the topologies are zero matrices. In this case, of course the upper bound on the duration of time epochs such that the network topology matrix is a zero matrix may be ignored.

- We notice that our results (Theorem 2.5 and Theorem 2.6) are connected with results in [6,45,46,54], however, we consider a broad network sensing model and local agent model. And our results differ from those, in the sense that we design the controller to achieve consensus rather than checking the stability of the existing algorithm.

Chapter 3

Consensus for Networks of Identical Agents

In Chapter 2, we have studied the consensus problem for network of general multiple-input linear time-invariant (LTI) identical agents and quite general time-invariant (fixed) network topologies, including the well-studied Laplacian topologies in the literature. The available information that we use to design the decentralized controller comes from the network. More specifically, each agent has access to a linear combination of multiple agents' states. However, in certain cases, such information is not available; instead, each agent has access to a linear combination of its own partial state output relative to that of neighboring agents. Li, Duan, Chen, and Huang [32] designed an observer-type protocol to solve consensus problem for such a case.

In this Chapter, we extend their result to the case that each agent has access to a linear combination of multiple agents' partial state output. We consider three problems, namely, the consensus problem, the model-reference consensus problem, and the regulation of consensus problem, for a network of identical linear time-invariant (LTI) multiple-input and multiple-output (MIMO) agents.

We propose a distributed LTI protocol to solve each problem for a broad class of time-invariant network topologies including not only Laplacian topologies, but a wide family of asymmetric topologies.

3.1 Background and Introduction

The consensus literature can be categorized according to various types of network observation models and internal models for each agent. Regarding the network observation model, the efforts on consensus have focused on Laplacian topologies. Along this line, most work assume that the relative state of the agent and its neighbors is available for each agent, see [44–46, 51, 52, 54]. A more realistic scenario, that is, the relative output rather than the relative state is available, has been considered in [32]. We refer the reader to [61, 62, 103] for a more general network model. Also, consensus for networks with time-varying topologies has been studied extensively; we refer the reader to Blondel's summary [6], which shows that general results in the time-varying case can be extracted from an early result of Tsitsiklis [82].

Regarding the internal model of each agent, the ongoing research is progressing toward increasing complexity. For a network of identical LTI agents, the consensus problem has been solved for first-order dynamics, [44–46, 54, 62], second-order dynamics [52, 61], integrator-chain dynamics [56] and general dynamics [32, 83–85, 103].

The consensus by itself does not impose any requirements on the consensus trajectory. In many applications, the goal is to design a protocol such that the states of each agent asymptotically approach an, a priori given, reference trajectory, generated by a reference model (virtual leader). This is called the model-reference consensus problem in the literature and has been considered in [56] for identical LTI agents with purely integrator dynamics and in [32] for identical LTI agents with general dynamics.

In this chapter, we extend the results given in [32] for the consensus problem and the model-reference consensus problem to general time-invariant network topologies. We also consider the regulation of the consensus problem, where the objective is to design a distributed protocol such that the controlled output of each agent tracks the same trajectory, generated by an arbitrary autonomous exosystem.

3.2 Preliminaries and Notations

Let us first give some notations which we use throughout the chapter. For a set of vectors $x_1, \ldots, x_n$, we denote by $\mathrm{col}(x_1, \ldots, x_n)$ the column vector obtained by stacking the elements of $x_1, \ldots, x_n$. $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$ represent the set of $n \times n$ real matrices and complex matrices, respectively. A^* denotes the conjugate transpose of the matrix $A \in \mathbb{C}^{n\times n}$. I_N denotes the identity matrix of dimension $N \times N$; we sometimes drop the subscript if the dimension is clear in the context. Similarly, 0_N represents the square matrix of dimension $N \times N$ with all entries equal to zero. $\mathbf{1}$ denotes the column vector with all entries equal to one. A matrix $A \in \mathbb{C}^{n\times n}$ is Hurwitz stable if all its eigenvalues have negative real parts. $\lambda(A)$ is an eigenvalue of the matrix $A \in \mathbb{C}^{n\times n}$. $\mathrm{diag}(a_1, \ldots, a_n)$ denotes the diagonal matrix with diagonal entries $a_1, \ldots, a_n$. In this chapter, we use some known results on stabilizing a matrix by scaling. A useful result has been given in Fisher and Fuller's paper [13], which was quoted as Lemma 2.1 in Chapter 2. Recently, generalizations of Lemma 2.1 were given in [57]. Also, note that the proofs of Lemma 2.1 given in [13] and the generalizations of Lemma 2.1 given in [57] are constructive.

3.3 Consensus Problem

In this section, we will consider the consensus problem for a network of multiple input multiple output linear time-invariant (LTI) agents under time-invariant (fixed) network topology.

3.3.1 Problem Formulation

We consider a network of N identical linear time-invariant (LTI) multiple-input multiple-output (MIMO) agents of the form

$$\dot{x}_i = Ax_i + Bu_i, \tag{3.1a}$$

$$y_i = Cx_i, \tag{3.1b}$$

for $i \in \{1, \ldots, N\}$, where $x_i \in \mathbb{R}^n$ is agent i's *local state*, $u_i \in \mathbb{R}^m$ is agent i's *local input*, and $y_i \in \mathbb{R}^q$ is agent i's output. Our goal is to achieve (state) consensus among the agents asymptotically; that is, to ensure that $\lim_{t\to\infty}(x_i(t) - x_j(t)) = 0$ for all $i, j \in \{1, \ldots, N\}$.

The available information comes from the network, which provides each agent with a linear combination of multiple agents' partial state outputs. In particular, agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j, \tag{3.2}$$

where $g_{ij} \in \mathbb{R}$ are scalars, referred as *observation weights*. The observation weight g_{ij} represents the influence (through sensing or networked communication) of agent j's output on agent i's observation. $g_{ij} \neq 0$ if and only if agent i can obtain information from agent j, and $g_{ij} = 0$ if and only if agent i cannot obtain information from agent j. We find it natural to assemble the observation weights into an $N \times N$ network topology matrix $G = [g_{ij}]$.

We also assume that the agents can exchange the information about their protocols' states using the network's communication infrastructure. Specifically, agent i is presumed to have access to the quantity

$$\zeta_i^c = \sum_{j=1}^{N} g_{ij}\eta_j, \tag{3.3}$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced by agent j as part of the consensus protocol. This variable will be specified as we proceed with the protocol design.

3.3.1.1 Assumptions

We make the following assumptions about the network topology and the identical agent model.

Assumption 3.1. *The matrix G has only one zero eigenvalue, with the right eigenvector* $\mathbf{1}$.

Remark 3.1. *If the matrix G is a Laplacian matrix corresponding to a digraph which contains a directed spanning tree, then Assumption 3.1 is satisfied. However, Assumption 3.1 includes other zero-row-sum matrices whose off-diagonal entries have arbitrary sign patterns.*

Assumption 3.2. *For the identical agent model* (3.1)*,*

1) the pair (A, B) is stabilizable; and

2) the pair (C, A) is detectable.

3.3.2 Protocol Design

In this section, we design a slightly different distributed observer-type protocol based on the proposed protocol in [32] for solving the consensus problem for general network topologies.

For each agent $i \in \{1, \ldots, N\}$, we construct the distributed observer-type protocol

$$\dot{\hat{x}}_i = (A + BF)\hat{x}_i + Kd_i(\zeta_i^c - \zeta_i), \tag{3.4a}$$

$$u_i = F\hat{x}_i, \tag{3.4b}$$

where $\hat{x}_i \in \mathbb{R}^n$ is the state of the protocol of agent i, which is an estimate of the deviation of the state x_i from the consensus trajectory. The matrices $F \in \mathbb{R}^{m\times n}$, $K \in \mathbb{R}^{n\times q}$, and $d_i \in R$ are parameters to be designed shortly.

In the protocol (3.4), we have made use of the quantity $\zeta_i^c = \sum_{j=1}^{N} g_{ij}\eta_j$, which is presumed to be available to agent i via the network, as described in Section 3.3.1. To complete the protocol design, we must define the values η_i, $i \in \{1,\dots,N\}$. We do this by setting $\eta_i = C\hat{x}_i$ for $i \in \{1,\dots,N\}$.

The following lemma gives a sufficient condition under which the consensus problem is solvable with the distributed observer-type protocol of the form (3.4). Moreover, it also gives the consensus trajectory – the dependence of the asymptotic dynamics on the initial conditions.

Lemma 3.1. *Consider a homogeneous network of N agents of the form* (3.1)*. Let Assumption 3.1 and condition 1) of Assumption 3.2 hold. Then the distributed observer-type protocol* (3.4)*, where F is chosen such that the matrix $A+BF$ is Hurwitz stable, and matrices K and $D = \mathrm{diag}(d_1,\dots,d_N)$ are chosen such that all the matrices $A+\lambda_i KC$ are Hurwitz stable for all λ_i, $i \in \{2,\dots,N\}$ which are nonzero eigenvalues of the matrix DG,* [1] *solves the consensus problem. Moreover,*

$$\lim_{t\to\infty}\left(\xi_i(t) - (\omega_0^T \otimes e^{\tilde{A}t})\xi(0)\right) = 0, \tag{3.5}$$

where $\tilde{A} = \begin{bmatrix} A & BF \\ 0 & A+BF \end{bmatrix}$, $\xi_i = \mathrm{col}(x_i,\hat{x}_i)$, $\xi = \mathrm{col}(\xi_1,\dots,\xi_N)$, *and ω_0^T is the normalized left eigenvector of the matrix DG associated with the zero eigenvalue.*

Proof. We first define a relative state vector as

$$\bar{x}_i = x_i - x_N, \quad \forall i \in \{1,\dots,N-1\}.$$

To prove the first part of the theorem, we need to prove the asymptotic stability of the manifold $\bar{x}_1 = \ldots = \bar{x}_{N-1} = 0$.

[1] We assume without loss of generality that $\lambda_1 = 0$

Since Assumption 3.1 is satisfied, we get $G\mathbf{1} = 0$, thus it is clear that $DG\mathbf{1} = 0$. Next, let us define $\bar{\hat{x}}_i = \hat{x}_i - \hat{x}_N$ for $i \in \{1, \ldots, N-1\}$. We then assemble all variables $\bar{x}_i$ and $\bar{\hat{x}}_i$ into single vectors $\bar{x} = \mathrm{col}(\bar{x}, \ldots, \bar{x}_{N-1})$, $\bar{\hat{x}} = \mathrm{col}(\bar{\hat{x}}_1, \ldots, \bar{\hat{x}}_{N-1})$, and define a single relative state vector $q = \mathrm{col}(\bar{x}, \bar{\hat{x}})$. With some algebra, we find that the dynamics of the relative state vector q is governed by

$$\dot{q} = \begin{bmatrix} I_{N-1} \otimes A & I_{N-1} \otimes (BF) \\ -(\overline{DG}) \otimes (KC) & I_{N-1} \otimes (A + BF) + (\overline{DG}) \otimes (KC) \end{bmatrix} q, \tag{3.6}$$

where $\overline{DG}$ is formed by removing the last row and column from $DG - d_N \mathbf{1} g_N^{\mathrm{T}}$, and g_N^{T} is the last row of the matrix G. We see that $\overline{DG}$ has $N-1$ eigenvalues, which are the nonzero eigenvalues of DG.

Consider a state transformation, $\bar{q} = Tq$, where

$$T = \begin{bmatrix} I_{(N-1)n} & -I_{(N-1)n} \\ 0_{(N-1)n} & I_{(N-1)n} \end{bmatrix}.$$

With just a little bit algebra, we find that the dynamics of $\bar{q}$ satisfy the following equation

$$\dot{\bar{q}} = \begin{bmatrix} I_{N-1} \otimes A + (\overline{DG}) \otimes (KC) & 0 \\ -(\overline{DG}) \otimes (KC) & I_{N-1} \otimes (A + BF) \end{bmatrix} \bar{q}. \tag{3.7}$$

Since the system matrix of the closed-loop dynamics (3.7) is a block lower-triangular matrix, its eigenvalues are the union of the eigenvalues of the matrices $I_{N-1} \otimes A + (\overline{DG}) \otimes (KC)$ and $I_{N-1} \otimes (A + BF)$. It is clear that the eigenvalues of the matrix $I_{N-1} \otimes (A + BF)$ are the eigenvalues of the matrix $A + BF$ repeated $N-1$ times, which are in the open left-half complex plane due to the choice of F. With some algebra, we can show that the eigenvalues of the matrix$I_{N-1} \otimes A + (\overline{DG}) \otimes (KC)$ are the union of the eigenvalues of $A + \lambda_i KC$ for all the eigenvalues λ_i of the matrix $\overline{DG}$ (that is, all the nonzero eigenvalues of DG). It then follows that all the poles of the closed-loop system (3.6) are in the open left-half complex plane, thus, asymptotic stabilization of the closed-loop system (3.6) is achieved. Hence, consensus is achieved.

Next let us try to figure out the consensus trajectory. From (3.1), (3.2), and (3.4), we obtain that

$$\dot{\xi}_i = \tilde{A}\xi_i + \sum_{j=1}^{N} d_i g_{ij} \tilde{H}\xi_j,$$

where

$$\tilde{A} = \begin{bmatrix} A & BF \\ 0 & A+BF \end{bmatrix}, \quad \text{and} \quad \tilde{H} = \begin{bmatrix} 0 & 0 \\ -KC & KC \end{bmatrix}.$$

It is easy to see that ξ satisfy the following dynamics:

$$\dot{\xi} = (I_N \otimes \tilde{A} + (DG) \otimes \tilde{H})\xi.$$

Using Jordan canonical representation, DG can be written as

$$\tilde{J} = V^{-1}(DG)V = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix},$$

where J are the Jordan blocks associated with the nonzero eigenvalues of DG. Note that

$$V = \begin{bmatrix} \mathbf{1} & Y \end{bmatrix}, \quad V^{-1} = \begin{bmatrix} \omega_0^{\mathrm{T}} \\ W \end{bmatrix},$$

where ω_0^{T} is the normalized left eigenvector of DG associated with zero eigenvalue, that is, $\omega_0^{\mathrm{T}} DG = 0$ and $\omega_0^{\mathrm{T}} \mathbf{1} = 1$, $Y \in \mathbb{R}^{N\times(N-1)}$ is a matrix whose columns are right eigenvectors and generalized right eigenvectors associated with the nonzero eigenvalues of DG, and $W \in \mathbb{R}^{(N-1)\times N}$ is a matrix whose rows are left eigenvectors and generalized left eigenvectors associated with the nonzero eigenvalues of DG, With some algebra, we get

$$\begin{aligned} \xi(t) &= e^{(I_N \otimes \tilde{A} + (DG)\otimes \tilde{H})t}\xi(0) \\ &= (V \otimes I_{2n}) \begin{bmatrix} e^{\tilde{A}t} & 0 \\ 0 & e^{(I_{N-1}\otimes \tilde{A} + J \otimes \tilde{H})t} \end{bmatrix} (V^{-1} \otimes I_{2n})\xi(0) \\ &= \Big((\mathbf{1} \otimes I_{2n}) e^{\tilde{A}t}(\omega_0^{\mathrm{T}} \otimes I_{2n}) + (Y \otimes I_{2n}) e^{(I_{N-1}\otimes \tilde{A} + J\otimes \tilde{H})t}(W \otimes I_{2n})\Big)\, \xi(0). \end{aligned} \tag{3.8}$$

Notice that the eigenvalues of $I_{N-1} \otimes \tilde{A} + J \otimes \tilde{H}$ are the union of the eigenvalues of $\tilde{A} + \lambda_i \tilde{H}$ for all the eigenvalues λ_i of the matrix J (that is, all the nonzero eigenvalues of DG). Also note that

$$\tilde{A} + \lambda_i \tilde{H} = \begin{bmatrix} A & BF \\ -\lambda_i KC & A + BF + \lambda_i KC \end{bmatrix}.$$

We also note that

$$\begin{bmatrix} I_n & -I_n \\ 0 & I_n \end{bmatrix} \begin{bmatrix} A & BF \\ -\lambda_i KC & A + BF + \lambda_i KC \end{bmatrix} \begin{bmatrix} I_n & I_n \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} A + \lambda_i KC & 0 \\ -\lambda_i KC & A + BF \end{bmatrix}.$$

It follows that the matrix $\tilde{A} + \lambda_i \tilde{H}$ is similar to the matrix

$$\begin{bmatrix} A + \lambda_i KC & 0 \\ -\lambda_i KC & A + BF \end{bmatrix}.$$

Therefore, it is easy to see that all the eigenvalues of the matrix $\tilde{A} + \lambda_i \tilde{H}$ are in the open left-half complex plane, which implies that all the eigenvalues of $I_{N-1} \otimes \tilde{A} + J \otimes \tilde{H}$ are in the open left-half complex plane, which implies that

$$\lim_{t \to \infty} e^{(I_{N-1}\tilde{A} + J \otimes \tilde{H})t} = 0.$$

Therefore, from (3.8), we obtain that

$$\lim_{t \to \infty} \left(\xi(t) - ((\mathbf{1}\omega_0^{\mathrm{T}}) \otimes e^{\tilde{A}t})\xi(0) \right) = 0.$$

This implies that for all $i \in \{1, \ldots, N\}$,

$$\lim_{t \to \infty} \left(\xi_i(t) - (\omega_0^{\mathrm{T}} \otimes e^{\tilde{A}t})\xi(0) \right) = 0.$$

Also notice that given the upper block-triangular structure of $\tilde{A}$ and asymptotic stability of $A+BF$, $\lim_{t \to \infty} \hat{x}_i(t) = 0$. $\square$

Next, we show that how to choose the matrices K and D such that all the matrices $A+\lambda_i KC$, for $i \in \{2,\dots,N\}$, are Hurwitz stable. In order to present our result, let us first recall the following lemma, which is Proposition 1 of [32] and then we give an alternative proof.

Lemma 3.2. *Given the agent dynamics* (3.1)*, there exists a matrix K such that $A+(x+yj)KC$ is Hurwitz stable for all $x \in [1,\infty)$ and $y \in (-\infty,\infty)$, if and only if the pair (C,A) is detectable.*

Proof. The necessity is trivial by setting $x = 1$ and $y = 0$.

Now, let us show the sufficiency. Since the pair (C,A) is detectable, we know that the continuous-time algebraic Riccati equation (CARE) defined as

$$AP + PA^{\mathrm{T}} - PC^{\mathrm{T}}CP + I_n = 0 \tag{3.9}$$

has a unique solution $P = P^{\mathrm{T}} > 0$.

Now, choose $K = -PC^{\mathrm{T}}$, we then get that

$$\begin{aligned}
[A+(x+yj)KC]\,P + P\,[A+(x+yj)KC]^* &= AP + PA^{\mathrm{T}} - 2xPC^{\mathrm{T}}CP \\
&= AP + PA^{\mathrm{T}} - PC^{\mathrm{T}}CP + (1-2x)PC^{\mathrm{T}}CP < 0,
\end{aligned}$$

where the last inequality follows from (3.9), $PC^{\mathrm{T}}CP \geq 0$, and $x \geq 1$.

Thus, $A+(x+yj)KC$, where $K = -PC^{\mathrm{T}}$ and the matrix $P > 0$ is the solution of (3.9), is Hurwitz stable. □

Combining the results of Lemma 2.1, Lemma 3.1, and Lemma 3.2, we see that the consensus problem is solvable by the distributed protocol (3.4) if the following Assumption 3.3 is satisfied.

Assumption 3.3. *There exists a permutation matrix P_1 such that all the leading principal minors of $P_1 G P_1^{-1}$ of size less than N are nonzero.*

The following theorem formally states such a result and moreover gives design procedure on how to choose the parameters of the distributed protocol (3.4).

Theorem 3.1. *Consider a homogeneous network of N agents of the form* (3.1)*. Let Assumptions 3.1, 3.2 and 3.3 hold. Then the distributed observer-type protocol* (3.4)*, where F is chosen such that the matrix $A + BF$ is Hurwitz stable, a diagonal matrix $D = \operatorname{diag}(d_1, \dots, d_N)$ is chosen such that all the nonzero eigenvalues of DG have the real parts greater than or equal to* 1*, $K = -PC^T$, and $P = P^T > 0$ is the unique solution of the continuous-time algebraic Riccati equation (CARE)* (3.9)*, solves the consensus problem.*

Proof. Since Assumption 3.3 is satisfied, following the constructive proof of Lemma 2.1 given in [13], we design a diagonal matrix D such that DG has all its eigenvalue in the closed right-half complex plane, except only one eigenvalue at the origin. We can further place all the nonzero eigenvalues of DG with real parts greater than or equal to 1 while the single zero eigenvalue is unchanged by positively scaling the matrix D. Since the pair (C, A) is detectable, and $K = -PC^{\mathrm{T}}$, where the matrix $P = P^{\mathrm{T}} > 0$ is the unique solution of (3.9), following Lemma 3.2, we see that $A+(x+yj)KC$ is Hurwitz stable for all $x \in [1, \infty)$ and $y \in (-\infty, \infty)$. Finally, we choose F such that $A + BF$ is Hurwitz, the rest of the proof follows from Lemma 3.1. $\square$

Let us make several comments regarding to Theorem 3.1:

- As a special case, it is easy to see that the condition given in Theorem 3.1 is satisfied when G is a Laplacian matrix associated with a directed graph which contains a directed spanning tree. Hence, the above theorem recovers the result in [32].
- The condition given in Theorem 3.1 is satisfied for a broad class of matrices known as D-*semistable matrices* with additional property that the matrix has no eigenvalues on the $j\omega$-axis

other than the single eigenvalue at the origin with the corresponding right eigenvector $\mathbf{1}$. For the definition of D-semistability, please see [19, 61]. It is clear that D-semistable matrices includes a wide family of matrices with more general entry sign pattern than the Laplacian matrix, and hence admits consensus control for a wider set of observation capabilities.

- Furthermore, the condition can be weakened if we use the generalizations of Fisher and Fuller's Theorem given in [57].

3.4 Model-reference Consensus Problem

In Section 3.3, we considered the state consensus problem. State consensus by itself does not impose any requirements on the consensus trajectory. In other words, we do not impose any conditions on the asymptotic behavior of the state of an individual agent as long as the asymptotic behavior is the same for all agents.

However, in many applications, the scenario is that given a reference trajectory, we are trying to design a distributed protocol such that all agents states asymptotically approach such a prior given reference trajectory. This is called the model-reference consensus problem in the literature. Such a problem was first solved in [10] for a network of identical agents with a purely integrator dynamics under Laplacian topologies. Recently, Li and co-worker [32] solved such a problem for a network of general identical agents internal dynamics under Laplacian topologies. In this section, we extend the results in [32] to general time-invariant (fixed) topologies.

3.4.1 Problem Formulation

For the model-reference consensus problem, our goal is to make the state of each agent asymptotically approach the reference state of the reference model (virtual leader), which has the same dynamics as each individual agent, given by

$$\dot{x}_r = Ax_r + Bu_r, \tag{3.10a}$$

$$y_r = Cx_r, \tag{3.10b}$$

where $x_r \in \mathbb{R}^n$ is a reference trajectory, which all the x_i need to approach asymptotically, $u_r \in \mathbb{R}^m$ is the input variable and $y_r \in \mathbb{R}^q$ is the output variable of the reference model, respectively. That is, we want to ensure that $\lim_{t\to\infty}(x_i(t) - x_r(t)) = 0$ for each $i \in \{1, \ldots, N\}$. Equivalently, we wish to regulate the synchronization error variable

$$x_{e,i} = x_i - x_r$$

to zero.

In order to achieve our goal, in addition to ζ_i given by (3.2) and ζ_i^c given by (3.3) provided by the network, some information must be available to the agents about their outputs relative to that of the reference trajectory. Specifically, let $\mathcal{I} \subset \{1, \ldots, N\}$ be a set of indices corresponding to a subset of agents in the network which observe its output relative to that of the reference-model. That is, we assume that each agent $i \in \{1, \ldots, N\}$ has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \quad \text{where } \iota_i = \begin{cases} 1, & i \in \mathcal{I}, \\ 0, & i \notin \mathcal{I}. \end{cases} \tag{3.11}$$

3.4.2 Protocol Design

In this section, we design a slight different distributed observer-type protocol based on the proposed protocol in [32] for solving the model-reference consensus problem for general time-invariant (fixed) network topologies.

Consider a distributed observer-type protocol given by

$$\dot{\hat{x}}_{e,i} = (A + BF)\hat{x}_{e,i} + Kd_i(\zeta_i^c - \zeta_i) + Kd_i(\iota_i C\hat{x}_{e,i} - \psi_i), \tag{3.12a}$$

$$u_i = F\hat{x}_{e,i} + u_r, \tag{3.12b}$$

where $\hat{x}_{e,i} \in \mathbb{R}^n$ is the state of the protocol of the agent i, which is an estimate for $x_{e,i} = x_i - x_r$. The matrices $F \in \mathbb{R}^{m\times n}$, $K \in \mathbb{R}^{n\times q}$, and $d_i \in R$ are parameters to be designed shortly.

In the protocol (3.12), we have made use of the quantity $\zeta_i^c = \sum_{j=1}^{N} g_{ij}\eta_j$, where $\eta_j = C\hat{x}_{e,j}$. We also notice in our protocol design, each agent has to have the access to the input u_r of the reference model (3.10).

Let us make the following assumption regarding the network topology.

Assumption 3.4. *There exists a permutation matrix P_1 such that all the leading principal minors of $P_1(G + \mathrm{diag}(\iota_1, \ldots, \iota_N))P_1^{-1}$ are nonzero.*

The following theorem states that the model-reference consensus problem is solvable by the distributed observer-type protocol. of the form (3.12) if Assumption 3.4 is satisfied.

Theorem 3.2. *Consider a homogeneous network of N agents of the form* (3.1) *and the reference model given by* (3.10). *Let Assumptions 3.1, 3.2 and 3.4 hold. Then the distributed observer-type protocol* (3.12)*, where F is chosen such that the matrix $A + BF$ is Hurwitz stable, $D = \mathrm{diag}(d_1, \ldots, d_N)$ is a diagonal matrix chosen such that all the nonzero eigenvalues of $D(G+\mathrm{diag}(\iota_1, \ldots, \iota_N))$ have the real parts greater than or equal to 1, $K = -PC^T$, and $P = P^T > 0$ is the unique solution of the continuous-time algebraic Riccati equation (CARE)* (3.9)*, solves the model-reference consensus problem.*

Proof. Let us first assemble all variables $x_{e,i}$ and $\hat{x}_{e,i}$ for $i = 1, \ldots, N$ into single vectors

$$x_e = \begin{bmatrix} x_{e,1} \\ \vdots \\ x_{e,N} \end{bmatrix}, \quad \hat{x}_e = \begin{bmatrix} \hat{x}_{e,1} \\ \vdots \\ \hat{x}_{e,N-1} \end{bmatrix},$$

and define a single relative state vector

$$q = \begin{bmatrix} x_e \\ \hat{x}_e \end{bmatrix}.$$

Since $G\mathbf{1} = 0$, we obtain that

$$\zeta_i = \sum_{j=1}^{N} g_{ij} C x_j = \sum_{j=1}^{N} g_{ij} C(x_{e,j} + x_r) = \sum_{j=1}^{N} g_{ij} C x_{e,j}. \tag{3.13}$$

Using (3.13) and properties of the Kronecker product, we find the dynamics of the state vector q[1] is governed by

$$\dot{q} = \begin{bmatrix} I_N \otimes A & I_N \otimes BF \\ -Q & I_N \otimes (A + BF) + Q \end{bmatrix} q, \tag{3.14}$$

where $Q = (D(G + \mathrm{diag}(\iota_1, \ldots, \iota_N))) \otimes KC$. Now, let us consider the state transformation $\bar{q} = Tq$, where

$$T = \begin{bmatrix} I_{Nn} & -I_{Nn} \\ 0_{Nn} & I_{Nn} \end{bmatrix}.$$

With just a little bit algebra, we find that the dynamics of $\bar{q}$ satisfy the following equation:

$$\dot{\bar{q}} = \begin{bmatrix} I_N \otimes A + Q & 0 \\ -Q & I_N \otimes (A + BF) \end{bmatrix} \bar{q}. \tag{3.15}$$

Since the system matrix of the above system (3.15) is a block lower-triangular matrix, its eigenvalues are the union of the eigenvalues of $I_N \otimes A + Q$ and $I_N \otimes (A + BF)$. It is easy to see that the eigenvalues of $I_N \otimes (A+BF)$ are the eigenvalues of $A+BF$, which are in the open left-half complex plane, repeated N times. Similar to the proof of Lemma 3.1, we can show that the eigenvalues of $I_N \otimes A + Q$ are the union of the eigenvalues of $A + \lambda_i KC$ for all the eigenvalues λ_i of the matrix $D(G + \mathrm{diag}(\iota_1, \ldots, \iota_N))$.

Finally, since Assumption 3.4 is satisfied, following the constructive proof of Lemma 2.1 given in [13], we design a diagonal matrix D such that $D\,(G + \mathrm{diag}(\iota_1, \ldots, \iota_N))$ has all its eigenvalue in the open right-half complex plane. We can further place all the eigenvalues of $D\,(G + \mathrm{diag}(\iota_1, \ldots, \iota_N))$ with real parts greater than or equal to 1 by positively scaling the matrix D. It then follows from the similar analysis as the proof of Theorem 3.1 that all the eigenvalues of $A + \lambda_i KC$, where $K = -PC^{\mathrm{T}}$ and the matrix $P = P^{\mathrm{T}} > 0$ is the unique solution of (3.9) are in the open left-half complex plane. Therefore, all the eigenvalues of the system (3.14) are in the open left-half complex plane. Hence, the model-reference consensus problem is solved. □

Remark 3.2. *Note that if the network topology matrix G is a Laplacian matrix associated with a digraph which contains a directed spanning tree with a root agent K, and $\iota_K = 1$ for the root agent agent K, then Assumption 3.4 is satisfied. Hence, Theorem 3.2 recovers the result in [32].*

3.5 Regulation of Consensus Problem

The main disadvantage of the result in Section 3.4 is that the input to the reference model needs to be known by each agent. Moreover, the reference model has to have the same dynamics as all the agents. Both of these conditions are quite restrictive. In this section, we consider the regulation of consensus problem, where the objective is to design a distributed protocol to make the output of each agent asymptotically track the same polynomial, sinusoidal signal, or the combination of these generated by an arbitrary autonomous.

3.5.1 Problem Formulation

Consider a network of N identical linear time-invariant (LTI) multiple-input multiple-output (MIMO) agents of the form

$$\dot{x}_i = Ax_i + Bu_i, \tag{3.16a}$$

$$y_i = Cx_i, \tag{3.16b}$$

$$z_i = C_z x_i, \tag{3.16c}$$

for $i \in \{1, \ldots, N\}$, where the new term $z_i \in \mathbb{R}^p$ compared to (3.1) is agent i's *local controlled output*.

Our goal is to make the controlled output z_i of each agent asymptotically track the same trajectory, generated by an arbitrary autonomous exosystem given by

$$\dot{\omega} = S\omega, \quad \omega(0) = \omega_0, \tag{3.17a}$$

$$z_r = C_r\omega, \tag{3.17b}$$

where $\omega \in \mathbb{R}^r$ is the state of the exosystem, and $z_r \in \mathbb{R}^p$ is the output of the exosystem, which is the consensus trajectory. That is, we want to ensure that $\lim_{t\to\infty}(z_i(t) - z_r(t)) = 0$ for each $i \in \{1, \ldots, N\}$,

Note that the regulation of consensus problem is different from classical consensus problem in that it does not strive to regulate the state of each agent but only the output of each agent. In that sense, the requirements are weaker than classical consensus problems. On the other hand, the conditions are stronger in the sense that instead of only requiring the outputs of each agent having the same asymptotic behavior, we actually impose the asymptotic behavior of each agent's output through the exosystem.

Assumption 3.5. *For the exosystem* (3.17)*, we make the classical assumptions that S is anti-Hurwitz stable, and the pair (C_r, S) is detectable.*

Similarly to the model-reference consensus problem considered in Section 3.4, in order to solve the regulation of consensus problem, in addition to ζ_i given by (3.2) and ζ_i^c given by (3.3) provided by the network, it is clear that a non-empty subset of agents should observe its output relative to the output variable z_r of the exosystem (3.17) in order for the network of agents to follow the reference trajectory. Specifically, let $\mathcal{I} \subset \{1, \ldots, N\}$ denotes such a subset. Then, agent $i \in \{1, \ldots, N\}$ has access to the quantity

$$\psi_i = \iota_i(z_i - z_r), \quad \text{where} \iota_i = \begin{cases} 1, & i \in \mathcal{I}, \\ 0, & i \notin \mathcal{I}. \end{cases} \tag{3.18}$$

Let us first check whether it is even possible for the controlled output of one individual agent of the form (3.16) to track the output of the exosystem when the agent has access to both its own state and the state of the exosystem. The following lemma, recalled from [70, Theorem 2.3.1], gives a sufficient condition under which such a problem is solvable.

Lemma 3.3. *Consider one agent of the form* (3.16) *and the exosystem* (3.17)*. If the following equations with unknown $\Pi \in \mathbb{R}^{n\times r}$ and $\Gamma \in \mathbb{R}^{m\times r}$, commonly known as the* regulator equations*:*

$$\Pi S = A\Pi + B\Gamma, \tag{3.19a}$$

$$C_r = C_z\Pi, \tag{3.19b}$$

are solvable, then the state feedback controller

$$u = Fx_i + (\Gamma - F\Pi)\omega,$$

where F is a matrix chosen such that $A + BF$ is Hurwitz stable, ensures that

$$\lim_{t\to\infty} (z_i(t) - z_r(t)) = 0.$$

Moreover,

$$\Pi\omega(t) - x_i(t) \to 0, \quad \text{and} \quad \Gamma\omega(t) - u_i(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Lemma 3.3 implies that, whenever the regulator equations (3.19) are solvable, output consensus, where all the outputs z_i must converge to the output z_r, is equivalent to state consensus, where all the states x_i must converge to $\Pi\omega$. However, for cases where the regulator equations (3.19) are not solvable, output consensus does not require state consensus.

Remark 3.3. *We note that our problem – to design a protocol such that the controlled output z_i tracks the output z_r of the exosystem* (3.17) *– is different from the output regulation problem considered in [70] in that the internal stability of the closed-loop system when the ω dynamics is non-existence is not required in our problem. Therefore, Lemma 3.3 only gives only a sufficient condition under which our problem is solvable, while [70, Theorem 2.3.1] gives the necessary and sufficient conditions under which the regulation problem is solvable.*

The following lemma gives the conditions under which the regulator equations (3.19) are solvable.

Lemma 3.4. *In addition to Assumptions 3.2 and 3.5, if we assume that the triplet (C_z, A, B) is right-invertible, and has no invariant zeros in the closed right-half complex plan that coincide with the eigenvalues of S, then the regulator equations* (3.19) *are solvable.*

Proof. From [70, Corollary 2.5.1], the regulator equations are solvable if, for each λ that is an eigenvalue of S,

$$\text{rank} \begin{bmatrix} A - \lambda I & B \\ C_z & 0 \end{bmatrix} = n + p. \tag{3.20}$$

The matrix in (3.20) is the Rosenbrock system matrix of the triplet (C_z, A, B), which has normal rank $n + p$ due to right-invertibility (see [68, Property 3.6.1]). Since this triplet has no invariant zeros in the closed right-half complex plane that coincide with eigenvalues of S, it follows that the rank of the Rosenbrock system matrix is equal to the normal rank for each λ that is an eigenvalue of S. $\square$

We make the following additional assumption for the rest of Section 3.5.

Assumption 3.6. *The triplet (C_z, A, B) is right-invertible, and has no invariant zeros in the closed right-half complex plan that coincide with the eigenvalues of S.*

3.5.2 Protocol Design

Our protocol design for the regulation of consensus will achieve state consensus. However, as noted before, this is not necessary. On the other hand, we will show that solving the regulation of consensus problem including state consensus requires only weak additional conditions in addition to the solvability condition of the regulator equations (3.19) given by Assumption 3.6.

Consider a distributed observer-type protocol given by

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + Kd_i(\zeta_i^c - \zeta_i) + Kd_i(\iota_i C_r \hat{\omega}_i - \psi_i), \tag{3.21a}$$

$$\dot{\hat{\omega}}_i = S\hat{\omega}_i + F_1\hat{\omega}_i + Ld_i(\zeta_i^c - \zeta_i) + Ld_i(\iota_i C_r \hat{\omega}_i - \psi_i), \tag{3.21b}$$

$$u_i = F_2\hat{x}_i + \Gamma\hat{\omega}_i, \tag{3.21c}$$

where $\hat{x}_i \in \mathbb{R}^n$ and $\hat{\omega}_i \in \mathbb{R}^r$ are states of the protocol, which are the estimates for $x_i - x_r$ and $\omega_i - \omega$, and $F_1 \in \mathbb{R}^{r\times r}, F_2 \in \mathbb{R}^{m\times n}$, $K \in R^{n\times q}$, $L \in \mathbb{R}^{r\times q}$ and $d_i \in \mathbb{R}$ are designed parameters to be determined shortly.

Note that the fact that in addition to a differential equation for the state $\hat{x}_i$, we also need the differential equation for $\hat{\omega}_i$ actually follows directly from the internal model principle [70].

In the protocol (3.21), we have made use of the quantity $\zeta_i^c = \sum_{j=1}^N g_{ij}\eta_j$, where $\eta_j = C\hat{x}_j$.

The following theorem gives an implicit condition under which the regulation of consensus problem can be solved by protocol of the form (3.21).

Theorem 3.3. *Consider a homogeneous network of N agents of the form* (3.1) *and the exosystem given by* (3.17). *Let Assumptions 3.1, 3.2, 3.5, and 3.6 hold. If A and S have no common eigenvalues, and the pair $(C\Pi, S)$ is detectable, where Π is the solution of the regulator equations* (3.19), *then the distributed observer-type protocol* (3.21), *where F_1 and F_2 are chosen such that $S + F_1$ and $A + BF_2$ are both Hurwitz stable, matrices K, L, and $D = \operatorname{diag}(d_1, \ldots, d_N)$ are chosen such that*

$$I_N \otimes A_f + (DG) \otimes (K_f C_f) + (D \operatorname{diag}(\iota_1, \ldots, \iota_N)) \otimes (K_f C_{z,f}) \tag{3.22}$$

is Hurwitz stable, where

$$A_f = \begin{bmatrix} S & 0 \\ B\Gamma & A \end{bmatrix}, \quad C_f = \begin{bmatrix} 0 & C \end{bmatrix}, \quad C_{z,f} = \begin{bmatrix} 0 & C_z \end{bmatrix}, \quad \text{and} \quad K_f = \begin{bmatrix} L \\ K \end{bmatrix}, \tag{3.23}$$

solves the regulation of consensus problem.

Proof. Let us first expand the exosystem (3.17):

$$\dot{\omega} = S\omega, \tag{3.24a}$$

$$\dot{x}_r = Ax_r + B\Gamma\omega, \tag{3.24b}$$

$$z_r = C_z x_r, \tag{3.24c}$$

with $x_r(0) = \Pi\omega(0)$. It is then easy to verify that $x_r(t) = \Pi\omega(t)$ for all t and

$$z_r(t) = C_r\omega(t).$$

The reason behind this expansion is that the exosystem also contains a target for the state of the individual agents.

Next, for each $i \in \{1, \dots, N\}$, we expand its dynamics:

$$\dot{\omega}_i = S\omega_i + u_{i,1}, \tag{3.25a}$$

$$\dot{x}_i = Ax_i + B\Gamma\omega_i + Bu_{i,2}, \tag{3.25b}$$

$$y_i = Cx_i, \tag{3.25c}$$

$$z_i = C_z x_i, \tag{3.25d}$$

where we have used $u_i = \Gamma\omega_i + u_{i,2}$.

Note that the first state equation for ω_i is basically part of our consensus protocol, but it is useful to write this in the above manner since we now have a reference model which is identical to the individual agent's dynamics.

Define

$$x_{f,i} = \begin{bmatrix} \omega_i \\ x_i \end{bmatrix}, \quad x_{f,r} = \begin{bmatrix} \omega \\ x_r \end{bmatrix}, \quad u_{f,i} = \begin{bmatrix} u_{i,1} \\ u_{i,2} \end{bmatrix}, \quad \text{and} \quad u_{f,r} = 0,$$

we then obtain an expanded reference model

$$\dot{x}_{f,r} = A_f x_{f,r} + B_f u_{f,r}, \tag{3.26a}$$

$$y_r = C_f x_{f,r} \tag{3.26b}$$

$$z_r = C_{z,f} x_{f,r}, \tag{3.26c}$$

where

$$B_f = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix},$$

and N individual agents

$$\dot{x}_{f,i} = A_f x_{f,i} + B_f u_{f,i}, \tag{3.27a}$$

$$y_i = C_f x_{f,i}, \tag{3.27b}$$

$$z_i = C_{z,f} x_{f,i}, \tag{3.27c}$$

where we have used in the above the following relationship

$$C_{z,f} x_{f,r} = C_z x_r = C_z \Pi \omega = C_r \omega = z_r.$$

Note that (C_f, A_f) is detectable since A and S have no common eigenvalues, and (C, A) and $(C\Pi, S)$ are detectable. This follows from the fact that

$$\begin{bmatrix} I & 0 \\ -\Pi & I \end{bmatrix} A_f \begin{bmatrix} I & 0 \\ \Pi & I \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & A \end{bmatrix}, \quad \text{and} \quad C_f \begin{bmatrix} I & 0 \\ \Pi & I \end{bmatrix} = \begin{bmatrix} C\Pi & C \end{bmatrix}.$$

It is also clear that (A_f, B_f) is stabilizable since (A, B) is stabilizable.

With just a bit algebra, the distributed observer-type protocol (3.21) can be rewritten as

$$\dot{\hat{x}}_{e,i} = (A_f + B_f F_f)\hat{x}_{e,i} + K_f d_i(\zeta_i^c - \zeta_i) + K_f d_i(\iota_i C_{z,f} \hat{x}_{e,i} - \psi_i), \tag{3.28a}$$

$$u_{f,i} = F_f x_{e,i}, \tag{3.28b}$$

where $\hat{x}_{e,i} = \text{col}(\omega_i, x_i)$ is the state of the protocol, which is an estimate for

$$x_{e,i} = x_{f,i} - x_{f,r},$$

and

$$F_f = \begin{bmatrix} F_1 & 0 \\ 0 & F_2 \end{bmatrix}.$$

Define $x_e = \operatorname{col}(x_{e,1}, \ldots, x_{e,N})$ and $\hat{x}_e = \operatorname{col}(\hat{x}_{e,1}, \ldots, \hat{x}_{e,N})$. Using the fact that $G\mathbf{1} = 0$ and some other algebra, we obtain that

$$\dot{x}_e - \dot{\hat{x}}_e = \left(I_N \otimes A_f + (DG) \otimes (K_f C_f) + (D \operatorname{diag}(\iota_1, \ldots, \iota_N)) \otimes (K_f C_{z,f})\right)(x_e - \hat{x}_e)\,.$$

Therefore, if the matrix (3.22) is Hurwitz stable, then $x_e(t) - \hat{x}_e(t) \to 0$ exponentially as $t \to \infty$. We note that

$$\dot{x}_e = (I_N \otimes (A_f + B_f F_f))\, x_e - (I_N \otimes (B_f F_f))\,(x_e - \hat{x}_e)\,.$$

It follows that $x_e(t) \to 0$ as $t \to \infty$ since $A_f + B_f F_f$ is Hurwitz stable. This yields that

$$\lim_{t\to\infty} (z_i(t) - z_r(t)) = \lim_{t\to\infty} C_{z,f}\, x_{e,i}(t) = 0,$$

which implies that the output of each individual agent asymptotically tracks the output of the exosystem. Hence, the regulation of consensus problem is solved. Note that the above also implies that the state consensus is achieved in that $\lim_{t\to\infty}(x_i(t) - x_r(t)) = 0$ for all $i \in \{1, \ldots, N\}$. $\square$

In general it is quite hard to design matrices K, L, and $D = \operatorname{diag}(d_1, \ldots, d_N)$ such that the matrix (3.22) is Hurwitz stable. However, there is one case where this can be verified quite easily and we have a constructive proof. Moreover in this case, the detectability of the pair $(C\Pi, S)$ is equivalent to the detectability of the pair (C_r, S).

Theorem 3.4. *Consider a homogeneous network of N agents of the form* (3.1) *and the exosystem given by* (3.17). *Let Assumptions 3.1, 3.2, 3.4, 3.5, and 3.6 hold. For the case $C_z = C$, the distributed observer-type protocol* (3.21)*, where F_1 and F_2 are chosen such that $S + F_1$ and $A + BF_2$ are both Hurwitz stable, $D = \operatorname{diag}(d_1, \ldots, d_N)$ is a diagonal matrix chosen such that all the eigenvalues of $D(G + \operatorname{diag}(\iota_1, \ldots, \iota_N))$ have the real parts greater than or equal to* 1*, $K_f = -P_f C_f^T$, and $P_f = P_f^T > 0$ is the unique solution of the continuous-time algebraic Riccati equations (CARE)* *defined as:*

$$A_f P_f + P_f A_f^T - P_f C_f^T C_f P + I_{n+r} = 0. \tag{3.29}$$

Proof. Since $C_z = C$, it is clear that $C_{z,f} = C_f$. We then find that (3.22) is equal to

$$I_N \otimes A_f + (D(G + \operatorname{diag}(\iota_1, \ldots, \iota_N)) \otimes (K_f C_f). \tag{3.30}$$

In order to solve the regulation of consensus problem, we need to show that the matrix (3.30) is Hurwitz stable.

Similar to the analysis in proof of Lemma 3.1, it is easy to show that the eigenvalues of the matrix (3.30) are the union of the eigenvalues $A_f + \lambda_i K_f C_f$, for all λ_i which are eigenvalues of $(D(G + \operatorname{diag}(\iota_1, \ldots, \iota_N))$.

Since Assumption 3.4 is satisfied, following the constructive proof of Lemma 2.1 given in [13], we design a diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_N)$ such that all the eigenvalues of $D(G + \operatorname{diag}(\iota_1, \ldots, \iota_N))$ have real parts greater than or equal to 1.

Also note that the pair (C_f, A_f) is detectable since A and S have no common eigenvalues and (C, A) and (C_r, S) are detectable.

Since $K = -P_f C_f^{\mathrm{T}}$, where $P_f = P_f^{\mathrm{T}} > 0$ is the unique is the unique solution of the continuous-time algebraic Riccati equations (CARE) (3.29), it follows from Lemma 3.2 that all the eigenvalues of the matrix (3.30) are in the open left-half complex plane. The rest of the proof follows from Theorem 3.3. □

Chapter 4

Output Synchronization for Heterogeneous Networks of Introspective Right-invertible Agents

Our focus so far (Chapter 2 and Chapter 3) has been on achieving consensus for networks of identical agents. In this chapter, we consider the case that the agents in the network have different internal dynamics.

4.1 Introduction

The synchronization problem in a network has received substantial attention in recent years (see [2, 44, 55, 100] and references therein). Active research is being conducted in this context and numerous results have been reported in the literature, to name a few see [32, 45, 46, 48, 51, 52, 54, 73, 83, 84].

Much of the attention has been devoted to achieving *state synchronization* in *homogeneous* networks (i.e., networks where the agent models are identical), where each agent has access to a linear combination of its own state relative to that of neighboring agents (e.g., [44–46, 52, 54, 83]). Roy, Saberi, and Herlugson [62], Tuna [83], and Yang, Roy, Wan, and Saberi [103] considered the state synchronization problem for more general network topologies. A more realistic scenario—that is, each agent receives a linear combination of its own partial-state output relative to that of neighboring agents—has been considered in [32, 49, 84, 85]. The results of [32] were expanded by [107] to more general network topologies. Many of the results on the synchronization problem are rooted in the seminal work of Wu and Chua [98, 99].

4.1.1 Heterogeneous Networks and Output Synchronization

Recent activities in the synchronization literature have been focused on achieving synchronization in *heterogeneous* networks (i.e., networks where the agent models are non-identical). This problem is challenging and only some partial results are available, see for instance [10,20,25,97,101].

In heterogeneous networks, the agents' states may have different dimensions. In this case, the state synchronization is not even properly defined, and it is more natural to aim for *output synchronization*—that is, asymptotic agreement on the agents' partial-state outputs. Chopra and Spong [10] studied the output synchronization problem for weakly minimum-phase nonlinear systems of relative degree one, using a pre-feedback to create a single-integrator system with decoupled zero dynamics. Kim, Shim, and Seo [25] considered the output synchronization problem for uncertain single-input single-output, minimum-phase linear systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output.

The designs mentioned in this section generally rely on some sort of self-knowledge that is separate from the information transmitted over the network. More specifically, the agents know their own state, their own output, or their own state/output relative to that of the reference trajectory. We shall refer to agents that possess this type of self-knowledge as *introspective* agents, to distinguish them from *non-introspective* agents – that is, agents that have no knowledge of their own state or output separate from what is received via the network. The output synchronization problem for a heterogeneous network of *non-introspective* agents have been considered in [17] and [113].

4.1.2 Organization of this Chapter

The remainder of this chapter is organized as follows. In this rest of Section 4.1, we introduce some notations and recall some results of algebraic graph theory. Section 4.2 presents the heterogeneous network considered in this chapter. In Section 4.3, we propose a decentralized controller to solve the output synchronization problem. The design is applied for solving the output formation problem in Section 4.4. The regulation of output synchronization problem is considered in Section 4.5. The results are illustrated by examples in Section 4.6.

4.1.3 Preliminaries and Notations

Given a matrix $A \in \mathbb{C}^{m\times n}$, A^* denotes its conjugate transpose, and $\lambda_i(A)$ is its i'th eigenvalue. $A \in \mathbb{C}^{n\times n}$ is said to be Hurwitz stable if all its eigenvalues are in the open left-half plane. $\otimes$ denotes the Kronecker product between two matrices of appropriate dimensions. Given a matrix $A \in \mathbb{C}^{m\times n}$ and a matrix $B \in \mathbb{C}^{p\times q}$ the Kronecker product $A \otimes B$ is defined as the $\mathbb{C}^{mp\times nq}$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

where a_{ij} denotes element (i,j) of A. I_n denotes the identity matrix of dimension n, similarly, 0_n denotes the square matrix of dimension n with all zero elements; we sometimes drop the subscript if the dimension is clear in the context. $\mathbf{1}$ denotes a column vector with all entries equal to one whose dimension should be clear from the context. For column vectors $x_1, \ldots, x_n$, $[x_1; \cdots; x_n]$ denotes the column vector by stacking the elements of $x_1, \ldots, x_n$.

4.2 Heterogeneous Network Structure

Consider a heterogeneous network of N linear agents

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i \end{cases} \tag{4.1}$$

for $i \in \{1, \ldots, N\}$, where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^p$.

The agents are introspective, meaning that the agents have access to their own local information. Specifically, each agent has access to the quantity

$$z_i = C_i^m x_i. \tag{4.2}$$

where $z_i \in \mathbb{R}^{q_i}$.

The network infrastructure provides each agent with a linear combination of its own output relative to that of other agents. In particular, each agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j), \tag{4.3}$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ with $i, j \in \{1, \ldots, n\}$. This network can be described by a weighted directed graph (digraph) $\mathcal{G}$ with nodes corresponding to the agents in the network and edges with weight given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ means that there exists an edge with weight a_{ij} from agent j to agent i.

We also define a matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^N a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. The matrix G, known as the *weighted Laplacian* matrix of the digraph $\mathcal{G}$ has the property that the sum of the coefficients on each row is equal to zero. In terms of the coefficients g_{ij} of G, ζ_i given by (4.3) can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j. \tag{4.4}$$

With the local information z_i given by (4.2) and the information ζ_i given by (4.4) provided by the network, the agent i, where $i \in \{1, \ldots, N\}$, has the following dynamical equations:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \\ z_i = C_i^m x_i, \\ \zeta_i = \sum_{j=1}^N g_{ij} y_j. \end{cases} \tag{4.5}$$

We make the following assumption regarding the network communication topology:

Assumption 4.1. *The digraph $\mathcal{G}$ has a directed spanning tree.*

From [53, Lemma 3.3], it is well known that under Assumption 4.1, the weighted Laplacian matrix G associated with network topology $\mathcal{G}$ has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$, and all the other eigenvalues are in the open right-half complex plane. We then let $\lambda_1, \ldots, \lambda_N$ denote the eigenvalues of G, such that $\lambda_1 = 0$ and $0 < \mathrm{re}(\lambda_2) \leq \ldots \leq \mathrm{re}(\lambda_N)$.

Let us now introduce the following definition to characterize a set of network communication topologies:

Definition 4.1. *For any given $\gamma \geq \beta > 0$, let $\Gamma_{\beta,\gamma}$ denote the set of digraphs that satisfy Assumption 4.1 and for which the corresponding Laplacian matrix has the following properties:* $\mathrm{re}(\lambda_2) \geq \beta$, *and* $\max_{i=2,\ldots,N} |\lambda_i| < \gamma$ *for $i \in \{2, \ldots, N\}$.*

Assumption 4.2. *For each agent $i \in \{1, \dots, N\}$, we make the following assumption:*

1) (A_i, B_i) is stabilizable;

2) (C_i, A_i) is detectable;

3) (C_i, A_i, B_i) is right-invertible; and

4) (C_i^m, A_i) is detectable.

Remark 4.1. *Right-invertibility of a triple (C_i, A_i, B_i) means that, given a reference output $y_r(t)$, there exist an initial condition $x_i(0)$ and an input $u_i(t)$ such that $y_i(t) = y_r(t)$ for all $t \geq 0$. For example, every single-input single-output system is right-invertible, unless its transfer function is identically zero.*

4.3 Output Synchronization

In this section, we consider the output synchronization problem for a heterogeneous network. The output synchronization is defined as follows:

Definition 4.2. *A heterogeneous network of N agents is said to achieve output synchronization if*

$$\lim_{t\to\infty} (y_i(t) - y_j(t)) = 0, \quad \forall i, j \in \{1, \dots, N\}.$$

Let us now formally formulate the output synchronization problem for a heterogeneous network.

Problem 4.1 (Output Synchronization)**.** *Consider a heterogeneous network of N agents* (4.5)*. For any given $\gamma \geq \beta > 0$, and the resulting set $\Gamma_{\beta,\gamma}$ of communication topologies, the output synchronization problem is to find, if possible, a linear dynamical controller*

$$\begin{cases} \dot{x}_{i,c} = A_{i,c}x_{i,c} + B_{i,c}\zeta_i + E_{i,c}z_i, \\ u_i \ = C_{i,c}x_{i,c} + D_{i,c}\zeta_i + M_{i,c}z_i, \end{cases} \tag{4.6}$$

for each agent $i \in \{1, \ldots, N\}$, such that output synchronization is achieved for any network communication topology represented by the digraph $\mathcal{G} \in \Gamma_{\beta,\gamma}$.

Remark 4.2. *Since (C_i^m, A_i) is detectable for $i \in \{1, \ldots, N\}$, one can, without any communication among agents, simply asymptotically stabilize each individual agent by utilizing z_i, and hence achieve the output synchronization with zero synchronization trajectory, that is $\lim_{t\to\infty} y_i(t) = 0$, $i \in \{1, \ldots, N\}$. In this chapter, we are not interested in such a case. We are aiming to achieve output synchronization with non-trivial synchronization trajectories.*

Theorem 4.1. *Consider a heterogeneous network of N agents* (4.5)*. Let Assumptions 4.1 and 4.2 hold. Then the output synchronization problem with $\Gamma_{\beta,\gamma}$ for any $\gamma \geq \beta > 0$ as defined in Problem 4.1, is solvable via N decentralized controllers of the form* (4.6)*.*

We shall prove Theorem 4.1 by explicit construction of synchronization controllers for each agent. The fundamental challenge of the output synchronization problem for heterogeneous networks is that the agent models are non-identical. Therefore, we first design a local pre-compensator to make all the agents almost identical, which we refer to as *homogenization of network*. Next, we show that the output synchronization problem with respect to the new almost identical models can be converted into a simultaneous stabilization problem. Finally, we design controllers via a low-gain approach to solve the reformulated simultaneous stabilization problem in the homogenized network.

4.3.1 Homogenization of the Network

Since each agent is introspective, we use the local information z_i to manipulate the agent dynamics such that all the agents' models are almost identical to the rest of network. This is shown in the following lemma.

Lemma 4.1. *Consider a heterogeneous network of N agents* (4.5)*. Let Assumption 4.2 hold, and let $\bar{n}_d$ denote the maximal order of infinite zeros of (C_i, A_i, B_i), $i \in \{1, \ldots, N\}$. Suppose a triple (C, A, B) is given such that*

1) $\operatorname{rank}(C) = p$,

2) (C, A, B) is invertible, of uniform rank $n_q \geq \bar{n}_d$, and has no invariant zeros.

Then for each agent $i \in \{1, \ldots, N\}$, there exist a pre-compensator of the form

$$\begin{cases} \dot{\xi}_i = A_{i,h}\xi_i + B_{i,h}z_i + E_{i,h}v_i, \\ u_i = C_{i,h}\xi_i + D_{i,h}v_i, \end{cases} \tag{4.7}$$

such that the interconnection of (4.5) *and* (4.7) *can be written in the following form:*

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + B(v_i + \rho_i), \\ y_i = C\bar{x}_i, \\ \zeta_i = \sum_{j=1}^{N} g_{ij}y_j, \end{cases} \tag{4.8}$$

where ρ_i is given by

$$\begin{cases} \dot{\omega}_i = A_{i,s}\omega_i, \\ \rho_i = C_{i,s}\omega_i, \end{cases} \tag{4.9}$$

and $A_{i,s}$ is Hurwitz stable.

Proof. The proof of Lemma 4.1 will be given in Appendix 4.B by explicit construction of a pre-compensator of the form (4.7). □

Remark 4.3. *We would like to make several observations:*

1) The property that the triple (C, A, B) is invertible and has no invariant zero implies that (A, B) is controllable and (C, A) is observable.

2) The triple (C, A, B) is arbitrarily assignable as long as the conditions are satisfied. They play a role as design parameters. We shall use this freedom in various places in this chapter.

Remark 4.4. *Without loss of generality, we assume that the triple (C, A, B) has the following form:*[1]

$$A = A_0 + BH, \quad A_0 := \begin{bmatrix} 0 & I_{p(n_q-1)} \\ 0 & 0 \end{bmatrix}, \quad B = B_0 := \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C = C_0 := \begin{bmatrix} I_p & 0 \end{bmatrix}, \tag{4.10}$$

where H is such that the matrix $A_0 + B_0 H$ has desired eigenvalues. The existence of such an H is guaranteed by the fact that (A_0, B_0) is controllable.

Lemma 4.1 shows that we can design a pre-compensator based on local information z_i to transform each non-identical agent model given by (4.5) into a new model given by (4.8) and (4.9). The new agent models (4.8) are almost identical except for different exponentially decaying signals ρ_i in the range space of B, generated by (4.9). We shall solve the output synchronization problem with respect to the new almost identical models (4.8) and (4.9), and then combine the result with Lemma 4.1 to prove Theorem 4.1.

4.3.2 Connection to Simultaneous Stabilization Problem

In this section, we show that the exponentially decaying signals ρ_i are irrelevant for solving the output synchronization problem with respect to the new almost identical models (4.8) and (4.9), and that the problem is essentially reduced to a simultaneous stabilization problem.

For solving the synchronization problem for a network of N agents (4.8) and (4.9) with a set of possible communication topologies $\Gamma_{\beta,\gamma}$, we consider N general decentralized controllers of the

[1] If (C, A, B) is not in this form, from [71], which is also reviewed in Appendix 4.A.1, there exist nonsingular state and input transformations, such that the transformed system is in this form.

form (4.11)

$$\begin{cases} \dot{\chi}_i = A_k\chi_i + B_k\zeta_i, \\ v_i = C_k\chi_i, \end{cases} \tag{4.11}$$

for $i \in \{1, \ldots, N\}$, where $\chi_i \in \mathbb{R}^{n_c}$, which should be independent of the specific communication topology $\mathcal{G} \in \Gamma_{\beta,\gamma}$.

With $\tilde{x}_i := [\bar{x}_i; \chi_i]$, the closed-loop system of (4.8) and (4.11) for each individual agent can be written as

$$\begin{cases} \dot{\tilde{x}}_i = \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix} \tilde{x}_i + \begin{bmatrix} 0 \\ B_k \end{bmatrix} \zeta_i + \begin{bmatrix} B \\ 0 \end{bmatrix} \rho_i, \\ y_i = \begin{bmatrix} C & 0 \end{bmatrix} \tilde{x}_i, \\ \zeta_i = \sum_{j=1}^{N} g_{ij} y_j. \end{cases} \tag{4.12}$$

Define $\tilde{x} := [\tilde{x}_1; \cdots ; \tilde{x}_N]$, $\rho := [\rho_1; \cdots ; \rho_N]$,

$$\bar{A} = \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B_k \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \text{ and } \bar{E} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{4.13}$$

We then obtain the overall dynamics of N agents:

$$\dot{\tilde{x}} = [I_N \otimes \bar{A} + G \otimes (\bar{B}\bar{C})]\tilde{x} + (I_N \otimes \bar{E})\rho.$$

Let U be a nonsingular matrix such that $J = U^{-1}GU$ is the Jordan canonical form of G with $J(1,1) = \lambda_1 = 0$. Define $\eta = [\eta_1; \cdots ; \eta_N] = (U^{-1} \otimes I_{pn_q+n_c})\tilde{x}$. We then obtain the following dynamical equations for η:

$$\dot{\eta} = [I_N \otimes \bar{A} + J \otimes (\bar{B}\bar{C})]\eta + (U^{-1} \otimes \bar{E})\rho. \tag{4.14}$$

Lemma 4.2. *Let Assumption 4.1 hold. If $\bar{A} + \lambda_i\bar{B}\bar{C}$ is Hurwitz stable for all $i \in \{2, \ldots, N\}$, then the output synchronization problem for a network of N agents of the form* (4.8) *and* (4.9) *is solved via N decentralized controllers of the form* (4.11).

Proof. The proof is carried out in two stages. In the first stage, we shall show that the output synchronization problem for a network N agents (4.8) and (4.9) via controllers of the form (4.11) is solved if

$$\lim_{t\to\infty} \eta_i(t) = 0$$

for all $i \in \{2, \ldots, N\}$. We then show that this is guaranteed if $\bar{A} + \lambda_i \bar{B}\bar{C}$ is Hurwitz stable for all $i \in \{2, \ldots, N\}$.

Suppose that

$$\lim_{t\to\infty} \left(\eta(t) - \begin{bmatrix} \eta_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = 0,$$

for some $\eta_1(t) \in \mathbb{C}^{pn_q+n_c}$. Then

$$\begin{aligned}
\lim_{t\to\infty} (\tilde{x}(t) - \mathbf{1} \otimes \eta_1(t)) &= \lim_{t\to\infty} [(U \otimes I)\eta - (U \otimes I)(U^{-1} \otimes I)(\mathbf{1} \otimes \eta_1(t))] \\
&= (U \otimes I) \lim_{t\to\infty} [\eta(t) - (U^{-1}\mathbf{1}) \otimes \eta_1(t)] \\
&= (U \otimes I) \lim_{t\to\infty} \left(\eta(t) - \begin{bmatrix} \eta_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = 0,
\end{aligned}$$

where we have used that $U^{-1}\mathbf{1} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}'$, which follows from the fact that $U^{-1}U = I_N$ and that U consists of all the (generalized) right eigenvectors of G, with the first column being $\mathbf{1}$. Hence, the output synchronization is achieved. So far, we have shown that the output synchronization is achieved if $\lim_{t\to\infty} \eta_i(t) = 0$ for all $i \in \{2, \ldots, N\}$. Next, we shall show that this is ensured if $A + \lambda_i \bar{B}\bar{C}$ is Hurwitz stable for all $i \in \{2, \ldots, N\}$.

Define $\bar{\eta} := [\eta_2; \cdots; \eta_N]$ and $\omega := [\omega_1; \cdots; \omega_N]$, from (4.14) and (4.9), we obtain that

$$\begin{bmatrix} \dot{\bar{\eta}} \\ \omega \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \bar{A} + \bar{J} \otimes (\bar{B}\bar{C}) & ((\bar{I}U^{-1}) \otimes \bar{E})C_s \\ 0 & A_s \end{bmatrix} \begin{bmatrix} \bar{\eta} \\ \omega \end{bmatrix}, \tag{4.15}$$

where

$$\bar{I} = \begin{bmatrix} 0 & I_{N-1} \end{bmatrix}, \quad C_s = \text{blkdiag}\{C_{i,s}\}_{i=1}^N, \quad A_s = \text{blkdiag}\{A_{i,s}\}_{i=1}^N, \quad J = \text{blkdiag}\{0; \bar{J}\}.$$

Since $I_{N-1} \otimes \bar{A}$ and $\bar{J} \otimes (\bar{B}\bar{C})$ are block upper triangular, the eigenvalues of $I_{N-1} \otimes \bar{A} + \bar{J} \otimes (\bar{B}\bar{C})$ are the union of eigenvalues $\bar{A} + \lambda_i \bar{B}\bar{C}$ for $i \in \{2, \ldots, N\}$, which are in the open left-half complex plane by the assumption. Together with the fact that A_s is Hurwitz stable, it is clear that the system (4.15) is asymptotically stable, that is, $\lim_{t\to\infty} \bar{\eta}(t) = 0$ for any initial conditions $\bar{\eta}(0)$ and $\omega(0)$. □

Remark 4.5. *In view of Lemma 4.2, the dynamics of $\eta_1(t)$ is governed by*

$$\dot{\eta}_1(t) = \bar{A}\eta_1(t) + (v' \otimes \bar{E})\rho(t), \quad \eta_1(0) = (v' \otimes I_{pn_q+n_c})\tilde{x}(0),$$

where v' is the first row of the matrix U^{-1}, i.e., the left eigenvector corresponding to the eigenvalue of G at zero. Since $\rho(t)$ is exponentially decaying, from [96, Lemma B.1] and Lemma 4.2, we see that for each $i \in \{1, \ldots, N\}$,

$$\lim_{t\to\infty} (\tilde{x}_i(t) - e^{\bar{A}t}\tilde{\eta}_1) = \lim_{t\to\infty} [(\tilde{x}_i(t) - \eta_1(t)) + (\eta_1(t) - e^{\bar{A}t}\tilde{\eta}_1)] = 0, \tag{4.16}$$

for some $\tilde{\eta}_1 \in \mathbb{R}^{pn_q+n_c}$. □

From Lemma 4.2, we see that the output synchronization problem for a network of agents (4.8) and (4.9) is achieved if $\bar{A} + \lambda_i \bar{B}\bar{C}$ is Hurwitz stable, for all $i \in \{2, \ldots, N\}$, which is a simultaneous stabilization problem. More specifically, we need to design the parameters A_k, B_k, and C_k in (4.11), such that the following compensator

$$\begin{cases} \dot{\chi} = A_k \chi + B_k z, \\ u = C_k \chi, \end{cases} \tag{4.17}$$

simultaneously stabilizes all the $N-1$ systems given by

$$\begin{cases} \dot{x} = Ax + Bu, \\ z = \lambda_i Cx, \quad i \in \{2, \ldots, N\}. \end{cases} \tag{4.18}$$

Due to the linearity, it is easy to see that the compensator (4.17) simultaneously stabilizes (4.18) if it simultaneously stabilizes all the $N-1$ systems given by

$$\begin{cases} \dot{x} = Ax + \lambda_i Bu, \\ z = Cx, \quad i \in \{2, \ldots, N\}. \end{cases} \tag{4.19}$$

Lemma 4.3. *The output synchronization for a heterogeneous network of N agents* (4.5) *as defined in Problem 4.1 is solvable if* (4.17) *simultaneously stabilizes all the $N-1$ systems* (4.19).

Proof. If (4.17) simultaneously stabilizes all the $N-1$ systems (4.19), then the composition of (4.7) and (4.11), which is of the form (4.6), solves Problem 4.1. □

Lemma 4.3 converts the output synchronization problem for a heterogeneous network of N agents (4.5) as defined in Problem 4.1 to a simultaneous stabilization problem.

4.3.3 Simultaneous Stabilization via a Low-gain Approach

In this section, we design the parameters A_k, B_k and C_k of the compensator (4.11), such that the compensator (4.17) simultaneously stabilizes all the $N-1$ systems (4.19).

It is clear that we can choose the matrix H in (4.10) such that the matrix A has all the eigenvalues on the imaginary axis.

Given a $\beta > 0$, such that $\text{re}(\lambda_2(G)) \geq \beta$ for any $\mathcal{G} \in \Gamma_{\beta,\gamma}$, let $P(\epsilon) = P'(\epsilon) > 0$ be the unique solution of the continuous-time algebraic Riccati equation

$$A'P(\epsilon) + P(\epsilon)A - \beta P(\epsilon)BB'P(\epsilon) + \epsilon I_{pn_q} = 0. \tag{4.20}$$

We then design the controller of the form (4.11) as:

$$\begin{cases} \dot{\chi}_i = A_k\chi_i + B_k\zeta_i := (A + KC)\chi_i - K\zeta_i, \\ v_i = C_k\chi_i := B'P(\epsilon)\chi_i, \quad i \in \{1, \ldots, N\}, \end{cases} \tag{4.21}$$

where the matrix K is such that $A + KC$ is Hurwitz stable, and $\epsilon > 0$ is a low-gain parameter. Note that (4.21) is of CSS type observer, see [9].

Following the proof of [73, Theorem 4], with just a little bit modification, we see that there exists an ϵ^*, which depends on γ, such that for $\epsilon \in (0, \epsilon^*]$, the compensator (4.17) with the parameters A_k, B_k and C_k given by (4.21) simultaneously stabilizes all the $N - 1$ systems (4.19).

Remark 4.6. *Note that the matrix A_k in the controller* (4.21) *is Hurwitz stable and the matrix $\bar{A}$ given by* (4.13) *is block upper triangular. It is then follows from the result of [96, Lemma B.1] and Remark 4.5 that the output synchronization trajectory is given by*

$$\lim_{t\to\infty}(y_i(t) - Ce^{At}d) = 0, \quad \forall i \in \{1, \ldots, N\}.$$

for some $d \in \mathbb{R}^{pn_q}$.

4.4 Application to Output Formation

In this section, we consider the output formation problem to be formally defined shortly. We shall show that the output formation problem can be solved by slightly modifying the design procedure for solving the output synchronization problem as defined in Problem 4.1.

Definition 4.3. *An output formation is a family of vectors* $\{h_1, \cdots, h_N\}$, $h_i \in \mathbb{R}^p$, $i \in \{1, \ldots, N\}$. *The heterogeneous network of* N *agents* (4.5) *is said to achieve the output formation if*

$$\lim_{t\to\infty} [(y_i(t) - h_i) - (y_j(t) - h_j)] = 0, \quad \forall i, j \in \{1, \ldots, N\}. \tag{4.22}$$

For this problem, we assume that the network infrastructure provides each agent with the following information

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}[(y_i - h_i) - (y_j - h_j)] = \sum_{j=1}^{N} g_{ij}(y_j - h_j), \tag{4.23}$$

With the local information z_i given by (4.2) and the information $\hat{\zeta}_i$ given by (4.23) provided by the network, the agent i, where $i \in \{1, \ldots, N\}$, has the following dynamical equations:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \\ z_i = C_i^m x_i, \\ \hat{\zeta}_i = \sum_{j=1}^{N} g_{ij}(y_j - h_j). \end{cases} \tag{4.24}$$

Let us formally formulate the output formation problem for a heterogeneous network.

Problem 4.2 (Output formation). *Consider a heterogeneous network of* N *agents* (4.24). *For any given* $\gamma \geq \beta > 0$ *and the resulting set* $\Gamma_{\beta,\gamma}$, *and an arbitrarily given family of vectors* $\{h_1, \cdots, h_N\}$, *where* $h_i \in \mathbb{R}^p$ *for* $i \in \{1, \ldots, N\}$, *the output formation problem with a set of communication topologies* $\Gamma_{\beta,\gamma}$ *is to find, if possible, a linear dynamical controller*

$$\begin{cases} \dot{x}_{i,c} = A_{i,c} x_{i,c} + B_{i,c} \hat{\zeta}_i + E_{i,c} z_i, \\ u_i = C_{i,c} x_{i,c} + D_{i,c} \hat{\zeta}_i + M_{i,c} z_i, \end{cases} \tag{4.25}$$

such that the output formation as defined in Definition 4.3 is achieved for any network communication topology $\mathcal{G} \in \Gamma_{\beta,\gamma}$.

Theorem 4.2. *Consider a heterogeneous network of N agents* (4.24)*. Let Assumptions 4.1 and 4.2 hold. Then the output formation problem with a set of communication topologies $\Gamma_{\beta,\gamma}$ for any $\gamma \geq \beta > 0$, and any formation vectors $\{h_1, \cdots, h_N\}$, where $h_i \in \mathbb{R}^p$ for $i \in \{1, \ldots, N\}$, as defined in Problem 4.2, is solvable via N decentralized controllers of the form* (4.25)*.*

The proof of Theorem 4.2 is very similar to the proof of Theorem 4.1 by explicit construction of a formation controller of the form (4.25). We first design a local pre-compensator of the form (4.7) for each agent such that the resulting systems are almost identical, that is, all the resulting systems are characterized by the same triple (C, A, B) for which the output formation is always achievable. The following lemma shows the existence of such a triple (C, A, B).

Lemma 4.4. *For an arbitrarily given family of vectors $\{h_1, \cdots, h_N\}$, $h_i \in \mathbb{R}^p$, $i = 1, \ldots, N$ and an integer $n_q > 0$, there exist a triple (C, A, B) and another family of vectors $\{\bar{h}_1, \cdots, \bar{h}_N\}$ of appropriate dimensions, such that*

1) $\operatorname{rank}(C) = p$*,*

2) (C, A, B) *is invertible, of uniform rank n_q, and has no invariant zero,*

3) A has all its eigenvalues in the closed left-half complex plane,

4) $A\bar{h}_i = 0$*,*

5) $C\bar{h}_i = h_i$*.*

Proof. Since we have freedom to chose the triple (C, A, B) in Lemma 4.1, let us choose the triple (C, A, B) as follows:

$$A = A_0 + B_0 H, \quad B = B_0, \quad C = C_0,$$

where A_0, B_0, C_0 are given in (4.10), $H = \begin{bmatrix} 0 & H_0 \end{bmatrix}$, and the matrix H_0 is such that the matrix $\bar{A}_0 + \bar{B}_0 H_0$, where

$$\bar{A}_0 := \begin{bmatrix} 0 & I_{p(n_q-2)} \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_0 := \begin{bmatrix} 0 \\ I_p \end{bmatrix},$$

has all the eigenvalues in the closed left-half complex plane. Such an H_0 exists due to the fact that $(\bar{A}_0, \bar{B}_0)$ is controllable. It is then easy to see that the matrix $A_0 + B_0 H$ has $p(n_q - 1)$ eigenvalues, which are the eigenvalues of $\bar{A}_0 + \bar{B}_0 H_0$, and the remaining p eigenvalues are simple eigenvalues at zero. Therefore, the third condition is satisfied.

We then define a family of vectors $\{\bar{h}_1, \cdots, \bar{h}_N\}$ as follows:

$$\bar{h}_i = \begin{bmatrix} h_i \\ 0 \end{bmatrix}, \quad i = 1, \ldots, N.$$

It is then easy to see that

$$C\bar{h}_i = C_0 \begin{bmatrix} h_i \\ 0 \end{bmatrix} = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} h_i \\ 0 \end{bmatrix} = h_i,$$

and

$$A\bar{h}_i = (A_0 + B_0 H) \begin{bmatrix} h_i \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & I_{p(n_q-1)} \\ 0 & H_0 \end{bmatrix} \begin{bmatrix} h_i \\ 0 \end{bmatrix} = 0.$$

□

Proof of Theorem 4.2. For any triple (C, A, B) which satisfies the condition of Lemma 4.4, from Lemma 4.1, it is clear that we can design a pre-compensator of the form (4.7) for each agent, such that the interconnection of (4.5) and (4.7) can be written in the following form:

$$\begin{cases} \dot{\bar{x}}_i = A\bar{x}_i + B(v_i + \rho_i), \\ y_i = C\bar{x}_i, \\ \hat{\zeta}_i = \sum_{j=1}^N g_{ij}(y_j - h_j), \end{cases} \tag{4.26}$$

where ρ_i is given by (4.9).

Define $\bar{x}_{i,s} = \bar{x}_i - \bar{h}_i$, since $A\bar{h}_i = 0$ and $C\bar{h}_i = h_i$ for $i = 1, \ldots, N$, (4.26) can be rewritten in term of $\bar{x}_{i,s}$ as:

$$\begin{cases} \dot{\bar{x}}_{i,s} = A\bar{x}_{i,s} + B(v_i + \rho_i), \\ y_i \ = C\bar{x}_{i,s} + h_i, \\ \hat{\zeta}_i \ = \sum_{j=1}^{N} g_{ij}(y_j - h_j). \end{cases} \tag{4.27}$$

Following the design procedure given in Section 4.3.3, we then design the following decentralized controller for each agent

$$\begin{cases} \dot{\chi}_i = (A + KC)\chi_i - K\hat{\zeta}_i, \\ v_i = B'P(\epsilon)\chi_i, \end{cases} \tag{4.28}$$

where the matrix K is such that $A + KC$ is Hurwitz stable, $\epsilon > 0$ is a low-gain parameter, and $P(\epsilon) = P'(\epsilon) > 0$ is the unique solution of the algebraic Riccati equation (4.20).

It then follows from the analysis in Section 4.3.3 that there exists an ϵ^*, which depends on γ, such that for all $\epsilon \in (0, \epsilon^*]$, the controller (4.28) solve the output synchronization for a network of N the models (4.27). Hence, $\lim_{t\to\infty}[(y_i(t) - h_i) - (y_j(t) - h_j)] = \lim_{t\to\infty}(C\bar{x}_{i,s}(t) - C\bar{x}_{j,s}(t)) = 0$ for all $i, j \in \{1, \ldots, N\}$. $\square$

4.5 Regulation of Output Synchronization

Note that the output synchronization problem does not impose any conditions on asymptotic behavior of the outputs of the agent models as long as they are asymptotic identical. In this section, we consider the related problem of regulating the output towards a desired reference trajectory $y_r(t)$, generated by an autonomous exosystem

$$\begin{cases} \dot{x}_r = A_r x_r, \quad x_r(0) = x_{r0}, \\ y_r = C_r x_r, \end{cases} \tag{4.29}$$

where $x_r \in \mathbb{R}^r$ and $y_r \in \mathbb{R}^p$.

We make the following assumption about the exosystem (4.29):

Assumption 4.3. *For the exosystem* (4.29),

1) (C_r, A_r) is observable,

2) All the eigenvalues of A_r are on the imaginary axis.

Definition 4.4. *A heterogeneous network of N agents is said to achieve the regulation of output synchronization if*

$$\lim_{t\to\infty}(y_i(t) - y_r(t)) = 0, \quad \forall i = \{1, \ldots, N\}.$$

For solving the regulation of output synchronization problem, we consider a subset Γ_s of Γ, where Γ is the set of all the network topologies, each of which contains a directed spanning tree. We assume that all the topologies in the set Γ_s have a common root. Without loss of generality, we assume that the common root is node (agent) 1. This (root) agent 1 measures its own output relative to the output of the exosystem, that is, agent 1 has access to a quantity $\psi_1 = d(y_1 - y_r)$, where $d > 0$, while $\psi_i = 0$ for all $i \in \{2, \ldots, N\}$.

With the local information z_i given by (4.2), the information ζ_i given by (7.2) provided by the network, and information ψ_i, the agent i for $i \in \{1, \ldots, N\}$ has the following dynamical equations:

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i, \\ y_i = C_i x_i, \\ z_i = C_i^m x_i, \\ \bar{\zeta}_i = \sum_{j=1}^N g_{ij} y_j + \psi_i. \end{cases} \tag{4.30}$$

Let us now formally formulate the regulation of output synchronization problem.

Problem 4.3 (Regulation of Output Synchronization). *Consider a heterogeneous network of N agents* (4.30) *and the autonomous exosystem* (4.29). . *For any given set $\Gamma_s \subset \Gamma$, the regulation of output synchronization problem is to find, if possible, a linear dynamical controller*

$$\begin{cases} \dot{x}_{i,c} = A_{i,c}x_{i,c} + B_{i,c}\bar{\zeta}_i + E_{i,c}z_i, \\ u_i \ = C_{i,c}x_{i,c} + D_{i,c}\bar{\zeta}_i + M_{i,c}z_i, \end{cases} \tag{4.31}$$

for each agent $i \in \{1, \ldots, N\}$, such that regulation of output synchronization is achieved for any network communication topology represented by the digraph $\mathcal{G} \in \Gamma_s$.

We present some preliminary work which are needed for presenting the result for the regulation of output synchronization problem as defined in Problem 4.3. Let $\bar{\mathcal{G}}$ denote an expanded network constructed from $\mathcal{G} \in \Gamma_s$ by adding the exosystem as node 0 and the edge from exosystem to agent 1 with weight d. It is then easy to see that the Laplacian matrix of the network $\bar{\mathcal{G}}$ is given by

$$\bar{G} = [\bar{g}_{ij}] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -d & g_{11}+d & g_{12} & \cdots & g_{1N} \\ 0 & g_{21} & g_{22} & \cdots & g_{2N} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & g_{N1} & g_{N2} & \cdots & g_{NN} \end{bmatrix}. \tag{4.32}$$

In view of (4.32), $\bar{\zeta}_i$ in (4.30) can be rewritten as

$$\bar{\zeta}_i = \sum_{j=1}^{N} g_{ij}y_j + \psi_i = \sum_{j=0}^{N} \bar{g}_{ij}y_j. \tag{4.33}$$

Also note that the expanded network also contains a directed spanning tree rooted at the node 0. It is then easy to see from [53, Lemma 3.3] that all the eigenvalues of $\bar{G}$ are in the closed right-half complex plane. Let $\bar{\lambda}_1, \cdots, \bar{\lambda}_{N+1}$ denote the eigenvalues of $\bar{G}$, such that $\bar{\lambda}_1 = 0$ and $0 < \mathrm{re}(\bar{\lambda}_2) \leq \ldots \leq \mathrm{re}(\bar{\lambda}_{N+1})$.

Assumption 4.4. *There exist $\bar{\gamma} \geq \bar{\beta} > 0$, such that for each expanded network, the corresponding Laplacian matrix has following properties:*

1) $\mathrm{re}(\bar{\lambda}_2) \geq \bar{\beta} > 0$*;*

2) $\max_{i=2,\dots,N+1} |\bar{\lambda}_i| \leq \bar{\gamma}$.

We are now ready to present our result for the regulation of output synchronization.

Theorem 4.3. *Consider a heterogeneous network of N agents* (4.30) *and the autonomous exosystem* (4.29)*. Let Assumptions 4.1, 4.2, 4.3 and 4.4 hold. Then the regulation of output synchronization problem as defined in Problem 4.3 is solvable via N decentralized controllers of the form* (4.31)*.*

Proof. For an exosystem given by (4.29), it is shown in Appendix 4.C that there exist another exosystem given by

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r \tilde{x}_r, \quad \tilde{x}_r(0) = \tilde{x}_{r0}, \\ y_r = \tilde{C}_r \tilde{x}_r, \end{cases} \tag{4.34}$$

such that for all $x_{r0} \in R^r$ there exists $\tilde{x}_{r0} \in R^{\tilde{r}}$ for which (4.34) generates exact the same output y_r as the original exosystem (4.29). Furthermore, we can find a matrix $\tilde{B}_r$ such that the triple $(\tilde{C}_r, \tilde{A}_r, \tilde{B}_r)$ is invertible, of uniform rank n_q, and has no invariant zero, where n_q is an integer greater than or equal to maximal order of infinite zeros of (C_i, A_i, B_i), $i \in \{1, \dots, N\}$ and all the observability index (see [8] for the definition) of (C_r, A_r). Note that as seen from Appendix 4.C, the eigenvalues of $\tilde{A}_r$ consists of all the eigenvalues of A_r and additional zero eigenvalues, which are degenerate.

The new exosystem can be rewritten as:

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r \tilde{x}_r + \tilde{B}_r(v_r + \rho_r), \quad \tilde{x}_r(0) = \tilde{x}_{r0}, \\ y_r = \tilde{C}_r \tilde{x}_r, \end{cases} \tag{4.35}$$

where $v_r(t) = 0$ and $\rho_r(t) = 0$.

Following from the constructive proof of Lemma 4.1, we then design a pre-compensator (4.7) for each agent $i \in \{1, \ldots, N\}$ such that the interconnection of (4.5) and (4.7) are almost identical to the exosystem system (4.35), that is, for each agent $i \in \{1, \ldots, N\}$,

$$\begin{cases} \dot{\bar{x}}_i = \tilde{A}_r \bar{x}_i + \tilde{B}_r(v_i + \rho_i), \\ y_i = \tilde{C}_r \bar{x}_i, \\ \bar{\zeta}_i = \sum_{j=0}^{N} \bar{g}_{ij} y_j, \end{cases} \tag{4.36}$$

where ρ_i is given by (4.9).

It is then easy to see that regulation of output synchronization for a heterogeneous network of N agents is converted to the output synchronization problem for an expanded network of $N+1$ agents by adding the exosystem system as agent 0 and the edge from agent 0 to agent 1 with weight d. More specifically, define $\bar{x}_0 := \tilde{x}_r$, $y_0 := \tilde{y}_r$, $v_0 := v_r$, and $\rho_0 := \rho_r$, the agent i, where $i \in \{0, 1, \ldots, N\}$ has the following dynamics:

$$\begin{cases} \dot{\bar{x}}_i = \tilde{A}_r \bar{x}_i + \tilde{B}_r(v_i + \rho_i), \\ y_i = \tilde{C}_r \bar{x}_i, \\ \bar{\zeta}_i = \sum_{j=0}^{N} \bar{g}_{ij} y_j. \end{cases} \tag{4.37}$$

Following the design procedure given in Section 4.3.3, we design the following controller for each agent $i \in \{0, 1, \ldots, N\}$

$$\begin{cases} \dot{\chi}_i = (\tilde{A}_r + K\tilde{C}_r)\chi_i - K\bar{\zeta}_i, \\ v_i = \tilde{B}_r' P(\epsilon)\chi_i, \end{cases} \tag{4.38}$$

where the matrix K is such that $A + KC$ is Hurwitz stable, $\epsilon > 0$ is a low-gain parameter, and $P(\epsilon) = P'(\epsilon) > 0$ is the unique solution of the following continuous-time algebraic Riccati equation

$$\tilde{A}_r' P(\epsilon) + P(\epsilon)\tilde{A}_r - \bar{\beta} P(\epsilon)\tilde{B}_r \tilde{B}_r' P(\epsilon) + \epsilon I_{pn_q} = 0. \tag{4.39}$$

Note that in the controller (4.38), we choose $\chi_0(0) = 0$ for agent 0. It is then clear that $v_0(t) = 0$ as desired since $\bar{\zeta}_0(t) = 0$.

It then follows from the analysis in Section 4.3.3 that there exists an ϵ^*, which depends on $\tilde{\gamma}$, such that for all $\epsilon \in (0, \epsilon^*]$, the controller (4.38), solves the output synchronization for a set of the expanded network topologies. Hence, $\lim_{t\to\infty}(y_i(t) - y_r(t)) = 0$ for all $i \in \{1, \ldots, N\}$. $\square$

4.6 Illustrative Examples

4.6.1 Output Synchronization

We illustrate our design procedure on a network of four agents. The agents dynamics are of form (4.1) with

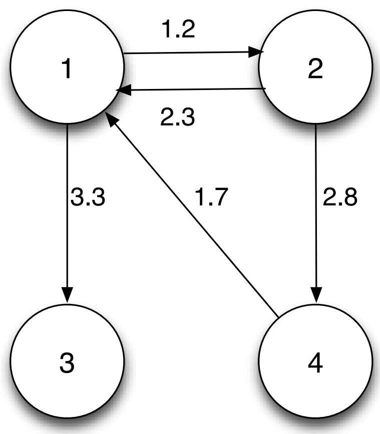
$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_1^m = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad C_2^m = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$

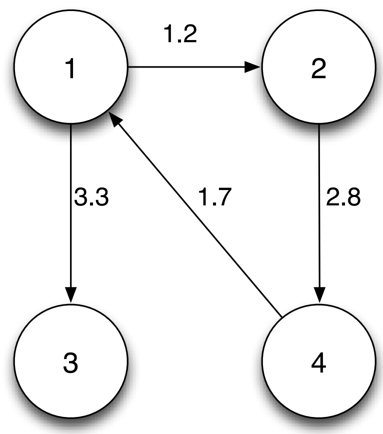
$$A_i = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}, B_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, C_i = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}, C_i^m = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

for $i = 3, 4$.

Given $\beta = 2.8$ and $\gamma = 4.1$, we have the resulting set $\Gamma_{2.8,4.1}$. Two network topologies in this set are given by Figure 9.1.



(a) Network 1



(b) Network 2

Figure 4.1: Network Topologies

Note that $\bar{n}_d = 3$, which is the degree of the infinite zeros of (C_2, A_2, B_2). We then choose $n_q = 3$, and matrices A, B, C as below

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

It is easy to see that the above matrices A, B, C satisfy the conditions of Lemma 4.1. Let us choose $\varepsilon = 0.01$ and

$$K = \begin{bmatrix} -6 \\ -10 \\ 0 \end{bmatrix}$$

and design the dynamic low-gain controller as follows:

$$\begin{cases} \dot{\chi}_i(t) = \begin{bmatrix} -6 & 1 & 0 \\ -10 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \chi_i(t) - \begin{bmatrix} -6 \\ -10 \\ 0 \end{bmatrix} \zeta_i(t) \\ v^i(t) = -\begin{bmatrix} 0.0598 & 0.0183 & 0.1423 \end{bmatrix} \chi_i(t) \end{cases}. \tag{4.40}$$

Figure 4.2 and Figure 4.3 show that the output synchronization is achieved for Network 1 and Network 2, respectively.

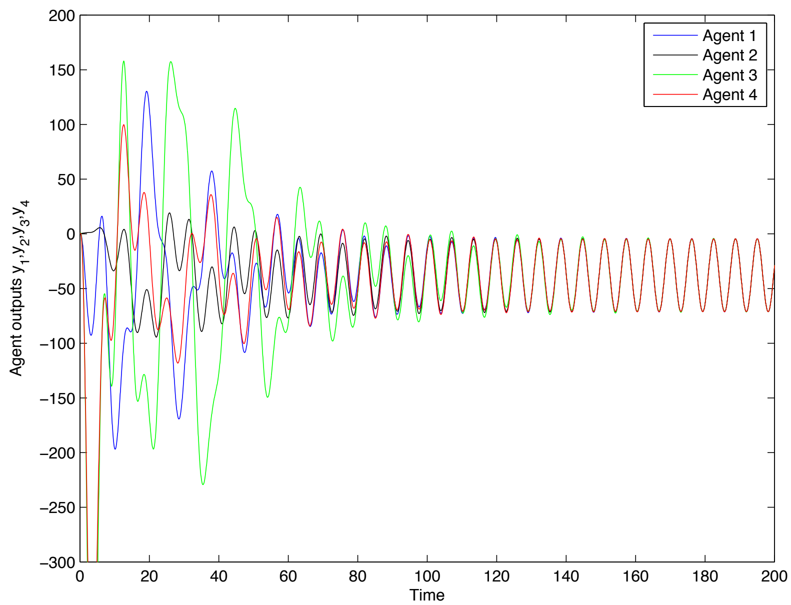


Figure 4.2: Outputs for Network 1

4.6.2 Output Formation

Consider the same two networks as in Section 4.6.1, our goal is to achieve output formation. We choose $h_1 = 10$, $h_2 = 20$, $h_3 = 30$, and $h_4 = 40$. Figure 4.4 and Figure 4.5 show that the output

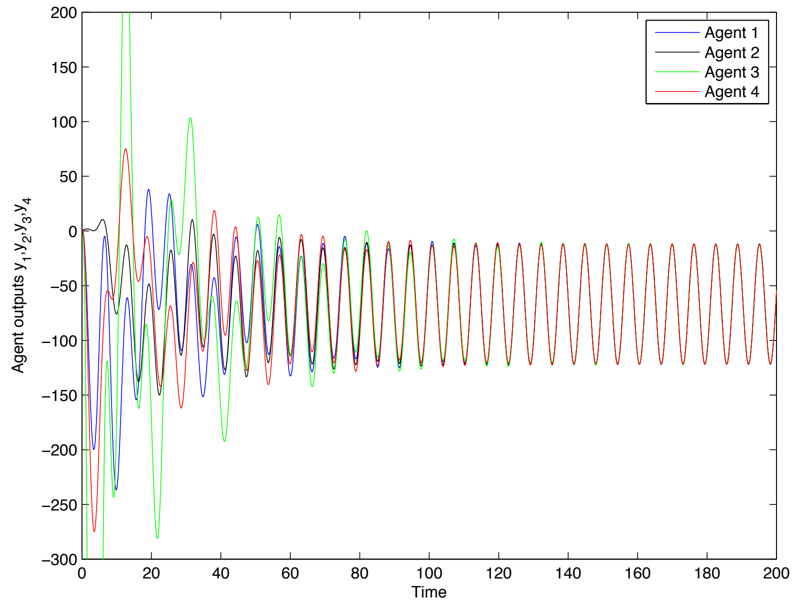


Figure 4.3: Outputs for Network 2

formation is achieved [2] for Network 1 and Network 2, respectively.

4.6.3 Regulation of Output Synchronization

Consider the same network as in Section 4.6.1, however, our goal now is to ensure that each agent's output follows the output y_r of the following exosystem

$$\begin{cases} \dot{x}_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_r, \\ y_r = \begin{bmatrix} 1 & 0 \end{bmatrix} x_r, \end{cases}$$

with $x_r(0) = [1; 1]$.

[2]Note that x_{d1}, x_{d2}, and x_{d3} is the coordinate where all the agents are almost identical

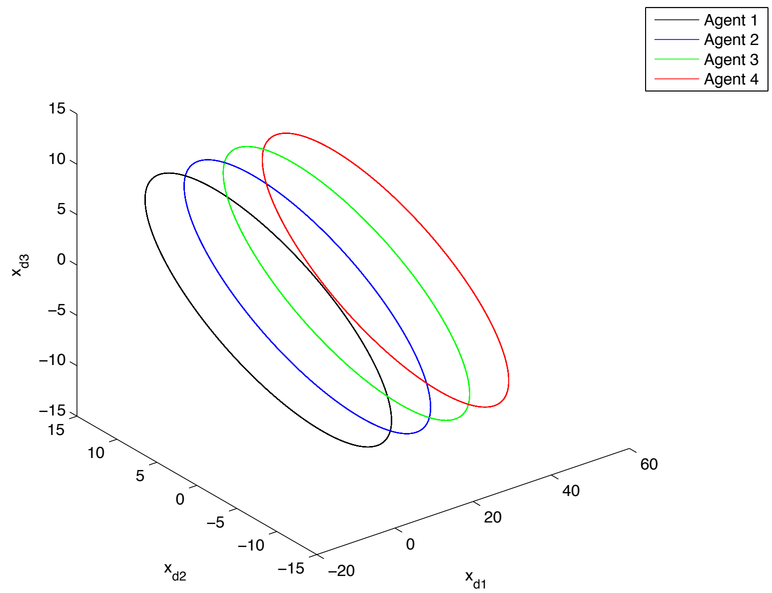


Figure 4.4: Formation for Network 1

We first expand the system to the following form

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r \tilde{x}_r := \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 0\,0\,0 \end{bmatrix} \tilde{x}_r, \\ y_r = \tilde{C}_r \tilde{x}_r := \begin{bmatrix} 1\,0\,0 \end{bmatrix} \tilde{x}_r, \end{cases}$$

with $\tilde{x}_r(0) = [1; 1; 0]$.

Let us now choose $\tilde{B}_r = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$. We then follow the same design procedure to design precompensator to make all the agents almost identical with different exponentially decaying signals. We then add a link with weight 10 from the exosystem to the root agent 1 for Networks 1 and 2 whose topologies are given by Figure 4.1. The resulting network topologies are shown in Figure 4.6.

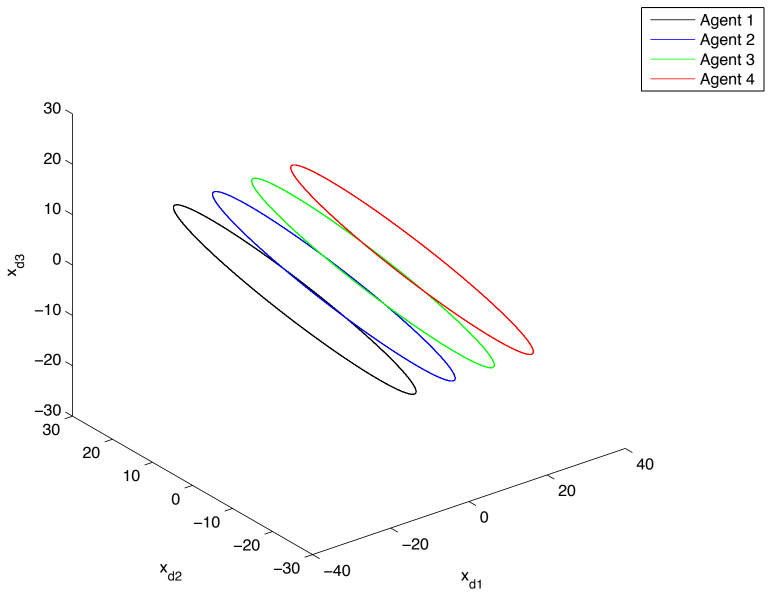
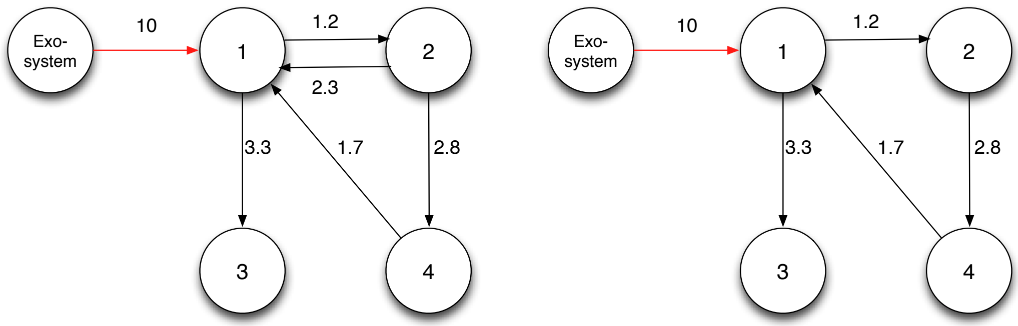


Figure 4.5: Formation for Network 2

Choose $\bar{\beta} = 0.77$, $\bar{\gamma} = 14.18$, and $\epsilon = 10^{-8}$. Figures 4.7 and 4.8 show the regulation of output synchronization is achieved for Network 1 and Network 2, respectively.

4.A Preliminary

In order to better understand our design methodology, the readers need to get familiar with special coordinate basis (SCB) [71], how to square down the right invertible system, and how to make the invertible system uniform rank. Therefore, we will briefly review these materials.



(a) Expanded Network 1 (b) Expanded Network 2

Figure 4.6: Expanded Network Topologies

4.A.1 Review of SCB

Consider a strictly proper linear system given by

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases} \tag{4.41}$$

with B injective where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$. There exist nonsingular transformations Γ_s, Γ_o, and Γ_i, such that

$$x = \Gamma_s \tilde{x}, \quad y = \Gamma_o \tilde{y}, \quad u = \Gamma_i \tilde{u},$$

$$\tilde{x} = \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} y_d \\ y_b \end{bmatrix}, \quad \tilde{u} = \begin{bmatrix} u_d \\ u_c \end{bmatrix},$$

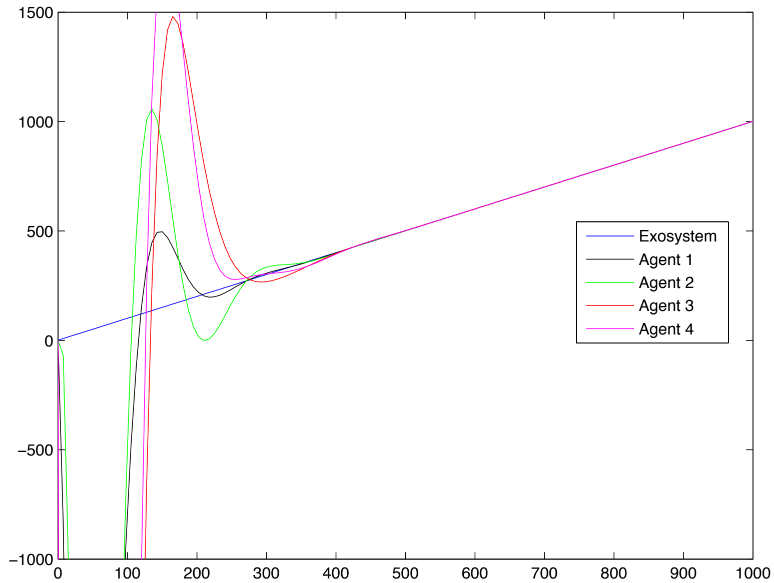


Figure 4.7: Outputs for expanded Network 1

$$x_d = \begin{bmatrix} x_{d,1} \\ x_{d,2} \\ \vdots \\ x_{d,m_d} \end{bmatrix}, \quad y_d = \begin{bmatrix} y_{d,1} \\ y_{d,2} \\ \vdots \\ y_{d,m_d} \end{bmatrix}, \quad u_d = \begin{bmatrix} u_{d,1} \\ u_{d,2} \\ \vdots \\ u_{d,m_d} \end{bmatrix},$$

and that in the new coordinate, (4.41) can be rewritten as

$$\begin{cases} \dot{x}_a &= A_{aa}x_a + L_{ab}y_b + L_{ad}y_d, \\ \dot{x}_b &= A_{bb}x_b + L_{bd}y_d, \\ \dot{x}_c &= A_{cc}x_c + B_c(u_c + E_{ca}x_a) + L_{cb}y_b + L_{cd}y_d, \\ \dot{x}_{d,j} &= A_{d,j}x_{d,j} + B_{d,j}(u_{d,j} + E_{d,j,a}x_a + E_{d,j,b}x_b + E_{d,j,c}x_c) + L_{d,j}y_d, \\ y_{d,j} &= C_{d,j}x_{d,j}, \quad j = 1, \ldots, m_d \\ y_b &= C_b x_b. \end{cases} \tag{4.42}$$

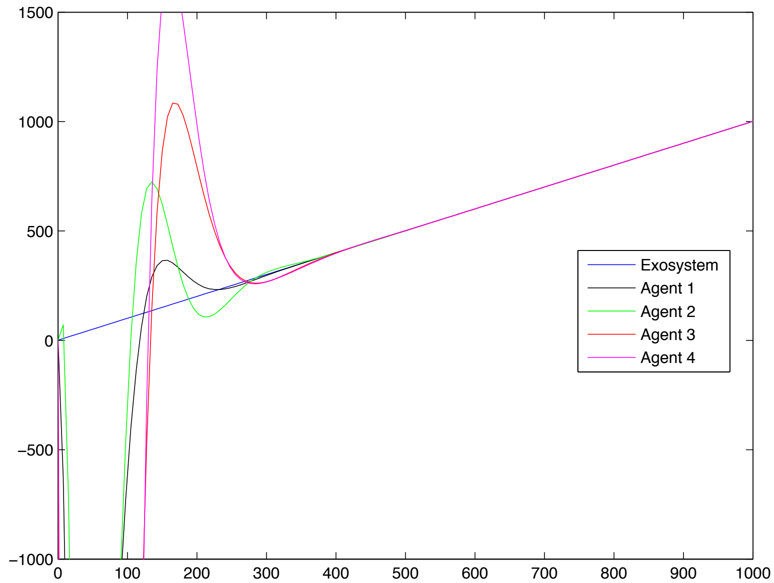


Figure 4.8: Outputs for expanded Network 2

Here the states x_a, x_b, x_c, and x_d are respectively of dimensions n_a, n_b, n_c, and $n_d = \sum_{j=1}^{m_d} n_{d,j}$, while the state $x_{d,j}$ is of dimension $n_{d,j}$ for each $j = 1, \ldots, m_d$. The inputs u_d and u_c are respectively of dimensions m_d and $m_c = m - m_d$, while the outputs y_d and y_c are respectively of dimensions m_d and $p - m_d$. The matrices $A_{d,j}$, $B_{d,j}$ and $C_{d,j}$ have the form

$$A_{d,j} = \begin{bmatrix} 0 & I_{n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_{d,j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{d,j} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Some important properties of SCB are summarized as follows:

1) The invariant zeros of the system (4.41) are the eigenvalues of A_{aa}.

2) (A_{cc}, B_c) is controllable, and (C_b, A_{bb}) is observable.

3) If the system (4.41) is right-invertible, then x_b, and hence y_b are nonexistence, and $\Gamma_o = I$.

4) If the system (4.41) is left-invertible, then x_c, and hence u_c are nonexistence, and $\Gamma_i = I$.

5) The system (4.41) has m_d zeros at the infinity with the order $n_{d,j}$, $j = 1, \ldots m_d$.

4.A.2 Squaring-down for a Right-invertible System

Let us now recall the following result from [66]:

Lemma 4.5. *Assume that for the system* (4.41)*,* (A, B) *is stabilizable,* (C, A) *is detectable, and* (C, A, B) *is right-invertible, then there exists a precompensator of the form*

$$\begin{cases} \dot{\chi}^1 = A^1\chi^1 + B^1 u^1, \\ u \ = C^1\chi^1 + D^1 u^1, \end{cases} \tag{4.43}$$

such that the resulting system of (4.41) *and* (4.43) *is invertible.*

The proof was given in [66] by explicit construction of such a precompensator. To be self contained, we briefly review such a design procedure.

If the system (4.41) is right-invertible, then x_b, and hence y_b are nonexistence. Therefore, with nonsingular transformations Γ_s and Γ_i, the system (4.41) can be transformed into the following SCB form:

$$\begin{cases} \dot{x}_a \ = A_{aa}x_a + L_{ad}y_d, \\ \dot{x}_c \ = A_{cc}x_c + B_c(u_c + E_{ca}x_a) + L_{cd}y_d, \\ \dot{x}_{d,j} = A_{d,j}x_{d,j} + B_{d,j}(u_{d,j} + E_{d,j,a}x_a + E_{d,j,c}x_c) + L_{d,j}y_d, \\ y_{d,j} = C_{d,j}x_{d,j}, \quad j = 1, \ldots, m_d \end{cases} \tag{4.44}$$

Consider the following precompensator for the system (4.44)

$$\begin{cases} \dot{\chi}^1 = N'\chi^1 + G'u^1, \\ \tilde{u} \ = [WE_c', M]'\chi^1 + [I, J]'u^1, \end{cases} \tag{4.45}$$

where $E_c = [E'_{d,1,c}, E'_{d,2,c}, \cdots, E'_{d,m_d,c}]' \in \mathbb{R}^{n_c \times m_d}$, $\chi^1 \in \mathbb{R}^{n_c - m_c}$,

$$WA'_{cc} = NW + MB'_c, \quad \text{rank} \begin{bmatrix} W \\ B'_c \end{bmatrix} = n_c, \quad [G, J]' = \begin{bmatrix} W \\ B'_c \end{bmatrix}^{-1} K,$$

and N and $A_{cc} - KE_c$ are Hurwitz stable. Such a matrix K exists since (A_{cc}, E_c) is detectable, which follows from the fact that (C, A) is detectable.

In [66], it is shown that the resulting system of (4.44) and (4.45) is invertible, and has the same infinize zero structure as the system (4.44). Moreover, the design procedure introduced additional invariant zeros, which are eigenvalues of N and $A_{cc} - KE_c$, and hence can be assigned to the open left-half plane.

It is then easy to see that the precompensator of the form (4.43) for the system (4.41) is given by

$$\begin{cases} \dot{\chi}^1 = N'\chi^1 + G'u^1, \\ u \ = \Gamma_i[WE'_c, M]'\chi^1 + \Gamma_i[I, J]'u^1. \end{cases} \tag{4.46}$$

4.A.3 Rank-equalization for a Invertible System

Let us now recall the following result from [65]:

Lemma 4.6. *Assume that the system* (4.41) *is invertible, then there exists a precompensator of the form*

$$\begin{cases} \dot{\chi}^2 = A^2\chi^2 + B^2u^2, \\ u \ = C^2\chi^2 + D^2u^2, \end{cases} \tag{4.47}$$

such that the resulting system of (4.41) *and* (4.47) *is uniform rank.*

The proof is given in [65]. The idea is to add an appropriate number of integrators to each scalar input $u_{d,j}$ for $j = 1, \ldots, m_d$. Let us briefly review such a design.

Since the system (4.41) is invertible, with a nonsingular transformation Γ_s, the system (4.41) can be transformed into the following SCB form:

$$\begin{cases} \dot{x}_a &= A_{aa}x_a + L_{ad}y_d, \\ \dot{x}_{d,j} &= A_{d,j}x_{d,j} + B_{d,j}(u_{d,j} + E_{d,j,a}x_a) + L_{d,j}y_d, \\ y_{d,j} &= C_{d,j}x_{d,j}, \quad j = 1, \ldots, m_d. \end{cases} \tag{4.48}$$

Let $\bar{r} \geq \max_{j=1,\ldots,m_d} n_{d,j}$. We then design the following pre-compensator for the system (4.48)

$$\begin{cases} \chi_j^2 &= A_j^2 x_j^2 + B_j^2 u_j^2, \\ u_{d,j} &= C_j^2 x_j^2 + D_j^2 u_j^2. \end{cases} \tag{4.49}$$

Here for the chain j where $n_{d,j} < \bar{r}$,

$$A_j^2 = \begin{bmatrix} 0 & I_{\bar{r}-n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_j^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_j^2 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_j^2 = 0,$$

while for the chain j where $n_{d,j} = \bar{r}$, χ_j^2, and hence A_j^2, B_j^2 and C_j^2 are nonexistence, while $D_j^2 = 1$, that is $u_{d,j} = u_j^2$.

It is then easy to see that the precompensator of the form (4.47) for the system (4.41) is given by

$$\begin{cases} \dot{\chi}^2 = A^2\chi^2 + B^2u^2, \\ u \;= C^2\chi^2 + D^2u^2, \end{cases}$$

where $\chi^2 = [\chi_1^2; \cdots; \chi_{m_d}^2]$, $u^2 = [u_1^2; \cdots; u_{m_d}^2]$,

$$A^2 = \text{blkdiag}\{A_j^2\}_{j=1}^{m_d},\ B^2 = \text{blkdiag}\{B_j^2\}_{j=1}^{m_d},\ C^2 = \text{blkdiag}\{C_j^2\}_{j=1}^{m_d},\ D^2 = \text{blkdiag}\{D_j^2\}_{j=1}^{m_d}.$$

4.B Proof of Lemma 4.1

We will prove Lemma 4.1 by explicit construction of the precompensator (4.7) for each agent. The design is carried in three steps.

Step 1: Squaring-down precompensator

In this step, we design a compensator for each agent $i \in \{1, \dots, N\}$ such that the resulting system is invertible. Since the triple (C_i, A_i, B_i) is right-invertible, in order to do so, we only need to design a pre-compensator of the form:

$$\begin{cases} \dot{\chi}_i^1 = A_i^1 \chi_i^1 + B_i^1 u_i^1, \\ u_i = C_i^1 \chi_i^1 + D_i^1 u_i^1, \end{cases} \tag{4.50}$$

where $u_i^1 \in \mathbb{R}^p$, such that the resulting system of (4.5) and (4.50) is invertible. The design procedure was developed in [71] and reviewed in Appendix 4.A.2.

Step 2: Rank-equalizing precompensator

It is clear that the resulting system of (4.5) and (4.50) is invertible. For a given $n_q \geq \bar{n}_d$, where $\bar{n}_d$ is the maximal order of infinite zero of (C_i, A_i, B_i) for all $i = 1, \dots, N$, we design a *rank-equalizing* precompensator of the form

$$\begin{cases} \dot{\chi}_i^2 = A_i^2 \chi_i^2 + B_i^2 u_i^2, \\ u_i^1 = C_i^2 \chi_i^2 + D_i^2 u_i^2, \end{cases} \tag{4.51}$$

where $u_i^2 \in \mathbb{R}^p$, such that the resulting system of (4.5), (4.50) and (4.51) is invertible and has uniform rank n_q. The design procedure was developed in [65] and reviewed in Appendix 4.A.3.

Step 3: Observer-based pre-feedback

The third stage is to design a observer-based controller such that the resulting system is given by (4.8) and (4.9).

It is clear that the resulting system of (4.5), (4.50) and (4.51) is invertible and has uniform rank

n_q. It is easy to see that there exists a nonsingular state transformation $\tilde{\Gamma}_{i,s}$ such that

$$\begin{bmatrix} x_i \\ \chi_i^1 \\ \chi_i^2 \end{bmatrix} = \tilde{\Gamma}_{i,s}\tilde{\chi}_i, \quad \tilde{\chi}_i = \begin{bmatrix} \tilde{\chi}_{i,a} \\ \tilde{\chi}_{i,d} \end{bmatrix},$$

such that the resulting system of (4.5), (4.50) and (4.51) can be written in the SCB form:

$$\begin{cases} \dot{\tilde{\chi}}_{i,a} = \tilde{A}_{i,a}\tilde{\chi}_{i,a} + \tilde{L}_{i,ad}y_i, \\ \dot{\tilde{\chi}}_{i,d} = \tilde{A}_d\tilde{\chi}_{i,d} + \tilde{B}_d(u_i^2 + D_{i,a}\tilde{\chi}_{i,a} + D_{i,d}\tilde{\chi}_{i,d}), \\ y_i \;= \tilde{C}_d\tilde{\chi}_{i,d}, \end{cases} \tag{4.52}$$

where

$$\tilde{A}_d = \begin{bmatrix} 0 & I_{p(n_q-1)} \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_d = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad \tilde{C}_d = \begin{bmatrix} I_p & 0 \end{bmatrix}.$$

Note that the information $\tilde{z}_i := [z_i; \chi_i^1; \chi_i^2]$ is available for agent i, and $\tilde{z}_i$ can be represented in terms of $\tilde{\chi}_{i,a}$ and $\tilde{\chi}_{i,d}$ as:

$$\tilde{z}_i = \tilde{C}_i \begin{bmatrix} \tilde{\chi}_{i,a} \\ \tilde{\chi}_{i,d} \end{bmatrix},$$

where

$$\tilde{C}_i = \begin{bmatrix} C_i^m & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{\Gamma}_{i,s}.$$

Define

$$\tilde{A}_i = \begin{bmatrix} \tilde{A}_{i,a} & \tilde{L}_{i,ad}\tilde{C}_d \\ \tilde{B}_d D_{i,a} & \tilde{A}_d + \tilde{B}_d D_{i,d} \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ \tilde{B}_d \end{bmatrix}$$

It is clear that $(\tilde{C}_i, \tilde{A}_i)$ is detectable which follows from the fact that (C_i^m, A_i) is detectable. We then design the observer-based pre-feedback for the system (4.52) as:

$$\begin{cases} \dot{\hat{\tilde{\chi}}}_i = \tilde{A}_i\hat{\tilde{\chi}}_i + \tilde{B}_i v_i - \tilde{K}_i(\tilde{z}_i - \tilde{C}_i\hat{\tilde{\chi}}_i) \\ u_i^2 = \begin{bmatrix} -D_{i,a} & \tilde{F}_d - D_{i,d} \end{bmatrix}\hat{\tilde{\chi}}_i + v_i, \end{cases} \tag{4.53}$$

where $v_i \in \mathbb{R}^p$ is a new input which will be designed in Section 4.3.3, $\tilde{K}_i$ is such that $\tilde{A}_i + \tilde{K}_i\tilde{C}_i$ is Hurwitz stable, and $\tilde{F}_d$ is such that $\tilde{A}_d + \tilde{B}_d\tilde{F}_d$ has desired eigenvalues. It is easy then to see that the observer error dynamics $\omega_i = \tilde{\chi}_i - \hat{\tilde{\chi}}_i$ is asymptotically stable, therefore, the injection term $\tilde{\chi}_{i,a}$ into the dynamics $\tilde{\chi}_{i,d}$ is asymptotically canceled. Hence, the mapping from v_i to y_i is given by

$$\begin{cases} \dot{\tilde{x}}_{i,d} = (\tilde{A}_d + \tilde{B}_d\tilde{F}_d)\tilde{\chi}_{i,d} + \tilde{B}_d(v_i + \rho_i), \\ y_i \ = \tilde{C}_d\tilde{\chi}_i, \end{cases} \tag{4.54}$$

where

$$\begin{cases} \dot{\omega}_i = (\tilde{A}_i + \tilde{K}_i\tilde{C}_i)\omega_i, \\ \rho_i = \begin{bmatrix} D_{i,a} & D_{i,d} - \tilde{F}_d \end{bmatrix}\omega_i. \end{cases} \tag{4.55}$$

It is clear that the system (4.54) is invertible, of uniform rank n_q, and has no invariant zero. Moreover, the system (4.54) is of the form (4.8) with $\bar{x}_i := \tilde{x}_{i,d}$, the parameters

$$A = \tilde{A}_d + \tilde{B}_d\tilde{F}_d, \quad B = \tilde{B}_d, \quad C = \tilde{C}_d,$$

and (4.55) is of the form (4.9) with $A_{i,s} = \tilde{A}_i + \tilde{K}_i\tilde{C}_i$ and $C_{i,s} = \begin{bmatrix} D_{i,a} & D_{i,d} - \tilde{F}_d \end{bmatrix}$.

Note that the observer-based pre-feedback (4.53) for the system in the original coordinate $[x_i; \chi_i^1; \chi_i^2]$ can be written as

$$\begin{cases} \dot{\hat{\tilde{\chi}}}_i = \tilde{A}_i\hat{\tilde{\chi}}_i + \tilde{B}_i v_i - \tilde{K}_i(\tilde{z}_i - \tilde{C}_i\hat{\tilde{\chi}}_i), \\ u_i^2 = \begin{bmatrix} -D_{i,a} & \tilde{F}_d - D_{i,d} \end{bmatrix}\tilde{\Gamma}_{i,s}\hat{\tilde{\chi}}_i + v_i. \end{cases} \tag{4.56}$$

It is easy to see that the composition of (4.50), (4.51), and (4.56), yields a pre-compensator of the form (4.7) with the parameters defined in obvious ways.

4.C Manipulation of Exosystem

Consider an arbitrary exosystem given by

$$\begin{cases} \dot{x} = Ax, \quad x(0) = x_0, \\ y = Cx, \end{cases} \tag{4.57}$$

where $x \in \mathbb{R}^r$, $y \in \mathbb{R}^p$, (C, A) is observable, and C is full column rank.

From [8, Theorem 4.3.1], we know that there exist nonsingular transformations $T_s \in \mathbb{R}^{r \times r}$ and $T_o \in \mathbb{R}^{p \times p}$, such that, in the transformed state and output, $x = T_s\tilde{x}$, $y = T_o\tilde{y}$, where

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{bmatrix}, \quad \tilde{x}_i = \begin{bmatrix} \tilde{x}_{i,1} \\ \vdots \\ \tilde{x}_{i,k_i} \end{bmatrix}, \quad i = 1, \ldots, p, \quad \tilde{y} = \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_p \end{pmatrix},$$

we have

$$\begin{cases} \dot{\tilde{x}}_i = A_i\tilde{x}_i + L_i\tilde{y}, \\ \tilde{y}_i = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}_i, \quad i = 1, \ldots, p, \end{cases} \tag{4.58}$$

with an initial condition $\tilde{x}_i(0)$ related to $x(0)$ in an obvious way, L_i is a constant matrix of an appropriate dimension and

$$A_i = \begin{bmatrix} 0 & I_{k_i-1} \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{k_i \times k_i}$$

The set of integers $\{k_1, k_2, \ldots, k_p\}$ is the observability index of (C, A). Note that k_i for $i = 1, \ldots, p$ are in general different. In order for the system to have uniform rank n_q, we then add an appropriate number of integrators to the bottom of each chain. In particular, define

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix}, \quad \bar{x}_i = \begin{bmatrix} \tilde{x}_i \\ x_i^2 \end{bmatrix} \in \mathbb{R}^{n_q}, \quad i = 1, \ldots, p, \quad x_i^2(0) = 0.$$

we then obtain that

$$\begin{cases} \dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{L}_i \tilde{y}, \quad \bar{x}_i(0) = [\tilde{x}_i(0); 0], \\ \tilde{y}_i = \bar{C}_i \bar{x}_i, \quad i = 1, \dots, p, \end{cases} \tag{4.59}$$

where

$$\bar{A}_i = \begin{bmatrix} 0 & I_{n_q-1} \\ 0 & 0 \end{bmatrix}, \quad \bar{L}_i = \begin{bmatrix} L_i \\ 0 \end{bmatrix}, \quad \bar{C}_i = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

It is easy to see that the system (4.58) and the system (4.59) generate exactly the same output $\tilde{y}$. The system (4.58) can be rewritten in a more compact form as follows:

$$\begin{cases} \dot{\bar{x}} = \bar{A} \bar{x}, \quad \bar{x}(0) = [\bar{x}_1(0); \cdots; \bar{x}_p(0)], \\ \tilde{y} = \bar{C} \bar{x}, \end{cases} \tag{4.60}$$

where

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{A}_p \end{bmatrix} + \begin{bmatrix} \bar{L}_1 \\ \vdots \\ \bar{L}_p \end{bmatrix} \bar{C}, \quad \bar{C} = \begin{bmatrix} \bar{C}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{C}_p \end{bmatrix}$$

and where $\star$ represents a matrix of less interest, generates the same output as (4.57). Note that the eigenvalues of the matrix $\bar{A}$ consists of all the eigenvalues of A and additional zero eigenvalues, which are degenerate.

Next, let us define

$$\bar{B} = \begin{bmatrix} \bar{B}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \bar{B}_p \end{bmatrix} \quad \text{where} \quad \bar{B}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^{n_q}.$$

It is then easy to see that $(\bar{C}, \bar{A}, \bar{B})$ is invertible, of uniform rank n_q, and has no invariant zero.

We then restore the output transformation T_o back to the system (4.60) as follows:

$$\begin{cases} \dot{\bar{x}} = \bar{A}\bar{x}, \quad \bar{x}(0) = [\bar{x}_1(0); \cdots; \bar{x}_p(0)], \\ y = T_o\bar{C}\bar{x}. \end{cases} \tag{4.61}$$

Note that the system (4.61) generate the same output as (4.57). Since the nonsingular output transformation does not change the zero structure and invertibility of the system, the system $(T_o\bar{C}, \bar{A}, \bar{B})$ is also invertible, of uniform rank n_q, and has no invariant zero. Finally, there exist a nonsingular state transformation that transforms the system (4.61) into the form of (4.10).

Chapter 5

Synchronization for Heterogeneous Networks of Discrete-time Introspective Right-invertible Agents

Chapter 4 studies synchronization problems for heterogeneous networks of non-identical introspective right-invertible linear time-invariant agents. The agent models are continuous-time. In this chapter, we will consider the same problems for discrete-time agent models. Let us first briefly review the literature on the synchronization problem for discrete-time agent models.

5.1 Literature Review

Although the research on synchronization is primarily focused on networks of continuous-time agent models, synchronization in homogeneous networks of discrete-time agents has been studied in [31,46,84] (also see the references therein). A distributed observer-based synchronization controller was developed in [31] which communicates the information about the protocol states over the same network communication topology. In [84], the author considered a very special case of neutrally stable agent with full actuation ($B = I$). A network of first-order agents with switching Laplacian communication topology was studied in [46]. All the aforementioned work only consider the case where all the agents models are identical, or in other word, homogeneous networks.

In this chapter, we will consider a heterogeneous network of non-identical introspective right-invertible linear time-invariant agents[1]. Both output synchronization and regulation of output synchronization problems are studied. We show that exchange of information among controllers is not needed. Depending on the desired frequencies in synchronization trajectories, different decentralized control schemes are proposed to achieve synchronization for a set of communication topologies. The chapter is organized as follows: In the remainder of this section, we declare several notations and recall some classical concept in graph theory. The network structure and preliminary assumptions and definitions are given in Section 5.2. The output synchronization and regulation of output synchronization problems are solved in Section 5.3 and 5.4 respectively. Technical development is given in the Appendix.

5.1.1 Notations and Preliminaries

The stacking column vector of column vectors $x_1, \ldots, x_n$, $x_1, \ldots, x_n$ is denoted by $[x_1; \ldots; x_n]$. For arbitrary matrix $X \in \mathbb{C}^{n\times m}$, X', X^{-1} and $\|X\|$ denote respectively the transpose, inverse and induced 2-norm of X. For square matrix $X \in \mathbb{C}^{n\times n}$, $\det(X)$ and $\lambda(A)$ represent its determinant and eigenvalue. The notation $\mathbf{1}$ denotes a column vector with element 1 whose dimension is indicated by the context and 0 represents a zero number, a zero row/column vector or a zero matrix also

[1]The definition of right-invertibility of a linear system can be found in [40].

depending on the context. We will specify their dimensions when the need arises.

For $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{p \times q}$, the Kronecker product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \vdots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}.$$

The following property of the Kronecker product will be particularly useful:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

A graph G is defined by a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, N\}$ is a vertex set and $\mathcal{E}$ is a set of pairs of vertices (i, j). Each pair in $\mathcal{E}$ is called an *arc*. A *directed path* from vertex i_1 to i_k is a sequence of vertices $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, ..., k-1$. A directed graph G contains a *directed spanning tree* if there is a node r such that a directed path exists between r and every other node.

A matrix $D = \{d_{ij}\}_{N \times N}$ is called a row stochastic matrix if

1) $d_{ij} \geq 0$ for any i, j;

2) $\sum_{j=1}^{N} d_{ij} = 1$ for $i = 1, ..., N$.

A row stochastic matrix D has at least one eigenvalue at 1 with right eigenvector $\mathbf{1}$. D can be associated with a graph $G = (\mathcal{N}, \mathcal{E})$. The number of nodes in $\mathcal{N}$ is the dimension of D and an arc $(j, i) \in \mathcal{E}$ if $d_{ij} > 0$. It is shown in [53] that 1 is a simple eigenvalue of D if and only if G contains a directed spanning tree. Moreover, the other eigenvalues are in the open unit disk if $d_{ii} > 0$ for all i.

5.2 Network Structure

Consider a heterogeneous network of N introspective agents

$$\begin{cases} x^i(k+1) = A^i x^i(k) + B^i u^i(k), \\ y^i(k) \quad = C_y^i x^i(k), \\ z^i(k) \quad = C_z^i x^i(k), \\ \zeta^i(k) \quad = \sum_{j=1}^N d_{ij}(y^i(k) - y^j(k)), \end{cases} \tag{5.1}$$

where $x^i \in \mathbb{R}^{n_i}$, $y^i \in \mathbb{R}^p$, $z^i \in \mathbb{R}^{q_i}$ and $u^i \in \mathbb{R}^{m_i}$. The matrix $D = \{d_{ij}\} \in \mathbb{R}^{N\times N}$ is a row-stochastic matrix that satisfies $d_{ii} > 0$, $d_{ij} \geq 0$ and $\sum_j d_{ij} = 1$. This D matrix defines a communication topology that can be captured by a directed graph $G = (\mathcal{N}, \mathcal{E})$. The set $\mathcal{N}$ contains all the node and $\mathcal{E}$ is the edge set such that an arc $(j,i) \in \mathcal{E}$ if $d_{ij} > 0$.

Assumption 5.1. *The communication topology G contains a directed spanning tree.*

Under Assumption 5.1, D has a simple eigenvalue at 1 associated with right eigenvector $\mathbf{1}$ and the other eigenvalues strictly inside the unit disk. Let $\lambda_1, ..., \lambda_N$ denote the eigenvalues of D such that $\lambda_1 = 1$ and $|\lambda_i| < 1$, $i = 2, ..., N$. We can define a set of communication topology as follows:

Definition 5.1. *For $\delta \in (0,1]$, let $\mathcal{G}_\delta$ denote a set of communication topologies such that for any topology $G \in \mathcal{G}_\delta$:*

1) Assumption 5.1 holds;

2) $|\lambda_i| < \delta$, $i = 2, ..., N$.

Remark 5.1. *For $\delta = 1$, $\mathcal{G}_1$ is the set of all communication topologies that satisfies Assumption 5.1. In this case, we shall drop the subscription 1 and simply denote it as $\mathcal{G}$ but it implies $\delta = 1$.*

In the network (5.1), each agent collects two measurements:

1) a network measurement $\zeta^i \in \mathbb{R}^p$ which is a combination of its own output relative to that of neighboring agents;

2) a local measurement $z^i \in \mathbb{R}^{q_i}$ of its internal dynamics.

For each agent, we make the following standard assumption.

Assumption 5.2. *The agents possess the following properties:*

1) (A^i, B^i) *is stabilizable;*

2) (A^i, C_z^i) *is detectable;*

3) (A^i, C_y^i) *is detectable;*

4) (A^i, B^i, C_y^i) *is right-invertible.*

5.3 Output Synchronization

The first problem studied in this chapter is the output synchronization problem. The output synchronization in a heterogeneous network of the form (5.1) is defined as follows:

Definition 5.2. *The agents in the network achieve output synchronization if*

$$\lim_{k\to\infty} (y^i(k) - y^j(k)) = 0, \quad \forall i, j \in \{1, \dots, N\}.$$

The output synchronization problem is formulated below:

Problem 5.1. *Consider a heterogeneous network of the form* (5.1)*. For* $\delta \in (0, 1]$ *and a given set* $\mathcal{G}_\delta$*, the output synchronization problem with a set of communication topologies* $\mathcal{G}_\delta$ *is to design a* *local linear dynamical controller*

$$\begin{cases} \hat{x}^i(k+1) = A_c^i \hat{x}^i(k) + B_c^i \zeta^i(k) + E_c^i z^i(k), \\ \quad u^i(k) \quad = C_c^i \hat{x}^i(k) + D_c^i \zeta^i(k) + M_c^i z^i(k), \end{cases} \tag{5.2}$$

such that the output synchronization can be achieved in the network with any communication topology belonging to $\mathcal{G}_\delta$.

Remark 5.2. *Since* (A^i, C_z^i) *is detectable, one can always design a local stabilizing measurement feedback controller so that the network achieves output synchronization in the sense that* $y^i(k) \to 0$ *as* $k \to \infty$. *Such a case is not interested in this chapter. We are aiming to reach synchronization with a non-trivial and possibly desirable synchronization trajectory.*

The synchronization trajectories considered in most applications are either bounded or polynomially increasing. We shall also present the main results respectively for these two cases. The first theorem is concerned with bounded synchronization trajectories.

Theorem 5.1. *For the set* $\mathcal{G}$, *Problem 5.1 with bounded synchronization trajectories is always solvable via a decentralized dynamic controller* (5.2).

Remark 5.3. *Theorem 5.1 indicates that in the case of bounded synchronization trajectories, a universal synchronization controller can be constructed which solves Problem 5.1 for any communication topology satisfying Assumption 5.1.*

If unbounded synchronization trajectories are demanded, the admissible set of communication topologies has to be more restricted. This is stated in the next theorem:

Theorem 5.2. *For* $\delta \in (0,1)$ *and a given set* $\mathcal{G}_\delta$, *Problem 5.1 with unbounded increasing synchronization trajectories is solvable via a decentralized dynamic consensus controller* (5.2).

We shall prove Theorem 5.1 and 5.2 by explicitly constructing the synchronization controllers. The design and analysis is done in the next three subsections. First, by exploiting the self-measurement of each agent, we design a local pre-compensator such that the agent model can be re-shaped as asymptotically identical, which we refer to *network homogenizing*. Next, in the resulting (asymptotically) homogeneous network, solvability of the output synchronization problem can be connected to that of a robust stabilization problem. Finally, the last step is to solve this robust stabilization problem by designing a compensator using a low-gain approach. In this stage, depending on different types of synchronization trajectories, two controllers can be constructed.

5.3.1 Homogenization of the Network

For introspective agents, their self-reflection of internal dynamics provides us with additional freedom to manipulate the agent models so as to disguise them as being almost identical to the rest of the network viewed from their output. This is shown in the next lemma.

Lemma 5.1. *Consider a heterogeneous network of the form* (5.1)*. Let n_d denote the maximum order of infinite zeros of (A^i, B^i, C^i). Suppose a triple (A, B, C) is given such that*

1) $\operatorname{rank}(C) = p$.

2) (A, B, C) is invertible, of uniform rank $n_q \geq n_d$ and has no invariant zeros.

There exists a compensator

$$\begin{cases} \xi^i(k+1) = A_h^i \xi^i(k) + B_h^i z^i(k) + E_h^i v^i(k), \\ \quad u^i(k) \quad = C_h^i \xi^i(k) + D_h^i v^i(k), \end{cases} \tag{5.3}$$

such that the closed-loop system of (5.1) *and* (5.3) *can be written in the following form:*

$$\begin{cases} \bar{x}^i(k+1) = A\bar{x}^i(k) + B\left(v^i(k) + d^i(k)\right), \\ y^i(k) \quad = C\bar{x}^i(k), \\ \zeta^i(k) \quad = \sum_{j=1}^N d_{ij}(y^i(k) - y^j(k)), \end{cases} \tag{5.4}$$

where d^i are generated by

$$\begin{cases} e^i(k+1) = A_s^i e^i(k), \quad i = 1, ..., N, \\ d^i(k) \quad = C_s^i e^i(k). \end{cases} \tag{5.5}$$

and A_s^i are Schur stable.

Proof. Proof and detailed design procedure can found in Appendix 5.A. □

Remark 5.4. *We have the following observations*

1) *The first condition of Lemma 5.1 is natural in the sense that the new model much maintain the same interface with the network.*

2) *The condition that (A, B, C) is invertible and has no invariant zero implies that (A, B) is controllable and (A, C) is observable.*

3) *We have a substantial freedom in choosing the eigenvalues of A which, as will be seen, determine the modes in the synchronization trajectories.*

Remark 5.5. *It should also be noted that such a triple (A, B, C) always exists and, without loss of generality, takes the following form:*

$$A = A_0 + BF, \quad A_0 = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C = \begin{bmatrix} I_p & 0 \end{bmatrix}, \tag{5.6}$$

and F is such that $A_0 + B_0F$ has desired eigenvalues. Such an F exists due to the fact that (A_0, B_0) is controllable.

5.3.2 Connection to Robust Simultaneous Stabilization Problem

In this subsection, we shall show that the output synchronization in an (asymptotically) homogeneous network (5.4) and (5.5) can be solved by equivalently solving a robust stabilization problem.

Suppose the synchronization problem for network (5.4) and (5.5) with any communication topology in $\mathcal{G}_\delta$ can be solved by a compensator

$$\begin{cases} \chi^i(k+1) = A_c\chi^i(k) + B_c\zeta^i(k), \\ v^i(k) \quad = C_c\chi^i(k). \end{cases} \tag{5.7}$$

Let $\tilde{x}^i = [\bar{x}^i; \chi^i]$. Then the closed-loop of each agent can be written as

$$\begin{cases} \tilde{x}^i(k+1) = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \tilde{x}^i(k) + \begin{bmatrix} 0 \\ B_c \end{bmatrix} \zeta^i(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} d^i(k), \\ y^i(k) \quad = \begin{bmatrix} C & 0 \end{bmatrix} \bar{x}^i(k), \\ \zeta^i(k) \quad = y^i(k) - \sum_{j=1}^N d_{ij}y^j(k). \end{cases} \tag{5.8}$$

Define $\tilde{x} = [\tilde{x}^1; \cdots; \tilde{x}^N]$,

$$\bar{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \text{ and } \bar{E} = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

The overall dynamics of the N agents can be written as

$$\tilde{x}(k+1) = \left[I_N \otimes \bar{A} + (I_N - D) \otimes \bar{B}\bar{C}\right] \tilde{x}(k) + (I_N \otimes \bar{E})d(k).$$

Define $\eta = [\eta^1; \cdots; \eta^N] = (T \otimes I_n)\tilde{x}$, where $\eta^i \in \mathbb{C}^n$ and T is such that $J_L = T(I_N - D)T^{-1}$ is in the Jordan canonical form and $J_L(1,1) = 0$. In the new coordinates, the dynamics of η can be written as

$$\eta(k+1) = \left[I_N \otimes \bar{A} + J_L \otimes \bar{B}\bar{C}\right] \eta(k) + (T \otimes \bar{E})d(k).$$

Lemma 5.2. *The network of the form* (5.8) *achieves output synchronization if* $\eta^i(k) \to 0$ *as* $k \to \infty$ *for* $i = 2, ..., N$.

Proof. Let

$$\pi(k) = \begin{bmatrix} \eta^1(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \eta^1(k)$$

If $\eta(k) \to \pi(k)$, then $\tilde{x}(k) \to (T^{-1} \otimes I_n)\pi(k)$. Note that the columns of T^{-1} comprise all the right eigenvectors and generalized eigenvectors of $I - D$. The first column of T^{-1} is vector $\mathbf{1}$. Hence the fact that $\tilde{x}(k) \to (T^{-1} \otimes I_n)\pi(k)$ implies that

$$\tilde{x}(k) \to \mathbf{1} \otimes \eta^1(k).$$

□

Remark 5.6. *It also becomes clear from Lemma 5.2 that the synchronization trajectory is given by* $\eta^1(k)$ *which is governed by*

$$\eta^1(k+1) = A\eta^1(k) + (w \otimes \bar{E})d(k), \quad \eta^1(0) = (w \otimes I_n)\tilde{x}(0),$$

where w *is the first row of* T*, that is, the left eigenvector associated with eigenvalue* 1*. Note that* $d(k) \to 0$ *as* $k \to \infty$*. This shows that the modes of the synchronization trajectory are determined by the eigenvalues of* A *and the complete dynamics depends on both* A *and a weighted average of the agents' initial conditions.*

Define $\bar{\eta} = [\eta^2; \cdots; \eta^N]$. Taking the dynamics of d into account, we can write

$$\begin{bmatrix} \bar{\eta}(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \bar{A} + \bar{J}_L \otimes \bar{B}\bar{C} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} \begin{bmatrix} \bar{\eta}(k) \\ e(k) \end{bmatrix}, \tag{5.9}$$

where $e = [e^1; \ldots; e^N]$,

$$\bar{C}_s = \text{blkdiag}\{C_s^i\}_{i=1}^N, \quad \bar{I} = [0, I_{N-1}], \quad \bar{A}_s = \text{blkdiag}\{A_s^i\}_{i=1}^N,$$

and $\bar{J}_L$ is such that

$$J_L = \begin{bmatrix} 0 & \\ & \bar{J}_L \end{bmatrix}.$$

Clearly $\bar{\eta} \to 0$ for any initial condition if the system (5.9) is globally asymptotically stable. Since $\bar{A}_s$ is Schur stable, the next lemma is straightforward:

Lemma 5.3. *The network of the form* (5.8) *achieves output synchronization if the system*

$$\tilde{\eta}(k+1) = (I_{N-1} \otimes \bar{A} + \bar{J}_L \otimes \bar{B}\bar{C})\tilde{\eta}(k), \tag{5.10}$$

is globally asymptotically stable.

Due to upper-triangular structure of $I_{N-1} \otimes \bar{A}$ and $\bar{J}_L \otimes \bar{B}\bar{C}$, the system (5.10) is essentially a family of $N-1$ subsystems:

$$\tilde{\eta}^i(k+1) = (\bar{A} + (1-\lambda_i)\bar{B}\bar{C})\tilde{\eta}^i(k), \quad i = 2, ..., N, \tag{5.11}$$

where λ_i, $i = 2, ..., N$ are those eigenvalues of D that are not equal to 1.

Lemma 5.4. *The network of the form* (5.8) *achieves output synchronization if* (5.11) *is globally asymptotically stable for* λ_i, $i = 2, ..., N$.

Note that (5.11) can be viewed as the closed-loop of

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ \quad z(k) \quad = (1-\lambda)Cx(k). \end{cases} \tag{5.12}$$

and a compensator

$$\begin{cases} \chi(k+1) = A_c\chi(k) + B_c z(k), \\ \quad u(k) \quad = C_c\chi(k), \end{cases} \tag{5.13}$$

with unknown λ satisfying $|\lambda| < \delta$. It is easy to see that owing to linearity, (5.13) stabilizes (5.12) if it stabilizes

$$\begin{cases} x(k+1) = Ax(k) + (1-\lambda)Bu(k), \\ \quad z(k) \quad = Cx(k). \end{cases} \tag{5.14}$$

Therefore, we arrive at the following conclusion by the end of this subsection.

Lemma 5.5. *Problem 5.1 is solved via a composite controller of* (5.3) *and* (5.7) *if the closed-loop of* (5.14) *and* (5.13) *is globally asymptotically stable for all* $|\lambda_i| < \delta$.

Proof. By establishing Lemma 5.2-5.4, we have shown that if the closed-loop of (5.14) and (5.13) is globally asymptotically stable for all $|\lambda| < \delta$, then the interconnections of the closed-loop of compensator (5.7) and (5.4), which is the network (5.8), will reach synchronization. This implies that the composite controller of (5.3) and (5.7) solves Problem 5.1. □

So far, we have converted the output synchronization problem to a simultaneous stabilization problem. Next, depending on different types of synchronization trajectories, the design bifurcates into two approaches.

5.3.3 Bounded Synchronization Trajectories

It has been shown that the eigenvalues of A dictate the modes in the synchronization trajectories. If the trajectories are required to be bounded, we can choose A matrix in Lemma 5.1 to have only semi-simple eigenvalues on the unit circle. This can be done by choosing proper F matrix in (5.6). Note that in this case, we can always assume without loss of generality that $A'A = I$. The controller designed based on this type of A matrix can be easily modified by a state transformation so as to be applicable to the agents with a more general form.

Based on the analysis in the preceding subsection, to prove Theorem 5.1, we need to design data (A_c, B_c, C_c) in the compensator (5.7) for which the closed-loop of (5.14) and (5.13) is globally asymptotically stable with any $|\lambda| < 1$. To do this, we construct (5.7) in the following form:

$$\begin{cases} \chi^i(k+1) = (A+KC)\chi^i(k) - K\zeta^i(k) \\ \quad v^i(k) \quad = -\epsilon B'A\chi^i(k), \end{cases} \tag{5.15}$$

where K is such that $A+KC$ is Schur stable and $\epsilon > 0$ is a design parameter to be chosen later. In other words, we choose $A_c = A+KC$, $B_c = -K$ and $C_c = -\epsilon B'A$. With this set of data, the closed-loop (5.14) and (5.13) can be written as

$$\begin{cases} x(k+1) = Ax(k) - (1-\lambda)\epsilon BB'A\chi(k), \\ \chi(k+1) = (A+KC)\chi(k) - KCx(k). \end{cases} \tag{5.16}$$

Lemma 5.6. *There exists an $\epsilon^* > 0$ such that for $\epsilon \in (0, \epsilon^*]$, (5.16) is globally asymptotically stable for $|\lambda| \in (0,1)$.*

Proof. Define $e(k) = x(k) - \chi(k)$. The system (5.16) can be rewritten in terms of x and e as follows:

$$\begin{cases} x(k+1) = (A-(1-\lambda)\epsilon BB'A)x(k) + (1-\lambda)\epsilon BB'Ae(k), \\ e(k+1) = (A+KC+(1-\lambda)\epsilon BB'A)e(k) - (1-\lambda)\epsilon BB'Ax(k). \end{cases} \tag{5.17}$$

Let Q be the positive definite solution of Lyapunov equation

$$(A+KC)'Q(A+KC) - Q + 4I = 0.$$

Since $|\lambda| \in (0, \delta)$, there exists an ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$,

$$(A+KC+(1-\lambda)\epsilon BB'A)^*Q(A+KC+(1-\lambda)\epsilon BB'A) - Q + 3I \leq 0.$$

Consider $V_1(k) = e(k)^*Qe(k)$. Let $\mu(k) = \epsilon B'Ax(k)$. To ease our presentation, we shall omit the time label (k) whenever this causes no confusion.

$$\begin{aligned}
&V_1(k+1) - V_1(k) \\
\le& -3\|e\|^2 + 2\left|\mathrm{re}\Big((1-\lambda)^*\mu^*B'Q[A+KC+(1-\lambda)BB'A]e\Big)\right| + |1-\lambda|^2\mu^*B'QB\mu \\
\le& -3\|e\|^2 + |1-\lambda|M_1\|\mu\|\|e\| + |1-\lambda|^2M_2\|\mu\|^2,
\end{aligned}$$

where

$$M_1 = 2\|B'Q\|\|A+KC\| + 4\|B'Q\|\|BB'A\|, \quad M_2 = \|B'QB\|.$$

It should be noted that M_1 and M_2 are independent of ϵ and λ.

Consider $V_2(k) = \|x(k)\|^2$. Note that

$$\begin{aligned}
&[A-(1-\lambda)\epsilon BB'A]'[A-(1-\lambda)\epsilon BB'A] - I \\
=& -2\,\mathrm{re}(1-\lambda)\epsilon A'BB'A + |1-\lambda|^2\epsilon^2A'BB'BB'A
\end{aligned}$$

There exists an ϵ_2 such that for $\epsilon \in (0, \epsilon_2]$, $\epsilon B'B \le \frac{1}{2}I$. Since $|1-\lambda|^2 \le 2\,\mathrm{re}(1-\lambda)$ for $|\lambda| < 1$, we get for $\epsilon \in (0, \epsilon_2]$,

$$[A-(1-\lambda)\epsilon BB'A]^*[A-(1-\lambda)\epsilon BB'A] - A'A \le -\tfrac{1}{2}|1-\lambda|^2\epsilon ABB'A.$$

Hence

$$\begin{aligned}
&V_2(k+1) - V_2(k) \\
\le& -\tfrac{1}{2\epsilon}|1-\lambda|^2\|\mu\|^2 + 2\,\mathrm{re}\left((1-\lambda^*)e^*A'B\mu - |1-\lambda|^2\epsilon e^*A'BB'B\mu\right) + |1-\lambda|^2\epsilon^2e^*A^*BB'BB'Ae \\
\le& -\tfrac{1}{2\epsilon}|1-\lambda|^2\|\mu\|^2 + \theta_1|1-\lambda|\|e\|\|\mu\| + \theta_3|1-\lambda|^2\|e\|\|\mu\| + \theta_2\|e\|^2,
\end{aligned}$$

where

$$\theta_1 = 2\|A'B\|, \quad \theta_3 = 2\|A'BB'B\|, \quad \theta_2 = 4\|A'BB'B'A\|.$$

Define a Lyapunov candidate $V(k) = V_1(k) + \epsilon\kappa V_2(k)$ with

$$\kappa = 4 + 2M_2 + 2M_1^2.$$

We get that

$$V(k+1) - V(k) \leq -(3 - \epsilon\theta_2\kappa)\|e\|^2 - (2 + M_1^2)|1-\lambda|^2\|\mu\|^2 \\ + (M_1 + \epsilon\theta_1\kappa)|1-\lambda|\|\mu\|\|e\| + \epsilon\theta_3\kappa|1-\lambda|^2\|\mu\|\|e\|.$$

There exists an ϵ_3 such that for $\epsilon \in (0, \epsilon_3]$,

$$3 - \epsilon\theta_2\kappa \geq 2.5, \quad M_1 + \epsilon\theta_1\kappa \leq 2M_1, \text{ and } \epsilon\theta_3\kappa \leq 1.$$

This yields that

$$\begin{aligned} & V(k+1) - V(k) \\ \leq & -2.5\|e\|^2 - (2 + M_1^2)|1-\lambda|^2\|\mu\|^2 + (2M_1|1-\lambda| + |1-\lambda|^2)\|\mu\|\|e\| \\ \leq & -0.5\|e\|^2 - |1-\lambda|^2\|\mu\|^2 - (\|e\| - M_1|1-\lambda|\|\mu\|)^2 - |1-\lambda|^2(\tfrac{1}{2}\|e\| - \|\mu\|)^2 \\ \leq & -0.5\|e\|^2 - |1-\lambda|^2\|\mu\|^2. \end{aligned}$$

Since (A, B) is controllable, it follows from LaSalle's invariance principle that system (5.17) is globally asymptotically stable. □

5.3.4 Unbounded Synchronization Trajectories

We proceed to consider synchronization trajectories that are possibly unbounded. In most applications, those unbounded synchronization trajectories are normally polynomially increasing. This can be achieved by choosing an A matrix that has all the eigenvalues on the unit circle, some of which may be degenerate. It should be pointed out we can not only assign the eigenvalues of A

to arbitrary locations, but we are also able to assign the multiplicity structures of the eigenvalues as long as they are compatible, that is, the summation of all algebraic multiplicities equals to the dimension of A.

Our design is built upon the solution of the following discrete algebraic Riccati equation (DARE) which is also used in [31]:

$$P_\epsilon = A' P_\epsilon A + \epsilon I - (1-\delta^2) A' P_\epsilon B (B' P_\epsilon B + I)^{-1} B' P_\epsilon A. \tag{5.18}$$

The next lemma can be proven following the work in [72, 77] (see also [31]).

Lemma 5.7. *For any $\epsilon > 0$ and $\delta \in (0,1)$, the DARE (5.18) has a unique positive definite solution P_ϵ and moreover $A - (1-\lambda)B(B'P_\epsilon B + I)^{-1} B^* P_\epsilon A$ is Schur stable for $|\lambda| < \delta$.*

The compensator (5.7) can be designed as follows

$$\begin{cases} \chi^i(k+1) = (A+KC)\chi^i(k) - K\zeta^i(k), \\ \quad v^i(k) \quad = F_\epsilon \chi^i(k), \end{cases} \tag{5.19}$$

where K is such that $A + KC$ is Schur stable and

$$F_\epsilon = -(B' P_\epsilon B + I)^{-1} B' P_\epsilon A. \tag{5.20}$$

In this case, $A_c = A + KC$, $B_c = -K$ and $C_c = F_\epsilon$. We shall prove that with this set of data, the closed-loop of (8.34) and (5.13) is globally asymptotically stable for $|\lambda| < \delta$. The closed-loop system can be written as:

$$\begin{cases} x(k+1) = Ax(k) + (1-\lambda) B F_\epsilon \chi(k), \\ \chi(k+1) = (A+KC)\chi(k) - KCx(k). \end{cases} \tag{5.21}$$

Lemma 5.8. *Let $\delta \in (0,1)$ be given. There exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$, the system (5.21) is globally asymptotically stable for $|\lambda| < \delta$.*

Proof. Define $e = x - \chi$. The (5.21) can be written in terms of x and e as follows:

$$\begin{cases} x(k+1) = [A + (1-\lambda)BF_\epsilon]x(k) - (1-\lambda)BF_\epsilon e(k), \\ e(k+1) = [A + KC - (1-\lambda)BF_\epsilon]e(k) + (1-\lambda)BF_\epsilon x(k). \end{cases}$$

Let Q be the positive definite solution of Lyapunov equation

$$(A + KC)'Q(A + KC) - Q + 4I = 0.$$

Since $F_\epsilon \to 0$ as $\epsilon \to 0$, there exists an ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$

$$(A + KC - (1-\lambda)BF_\epsilon)'Q(A + KC - (1-\lambda)BF_\epsilon) - Q + 3I \leq 0.$$

Consider $V_1(k) = e(k)^* Q e(k)$. Let $\mu(k) = F_\epsilon x(k)$. To ease our presentation, we shall omit the time label (k) whenever this causes no confusion.

$$\begin{aligned} & V_1(k+1) - V_1(k) \\ \leq & -3\|e\|^2 + 2\,\mathrm{re}\Big((1-\lambda)^* \mu^* B'Q[A + KC - (1-\lambda)BF_\epsilon]e\Big) + |1-\lambda|^2 \mu^* B'QB\mu \\ \leq & -3\|e\|^2 + M_1\|\mu\|\|e\| + M_2\|\mu\|^2, \end{aligned} \tag{5.22}$$

where

$$M_1 = 4\|B'Q\|\|A + KC\| + 8\|B'Q\| \max_{\epsilon \in [0,1]}\{\|BF_\epsilon\|)\}, \quad M_2 = 4\|B'QB\|.$$

It should be noted that M_1 and M_2 are independent of ϵ and λ provided that $\|\lambda\| < \delta$.

Consider $V_2(k) = x(k)'P_\epsilon x(k)$. We have that

$$\begin{aligned} V_2(k+1) - V_2(k) \leq -\epsilon\|x\|^2 - (1-\delta)^2\|\mu\|^2 + 2\,\mathrm{re}\Big((1-\lambda)^* e^* F_\epsilon' B' P_\epsilon [A + (1-\lambda)BF_\epsilon]x\Big) \\ + |1-\lambda|^2 e^* F_\epsilon' B P_\epsilon B F_\epsilon e. \end{aligned}$$

Note that

$$
\begin{aligned}
e^* F_\epsilon' B' P_\epsilon [A + (1-\lambda) B F_\epsilon] x &= e^* F_\epsilon' B' P_\epsilon A x + (1-\lambda) e^* F_\epsilon' B' P_\epsilon B \mu \\
&= -e^* F_\epsilon' (B' P_\epsilon B + I) \mu + (1-\lambda) e^* F_\epsilon' B' P_\epsilon B \mu \\
&= e^* [F_\epsilon' + -\lambda F_\epsilon' B' P_\epsilon B] \mu,
\end{aligned}
$$

and hence

$$
V_2(k+1) - V_2(k) \leq -\epsilon \|x\|^2 - (1-\delta)^2 \|\mu\|^2 + \theta_1(\epsilon) \|e\| \|\mu\| + \theta_2(\epsilon) \|e\|^2, \tag{5.23}
$$

where

$$
\theta_1(\epsilon) = 4(\|F_\epsilon'\| + 3\|F_\epsilon' B' P_\epsilon B\|), \quad \theta_2(\epsilon) = 4\|F_\epsilon' B P_\epsilon B F_\epsilon\|.
$$

Consider a Lyapunov candidate $V(k) = V_1(k) + \kappa V_2(k)$ with

$$
\kappa = \frac{M_2 + M_1^2}{1 - \delta^2}.
$$

In view of (5.22) and (5.23), we get

$$
V(k+1) - V(k) \leq -\epsilon\kappa \|x\|^2 - M_1^2 \|\mu\|^2 - [3 - \kappa\theta_2(\epsilon)] \|e\|^2 + [M_1 + \kappa\theta_1(\epsilon)] \|\mu\| \|e\|.
$$

There exists an $\epsilon^* \leq \epsilon_1$ such that for $\epsilon \in (0, \epsilon^*]$,

$$
3 - \kappa\theta_2(\epsilon) \geq 2, \quad M_1 + \kappa\theta_1(\epsilon) \leq 2M_1.
$$

This yields that

$$
V(k+1) - V(k) \leq -\epsilon\kappa \|x\|^2 - \|e\|^2 - (\|e\| - M_1 \|\mu\|)^2.
$$

Therefore, for $\epsilon \in (0, \epsilon^*]$, the system (5.21) is globally asymptotically stable. □

5.3.5 Application to Output Formation

The formation problem is closely related to synchronization. The design procedure in preceding subsections can be easily modified to solve the output formation problem.

Definition 5.3. *An output formation is a family of vectors* $\{h_1, \ldots, h_N\}$*,* $h_i \in \mathbb{R}^p$*. The network of the form* (5.1) *is said to achieve output formation if*

$$\lim_{k\to\infty} [(y_i(k) - h_i) - (y_j(k) - h_j)] = 0.$$

Theorem 5.3. *Consider a heterogeneous network of the form* (5.1)*. For any* $\delta \in (0, 1]$*, a given set of communication topologies* $\mathcal{G}_\delta$ *and a formation vector* $\{h_1, ..., h_N\}$*, the output formation is always achievable via a local compensator in the form* (5.2) *in the network with any communication topology in* $\mathcal{G}_\delta$*.*

The proof and controller design follows a similar procedure as in the output synchronization problem. First, we design a pre-compensator in the form of (5.3) for each agent to homogenize the network utilizing its local measurements so that the agents are asymptotically identical to a new model characterized by (A, B, C) for which the output formation is always achievable. The existence of such a triple (A, B, C) is shown in the next lemma.

Lemma 5.9. *For a given family of vectors* $\{h_1, ..., h_N\}$ *and integer* $n_q > 0$*,* $h_i \in \mathbb{R}^p$*, there exists a triple* (A, B, C) *and another set of vectors* $\{\bar{h}_1, ..., \bar{h}_N\}$ *of appropriate dimensions such that*

1) $\text{rank}(C) = p$*,*

2) (A, B, C) *is invertible, of uniform rank* n_q *and has no invariant zero,*

3) $C\bar{h}_i = h_i$*,* $i = 1, ..., N$*,* *4) $A\bar{h}_i = \bar{h}_i$, $i = 1, ..., N$, i.e. A has some semi-simple eigenvalues at 1 with eigenvectors $\bar{h}_i$,*

5) the other eigenvalues of A are at desired locations on the unit circle.

Proof. Choose $\bar{h}_i = h_i \otimes \mathbf{1}$ with $\mathbf{1} \in \mathbb{R}^{n_q}$ and let

$$A_0 = \text{blkdiag}\{A_{ii}\}_{i=1}^p, \quad B = \text{blkdiag}\{B_{ii}\}_{i=1}^p, \quad C = \text{blkdiag}\{C_{ii}\}_{i=1}^p,$$

and

$$A_{ii} = \begin{bmatrix} 0 & I_{(n_q-1)} \\ 0 & 0 \end{bmatrix}, \quad B_{ii} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{ii} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and $A = A_0 + BF$ for some matrix F of appropriate dimension. Obviously, Conditions 1,2 and 3 are satisfied. What remains is to choose an F such that conditions 4 and 5 can be satisfied.

Let $F = \text{blkdiag}\{F_{11}, \ldots, F_{pp}\}$ where F_{ii} is such that $A_i + B_i F_{ii}$ has all its eigenvalues on the unit circle and at least one eigenvalue at 1. Then we get

$$A = A_0 + B_0 F = \text{blkdiag}\{A_{ii} + B_{ii}F_{ii}\}_{i=1}^p. \tag{5.24}$$

and hence Condition 5 is satisfied.

It remains to show Condition 4. In view of the structure of $\bar{h}_i$ and A, we find that $A\bar{h}_i = \bar{h}_i$ if $\mathbf{1}$ is an eigenvector of $A_{ii} + B_{ii}F_{ii}$ associated with eigenvalue 1. Suppose $F_{ii} = [f_{n_q}, ..., f_1]$. We observe that

$$(A_{ii} + B_{ii}F_{ii})\mathbf{1} = \mathbf{1} \Leftrightarrow \sum_{i=1}^{n_q} f_i = 1.$$

On the other hand, the characteristic polynomial of $A_{ii} + B_{ii}F_{ii}$ is given by

$$C(\lambda) = \det(\lambda I - A_{ii} - B_{ii}F_{ii}) = \lambda^{n_q} - f_1\lambda^{n_q-1} - \cdots - f_{n_q-1}\lambda - f_{n_q}$$

Since $A_{ii} + B_{ii}F_{ii}$ has at least one eigenvalue at 1, we get $C(1) = 1 - \sum_{i=1}^{n_q} f_i = 0$ and $\sum_{i=1}^{n_q} f_i = 1$. Therefore, $\mathbf{1}$ is an eigenvector of $A_{ii} + B_{ii}F_{ii}$ for $i = 1, ..., p$ and Condition 4 is satisfied. $\square$

Based on Lemma 5.9 and its proof, we can place p semi-simple eigenvalues of A at 1 with eigenvectors $\{\bar{h}_1, ..., \bar{h}_N\}$ (at most p of them can be linearly independent). We have have a complete freedom to choose the locations of the other eigenvalues and a relative freedom to assign their multiplicity structures. Similarly as in preceding subsections, we can put only semi-simple eigenvalues on the unit circle to ensure bounded synchronization trajectories or allow degenerate eigenvalues to have unbounded synchronization trajectories. Next, depending on the type of synchronization trajectories and the resulting choice of A, a local formation controller can be constructed for the new model as follows:

$$\begin{cases} \chi^i(k+1) = (A+KC)\chi^i(k) - K\left[\sum_{j=1}^N d_{ij}[(y^i(k)-h_i)-(y^j(k)-h_j)]\right], \\ v^i(k) \quad = C_c\chi^i(k), \end{cases} \tag{5.25}$$

where

$$C_c = \begin{cases} -\epsilon B'A, & A \text{ only has semi-simple eigenvalues on the unit circle;} \\ -(B'P_\epsilon B + I)^{-1}B'P_\epsilon A, & A \text{ has degenerate eigenvalues on the unit circle,} \end{cases}$$

where P_ϵ is the positive definite solution of (5.18).

Proof of Theorem 5.3. For any triple (A, B, C) satisfying the conditions in Lemma 5.9, there exists a shaping pre-compensator in the form of (5.3) such that the interconnection of the agents and shaping pre-compensator can be written in the following form:

$$\begin{cases} \bar{x}^i(k+1) = A\bar{x}^i(k) + B\left(v^i(k) + d^i(k)\right), \\ y^i(k) \quad = C\bar{x}^i(k). \end{cases} \tag{5.26}$$

Let $\bar{x}_s^i = \bar{x}^i - \bar{h}_i$. In view of Condition 3 and 4 in Lemma 5.9, the closed-loop system of (5.26) and controller (5.25) can be written in terms of $\bar{x}_s^i$ and χ^i as

$$\begin{cases} \bar{x}_s^i(k+1) = A\bar{x}_s^i(k) + BC_c\chi^i(k) + Bd^i(k), \\ \chi^i(k) \quad = A_c\chi^i(k) + B_cC\left[\sum_{j=1}^N d_{ij}(\bar{x}_s^i(k) - \bar{x}_s^j(k))\right] \end{cases}$$

We have already proved that the above network synchronizes. Hence

$$\lim_{k\to\infty}\left[C\bar{x}_s^i(k) - C\bar{x}_s^j(k)\right] = \lim_{k\to\infty}\left[(y_i(k) - h_i) - (y_j(k) - h_j)\right] = 0. \qquad \square$$

Remark 5.7. *We would like to emphasize that thanks to the freedom we have in choosing appropriate (A, B, C), no restriction on formation vector needs to be imposed.*

5.4 Output Regulation

Despite a freedom in choosing the mode or frequencies of the synchronization trajectories, we can not plan the trajectories arbitrarily because they are partially determined by the weighted average of initial conditions. On the other hand, it is important in some scenario to regulate the output of the agents to desired trajectories when the output synchronization is reached. Suppose the objective trajectories are generated by an exo-system

$$\begin{cases} x^0(k+1) = A^0 x^0(k), \quad x^0(0) = x_r, \\ \quad y^0(k) \quad = C^0 x^0(k), \end{cases} \tag{5.27}$$

where A^0 has all its eigenvalues in the closed unit disc and (A^0, C^0) is observable. It is reasonable to assume that the synchronization trajectories are not geometrically increasing.

We want to regulate each agent's output to y^0. Instead of disseminating the information of exo-system to every agent, we assume that only the *root*, which is agent 1, receives such information. In this case, the root measures its output relative to y^0 besides what it originally receives from the network. To be precise, agent 1 takes the following form:

$$\begin{cases} x^1(k+1) = A^1 x^1(k) + B^1 u^1(k), \\ z^1(k) \quad = C_z^1 x^1(k), \\ y^1(k) \quad = C_y^1 x^1(k), \\ \zeta^1(k) \quad = \sum_{j=1}^N d_{1j}(y^1(k) - y^j(k)) + \delta(y^1(k) - y^0(k)), \end{cases} \tag{5.28}$$

where $\delta = \frac{d_{11}}{2} > 0$.

Definition 5.4. *The agents in the network achieve output regulation if*

$$\lim_{k\to\infty} (y^i(k) - y^0(k)) = 0, \quad \forall i \in \{1, \dots, N\}.$$

We then formulate the regulation problem as follows:

Problem 5.2. *Consider a heterogeneous network of the form* (5.1)*. For a given exo-system* (5.27)*, a set $\mathcal{G}_\delta$, the output regulation problem with exo-system* (5.27) *and a set of communication topologies $\mathcal{G}_\delta$ is to design a local linear dynamical controller* (5.2) *such that the output regulation can be achieved in the network with any communication topology belonging to $\mathcal{G}_\delta$.*

Before we present the result for output regulation problem, some preparatory work needs to be done. First, we augment the network by including the exo-system as agent 0. In this augmented network, the agent 0 does not have any network measurement. We can write network measurement of all the agents uniformly as

$$\tilde{\zeta}^i(k) = \sum_{j=0}^{N} \tilde{d}_{ij}(y^i(k) - y^j(k)), i = 0, ..., N,$$

where

$$\bar{D} = \{\bar{d}_{ij}\} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{d_{11}}{2} & \frac{d_{11}}{2} & d_{12} & \cdots & d_{1N} \\ 0 & d_{21} & d_{22} & \cdots & d_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & d_{N1} & d_{N2} & \cdots & d_{NN} \end{bmatrix}. \tag{5.29}$$

This $\bar{D}$ is also an row stochastic matrix and defines an augmented topology $\bar{G}$. Moreover, since agent 0 (exo-system) is connected to the root of the original network via an out-coming arc $(0, 1)$, this $\bar{G}$ also has a directed spanning tree with a new root at agent 0 (exo-system). Suppose eigenvalues of $\bar{D}$ are denoted by $\bar{\lambda}_i,\ i = 0, ..., N$ with $\bar{\lambda}_0 = 1$ and $\bar{\lambda}_i,\ i = 1, ..., N$ are in open unit disc. For a given set $\mathcal{G}_\delta$, the set of augmented topologies by including agent 0 and the arc $(0, 1)$ can be denoted by $\bar{\mathcal{G}}_{\bar{\delta}}$ such that for any $\bar{G} \in \bar{\mathcal{G}}_{\bar{\delta}}$,

$$|\bar{\lambda}_i| < \bar{\delta},\ i = 1, ..., N.$$

We have the following theorem:

Theorem 5.4. *Consider a heterogeneous network of the form* (5.1) *and an exo-system* (5.27). *For a given set $\mathcal{G}_\delta$, Problem 5.2 is solvable via a decentralized dynamic consensus controller* (5.2).

Proof. For a given exo-system (5.27), it is shown in Appendix 5.B that there exists another system and initial condition:

$$\begin{cases} \tilde{x}^0(k+1) = \tilde{A}^0 \tilde{x}^0(k), \quad \tilde{x}^0(0) = \tilde{x}^0_0 \\ \quad y^0(k) \quad = \tilde{C}^0 \tilde{x}^0(k) \end{cases} \tag{5.30}$$

that produces the same output as the original exo-system (5.27). Moreover, we can find a $\tilde{B}^0$ matrix such that $(\tilde{A}^0, \tilde{B}^0, \tilde{C}^0)$ is invertible, of uniform rank n_q and has no invariant zero where n_q

is an integer greater than the maximal order of infinite zeros of all the agent and the observability index of (C^0, A^0). We can view this system as the exo-system.

It is shown in Lemma 5.1 that a pre-compensator of the form (5.3) can be designed for agent $1, ..., N$ such that their internal model can be shaped as asymptotically identical to the exo-system (5.30).

Then, depending on the eigenvalues of $\tilde{A}^0$, a synchronization controller (5.15) or (5.19) can be designed for all the agents in the homogenized augmented network with communication topologies belonging to $\bar{\mathcal{G}}_{\bar{\delta}}$. If such a controller were applied to all the agents, output synchronization could be achieved. However, the exo-system does not have any input. In fact, this is not a problem. We can assume the controller that should be applied to exo-system has zero initial condition and the network should still synchronize regardless. Moreover, it should be noted that the exo-system is not associated with any network measurement either. Consequently, the controller would produce zero input if it were applied to the exo-system. This implies that we actually need not apply controller to the exo-system but only to the agents $1, ..., N$ in order for the network to synchronize. Since all the agent output y^i, $i = 1, ..., N$ will synchronize with the output of exo-system, the output regulation is achieved. $\square$

5.5 Conclusion

In this chapter, a decentralized control scheme is developed to solve the output synchronization and output regulation problems in a heterogeneous network of discrete-time introspective right-invertible agents. The essence of the proposed design is two-folds: first, by exploiting the introspection and right-invertibility property of each agent, we design a local shaping pre-compensator to manipulate the agent's internal dynamics as being asymptotically identical to a new model in which we enjoy a substantial freedom in assigning its eigenstructures. Then, different synchronization controllers depending on the two types of synchronization trajectories can be constructed on top of the new model so that the output synchronization can be achieved for a set of communication topologies.

5.A Shaping Pre-compensator Design

In the appendix, we develop the proof of Lemma 5.1 and present a detailed procedure for the design of the shaping pre-compensator. In the venture to achieve this, a Special Coordinate Basis (SCB) of linear system developed in [71] plays a fundamental role. We shall first review this canonical decomposition form and based on that develop some technical results that are instrumental to our proof.

5.A.1 Review of Special Coordinate Basis (SCB)

Consider a discrete-time strictly proper system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ y(k) \quad = Cx(k). \end{cases} \tag{5.31}$$

There exist state, input and output transformation

$$\Gamma_s x = \tilde{x} = \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}, \quad \Gamma_o y = \tilde{y} = \begin{bmatrix} y_b \\ y_d \end{bmatrix}, \quad \Gamma_i u = \tilde{u} = \begin{bmatrix} u_c \\ u_d \end{bmatrix}, \tag{5.32}$$

such that in the new coordinate, (5.31) can be rewritten as

$$
\begin{aligned}
&x_a(k+1) = A_{aa}x_a(k) + L_{ab}y_b(k) + L_{ad}y_d(k),\\
&x_b(k+1) = A_{bb}x_b(k) + L_{bd}y_d(k),\\
&x_c(k+1) = A_{cc}x_c(k) + B_c u_c(k) + E_{ca}x_a(k) + L_{cb}y_b(k) + L_{cd}y_d(k),\\
&x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y_d(k) + B_{d,j}[u_{d,j}(k) + G_j\tilde{x}(k)],\\
&y_{d,j}(k) = C_{d,j}x_{d,j}(k), \quad j = 1, ..., r,\\
&y_b = C_b x_b,
\end{aligned}
\tag{5.33}
$$

and

$$
x_d = \begin{bmatrix} x_{d,1} \\ \vdots \\ x^i_{d,r} \end{bmatrix}, \quad u_d = \begin{bmatrix} u_{d,1} \\ \vdots \\ u^i_{d,r} \end{bmatrix}, y_d = \begin{bmatrix} y_{d,1} \\ \vdots \\ y^i_{d,r} \end{bmatrix},
$$

where the dimension of x_a, x_b, x_c, x_d, $x_{d,j}$, y_b, y_d, u_c and u_d have dimensions n_a, n_b. n_c, n_d, $n_{d,j}$, $p-r$, r, $m-r$ and r ($n_d = \sum_{j=1}^{r} n_{d,j}$) respectively and

$$
A_{d,j} = \begin{bmatrix} 0 & I_{n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_{d,j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{d,j} = \begin{bmatrix} 1 & 0 \end{bmatrix}.
$$

Some important properties of SCB are summarized as follows:

1) The invariant zeros of system (5.31) are given by the eigenvalues of A_{aa}.

2) x_b is nonexistent and $\Gamma_o = I$ if (5.31) is right-invertible.

3) x_c is nonexistent and $\Gamma_i = I$ if (5.31) is left-invertible.

4) (A_{cc}, B_c) is controllable and (A_{bb}, C_b) is observable.

5) system (5.31) has r zeros at infinity with order $n_{d,j}$, $j = 1, ..., r$.

5.A.2 Technical Lemmas

Lemma 5.10 (Squaring down of right-invertible systems)**.** *Suppose that system* (5.31) *is right-invertible, then there exists a pre-compensator*

$$\begin{cases} \chi_1(k+1) = A_1\chi_1(k) + B_1u_1(k), \\ \quad u(k) \quad = C_1\chi_1(k) + D_1u_1(k), \end{cases} \tag{5.34}$$

such that the interconnection of (5.31) *and* (5.34) *is invertible.*

Proof. Since (5.31) is right invertible, there exist state and input transformations Γ_s and Γ_i such that (5.31) can be transformed into its SCB form (5.33) while x_b is nonexistent. To make this explicit, we write its SCB form as follows

$$\begin{aligned} &x_a(k+1) = A_{aa}x_a(k) + L_ay(k), \\ &x_c(k+1) = A_{cc}x_c(k) + B_cu_c(k) + E_ax_a(k) + L_cy(k), \\ &x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y(k) + B_{d,j}[u_{d,j}(k) + G_{a,j}x_a(k) + G_{c,j}x_c(k) + G_{d,j}x_d(k)], \\ &y_j(k) = C_{d,j}x_{d,j}(k), \quad j = 1, ..., p, \end{aligned} \tag{5.35}$$

Let $G_c = [G_{c,1}; \cdots ; G_{c,p}]$. Since (A, C) is observable, we find that (A_{cc}, G_c) is observable. Let F and K be such that $A_{cc} + B_cF$ and $A_{cc} + KG_c$ are Schur stable. A pre-compensator can be constructed as

$$\begin{cases} \chi_1(k+1) = (A_{cc} + B_cF)\chi_1(k) + K\nu(k), \\ \begin{bmatrix} u_c(k) \\ u_d(k) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix} \nu(k) + \begin{bmatrix} F \\ -G_c \end{bmatrix} \chi_1(k) \end{cases} \tag{5.36}$$

We proceed to show that the interconnection (5.36) and (5.35) is invertible. Suppose $\nu =$

$[\nu_1; \cdots ; \nu_p]$ and define $e_1 = x_c - \chi_1$. Then

$$x_a(k+1) = A_{aa}x_a(k) + L_a y(k), \tag{5.37}$$

$$x_c(k+1) = (A_{cc} + B_c F)x_c(k) - B_c F e_1(k) + E_a x_a(k) + L_c y(k), \tag{5.38}$$

$$e_1(k+1) = A_{cc}e_1(k) + E_a x_a(k) + L_c y(k) - K\nu(k), \tag{5.39}$$

and

$$x_{d,j}(k+1) \;=\; A_{d,j}x_{d,j}(k) + L_{d,j}y(k) + B_{d,j}[\nu_j^i(k) + G_{a,j}x_a(k) + G_{c,j}e_1(k) + G_{d,j}x_d(k)], \tag{5.40}$$

while

$$y_j(k) = C_{d,j}x_{d,j}(k). \tag{5.41}$$

Note that (5.37), (5.38), (5.40) and (5.41) are already in SCB form, but (5.39) is not. We need to eliminate ν from (5.39). Define $\tilde{x}_d = [x_{d,1,n_{d,1}}; \cdots ; x_{d,p,n_{d,p}}]$. We get

$$\tilde{x}_d(k+1) = \nu(k) + L_1 y(k) + G_a x_a^i(k) + G_c e_1^i(k) + G_d x_d(k),$$

where $G_a = [G_{a,1}; \cdots ; G_{a,p}]$, $G_d = [G_{d,1}; \cdots ; G_{d,p}]$ and L_1 is of appropriate dimension. Let $e_2 = e_1 + K\tilde{x}_d$ whose dynamics are given by

$$e_2(k+1) = (A_{cc} + KG_c)e_2(k) + L_2 y(k) + KG_a x_a(k) + \tilde{G}_d x_d(k),$$

where $\tilde{G}$ is an appropriate matrix. Define $\bar{x}_a = [x_c; e_2; x_a]$. We get

$$\bar{x}_a(k+1) = \bar{A}_{aa}\bar{x}_a(k) + \bar{L}_a y(k) + \bar{G}_d x_d(k), \tag{5.42}$$

where

$$\bar{A}_{aa} = \begin{bmatrix} A_{cc} + B_c F & -B_c F & E_a \\ 0 & A_{cc} + KG_c & KG_a \\ 0 & 0 & A_{aa} \end{bmatrix}, \quad \bar{L}_a = \begin{bmatrix} L_c \\ L_c \\ L_2 \end{bmatrix}$$

and $\bar{G}_d$ is an appropriate matrices. Finally, we need to eliminate x_d from (5.42). According to [71], there exists a matrix M_d such that

$$\tilde{x}_a = \bar{x}_a + M_d x_d$$

and x_d satisfy

$$\begin{aligned} &\tilde{x}_a(k+1) = \bar{A}_{aa}\tilde{x}_a(k) + \bar{L}_a y(k), \\ &x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y(k) + B_{d,j}[\nu_j(k) + \tilde{G}_a\tilde{x}_a(k) + G_{d,j}x_d(k)], \\ &y_j(k) = C_{d,j}x_{d,j}(k), \end{aligned} \tag{5.43}$$

with appropriate $\tilde{G}_a$. Clearly, (5.43) is in the SCB form and is square invertible. Note that in the original coordinate, the pre-compensator takes the form

$$\begin{cases} \chi_1(k+1) = (A_{cc} + B_c F)\chi_1(k) + K\nu(k), \\ \quad u(k) \quad = \Gamma_i^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}\nu(k) + \Gamma_i^{-1}\begin{bmatrix} F \\ -G_c \end{bmatrix}\chi_1(k). \end{cases} \tag{5.44}$$

$\square$

Lemma 5.11 (Rank equalizing of invertible system)**.** *Suppose that system* (5.31) *is invertible, then there exists a pre-compensator*

$$\begin{cases} \chi_2(k+1) = A_2\chi_2(k) + B_2 v(k), \\ \quad w(k) \quad = C_2\chi_2(k), \end{cases} \tag{5.45}$$

such that the interconnection of (5.45) *and* (5.31) *is of uniform rank.*

Proof. Since (5.31) is invertible, with only a nonsingular state transformation Γ_s, we can put it in the following form:

$$\begin{aligned} &x_a(k+1) = A_{aa}x_a(k) + L_a y(k), \\ &x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y(k) + B_{d,j}[u_j(k) + G_{a,j}x_a(k) + G_{d,j}x_d(k)], \\ &y_j(k) = C_{d,j}x_{d,j}(k), \quad j = 1, ..., p. \end{aligned} \tag{5.46}$$

We can add more delays to u_j so that all the infinite zeros have the same order. Let $r > \max_j n_{d,j}$. For $x_{d,j}$ with $n_{d,j} < r$, a pre-compensator can be constructed as

$$\begin{cases} \chi_{2,j}(k+1) = A_{2,j}\chi_{2,j}(k) + B_{2,j}v_j(k), \\ \quad u_j(k) \quad\; = C_{2,j}\chi_{2,j}, \end{cases}$$

where

$$A_{2,j} = \begin{bmatrix} 0 & I_{r-n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_{2,j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{2,j} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

By adding these pre-compensators to u_j, all the infinite zeros will have the same order of r. We can write them together as

$$\chi_2(k+1) = A_2\chi_2(k) + B_2v(k),$$

where $\chi_2 = [\chi_{2,1}; \cdots ; \chi_{2,p}]$, $v = [v_1; \cdots ; v_p]$, $u = [u_1; \cdots ; u_j]$ and

$$A_2 = \text{blkdiag}\{A_{2,j}\}, \quad B_2 = \text{blkdiag}\{B_{2,j}\}, \quad C_2 = \Gamma_i \,\text{blkdiag}\{C_{2,j}\}. \qquad \square$$

5.A.3 Proof of Lemma 1

Proof. Given Assumption 5.1, there exist non-singular state transformation Γ_s^i and input transformation Γ_c^i such that (A^i, B^i, C_y^i) can be transformed into SCB. Without loss of generality, we assume in the first place that the agent i is of the following form

The design of compensator (5.3) can be accomplished in three steps.

Step 1: Squaring down.

It follows from Lemma 5.10 that for each agent i, there exists a pre-compensator

$$\begin{cases} \chi_1^i(k+1) = A_1^i\chi_1^i(k) + B_1^iu_1^i(k), \\ \quad u^i(k) \quad = C_1^i\chi_1^i(k) + D_1^iu_1^i(k), \end{cases} \tag{5.47}$$

such that the interconnection of (5.1) and (5.47) is invertible.

Step 2: Rank equalizing.

Lemma 5.11 shows that a rank-equalizing compensator can be constructed as

$$\begin{cases} \chi_2^i(k+1) = A_2^i \chi_2^i(k) + B_2^i u_2^i(k), \\ u_1^i(k) \quad = C_2^i \chi_2^i(k), \end{cases} \tag{5.48}$$

such that the interconnection of (5.1), (5.47) and (5.48) is invertible and of uniform rank $n_q > n_d$.

Step 3: Zero decoupling and Pole placement.

Using a non-singular state transformation

$$\Gamma_s^i \begin{bmatrix} x^i \\ \chi_1^i \\ \chi_2^i \end{bmatrix} = \tilde{\chi}^i = \begin{bmatrix} \tilde{\chi}_0^i \\ \tilde{\chi}_d^i \end{bmatrix},$$

the interconnection of (5.1), (5.47) and (5.48) can be written in the following form:

$$\begin{cases} \tilde{\chi}_0^i(k+1) = \tilde{A}_0^i \tilde{\chi}_0^i(k) + \tilde{L}_0^i y^i(k), \\ \tilde{\chi}_d^i(k+1) = \tilde{A}_d^i \tilde{\chi}_d^i(k) + \tilde{B}_d^i [u_2^i(k) + D_0^i \tilde{\chi}_0^i(k) + D_d^i \tilde{\chi}_d^i(k)], \\ y^i(k) \quad = \tilde{C}_d^i \tilde{\chi}_d^i(k), \end{cases}$$

where

$$\tilde{A}_d^i = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_d^i = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad \tilde{C}_d^i = \begin{bmatrix} I_p & 0 \end{bmatrix}.$$

Note that each agent also has a local measurement $\tilde{z}^i$ which consists of the original z^i and compensator states χ_1^i and χ_2^i. This $\tilde{z}^i$ can be written in terms of $\tilde{\chi}_0$ and $\tilde{\chi}_d$ as

$$\tilde{z}^i(k) = \tilde{C}_1^i \tilde{\chi}_0^i(k) + \tilde{C}_2^i \tilde{\chi}_d^i(k).$$

Let

$$\tilde{A}^i = \begin{bmatrix} \tilde{A}_0^i & \tilde{C}^i \tilde{L}_0^i \\ \tilde{B}_d^i D_0^i & \tilde{A}_d^i + \tilde{B}_d^i D_d^i \end{bmatrix}, \quad \tilde{B}^i = \begin{bmatrix} 0 \\ \tilde{B}_d^i \end{bmatrix}, \quad \tilde{C}^i = \begin{bmatrix} \tilde{C}_1^i & \tilde{C}_2^i \end{bmatrix}.$$

Clearly, $(\tilde{A}^i, \tilde{C}^i)$ is detectable. Then an observer based pre-feedback is designed as follows

$$\begin{cases} \hat{\chi}^i(k+1) = \tilde{A}^i\hat{\chi}^i(k) + \tilde{B}^i v^i(k) - \tilde{K}^i(\tilde{z}^i(k) - \tilde{C}^i\hat{\chi}^i(k)), \\ u_2^i(k) \quad = -\tilde{D}^i\hat{\chi}^i(k) + \tilde{F}^i\hat{\chi}^i(k) + v^i(k), \end{cases} \tag{5.49}$$

where $\tilde{A}^i + \tilde{K}^i\tilde{C}^i$ is Schur stable, $\tilde{D}^i = [D_0^i \quad D_d^i]$, $\tilde{F}^i = [0 \quad \tilde{F}_d^i]$ and $\tilde{A}_d^i + \tilde{B}_d^i\tilde{F}_d^i$ has a set of pre-specified eigenvalues. It is easy to see that the error dynamics $e^i = \tilde{\chi}^i - \hat{\chi}^i$ is asymptotically stable. Therefore, the coupling between $\tilde{\chi}_0$ and $\tilde{\chi}_d$ is canceled asymptotically. The mapping from v^i to y^i is described by the following dynamics:

$$\begin{cases} \tilde{\chi}_d^i(k+1) = (\tilde{A}_d^i + \tilde{B}_d^i\tilde{F}_d^i)\tilde{\chi}_d^i(k) + \tilde{B}_d^i v^i(k) + \tilde{B}_d^i d^i(k), \\ y^i(k) \quad = \tilde{C}_d^i\tilde{\chi}(k), \end{cases}$$

and

$$\begin{cases} e^i(k+1) = (\tilde{A}^i + \tilde{K}^i\tilde{C}^i)e^i(k), \\ d^i(k) \quad = (\tilde{D}^i - \tilde{F}^i)e^i(k). \end{cases}$$

Note that $(\tilde{A}_d^i + \tilde{B}^i\tilde{F}^i, \tilde{B}_d^i, \tilde{C}_d^i)$ is invertible, of uniform rank n_q and has no invariant zeros. Moreover, $\tilde{A}_d^i + \tilde{B}_d^i\tilde{F}_d^i$ has pre-selected eigenvalues.

In the original coordinate, (5.49) can be written as

$$\begin{cases} \hat{\chi}^i(k+1) = \tilde{A}^i\hat{\chi}^i(k) + \tilde{B}^i\tilde{K}^i(\tilde{z}^i(k) - \tilde{C}^i\hat{\chi}^(k)), \\ u_2^i(k) \quad = -\tilde{D}^i\Gamma_s\hat{\chi}^i(k) + \tilde{F}^i\Gamma_s\hat{\chi}^i(k) + v^i(k). \end{cases} \tag{5.50}$$

□

5.B Manipulation of Exo-system

Consider an arbitrary exo-system

$$\begin{cases} x(k+1) = Ax(k), \quad x(0) = x_0, \\ y(k) \quad = Cx(k), \end{cases} \tag{5.51}$$

There exists a non-singular state transformation $x = T_s\tilde{x}$ and $y = T_o\tilde{y}$ where

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{bmatrix}, \quad \tilde{x}_i = \begin{bmatrix} \tilde{x}_{i,1} \\ \vdots \\ \tilde{x}_{i,n_i} \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_p \end{bmatrix},$$

we have

$$\tilde{x}_i(k+1) = A_i\tilde{x}_i(k) + L_i\tilde{y}, \quad \tilde{y}_i(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}_i(k), \tag{5.52}$$

where

$$A_i = \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}$$

and L_i is of appropriate dimension. The set of integers $\{n_1, ..., n_p\}$ is the observability index of (C, A) (see [8]).

Note that we can equalize the size of A_i to n_q by adding shift registers to the bottom of each chain $\tilde{x}_i$ with zero initial conditions. Define

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix}, \quad \bar{x}_i = \begin{bmatrix} \tilde{x}_i \\ s_i \end{bmatrix} \in \mathbb{R}^{n_q}, \quad s_i(0) = 0,$$

and

$$\bar{x}_i(k+1) = \bar{A}_i\bar{x}_i(k) + \bar{L}_i\tilde{y}, \quad \tilde{y}_i(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}_i(k), \tag{5.53}$$

with

$$\bar{A}_i = \begin{bmatrix} 0 & I_{n_q-1} \\ 0 & 0 \end{bmatrix}, \quad \bar{L}_i = \begin{bmatrix} L_i \\ 0 \end{bmatrix}.$$

By adding s_i, we introduce several zero eigenvalues to the system. It is easy to see that (5.52) and (5.53) generate exactly the same output $\tilde{y}$. We can write system (5.53) in a compact form as

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k), \\ \tilde{y}(k) \quad = \bar{C}\bar{x}(k), \end{cases} \quad \bar{x}(0) = \begin{bmatrix} \bar{x}_1(0) \\ \vdots \\ \bar{x}_p(0) \end{bmatrix}. \tag{5.54}$$

where

$$\bar{A} = \begin{bmatrix} \star & I_{n_q-1} & \cdots & \star & 0 \\ \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & 0 & \cdots & \star & I_{n_q-1} \\ \star & 0 & \cdots & \star & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Choose

$$\bar{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}.$$

Then $(\bar{A}, \bar{B}, \bar{C})$ is invertible, of uniform rank n_q and has no invariant zero.

Finally we restore the output transformation in system (5.54) as

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k), \\ y(k) \quad = T_o\bar{C}\bar{x}(k), \end{cases} \quad \bar{x}(0) = \begin{bmatrix} \bar{x}_1(0) \\ \vdots \\ \bar{x}_p(0) \end{bmatrix}. \tag{5.55}$$

This system produces the same output as (5.52). Since non-singular output transformation does not change invertibility and zero structure. Therefore, the triple $(\bar{A}, \bar{B}, T_o\bar{C})$ is still invertible, of uniform rank n_q and has no invariant zero. According to the property of SCB, there exists a state transformation that put it in the form of (5.6).

Chapter 6

Output Synchronization for Heterogeneous Networks of Non-introspective Right-invertible Agents

6.1 Introduction

The problem of achieving *synchronization* among agents in a network—that is, asymptotic agreement on the agents' state or output trajectories—has received substantial attention in recent years. The essential difficulty of the synchronization problem is the lack of a central authority with the ability to control the network as a whole. Instead, each agent must implement a controller based on limited information about itself and its surroundings—typically in the form of measurements of its own state or output relative to that of neighboring agents in the network.

Much of the attention has been directed toward *state synchronization* in *homogeneous* networks (i.e., networks where the agent models are identical), with each agent receiving information about its own state relative to that of neighboring agents (e.g., [44–46, 52, 54, 56, 61, 83]). Roy, Saberi, and Herlugson [62], Tuna [83], and Yang, Roy, Wan, and Saberi [103] considered this type of problem for more general observation topologies and more complex identical agent models than previously considered. Others have studied the case where the agents receive relative information about their own partial-state output, see for example, [32, 48, 49, 84]. A key idea in the work of [32], which was expanded upon by [107], is the development of a distributed observer. This observer makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. Many of the results on the synchronization problem are rooted in the seminal work of [98, 99].

6.1.1 Heterogeneous Networks and Output Synchronization

A limited amount of research has also been conducted on *heterogeneous* networks (i.e., networks where the agent models are non-identical). [50] presented a robust state-synchronization design for networks of nonlinear systems with relative degree one, where each agent implements a sufficiently strong feedback based on the difference between its own state and that of a common reference model. In the work of [101] it is assumed that a common Lyapunov function candidate is available, which is used to analyze stability with respect to a common equilibrium point. Depending on the system, some agents may also implement feedbacks to ensure stability, based on the difference between those agents' states and the equilibrium point. [114] analyzed state synchronization in a network of nonlinear agents based on the network topology and the existence of certain time-varying matrices. Controllers can be designed based on this analysis, to the extent that the available information and actuation allows for the necessary manipulation of the network topology.

The above-cited works focus on synchronizing the agents' internal states. In heterogeneous networks, however, the physical interpretation of one agent's state may be different from that of another agent. Indeed, the agents may be governed by models of different dimensions. In this case, comparing the agents' internal states is not meaningful, and it is more natural to aim for *output synchronization*—that is, agreement on some partial-state output from each agent. Chopra and Spong [10] focused on output synchronization for weakly minimum-phase systems of relative degree one, using a pre-feedback within each agent to create a single-integrator system with decoupled zero dynamics. Pre-feedbacks were also used by [2] to facilitate passivity-based designs. The authors have previously considered output synchronization for right-invertible agents, using pre-compensators and an observer-based pre-feedback within each agent to yield a network of asymptotically identical agents [104]. Kim, Shim, and Seo [25] studied output synchronization for uncertain single-input single-output, minimum-phase systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output. A similar approach was taken by [97], which showed that a necessary condition for output synchronization in heterogeneous networks is the existence of a *virtual exosystem* that produces a trajectory to which all the agents asymptotically converge. If one knows the model of an observable virtual exosystem without exponentially unstable modes, which each agent is capable of tracking, then it can be implemented within each agent and synchronized via the network. The agent can then be made to track the model with the help of a local observer estimating the agent's states.

6.1.2 Introspective Versus Non-introspective Agents

The designs mentioned in the previous section rely—explicitly or implicitly—on some sort of self-knowledge that is separate from the information transmitted over the network. In particular, the agents may be required to know their own state, their own output, or their own state/output relative to that of a reference trajectory. In this chapter, we shall refer to agents that possess this type of self-knowledge as *introspective* agents, to distinguish them from *non-introspective* agents—that is, agents that have no knowledge of their own state or output separate from what is received via the network. This distinction is significant because introspective agents have much greater freedom to manipulate their internal dynamics (e.g., through the use of pre-feedbacks) and thus change the way that they present themselves to the rest of the network. The notion of a non-introspective agent is also practically relevant; for example, two vehicles in close proximity may be able to measure their relative distance without either of them having knowledge of their absolute position.

To the authors' knowledge, the only result that solves the output synchronization problem for a well-defined class of heterogeneous networks of non-introspective agents is by [113]. In their work, the only information available to each agent is a linear combination of outputs received over the network. However, the agents are assumed to be passive—a strict requirement that, among other things, requires the agents to be weakly minimum-phase and of relative degree one.

6.1.3 Contributions of this Chapter

In this chapter we consider heterogeneous networks of non-introspective linear agents that receive, via the network, a linear combination of their own output relative to that of neighboring agents. In the spirit of [32], we also assume that the agents can exchange information about their internal estimates using the network's communication infrastructure. We design decentralized controllers for achieving output synchronization under a set of straightforward assumptions about the agents and the topology of the network.

Based on the output-synchronization results we also consider the slightly different problem of *regulated output synchronization*. Here, the goal is not only to achieve output synchronization, but to make the synchronization trajectory follow an *a priori* given reference. When considering this problem we assume that some of the agents are introspective in the sense that they know their own output relative to that of the reference output.

6.1.4 Notation

Given a matrix A, A' denotes its transpose and A^* denotes its conjugate transpose. We denote by $A \otimes B$ the Kronecker product between matrices A and B. When clear from the context, 0 denotes a matrix of appropriate dimensions with all zero elements.

6.2 Problem Formulation

We consider a network of N multiple-input multiple-output agents of the form

$$\dot{x}_i = A_i x_i + B_i u_i, \tag{6.1a}$$

$$y_i = C_i x_i + D_i u_i, \tag{6.1b}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, and $y_i \in \mathbb{R}^p$. Our goal is to achieve output synchronization among the agents, meaning that $\lim_{t\to\infty}(y_i - y_j) = 0$ for all $i, j \in \{1, \ldots, N\}$.

The agents are non-introspective; hence, agent i does not have access to its own output y_i. The only available information comes from the network, which provides each agent with a linear combination of its own output relative to that of the other agents. In particular, agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j),$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$. The topology of the network can be described by a directed graph (digraph) $\mathcal{G}$ with nodes corresponding to the agents in the network and edges given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ implies that an edge exists from agent j to i. Agent j is then called a *parent* of agent i, and agent i is called a *child* of agent j. The weight of the edge equals the magnitude of a_{ij}.

We shall frequently make use of the matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{N} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. This matrix is known as the *Laplacian* matrix of the digraph $\mathcal{G}$ and has the property that all the row sums are zero. In terms of the coefficients of G, ζ_i can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j.$$

We also assume that the agents can exchange relative information about their internal estimates using the network's communication infrastructure. Specifically, agent i is presumed to have access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{N} g_{ij}\eta_j,$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced internally by agent j as part of the controller. This variable will be specified as we proceed with the control design.

6.2.1 Assumptions

We make the following assumptions about the network topology and the individual agents.

Assumption 6.1. *The digraph $\mathcal{G}$ has a directed spanning tree with root agent $K \in \{1, \ldots, N\}$, such that for each $i \in \{1, \ldots, N\} \setminus K$,*

1) (A_i, B_i) is stabilizable

2) (A_i, C_i) is observable

3) (A_i, B_i, C_i, D_i) is right-invertible

4) (A_i, B_i, C_i, D_i) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of A_K

Remark 6.1. *A* directed tree *is a directed subgraph of $\mathcal{G}$, consisting of a subset of the nodes and edges, such that every node has exactly one parent, except a single root node with no parents. Furthermore, there must exist a directed path from the root to every other agent. A* directed spanning tree *is a directed tree that contains all the nodes of $\mathcal{G}$. A digraph may contain many directed spanning trees, and thus there may be several choices of root agent K. Fig. 6.1 illustrates a digraph containing multiple directed spanning trees.*

Remark 6.2. *Right-invertibility of a quadruple (A_i, B_i, C_i, D_i) means that, given a reference output $y_r(t)$, there exist an initial condition $x_i(0)$ and an input $u_i(t)$ such that $y_i(t) = y_r(t)$ for all $t \geq 0$. For example, every single-input single-output system is right-invertible, unless its transfer function is identically zero.*

Let the matrix $\bar{G}_K = [g_{ij}]_{i,j \neq K}$ be defined from G by removing row and column number K, corresponding to the root of a directed spanning tree of $\mathcal{G}$. We shall need the following result, which is proven in Appendix 6.A.

Lemma 6.1. *All the eigenvalues of $\bar{G}_K$ are in the open right-half complex plane.*

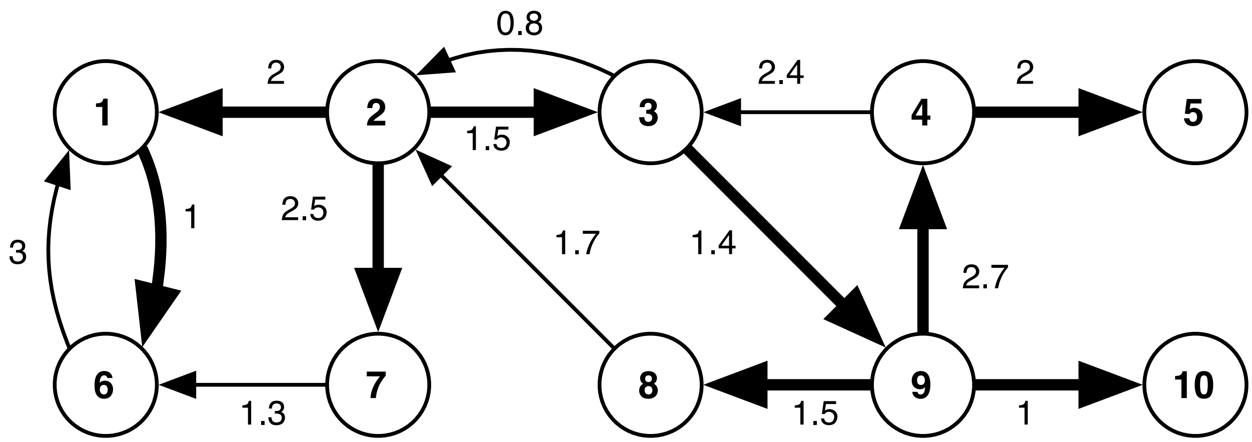


Figure 6.1: The depicted digraph contains multiple directed spanning trees, rooted at nodes 2, 3, 4, 8, and 9. One of these, with root node 2, is illustrated by bold arrows.

6.3 Control Design

In this section we describe the construction of decentralized controllers that achieve output synchronization. Before embarking on the actual design procedure, however, we shall describe the motivation behind the design.

The main idea is to set the control input of the root agent K to zero (i.e., $u_K = 0$) and to also set $\eta_K = 0$. We then design controllers for all the other agents such that their outputs asymptotically synchronize with the trajectory $y_K(t)$. That is, for each $i \in \{1, \ldots, N\} \setminus K$ we wish to achieve $\lim_{t\to\infty}(y_i - y_K) = 0$. Equivalently, we wish to regulate the synchronization error variable

$$e_i := y_i - y_K$$

to zero, where the dynamics of e_i is governed by

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_K \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & A_K \end{bmatrix} \begin{bmatrix} x_i \\ x_K \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{6.2a}$$

$$e_i = \begin{bmatrix} C_i & -C_K \end{bmatrix} \begin{bmatrix} x_i \\ x_K \end{bmatrix} + D_i u_i. \tag{6.2b}$$

The system (6.2) is in general not stabilizable. If x_i and x_K were available to agent i as measurements, then the problem of making e_i converge to zero would nevertheless be solvable by standard output-regulation methods (see, e.g., [70]). But alas, the only information available to agent i is ζ_i and $\hat{\zeta}_i$. To achieve our objective with such limited information, we carry out our design for each agent $i \in \{1, \ldots, N\} \setminus K$ in three steps.

In Step 1 we construct a new state $\bar{x}_i$, via a transformation of x_i and x_K, so that the dynamics of the synchronization error variable e_i can be described by the alternative equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i u_i := \begin{bmatrix} A_i & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} u_i, \tag{6.3a}$$

$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i u_i := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i u_i. \tag{6.3b}$$

The purpose of this state transformation is to reduce the dimension of the model underlying e_i by removing redundant modes that have no effect on e_i. In particular, the model (6.2) may be unobservable, but the model (6.3) is always observable.

The properties of the model (6.3) also allow us, in Step 2 of the design, to construct a controller that regulates e_i to zero by using state feedback from $\bar{x}_i$. This controller is not directly implementable, however, because $\bar{x}_i$ is not known to agent i. This brings us to Step 3 of the design, where we construct an observer that makes an estimate of $\bar{x}_i$ available to agent i. This observer is based on the information ζ_i and $\hat{\zeta}_i$ received via the network, and it works in a distributed manner together with the observers for the other agents to achieve convergence. The observer design is based on previous results on distributed observer design for *homogeneous* networks. Since our network is heterogeneous, we first perform a second state transformation of $\bar{x}_i$ to χ_i, in order to obtain a dynamical model that is substantially the same as for the other agents. In particular, the model differences now occur only in particular locations where they can be suppressed by using high-gain observer techniques. By combining the observer estimates with the state-feedback controller designed in Step 2, we achieve output synchronization.

6.3.1 Design Preliminaries

Due to the design strategy of setting $u_K = 0$, the trajectory $y_K(t)$ becomes the unforced response of agent K, consisting of a linear combination of the observable modes of the pair (A_K, C_K). Asymptotically stable modes vanish as $t \to \infty$, and they therefore play no role asymptotically. For simplicity of presentation, we therefore assume that all the eigenvalues of A_K are in the closed right-half complex plane and that (A_K, C_K) is observable. We make this assumption without any loss of generality since, if A_K does contain unobservable or asymptotically stable modes, we can always create an auxiliary model excluding those modes for the purpose of control design (see Appendix 6.C for details).

Below we describe the three steps of the design procedure that must be carried out for each agent $i \in \{1, \dots, N\} \setminus K$. In addition to agent i's system matrices (A_i, B_i, C_i, D_i), the information needed to carry out these three steps for agent i is as follows:

- the matrices A_K and C_K of the root agent

- a common integer $\bar{n}$ such that $\bar{n} \geq n_i + n_K$ for all $i \in \{1, \ldots, N\} \setminus K$[1]

- a common matrix $L \in \mathbb{R}^{p \times p\bar{n}}$, freely chosen[2]

- a common high-gain parameter $\varepsilon \in (0, 1]$

- a common number $\tau > 0$ that is a lower bound on the real part of the eigenvalues of the matrix $\bar{G}_K$

Based on this information, we can define the matrices $\mathcal{A} \in \mathbb{R}^{p\bar{n} \times p\bar{n}}$, $\mathcal{C} \in \mathbb{R}^{p \times p\bar{n}}$, $\Omega_\varepsilon \in \mathbb{R}^{p\bar{n} \times p\bar{n}}$, and $\mathcal{L}_\varepsilon \in \mathbb{R}^{p\bar{n} \times p\bar{n}}$ as

$$\mathcal{A} = \begin{bmatrix} 0 & I_{p(\bar{n}-1)} \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} I_p & 0 \end{bmatrix},$$

$$\Omega_\varepsilon = \begin{bmatrix} I_p \varepsilon^{-1} & & \\ & \ddots & \\ & & I_p \varepsilon^{-\bar{n}} \end{bmatrix}, \quad \mathcal{L}_\varepsilon = \begin{bmatrix} 0 \\ \varepsilon^{\bar{n}+1} L \Omega_\varepsilon \end{bmatrix}.$$

The pair $(\mathcal{A} + \mathcal{L}_\varepsilon, \mathcal{C})$ is always observable; hence, we can define a matrix $\mathcal{P}_\varepsilon = \mathcal{P}_\varepsilon' > 0$ as the unique solution of the algebraic Riccati equation

$$(\mathcal{A} + \mathcal{L}_\varepsilon)\mathcal{P}_\varepsilon + \mathcal{P}_\varepsilon(\mathcal{A} + \mathcal{L}_\varepsilon)' - 2\tau \mathcal{P}_\varepsilon \mathcal{C}' \mathcal{C} \mathcal{P}_\varepsilon + I_{p\bar{n}} = 0. \tag{6.4}$$

The matrices $\mathcal{A}$, $\mathcal{C}$, Ω_ε, $\mathcal{L}_\varepsilon$, and $\mathcal{P}_\varepsilon$ will be used during the design procedure.

[1]The integer $\bar{n}$ can be defined less conservatively as a bound on $n_i + r_i$ for $i \in \{1, \ldots, N\} \setminus K$, where r_i is defined during Step 1 of the design procedure for each agent.

[2]See Section 6.3.4 for an explanation of the purpose of L.

6.3.2 Design Procedure for Agent $i \in \{1, \ldots, N\} \setminus K$

Step 1: State Transformation

Let O_i be the observability matrix corresponding to the system (6.2):

$$O_i = \begin{bmatrix} C_i & -C_K \\ \vdots & \vdots \\ C_i A_i^{n_i+n_K-1} & -C_K A_K^{n_i+n_K-1} \end{bmatrix}. \tag{6.5}$$

Let q_i denote the dimension of the null space of O_i, and define $r_i = n_K - q_i$. Next, define $\Lambda_{iu} \in \mathbb{R}^{n_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{n_K \times q_i}$ such that

$$O_i \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0, \quad \text{rank} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = q_i. \tag{6.6}$$

Because (A_i, C_i) and (A_K, C_K) are observable, Λ_{iu} and Φ_{iu} have full column rank (see Appendix 6.D). Let therefore Λ_{io} and Φ_{io} be defined such that $\Lambda_i := [\Lambda_{iu}, \Lambda_{io}] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i := [\Phi_{iu}, \Phi_{io}] \in \mathbb{R}^{n_K \times n_K}$ are nonsingular. We define a new state variable $\bar{x}_i \in \mathbb{R}^{n_i + r_i}$ as

$$\bar{x}_i = \begin{bmatrix} x_i - \Lambda_i M_i \Phi_i^{-1} x_K \\ -N_i \Phi_i^{-1} x_K \end{bmatrix},$$

where $M_i \in \mathbb{R}^{n_i \times n_K}$ and $N_i \in \mathbb{R}^{r_i \times n_K}$ are defined as

$$M_i = \begin{bmatrix} I_{q_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad N_i = \begin{bmatrix} 0 & I_{r_i} \end{bmatrix}.$$

The following lemma, which is proven in Appendix 6.A, shows how the synchronization error e_i is given in terms of $\bar{x}_i$.

Lemma 6.2. *The synchronization error variable e_i is governed by dynamical equations of the form (6.3), where $(\bar{A}_i, \bar{C}_i)$ is observable and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K.*

Step 2: State-Feedback Control Design

We now design a controller as a function of $\bar{x}_i$ to regulate e_i to zero. Consider the following equations with unknowns $\Pi_i \in \mathbb{R}^{n_i \times r_i}$ and $\Gamma_i \in \mathbb{R}^{m_i \times r_i}$, commonly known as the *regulator equations*:

$$\Pi_i \bar{A}_{i22} = A_i \Pi_i + \bar{A}_{i12} + B_i \Gamma_i, \tag{6.7a}$$

$$C_i \Pi_i - \bar{C}_{i2} + D_i \Gamma_i = 0. \tag{6.7b}$$

Based on Π_i and Γ_i, we define a matrix

$$\bar{F}_i = \begin{bmatrix} F_i & \Gamma_i - F_i \Pi_i \end{bmatrix}, \tag{6.8}$$

where F_i is chosen such that $A_i + B_i F_i$ is Hurwitz. The following lemma, which is proven in Appendix 6.A, shows that the regulator equations (6.7) are always solvable and that the matrix $\bar{F}_i$ can be used to define a state-feedback controller.

Lemma 6.3. *The regulator equations* (6.7) *are solvable, and the state-feedback controller* $u_i = \bar{F}_i \bar{x}_i$ *ensures that* $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(y_i - y_K) = 0$.

Step 3: Observer-Based Implementation

Our last step is to design an observer to produce an estimate of $\bar{x}_i$, denoted by $\hat{\bar{x}}_i$. Define $\chi_i = T_i \bar{x}_i$, where

$$T_i = \begin{bmatrix} \bar{C}_i \\ \vdots \\ \bar{C}_i \bar{A}_i^{\bar{n}-1} \end{bmatrix}.$$

Note that T_i is not necessarily a square matrix; however, due to observability of $(\bar{A}_i, \bar{C}_i)$, T_i is injective, which implies that $T_i'T_i$ is nonsingular. In terms of χ_i, we can write the system equations as

$$\dot{\chi}_i = (\mathcal{A} + \mathcal{L}_i)\chi_i + \mathcal{B}_i u_i, \quad \chi_i(0) = T_i\bar{x}_i(0), \tag{6.9a}$$

$$e_i = \mathcal{C}\chi_i + \mathcal{D}_i u_i, \tag{6.9b}$$

where

$$\mathcal{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}, \quad \mathcal{B}_i = T_i \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \mathcal{D}_i = D_i,$$

and where $L_i = \bar{C}_i \bar{A}_i^{\bar{n}} (T_i'T_i)^{-1}T_i'$. We construct the observer

$$\dot{\hat{\chi}}_i = (\mathcal{A} + \mathcal{L}_i)\hat{\chi}_i + \mathcal{B}_i u_i + \Omega_\varepsilon \mathcal{P}_\varepsilon \mathcal{C}'(\zeta_i - \hat{\zeta}_i), \tag{6.10a}$$

$$\hat{\bar{x}}_i = (T_i'T_i)^{-1}T_i'\hat{\chi}_i. \tag{6.10b}$$

Based on the observer estimate, we define the variable $\eta_i = \mathcal{C}\hat{\chi}_i + \mathcal{D}_i u_i$ to be shared with the other agents via the network's communication infrastructure as described in Section 6.2, and the observer-based control law

$$u_i = \bar{F}_i \hat{\bar{x}}_i. \tag{6.11}$$

Together, the observers for agents $i \in \{1, \ldots, N\} \setminus K$ form a distributed observer parameterized by a common high-gain parameter ε. The following lemma, which is proven in Appendix 6.A, shows that all the observation errors vanish asymptotically if ε is chosen sufficiently small.

Lemma 6.4. *There exists an $\varepsilon^* \in (0, 1]$ such that, if ε is chosen such that $\varepsilon \in (0, \varepsilon^*]$, then for each $i \in \{1, \ldots, N\} \setminus K$, $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$.*

6.3.3 Main Result

By implementing the observer-based control law (6.11) for each agent $i \in \{1, \ldots, N\} \setminus K$, we obtain a decentralized controller structure that achieves output synchronization. The following theorem formalizes this result.

Theorem 6.1. *There exists an $\varepsilon^* \in (0, 1]$ such that, if ε is chosen such that $\varepsilon \in (0, \varepsilon^*]$, then for each $i, j \in \{1, \ldots, N\}$, $\lim_{t\to\infty}(y_i - y_j) = 0$.*

Proof. Since the systems are linear, the result follows from Lemmas 6.3 and 6.4 and the separation principle. □

6.3.4 Remarks on the Design Procedure

Having presented the design procedure, some remarks are in order.

The purpose of Step 1 is to reduce the dimension of the model (6.2) by removing redundant modes that cannot be observed from e_i. Such modes exist if agent i and agent K share particular unforced solutions. Consider, for example, the case where agents i and K are identical. Then the states x_i and x_K cannot be individually observed from $e_i = y_i - y_K$, since there are infinitely many initial conditions that yield the unforced solution $e_i = 0$. If, on the other hand, we define the state $\bar{x}_i = x_i - x_K$, then we obtain the model $\dot{\bar{x}}_i = A_i\bar{x}_i + B_iu_i$, $e_i = C_i\bar{x}_i + D_iu_i$, which is observable. Indeed, it is easily verified that in our design procedure, identical agents yield $q_i = n_i = n_K$ and $r_i = 0$, and that $\Lambda_i = I_{n_i}$ and $\Phi_i = I_{n_K}$ are valid choices; thus, one obtains precisely $\bar{x}_i = x_i - x_K$. In the general case, Step 1 yields a model (6.3) that incorporates the difference between modes that are shared between agents i and K in addition modes from both agent i and K that are not shared.

In Step 2 we must find the solutions Π_i and Γ_i of the regulator equations (6.7). A special situation arises when $r_i = 0$, which implies that $\bar{A}_{i22}$, $\bar{A}_{i12}$, and $\bar{C}_{i2}$ are empty matrices. In this case, Π_i and Γ_i are also empty matrices, and the need to solve the regulator equations vanishes. This situation occurs, in particular, if agent i and agent K are identical.

In Step 3, we introduce a state transformation from $\bar{x}_i$ to χ_i, where χ_i has dimension $p\bar{n}$. Since the dimension of $\bar{x}_i$ may be less than $p\bar{n}$, the transformation to χ_i may involve an over-parameterization. In this case, (6.9) is not the only possible dynamical model of χ_i, but it is always *one* of the possible representations. After performing the state transformation, we proceed to construct an observer that depends on a high-gain parameter ε. Following the proof of Lemma 6.4, it can be seen that ε must be chosen to stabilize the dynamics (6.15) by making the matrix $I_{N-1} \otimes (\mathcal{A} + \mathcal{L}_\varepsilon) - \bar{G}_K \otimes (\mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}) - \tilde{\mathcal{L}}_\varepsilon$ Hurwitz. This works because the nonzero elements of $\tilde{\mathcal{L}}_\varepsilon$ are on the form $\varepsilon^{\bar{n}+1}(L - L_i)\Omega_\varepsilon$ (meaning that $\|\tilde{\mathcal{L}}_\varepsilon\| = O(\varepsilon)$), and $\tilde{\mathcal{L}}_\varepsilon$ is therefore dominated by the Hurwitz matrix $I_{N-1} \otimes (\mathcal{A} + \mathcal{L}_\varepsilon) - \bar{G}_K \otimes (\mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C})$ as $\varepsilon \to 0$. The freely chosen matrix L plays a role in determining how small ε needs to be chosen, because the difference $L - L_i$ affects the nonzero elements of $\tilde{\mathcal{L}}_\varepsilon$. If sufficient information is available about the agent models, L can be chosen to make the differences $L - L_i$ small, in order to reduce the need for high gain. If all the agents are identical, then L_i is the same for all the agents and one can choose $L = L_i$. In this case, $\tilde{\mathcal{L}}_\varepsilon$ vanishes and ε can be chosen arbitrarily. It is therefore evident that the role of ε is to suppress the differences in agent models that exist in heterogeneous networks.

6.3.4.1 Information Required About the Network

When designing the controller for agent i, it is necessary to know the model (A_i, B_i, C_i, D_i) of agent i, but it is not necessary to know the models of all the other agents or the exact topology of the network. Some additional information is nevertheless required, as specified in Section 6.3.1. To justify the required level of information, we note that most of the required information is also assumed available in the literature on homogeneous networks, albeit implicitly. In a homogeneous network, knowledge of A_i and C_i implies knowledge of A_K and C_K, since the models are identical. Moreover, $\bar{n} = 2n_i$ is a known bound on $n_i + n_K$, since the agents are of the same order.[3] As described above, the matrices L_i are all the same in a homogeneous network; hence one can choose $L = L_i$, which means that $\varepsilon = 1$ is always a valid choice. The lower bound $\tau > 0$ on the real part of the eigenvalues of $\bar{G}_K$ can be viewed as a measure of the connectivity of the network. Similar measures of connectivity are typically assumed available in the literature on general homogeneous networks [32, 83, 103].

Even though exact information about the network is not required in the design process, it is nevertheless useful, as it is then possible to search for a non-conservative ε that makes $I_{N-1} \otimes (\mathcal{A} + \mathcal{L}_\varepsilon) - \bar{G}_K \otimes (\mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}) - \tilde{\mathcal{L}}_\varepsilon$ Hurwitz. One can also define τ as a tight lower bound on the real part of the eigenvalues of $\bar{G}_K$ and $\bar{n}$ as a tight bound on $n_i + r_i$ in accordance with footnote [1] on page 154.

6.3.5 Computational Complexity

The controllers constructed in this chapter contain internal dynamics in the form of an observer for χ. The internal dynamics introduces additional computational complexity compared to the static control laws that have previously been used for synchronization of of single and double integrators (e.g., [45, 45, 52]) to general homogeneous networks with relative-state information (e.g., [83, 103]). The need for internal dynamics arises for two reasons. First, since only relative-output information is exchanged, the agents need internal observer dynamics to estimate unmeasured

[3] In fact, according to footnote [1] on page 154 one can choose $\bar{n}$ less conservatively as $\bar{n} = n_i$ in the case of identical agents, since one always has $r_i = 0$.

states. Second, since the agents are non-identical, the agreement manifold may contain modes that are not contained within all the agents, and which must therefore be replicated by internal dynamics according to the internal model principle.

The order of the internal dynamics is $\bar{n}$, which is an upper bound on $n_i + n_K$ for $i \in \{1, \ldots, N\} \setminus K$. Alternatively, as remarked in footnote [1] on page 154. $\bar{n}$ can be defined less conservatively as a bound on $n_i + r_i$. The integer r_i can be viewed as representing the order of the part of the root agent dynamics that is not contained within agent i. Hence, the computational complexity is in this case dependent on how similar the agents are to one another. Indeed, in the case of identical agents, one always has $r_i = 0$, so $\bar{n} = n_i$, meaning that each agent implements an observer of order equal to that of its own dynamics.

An interesting topic of future work is the reduction of computational complexity by finding ways to reduce the order of the internal dynamics within each agent.

6.4 Regulated Output Synchronization

Our focus so far has been on achieving agreement on a common output trajectory, without regard to the particular properties of that trajectory. In this section we consider the related problem of regulating the outputs toward a desired reference trajectory $y_r(t)$, which is defined as the output of an autonomous exosystem

$$\dot{\omega} = S\omega, \tag{6.12a}$$

$$y_r = R\omega, \tag{6.12b}$$

where $\omega \in \mathbb{R}^{n_\omega}$ and $y_r \in \mathbb{R}^p$. Our goal is to achieve $\lim_{t\to\infty} e_i = 0$ for each $i \in \{1,\ldots,N\}$, where e_i is now defined as

$$e_i := y_i - y_r.$$

By the same argument as in Section 6.3.1, we assume without loss of generality that (S, R) is observable and that all the eigenvalues of S are in the closed right-half complex plane.

In order for the agents to follow the reference trajectory, some information must be available to the network about agent outputs relative to the reference trajectory. In particular, let $\mathcal{I} \subset \{1,\ldots,N\}$ be a set of indices corresponding to a subset of agents in the network. We assume that each agent $i \in \{1,\ldots,N\}$ has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \quad \iota_i = \begin{cases} 1, & i \in \mathcal{I}, \\ 0, & i \notin \mathcal{I}. \end{cases}$$

That is, each agent in the index set $\mathcal{I}$ knows the difference between its own output and that of the reference trajectory. To proceed with the design, we need to replace Assumption 6.1 with a slightly modified assumption.

Assumption 6.2. *Every node of $\mathcal{G}$ is a member of a directed tree with the root contained in $\mathcal{I}$. Furthermore, for each $i \in \{1,\ldots,N\}$,*

1) (A_i, B_i) is stabilizable

2) (A_i, C_i) is observable

3) (A_i, B_i, C_i, D_i) is right-invertible

4) (A_i, B_i, C_i, D_i) has no invariant zeros in the closed right-half complex plane that coincide with the eigenvalues of S

We define the matrix $\bar{G} := G + \text{diag}(\iota_1, \ldots, \iota_N)$. It then follows from Lemma 6.7 in Appendix 6.B that all the eigenvalues of $\bar{G}$ are in the open right-half complex plane.

6.4.1 Control Design

The control design starts in the same way as in Section 6.3.2, except that the exosystem now plays the role of agent K, and we carry out three steps for each agent $i \in \{1, \ldots, N\}$. In addition to agent i's system matrices (A_i, B_i, C_i, D_i), the information needed to carry out these three steps is as follows:

- the matrices S and R of the exosystem
- a common integer $\bar{n}$ such that $\bar{n} \geq n_i + n_\omega$ for all $i \in \{1, \ldots, N\}$ (see footnote [1] on Page 4 for a less conservative definition).
- a common matrix $L \in \mathbb{R}^{p \times p\bar{n}}$, freely chosen
- a common high-gain parameter $\varepsilon \in (0, 1]$
- a common number $\tau > 0$ that is a lower bound on the real part of the eigenvalues of the matrix $\bar{G}$

Based on this information, the matrices $\mathcal{A}$, $\mathcal{C}$, Ω_ε, $\mathcal{L}_\varepsilon$, and $\mathcal{P}_\varepsilon$ can be defined in the same way as in Section 6.3.1.

6.4.1.1 Design Procedure for Agent $i \in \{1, \ldots, N\}$

We follow the exact procedure of Steps 1 and 2 in Section 6.3.2, with $x_K = \omega$, $y_K = y_r$, and $(A_K, C_K) = (S, R)$.[4] This yields a state $\bar{x}_i$ such that the dynamics of the synchronization error e_i

[4]We note that Assumption 6.2 ensures that Properties 1–4 of Assumption 6.1 now hold for each $i \in \{1, \ldots, N\}$, which facilitates the design in Steps 1 and 2.

is governed by the system (6.3), with the same properties as those shown in Lemma 6.2. Similar to Lemma 6.3, we can therefore state the following result.

Lemma 6.5. *The regulator equations* (6.7) *are solvable, and the state-feedback controller* $u_i = \bar{F}_i \bar{x}_i$*, where* $\bar{F}_i = [F_i, \Gamma_i - F_i \Pi_i]$ *and* F_i *is chosen such that* $A_i + B_i F_i$ *is Hurwitz, ensures that* $\lim_{t\to\infty} e_i = \lim_{t\to\infty}(y_i - y_r) = 0$*.*

We continue by constructing an observer. Let χ_i be defined in the same way as in Step 3 of Section 6.3.2, to obtain the dynamic equations (6.9). We construct the observer

$$\dot{\hat{\chi}}_i = (\mathcal{A} + \mathcal{L}_i)\hat{\chi}_i + \mathcal{B}_i u_i + \Omega_\varepsilon \mathcal{P}_\varepsilon \mathcal{C}'(\zeta_i - \hat{\zeta}_i) + \Omega_\varepsilon \mathcal{P}_\varepsilon \mathcal{C}'(\psi_i - \iota_i(\mathcal{C}\hat{\chi}_i + \mathcal{D}_i u_i)), \tag{6.13a}$$

$$\hat{\bar{x}}_i = (T_i' T_i)^{-1} T_i' \hat{\chi}_i. \tag{6.13b}$$

Finally, we define $\eta_i = \mathcal{C}\hat{\chi}_i + \mathcal{D}_i u_i$ and $u_i = \bar{F}_i \hat{\bar{x}}_i$ as before.

The following lemma, which is proven in Appendix 6.A, shows that all the estimation errors vanish asymptotically if the high-gain parameter ε is chosen sufficiently small.

Lemma 6.6. *There exists an* $\varepsilon^* \in (0,1]$ *such that, if* ε *is chosen such that* $\varepsilon \in (0, \varepsilon^*]$*, then for each* $i \in \{1, \ldots, N\}$ *we have* $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$*.*

Based on Lemmas 6.5 and 6.6, we can state the following result, which shows that regulated output synchronization is achieved.

Theorem 6.2. *There exists an* $\varepsilon^* \in (0,1]$ *such that, if* ε *is chosen such that* $\varepsilon \in (0, \varepsilon^*]$*, then for each* $i \in \{1, \ldots, N\}$*,* $\lim_{t\to\infty}(y_i - y_r) = 0$*.*

6.5 Example

We illustrate the results from Section 6.3 on a network of ten agents. Agents 1 and 2 are composed as the cascade of a second-order oscillator and a single integrator:

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_i = 0.$$

Agents 3, 4, and 5 are double integrators:

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

Agents 6, 7, and 8 are single integrators: $A_i = 0$, $B_i = 1$, $C_i = 1$, $D_i = 0$. Finally, agents 9 and 10 are second-order mass-spring-damper systems:

$$A_i = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

The topology of the network is given by the digraph depicted in Fig. 6.1, which contains multiple directed spanning trees. One of these is rooted at node 2, and we therefore choose $K = 2$ for our design. The real part of the eigenvalues of the matrix $\bar{G}_2$, constructed by removing row and column 2 from the Laplacian of the digraph in Fig. 6.1, are lower bounded by approximately 0.33. We assume that a bound $\tau = 0.3$ is known during the design process. We also assume that a bound $\bar{n} = 6$ on $n_2 + n_i$, $i \in \{1, \ldots, 10\} \setminus 2$, is known. The matrix L is chosen as the zero matrix. Following the design procedure in Section 6.3.2, we set $u_2 = 0$ and proceed with Steps 1–3 for each of the other agents.

For illustrative purposes, we give the details for agent 3. In Step 1, we first compute O_3 as

$$O_3 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow q_3 = 1,\ r_3 = 2.$$

We may choose $\Lambda_{3u} = [1, 0]'$ and $\Phi_{3u} = [1, 0, 0]'$, and hence we can set $\Lambda_3 = I_2$ and $\Phi_3 = I_3$. It follows that

$$\bar{x}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x_3 - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x_2.$$

It can be confirmed that the dynamics of $\bar{x}_i$ with output e_i takes the form of (6.3) with

$$\bar{A}_{312} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A}_{322} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \bar{C}_{32} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

In Step 2, the regulator equations (6.7) are found to have the solution

$$\Pi_3 = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad \Gamma_3 = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

We select the matrix $F_3 = [-2 - 3]$ to place the poles of $A_3 + B_3F_3$ at -1 and -2. Thus, we obtain the matrix $\bar{F}_3 = [-2, -3, -3, -1]$.

In Step 3 we design the observer according to the procedure in Section 6.3.2, with the high-gain

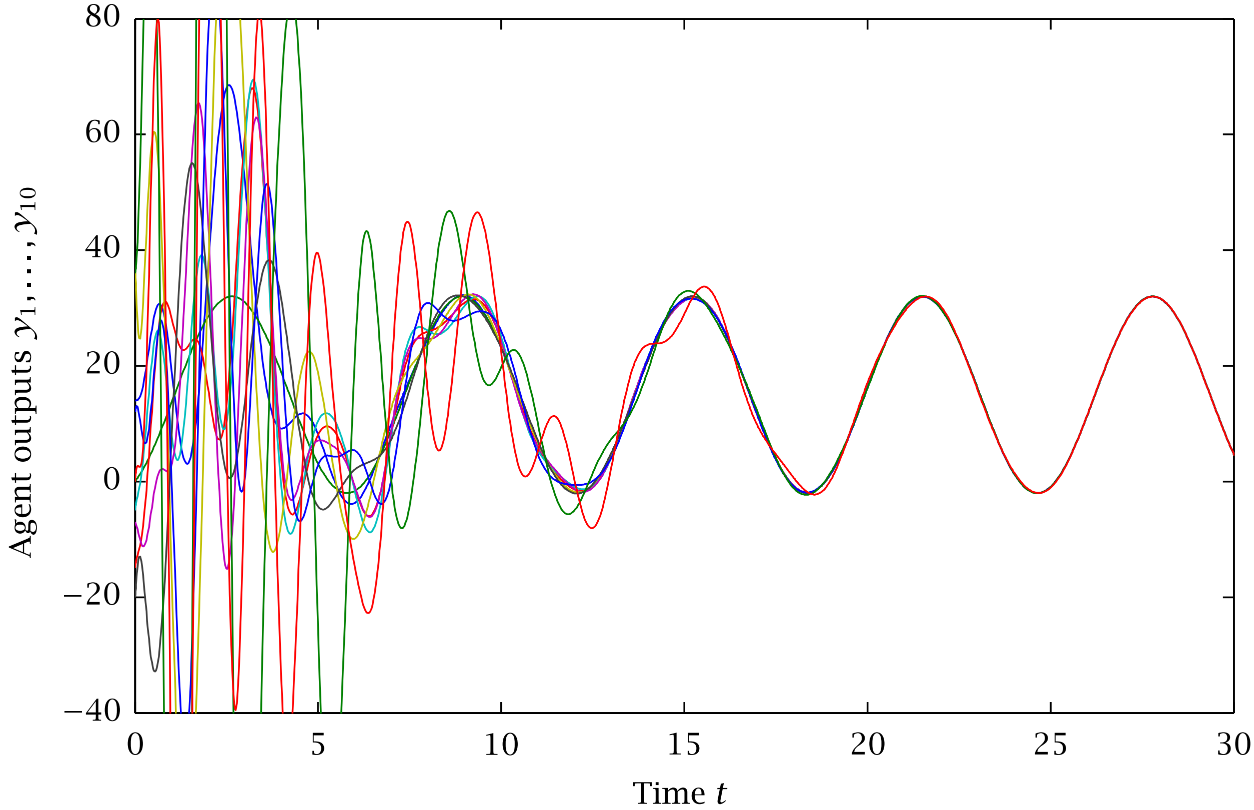


Figure 6.2: Outputs from the simulation example

parameter $\varepsilon = 0.3$. The relevant matrices for the model (6.9) are

$$\mathcal{A} = \begin{bmatrix} 0 & I_5 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{B}_3 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix}', \quad L_3 = \begin{bmatrix} 0 & 0 & 0.5 & 0 & -0.5 & 0 \end{bmatrix}.$$

We perform the same procedure for the other agents. For agent 1, we obtain $q_i = 3$ and $r_i = 0$; for agents 6, 7, and 8, we obtain $q_i = 1$ and $r_i = 2$; and for agents 9 and 10, we obtain $q_i = 0$ and $r_i = 3$. Fig. 6.2 shows the resulting simulated output for all ten agents.

6.6 Concluding Remarks

The designs presented in this chapter rely on a set of conditions about the agents and the network that are straightforward to verify. However, they are not all strictly necessary. Inspecting the proofs of our results we see, for example, that the condition on the invariant zeros in Assumption 6.1 (and $1'$) is used only in the proof of Lemma 6.3 (6.5) to guarantee that no invariant zeros of (A_i, B_i, C_i, D_i) coincide with the eigenvalues of $\bar{A}_{i22}$. Since the eigenvalues of $\bar{A}_{i22}$ are only a subset of the eigenvalues of A_K (S), the quadruple (A_i, B_i, C_i, D_i) can be allowed to contain certain invariant zeros of A_K (S). Indeed, in the special case of identical agents, the matrix $\bar{A}_{i22}$ vanishes, so the condition on the invariant zeros is not needed. Similarly, the condition of right-invertibility is used only to guarantee solvability of the regulator equations (6.7), which vanish for identical agents. Hence, if agent i is identical to A_K, then it does not need to be right-invertible.

Finally, we also note that by choosing $u_K = 0$ and $\eta_K = 0$ in the design for output synchronization, we discard agent K's actuation capability and the information that it receives from the network. It is possible that the assumptions made in this chapter can be relaxed by letting all the agents participate actively in the synchronization process (as is done in the regulated output synchronization problem), although this is yet to be investigated. Current research is focused on relaxing the assumptions with respect to right-invertibility and invariant zeros.

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6.A Proof of Lemmas 6.1, 6.2, 6.3, 6.4, and 6.6

Lemma 6.1. The set of nodes $\{1,\ldots,N\}\setminus K$ can be grouped into directed subgraphs $\mathcal{G}_1,\ldots,\mathcal{G}_M$, each of which has a directed spanning tree rooted at a child of node K. We can assume that there are no edges from $\mathcal{G}_k$ to $\mathcal{G}_j$ if $k>j$ (if such an edge exists, then the child node in $\mathcal{G}_j$ can be moved to $\mathcal{G}_k$). With this permutation, the matrix $\bar{G}_K$ takes the block-triangular form

$$\bar{G}_K = \begin{bmatrix} \tilde{G}_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \tilde{G}_{M1} & \cdots & \tilde{G}_{MM} \end{bmatrix}.$$

Each submatrix $\tilde{G}_{ii}$, $i \in 1,\ldots,M$, can be written as $\tilde{G}_{ii} = G_i + D_i$, where G_i is the Laplacian of $\mathcal{G}_i$ and D_i is a diagonal matrix whose j'th entry is the total weight of all the edges to node j of $\mathcal{G}_i$ from nodes outside of $\mathcal{G}_i$. Since $\mathcal{G}_i$ contains a directed spanning tree whose root is the child of node K, the diagonal element in D_i corresponding to that root is positive. It therefore follows from Lemma 6.7 in Appendix 6.B that all the eigenvalues of $\tilde{G}_{ii}$ are in the open right-half complex plane. The same is true for $\bar{G}_K$, due to its block-triangular form. □

Lemma 6.2. The definitions of Λ_{iu} and Φ_{iu} imply that the columns of $[\Lambda_{iu}', \Phi_{iu}']'$ span the unobservable subspace of the model (6.2), which is invariant with respect to $\mathrm{blkdiag}(A_i, A_K)$. Hence, there exists a matrix $U_i \in \mathbb{R}^{q_i\times q_i}$ such that

$$\begin{bmatrix} A_i & 0 \\ 0 & A_K \end{bmatrix}\begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} U_i, \quad \begin{bmatrix} C_i & -C_K \end{bmatrix}\begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0. \tag{6.14}$$

Let $\bar{x}_i$ be partitioned as $\bar{x}_i = [\bar{x}_{i1}', \bar{x}_{i2}']'$, where

$$\bar{x}_{i1} = x_i - \Lambda_i M_i \Phi_i^{-1} x_K, \quad \bar{x}_{i2} = -N_i \Phi_i^{-1} x_K.$$

Using the equality $C_i \Lambda_{iu} = C_K \Phi_{iu}$, derived from (6.14), we calculate e_i in terms of $\bar{x}_{i1}$ and $\bar{x}_{i2}$:

$$\begin{aligned}
e_i &= C_i x_i - C_K x_K + D_i u_i \\
&= C_i x_i - C_K \begin{bmatrix} \Phi_{iu} & \Phi_{io} \end{bmatrix} \Phi_i^{-1} x_K + D_i u_i \\
&= C_i x_i - \begin{bmatrix} C_i \Lambda_{iu} & C_K \Phi_{io} \end{bmatrix} \Phi_i^{-1} x_K + D_i u_i \\
&= C_i x_i - (C_i \Lambda_i M_i + C_K \Phi_i N_i' N_i) \Phi_i^{-1} x_K + D_i u_i \\
&= C_i (x_i - \Lambda_i M_i \Phi_i^{-1} x_K) - C_K \Phi_i N_i' N_i \Phi_i^{-1} x_K + D_i u_i \\
&= C_i \bar{x}_{i1} + C_K \Phi_i N_i' \bar{x}_{i2} + D_i u_i.
\end{aligned}$$

From (6.14), we also have that $A_i \Lambda_{iu} = \Lambda_{iu} U_i$ and $A_K \Phi_{iu} = \Phi_{iu} U_i$. We therefore easily derive that there exist matrices Q_i and R_i on the form

$$Q_i = \begin{bmatrix} U_i & Q_{i12} \\ 0 & Q_{i22} \end{bmatrix}, \quad R_i = \begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix},$$

such that $A_i \Lambda_i = \Lambda_i Q_i$ and $A_K \Phi_i = \Phi_i R_i$. For $\bar{x}_{i1}$ we can now calculate the state equations as

$$\begin{aligned}
\dot{\bar{x}}_{i1} &= A_i x_i - \Lambda_i M_i \Phi_i^{-1} A_K x_K + B_i u_i \\
&= A_i x_i - \Lambda_i M_i R_i \Phi_i^{-1} x_K + B_i u_i \\
&= A_i x_i - \Lambda_i \begin{bmatrix} U_i & R_{i12} \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K + B_i u_i \\
&= A_i x_i - \Lambda_i \begin{bmatrix} U_i & 0 \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K - \Lambda_i \begin{bmatrix} 0 & R_{i12} \\ 0 & 0 \end{bmatrix} \Phi_i^{-1} x_K + B_i u_i \\
&= A_i x_i - \Lambda_i Q_i M_i \Phi_i^{-1} x_K - \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} N_i \Phi_i^{-1} x_K + B_i u_i \\
&= A_i \left(x_i - \Lambda_i M_i \Phi_i^{-1} x_K \right) - \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} N_i \Phi_i^{-1} x_K + B_i u_i \\
&= A_i \bar{x}_{i1} + \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix} \bar{x}_{i2} + B_i u_i .
\end{aligned}$$

For $\bar{x}_{i2}$ we have $\dot{\bar{x}}_{i2} = -N_i \Phi_i^{-1} A_K x_K = -N_i R_i \Phi_i^{-1} x_K = -R_{i22} N_i \Phi_i^{-1} x_K = R_{i22} \bar{x}_{i2}$. Defining

$$\bar{A}_{i12} = \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix}, \quad \bar{A}_{i22} = R_{i22}, \quad \bar{C}_{i2} = -C_K \Phi_i N_i',$$

we see that e_i is governed by the dynamical equations (6.3). To see that $(\bar{A}_i, \bar{C}_i)$ is observable, note that the observability matrix O_i of the system (6.2) has rank $n_i + r_i$, which is precisely the order of the system (6.3). To see that the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K, note that, due to the block-triangular form of R_i, the eigenvalues of $\bar{A}_{i22} = R_{i22}$ are a subset of the eigenvalues of R_i. Since R_i is obtained from A_K via a similarity transform $R_i = \Phi_i^{-1} A_K \Phi_i$, it has the same eigenvalues as A_K. □

Lemma 6.3. Using the notation of the proof of Lemma 6.2, the task of achieving $\lim_{t\to\infty} e_i = 0$ can be viewed as an output regulation problem, where the subsystem $\dot{\bar{x}}_{i2} = \bar{A}_{i22}\bar{x}_{i2}$ is the exosystem and $\dot{\bar{x}}_{i1} = A_i\bar{x}_{i1} + \bar{A}_{i12}\bar{x}_{i2} + B_i u_i$ is the system to be regulated to achieve $e_i = C_i\bar{x}_{i1} - \bar{C}_{i2}\bar{x}_{i2} + D_i u_i = 0$. Since (A_i, B_i) is stabilizable and the eigenvalues of $\bar{A}_{i22}$ are in the closed right-half complex plane, [70, Theorem 2.3.1] shows that the state-feedback controller $u_i = \bar{F}_i\bar{x}_i$ solves the regulation problem, assuming the regulator equations (6.7) are solvable. From [70, Corollary 2.5.1], the regulator equations are solvable if, for each λ that is an eigenvalue of $\bar{A}_{i22}$, the Rosenbrock system matrix $\left[\begin{smallmatrix} A_i-\lambda I & B_i \\ C_i & D_i \end{smallmatrix}\right]$ has rank $n_i + p$. The Rosenbrock system matrix has normal rank $n_i + p$ due to right-invertibility of the quadruple (A_i, B_i, C_i, D_i) (see [68, Property 3.1.6]). Since this quadruple has no invariant zeros coinciding with eigenvalues of A_K and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of A_K, it follows that the rank of the Rosenbrock system matrix is equal to the normal rank for each λ that is an eigenvalue of $\bar{A}_{i22}$. □

Lemma 6.4. Let $\tilde{\chi}_i = \chi_i - \hat{\chi}_i$. Then

$$\begin{aligned}\dot{\tilde{\chi}}_i &= (\mathcal{A} + \mathcal{L}_i)\tilde{\chi}_i - \Omega_\varepsilon \mathcal{P}_\varepsilon \mathcal{C}'(\zeta_i - \hat{\zeta}_i) \\ &= (\mathcal{A} + \mathcal{L})\tilde{\chi}_i - \tilde{\mathcal{L}}_i\tilde{\chi}_i - \Omega_\varepsilon \mathcal{P}_\varepsilon \mathcal{C}'(\zeta_i - \hat{\zeta}_i),\end{aligned}$$

where $\mathcal{L} = [0, L']'$ and $\tilde{\mathcal{L}}_i := \mathcal{L} - \mathcal{L}_i$. Noting that for each $i \in \{1, \ldots, N\}$, $\sum_{j=1}^{N} g_{ij} = 0$, we have

$$\begin{aligned}\zeta_i &= \sum_{j=1}^{N} g_{ij} y_j = \sum_{j=1}^{N} g_{ij}(y_j - y_K) \\ &= \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij} e_j = \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij}(\mathcal{C}\chi_j + \mathcal{D}_j u_j).\end{aligned}$$

Also, since $\eta_K = 0$, $\hat{\zeta}_i = \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij}(\mathcal{C}\hat{\chi}_j + \mathcal{D}_j u_j)$. It follows that

$$\dot{\tilde{\chi}}_i = (\mathcal{A} + \mathcal{L})\tilde{\chi}_i - \tilde{\mathcal{L}}_i\tilde{\chi}_i - \Omega_\varepsilon \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij} \mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}\tilde{\chi}_j.$$

Introducing the state transformation $\xi_i = \varepsilon^{-1}\Omega_\varepsilon^{-1}\tilde{\chi}_i$, it can be confirmed that

$$\varepsilon\dot{\xi}_i = (\mathcal{A} + \mathcal{L}_\varepsilon)\xi_i - \tilde{\mathcal{L}}_{i\varepsilon}\xi_i - \sum_{j\in\{1,\ldots,N\}\setminus K} g_{ij}\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}\xi_j,$$

where

$$\tilde{\mathcal{L}}_{i\varepsilon} = \begin{bmatrix} 0 \\ \varepsilon^{\bar{n}+1}(L - L_i)\Omega_\varepsilon \end{bmatrix}.$$

Define

$$\xi = \begin{bmatrix} \xi_1' & \ldots & \xi_{K-1}' & \xi_{K+1}' & \ldots & \xi_N' \end{bmatrix}',$$

$$\tilde{\mathcal{L}}_\varepsilon = \operatorname{blkdiag}(\tilde{\mathcal{L}}_{1\varepsilon}, \ldots, \tilde{\mathcal{L}}_{(K-1)\varepsilon}, \tilde{\mathcal{L}}_{(K+1)\varepsilon}, \ldots, \tilde{\mathcal{L}}_{N\varepsilon}),$$

and note that $\|\tilde{\mathcal{L}}_\varepsilon\| = O(\varepsilon)$. The overall dynamics of ξ is

$$\varepsilon\dot{\xi} = (I_{N-1} \otimes (\mathcal{A} + \mathcal{L}_\varepsilon) - \bar{G}_K \otimes (\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}) - \tilde{\mathcal{L}}_\varepsilon)\xi. \tag{6.15}$$

We shall show that the dynamics in (6.15) can be stabilized by making ε small, in order to diminish $\tilde{\mathcal{L}}_\varepsilon$.

Following the methodology of [98], we define U such that $J = U^{-1}\bar{G}_K U$, where J is the Jordan form of $\bar{G}_K$, and introduce the transformation $\xi = (U \otimes I_{p\bar{n}})\nu$. Then

$$\varepsilon\dot{\nu} = (I_{N-1} \otimes (\mathcal{A} + \mathcal{L}_\varepsilon) - J \otimes (\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}) - \tilde{\mathcal{W}}_\varepsilon)\nu, \tag{6.16}$$

where $\tilde{\mathcal{W}}_\varepsilon := (U^{-1} \otimes I_{p\bar{n}})\tilde{\mathcal{L}}_\varepsilon(U \otimes I_{p\bar{n}})$. Note that $\|\tilde{\mathcal{W}}_\varepsilon\| = O(\varepsilon)$. Partitioning $\nu = [\nu_1^*, \ldots, \nu_{N-1}^*]^*$ in the same way as ξ, we have that

$$\begin{aligned}
\varepsilon\dot{\nu}_i &= \mathcal{R}_i\nu_i - \rho_i\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}\nu_{i+1} - \sum_{j=1}^{N-1}\tilde{w}_{\varepsilon ij}\nu_j, \quad i \in 1, \ldots, N-2, \\
\varepsilon\dot{\nu}_{N-1} &= \mathcal{R}_{N-1}\nu_{N-1} - \sum_{j=1}^{N-1}\tilde{w}_{\varepsilon(N-1)j}\nu_j,
\end{aligned}$$

where $\mathcal{R}_i = \mathcal{A}+\mathcal{L}_\varepsilon - \lambda_i \mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}$; λ_i is the i'th eigenvalue along the diagonal of J; $\rho_i \in \{0,1\}$; and $\tilde{w}_{\varepsilon ij}$ is the (i,j)'th $p\bar{n} \times p\bar{n}$ block of $\tilde{\mathcal{W}}_\varepsilon$. Following the results of [104], we can show that $\mathcal{R}_i$ is Hurwitz:

$$\begin{aligned}\mathcal{R}_i\mathcal{P}_\varepsilon + \mathcal{P}_\varepsilon\mathcal{R}_i^* &= (\mathcal{A}+\mathcal{L}_\varepsilon)\mathcal{P}_\varepsilon + \mathcal{P}_\varepsilon(\mathcal{A}+\mathcal{L}_\varepsilon)' - 2\mathrm{Re}(\lambda_i)\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}\mathcal{P}_\varepsilon \\ &= (\mathcal{A}+\mathcal{L}_\varepsilon)\mathcal{P}_\varepsilon + \mathcal{P}_\varepsilon(\mathcal{A}+\mathcal{L}_\varepsilon)' - 2\tau\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}\mathcal{P}_\varepsilon - 2(\mathrm{Re}(\lambda_i)-\tau)\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}\mathcal{P}_\varepsilon \le -I_{p\bar{n}}.\end{aligned}$$

Next, note that there exists an $M_P > 0$ such that for all sufficiently small $\varepsilon > 0$, $\|\mathcal{P}_\varepsilon\| < M_P$. To see this, let $\mathcal{P}$ be the solution of the Riccati equation $\mathcal{A}\mathcal{P}+\mathcal{P}\mathcal{A}' - 2\tau\mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P} + 2I_{p\bar{n}} = 0$ and let ε be small enough that $\mathcal{L}_\varepsilon\mathcal{P}+\mathcal{P}\mathcal{L}_\varepsilon' \le I_{p\bar{n}}$. Then clearly $(\mathcal{A}+\mathcal{L}_\varepsilon)\mathcal{P} + \mathcal{P}(\mathcal{A}+\mathcal{L}_\varepsilon)' - 2\tau\mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P} + I_{\bar{n}p} \le 0$ and it then follows from standard LQ theory that $\mathcal{P}_\varepsilon \le \mathcal{P}$ (see, e.g., [27]).

Define a Lyapunov function $V = \varepsilon\sum_{i=1}^{N-1}\ell_i v_i^*\mathcal{P}_\varepsilon^{-1}v_i$, where the ℓ_i's are defined recursively by $\ell_{N-1} = 1$ and $\ell_i = \ell_{i+1}/(9M_P^4)$ for $i \in 1,\dots,N-2$. Then

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{N-1}\ell_i\nu_i^*\mathcal{P}_\varepsilon^{-1}(\mathcal{R}_iP_\varepsilon + P_\varepsilon\mathcal{R}_i^*)\mathcal{P}_\varepsilon^{-1}\nu_i - 2\mathrm{Re}\left(\sum_{i=1}^{N-2}\ell_i\rho_i\nu_i^*\mathcal{P}_\varepsilon^{-1}(\mathcal{P}_\varepsilon\mathcal{C}'\mathcal{C}\mathcal{P}_\varepsilon)\mathcal{P}_\varepsilon^{-1}\nu_{i+1}\right) \\ &\quad - 2\mathrm{Re}\left(\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\ell_i\nu_i^*\mathcal{P}_\varepsilon^{-1}(\tilde{w}_{\varepsilon ij}\mathcal{P}_\varepsilon)\mathcal{P}_\varepsilon^{-1}\nu_j\right) \\ &\le -\sum_{i=1}^{N-1}\ell_i\upsilon_i^2 + 2\sum_{i=1}^{N-2}\ell_iM_P^2\upsilon_i\upsilon_{i+1} + 2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\ell_i\|\tilde{w}_{\varepsilon ij}\mathcal{P}_\varepsilon\|\upsilon_i\upsilon_j,\end{aligned}$$

where $\upsilon_i := \|\mathcal{P}_\varepsilon^{-1}\nu_i\|$. Note that the first two terms can be written as

$$-\frac{1}{3}\sum_{i=1}^{N-1}\ell_i\upsilon_i^2 - \frac{1}{3}\ell_1\upsilon_1^2 - \frac{1}{3}\ell_{N-1}\upsilon_{N-1}^2 - \sum_{i=1}^{N-2}\left(\frac{\ell_iM_P^2}{\sqrt{\frac{1}{3}\ell_{i+1}}}\upsilon_i - \sqrt{\frac{1}{3}\ell_{i+1}}\upsilon_{i+1}\right)^2 - \sum_{i=1}^{N-2}\left(\frac{1}{3}\ell_i - \frac{\ell_i^2M_P^4}{\frac{1}{3}\ell_{i+1}}\right)\upsilon_i^2.$$

From the definition of ℓ_i, it can be confirmed that the last term is zero. It follows that $\dot{V} \le -\frac{1}{3}\sum_{i=1}^{N-1}\ell_i\upsilon_i^2 + 2\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\ell_iM_P\|\tilde{w}_{\varepsilon ij}\|\upsilon_i\upsilon_j$, Since $\|\tilde{w}_{\varepsilon ij}\| = O(\varepsilon)$ and the ℓ_i's are independent of ε, the first quadratic term dominates the second quadratic term for all sufficiently small ε, and hence $\dot{V}$ is negative definite. It now follows that $\lim_{t\to\infty}\nu = 0$, which implies $\lim_{t\to\infty}\xi = 0$. This in turn implies that $\hat{\chi}_i$ converges to $\chi_i = T_i\bar{x}_i$, and hence $\hat{\bar{x}}_i$ converges to $(T_i'T_i)^{-1}T_i'T_i\bar{x}_i = \bar{x}_i$. $\square$

Lemma 6.6. Let $\tilde{\chi}_i = \chi_i - \hat{\chi}_i$. Then

$$\dot{\tilde{\chi}}_i = (\mathcal{A} + \mathcal{L})\tilde{\chi}_i - \tilde{\mathcal{L}}_i\tilde{\chi}_i - \Omega_\varepsilon \mathcal{P}_\varepsilon \mathcal{C}'(\zeta_i - \hat{\zeta}_i) - \Omega_\varepsilon \mathcal{P}_\varepsilon \mathcal{C}'(\psi_i - \iota_i(\mathcal{C}\hat{\chi}_i + \mathcal{D}_i u_i)),$$

where $\mathcal{L} = [0, L']'$ and $\tilde{\mathcal{L}}_i := \mathcal{L} - \mathcal{L}_i$. Note that

$$\sum_{j=1}^{N} g_{ij} y_j = \sum_{j=1}^{N} g_{ij}(y_j - y_r) = \sum_{j=1}^{N} g_{ij}(\mathcal{C}\chi_j + \mathcal{D}_j u_j).$$

Also, $\hat{\zeta}_i = \sum_{j=1}^{N} g_{ij}(\mathcal{C}\hat{\chi}_j + \mathcal{D}_j u_j)$ and $\psi_i = \iota_i e_i = \iota_i(\mathcal{C}\chi_i + \mathcal{D}_i u_i)$. It follows that

$$\dot{\tilde{\chi}}_i = (\mathcal{A} + \mathcal{L})\tilde{\chi}_i - \tilde{\mathcal{L}}_i\tilde{\chi}_i - \Omega_\varepsilon \left(\sum_{j=1}^{N} g_{ij} \mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}\tilde{\chi}_j + \iota_i \mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}\tilde{\chi}_i \right),$$

or, after introducing the state transformation $\xi_i = \varepsilon^{-1}\Omega_\varepsilon^{-1}\tilde{\chi}_i$,

$$\varepsilon\dot{\xi}_i = (\mathcal{A} + \mathcal{L}_\varepsilon)\xi_i - \tilde{\mathcal{L}}_{i\varepsilon}\xi_i - \left(\sum_{j=1}^{N} g_{ij} \mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}\xi_j + \iota_i \mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}\xi_i \right),$$

where $\tilde{\mathcal{L}}_{i\varepsilon}$ is defined in the same way as in the proof of Lemma 6.4. Defining $\xi = [\xi_1', \ldots, \xi_N']'$ and $\tilde{\mathcal{L}}_\varepsilon = \text{blkdiag}(\tilde{\mathcal{L}}_{1\varepsilon}, \ldots, \tilde{\mathcal{L}}_{N\varepsilon})$, the overall dynamics becomes

$$\varepsilon\dot{\xi} = (I_N \otimes (\mathcal{A} + \mathcal{L}_\varepsilon) - \bar{G} \otimes (\mathcal{P}_\varepsilon \mathcal{C}'\mathcal{C}) - \tilde{\mathcal{L}}_\varepsilon)\xi.$$

The proof can now be completed in the same way as the proof of Lemma 6.4. □

6.B A Useful Lemma

We here give a slightly extended version of [32, Lemma 5].

Lemma 6.7. *Suppose that $\mathcal{G}$ is a weighted digraph with N nodes, and suppose that $\mathcal{I} \subset \{1, \ldots, N\}$ represents a subset of nodes such that every node of $\mathcal{G}$ is a member of a directed tree with its root contained in $\mathcal{I}$.*[5] *Let G be the Laplacian of $\mathcal{G}$ and let $D = \text{diag}(d_1, \ldots, d_N)$ be a diagonal matrix*

[5] A special case is when $\mathcal{I}$ consists of a single element corresponding to the root of a directed spanning tree of $\mathcal{G}$.

with non-negative elements. If for each $i \in \mathcal{I}$, $d_i > 0$, then all the eigenvalues of $\bar{G} := G + D$ are in the open right-half complex plane.

Proof. Let $\hat{\mathcal{G}}$ denote an expanded digraph constructed from $\mathcal{G}$ by adding a node 0 and edges from node 0 to node $i \in \{1, \ldots, N\}$ with weigth d_i, whenever $d_i > 0$. Then the Laplacian of $\hat{\mathcal{G}}$ is given by $\hat{G} = \left[\begin{smallmatrix} 0 & 0 \\ -d & \bar{G} \end{smallmatrix}\right]$, where $d = [d_1, \ldots, d_N]'$. Since $\hat{\mathcal{G}}$ contains edges from 0 to every node in $\mathcal{I}$, it contains a directed spanning tree rooted at node 0. Hence, from [53, Lemma 3.3], $\hat{G}$ has a simple eigenvalue at the origin, and all the other eigenvalues are in the open right-half complex plane. Due to the block-triangular form of $\hat{G}$, its eigenvalues consist of the zero element $(1, 1)$ and the eigenvalues of $\bar{G}$. It therefore follows that the eigenvalues of $\bar{G}$ must be in the open right-half complex plane. $\square$

6.C Auxiliary Model for (A_K, C_K)

Suppose that the model $\dot{x}_K = A_K x_K$, $y_K = C_K x_K$ contains unobservable or asymptotically stable modes. We show here how to construct an observable auxiliary model without asymptotically stable modes, whose output converges to that of the original model. Let Γ_1 be a nonsingular matrix such that the state transformation $\Gamma_1 z_K = x_K$ yields the *stability structural decomposition* ([8])

$$\begin{bmatrix} \dot{z}_{K1} \\ \dot{z}_{K2} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} & 0 \\ 0 & \hat{A}_{22} \end{bmatrix} \begin{bmatrix} z_{K1} \\ z_{K2} \end{bmatrix}, \quad y_K = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \begin{bmatrix} z_{K1} \\ z_{K2} \end{bmatrix},$$

where $\hat{A}_{11}$ has all its eigenvalues in the closed right-half complex plane and $\hat{A}_{22}$ has all its eigenvalues in the open left-half complex plane. Since z_{K2} vanishes asymptotically, the system $\dot{z}_{K1} = \hat{A}_{11} z_{K1}$, $y_{K1} = \hat{C}_1 z_{K1}$ has the property that $\lim_{t\to\infty}(y_{K1} - y_K) = 0$ for $z_{K1}(0) = [I, 0]\Gamma_1^{-1} x_K(0)$. Next, let Γ_2 be a nonsingular matrix such that the state transformation $\Gamma_2 q_K = z_{K1}$ yields the Kalman observable canonical form:

$$\begin{bmatrix} \dot{q}_{K1} \\ \dot{q}_{K2} \end{bmatrix} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix} \begin{bmatrix} q_{K1} \\ q_{K2} \end{bmatrix}, \quad y_{K1} = \begin{bmatrix} 0 & \tilde{C}_2 \end{bmatrix} \begin{bmatrix} q_{K1} \\ q_{K2} \end{bmatrix}.$$

The reduced-order system $\dot{q}_{K2} = \tilde{A}_{22} q_{K2}$, $y_{K1} = \tilde{C}_2 q_{K2}$ is clearly observable and yields the same output for $q_{K2}(0) = [0, I]\Gamma_2^{-1} z_{K1}(0)$.

6.D Proof of Column Rank of Λ_{iu} and Φ_{iu}

In this section we demonstrate that the matrices Λ_{iu} and Φ_{iu} must have full column rank. For the sake of establishing a contradiction, suppose that one of the matrices, say Λ_{iu}, has linearly dependent columns. Then there are nonzero vectors $z \in \mathbb{R}^{q_i}$ and $\bar{z} \in \mathbb{R}^{n_K}$ such that

$$\begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} z = \begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} \implies O_i \begin{bmatrix} 0 \\ \bar{z} \end{bmatrix} = 0 \implies \begin{bmatrix} C_K \\ \vdots \\ C_K A_K^{n_K - 1} \end{bmatrix} \bar{z} = 0.$$

The last statement implies that (A_K, C_K) is unobservable, thus establishing the contradiction.

Chapter 7

Semi-global Regulation of Output Synchronization for Heterogeneous Networks of Non-introspective, Invertible Agents subject to Actuator Saturation

7.1 Introduction

The synchronization problem in a network has received substantial attention in recent years (see [2, 44, 55, 100] and references therein). Active research is being conducted in this context and numerous results have been reported in the literature, to name a few see [32, 45, 46, 48, 51, 52, 54, 73, 83, 84].

Much of the attention has been devoted to achieving *state synchronization* in *homogeneous* networks (i.e., networks where the agent models are identical), where each agent has access to a linear combination of its own state relative to that of neighboring agents (e.g., [44–46, 52, 54, 56, 61, 83]). Roy, Saberi, and Herlugson [62] and Yang, Roy, Wan, and Saberi [103] considered the state synchronization problem for more general network topologies. A more realistic case—that is, each agent receives a linear combination of its own partial-state output relative to that of neighboring agents—has been considered in [32,48,49,84,85]. A key idea in the work of [32], which was expanded upon by Yang, Stoorvogel, and Saberi [107], is the development of a distributed observer. This observer makes additional use of the network by allowing the agents to exchange information with their neighbors about their own internal estimates. Many results on the synchronization problem are rooted in the seminal work [98, 99].

7.1.1 Heterogeneous Networks and Output Synchronization

Recent activities in the synchronization literature have been focused on achieving synchronization for heterogeneous networks (i.e., networks where the agent models are non-identical). This problem is challenging and only some results are available, see for instance [10, 20, 25, 39, 97, 101].

In heterogeneous networks, the agents' states may have different dimensions. In this case, the state synchronization is not even properly defined, and it is more natural to aim for *output synchronization*—that is, asymptotic agreement on some partial-state output from each agent. Chopra and Spong [10] studied the output synchronization for weakly minimum-phase nonlinear systems of relative degree one, using a pre-feedback to create a single-integrator system with decoupled zero dynamics. Kim, Shim, and Seo [25] considered the output synchronization for uncertain single-input single-output, minimum-phase linear systems, by embedding an identical model within each agent, the output of which is tracked by the actual agent output. The authors have previously considered the output synchronization problem for right-invertible linear agents, using pre-compensators and an observer-based pre-feedback within each agent to yield a network of substantially identical agents [104].

7.1.2 Introspective versus Non-introspective Agents

The designs mentioned in Section 7.1.1 generally rely on some sort of self-knowledge that is separate from the information transmitted over the network. More specifically, the agents may be required to know their own states or their own outputs. In [16, 17], we refer to agents that possess this type of self-knowledge as *introspective* agents to distinguish them from *non-introspective* agents—that is, agents that have no knowledge about their own states or outputs separate from what is received via the network.

To our best knowledge, the only result besides [16, 17] that clearly applies to heterogeneous networks of non-introspective agents is by Zhao, Hill and Liu [113]. However, the agents are assumed to be passive—a strict requirement that, among other things, requires that the agents are weakly minimum-phase and of relative degree one.

7.1.3 Contributions of this Chapter

The *regulation of output synchronization* problem, where the objective is not only to achieve output synchronization, but to make the synchronization trajectory follow an *a priori* given reference trajectory generated by an arbitrary autonomous exosystem, has been considered in [17]. In [17], we assume that the agents in the network are non-introspective except for some of the agents, which know their own outputs relative to the reference trajectory. However, we do not have any constraints on the magnitude of the agent's input. In the real world, every physically conceivable actuator has bounds on its input, and thus actuator saturation is a common phenomenon. In this chapter, we extend the results in [17] to the case where all the agents are subject to actuator saturation, which introduces significant complexities in terms of the analysis and design.

7.1.4 Notations

Let $A \in \mathbb{R}^{m\times n}$ denote the matrix with complex entries. Given a matrix $A \in \mathbb{R}^{m\times n}$, A' denotes its transpose. $A \in \mathbb{R}^{n\times n}$ is said to be Hurwitz stable if all its eigenvalues are in the open left-half complex plane. The Kronecker product between $A \in \mathbb{R}^{m\times n}$ and a matrix $B \in \mathbb{R}^{p\times q}$ is defined as the $\mathbb{R}^{mp\times nq}$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mn}B \end{bmatrix},$$

where a_{ij} denotes element (i,j) of A. I_n denotes the identity matrix of dimension n. Similarly, 0_n denotes the square matrix of dimension n with all zero elements. We sometimes drop the subscript if the dimension is clear in the context. When clear form the context, $\mathbf{1}$ denotes the column vector with all entries equal to one.

7.2 Problem Formulation and Main Result

7.2.1 Problem Formulation

Consider a network of N multiple-input multiple-output invertible agents of the form

$$\dot{x}_i = A_i x_i + B_i \sigma(u_i), \tag{7.1a}$$

$$y_i = C_i x_i + D_i \sigma(u_i), \tag{7.1b}$$

for $i \in \{1, \ldots, N\}$, where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^p$, $y_i \in \mathbb{R}^p$, and

$$\sigma(u_i) = [\sigma_1(u_{i,1}), \ldots, \sigma_1(u_{i,p})]',$$

where $\sigma_1(u)$ is the standard saturation function

$$\sigma_1(u) = \operatorname{sgn}(u) \min\{1, |u|\},$$

and where the quadruple (A_i, B_i, C_i, D_i) is invertible.

The network provides each agent with a linear combination of its own output relative to that of other agents. In particular, each agent i has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j), \tag{7.2}$$

where $a_{ij} \geq 0$ and $a_{ii} = 0$ with $i, j \in \{1, \ldots, N\}$. This network can be described by a weighted directed graph (digraph) $\mathcal{G}$ with nodes corresponding to the agents in the network and edges with weight given by the coefficients a_{ij}. In particular, $a_{ij} > 0$ means that there exists an edge with weight a_{ij} from agent j to agent i, where agent j is called a parent of agent i, and agent i is called a child of agent j.

We also define a matrix $G = [g_{ij}]$, where $g_{ii} = \sum_{j=1}^{N} a_{ij}$ and $g_{ij} = -a_{ij}$ for $j \neq i$. The matrix G, known as the *weighted Laplacian* matrix of the digraph $\mathcal{G}$ has the property that the sum of the coefficients on each row is equal to zero. In terms of the coefficients g_{ij} of G, ζ_i given by (7.2) can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} g_{ij} y_j. \tag{7.3}$$

In addition to ζ_i given by (7.3), we assume that the agents exchange information about their own internal estimates via the same network. That is, agent i has access to the quantity

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\eta_i - \eta_j) = \sum_{j=1}^{N} g_{ij} \eta_j, \tag{7.4}$$

where $\eta_j \in \mathbb{R}^p$ is a variable produced internally by agent j. This value will be specified as we proceed with the design.

Our goal is to regulate the outputs of all agents towards an *a priori* specified reference trajectory $y_r(t)$, generated by an arbitrary autonomous exosystem

$$\dot{\omega} = S\omega, \quad \omega(0) = \omega_0 \in \Omega_0, \tag{7.5a}$$

$$y_r = C_r \omega, \tag{7.5b}$$

where $\omega \in \mathbb{R}^r$, $y_r \in \mathbb{R}^p$, and Ω_0 is a compact set of possible initial conditions for the exosystem. That is, for each agent $i \in \{1, \ldots, N\}$, we wish to achieve $\lim_{t\to\infty}(y_i - y_r) = 0$. Equivalently, we wish to regulate the synchronization error variable

$$e_i := y_i - y_r$$

to zero asymptotically, where the dynamics of e_i is governed by

$$\begin{bmatrix} \dot{x}_i \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} A_i & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} x_i \\ \omega \end{bmatrix} + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \sigma(u_i), \tag{7.6a}$$

$$e_i = \begin{bmatrix} C_i & -C_r \end{bmatrix} \begin{bmatrix} x_i \\ \omega \end{bmatrix} + D_i \sigma(u_i). \tag{7.6b}$$

In order to achieve our goal, in addition to ζ_i given by (7.3) and $\hat{\zeta}_i$ given by (7.4) provided by the network, it is clear that a non-empty subset of agents should observe its output relative to the reference trajectory y_r generated by (7.5) in order for the network of agents to follow the reference trajectory. Specifically, let $\mathcal{I} \subset \{1, \ldots, N\}$ denotes such a subset. Then, each agent $i \in \{1, \ldots, N\}$ has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \quad \iota_i = \begin{cases} 1, & i \in \mathcal{I}, \\ 0, & i \notin \mathcal{I}. \end{cases} \tag{7.7}$$

Clearly, we need to restrict the initial conditions of the exosystem since, due to the input saturation, the agents will only be able to track a limited set of reference trajectories. This is formulated in the above by assuming that $\omega(0) \in \Omega_0$ with the set Ω_0 a prior known. Regarding the initial conditions of the agents, we would ideally like to design a controller that achieves $\lim_{t\to\infty} e_i(t) = 0$ for all initial conditions subject to $\omega(0) \in \Omega_0$, a problem that can be referred to as *global regulation of output synchronization*. However, from the literature on linear systems subject to actuator saturation, we know that global regulation of output synchronization in general requires nonlinear controllers. In this chapter, we would like to use linear dynamical controllers

$$\dot{x}_i^c = A_{i,c}x_i^c + B_{i,c}\begin{bmatrix} \zeta_i \\ \hat{\zeta}_i \\ \psi_i \end{bmatrix}, \tag{7.8a}$$

$$u_i = C_{i,c}x_i^c, \quad \forall i \in \{1, \ldots, N\}, \tag{7.8b}$$

where $x_i^c \in \mathbb{R}^{q_i}$ is the state of the controller for agent i. Thus, we restrict attention to the *semi-global regulation of output synchronization* problem, which is defined as follows.

Problem 7.1 (Semi-global regulation of output synchronization)**.** *Consider a network of N agents as given by* (7.1) *and the reference model given by* (7.5) *with initial conditions in an* a priori *given* *compact set $\Omega_0 \subset \mathbb{R}^r$. The semi-global regulation of output synchronization problem is to find, if possible, integers q_i, $i \in \{1, \dots, N\}$, such that for any arbitrarily large bounded sets $\mathcal{X}_i \subset \mathbb{R}^{n_i}$ and $\mathcal{P}_i \subset \mathbb{R}^{q_i}$, $i \in \{1, \dots, N\}$, there exist linear dynamical controllers* (7.8) *such that*

$$\lim_{t\to\infty} e_i(t) = 0, \quad \forall i \in \{1, \dots, n\}, \tag{7.9}$$

for all initial conditions $x_i(0) \in \mathcal{X}_i$, $x_i^c(0) \in \mathcal{P}_i$, and $\omega(0) \in \Omega_0$.

Remark 7.1. *We would like to emphasize that our definition of the above semi-global regulation of output synchronization problem does not view the set of initial conditions of the agents' model* (7.1) *and their controllers* (7.8) *as given data. The set of given data consists of the models of the agent* (7.1)*, the exosystem* (7.5)*, and the set Ω_0 of possible initial conditions for the exosystem. Therefore, the solvability conditions must be independent of the set of initial conditions of the agents, $\mathcal{X}_i$, and the set of initial conditions for the controllers, $\mathcal{P}_i$.*

7.2.2 Assumptions

In this section, we present the assumptions about the network topology, the individual agents, and the reference model for solving the semi-global regulation of output synchronization problem as defined in Problem 7.1.

Assumption 7.1. *Every node of the digraph $\mathcal{G}$ is a member of a directed tree with the root contained in $\mathcal{I}$.*

Remark 7.2. *It is possible for $\mathcal{I}$ to consist of a single node, in which case Assumption 7.1 requires this node to be the root of a directed spanning tree of $\mathcal{G}$.*

Assumption 7.2. *For each agent $i \in \{1, \dots, N\}$ as given in* (7.1)

1) all the eigenvalues of A_i are in the closed left-half complex plane;

2) the pair (A_i, B_i) is stabilizable; and

3) the pair (C_i, A_i) is observable.

Remark 7.3. *Conditions 2 and 3 are natural assumptions. Condition 1 is a necessary condition, since if A_i has one observable eigenvalue in the open right-half complex plane for some $i \in \{1, \ldots, N\}$, then for sufficiently large initial conditions $x_i(0)$, the output of that system y_i will be exponentially growing, and the bounded input $\sigma(u_i)$ can influence this exponentially growing signal only in a limited sense. Therefore, we cannot substantially alter that output to track y_r.*

Assumption 7.3. *For the reference model* (7.5)*,*

1) the pair (C_r, S) is observable;

2) all the eigenvalues of S are in the closed right-half complex plane; and

3) the matrix S is neutrally stable.

Remark 7.4. *Condition 1 is a natural assumption. Condition 2 is made without loss of generality because, asymptotically stable modes vanish asymptotically, and they therefore play no role asymptotically. Condition 3 is reasonable since the output of an agent cannot be expected to track exponentially growing signals with a bounded input. Polynomially growing reference signals can be easily included but it requires very restrictive solvability conditions and hence, for ease of presentation, we have excluded this case.*

Assumption 7.4. *The following equations with unknowns $\Pi_i \in \mathbb{R}^{n_i \times r}$ and $\Gamma_i \in \mathbb{R}^{p \times r}$, commonly known as the* regulator equations

$$\Pi_i S = A_i \Pi_i + B_i \Gamma_i, \tag{7.10a}$$

$$C_r = C_i \Pi_i + D_i \Gamma_i \tag{7.10b}$$

are solvable, and there exists a $\delta > 0$ such that for each agent $i \in \{1, \dots, n\}$,

$$\|\Gamma_i \omega(t)\|_\infty \le 1 - \delta, \tag{7.11}$$

for all $t > 0$ and all $\omega(t)$ with $\omega(0) \in \Omega_0$.

Remark 7.5. *Note that if the regulator equations* (7.10) *have a solution, then the solution is unique, as a consequence of invertibility of the quadruple (A_i, B_i, C_i, D_i). Therefore, one can easily verify* (7.11)*.*

7.2.3 Necessity of Assumption 7.4

All Assumptions 7.1, 7.2, and 7.3 are natural as discussed in Remarks 7.3 and 7.4, however Assumption 7.4 is critical. Essentially, this assumption is necessary for solving the semi-global regulation of output synchronization problem as defined in Problem 7.1. The following lemma, which is proven in Appendix 7.A, shows this fact and gives the necessary condition for solving Problem 7.1.

Lemma 7.1. *Suppose that each agent $i \in \{1, \dots, N\}$ has access to full information. Assume that Ω_0 contains 0 in its interior. Then for any initial condition $\omega(0) \in \Omega_0$, there exist initial conditions $x_i(0)$ and an input $u_i(t)$ that leads to $e_i(t) \to 0$ as $t \to \infty$ exists only if the regulator equations* (7.10) *are solvable, and moreover the solution must satisfy*

$$\|\Gamma_i \omega(t)\|_\infty \le 1 \tag{7.12}$$

for all $t > 0$.

7.2.4 Main Result

Theorem 7.1. *Consider a network of N agents as given by* (7.1) *and the reference model given by* (7.5)*. Let Assumptions 7.1, 7.2, 7.3, and 7.4 hold. Then the semi-global regulation of output synchronization problem as defined in Problem 7.1 is solvable.*

Proof. The proof of Theorem 7.1 is given in Section 7.3 by explicit construction of a controller for each agent. □

7.3 Design of Control Law for Each Agent

In this section, we describe the construction of a controller for each agent to solve the semi-global regulation of output synchronization problem as defined in Problem 7.1. The construction is carried out in three steps.

In Step 1, we construct a new state $\bar{x}_i$, via a transformation of x_i and ω, such that the dynamics of the synchronization error variable e_i can be described by equations

$$\dot{\bar{x}}_i = \bar{A}_i\bar{x}_i + \bar{B}_i\sigma(u_i) := \begin{bmatrix} A_i & 0 \\ 0 & \bar{A}_{i22} \end{bmatrix}\bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix}\sigma(u_i), \tag{7.13a}$$

$$e_i = \bar{C}_i\bar{x}_i + \bar{D}_i\sigma(u_i) := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix}\bar{x}_i + D_i\sigma(u_i). \tag{7.13b}$$

The purpose of this state transformation is to reduce the dimension of the model underlying e_i—the dimension of $\bar{x}_i$ is generally lower than that of $[x_i', \omega']'$—by removing redundant modes that have no effect on e_i. In particular, the model (7.6) may be unobservable, but the model (7.13) is always observable.

In Step 2, we construct a low-gain state feedback from $\bar{x}_i$, parameterized in ϵ, that regulates e_i to zero for any arbitrarily large bounded set of initial conditions of the agent's models by suitably choosing the low-gain parameter ϵ. Moreover, by properly choosing the low-gain parameter ϵ, the amplitude of the control law can be made to be less than any given α, where $1-\delta < \alpha < 1$. Since the agent i has neither the internal state x_i nor the state ω of the exosystem available, this controller is not directly implementable. This brings us to Step 3 of the design.

In Step 3, we follow the procedure as given in our previous paper [17], that is, we construct a decentralized high-gain observer that makes an estimate of $\bar{x}_i$ available to agent i. However, as we shall see later that our state feedback design and high-gain observer are coupled. This will be illustrated in Section 7.3.1.

7.3.1 Design Procedure for Agent i

Step 1: State transformation

Let O_i be the observability matrix corresponding to the system (7.6).

$$O_i = \begin{bmatrix} C_i & -C_r \\ \vdots & \vdots \\ C_i A_i^{n_i+r-1} & -C_r S^{n_i+r-1} \end{bmatrix}.$$

Let q_i denote the dimension of the null space of matrix O_i, and define $r_i = r - q_i$. Next, define $\Lambda_{iu} \in \mathbb{R}^{n_i \times q_i}$ and $\Phi_{iu} \in \mathbb{R}^{r \times q_i}$ such that

$$O_i \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = 0, \quad \operatorname{rank} \begin{bmatrix} \Lambda_{iu} \\ \Phi_{iu} \end{bmatrix} = q_i.$$

Since the pair (C_i, A_i) and the pair (C_r, S) are observable, it is easy to see that Λ_{iu} and Φ_{iu} have full column rank (see [17, Appendix A]). Let therefore Λ_{io} and Φ_{io} be defined such that $\Lambda_i := [\Lambda_{iu}, \Lambda_{io}] \in \mathbb{R}^{n_i \times n_i}$ and $\Phi_i := [\Phi_{iu}, \Phi_{io}] \in \mathbb{R}^{r \times r}$ are nonsingular.

From the proof of [16, Lemma 1], we know that

$$S\Phi_i = \Phi_i R_i, \tag{7.14}$$

where

$$R_i = \begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix}.$$

Since S is anti-Hurwitz stable and neutrally stable, we know that S is diagonalizable, and hence R_i is diagonalizable. This implies that R_i has r independent right eigenvectors. Let $v_{i,1}, \cdots, v_{i,r}$ be r independent right eigenvectors of R_i, such that

$$v_{i,j} = \begin{bmatrix} \tilde{v}_{i,j} \\ 0 \end{bmatrix}$$

for $j = 1, \ldots, q_i$, where $\tilde{v}_{i,j}$ are right eigenvectors of U_i. In that case we choose $V_{i11} \in \mathbb{R}^{q_i \times q_i}$ such that

$$\operatorname{Im} V_{i11} = \operatorname{span}\{\text{re } v_{i,j}, \text{ im } v_{i,j} \mid j = 1, \cdots, q_i\}$$

and we choose $V_{i12} \in \mathbb{R}^{q_i \times r_i}$ and $V_{i22} \in \mathbb{R}^{r_i \times r_i}$ such that

$$\operatorname{Im} \begin{bmatrix} V_{i12} \\ V_{i22} \end{bmatrix} = \operatorname{span}\{\text{re } v_{i,j}, \text{ im } v_{i,j} \mid j = q_i + 1, \cdots, r\}.$$

We then construct:

$$V_i = \begin{bmatrix} V_{i11} & V_{i12} \\ 0 & V_{i22} \end{bmatrix},$$

We then have

$$R_i V_i = V_i \begin{bmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{bmatrix} = V_i \Lambda_i \tag{7.15}$$

From (7.15), we obtain that

$$V_{i11}^{-1} U_i V_{i11} = \Lambda_{i1}, \quad V_{i22}^{-1} R_{i22} V_{i22} = \Lambda_{i2}, \tag{7.16}$$

and

$$U_i V_{i12} - V_{i12}\Lambda_{i2} = -R_{i12}V_{i22}. \tag{7.17}$$

We then define

$$\bar{\Phi}_i := [\bar{\Phi}_{iu}, \bar{\Phi}_{io}] = \Phi_i \begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix}. \tag{7.18}$$

We then define a new state variable $\bar{x}_i \in \mathbb{R}^{n_i+r_i}$ as

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{i1} \\ \bar{x}_{i2} \end{bmatrix} := \begin{bmatrix} x_i - \Lambda_i M_i \bar{\Phi}_i^{-1}\omega \\ N_i \bar{\Phi}_i^{-1}\omega \end{bmatrix},$$

where $M_i \in \mathbb{R}^{n_i \times r}$ and $N_i \in \mathbb{R}^{r_i \times r}$ are defined as

$$M_i = \begin{bmatrix} I_{q_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad N_i = \begin{bmatrix} 0 & I_{r_i} \end{bmatrix}.$$

With this state transformation, the system (7.6) can be transformed into the system (7.13). The following lemma, which is proven in Appendix 7.B, shows this.

Lemma 7.2. *The synchronization error variable e_i is governed by dynamical equations of* (7.13), *where the pair $(\bar{C}_i, \bar{A}_i)$ is observable and the eigenvalues of $\bar{A}_{i22}$ are a subset of the eigenvalues of S.*

Remark 7.6. *If the unforced system for an agent i is the same as the exosystem, i.e., if $C_i = C_r$ and $A_i = S$, then it is easy to see that the dynamics of system* (7.13) *reduces to the dynamics of system* (7.1).

Step 2: State feedback control design

For any arbitrarily large bounded set $\mathcal{X}_i$, we design a controller as function of $\bar{x}_i$ such that

$\lim_{t\to\infty} e_i(t) = 0$ for all $x_i(0) \in \mathcal{X}_i$ and $\omega(0) \in \Omega_0$. Consider the following regulator equations with unknowns $\Pi_i^r \in \mathbb{R}^{n_i \times r_i}$ and $\Gamma_i^r \in \mathbb{R}^{p \times r_i}$ for system (7.13)

$$\Pi_i^r \bar{A}_{i22} = A_i \Pi_i^r + B_i \Gamma_i^r, \tag{7.19a}$$

$$\bar{C}_{i2} = C_i \Pi_i^r + D_i \Gamma_i^r. \tag{7.19b}$$

The following lemma shows that the regulator equations (7.19) are solvable if and only if the regulator equations (7.10) are solvable, and gives the mapping between the solutions of the two regulator equations. Note that if the regulator equations (7.19) (or the regulator equations (7.10)) have a solution, then it is unique due to the invertibility of the quadruple (A_i, B_i, C_i, D_i).

Lemma 7.3. *The regulator equations* (7.19) *have a solution* (Π_i^r, Γ_i^r) *if and only if the regulator equations* (7.10) *have a solution* (Π_i, Γ_i)*. Moreover,*

$$\Pi_i = \Pi_i^r N_i \bar{\Phi}_i^{-1} + \Lambda_i M_i \bar{\Phi}_i^{-1}, \quad \Gamma_i = \Gamma_i^r N_i \bar{\Phi}_i^{-1}, \tag{7.20}$$

or equivalently

$$\Pi_i^r = \Pi_i \bar{\Phi}_{io}, \quad \Gamma_i^r = \Gamma_i \bar{\Phi}_{io}. \tag{7.21}$$

Proof. To show the sufficiency, suppose that the regulator equations (7.10) have a solution and (7.20) is satisfied.

Define $W_i = \begin{bmatrix} I_{q_i} & 0 \end{bmatrix}$, from (7.20), it is easy to see that

$$\Pi_i = \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \bar{\Phi}_i^{-1} + \Lambda_i M_i \bar{\Phi}_i^{-1}, \quad \Gamma_i = \begin{bmatrix} \Gamma_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \bar{\Phi}_i^{-1}.$$

With some algebra, we obtain that

$$\begin{aligned}
\Pi_i S\bar{\Phi}_i &= \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \bar{\Phi}_i^{-1} S\bar{\Phi}_i + \Lambda_i M_i \bar{\Phi}_i^{-1} S\bar{\Phi}_i \\
&= \begin{bmatrix} 0 & \Pi_i^r \end{bmatrix} \begin{bmatrix} U_i & 0 \\ 0 & \bar{A}_{i22} \end{bmatrix} + \begin{bmatrix} \Lambda_{iu} & 0 \end{bmatrix} \begin{bmatrix} U_i & 0 \\ 0 & \bar{A}_{i22} \end{bmatrix} \\
&= \begin{bmatrix} \Lambda_{iu} U_i & \Pi_i^r \bar{A}_{i22} \end{bmatrix}, \qquad (7.22)
\end{aligned}$$

where we have used that $S\bar{\Phi}_i = \bar{\Phi}_i R$, and that

$$\begin{aligned}
(A_i\Pi_i + B_i\Gamma_i)\bar{\Phi}_i &= A_i \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} + A_i\Lambda_i M_i + B_i \begin{bmatrix} \Gamma_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \\
&= \begin{bmatrix} 0 & A_i\Pi_i^r \end{bmatrix} + \begin{bmatrix} A_i\Lambda_{iu} & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_i\Gamma_i^r \end{bmatrix} \\
&= \begin{bmatrix} \Lambda_{iu} U_i & A_i\Pi_i^r + B_i\Gamma_i^r \end{bmatrix}, \qquad (7.23)
\end{aligned}$$

where we have used that $A_i\Lambda_{iu} = \Lambda_{iu} U_i$.

From (7.10a), (7.22), and (7.23), it is then easy to see that $\Pi_i^r \bar{A}_{i22} = A_i\Pi_i^r + B_i\Gamma_i^r$, that is, (7.19a) is satisfied.

With a little bit algebra, we also obtain that

$$C_r\bar{\Phi}_i = \begin{bmatrix} C_r\bar{\Phi}_{iu} & C_r\bar{\Phi}_{io} \end{bmatrix} = \begin{bmatrix} C_i\Lambda_{iu} & \bar{C}_{i2} \end{bmatrix}, \qquad (7.24)$$

where we have used that $C_r\bar{\Phi}_{iu} = C_i\Lambda_{iu}$ and $\bar{C}_{i2} = C_r\bar{\Phi}_i N_i' = C_r\bar{\Phi}_{io}$ and that

$$\begin{aligned}
(C_i\Pi_i + D_i\Gamma_i)\bar{\Phi}_i &= C_i \begin{bmatrix} \Pi_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} + C_i\Lambda_i M_i + D_i \begin{bmatrix} \Gamma_i^r & 0 \end{bmatrix} \begin{bmatrix} N_i \\ W_i \end{bmatrix} \\
&= \begin{bmatrix} C_i\Lambda_{iu} & C_i\Pi_i^r + D_i\Gamma_i^r \end{bmatrix}. \qquad (7.25)
\end{aligned}$$

From (7.10b), (7.24), and (7.25), it is then easy to see that $\bar{C}_{i2} = C_i\Pi_i^r + D_i\Gamma_i^r$, that is, (7.19b) is satisfied. Hence, (Π_i^r, Γ_i^r) is the solution of the regulator equations (7.19).

Next we show the necessity. Suppose that the regulator equations (7.19) have a solution and (7.21) is satisfied. With just a little bit algebra, we obtain that

$$A_i\Pi_i^r + B_i\Gamma_i^r = A_i\Pi\bar{\Phi}_{io} + B_i\Gamma_i\bar{\Phi}_{io} \tag{7.26}$$

and

$$\Pi_i^r\bar{A}_{i22} = \Pi_i\bar{\Phi}_{io}\bar{A}_{i22} = \Pi_i S\bar{\Phi}_{io}, \tag{7.27}$$

where we have used that $S\bar{\Phi}_{io} = \bar{\Phi}_{io}\bar{A}_{i22}$, which follows from the fact that $S\bar{\Phi}_i = \bar{\Phi}_i R_i$.

From (7.19a), (7.26), and (7.27), it is easy to see that $\Pi_i S = A_i\Pi_i + B_i\Gamma_i$, that is, (7.10a) is satisfied.

With just a little bit algebra, we also obtain that

$$C_i\Gamma_i^r + D_i\Gamma_i^r = C_i\Pi_i\bar{\Phi}_{io} + D_i\Gamma_i\bar{\Phi}_{io}. \tag{7.28}$$

This together with the fact that $\bar{C}_{i2} = C_r\bar{\Phi}_i N_i' = C_r\bar{\Phi}_{io}$ and (7.19b) yields $C_r = C_i\Pi_i + D_i\Gamma_i$, that is, (7.10b) is satisfied. Hence, (Π_i, Γ_i) is the solution of the regulator equations (7.10). $\square$

Remark 7.7. *In view of Lemma 7.3 and* (7.11) *of Assumption 7.4, we see that* $\|\Gamma_i^r\bar{x}_{i2}\| = \|\Gamma_i\omega\| \leq 1 - \delta$. $\square$

Since agent i is subject to actuator saturation, we design the state feedback controller by using a *low-gain* technique, which is widely used for the semi-global stabilization problem for linear systems subject to actuator saturation, see for instance, [35, 70]. There exist in the literature several low-gain design algorithms. For conceptual clarity, we use here the one based on the solution of a continuous-time algebraic Riccati equation, parameterized in a low-gain parameter $\epsilon \in (0, 1]$. More specifically, we form a family of parameterized state feedback gain matrices $F_{i,\epsilon}$ for $\bar{x}_{i1}$ as

$$F_{i,\epsilon} = -B_i' P_{i,\epsilon},$$

where $P_{i,\epsilon} = P_{i,\epsilon}' > 0$ is the unique solution of the continuous-time algebraic Riccati equation defined as

$$P_{i,\epsilon} A_i + A_i' P_{i,\epsilon} - P_{i,\epsilon} B_i B_i' P_{i,\epsilon} + \epsilon I_{n_i} = 0. \tag{7.29}$$

It follows from Lemma 7.3 and and Condition 1 of Assumption 7.4 that the regulator equations (7.19) have a unique solution (Π_i^r, Γ_i^r). We use the unique (Π_i^r, Γ_i^r) and the feedback gain matrix $F_{i,\epsilon}$ to define a family of parameterized state feedback controllers in terms of $\bar{x}_i$ as

$$u_i = \begin{bmatrix} F_{i,\epsilon} & \Gamma_i^r - F_{i,\epsilon}\Pi_i^r \end{bmatrix} \bar{x}_i. \tag{7.30}$$

Then for any given arbitrarily large bounded set of initial conditions, there exists an $\epsilon^* \in (0, 1]$, such that for all $\epsilon \in (0, \epsilon^*]$, the family of linear state feedback controllers of the form (7.30) ensures that $\lim_{t\to\infty} e_i(t) = 0$ for all initial conditions belong to the given arbitrarily large bounded set and $\omega(0) \in \Omega_0$. This is a well known result, see [70, Theorem 3.3.2].

Remark 7.8. *If the unforced system for an agent i is the same as the exosystem, i.e., if $C_i = C_r$ and $A_i = S$, then it is easy to see that $\Pi_i = I$ and $\Gamma_i = 0$ is the solution of regulator equations* (7.10)*. Thus, Assumption 7.4 is always satisfied for that agent.*

Step 3: Observer-based implementation

Following the design procedure given in the proof of [70, Theorem 3.3.4], one can obtain, for a given set of initial conditions, suitable state feedback controllers for which input saturation is not active. This is done by properly choosing the low-gain parameter ϵ. Then such a state feedback law must be implemented by a suitable designed distributed observer. This will be done next.

We will design a high-gain decentralized observer to produce an estimate of $\bar{x}_i$, denoted by $\hat{\bar{x}}_i$. We follow the procedure as given in our previous paper [17], to be self-contained, we reproduce the design here.

Let $\bar{n}$ denotes the maximum order among the all the systems (7.13) for $i \in \{1, \dots, N\}$, that is, $\bar{n} = \max_{i=1,\dots,n}(n_i + r_i)$. Define $\chi_i = T_i \bar{x}_i$, where

$$T_i = \begin{bmatrix} \bar{C}_i \\ \vdots \\ \bar{C}_i \bar{A}_i^{\bar{n}-1} \end{bmatrix}.$$

Note that T_i is injective since the pair $(\bar{C}_i, \bar{A}_i)$ is observable, which implies that $T_i' T_i$ is nonsingular.

In term of χ_i, we can write the system equations

$$\dot{\chi}_i = (\mathcal{A} + \mathcal{L}_i)\chi_i + \mathcal{B}_i \sigma(u_i), \quad \chi_i(0) = T_i \bar{x}_i(0), \tag{7.31a}$$

$$e_i = \mathcal{C}\chi_i + \mathcal{D}_i \sigma(u_i), \tag{7.31b}$$

where

$$\mathcal{A} = \begin{bmatrix} 0 & I_{p(\bar{n}-1)} \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} I_p & 0 \end{bmatrix}, \quad \mathcal{L}_i = \begin{bmatrix} 0 \\ L_i \end{bmatrix}, \quad \mathcal{B}_i = T_i \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \quad \mathcal{D}_i = D_i,$$

for some matrix $L_i \in \mathbb{R}^{p \times \bar{n}p}$. Note that the matrices $\mathcal{A}$ and $\mathcal{C}$ are the same for all the agents $i \in \{1, \dots, n\}$, and the special form of these matrices implies that $(\mathcal{C}, \mathcal{A})$ is observable.

Next, define the matrix $\bar{G} = G + \operatorname{diag}(\iota_1, \dots, \iota_n)$ and $\tau = \min_{i=1,\dots,n} \operatorname{Re}(\lambda_i(\bar{G})) > 0$. Let $\mathcal{P} = \mathcal{P}' > 0$ be the unique solution of the algebraic Riccati equation

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}' - \tau \mathcal{P}\mathcal{C}'\mathcal{C}\mathcal{P} + I_{\bar{n}p} = 0. \tag{7.32}$$

We then we design the observer

$$\dot{\hat{\chi}}_i = (\mathcal{A} + \mathcal{L}_i)\hat{\chi}_i + \mathcal{B}_i\sigma(u_i) + S(\ell)\mathcal{P}\mathcal{C}'(\zeta_i - \hat{\zeta}_i) + S(\ell)\mathcal{P}\mathcal{C}'(\psi_i - \iota_i(\mathcal{C}\hat{\chi}_i + D_i\sigma(u_i))), \tag{7.33a}$$

$$\hat{\bar{x}}_i = (T_i'T_i)^{-1}T_i'\hat{\chi}_i, \tag{7.33b}$$

where $S(\ell) = \text{blkdiag}(I_p\ell, I_p\ell^2, \ldots, I_p\ell^{\bar{n}})$ and $\ell > 1$ is a high-gain parameter.

Based on the observer estimate, we define the variable $\eta_i = \mathcal{C}\hat{\chi}_i + \mathcal{D}_i\sigma(u_i)$ to be shared with the other agents via the networks communication infrastructure as described in Section 7.2.1, and the observer-based control law

$$u_i = \begin{bmatrix} F_{i,\epsilon} & \Gamma_i^r - F_{i,\epsilon}\Pi_i^r \end{bmatrix} \hat{\bar{x}}_i. \tag{7.34}$$

Together, the observers for agents $i \in \{1, \ldots, N\}$ form a distributed observer parameterized by a high-gain parameter ℓ. It has been shown in [17, Lemma 4] that the estimation errors dynamics are globally exponentially stable, that is, $\lim_{t\to\infty}(\bar{x}_i - \hat{\bar{x}}_i) = 0$, by choosing the high-gain parameter ℓ sufficiently large.

Remark 7.9. *If all the agents have the same dynamics, it is not necessary to design an observer based on the high-order system* (7.31) *and one can design an observer based on the original system* (7.13)*.*

In summary, for any given arbitrarily large bounded sets $\mathcal{X}_i \subset \mathbb{R}^{n_i}$ and $\mathcal{P}_i \subset \mathbb{R}^{p\bar{n}}$, there exist ϵ^* with the property that for any $\epsilon \in (0, \epsilon^*]$ there exists ℓ^* such that for $\ell \geq \ell^*$, the observer-based implementation (7.33) and (7.34), ensure that

$$\lim_{t\to\infty} e_i(t) = 0, \quad \forall i \in \{1, \ldots, N\}, \tag{7.35}$$

for all initial conditions $x_i(0) \in \mathcal{X}_i$, $\hat{\chi}_i(0) \in \mathcal{P}_i$, and $\omega(0) \in \Omega_0$.

7.3.2 Comparison with the case where the agents have no actuator magnitude constraints

Let us make several comments to compare our result to the case where the agents do not have actuator saturation.

- The regulator equations (7.10) have to be solvable for the case with actuator magnitude constraints. In our previous work for the case without saturation we assumed existence of a solution of the regulator equations but in that case this existence is not necessary.
- For the case with actuator magnitude constraints, we only achieve semi-global regulation of output synchronization.
- For the case with actuator magnitude constraints, it is required that all the eigenvalues of agents' system matrices are in the closed left-half complex plane.
- For the case with actuator magnitude constraints, we have constraints on the size of the synchronized output trajectory as given by (7.11).

7.4 Example

In this section, we illustrate our design procedure by considering a network of ten agents. Agents 1 and 2 are composed as the cascade of a second-order oscillator and a single integrator:

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad D_i = 0.$$

Agents 3, 4, and 5 have the following dynamics:

$$A_i = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 2.$$

Agents 6, 7, and 8 have the following dynamics:

$$A_i = 0, \quad B_i = 1, \quad C_i = 1, \quad D_i = 1.$$

Finally, Agents 9 and 10 are second-order mass-spring-damper systems:

$$A_i = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_i = 0.$$

The reference trajectory y_r is generated by an exosystem with

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad C_r = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

and initial conditions $\Omega_0 = \left\{\omega \in \mathbb{R}^3 : \|\omega\| \leq 0.1\right\}$.

The communication topology of the network is given by the digraph depicted in Figure 7.1, and the agent 2 has access to the information $y_2 - y_r$.

Step 1

For illustrative purpose, we give the details for agent 3. In Step 1,

$$O_3 = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \implies q_3 = 1, \ r_3 = 2,$$

We may choose

$$\Lambda_{3u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Phi_{3u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and hence we can set $\Lambda_3 = I_2$ and $\Phi_3 = I_3$. Following the design procedure, we have

$$V_{311} = 1,\, V_{312} = \begin{bmatrix} 0 & 1 \end{bmatrix}, \text{ and } V_{322} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

for (7.16) and (7.17). Therefore, from (7.18), we obtain that

$$\bar{\Phi}_3 = \Phi_i \begin{bmatrix} I_{q_3} & V_{312}V_{322}^{-1} \\ 0 & I_{r_i} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

thus, it follows that

$$\bar{x}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} x_3 - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \omega,$$

then the dynamics of $\bar{x}_i$ with output e_i takes the form of (7.13) with

$$\bar{A}_{322} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \bar{C}_{32} = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

Step 2

We now need to solve the regulator equations (7.19), which are easily found to have the unique solution

$$\Pi_3^r = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Gamma_3^r = \begin{bmatrix} 0 & -1 \end{bmatrix}.$$

We then select the matrix $F_{3,\epsilon} = -B_3' P_{3,\epsilon}$, where $P_{3,\epsilon} = P_{3,\epsilon}'$ is the unique solution of (7.29), and the value of ϵ will be determined later.

We perform the same procedure for the other agents, to identify appropriate state feedbacks. For agents 1 and 2, there is no need for solving the regulator equations (7.19); for agents 6, 7, and 8, we obtain

$$\Pi_6^r = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad \Gamma_6^r = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

and for agents 9 and 10, the system (7.6) is observable, moreover $\bar{x}_{i2} = \omega$. We then find the unique solution of the regulator equations (7.10) as

$$\Pi_9 = \Pi_9^r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \Gamma_9 = \Gamma_9^r = \begin{bmatrix} 2 & 2 & 1 \end{bmatrix}.$$

Note that

$$\Gamma_9 \omega \leq 0.5,$$

therefore, we choose $\delta = 0.5$, such that

$$\Gamma_9 \omega \leq 1 - \delta$$

for all $\omega(0) \in \Omega_0$. It is also easy to check that $\delta = 0.5$ works for all other agents.

Step 3

In Step 3 we design the decentralized observer that allows the feedbacks to be implemented based on observer estimates. It is easy to check that $\bar{n} = 5$, then we have

$$\mathcal{A} = \begin{bmatrix} 0 & I_4 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that in order to implement the observer-based feedback (7.33) and (7.34), we need to determine the value of the low-gain parameter $\epsilon \in (0, \epsilon^*]$, for the set given by $\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} : \|x_i\| \leq 1\}$ and

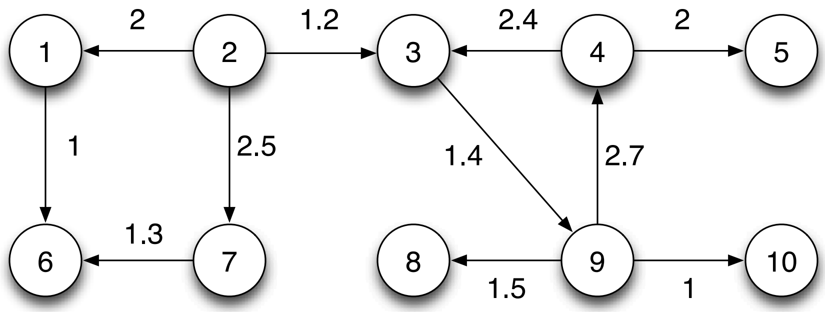

Figure 7.1: Network topology

$\mathcal{P}_i = \{x_i \in \mathbb{R}^{q_i} : \|x_i^c\| \leq 1\}$, we can confirm that $\epsilon^* = 0.1$, thus we choose $\epsilon = \epsilon^* = 0.1$. Now, we construct the weighted Laplacian G from the digraph in Figure 7.1, note that the digraph contains a directed spanning tree with agent 2 being the root. Given fact that $\iota_2 = 1$ while $\iota_i = 0$ for all other i. we find that $\tau = \min_{i=1,\ldots,10} \mathrm{re}(\lambda_i(G + \mathrm{diag}(\iota_1, \ldots, \iota_{10}))) \approx 0.2749$. Solving the algebraic Riccati equation (7.32) and implementing observer-based feedback (7.33) and (7.34), we find that we achieve stability with $\ell = 2$. Figure 7.2 shows the resulting simulated output of four agents and the synchronization trajectory, while Figure 7.3 shows the resulting simulated input of four agents.

7.A Proof of Lemma 7.1

Proof. If the quadruple (A_i, B_i, C_i, D_i) has no invariant zeros which are eigenvalues of the matrix S, then the existence of solutions to the regulator equations follows from the fact that the system is right-invertible (see Corollary 2.5.1 of [70]).

On the other hand, assume that the quadruple (A_i, B_i, C_i, D_i) has an invariant zero λ which is

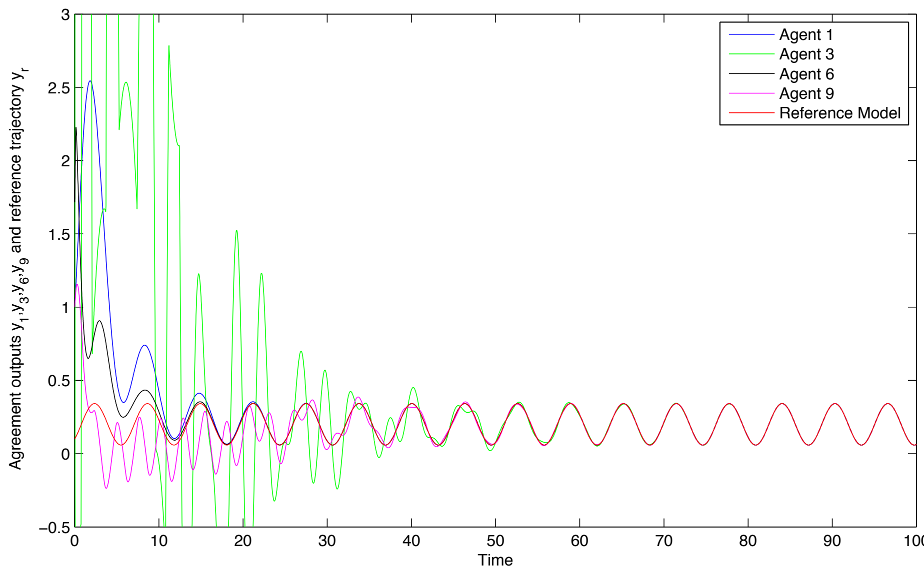

Figure 7.2: Output trajectories for agents 1, 3, 6, 9 and reference model

an eigenvalue of the matrix S. In that case let (v, w) be such that

$$\begin{pmatrix} v' & w' \end{pmatrix} \begin{pmatrix} A_i - \lambda I & B_i \\ C_i & D_i \end{pmatrix} = 0 \tag{7.36}$$

and ω_0 such that

$$S\omega_0 = \lambda\omega_0.$$

Since Ω_0 contains 0 in its interior, we can, without loss of generality, assume that $\omega_0 \in \Omega_0$.

We first assume that $w'C_r\omega_0 \neq 0$ and we will establish a contradiction with the fact that there exists for $\omega(0) = \omega_0$, an input u_i and an appropriate initial condition $x_i(0)$ such that $e_i(t) \to 0$ as $t \to \infty$.

Since (A_i, B_i, C_i, D_i) is right-invertible, we note that the subsystem from u to $z = w'y$ (which has a scalar output) can be described by a polynomial description:

$$d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z(t) = N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) u(t),$$

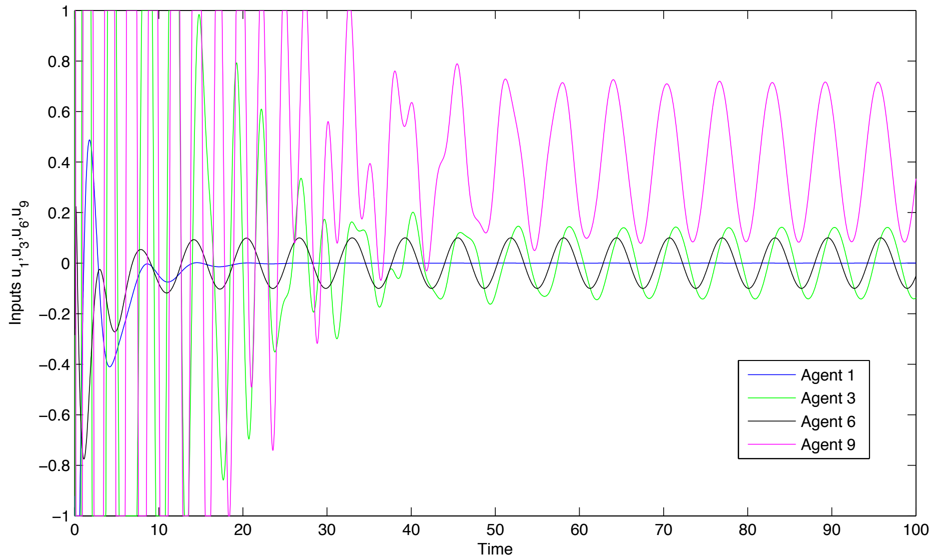

Figure 7.3: Input trajectories for agents 1, 3, 6 and 9

where $N(s)$ is a non-zero polynomial row vector while $d(s)$ is a scalar polynomial. Since, the subsystem from u to z is right-invertible and has a zero in λ, we find that N has a zero in λ. Moreover, if d also has a zero in λ then N has a zero in λ whose order is at least one higher than the zero in λ of d. We define:

$$\bar{z}(t) = e^{-\lambda t} z(t), \qquad \bar{u}(t) = e^{-\lambda t} u(t),$$

and

$$\bar{d}(s) = d(s+\lambda), \qquad \bar{N}(s) = N(s+\lambda).$$

We note that

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda \right) \bar{z}(t) = e^{-\lambda t} \frac{\mathrm{d}}{\mathrm{d}t} z(t),$$

and similarly for $u, \bar{u}$. Hence,

$$\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t} \right) \bar{z}(t) = e^{-\lambda t} d\left(\frac{\mathrm{d}}{\mathrm{d}t} \right) z(t),$$

and

$$\bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{u}(t) = e^{-\lambda t} N\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u(t).$$

Assume that the input u is such that tracking is achieved, then we have:

$$z(t) \to w' C_r \omega(t) = e^{\lambda t} w' C_r \omega_0$$

as $t \to \infty$ and hence

$$\bar{z}(t) \to w' C_r \omega_0$$

as $t \to \infty$. Without loss of generality we assume that $w' C_r \omega_0 = \delta > 0$. In that case, there exists $t_0 > 0$ such that we have

$$\tfrac{1}{2}\delta \leq \bar{z}(t) \leq \tfrac{3}{2}\delta$$

for all $t > t_0$. On the other hand, given that λ is on the imaginary axis and that $u(t)$ is bounded, we have that there exists an $M > 0$ such that

$$\|\bar{u}(t)\| \leq M$$

for all $t > 0$. We have

$$\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{z}(t) = \bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{u}(t)$$

Define

$$\bar{d}(s) = d_i s^i + d_{i+1} s^{i+1} + \cdots + d_n s^n,$$

and

$$\bar{N}(s) = N_{i+1} s^{i+1} + \cdots + N_n s^n,$$

such that $d_i \neq 0$. Here we used that N had a zero in λ and, if d has a zero as well in λ then it is of strictly lower order. We find that

$$\left|\underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{z}(t)\right| \geq \left(|d_i|(t_2-t_1)^{n-i} - 3\sum_{j=i+1}^{n}|d_j|(t_2-t_1)^{n-j}\right)\tfrac{1}{2}\delta$$

for all $t_2, t_1 > t_0$. On the other hand,

$$\left|\underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{u}(t)\right| \leq M\sum_{j=i+1}^{n}\|N_i\|(t_2-t_1)^{n-j}$$

for all $t_2, t_1 > t_0$. This yields a contradiction as $t_2 \to \infty$ since we have:

$$\underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{z}(t) = \underbrace{\int_{t_1}^{t_2}\int_{t_1}^{t_2}\cdots\int_{t_1}^{t_2}}_{n}\bar{N}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bar{u}(t)$$

and our inequalities imply that the left-hand side grows like $(t_2-t_1)^{n-i}$ while the right-hand side can at most grow like $(t_2-t_1)^{n-i-1}$.

Since, assuming that $w'C_r\omega_0 \neq 0$, we obtain a contradiction we must have that $w'C_r\omega_0 = 0$.

Using this property we will establish that (7.10) has a solution. Without loss of generality and using Assumption 7.3, we can assume that:

$$S = \begin{pmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \omega_r \end{pmatrix}, \quad C_r = \begin{pmatrix} C_{r,1} \cdots C_{r,r} \end{pmatrix},$$

and we also decompose the potential solutions of the regulator equations as:

$$\Pi_i = \begin{pmatrix} \Pi_{i,1} \cdots \Pi_{i,r} \end{pmatrix}, \quad \Gamma_i = \begin{pmatrix} \Gamma_{i,1} \cdots \Gamma_{i,r} \end{pmatrix}.$$

We obtain that (7.10) is equivalent to:

$$\Pi_{i,j}\omega_j = A_i\Pi_{i,j} + B_i\Gamma_{i,j},$$

$$C_{r,j} = C_i\Pi_{i,j} + D_i\Gamma_{i,j}$$

for $j = 1, \dots, r$. This can be rewritten as:

$$\begin{pmatrix} A_i - \omega_j I & B_i \\ C_i & D_i \end{pmatrix} \begin{pmatrix} \Pi_{i,j} \\ \Gamma_{i,j} \end{pmatrix} = \begin{pmatrix} 0 \\ C_{r,j} \end{pmatrix},$$

which is solvable if

$$\operatorname{Im} \begin{pmatrix} 0 \\ C_{r,j} \end{pmatrix} \subset \operatorname{Im} \begin{pmatrix} A_i - \omega_j I & B_i \\ C_i & D_i \end{pmatrix},$$

and the latter condition is equivalent to:

$$\begin{pmatrix} v' & w' \end{pmatrix} \begin{pmatrix} A_i - \omega_j I & B_i \\ C_i & D_i \end{pmatrix} = 0 \Longrightarrow \begin{pmatrix} v' & w' \end{pmatrix} \begin{pmatrix} 0 \\ C_{r,j} \end{pmatrix} = 0.$$

Since the latter is equivalent to $w' C_r e_j = 0$ where $S e_j = \omega_j e_j$, we note that this implication is exactly the condition that we have proven above.

The fact that we need (7.12) is a consequence of Corollary 3.3.1 in [70]. □

7.B Proof of Lemma 7.2

Proof. Let us first assume that $\bar{\Phi}_i = \Phi_i$, then from the proof of [16, Lemma 1], we know that e_i is governed by the following dynamical equations

$$\dot{\bar{x}}_i = \bar{A}_i \bar{x}_i + \bar{B}_i \sigma(u_i) := \begin{bmatrix} A_i & \bar{A}_{i12} \\ 0 & \bar{A}_{i22} \end{bmatrix} \bar{x}_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} \sigma(u_i), \tag{7.38a}$$

$$e_i = \bar{C}_i \bar{x}_i + \bar{D}_i \sigma(u_i) := \begin{bmatrix} C_i & -\bar{C}_{i2} \end{bmatrix} \bar{x}_i + D_i \sigma(u_i), \tag{7.38b}$$

where

$$\bar{A}_{i12} = \Lambda_i \begin{bmatrix} R_{i12} \\ 0 \end{bmatrix}, \quad \bar{A}_{i22} = R_{i22}, \quad \bar{C}_{i2} = C_r \Phi_i N_i'.$$[1]

[1] Note that the variable $\bar{x}_{i2}$ has a sign difference from that of [17].

The matrices R_{i12} and R_{i22} are such that (7.14) holds. Let

$$V_i = \begin{bmatrix} V_{i11} & V_{i12} \\ 0 & V_{i22} \end{bmatrix}.$$

Note that $\bar{A}_i$ of the system (7.38) is block-upper triangular, however, $\bar{A}_i$ in the system (7.13) is block-diagonal. In order to obtain the system (7.13), we need to show that we can make $R_{i12} = 0$ such that $S\bar{\Phi}_i = \bar{\Phi}_i R_i$ by using $\bar{\Phi}_i$ given by (7.18).

From (7.16) and (7.17), it is easy to obtain that

$$V_i^{-1} \begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix} V_i = \begin{bmatrix} \Lambda_{i1} & 0 \\ 0 & \Lambda_{i2} \end{bmatrix}. \tag{7.39}$$

From (7.16) and (7.39), it is easy to show that

$$\begin{bmatrix} U_i & R_{i12} \\ 0 & R_{i22} \end{bmatrix} \begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix} = \begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix} \begin{bmatrix} U_i & 0 \\ 0 & R_{i22} \end{bmatrix}. \tag{7.40}$$

Now post multiplying both sides of (7.14) by

$$\begin{bmatrix} I_{q_i} & V_{i12}V_{i22}^{-1} \\ 0 & I_{r_i} \end{bmatrix},$$

we obtain that $S\bar{\Phi}_i = \bar{\Phi}_i R_i$, where

$$R_i = \begin{bmatrix} U_i & 0 \\ 0 & R_{i22} \end{bmatrix}. \tag{7.41}$$

Hence, $R_{i12} = 0$ and $\bar{A}_{i12} = 0$. □

Chapter 8

Consensus for Homogeneous Networks with Uniform Constant Communication Delay

8.1 Introduction

The consensus problem in a network has received substantial attention in recent years, partly due to the wide applications in areas such as sensor networks and autonomous vehicle control. A relatively complete coverage of earlier work can be found in the survey paper of [44], the recent books by [55, 100] and references therein.

Consensus in the network with time delay has been extensively studied in the literature. Most results consider the agent model as described by single-integrator dynamics [5, 46, 80], or double-integrator dynamics [7, 34, 81]. Specifically, it is shown by [46] that a network of single-integrator agents subject to uniform constant communication delay can achieve consensus with a particular linear local control protocol if and only if the delay is bounded by a maximum that is inversely proportional to the largest eigenvalue of the graph Laplacian associated with the network. This result was later on generalized in [5] to non-uniform constant or time-varying delays. Sufficient conditions for consensus among agents with first order dynamics were also obtained in [80]. The results in [46] were extended in [7, 34] to double integrator dynamics. An upper bound on the maximum network delay tolerance for second-order consensus of multi-agent systems with any given linear control protocol was obtained.

In this chapter, we study the multi-agent consensus problem with uniform constant communication delay. The agents are assumed to be at most critically unstable, i.e. each agent has all its eigenvalues in the closed left half plane. The contribution of this chapter with respect to [5,7,34,46] is twofold: first, we find a sufficient condition on the tolerable communication delay for agents with high-order dynamics, which has an explicit dependence on the agent dynamics and network topology. For undirected network, this upper bound can be independent of network topology provided that the network is connected. Moreover, in a special case where the agents only have zero eigenvalues, such as single- and double-integrator dynamics, arbitrarily large but bounded delay can be tolerated. Another layer of contribution is that for delay satisfying the proposed upper bound, we present a controller design methodology without exact knowledge of network topology so that the multi-agent consensus in a set of unknown networks can be achieved. When the network topology is precisely known, the controller design can be modified to be topology-dependent and a larger delay tolerance is attainable.

The rest of the chapter is organized as follows: notations and some preliminary results are declared in the remainder of Section 8.1. System and network configuration and consensus problem formulations are given in Section 8.2. The consensus problems with full-state coupling are solved in

Section 8.3 both for a set of unknown communication topologies and for a known communication topology. Corresponding problems with partial-state coupling are dealt with in Section 8.4. Some illustrative examples are given in Section 8.5. A technical lemma is appended at the end of this chapter.

8.1.1 Notations and Preliminaries

The following notations will be used in this chapter. For a matrix $X \in \mathbb{C}^{n\times m}$,

$$X' : \text{transpose of } X;$$

$$X^* : \text{conjugate transpose of } X;$$

$$X^{-1} : \text{inverse of } X \text{ if it exists}$$

$$\bar{\sigma}(X) : \text{maximal singular value of } X;$$

$$\underline{\sigma}(X) : \text{minimal singular value of } X;$$

$$\|X\| : \text{induced 2 norm};$$

$$\det(X) : \text{determinant of } X.$$

For a transfer function $H(s) : \mathbb{C} \to \mathbb{C}^{n\times m}$,

$$\|H(s)\|_\infty : \mathcal{H}_\infty \text{ norm of } H(s).$$

For a vector d, we denote a diagonal matrix by D=diag$\{d\}$ whose diagonal is specified by d. For column vectors $x_1, \ldots, x_n$, the stacking column vector of $x_1, \ldots, x_n$ is denoted by $[x_1; \ldots; x_n]$.

For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{p\times q}$, the Kronecker product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \vdots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$$

The following property of the Kronecker product will be used in this chapter:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

A graph G is defined by a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, N\}$ is a vertex set and $\mathcal{E}$ is a set of pairs of vertices (i, j). Each pair in $\mathcal{E}$ is called an *arc*. G is *undirected* if $(i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$. Otherwise, G is *directed*. A *directed path* from vertex i_1 to i_k is a sequence of vertices $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, ..., k-1$. A directed graph G contains a *directed spanning tree* if there is a node r such that a directed path exists between r and every other node.

The graph G is *weighted* if each arc (i, j) is assigned with a real number a_{ij}. For a weighted graph G, a matrix $L = \{\ell_{ij}\}$ with

$$\ell_{ij} = \begin{cases} \sum_{j=1}^{N} a_{ij}, & i = j \\ -a_{ij}, & i \neq j, \end{cases}$$

is called *Laplacian matrix* associated with graph G. In the case where G has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$, see for example [63]. If G has a directed spanning tree, L has a simple eigenvalue at zero and all the other eigenvalues have strictly positive real parts, see for example [53].

8.2 Problem Formulation

Consider a network of N identical agents

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) + Bu^i(t), \quad i = 1, ..., N, \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} x^j(t - \tau). \end{cases} \tag{8.1}$$

where $x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ and $z^i \in \mathbb{R}^n$, $\tau > 0$ is an unknown constant satisfying $\tau \in [0, \bar{\tau}]$. The coefficients ℓ_{ij} are such that $\ell_{ij} \leq 0$ for $i \neq j$ and $\ell_{ii} = -\sum_{j \neq i}^{N} \ell_{ij}$. In (8.1), each agent collects a delayed information z^i of the state of neighboring agents through the network, which we refer to as *full-state coupling.*

It is also common that z_i may consist of the output of neighboring agents instead of the complete state which can be formulated as follows:

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) + Bu^i(t), \\ y^i(t) = Cx^i(t), \qquad\qquad i = 1, ..., N, \\ z^i(t) = -\sum_{j=1}^N \ell_{ij} y^j(t-\tau), \end{cases} \tag{8.2}$$

where $x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ and $y^i, z^i \in \mathbb{R}^p$. We refer to the agents in this case as having *partial-state coupling.*

The matrix $L = \{\ell_{ij}\} \in \mathbb{R}^{N\times N}$ defines the *communication topology* which can be captured by a weighted graph $G = (\mathcal{N}, \mathcal{E})$ where $(j,i) \in \mathcal{E} \Leftrightarrow \ell_{ij} < 0$ and $a_{ii} = 0$ and $a_{ij} = -\ell_{ij}$ for $i \neq j$. The G is directed in general. However, in a special case where L is symmetric, G is undirected. This L is the the Laplacian matrix associated with G.

Assumption 8.1. *The following assumptions are made throughout the chapter:*

1) *The agents are at most critically unstable, that is, A has all its eigenvalues in the closed left half plane;*

2) *(A,B) is stabilizable and (A,C) is detectable;*

3) *The communication topology G contains a directed spanning tree.*

It should be noted that in practice, perfect information of the communication topology is usually not available for controller design and that only some rough characterization of the network can be obtained. Using the non-zero eigenvalues of L as a "measure" for the graph, we can introduce the following definition to characterize a set of unknown communication topologies. Let $\lambda_1, \cdots, \lambda_N$ denoted the eigenvalues of L and assume $\lambda_1 = 0$.

Definition 8.1. *For any $\gamma \geq \beta \geq 0$ and $\frac{\pi}{2} > \varphi \geq 0$, $\mathcal{G}_{\beta,\gamma,\varphi}$ is the set of directed graphs whose associated Laplacian satisfies that*

$$|\lambda_i| \in (\beta, \gamma) \text{ and } \arg \lambda_i \in [-\varphi, \varphi]$$

for $i = 2, ..., N$.

Definition 8.2. *The agents in the network achieve consensus if*

$$\lim_{t\to\infty} (x^i(t) - x^j(t)) = 0, \quad \forall i, j \in \{1, \ldots, N\}.$$

Two consensus problems for agents with full-state coupling (8.1) and partial-state coupling (8.2) respectively can be formulated in this set of networks as follows:

Problem 8.1. *Consider a network of agents* (8.1) *with full state coupling. The consensus problem given a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$ is to design linear static controllers $u^i = Fz^i$ for $i = 1, \ldots, N$ such that the agents* (8.1) *with $u^i = Fz^i$ achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.*

Problem 8.2. *Consider a network of agents* (8.2) *with partial state coupling. The consensus problem with a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$ is to design linear dynamical control protocols*

$$\begin{cases} \dot{\chi}^i = A_k \chi^i + B_k z^i, \\ u^i = C_k \chi^i, \end{cases} \tag{8.3}$$

for $i = 1, \ldots, N$ such that the agents (8.2) *with controller* (8.3) *achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.*

As will be shown in the sequel, under Assumption 8.1 and certain condition on the upper bound $\bar{\tau}$, we are able to design a decentralized local controller without exact knowledge of the network topology so that Problem 8.1 and 8.2 can be solved. On the other hand, we shall also observe that such robustness against uncertainties in the communication topology is not free. In general, a cost is incurred in terms of a more conservative condition on $\bar{\tau}$. When perfect information of the network topology is known, a larger delay tolerance may be attainable. Therefore, we also formulate the consensus problems in a known network as follows:

Problem 8.3. *Consider a network of agents* (8.1) *with full-state coupling. The consensus problem with a known communication topology and a delay upper bound $\bar{\tau}$ is to design local linear static consensus controllers $u^i = Fz^i$ for $i = 1, \ldots, N$, such that the agents* (8.1) *with $u^i = Fz^i$ achieve consensus with the given topology for all $\tau \leq \bar{\tau}$.*

Problem 8.4. *Consider agents* (8.2) *with partial-state coupling. The consensus problem with a known communication topology and a delay upper bound $\bar{\tau}$ is to design a local linear dynamical consensus controllers* (8.3) *for $i = 1, \ldots, N$ such that the agents* (8.2) *with controller* (8.3) *and with the given topology achieve consensus for all $\tau \leq \bar{\tau}$.*

8.3 Consensus with Full-state Coupling

In this section, we with consider agents with full-state coupling as given in (8.1) and solve Problems 8.1 and 8.3.

8.3.1 Consensus in Networks with Unknown Communication Topology

We first consider Problem 8.1.

8.3.1.1 Consensus Controller Design and Main Results

For a given set of networks $\mathcal{G}_{\beta,\gamma,\varphi}$, we design a decentralized local consensus controller for any network in $\mathcal{G}_{\beta,\gamma,\varphi}$ as follows:

$$u^i = \tfrac{1}{\beta} F_\epsilon z^i, \tag{8.4}$$

where $F_\epsilon = B' P_\epsilon$. Here P_ϵ is the positive definite solution of the Algebraic Riccati Equation (ARE)

$$A' P_\epsilon + P_\epsilon A - P_\epsilon B B' P_\epsilon + \epsilon I = 0. \tag{8.5}$$

and ϵ is a tuning parameter which will be chosen according to β and γ so that the multi-agent consensus can be achieved with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. Let

$$\omega_{\max} = \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A) = 0\}.$$

The first main result of this chapter is stated in the next theorem which solves the network consensus problem with respect to $\mathcal{G}_{\beta,\gamma,\varphi}$.

Theorem 8.1. *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (8.1) *and any coupling network belonging to the set $\mathcal{G}_{\beta,\gamma,\varphi}$. In that case Problem 8.1 is solvable if,*

$$\varphi < \pi/3 \quad \textit{and} \quad \bar{\tau} < \frac{\frac{\pi}{3} - \varphi}{\omega_{\max}}. \tag{8.6}$$

Moreover, it can be solved by the consensus controller (8.4) *if* (8.6) *holds. Specifically, for given $0 \le \beta \le \gamma$ and given φ and $\bar{\tau}$ satisfying* (8.6)*, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.1) *with controller* (8.4) *achieve consensus for any communication topologies in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

Remark 8.1. *In order to have non-zero delay tolerance, here we require $\varphi < \pi/3$. However, if $\bar{\tau} = 0$, in other words, the communication delay is absent from the network, $\varphi < \pi/3$ is not* *necessary. In that case, it is a known result that the agents* (8.1) *with consensus controller* (8.4) *achieve consensus for any communication topology whose Laplacian eigenvalues satisfy* $\mathrm{re}(\lambda_i) \geq \beta/2$, $i = 2, ..., N$, *see for example [111]. In this chapter, we restrict ourselves to the case* $\bar{\tau} > 0$. *However, it remains an interesting open question that whether* $\varphi < \pi/3$ *is indeed needed for* $\bar{\tau} > 0$.

Remark 8.2. *The consensus controller design depends only on agents and parameters* $\bar{\tau}, \beta, \gamma$ *and* φ *and is independent of specific network topology provided that the network satisfies Assumption 8.1.*

In the special case where A has all the eigenvalues at zero, an arbitrarily bounded communication delay can be tolerated.

Corollary 8.1. *For a given set* $\mathcal{G}_{\beta,\gamma,\varphi}$ *and* $\bar{\tau} > 0$, *consider the agents* (8.1) *and any communication topology belonging to the set* $\mathcal{G}_{\beta,\gamma,\varphi}$. *Suppose* A *has all the eigenvalues at zero. In that case, Problem 8.1 is solvable via the consensus controller* (8.4) *if* $\varphi < \frac{\pi}{3}$. *Specifically, for given* $0 \leq \beta \leq \gamma$, $\varphi < \frac{\pi}{3}$ *and* $\bar{\tau} > 0$, *there exists an* ϵ^* *such that for any* $\epsilon \in (0, \epsilon^*]$, *the agents* (8.1) *with controller* (8.4) *achieve consensus for any communication topologies in* $\mathcal{G}_{\beta,\gamma,\varphi}$ *and* $\tau \in [0, \bar{\tau}]$.

Remark 8.3. *In the previous study of network consensus problem, agents are normally assumed as having single- or double-integrator type dynamics. Based on Corollary 8.1, we find that the delay tolerance in such cases is independent of network topology and can be made arbitrarily large provided that* $\varphi < \frac{\pi}{3}$. *This result in no way contradicts that in [7, 34, 46] since the goal here is to find the maximal achievable delay tolerance by controller design whereas obtained in [7, 34, 46] are the conditions on delay for which the consensus with certain given controller is not spoiled.*

If the communication topology is undirected, the Laplacian associated with G is symmetric and has only real eigenvalues, i.e. $\varphi = 0$. A set of undirected networks can be denoted as $\mathcal{G}_{\beta,\gamma,0}$. From

Theorem 8.1, we immediately have the following result:

Corollary 8.2. *For a given set of undirected networks $\mathcal{G}_{\beta,\gamma,0}$ and $\bar{\tau} > 0$, consider the agents* (8.1) *and any communication topology belonging to the set $\mathcal{G}_{\beta,\gamma,0}$. In that case, Problem 8.1 is solvable if*

$$\bar{\tau} < \frac{\pi}{3\omega_{\max}}. \tag{8.7}$$

Moreover, it can be solved by the consensus controller (8.4) *if* (8.7) *holds. Specifically, for given $0 \leq \beta \leq \gamma$ and given $\bar{\tau}$ satisfying* (8.7)*, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.1) *with controller* (8.4) *achieve consensus for any communication topologies in $\mathcal{G}_{\beta,\gamma,0}$ and $\tau \in [0, \bar{\tau}]$.*

Remark 8.4. *Note that for undirected and connected networks, the upper bound of tolerable delay is independent of network topology. However, in directed networks, we have to sacrifice some robustness in the delay tolerance in order to cope with the complex part of Laplacian eigenvalues.*

The proof of Theorem 8.1 will be carried out in the next two subsections. We will proceed in two steps: first, we convert the network consensus problem to the robust stabilization problem subject to input delay via state feedbacks; then we shall show that controller (8.4) solves this equivalent stabilization problem and thus solves the network consensus problem.

8.3.1.2 Connection to a Robust Stabilization Problem

Assume that the condition (8.6) in Theorem 8.1 holds. We choose a consensus controller for agent i as

$$u^i = Fz^i.$$

for some matrix $F \in \mathbb{R}^{m\times n}$. Define $\tilde{x} = [x^1; \cdots ; x^N]$. The overall dynamics of N agents can be written as

$$\dot{\tilde{x}}(t) = (I_N \otimes A)\tilde{x}(t) - (L \otimes BF)\tilde{x}(t-\tau).$$

Define $\xi = [\xi^1; \cdots ; \xi^N] = (T \otimes I_n)\tilde{x}$ where $\xi^i \in \mathbb{C}^n$ and T is such that $J_L = TLT^{-1}$ is in the Jordan canonical form and $J_L(1,1) = 0$. In the new coordinates, the dynamics of ξ can be written as

$$\dot{\xi}(t) = (I_N \otimes A)\xi(t) - (J_L \otimes BF)\xi(t-\tau).$$

Lemma 8.1. *The network consensus problem is solved if $\xi^i \to 0$ as $t \to \infty$ for $i = 2, ..., N$.*

Proof. Let $\eta(t) = [\xi^1(t); 0; \cdots ; 0]$. If $\xi(t) \to \eta(t)$, then $\tilde{x}(t) \to (T^{-1} \otimes I_n)\eta(t)$. Note that the columns of T^{-1} comprise all the right eigenvectors and generalized eigenvectors of L. The first column of T^{-1} is vector $\mathbf{1}$. This implies that for $i = 1, ..., N$

$$x^i(t) \to \xi^1(t).$$

The result of Lemma 8.1 follows. $\square$

The sub-dynamics of $\bar{\xi}(t) = [\xi^2(t); \cdots ; \xi^N(t)]$ are

$$\dot{\bar{\xi}}(t) = (I_{N-1} \otimes A)\bar{\xi}(t) - (\bar{J}_L \otimes BF)\bar{\xi}(t-\tau) \tag{8.8}$$

where $\bar{J}_L$ is such that

$$J_L = \begin{bmatrix} 0 & \\ & \bar{J}_L \end{bmatrix}.$$

The eigenvalues of system (8.8) are given by the roots of its characteristic equation

$$H(s) = \det\left\{sI - (I_{N-1} \otimes A) + e^{-s\tau}(\bar{J}_L \otimes BF)\right\} = 0,$$

which, due to the upper-triangular structure of $I_{N-1} \otimes A$ and $\bar{J}_L \otimes BF$, are the union of the eigenvalues of the $N-1$ systems:

$$\dot{\xi}^i(t) = A\xi^i(t) - \lambda_i BF\xi^i(t-\tau), \quad i = 2, ..., N.$$

We immediately have the following conclusion

Lemma 8.2. *For any given positive β, γ, φ and $\bar{\tau}$, Problem 8.1 is solvable if there exists an F such that the system*

$$\dot{x}(t) = Ax(t) - \lambda e^{j\psi} BFx(t-\tau) \tag{8.9}$$

is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

8.3.1.3 Robust Stabilization via a Low-Gain State Feedback

Next, we shall show that the consensus controller in (8.4) satisfies the condition from Lemma 8.2. It remains to show that with properly chosen ϵ, the system (8.9) with $F = \frac{1}{\beta}F_\epsilon$ given by (8.4) is globally asymptotically stable for all $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. The following (slightly modified) lemma of [112] is fundamental.

Lemma 8.3. *Consider a linear time-delay system*

$$\dot{x} = Ax + e^{j\psi} A_d x(t-\tau). \tag{8.10}$$

Assume $A + A_d$ is Hurwitz and $A_d = BF$ where B and F have full rank. We have that (8.10) *is globally asymptotically stable for $\tau \in [0, \bar{\tau}]$ and $|\psi| \in [0, \bar{\psi}]$ if*

$$\det\left[I - G(j\omega)(D_\psi(j\omega) - I)\right] \neq 0$$

for all $\omega \in \mathbb{R}$, $\tau \in [0, \bar{\tau}]$ and $|\psi| \in [0, \bar{\psi}]$ where

$$G(s) = F(sI - A - A_d)^{-1}B \text{ and } D_\psi(s) = e^{-\tau s + j\psi}I.$$

Using Lemma 8.3, we can prove the following result:

Lemma 8.4. *For any $\gamma \geq \beta > 0$, $\bar{\tau} > 0$ and φ such that*

$$\bar{\tau} < \frac{\frac{\pi}{3} - \varphi}{\omega_{\max}}, \tag{8.11}$$

there exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$, the closed-loop system (8.9) *with $F = \frac{1}{\beta} F_\epsilon$ given by* (8.4) *is asymptotically stable for all $\tau \in [0, \bar{\tau}]$, $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$.*

Proof. With $F = \frac{1}{\beta} F_\epsilon$, (8.9) can be rewritten as

$$\dot{x} = Ax - \tfrac{\lambda}{\beta} e^{j\psi} BB' P_\epsilon x(t - \tau). \tag{8.12}$$

Since $\frac{\lambda}{\beta} \geq 1$, (9.17) can be rewritten as

$$A' P_\epsilon + P_\epsilon A - \tfrac{\lambda}{\beta} P_\epsilon BB' P_\epsilon + Q_\epsilon = 0, \tag{8.13}$$

where

$$Q_\epsilon = \epsilon I + \left(\tfrac{\lambda}{\beta} - 1 \right) P_\epsilon BB' P_\epsilon > 0.$$

Let

$$F_{\varepsilon,\lambda} = -\lambda F_\epsilon = -\tfrac{\lambda}{\beta} B' P_\epsilon, \tag{8.14}$$

$$G_{\varepsilon,\lambda}(s) = F_{\varepsilon,\lambda}(sI - A - BF_{\varepsilon,\lambda})^{-1} B. \tag{8.15}$$

Note that $F_{\varepsilon,\lambda}$ is a linear quadratic regulator with associated Riccati equation (8.13), and

$$\lim_{\epsilon \downarrow 0} F_{\varepsilon,\lambda} = 0$$

for any $\lambda > \beta$. Moreover, from [1], the following holds, for any $\lambda > \beta$ and $\epsilon > 0$

$$\underline{\sigma}(I - F_{\varepsilon,\lambda}(j\omega I - A)^{-1} B) \geq 1, \quad \forall \omega \in \mathbb{R}.$$

By applying Sherman-Morrison-Woodbury formula, also known as the matrix inversion lemma, see for example [15], to the left-hand side, we obtain that for any $\lambda > \beta$ and $\epsilon > 0$

$$\bar{\sigma}(I + G_{\varepsilon,\lambda}(j\omega)) \leq 1, \quad \forall \omega \in \mathbb{R}, \tag{8.16}$$

and hence

$$\bar{\sigma}(G_{\varepsilon,\lambda}(j\omega)) \leq 2, \quad \forall \omega \in \mathbb{R}. \tag{8.17}$$

The closed-loop system (8.12) can also be written as

$$\dot{x} = Ax + e^{j\psi} BF_{\varepsilon,\lambda} x(t - \tau). \tag{8.18}$$

It follows from Lemma 8.3 that (8.18) is globally asymptotically stable if

$$\det \left[I - G_{\varepsilon,\lambda}(j\omega)(D_{\psi}(j\omega) - I)\right] \neq 0, \tag{8.19}$$

for all $\omega \in \mathbb{R}$, $\tau \in [0, \bar{\tau}]$, $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$.

Although (8.19) has to hold for all $\omega \in \mathbb{R}$, thanks to Lemma 8.9, we only need to verify (8.19) for those ω's in a finite number of small intervals if ϵ is chosen sufficiently small.

Assume A has r distinct eigenvalues on the imaginary axis which are denoted by $j\omega_k$, $k = 1, ..., r$. Given (8.11), there exists a $\delta > 0$ such that

1) The neighborhoods $\mathcal{E}_k := (\omega_k - \delta, \omega_k + \delta)$, $k = 1, ..., r$ around these eigenfrequencies, are mutually disjoint;

2) We have:

$$-\tfrac{\pi}{3} < \omega\bar{\tau} + \psi < \tfrac{\pi}{3}$$

for $\psi \in [-\varphi, \varphi]$ and $\omega \in \cup_{k=1}^{r} \mathcal{E}_k$.

It is shown in Lemma 8.9 that there exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$,

$$\|G_{\varepsilon,\lambda}(j\omega)\| \leq \tfrac{1}{3},\ \forall \omega \in \Omega := \mathbb{R} \setminus \cup_{k=1}^{r} \mathcal{E}_k,\ \lambda \in (\beta, \gamma).$$

Since $\|D_\psi(j\omega) - I\| \leq 2$ for any $\omega, \psi \in \mathbb{R}$, the above inequality implies that (8.19) holds if

$$\det\left[I - G_{\varepsilon,\lambda}(j\omega)(D_\psi(j\omega) - I)\right] \neq 0, \tag{8.20}$$

for all $\omega \in \cup_{k=1}^{r} \mathcal{E}_k$, $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$. Note that

$$I - G_{\varepsilon,\lambda}(j\omega)(D_\psi(j\omega) - I) = D_\psi(j\omega) - (I + G_{\varepsilon,\lambda}(j\omega))(D_\psi(j\omega) - I)$$

and $D_\psi(j\omega)$ is unitary. Combined with (8.16), we have that (8.20) holds if

$$\bar{\sigma}(D_\psi(j\omega) - I) < 1,\ \forall \omega \in \cup_{k=1}^{r} \mathcal{E}_k,\ \psi \in [-\varphi, \varphi]. \tag{8.21}$$

This is guaranteed by definition of $\mathcal{E}_k$. Therefore, we conclude that for any $\epsilon \in (0, \epsilon^*]$, the closed-loop system is asymptotically stable for all $\tau \in [0, \bar{\tau}]$, $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$. □

8.3.1.4 Neutrally Stable Agents

We observe that the consensus controller design in Theorem 8.1 for general critically unstable agents depends on β explicitly. We next consider a special case where the agents have neutrally stable dynamics, i.e. those eigenvalues of A on the imaginary axis are semi-simple. Without loss of generality, we assume that $A' + A = 0$. In this case, we shall show that the consensus controller design no longer requires the knowledge of β and hence allows us to deal with a larger set of unknown communication topologies. Moreover, less restrictive conditions on φ and $\bar{\tau}$ can be obtained.

Consider the agents (8.1). Assume $A' + A = 0$. A local consensus controller can be constructed as

$$u^i = \epsilon B' z^i. \tag{8.22}$$

We have the following theorem:

Theorem 8.2. *For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (8.1) *and any communication topology belonging to the set $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A' + A = 0$. In that case, Problem 8.1 is solvable if,*

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{8.23}$$

Moreover, it can be solved by the consensus controller (8.22) *if* (8.23) *holds. Specifically, for given γ and given φ and $\bar{\tau}$ satisfying* (8.23)*, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.1) *with controller* (8.22) *achieve consensus for any communication topology in $\mathcal{G}_{0,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

Remark 8.5. *Note that we always have $\phi < \frac{\pi}{2}$ provided that Assumption 8.1 holds. Hence, we find that the restriction $\varphi < \pi/3$ in* (8.6) *can actually be removed for neutrally stable agents and a larger $\bar{\tau}$ is permitted.*

Proof of Theorem 8.2. We have proved in Section 8.3.1.2 that (8.22) solves Problem 8.1 with a set of unknown communication topology $\mathcal{G}_{0,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$ if the system

$$\dot{x} = Ax - \lambda\epsilon e^{j\psi} BB' x(t - \tau) \tag{8.24}$$

is globally asymptotically stable for $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

To show this, we shall need the following lemma adapted from [112].

Lemma 8.5. *The time-delayed system* (8.10) *is globally asymptotically stable for $\tau \in [0, \bar{\tau}]$ if and only if*

$$\det(j\omega I - A - A_d e^{-j\omega\tau}) \neq 0, \ \forall \omega \in \mathbb{R}, \ \tau \in [0, \bar{\tau}].$$

It can be concluded from Lemma 8.5 that (8.24) is globally asymptotically stable for $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$ if and only if

$$\det\left[j\omega I - A + \lambda\epsilon e^{j\psi - j\omega\tau} BB'\right] \neq 0, \ \forall \omega \in \mathbb{R}, \ \lambda \in (0, \gamma), \ \psi \in [-\varphi, \varphi], \ \tau \in [0, \bar{\tau}]. \tag{8.25}$$

Assume A has r distinct eigenvalues on the imaginary axis which are denoted by $j\omega_k$, $k = 1, ..., r$. Given (8.23), there exists a $\delta > 0$ such that

1) The neighborhoods $\mathcal{E}_k := (\omega_k - \delta, \omega_k + \delta)$, $k = 1, ..., r$ around these eigenfrequencies, are mutually disjoint;

2) We have:

$$-\tfrac{\pi}{2} < \omega\bar{\tau} + \psi < \tfrac{\pi}{2}$$

for $\psi \in [-\varphi, \varphi]$ and $\omega \in \cup_{k=1}^{r} \mathcal{E}_k$.

For given γ, we can show with a similar argument as we use to prove Lemma 8.9 in the Appendix that there exists a $\mu > 0$ and a ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$ and $\lambda \in (0, \gamma)$

$$\underline{\sigma}(j\omega I - A + \lambda\epsilon e^{j\psi - j\omega\tau} BB) > \mu, \, \forall \omega \in \mathbb{R} / \cup_{r=1}^{k} \mathcal{E}_k.$$

Hence, (8.25) is satisfied for $\epsilon \in (0, \epsilon_1]$ and $\omega \in \mathbb{R} / \cup_{r=1}^{k} \mathcal{E}_k$.

It remains to show (8.25) for $\omega \in \cup_{r=1}^{k} \mathcal{E}_k$. Note that $\psi - \omega\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by definition of $\mathcal{E}_k$ and hence $\cos(\psi - \omega\tau) > 0$. Then

$$[A - \lambda\epsilon e^{j\psi - j\omega\tau} BB']^* + [A - \lambda\epsilon e^{j\psi - j\omega\tau} BB'] = -2\lambda\epsilon\cos(\psi - \omega\tau) BB' \leq 0$$

Since (A, B) is controllable, we conclude that $A - \lambda\epsilon e^{j\psi - j\omega\tau} BB'$ is Hurwitz for $\omega \in \cup_{k=1}^{r} \mathcal{E}_k$. Hence (8.25) also holds for $\omega \in \cup_{r=1}^{k} \mathcal{E}_k$. □

Similar results as in Corollaries 8.1 and 8.2 also carry over here for undirected communication topologies and neutrally stable agents that only have zero eigenvalues (a collection of single integrators).

Corollary 8.3. *For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (8.1) *and any communication topology belonging to the set $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A + A' = 0$ and A has all the eigenvalues at zero. In* *that case, Problem 8.1 is always solvable via the consensus controller* (8.22)*. Specifically, for given* $\mathcal{G}_{0,\gamma,\varphi}$ *and* $\bar{\tau} > 0$*, there exists an* ϵ^* *such that for any* $\epsilon \in (0, \epsilon^*]$*, the agents* (8.1) *with controller* (8.22) *achieve consensus for any communication topologies in* $\mathcal{G}_{0,\gamma,\varphi}$ *and* $\tau \in [0, \bar{\tau}]$*.*

Corollary 8.4. *For a given set of undirected networks* $\mathcal{G}_{0,\gamma,0}$ *and* $\bar{\tau} > 0$*, consider the agents* (8.1) *and any communication topology belonging to the set* $\mathcal{G}_{0,\gamma,0}$*. Suppose* $A + A' = 0$*. In that case, Problem 8.1 is solvable if*

$$\bar{\tau} < \frac{\pi}{2\omega_{\max}}. \tag{8.26}$$

Moreover, it can be solved by the consensus controller (8.22) *if* (8.26) *holds. Specifically, for given* $\mathcal{G}_{0,\gamma,0}$ *and given* $\bar{\tau}$ *satisfying* (8.26)*, there exists an* ϵ^* *such that for any* $\epsilon \in (0, \epsilon^*]$*, the agents* (8.1) *with controller* (8.22) *achieve consensus for any communication topologies in* $\mathcal{G}_{0,\gamma,0}$ *and* $\tau \in [0, \bar{\tau}]$*.*

8.3.2 Consensus in Networks with Known Communication Topology

It is clear from Theorem 8.1 that in the case where the network is directed and its associated Laplacian has complex eigenvalues, we have to yield some delay tolerance to accommodate the uncertainty of complex Laplacian eigenvalues. However, it will be shown in this section that if the network topology is exactly known, we can achieve a larger delay tolerance as in the undirected network case shown in Corollary 8.2. In this section, we shall study the multi-agent consensus problem in a known network which can be either directed or undirected.

From the work of [13], we have that for the given Laplacian matrix L and any positive real number β, there exists a diagonal matrix $D = \operatorname{diag}\{d_i\}$, where d_i is a scalar for $i = 1, \ldots, N$, such that the eigenvalues of DL, denoted by $\lambda_i(DL)$, $i = 1, ..., N$, are real and satisfy

$$\lambda_1(DL) = 0, \quad \lambda_i(DL) > \beta, \quad i = 2, ..., N.$$

Note that D depends on L and hence the network topology. Let $\gamma > 0$ be such that $\lambda_i(DL) < \gamma$, $i = 2, ..., N$. For each agent i, we apply a controller

$$u^i = \tfrac{d_i}{\beta} F_\epsilon z^i, \tag{8.27}$$

where d_i is the ith diagonal element of D and F_ϵ is the same as in (8.4). The design depends on the network G through the choice of D.

Theorem 8.3. *Consider a network of agents* (8.1). *Suppose the network topology G defined by matrix L is known. Problem 8.3 is solvable with consensus controller* (8.27) *if $\bar{\tau} < \frac{\pi}{3\omega_{\max}}$. Specifically, for the given L, there exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$,* (8.1) *with controller* (8.27) *achieve consensus for any $\tau \in [0, \bar{\tau}]$.*

Proof. Using the same notations declared in the preceding section, we can write the overall dynamics of N agents as

$$\dot{\tilde{x}}(t) = (I_N \otimes A)\tilde{x}(t) - \tfrac{1}{\beta}(DL \otimes BF_\epsilon)\tilde{x}(t - \tau).$$

Since all the eigenvalues of DL are real and $\lambda_i(DL) \in (\beta, \gamma)$, $i = 2, ..., N$, the rest follows from the same argument as used in the proof of Theorem 1. Therefore, given $\bar{\tau} < \frac{\pi}{3\omega_{\max}}$, we can find an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$, consensus is achieved with respect to G for all $\tau < \bar{\tau}$. □

8.3.2.1 Neutrally Stable Agents

For neutrally stable agents, where the communication topology is known, the design in Section 8.3.1.4 can also be modified to be topology-dependent so that the impact of φ on $\bar{\tau}$ can be removed.

Consider the agents (8.1) with a known communication topology defined by matrix L. Suppose $A' + A = 0$. For the given L and any $\beta > 0$, choose the same $D = \text{diag}\{d_i\}$, where d_i is a scalar for

$i = 1, \ldots, N$ as in the preceding subsection so that DL only has real eigenvalues and

$$\lambda_1(DL) = 0, \quad \lambda_i(DL) > \beta, \quad i = 2, ..., N.$$

The consensus controller for each agent can be designed as

$$u^i = d_i \epsilon B' z^i, \tag{8.28}$$

where d_i is the ith diagonal element of the matrix D.

Theorem 8.4. *Consider a network of agents* (8.1). *Suppose $A' + A = 0$ and the communication topology G defined by matrix L is known. Problem 8.3 is solvable with consensus controller* (8.28) *if $\bar{\tau} < \frac{\pi}{2\omega_{\max}}$. Specifically, for the given L, there exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$,* (8.1) *with controller* (8.28) *achieve consensus for any $\tau \in [0, \bar{\tau}]$.*

8.4 Consensus with Partial-state Coupling

Next, we proceed to the case of partial-state coupling and design a dynamic consensus controller (8.3) which solves Problem 8.2 and 8.4.

8.4.1 Consensus in Networks with Unknown Communication Topology

8.4.1.1 Controller Design and Main Result

In this subsection, a dynamic low-gain consensus controller in the form of (8.3) is constructed to solve the consensus problem for agents (8.2) with a set of unknown communication topologies.

For $\epsilon > 0$, let P_ϵ be the positive definite solution of the ARE (8.5). A dynamic low-gain consensus controller can be constructed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - K z^i, \\ u^i = \frac{1}{\beta} B' P_\epsilon \chi^i, \end{cases} \tag{8.29}$$

where K is such that $A+KC$ is Hurwitz stable. We shall prove that the consensus controller solves Problem 8.2.

Theorem 8.5. *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (8.2) *with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. In that case, Problem 8.2 is solvable if,*

$$\varphi < \frac{\pi}{3} \quad \textit{and} \quad \bar{\tau} < \frac{\frac{\pi}{3} - \varphi}{\omega_{\max}}. \tag{8.30}$$

Moreover, it can be solved by the consensus controller (8.29) *if* (8.30) *holds. Specifically, for given β and γ and given φ and $\bar{\tau}$ satisfying* (8.30)*, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.2) *with controller* (8.29) *achieve consensus for any communication topology in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

The next corollary is concerned with the case where A only has zero eigenvalues.

Corollary 8.5. *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (8.2) *with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. Suppose A has all the eigenvalues at zero. In that case, Problem 8.2 is solvable if,*

$$\varphi < \frac{\pi}{3}. \tag{8.31}$$

Moreover, it can be solved by the consensus controller (8.29) *if* (8.31) *holds. Specifically, for given β, γ, $\bar{\tau} > 0$ and $\varphi < \frac{\pi}{3}$, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.2) *with controller* (8.29) *achieve consensus for any communication topology in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

If we are only interested in undirected graphs, the following corollary can be utilized:

Corollary 8.6. *For a given set of undirected topologies $\mathcal{G}_{\beta,\gamma,0}$ and $\bar{\tau} > 0$, consider the agents* (8.2) *with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,0}$. In that case, Problem 8.2 is solvable if,*

$$\bar{\tau} < \frac{\pi}{3\omega_{\max}}. \tag{8.32}$$

Moreover, it can be solved by the consensus controller (8.29) *if* (8.32) *holds. Specifically, for given* $0 < \beta \leq \gamma$, $\bar{\tau} > 0$ *and given* $\varphi < \frac{\pi}{3}$, *there exists an* ϵ^* *such that for any* $\epsilon \in (0, \epsilon^*]$, *the agents* (8.2) *with controller* (8.29) *achieve consensus with any communication topology in* $\mathcal{G}_{\beta,\gamma,0}$ *and* $\tau \in [0, \bar{\tau}]$.

We adopt a similar two-step strategy as we proved Theorem 8.1, which will be presented in the next two subsections.

8.4.1.2 Connection to a Robust Stabilization Problem

In the partial-state coupling case, the network information z^i of each agent can be fed into the input u^i through a compensator (8.3).

Let $\bar{x}^i = [x^i; \chi^i]$. Then for each agent, the closed-loop dynamics are

$$\begin{cases} \dot{\bar{x}}^i(t) = \begin{bmatrix} A\ BC_k \\ 0 \quad A_k \end{bmatrix} \bar{x}^i(t) + \begin{bmatrix} 0 \\ B_k \end{bmatrix} z^i(t), \\ y^i(t) = \begin{bmatrix} C\ 0 \end{bmatrix} \bar{x}^i, \\ z^i(t) = -\sum_{j=1}^N \ell_{ij} y^j(t-\tau). \end{cases}$$

Define $\tilde{x} = [\bar{x}^1; \cdots; \bar{x}^N]$, $\tilde{y} = [y^1; \cdots; y^N]$,

$$\mathcal{A} = \begin{bmatrix} A & BC_k \\ 0 & A_k \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ B_k \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}.$$

The overall dynamics of the N agents can be written as

$$\dot{\tilde{x}}(t) = (I_N \otimes \mathcal{A})\tilde{x}(t) - (L \otimes \mathcal{B}\mathcal{C})\tilde{x}(t-\tau).$$

Similarly as in Section 8.3.1.2, we can prove the following lemma.

Lemma 8.6. *For any given set* $\mathcal{G}_{\beta,\gamma,\varphi}$ *and* $\bar{\tau} \geq 0$, *Problem 8.2 is solvable via consensus controller* (9.2) *if the system*

$$\dot{\xi}(t) = \mathcal{A}\xi(t) - \lambda e^{j\psi}\mathcal{B}\mathcal{C}\xi(t-\tau) \tag{8.33}$$

is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ *and* $\tau \in [0, \bar{\tau}]$.

Define an auxiliary system

$$\begin{cases} \dot{x}(t) = Ax - \lambda e^{j\psi} Bu(t-\tau), \\ z(t) = Cx(t), \end{cases} \tag{8.34}$$

and a compensator

$$\begin{cases} \dot{\chi}(t) = A_k \chi(t) + B_k z(t), \\ u(t) = C_k \chi(t), \end{cases} \tag{8.35}$$

where A_k, B_k and C_k are the same as in (8.3).

Lemma 8.7. *For any given set* $\mathcal{G}_{\beta,\gamma,\varphi}$ *and* $\bar{\tau} \geq 0$, *Problem 8.2 is solvable via consensus controller* (9.2) *if the closed-loop system of* (8.34) *and* (8.35) *is globally asymptotically stable for any* $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ *and* $\tau \in [0, \bar{\tau}]$.

Proof. It is easy to see from the definitions of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ that (8.33) is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$ if the closed-loop system of (8.35) and

$$\begin{cases} \dot{\zeta}(t) = A\zeta(t) + Bu(t), \\ z(t) = -\lambda e^{j\psi} Cx(t-\tau), \end{cases} \tag{8.36}$$

is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$.

The closed-loop system of (8.36) and (8.35) has a set of eigenvalues determined by

$$\det \begin{bmatrix} sI - A & -BC_k \\ \lambda e^{s\tau} B_k C & sI - A_k \end{bmatrix} = 0. \tag{8.37}$$

On the other hand, the eigenvalues of closed-loop system of (8.34) and (8.35) are given by

$$\det \begin{bmatrix} sI - A & \lambda e^{s\tau} BC_k \\ -B_k C & sI - A_k \end{bmatrix} = 0. \tag{8.38}$$

Note that

$$\begin{bmatrix} -\frac{1}{\lambda}e^{-s\tau}I & \\ & I \end{bmatrix} \begin{bmatrix} sI-A & \lambda e^{s\tau}BC_k \\ -B_kC & sI-A_k \end{bmatrix} \begin{bmatrix} -\lambda e^{s\tau}I & \\ & I \end{bmatrix} = \begin{bmatrix} sI-A & -BC_k \\ \lambda e^{s\tau}B_kC & sI-A_k \end{bmatrix}.$$

We find that the two closed-loop systems have the same set of eigenvalues. Therefore, (8.33) is globally asymptotically stable if the closed-loop of (8.35) and (8.34) is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. The result in Lemma 8.7 then follows from Lemma 8.6. □

8.4.1.3 Robust Stabilization via a Low-gain Compensator

Let A_k, B_k and C_k be given by the consensus controller (8.29). Then (8.35) can be written as

$$\begin{cases} \dot{\chi}(t) = (A+KC)\chi(t) - Kz(t), \\ u(t) = \frac{1}{\beta}B'P_\epsilon\chi(t). \end{cases} \tag{8.39}$$

In this subsection, we show that (8.39) globally asymptotically stabilizes (8.34) for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. Hence the low-gain consensus controller (8.29) solves Problem 8.2.

Lemma 8.8. *Let $\omega_{\max} = \max\{\omega \mid \det(j\omega I - A) = 0\}$. For given $0 < \beta \leq \gamma$ and $\varphi, \bar{\tau}$ satisfying the conditions* (8.30)*, there exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$, the closed-loop system of* (8.34) *and* (8.39) *is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \leq \bar{\tau}$.*

Proof. Let $F_{\varepsilon,\lambda} = -\frac{\lambda}{\beta}B'P_\epsilon$ and

$$\bar{A}_1 = \begin{bmatrix} A & BF_{\varepsilon,\lambda} \\ -KC & A+KC \end{bmatrix}.$$

It follows from the work of [73] that there exists an ϵ_0 such that for $\epsilon \in (0, \epsilon_0]$, $\bar{A}_1$ is Hurwitz. Define

$$G^m_{\varepsilon,\lambda}(s) = \begin{bmatrix} 0 & F_{\varepsilon,\lambda} \end{bmatrix} (sI - \bar{A}_1)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

Lemma 8.3 shows that the closed-loop of (8.34) and (8.39) is globally asymptotically stable for all $\tau \leq \bar{\tau}$, $\psi \in [-\varphi, \varphi]$ and $\lambda \in (\beta, \gamma)$ if

$$\det\left\{I - G^m_{\varepsilon,\lambda}(j\omega)[D_\psi(j\omega) - I]\right\} \neq 0,\ \forall \omega \in \mathbb{R},\ \tau \leq \bar{\tau},\ \lambda \in (\beta, \gamma),\ \psi \in [-\varphi, \varphi]. \tag{8.40}$$

Let's consider $G^m_{\varepsilon,\lambda}(s)$. Define

$$\bar{A}_2 = \begin{bmatrix} A & BF_{\varepsilon,\lambda} \\ -KC & A + KC + BF_{\varepsilon,\lambda} \end{bmatrix}.$$

Note that $\bar{A}_1 = \bar{A}_2 + \Gamma$, where

$$\Gamma = \begin{bmatrix} 0 & 0 \\ 0 & -BF_{\varepsilon,\lambda} \end{bmatrix} = -\begin{bmatrix} 0 \\ B \end{bmatrix} \begin{bmatrix} 0 & F_{\varepsilon,\lambda} \end{bmatrix}.$$

Then

$$(sI - \bar{A}_1)^{-1} = \left[I - (sI - \bar{A}_2)^{-1}\Gamma\right]^{-1} (sI - \bar{A}_2)^{-1}.$$

Hence

$$\begin{aligned} G^m_{\varepsilon,\lambda}(s) &= \begin{bmatrix} 0 & F_{\varepsilon,\lambda} \end{bmatrix} (sI - \bar{A}_1)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & F_{\varepsilon,\lambda} \end{bmatrix} \left[I - (sI - \bar{A}_2)^{-1}\Gamma\right]^{-1} (sI - \bar{A}_2)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix} \\ &= \left[I + G^d_{\varepsilon,\lambda}(s)\right]^{-1} G^f_{\varepsilon,\lambda}(s), \end{aligned}$$

where we use the definition of Γ and the property that $(I + AB)^{-1}A = A(I + BA)^{-1}$ to obtain the last equality and

$$G^f_{\varepsilon,\lambda}(s) = \begin{bmatrix} 0, F_{\varepsilon,\lambda} \end{bmatrix} (sI - \bar{A}_2)^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$G^d_{\varepsilon,\lambda}(s) = \begin{bmatrix} 0 & F_{\varepsilon,\lambda} \end{bmatrix} (sI - \bar{A}_2)^{-1} \begin{bmatrix} 0 \\ B \end{bmatrix}.$$

We next evaluate $G^d_{\varepsilon,\lambda}(s)$ and $G^f_{\varepsilon,\lambda}(s)$. By a simple transformation of $\bar{A}_2$, we obtain

$$G^f_{\varepsilon,\lambda}(s) = \begin{bmatrix} 0 & F_{\varepsilon,\lambda} \end{bmatrix} \begin{bmatrix} sI - A - KC & 0 \\ KC & sI - A - BF_{\varepsilon,\lambda} \end{bmatrix}^{-1} \begin{bmatrix} B \\ 0 \end{bmatrix},$$

$$G^d_{\varepsilon,\lambda}(s) = \begin{bmatrix} 0 & F_{\varepsilon,\lambda} \end{bmatrix} \begin{bmatrix} sI - A - KC & 0 \\ KC & sI - A - BF_{\varepsilon,\lambda} \end{bmatrix}^{-1} \begin{bmatrix} -B \\ B \end{bmatrix}.$$

It is not difficult to get that

$$G^f_{\varepsilon,\lambda}(s) = -F_{\varepsilon,\lambda}(sI - A - BF_{\varepsilon,\lambda})^{-1} KC (sI - A - KC)^{-1} B,$$

$$G^d_{\varepsilon,\lambda}(s) = F_{\varepsilon,\lambda}(sI - A - BF_{\varepsilon,\lambda})^{-1} B - G^f_{\varepsilon,\lambda}(s) = G_{\varepsilon,\lambda}(s) - G^f_{\varepsilon,\lambda}(s),$$

where $G_{\varepsilon,\lambda}$ has been defined in (8.15). Therefore,

$$G^m_{\varepsilon,\lambda}(s) = \left[I + \left(G_{\varepsilon,\lambda}(s) - G^f_{\varepsilon,\lambda}(s) \right) \right]^{-1} G^f_{\varepsilon,\lambda}(s). \tag{8.41}$$

It is clear that if $\|G^f_{\varepsilon,\lambda} - G_{\varepsilon,\lambda}\|_\infty \to 0$ as $\epsilon \to 0$, then $\|G^m_{\varepsilon,\lambda} \to G_{\varepsilon,\lambda}\|_\infty \to 0$ as $\epsilon \to 0$. In other words, $G^m_{\varepsilon,\lambda}(j\omega)$ converges to $G_{\varepsilon,\lambda}(j\omega)$ uniformly in ω as $\epsilon \to 0$. Let's consider $G^f_{\varepsilon,\lambda}$. We have:

$$\begin{aligned}
G_{\varepsilon,\lambda}(s) - G^f_{\varepsilon,\lambda}(s) =& F_{\varepsilon,\lambda}(sI - A - BF_{\varepsilon,\lambda})^{-1}\left[I + KC(sI - A - KC)^{-1}\right]B \\
=& F_{\varepsilon,\lambda}(sI - A - BF_{\varepsilon,\lambda})^{-1}\left[I - KC(sI - A)^{-1}\right]^{-1}B \\
=& F_{\varepsilon,\lambda}(sI - A - BF_{\varepsilon,\lambda})^{-1}(sI - A)(sI - A - KC)^{-1}B \\
=& F_{\varepsilon,\lambda}\left[I - (sI - A)^{-1}BF_{\varepsilon,\lambda}\right]^{-1}(sI - A - KC)^{-1}B \\
=& F_{\varepsilon,\lambda}\left[I + (sI - A - BF_{\varepsilon,\lambda})^{-1}BF_{\varepsilon,\lambda}\right](sI - A - KC)^{-1}B \\
=& \left[I + F_{\varepsilon,\lambda}(sI - A - BF_{\varepsilon,\lambda})^{-1}B\right]F_{\varepsilon,\lambda}(sI - A - KC)^{-1}B \\
=& \left[I + G_{\varepsilon,\lambda}(s)\right]F_{\varepsilon,\lambda}(sI - A - KC)^{-1}B.
\end{aligned}$$

Since $A + KC$ is Hurwitz stable and (8.16) is satisfied, we have

$$\|G_{\varepsilon,\lambda} - G^f_{\varepsilon,\lambda}\|_\infty \to 0 \text{ as } \epsilon \to 0. \tag{8.42}$$

Hence $\|G^m_{\varepsilon,\lambda} \to G_{\varepsilon,\lambda}\|_\infty \to 0$ as $\epsilon \to 0$. From the proof of Lemma 8.4, for given φ and $\bar{\tau}$ satisfying (8.30), there exists an ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$, (8.19) holds. With (8.42), this implies that there exists an $\epsilon^* \le \epsilon_1$ such that for $\epsilon \in (0, \epsilon^*]$, (8.40) also holds. This completes the proof. $\square$

8.4.1.4 Neutrally stable agents

Corresponding to Section 8.3.1.4, we shall also consider here the special case where agents have neutrally stable dynamics. Assume $A' + A = 0$. In this case, a low-gain consensus controller can be designed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - Kz^i \\ u^i = \epsilon B'\chi^i, \end{cases} \tag{8.43}$$

where K is such that $A + KC$ is Hurwitz.

Theorem 8.6. *For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (8.2) *with any communication topology belonging to $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A + A' = 0$. In that case, Problem 8.2 is solvable if,*

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{8.44}$$

Moreover, it can be solved by the consensus controller (8.43) *if* (8.44) *holds. Specifically, for given γ and given φ and $\bar{\tau}$ satisfying* (8.44)*, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.2) *with controller* (8.43) *achieve consensus for any communication topology in $\mathcal{G}_{0,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

Remark 8.6. *As expected, the restriction $\varphi < \frac{\pi}{3}$ in* (8.30) *for general critically unstable agents disappears and a larger $\bar{\tau}$ is obtained.*

Proof of Theorem 8.6. Define

$$\mathcal{A}_k = \begin{bmatrix} A & 0 \\ -KC & A + KC \end{bmatrix}, \quad \mathcal{A}_{d,\epsilon} = \begin{bmatrix} 0 & -\lambda\epsilon e^{j\psi} BB' \\ 0 & 0 \end{bmatrix}.$$

It has been proven that (8.43) solves Problem 8.2 if the system

$$\dot{\xi}(t) = \mathcal{A}_k \xi(t) + \mathcal{A}_{d,\epsilon} \xi(t - \tau), \tag{8.45}$$

is globally asymptotically stable for $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. From Lemma 8.5, (8.45) is globally asymptotically stable if and only if

$$\det\left(j\omega I - \mathcal{A}_k - \mathcal{A}_{d,\epsilon} e^{-j\omega\tau}\right) \neq 0, \forall \omega \in \mathbb{R},\ \lambda \in (0, \gamma),\ \psi \in [-\varphi, \varphi],\ \tau \in [0, \bar{\tau}]. \tag{8.46}$$

Let δ and $\mathcal{E}_k$ be defined as in the proof of Theorem 8.2. For given γ, we can show with a similar argument as we use to prove Lemma 8.9 that there exists a $\mu > 0$ and an ϵ_1 such that for $\epsilon \in (0, \epsilon_1]$ and $\gamma \in (0, \gamma)$

$$\underline{\sigma}\left(j\omega I - \mathcal{A}_k - \mathcal{A}_{d,\epsilon} e^{j\omega\tau}\right) > \mu, \forall \omega \in \mathbb{R} / \cup_{r=1}^{k} \mathcal{E}_k.$$

Hence, (8.46) is satisfied for $\epsilon \in (0, \epsilon_1]$, $\omega \in \mathbb{R}/\cup_{r=1}^{k} \mathcal{E}_k$ and $\lambda \in (0, \gamma)$.

It remains to show that (8.46) holds for $\omega \in \cup_{r=1}^{k} \mathcal{E}_k$. It suffices to show that there exists an $\epsilon^* \leq \epsilon_1$ such that for $\epsilon \in (0, \epsilon^*]$, $\mathcal{A}_k + \mathcal{A}_{d,\epsilon} e^{-j\omega\tau}$ is Hurwitz for $\omega \in \cup_{k=1}^{r} \mathcal{E}_k$.

Note that there exists a non-singular transformation Γ such that

$$\bar{\mathcal{A}}_\psi(\omega) = \Gamma^{-1}(\mathcal{A}_k + \mathcal{A}_d e^{-j\omega\tau})\Gamma = \begin{bmatrix} A - \lambda\epsilon e^{j\psi - j\omega\tau} BB' & \lambda\epsilon e^{j\psi - j\omega\tau} BB' \\ -\lambda\epsilon e^{j\psi - j\omega\tau} BB' & A + KC + \lambda\epsilon e^{j\psi - j\omega\tau} BB' \end{bmatrix}.$$

Let Q be the positive definite solution of Lyapunov equation

$$(A + KC)'Q + Q(A + KC) = -2I.$$

There exists an ϵ_2 such that for $\epsilon \in (0, \epsilon_2]$,

$$(A + KC + \lambda\epsilon e^{j\psi - j\omega\tau} BB')^* Q + Q(A + KC + \lambda\epsilon e^{j\psi - j\omega\tau} BB') \leq -I.$$

Define

$$P = \begin{bmatrix} I & 0 \\ 0 & Q \end{bmatrix}.$$

We have that

$$\bar{\mathcal{A}}_\psi(\omega)^* P + P\bar{\mathcal{A}}_\psi(\omega) \leq \begin{bmatrix} -2\lambda\epsilon\cos(\psi - \omega\tau)BB' & -\lambda\epsilon BB' M_\psi(\omega) \\ -\lambda\epsilon M_\psi^*(\omega) BB' & -I \end{bmatrix},$$

where

$$M_\psi(\omega) = e^{-j\psi + j\omega\tau} Q - e^{j\psi - j\omega\tau} I.$$

Note that $\|M_\psi(\omega)\| \leq \|Q\| + 1$ and $\cos(\psi - \omega\tau) > 0$ for $\omega \in \cup_{k=1}^{r} \mathcal{E}_k$. Let ϵ_3 be such that for $\epsilon \in (0, \epsilon_3]$,

$$\frac{\lambda\epsilon}{\cos(\psi - \omega\tau)} M_\psi^*(\omega) BB' M_\psi(\omega) \leq \tfrac{1}{2} I.$$

Then, for $\epsilon \in (0, \epsilon_3]$,

$$\bar{\mathcal{A}}_\psi^*(\omega)P + P\bar{\mathcal{A}}_\psi(\omega) \leq \begin{bmatrix} -\lambda\epsilon\cos(\psi - \omega\tau)BB' & 0 \\ 0 & -\frac{1}{2}I \end{bmatrix} \leq 0.$$

Since $\cos(\psi - \omega\tau) > 0$ for $\omega \in \omega \in \cup_{k=1}^r \mathcal{E}_k$ and (A, B) is controllable, we find that for $\epsilon \in (0, \epsilon_3]$, $\bar{\mathcal{A}}_\psi(\omega)$ is Hurwitz for $\omega \in \cup_{k=1}^r \mathcal{E}_k$. Let $\epsilon^* = \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. Therefore, (8.46) holds for $\epsilon \in (0, \epsilon^*]$. □

The following corollaries follow immediately:

Corollary 8.7. *For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents* (8.2) *with any communication topology belonging to $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A + A' = 0$ and A has all its eigenvalues at zero. In that case, Problem 8.2 is always solvable. Moreover, it can be solved by the consensus controller* (8.43)*. Specifically, for given γ, φ and $\bar{\tau}$, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.2) *with controller* (8.43) *achieve consensus for any communication topology in $\mathcal{G}_{0,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.*

Corollary 8.8. *For a given set of undirected communication topologies $\mathcal{G}_{0,\gamma,0}$ and $\bar{\tau} > 0$, consider the agents* (8.2) *with any communication topology belonging to $\mathcal{G}_{0,\gamma,0}$. Suppose $A + A' = 0$. In that case, Problem 8.2 is solvable if*

$$\bar{\tau} < \frac{\pi}{2\omega_{\max}}. \tag{8.47}$$

Moreover, it can be solved by the consensus controller (8.43) *if* (8.47) *holds. Specifically, for given γ and given φ and $\bar{\tau}$ satisfying* (8.47)*, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.2) *with controller* (8.43) *achieve consensus for any communication topology in $\mathcal{G}_{0,\gamma,0}$ and $\tau \in [0, \bar{\tau}]$.*

8.4.2 Consensus in Networks with Known Communication Topology

When the perfect information of communication topology is available, the design in preceding subsection can be modified to be topology-dependent, so that a less restrictive condition on $\bar{\tau}$

compared with Theorem 8.5 can be achieved.

Similar as in Section 8.3.2, for a given Laplacian matrix L and a positive real number $\beta > 0$, we can find a diagonal matrix D such that the eigenvalues of DL, denoted by $\lambda_i(DL)$, $i = 1, ..., N$, are real and satisfy

$$\lambda_1(DL) = 0, \quad \lambda_i(DL) > \beta, \quad i = 2, ..., N.$$

Let $\gamma > 0$ be such that $\lambda_i(DL) < \gamma$, $i = 2, ..., N$. For each agent i, we apply a controller

$$\begin{cases} \dot{\chi}^i(t) = (A + KC)\chi^i(t) - Kz^i(t), \\ u^i(t) = \frac{d_i}{\beta} B' P_\epsilon \chi^i(t), \end{cases} \tag{8.48}$$

where d_i is the ith diagonal element of D and P_ϵ is the positive definite solution of (9.17). We can prove the following theorem:t

Theorem 8.7. *For given $\bar{\tau} > 0$, consider a network of agents* (8.2) *with a known communication topology defined by matrix L. Problem 8.4 is solvable if,*

$$\bar{\tau} < \frac{\pi}{3\omega_{\max}}. \tag{8.49}$$

Moreover, it can be solved by the consensus controller (8.48) *if* (8.49) *holds. Specifically, for a given L and a $\bar{\tau} \geq 0$ satisfying* (8.49)*, there exists an ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, the agents* (8.2) *with controller* (8.48) *achieve consensus in the network.*

8.4.2.1 Neutrally stable agents

Consider the agents (8.2) with a known communication topology defined by matrix L. Suppose $A' + A = 0$. For the given L and any $\beta > 0$, choose the same $D = \text{diag}\{d\}$ as in the preceding subsection so that DL only has real eigenvalues and

$$\lambda_1(DL) = 0, \quad \lambda_i(DL) > \beta, \quad i = 2, ..., N.$$

The consensus controller for each agent can be designed as

$$\begin{cases} \dot{\chi}^i(t) = (A + KC)\chi^i(t) - Kz^i(t), \\ u^i(t) = d_i \epsilon B' \chi^i(t). \end{cases} \tag{8.50}$$

Theorem 8.8. *Consider a network of agents* (8.2)*. Suppose $A' + A = 0$ and the communication topology G defined by matrix L is known. Problem 8.4 is solvable with consensus controller* (8.50) *if $\bar{\tau} < \frac{\pi}{2\omega_{\max}}$. Specifically, for the given L, there exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$,* (8.2) *with controller* (8.50) *achieve consensus for any $\tau \in [0, \bar{\tau}]$.*

8.5 Illustrative Examples

8.5.1 Consensus with Full-state Coupling with a Set of Communication Topologies

Consider the 4 identical agents

$$\begin{cases} \dot{x}^i(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x^i(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u^i(t), \; i = 1, ..., 4, \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} x^j(t - \tau), \end{cases} \tag{8.51}$$

with full-state coupling as given in (8.1). The communication topologies defined by $L = \{\ell_{ij}\}$ belong to the set $\mathcal{G}_{3,5,\pi/6}$. We have $\omega_{max} = 1$ in this case. For this given set of data, we can choose the parameter ϵ to be 2×10^{-4} and design a consensus controller according to (8.4) and (8.5)

$$u^i = \frac{1}{\beta} F_\epsilon z^i = \begin{bmatrix} 0.0045 & 0.0540 & -0.0536 & 0.0091 \\ 0.0015 & 0.0091 & -0.0087 & 0.0155 \end{bmatrix} z^i.$$

We apply this u^i to two networks in the set $\mathcal{G}_{3,5,\pi/6}$ as shown in Fig. 8.1 and 8.2. The communication delay in these two networks is $\tau = 0.5$. The corresponding simulation results are shown in Fig 8.3 and Fig 8.4.

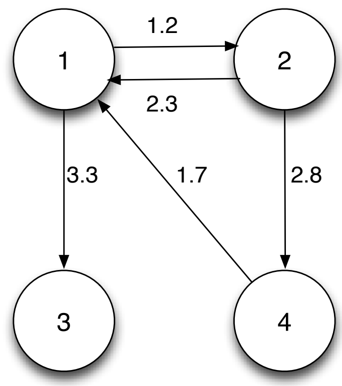

Figure 8.1: Network 1

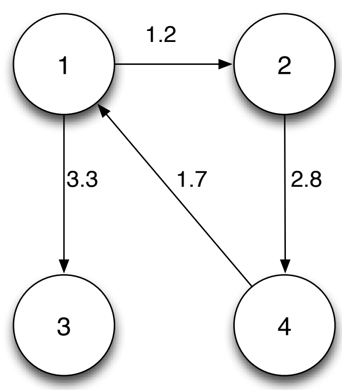

Figure 8.2: Network 2

8.5.2 Consensus with Full-state Coupling with a Known Communication Topology

Consider a network of 4 identical agents given by (8.51). The network topology is described by a weighted and direct graph shown in Fig. 8.5. The delay in this network is $\tau = 1$. Note that for

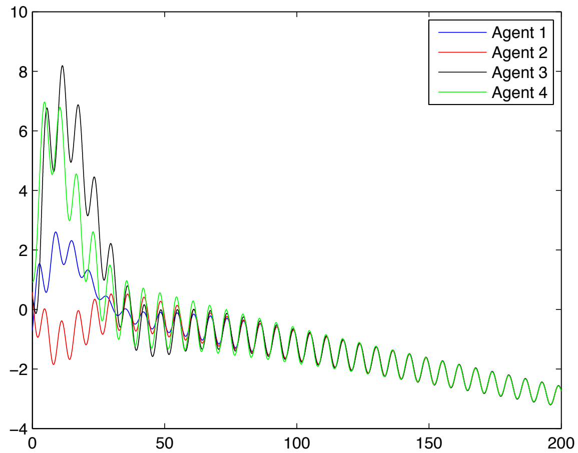

Figure 8.3: Evolution of the first state element of all four agents in Network 1

this network, we have $\varphi = 0.4086$. It is easy to see that the condition in Theorem 8.1 does not hold.

However, since the network topology is known, we can choose $\beta = 1$ and a diagonal matrix

$$D = \text{diag}\{[1, 1, 1, 0.01]\}$$

such that DL only has real eigenvalues and the non-zero eigenvalues of DL are greater than β. Then for this D, we get $\gamma = 4$. The parameter ϵ is chosen to be 10^{-3}. We can construct the following consensus controller for Network 3:

$$u^i = \frac{d_i}{\beta} F_\epsilon z^i = d_i \begin{bmatrix} 0.0281 & 0.2319 & -0.2262 & 0.0587 \\ 0.0145 & 0.0587 & -0.0512 & 0.1120 \end{bmatrix} z^i.$$

The simulation result is given in Fig 8.6.

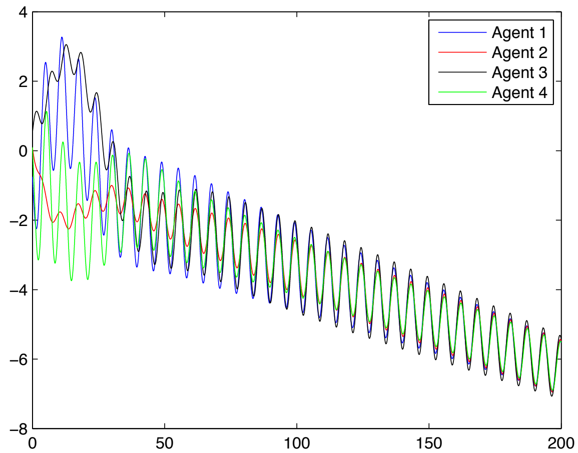

Figure 8.4: Evolution of the first state element of all four agents in Network 2

8.5.3 Consensus with Partial-state Coupling with a Set of Communication Topologies

Consider a network of 4 identical agents

$$\begin{cases} \dot{x}^i(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x^i(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u^i(t), \\ y^i(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x^i(t), \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} y^j(t-\tau), \end{cases} \tag{8.52}$$

with $\tau = 0.5$ and the same set of communication topology $\mathcal{G}_{3,5,\pi/6}$ as in Section 8.5.1.

We can choose $\epsilon = 10^{-6}$ and

$$K = \begin{bmatrix} -10 & -29 & -5 & -20 \end{bmatrix}', \tag{8.53}$$

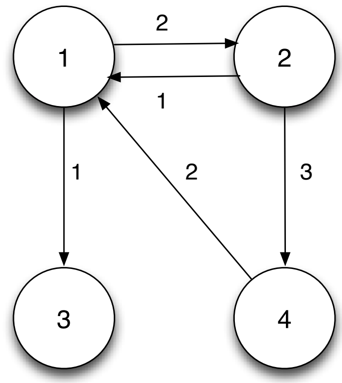

Figure 8.5: Network 3

and design the dynamic low-gain consensus controller as follows:

$$\begin{cases} \dot{\chi}^i(t) = \begin{bmatrix} -10 & 1 & 1 & 0 \\ -29 & 0 & 0 & 1 \\ -5 & 0 & 0 & 1 \\ -20 & 0 & -1 & 0 \end{bmatrix} \chi^i(t) + \begin{bmatrix} -10 \\ -29 \\ -5 \\ -20 \end{bmatrix} z^i(t) \\ u^i(t) = \begin{bmatrix} 0.0003 & 0.0149 & -0.0149 & 0.0007 \\ 0.0000 & 0.0007 & -0.0007 & 0.0009 \end{bmatrix} \chi^i(t). \end{cases}$$

Consider the same two communication topologies depicted in Fig. 8.1 and 8.2. The respective simulation results are shown in Fig. 8.7 and Fig. 8.8.

8.5.4 Consensus with Partial-state Coupling with a Known Communication Topology

Consider the agents (8.52) with $\tau = 1$ and the communication topology is given in Fig. 8.5. Let $\beta = 1$. Following the same procedure as in Section 8.5.2, we can design

$$D = \text{diag}\{1, 1, 1, 0.01\} \tag{8.54}$$

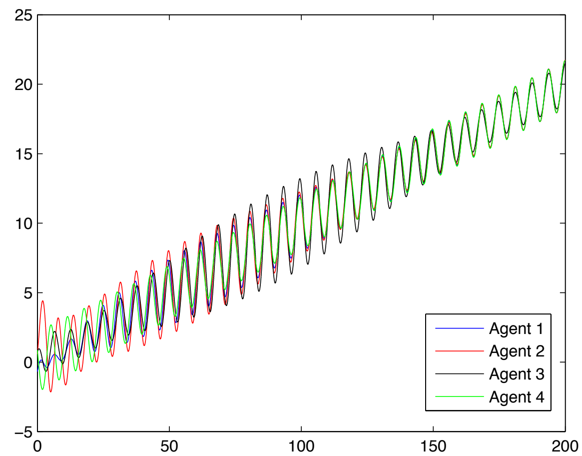

Figure 8.6: Evolution of x_1^i

such that DL has real eigenvalues and non-zero eigenvalues of DL are greater than β. In this case, $\gamma = 4$. We choose $\epsilon = 10^{-4}$ and K the same as in (8.53). The dynamic low-gain consensus controller is

$$\begin{cases} \dot{\chi}^i(t) = \begin{bmatrix} -10 & 1 & 1 & 0 \\ -29 & 0 & 0 & 1 \\ -5 & 0 & 0 & 1 \\ -20 & 0 & -1 & 0 \end{bmatrix} \chi^i(t) + \begin{bmatrix} -10 \\ -29 \\ -5 \\ -20 \end{bmatrix} z^i(t), \\ u^i(t) = d_i \begin{bmatrix} 0.0096 & 0.1377 & -0.1371 & 0.0195 \\ 0.0027 & 0.0195 & -0.0189 & 0.0319 \end{bmatrix} \chi^i(t), \end{cases}$$

where d_i is given by (8.54). The simulation result is shown in Fig. 8.9.

8.6 Conclusion

In this chapter, we study the multi-agent consensus with uniform constant communication delay for agents with high-order dynamics. A sufficient condition on delay is derived under which the

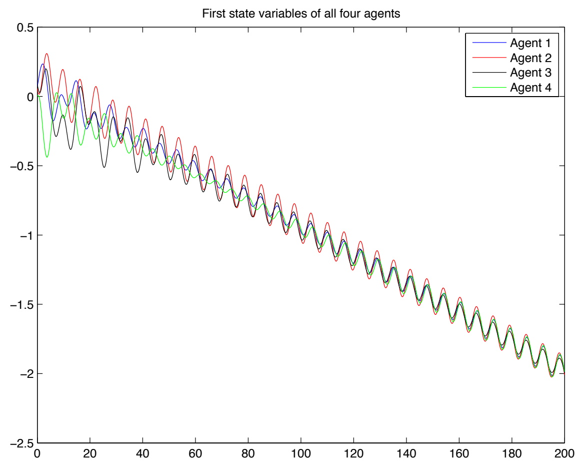

Figure 8.7: Evolution of x_1^i

multi-agent consensus is attainable. Whenever this condition is satisfied a consensus controller without the exact knowledge of network topology can be constructed such that consensus can be achieved in a set of network. Furthermore, a larger delay tolerance is possible if the topology information is available in the controller design.

We consider identical agents with uniform constant delay in this chapter. Future research will continuous in two directions: 1. extend the results to non-identical agents; 2. consider non-uniform and time-varying delay.

8.A A Useful Lemma

Lemma 8.9. *There exists* ϵ^* *such that for* $\epsilon \in (0, \epsilon^*]$,

$$\|G_{\varepsilon,\lambda}(j\omega)\| \leq \tfrac{1}{3},\ \forall \omega \in \Omega := \mathbb{R} \setminus \cup_{k=1}^{r} \mathcal{E}_k,\ \lambda \in (\beta, \gamma).$$

Proof. By definition, $\det(j\omega I - A) \neq 0$ for all $\omega \in \Omega$. There exists a μ such that

$$\underline{\sigma}(j\omega I - A) > \mu,\ \forall \omega \in \Omega.$$

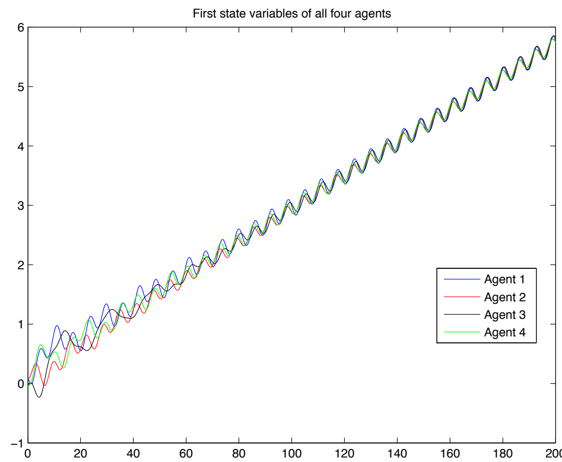

Figure 8.8: Evolution of x_1^i

After all assume this is not the case. Then there exists a sequence $\omega^i \in \Omega$ such that

$$\underline{\sigma}(j\omega^i I - A) \to 0$$

as $i \to \infty$. We can ensure that this sequence ω^i is bounded since for ω satisfying $|\omega| > \|A\| + 1$ we have:

$$\underline{\sigma}(j\omega I - A) > |\omega| - \|A\| > 1.$$

But it follows from the Bolzano-Weierstrass theorem that a bounded sequence ω^i has a convergent subsequence whose limit, denoted by $\bar{\omega}$, is in Ω (since Ω is closed). The limit $\bar{\omega}$ would have the property

$$\underline{\sigma}(j\bar{\omega} I - A) = 0.$$

This implies $\bar{\omega}$ is an eigenvalue of A which is in contradiction with definition of Ω. Choose ϵ^* such that for any $\epsilon \in (0, \epsilon^*]$, $\|F_{\varepsilon,\lambda}\| \leq \frac{\mu}{4}\|B\|^{-1}$ for any $\lambda \in (\beta, \gamma)$. In that case:

$$\underline{\sigma}(j\omega I - A - BF_{\varepsilon,\lambda}) > \mu - \|B\|\|F_{\varepsilon,\lambda}\| > \tfrac{3\mu}{4},\ \forall \omega \in \Omega,\ \lambda \in (\beta, \gamma),$$

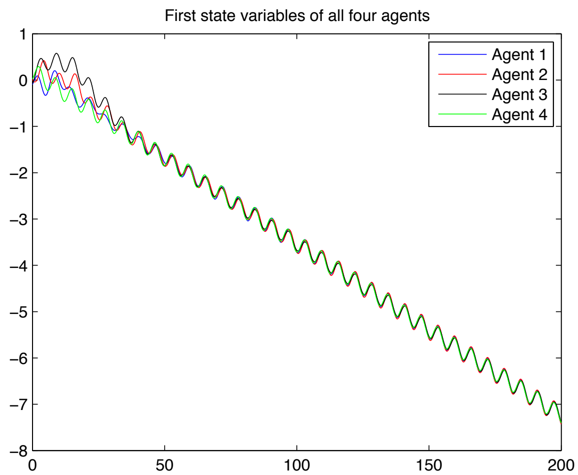

Figure 8.9: Evolution of x_1^i

and hence

$$\|(j\omega I - A - BF_{\varepsilon,\lambda})^{-1}\| < \tfrac{4}{3\mu}, \forall \omega \in \Omega,\ \lambda \in (\beta, \gamma),$$

but then

$$\|F_{\varepsilon,\lambda}(j\omega I - A - BF_{\varepsilon,\lambda})^{-1}B\| \leq \|F_{\varepsilon,\lambda}\|\|(j\omega I - A - BF_{\varepsilon,\lambda})^{-1}\|\|B\| \leq \tfrac{1}{3}$$

for all $\omega \in \Omega$ and $\lambda \in (\beta, \gamma)$. $\square$

Chapter 9

Synchronization for Heterogeneous Networks of Introspective Right-invertible Agents with Uniform Constant Communication Delay

9.1 Introduction

The synchronization analysis and design in networks has received substantial attention from researchers in recognition of its wide applications in a variety of areas (see [44,55,100] and references therein).

The study on state synchronization in homogeneous network, that is–network consists of identical agents, has been quite fruitful. Depending on what information the agents collect from the network, synchronization in homogeneous networks can be classified into two categories. In some networks, each agent measures its own state relative to that of neighbors, which is referred to as *full-state coupling* [45, 46, 51, 52, 54, 83]; In other networks, the agents may collect information of its output relative to that of its neighboring agents, which we refer to as *partial-state coupling* [32, 48, 73, 84].

In contrast to the flourishing research on synchronization in homogeneous networks, relatively limited results have been obtained for *heterogeneous networks*, as we refer to networks comprising non-identical agents. For heterogeneous networks, the notion of state synchronization may no longer make sense as each agent possesses a set of state information which may be inherently different from others. In this case, it is more natural to study an alternative problem of *output synchronization*, that is, all the agents should agree on a set of pre-selected outputs (see, for example, [10, 25, 104]). In this body of work, it is commonly assumed that each agent has a local measurement of its own states, which we refer to as introspective agents. The synchronization problem is more difficult for non-introspective agents, yet effort has already been made in this direction [17, 113].

Due to the ubiquity of communication delay during the transmission of information, the research has also been directed to synchronization in networks with time delay. Most work assumes the agents as described by first order [5,33,43,46,74,80] or second order dynamics [21,34,41,47,81,110]. Single-input and single-output agents are considered in [42]. The authors also previously study multi-input multi-output agents that are at most critically unstable in [94]. Both time and frequency domain approaches have been utilized. In the time-domain, the design and analysis are usually based on the Lyapunov-Krasovskii or Lyapunov-Razumikhin function [22, 33, 41, 47, 74]. In the frequency domain, they rely on the Nyquist criteria or small gain theorem [7, 34, 41, 42, 80, 81, 94]. Despite aforementioned advances, this research is still largely situated in a limited framework – that is, homogeneous networks of simple agents mostly with first order or second order dynamics.

To the best of the authors' knowledge, the results that explicitly consider heterogeneous networks of higher-order agents and time-delay are [29, 30]. The single-output synchronization is studied in [29]. A frequency-domain approach based on Geršgorin's theorem and spectral radius stability theorem is proposed to design a decentralized linear consensus controller. However, the consensus condition obtained in [29] is very conservative (see [80]). [30] studies single-input single-output agents and undirected communication topologies. A consensus condition is derived based on the notion of *S-hull*.

9.1.1 Contribution

In this chapter, we consider heterogeneous networks of introspective multi-input multi-output agents with uniform constant communication delay. We assume that the agents are right-invertible. Two problems are studied in this chapter, namely the output synchronization problem and output regulation problem. The underlying idea is to shape the agent dynamics into a particular form by exploiting the self-knowledge and right-invertibility property of the agents. Specifically, in the output synchronization problem, the agents are manipulated to imitate a neutrally stable system and as such can tolerate *arbitrary* bounded delay and accommodate more network uncertainties under standard assumption on the communication topology. However, when one is more concerned with the synchronization trajectories as in the output regulation problem, we can re-shape the agent to be the same with exo-system and regulate the agents' outputs by providing relative output measurement of the exo-system only to one particular agent. Moreover, we propose a *decentralized* controller design methodology that does not require exact knowledge of communication topologies so that these two problems can be solved for a set of unknown networks. Finally, we show that the design proposed in this chapter also applies to a formation control problem.

9.1.2 Notations and Preliminaries

The following notations will be used in this chapter. For column vectors $x_1, \ldots, x_n$, the stacking column vector of $x_1, \ldots, x_n$ is denoted by $[x_1; \ldots; x_n]$. For arbitrary matrix $X \in \mathbb{C}^{n\times m}$, X', X^{-1} and $\|X\|$ denote respectively the transpose, inverse and induced 2-norm of X. For square matrix $X \in \mathbb{C}^{n\times n}$, $\det(X)$ and $\lambda(A)$ represent its determinant and eigenvalue.

For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{p\times q}$, the Kronecker product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \vdots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$$

The following property of the Kronecker product will be used in this chapter:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

A graph G is defined by a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, N\}$ is a vertex set and $\mathcal{E}$ is a set of pairs of vertices (i, j). Each pair in $\mathcal{E}$ is called an *arc*. G is *undirected* if $(i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$. Otherwise, G is *directed*. A *directed path* from vertex i_1 to i_k is a sequence of vertices $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, ..., k-1$. A directed graph G contains a *directed spanning tree* if there is a node r such that a directed path exists between r and every other node.

The graph G is *weighted* if each arc (i, j) is assigned with a real number a_{ij}. For a weighted graph G, a matrix $L = \{\ell_{ij}\}$ with

$$\ell_{ij} = \begin{cases} \sum_{j=1}^{N} a_{ij}, & i = j \\ -a_{ij}, & i \neq j, \end{cases}$$

is called *Laplacian matrix* associated with graph G. In the case where G has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ (see e.g. [63]). If G has a directed spanning tree, L has a simple eigenvalue at zero and all the other eigenvalues have strictly positive real parts (see e.g. [53]).

9.2 Problem Formulation

Consider a heterogeneous network of N introspective agents

$$\begin{cases} \dot{x}^i(t) = A^i x^i(t) + B^i u^i(t), \\ y^i(t) = C_y^i x^i(t), \\ z^i(t) = C_z^i x^i(t), \\ \zeta^i(t) = \sum_{j=1}^N \ell_{ij} y^j(t-\tau), \end{cases} \tag{9.1}$$

where $x^i \in \mathbb{R}^{n_i}$, $u^i \in \mathbb{R}^{m_i}$, $y^i, \zeta^i \in \mathbb{R}^p$, $z^i \in \mathbb{R}^{q_i}$ and $\tau > 0$ is an unknown constant satisfying $\tau \in [0, \bar{\tau}]$. The coefficients ℓ_{ij} are such that $\ell_{ij} \leq 0$ for $i \neq j$ and $\ell_{ii} = -\sum_{j \neq i}^N \ell_{ij}$.

The matrix $L = \{\ell_{ij}\} \in \mathbb{R}^{N \times N}$ defines the *communication topology* which can be captured by a weighted graph $G = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ where $(j, i) \in \mathcal{E} \Leftrightarrow \ell_{ij} < 0$ and $a_{ii} = 0$ and $a_{ij} = -\ell_{ij}$ for $i \neq j$.

Assumption 9.1. The communication topology G contains a directed spanning tree whose root (without loss of generality) is agent N.

In this case, L has a simple eigenvalue at zero and the rest are located in the open right half plane. Let $\lambda_1, \cdots, \lambda_N$ denote the eigenvalues of L and assume $\lambda_1 = 0$. When the perfect information of the communication topology is not available, we can use the non-zero eigenvalues of L as a rough "metric" of the graph and introduce the following definition to characterize a set of unknown communication topologies.

Definition 9.1. *For any $\gamma \geq \beta > 0$ and $\frac{\pi}{2} > \varphi \geq 0$, $\mathcal{G}_{\beta,\gamma,\varphi}$ is the set of networks whose Laplacian* *eigenvalues satisfy that*

$$|\lambda_i| \in (\beta, \gamma),\ \arg \lambda_i \in [-\varphi, \varphi] \ \ \textit{for } i = 2, \dots, N.$$

In this network, each agent collects two measurements:

1) a network measurement $\zeta^i \in \mathbb{R}^p$ which is a combination of its own output relative to that of neighboring agents and is subject to a uniform constant communication delay;

2) a local measurement $z^i \in \mathbb{R}^{q_i}$ of its internal dynamics to which the agent has an instantaneous access.

Assumption 9.2. *The agents satisfy the following properties:*

1) (A^i, B^i) *is stabilizable;*

2) (A^i, C_y^i) *is detectable;*

3) (A^i, B^i, C_y^i) *is right-invertible;*

4) (A^i, C_z^i) *is detectable.*

The output synchronization in a heterogeneous network of agents (9.1) can be defined as follows:

Definition 9.2. *The agents in the network achieve output synchronization if*

$$\lim_{t\to\infty} (y^i(t) - y^j(t)) = 0, \quad \forall i, j \in \{1, \dots, N\}.$$

With the above defined notations, the first problem studied in this chapter is formally stated below:

Problem 9.1. *Consider a heterogeneous network of the form* (9.1). *For a given set* $\mathcal{G}_{\beta,\gamma,\varphi}$ *and* $\bar{\tau} \geq 0$, *the output synchronization problem with a set of communication topologies* $\mathcal{G}_{\beta,\gamma,\varphi}$ *for all*

$\tau \leq \bar{\tau}$ is to design a local linear dynamical controller

$$\begin{cases} \dot{\chi}^i = A_c^i \chi^i + B_c^i \zeta^i + E_c^i z^i, \\ u^i = C_c^i \chi^i + D_c^i \zeta^i + M_c^i z^i, \end{cases} \tag{9.2}$$

such that the synchronization can be achieved in the network with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.

Note that the above synchronization problem does not impose any restriction on the synchronization trajectories. The focus here is to solve this problem for as a large set of communication topologies and delay as possible. On the other hand, it is important in some scenario to regulate the output of the agents to desired trajectories when the output synchronization is reached. Let an exo-system be given as

$$\begin{cases} \dot{x}_r = A_r x_r, \quad x_r(0) = x_r, \\ y_r = C_r x_r, \end{cases} \tag{9.3}$$

where A_r has all its eigenvalues in the closed left half complex plane and (A_r, C_r) is observable. We want to regulate each agent's output to y_r. It is reasonable to assume that the synchronization trajectories are not exponentially increasing. In this case, we assume *the root* of network also measures its own output relative to y_r of the exo-system. To be precise, the root agent, which is the agent N, takes the following form:

$$\begin{cases} \dot{x}^N = A^N x^N(t) + B^N u^N(t), \\ z^N = C_z^N x^N(t), \\ y^N = C_y^N x^N(t), \\ \zeta^N = \sum_{j=1}^N \ell_{Nj} y^j(t-\tau) + \delta \left[y^N(t-\tau) - y_r(t-\tau) \right], \end{cases} \tag{9.4}$$

with $\delta > 0$.

Definition 9.3. *The agents in the network achieve output regulation if*

$$\lim_{t\to\infty}(y^i(t) - y_r(t)) = 0, \quad \forall i \in \{1, \dots, N\}.$$

We can formulate the regulation problem as follows:

Problem 9.2. *Consider a heterogeneous network of the form* (9.1)*. For a given exo-system* (9.3)*, a set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, the output regulation problem with exo-system* (9.3) *and a set of communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ for all $\tau \leq \bar{\tau}$ is to design a local linear dynamical controller* (9.2) *such that the output regulation can be achieved in the network with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.*

9.3 Main Result

The first main result of this chapter is stated in the following theorem:

Theorem 9.1. *For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, the Problem 9.1 is always solvable via a decentralized dynamic consensus controller* (9.2)*.*

Before we present the result for output regulation problem, some preparatory work needs to be done. For any communication topology G, an augmented graph $\bar{G}$ can be defined by including the exo-system denoted by e and an arc (e, N) with weight δ into the topology. The Laplacian associated with $\bar{G}$ is

$$\bar{L} = \{\bar{\ell}_{ij}\} = \begin{bmatrix} \ell_{11} & \ell_{12} & \cdots & \ell_{1N} & 0 \\ \ell_{21} & \ell_{22} & \cdots & \ell_{2N} & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 \\ \ell_{N1} & \ell_{N2} & \cdots & \ell_{NN}+\delta & -\delta \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{9.5}$$

whose eigenvalues are denoted by $\bar{\lambda}_i$, $i = 1, ..., N+1$ with $\bar{\lambda}_1 = 0$. Obviously, this $\bar{G}$ also has a directed spanning tree and thus $\bar{\lambda}_i$, $i = 2, ..., N+1$ are in the open right half plane. For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$, the set of augmented topologies can be denoted by $\bar{\mathcal{G}}_{\bar{\alpha},\bar{\beta},\bar{\varphi}}$ such that for any $\bar{G} \in \bar{\mathcal{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$,

$$|\bar{\lambda}_i| \in (\bar{\beta}, \bar{\gamma}),\ \arg(\bar{\lambda}_i) \in [-\bar{\varphi}, \bar{\varphi}],\ i = 2, ..., N.$$

We have the following theorem:

Theorem 9.2. *For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, the Problem 9.2 is solvable via a decentralized dynamic consensus controller* (9.2) *if the set of augmented topologies $\bar{\mathcal{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ satisfies:*

1) $\bar{\varphi} < \frac{\pi}{3}$;

2) $\bar{\tau} < \frac{\frac{\pi}{3} - \bar{\varphi}}{\omega_{\max}}$,

where $\omega_{\max} = \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A_r) = 0\}$.

We shall prove Theorem 9.1 and 9.2 by explicitly constructing a synchronization or regulation controller in the form of (9.2) via a progressive design approach. First, we design a local pre-compensator to make the agents quasi-identical to a new common model, which we refer to as *homogenization of network*; Next, we show that for this new network, both problems can be reduced to a robust stabilization problem. Finally, we shall design a controller that solves the reformulated stabilization problem so that synchronization or output regulation can be achieved in the homogenized network.

9.3.1 Homogenization of the Network

For introspective agents, their self-reflection of internal dynamics provides us with additional freedom to manipulate the agent models so as to disguise them as being almost identical to the rest of the network viewed from their output. This is shown in the next lemma.

Lemma 9.1. *Consider a heterogeneous network of the form* (9.1) *with communication topologies given by* $\mathcal{G}_{\beta,\gamma,\varphi}$ *and communication delay* $\tau \leq \bar{\tau}$. *Let* n_d *denote the maximum order of infinite zeros of* (A^i, B^i, C^i). *Suppose a triple* (A, B, C) *is given such that*

1) $\operatorname{rank}(C) = p$.

2) (A, B, C) *is invertible, of uniform rank* $n_q \geq n_d$ *and has no invariant zero.*

There exists a compensator

$$\begin{cases} \dot{\xi}^i(t) = A_H^i \xi^i(t) + B_H^i z^i(t) + E_H^i v^i(t), \\ u^i(t) = C_H^i \xi^i(t) + D_H^i v^i(t), \end{cases} \tag{9.6}$$

such that the closed-loop system of (9.1) *and* (9.6) *can be written in the following form:*

$$\begin{cases} \dot{\bar{x}}^i(t) = A\bar{x}^i(t) + B\left(v^i(t) + d^i(t)\right), \\ y^i(t) = C\bar{x}^i(t), \\ \zeta^i(t) = \sum_{j=1}^N \ell_{ij} y^j(t-\tau), \end{cases} \tag{9.7}$$

where d^i *are generated by*

$$\begin{cases} \dot{\omega}^i(t) = A_s^i \omega^i(t), \quad i = 1, ..., N, \\ d^i(t) = C_s^i \omega^i(t), \end{cases} \tag{9.8}$$

and A_s^i *are Hurwitz stable.*

Proof. See [104]. □

Remark 9.1. *Lemma 1 shows that we can design a compensator* (9.6) *to make the agent identical to a new common model characterized by* a priori *given triple* (A, B, C) *except for an exponentially decaying exogenous signal injected in the range space of* B. *Moreover, we have a complete freedom to choose the modes of* A *which is fundamental in proving Theorems 9.1 and 9.2.*

The resulting network (9.7) can be viewed as a homogeneous network affected by the exponentially decaying disturbances d_i generated by (9.8). The injection of such exponentially decaying d^i turns out to be irrelevant and the output synchronization problem in the original heterogeneous network of agents (9.1) can be reduced to the output synchronization problem in a homogeneous network with the same communication topology.

9.3.2 Synchronization in Homogeneous Networks

Next, we consider the synchronization problem for the agents (9.7) as formulated in Problem 9.1. We can choose in Lemma 9.1 the triple (A, B, C) satisfying additional properties

$$A + A' = 0, \quad |\lambda(A)| < \frac{\frac{\pi}{2} - \varphi}{\bar{\tau}}. \tag{9.9}$$

Such a triple (A, B, C) always exists and in fact can be chosen in the following form:

$$A = \Gamma(A_0 + B_0 H)\Gamma^{-1}, \quad B = \Gamma B_0, \quad C = C_0\Gamma^{-1},$$

and

$$A_0 = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_0 = \begin{bmatrix} I_p & 0 \end{bmatrix},$$

where H is such that $A_0 + B_0 H$ only has semi-simple eigenvalues on the imaginary axis satisfying (9.9). H exists due to the fact that (A_0, B_0) is controllable. Then a transformation Γ can be found such that $\Gamma(A_0 + B_0 H)\Gamma^{-1}$ is in the real Jordan canonical form and thus $A + A' = 0$.

For the above (A, B, C), a low-gain compensator can be constructed as

$$\begin{cases} \dot{\chi}^i(t) = (A + KC)\chi^i(t) - K\zeta^i(t), \\ v^i(t) = -\epsilon B'\chi^i(t), \end{cases} \tag{9.10}$$

where K is such that $A + KC$ is Hurwitz stable. The existence of K is due to the fact that (A, C) is observable.

Define $\tilde{x}^i = [\bar{x}^i; \chi^i]$. Then for each agent, the closed-loop dynamics of (9.7) and (9.10) are

$$\begin{cases} \dot{\tilde{x}}^i(t) = \bar{A}\tilde{x}^i(t) + \bar{B}\zeta^i(t) + \bar{E}d^i(t), \\ y^i(t) = \bar{C}\tilde{x}^i(t), \\ \zeta^i(t) = \sum_{j\in\mathcal{N}} \ell_{ij} y^j(t-\tau), \end{cases}$$

where

$$\bar{A} = \begin{bmatrix} A & -\epsilon BB' \\ 0 & A+KC \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ -K \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{9.11}$$

Define $\tilde{x} = [\tilde{x}^1; \cdots; \tilde{x}^N]$ and $d = [d^1; \cdots; d^N]$. The overall dynamics of N agents can be written as

$$\dot{\tilde{x}}(t) = (I_N \otimes \bar{A})\tilde{x}(t) + (L \otimes \bar{B}\bar{C})\tilde{x}(t-\tau) + (I_N \otimes \bar{E})d.$$

Let T be a non-singular matrix such that $J = TLT^{-1}$ is in the Jordan Canonical Form with $J(1,1) = \lambda_1 = 0$ and $\eta = [\eta^1; \cdots; \eta^N] = (T \otimes I_n)\tilde{x}$ where n is the dimension of A. The dynamics of η are governed by

$$\dot{\eta}(t) = (I_N \otimes \bar{A})\eta(t) + (J \otimes \bar{B}\bar{C})\eta(t-\tau) + (T \otimes \bar{E})d.$$

Lemma 9.2. *The interconnections of* (9.7) *and* (9.10) *reach output synchronization if* $\eta^i \to 0$ *as* $t \to \infty$ *for* $i = 2, ..., N$.

Proof. Let $\pi(t) = [\eta^1(t); 0; \cdots; 0]$. If $\eta(t) \to \pi(t)$, then $\tilde{x}(t) \to (T^{-1} \otimes I_n)\pi(t)$ where n is the dimension of A. Note that the columns of T^{-1} comprise all the right eigenvectors and generalized eigenvectors of L. The first column of T^{-1} is vector $\mathbf{1}$. This implies that for $i = 1, ..., N$

$$\tilde{x}^i(t) \to \eta^1(t).$$

□

Define $\bar{\eta} = [\eta^2; \cdots; \eta^N]$ and take the dynamics of d into account. We will get

$$\begin{bmatrix} \dot{\bar{\eta}}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \bar{A} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} \begin{bmatrix} \bar{\eta}(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} \bar{J} \otimes \bar{B}\bar{C} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\eta}(t-\tau) \\ \omega(t-\tau) \end{bmatrix}, \tag{9.12}$$

where $\omega = [\omega_1; \ldots; \omega_N]$, $\bar{C}_s = \text{blkdiag}\{C_s^i\}_{i=1}^N$, $\bar{I} = [0, I_{N-1}]$ and $\bar{A}_s = \text{blkdiag}\{A_s^i\}_{i=1}^N$ is Hurwitz. Clearly $\bar{\eta} \to 0$ for any initial condition if the system (9.12) is globally asymptotically stable. Note that the system (9.12) is globally asymptotically stable if and only if

$$\det \left[sI - \begin{bmatrix} I_{N-1} \otimes \bar{A} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} - \begin{bmatrix} \bar{J} \otimes \bar{B}\bar{C} & 0 \\ 0 & 0 \end{bmatrix} e^{-s\tau} \right] \neq 0, \quad \forall s \in \mathbb{C}^+. \tag{9.13}$$

Due to the upper triangular structure of both matrices in (9.13) and the fact that $\bar{A}_s$ is Hurwitz, it is easy to see that (9.13) holds if and only if

$$\det \left[sI - (I_{N-1} \otimes \bar{A}) - (\bar{J} \otimes \bar{B}\bar{C}) e^{-s\tau} \right] \neq 0, \quad \forall s \in \mathbb{C}^+. \tag{9.14}$$

Therefore, we have the following lemma.

Lemma 9.3. *The interconnections of agents* (9.7) *and* (9.10) *achieve output synchronization if the system*

$$\dot{\tilde{\eta}}(t) = \bar{A}\tilde{\eta}(t) + \lambda \bar{B}\bar{C}\tilde{\eta}(t - \tau), \tag{9.15}$$

is globally asymptotically stable for $|\lambda| \in (0, \gamma)$, $\arg \lambda \in [-\varphi, \varphi]$ *and* $\tau \in [0, \bar{\tau}]$.

The next lemma is proved in [94].

Lemma 9.4. *Let* $\bar{A}$, $\bar{B}$ *and* $\bar{C}$ *be given in* (9.11). *For* $\gamma \geq \beta > 0, \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $\bar{\tau} > 0$, *there exists an* ϵ^* *such that for* $\epsilon \in (0, \epsilon^*]$, *the systems* (9.15) *are globally asymptotically stable for* $|\lambda| \in (0, \gamma)$, $\arg \lambda \in [-\varphi, \varphi]$ *and* $\tau \in [0, \bar{\tau}]$.

Proof of Theorem 9.1. For given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, it follows from Lemma 9.1, 9.3 and 9.4 that there exists an ϵ^* such that for $\epsilon \in (0, \epsilon^*]$, the composition of (9.6) and (9.10) will solve Problem 1. $\square$

9.3.3 Output Regulation in Homogeneous Networks

Now we consider the output regulation problem. It is shown in the Appendix that without loss of generality, we can always manipulate the internal dynamics of exo-system (9.3) and find a matrix B_r such that (A_r, B_r, C_r) is invertible, of uniform rank $n_q > n_d$ and has no invariant zero. Therefore, according to Lemma 9.1, there exists a pre-compensator (9.6) such that the interconnection of (9.6) and agent (9.1) can be written in the form of (9.7) with A, B and C replaced by A_r, B_r and C_r.

Next, we design a controller for the homogenized network (9.7). By the definition of $\bar{L}$ in (9.5), we can define an augmented homogenized network by including the exo-system into (9.7) as follows:

$$\begin{cases} \dot{\bar{x}}^i(t) = A_r\bar{x}^i(t) + B_r\left(v^i(t) + d^i(t)\right), i = 1, ..., N+1 \\ y^i(t) = C_r\bar{x}^i(t), \\ \zeta^i(t) = \sum_{j=1}^{N+1} \bar{\ell}_{ij} y^j(t-\tau), \end{cases} \tag{9.16}$$

where agent $N+1$ is the exo-system and $d_{N+1}(t) = 0$. We can not control the exo-system, i.e. $v^{N+1}(t) = 0$. Obviously, the output regulation problem is solved if this augmented network reaches synchronization for any communication topology in $\bar{\mathcal{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ and $\tau \le \bar{\tau}$. We shall design a controller to achieve this goal.

For $\epsilon > 0$, let P_ϵ be the positive definition solution of Algebraic Riccati Equation (ARE)

$$A_r'P_\epsilon + P_\epsilon A_r - B_\epsilon B_r B_r' P_\epsilon + \epsilon I = 0, \tag{9.17}$$

and K be such that $A_r + KC_r$ is Hurwitz stable. A low-gain compensator can be constructed for agent $i = 1, ..., N$ as

$$\begin{cases} \dot{\chi}^i(t) = (A_r + KC_r)\chi^i(t) - K\zeta^i(t), i = 1, ..., N. \\ v^i(t) = -\frac{1}{\beta}B_r'P_\epsilon\chi^i(t). \end{cases} \tag{9.18}$$

We can imagine that (9.18) also apply to agent $N+1$ (exo-system) but with initial condition $\chi^{N+1}(0) = 0$. Since $\zeta^{N+1}(t) = 0$, we shall have $v^{N+1}(t) = 0$. In view of this, we can write the dynamics of the whole augmented network as

$$\dot{\tilde{x}}(t) = (I_N \otimes \bar{A})\tilde{x}(t) + (\bar{L} \otimes \bar{B}\bar{C})\tilde{x}(t-\tau) + (I_N \otimes \bar{E})d,$$

where

$$\bar{A} = \begin{bmatrix} A_r & -\frac{1}{\beta}B_rB_r'P_\epsilon \\ 0 & A_r + KC_r \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ -K \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_r & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} B_r \\ 0 \end{bmatrix}. \tag{9.19}$$

Similarly as in preceding subsection, we can prove

Lemma 9.5. *The interconnections of agents* (9.16) *and controller* (9.18) *achieve synchronization for any communication topology in* $\bar{\mathcal{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ *and* $\tau \leq \bar{\tau}$ *if the following system*

$$\dot{\tilde{\eta}}(t) = \bar{A}\tilde{\eta}(t) + \bar{\lambda}\bar{B}\bar{C}\tilde{\eta}(t-\tau) \tag{9.20}$$

is globally asymptotically stable for $|\bar{\lambda}| \in (\bar{\beta}, \bar{\gamma})$*,* $\arg(\bar{\lambda}) \in [-\bar{\varphi}, \bar{\varphi}]$ *and* $\tau \in [0, \bar{\tau}]$*, where* $\bar{A}$*,* $\bar{B}$ *and* $\bar{C}$ *is given by* (9.19)*.*

The next lemma is shown in [94]:

Lemma 9.6. *For a given set* $\bar{\mathcal{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ *and* $\bar{\tau} > 0$*, let the conditions in Theorem 9.2 be satisfied. There exists an* ϵ^* *such that for* $\epsilon \in (0, \epsilon^*]$*, the system* (9.20) *with* (9.19) *is globally asymptotically stable for* $|\bar{\lambda}| \in (\bar{\beta}, \bar{\gamma})$*,* $\arg(\bar{\lambda}) \in [-\bar{\varphi}, \bar{\varphi}]$ *and* $\tau \in [0, \bar{\tau}]$*.*

9.4 Application to Formation

In this section, we show that the design method presented in preceding sections is also applicable to formation problem.

Definition 9.4. *A* formation *is a family of vectors* $\{h_1, \dots, h_N\}$, $h_i \in \mathbb{R}^p$. *The agents are said to achieve output formation if*

$$\lim_{t\to\infty} [(y_i(t) - h_i) - (y_j(t) - h_j)] = 0.$$

Suppose a set of communication topologies $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$ are given. Let n_q be the maximum order of infinite zeros of all the agents. The controller design follows a similar procedure as in the synchronization problem. First, we design a pre-compensator (9.6) for each agent to homogenize the network utilizing its local measurement so that the agents are quasi-identical to a new common model characterized by a given trip (A, B, C) which satisfies the following conditions:

1. $\operatorname{rank}(C) = p$.
2. (A, B, C) is invertible, of uniform rank n_q and has no invariant zero,
3. $A + A' = 0$,
4. The eigenvalues of A satisfy
$$|\lambda(A)| < \frac{\frac{\pi}{2} - \varphi}{\bar{\tau}}.$$

Moreover, there exists a family of vectors $\{\bar{h}_1, \dots, \bar{h}_N\}$ of appropriate dimension such that for $i = 1, ..., N$,

5. $C\bar{h}_i = h_i$,
6. $A\bar{h}_i = 0$.

Remark 9.2. *For arbitrary given vectors* $\{h_1, \dots, h_N\}$, *such a triple* (A, B, C) *always exists. One particular choice which satisfies the above conditions is the following*

$$A = T(A_0 + B_0 H)T^{-1}, \quad B = TB_0, \quad C = C_0 T^{-1},$$

and

$$A_0 = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_0 = \begin{bmatrix} I_p & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & H_0 \end{bmatrix},$$

where H_0 *is such that*

$$\bar{A}_0 + \bar{B}_0 H_0 = \begin{bmatrix} 0 & I_{(n_q-2)p} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_p \end{bmatrix} H_0$$

is non-singular and only has semi-simple eigenvalues on the imaginary axis. H_0 *exists due to the fact that* $(\bar{A}_0, \bar{B}_0)$ *is controllable. It is easy to see that* $A_0 + B_0 H$ *has* $(n_q - 1)p$ *semi-simple non-zero eigenvalues on the imaginary axis and* p *semi-simple eigenvalues at zero. Then a transformation* T *can be found such that* $T(A_0 + B_0 H)T^{-1}$ *is in the real Jordan canonical form and thus* $A + A' = 0$. *For this triple* (A, B, C), *a family of vector* $\{\bar{h}_1, \ldots, \bar{h}_N\}$ *can be found as*

$$\bar{h}_i = T \begin{bmatrix} h_i \\ 0 \end{bmatrix}$$

so that

$$C\bar{h}_i = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} h_i \\ 0 \end{bmatrix} = h_i.$$

Next, a local formation controller can be designed as follows:

$$\begin{cases} \dot{\chi}^i(t) = (A + KC)\chi^i(t) - K\left[\sum_{j=1}^N \ell_{ij}(y_j(t-\tau) - h_j)\right], \\ v^i(t) = -\epsilon B'\chi^i(t). \end{cases} \tag{9.21}$$

We can prove the following result:

Theorem 9.3. *For a given set* $\mathcal{G}_{0,\gamma,\varphi}$, *a formation* $\{h_1, \ldots, h_N\}$ *and* $\bar{\tau} \geq 0$, *there exists* ϵ^* *such that for* $\epsilon \in (0, \epsilon^*]$, *the agents* (9.1) *with controller* (9.6) *and* (9.21) *achieve formation for any communication topology belonging to* $\mathcal{G}_{0,\gamma,\varphi}$ *and* $\tau \leq \bar{\tau}$.

Proof. It follows from Lemma 9.1 that the interconnection of the agents and (9.6) can be written in the following form:

$$\begin{cases} \dot{\bar{x}}^i(t) = A\bar{x}^i(t) + B\left(v^i(t) + d^i(t)\right), \\ y^i(t) = C\bar{x}^i(t). \end{cases} \tag{9.22}$$

Let $\bar{x}_s^i = \bar{x}^i - \bar{h}_i$. Then the closed-loop system of (9.22) and controller (9.21) can be written in terms of $\bar{x}_s^i$ and χ^i as

$$\begin{cases} \dot{\bar{x}}_s^i(t) = A\bar{x}_s^i(t) + B\left(v^i(t) + d^i(t)\right) + A\bar{h}_i, \\ \chi^i(t) = (A + KC)\chi^i(t) - K\left[\sum_{j=1}^N \ell_{ij}(C\bar{x}_s^j(t-\tau))\right]. \end{cases}$$

Since $A\bar{h}_i = 0$, $i = 1, ..., N$, the rest of the proof is exactly the same as in the preceding section. □

Remark 9.3. *One thing that should be noted is that owing to the freedom we have in choosing appropriate (A, B, C), no restriction on formation vector needs to be imposed.*

9.5 Illustrative Examples

9.5.1 Output Synchronization

We illustrate our design procedure on a network of four agents. The agents dynamics are of form (1) with

$$A^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B^1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_y^1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_z^1 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix},$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_y^2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad C_z^2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix},$$

$$A^i = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{bmatrix}, B^i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad C_y^i = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \end{bmatrix}, C_z^i = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \end{bmatrix}, \quad i = 3, 4.$$

The topology of the network is given by Figure 9.1. The delay in this network is $\tau = 1$. Note that for this network, we have $\phi = 0.1913$. It is easy to see that we need to choose A such that $|\lambda(A)| < 1.3795$. Note that $n_d = 3$, which is the degree of the infinite zeros of (C_y^2, A^2, B^2). It is

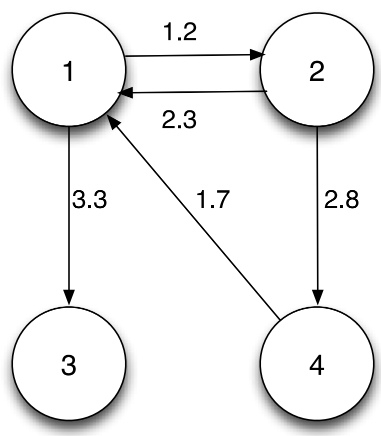

Figure 9.1: Network topology

easy to check that the following matrices A, B, C satisfy the conditions of Lemma 9.1.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

For Agent 1, we design the following compensator of the form (3) with

$$A_H^1 = \begin{bmatrix} 0 & 1 & -59 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -3 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -5.0911 & -4.9967 & -6.3138 & -8.9545 & -660 & 96 & -12 \\ 15.0583 & 70.2666 & -2.3301 & -0.3403 & 0 & -5.8764 & 2.3364 & 6.0692 & -4.2506 & -10.6208 & -1.7534 & 0.3403 \\ 1.6903 & 74.5116 & 6.9615 & 0.5860 & 0.1601 & -2.4165 & -0.5098 & -0.4044 & -0.7834 & -0.8056 & 0.0704 & -0.5860 \\ 1.7525 & 25.5179 & -6.8358 & -0.5645 & 0.3925 & -2.3798 & -5.4126 & 0.0499 & -1.4955 & -2.6122 & 0.0613 & 0.5645 \\ -12.0815 & -84.2475 & 34.0778 & 2.9156 & -0.8085 & 5.6036 & -3.3072 & -7.9601 & 4.1533 & 6.8159 & 0.9090 & -2.9156 \\ 17.9032 & 8.5467 & -61.9838 & -5.3051 & -1.0801 & -12.8002 & 2.5078 & 1.7113 & -8.4662 & -2.2577 & -1.6775 & 5.3051 \\ 0.9197 & 4.0411 & -4.3035 & -0.4548 & 0 & 0.0877 & -0.2802 & 0.5091 & -0.0489 & -4.0878 & -0.1545 & 0.4548 \\ 0.7415 & 11.8220 & 16.7633 & 0.9295 & 0 & -1.2706 & 0.8942 & -0.0789 & -0.6650 & -0.9267 & -5.6090 & 0.0705 \\ 9.2182 & 56.1777 & 73.4222 & 14.0479 & 0 & -11.4014 & -2.8270 & -3.6974 & -12.9563 & -666.0774 & 94.1525 & -14.0479 \end{bmatrix}$$

$$B_H^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 60.9069 \\ 73.1961 \\ 17.7097 \\ -79.1320 \\ 6.7245 \\ 2.3336 \\ 13.8082 \\ 54.2374 \end{bmatrix}, \quad E_H^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_H^1 = \begin{bmatrix} -7 & -7 & -35 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_H^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then the interconnection of (1) and (3) is in the form of (4) with d^1 generated by (5) with

$$A_s^1 = \begin{bmatrix} 0 & -5.8764 & 2.3364 & 6.0692 & -4.2506 & -10.6208 & -1.7534 & 0.3403 \\ 0.1601 & -2.4165 & -0.5098 & -0.4044 & -0.7834 & -0.8056 & 0.0704 & -0.5860 \\ 0.3925 & -2.3798 & -5.4126 & 0.0499 & -1.4955 & -2.6122 & 0.0613 & 0.5645 \\ -0.8085 & 5.6036 & -3.3072 & -7.9601 & 4.1533 & 6.8159 & 0.9090 & -2.9156 \\ -1.0801 & -12.8002 & 2.5078 & 1.7113 & -8.4662 & -2.2577 & -1.6775 & 5.3051 \\ 0 & 0.0877 & -0.2802 & 0.5091 & -0.0489 & -4.0878 & -0.1545 & 0.4548 \\ 0 & -1.2706 & 0.8942 & -0.0789 & -0.6650 & -0.9267 & -5.6090 & 0.0705 \\ 1.0000 & -6.3103 & 2.1697 & 2.6163 & -4.0018 & -6.0774 & -1.8475 & -2.0479 \end{bmatrix},$$

$$C_s^1 = \begin{bmatrix} 1 & 5.0911 & 4.9967 & 6.3138 & 8.9545 & 660 & -96 & 12 \end{bmatrix}.$$

For agent 2, we design the following compensator of the form (3) with

$$A_H^2 = \begin{bmatrix} -721 & -720 & 0 \\ 691 & 691 & 1 \\ -990 & -991 & 0 \end{bmatrix}, \quad B_H^2 = \begin{bmatrix} 721 \\ -691 \\ 990 \end{bmatrix}, \quad E_H^2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_H^2 = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}, \quad D_H^2 = 1.$$

Then the interconnection of (1) and (3) is in the form of (4) with d^2 generated by (5) with

$$A_s^2 = \begin{bmatrix} -721 & -720 & 0 \\ 691 & 691 & 1 \\ -990 & -990 & 0 \end{bmatrix}, \quad C_s^2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

For agent 3 and agent 4, we design the following compensator of the form (3) with

$$A_H^i = \begin{bmatrix} 0 & 1.2 & -0.2 & 0.2 & 4 & -3 & -2 \\ -170.2 & -179.7 & -65.9 & 112.9 & -97.1 & -340.4 & 170.2 \\ 758.9 & 727.5 & 220.4 & -504.4 & 577.2 & 1517.8 & -758.9 \\ 233. & 172.7 & 16 & -156 & 280 & 466 & -233 \\ 300.9 & 284.2 & 83.5 & -200.6 & 234.2 & 602.9 & -300.9 \\ 740.8 & 680.4 & 186.5 & -493.9 & 614.7 & 1481.6 & -739.8 \\ 1623.4 & 1514.4 & 432.1 & -1082.3 & 1300.3 & 3245.9 & -1623.4 \end{bmatrix},$$

$$B_H^i = \begin{bmatrix} 0 \\ -122.5903 \\ 474.2431 \\ 93.5599 \\ 183.8460 \\ 433.4444 \\ 973.2716 \end{bmatrix}, \quad E_H^i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C_H^i = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_H^i = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Then the interconnection of (1) and (3) is in the form of (4) with d^i generated by (5) with

$$A_s^i = \begin{bmatrix} -179.7 & -65.9 & 112.9 & -97.1 & -340.4 & 170.2 \\ 727.5 & 220.4 & -504.4 & 577.2 & 1517.8 & -758.9 \\ 172.7 & 16 & -156 & 280 & 466 & -233 \\ 284.2 & 83.5 & -200.6 & 234.2 & 602.9 & -300.9 \\ 680.4 & 186.5 & -493.9 & 614.7 & 1481.6 & -739.8 \\ 1513.2 & 432.4 & -1082.5 & 1296.3 & 3248.9 & -1621.4 \end{bmatrix},$$

$$C_s^i = \begin{bmatrix} -1.2440 & 0.2440 & -0.2440 & -4 & 3 & 2 \end{bmatrix}.$$

Note that the following state transformation

$$V = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

such that

$$VAV^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad VB = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad CV^{-1} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}.$$

Let us choose $\varepsilon = 0.02$ and

$$K = \begin{bmatrix} -6 \\ -10 \\ 0 \end{bmatrix},$$

and design the dynamic low-gain controller as follows:

$$\begin{cases} \dot{\chi}^i(t) = \begin{bmatrix} -6 & 1 & 0 \\ -10 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \chi^i(t) + \begin{bmatrix} -6 \\ -10 \\ 0 \end{bmatrix} \zeta^i(t), \\ v^i(t) = -\begin{bmatrix} 0.02 & 0.02 & 0 \end{bmatrix} \chi^i(t) \end{cases}. \tag{9.23}$$

Figure 9.2 shows the resulting simulated output for all four agents.

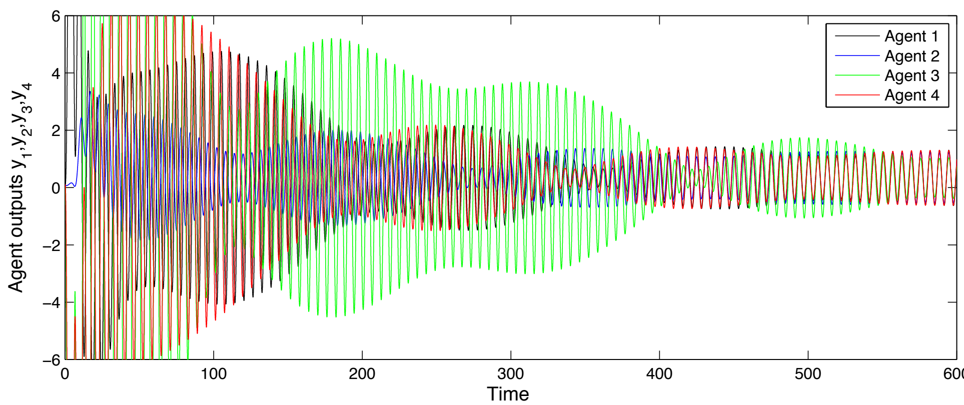

Figure 9.2: Outputs from the simulation example

9.5.2 Output Regulation

Consider the same network as in Section 9.5.1, however, our goal now is to ensure that each agent's output follows the output y_r of the following exosystem

$$\begin{cases} \dot{x}_r = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x_r, \\ y_r = \begin{bmatrix} 1 & 0 \end{bmatrix} x_r, \end{cases}$$

with $x_r(0) = [1; 1]$.

We first expand the system to the following form

$$\begin{cases} \dot{\bar{x}}_r = A\bar{x}_r := \begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 0\,0\,0 \end{bmatrix} \bar{x}_r, \\ y_r = C\bar{x}_r := \begin{bmatrix} 1\,0\,0 \end{bmatrix} \bar{x}_r, \end{cases}$$

with $\bar{x}_r(0) = [1; 1; 0]$.

Let us now choose $B = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}'$. We then follow the same design procedure to design precompensator to make all the agents almost identical with different exponentially decaying signals. We then add a link with weight $\delta = 10$ from the exosystem to the root agent 1 in Figure 9.1. The resulting network topology is shown in Figure 9.3.

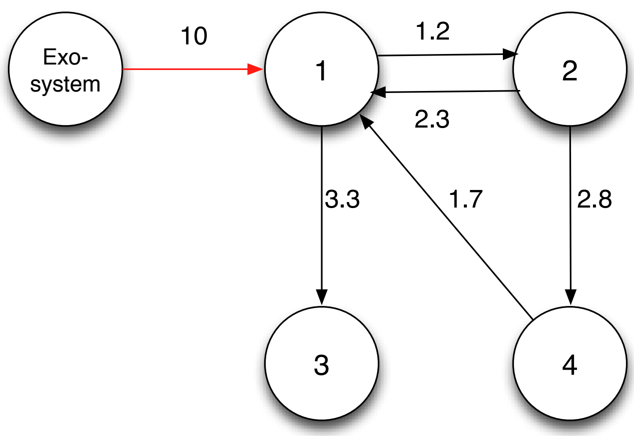

Figure 9.3: Network topology with the exosystem

Choose $\bar{\beta} = 0.77$, $\tau = 1$, and $\epsilon = 10^{-11}$. Figure 9.4 shows the resulting simulated output for all four agents and the exosystem.

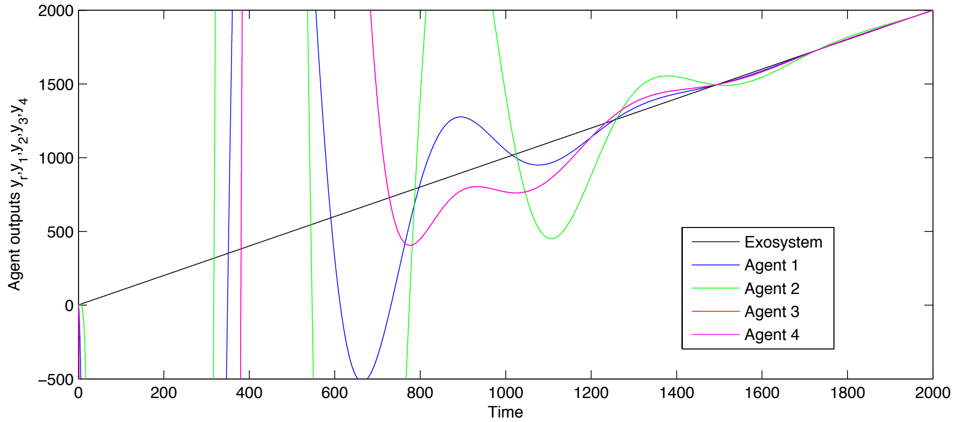

Figure 9.4: Outputs from the simulation example

9.5.3 Output Formation

Consider the same network as in Section 9.5.1, our goal is to achieve output formation. We choose $h_1 = 10$, $h_2 = 20$, $h_3 = 30$, and $h_4 = 40$. Figure 9.5 shows that the output formation is achieved [1].

9.A Manipulation of Exo-system

For a given exo-system (9.3), there exists a non-singular transformation $\tilde{x}_r = Tx_r$ such that (9.3) can be transformed in the following canonical form [36]:

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r \tilde{x}_r, \\ y_r = \tilde{C}_r \tilde{x}_r, \end{cases} \tag{9.24}$$

[1] Note that x_{d1}, x_{d2}, and x_{d3} is the coordinate where all the agents are almost identical.

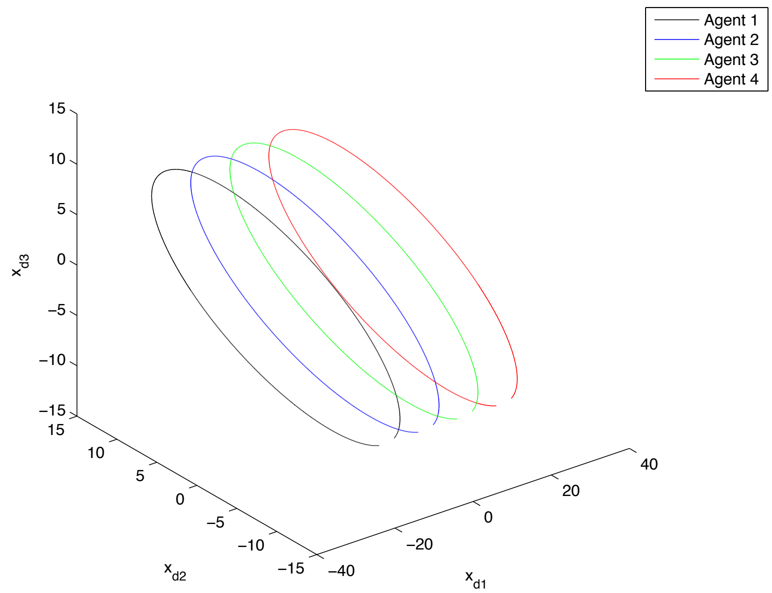

Figure 9.5: Output formation

where

$$\tilde{A}_r = TA_rT^{-1} = \begin{bmatrix} \tilde{A}_1 & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & \star & \star & \cdots & \star \\ 0 & \tilde{A}_2 & 0 & 0 & \cdots & 0 \\ \star & \star & \star & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \tilde{A}_p \\ \star & \star & \star & \star & \cdots & \star \end{bmatrix}, \tag{9.25}$$

$$\tilde{C}_r = C_rT^{-1} = \begin{bmatrix} \tilde{C}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{C}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{C}_p \end{bmatrix}, \tag{9.26}$$

and

$$\tilde{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$\tilde{C}_i = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Here $\star$ denotes a possible non-zero row. Note that for the original system (9.3), $\tilde{A}_i$ may not have the same size. However, by adding integrators to the bottom of each block and setting the initial conditions of those extended states to zero, we can extend the dimension of $\tilde{A}_i$ to $n_q > n_d$ while system (9.24) still produces the same output as the original exo-system (9.3).

Eventually, we can choose

$$\tilde{B}_r = \begin{bmatrix} \tilde{B}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{B}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{B}_p \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

We find that $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r)$ is invertible, of uniform rank $n_q > n_d$ and has no invariant zero.

Part II

Issues regarding Internal Stability of Linear Systems subject to Actuator Saturation

Chapter 10

Issues on Global Stabilization of Linear Systems Subject to Actuator Saturation

10.1 Introduction and Contribution

Linear systems subject to actuator saturation are ubiquitous and have been the subject of extensive study. See for instance two special issues [4, 69], and references therein. Internal stabilization for this class of systems has a long history. Fuller [14] established that a chain of integrators with order greater or equal to three cannot be globally stabilized by any saturating linear static state feedback control law with only one input channel. Sontag and Sussmann [78] established that, global stabilization of continuous-time linear systems with bounded input can be achieved if and only if the linear system, in the absence of actuator saturation, is stabilizable and critically unstable (equivalently, *asymptotically null controllable with bounded control*). In general, this requires nonlinear control laws. However, for certain cases, global stabilization can be achieved by linear control laws. More precisely, the paper [87] noted that systems which are asymptotically null controllable with bounded inputs can be globally stabilized by linear static state feedback control laws if all non-zero eigenvalues on the imaginary axis are semi-simple (geometric and algebraic multiplicities are equal) while zero is allowed to be an eigenvalue whose Jordan blocks can be at most of size 2×2 (which are associated with double integrators). The quoted paper [87] does not give a full proof of this result. In this chapter we prove this result. Moreover, our proof is constructive.

Another issue is that in the literature, there is this general belief that if there are Jordan blocks of size greater or equal to three associated to an eigenvalue in zero then one need nonlinear control laws to globally stabilize such linear systems subject to actuator saturation. This is a misconception. Such a misconception is possibly based on a misreading of the result of [14]. One should emphasize that the beautiful result of Fuller does not claim anything beyond linear static state feedback control laws for chains of integrators. In this chapter we illustrate this issue by showing that a triple-integrator with multiple inputs subject to actuator saturation can be globally stabilized by linear static state feedback control laws. This is clearly a first step towards a better understanding when nonlinear static state feedback control laws are needed.

Two general open problems are still unresolved: (1) under what conditions one can utilize a linear *static* state feedback control law to globally stabilize a linear system subject to actuator saturation?, and (2) under what conditions one can utilize a linear *dynamic* state feedback control law to globally stabilize a linear system subject to actuator saturation?

10.2 Design Linear Static State Feedback Control Laws for Mixed Case

In this section, we investigate when a linear static state feedback control law can globally asymptotically stabilize linear systems subject to actuator saturation. It is well known that both double-integrator and neutrally stable systems subject to actuator saturation can be globally stabilized by linear static state feedback control laws. However, the mixture of these cases is not well studied. For the mixed case, Tyan and Bernstein [87] gave a sufficient condition that guarantees global stability of the closed-loop system by using linear static state feedback control laws, but this result is not studied from a design point of view. In contrast, we design a linear static state feedback control law to globally stabilize the mixed system subject to actuator saturation.

In this section, we first formulate our problem—to design a linear static state feedback control law to globally stabilize the mixed systems consisting of double integrators, single integrators and neutrally stable dynamics subject to actuator saturation. We then present an algorithm which gives us a methodology for designing such globally stabilizing linear static state feedbacks in Section 10.2.2. Next, we prove that such a control law globally stabilizes the mixed system subject to actuator saturation via a Lyapunov argument in Section 10.2.3. Finally, an illustrative example is given in Section 10.2.4.

10.2.1 Problem Formulation

Consider a continuous-time linear system subject to actuator saturation described by

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\sigma(u) \tag{10.1}$$

where $\tilde{x} \in \mathbb{R}^n$, $\tilde{u} \in \mathbb{R}^m$, and

$$\sigma(u) = \begin{bmatrix} \sigma_1(u_1) \\ \sigma_1(u_2) \\ \vdots \\ \sigma_1(u_m) \end{bmatrix},$$

where $\sigma_1(u_i)$ is the standard saturation function

$$\sigma_1(u_i) = \operatorname{sgn}(u_i) \min\{1, |u_i|\}.$$

We assume that the pair $(\tilde{A}, \tilde{B})$ is stabilizable, and eigenvalues of $\tilde{A}$ are all located in the closed left-half complex plane (i.e., the pair $(\tilde{A}, \tilde{B})$ is asymptotically null controllable with bounded control). Furthermore, we assume that $\tilde{A}$ has eigenvalue zero with geometric multiplicity m and algebraic multiplicity $m+q$ with no Jordan blocks of size larger than 2 while the remaining eigenvalues are simple purely imaginary eigenvalues. Obviously, for such systems, stabilizability of the pair $(\tilde{A}, \tilde{B})$ is equivalent to controllability of the pair $(\tilde{A}, \tilde{B})$.

10.2.2 Algorithm

The algorithm for designing a linear static state feedback control law to globally stabilize the system described in (10.1) is carried out in three steps:

Step 1: We can obviously find a basis transformation Γ_x such that

$$A = \Gamma_x^{-1} \tilde{A} \Gamma_x = \begin{bmatrix} A_d & 0 & 0 \\ 0 & A_s & 0 \\ 0 & 0 & A_\omega \end{bmatrix}$$

with

$$A_d = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}$$

while $A_s = 0$ and A_ω satisfies $A_\omega + A_\omega^{\mathrm{T}} = 0$. With respect to this basis transformation we obtain:

$$B = \Gamma_x^{-1} \tilde{B} = \begin{bmatrix} B_d \\ B_s \\ B_\omega \end{bmatrix}$$

with

$$B_d = \begin{bmatrix} B_{d,1} \\ B_{d,2} \end{bmatrix}$$

compatible with the structure of A_d. The system in the new coordinates is given by:

$$\dot{x} = Ax + B\sigma(u). \tag{10.2}$$

Step 2: Design K such that

$$K = \begin{bmatrix} K_1 \\ K_2 \\ \vdots \\ K_m \end{bmatrix}$$

satisfies

$$KA + B^{\mathrm{T}} \Lambda = 0,$$

$$KB + (KB)^{\mathrm{T}} < 0,$$

where Λ is a diagonal matrix such that:

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

with

$$\Lambda_1 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}$$

compatible with the structure of A_d. The existence of such K is shown in the proof of Theorem 10.1.

Step 3: The linear static state feedback control law

$$u = \tilde{K}\tilde{x}, \tag{10.3}$$

where $\tilde{K} = K\Gamma_x^{-1}$ globally stabilizes the system described in (10.1).

10.2.3 Theorem

In order to show our main theorem, we need the following lemmas. These two lemmas are very well-known and can be found in [28] and [24] respectively.

Lemma 10.1. *Given two matrices X and Y, there exists a matrix Z such that*

$$ZX = Y,$$

if and only if

$$\ker X \subset \ker Y,$$

where $\ker A$ *is the null space of a matrix* $A \in \mathbb{R}^{m\times n}$ *defined as*

$$\ker A := \{x \in \mathbb{R}^n \mid Ax = 0\}. \tag{10.4}$$

Here, we present a special case of LaSalle's invariance principle, where $V(x)$ is positive definite, which is also known as Krasovskii Theorem.

Lemma 10.2. *Consider the system*

$$\dot{x} = f(x)$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$.

Let $x = 0$ *be an equilibrium point. Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable, radially unbounded, positive definite function such that* $\dot{V}(x) \leq 0$ *for all* $x \in \mathbb{R}^n$. *Let* $\mathcal{S} = \{ x \in \mathbb{R}^n \mid \dot{V}(x) = 0 \}$ *and suppose that no solution can stay in* $\mathcal{S}$ *for all* $t \geq 0$, *other than the trivial solution* $x(t) = 0$ *for all* $t \geq 0$. *Then, the origin is globally asymptotically stable.*

Now, we show that the control law constructed above in (10.3) can globally stabilize the system (10.1).

Theorem 10.1. *Consider a linear system as given in* (10.1). *Assume that the pair* $(\tilde{A}, \tilde{B})$ *is controllable. Moreover, we assume that* $\tilde{A}$ *has eigenvalue zero with geometric multiplicity* m *and algebraic multiplicity* $m + q$ *with no Jordan blocks of size larger than* 2 *while the remaining eigenvalues are simple purely imaginary eigenvalues. The linear state feedback control law* $u = \tilde{K}\tilde{x}$ *given in* (10.3) *with appropriate gain matrix* $\tilde{K}$ *can globally stabilize the system* (10.1).

Proof. Through step 1 of the algorithm, we can transfer the system (10.1) into (10.2) as

$$\dot{x} = Ax + B\sigma(u).$$

The state vector has a decomposition:

$$x = \begin{bmatrix} x_d \\ x_s \\ x_\omega \end{bmatrix}.$$

compatible to the decomposition of A. Moreover,

$$x_d = \begin{bmatrix} x_{d,1} \\ x_{d,2} \end{bmatrix}.$$

We prove the theorem via Lyapunov argument, consider a Lyapunov candidate

$$V(x) = \frac{1}{2}x_\omega^{\mathrm{T}} x_\omega + \frac{1}{2}x_{d,2}^{\mathrm{T}} x_{d,2} + \sum_{i=1}^{m} \int_0^{K_i x} \sigma_1(y)dy. \tag{10.5}$$

The evaluation of $\dot{V}(x)$ along the trajectories of the closed-loop system, yields,

$$\dot{V}(x) = x_\omega^{\mathrm{T}} \dot{x}_\omega + x_{d,2}^{\mathrm{T}} \dot{x}_{d,2} + \sigma(Kx)K\dot{x}.$$

With some algebra, we can write the above equation in the matrix form as

$$\dot{V}(x) = \sigma^{\mathrm{T}}(Kx)(KAx + B^{\mathrm{T}}\Lambda x) + \sigma^{\mathrm{T}}(Kx)KB\sigma(Kx). \tag{10.6}$$

In order to make $\dot{V}(x)$ non-positive, it is sufficient to guarantee that the gain matrix K satisfies

$$KA + B^{\mathrm{T}}\Lambda = 0, \tag{10.7a}$$

$$KB + (KB)^{\mathrm{T}} < 0. \tag{10.7b}$$

Let us write the equation (10.7) in matrix equality form

$$K \begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} -B^{\mathrm{T}}\Lambda & S \end{bmatrix} \tag{10.8}$$

where S is any matrix satisfying $S + S^{\mathrm{T}} < 0$, we get:

$$KB + (KB)^{\mathrm{T}} = S + S^{\mathrm{T}} < 0.$$

Now, let us show that a K which satisfies the equation (10.8) exists. From Lemma 10.1, we see that to show the solvability of equation (10.8) is equivalent to show

$$\begin{bmatrix} -B^{\mathrm{T}}\Lambda & S \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0, \tag{10.9}$$

given

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ \\ u \end{bmatrix} = 0. \tag{10.10}$$

Since the pair (A, B) is controllable, from the Hautus test [18], we know that

$$\operatorname{rank} \begin{bmatrix} A & B \end{bmatrix} = n,$$

Moreover, from the structure of matrices A and B, we know $\operatorname{rank} A = n - m$ and $\operatorname{rank} B = m$. Thus

$$\operatorname{Im} A \cap \operatorname{Im} B = 0,$$

where $\operatorname{Im} Z$ is the range space of a matrix $Z \in \mathbb{R}^{m \times n}$ defined as

$$\operatorname{Im} Z := \{Zx \mid x \in \mathbb{R}^n\}.$$

Therefore, the equation (10.10), implies $Ax = 0$ and $Bu = 0$. Clearly $Ax = 0$ implies $x_{d,2} = 0$ and $x_\omega = 0$ which yields $\Lambda x = 0$. Hence

$$-B^{\mathrm{T}} \Lambda x = 0.$$

Moreover $\operatorname{rank} B = m$ while B has m columns yields that B is injective. Therefore $Bu = 0$ implies $u = 0$ and therefore $Su = 0$. Hence (10.9) is satisfied and we have shown that the equation (10.8) is solvable. Since $\begin{bmatrix} A & B \end{bmatrix}$ is surjective, for any given S, we have a unique solution K for the equation (10.8) such that

$$\dot{V}(x) = \sigma^{\mathrm{T}}(Kx) K B \sigma(Kx) \leq 0$$

provided $S + S^{\mathrm{T}} < 0$. In order to prove asymptotic stability we apply Lemma 10.2. Clearly our Lyapunov candidate function $V(x)$ given in (10.5) is continuously differentiable, radially unbounded, positive definite function.

Next, we note that the $\dot{V}(x) = 0$ if and only if $Kx = 0$. When $Kx = 0$, the dynamics obviously becomes $\dot{x} = Ax$. We need to show that there exists no initial condition $x(0) = x_0 \neq 0$ such that $Kx(t) = 0$ for all $t > 0$ while $\dot{x}(t) = Ax(t)$. We have:

$$x(t) = \begin{bmatrix} x_{d,1}(0) + t x_{d,2}(0) \\ x_{d,2}(0) \\ x_s(0) \\ x_\omega(t) \end{bmatrix}.$$

Since $x_\omega(t)$ is only related to non-zero eigenvalues, we get from $Kx(t) = 0$ for all $t \geq 0$ that:

$$K \begin{bmatrix} x_{d,2}(0) \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0, \quad K \begin{bmatrix} x_{d,1}(0) \\ x_{d,2}(0) \\ x_s(0) \\ 0 \end{bmatrix} = 0. \tag{10.11}$$

The first equality in (10.11) implies:

$$0 = K \begin{bmatrix} x_{d,2}(0) \\ 0 \\ 0 \\ 0 \end{bmatrix} = KA \begin{bmatrix} 0 \\ x_{d,2}(0) \\ 0 \\ 0 \end{bmatrix} = -B^{\mathrm{T}} \begin{bmatrix} 0 \\ x_{d,2}(0) \\ 0 \\ 0 \end{bmatrix},$$

which yields $B_{d,2}^{\mathrm{T}} x_{d,2}(0) = 0$. Controllability of the pair (A, B) implies that $B_{d,2}$ must be surjective. Hence $B_{d,2}^{\mathrm{T}}$ is injective and we obtain $x_{d,2}(0) = 0$. Next, we note that the second equality in (10.11) yields:

$$\begin{bmatrix} x_{d,1}(0) \\ 0 \\ x_s(0) \\ 0 \end{bmatrix} = Ax + Bu, \text{ where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

for suitably chosen x and u since $\begin{bmatrix} A & B \end{bmatrix}$ is surjective (because of controllability). Obviously, this implies that $0 = B_{d,2}u$ while x_4 satisfies:

$$x_4 = -A_\omega^{-1} B_\omega u.$$

We find:

$$\begin{aligned}
0 = u^{\mathrm{T}} K \begin{bmatrix} x_{d,1}(0) \\ 0 \\ x_s(0) \\ 0 \end{bmatrix} &= u^{\mathrm{T}} K \left[Ax + Bu \right] \\
&= -u^{\mathrm{T}} B^{\mathrm{T}} \Lambda x + u^{\mathrm{T}} S u \\
&= -u^{\mathrm{T}} B_{d,2}^{\mathrm{T}} x_2 - u^{\mathrm{T}} B_\omega^{\mathrm{T}} x_4 + u^{\mathrm{T}} S u \\
&= u^{\mathrm{T}} B_\omega^{\mathrm{T}} A_\omega^{-1} B_\omega u + u^{\mathrm{T}} S u \\
&= u^{\mathrm{T}} S u
\end{aligned}$$

where we used that $B_{d,2}u = 0$ and the fact that A_ω^{-1} is skew-symmetric. Since $S + S^{\mathrm{T}} < 0$, we find $u = 0$. But this immediately yields that $x_s(0) = 0$. Using that $x_s(0) = 0$ and $x_{d,2}(0) = 0$, we get from the second equality in (10.11) that

$$0 = K \begin{bmatrix} x_{d,1}(0) \\ 0 \\ 0 \\ 0 \end{bmatrix} = KA \begin{bmatrix} 0 \\ x_{d,1}(0) \\ 0 \\ 0 \end{bmatrix} = -B^{\mathrm{T}} \begin{bmatrix} 0 \\ x_{d,1}(0) \\ 0 \\ 0 \end{bmatrix},$$

which yields $B_{d,2}^{\mathrm{T}} x_{d,1}(0) = 0$. As noted before, $B_{d,2}^{\mathrm{T}}$ is injective and therefore $x_{d,1}(0) = 0$.

It remains to show that $x_\omega(0) = 0$. We note that

$$K \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_\omega(0) \end{bmatrix} = K_\omega x_\omega(0) = 0,$$

where K_ω is the gain matrix associated with neutrally stable dynamics. We know that $x(t)$ remains in the kernel of K with $u(t) = 0$. Hence:

$$KA \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_\omega(0) \end{bmatrix} = K_\omega A_\omega x_\omega(0) = 0.$$

But since $KA = -B^{\mathrm{T}}\Lambda$ this yields:

$$B_\omega^{\mathrm{T}} x_\omega(0) = 0.$$

Hence if $x_\omega(0) \neq 0$, we have a non-trivial A_ω-invariant subspace which is contained in $\ker B_\omega^{\mathrm{T}}$. Using the skew-symmetry of A_ω we find that this subspace is also A_ω^{T}-invariant. However, the existence of a non-trivial A_ω^{T} invariant subspace contained in $\ker B_\omega^{\mathrm{T}}$ yields a contradiction with the observability of the pair $(B_\omega^{\mathrm{T}}, A_\omega^{\mathrm{T}})$ or, equivalently a contradiction with the controllability of the pair (A_ω, B_ω). Therefore $x_\omega(0) = 0$.

Hence, the origin is the only solution within the subset of $\mathbb{R}^n$ for which $\dot{V}(x) = 0$. Hence, the global asymptotic stability of the closed-loop system follows from LaSalle's Invariance Principle. $\square$

10.2.4 Example

In this section, our methodology for designing a globally stabilizing linear static state feedback control law will be illustrated by an example. Let us consider an example in form of (10.2), which contains two double integrators, one single integrator and neutrally stable dynamics with A and B as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 5 \\ 1 & 2 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

It is easy to check that the pair (A, B) is controllable. Also we notice that since S is an arbitrary such that $S + S^{\mathrm{T}} < 0$, the solution for the equation (10.8) is not unique, therefore, the linear static state feedback control laws which can globally stabilize the closed-loop system is not unique either. However, for a given S, we have a unique solution for the equation (10.8), therefore, we have a unique linear static state feedback control laws which can globally stabilize the closed-loop system.

For this example, we choose

$$S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -3 & 1 \\ -1 & -1 & -53 \end{bmatrix}.$$

Then the unique possible globally stabilizing linear static state feedback control law is $u = Kx$,

where

$$K = \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$$

with K_1, K_2 and K_3 given as below:

$$\begin{aligned} K_1 &= \begin{bmatrix} -1 & 0 & -1 & 3 & -11 & -1 & 1 \end{bmatrix}, \\ K_2 &= \begin{bmatrix} -2 & -1 & 0 & -1 & 17 & 0 & 1 \end{bmatrix}, \\ \text{and} \quad K_3 &= \begin{bmatrix} -4 & -6 & 0 & 4 & -35 & -1 & 0 \end{bmatrix}. \end{aligned}$$

For the initial condition

$$x_0 = \begin{bmatrix} 100 & -100 & -100 & 100 & 100 & -100 & 100 \end{bmatrix}^{\mathrm{T}},$$

the dynamics are shown in Figure 10.1, which clearly shows that the closed-loop system is asymptotically stable.

10.3 Triple Integrator with Multiple Inputs

In the saturation literature, it is a general belief that for a system which has an eigenvalue in zero with associated Jordan block of size greater or equal to three, there does not exist a saturating linear static state feedback control law which can globally stabilize the system. We claim that this is a misconception. More precisely, whether a saturating linear static state feedback control law exists does not only depend on the size of Jordan block. The following triple-integrator example

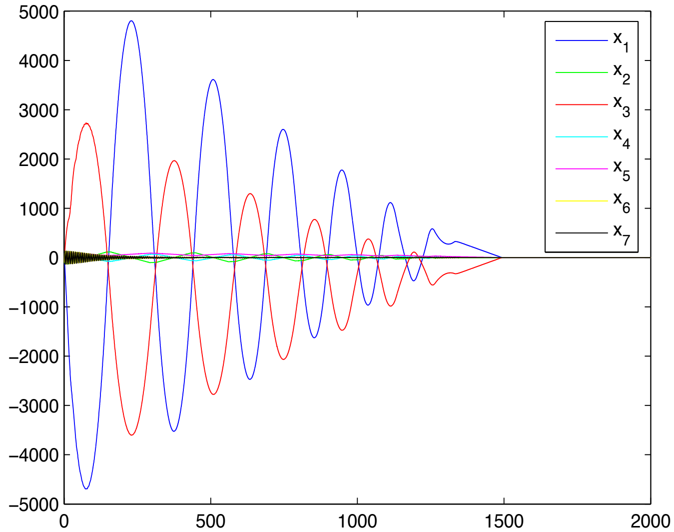

Figure 10.1: Global stabilization via a linear static state feedback

illustrate this.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1(u_1) \\ \sigma_1(u_2) \end{bmatrix} \tag{10.12}$$

We will prove that there exists a saturating linear static state feedback control law which can globally asymptotically stabilize the system. However, we will first present a useful lemma, see [78, 79], that we will use later:

Lemma 10.3. *Assume that $\dot{\zeta} = f(\zeta, v)$ has a globally Lipschitz right-hand side, and that the origin is a globally asymptotically stable state for $\dot{\zeta} = f(\zeta, 0)$. Then there exists some $\lambda > 0$ such that every solution of $\dot{\zeta} = f(\zeta, v)$ converges to zero, for every v such that $||v(t)|| \leq \kappa e^{-\lambda t}$.*

Theorem 10.2. *Consider a triple-integrator subject to actuator saturation, with two input channels, described by* (10.12) *A linear static state feedback control law can globally asymptotically stabilize the system* (10.12).

We will present two proofs for Theorem 10.2. Let us present the first one based on Lemma 10.3.

Proof. Consider a linear static state feedback control law

$$u_1 = -\gamma x_3,$$

$$u_2 = -\alpha x_1 - \beta x_2,$$

where $\alpha > 0$, $\beta > 0$, and $\gamma >> 0$. Applying this particular state feedback control law, yields the closed-loop system

$$\dot{x}_1 = x_2, \tag{10.13a}$$

$$\dot{x}_2 = x_3 + \sigma_1(-\alpha x_1 - \beta x_2), \tag{10.13b}$$

$$\dot{x}_3 = \sigma_1(-\gamma x_3). \tag{10.14}$$

The asymptotic stability of the closed-loop system follows from the fact that the poles of the linearized system are in the open left half complex plane. In order to prove global asymptotic stability of the closed-loop system, we need to show that the closed-loop system is globally attractive.

We can view the system as two subsystems, where the dynamics of subsystem 2 (equation 10.14) $x_3(t)$ is decoupled from the dynamics of subsystem 1, $x_1(t)$ and $x_2(t)$, and the dynamics of subsystem 2, $x_3(t)$ is viewed as a disturbance into subsystem 1.

We further proceed our proof by applying Lemma 10.3. Let us first check all the conditions of the Lemma 10.3.

Consider the subsystem 1. let us define

$$f(x) = \begin{bmatrix} x_2 \\ x_3 + \sigma_1(-\alpha x_1 - \beta x_2) \end{bmatrix}.$$

Thus, we can write the dynamics of subsystem 1 as $\dot{\zeta} = f(\zeta, v)$, where

$$\zeta = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and $v = x_3$.

Clearly, the origin is globally asymptotically stable for $\dot{\zeta} = f(\zeta, 0)$, since for $v = 0$, it becomes a double-integrator subject to actuator saturation with arbitrary negative linear state feedback control law, for which is well known that the closed-loop system is globally asymptotically stable. It is also easily verified that f is globally Lipschitz.

In order to apply lemma 10.3, we know that we must guarantee $\|x_3(t)\| \leq \kappa e^{-\lambda t}$ for some λ determined by the dynamics of subsystem 1.

Obviously, we see that for big initial condition $\|x_3(0)\|$, i.e. the subsystem 2 given by (10.14) is subject to actuator saturation at the beginning, $\|x_3(t)\|$ decays linearly up to certain point, once it gets out of saturation region, $\|x_3(t)\|$ decays exponentially fast to zero, depending on γ, thus, we can design $\gamma > \lambda$, such that $\|x_3(t)\| \leq \kappa e^{-\lambda t}$. We automatically see that all the conditions of Lemma 10.3 are satisfied, therefore, every solution of the first subsystem converges to zero. Thus, the closed-loop system is globally attractive. Hence, we have proved that the closed-loop system is globally asymptotically stable. □

Now, we present the second proof by constructing a Lyapunov function. This also demonstrates the fact that searching for the Lyapunov function even for simple (low order) linear system subject to actuator saturation is very complicated. Also, the Lyapunov approach guarantees stability for all $\alpha, \beta, \gamma > 0$ while Lemma 10.3 only proves stability for γ sufficiently large.

Proof. Let us partitioning the $\mathbb{R}^3$ into 4 regions. The partitions are:

$$R_1 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 x_3 > 0, |\gamma x_3| > 1\right\},$$

$$R_2 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 x_3 < 0, |\gamma x_3| > 1\right\},$$

$$R_3 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 x_3 > 0, |\gamma x_3| < 1\right\},$$

$$\text{and} \quad R_4 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_2 x_3 < 0, |\gamma x_3| < 1\right\}.$$

Consider a Lyapunov candidate

$$V(x) = \int_0^{\alpha x_1 + \beta x_2} \sigma_1(y) dy + \frac{\alpha}{2} x_2^2 + \tfrac{\alpha}{\gamma} \max\{0, x_2 x_3, x_2 x_3 |\gamma x_3|\} + r \max\left\{(\gamma x_3)^2, (\gamma x_3)^4\right\}. \quad (10.15)$$

We want to show that the Lyapunov candidate shown in (10.15) is indeed a Lyapunov function, thus, the global asymptotic stability of the closed-loop system follows. First, it is easy to see that $V(x)$ is continuous and positive definite. Also, $V(x)$ is radially unbounded.

In order to show globally asymptotically stable of the closed-loop system, we need to show that $\dot{V}(x)$ in each region is negative.

In regions R_1 and R_2, the system is described by

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = x_3 + \sigma_1(-\alpha x_1 - \beta x_2),$$

$$\dot{x}_3 = -\operatorname{sgn}(\gamma x_3).$$

For region R_1, the evaluation of $\dot{V}(x)$ along the trajectories of the closed-loop system, yields:

$$\begin{aligned}\dot{V} =& \sigma_1(\alpha x_1 + \beta x_2)[\alpha x_2 + \beta(x_3 - \sigma_1(\alpha x_1 + \beta x_2))] + \alpha x_2[x_3 - \sigma_1(\alpha x_1 + \beta x_2)] \\ & + \alpha[x_3 - \sigma_1(\alpha x_1 + \beta x_2)]x_3|x_3| - 2\alpha x_2 x_3 - 4r\gamma|\gamma x_3|^3 \\ =& \beta x_3\sigma_1(\alpha x_1 + \beta x_2) - \beta\sigma_1^2(\alpha x_1 + \beta x_2) - \alpha x_2 x_3 \\ & + \alpha|x_3|^3 - \alpha\sigma_1(\alpha x_1 + \beta x_2)x_3|x_3| - 4r\gamma|\gamma x_3|^3.\end{aligned}$$

Since for $|\gamma x_3| > 1$, the following hold

$$x_3\sigma(\alpha x_1 + \beta x_2) \leq |x_3| \leq \frac{1}{\gamma}|\gamma x_3| \leq \frac{1}{\gamma}|\gamma x_3|^3,$$
$$-\sigma_1(\alpha x_1 + \beta x_2)x_3|x_3| \leq |x_3|^2 = \frac{1}{\gamma^2}|\gamma x_3|^2 \leq \frac{1}{\gamma^2}|\gamma x_3|^3.$$

We get

$$\dot{V} \leq -\beta\sigma_1^2(\alpha x_1 + \beta x_2) - \alpha x_2 x_3 - (4r\gamma - \frac{\alpha}{\gamma^2} - \frac{\alpha}{\gamma^3} - \frac{\beta}{\gamma})|\gamma x_3|^3.$$

Choosing r such that:

$$4r\gamma^4 > \alpha\gamma + \alpha + \beta\gamma^2, \tag{10.16}$$

then yields $\dot{V} < 0$.

For region R_2, the evaluation of $\dot{V}_2$ along the trajectories of the closed-loop system, yields,

$$\begin{aligned}\dot{V} &= \sigma_1(\alpha x_1 + \beta x_2)[\alpha x_2 + \beta(x_3 - \sigma_1(\alpha x_1 + \beta x_2))] \\ &\quad + \alpha x_2[x_3 - \sigma_1(\alpha x_1 + \beta x_2)] - 4r\gamma|\gamma x_3|^3 \\ &= \beta x_3\sigma_1(\alpha x_1 + \beta x_2) - \beta\sigma_1^2(\alpha x_1 + \beta x_2) + \alpha x_2 x_3 - 4r\gamma|\gamma x_3|^3 \\ &\leq -\beta\sigma_1^2(\alpha x_1 + \beta x_2) + \alpha x_2 x_3 - (4r\gamma - \frac{\beta}{\gamma})|\gamma x_3|^3.\end{aligned}$$

Choosing r such that:

$$4r\gamma^2 > \beta, \tag{10.17}$$

then yields $\dot{V} < 0$.

In regions R_3 and R_4, the system is described by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = x_3 + \sigma_1(\alpha x_1 - \beta x_2),$$
$$\dot{x}_3 = -\gamma x_3.$$

For region R_3, the evaluation of $\dot{V}_3$ along the trajectories of the closed-loop system, yields,

$$\begin{aligned}\dot{V} =& \sigma_1(\alpha x_1 + \beta x_2)[\alpha x_2 + \beta(x_3 - \sigma_1(\alpha x_1 + \beta x_2))] + \alpha x_2[x_3 - \sigma_1(\alpha x_1 + \beta x_2)] \\ &+ \frac{\alpha}{\gamma}[x_3 - \sigma_1(\alpha x_1 + \beta x_2)]x_3 - \alpha x_2 x_3 - 2r\gamma^3 x_3^2 \\ =& (\beta - \frac{\alpha}{\gamma})x_3\sigma_1(\alpha x_1 + \beta x_2) - \beta\sigma_1^2(\alpha x_1 + \beta x_2) + (\frac{\alpha}{\gamma} - 2r\gamma^3)x_3^2.\end{aligned}$$

In this case, we choose ϵ small enough such that:

$$\frac{\epsilon}{2}|\beta - \frac{\alpha}{\gamma}| \leq \frac{\beta}{2}.$$

Next, using

$$|x_3\sigma_1(\alpha x_1 + \beta x_2)| \leq \frac{\epsilon}{2}\sigma_1^2(\alpha x_1 + \beta x_2) + \frac{1}{2\epsilon}x_3^2,$$

we get

$$\dot{V} \leq -\frac{\beta}{2}\sigma_1^2(\alpha x_1 + \beta x_2) + \left(\frac{1}{2\epsilon}|\beta - \frac{\alpha}{\gamma}| + \frac{\alpha}{\gamma} - 2r\gamma^3\right)x_3^2.$$

Choosing r such that:

$$2r\gamma^3 > \frac{1}{2\epsilon}|\beta - \frac{\alpha}{\gamma}| + \frac{\alpha}{\gamma}, \tag{10.18}$$

then yields $\dot{V} < 0$.

For region R_4, the evaluation of $\dot{V}_4$ along the trajectories of the closed-loop system, yields,

$$\begin{aligned}
\dot{V} &= \sigma_1(\alpha x_1 + \beta x_2)[\alpha x_2 + \beta(x_3 - \sigma_1(\alpha x_1 + \beta x_2))] + \alpha x_2[x_3 - \sigma_1(\alpha x_1 + \beta x_2)] - 2r\gamma^3 x_3^2 \\
&= \beta x_3 \sigma_1(\alpha_1 + \beta x_2) - \beta \sigma_1^2(\alpha x_1 + \beta x_2) + \alpha x_2 x_3 - 2r\gamma^3 x_3^2 \\
&\le \frac{\beta}{2}[x_3^2 + \sigma_1^2(\alpha x_1 + \beta x_2)] - \beta \sigma_1^2(\alpha x_1 + \beta x_2) + \alpha x_2 x_3 - 2r\gamma^3 x_3^2 \\
&= -\frac{\beta}{2}\sigma_1^2(\alpha x_1 + \beta x_2) + \alpha x_2 x_3 - (2r\gamma^3 - \frac{\beta}{2})x_3^2.
\end{aligned}$$

Choosing r such that:

$$4r\gamma^3 > \beta, \tag{10.19}$$

then yields $\dot{V} < 0$.

Thus, choosing r sufficiently large such that all inequalities (10.16)-(10.19) are satisfied, then yields $\dot{V}(x) < 0$ for all four different regions. Therefore, $\dot{V}(x)$ is negative along all trajectories unequal to zero. And for the origin, from the analysis above, we see that $\dot{V}(\mathbf{0}) = 0$. Thus, the closed-loop system is globally asymptotically stable. Hence, the triple-integrator subject to actuator saturation can be globally asymptotically stabilized via linear static state feedback control laws. □

10.4 Conclusions

In this chapter, we reexamine the classical issue of requiring linear or nonlinear static state feedback control laws for globally stabilizing asymptotically null controllable with bounded control systems subject to actuator saturation. We resolve here the general misconception that the size of Jordan block associated with a zero eigenvalue determines whether linear control laws or nonlinear control laws are needed for globally stabilizing a linear system subject to actuator saturation. Also, we present here constructive linear static globally stabilizing saturated state feedback control laws for linear systems mixed with double integrators, single integrators, and neutrally stable dynamics. This is our first step towards our future goals of (1) completely characterizing the class of linear systems subject to actuator saturation which can be globally stabilized by linear *static* state feedback control laws, and similarly (2) completely characterizing the class of linear systems subject to actuator saturation which can be globally stabilized by linear *dynamic* state feedback control laws.

Chapter 11

Dynamic Behavior of the Discrete-time Double Integrator with Saturated Locally Stabilizing Linear State Feedback Laws

11.1 Introduction

Linear systems subject to actuator saturation are ubiquitous and have been the subject of extensive study, see for instance two special issues, [4, 69], and references therein.

Internal stabilization for this class of systems has a long history. Let us briefly review the literature on linear systems subject to actuator saturation. Sontag and Sussmann [78] and Yang, Sontag and Sussmann [109] established that, global stabilization of linear systems subject to actuator saturation can be achieved if and only if the linear system in the absence of actuator saturation is stabilizable, and has all its open-loop poles in the closed left-half plane for continuous-time linear systems and in the closed unit disc for discrete-time linear systems (equivalently, *asymptotically null controllable with bounded control*). In general, this requires nonlinear feedback control laws.

We have only very limited insight into which linear controller yields global stability and which one does not. For certain cases, global stabilization can be achieved by linear static state feedback control laws. For example, in both continuous-time and discrete-time settings, it is well-known that there exist linear static state feedback control laws which globally stabilize neutrally stable linear systems subject to actuator saturation, see for instance [76]. Some extensions have been established in [86, 106].

In discrete-time setting, the stabilization in a specific region subject to actuator saturation was studied in [11]. Also, the anti-windup design for linear discrete-time control systems guaranteeing regional and global stability and performance was addressed in [38]. In contrast, in this chapter we consider the global stabilization problem for a simple discrete-time linear system, namely, double-integrator, subject to actuator saturation with linear static feedbacks.

In continuous-time setting, it is well-known that a linear static state feedback law which locally stabilizes the double integrator subject to actuator saturation[1] also globally stabilizes the system in the presence of actuator saturation, see for instance, [75, 87]. However, similar result has not yet been obtained for the discrete-time case. The goal of this chapter is to investigate whether the equivalent of the double integrator subject to actuator saturation in discrete-time is globally asymptotically stable when a locally stabilizing linear state feedback law is used. The answer turns out to be no.

In this chapter we completely characterize the global behavior of the discrete-time double in-

[1] Note that a linear state feedback law with arbitrary negative feedback gains locally stabilizes the double integrator.

tegrator subject to actuator saturation under all possible linear, locally stabilizing state feedback laws. We establish that the class of linear controllers which achieve local asymptotic stability splits into two parts. One part does not yield global stability of the closed-loop system, which is shown by explicitly constructing nontrivial limit cycles. The other part yields global stability of the closed-loop system. Although we only study a specific system, this discrete-time double integrator is well-known as a key benchmark for the global internal stabilization problem for linear systems subject to actuator saturation. By fully understanding this system we make a key step in understanding the abilities of linear controllers for global stabilization.

11.2 Problem Formulation

Consider the discrete-time double integrator subject to actuator saturation described by

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k), \\ x_2(k+1) = x_2(k) + \sigma(u(k)), \end{cases} \tag{11.1}$$

where $\sigma(u)$ is the standard saturation function: $\sigma(u) = \text{sgn}(u) \min\{1, |u|\}$. We consider linear state feedbacks with feedback gains f_1 and f_2 of the form:

$$u(k) = f_1 x_1(k) + f_2 x_2(k). \tag{11.2}$$

Let us first consider system (11.1) with a feedback control law (11.2) in the absence of actuator saturation. From Jury's test (see [23] and references therein), we see that any feedback control law (11.2), where f_1 and f_2 satisfy the following condition

$$\tfrac{1}{2} f_1 - 2 < f_2 < f_1 < 0, \tag{11.3}$$

stabilizes the system (11.1) in the absence of actuator saturation or, in other words, achieves local asymptotic stability for the closed-loop system. The question arises is whether such locally stabilizing linear state feedbacks also globally stabilize system (11.1) in the presence of actuator saturation. In this chapter, we will establish that this question has a negative answer. Based on this negative result, we then completely characterize which feedback laws achieve global asymptotic stability for the closed-loop system.

11.3 Main Results

In this section, we present our results on the dynamic behavior of the discrete-time double integrator with saturated locally stabilizing linear state feedback laws.

Theorem 11.1. *If f_1 and f_2 satisfy the Jury's condition* (11.3) *plus the following condition*

$$f_2 > \tfrac{3}{2} f_1, \tag{11.4}$$

then the closed-loop system exhibits non-zero limit cycles, that is, there exist initial conditions that yield non-zero periodic solutions, hence, the closed-loop system is not globally asymptotically stable.

Proof. We will prove Theorem 11.1 by explicitly constructing non-zero periodic solutions. The periodic solution with an even period $T = 2m$ that we will construct is such that the system is always in saturation, and the saturated input sequence is composed of 1 for the first m steps, followed by -1 for the next m steps. For such a solution, we always have $x_2(T) = x_2(0)$. In order to have $x_1(T) = x_1(0)$, it is easily verified that we must have that $x_2(0) = -\frac{m}{2}$.

Clearly, this will yield the required periodic solution if $x_1(0)$, f_1 and f_2 satisfy the following $2m$ inequalities: $u(k) \geq 1$ for $k = 0, \ldots, m-1$ and $u(k) \leq -1$ for $k = m, \ldots, 2m-1$, which guarantee that the periodic solution has the required characteristic of the saturated input being 1 for the first m steps and -1 for the next m steps. We basically have three unknowns $x_1(0)$, f_1 and f_2. However, if we view $f_1 x_1(0)$, f_1 and f_2 as the unknown variables, then the above inequalities become linear inequalities. Next, we note that the above $2m$ inequalities can be reduced to only two inequalities. Note that for $k = 1, \ldots, m$,

$$x_1(k) = x_1(0) + kx_2(0) + \tfrac{k(k-1)}{2},$$

$$x_2(k) = x_2(0) + k,$$

since by construction the input saturate to 1 for the first m steps. We then note that the Jury conditions (11.3) imply that $f_2 < f_1$ and $f_1 < 0$. This implies that $f_2 < f_1 - \frac{1}{2}kf_1$ for $k = 0, \ldots, m-1$. This yields $f_1x_2(0) + f_2 < \frac{1}{2}f_1(-m-k+2)$ since $x_2(0) = -\frac{m}{2}$. We know $m-k-1 \geq 0$ and multiplying the above inequality on both sides with $m-k-1$ yields:

$$f_1(m-k-1)x_2(0) + f_2(m-k-1) \leq f_1\left[\tfrac{k(k-1)}{2} - \tfrac{(m-1)(m-2)}{2}\right].$$

This is equivalent to

$$u(m-1) = f_1x_1(m-1) + f_2x_2(m-1) \leq f_1x_1(k) + f_2x_2(k) = u(k)$$

for $k = 0, \ldots, m-1$. Hence $u(m-1) \geq 1$ implies that $u(k) \geq 1$ for $k = 0, \ldots, m-1$. A similar argument shows that $u(2m-1) \leq -1$ implies that $u(k) \leq -1$ for $k = m, \ldots, 2m-1$.

Therefore, we have a periodic solution for given f_1 and f_2 provided there exists $x_1(0)$ such that

$$u(m-1) = f_1[x_1(0) + (m-1)x_2(0) + \tfrac{(m-1)(m-2)}{2}] + f_2[x_2(0) + (m-1)] \geq 1,$$

$$u(2m-1) = f_1[x_1(0) - x_2(0) - 1] + f_2[x_2(0) + 1] \leq -1.$$

Using that $x_2(0) = -\frac{m}{2}$ we find a periodic solution if we can find $x_1(0)$ such that the following two inequalities are satisfied:

$$f_1[x_1(0) - (m-1)] + f_2[\tfrac{m}{2} - 1] \geq 1,$$

$$f_1[x_1(0) + \tfrac{m}{2} - 1] + f_2[-\tfrac{m}{2} + 1] \leq -1,$$

which is equivalent to:

$$1 + f_1(m-1) - f_2[\tfrac{m}{2} - 1] \le f_1 x_1(0) \le -1 - f_1[\tfrac{m}{2} - 1] + f_2[\tfrac{m}{2} - 1]. \tag{11.5}$$

Clearly, a suitable $x_1(0)$ exists if and only if

$$1 + f_1(m-1) - f_2[\tfrac{m}{2} - 1] \le -1 - f_1[\tfrac{m}{2} - 1] + f_2[\tfrac{m}{2} - 1].$$

This implies that

$$f_1(3m-4) - f_2(2m-4) \le -4,$$

which, for $m > 2$, is equivalent to

$$\tfrac{3m-4}{2m-4} f_1 + \tfrac{2}{m-2} \le f_2. \tag{11.6}$$

From (11.4), it is clear that

$$\lim_{m \to \infty} (\tfrac{3m-4}{2m-4} f_1 + \tfrac{2}{m-2}) = \tfrac{3}{2} f_1 \le f_2.$$

Therefore for any f_1, f_2 which satisfy Jury's condition (11.3) and the additional condition (11.4), there exists m sufficiently large such that (11.6) is satisfied. But in the above we have seen that this implies that the system (11.1) with a feedback control law (11.2) exhibits periodic behavior for certain initial conditions with period $2m$. Hence, the system (11.1) can never be globally asymptotically stabilized by the feedback control law (11.2) if f_1 and f_2 satisfy (11.3) and (11.4). $\square$

Theorem 11.2. *If f_1 and f_2 satisfy the Jury's condition* (11.3) *plus the following condition*

$$f_2 < \tfrac{3}{2} f_1, \tag{11.7}$$

then the closed-loop system is globally asymptotically stable.

The above results can be illustrated by Figure 11.1. Note that in Figure 11.1, line AB is $f_2 = f_1$, line BC is $f_2 = \frac{1}{2} f_1 - 2$, line AD is $f_2 = \frac{3}{2} f_1$ and line AC is $f_1 = 0$. The Jury test establishes that

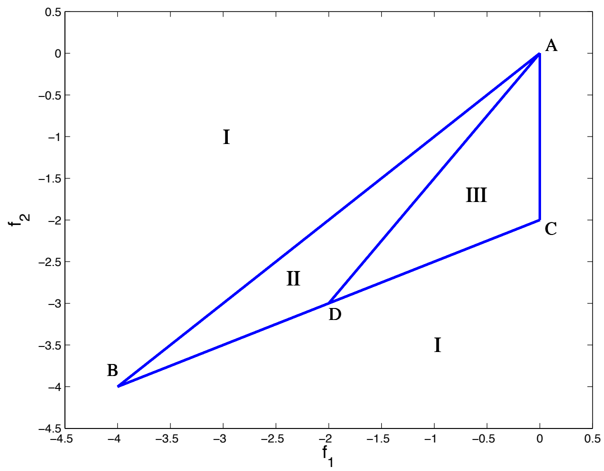

Figure 11.1: Stability characteristics as a function of f_1 and f_2

whenever f_1 and f_2 take their values within the triangle ABC, the closed-loop system is locally asymptotically stable; otherwise unstable. The triangle ABC can be bisected into two regions, triangle ABD (Region II) and triangle ADC (Region III). As shown by Theorem 11.1, whenever f_1 and f_2 take their values within the triangle ABD, there exist initial conditions that lead to non-zero periodic solutions and hence the closed-loop system is not globally asymptotically stable. On the other hand, as shown by Theorem 11.2, whenever f_1 and f_2 take their values within the triangle ADC, the closed-loop system is globally asymptotically stable. The results of Theorem 11.1 and Theorem 11.2 will be illustrated by simulation examples in Section 11.4.

In order to prove Theorem 11.2, we need to establish asymptotic stability of the closed-loop system whenever f_1 and f_2 take their values in the Region III depicted in Figure 11.1. A basis transformation turns out to be useful for establishing this result. We define $y_1(k) = u(k)$ and $y_2(k) = f_1 x_2(k)$. The closed-loop system is then given by:

$$\begin{cases} y_1(k+1) = y_1(k) + y_2(k) + f_2\sigma(y_1(k)), \\ y_2(k+1) = y_2(k) + f_1\sigma(y_1(k)). \end{cases} \tag{11.8}$$

We sometimes denote:

$$y(k) = \begin{pmatrix} y_1 \\ \\ y_2 \end{pmatrix}$$

and y, y_1 or y_2 without explicitly indicating time will refer to $y(k)$, $y_1(k)$ or $y_2(k)$ respectively. Without loss of generality, we assume the closed-loop system is given by (11.8). We will prove Theorem 11.2 by classical and modified Lyapunov argument by splitting the Region III into the following three parts:

- Case 1: $\{(f_1, f_2)|(f_2 - f_1 + 1)^2 - 1 < f_1\}$, which is Region IV depicted in Figure 11.2.
- Case 2: $\{(f_1, f_2)|(f_2 - f_1 + 1)^2 - 1 > f_1, f_1 \geq -1.6\}$,
- Case 3: $\{(f_1, f_2)|f_1 < -1.6\}$.

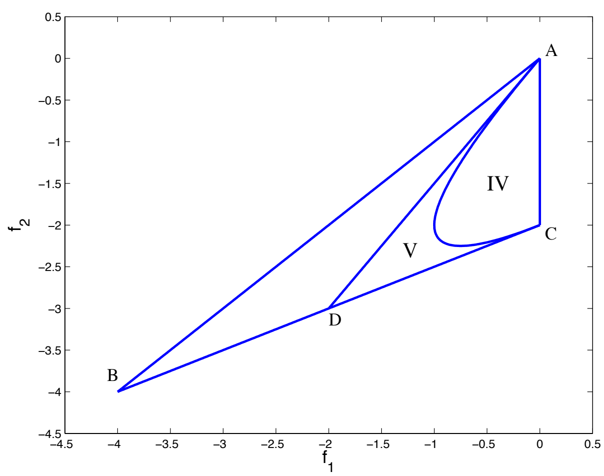

Figure 11.2: Stability characteristics as a function of f_1 and f_2

Proof of Theorem 11.2 in case $(f_2 - f_1 + 1)^2 - 1 < f_1$. We consider the following Lyapunov candidate

$$V_k = V(y(k)) = 2y_1(k)\sigma(y_1(k)) - \sigma^2(y_1(k)) - 2\sigma(y_1(k))y_2(k) - \tfrac{1}{f_1}y_2^2(k).$$

Then with some algebra, we get

$$V_{k+1} = 2y_1\sigma(\tilde{y}_1) + 2(f_2 - f_1)\sigma(y_1)\sigma(\tilde{y}_1) - \sigma(\tilde{y}_1)^2 - \tfrac{1}{f_1}y_2^2 - 2y_2\sigma(y_1) - f_1\sigma^2(y_1),$$

where to simplify notation we have used $\tilde{y}_1 = y_1(k+1)$, while $y_1(k)$ and $y_2(k)$ are abbreviated to y_1 and y_2 respectively. We find:

$$\Delta V = V_{k+1} - V_k = 2y_1\sigma(\tilde{y}_1) + 2(f_2 - f_1)\sigma(y_1)\sigma(\tilde{y}_1) - \sigma(\tilde{y}_1)^2 - 2y_1\sigma(y_1) - (f_1 - 1)\sigma^2(y_1).$$

Next, we show $\Delta V \leq 0$ by considering three different cases: Case 1.1: $y_1 \geq 1$, Case 1.2: $y_1 \leq -1$, and Case 1.3: $|y_1| < 1$.

In Case 1.1, we get:

$$\begin{aligned}
\Delta V &= 2y_1\sigma(\tilde{y}_1) + 2(f_2 - f_1)\sigma(\tilde{y}_1) - \sigma(\tilde{y}_1)^2 - 2y_1 - (f_1 - 1) \\
&= 2(y_1 - 1)(\sigma(\tilde{y}_1) - 1) - (\sigma(\tilde{y}_1) - f_2 + f_1 - 1)^2 + (f_2 - f_1 + 1)^2 - (f_1 + 1) \\
&\leq (f_2 - f_1 + 1)^2 - (f_1 + 1) < 0.
\end{aligned}$$

In Case 1.2, we get:

$$\begin{aligned}
\Delta V &= 2y_1\sigma(\tilde{y}_1) - 2(f_2 - f_1)\sigma(\tilde{y}_1) - \sigma(\tilde{y}_1)^2 + 2y_1 - (f_1 - 1) \\
&= 2(y_1 + 1)(\sigma(\tilde{y}_1) + 1) - (\sigma(\tilde{y}_1) + f_2 - f_1 + 1)^2 + (f_2 - f_1 + 1)^2 - (f_1 + 1) \\
&\leq (f_2 - f_1 + 1)^2 - (f_1 + 1) < 0.
\end{aligned}$$

Finally, in Case 1.3, we get:

$$\begin{aligned}
\Delta V &= 2(f_2 - f_1 + 1)y_1\sigma(\tilde{y}_1) - \sigma(\tilde{y}_1)^2 - (f_1 + 1)y_1^2 \\
&= -[\sigma(\tilde{y}_1) - (f_2 - f_1 + 1)y_1]^2 + [(f_2 - f_1 + 1)^2 - (f_1 + 1)]y_1^2 \leq 0.
\end{aligned}$$

We also see that equality holds only if $y_1 = 0$ and $\tilde{y}_1 = 0$ which implies that $x_1 = x_2 = 0$. Hence, the global asymptotic stability of the closed-loop system follows. □

In remains to prove Theorem 11.2 for Cases 2 and 3, that is, Region V as depicted in Figure 11.2. Let us first derive a candidate Lyapunov candidate for the closed-loop system. To do so, we consider a Lyapunov candidate in the presence of saturation, which is based on the linearized system as follows:

$$V_k = V(y(k)) = 2y_1(k)\sigma(y_1(k)) - \sigma^2(y_1(k)) + 2b\sigma(y_1(k))y_2(k) - \tfrac{1}{f_1}y_2^2(k), \tag{11.9}$$

where

$$b = \begin{cases} \frac{2}{f_2} & f_2^2 + 4f_1 \geq 0, \\ -\frac{f_2}{2f_1} & f_2^2 + 4f_1 < 0. \end{cases} \tag{11.10}$$

It is easily verified that in the triangle ADC of Figure 11.2 we have $b \in [-1, -0.5)$ while for $b = -1$ we get the Lyapunov function used in Case 1, and $b \in (-1, -0.5)$ for Case 2 and Case 3 We sometimes refer to the first case, when $f_2^2 + 4f_1 \geq 0$ as the real case since in that case the linearized system has real eigenvalues while the second case, when $f_2^2 + 4f_1 < 0$, is referred to as the complex case since in that case the linearized system has complex eigenvalues.

It is easy to see that the Lyapunov candidate (11.9) works for the linearized closed-loop system. In order to be a valid Lyapunov function, it is necessary that it must work when $\sigma(y_1)$ stays at 1 or at -1 in two consecutive time instants. It is easy to verify that in that case:

$$\Delta V = (2b - 1)f_1 + 2f_2, \tag{11.11}$$

where $(\Delta V)(k) = V_{k+1} - V_k$, while $V_k = V(y(k))$. Thus, $\Delta V = (f_2^2 + 4f_1)/f_2 + (f_2 - f_1) < 0$ in the real case while $\Delta V = f_2 - f_1 < 0$ in the complex case.

Therefore, the Lyapunov candidate (11.9) has the required properties when $\sigma(y_1)$ is in saturation for two consecutive time instants or is out of saturation for two consecutive time instants. Note that for a continuous-time problem, we would be done, since y_1 is continuous. However, for discrete-time systems, y_1 obviously jumps from one time to the other and hence if $\sigma(y_1(k))$ saturates then it might well be that $\sigma(y_1(k+1))$ is out of saturation or conversely. This is intrinsically different from the continuous-time case. Thus, we have to show that the Lyapunov candidate (11.9) also decreases when y_1 jumps. The traditional Lyapunov argument is to show that $V_{k+1} - V_k < 0$ for all initial conditions. However, this approach does not work here. For the real case, if $f_2 < -2$, there exist initial conditions, such that $V_{k+1} - V_k > 0$. A similar problem can arise in the complex case. Thus, we need a different technique. The main idea is to show that V decreases over a specifically chosen number of time steps, and V is bounded in the interim. In order to proceed with this idea, we first choose suitable time instants k_i. The formal definition of k_i is given below:

Definition 11.1. $k_0 = 0$, *and* k_i *is the smallest integer larger than* k_{i-1}, *such that either*

- $|y_1(k_i)| < 1$; *or*
- $y_1(k_i)y_1(k_i+1) < 0$ *and* $|y_1(k_i+1)| \geq 1$.

In other words, k_i is defined as the first time instant $k > k_{i-1}$ where $y_1(k)$ either gets out of saturation, or where $y_1(k)$ switches the sign. It is easily seen that k_i is well defined for a given k_{i-1} since the only way k_i would not be well defined is if $y_1(k) > 1$ for all $k > k_{i-1}$ or if $y_1(k) < -1$ for all $k > k_{i-1}$. It is easily seen from the dynamics (9.15) that this is not possible.

Proof of Theorem 11.2 in case $(f_2 - f_1 + 1)^2 - 1 > f_1$ *and* $f_1 \geq -1.6$. We consider the Lyapunov candidate V defined in (11.9) and assume the feedback gains f_1 and f_2 are in the region $\{(f_1, f_2) | (f_2 - f_1 + 1)^2 - 1 > f_1, f_1 \geq -1.6\}$. In appendix 11.A, it is established that if $V_{k_{i-1}} \neq 0$, then

$$V_{k_i} - V_{k_{i-1}} < 0. \tag{11.12}$$

We already know that the system is locally asymptotically stable from Jury's test. It remains to show global attractivity of the origin.

We first note that (11.9) can be rewritten as:

$$V(y) = 2\sigma(y_1)\,[y_1 - \sigma(y_1)] + [\sigma(y_1) + by_2]^2 - \left(b^2 + \tfrac{1}{f_1}\right) y_2^2.$$

It is easy to show that for $f_2^2 + 4f_1 \neq 0$ we have $b^2 + \frac{1}{f_1} > 0$. This immediately implies $V(y) > 0$ if $y \neq 0$. Using (11.12) implies that we have a sequence $\{k_i\}$ such that $V_{k_{i+1}} < V_{k_i}$ for all i. This clearly implies that V_{k_i} is bounded and hence $k_{i+1} - k_i$ is bounded as well. This implies that $V_{k_i} \to 0$ as $i \to \infty$. Local asymptotic stability implies that if V_{k_i} is small enough for some i then $y(k) \to 0$ as $k \to \infty$ and therefore we have global attractivity.

For the case that $b^2 + \frac{1}{f_1} = 0$, we have $V(y) \geq 0$ and $V(y) = 0$ implies $y_1 + by_2 = 0$ and $y_1 \in [-1, 1]$. We still have that (11.12) is satisfied and hence $V(k) \to 0$ as $k \to \infty$. Let $\mathcal{W}$ denote the compact set of $y \in \mathbb{R}^2$ for which $V(y) = 0$. Then it is easily verified that $y(k) \in \mathcal{W}$ implies that $y(k+1) \in \mathcal{W}$ and, moreover $y(k_0) \in \mathcal{W}$ implies that $y(k) \to 0$ as $k \to \infty$. Then a minor variation of the classical LaSalle argument implies that the system is globally attractive. □

Proof of Theorem 11.2 in case $f_1 < -1.6$. Again, we consider the Lyapunov candidate V defined in (11.9) and assume the feedback gains f_1 and f_2 satisfy $f_1 < -1.6$. In that case it is proven in Appendix 11.B that if $V_{k_{i-1}} \neq 0$, then

$$V_{k_i} - V_{k_{i-1}} < 0 \quad \text{or} \quad V_{k_{i+1}} - V_{k_{i-1}} < 0. \tag{11.13}$$

As before, we already know that the system is locally asymptotically stable from Jury's test. It remains to show global attractivity of the origin which can be done in a similar way as in Case 2. □

Remark 11.1. *Note that if the feedback gains f_1 and f_2 take their values with the triangle ABD (Region II) in Figure 11.1, there actually exist initial conditions for which $V_{k_{i+1}} - V_{k_{i-1}} = 0$ since*

$k_{i+1} - k_{i-1}$ *is precisely the period of the periodic behavior as constructed in the proof of Theorem 11.1.*

11.4 Illustrative Examples

In this section, we give two simulation examples to illustrate our results. The first example is the simulation example where the gain parameters f_1 and f_2 take their values within triangle ABD (Region II) of Figure 11.1, which clearly shows the periodic behavior of the closed-loop system for some initial conditions. The second example is the simulation example where the gain parameters f_1 and f_2 take their values within triangle ADC (Region III) of Figure 11.1, which clearly shows that the closed-loop system is globally asymptotically stable.

Example 11.1. *Consider system* (11.1) *with a state feedback control law* (11.2) *with gain parameters* $f_1 = -1$ *and* $f_2 = -1.2$. *The system has a periodic solution of period* $T = 56$ *for the following initial conditions:* $3.6 \leq x_1(0) \leq 10.4$ *and* $x_2(0) = -14$. *The state trajectory for* $x_1(0) = 10.4$ *and* $x_2(0) = -14$ *is given in Figure 11.3, where we clearly see the symmetric period orbit. Note that*

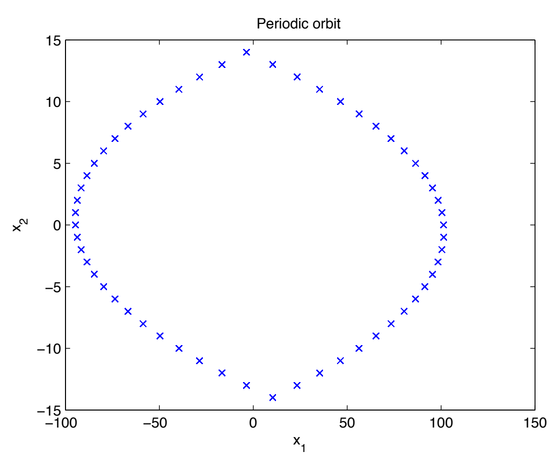

Figure 11.3: Periodic orbit of period 56

the state trajectory moves clockwise along the periodic orbit shown in Figure 11.3.

Example 11.2. *Consider system* (11.1) *with a state feedback control law* (11.2) *with gain parameters* $f_1 = -1.5$ *and* $f_2 = -2.5$. *The simulation result for initial condition* $x_1(0) = 10$ *and* $x_2(0) = -8$ *is given in Figure 11.4, where clearly shows that the closed-loop system is globally asymptotically stable.*

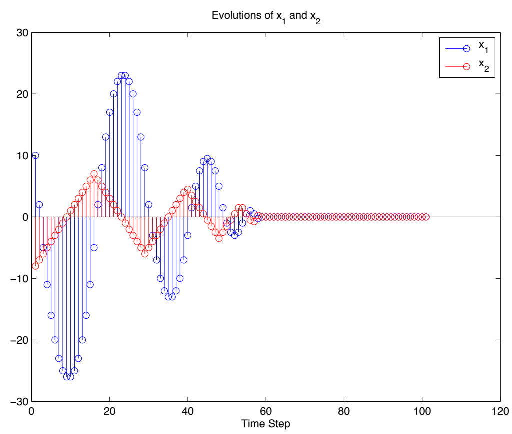

Figure 11.4: Simulation Example

11.A Proof of Theorem 11.2 in case $(f_2 - f_1 + 1)^2 - 1 > f_1$ and $f_1 \geq -1.6$

In this appendix, we will establish that (11.12) is satisfied. For simplicity we denote $y_1(k_{i-1})$ and $y_2(k_{i-1})$ by y_1 and y_2 respectively while $y_1(k_i)$ and $y_2(k_i)$ are denoted by $\tilde{y}_1$ and $\tilde{y}_2$ respectively.

We will prove that Lyapunov candidate will decay for the following cases depending on whether y_1 and $\tilde{y}_1$ is saturated or not.

- Case 2.1: $|y_1| \geq 1$ and $\tilde{y}_1 \in [-1, 1]$,

- Case 2.2: $y_1 \in [-1, 1]$ and $\tilde{y}_1 \in [-1, 1]$,
- Case 2.3: $y_1 \in [-1, 1]$ and $|\tilde{y}_1| \geq 1$.
- Case 2.4: $|y_1| \geq 1$ and $|\tilde{y}_1| \geq 1$.

We first note that (11.3), (11.7) and the following equation (11.14)

$$(f_2 - f_1 + 1)^2 - 1 > f_1 \tag{11.14}$$

imply that b as defined in (11.10) satisfies $b \in (-1, -\frac{2}{3})$ in the real case and $b \in (-1, -\frac{3}{4})$ in the complex case.

11.A.1 Case 2.1

Without loss of generality, we only consider the case $y_1 \geq 1$ (the other case where $y_1 \leq -1$ is completely symmetric).

In the case where $y_1 \geq 1$ we have

$$\tilde{y}_1 = y_1 + ky_2 + e_1,$$

$$\tilde{y}_2 = y_2 - (k-2)f_1,$$

where we denote $k = k_i - k_{i-1}$ while

$$e_1 = f_2 + (k-1)(f_1 - f_2) - \tfrac{f_1}{2}(k-1)(k-2). \tag{11.15}$$

We will prove the Lyapunov function defined in (11.9) will decay if $y_1 \geq 1$ and $\tilde{y}_1 \in [-1, 1]$. In doing this, we ignore the other constraints which follow from the definition of k_i, namely that $y_1(k_{i-1} + j) \leq -1$ for $j = 1, \ldots, k-1$. However, if the Lyapunov function always decays without these constraints then it will definitely still decay when these additional constraints are imposed.

We get

$$V_{k_i} - V_{k_{i-1}} = \tilde{y}_1^2 + 2b\tilde{y}_1\tilde{y}_2 - \frac{1}{f_1}\tilde{y}_2^2 - 2y_1 + 1 - 2by_2 + \frac{1}{f_1}y_2^2.$$

This can be rewritten completely in terms of $\tilde{y}_1$ and y_1. We obtain:

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} = {} & (1 + 2\tfrac{b}{k})\tilde{y}_1^2 + \left[-2\tfrac{b}{k}(\tilde{y}_1 - 1) - 4\tfrac{k-1}{k}\right] y_1 \\ & + \left[-2\tfrac{b}{k}e_1 + 2(2-k)bf_1 + 2 - 2(2+b)\tfrac{1}{k}\right] \tilde{y}_1 - 2e_1 + 2(2+b)\tfrac{1}{k}e_1 - (k-2)^2 f_1 + 1. \end{aligned}$$

We need to show this is negative for all $y_1 \geq 1$ and all $\tilde{y}_1 \in [-1, 1]$. However, this is a linear function of y_1 whose coefficient is negative and hence $V_{k_i} - V_{k_{i-1}}$ is maximal for $y_1 = 1$. We find:

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} \leq {} & (1 + 2\tfrac{b}{k})\tilde{y}_1^2 + \left[-2\tfrac{b}{k}e_1 + 2(2-k)bf_1 + 2 - 4(1+b)\tfrac{1}{k}\right] \tilde{y}_1 \\ & - 2e_1 + 2(2+b)\tfrac{1}{k}e_1 + 2\tfrac{b}{k} - 4\tfrac{k-1}{k} - (k-2)^2 f_1 + 1. \end{aligned} \tag{11.16}$$

The upper bound is a quadratic function which we need to maximize. Clearly, the sign of the quadratic term is crucial here. For $k = 1$ the coefficient of the quadratic term is negative and the maximum is obtained by setting the derivative equal to zero (if we ignore that $\tilde{y}_1 \in [-1, 1]$). We obtain for $k = 1$:

$$V_{k_i} - V_{k_{i-1}} \leq (1 + 2b)\tilde{y}_1^2 + [2b(f_1 - f_2) - 2(1 + 2b)]\tilde{y}_1 + 2b + 1 - f_1 + 2(1 + b)f_2. \tag{11.17}$$

For the real case ($f_2^2 + 4f_1 > 0$), since $b = 2f_2^{-1}$ and using (11.3), we obtain from (11.17) that

$$V_{k_i} - V_{k_{i-1}} \leq (f_2^2 + 4f_1)(4 + 2f_2 - f_1)f_2^{-1}/(f_2 + 4) < 0.$$

For the complex case ($f_2^2 + 4f_1 < 0$), since $b = -f_2 f_1^{-1}/2$ and again using (11.3), we obtain from (11.17) that

$$V_{k_i} - V_{k_{i-1}} \leq (f_2 - f_1)(f_2^2 + 4f_1)f_1^{-1}/4 < 0.$$

Finally, if $f_2^2 + 4f_1 = 0$ then it is easily verified from (11.17) that $V_{k_i} - V_{k_{i-1}} < 0$ unless $y_1 = 1$ and $\tilde{y}_1 = 1 + f_2/2$. However, it can be seen that in that case $V_{k_{i-1}} = 0$.

Next, we return to the case where $k > 1$. In that case, the upper bound (11.16) has a quadratic term with a positive coefficient. Therefore, the maximum is attained on the boundary, i.e. $\tilde{y}_1 = 1$ or $\tilde{y}_1 = -1$. For $\tilde{y}_1 = 1$ we obtain:

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} &\leq (k-2)\left[-2bf_1 - \tfrac{4}{k}(f_2 - f_1) - 3f_1 + 2f_2\right] \\ &\leq (k-2)\left[-2bf_1 - 2(f_2 - f_1) - 3f_1 + 2f_2\right] \\ &\leq (2-k)(2b+1)f_1 \\ &\leq 0, \end{aligned}$$

where we used $k \geq 2$ and $b < -0.5$. Note that we have that the upper bound is negative unless $k = 2$ in which case it easily verified that the decay equals zero only if $\tilde{y}_1 = 1$ and $y_1 = 1$. The latter is inconsistent with $k = 2$ since we then get

$$y(k_{i-1} + 1) = y_1 + y_2 + f_2 = 1 - \tfrac{1}{2}f_1 + f_2 \geq -1,$$

where we used $f_2 > \frac{1}{2}f_1 - 2$ for the inequality, while we should have $y(k_{i-1} + 1) \leq -1$. Next, we need to investigate the other boundary where $\tilde{y}_1 = -1$. We get

$$V_{k_i} - V_{k_{i-1}} \leq [2b - (k-2)]\left[(3f_1 - 2f_2) - \tfrac{4}{k}(f_1 - f_2 - 1)\right] < 0. \tag{11.18}$$

The first inequality is a simple rewriting of our upper bound for $\tilde{y}_1 = -1$. The second inequality is more subtle. It is easy to see that $2b - (k-2) < 0$. If $f_1 - f_2 - 1 \leq 0$, we immediately find that the expression is negative since we know from (11.7) that $3f_1 - 2f_2 > 0$. On the other hand if $f_1 - f_2 - 1 > 0$ then we find that

$$(3f_1 - 2f_2) - \tfrac{4}{k}(f_1 - f_2 + 1) \geq (3f_1 - 2f_2) - 2(f_1 - f_2 - 1) = f_1 + 2 > 0,$$

and the inequality (11.18) is satisfied. The fact that $f_1 > -2$ follows from (11.3) and (11.7).

11.A.2 Case 2.2

In this case we have:

$$y_1 \in [-1, 1] \quad \text{and} \quad \tilde{y}_1 \in [-1, 1]. \tag{11.19}$$

The proof is split into two cases: the real case where $f_2^2 + 4f_1 \geq 0$ and the complex case where $f_2^2 + 4f_1 \leq 0$.

11.A.2.1 The real case: $f_2^2 + 4f_1 \geq 0$

In this case we have

$$\tilde{y}_1 = y_1(k_i) = d_4 y_1 + k y_2 + e_4, \tag{11.20}$$

$$\tilde{y}_2 = y_2(k_i) = f_1 y_1 + y_2 - (k-1) f_1, \tag{11.21}$$

where we denote $k = k_i - k_{i-1}$ and

$$d_4 = 1 + f_2 + (k-1) f_1, \tag{11.22}$$

$$e_4 = -(k-1)\left(f_2 - f_1 + \tfrac{1}{2} f_1 k \right). \tag{11.23}$$

Given (11.19), we find that:

$$V_{k_i} - V_{k_{i-1}} = \tilde{y}_1^2 + 2b\tilde{y}_1\tilde{y}_2 - \tfrac{1}{f_1}\tilde{y}_2^2 - y_1^2 - 2by_1y_2 + \tfrac{1}{f_1}y_2^2.$$

We can eliminate $\tilde{y}_2$ and y_2 from the above expression by using (11.20) and (11.21):

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} = {} & \tilde{y}_1^2 + 2b\tilde{y}_1 \left[f_1 y_1 + \tfrac{1}{k}(\tilde{y}_1 - d_4 y_1 - e_4) - (k-1) f_1 \right] \\ & - \tfrac{1}{f_1} \left[f_1 y_1 + \tfrac{1}{k}(\tilde{y}_1 - d_4 y_1 - e_4) - (k-1) f_1 \right]^2 - y_1^2 - 2\tfrac{b}{k} y_1 \left[\tilde{y}_1 - d_4 y_1 - e_4 \right] \\ & + \tfrac{1}{k^2 f_1} \left[\tilde{y}_1 - d_4 y_1 - e_4 \right]^2. \end{aligned} \tag{11.24}$$

Our objective is now to prove that (11.24) is negative. We first note that for $k = 1$ we only need to study the unsaturated linear system and it is easily verified that we have:

$$V_{k_i} - V_{k_{i-1}} < 0 \tag{11.25}$$

provided $V_{k_{i-1}} \neq 0$. For $k = 2$ we will show that (11.24) is negative for all

$$-1 \leq y_1 \leq 1, \qquad -1 \leq \tilde{y}_1 \leq -1 - (1 + f_2 - f_1)(y_1 + 1) - f_1, \tag{11.26}$$

where the upper bound for $\tilde{y}_1$ follows from the constraint that $y_1(k_i - 1) \leq -1$. For $k > 2$ we consider all

$$-1 \leq y_1 \leq 1, \qquad -1 \leq \tilde{y}_1 \leq -1, \tag{11.27}$$

and we ignore all other constraints which follow from the definition of k_i namely that $y_1(k_{i-1}+j) \leq -1$ for $j = 1, \ldots, k-1$.

The quadratic term in $\tilde{y}_1$ in (11.24) is equal to

$$1 + 2b\tfrac{1}{k}$$

which is positive for $k \geq 2$ since $b > -1$. Therefore, we know (11.24) is maximal in a boundary point, that is,

$$\tilde{y}_1 = -1 \text{ or } \tilde{y}_1 = -1 - (1 + f_2 - f_1)(y_1 + 1) - f_1$$

for $k = 2$ and

$$\tilde{y}_1 = -1 \quad \text{ or } \quad \tilde{y}_1 = 1$$

for $k > 2$. For the lower bound $\tilde{y}_1 = -1$ we do not need to distinguish between $k = 2$ and $k > 2$ and we obtain that

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} = 1 - 2bf_1\left[y_1 - (k-1)\right] - \tfrac{2b}{k}(1 + y_1)\left[-1 - d_4y_1 - e_4\right] \\ - \tfrac{2}{k}\left[y_1 - (k-1)\right]\left[-1 - d_4y_1 - e_4\right] - f_1\left[y_1 - (k-1)\right]^2 - y_1^2 \end{aligned} \tag{11.28}$$

and we need to show this expression is negative. The expression has the form:

$$\bar{a}y_1^2 + \bar{b}y_1 + \bar{c}. \tag{11.29}$$

Here we have:

$$\begin{aligned}
\bar{a} &= (2b+1)f_1 - 1 + \tfrac{2}{k}(b+1)(1+f_2-f_1), \qquad (11.30)\\
\bar{b} &= -(1+b)f_1k + [(1+b)(3f_1-2f_2) - 2(1+f_2-f_1)] + \tfrac{4}{k}(1+b)(1+f_2-f_1),\\
\bar{c} &= [(1+b)f_1 - (3f_1-2f_2)]k + [1-(2b+1)f_1 + (b+1)(3f_1-2f_2) - 2(1+f_2-f_1)]\\
&\qquad + \tfrac{2}{k}(1+b)(1+f_2-f_1).
\end{aligned}$$

We note that $\bar{a} < 0$. After all if $1+f_2-f_1 \le 0$ we have that:

$$\bar{a} \le (2b+1)f_1 - 1 \le (2b+1)(-\tfrac{1}{4}f_2^2) - 1 = -(\tfrac{1}{2}f_2+1)^2 < 0$$

where we used that $2b+1<0$, the fact that in the real case $f_2^2+4f_1 \ge 0$ and the definition of b. If $1+f_2-f_1>0$ then we obtain that $\bar{a}$ is maximal for $k=2$ and we obtain:

$$\bar{a} \le bf_1 + (1+b)f_2 + b \le b(-\tfrac{1}{4}f_2^2) + (1+b)f_2 + b = \tfrac{1}{2f_2}(f_2+2)^2 < 0.$$

We need to verify that (11.29) is negative for all $y_1 \in [-1,1]$. We first verify it is negative in the boundary points. We get for $y_1=-1$ that (11.29) equals:

$$\bar{a} - \bar{b} + \bar{c} = [2(1+b)f_1 - (3f_1-2f_2)]\,k < 0.$$

In the proof of Case 2.1, we already established that for $y_1 = 1$,

$$V_{k_i} - V_{k_{i-1}} < 0.$$

Finally, (11.29) may attain its maximum in the interior where

$$y_1 = -\frac{\bar{b}}{2\bar{a}} \quad \text{with} \quad \left|\frac{\bar{b}}{2\bar{a}}\right| < 1,$$

but then the maximum is less than $\bar{c} - \bar{a}$ and we get

$$\bar{c} - \bar{a} = [(1+b)f_1 - (3f_1 - 2f_2)]\, k + (3-b)f_1 - (2b+4)f_2.$$

Note that the above expression is a linear function of k whose coefficient is negative since $b > -1$, $f_1 < 0$, and $3f_1 - 2f_2 > 0$, and hence it is maximal for all $k \geq 2$ when $k = 2$ and we get:

$$\bar{c} - \bar{a} \leq (b-1)f_1 - 4 = \tfrac{1}{f_2}\left[2f_1 - f_2(f_1+4)\right] \leq \tfrac{1}{f_2}\left[2f_1 + 2(f_1+4)\right] = \tfrac{4}{f_2}(f_1+2) < 0.$$

In other words, for $\tilde{y}_1 = -1$ we have that (11.24) is negative.

It remains to check whether (11.24) is negative for the upper bound for $\tilde{y}_1$. Unfortunately, here we have to distinguish between $k = 2$ and $k > 2$. For $k = 2$ we have

$$\tilde{y}_1 = -1 - (1 + f_2 - f_1)(y_1 + 1) - f_1$$

for the upper bound. We obtain that

$$V_{k_i} - V_{k_{i-1}} = \hat{a}y_1^2 + \hat{b}y_1 + \hat{c}. \tag{11.31}$$

Here we have:

$$\hat{a} = (1+2b)(f_1 - f_2 - 1)^2 - (f_1 + 1) + 2(1+b)(f_2+1),$$

$$\hat{b} = -2(1+b)(f_1 - f_2 - 1)(f_2 + 2) - 2b(f_1 - f_2 - 1)(1 + f_1) + 2(1+b) + 2(f_1 - f_2 - 1),$$

$$\hat{c} = (f_2 + 2)^2 + 2b(f_2+2)(f_1+1) + 2(f_1 - f_2 - 1) + 2f_2 - 3f_1.$$

Using $-4f_1 \leq f_2^2$ we get:

$$\hat{a} \leq (1+2b)(f_1 - f_2 - 1)^2 + \tfrac{1}{4f_2}(f_2+4)(f_2+2)^2 < 0$$

since $1 + 2b < 0$ and $-3 < f_2 < -2$. Therefore the maximum is attained on the boundary or in the interior. We show that $-\frac{\hat{b}}{2\hat{a}} \geq 1$. Since $\hat{a} < 0$, it is equivalent to show that $\hat{b} + 2\hat{a} > 0$ and, with some algebra, we get:

$$\hat{b} + 2\hat{a} = 2(2f_2 - f_1 + 4)[-(f_1 - f_2 - 1) + 1](1 + \tfrac{2}{f_2}) > 0.$$

Thus, for case $k = 2$, $V_{k_i} - V_{k_{i-1}}$ in equation (11.29) is maximal for all $y_1 \in [-1, 1]$ when $y_1 = 1$, and the maximum is $\hat{a} + \hat{b} + \hat{c}$. Next, we show that this is indeed negative. We get:

$$\begin{aligned} \hat{a} + \hat{b} + \hat{c} &= 4(f_2 - \tfrac{1}{2}f_1 + 2)(f_2 + \tfrac{4}{f_2} - \tfrac{1}{2}f_1 + 3) \\ &< 4(f_2 - \tfrac{1}{2}f_1 + 2)(-\tfrac{1}{2}f_1 - 1) < 0. \end{aligned}$$

The following step is to check the upper bound $\tilde{y} = 1$ for $k > 2$. We obtain that

$$\begin{aligned} V_{k_i} - V_{k_{i-1}} = 1 &+ 2bf_1\left[y_1 - (k-1)\right] + \tfrac{2b}{k}(1 - y_1)\left[1 - d_4y_1 - e_4\right] \\ &- \tfrac{2}{k}\left[y_1 - (k-1)\right]\left[1 - d_4y_1 - e_4\right] - f_1\left[y_1 - (k-1)\right]^2 - y_1^2, \end{aligned} \tag{11.32}$$

and we need to show this expression is negative. The expression has the form:

$$\tilde{a}y_1^2 + \tilde{b}y_1 + \tilde{c}. \tag{11.33}$$

Here we have:

$$\begin{aligned} \tilde{a} &= (2b+1)f_1 - 1 + \tfrac{2}{k}(b+1)(1 + f_2 - f_1), \\ \tilde{b} &= -(b+1)kf_1 + b(3f_1 - 2f_2) - 2 - 4f_2 + 5f_1 + \tfrac{1}{k}(-4b - 4f_1 + 4f_2), \\ \tilde{c} &= k(-bf_1 + 2f_2 - 2f_1) + 3 + 4f_1 - 4f_2 + b(2f_2 - f_1) + \tfrac{2}{k}(b-1)(1 - f_2 + f_1). \end{aligned}$$

Since $\tilde{a} = \bar{a}$ and we already showed that $\bar{a}$ is negative, $\tilde{a}$ is negative. In Subsection 11.A.1 we already established that for $y_1 = 1$ we have:

$$V_{k_i} - V_{k_{i-1}} < 0.$$

On the other hand for $y_1 = -1$ we have:

$$\tilde{a} - \tilde{b} + \tilde{c} = k(2f_2 - f_1) + 2b(2f_2 - f_1) + 4 + \frac{8b}{k}.$$

For $k = 3$ we get:

$$6f_2 - 3f_1 + 12 - \tfrac{4}{f_2}f_1 + \tfrac{16}{3f_2} = 6f_2 - (3 + \tfrac{4}{f_2})f_1 + 12 + \tfrac{16}{3f_2} < 6f_2 - 2f_1 + 12 + \tfrac{16}{3f_2}.$$

This upper bound equals to:

$$\tfrac{2}{3}(2f_2 - 3f_1) + \tfrac{1}{3f_2}(f_2 + 2)(14f_2 + 8) < 0.$$

For $k > 3$ we have:

$$\begin{aligned}\tilde{a} - \tilde{b} + \tilde{c} &< k(2f_2 - f_1) + 2b(2f_2 - f_1) + 4 \\ &\leq 4(2f_2 - f_1) + 2b(2f_2 - f_1) + 4 = 8f_2 - 4(1 + \tfrac{1}{f_2})f_1 + 12 \\ &< 8f_2 - \tfrac{8}{3}f_1 + 12 < 0.\end{aligned}$$

It remains to show that if the maximum of (11.33) is attained in the interior, i.e. $y_1 \in (-1, 1)$, the maximum of (11.33) is also negative. As before, we note that the maximum is less than $\tilde{c} - \tilde{a}$ and we get

$$\tilde{c} - \tilde{a} = k[-(b+2)f_1 + 2f_2] + 3(1-b)f_1 + 2(b-2)f_2 + 4 + \tfrac{4}{k}[b(f_1 - f_2) - 1].$$

For $k = 3$ we get:

$$\tilde{c} - \tilde{a} = -(\tfrac{28}{3f_2} + 3)f_1 + 2f_2 + 4 < \tfrac{1}{9}f_1 + 2f_2 + 4 < 0,$$

while for $k > 3$ we get:

$$\tilde{c} - \tilde{a} < 4[-(b+2)f_1 + 2f_2] + 3(1-b)f_1 + 2(b-2)f_2 + 4 = -(7b+5)f_1 + 4(f_2 + 2).$$

If $7b+5<0$ then this expression is negative since $f_1<0$ and $f_2<-2$. On the other hand, if $7b+5>0$ then

$$-(7b+5)f_1+4f_2+8<\tfrac{28}{f_2}+4f_2+18<\tfrac{22}{f_2}+4f_2+16=\tfrac{2}{f_2}\left(2(f_2+2)^2+3\right)<0$$

since $f_1\in(-2,0)$ and $f_2\in(-3,-2)$.

11.A.2.2 The complex case: $f_2^2+4f_1\leq 0$

Next, we study the complex case. We again want to establish that $V_{k_i}-V_{k_{i-1}}<0$. However, in this case, it is not sufficient to consider the case $\tilde{y}_1\in[-1,1]$ and $y_1\in[-1,1]$ since in that case the result is simply not true for f_1 and f_2 sufficiently small. But recall that we ignored the constraints that $y_1(k_{i-1}+j)\leq -1$ for $j=1,\ldots,k-1$. In this case we actually ignore the constraint that $\tilde{y}_1<1$ and replace it by the constraint that $y_1(k_i-1)<-1$. Note that

$$y_1(k_i-1)=d_3y_1+(k-1)y_2+e_3\leq -1, \tag{11.34}$$

where

$$\begin{aligned} d_3&=1+f_2+(k-2)f_1,\\ e_3&=-(k-2)[f_2-f_1+\tfrac{1}{2}f_1(k-1)]. \end{aligned}$$

We also recall (11.20) and (11.21) and we again obtain (11.24). However, this time we want to prove that (11.24) is negative for all y_1 and $\tilde{y}_1$ for which $y_1\in[-1,1]$ and

$$-1\leq\tilde{y}_1\leq -1-(1+f_2-f_1)\tfrac{1}{k-1}(y_1+1)-\tfrac{1}{2}f_1k,$$

where the upper bound for $\tilde{y}_1$ follows from (11.34). We note that (11.24) is a quadratic function in $\tilde{y}_1$ and the quadratic term has coefficient $1+2b\frac{1}{k}$ which is positive for $k\geq 2$ (note that, like in the real case, for $k=1$ we have a linear system without saturation and hence we can trivially verify

$V_{k_i} - V_{k_{i-1}} < 0$). Hence for $k \geq 2$, (11.24) attains its maximum on the boundary where either $\tilde{y}_1 = -1$ or

$$\tilde{y}_1 \leq -1 - (1 + f_2 - f_1)\tfrac{1}{k-1}(y_1 + 1) - \tfrac{1}{2} f_1 k.$$

On the boundary $\tilde{y}_1 = -1$ we have that (11.24) is equal to:

$$\tilde{a}_1 y_1^2 + \tilde{b}_1 y_1 + \tilde{c}_1, \tag{11.35}$$

where

$$\begin{aligned}
\tilde{a}_1 &= (2b+1)f_1 - 1 + \tfrac{2}{k}(b+1)(1 + f_2 - f_1), \\
\tilde{b}_1 &= -kf_1(1+b) - 2(1 - f_1 + f_2) + (b+1)(3f_1 - 2f_2) + \tfrac{4}{k}(b+1)(1 + f_2 - f_1), \\
\tilde{c}_1 &= 2k(f_2 - f_1) + kbf_1 - 1 + 4f_1 - 4f_2 + b(f_1 - 2f_2) + \tfrac{2}{k}(1+b)(1 + f_2 - f_1).
\end{aligned} \tag{11.36}$$

We have

$$\tilde{a}_1 \leq (1+b)f_2 + bf_1 + b = \tfrac{f_2}{2f_1}(f_1 - f_2 - 1) < 0,$$

where we used that $\tilde{a}_1$ is maximal for $k = 2$ and that in the complex case, where $f_2^2 + 4f_1 \leq 0$, we have that $2bf_1 = -f_2$ and

$$1 + f_2 - f_1 > 0. \tag{11.37}$$

We note that

$$\tilde{b}_1 - 2\tilde{a}_1 = -kf_1(1+b) + (1+b)(3f_1 - 2f_2) > 0$$

since $b > -1$, $f_1 < 0$, $k > 0$ and $3f_1 - 2f_2 > 0$. This implies that

$$-\tfrac{\tilde{b}_1}{2\tilde{a}_1} > -1,$$

which implies that (11.35) attains its maximum for $y_1 > -1$ (recall that $y_1 \in [-1, 1]$). Next we assume that the maximum is attained for $y_1 \in (-1, 1)$ which implies that $y_1 = -\frac{\tilde{b}_1}{2\tilde{a}_1}$. The maximum is then equal to

$$\tilde{c}_1 - \frac{\tilde{b}_1^2}{4\tilde{a}_1} < \tilde{c}_1 - \tilde{a}_1,$$

where we used that $\tilde{a}_1 < 0$ and $|\frac{\tilde{b}_1}{2\tilde{a}_1}| < 1$. In that case, we obtain:

$$\tilde{c}_1 - \tilde{a}_1 = k[(1+b)f_1 - (3f_1 - 2f_2)] + [-1 + 3f_1 - 4f_2 - b(f_1 + 2f_2)],$$

which is maximal for $k = 2$ and hence we obtain:

$$\tilde{c}_1 - \tilde{a}_1 \leq -1 - f_1 + b(f_1 - 2f_2) = -1 - f_1 - \tfrac{1}{2}f_2 + \tfrac{f_2}{f_1}f_2 < -1 - f_1 + f_2 < 0,$$

where we used that $2f_2 < 3f_1$. It remains to show that (11.35) is negative if the maximum is attained for $y_1 = 1$. In that case the maximum equals to

$$\tilde{a}_1 + \tilde{b}_1 + \tilde{c}_1 = k(2f_2 - 3f_1) - 4 + 10f_1 - 8f_2 + 2b(3f_1 - 2f_2) + \tfrac{8}{k}(1+b)(1 + f_2 - f_1),$$

which is maximal for $k = 2$ and hence less than or equal to:

$$b(4 + 2f_1)$$

which is negative. The above establishes that $V_{k_i} - V_{k_{i-1}} < 0$ if it attains its maximum on the boundary where $\tilde{y}_1 = -1$. It remains to show that $V_{k_i} - V_{k_{i-1}} < 0$ if it attains its maximum on the other boundary where

$$\tilde{y}_1 = -1 - (1 + f_2 - f_1)\tfrac{1}{k-1}(y_1 + 1) - \tfrac{1}{2}f_1 k.$$

In that case we get that $V_{k_i} - V_{k_{i-1}}$ as given in (11.24) is equal to:

$$\tilde{a}_2(y_1 + 1)^2 + \tilde{b}_2(y_1 + 1) + \tilde{c}_2, \tag{11.38}$$

where

$$\tilde{a}_2 = (1+2b)\left[\tfrac{1}{k-1}(1+f_2-f_1)\right]^2 + \tfrac{2}{k-1}(b+1)(1+f_2-f_1) - (1+f_2-f_1),$$

$$\tilde{b}_2 = \tfrac{1}{k-1}(1+f_2-f_1)\left[(1+b)f_1 + b(3f_1-2f_2)\right] + (1+f_2-f_1)f_1(1+2b)$$

$$+ (b+1)(3f_1-2f_2) - (1+b)f_1(k-1),$$

$$\tilde{c}_2 = (f_2-f_1)\left[-\tfrac{1}{4}k^2 f_1 + k(1+\tfrac{1}{2}f_2)\right].$$

It is easily verified that in the region of interest we have that $\tilde{c}_2 < 0$ since:

$$-\tfrac{1}{4}k^2 f_1 + k(1+\tfrac{1}{2}f_2) > -\tfrac{1}{4}k^2 f_1 + k(1+f_1) = -\tfrac{1}{4}(k-2)^2 f_1 + k + f_1 > 0.$$

Secondly, for $\tilde{b}_2$ it is easily verified that the coefficient of $1/(k-1)$ is negative while the coefficient of $(k-1)$ is positive which implies that $\tilde{b}_2$ is increasing in k and attains its maximum for $k=2$. Moreover for $k=2$ we find that $\tilde{b}_2$ is equal to:

$$2(b+1)(f_1-f_2)(2+f_2-f_1) > 0.$$

We therefore note that $\tilde{b}_2 > 0$ for $k \geq 2$. Finally, $\tilde{a}_2 < 0$ in the region of interest since $1+2b<0$ while $b+1>0$ and $1+f_2-f_1>0$ imply:

$$\tfrac{2}{k-1}(b+1)(1+f_2-f_1) - (1+f_2-f_1) < 2(b+1)(1+f_2-f_1) - (1+f_2-f_1) < 0.$$

We need to show that (11.38) is negative for all $y_1 \in [-1,1]$. We use the following two bounds:

$$\tilde{a}_2 < \bar{a}_2 = -\tfrac{1}{2}\left[\tfrac{1}{k-1}(1+f_2-f_1)\right]^2 - \tfrac{1}{2}(1+f_2-f_1),$$

$$\tilde{b}_2 < \bar{b}_2 = (1+f_2-f_1)\tfrac{1}{k-1}(f_1-\tfrac{1}{2}f_2) + (1+f_2-f_1)(f_1-f_2) - 2f_2 + 3f_1 - (f_1-\tfrac{1}{2}f_2)(k-1),$$

where we used that $2bf_1 = -f_2$ and we note that $2f_1 - f_2 < 0$ in our region of interest.

We get that (11.38) is negative for $y_1 = -1$ since $\tilde{c}_2 < 0$. For $k=2$, it is also negative for $y_1 = 1$ since:

$$4\tilde{a}_2 + 2\tilde{b}_2 + \tilde{c}_2 = \tfrac{1}{f_1}(f_1-f_2)(f_1-f_2-2)(f_1-2f_2-4) < 0.$$

For $y_1 = 1$ and $k = 3$ we get

$$\begin{aligned}
4\tilde{a}_2 + 2\tilde{b}_2 + \tilde{c}_2 &= \tfrac{1}{4}(4 - 2f_2^2 + f_2 f_1 + f_1^2) - \tfrac{1}{f_1}(3f_2 + f_2^2) \\
&< \tfrac{1}{4}(-14 - 6f_2 - 2f_2^2 + f_2 f_1 + f_1^2) \\
&< \tfrac{1}{4}(-14 - 8f_2 - 2f_2^2 + f_1^2) \\
&= \tfrac{1}{4}(f_1^2 - 4) - \tfrac{1}{2}(1 + (f_2 + 2)^2) < 0,
\end{aligned}$$

where in the first inequality we used that $3f_1 > 2f_2$ and $f_2 > -3$ while for $k > 3$ we have:

$$4\tilde{a}_2 + 2\tilde{b}_2 + \tilde{c}_2 \leq 4\bar{a}_2 + 2\bar{b}_2 + \tilde{c}_2$$

and

$$\begin{aligned}
4\bar{a}_2 + 2\bar{b}_2 + \tilde{c}_2 = -2\left[\tfrac{1}{k-1}(1 + f_2 - f_1)\right]^2 - \tfrac{1}{4}k(f_2 - f_1)(kf_1 - 2f_2) + (k-2)(2f_2 - 3f_1) \\
+ 2(f_1 - \tfrac{1}{2}f_2) + 2(1 + f_2 - f_1)\left[\tfrac{1}{k-1}(f_1 - \tfrac{1}{2}f_2) - (1 + f_2 - f_1)\right] < 0.
\end{aligned}$$

It remains to prove that if the maximum of (11.38) is also negative if (11.38) attains its maximum in the interior of the interval $(-1, 1)$. We get

$$\tilde{a}_2(y_1 + 1)^2 + \tilde{b}_2(y_1 + 1) + \tilde{c}_2 \leq \tilde{c}_2 - \frac{\tilde{b}_2^2}{4\tilde{a}_2} < \tilde{c}_2 + \tilde{b}_2 < \tilde{c}_2 + \bar{b}_2,$$

where we used that the maximum is attained in the interior and hence

$$-\frac{\tilde{b}_2}{2\tilde{a}_2} < 2.$$

We find that

$$\begin{aligned}
\tilde{c}_2 + \bar{b}_2 = -\tfrac{1}{4}(f_2 - f_1)(k^2 f_1 - 2kf_2) - 2k(f_1 - \tfrac{3}{4}f_2) + (1 + f_2 - f_1)\tfrac{1}{k-1}(f_1 - \tfrac{1}{2}f_2) \\
- (f_1 - f_2)^2 + (5f_1 - \tfrac{7}{2}f_2).
\end{aligned}$$

The first term is decreasing in k for $k \geq 2$ since $2f_1 - f_2 < 0$ and $f_2 - f_1 < 0$. It is then easy to verify that this complete upper bound is decreasing in k and therefore is maximal for $k = 2$ and we get:

$$\tilde{c}_2 + \bar{b}_2 \leq (f_1 - \tfrac{1}{2}f_2)(2 - f_1 + f_2) < 0.$$

11.A.3 Case 2.3

We consider the case when $|\tilde{y}_1| \geq 1$. Due to the symmetry, we only need to consider the case that $\tilde{y}_1 \leq -1$. The case where $\tilde{y}_1 \geq 1$ then follows trivially. As before we define

$$k = k_i - k_{i-1}.$$

By definition, we have $k \geq 1$. On the other hand, $k = 1$ would imply:

$$y_1(k_i - 1) \geq -1, \quad y_1(k_i) \leq -1, \quad y_1(k_i + 1) \geq 1$$

and it is easily verified, given the system dynamics (9.15), that this can only happen if $f_1 > 2f_2 + 4$ which contradicts the Jury conditions (11.3). Therefore we only need to address the case where $y_1(k_{i-1} + j) = -1$ for $j = 1, \ldots, k$ with $k \geq 2$.

The proof is split into two cases: the real case where $f_2^2 + 4f_1 \geq 0$ and the complex case where $f_2^2 + 4f_1 \leq 0$.

11.A.3.1 The real case: $f_2^2 + 4f_1 \geq 0$

From the system equations (9.15), we have

$$y_1(k_i + 1) = y_1(k_{i-1} + k + 1) = d_5 y_1 + (k+1)y_2 + e_5,$$

where $d_5 = 1 + f_2 + kf_1$ and $e_5 = -k(f_2 + \frac{1}{2}(k-1)f_1)$. Since $y_1(k_i + 1) \geq 1$, we get

$$y_2 \geq \tfrac{1}{k+1}(1 - e_5 - d_5 y_1). \tag{11.39}$$

As argued before, our Lyapunov has a constant decay, which is given in (11.11), if we are in -1 for two consecutive time instants, thus we analyse $V_{k_i} - V_{k_{i-1}}$ as:

$$
\begin{aligned}
V_{k_i} - V_{k_{i-1}} &= V_{k_{i-1}+1} - V_{k_{i-1}} + (k-1)[(2b-1)f_1 + 2f_2] \\
&= -2(1 + bf_1 + f_2)y_1 - 2(1+b)(1+y_1)y_2 - 1 - (f_1+1)y_1^2 \\
&\qquad + (k-1)[(2b-1)f_1 + 2f_2].
\end{aligned}
$$

Since $b > -1$ and $-1 < y_1 < 1$, the term $-2(1+b)(1+y_1)y_2$ is maximal for minimal y_2 and using the bound (11.39), we get:

$$
\begin{aligned}
V_{k_i} - V_{k_{i-1}} &\leq -2(1 + bf_1 + f_2)y_1 - 2(1+b)(1+y_1)\tfrac{1}{k+1}(1 - e_5 - d_5 y_1) \\
&\qquad - 1 - (f_1+1)y_1^2 + (k-1)[(2b-1)f_1 + 2f_2] \\
&= \bar{a}_3 y_1^2 + \bar{b}_3 y_1 + \bar{c}_3,
\end{aligned}
\tag{11.40}
$$

where

$$
\begin{aligned}
\bar{a}_3 &= [(2b+1)f_1 - 1] + \tfrac{2}{k+1}(1+b)(f_2 - f_1 + 1), \\
\bar{b}_3 &= -(1+b)f_1 k + (1+b)(4f_1 - 2f_2) - 2(1 + bf_1 + f_2) + \tfrac{4}{k+1}(1+b)(f_2 - f_1), \\
\bar{c}_3 &= [(1+b)f_1 + 2f_2 - 3f_1]k + 2(1+b)(f_1 - f_2) - (2b-1)f_1 - 2f_2 - 1 \\
&\qquad + \tfrac{2}{k+1}(1+b)(f_2 - f_1 - 1).
\end{aligned}
$$

Note that $\bar{a}_3 < 0$ since it is the same as $\bar{a}$ given in (11.30) (with k replaced by $k+1$), which has been shown to be negative. Next, let us show that $V_{k_i} - V_{k_{i-1}} < 0$.

We show that $-\frac{\bar{b}_3}{2\bar{a}_3} > 1$ for all $k \geq 1$, which implies that our bound for $V_{k_i} - V_{k_{i-1}}$ for all $k \geq 1$

is maximal at $y_1 = 1$. Since $\bar{a}_3 < 0$ this implies that we need to show that $\bar{b}_3 + 2\bar{a}_3 > 0$. We get

$$\bar{b}_3 + 2\bar{a}_3 = -(1+b)f_1 k + (1+b)(4f_1 - 2f_2) - 2(1 + bf_1 + f_2) + 2[(1+2b)f_1 - 1] + \frac{4}{k+1}(1+b)(2f_2 - 2f_1 + 1).$$

Since $2f_2 - 2f_1 + 1 < 0$ and $1 + b > 0$ in the region of interest we find:

$$\begin{aligned} \bar{b}_3 + 2\bar{a}_3 &> -(1+b)f_1 + (1+b)(4f_1 - 2f_2) - 2(1 + bf_1 + f_2) + 2[(1+2b)f_1 - 1] \\ &\quad + 2(1+b)(2f_2 - 2f_1 + 1) \\ &= (1+b)(2+f_1) > 0, \end{aligned}$$

where we used that $bf_2 = 2$ and $1 + b > 0$ and $f_1 > -2$. We find that our bound for $V_{k_i} - V_{k_{i-1}}$ for all $k \geq 1$ is maximal at $y_1 = 1$ and hence:

$$V_{k_i} - V_{k_{i-1}} \leq \bar{a}_3 + \bar{b}_3 + \bar{c}_3,$$

and we find:

$$\bar{a}_3 + \bar{b}_3 + \bar{c}_3 = k(2f_2 - 3f_1) + 4(2f_1 - 2f_2 - 1) + 4b(f_1 - f_2) - \frac{8}{k+1}(1+b)(f_1 - f_2). \quad (11.41)$$

It can be verified that it is negative provided $f_1 \geq -1.6$.

11.A.3.2 The complex case: $f_2^2 + 4f_1 \leq 0$

Similarly as in the proof for the real case we obtain (11.40) with the same expressions for $\bar{a}_3$, $\bar{b}_3$ and $\bar{c}_3$. This time the fact that $\bar{a}_3 < 0$ follows from the fact that it is the same as $\tilde{a}_1$ as defined in (11.36) (with k replaced by $k+1$) which has been shown to be negative.

We first show that the bound in (11.40) is negative for $y_1 = -1$ and $y_1 = 1$ respectively. We get for $y_1 = -1$:

$$\bar{a}_3 - \bar{b}_3 + \bar{c}_3 = (f_2 - f_1)k < 0.$$

On the other hand for $y_1 = 1$ we get the same expression (11.41) as in the real case. It can be verified that it is negative provided $f_1 \geq -1.6$ since (11.40) is less than:

$$3(2f_2 - 3f_1) + 4(2f_1 - 2f_2 - 1) + 4b(f_1 - f_2) = 2f_2 - f_1 - 4 + 4b(f_1 - f_2)$$

which is negative for $f_1 \geq -1.6$. It remains to check that, if $f_1 \geq -1.6$, then (11.40) is also negative if the maximum is attained in the interior. Using the same arguments as before and the fact that $\bar{a}_3 < 0$, we find that the maximum is less than:

$$\bar{c}_3 - \bar{a}_3 = k(\tfrac{3}{2}f_2 - 2f_1) + 2f_1 - 3f_2 - 2bf_2 - \frac{4}{k+1}(1+b)$$

and we find that:

$$\bar{c}_3 - \bar{a}_3 < (\tfrac{3}{2}f_2 - 2f_1) + 2f_1 - 3f_2 - 2bf_2 = b(3f_1 - 2f_2) < 0.$$

11.A.4 Case 2.4

We again investigate the real and complex case separately:

11.A.4.1 The real case: $f_2^2 + 4f_1 \geq 0$

From the system equations (9.15), we have

$$y_1(k_i + 1) = y_1(k_{i-1} + k + 1) = y_1 + (k+1)y_2 + e_2.$$

where $e_2 = f_2 + k(f_1 - f_2) - \frac{f_1}{2}k(k-1)$ and, for ease of presentation, we denote $y_1(k_{i-1}) = y_1$ and $y_2(k_{i-1}) = y_2$. Since $y_1(k_i + 1) \geq 1$, we get

$$y_2 \geq \tfrac{1}{k+1}(1 - y_1 - e_2). \tag{11.42}$$

As noted before, if $\sigma(y_1)$ stays at 1 for two consecutive time instants, our Lyapunov candidate actually have a constant decay, which is given in (11.11). Therefore, we obtain:

$$\begin{aligned}V_{k_i} - V_{k_{i-1}} &= V_{k_{i-1}+1} - V_{k_{i-1}} + (k-1)[(2b-1)f_1 + 2f_2] \\ &= -4y_1 - 4(1+b)y_2 - (1+2b)f_1 - 2f_2 + (k-1)[(2b-1)f_1 + 2f_2].\end{aligned}$$

Since $b > -1$, the term $-4(1+b)y_2$ is maximal for minimal y_2, i.e. (11.42), thus, we get:

$$\begin{aligned}V_{k_i} - V_{k_{i-1}} \leq{} & [-4 + 4(1+b)\tfrac{1}{k+1}]y_1 - 4(1+b)\tfrac{1}{k+1}(1-c) \\ & - (1+2b)f_1 - 2f_2 + (k-1)[(2b-1)f_1 + 2f_2].\end{aligned}$$

Since $b < 0$, we have that $-4+4(1+b)\frac{1}{k+1} < 0$ for all $k \geq 2$ and hence the upper bound is maximal for minimal y_1, i.e., $y_1 = 1$. With some algebra, we get

$$V_{k_i} - V_{k_{i-1}} \leq \bar{a}_4 k + \bar{b}_4 + \bar{c}_4 \tfrac{1}{k+1}, \tag{11.43}$$

where

$$\begin{aligned}\bar{a}_4 &= 2f_2 - 3f_1, \\ \bar{b}_4 &= -4 + 4(b+2)(f_1 - f_2), \\ \bar{c}_4 &= 8(1+b)(f_2 - f_1).\end{aligned}$$

This is equal to the upper bound found in (11.41) and it can be verified that, in the real case, this bound is negative provided $f_1 \geq -1.6$.

11.A.4.2 The complex case: $f_2^2 + 4f_1 \leq 0$

We get the same expressions as in the proof for the real case $f_2^2 + 4f_1 \geq 0$ resulting in (11.43). This is equal to the upper bound found in (11.41) and it can be verified that, in the complex case, this bound is negative provided $f_1 \geq -1.6$.

11.B Proof of Theorem 11.2 in case $f_1 < -1.6$

In this appendix, we will establish that (11.13) is satisfied. For simplicity we denote $y_1(k_{i-1})$ and $y_2(k_{i-1})$ by y_1 and y_2 respectively while $y_1(k_i)$ and $y_2(k_i)$ are denoted by $\tilde{y}_1$ and $\tilde{y}_2$ respectively.

We will prove that Lyapunov candidate will decay for the following cases depending on whether y_1, $\tilde{y}_1$, and $y_1(k_{i+1})$ is saturated or not.

- Case 3.1: $|y_1| \geq 1$ and $|\tilde{y}_1| \leq 1$,
- Case 3.2: $|y_1| \leq 1$ and $|\tilde{y}_1| \leq 1$,
- Case 3.3: $|y_1| \geq 1$, $|\tilde{y}_1| \geq 1$ and $|y_1(k_{i+1})| \geq 1$.
- Case 3.4: $|y_1| \geq 1$, $|\tilde{y}_1| \geq 1$ and $|y_1(k_{i+1})| \leq 1$.
- Case 3.5: $|y_1| \leq 1$, $|\tilde{y}_1| \geq 1$ and $|y_1(k_{i+1})| \geq 1$.
- Case 3.6: $|y_1| \leq 1$, $|\tilde{y}_1| \geq 1$ and $|y_1(k_{i+1})| \leq 1$.

For Case 3.1 and 3.2 we can establish that (11.12) is satisfied using the arguments in Subsections 11.A.1 and 11.A.2. This immediately implies (11.13). However, we still need to address the last four cases.

However, in the last four cases, where $f_1 < -1.6$, the problem we have is that the Lyapunov candidate given in (11.9) increases during the interval $[k_{i-1}, k_i]$, for some initial conditions y_1 and y_2, i.e.

$$V_{k_i} - V_{k_{i-1}} = -4y_1 - 4(1+b)y_2 - (1+2b)f_1 - 2f_2 + (k-1)[(2b-1)f_1 + 2f_2] > 0 \qquad (11.44)$$

when $y_1 = y_1(k_{i-1}) > 1$ or

$$V_{k_i} - V_{k_{i-1}} = -2(1 + bf_1 + f_2)y_1 - 2(1 + b)(1 + y_1)y_2 - 1 - (f_1 + 1)y_1^2 \\ + (k - 1)[(2b - 1)f_1 + 2f_2] > 0 \quad (11.45)$$

when $y_1 = y_1(k_{i-1}) \in [-1, 1]$, where $k = k_i - k_{i-1}$. We will show that if (11.44) or (11.45) is positive then $V(k_{i+1}) - V(k_{i-1}) < 0$. We proceed to show this all the last four cases and we will use the notation that $\ell = k_{i+1} - k_i$.

11.B.1 Case 3.3

Due to the symmetry, we only need to consider the case where $y_1(k_i) \leq -1$ (or equivalently $y_1(k_{i-1}) \geq 1$), and then $\sigma(y_1)$ switches from $+1$ to -1, and stays at -1 for k steps, after which it switches to $+1$ and stays at $+1$ for ℓ steps, and finally $y_1(k_{i+1} + 1) \leq -1$. Clearly for this case, we need $k \geq 2$, and $\ell \geq 2$.

We obtain

$$V_{k_{i+1}} - V_{k_i} = 4y_1(k_i) + 4(1 + b)y_2(k_i) - (1 + 2b)f_1 - 2f_2 + (\ell - 1)[(2b - 1)f_1 + 2f_2]. \quad (11.46)$$

Combining (11.44) and (11.46), we get

$$V_{k_{i+1}} - V_{k_{i-1}} = 4[ky_2 + e_1] - 4(1 + b)(k - 2)f_1 - 2(1 + 2b)f_1 - 4f_2 \\ + (k + \ell - 2)[(2b - 1)f_1 + 2f_2]. \quad (11.47)$$

where e_1 is defined in (11.15). We need to show that $V_{k_{i+1}} - V_{k_{i-1}} < 0$ given (11.44) and the following constraints on y_1 and y_2:

$$\begin{aligned}
y_1(k_{i-1}) &= y_1 && \geq 1 && (11.48)\\
y_1(k_{i-1}+1) &= y_1 + y_2 + f_2 && \leq -1 && (11.49)\\
\vdots \quad & \quad \vdots && && \\
y_1(k_i) &= y_1 + ky_2 + e_1 && \leq -1 && (11.50)\\
y_1(k_i+1) &= y_1 + (k+1)y_2 + e_5 && \geq 1 && (11.51)\\
\vdots \quad & \quad \vdots && && \\
y_1(k_{i+1}) &= y_1 + (k+\ell)y_2 + e_1 + e_6 - (k-2)\ell f_1 && \geq 1 && (11.52)\\
y_1(k_{i+1}+1) &= y_1 + (k+\ell+1)y_2 + e_1 + e_7 - (k-1)(\ell+1)f_1 && \leq -1 && (11.53)
\end{aligned}$$

where $k = k_i - k_{i-1}$ and $\ell = k_{i+1} - k_i$ and

$$\begin{aligned}
e_5 &= f_2 + k(f_1 - f_2) - \tfrac{f_1}{2}k(k-1),\\
e_6 &= -f_2 - (\ell-1)(f_1 - f_2) + \tfrac{f_1}{2}(\ell-1)(\ell-2),\\
e_7 &= -f_2 - \ell(f_1 - f_2) + \tfrac{f_1}{2}\ell(\ell-1),
\end{aligned} \tag{11.54}$$

We first note that if $k = 2$ we get

$$y_1(k_{i-1}) = y_1 \geq 1, \qquad y_1(k_i+2) = y_1 + 4y_2 \geq 1$$

and then

$$-1 \geq y_1(k_i) = y_1 + 2y_2 + f_1 > 1 + f_1,$$

which yields a contradiction with $f_1 > -2$. Therefore we have $k \geq 3$. We claim that $\ell \geq k - 4$.

Since $\ell \geq 2$ we only need to prove this property for $k \geq 6$. We have:

$$y_1(k_i + j) = y_1 + (k+j)y_2 + (k-j)(f_1 - f_2) - \tfrac{f_1}{2}(k-1)(k-2) + \tfrac{f_1}{2}(j-1)(j-2)$$

$$- (k-2)jf_1$$

$$= \tfrac{k+j-1}{k}(y_1 + (k+1)y_2) - \tfrac{j-1}{k}(y_1 + y_2) + (k-j)(f_1 - f_2)$$

$$- \tfrac{f_1}{2}(k-1)(k-2) + \tfrac{f_1}{2}(j-1)(j-2) - (k-2)jf_1.$$

Using the inequalities (11.49) and (11.51) in the above we get:

$$y_1(k_i + j) \geq \tfrac{k+j-1}{k}(1 - e_5) + \tfrac{j-1}{k}(1 + f_2) + (k-j)(f_1 - f_2) - \tfrac{f_1}{2}(k-1)(k-2)$$

$$+ \tfrac{f_1}{2}(j-1)(j-2) - (k-2)jf_1$$

$$= \tfrac{k+2j-2}{k} - f_2 - (2j-1)(f_1 - f_2) + \tfrac{1}{2}f_1 k + \tfrac{1}{2}f_1 j^2 - \tfrac{1}{2}f_1 kj + \tfrac{1}{2}f_1.$$

Note that this lower bound is a concave function in j. Therefore, if this larger bound is larger than or equal to 1 for $j = 1$ and $j = k-4$ then it is larger than for all j satisfying $1 \leq j \leq k-4$ and this implies that $\ell \geq k-4$. For $j = 1$ the lower bound is actually equal to 1, while for $j = k-4$ we find:

$$y_1(k_i + k - 4) \geq \tfrac{3k-10}{k} + (k-5)(2f_2 - \tfrac{7}{2}f_1).$$

For $f_1 < -1.6$, we have $2f_2 - \frac{7}{2}f_1 > 0$ and hence this expression is increasing in k. The minimum is achieved for $k = 6$ and we get

$$y_1(k_i + k - 4) \geq \tfrac{4}{3} + 2f_2 - \tfrac{7}{2}f_1,$$

which is larger than 1 in the critical region. This completes the proof that $\ell \geq k-4$.

Using (11.48) and (11.50) in (11.47) we get:

$$V_{k_{i+1}} - V_{k_{i-1}} \leq -8 - 4(1+b)(k-2)f_1 - 2(1+2b)f_1 - 4f_2$$

$$+ (k + \ell - 2)[(2b-1)f_1 + 2f_2].$$

If we use $\ell \geq k$ we get:

$$\begin{aligned}
V_{k_{i+1}} - V_{k_{i-1}} &\leq -8 - 4(1+b)(k-2)f_1 - 2(1+2b)f_1 - 4f_2 \\
&\quad + (2k-2)[(2b-1)f_1 + 2f_2] \\
&= -8 + 8f_1 - 8f_2 + k(-6f_1 + 4f_2) \\
&< -8 - 4f_1 \\
&< 0
\end{aligned}$$

for $k \geq 2$. Therefore, it only remains $h = k - \ell \in \{1, 2, 3, 4\}$. Inequality (11.53) combined with inequality (11.48) yields:

$$(k + \ell + 1)y_2 \leq -2 + (k-1)(\ell+1)f_1 - (e_1 + e_7)$$

and working this out using the definitions of e_1 and e_7 we get:

$$(2k - h + 1)y_2 \leq -2 + \left[k^2 - k + 1 - \tfrac{1}{2}h(h+1)\right] f_1 + (h-1)f_2.$$

Using this bound in (11.47) we get:

$$\begin{aligned}
V_{k_{i+1}} - V_{k_{i-1}} &\leq -2k^2 f_1 + 4kf_1 + hf_1 - 2hf_2 - 2hbf_1 \\
&\qquad + \tfrac{4k}{2k-h+1}\left[-2 + \left[k^2 - k + 1 - \tfrac{1}{2}h(h+1)\right] f_1 + (h-1)f_2\right].
\end{aligned}$$

Rewriting this equation we obtain:

$$\begin{aligned}
V_{k_{i+1}} - V_{k_{i-1}} &\leq -4 + 2kf_1 + (2-h^2)f_1 - 2f_2 - 2hbf_1 \\
&\qquad + \tfrac{2h-2}{2k-h+1}\left[-2 + \left[k^2 - k + 1 - \tfrac{1}{2}h(h+1)\right] f_1 + (h-1)f_2\right].
\end{aligned}$$

It is easily verified that this upper bound is negative in the specified region for $h = 1$. For $h = \{2, 3, 4\}$ we want to show this upper bound is decreasing in k and therefore we differentiate the upper bound with respect to k. This results in:

$$\frac{2(h-1)}{(2k-h+1)^2}\left[\left(\tfrac{4}{h-1}+2\right)k^2 f_1-(2+2h)kf_1+4+(2-2h)f_2+(h^2+3h-4)f_1\right].$$

This is clearly negative provided that

$$\left(\tfrac{4}{h-1}+2\right)k^2 f_1-(2+2h)kf_1+4+(2-2h)f_2+(h^2+3h-4)f_1 \tag{11.55}$$

is negative. This is a simple quadratic function in k which achieves its maximum for $k=(h-1)/2<3$. In our case, we know $k=\ell+h\geq 2+h$. Using that in (11.55) we get:

$$4-2(h-1)f_2+\left[(h+4)(h+5)+\frac{36}{h-1}\right]f_1,$$

which is easily checked to be negative in the specified region for $h=2,3,4$. This proves that our upper bound for $V_{k_{i+1}}-V_{k_{i-1}}$ is decreasing in k and hence we only need to verify the worst case when $k=h+2$. In that case, we find

$$V_{k_{i+1}}-V_{k_{i-1}}\leq\frac{1}{h+5}\Big[-8h-16+(h^2+17h+24)f_1+(2h^2-6h-8)f_2-2(h^2+5h)f_1 b\Big].$$

It is then straightforward to verify that for the three remaining cases $h=2,3,4$ we have that $V_{k_{i+1}}-V_{k_{i-1}}$ is negative. This completes the proof.

11.B.2 Case 3.4

Again, due to the symmetry, we only need to consider the case where $y_1(k_i)\leq -1$ (or equivalently $y_1(k_{i-1})\geq 1$), and then $\sigma(y_1)$ switches from $+1$ to -1 and stays at -1 for k steps, then it switches to $+1$ and stays at $+1$ for $\ell-1$ steps, and finally $y_1(k_{i+1})\in[-1,1]$. Clearly for this case, we need $k\geq 2$, and $\ell\geq 1$.

The arguments used in the case where $y_1(k_{i-1})$ and $y_1(k_{i+1})$ are both saturated immediately imply that we have $\ell\geq k-3$ for $k\geq 4$. Note that $\ell=k-4$ is in our case not possible since in the earlier argument it was shown that $y_1(k_i+k-4)>1$ while we currently consider the case that $y_1(k_{i+1})$ is unsaturated. We claim that we also have that $\ell \leq k$. We first note that the bounds (11.48)-(11.51) are still valid when $y(k_{i+1})$ is unsaturated. However (11.52) and (11.53) no longer hold and instead we have:

$$y_1(k_{i+1}-1) = y_1 + (k+\ell-1)y_2 + e_1 + e_8 - (k-2)(\ell-1)f_1 \quad \geq 1$$

$$y_1(k_{i+1}) = y_1 + (k+\ell)y_2 + e_1 + e_6 - (k-2)\ell f_1 \quad \geq -1 \tag{11.56}$$

$$y_1(k_{i+1}) = y_1 + (k+\ell)y_2 + e_1 + e_6 - (k-2)\ell f_1 \quad \leq 1 \tag{11.57}$$

where

$$e_8 = -f_2 - (\ell-2)(f_1-f_2) + \tfrac{f_1}{2}(\ell-2)(\ell-3).$$

Now if we assume that $\ell > k$ then we have:

$$y_1(k_i) = y_1 + ky_2 + e_1 \quad \leq -1 \tag{11.58}$$

$$y_1(k_i+k) = y_1 + 2ky_2 - (k-2)kf_1 \quad \geq 1 \tag{11.59}$$

together with (11.48). We obtain from (11.59) that:

$$ky_2 \geq \tfrac{1}{2}(1 - y_1 + (k-2)kf_1).$$

Using this combined with (11.48) in (11.58) we get that

$$1 + \tfrac{1}{2}(k-2)kf_1 + e_1 \leq -1,$$

which yields that

$$f_1 + \tfrac{1}{2}(k-2)(3f_1 - 2f_2) \leq -2.$$

We obtain a contradiction since $k \geq 2$, $f_1 > -2$ and $3f_1 - 2f_2 > 0$ and hence we must have that

$\ell \leq k$. We obtain

$$V_{k_{i+1}} - V_{k_i} = (y_1(k_i) + \ell y_2(k_i) + e_6)^2 + 2b(y_1(k_i) + \ell y_2(k_i) + e_6)[y_2(k_i) + (\ell - 2)f_1]$$
$$- 2(\ell - 2)y_2(k_i) - (\ell - 2)^2 f_1 + 2y_1(k_i) + 1 + 2by_2(k_i),$$

which yields:

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} = {} & \{y_1 + ky_2 + e_1 + \ell[y_2 - (k-2)f_1] + e_6\}^2 \\ & + 2b\{y_1 + ky_2 + e_1 + \ell[y_2 - (k-2)f_1] + e_6\}[y_2 + (\ell - k)f_1] \\ & - 2(\ell - 2)[y_2 - (k-2)f_1] - (\ell - 2)^2 f_1 \\ & + 2[y_1 + ky_2 + e_1] + 1 + 2b[y_2 - (k-2)f_1] \\ & - 4y_1 - 4(1+b)y_2 - (1+2b)f_1 - 2f_2 + (k-1)[(2b-1)f_1 + 2f_2]. \end{aligned} \tag{11.60}$$

We will show that $V_{k_{i+1}} - V_{k_{i-1}} < 0$ for all y_1 and y_2 satisfying (11.48), (11.56) and (11.57). By ignoring some of the constraints we actually prove that $V_{k_{i+1}} - V_{k_{i-1}} < 0$ for a larger class of y_1 and y_2.

Note that the coefficient of y_2^2 term in (11.60) is $(k+\ell)^2 + 2b(k+\ell)$ which is positive since $b > -1$ and $k + \ell \geq 3$. Thus, $V_{k_{i+1}} - V_{k_{i-1}}$ is maximal as a function of y_2 if y_2 takes a boundary value. Recall that we ignore all contraints on y_1 and y_2 except (11.48), (11.56) and (11.57). Hence a boundary value for y_2 implies that either (11.56) or (11.57) is an equality.

In case (11.56) is an equality we get:

$$(2k - h)y_2 = -1 + (-\tfrac{1}{2}h^2 - \tfrac{1}{2}h + k^2 - 2k)f_1 + hf_2 - y_1,$$

where $\ell = k - h$ with $h \in \{0, 1, 2, 3\}$. This yields:

$$V_{k_{i+1}} - V_{k_{i-1}} = 2 - 2y_1 + \frac{2h - 4b}{2k - h}\left[-1 + (-\tfrac{1}{2}h^2 - \tfrac{1}{2}h + k^2 - 2k)f_1 + hf_2 - y_1\right] + 2hbf_1 - h^2 f_1.$$

We note that this expression is linear in y_1 with a negative coefficient and hence it is maximal for $y_1 = 1$ given (11.48). We obtain:

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} &\leq \frac{2h-4b}{2k-h}\left[-2 + (-\tfrac{1}{2}h^2 - \tfrac{1}{2}h + k^2 - 2k)f_1 + hf_2\right] - (h-2b)if_1 \\ &\leq \frac{h-2b}{2k-h}\left[-4 + (2k^2 - 2(2+h)k - h)f_1 + 2hf_2\right]. \end{aligned}$$

The sign of the above upper bound is determined by the sign of

$$-4 + (2k^2 - 2(2+h)k - h)f_1 + 2hf_2.$$

This expression is decreasing in k given that $k \geq 2$ and $k \geq h+1$ and hence the maximum is obtained for $k = \max\{2, h+1\}$ and it is then easily verified that this expression is negative in the region of interest for $h \in \{0, 1, 2, 3\}$ which establishes that

$$V_{k_{i+1}} - V_{k_{i-1}} < 0 \tag{11.61}$$

if (11.56) is an equality. The only other possible alternative was that (11.57) is an equality. In that case we obtain:

$$(2k-h)y_2 = 1 + (-\tfrac{1}{2}h^2 - \tfrac{1}{2}h + k^2 - 2k)f_1 + hf_2 - y_1,$$

where $\ell = k - h$ with $h \in \{0, 1, 2, 3\}$. This yields:

$$V_{k_{i+1}} - V_{k_{i-1}} = 2 - 2y_1 + \frac{2h}{2k-h}\left[1 + (-\tfrac{1}{2}h^2 - \tfrac{1}{2}h + k^2 - 2k)f_1 + hf_2 - y_1\right] - 2hbf_1 - h^2 f_1.$$

We note that this expression is linear in y_1 with a negative coefficient and hence it is maximal for $y_1 = 1$ given (11.48). We obtain:

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} &\leq \frac{2h}{2k-h}\left[(-\tfrac{1}{2}h^2 - \tfrac{1}{2}h + k^2 - 2k)f_1 + hf_2\right] - 2hbf_1 - h^2 f_1 \\ &\leq \frac{h}{2k-h}\left[(2k^2 - 2(2b+h+2)k + (2b-1)h)f_1 + 2hf_2\right]. \end{aligned} \tag{11.62}$$

For $h = 0$ this establishes that

$$V_{k_{i+1}} - V_{k_{i-1}} \leq 0. \tag{11.63}$$

An equality would imply $y_1 = 1$ and $2y_2 = (k-2)f_1$ in which case we obtain that:

$$y_1(k_i) = 2(f_2 - f_1 + 1) - 1 + \tfrac{k}{2}(3f_1 - 2f_2) \geq f_1 + 1 > -1,$$

where in the first inequality we used that $3f_1 - 2f_2 > 0$ and $k \geq 2$. This yields a contradiction with (11.50) and hence we must have a strict inequality in (11.63) for $h = 0$.

The sign of the upper bound in (11.62) for $h \in \{1, 2, 3\}$ is determined by the sign of

$$(2k^2 - 2(2b + h + 2)k + (2b - 1)h)f_1 + 2hf_2.$$

This expression is decreasing in k given that $k \geq 2$ and $k \geq h + 1$ and hence the maximum is obtained for $k = h + 1$ and it is then easily verified that for $h \in \{1, 2, 3\}$ the choice $k = h + 1$ yields $2f_2 - (5 + 6b)f_1$, $4f_2 - (8 + 8b)f_1$ and $6f_2 - (11 + 10b)f_1$ respectively which are all negative in the area of interest. This establishes that

$$V_{k_{i+1}} - V_{k_{i-1}} < 0$$

which completes the proof.

11.B.3 Case 3.5

Clearly in this case, we have $k \geq 2$ and $\ell \geq 2$. The following constraints are satisfied.

$$y_1(k_{i-1}) = y_1 \qquad \in (-1,1) \tag{11.64}$$

$$y_1(k_{i-1}+1) = (1+f_2)y_1 + y_2 \qquad \leq -1 \tag{11.65}$$

$$\vdots \qquad \vdots$$

$$y_1(k_i) = d_4 y_1 + k y_2 + e_4 \qquad \leq -1 \tag{11.66}$$

$$y_1(k_i+1) = (d_4+f_1)y_1 + (k+1)y_2 + e_4 - (k-1)f_1 - f_2 \qquad \geq 1 \tag{11.67}$$

$$\vdots \qquad \vdots$$

$$y_1(k_{i+1}) = y_1(k_i) + \ell y_2(k_i) + e_6 \qquad \geq 1 \tag{11.68}$$

$$y_2(k_{i+1}) = y_2(k_i) + (\ell-2)f_1 \tag{11.69}$$

$$y_1(k_{i+1}+1) = y_1(k_i) + \ell y_2(k_i) + e_6 + y_2(k_i) + (\ell-2)f_1 - f_2 \qquad \leq -1 \tag{11.70}$$

Notice that $y_1(k_i)$ and $y_2(k_i)$ are given in equation (11.20) and (11.21) respectively, d_4 and e_4 are defined in (11.22) and (11.23), while e_6 is defined in (11.54).

We will show that ℓ satisfies $k-4 \leq \ell < k$. We first establish that $\ell \geq k-4$. Since $\ell \geq 2$, we only need to show this property for $k \geq 6$.

Using (11.67) to obtain a lower bound for y_2, we get

$$\begin{aligned} y_1(k_i+j) &= y_1(k_i) + j y_2(k_i) - f_2 - (j-1)(f_1-f_2) + \tfrac{f_1}{2}(j-1)(j-2) \\ &\geq \tfrac{(f_1-f_2-1)(j-1)}{k+1} y_1 + (k+j)\left(\tfrac{f_1-f_2+1}{k+1} + f_2 - f_1 + \tfrac{f_1}{2}k\right) + e_4 - j(k-1)f_1 - f_2 \\ &\qquad - (j-1)(f_1-f_2) + \tfrac{f_1}{2}(j-1)(j-2). \end{aligned}$$

Note that this lower bound is concave function in j. Therefore, if this lower bound is larger than or equal to 1 for $j = 1$ and $j = k-4$, then it is larger than 1 for all j satisfying $1 \leq j \leq k-4$ and this implies that $\ell \geq k-4$. For $j = 1$, the lower bound is actually equal to 1, while for $j = k-4$, we find using that $y_1 \in (-1, 1)$ that:

$$y_1(k_i + j) \geq -\left|\tfrac{(f_1 - f_2 - 1)(k-5)}{k+1}\right| + \tfrac{2k-4}{k+1}(f_1 - f_2 + 1) - f_1(4k - 19) + f_2(2k - 9).$$

If $f_1 - f_2 - 1 > 0$ we get:

$$\begin{aligned} y_1(k_i + j) &\geq 2(f_2 - 2f_1)k + 20f_1 - 10f_2 + 3 - \tfrac{12}{k+1} \\ &\geq 2(f_2 - 2f_1) + \tfrac{9}{7} > 1, \end{aligned}$$

where we have used that $f_2 > 2f_1$ and that $k \geq 6$. On the other hand, if $f_1 - f_2 - 1 < 0$ we get:

$$\begin{aligned} y_1(k_i + j) &\geq 2(f_2 - 2f_1)k + 22f_1 - 12f_2 + 1 - \tfrac{12(f_1 - f_2)}{k+1} \\ &\geq \tfrac{12}{7}(f_2 - 2f_1) - \tfrac{2}{7}f_1 + 1 > 1, \end{aligned}$$

where we have used that $2f_1 < f_2 < f_1$ and that $k \geq 6$. Next, we establish that $\ell \leq k$. We show this by contradiction. Assume that $\ell \geq k$, then we have

$$y_1(k_i + k) = y_1(k_i) + ky_2(k_i) - f_2 - (k-1)(f_1 - f_2) + \tfrac{f_1}{2}(k-1)(k-2) \geq 1.$$

Using this we obtain that:

$$2ky_2 \geq 1 - (d_4 + kf_1)y_1 - e_4 + k(k-1)f_1 + f_2 + (k-1)(f_1 - f_2) - \tfrac{f_1}{2}(k-1)(k-2).$$

Applying this lower bound in (11.66) we get that

$$(1 + f_2 - f_1)y_1 + 3(f_2 - f_1) + 1 - k(2f_2 - 3f_1) \leq -2. \tag{11.71}$$

Now, let us show that from the above equation, we obtain a contradiction. Let us first consider the case $1 + f_2 - f_1 < 0$. Since $y_1 \leq 1$, we obtain that

$$(1 + f_2 - f_1)y_1 + 3(f_2 - f_1) + 1 - k(2f_2 - 3f_1) \geq 2(f_1 + 1) + (k-2)(3f_1 - 2f_2) > -2,$$

where we used that $k \geq 2$, $f_1 > -2$ and $3f_1 - 2f_2 > 0$ and we obtain a contradiction with (11.71). Next let us consider the case $1 + f_2 - f_1 > 0$. Using that $y_1 \geq -1$, we obtain that

$$(1 + f_2 - f_1)y_1 + 3(f_2 - f_1) + 1 - k(2f_2 - 3f_1) \geq f_1 + (k-1)(3f_1 - 2f_2) > -2$$

because $k \geq 2$, $f_1 > -2$ and $3f_1 - 2f_2 > 0$ and we again obtain a contradiction with (11.71). Therefore we can conclude that $\ell < k$.

Returning to our Lyapunov function, we note that, for this case, we have

$$\begin{aligned} V_{k_{i+1}} - V_{k_i} &= 2[y_1(k_i) + \ell y_2(k_i) + e_6] + 2b[y_2(k_i) + (\ell - 2)f_1] \\ &\quad - 2(\ell - 2)y_2(k_i) - (\ell - 2)^2 f_1 + 2y_1(k_i) + 2by_2(k_i). \end{aligned}$$

Therefore, together with equation (11.45), we have

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} &= 2[y_1(k_i) + \ell y_2(k_i) + e_6] + 2b[y_2(k_i) + (\ell - 2)f_1] - 2(\ell - 2)y_2(k_i) \\ &\quad - (\ell - 2)^2 f_1 + 2y_1(k_i) + 2by_2(k_i) - 2(1 + bf_1 + f_2)y_1 \\ &\quad - 2(1 + b)(1 + y_1)y_2 - 1 - (f_1 + 1)y_1^2 + (k - 1)[(2b - 1)f_1 + 2f_2]. \end{aligned}$$

Note that the coefficient of y_2 term is

$$2(k + \ell) + 2b - 2(\ell - 2) + 2k + 2b - 2(1 + b)(1 + y_1) = 4k + 2(1 + b)(1 - y_1) > 0,$$

where we have used that $b > -1$ and $y_1 < 1$. Therefore, $V_{k_{i+1}} - V_{k_{i-1}}$ is maximal for maximal value of y_2.

For the upper bound for y_2 we use (11.70) while ignore all other constraints except for $y_1 \in [-1, 1]$. We note that (11.70) implies that:

$$y_2 \leq -f_1 y_1 + \tfrac{1}{2(2k-h+1)}\left[-2(1 + f_2 - f_1)y_1 - 2 + (2k^2 - h - h^2)f_1 + (4 + 2h)f_2\right],$$

where $h = k - \ell \in \{1, 2, 3, 4\}$.

Since we know $V_{k_{i+1}} - V_{k_{i-1}}$ is maximal for maximal value of y_2 we can replace y_2 by its upper bound to obtain:

$$V_{k_{i+1}} - V_{k_{i-1}} \leq a_5 y_1^2 + b_5 y_1 + c_5, \tag{11.72}$$

where

$$a_5 = (1+2b)f_1 - 1 + \tfrac{2}{2k-h+1}(b+1)(1+f_2-f_1),$$

$$b_5 = \tfrac{2(h-1)}{2k-h+1}(f_1-f_2-1) + \tfrac{1+b}{2k-h+1}\left[2(f_1-f_2) - (2k^2-h-h^2)f_1 - (4+2h)f_2\right],$$

$$c_5 = f_1[1-2b(h+1)-h^2] + 2f_2 - 3 + \tfrac{b+h}{2k-h+1}[-2+(2k^2-h-h^2)f_1 + (4+2h)f_2].$$

We first note that a_5 is equal to $\bar{a}$ defined in (11.30) witb k replaced by $2k-h+1 \geq 3$. Therefore the earlier argument also implies that $a_5 < 0$. The upper bound (11.72) is a quadratic function in y_1. We will show that $2a_5 + b_5 > 0$. This implies that the upper bound given $y_1 \in [-1,1]$ takes it maximum for $y_1 = 1$. On the other hand, we have already shown that $V_{k_{i+1}} - V_{k_{i-1}}$ only subject to $y_1(k_{i+1}+1) \leq -1$ and $y_1 \geq 1$ is negative in Appendix 11.B.1. It remains to establish that $2a_5 + b_5 > 0$. We have:

$$\begin{aligned} 2a_5 + b_5 = \tfrac{1}{2k-h+1} \big[&-2(1+b)f_1(k-1)^2 + 4(bf_1-1)k \\ &+ f_1\big((1+b)(h^2-3h+4) + 4(h-1)\big) \\ &- f_2\big(2(1+b)(1+h) - 2(1-h)\big) + 4(1+b)\big]. \end{aligned}$$

Clearly the factor $1/(2k-h+1)$ is irrelevant for the sign of $2a_5+b_5$. Remains to establish that:

$$\begin{aligned} &-2(1+b)f_1(k-1)^2 + 4(bf_1-1)k + f_1\big((1+b)(h^2-3h+4) + 4(h-1)\big) \\ &\qquad - f_2\big(2(1+b)(1+h) - 2(1-h)\big) + 4(1+b) > 0. \end{aligned} \tag{11.73}$$

We note that $k = h + \ell \geq h+2$. By taking the derivative of (11.73) with respect to k, we obtain:

$$-4(1+b)f_1(k-1) + 4(bf_1-1) > 0$$

since $1+b>0,\ f_1<0,\ k>1$, and

$$bf_1 - 1 > -\tfrac{3}{4}f_1 - 1 = -\tfrac{3}{4}(f_1 + \tfrac{4}{3}) > 0,$$

where we have used that $b \geq -\frac{3}{4}$ and $f_1 < -1.6$ in the region of interest. This implies that (11.73) is minimal for the smallest possible k, i.e. $k = h+2$. Setting $k = h+2$ in (11.73) we get:

$$f_1\left((1+b)(-h^2-3h+10)-12\right) - f_2\left(2(1+b)(1+h)-2(1-h)\right) - 4h - 8 + 4(1+b) > 0$$

Next, we note that the derivative with respect to h equals

$$-3f_1(1+b) - 2(f_2+2) - 2hf_1(1+b) - 2(1+b)f_2 > 0,$$

where we have used that $f_1 < 0$, $f_2 < -2$, $1+b > 0$, and $h > 0$. Therefore the expression is minimal for $h = 1$ and we obtain:

$$6(b-1)f_1 - 4(1+b)f_2 + 4(b-2),$$

which is positive in the region of interest. Therefore, we conclude that $b_5 + 2a_5 > 0$. As noted before this yields that $V_{k_{i+1}} - V_{k_{i-1}}$ is negative and the proof is complete.

11.B.4 Case 3.6

Clearly in this case, we have $k \geq 2$ and $\ell \geq 1$. The following constraints are satisfied.

$$
\begin{aligned}
y_1(k_{i-1}) &= y_1 & &\in [-1,1] & (11.74)\\
y_1(k_{i-1}+1) &= (1+f_2)y_1 + y_2 & &\leq -1 & (11.75)\\
&\vdots \qquad\qquad \vdots \\
y_1(k_i) &= d_4 y_1 + k y_2 + e_4 & &\leq -1 & (11.76)\\
y_1(k_i+1) &= (d_4+f_1)y_1 + (k+1)y_2 + e_4 - (k-1)f_1 - f_2 & &\geq 1 & (11.77)\\
&\vdots \qquad\qquad \vdots \\
y_1(k_{i+1}) &= y_1(k_i) + \ell y_2(k_i) + e_6 & &\geq -1 & (11.78)\\
y_1(k_{i+1}) &= y_1(k_i) + \ell y_2(k_i) + e_6 & &\leq 1 & (11.79)
\end{aligned}
$$

Notice that $y_1(k_i)$ and $y_2(k_i)$ are given in equation (11.20) and (11.21) respectively, d_4 and e_4 are defined in (11.22) and (11.23), while e_6 is defined in (11.54). Finally

$$y_2(k_{i+1}) = y_2(k_i) + (\ell - 2)f_1$$

The argument used in the case where $y_1(k_{i-1})$ is unsaturated and $y_1(k_{i+1})$ is saturated immediately imply that we have $\ell \geq k-3$ for $k \geq 4$, and $\ell \leq k$. We will show that

$$V_{k_{i+1}} - V_{k_{i-1}} < 0.$$

Let us define

$$
\begin{aligned}
\hat{y}_1 &= y_1(k_{i+1}) = (d_4 + \ell f_1)y_1 + (k+\ell)y_2 + e_4 + e_6 - \ell(k-1)f_1,\\
\hat{y}_2 &= y_2(k_{i+1}) = f_1 y_1 + y_2 + (\ell - k - 1)f_1. \qquad (11.80)
\end{aligned}
$$

Then, we have

$$V_{k_{i+1}} - V_{k_i} = \hat{y}_1^2 + 2b\hat{y}_1\hat{y}_2 - \tfrac{1}{f_1}\hat{y}_2^2 + 2y_1(k_i) + 1 + 2by_2(k_i) + \tfrac{1}{f_1}y_2^2(k_i).$$

Combining this with equation (11.45) and eliminating $y_1(k_i)$ and $y_2(k_i)$ by using equation (11.20) and (11.21) and eliminating $\hat{y}_2$ using (11.80) we obtain:

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} = \hat{y}_1^2 &+ 2b\hat{y}_1\left[f_1y_1 + y_2 + (\ell - k - 1)f_1\right] - (\ell-2)^2 f_1 \\ &- 2(\ell-2)\left[f_1y_1 + y_2 - (k-1)f_1\right] + 2(d_4y_1 + ky_2 + e_4) - 2(b+1)y_1y_2 \\ &- 2(1+f_2)y_1 - 2y_2 - (f_1+1)y_1^2 + (k-1)\left[2f_2 - f_1\right]. \end{aligned}$$

Let us write this expression in terms of y_1 and $\hat{y}_1$ by eliminating y_2 using that:

$$(k+\ell)y_2 = \hat{y}_1 - (d_4 + \ell f_1)y_1 + \ell(k-1)f_1 - e_4 - e_6$$

We get:

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} = \hat{y}_1^2 &+ 2b\hat{y}_1\left[f_1y_1 + (\ell-k-1)f_1\right] - (\ell-2)^2 f_1 + 2(d_4y_1 + e_4) - (f_1+1)y_1^2 \\ &- 2(\ell-2)f_1\left[y_1 - (k-1)\right] - 2(1+f_2)y_1 + (k-1)\left[2f_2 - f_1\right] \\ &+ \tfrac{1}{k+\ell}\left[2b\hat{y}_1 + 2(k-\ell+1) - 2(1+b)y_1\right]\left[\hat{y}_1 - (d_4+\ell f_1)y_1 + \ell(k-1)f_1 - e_4 - e_6\right]. \end{aligned} \tag{11.81}$$

Our objective is now to prove that this expression is always negative. We only consider the constraints (11.74), (11.78) and (11.79), that is,

$$-1 \le y_1 \le 1, \quad -1 \le \hat{y}_1 \le 1. \tag{11.82}$$

while we ignore all other constraints. Note that the coefficient of the term $\hat{y}_1^2$ is equal to

$$1 + \tfrac{2b}{k+\ell},$$

which is positive for $k+\ell \geq 2$ and $b > -1$. Therfore, we know (11.81) is maximal at the boundary of $\hat{y}_1$, that is, either $\hat{y}_1 = -1$ or $\hat{y}_1 = 1$. Define $h = k - \ell = \{0, 1, 2, 3\}$.

Let us first consider the boundary $\hat{y}_1 = -1$. We obtain that

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} = 1 + 2b(h+1)f_1 - (h+1)^2 f_1 + \left[(1+2b)f_1 - 1\right] y_1^2 \\ - \tfrac{1}{2k-h}\left[(1+b)(y_1+1) - (h+2)\right] \times \\ \left[-2 - 2(1+f_2-f_1)y_1 + 2f_2(h+1) + f_1(2k^2 - h^2 - 3h - 2)\right]. \quad (11.83) \end{aligned}$$

Note that the coefficient of the term y_1^2 is

$$a_6 = (1+2b)f_1 - 1 + \tfrac{2}{2k-h}(b+1)(1+f_2-f_1),$$

which is negative as it is the same as $\bar{a}$ given in (11.30), with k replaced by $k+\ell$. Let us now derive the coefficient of the term y_1:

$$b_6 = -\tfrac{2(1+f_2-f_1)}{2k-h}(h+2) - \tfrac{1+b}{2k-h}\left[h(2f_2 - 3f_1) + f_1(2k^2 - h^2) - 4\right].$$

We will show that $2a_6 + b_6 > 0$. This implies that the upper bound given the constraints (11.82) takes it maximum for $y_1 = 1$ and $\hat{y}_1 = -1$. On the other hand, we have already shown that $V_{k_{i+1}} - V_{k_{i-1}}$ only subject to $y_1(k_{i+1}) \in [-1, 1]$ and $y_1 \geq 1$ is negative in Appendix 11.B.2. Therefore, we have

$$\begin{aligned} b_6 + 2a_6 = \tfrac{1}{2k-h}[-2(1+f_2-f_1)(h+2) - (1+b)[h(2f_2 - 3f_1) + f_1(2k^2 - h^2) - 4] \\ + 2(2b+1)f_1(2k-h) - 2(2k-h) + 4(1+b)(1+f_2-f_1)]. \end{aligned}$$

Note that the term $\frac{1}{2k-h}$ does not affect the sign of $b_6 + 2a_6$. Therefore, to show $b_6 + 2a_6 > 0$ is equivalent to show that

$$\begin{aligned} & -2(1+f_2-f_1)(h+2) - (1+b)[h(2f_2 - 3f_1) + f_1(2k^2 - h^2) - 4] \\ & + 2(2b+1)f_1(2k-h) - 2(2k-h) + 4(1+b)(1+f_2-f_1) > 0. \quad (11.84) \end{aligned}$$

We first show that the left-hand side of the above inequality is increasing in k and therefore we differentiate the left-hand side with respect to k.This results in:

$$-4(1+b)f_1(k-1)+4(bf_1-1),$$

which is positive since $1+b>0$, $f_1<0$, $k>1$, and $bf_1-1>0$. Thus, the left-hand side of (11.84) achieves its minimum for

$$k=1+\tfrac{bf_1-1}{(1+b)f_1}<1,$$

where we have used that $1+b>0$, $f_1<0$ and $bf_1-1>0$. In our case, we know that $k=\ell+h\geq h+1$ and $k\geq 2$. Therefore, we have $k\geq\max\{2,h+1\}$. Thus, the left-hand side of (11.84) achieves its minimal for $k=2$ when $h=0$, while for $h=1,2,3$, it achieves it minimal when $k=h+1$.

For the case where $k=2$ and $h=0$, the left-hand side of (11.84) is

$$4bf_2+4bf_1+4(2b-1)=4b(f_2+2)+4(bf_1-1)>0,$$

where we have used that $b<0$, $f_2<-2$, and $bf_1-1>0$.

For the case where $k=h+1$ and $h=1,2,3$. Using $k=h+1$ in the left-hand sider of (11.84) yields

$$\begin{aligned}-2(1+f_2-f_1)(h+2)-(1+b)[h(2f_2-3f_1)+f_1(h^2+4h+2)-4]\\+2(2b+1)f_1(h+2)-2(h+2)+4(1+b)(1+f_2-f_1).\end{aligned}\tag{11.85}$$

We first show that this is increasing in h and therefore we differentiate with respect to h. This results in:

$$-4+3(1+b)f_1-2(b+2)f_2-2(1+b)f_1h>(1+b)(3f_1-2f_2)-2(f_2+2)>0,$$

where for the first inequality we have used that $1+b>0$, $f_1<0$, and $h>0$, while for the second inequality we have used that $1+b>0$, $3f_1-2f_2>0$ and $f_2<-2$. Therefore, (11.85) is minimal for minimal value of h, that is $h = 1$. When $h = 1$, we have

$$-4(1-2b) + 4(1+b)f_1 + 2(b-2)f_2$$

which is positive in the region of interest. Therefore, we conclude that $b_6 + 2a_6 > 0$, which as argued before implies that $V_{k_{i+1}} - V_{k_{i-1}} < 0$.

The only other possible alternative is that $\hat{y}_1 = 1$. In that case, we obtain that

$$\begin{aligned} V_{k_{i+1}} - V_{k_{i-1}} = 1 - 2b(h+1)f_1 - (h+1)^2 f_1 + [(1+2b)f_1 - 1]\, y_1^2 \\ + \tfrac{1}{2k-h}\left[(1+b)(1-y_1) + h\right] \times \\ \left[2 - 2(1+f_2-f_1)y_1 + 2f_2(h+1) + f_1(2k^2 - h^2 - 3h - 2)\right]. \quad (11.86) \end{aligned}$$

Note that the coefficient of the term y_1^2 is

$$a_7 = (1+2b)f_1 - 1 + \tfrac{2}{2k-h}(b+1)(1+f_2-f_1),$$

which is equal to a_6, and thus, it is negative. Let us now derive the coefficient of the term y_1:

$$b_7 = \tfrac{1}{2k-h}\left[-2(1+f_2-f_1)(h+2+2b) - (1+b)h(2f_2-3f_1) - (1+b)f_1(2k^2-h^2)\right].$$

We will show that $2a_7 + b_7 > 0$. This implies that the upper bound given the constraints (11.82) takes it maximum for $y_1 = 1$ and $\hat{y}_1 = 1$. As argued before, we have already shown that $V_{k_{i+1}} - V_{k_{i-1}}$ only subject to $y_1(k_{i+1}) \in [-1, 1]$ and $y_1 \geq 1$ is negative in Appendix 11.B.2.

With just a little bit algebra, we obtain that

$$\begin{aligned} b_7 + 2a_7 = \tfrac{1}{2k-h}[(2f_1(1+2b) - 2)(2k-h) \\ - 2(1+f_2-f_1)h - (1+b)h(2f_2-3f_1) - (1+b)f_1(2k^2-h^2)]. \end{aligned}$$

Note that the term $\frac{1}{2k-h}$ does not affect the sign of $b_7 + 2a_7$. Therefore, to show $b_7 + 2a_7 > 0$ is equivalent to show that

$$[2f_1(1+2b) - 2](2k-h) - 2(1+f_2-f_1)h - (1+b)h(2f_2-3f_1) - (1+b)f_1(2k^2-h^2) > 0. \quad (11.87)$$

We first show that the left-hand side of the above inequality is increasing in k and therefore we differentiate the left-hand side with respect to k. This results in:

$$-4(1+b)f_1(k-1)+4(bf_1-1),$$

which is positive since $1+b>0$, $f_1<0$, $k>1$, and $bf_1-1>0$. Thus, the left-hand side of (11.87) achieves its minimum for

$$k=1+\frac{bf_1-1}{(1+b)f_1}<1,$$

where we have used that $1+b>0$, $f_1<0$ and $bf_1-1>0$. In our case, we know that $k=\ell+h\geq h+1$ and $k\geq 2$. Therefore, we have $k\geq \max\{2,h+1\}$. Thus, the left-hand side of (11.84) achieves its minimal for $k=2$ when $h=0$, while for $h=1,2,3$, it achieves it minimal when $k=h+1$.

For the case where $k=2$ and $h=0$, the left-hand side of (11.87) is

$$8(bf_1-1)>0,$$

where we have used that $bf_1-1>0$.

For the case where $k=h+1$ and $h=1,2,3$. Using $k=h+1$ in the left-hand side of (11.87) yields

$$[2f_1(1+2b)-2](h+2)-2(1+f_2-f_1)h-(1+b)h(2f_2-3f_1)-(1+b)f_1(h^2+4h+2). \quad (11.88)$$

We first show that this is increasing in h and therefore we differentiate with respect to h. This results in:

$$-4+3(1+b)f_1-2(b+2)f_2-2(1+b)f_1h>(1+b)(3f_1-2f_2)-2(f_2+2)>0,$$

where for the first inequality we have used that $1+b>0$, $f_1<0$, and $h>0$, while for the second inequality we have used that $1+b>0$, $3f_1-2f_2>0$ and $f_2<-2$. Therefore, (11.88) is minimal for minimal value of h, that is $h = 1$. When $h = 1$, we have

$$-8 + 4(1 + 2b)f_1 - 2(b + 2)f_2,$$

which is positive in the region of interest. Therefore, we conclude that $b_7 + 2a_7 > 0$, , which as argued before implies that $V_{k_{i+1}} - V_{k_{i-1}} < 0$. This completes the proof for the case when both $y_1(k_{i-1})$ and $y_1(k_{i+1})$ unsaturated.

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