Robust stability analysis of linear time-varying feedback systems

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Abstract

Feedback interconnections often arise in the modelling and control of dynamical systems. This thesis considers the robustness of uncertain linear time-varying feedback systems in continuous time.

Robust stability results are developed in terms of a generalisation of the ν-gap metric for causal linear time-varying systems, which in the special case of time-invariant systems reduces to the well-known metric introduced by Vinnicombe. The robust stability results motivate the ν-gap metric as a measure of distance between open-loop systems from the perspective of quantifying the difference in closed-loop behaviour. The approach taken is operator-theoretic and it is underpinned by the existence of normalised strong representations of open-loop system graphs; this is known to hold given a stabilisable and detectable time-varying state-space realisation, for example. It is shown that the robust stability conditions are also necessary when the variation with time is periodic.

While the ν-gap alone offers a sensible measure of uncertainty in a feedback interconnection, the use of integral quadratic constraints to encapsulate known aspects of the uncertainty is shown to lead to a potential reduction in conservatism in robustness analysis. Formalising this involves establishing the pathwise connectedness of sufficiently small ν-gap metric balls in the graph topology, using a linear fractional transformation (LFT) characterisation of the metric. Central to the LFT characterisation is the existence of a J-spectral factorisation, which is also shown herein for time-varying systems with stabilisable and detectable state-space models and for distributed-parameter transfer functions in the constantly proper subclass of the Callier-Desoer algebra.

As an application of the theoretical developments, a sampled-data approximation problem is rigorously formulated with respect to the ν-gap measure of error and optimally solved. The formulation involves development of a modelling framework within which it is possible to represent both types of system of interest and have a well-defined ν-gap distance. The aforementioned LFT characterisation of the ν-gap plays an important role in recasting the approximation problem as a sequence of convex feasibility problems solvable by standard $\boldsymbol{H}^\infty$ synthesis tools.

Declaration

This is to certify that:

(i) the thesis comprises only my original work towards the PhD except where indicated in the Preface,

(ii) due acknowledgement has been made in the text to all other material used,

(iii) the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Sei Zhen Khong
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Preface

Unless otherwise stated, the material presented in this thesis, submitted for the degree of Doctor of Philosophy, is the result of collaboration between Associate Professor Michael Cantoni and me during postgraduate studies undertaken at the University of Melbourne, Australia. The following refereed publications have appeared during the course of my research:

1. S. Z. Khong, M. Cantoni, and U. T. Jönsson. Robust stability properties of the ν-gap metric for time-varying systems. In *Proceedings of the 50th IEEE Conference Decision Control and European Control Conference*, pages 2028–2033, Orlando, FL, USA, Dec. 2011.

2. S. Z. Khong and M. Cantoni. On stability analysis with the ν-gap metric and integral quadratic constraints. In *Proceedings of the 1st Australian Control Conference*, pages 519–524, Melbourne, Australia, Nov. 2011 (awarded the Best Student Paper High Commendation Award).

3. S. Z. Khong, M. Cantoni, and U. T. Jönsson. Path-connectedness of frequency-domain uncertainty sets in the graph topology. In *Proceedings of the 18th IFAC World Congress*, pages 3366–3371, Milan, Italy, Aug. 2011.

4. S. Z. Khong and M. Cantoni. Optimal sampled-data approximation in the ν-gap metric. In *Proceedings of the 8th Asian Control Conference*, pages 857–862, Kaohsiung, Taiwan, May. 2011.

5. S. Z. Khong and M. Cantoni. Shift-invariant representation of two periodic system classes defined over doubly-infinite continuous time. In *Proceedings of the 49th SICE Annual Conference*, pages 197–204, Taipei, Taiwan, Aug. 2010 (invited).

Several journal papers are in preparation at the time of writing.

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Contents

Chapter 1

Introduction

Uncertainty is inherent in system modelling inasmuch as any mathematical model can at best approximate the behaviour of a real-world process. It may arise due to aspects of the dynamics of real systems not being representable by the class of models used or the precise numerical values of the various parameters of the model only being determinable to some degree of certainty. This dissertation is mainly concerned with quantifying uncertainty within the context of *feedback interconnections*. These commonly arise in modelling and control of dynamical systems, for both analysis and synthesis purposes [DFT92, ZDG96, ZD98, DP00, Vin01, Kha02, ÅM08]. In practice, physical systems can often be modelled as feedback interconnections of a component with a simple structure, linear time-invariant (LTI) for example, with another component which may be time-varying or uncertain. Feedback control, on the other hand, is widely applied for alleviating the effects of any unmeasured disturbances acting on the system and uncertainty about the system dynamics. Robust stability and performance of uncertain feedback interconnection is therefore an important problem to investigate.

The need to quantify the uncertainty to which a feedback interconnection is insensitive spawned the development of the gap metrics in the literature [ZES80, ES85, GS90, QD92, Vin93]. Of these, the ν-gap metric [Vin93, Vin01] emerges to be the least conservative for finite-dimensional LTI systems, as far as a generalised robust stability margin is concerned. In addition, it has a clear frequency response interpretation. A unified ν-gap metric and integral quadratic constraint (IQC) based robust stability framework, established for linear time-invariant (LTI) systems in the series of publications [CJK12, CJK10, CJK09, JCK08], is generalised in [JC10, JC11] to accommodate *causal* linear systems that are *time-varying* with potentially *unbounded gain* over the space of *doubly infinite* finite-energy signals. In particular, a generalised ν-gap distance is defined, assuming the existence of certain normalised strong representations of the system graphs, which is the case for various classes of linear systems, including those with time-varying state-space realisations that are stabilisable and detectable. The main results in [JC10, JC11] establish that the generalised ν-gap metric enjoys homotopy-type robustness properties when combined with IQC conditions. It remained unclear whether the metric could be used to quantify feedback robustness non-conservatively, as is the case for LTI systems [Vin93, Vin01]. In this thesis, aspects of this issue are addressed. The definition of closed-loop stability is first formulated in a manner that is seamlessly compatible with the generalised ν-gap metric, whereby the double-axis Paradox of Georgiou & Smith [GS95] does not arise via the implicit encapsulation of an arrow of time [GS10]. The following are developed by combining the ideas in [Vin93, Vin01] and [JC10, JC11]: (i) sufficient conditions for robust stability; (ii) properties of the topology induced by the ν-gap metric; and, (iii) for a class of linear periodically time-varying (LPTV) systems, the so-called *strong necessity* robustness condition. These motivate the generalised ν-gap metric as a tool for studying the robustness of linear time-varying (LTV) feedback systems.

An operator-theoretic robust stability theory for LTV systems defined on singly infinite (i.e. with a fixed 'initial' time) *discrete* time is reported in [Fei98] and the references therein. This setting has the advantage that the inverses of causal systems are causal and closed-loop stabilisability is equivalent to the existence of strong left and right graph representations/symbols [DS93]. These are not known to hold in continuous time. Robust stability is studied in [Fei98] in terms of a discrete time-varying generalisation of the Georgiou's formula of the gap metric [Geo88]. By contrast, this thesis considers *continuous-time* systems on doubly infinite signal spaces in a manner consistent with the original development of the ν-gap metric in a time-invariant setting [Vin93, Vin01].

The ν-gap metric is closely related to the standard gap metric, in that they are both useful measures of feedback uncertainty and they induce the same topology (at least on LTI systems), which is often called the graph topology. Variants of the robustness properties of the ν-gap metric can also be found expressed in terms of the gap metric [GS90, FGS93, CV02]. A few notes are in order. Firstly, the gap metric theory is predominantly developed for systems on signal spaces with support on a positive half-line. On the contrary, the definitions of the generalised ν-gap metric and feedback stability used in this thesis do not attribute special significance to a particular time instant separating the past and the future, as is consistent within a *time-varying* context. Secondly, the issue of causality of feedback interconnections is addressed upfront, amounting to a proper treatment of closed-loop stability [GS10]. Thirdly, the generalised ν-gap metric is amenable to a characterisation in terms of a linear fractional transformation (LFT) [Can06], which is useful in the study of the pathwise connectedness of ν-gap metric balls in the graph topology, as well as the numerical synthesis of a system to lie within a specified ν-gap distance from another, as discussed further below.

Feedback stability analysis via integral quadratic constraints (IQCs) is known to offer generality over small-gain [Zam66], passivity [Wil72], and circle/Popov criterion [HM68] based approaches, in terms of exploiting known system structure to reduce conservatism in stability analysis [MR97, Jön01]. The IQC framework was first developed for the feedback interconnection of a *stable* LTI system and a *stable* uncertain system that may be non-stationary and even nonlinear. In [RM97], nonlinear open-loop *unstable* systems are accommodated *directly*, without recourse to methods such as loop transformations, in conjunction with homotopies that are continuous with respect to the generalised gap metric from [GS97, JSV05]. The work [JC10, JC11] is similar in spirit, but rather involves the generalised definition of closed-loop stability and the generalised ν-gap metric for *linear* time-varying systems. In this thesis, the additional flexibility engendered by IQCs is reconciled with a ν-gap ball based robustness result described above, by establishing that the latter can be recovered within the unified IQC / ν-gap homotopy framework of [JC10, JC11]. This involves a proof that any ν-gap metric ball of a radius less than the maximal stability margin of its centre is path-connected with respect to the graph topology. Towards this end, the aforementioned linear fractional characterisation of the ν-gap metric [Can06] plays a crucial role, in that it provides a bijective continuous mapping between a unit ball of stable systems and a ν-gap ball of a specified size, as already seen in a finite-dimensional time-invariant context [CJK12, CJK10]. The LFT is contingent on the existence of a certain J-spectral factorisation of a graph symbol expression. In this thesis, the results are established for more general classes of linear systems; specifically, time-varying finite-dimensional state-space systems and time-invariant distributed-parameter systems in the Callier-Desoer algebra.

To illustrate an application of the theory developed in this thesis, the problem of model approximation is considered in terms of the ν-gap metric as the measure of error. A systematic approach to approximation is to formulate it in terms of a constrained optimisation problem, where the constraints reflect the structure to be imposed on the systems involved and the cost is a measure of approximation error. Here, attention is restricted to the problem of approximating a given LTI system by a periodic sampled-data system, which arises commonly in situations where feedback controllers designed using continuous-time algorithms are to be implemented using digital hardware. This problem has been considered in [CV04], in which the pointwise gap metric is the measure of approximation error. While a solution is identified therein, it remained unclear whether the class of systems employed completely characterise behaviour as required to rigorously introduce the ν-gap metric. This issue is addressed in this thesis via the development of a framework that is sufficiently rich to equivalently represent all of the system types involved.

A direct formulation of the solution to the problem of optimal sampled-data approximation with dynamic input-output weightings is developed. The approach involves the aforementioned LFT characterisation, constructed from a single J-spectral factorisation; by contrast, a more complicated three-term factorisation, which underlies the DGKF solution to the standard $\boldsymbol{H}^\infty$ control problem [DGKF89, ZDG96], is utilised in [CV04]. The proposed algorithm is a bisection search yielding a finite sequence of convex LFT synthesis problems which can be solved using standard $\boldsymbol{H}^\infty$ sampled-data methods [BP92, Yam94, CF95, CG97]. For completeness, a corresponding system of linear matrix inequality (LMI) feasibility conditions, which can be efficiently tested via standard optimisation tools [BV04], is provided based on [GA94].

Overview of contents

Chapter 2 introduces the notation and reviews necessary preliminaries on basic signals and systems theory. In particular, notions of operator equivalence, causality, and Wiener-Hopf / Hankel operator classes are presented. Importantly, this chapter states the list of assumptions made on the operators considered in this thesis, regarding in particular the existence of normalised strong graph representations/symbols. These are verified for three generic classes of linear systems — (i) time-varying finite-dimensional state-space systems, (ii) distributed-parameter time-invariant systems, and (iii) periodic systems with finite-dimensional transfer function 'realisations' in the lifted domain, which include systems with a sampled-data structure. All these classes are reconsidered in future chapters. Specifically, (i) and (ii) appear in Chapter 4, while (iii) in Chapters 3 and 5.

The notion of closed-loop stability and the generalised ν-gap metric are presented in Chapter 3 for causal LTV systems having normalised strong/coprime graph symbols. Before this, some results on Fredholm / Wiener-Hopf / Hankel operators are provided. Several ν-gap based robustness results are derived, including a bound on the residual stability margin in the presence of perturbations on a component of a feedback interconnection and bilateral bounds on the induced norm of the difference between the closed-loop operators. For the class of periodic systems considered in Chapter 2, a strong necessity condition for robust stability is also established in terms of the generalised ν-gap metric.

In Chapter 4, a unifying framework encompassing both the ν-gap metric and IQC based analysis is reviewed. Therein, known structure of the uncertain components of a feedback connection can be characterised by IQCs and exploited to reduce conservatism in stability analysis. The associated additional flexibility is reconciled with the ν-gap ball based stability results of Chapter 3 by showing that the latter can be recovered within the combined IQC and ν-gap homotopy based framework. To this end, path-connectedness of ν-gap balls in the graph topology plays a central role. This is established by exploiting a linear fractional characterisation of the ν-gap metric and the existence of a certain J-spectral factorisation, which is shown to be true for the classes of finite-dimensional LTV systems and distributed-parameter LTI systems introduced in Chapter 2.

Chapter 5 considers the problem of model approximation with respect to the ν-gap as the measure of error. Part of this effort involves developing appropriate system representations via the Fourier transform and time-lifting isomorphisms, so as to be consistent with the definition of the ν-gap. The LFT characterisation of the metric is again employed, in this case to transform the approximation problem into a finite sequence of feasibility problems, from which an optimal solution may be found. More specifically, the problem of optimal sampled-data approximation of continuous-time LTI systems is examined and an LMI-based numerical algorithm proposed. The class of periodic systems considered in Chapter 2 plays a crucial role in this respect.

Finally, in Chapter 6, the main contributions of this work are summarised. Potential directions for future research are also identified.

Chapter 2

Signals and Systems

This chapter establishes the basic notation and operator-theoretic setting for the thesis. A recurring theme of this thesis is the study of linear operators via their graphs. The notion of equivalence of operators in terms of an isomorphic relation between their graphs is introduced in Section 2.1. In addition, generalised Wiener-Hopf and Hankel operators, which are important in the study of *time-varying* systems as demonstrated in [JC10, JC11], are defined.

In Section 2.2, we identify assumptions on the class of linear operators considered throughout this thesis in accordance with [JC10, JC11]. Loosely speaking, we assume that a system graph can be represented as the range of (resp. kernel) of a causal, bounded operator which is causally left (resp. right) invertible. To justify these assumptions, each is verified for three general classes of linear systems, which also arise later in the thesis. In Section 2.3, time-varying systems with finite-dimensional state-space realisations that are stabilisable and detectable [IS04, MC10] are considered. Distributed-parameter time-invariant systems in the Callier-Desoer algebra [CD78, CZ95] are examined in Section 2.4. Finally, Section 2.5 considers a class of periodic systems taken from [CV04], which include those with a sampled-data structure.

The development of this chapter is based on [JC10, JC11]. Standard references for the functional analysis concepts used here are [Kat80, Kre89, GGK90, Rud91].

2.1 Notation and basic operator theory

The real and complex numbers are denoted $\mathbb{R}$ and $\mathbb{C}$, respectively. The transpose of a matrix $M \in \mathbb{R}^{p\times m}$ is denoted $M^T \in \mathbb{R}^{m\times p}$ and the complex conjugate transpose of $M \in \mathbb{C}^{p\times m}$ is denoted $M^* \in \mathbb{C}^{m\times p}$. The maximum and minimum singular values of an

$M \in \mathbb{C}^{p\times m}$ are denoted $\bar{\sigma}(M)$ and $\underline{\sigma}(M)$, respectively. We denote by $\mathbb{Z}$, $\mathbb{T}$ and $\mathbb{D}$ the integers, the unit circle and the open unit disc in the complex plane, respectively.

For a linear operator mapping between Hilbert spaces $\mathbf{X} : \mathrm{dom}\,(\mathbf{X}) \subset \mathcal{H}_1 \to \mathcal{H}_2$, its kernel is denoted by

$$\ker(\mathbf{X}) := \{x \in \mathrm{dom}\,(\mathbf{X}) \mid \mathbf{X}x = 0\},$$

its image by

$$\mathrm{img}\,(\mathbf{X}) := \{y \in \mathcal{H}_2 \mid y = \mathbf{X}x \text{ for some } x \in \mathrm{dom}\,(\mathbf{X})\},$$

and its graph by

$$\mathscr{G}_{\mathbf{X}} := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathcal{H}_2 \times \mathcal{H}_1 : u \in \mathrm{dom}\,(\mathbf{X}) \text{ and } y = \mathbf{X}u \right\}.$$

Given $\mathcal{X} \subset \mathrm{dom}\,(\mathbf{X})$, $\mathbf{X}|_{\mathcal{X}}$ denotes the restriction of $\mathbf{X}$ to $\mathcal{X}$. We define, respectively, the upper and lower gains of $\mathbf{X}$ as

$$\bar{\gamma}(\mathbf{X}) := \sup_{\|w\|_{\mathcal{H}_1}=1} \|\mathbf{X}w\|_{\mathcal{H}_2} \quad \text{and} \quad \underline{\gamma}(\mathbf{X}) := \inf_{\|w\|_{\mathcal{H}_1}=1} \|\mathbf{X}w\|_{\mathcal{H}_2}.$$

$\mathbf{X}$ is said to be bounded if $\mathrm{dom}\,(\mathbf{X}) = \mathcal{H}_1$ and $\bar{\gamma}(\mathbf{X}) < \infty$.

We denote by $\mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ the Banach space of all bounded linear operators mapping between the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. An operator $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is compact if for any bounded sequence $\{x_k\}$ in $\mathcal{H}_1$, $\{\mathbf{X}x_k\}$ has a convergent subsequence in $\mathcal{H}_2$. A compact operator $\mathbf{X} \in \mathscr{L}(\mathcal{H}, \mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ is of the Hilbert-Schmidt class [GGK90, Chapter VIII] if

$$\sum_{i=1}^{\infty} \|\mathbf{X}e_i\|_{\mathcal{H}}^2 < \infty \text{ for any orthonormal basis } \{e_n\}_{n=1}^{\infty} \text{ of } \mathcal{H}.$$

An important result on compact operators, exploited several times in this thesis, is stated below.

Lemma 2.1.1 ([Kat80, Thm III.4.8])**.** *Given two bounded linear operators* $\mathbf{X} \in \mathscr{L}(\mathcal{H}_0, \mathcal{H}_1)$, $\mathbf{Z} \in \mathscr{L}(\mathcal{H}_2, \mathcal{H}_3)$ *and a compact operator* $\mathbf{Y} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$*, the composition* $\mathbf{ZYX} \in \mathscr{L}(\mathcal{H}_0, \mathcal{H}_3)$ *is compact.*

The unique Hilbert adjoint of an $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is denoted $\mathbf{X}^* \in \mathscr{L}(\mathcal{H}_2, \mathcal{H}_1)$ and

$$\langle \mathbf{X}w, v\rangle_{\mathcal{H}_2} = \langle w, \mathbf{X}^* v\rangle_{\mathcal{H}_2} \, \forall w \in \mathcal{H}_1, v \in \mathcal{H}_2.$$

It holds that $\mathrm{img}\,(\mathbf{X})^{\perp} = \ker(\mathbf{X}^*)$ and $\ker(\mathbf{X})^{\perp} = \mathrm{cl}\,\mathrm{img}\,(\mathbf{X}^*)$, where $\perp$ denotes the orthogonal complement and $\mathrm{cl}(\cdot)$ the closure of a subspace. As such, $\ker(\mathbf{X}^*\mathbf{X}) = \ker(\mathbf{X})$. Furthermore, the following identities hold: $\bar{\gamma}(\mathbf{X}^*) = \bar{\gamma}(\mathbf{X})$; when $\mathbf{X}$ has a bounded inverse, $\bar{\gamma}(\mathbf{X}^{-1}) = 1/\underline{\gamma}(\mathbf{X})$ and $\underline{\gamma}(\mathbf{X}^*) = \underline{\gamma}(\mathbf{X})$. Given any $\mathbf{X}, \mathbf{Y} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$, $\underline{\gamma}(\mathbf{X}+\mathbf{Y}) \geq \underline{\gamma}(\mathbf{X}) - \bar{\gamma}(\mathbf{Y})$. Finally, if $\left[\begin{smallmatrix} \mathbf{X} \\ \mathbf{Y} \end{smallmatrix}\right] : \mathcal{H}_1 \to \mathcal{H}_2 \times \mathcal{H}_3$ satisfies $\mathbf{X}^*\mathbf{X} + \mathbf{Y}^*\mathbf{Y} = \mathbf{I}$ (i.e. it is an isometry) or if $\begin{bmatrix} \mathbf{X} & \mathbf{Y} \end{bmatrix} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_3$ is such that $\mathbf{X}\mathbf{X}^* + \mathbf{Y}\mathbf{Y}^* = \mathbf{I}$ (i.e. it is a co-isometry), then $\bar{\gamma}^2(\mathbf{Y}) = 1 - \underline{\gamma}^2(\mathbf{X})$.

2.1.1 Equivalence of operators

Two normed spaces $\mathcal{V}_1$ and $\mathcal{V}_2$ are said to be isometrically isomorphic if there exists a bijective bounded linear operator $\mathbf{\Phi} : \mathcal{V}_1 \to \mathcal{V}_2$ such that $\|\mathbf{\Phi} v_1\|_{\mathcal{V}_2} = \|v_1\|_{\mathcal{V}_1}, \forall v_1 \in \mathcal{V}_1$. When this is the case, we denote the isomorphic relationship between $\mathcal{V}_1$ and $\mathcal{V}_2$ via the isomorphism $\mathbf{\Phi}$ by $\mathcal{V}_1 \overset{\mathbf{\Phi}}{\sim} \mathcal{V}_2$. In line with the input-output approach adopted in this thesis, we study the relationships between operators by way of their graphs.

Definition 2.1.2. Two operators $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \mathcal{X}_1 \to \mathcal{X}_2$ and $\mathbf{Y} : \mathrm{dom}(\mathbf{Y}) \subset \mathcal{Y}_1 \to \mathcal{Y}_2$ are said to be equivalent if there exists an isomorphism

$$\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1 := \begin{bmatrix} \mathbf{\Phi}_2 & 0 \\ 0 & \mathbf{\Phi}_1 \end{bmatrix} : \begin{bmatrix} \mathcal{X}_2 \\ \mathcal{X}_1 \end{bmatrix} \to \begin{bmatrix} \mathcal{Y}_2 \\ \mathcal{Y}_1 \end{bmatrix}$$

such that $\mathscr{G}_{\mathbf{X}} \overset{\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1}{\sim} \mathscr{G}_{\mathbf{Y}}$. When this is the case, we denote it as $\mathbf{X} \overset{\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1}{\sim} \mathbf{Y}$. When $\mathcal{X}_1 = \mathcal{X}_2$, $\mathcal{Y}_1 = \mathcal{Y}_2$ and $\mathbf{\Phi}_1 = \mathbf{\Phi}_2 = \mathbf{\Phi}$, we use the shorthand notation $\mathbf{\Phi}$ to denote $\mathbf{\Phi} \oplus \mathbf{\Phi}$ and say $\mathbf{X} \overset{\mathbf{\Phi}}{\sim} \mathbf{Y}$ if $\mathscr{G}_{\mathbf{X}} \overset{\mathbf{\Phi}}{\sim} \mathscr{G}_{\mathbf{Y}}$. Note that if $\mathbf{X}$ is a bounded linear operator, then $\mathbf{X} \overset{\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1}{\sim} \mathbf{Y}$ implies $\bar{\gamma}(\mathbf{Y}) = \bar{\gamma}(\mathbf{X})$ and $\underline{\gamma}(\mathbf{Y}) = \underline{\gamma}(\mathbf{X})$.

Remark 2.1.3. Given a linear operator $\mathbf{X} : \mathrm{dom}(\mathbf{X}) \subset \mathcal{X}_1 \to \mathcal{X}_2$ and an isomorphism $\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1 : \left[\begin{smallmatrix} \mathcal{X}_2 \\ \mathcal{X}_1 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mathcal{Y}_2 \\ \mathcal{Y}_1 \end{smallmatrix}\right]$, note that $\left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right] \in (\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1)\mathscr{G}_{\mathbf{X}} \implies y = 0$. Thus, $(\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1)\mathscr{G}_{\mathbf{X}}$ is the graph of a linear operator $\mathbf{Y} : \mathrm{dom}(\mathbf{Y}) \subset \mathcal{Y}_1 \to \mathcal{Y}_2$ defined by $\mathbf{Y} := u \mapsto \mathbf{\Phi}_2 \mathbf{X} \mathbf{\Phi}_1^{-1} u$, for $u \in \mathrm{dom}(\mathbf{Y}) := \mathbf{\Phi}_1 \mathrm{dom}(\mathbf{X})$. Indeed, $\mathbf{Y}$ is such that $\mathbf{X} \overset{\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1}{\sim} \mathbf{Y}$, since clearly $\mathscr{G}_{\mathbf{X}} \overset{\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1}{\sim} \mathscr{G}_{\mathbf{Y}}$. It is therefore the case that $\mathscr{G}_{\mathbf{X}} \overset{\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1}{\sim} \mathscr{G}_{\mathbf{Y}}$ is equivalent to the following collection of conditions:

1. $\mathrm{dom}(\mathbf{X}) \overset{\mathbf{\Phi}_1}{\sim} \mathrm{dom}(\mathbf{Y})$;
2. $\mathrm{img}(\mathbf{X}) \overset{\mathbf{\Phi}_2}{\sim} \mathrm{img}(\mathbf{Y})$;
3. $\mathbf{Y}u = \mathbf{\Phi}_2 \mathbf{X} \mathbf{\Phi}_1^{-1} u, \forall u \in \mathrm{dom}(\mathbf{Y})$.

Definition 2.1.4. We use the notation $\mathbf{X} \overset{\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1}{\hookleftarrow} \mathbf{Y}$ to denote $\mathbf{X}$ is defined by and equivalent to $\mathbf{Y}$ via $\mathbf{\Phi}_2 \oplus \mathbf{\Phi}_1$.

Remark 2.1.5. Given two operators $\mathbf{A} : \operatorname{dom}(\mathbf{A}) \subset \mathcal{V}_2 \to \mathcal{V}_3$ and $\mathbf{B} : \operatorname{dom}(\mathbf{B}) \subset \mathcal{V}_1 \to \mathcal{V}_2$, note that

$$\operatorname{dom}(\mathbf{AB}) = \{x \in \operatorname{dom}(\mathbf{B}) \mid \mathbf{B}x \in \operatorname{dom}(\mathbf{A})\}.$$

In the special case where $\operatorname{img}(\mathbf{B}) \subset \operatorname{dom}(\mathbf{A})$, we have then $\operatorname{dom}(\mathbf{AB}) = \operatorname{dom}(\mathbf{B})$.

Lemma 2.1.6. *Given linear operators* $\mathbf{A} : \operatorname{dom}(\mathbf{A}) \subset \mathcal{V}_2 \to \mathcal{V}_3$*,* $\mathbf{B} : \operatorname{dom}(\mathbf{B}) \subset \mathcal{V}_1 \to \mathcal{V}_2$*,* $\mathbf{C} : \operatorname{dom}(\mathbf{C}) \subset \mathcal{X}_2 \to \mathcal{X}_3$*, and* $\mathbf{D} : \operatorname{dom}(\mathbf{D}) \subset \mathcal{X}_1 \to \mathcal{X}_2$ *and isomorphisms* $\boldsymbol{\Phi}_1 : \mathcal{V}_1 \to \mathcal{X}_1$*,* $\boldsymbol{\Phi}_2 : \mathcal{V}_2 \to \mathcal{X}_2$ *and* $\boldsymbol{\Phi}_3 : \mathcal{V}_3 \to \mathcal{X}_3$*, suppose that* $\operatorname{img}(\mathbf{B}) \subset \operatorname{dom}(\mathbf{A})$*,* $\mathbf{A} \overset{\boldsymbol{\Phi}_3 \oplus \boldsymbol{\Phi}_2}{\sim} \mathbf{C}$ *and* $\mathbf{B} \overset{\boldsymbol{\Phi}_2 \oplus \boldsymbol{\Phi}_1}{\sim} \mathbf{D}$*. Then* $\operatorname{img}(\mathbf{D}) \subset \operatorname{dom}(\mathbf{C})$ *and* $\mathbf{AB} \overset{\boldsymbol{\Phi}_3 \oplus \boldsymbol{\Phi}_1}{\sim} \mathbf{CD}$*.*

Proof. From Remark 2.1.3, $\mathbf{A} \overset{\boldsymbol{\Phi}_3 \oplus \boldsymbol{\Phi}_2}{\sim} \mathbf{C}$ is equivalent to

(a) $\operatorname{dom}(\mathbf{A}) \overset{\boldsymbol{\Phi}_2}{\sim} \operatorname{dom}(\mathbf{C})$;

(b) $\operatorname{img}(\mathbf{A}) \overset{\boldsymbol{\Phi}_3}{\sim} \operatorname{img}(\mathbf{C})$;

(c) $\mathbf{C}u = \boldsymbol{\Phi}_3 \mathbf{A} \boldsymbol{\Phi}_2^{-1} u, \forall u \in \operatorname{dom}(\mathbf{C})$;

and $\mathbf{B} \overset{\boldsymbol{\Phi}_2 \oplus \boldsymbol{\Phi}_1}{\sim} \mathbf{D}$ is equivalent to

(d) $\operatorname{dom}(\mathbf{B}) \overset{\boldsymbol{\Phi}_1}{\sim} \operatorname{dom}(\mathbf{D})$;

(e) $\operatorname{img}(\mathbf{B}) \overset{\boldsymbol{\Phi}_2}{\sim} \operatorname{img}(\mathbf{D})$;

(f) $\mathbf{D}u = \boldsymbol{\Phi}_2 \mathbf{B} \boldsymbol{\Phi}_1^{-1} u, \forall u \in \operatorname{dom}(\mathbf{D})$.

Now, suppose $u \in \operatorname{img}(\mathbf{D})$. Then $\boldsymbol{\Phi}_2^{-1} u \in \operatorname{img}(\mathbf{B})$ by (e). Since $\operatorname{img}(\mathbf{B}) \subset \operatorname{dom}(\mathbf{A})$, we have $\boldsymbol{\Phi}_2^{-1} u \in \operatorname{dom}(\mathbf{A})$. This implies $\boldsymbol{\Phi}_2 \boldsymbol{\Phi}_2^{-1} u = u \in \operatorname{dom}(\mathbf{C})$ by (a), and hence $\operatorname{img}(\mathbf{D}) \subset \operatorname{dom}(\mathbf{C})$. By Remark 2.1.5,

$$\operatorname{dom}(\mathbf{AB}) = \operatorname{dom}(\mathbf{B}) \quad \text{and} \quad \operatorname{dom}(\mathbf{CD}) = \operatorname{dom}(\mathbf{D}),$$

and therefore $\operatorname{dom}(\mathbf{AB}) \overset{\boldsymbol{\Phi}_1}{\sim} \operatorname{dom}(\mathbf{CD})$ by (d).

We show now $\operatorname{img}(\mathbf{AB}) \overset{\boldsymbol{\Phi}_3}{\sim} \operatorname{img}(\mathbf{CD})$. Note that $\operatorname{img}(\mathbf{AB}) = \mathbf{AB} \operatorname{dom}(\mathbf{B}) = \mathbf{A} \operatorname{img}(\mathbf{B})$ and similarly $\operatorname{img}(\mathbf{CD}) = \mathbf{C} \operatorname{img}(\mathbf{D})$. Suppose $y \in \mathbf{A} \operatorname{img}(\mathbf{B})$, whereby $y = \mathbf{A}u$, for a $u \in \operatorname{img}(\mathbf{B}) \subset \operatorname{dom}(\mathbf{A})$. By (c), we then have $y = \boldsymbol{\Phi}_3^{-1} \mathbf{C} \boldsymbol{\Phi}_2 u \implies \boldsymbol{\Phi}_3 y = \mathbf{C} \boldsymbol{\Phi}_2 u$. Note that $\boldsymbol{\Phi}_2 u \in \operatorname{img}(\mathbf{D})$ by (e), and hence $\boldsymbol{\Phi}_3 y \in \mathbf{C} \operatorname{img}(\mathbf{D})$. Therefore, $\boldsymbol{\Phi}_3 \operatorname{img}(\mathbf{AB}) \subset \operatorname{img}(\mathbf{CD})$. The fact that $\boldsymbol{\Phi}_3^{-1} \operatorname{img}(\mathbf{CD}) \subset \operatorname{img}(\mathbf{AB})$ follows by reversing the preceding argument.

Finally, note that for any $u \in \mathrm{dom}\,(\mathbf{CD}) = \mathrm{dom}\,(\mathbf{D})$,

$$\mathbf{CD}u = \mathbf{\Phi}_3\mathbf{A}\mathbf{\Phi}_2^{-1}\mathbf{D}u = \mathbf{\Phi}_3\mathbf{A}\mathbf{\Phi}_2^{-1}\mathbf{\Phi}_2\mathbf{B}\mathbf{\Phi}_1^{-1}u = \mathbf{\Phi}_3\mathbf{AB}\mathbf{\Phi}_1^{-1}u,$$

where the first equality follows from (c) and the proven fact that $\mathrm{img}\,(\mathbf{D}) \subset \mathrm{dom}\,(\mathbf{C})$, and the second equality from (f). Putting all the results together, we have, by Remark 2.1.3, $\mathscr{G}_{\mathbf{AB}} \overset{\mathbf{\Phi}_3\oplus\mathbf{\Phi}_1}{\sim} \mathscr{G}_{\mathbf{CD}}$. □

2.1.2 Causal mappings of finite-energy signals

In line with the ν-gap metric based analysis [Vin93, Vin01, CJK12, JC10, JC11], operators mapping between *finite-energy continuous-time* signals are of central concern. Define the Hilbert space

$$\boldsymbol{L}^2_{\mathbb{R}} := \left\{ \phi : \mathbb{R} \to \mathbb{R}^m \;\middle|\; \|\phi\|_2 := \langle \phi, \phi \rangle_2^{\frac{1}{2}} < \infty,\ \text{where } \langle u, v \rangle_2 := \int_{-\infty}^{\infty} u(t)^T v(t)\, dt \right\}.$$

In general, the codomain of functions in $\boldsymbol{L}^2_{\mathbb{R}}$ may also be taken as $\mathbb{C}^m$ or any Hilbert space. In the sequel, we assume compatibility between the input-output spaces of component operator mappings in compositions. Define the following two subsets of $\boldsymbol{L}^2_{\mathbb{R}}$:

$$\begin{aligned} \boldsymbol{L}^2_{\mathbb{I}} &:= \left\{ \phi \in \boldsymbol{L}^2_{\mathbb{R}} \mid \phi(t) = 0\, \forall t \in \mathbb{R} \setminus \mathbb{I} \right\} \text{ for some } \mathbb{I} \subset \mathbb{R}; \\ \boldsymbol{L}^{2+} &:= \left\{ \phi \in \boldsymbol{L}^2_{\mathbb{R}} \mid \mathbf{\Pi}_\tau \phi = 0,\ \text{for some } \tau \in \mathbb{R} \right\} = \bigcup_{\tau \in \mathbb{R}} \boldsymbol{L}^2_{[\tau,\infty)}, \end{aligned}$$

where $\mathbf{\Pi}_\tau$ denotes the truncation operator at time $\tau \in \mathbb{R}$ defined by

$$\mathbf{\Pi}_\tau : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{(-\infty,\tau)}; \quad \mathbf{\Pi}_\tau x(t) := \begin{cases} x(t) & t < \tau \\ 0 & t \geq \tau. \end{cases}$$

Clearly, $\boldsymbol{L}^2_{\mathbb{R}}$ is the orthogonal direct sum of $\boldsymbol{L}^2_{(-\infty,\tau)}$ and $\boldsymbol{L}^2_{[\tau,\infty)}$ for any $\tau \in \mathbb{R}$. Note that the closure of $\boldsymbol{L}^{2+}$ in $\boldsymbol{L}^2_{\mathbb{R}}$ is equal to $\boldsymbol{L}^2_{\mathbb{R}}$. For a linear operator $\mathbf{X} : \mathrm{dom}\,(\mathbf{X}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$, define respectively its graph and truncated graph as

$$\begin{aligned} \mathscr{G}_{\mathbf{X}} &:= \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \boldsymbol{L}^2_{\mathbb{R}} : u \in \mathrm{dom}\,(\mathbf{X}) \text{ and } y = \mathbf{X}u \right\}; \\ \text{and} \quad \mathscr{G}^{\tau}_{\mathbf{X}} &:= \mathscr{G}_{\mathbf{X}} \cap \boldsymbol{L}^2_{[\tau,\infty)}. \end{aligned}$$

Similarly, its inverse graph and truncated inverse graph are defined respectively as

$$\begin{aligned} \mathscr{G}'_{\mathbf{X}} &:= \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \boldsymbol{L}^2_{\mathbb{R}} : y \in \operatorname{dom}(\mathbf{X}) \text{ and } u = \mathbf{X}y \right\}; \\ \text{and} \quad \mathscr{G}'^{\tau}_{\mathbf{X}} &:= \mathscr{G}'_{\mathbf{X}} \cap \boldsymbol{L}^2_{[\tau,\infty)}. \end{aligned}$$

We conclude this subsection with the definition of *causality*, which is an important notion in systems theory and has been widely studied the literature; see, for example, [Wil69, Sae70, FS82, GS93, CV02, GS10].

Definition 2.1.7. A linear operator $\mathbf{X} : \operatorname{dom}(\mathbf{X}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ is said to be *causal* if for all $\tau \in \mathbb{R}$, $\mathbf{\Pi}_\tau \mathscr{G}_{\mathbf{X}}$ is the graph of a linear operator, i.e.,

$$\forall \tau \in \mathbb{R}, \forall \begin{bmatrix} y_\tau \\ u_\tau \end{bmatrix} \in \mathbf{\Pi}_\tau \mathscr{G}_{\mathbf{X}}, \text{ we have that } u_\tau = 0 \implies y_\tau = 0.$$

A similar definition can be made in terms of the inverse graph.

If $\operatorname{dom}(\mathbf{X}) = \boldsymbol{L}^2_{\mathbb{R}}$, then the above definition is equivalent to

$$\mathbf{\Pi}_\tau \mathbf{X}(\mathbf{I} - \mathbf{\Pi}_\tau) = 0 \; \forall \tau \in \mathbb{R}.$$

By contrast, $\mathbf{X}$ is said to be *anti-causal* if $(\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}\mathbf{\Pi}_\tau = 0 \; \forall \tau \in \mathbb{R}$. If $\mathbf{X}$ is simultaneously causal and anti-causal, it is called *memoryless*. Note that $\mathbf{X} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ is causal if, and only if, $\mathbf{X}^*$ is anti-causal.

Lemma 2.1.8. *An operator $\mathbf{X} : \operatorname{dom}(\mathbf{X}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ for which $\boldsymbol{L}^{2+} \subset \operatorname{dom}(\mathbf{X})$ is such that $\mathbf{\Pi}_\tau \mathbf{X}(\mathbf{I} - \mathbf{\Pi}_\tau) = 0 \, \forall \tau \in \mathbb{R}$ if, and only if, $\mathbf{X}\boldsymbol{L}^2_{[\tau,\infty)} \subset \boldsymbol{L}^2_{[\tau,\infty)} \, \forall \tau \in \mathbb{R}$.*

Proof. ($\Longrightarrow$) For any $\tau \in \mathbb{R}$ and $u \in \boldsymbol{L}^2_{[\tau,\infty)}$, note that $\mathbf{X}u = \mathbf{X}(\mathbf{I} - \mathbf{\Pi}_\tau)u$, and therefore $\mathbf{\Pi}_\tau \mathbf{X}u = \mathbf{\Pi}_\tau \mathbf{X}(\mathbf{I} - \mathbf{\Pi}_\tau)u = 0$, where the last equality holds by hypothesis. Consequently, $\mathbf{X}u \in \boldsymbol{L}^2_{[\tau,\infty)}$. ($\Longleftarrow$) Suppose, to the contrary, that there exist a $\tau \in \mathbb{R}$ and a $u \in \boldsymbol{L}^2_{[\tau,\infty)}$ such that $\mathbf{\Pi}_\tau \mathbf{X}(\mathbf{I} - \mathbf{\Pi}_\tau)u \neq 0$. Let $v := (\mathbf{I} - \mathbf{\Pi}_\tau)u \in \boldsymbol{L}^2_{[\tau,\infty)}$, so that $\mathbf{\Pi}_\tau \mathbf{X}v \neq 0$. This implies $\mathbf{X}v \notin \boldsymbol{L}^2_{[\tau,\infty)}$, contradicting the hypothesis. $\square$

2.1.3 Wiener-Hopf and Hankel operators

Generalised Wiener-Hopf (a.k.a. Toeplitz) and Hankel operators are defined here. These operators are useful in the study of linear time-varying systems; see e.g. [IS04]. As reported in [JC10, JC11], they are crucial in ν-gap metric based stability analysis of time-varying feedback systems, in conjunction with the theory of Fredholm operators, as detailed in the next chapter.

Definition 2.1.9. Given $\mathbf{X} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$, define

1. the Wiener-Hopf operator relative to the 'initial' time τ as

$$\mathbf{T}_{\mathbf{X},\tau} := (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}} : \boldsymbol{L}^2_{[\tau,\infty)} \to \boldsymbol{L}^2_{[\tau,\infty)};$$

2. the forward Hankel operator relative to the 'initial' time τ as

$$\mathbf{H}^{+-}_{\mathbf{X},\tau} := (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}|_{\boldsymbol{L}^2_{(-\infty,\tau)}} : \boldsymbol{L}^2_{(-\infty,\tau)} \to \boldsymbol{L}^2_{[\tau,\infty)};$$

3. the backward Hankel operator relative to the 'initial' time τ as

$$\mathbf{H}^{-+}_{\mathbf{X},\tau} := \mathbf{\Pi}_\tau\mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}} : \boldsymbol{L}^2_{[\tau,\infty)} \to \boldsymbol{L}^2_{(-\infty,\tau)},$$

It is straightforward to show from first principles that the adjoints of these operators are given by

$$(\mathbf{T}_{\mathbf{X},\tau})^* = \mathbf{T}_{\mathbf{X}^*,\tau};\ (\mathbf{H}^{+-}_{\mathbf{X},\tau})^* = \mathbf{H}^{-+}_{\mathbf{X}^*,\tau};\ (\mathbf{H}^{-+}_{\mathbf{X},\tau})^* = \mathbf{H}^{+-}_{\mathbf{X}^*,\tau};$$

see for e.g., [JC10, Lem. 2] for a standard proof.

Definition 2.1.10. A causal $\mathbf{X} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ is said to have *non-singular instantaneous gain* if

$$\inf_{\tau\in\mathbb{R}} \inf_{\delta>0} \underline{\gamma}\left(\mathbf{\Pi}_{\tau+\delta}\mathbf{X}|_{\mathrm{img}(\mathbf{\Pi}_{\tau+\delta}-\mathbf{\Pi}_\tau)}\right) > 0.$$

Remark 2.1.11. The concept of instantaneous gain arises in the earlier study of well-posedness of feedback interconnections for possibly nonlinear systems [Wil71, DV75]. As observed in [JC10, JC11], it is useful for defining graphs of causal operators; see Lemma 2.1.13 after the following.

Lemma 2.1.12 ([JC10, Lem. 5])**.** *If a* causal $\mathbf{X} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ *has non-singular instantaneous gain, then for all* $\tau \in \mathbb{R}$*,*

1. *the Wiener-Hopf operator* $\mathbf{T}_{\mathbf{X},\tau}$ *is injective; and*
2. *the algebraic inverse* $\mathbf{T}^{-1}_{\mathbf{X},\tau} : \mathrm{img}\,(\mathbf{T}_{\mathbf{X},\tau}) \to \boldsymbol{L}^2_{[\tau,\infty)}$ *is causal.*

Proof. Suppose for some $\tau \in \mathbb{R}$, there exists a non-zero $w \in \ker(\mathbf{T}_{\mathbf{X},\tau}) \subset \boldsymbol{L}^2_{[\tau,\infty)}$. Then for sufficiently large $\delta t > 0$, we have $(\mathbf{\Pi}_{\tau+\delta t} - \mathbf{\Pi}_\tau)w = \mathbf{\Pi}_{\tau+\delta t}w \neq 0$. It follows that

$$\mathbf{\Pi}_{\tau+\delta t}\mathbf{X}(\mathbf{\Pi}_{\tau+\delta t} - \mathbf{\Pi}_\tau)w = \mathbf{\Pi}_{\tau+\delta t}\mathbf{X}\mathbf{\Pi}_{\tau+\delta t}w = \mathbf{\Pi}_{\tau+\delta t}\mathbf{X}w = 0,$$

where the second equality holds by the causality of $\mathbf{X}$. This contradicts the hypothesis of non-singular instantaneous gain and thus $\mathbf{T}_{\mathbf{X},\tau}$ must be injective for all $\tau \in \mathbb{R}$, whereby the algebraic inverse $\mathbf{T}_{\mathbf{X},\tau}^{-1} : \operatorname{img}(\mathbf{T}_{\mathbf{X},\tau}) \to \boldsymbol{L}^2_{[\tau,\infty)}$ is well-defined.

Now suppose to the contrary that $\mathbf{T}_{\mathbf{X},\tau}^{-1}$ is not causal for some $\tau \in \mathbb{R}$, i.e. there exist $\delta > 0$ and $y \in \operatorname{img}(\mathbf{T}_{\mathbf{X},\tau}) \cap \boldsymbol{L}^2[\tau+\delta,\infty)$ such that with $u := \mathbf{T}_{\mathbf{X},\tau}^{-1} y \in \boldsymbol{L}^2_{[\tau,\infty)}$, we have $\boldsymbol{\Pi}_{\tau+\delta} u \neq 0$. Observe that

$$\begin{aligned} 0 = \boldsymbol{\Pi}_{\tau+\delta} y = \boldsymbol{\Pi}_{\tau+\delta}\mathbf{T}_{\mathbf{X},\tau} u &= \boldsymbol{\Pi}_{\tau+\delta}\mathbf{X}\boldsymbol{\Pi}_{\tau+\delta} u \\ &= \boldsymbol{\Pi}_{\tau+\delta}\mathbf{X}(\boldsymbol{\Pi}_{\tau+\delta} - \boldsymbol{\Pi}_{\tau})u, \end{aligned}$$

where the second last equality follows from the causality of $\mathbf{X}$ and the last from $\boldsymbol{\Pi}_\tau u = 0$. This contradicts the hypothesis that $\mathbf{X}$ has non-singular instantaneous gain. Therefore, $\mathbf{T}_{\mathbf{X},\tau}^{-1}$ must be causal for all $\tau \in \mathbb{R}$. ☐

Lemma 2.1.13 ([JC10, Rem. 3])**.** *Given a causal operator* $\mathbf{G} := \left[\begin{smallmatrix}\mathbf{N}\\ \mathbf{M}\end{smallmatrix}\right] \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$*, if* $\mathbf{M}$ *has non-singular instantaneous gain, then*

$$\mathbf{G}\boldsymbol{L}^{2+} = \{v \in \boldsymbol{L}^2_{\mathbb{R}} \mid v = \mathbf{G}w;\ w \in \boldsymbol{L}^{2+}\} = \bigcup_{\tau\in\mathbb{R}} \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau})$$

is the graph of a causal *operator* $\mathbf{P} : \operatorname{img}(\mathbf{M}|_{\boldsymbol{L}^{2+}}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ *which satisfies*

$$\mathbf{P}|_{\operatorname{img}\left(\mathbf{M}|_{\boldsymbol{L}^2_{[\tau,\infty)}}\right)} = \mathbf{T}_{\mathbf{N},\tau}\mathbf{T}_{\mathbf{M},\tau}^{-1} \text{ for all } \tau \in \mathbb{R}.$$

Proof. Note for any $\tau \in \mathbb{R}$ and $\left[\begin{smallmatrix}y\\ u\end{smallmatrix}\right] \in \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau})$, there exists by definition a $v \in \boldsymbol{L}^2_{[\tau,\infty)}$ for which $y = \mathbf{T}_{\mathbf{N},\tau} v$ and $u = \mathbf{T}_{\mathbf{M},\tau} v$. If $u = 0$, then $v = \mathbf{T}_{\mathbf{M},\tau}^{-1} u = 0$, where $\mathbf{T}_{\mathbf{M},\tau}^{-1}$ exists by Lemma 2.1.12. This implies $y = 0$ and hence $\mathbf{G}\boldsymbol{L}^{2+}$ is the graph of an operator; call it $\mathbf{P}$. To show causality of $\mathbf{P}$, suppose $\boldsymbol{\Pi}_{\tau_1} u = 0$ for some $\tau_1 > \tau$, then $\boldsymbol{\Pi}_{\tau_1} v = \boldsymbol{\Pi}_{\tau_1}\mathbf{T}_{\mathbf{M},\tau}^{-1} u = 0$, since by Lemma 2.1.12 $\mathbf{T}_{\mathbf{M},\tau}^{-1}$ is causal. This implies $\boldsymbol{\Pi}_{\tau_1} y = \boldsymbol{\Pi}_{\tau_1}\mathbf{T}_{\mathbf{N},\tau} v = 0$, where the last equality follows by the causality of $\mathbf{T}_{\mathbf{N},\tau}$, as required for $\mathbf{P}$ to be causal. ☐

2.2 Representations of system graphs

Following [JC10, JC11], the developments of closed-loop robustness results in the forthcoming chapters are underpinned by the following assumptions on the existence of 'strong graph representations/symbols' for causal linear operators. In Sections 2.3, 2.4, and 2.5, we present classes of systems for which the assumptions are satisfied; namely, finite-dimensional time-varying state-space systems, distributed-parameter time-invariant systems, and generic periodic systems with finite-dimensional 'realisations'; of these, the first two classes mentioned have already been examined in [JC10, JC11], and the last is adapted from [CV04].

Assumption 2.2.1. *Given a* causal *operator* $\mathbf{P} : \mathrm{dom}\,(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$*, there exist* causal *operators* $\mathbf{N}, \mathbf{M}, \tilde{\mathbf{N}}, \tilde{\mathbf{M}}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ *satisfying the following properties:*

1. *the double Bezout identity*

$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} \begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} = \mathbf{I};$$

2. $\mathrm{img}\,(\mathbf{G}) = \ker\left(\tilde{\mathbf{G}}\right)$ *and* $\mathscr{G}^\tau_{\mathbf{P}} := \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \mathrm{img}\,(\mathbf{T}_{\mathbf{G},\tau}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right)$ *for all* $\tau \in \mathbb{R}$*, where*

$$\mathbf{G} := \begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} \quad \textit{and} \quad \tilde{\mathbf{G}} := \begin{bmatrix} -\tilde{\mathbf{M}} & \tilde{\mathbf{N}} \end{bmatrix}$$

are respectively called right and left strong graph symbols/representations *for* $\mathbf{P}$*, respectively.*

Assumption 2.2.2. $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ *and* $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$*, i.e. the strong right and left graph symbols are normalised.*

Assumption 2.2.3. $\mathbf{H}^{+-}_{\mathbf{G},\tau}$ *and* $\mathbf{H}^{+-}_{\tilde{\mathbf{G}},\tau}$ *are compact for all* $\tau \in \mathbb{R}$.

The term 'strong' in part 2 of Assumption 2.2.1 is borrowed from [DS93] to emphasise that right (resp. left) graph symbols have left (resp. right) bounded causal inverses. In this thesis, graph symbols are always taken to be strong.

Definition 2.2.4. We denote by $\mathbb{S}$ the set of causal operators for which all of Assumptions 2.2.1, 2.2.2, and 2.2.3 are satisfied.

2.3 Finite-dimensional time-varying systems

Here we summarise several basic notions for LTV systems, and refer to [IS04] for more details. Consider the finite-dimensional continuous time-varying linear system $u \mapsto y$ described by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t), \end{aligned} \tag{2.1}$$

where $t \in \mathbb{R}$ and A, B, C, and D are continuous and bounded matrix-valued functions with $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times m}$, $C(t) \in \mathbb{R}^{p\times n}$, and $D(t) \in \mathbb{R}^{p\times m}$. We let $X : \mathbb{R} \to \mathbb{R}^{n\times n}$ denote the (invertible) principal fundamental matrix defined by the solution of $\dot{X}(t) = A(t)X(t)$ with $X(0) = I$, which exists by the assumptions on A [BAG92]. It follows that the state transition matrix $\Phi_A(t, s) := X(t)X(s)^{-1}$ satisfies, for all $t, s, \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_A(t,s) = A(t)\Phi_A(t,s),\ \Phi_A(t,t) = I,\ \text{ and } \Phi_A(t,\tau)\Phi_A(\tau,s) = \Phi_A(t,s)\,\forall\tau \in \mathbb{R}.$$

Definition 2.3.1 ([Cop78])**.** A continuous, bounded function $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is said to define an *exponentially dichotomic* evolution if there exist $\tau \in \mathbb{R}$, an associated projection $P = P^2 \in \mathbb{R}^{n\times n}$, $\rho \geq 1$, and $\sigma > 0$ such that

$$\begin{aligned} \|\Phi_A(t,\tau)P\Phi_A(\tau,s)\| &\leq \rho e^{-\sigma(t-s)}, \forall t \geq s \\ \|\Phi_A(t,\tau)(I_n - P)\Phi_A(\tau,s)\| &\leq \rho e^{-\sigma(s-t)}, \forall s \geq t, \end{aligned} \tag{2.2}$$

where $\|\cdot\|$ denotes the spectral norm. This implies

$$\begin{aligned} \operatorname{img}(P) &= \{x \in \mathbb{R}^n : \Phi_A(\cdot,\tau)x \in \boldsymbol{L}^2_{[\tau,\infty)}\} \quad \text{and} \\ \ker(P) &= \{x \in \mathbb{R}^n : \Phi_A(-\cdot,\tau)x \in \boldsymbol{L}^2_{(-\infty,\tau)}\}. \end{aligned}$$

In particular, if $P = I_n$, then A is said to define an *exponentially stable* evolution.

The pair (A, B) is said to be *stabilisable* if, and only if, there exists a continuous and bounded F such that $A + BF$ defines an exponentially stable evolution. In contrast, (C, A) is said to be *detectable* if, and only if, there exists a continuous and bounded L such that $A + LC$ defines an exponentially stable evolution. We assume throughout this section that all state-space realisations are stabilisable and detectable.

When the matrix function A in a system of the form (2.1) defines an exponentially dichotomous evolution, the state-space system can be equivalently interpreted as a bounded convolution operator $\mathbf{Z} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ given by

$$(\mathbf{Z}u)(t) = D(t)u(t) + C(t)\Phi_A(t,\tau)P\int_{-\infty}^{t}\Phi_A(\tau,s)B(s)u(s)\,ds \tag{2.3}$$

$$- C(t)\Phi_A(t,\tau)(I-P)\int_{t}^{\infty}\Phi_A(\tau,s)B(s)u(s)\,ds, \tag{2.4}$$

where $\tau \in \mathbb{R}$, $P \in \mathbb{R}^{n\times n}$ are such that (2.2) is satisfied for A [BAG92, Thm 1.1]. It follows that A defines an exponentially stable evolution (i.e. $P = I$) if, and only if, the corresponding $\mathbf{Z}$ is bounded and causal (on the entirety of $\boldsymbol{L}^2_{\mathbb{R}}$). When this is the case, it follows that the associated forward Hankel operator, $\mathbf{H}^{+-}_{\mathbf{Z},\tau}$ is compact for all $\tau \in \mathbb{R}$. To see this, note the Hankel factorisation $\mathbf{H}^{+-}_{\mathbf{Z},\tau} = \mathbf{L}_{O,\tau}\mathbf{L}_{C,\tau}$, where the observability operator $\mathbf{L}_{O,\tau} \in \mathscr{L}(\mathbb{R}^n, \boldsymbol{L}^2_{[\tau,\infty)})$ and controllability operator $\mathbf{L}_{C,\tau} \in \mathscr{L}(\boldsymbol{L}^2(-\infty,\tau], \mathbb{R}^n)$ are defined by

$$(\mathbf{L}_{O,\tau}x)(t) := C(t)\Phi_A(t,\tau)x \quad \text{and} \quad \mathbf{L}_{C,\tau}u := \int_{-\infty}^{\tau} \Phi_A(\tau,s)B(s)u(s)\,ds.$$

Since $\mathbf{L}_{C,\tau}$ has finite-dimensional image and $\mathbf{L}_{O,\tau}$ has finite-dimensional domain, both operators are compact [Kre89, Thm. 8.1-4]. As such, $\mathbf{H}^{+-}_{\mathbf{Z},\tau}$ is compact by Lemma 2.1.1.

Henceforth we denote $\mathbf{Z}$ in terms of its state-space realisation:

$$\mathbf{Z} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = (A, B, C, D).$$

Elementary operations on input-output convolution operators are briefly described below; see [IS04, Section 2.1] for more details. First of all, the Hilbert adjoint of $\mathbf{Z}$ is given by $\mathbf{Z}^* = (-A^T, -C^T, B^T, D^T)$. If D is boundedly invertible and $A - BD^{-1}C$ defines an exponentially dichotomic evolution, then $\mathbf{Z}$ is boundedly invertible and

$$\mathbf{Z}^{-1} = \left(\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right).$$

Let $\mathbf{Z}_1 := (A_1, B_1, C_1, D_1)$ and $\mathbf{Z}_2 := (A_2, B_2, C_2, D_2)$. A realisation for their composition is given by

$$\mathbf{Z}_1\mathbf{Z}_2 = \left(\begin{array}{c|c} \begin{bmatrix} A_2 & 0 \\ B_1C_2 & A_1 \end{bmatrix} & \begin{bmatrix} B_2 \\ B_1D_2 \end{bmatrix} \\ \hline \begin{bmatrix} D_1C_2 & C_1 \end{bmatrix} & D_1D_2 \end{array}\right). \tag{2.5}$$

For any $\tau \in \mathbb{R}$, let

$$\mathscr{G}^\tau := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \boldsymbol{L}^2_{[\tau,\infty)} \;\middle|\; \exists x \in \boldsymbol{L}^2_{[\tau,\infty)} \text{ for which (2.1) is satisfied} \right\}.$$

Without assuming that A defines an exponentially dichotomic evolution, it is shown in [MC10] that if the pairs (A, B) and (C, A) are respectively stabilisable and detectable, then

$$\begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} := \left(\begin{array}{c|c} A+BF & \begin{bmatrix} BR^{-1/2} & -L\tilde{R}^{1/2} \end{bmatrix} \\ \hline \begin{bmatrix} C+DF \\ F \end{bmatrix} & \begin{bmatrix} DR^{-1/2} & \tilde{R}^{1/2} \\ R^{-1/2} & 0 \end{bmatrix} \end{array} \right);$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} := \left(\begin{array}{c|c} A+LC & \begin{bmatrix} L & -(B+LD) \end{bmatrix} \\ \hline \begin{bmatrix} R^{1/2}F \\ \tilde{R}^{-1/2}C \end{bmatrix} & \begin{bmatrix} 0 & R^{1/2} \\ \tilde{R}^{-1/2} & -\tilde{R}^{-1/2}D \end{bmatrix} \end{array} \right),$$

where $R := I+D^TD$, $\tilde{R} := I+DD^T$, $F := -R^{-1}(D^TC+B^TX)$, $L := -(BD^T+YC^T)\tilde{R}^{-1}$, $X(t) := \lim_{t_f\to\infty} \tilde{X}(t;t_f)$, $Y(t) := \lim_{t_i\to-\infty} \tilde{Y}(t;t_i)$, and $\tilde{X} = \tilde{X}^T$ and $\tilde{Y} = \tilde{Y}^T$ are respectively the solutions to the time-varying single-point boundary-value differential Riccati equations

$$-\dot{\tilde{X}} = \tilde{X}(A - BR^{-1}D^TC) + (A - BR^{-1}D^TC)^T\tilde{X} - \tilde{X}BR^{-1}B^T\tilde{X} + C^T\tilde{R}^{-1}C; \tilde{X}(t_f;t_f) = 0$$
$$\dot{\tilde{Y}} = (A - BD^T\tilde{R}^{-1}C)\tilde{Y} + \tilde{Y}(A - BD^T\tilde{R}^{-1}C)^T - \tilde{Y}C^T\tilde{R}^{-1}C\tilde{Y} + BR^{-1}B^T; \tilde{Y}(t_i;t_i) = 0,$$

are such that:

$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} \begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} = \mathbf{I},$$

$\operatorname{img}(\mathbf{G}) = \ker\left(\tilde{\mathbf{G}}\right)$, $\mathscr{G}^\tau = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right)$ for all $\tau \in \mathbb{R}$, $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$, where

$$\mathbf{G} := \begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{G}} := \begin{bmatrix} -\tilde{\mathbf{M}} & \tilde{\mathbf{N}} \end{bmatrix}.$$

Also see [ABB02, TV92], in which the same result is obtained under stronger assumptions.

As such, a stabilisable and detectable state-space system of the form (2.1) may be associated with a linear mapping $\mathbf{P} : \operatorname{img}(\mathbf{M}|_{\boldsymbol{L}^{2+}}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ (as in [JC10, JC11]), with graph

$$\mathscr{G}_{\mathbf{P}} := \mathbf{G}\boldsymbol{L}^{2+} = \{w \mid w = \mathbf{G}v \text{ for some } v \in \boldsymbol{L}^{2+}\} \subset \boldsymbol{L}^{2+}. \tag{2.6}$$

As discussed in Lemma 2.1.13, since the instantaneous gain of $\mathbf{M}$ is $\inf_{t\in\mathbb{R}} \underline{\sigma}(R^{-1/2}(t)) > 0$, $\mathbf{P}$ is a causal operator by the above definition of its graph. Importantly, Assumptions 2.2.1, 2.2.2, and 2.2.3 hold from the preceding developments.

Definition 2.3.2. We denote by $\mathbb{V} \subset \mathbb{S}$ the set of causal operators with stabilisable and detectable state-space realisations of the form (2.1) and graphs defined as in (2.6).

2.4 Infinite-dimensional time-invariant systems

In this section, we consider the constantly proper subclass of the Callier-Desoer algebra [CD78, CZ95]. Time-domain systems can be defined via multiplication operators with symbols in this class using the continuous-time Fourier transform isomorphism, as discussed below. For this class of systems, the ν-gap metric introduced in Chapter 3 will be shown in Section 4.6.2 to reduce to the original expression of [Vin93, Vin01, CJK12, CJK10, CJK09].

Define $\mathbb{C}_{\sigma+} := \{a + jb \mid a, b \in \mathbb{R}, a > \sigma\}$ with closure $\overline{\mathbb{C}}_{\sigma+}$.

Frequency-domain signal spaces

We define the frequency-domain signal spaces $\boldsymbol{L}^2_{j\mathbb{R}}$ (resp. $\boldsymbol{H}^2_{\mathbb{C}_+}$ and $\boldsymbol{H}^2_{\mathbb{C}_-}$) to comprise of continuous-time Fourier transforms $\mathscr{F}$ of functions in $\boldsymbol{L}^2_{\mathbb{R}}$ (resp. $\boldsymbol{L}^2_{[0,\infty)}$ and $\boldsymbol{L}^2_{(-\infty,0)}$), so that

$$\boldsymbol{L}^2_{\mathbb{R}} \overset{\mathscr{F}}{\sim} \boldsymbol{L}^2_{j\mathbb{R}} \text{ (resp. } \boldsymbol{L}^2_{[0,\infty)} \overset{\mathscr{F}}{\sim} \boldsymbol{H}^2_{\mathbb{C}_+} \text{ and } \boldsymbol{L}^2_{(-\infty,0)} \overset{\mathscr{F}}{\sim} \boldsymbol{H}^2_{\mathbb{C}_-}),$$

with $\mathscr{F} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{j\mathbb{R}}$ defined by $(\mathscr{F}f)(j\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$. The inner product and norm on $\boldsymbol{L}^2_{j\mathbb{R}}$ are given by $\langle \hat{\phi}, \hat{\psi} \rangle_2 := \int_{-\infty}^{\infty} \hat{\phi}(j\omega)^* \hat{\psi}(j\omega)\, d\omega$ and $\|\hat{\phi}\|_2 := \langle \hat{\phi}, \hat{\phi} \rangle_2^{\frac{1}{2}}$, respectively. Let $\mathbf{S}_\tau : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ denote the continuous-time shift operator defined by $(\mathbf{S}_\tau u)(t) := u(t - \tau)$. Observe that for any $u \in \boldsymbol{L}^2_{\mathbb{R}}$,

$$(\mathscr{F}\mathbf{S}_\tau u)(j\omega) = e^{-j\omega\tau}(\mathscr{F}u)(j\omega),$$

whereby $\boldsymbol{L}^{2+} = \bigcup_{\tau\in\mathbb{R}} \mathbf{S}_\tau \boldsymbol{L}^2_{[0,\infty)} \overset{\mathscr{F}}{\sim} \bigcup_{\tau\in\mathbb{R}} e^{-j\omega\tau} \boldsymbol{H}^2_{\mathbb{C}_+}$.

A transfer function algebra

Suppose $u : [0, \infty) \to \mathbb{R}^{p\times m}$ satisfies $e^{-\sigma\cdot}u(\cdot) \in \boldsymbol{L}^1$ for some $\sigma \in \mathbb{R}$, where

$$\boldsymbol{L}^1 := \left\{ \phi : [0, \infty) \to \mathbb{R}^{p\times m} \;\middle|\; \int_0^{\infty} |\phi_{jk}(t)|\, dt < \infty \right\}.$$

Then for $s \in \overline{\mathbb{C}}_{\sigma+}$, the Laplace transform of u is defined by

$$\hat{u}(s) := \int_0^{\infty} e^{-st} u(t)\, dt.$$

Given $\sigma \in \mathbb{R}$, let $\boldsymbol{\mathcal{A}}(\sigma)$ denote the causal convolution algebra of integral operator kernels

$$\phi(t) = \begin{cases} 0 & \text{for } t < 0 \\ \phi_a(t) + \sum_{i=1}^{\infty} \phi_i \delta(t - t_i) & \text{for } t \geq 0 \end{cases}, \tag{2.7}$$

where

1. $e^{-\sigma \cdot}\phi_a(\cdot) \in \boldsymbol{L}^1$;

2. $t_1 = 0$ and $t_i > 0$ for $i > 1$;

3. $\delta(\cdot - t_i)$ is the Dirac delta centred at t_i;

4. $\sum_{i=1}^{\infty} |\phi_i[j,k]| e^{-\sigma t_i} < \infty$, for all $j = 1, 2, \dots, p$ and $k = 1, 2, \dots, m$, where $\phi_i[j,k]$ denotes the $(j,k)^{\text{th}}$ entry of $\phi_i \in \mathbb{R}^{p \times m}$; and

5. $\|\phi[j,k]\|_\sigma := \int_0^\infty e^{-\sigma t} |\phi_a[j,k](t)|\, dt + \sum_{i=1}^{\infty} |\phi_i[j,k]| e^{-\sigma t_i}$.

Each $\phi \in \boldsymbol{\mathcal{A}}(\sigma)$ has Laplace transform

$$\hat{\phi}(s) = \int_0^\infty \phi_a(t) e^{-st} dt + \sum_{i=1}^{\infty} \phi_i e^{-st_i},$$

defined for all $s \in \overline{\mathbb{C}}_{\sigma+}$. The class $\hat{\boldsymbol{\mathcal{A}}}(\sigma)$ comprises functions that are Laplace transforms of kernels in $\boldsymbol{\mathcal{A}}(\sigma)$. Every entry $\hat{\phi}[j,k]$ of a $\hat{\phi} \in \hat{\boldsymbol{\mathcal{A}}}(\sigma)$ is [CZ95, Lem. A.7.47]:

1. analytic on $\mathbb{C}_{\sigma+}$;

2. continuous on $s = \sigma + j\omega$ for $\omega \in \mathbb{R}$; and

3. bounded on $\overline{\mathbb{C}}_{\sigma+}$ with $\sup_{s \in \overline{\mathbb{C}}_{\sigma+}} |\hat{\phi}[j,k](s)| \leq \|\phi[j,k]\|_\sigma$.

Moreover,

$$\widehat{\phi \circledast \psi} = \hat{\phi}\hat{\psi}$$

for any $\psi \in \boldsymbol{\mathcal{A}}(\sigma)$, where $\circledast$ denotes convolution product; see [CZ95, Lem. A.7.46].

Let $\hat{\boldsymbol{\mathcal{A}}} := \hat{\boldsymbol{\mathcal{A}}}(0)$ and define:

$$\hat{\boldsymbol{\mathcal{A}}}_- := \left\{ \hat{\phi} \;\middle|\; \hat{\phi} \in \hat{\boldsymbol{\mathcal{A}}}(\sigma) \text{ for some } \sigma < 0 \right\}; \text{ and}$$

$$\hat{\boldsymbol{\mathcal{A}}}_\infty := \left\{ \hat{\phi} \in \hat{\boldsymbol{\mathcal{A}}}_- \;\middle|\; \inf_{\{s \in \overline{\mathbb{C}}_{0+} \,|\, |s| \geq \rho\}} \underline{\sigma}(\hat{\phi}(s)) > 0 \text{ for sufficiently large } \rho > 0 \right\}.$$

The Callier-Desoer class $\hat{\boldsymbol{\mathcal{B}}}$ [CD78, CZ95] consists of transfer functions in which each entry belongs to the quotient algebra

$$\hat{\boldsymbol{\mathcal{A}}}_{-}\left[\hat{\boldsymbol{\mathcal{A}}}_{\infty}\right]^{-1} := \left\{\hat{\phi} = \hat{\nu}\hat{\mu}^{-1} \;\middle|\; \hat{\nu} \in \hat{\boldsymbol{\mathcal{A}}}_{-}, \hat{\mu} \in \hat{\boldsymbol{\mathcal{A}}}_{\infty}\right\}.$$

Indeed, $\hat{\boldsymbol{\mathcal{B}}}$ encompasses all real-rational proper transfer functions [CZ95, Thm. 7.1.16].

All $\hat{\Phi} \in \hat{\boldsymbol{\mathcal{B}}}$ are proper in that for sufficiently large $\rho > 0$,

$$\sup_{\{s \in \overline{\mathbb{C}}_{0+} \mid |s| \geq \rho\}} \bar{\sigma}(\hat{\Phi}(s)) < \infty;$$

see [CZ95, Lem. 7.2.5]. $\hat{\Phi}$ is called *constantly proper* if for some matrix $\hat{\Phi}_{\infty} \in \mathbb{C}^{p \times m}$,

$$\lim_{\rho \to \infty}\left[\sup_{\{s \in \overline{\mathbb{C}}_{0+} \mid |s| \geq \rho\}} \bar{\sigma}\left(\hat{\Phi}(s) - \hat{\Phi}_{\infty}\right)\right] = 0.$$

We use the superscript $^{\mathrm{cp}}$ to denote the sub-algebra of transfer functions which are constantly proper. Note, the sub-algebra $\hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}(\sigma)$ corresponds to the Laplace transforms of distributions in $\boldsymbol{\mathcal{A}}(\sigma)$ for which $\phi_i = 0, \forall i > 1$ in (2.7); see [CZ95, Lem. 7.2.5]. A symbol $\hat{\phi} \in \hat{\boldsymbol{\mathcal{A}}}_{\infty}^{\mathrm{cp}}$ satisfies $\hat{\phi}(s) \to \phi_1$ as $|s| \to \infty$ in $\overline{\mathbb{C}}_{0+}$ [CW90, Fact 1(c)], whereby ϕ_1 is invertible by the definition of $\hat{\boldsymbol{\mathcal{A}}}_{\infty}^{\mathrm{cp}}$ and hence convolution with ϕ has non-singular instantaneous gain (cf. Definition 2.1.10).

Importantly, every transfer function in $\hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$ admits normalised doubly coprime factorisation, as shown in the following.

Proposition 2.4.1. *Given any $P \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$, there exist $N, \tilde{N}, X, Y, \tilde{X}, \tilde{Y} \in \hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}}$ and $\tilde{M}, M \in \hat{\boldsymbol{\mathcal{A}}}_{\infty}^{\mathrm{cp}}$ such that*

$$\begin{gathered} NM^{-1} = \tilde{M}^{-1}\tilde{N} = P; \\ \begin{bmatrix} Y & X \\ \tilde{M} & -\tilde{N} \end{bmatrix}\begin{bmatrix} N & \tilde{X} \\ M & -\tilde{Y} \end{bmatrix} = I \quad \text{on } \overline{\mathbb{C}}_{0+}; \\ M^*M + N^*N = I \quad \text{and} \quad \tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I \quad \text{on } j\mathbb{R}. \end{gathered}$$

Proof. The proof is based on [CZ95, Thm. 7.2.8 and 7.2.14] and the spectral factorisation result [CW99, Thm. 2.2]. See [CJK09, Thm. 1] for details. □

Frequency-domain multiplication operators

Given any $P \in \hat{\mathcal{B}}^{\mathrm{cp}} \setminus \hat{\mathcal{A}}_-^{\mathrm{cp}}$, we define the multiplication operator $\boldsymbol{M}_P : \mathrm{dom}\,(\boldsymbol{M}_P) \subset \boldsymbol{L}^2_{j\mathbb{R}} \to \boldsymbol{L}^2_{j\mathbb{R}}$ associated with symbol P, by $(\boldsymbol{M}_P u)(s) := P(s)u(s)$, for u in

$$\begin{aligned}\mathrm{dom}\,(\boldsymbol{M}_P) &:= \left\{ u \in \bigcup_{\tau\in\mathbb{R}} e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+} \;\middle|\; \forall \tau \in \mathbb{R}, u \in e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+} \implies P(\cdot)u(\cdot) \in e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+} \right\} \\ &= \bigcup_{\tau\in\mathbb{R}} e^{j\omega\tau}\{u \in \boldsymbol{H}^2_{\mathbb{C}_+} \mid P(\cdot)u(\cdot) \in \boldsymbol{H}^2_{\mathbb{C}_+}\}.\end{aligned} \tag{2.8}$$

If $P \in \hat{\mathcal{A}}_-^{\mathrm{cp}}$, $\boldsymbol{M}_P : \mathrm{dom}\,(\boldsymbol{M}_P) := \boldsymbol{L}^2_{j\mathbb{R}} \to \boldsymbol{L}^2_{j\mathbb{R}}$ is defined to be $(\boldsymbol{M}_P u)(s) := P(s)u(s)$. Note that given a $P \in \hat{\mathcal{B}}^{\mathrm{cp}}$, $P \in \hat{\mathcal{A}}_-^{\mathrm{cp}}$ if, and only if,

$$\boldsymbol{M}_P \boldsymbol{H}^2_{\mathbb{C}_+} \subset \boldsymbol{H}^2_{\mathbb{C}_+} \text{ and } \bar{\gamma}(\boldsymbol{M}_P) := \sup_{u\in\boldsymbol{L}^2_{j\mathbb{R}}} \frac{\|\boldsymbol{M}_P u\|_2}{\|u\|_2} = \sup_{s\in\mathbb{C}_{0+}} \bar{\sigma}(P(s)) =: \|P\|_\infty < \infty, \tag{2.9}$$

in which case we say P is a stable transfer function [CZ95, Thm. A.6.26].

Proposition 2.4.2. *Given $P \in \hat{\mathcal{B}}^{\mathrm{cp}}$, let $N, M, \tilde{N}, \tilde{M} \in \hat{\mathcal{A}}_-^{\mathrm{cp}}$ be coprime factors for P as in Proposition 2.4.1. Define $G := \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \in \hat{\mathcal{A}}_-^{\mathrm{cp}}$ and $\tilde{G} := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix} \in \hat{\mathcal{A}}_-^{\mathrm{cp}}$. Then* $\mathrm{img}\,(\boldsymbol{M}_G) = \ker\left(\boldsymbol{M}_{\tilde{G}}\right)$ *and*

$$\mathscr{G}_{\boldsymbol{M}_P} \cap e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+} = \mathrm{img}\left(\boldsymbol{M}_G|_{e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+}}\right) = \ker\left(\boldsymbol{M}_{\tilde{G}}|_{e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+}}\right) \; \forall \tau \in \mathbb{R}, \tag{2.10}$$

By virtue of the above equation, G and $\tilde{G}$ are respectively called right and left normalised graph symbols for P.

Proof. We follow the standard argument in [Vin01, Prop. 1.33] and first prove (2.10) for $\tau = 0$. If $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{M}_P} \cap \boldsymbol{H}^2_{\mathbb{C}_+}$, then $y = NM^{-1}u$. Let $q := M^{-1}u$, we obtain $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = Gq$. By Proposition 2.4.1, there exists $\left[\begin{smallmatrix} Y & X \end{smallmatrix}\right] \in \hat{\mathcal{A}}_-^{\mathrm{cp}}$ such that $\left[\begin{smallmatrix} Y & X \end{smallmatrix}\right] G = I$. So $q = \left[\begin{smallmatrix} Y & X \end{smallmatrix}\right]\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \boldsymbol{H}^2_{\mathbb{C}_+}$ and $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathrm{img}\left(\boldsymbol{M}_G|_{\boldsymbol{H}^2_{\mathbb{C}_+}}\right)$. Conversely, if $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in Gq : q \in \boldsymbol{H}^2_{\mathbb{C}_+}$, then $y = NM^{-1}u = Pu$ and $u, y \in \boldsymbol{H}^2_{\mathbb{C}_+}$, so $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{M}_P} \cap \boldsymbol{H}^2_{\mathbb{C}_+}$. As such, $\mathscr{G}_{\boldsymbol{M}_P} \cap \boldsymbol{H}^2_{\mathbb{C}_+} = \mathrm{img}\left(\boldsymbol{M}_G|_{\boldsymbol{H}^2_{\mathbb{C}_+}}\right)$. Now note that by Proposition 2.4.1, $\tilde{G}G = 0$, whereby

$$\mathscr{G}_{\boldsymbol{M}_P} \cap \boldsymbol{H}^2_{\mathbb{C}_+} = \mathrm{img}\left(\boldsymbol{M}_G|_{\boldsymbol{H}^2_{\mathbb{C}_+}}\right) \subset \ker\left(\boldsymbol{M}_{\tilde{G}}|_{\boldsymbol{H}^2_{\mathbb{C}_+}}\right).$$

Conversely, if $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \boldsymbol{H}^2_{\mathbb{C}_+}$ and $\tilde{G}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = 0$, then $y = \tilde{M}^{-1}\tilde{N}u = Pu$ and thus $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{M}_P} \cap \boldsymbol{H}^2_{\mathbb{C}_+}$.

The case for $\tau \neq 0$ follows the same line of arguments above by replacing throughout $\boldsymbol{H}^2_{\mathbb{C}_+}$ with $e^{-j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+}$, while noting that given any $\Phi \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}_{-}$,

$$\boldsymbol{M}_\Phi(e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+}) = e^{j\omega\tau}\boldsymbol{M}_\Phi\boldsymbol{H}^2_{\mathbb{C}_+} \subset e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+}\ \forall\tau \in \mathbb{R}.$$

To see that $\mathrm{img}\,(\boldsymbol{M}_G) = \ker\left(\boldsymbol{M}_{\tilde{G}}\right)$, note that $\tilde{G}G = 0$ implies $\mathrm{img}\,(\boldsymbol{M}_G) \subset \ker\left(\boldsymbol{M}_{\tilde{G}}\right)$. Conversely, suppose that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \boldsymbol{L}^2_{j\mathbb{R}}$ and $\tilde{G}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = 0$, then $y = \tilde{M}^{-1}\tilde{N}u = NM^{-1}u$. Letting $q := M^{-1}u$ yields $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = Gq$. As above, since $\left[\begin{smallmatrix} Y & X \end{smallmatrix}\right]G = I$, $q = \left[\begin{smallmatrix} Y & X \end{smallmatrix}\right]\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \boldsymbol{L}^2_{j\mathbb{R}}$, from which it follows that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathrm{img}\,(\boldsymbol{M}_G)$. $\square$

By the above proposition, given any $P \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$, we have

$$\mathscr{G}_{\boldsymbol{M}_P} = \bigcup_{\tau\in\mathbb{R}} \mathrm{img}\left(\boldsymbol{M}_G|_{e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+}}\right) = \bigcup_{\tau\in\mathbb{R}} \ker\left(\boldsymbol{M}_{\tilde{G}}|_{e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+}}\right).$$

Remark 2.4.3. If $P \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}_{-}$ is as in Proposition 2.4.2, then

$$\mathscr{G}_{\boldsymbol{M}_P} = \mathrm{img}\,(\boldsymbol{M}_G) = \ker\left(\boldsymbol{M}_{\tilde{G}}\right),$$

which is a closed subspace in $\boldsymbol{L}^2_{j\mathbb{R}}$, since $\boldsymbol{M}_G$ has a left stable inverse $\boldsymbol{M}_{[\,Y\ X\,]}$ [Kat80, Thm. IV.5.2]. The proof is the same as that of Proposition 2.4.2 after replacing $\boldsymbol{H}^2_{\mathbb{C}_+}$ with $\boldsymbol{L}^2_{j\mathbb{R}}$ throughout.

Equivalent time-domain systems

Definition 2.4.4. Define the set of continuous-time operators

$$\mathbb{W} := \left\{\mathbf{P} \overset{\mathscr{F}}{\hookleftarrow} \boldsymbol{M}_P \;\middle|\; P \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}\right\},$$

where the notation $\hookleftarrow$ is defined in Definition 2.1.4.

From (2.8), we have that each $\mathbf{P} \in \mathbb{W}$ is *time-invariant* in that $\mathbf{S}_\tau\mathscr{G}_{\mathbf{P}} \subset \mathscr{G}_{\mathbf{P}}\ \forall\tau \in \mathbb{R}$, where $\mathbf{S}_\tau$ denotes the continuous-time shift operator. Furthermore, $\mathbf{P}$ is causal because if $u \in \mathrm{dom}\,(\mathbf{P}) = \mathscr{F}^{-1}\mathrm{dom}\,(\boldsymbol{M}_P)$ is an element of $\boldsymbol{L}^2_{[\tau,\infty)}$ for some $\tau \in \mathbb{R}$, then $y := \mathbf{P}u \in \boldsymbol{L}^2_{[\tau,\infty)}$, whereby $\mathbf{\Pi}_\tau u = 0 \Longrightarrow \mathbf{\Pi}_\tau y = 0$. When $\Phi \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}_{-}$, the element $\mathbf{\Phi} \overset{\mathscr{F}}{\hookleftarrow} \boldsymbol{M}_\Phi$ of $\mathbb{W}$ can be identified with several properties. First of all, in view of (2.9), $\mathbf{\Phi}$ is bounded on $\boldsymbol{L}^2_{\mathbb{R}}$ with $\bar{\gamma}(\mathbf{\Phi}) = \bar{\gamma}(\boldsymbol{M}_\Phi) = \|\Phi\|_\infty$. Second, $\mathbf{\Phi}$ is causal by Lemma 2.1.8, since $\mathbf{\Phi}\boldsymbol{L}^2_{[\tau,\infty)} \subset \boldsymbol{L}^2_{[\tau,\infty)}$ for all $\tau \in \mathbb{R}$ by (2.9) and time-invariance. Also, the Hankel operator $\mathbf{H}^{+-}_{\mathbf{\Phi},0}$ is compact [CZ95, Lem. 8.2.4]. Exploiting the fact that $\mathbf{\Phi S}_\tau = \mathbf{S}_\tau \mathbf{\Phi}$ for all $\tau \in \mathbb{R}$, note

$$\begin{aligned}\mathbf{H}^{+-}_{\mathbf{\Phi},\tau} := (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{\Phi}|_{\boldsymbol{L}^2_{(-\infty,\tau)}} &= (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{S}_\tau \mathbf{\Phi} \mathbf{S}_{-\tau}|_{\boldsymbol{L}^2_{(-\infty,\tau)}} \\ &= \mathbf{S}_\tau(\mathbf{I} - \mathbf{\Pi}_0)\mathbf{\Phi}|_{\boldsymbol{L}^2_{(-\infty,0)}} \mathbf{S}_{-\tau}|_{\boldsymbol{L}^2_{(-\infty,\tau)}} \\ &= \mathbf{S}_\tau \mathbf{H}^{+-}_{\mathbf{\Phi},0} \mathbf{S}_{-\tau}|_{\boldsymbol{L}^2_{(-\infty,\tau)}},\end{aligned}$$

from which it follows by Lemma 2.1.1 that $\mathbf{H}^{+-}_{\mathbf{\Phi},\tau}$ is compact for all $\tau \in \mathbb{R}$.

Given any $P \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$ and the corresponding linear time-invariant (LTI) operator $\mathbf{P} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_P$, define the bounded causal operators on $\boldsymbol{L}^2_{\mathbb{R}}$:

$$\mathbf{N} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_N, \mathbf{M} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_M, \tilde{\mathbf{N}} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_{\tilde{N}}, \tilde{\mathbf{M}} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_{\tilde{M}}, \mathbf{X} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_X, \mathbf{Y} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_Y, \tilde{\mathbf{X}} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_{\tilde{X}}, \tilde{\mathbf{Y}} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_{\tilde{Y}},$$

where $N, \tilde{N}, X, Y, \tilde{X}, \tilde{Y} \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}_{-}$ and $\tilde{M}, M \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}_{\infty}$ are as in Proposition 2.4.1. By Proposition 2.4.2, $\mathrm{img}\,(\mathbf{G}) = \ker\left(\tilde{\mathbf{G}}\right)$ and

$$\mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \mathrm{img}\,(\mathbf{T}_{\mathbf{G},\tau}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right) \text{ for all } \tau \in \mathbb{R},$$

where $\mathbf{G} := \left[\begin{smallmatrix}\mathbf{N}\\ \mathbf{M}\end{smallmatrix}\right]$ and $\tilde{\mathbf{G}} := \begin{bmatrix}-\tilde{\mathbf{M}} & \tilde{\mathbf{N}}\end{bmatrix}$. $\mathbf{H}^{+-}_{\mathbf{G},\tau}$ and $\mathbf{H}^{+-}_{\tilde{\mathbf{G}},\tau}$ are compact for all $\tau \in \mathbb{R}$ as noted before. Furthermore, by Proposition 2.4.1, it holds that

$$\begin{bmatrix}\mathbf{Y} & \mathbf{X}\\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}}\end{bmatrix}\begin{bmatrix}\mathbf{N} & \tilde{\mathbf{X}}\\ \mathbf{M} & -\tilde{\mathbf{Y}}\end{bmatrix} = \mathbf{I}$$

and $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$.

In a nutshell, elements in $\mathbb{W}$ satisfy all Assumptions 2.2.1, 2.2.2, and 2.2.3, i.e. $\mathbb{W} \subset \mathbb{S}$.

Remark 2.4.5. In [JC10, JC11], each $P \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$ is associated with a time-domain operator $\mathbf{P} : \mathrm{dom}\,(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$, with graph

$$\mathscr{G}_{\mathbf{P}} := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \;\middle|\; \hat{y} = N\hat{w};\ \hat{u} = M\hat{w};\ \hat{w} \in \bigcup_{\tau\in\mathbb{R}} e^{j\omega\tau}\boldsymbol{H}^2_{\mathbb{C}_+} \right\},$$

where N and M are as in Proposition 2.4.1. This association is consistent with the more general LTV case considered at the end of Section 2.3. Since $M \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}_{\infty}$, convolution with M (i.e. the operator $\mathbf{M} \overset{\mathscr{F}}{\multimapinv} \boldsymbol{M}_M$) has non-singular instantaneous gain in the time domain. As such, $\mathbf{P}$ is linear and causal by Lemma 2.1.13. Furthermore, $\mathbf{P} \overset{\mathscr{F}}{\sim} \boldsymbol{M}_P$, i.e. the above definition results in the same time-domain objects as those in $\mathbb{W}$. The use of multiplication operators as the building block for time-domain operators in this section is motivated by the fact that in Chapter 5, systems with sampled-data structure have to be constructed in the absence of the coprime factors N and M.

2.5 Periodic systems

This section considers a class of periodic systems adapted from [CV04], via the time-lifting technique [BPFT91, BP92, Yam94, CF95]. Time-domain systems are defined in terms of frequency-domain multiplication operators as in the previous section.

Signal spaces

The following Hilbert spaces, with $h > 0$ as a parameter, play a central role in our study of h-periodic systems:

$$\boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)}) := \left\{ f : \mathbb{Z} \to \boldsymbol{L}^2_{[0,h)} \;\middle|\; \|f\|^2_{\boldsymbol{\ell}^2_{\mathbb{Z}}} := \langle f, f \rangle_{\boldsymbol{\ell}^2_{\mathbb{Z}}} < \infty, \text{ where } \langle f, g \rangle_{\boldsymbol{\ell}^2_{\mathbb{Z}}} := \sum_{i=-\infty}^{\infty} \langle f_i, g_i \rangle_2 \right\};$$
$$\boldsymbol{\ell}^2_{\mathbb{Z}_+}(\boldsymbol{L}^2_{[0,h)}) := \left\{ f \in \boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)}) \mid f_i = 0, \forall i < 0 \right\}.$$

We define $\boldsymbol{L}^2_{\mathbb{T}}(\boldsymbol{L}^2_{[0,h)})$ (resp. $\boldsymbol{H}^2_{\mathbb{D}}(\boldsymbol{L}^2_{[0,h)})$) to comprise of the discrete-time Fourier transform $\mathscr{Z}$ of the signals in $\boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)})$ (resp. $\boldsymbol{\ell}^2_{\mathbb{Z}_+}(\boldsymbol{L}^2_{[0,h)})$) so that $\boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)}) \overset{\mathscr{Z}}{\sim} \boldsymbol{L}^2_{\mathbb{T}}(\boldsymbol{L}^2_{[0,h)})$ and $\boldsymbol{\ell}^2_{\mathbb{Z}_+}(\boldsymbol{L}^2_{[0,h)}) \overset{\mathscr{Z}}{\sim} \boldsymbol{H}^2_{\mathbb{D}}(\boldsymbol{L}^2_{[0,h)})$, where the isomorphism [SNF70, Chapter 5]

$$\mathscr{Z} : \boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)}) \to \boldsymbol{L}^2_{\mathbb{T}}(\boldsymbol{L}^2_{[0,h)}); \quad (\mathscr{Z} f)(z) := \sum_{i \in \mathbb{Z}} z^i f_i.$$

The inner product and norm on $\boldsymbol{L}^2_{\mathbb{T}}(\boldsymbol{L}^2_{[0,h)})$ are given by $\langle f, g \rangle_{\boldsymbol{L}^2_{\mathbb{T}}} := \int_{z \in \mathbb{T}} \langle f(z), g(z) \rangle_2 \, dz$ and $\|f\|^2_{\boldsymbol{L}^2_{\mathbb{T}}} := \int_{z \in \mathbb{T}} \|f(z)\|^2_2 \, dz$, respectively. Similarly, we make use of the relation $\boldsymbol{L}^2_{\mathbb{R}} \overset{\mathscr{W}_h}{\sim} \boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)})$, where $\mathscr{W}_h$ denotes the *time-lifting* isomorphism [BPFT91, BP92, Yam94, CF95], defined by

$$\mathscr{W}_h : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)}); \quad (\mathscr{W}_h f)_i(t) = f(hi + t), t \in [0, h).$$

Together,

$$\boldsymbol{L}^2_{\mathbb{R}} \overset{\mathscr{Z}\mathscr{W}_h}{\sim} \boldsymbol{L}^2_{\mathbb{T}}(\boldsymbol{L}^2_{[0,h)}) \quad \text{and} \quad \boldsymbol{L}^2_{[0,\infty)} \overset{\mathscr{Z}\mathscr{W}_h}{\sim} \boldsymbol{H}^2_{\mathbb{D}}(\boldsymbol{L}^2_{[0,h)}).$$

Now let $\mathbf{D}_i : \boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)}) \to \boldsymbol{\ell}^2_{\mathbb{Z}}(\boldsymbol{L}^2_{[0,h)})$ denotes the discrete-time shift operator, i.e. $(\mathbf{D}_i u)(k) := u(k - i)$ for any $i \in \mathbb{Z}$. Similarly, let $\mathbf{S}_\tau : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ denote the continuous-time shift operator defined by $(\mathbf{S}_\tau u)(t) := u(t - \tau)$ for any $\tau \in \mathbb{R}$. Observe that for any

$h > 0$, $k \in \mathbb{Z}$, and $u \in \boldsymbol{L}^2_{\mathbb{R}}$,

$$(\mathscr{Z}\mathscr{W}_h \mathbf{S}_{kh} u)(z) = (\mathscr{Z}\mathbf{D}_i \mathscr{W}_h u)(z) = z^k (\mathscr{Z}\mathscr{W}_h u)(z).$$

Noting $\boldsymbol{L}^{2+} = \bigcup\limits_{k\in\mathbb{Z}} \mathbf{S}_{kh}\boldsymbol{L}^2_{[0,\infty)}$ for any $h > 0$, it follows $\bigcup\limits_{k\in\mathbb{Z}} z^k \boldsymbol{H}^2_{\mathbb{D}}(\boldsymbol{L}^2_{[0,h)}) = \mathscr{Z}\mathscr{W}_h \boldsymbol{L}^{2+}$. We may often drop the codomain of the functions in these spaces, i.e. $\boldsymbol{L}^2_{[0,h)}$, for notational convenience.

Transfer functions

The following class of rational transfer functions [CV04] is the central object of study:

$$\boldsymbol{\mathcal{L}} := \left\{ \begin{array}{l} P = z \mapsto zC(I - zA)^{-1}B + D \\ \in \mathscr{L}(\mathbb{C}, \mathscr{L}(\boldsymbol{L}^2_{[0,h)}, \boldsymbol{L}^2_{[0,h)})) \text{ (a.e.)} \end{array} \;\middle|\; \begin{array}{l} A \in \mathbb{C}^{n\times n};\ B \in \mathscr{L}(\boldsymbol{L}^2_{[0,h)}, \mathbb{C}^n); \\ C \in \mathscr{L}(\mathbb{C}^n, \boldsymbol{L}^2_{[0,h)}); \\ D \in \mathscr{L}(\boldsymbol{L}^2_{[0,h)}, \boldsymbol{L}^2_{[0,h)}) \text{ is causal} \end{array} \right\},$$

where by causality of D we mean $\boldsymbol{\Pi}_\tau D(\mathbf{I} - \boldsymbol{\Pi}_\tau) = 0\,\forall \tau \in [0, h)$. For any transfer function $P(z) = zC(I - zA)^{-1}B + D \in \boldsymbol{\mathcal{L}}$, we denote its non-unique realisation by (A, B, C, D). The *order/complexity* of a realisation of a transfer function is defined to be the dimension of its 'A' matrix. A minimal realisation is one of minimal order [Can98, Section 2.4.1]. Define the stable subclass of $\boldsymbol{\mathcal{L}}$ as

$$\boldsymbol{\mathcal{L}}\boldsymbol{H}^\infty_{\mathbb{D}} = \{P \in \boldsymbol{\mathcal{L}} \mid \text{if } (A, B, C, D) \text{ is a minimal realisation for } P, \text{ then } \operatorname{spec}(A) \subset \mathbb{D}\},$$

where $\operatorname{spec}(\cdot)$ the spectrum of a matrix. Note that every $P \in \boldsymbol{\mathcal{L}}\boldsymbol{H}^\infty_{\mathbb{D}}$ is analytic on $\mathbb{D}$ and has finite infinity-norm $\|P\|_\infty := \sup_{z\in\mathbb{D}} \bar{\gamma}(P(z))$.

Given $P(z) = zC(I - zA)^{-1}B + D \in \boldsymbol{\mathcal{L}}$, its corresponding para-Hermitian conjugate function is given by $P^*(z) = B^*(zI - A^*)^{-1}C^* + D^*$. Note that unless D is memoryless, $P^* \notin \boldsymbol{\mathcal{L}}$. P is invertible as an element in $\boldsymbol{\mathcal{L}}$, i.e. $P^{-1} \in \boldsymbol{\mathcal{L}}$, if, and only if, its *feedthrough* term $D \in \mathscr{L}(\boldsymbol{L}^2_{[0,h)}, \boldsymbol{L}^2_{[0,h)})$ has a bounded causal inverse. In this case, a realisation for the inverse is given by

$$P^{-1} = \left(\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right). \tag{2.11}$$

Given $P_1 = (A_1, B_1, C_1, D_1) \in \boldsymbol{\mathcal{L}}$ and $P_2 = (A_2, B_2, C_2, D_2) \in \boldsymbol{\mathcal{L}}$ of compatible dimensions, a representation of their product is given by

$$P_1P_2 = \left(\begin{array}{c|c} \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} & \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} & D_1D_2 \end{array}\right) = \left(\begin{array}{c|c} \begin{bmatrix} A_2 & 0 \\ B_1C_2 & A_1 \end{bmatrix} & \begin{bmatrix} B_2 \\ B_1D_2 \end{bmatrix} \\ \hline \begin{bmatrix} D_1C_2 & C_1 \end{bmatrix} & D_1D_2 \end{array}\right) \in \boldsymbol{\mathcal{L}}.$$

Likewise, for addition,

$$P_1 + P_2 = \left(\begin{array}{c|c} \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 & C_2 \end{bmatrix} & D_1 + D_2 \end{array}\right) \in \boldsymbol{\mathcal{L}}.$$

Definition 2.5.1. Given $P \in \boldsymbol{\mathcal{L}}$, P is said to admit normalised doubly coprime factorisations if there exist coprime factors $N, M, \tilde{N}, \tilde{M}, X, Y, \tilde{X}, \tilde{Y} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ such that $M^{-1}, \tilde{M}^{-1} \in \boldsymbol{\mathcal{L}}$ and

$$\begin{bmatrix} Y & X \\ \tilde{M} & -\tilde{N} \end{bmatrix} \begin{bmatrix} N & \tilde{X} \\ M & -\tilde{Y} \end{bmatrix} = I; \quad NM^{-1} = \tilde{M}^{-1}\tilde{N} = P;$$
$$M^*M + N^*N = I; \quad \tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I.$$

Proposition 2.5.2. *Given any $P = (A, B, C, D) \in \boldsymbol{\mathcal{L}}$, if D^*D and DD^* are Hilbert-Schmidt operators, then P admits normalised doubly coprime factorisations as per Definition 2.5.1 by construction. To be specific, let $R := I + D^*D$ and $\tilde{R} := I + DD^*$. Suppose without loss of generality that the realisation (A, B, C, D) is minimal [Can98, Section 2.4.1], then one realisation of the required coprime factors is given by*

$$\begin{bmatrix} N & \tilde{X} \\ M & -\tilde{Y} \end{bmatrix} := \left(\begin{array}{c|c} A + BF & \begin{bmatrix} BV & -LS^{-1} \end{bmatrix} \\ \hline \begin{bmatrix} C + DF \\ F \end{bmatrix} & \begin{bmatrix} DV & S^{-1} \\ V & 0 \end{bmatrix} \end{array}\right);$$
$$\begin{bmatrix} Y & X \\ \tilde{M} & -\tilde{N} \end{bmatrix} := \left(\begin{array}{c|c} A + LC & \begin{bmatrix} L & -(B + LD) \end{bmatrix} \\ \hline \begin{bmatrix} V^{-1}F \\ SC \end{bmatrix} & \begin{bmatrix} 0 & V^{-1} \\ S & -SD \end{bmatrix} \end{array}\right),$$

where

$$F := -(R + B^*XB)^{-1}(B^*XA + D^*C); \quad L := -(BD^* + AYC^*)(\tilde{R} + CYC^*);$$

V and S are causal operators in $\mathscr{L}(\boldsymbol{L}^2_{[0,h)}, \boldsymbol{L}^2_{[0,h)})$ with bounded causal inverses satisfying $(R + B^*XB)^{-1} = VV^*$ and $(\tilde{R} + CYC^*)^{-1} = SS^*$; and $0 \le X = X^* \in \mathbb{C}^{n\times n}$, $0 \le Y = Y^* \in \mathbb{C}^{n\times n}$ are the stabilising solutions to the discrete-time finite-dimensional algebraic

Riccati equations[1]

$$X = (A - BR^{-1}D^*C)^*X(I + BR^{-1}B^*X)^{-1}(A - BR^{-1}D^*C) + C^*\tilde{R}^{-1}C;$$
$$Y = (A - BD^*\tilde{R}^{-1}C)Y(I + C^*\tilde{R}^{-1}CY)^{-1}(A - BD^*\tilde{R}^{-1}C)^* + BR^{-1}B^*$$

which satisfy $\operatorname{spec}(A + BF) \subset \mathbb{D}$ *and* $\operatorname{spec}(A + LC) \subset \mathbb{D}$.

Proof. This mainly follows from [Can98, Lem. 5.4], where verification of the conditions of Definition 2.5.1 is achieved by direct calculations. Additional effort is required in the setting here to show that the square-root factors V and S can be taken to be causal and causally invertible, as required for $N, M, \tilde{N}, \tilde{M}, X, Y, \tilde{X}, \tilde{Y}, M^{-1}, \tilde{M}^{-1}$ to reside in $\boldsymbol{\mathcal{L}}$. This follows by the so-called 'triangular' spectral factorisation results in [AHM09, Prop 2.2 and Thm. 3.7]. In particular, note that because B^*XB and CYC^* are finite-rank operators (specifically, compositions of three finite-rank operators), they are of the Hilbert-Schmidt class [GGK90, Cor. VIII.2.4]. Consequently, since D^*D and DD^* are Hilbert-Schmidt by hypothesis, $D^*D + B^*XB$ and $DD^* + CYC^*$ are also Hilbert-Schmidt [GGK90, Thm. 2.3]. As such, by [AHM09, Prop 2.2 and Thm. 3.7], $I + D^*D + B^*XB$ and $I + DD^* + CYC^*$ admit 'triangular' spectral factorisations, i.e. there exist causal $V, V^{-1}, S, S^{-1} \in \mathscr{L}(\boldsymbol{L}^2_{[0,h)}, \boldsymbol{L}^2_{[0,h)})$ such that

$$I + D^*D + B^*XB = V^{-1}(V^{-1})^* \quad \text{and} \quad I + DD^* + CYC^* = S^{-1}(S^{-1})^*,$$

as required. □

In light of the preceding development, we have the following definition.

Definition 2.5.3. Define the following subsets of $\boldsymbol{\mathcal{L}}$:

$$\boldsymbol{\mathcal{L}}_{CF} := \{P \in \boldsymbol{\mathcal{L}} \mid P \text{ admits normalised doubly coprime factorisations}\};$$
$$\boldsymbol{\mathcal{L}}_{HS} := \{P = (A, B, C, D) \in \boldsymbol{\mathcal{L}} \mid D^*D \text{ and } DD^* \text{ are Hilbert-Schmidt}\}.$$

Note that $\boldsymbol{\mathcal{L}}_{HS} \subset \boldsymbol{\mathcal{L}}_{CF}$ by Proposition 2.5.2.

Multiplication operators

Recall that $\bigcup_{k\in\mathbb{Z}} z^k \boldsymbol{H}^2_{\mathbb{D}} = \mathscr{Z}\mathscr{W}_h\boldsymbol{L}^{2+}$. Given a $P \in \boldsymbol{\mathcal{L}} \setminus \boldsymbol{\mathcal{L}}\boldsymbol{H}^\infty_{\mathbb{D}}$, define the associated multiplication operator, $\boldsymbol{M}_P : \operatorname{dom}(\boldsymbol{M}_P) \subset \boldsymbol{L}^2_{\mathbb{T}} \to \boldsymbol{L}^2_{\mathbb{T}}$, to be $(\boldsymbol{M}_P u)(z) := P(z)u(z)$,

[1] Note that the solutions X and Y exist by the minimality of (A, B, C, D) [Can98, Prop. 2.20].

where

$$\operatorname{dom}(\boldsymbol{M}_P) := \left\{ u \in \bigcup_{k\in\mathbb{Z}} z^k \boldsymbol{H}^2_{\mathbb{D}} \;\middle|\; \text{if } u \in z^j \boldsymbol{H}^2_{\mathbb{D}} \text{ for some } j \in \mathbb{Z}, \text{then } P(\cdot)u(\cdot) \in z^j \boldsymbol{H}^2_{\mathbb{D}} \right\}$$
$$= \bigcup_{k\in\mathbb{Z}} z^k \left\{ u \in \boldsymbol{H}^2_{\mathbb{D}} \;\middle|\; P(\cdot)u(\cdot) \in \boldsymbol{H}^2_{\mathbb{D}} \right\}.$$

When $P \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$, $\boldsymbol{M}_P : \operatorname{dom}(\boldsymbol{M}_P) := \boldsymbol{L}^2_{\mathbb{T}} \to \boldsymbol{L}^2_{\mathbb{T}}$ is defined to be $(\boldsymbol{M}_P u)(z) := P(z)u(z)$. Note that given a $P \in \boldsymbol{\mathcal{L}}$, $P \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ if, and only if,

$$\boldsymbol{M}_P \boldsymbol{H}^2_{\mathbb{D}} \subset \boldsymbol{H}^2_{\mathbb{D}} \quad \text{and} \quad \bar{\gamma}(\boldsymbol{M}_P) := \sup_{\|u\|_{\boldsymbol{L}^2_{\mathbb{T}}}=1} \|\boldsymbol{M}_P u\|_{\boldsymbol{L}^2_{\mathbb{T}}} = \sup_{z\in\mathbb{D}} \bar{\gamma}(P(z)) =: \|P\|_\infty < \infty,$$

in which case we say P is a stable transfer function [SNF70, Chapter 5].

Proposition 2.5.4. *Given a $P \in \boldsymbol{\mathcal{L}}_{CF}$, let $N, M, \tilde{N}, \tilde{M} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ be coprime factors for P as in Definition 2.5.1. Define $G := \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ and $\tilde{G} := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$. Then* $\operatorname{img}(\boldsymbol{M}_G) = \ker\left(\boldsymbol{M}_{\tilde{G}}\right)$ *and*

$$\mathscr{G}_{\boldsymbol{M}_P} \cap z^k \boldsymbol{H}^2_{\mathbb{D}} = \operatorname{img}\left(\boldsymbol{M}_G|_{z^k \boldsymbol{H}^2_{\mathbb{D}}}\right) = \ker\left(\boldsymbol{M}_{\tilde{G}}|_{z^k \boldsymbol{H}^2_{\mathbb{D}}}\right) \ \forall k \in \mathbb{Z}.$$

By virtue of the above equation, G and $\tilde{G}$ are respectively called right and left normalised graph symbols for P.

Proof. The proof follows the same line of argument as in Proposition 2.5.4. □

By the above proposition, given any $P \in \boldsymbol{\mathcal{L}}_{CF}$, we have

$$\mathscr{G}_{\boldsymbol{M}_P} = \bigcup_{k\in\mathbb{Z}} \operatorname{img}\left(\boldsymbol{M}_G|_{z^k \boldsymbol{H}^2_{\mathbb{D}}}\right) = \bigcup_{k\in\mathbb{Z}} \ker\left(\boldsymbol{M}_{\tilde{G}}|_{z^k \boldsymbol{H}^2_{\mathbb{D}}}\right).$$

Remark 2.5.5. If the $P \in \boldsymbol{\mathcal{L}}_{CF}$ in Proposition 2.5.4 is an element of $\boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$, then

$$\mathscr{G}_{\boldsymbol{M}_P} = \operatorname{img}(\boldsymbol{M}_G) = \ker\left(\boldsymbol{M}_{\tilde{G}}\right),$$

which is a closed subspace in $\boldsymbol{L}^2_{\mathbb{T}}$, since $\boldsymbol{M}_G$ has a left stable inverse $\boldsymbol{M}_{[\,Y\ X\,]}$ [Kat80, Thm. IV.5.2]. The proof is the same as that of Proposition 2.5.4 after replacing $\boldsymbol{H}^2_{\mathbb{D}}$ with $\boldsymbol{L}^2_{\mathbb{T}}$ throughout.

Time-domain systems

Given a $P \in \boldsymbol{\mathcal{L}}$, observe that for any $i \in \mathbb{Z}$, if $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{M}_P}$, i.e. $y, u \in z^k \boldsymbol{H}^2_{\mathbb{D}}$ for some $k \in \mathbb{Z}$ and $y = \boldsymbol{M}_P u$, then $\begin{bmatrix} z^i y \\ z^i u \end{bmatrix} \in \mathscr{G}_{\boldsymbol{M}_P}$ because $z^i y, z^i u \in z^{i+k}\boldsymbol{H}^2_{\mathbb{D}}$ and $\boldsymbol{M}_P(z^i u) = z^i y$. Consequently, the discrete-time operator $\mathbf{P}_d \overset{\mathscr{Z}}{\leftrightarrow} \boldsymbol{M}_P$ is a shift-invariant operator in the sense that for all $i \in \mathbb{Z}$, $\mathbf{D}_i \mathscr{G}_{\mathbf{P}_d} \subset \mathscr{G}_{\mathbf{P}_d}$. In light of the fact that $\mathscr{W}_h^{-1}\mathbf{D}_i = \mathbf{S}_{ih}\mathscr{W}_h^{-1}\,\forall i \in \mathbb{Z}$, it follows that with $\mathbf{P} \overset{\mathscr{W}_h}{\leftrightarrow} \mathbf{P}_d \overset{\mathscr{Z}}{\leftrightarrow} \boldsymbol{M}_P$ (cf. Definition 2.1.4), we have

$$\mathbf{S}_{ih}\mathscr{G}_{\mathbf{P}} \subset \mathscr{G}_{\mathbf{P}}\,\forall i \in \mathbb{Z}.$$

In other words, $\mathbf{P} \overset{\mathscr{Z}\mathscr{W}_h}{\leftrightarrow} \boldsymbol{M}_P$ is a continuous-time linear periodically time-varying (LPTV) operator with period h. Furthermore, $\mathbf{P}$ is a causal operator. To see this, suppose a realisation for $P \in \boldsymbol{\mathcal{L}}$ is (A, B, C, D). Now given any $\left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{M}_P}$, there exists by definition a $j \in \mathbb{Z}$ such that $\hat{u}, \hat{y} \in z^j\boldsymbol{H}^2_{\mathbb{D}}$ and $\hat{y} = \boldsymbol{M}_P\hat{u}$. In particular,

$$\hat{y}(z) = P(z)\hat{u}(z) = (zC(I - zA)^{-1}B + D)\hat{u}(z).$$

As such, $\left[\begin{smallmatrix} \bar{y} \\ \bar{u} \end{smallmatrix}\right] := \mathscr{Z}^{-1}\left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}_d} \cap \mathbf{D}_j\boldsymbol{\ell}^2_{\mathbb{Z}_+}$ can be described by the convolution operation [IOW99, Section 2.6]:

$$\bar{y}_k = \sum_{i=j}^{k-1} CA^{k-i-1}B\bar{u}_i + D\bar{u}_k, \forall k \geq j.$$

Since D is causal by definition, it follows that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] := \mathscr{W}_h^{-1}\left[\begin{smallmatrix} \bar{y} \\ \bar{u} \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2[j, \infty)$ is such that y depends causally on u. In consequence, $\mathbf{P}$ is causal, as claimed.

Definition 2.5.6. We define the following classes linear periodically time-varying (LPTV) systems with transfer function representations in $\boldsymbol{\mathcal{L}}$:

$$\begin{aligned}
\mathbb{P} &:= \left\{\mathbf{P} \overset{\mathscr{Z}\mathscr{W}_h}{\leftrightarrow} \boldsymbol{M}_P : P \in \boldsymbol{\mathcal{L}}\right\}; \\
\mathbb{P}_{CF} &:= \left\{\mathbf{P} \overset{\mathscr{Z}\mathscr{W}_h}{\leftrightarrow} \boldsymbol{M}_P : P \in \boldsymbol{\mathcal{L}}_{CF} \subset \boldsymbol{\mathcal{L}}\right\} \subset \mathbb{P}; \\
\mathbb{P}_{HS} &:= \left\{\mathbf{P} \overset{\mathscr{Z}\mathscr{W}_h}{\leftrightarrow} \boldsymbol{M}_P : P \in \boldsymbol{\mathcal{L}}_{HS} \subset \boldsymbol{\mathcal{L}}_{CF}\right\} \subset \mathbb{P}_{CF},
\end{aligned}$$

where $\boldsymbol{\mathcal{L}}_{CF}$ and $\boldsymbol{\mathcal{L}}_{HS}$ are as in Definition 2.5.3.

We verify below that all of Assumptions 2.2.1, 2.2.2, and 2.2.3 are satisfied by operators in $\mathbb{P}_{CF}$, i.e. $\mathbb{P}_{CF} \subset \mathbb{S}$.

Given any (stable) $\Phi = (A, B, C, D) \in \boldsymbol{\mathcal{L}}\boldsymbol{H}^\infty_{\mathbb{D}}$, any $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{\Phi}_d}$, where $\boldsymbol{\Phi}_d \overset{\mathscr{Z}}{\leftrightarrow} \boldsymbol{M}_\Phi$, can be described by the convolution operation [IOW99, Thm. 2.6.1]:

$$y_k = \sum_{i=-\infty}^{k-1} CA^{k-i-1}Bu_i + Du_k, \forall k \in \mathbb{Z}. \tag{2.12}$$

Since D is causal by definition, it follows that $\mathbf{\Phi} \overset{\mathscr{W}_h}{\hookleftarrow} \mathbf{\Phi}_d$ is a bounded causal operator on $\boldsymbol{L}^2_{\mathbb{R}}$ with $\bar{\gamma}(\mathbf{\Phi}) = \bar{\gamma}(\boldsymbol{M}_\Phi) = \|\Phi\|_\infty$ and $\underline{\gamma}(\mathbf{\Phi}) = \underline{\gamma}(\boldsymbol{M}_\Phi) = \inf_{z\in\mathbb{D}} \underline{\gamma}(\Phi(z))$. Also, for any $\tau \in \mathbb{R}$, the Hankel factorisation $\mathbf{H}^{+-}_{\mathbf{\Phi},\tau} = \mathbf{L}_{O,\tau}\mathbf{L}_{C,\tau}$ holds, where the observability operator $\mathbf{L}_{O,\tau} \in \mathscr{L}(\mathbb{C}^n, \boldsymbol{L}^2_{[\tau,\infty)})$ and the controllability operator $\mathbf{L}_{C,\tau} \in \mathscr{L}(\boldsymbol{L}^2_{(-\infty,\tau)}, \mathbb{C}^n)$ are defined by

$$(\mathbf{L}_{O,\tau}x)(t) := \left(CA^{j(t)}x\right)\big(t - \big(k + j(t)\big)h\big), t \geq \tau \text{ and } \mathbf{L}_{C,\tau}u := \sum_{i=-\infty}^{k-1} A^{k-i-1}B(\mathscr{W}_h u)_i,$$

in which $k := \lfloor \tau/h \rfloor$, $j(t) := \lfloor (t-kh)/h \rfloor$, and $\lfloor\cdot\rfloor$ denotes the floor function. Since $\mathbf{L}_{C,\tau}$ has finite-dimensional image and $\mathbf{L}_{O,\tau}$ has finite-dimensional domain, both operators are compact [Kre89, Thm. 8.1-4]. This implies that $\mathbf{H}^{+-}_{\mathbf{\Phi},\tau}$ is compact by Lemma 2.1.1.

Now, given any $P \in \boldsymbol{\mathcal{L}}_{CF}$ and the corresponding LPTV operator $\mathbf{P} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_P$, define the bounded causal operators

$$\begin{aligned}
&\mathbf{N} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_N, \mathbf{M} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_M, \tilde{\mathbf{N}} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{\tilde{N}}, \tilde{\mathbf{M}} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{\tilde{M}},\\
&\mathbf{X} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_X, \mathbf{Y} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_Y, \tilde{\mathbf{X}} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{\tilde{X}}, \tilde{\mathbf{Y}} \overset{\mathscr{X}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{\tilde{Y}},
\end{aligned}$$

where $N, M, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{Y} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ are as in Definition 2.5.1. By the properties of coprime factors and the preceding developments, it follows that

$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} \begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} = \mathbf{I},$$

$\mathbf{G}^*\mathbf{G} = \mathbf{I}$, $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$, and $\mathbf{H}^{+-}_{\mathbf{G},\tau}$ and $\mathbf{H}^{+-}_{\tilde{\mathbf{G}},\tau}$ are compact for all $\tau \in \mathbb{R}$, where $\mathbf{G} := \left[\begin{smallmatrix}\mathbf{N}\\ \mathbf{M}\end{smallmatrix}\right]$ and $\tilde{\mathbf{G}} := \left[\begin{smallmatrix}-\tilde{\mathbf{M}} & \tilde{\mathbf{N}}\end{smallmatrix}\right]$. Moreover, by Proposition 2.5.4, we have that $\operatorname{img}(\mathbf{G}) = \ker\left(\tilde{\mathbf{G}}\right)$ and $\mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2[kh,\infty) = \operatorname{img}(\mathbf{T}_{\mathbf{G},kh}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},kh}\right)$ for all $k \in \mathbb{Z}$. In fact, it can be shown that

$$\mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right) \text{ for all } \tau \in \mathbb{R}. \tag{2.13}$$

To be precise, given any $k \in \mathbb{Z}$ and $\tau \in [kh,(k+1)h)$, note that $\operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) \subset \boldsymbol{L}^2_{[\tau,\infty)}$ and $\operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) \subset \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2[kh,\infty)$ by (2.13), and hence $\operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) \subset \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)}$. Conversely, suppose that $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)}$, then from (2.13) there exists a $q \in \boldsymbol{L}^2[kh,\infty)$ such that $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right] = \mathbf{T}_{\mathbf{G},kh}q$. In fact, it can be shown that $q \in \boldsymbol{L}^2_{[\tau,\infty)}$, so that $\left[\begin{smallmatrix}y\\u\end{smallmatrix}\right] = \mathbf{T}_{\mathbf{G},\tau}q$, whereby

$\mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} \subset \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau})$. To see this, suppose to the contrary that $\mathbf{\Pi}_\tau q \neq 0$. Let D be the feedthrough D-term of M. Therefore, from (2.12), $u(t) = (Dq)(t)$ and $(D^{-1}u)(t) = q(t)$ for $t \in [kh,(k+1)h)$. Note that $\mathbf{\Pi}_\tau u = 0$ since $u \in \boldsymbol{L}^2_{[\tau,\infty)}$, from which it follows that D^{-1} is not causal, contradicting $M^{-1} \in \boldsymbol{\mathcal{L}}$. Similar arguments can be used to establish $\mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right)$.

In summary, all of Assumptions 2.2.1, 2.2.2 and 2.2.3 hold, leading to the fact that $\mathbb{P}_{CF} \subset \mathbb{S}$.

2.6 Summary

This chapter presents the preliminary material. The notion of operator equivalence is utilised to define time-domain linear operators based on frequency-domain objects. Three generic classes of linear systems are presented and shown to satisfy assumptions on the existence of normalised strong graph representations/symbols. Subsequent chapters develop robust stability analysis results for abstract operators assumed to admit such graph representations and the system classes here serve as examples of operators of this kind. Restrictions to these specific classes of systems from the abstract setting take place on several occasions later in the thesis, where certain additional results may be derived. In particular, see Sections 3.4, 4.6, and Chapter 5.

Chapter 3

Robust stability analysis via the ν-gap metric

In this chapter, we define a notion of closed-loop stability for continuous linear time-varying (LTV) systems, which implicitly contains an arrow of time [GS10]. Robust stability results for feedback systems are derived. The development is underpinned only by the assumptions on the existence of normalised strong graph symbols/representations stated in Section 2.2. Building on the initial work in [JC10, JC11], the definition of the ν-gap metric is motivated here via a necessary and sufficient Fredholm index condition for the stability of an uncertain feedback interconnection, contingent on a robust stability margin of the nominal closed-loop being sufficiently large. A lower bound on robust stability margin of the perturbed feedback systems is derived thereupon. We also consider the variation of a closed-loop mapping, used to gauge performance and robustness, as an open-loop component of the feedback interconnection is perturbed. Uniform upper and lower bounds on the induced norm of the difference are established in terms of the ν-gap distance between the perturbed and nominal systems. From these bounds it follows that the ν-gap metric induces the coarsest topology with respect to which closed-loop stability is maintained in small neighbourhoods and closed-loop performance varies continuously. All of the aforementioned results have their time-invariant roots in [Vin93, Vin01].

Towards addressing the issue of potential conservatism, with respect to the robust stability margin, in closed-loop analysis via the generalised ν-gap metric, the class of causal linear periodically time-varying (LPTV) systems introduced in Section 2.5 is considered via the well-known time-lifting [BPFT91, BP92, Yam94, CF95] isomorphism also discussed therein. Importantly, this system class includes from an operator-theoretic point of view finite-dimensional linear time-invariant (LTI) systems and periodic sampled-data systems, thereby laying the foundation for the work in Chapter 5 on sampled-data approximation. A necessary condition for robust stability, analogous to the LTI results in [Vin01], is then derived. This leads to a quantitative measure of the maximal ν-gap ball of causal LPTV perturbations a feedback system can sustain while preserving internal stability.

The chapter is organised as follows. In Section 3.1 we present some properties of Fredholm, Wiener-Hopf, and Hankel operators, which are useful in the modelling of system behaviour as in [JC10, JC11]. In Section 3.2 a generalised definition of feedback stability is stated and characterised in terms of system graph symbols as in [JC10, JC11]. Section 3.3 contains the definition of the ν-gap metric from [JC10, JC11], sufficient conditions for robust stability, and bilateral bounds on closed-loop errors. Finally, we consider a class of LPTV systems and derive a necessary and sufficient robust stability condition in Section 3.4.

3.1 Preliminaries

3.1.1 Fredholm operators

The use of Fredholm theory in ν-gap metric based analysis is initially suggested in [Vin93, Vin01] to extend the concept of winding number of a closed Nyquist curve for irrational transfer functions via the Fredholm index; see [CJK12, CJK10, CJK09] for a more complete development. This is further generalised in [JC10, JC11] to linear time-varying systems.

Definition 3.1.1. An operator $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ is of Fredholm type if both $\dim\ker(\mathbf{X})$ and $\operatorname{codim}\operatorname{img}(\mathbf{X}) = \dim\operatorname{coker}(\mathbf{X}) = \dim\ker(\mathbf{X}^*)$ are finite, where dim denotes the dimension of a subspace and coker denotes the quotient space of the codomain by the image. In this case, the Fredholm index of $\mathbf{X}$ is defined to be

$$\operatorname{ind}(\mathbf{X}) := \dim\ker(\mathbf{X}) - \operatorname{codim}\operatorname{img}(\mathbf{X}).$$

Note that a bijective $\mathbf{X}$ is necessarily Fredholm with

$$\dim\ker(\mathbf{X}) = \operatorname{codim}\operatorname{img}(\mathbf{X}) = \operatorname{ind}(\mathbf{X}) = 0.$$

Lemma 3.1.2. *Let $\mathbf{X} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathbf{Z} \in \mathscr{L}(\mathcal{H}_2, \mathcal{H}_3)$ be Fredholm operators. The following hold:*

(i) $\mathbf{X}^*$ *is Fredholm and* $\operatorname{ind}(\mathbf{X}^*) = -\operatorname{ind}(\mathbf{X})$*;*

(ii) $\mathbf{ZX}$ *is Fredholm and* $\operatorname{ind}(\mathbf{ZX}) = \operatorname{ind}(\mathbf{Z}) + \operatorname{ind}(\mathbf{X})$*;*

(iii) *if* $\mathbf{Y} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ *is such that* $\underline{\gamma}(\mathbf{X}) > \bar{\gamma}(\mathbf{Y})$*, then* $\mathbf{X} + \mathbf{Y}$ *is Fredholm and*

$$\operatorname{ind}(\mathbf{X} + \mathbf{Y}) = \operatorname{ind}(\mathbf{X});$$

(iv) *if* $\mathbf{K} \in \mathscr{L}(\mathcal{H}_1, \mathcal{H}_2)$ *is a* compact *operator, then* $\mathbf{X} + \mathbf{K}$ *is Fredholm and*

$$\operatorname{ind}(\mathbf{X} + \mathbf{K}) = \operatorname{ind}(\mathbf{X}).$$

Proof. (i) See [Kat80, Cor. IV.5.14];

(ii) See [GGK90, Thm. XI.3.2];

(iii) This follows from the more general [Kat80, Thm. IV.5.22], in which the relative gain term b can be taken to be 0 because $\mathbf{Y}$ is bounded. See also [JC10, Lem. 1];

(iv) See [GGK90, Thm. XI.4.2] or [Kat80, Thm. IV.5.26]. □

3.1.2 Wiener-Hopf and Hankel operators

We collect here a number of useful results on families of generalised Wiener-Hopf (a.k.a. Toeplitz) and Hankel operators defined in Definition 2.1.9 of Section 2.1. As reported in [JC10, JC11], these operators play an important role in ν-gap metric based stability analysis of time-varying feedback systems in conjunction with the theory of Fredholm operators.

Lemma 3.1.3. *Let* $\mathbf{X}, \mathbf{Y} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$*, the following hold:*

(i) *If* $\mathbf{X}$ *is causal then* $\mathbf{T}_{\mathbf{X},\tau} = \mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}}$ *and* $\mathbf{T}_{\mathbf{X},\tau}$ *is causal for all* $\tau \in \mathbb{R}$*;*

(ii) *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are causal and* $\mathbf{T}_{\mathbf{X},\tau} = \mathbf{T}_{\mathbf{Y},\tau}$ *for all* $\tau \in \mathbb{R}$*, then* $\mathbf{X} = \mathbf{Y}$*.*

(iii) *If* $\mathbf{T}_{\mathbf{X},\tau}$ *is causal for all* $\tau \in \mathbb{R}$*, then* $\mathbf{X}$ *is causal;*

(iv) *The mixed Toeplitz-Hankel composition identity*

$$\mathbf{T}_{\mathbf{YX},\tau} = \mathbf{T}_{\mathbf{Y},\tau}\mathbf{T}_{\mathbf{X},\tau} + \mathbf{H}^{+-}_{\mathbf{Y},\tau}\mathbf{H}^{-+}_{\mathbf{X},\tau}, \forall \tau \in \mathbb{R}.$$

If $\mathbf{X}$ *is causal or* $\mathbf{Y}$ *is anti-causal, then* $\mathbf{T}_{\mathbf{YX},\tau} = \mathbf{T}_{\mathbf{Y},\tau}\mathbf{T}_{\mathbf{X},\tau}, \forall \tau \in \mathbb{R}$*;*

(v) *If* $\mathbf{X}$ *is causal, then* $\bar{\gamma}(\mathbf{X}) = \sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{T}_{\mathbf{X},\tau})$ *and* $\underline{\gamma}(\mathbf{X}) = \inf_{\tau\in\mathbb{R}} \underline{\gamma}(\mathbf{T}_{\mathbf{X},\tau})$*.*

Proof. (i): See [JC10, Lem. 4]. In particular, since $\mathbf{X}$ is causal, $\mathbf{\Pi}_\tau \mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}} = 0$. Hence

$$\mathbf{T}_{\mathbf{X},\tau} = (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}} = \mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}}.$$

Causality of $\mathbf{T}_{\mathbf{X},\tau}$ then follows from that of $\mathbf{X}$.

(ii): Since $\boldsymbol{L}^{2+}$ is dense in $\boldsymbol{L}^2_{\mathbb{R}}$, given any $u \in \boldsymbol{L}^2_{\mathbb{R}}$, there exists a sequence $\{v_i\}$ in $\boldsymbol{L}^{2+}$ such that $v_i \to u$. Note that $\mathbf{X}$ and $\mathbf{Y}$ are continuous because they are bounded [Kre89, Thm. 2.7-9], whereby $\mathbf{X}v_i \to \mathbf{X}u$ and $\mathbf{Y}v_i \to \mathbf{Y}u$. But $\mathbf{X}v_i = \mathbf{Y}v_i \, \forall i$ by hypothesis and part (i) of the lemma, and hence $\mathbf{X}u = \mathbf{Y}u$, by the uniqueness of the limit of a convergent sequence. Since $u \in \boldsymbol{L}^2_{\mathbb{R}}$ is arbitrary, it follows that $\mathbf{X} = \mathbf{Y}$.

(iii): Suppose $\mathbf{X}$ is not causal, then there exists a $\tau \in \mathbb{R}$ such that $\mathbf{T}_{\mathbf{X},\tau}$ is not causal. In particular, non-causality of $\mathbf{X}$ implies the existence of a $\hat{\tau} \in \mathbb{R}$ and a $u \in \boldsymbol{L}^2[\hat{\tau},\infty)$ such that with $y := \mathbf{X}u$, $\mathbf{\Pi}_{\hat{\tau}} y \neq 0$. In other words, $\mathbf{\Pi}_{\hat{\tau}} y$ is a nonzero signal in $\boldsymbol{L}^2(-\infty,\hat{\tau})$. As such, there exists a $\tau < \hat{\tau}$ for which $(\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{\Pi}_{\hat{\tau}} y \neq 0$. It follows that

$$\mathbf{\Pi}_{\hat{\tau}} \mathbf{T}_{\mathbf{X},\tau} u = \mathbf{\Pi}_{\hat{\tau}}(\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{X}u = \mathbf{\Pi}_{\hat{\tau}}(\mathbf{I} - \mathbf{\Pi}_\tau)y = (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{\Pi}_{\hat{\tau}} y \neq 0,$$

i.e. $\mathbf{T}_{\mathbf{X},\tau}$ is not causal.

(iv): See [JC10, Lem. 4]. In particular, notice that

$$\begin{aligned}
\mathbf{T}_{\mathbf{YX},\tau} &= (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{YX}|_{\boldsymbol{L}^2_{[\tau,\infty)}} \\
&= (\mathbf{I} - \mathbf{\Pi}_\tau)\mathbf{Y}\big((\mathbf{I} - \mathbf{\Pi}_\tau) + \mathbf{\Pi}_\tau\big)\mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}} \\
&= \mathbf{T}_{\mathbf{Y},\tau}\mathbf{T}_{\mathbf{X},\tau} + \mathbf{H}^{+-}_{\mathbf{Y},\tau}\mathbf{H}^{-+}_{\mathbf{X},\tau}.
\end{aligned}$$

Furthermore, when $\mathbf{X}$ is causal, $\mathbf{H}^{-+}_{\mathbf{X},\tau} = 0$; whereas when $\mathbf{Y}$ is anti-causal, $\mathbf{H}^{+-}_{\mathbf{Y},\tau} = 0$.

(v): First note that

$$\bar{\gamma}(\mathbf{X}) \geq \sup_{\tau \in \mathbb{R}} \bar{\gamma}(\mathbf{T}_{\mathbf{X},\tau}) \quad \text{and} \quad \underline{\gamma}(\mathbf{X}) \leq \inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\mathbf{X},\tau}), \tag{3.1}$$

where the second inequality holds since $\mathbf{T}_{\mathbf{X},\tau} = \mathbf{X}|_{\boldsymbol{L}^2_{[\tau,\infty)}}$, by part (i) of the lemma. Also, by the definition of $\bar{\gamma}(\mathbf{X})$, for any $\epsilon > 0$, there exists a $u \in \boldsymbol{L}^2_{\mathbb{R}}$ such that $\|u\|_2 = 1$ and $\bar{\gamma}(\mathbf{X}) - \|\mathbf{X}u\|_2 < \frac{\epsilon}{2}$. Now recall that boundedness of $\mathbf{X}$ implies its continuity on $\boldsymbol{L}^2_{\mathbb{R}}$ [Kre89, Thm. 2.7-9], whereby there exists a $\delta > 0$ such that $\|\mathbf{X}(u - v)\|_2 < \frac{\epsilon}{2}$ for any $v \in \boldsymbol{L}^2_{\mathbb{R}}$ satisfying $\|u - v\|_2 < \delta$. Since $\boldsymbol{L}^{2+}$ is dense in $\boldsymbol{L}^2_{\mathbb{R}}$, there exists a $\hat{\tau} \in \mathbb{R}$ such that $v \in \boldsymbol{L}^2[\hat{\tau},\infty)$, $v(t) = u(t) \, \forall t \geq \hat{\tau}$ and $\|u - v\|_2 < \delta$. Note because $\|u\|_2 = 1$, $\|v\|_2 \leq 1$, whereby $\frac{\|\mathbf{T}_{\mathbf{X},\hat{\tau}}v\|_2}{\|v\|_2} \geq \|\mathbf{T}_{\mathbf{X},\hat{\tau}}v\|_2$. As such,

$$\begin{aligned}\bar{\gamma}(\mathbf{X}) - \frac{\|\mathbf{T}_{\mathbf{X},\hat{\tau}}v\|_2}{\|v\|_2} \leq \bar{\gamma}(\mathbf{X}) - \|\mathbf{T}_{\mathbf{X},\hat{\tau}}v\|_2 &= \bar{\gamma}(\mathbf{X}) - \|\mathbf{X}v\|_2 \\ &= \bar{\gamma}(\mathbf{X}) - \|\mathbf{X}u - \mathbf{X}(u-v)\|_2 \\ &\leq \bar{\gamma}(\mathbf{X}) - \|\mathbf{X}u\|_2 + \|\mathbf{X}(u-v)\|_2 \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Since ϵ was arbitrary, it follows from (3.1) that $\bar{\gamma}(\mathbf{X}) = \sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{T}_{\mathbf{X},\tau})$. The case for $\underline{\gamma}(\mathbf{X})$ can be shown using the same line of argument. □

Remark 3.1.4. The mixed Hankel-Toeplitz composition identity in Lemma 3.1.3(iv) has been explored in various contexts; see, for example, [IS04], [ZM88], and [BSK06, Chapter 2]. As demonstrated in [JC10, JC11], the identity is imperative to subsequent robustness analysis for feedback interconnections of *causal linear time-varying* systems.

Remark 3.1.5. By Lemma 3.1.3(iv), when $\mathbf{X}, \mathbf{X}^{-1} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ are both causal, we have $\mathbf{T}^{-1}_{\mathbf{X},\tau} = \mathbf{T}_{\mathbf{X}^{-1},\tau}$ for all $\tau \in \mathbb{R}$. This implies $\mathbf{T}_{\mathbf{X},\tau}$ is Fredholm with $\mathrm{ind}(\mathbf{T}_{\mathbf{X},\tau}) = 0\,\forall \tau \in \mathbb{R}$.

Lemma 3.1.6 ([JC10, Lem. 3])**.** *Given an $\mathbf{X} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ and any $\tau \in \mathbb{R}$, the Wiener-Hopf operator $\mathbf{T}_{\mathbf{X},\tau} : \boldsymbol{L}^2_{[\tau,\infty)} \to \boldsymbol{L}^2_{[\tau,\infty)}$ has a bounded inverse if, and only if,*

$$\underline{\gamma}(\mathbf{T}_{\mathbf{X},\tau}) > 0 \quad \textit{and} \quad \mathbf{T}_{\mathbf{X},\tau} \textit{ is Fredholm with } \mathrm{ind}(\mathbf{T}_{\mathbf{X},\tau}) = 0.$$

Proof. Necessity is straightforward since when $\mathbf{T}_{\mathbf{X},\tau}$ has a bounded inverse, it is bijective and $\underline{\gamma}(\mathbf{T}_{\mathbf{X},\tau}) = 1/\bar{\gamma}(\mathbf{T}^{-1}_{\mathbf{X},\tau}) > 0$. For sufficiency, note that $\underline{\gamma}(\mathbf{T}_{\mathbf{X},\tau}) > 0$ implies $\ker(\mathbf{T}_{\mathbf{X},\tau}) = \{0\}$, whereby $\mathrm{codim}\,\mathrm{img}\,(\mathbf{T}_{\mathbf{X},\tau}) = \dim\ker(\mathbf{T}_{\mathbf{X},\tau}) - \mathrm{ind}(\mathbf{T}_{\mathbf{X},\tau}) = 0$. Hence, $\mathbf{T}_{\mathbf{X},\tau}$ is bijective. That the inverse is bounded follows from the bounded inverse theorem [Kre89, Thm. 4.12-2]. □

3.2 Stability of feedback systems

In this section we introduce the generalised notion of closed-loop stability and derive a useful characterisation in terms of system graph symbols/representations. While initial versions of these appear in [JC10, JC11], some effort is required here to establish additional useful properties of graph symbols and to rigorously account for the slightly amended definition of feedback stability, which now implicitly embeds an arrow of time [GS10].

In Section 3.2.1, the stability of feedback interconnections of linear time-varying systems is defined. Discussions of causality and the well-known difficulties of classical double-axis input-output approaches to stability analysis are provided. Properties of graph symbols / representations are developed in Section 2.2, leading to a useful characterisation of closed-loop stability in Section 3.2.3. This is central to the development of the main robustness results.

3.2.1 Feedback stability

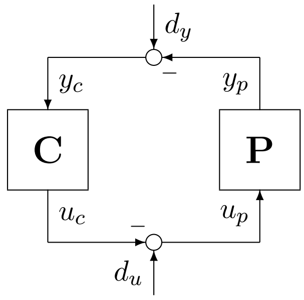

Figure 3.1: Standard feedback configuration

The feedback interconnection described by the following and illustrated in Figure 3.1 is the main object of study:

$$d_y = y_c + y_p; \quad d_u = u_p + u_c; \quad y_p = \mathbf{P}u_p; \quad u_c = \mathbf{C}y_c, \tag{3.2}$$

where $\mathbf{P} : \mathrm{dom}\,(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{C} : \mathrm{dom}\,(\mathbf{C}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ are two *causal* linear operators.

Definition 3.2.1. The feedback interconnection of $\mathbf{P}$ and $\mathbf{C}$, denoted $[\mathbf{P}, \mathbf{C}]$, is said to be internally *stable* if for all $\tau \in \mathbb{R}$, $\mathbf{F}_{\mathbf{P},\mathbf{C},\tau}$ is bijective, where

$$\begin{aligned} &\mathbf{F}_{\mathbf{P},\mathbf{C},\tau} : (\mathrm{dom}\,(\mathbf{C}) \times \mathrm{dom}\,(\mathbf{P})) \cap \boldsymbol{L}^2_{[\tau,\infty)} \to \boldsymbol{L}^2_{[\tau,\infty)} \\ &\begin{bmatrix} y_c \\ u_p \end{bmatrix} \mapsto \begin{bmatrix} d_y \\ d_u \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{P} \\ \mathbf{C} & \mathbf{I} \end{bmatrix} \begin{bmatrix} y_c \\ u_p \end{bmatrix} \end{aligned}$$

and $\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau}) < \infty$. We say that $\mathbf{P}$ is feedback stabilisable if there exists a causal $\mathbf{C}$ such that $[\mathbf{P}, \mathbf{C}]$ is stable.

Lemma 3.2.2. *If $[\mathbf{P}, \mathbf{C}]$ is stable in the sense of Definition 3.2.1, then $\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau}$ is necessarily causal for every $\tau \in \mathbb{R}$.*

Proof. According to the definition of feedback stability, $\mathbf{F}_{\mathbf{P},\mathbf{C},\tau} : \mathrm{dom}\,(\mathbf{F}_{\mathbf{P},\mathbf{C},\tau}) \to \boldsymbol{L}^2[\tau, \infty)$, where $\mathrm{dom}\,(\mathbf{F}_{\mathbf{P},\mathbf{C},\tau}) := (\mathrm{dom}\,(\mathbf{C}) \times \mathrm{dom}\,(\mathbf{P})) \cap \boldsymbol{L}^2[\tau,\infty)$, is bijective for every $\tau \in \mathbb{R}$. Note that, for real $\tau_2 \geq \tau_1$,

$$\mathbf{F}_{\mathbf{P},\mathbf{C},\tau_2} = \mathbf{F}_{\mathbf{P},\mathbf{C},\tau_1}|_{\mathrm{dom}\left(\mathbf{F}_{\mathbf{P},\mathbf{C},\tau_2}\right)}, \tag{3.3}$$

since $\mathrm{dom}\,(\mathbf{F}_{\mathbf{P},\mathbf{C},\tau_2}) \subset \mathrm{dom}\,(\mathbf{F}_{\mathbf{P},\mathbf{C},\tau_1})$. Moreover, note that

$$\mathrm{dom}\left(\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau_2}\right) = \boldsymbol{L}^2[\tau_2,\infty) \subset \boldsymbol{L}^2[\tau_1,\infty) = \mathrm{dom}\left(\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau_1}\right).$$

For a fixed $\tau_1 \in \mathbb{R}$, suppose to the contrapositive that there exist $x \in \boldsymbol{L}^2[\tau_1,\infty)$ and $\tau_2 > \tau_1$ for which $\mathbf{\Pi}_{\tau_2} x = 0$ (i.e. $x \in \boldsymbol{L}^2[\tau_2,\infty)$) and $\mathbf{\Pi}_{\tau_2}\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau_1} x \neq 0$; in other words, suppose that $\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau_1}$ is not causal. Let $z_1 := \mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau_1} x$ and $z_2 := \mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau_2} x$. Then

$$\mathbf{F}_{\mathbf{P},\mathbf{C},\tau_1} z_1 = x = \mathbf{F}_{\mathbf{P},\mathbf{C},\tau_2} z_2 = \mathbf{F}_{\mathbf{P},\mathbf{C},\tau_1} z_2,$$

where (3.3) has been used. As such, $\mathbf{F}_{\mathbf{P},\mathbf{C},\tau_1}(z_1 - z_2) = 0$, which implies $z_1 = z_2 \in \boldsymbol{L}^2[\tau_2,\infty)$, since $\ker\,(\mathbf{F}_{\mathbf{P},\mathbf{C},\tau_1}) = \{0\}$. This contradicts the hypothesis that $\mathbf{\Pi}_{\tau_2} z_1 \neq 0$. Thus, $\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau_1}$ must be causal, as claimed. □

Remark 3.2.3. Lemma 3.2.2 illustrates that causality of the closed-loop mapping is encapsulated in the definition of feedback stability. In other words, causality can be importantly viewed as a property that is preserved through feedback interconnections that are stable in the generalised sense of Definition 3.2.1. This is consistent with the viewpoint in [DN70, GS10], where it is argued that a proper definition of closed-loop stability must incorporate causality, or a so-called 'arrow of time'. We note a non-singular instantaneous gain assumption is used in [JC10, JC11] to guarantee causality; this appears to be redundant in light of Lemma 3.2.2.

Example 3.2.4. Consider the feedback interconnection (3.2) involving two causal LTI operators

$$\begin{aligned}
&\mathbf{P} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}} \\
&u_p \mapsto y_p, \quad y_p(t) = u_p(t-\delta) - u_p(t) \quad \text{and} \\
&\mathbf{C} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}} \\
&y_c \mapsto u_c, \quad u_c(t) = -y_c(t),
\end{aligned} \tag{3.4}$$

for some $\delta > 0$. This example is taken from [Wil71, Section 4.3.2] to demonstrate that a feedback interconnection of two bounded causal operators is *not* necessarily causal[1]. Indeed, let f be a function in $\boldsymbol{L}^2_{[0,\infty)}$ that is non-zero on $[0,\delta)$ and set $\left[\begin{smallmatrix} d_y \\ d_u \end{smallmatrix}\right] := \left[\begin{smallmatrix} 0 \\ f \end{smallmatrix}\right] \in \boldsymbol{L}^2_{[0,\infty)}$.

[1] The feedback interconnection of the two systems described in (3.4) is said to be *ill-posed* in [Wil71].

It follows that the corresponding u_p satisfying (3.2) is given by $u_p(\cdot) = d_u(\cdot + \delta) \notin \boldsymbol{L}^2_{[0,\infty)}$, whereby $\mathbf{F}_{\mathbf{P},\mathbf{C},0} = \left[\begin{smallmatrix} y_c \\ u_p \end{smallmatrix}\right] \in \boldsymbol{L}^2_{[0,\infty)} \mapsto \begin{bmatrix} d_y \\ d_u \end{bmatrix} \in \boldsymbol{L}^2_{[0,\infty)}$ is not bijective. As such, $[\mathbf{P}, \mathbf{C}]$ is *not* internally stable in accordance with Definition 3.2.1.

Remark 3.2.5. In Definition 3.2.1, bounded invertibility of the closed-loop operator is required on a *singly infinite* space $\boldsymbol{L}^2_{[\tau,\infty)}$, for all possible 'initial times' $\tau \in \mathbb{R}$. A necessary condition is that the truncated graphs $\mathscr{G}^\tau_{\mathbf{P}}$ and $\mathscr{G}'^\tau_{\mathbf{C}}$ are closed subspaces of $\boldsymbol{L}^2[\tau,\infty)$ [DS93, Thm 6.2][FGS93, Prop. 1]; this is the case under the assumptions of Section 2.2, whereby 'strong' graph symbols exist; see Remark 3.2.10. Invertibility over the *doubly infinite* $\boldsymbol{L}^2_{\mathbb{R}}$ is not considered since for *causal* systems of importance $\mathbf{P} : \mathrm{dom}\,(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{C} : \mathrm{dom}\,(\mathbf{C}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$, the $\boldsymbol{L}^2_{\mathbb{R}}$-graphs $\mathscr{G}_{\mathbf{P}}$ and $\mathscr{G}'_{\mathbf{C}}$ may not be closed subspaces of $\boldsymbol{L}^2_{\mathbb{R}}$ [GS95, JP00, Par04]. An example of such system, taken from [GS95], is

$$(\mathbf{P}_i u)(t) := \int_{-\infty}^{\infty} h_i(t-\tau)u(\tau)\, d\tau =: (h_i \circledast u)(t), \quad t \in \mathbb{R},$$

where

$$h_1(t) := \begin{cases} e^t & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h_2(t) := \begin{cases} -e^t & t \leq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Note that $\mathbf{P}_1$ is causal, $\mathbf{P}_2$ is anti-causal, and $\mathscr{G}_{\mathbf{P}_2} \subset \mathrm{cl}\,\mathscr{G}_{\mathbf{P}_1}$ [GS95], where cl denotes the closure of a subspace.

Our viewpoint is consistent with the 'behavioural' theory of [BFP08], where system behaviour is effectively defined on doubly-infinite time-axis while closed-loop stability requires boundedness mappings of signals of semi-infinite support.

Remark 3.2.6. [Fei98] presents a robust stability theory in a discrete and semi-infinite time setting (i.e. with a fixed 'initial' time), where invertibility and causal invertibility of causal systems are equivalent (lower triangular matrices have lower triangular inverses). This is not the case in general for continuous-time systems defined on the doubly infinite time axis, motivating the importance of Lemma 3.2.2.

Remark 3.2.7. By requiring $\mathbf{F}_{\mathbf{P},\mathbf{C},\tau}$ to be invertible for all $\tau \in \mathbb{R}$, Definition 3.2.1 does not attribute significance to any particular 'initial time' in line with the time-varying setting of this thesis. The definition generalises from LTI systems, in which case it is enough to only test bounded invertibility of $\mathbf{F}_{\mathbf{P},\mathbf{C},0}$, since this is equivalent to bounded invertibility at any initial time by shift-invariance.

Given a stable feedback interconnection $[\mathbf{P}, \mathbf{C}]$, for each $\tau \in \mathbb{R}$ let

$$\begin{aligned}\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} &:= \begin{bmatrix} d_y \\ d_u \end{bmatrix} \in \boldsymbol{L}^2_{[\tau,\infty)} \mapsto \begin{bmatrix} y_p \\ u_p \end{bmatrix} \in \mathscr{G}_{\mathbf{P}}^{\tau} = \begin{bmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \mathbf{F}_{\mathbf{P},\mathbf{C},\tau}^{-1} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix} \text{ and} \\ \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} &:= \begin{bmatrix} d_y \\ d_u \end{bmatrix} \in \boldsymbol{L}^2_{[\tau,\infty)} \mapsto \begin{bmatrix} y_c \\ u_c \end{bmatrix} \in \mathscr{G}_{\mathbf{C}}'^{\tau} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix} \mathbf{F}_{\mathbf{P},\mathbf{C},\tau}^{-1} + \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{bmatrix},\end{aligned} \tag{3.5}$$

where d_y, d_u, y_p, u_p, y_c, and u_c satisfy (3.1). The notation reflects that these are parallel projection operators onto and along the restricted graphs $\mathscr{G}_{\mathbf{P}}^{\tau}$ and $\mathscr{G}_{\mathbf{C}}'^{\tau}$, which are of importance in robust stability and performance analysis [DGS93, FGS93, CV02]. These operators satisfy the following identities for all $\tau \in \mathbb{R}$:

$$\begin{aligned}\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} + \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} &= \mathbf{I}; \\ \mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} \left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} x_1 + \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} x_2 \right) &= \mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} x_1 \; \forall x_1, x_2 \in \boldsymbol{L}^2_{[\tau,\infty)}; \\ \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} \left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} x_1 + \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} x_2 \right) &= \mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} x_1 \; \forall x_1, x_2 \in \boldsymbol{L}^2_{[\tau,\infty)}.\end{aligned} \tag{3.6}$$

We define the robust stability/performance margin of the feedback interconnection (3.2) as

$$b_{\mathbf{P},\mathbf{C}} := \begin{cases} \left(\sup_{\tau \in \mathbb{R}} \bar{\gamma} \left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} \right) \right)^{-1} & \text{if } [\mathbf{P}, \mathbf{C}] \text{ is stable;} \\ 0 & \text{otherwise.} \end{cases}$$

Similarly,

$$b_{\mathbf{C},\mathbf{P}} := \begin{cases} \left(\sup_{\tau \in \mathbb{R}} \bar{\gamma} \left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{C}}'^{\tau}\|\mathscr{G}_{\mathbf{P}}^{\tau}} \right) \right)^{-1} & \text{if } [\mathbf{P}, \mathbf{C}] \text{ is stable;} \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, $b_{\mathbf{P},\mathbf{C}} = b_{\mathbf{C},\mathbf{P}}$, since $\mathbf{P}$ and $\mathbf{C}$ are linear [DGS93]. Note that $b_{\mathbf{P},\mathbf{C}}$ is a generic measure of performance in the $\boldsymbol{H}^\infty$ loop-shaping paradigm for robust control design [MG90, MG92, Vin01]. Generalisation to possibly nonlinear systems can also be seen in [GS97, Vin99, JSV05]. The following lemma offers a geometric interpretation of the robustness margin in terms of system graphs along the lines of [FGS93, GS97].

Lemma 3.2.8. *If* $[\mathbf{P}, \mathbf{C}]$ *is stable,*

$$b_{\mathbf{P},\mathbf{C}} = \inf_{\tau \in \mathbb{R}} \inf_{v \in \mathscr{G}_{\mathbf{P}}^{\tau}, w \in \mathscr{G}_{\mathbf{C}}'^{\tau}} \frac{\|v + w\|_2}{\|v\|_2}.$$

Proof. Since $[\mathbf{P}, \mathbf{C}]$ is stable, we have

$$\sup_{\tau \in \mathbb{R}} \bar{\gamma} \left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} \right) = \sup_{\tau \in \mathbb{R}} \sup_{u \in \boldsymbol{L}^2_{[\tau,\infty)}} \frac{\left\| \mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}'^{\tau}} u \right\|_2}{\|u\|_2} < \infty$$

and $\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}^{\prime\tau}}$ is causal, whereby $\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}^{\prime\tau}}\boldsymbol{L}^2_{[\tau,\infty)} \subset \boldsymbol{L}^2_{[\tau,\infty)}$. It follows from the feedback configuration (3.2) that

$$\bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}^{\prime\tau}}\right) = \sup_{u\in\boldsymbol{L}^2_{[\tau,\infty)}} \frac{\left\|\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}^{\prime\tau}}u\right\|_2}{\|u\|_2} = \sup_{v\in\mathscr{G}_{\mathbf{P}}^{\tau},w\in\mathscr{G}_{\mathbf{C}}^{\prime\tau}} \frac{\|v\|_2}{\|v+w\|_2}.$$

Consequently,

$$\begin{aligned}
b_{\mathbf{P},\mathbf{C}} = \left(\sup_{\tau\in\mathbb{R}}\bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}^{\prime\tau}}\right)\right)^{-1} &= \inf_{\tau\in\mathbb{R}}\left(\bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^{\tau}\|\mathscr{G}_{\mathbf{C}}^{\prime\tau}}\right)\right)^{-1} \\
&= \inf_{\tau\in\mathbb{R}}\left(\sup_{v\in\mathscr{G}_{\mathbf{P}}^{\tau},w\in\mathscr{G}_{\mathbf{C}}^{\prime\tau}} \frac{\|v\|_2}{\|v+w\|_2}\right)^{-1} \\
&= \inf_{\tau\in\mathbb{R}}\inf_{v\in\mathscr{G}_{\mathbf{P}}^{\tau},w\in\mathscr{G}_{\mathbf{C}}^{\prime\tau}} \frac{\|v+w\|_2}{\|v\|_2},
\end{aligned}$$

as claimed. □

This section is concluded with the following well-known small-gain theorem, which can be found in one form or another in [Zam66, Wil69, DV75, MH92, Kha02].

Lemma 3.2.9. *Given causal* $\mathbf{P} : \mathrm{dom}(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ *and* $\mathbf{C} : \mathrm{dom}(\mathbf{C}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ *such that* $\mathbf{P}$ *and* $\mathbf{C}$ *are bounded on* $\boldsymbol{L}^{2+}$ *with* $\boldsymbol{L}^{2+} \subset \mathrm{dom}(\mathbf{P})$ *and* $\boldsymbol{L}^{2+} \subset \mathrm{dom}(\mathbf{C})$*, let* $\mathbf{P}_\tau := \mathbf{P}|_{\boldsymbol{L}^2_{[\tau,\infty)}}$ *and* $\mathbf{C}_\tau := \mathbf{C}|_{\boldsymbol{L}^2_{[\tau,\infty)}}$*. If* $\sup_{\tau\in\mathbb{R}}\bar{\gamma}(\mathbf{C}_\tau\mathbf{P}_\tau) < 1$*, then* $[\mathbf{P},\mathbf{C}]$ *is stable.*

Proof. Because $\bar{\gamma}(\mathbf{C}_\tau\mathbf{P}_\tau) < 1$ for every $\tau \in \mathbb{R}$, $(\mathbf{I} - \mathbf{C}_\tau\mathbf{P}_\tau)$ is boundedly invertible and $\bar{\gamma}\left((\mathbf{I} - \mathbf{C}_\tau\mathbf{P}_\tau)^{-1}\right) \leq (1 - \bar{\gamma}(\mathbf{C}_\tau\mathbf{P}_\tau))^{-1}$ [Kre89, Thm. 7.3-1]. It is straightforward to verify using (3.2) that

$$\mathbf{F}^{-1}_{\mathbf{P},\mathbf{C},\tau} = \begin{bmatrix} -\mathbf{P}_\tau \\ \mathbf{I} \end{bmatrix} (\mathbf{I} - \mathbf{C}_\tau\mathbf{P}_\tau)^{-1} \begin{bmatrix} -\mathbf{C}_\tau & \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{I} & 0 \\ 0 & 0 \end{bmatrix},$$

wherefrom the claimed result follows. □

3.2.2 Properties of graph symbols

Recall the three assumptions on the existence of 'strong graph representations/symbols' for causal linear operators stated in Section 2.2. These assumptions underly importantly the developments in this thesis. They are reproduced below, intertwined with several notes on useful properties. These are exploited in subsequent sections for closed-loop robustness analysis.

Assumption 2.2.1. Given a *causal* operator $\mathbf{P} : \operatorname{dom}(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$, there exist *causal* operators $\mathbf{N}, \mathbf{M}, \tilde{\mathbf{N}}, \tilde{\mathbf{M}}, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{X}}, \tilde{\mathbf{Y}} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ satisfying the following properties:

1. the double Bezout identity
$$\begin{bmatrix} \mathbf{Y} & \mathbf{X} \\ \tilde{\mathbf{M}} & -\tilde{\mathbf{N}} \end{bmatrix} \begin{bmatrix} \mathbf{N} & \tilde{\mathbf{X}} \\ \mathbf{M} & -\tilde{\mathbf{Y}} \end{bmatrix} = \mathbf{I};$$

2. $\operatorname{img}(\mathbf{G}) = \ker\left(\tilde{\mathbf{G}}\right)$ and $\mathscr{G}^\tau_{\mathbf{P}} := \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right)$ for all $\tau \in \mathbb{R}$, where
$$\mathbf{G} := \begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{G}} := \begin{bmatrix} -\tilde{\mathbf{M}} & \tilde{\mathbf{N}} \end{bmatrix}$$
are respectively called right and left *strong graph symbols/representations* for $\mathbf{P}$.

Remark 3.2.10. By the properties of graph symbols in Assumption 2.2.1 and Lemma 3.1.3(iv), it follows that $\mathbf{T}_{\mathbf{G},\tau}$ has a causal left inverse $\mathbf{T}_{\mathbf{Z},\tau}$, and $\mathbf{T}_{\tilde{\mathbf{G}},\tau}$ has a causal right inverse $\mathbf{T}_{\tilde{\mathbf{Z}},\tau}$, where $\mathbf{Z} := \begin{bmatrix} \mathbf{Y} & \mathbf{X} \end{bmatrix}$ and $\tilde{\mathbf{Z}} := \begin{bmatrix} -\tilde{\mathbf{X}} \\ \tilde{\mathbf{Y}} \end{bmatrix}$. Note that the left-bounded-invertibility of $\mathbf{T}_{\mathbf{G},\tau}$ implies $\mathscr{G}^\tau_{\mathbf{P}} = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau})$ is a closed subspace [Kat80, Thm. IV.5.2], as is necessary for feedback stability [DS93, Thm. 6.2][FGS93, Prop. 1]. This is also consistent with Remark 3.2.5 about the definition of closed-loop stability.

Note that graph symbols are not unique. The time-invariant version of the following lemma can be found in e.g. [Vin01, Prop. 1.4].

Lemma 3.2.11. *Right (resp. left) graph symbols are unique to within right (resp. left) composition with a bounded causal operator which has a bounded and causal inverse.*

Proof. We prove the result for right graph symbols; the case for left graph symbols can be established similarly. Suppose we have a right graph symbol $\mathbf{G}$ for $\mathbf{P}$ and $\mathbf{Z}$ is a left causal bounded inverse of $\mathbf{G}$, i.e. $\mathbf{ZG} = \mathbf{I}$, then $\mathbf{GQ}$ is also a right graph symbol for any $\mathbf{Q} \in \mathbb{Q}$, where

$$\mathbb{Q} := \left\{\mathbf{Q} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}}) \mid \mathbf{Q} \text{ is boundedly invertible with } \mathbf{Q}, \mathbf{Q}^{-1} \text{ causal}\right\}. \tag{3.7}$$

To see this, note that $\mathbf{Q}^{-1}\mathbf{Z}$ is a left bounded causal inverse of $\mathbf{GQ}$, and

$$\operatorname{img}(\mathbf{T}_{\mathbf{GQ},\tau}) = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}\mathbf{T}_{\mathbf{Q},\tau}) = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) = \mathscr{G}^\tau_{\mathbf{P}} \, \forall \tau \in \mathbb{R},$$

where the first equality follows from Lemma 3.1.3(iv) and the second from the fact that $\mathbf{T}^{-1}_{\mathbf{Q},\tau} = \mathbf{T}_{\mathbf{Q}^{-1},\tau} \, \forall \tau \in \mathbb{R}$; see Remark 3.1.5.

Conversely, suppose $\mathbf{G}_a$ is another right graph symbol for $\mathbf{P}$ and $\mathbf{Z}_a$ is a left causal bounded inverse of $\mathbf{G}_a$, i.e. $\mathbf{Z}_a\mathbf{G}_a = \mathbf{I}$. We show below that there exists a $\mathbf{Q} \in \mathbb{Q}$ such that

$$\mathbf{G}_a = \mathbf{G}\mathbf{Q}.$$

For any $\tau \in \mathbb{R}$ and $q \in \boldsymbol{L}^2_{[\tau,\infty)}$, let $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] := \mathbf{T}_{\mathbf{G},\tau} q$. Since by the definition of graph symbols $\operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) = \operatorname{img}(\mathbf{T}_{\mathbf{G}_a,\tau}) = \mathscr{G}^\tau_{\mathbf{P}}$, there exists a $q_a \in \boldsymbol{L}^2_{[\tau,\infty)}$ such that

$$\mathbf{T}_{\mathbf{G}_a,\tau} q_a = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] = \mathbf{T}_{\mathbf{G},\tau} q.$$

Left-composing the above with $\mathbf{T}_{\mathbf{Z}_a,\tau}$ yields $q_a = \mathbf{T}_{\mathbf{Z}_a,\tau}\mathbf{T}_{\mathbf{G},\tau} q$; see Remark 3.2.10. Thus,

$$\mathbf{T}_{\mathbf{G},\tau} = \mathbf{T}_{\mathbf{G}_a,\tau}(\mathbf{T}_{\mathbf{Z}_a,\tau}\mathbf{T}_{\mathbf{G},\tau}) = \mathbf{T}_{\mathbf{G}_a\mathbf{Z}_a\mathbf{G},\tau}, \tag{3.8}$$

where the last equality holds by Lemma 3.1.3(iv). Similarly,

$$\mathbf{T}_{\mathbf{G}_a,\tau} = \mathbf{T}_{\mathbf{G}\mathbf{Z}\mathbf{G}_a,\tau}. \tag{3.9}$$

Now composing (3.8) and (3.9) with $\mathbf{T}_{\mathbf{Z},\tau}$ and $\mathbf{T}_{\mathbf{Z}_a,\tau}$, respectively, yields

$$\mathbf{I} = \mathbf{T}_{\mathbf{Z}\mathbf{G}_a\mathbf{Z}_a\mathbf{G},\tau} = \mathbf{T}_{\mathbf{Z}_a\mathbf{G}\mathbf{Z}\mathbf{G}_a,\tau}.$$

Let $\mathbf{Q} := \mathbf{Z}\mathbf{G}_a$ and $\mathbf{S} := \mathbf{Z}_a\mathbf{G}$, we then have from the above that

$$\mathbf{T}_{\mathbf{G}_a,\tau} = \mathbf{T}_{\mathbf{G}\mathbf{Q},\tau} \quad \text{and} \quad \mathbf{T}_{\mathbf{Q}\mathbf{S},\tau} = \mathbf{T}_{\mathbf{S}\mathbf{Q},\tau} = \mathbf{I} \text{ for all } \tau \in \mathbb{R}.$$

By Lemma 3.1.3(ii), it follows that $\mathbf{G}_a = \mathbf{G}\mathbf{Q}$ and $\mathbf{Q}\mathbf{S} = \mathbf{S}\mathbf{Q} = \mathbf{I}$, i.e. $\mathbf{Q}^{-1} = \mathbf{S}$, whereby $\mathbf{Q} \in \mathbb{Q}$, as required. $\square$

Remark 3.2.12. In the semi-infinite discrete-time setting [Fei98], by exploiting Arveson's inner/outer factorisation in *nest algebras*, closed-loop stabilisability of an operator can be shown to be equivalent to the existence of strong right and left representations on semi-infinite time [DS93]. Therefrom, the set of all stabilising 'controllers' may be characterised by the so-called Youla parameterisation. As discussed in [DS93], the equivalence proof does not carry over to continuous-time systems since it exploits the fact that causal bounded operators admit lower triangular matrix representations; furthermore, Arveson's inner/outer factorisations may not exist in continuous nest algebras [DS93, Thm. 6.1].

Assumption 2.2.2. $\mathbf{G}^*\mathbf{G} = \mathbf{I}$ and $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$, i.e. the right and left graph symbols are normalised.

Remark 3.2.13. Normalised right (resp. left) graph symbols are unique to within right (resp. left) composition with a bounded *unitary* memoryless operator. To see this, suppose $\mathbf{G}$ is a normalised right graph symbol for $\mathbf{P}$. Then by Lemma 3.2.11, all other right graph symbols may be expressed as $\mathbf{GQ}$ for some $\mathbf{Q} \in \mathbb{Q}$, where $\mathbb{Q}$ is as defined in (3.7). These are normalised if, and only if, $\mathbf{Q}$ is unitary, i.e. $\mathbf{Q}^*\mathbf{Q} = \mathbf{Q}\mathbf{Q}^* = \mathbf{I}$. Note that since $\mathbf{Q}^* = \mathbf{Q}^{-1}$, $\mathbf{Q}$ is necessarily memoryless, i.e. simultaneously causal and anti-causal. The case for left graph symbols can be established similarly.

Assumption 2.2.3. $\mathbf{H}^{+-}_{\mathbf{G},\tau}$ and $\mathbf{H}^{+-}_{\tilde{\mathbf{G}},\tau}$ are compact for all $\tau \in \mathbb{R}$.

Lemma 3.2.14 ([JC10, Lem. 9])**.** *Given causal operators $\mathbf{P}_1$ and $\mathbf{P}_2$, suppose Assumptions 2.2.1 and 2.2.3 hold, then $\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is compact for all $\tau \in \mathbb{R}$.*

Proof. Since by hypothesis $\tilde{\mathbf{G}}_2$ causal, $\tilde{\mathbf{G}}_2^*$ must be anti-causal, and hence for any $\tau \in \mathbb{R}$

$$\mathbf{H}^{+-}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau} = (\mathbf{I} - \mathbf{\Pi}_\tau)\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*\mathbf{\Pi}_\tau = (\mathbf{I} - \mathbf{\Pi}_\tau)\tilde{\mathbf{G}}_1\mathbf{\Pi}_\tau\tilde{\mathbf{G}}_2^*\mathbf{\Pi}_\tau = \mathbf{H}^{+-}_{\tilde{\mathbf{G}}_1,\tau}(\mathbf{\Pi}_\tau\tilde{\mathbf{G}}_2^*\mathbf{\Pi}_\tau).$$

By lemma 2.1.1, $\mathbf{H}^{+-}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau}$ is compact. This implies $\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau} = \left(\mathbf{H}^{+-}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau}\right)^*$ is compact, since the adjoint of a compact operator is compact [Kat80, Thm. III.4.10]. □

By convention throughout this thesis, $\mathbf{G} := \left[\begin{smallmatrix}\mathbf{N}\\ \mathbf{M}\end{smallmatrix}\right]$ denote as above a right strong graph symbol for $\mathbf{P}$, and $\tilde{\mathbf{G}} := \left[\begin{smallmatrix}-\tilde{\mathbf{M}} & \tilde{\mathbf{N}}\end{smallmatrix}\right]$ a left strong graph symbol. Similarly, $\left[\begin{smallmatrix}\mathbf{V}\\ \mathbf{U}\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-\tilde{\mathbf{U}} & \tilde{\mathbf{V}}\end{smallmatrix}\right]$ respectively denote a right strong (inverse) graph symbol and a left strong (inverse) graph symbol for $\mathbf{C}$. Specifically, we define

$$\mathbf{K} := \begin{bmatrix}\mathbf{V}\\ \mathbf{U}\end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{K}} := \begin{bmatrix}-\tilde{\mathbf{U}} & \tilde{\mathbf{V}}\end{bmatrix},$$

so that for every $\tau \in \mathbb{R}$, $\mathscr{G}'^{\tau}_{\mathbf{C}} := \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \operatorname{img}(\mathbf{T}_{\mathbf{K},\tau}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right)$.

3.2.3 Characterisation of feedback stability via graph symbols

The stability of a feedback interconnection $[\mathbf{P}, \mathbf{C}]$ can be characterised in terms of graph symbols for $\mathbf{P}$ and $\mathbf{C}$. This is an important ingredient in the development of ν-gap based robustness analysis, as developed in the time-invariant setting of [Vin93, Vin01, CJK12, CJK10, CJK09, JCK08] and generalised to time-varying systems in [JC10, JC11]. We suppose the standing Assumption 2.2.1 holds for every causal operator in this subsection.

The notation is as introduced in the preceding subsections. In particular, given causal operators $\mathbf{P} : \operatorname{dom}(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ and $\mathbf{C} : \operatorname{dom}(\mathbf{C}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$, $\mathbf{G}$ and $\tilde{\mathbf{G}}$ denote respectively right and left graph symbols for $\mathbf{P}$ while $\mathbf{K}$ and $\tilde{\mathbf{K}}$ denote respectively right and left graph symbols for $\mathbf{C}$. For the subsequent lemma, the definition of Wiener-Hopf operators is as given in Definition 2.1.9.

Lemma 3.2.15. *Given $\tau \in \mathbb{R}$, the following are equivalent:*

(i) $\mathbf{F}_{\mathbf{P},\mathbf{C},\tau}$*, as defined in Definition 3.2.1, has a bounded inverse on* $\boldsymbol{L}^2_{[\tau,\infty)}$*;*

(ii) $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ *has a bounded inverse;*

(iii) $\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}$ *has a bounded inverse.*

When any of the above holds,

$$\mathbf{\Pi}_{\mathscr{G}^\tau_\mathbf{P} \| \mathscr{G}'^\tau_\mathbf{C}} = \mathbf{T}_{\mathbf{G},\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau} \quad \textit{and} \quad \mathbf{\Pi}_{\mathscr{G}'^\tau_\mathbf{C} \| \mathscr{G}^\tau_\mathbf{P}} = \mathbf{T}_{\mathbf{K},\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{G}}\mathbf{K},\tau}\mathbf{T}_{\tilde{\mathbf{G}},\tau},$$

where $\mathbf{\Pi}_{\mathscr{G}^\tau_\mathbf{P} \| \mathscr{G}'^\tau_\mathbf{C}}$ *and* $\mathbf{\Pi}_{\mathscr{G}'^\tau_\mathbf{C} \| \mathscr{G}^\tau_\mathbf{P}}$ *are parallel projections in (3.5) and (3.6).*

Proof. The proof can be found in [Can06, Prop. 3]; it is included here for completeness. We establish the equivalence between (i) and (ii); (i) $\Longleftrightarrow$ (iii) follows similarly.

Since $\mathscr{G}_\mathbf{P} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau})$ and $\mathscr{G}'_\mathbf{C} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right)$, we have by [FGS93, Prop. 1 & 2] that (i) is equivalent to

$$\operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) + \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right) = \boldsymbol{L}^2_{[\tau,\infty)} \quad \text{and} \quad \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) \cap \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right) = \{0\}. \tag{3.10}$$

Observe that since $\mathbf{T}_{\tilde{\mathbf{K}},\tau}$ has a right causal inverse and $\mathbf{T}_{\mathbf{G},\tau}$ has a left causal inverse, as in Remark 3.2.10, we have $\operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right) = \boldsymbol{L}^2_{[\tau,\infty)}$ and $\ker(\mathbf{T}_{\mathbf{G},\tau}) = \{0\}$. With the co-ordinatisation identities in (3.10), this implies

$$\operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}\right) = \operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\mathbf{T}_{\mathbf{G},\tau}\right) = \boldsymbol{L}^2_{[\tau,\infty)} \text{ and } \ker\left(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}\right) = \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\mathbf{T}_{\mathbf{G},\tau}\right) = \{0\}.$$

That $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ has a bounded inverse then follows from the open mapping theorem [Kre89, Thm. 4.12-2]. To establish the converse of this implication, assume $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ boundedly invertible. Then the fact that $\ker\left(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}\right) = \{0\}$ implies $\operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) \cap \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right) = \{0\}$. Now for any $h \in \boldsymbol{L}^2_{[\tau,\infty)}$, define $e := \mathbf{T}_{\tilde{\mathbf{K}},\tau}h \in \boldsymbol{L}^2_{[\tau,\infty)}$. By the bounded invertibility of $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$, there exists a unique $q \in \boldsymbol{L}^2_{[\tau,\infty)}$ such that $e = \mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}q$. Putting these together, we thus have

$$\mathbf{T}_{\tilde{\mathbf{K}},\tau}(h - \mathbf{T}_{\mathbf{G},\tau}q) = 0,$$

i.e. $(h - \mathbf{T}_{\mathbf{G},\tau}q) \in \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right)$. Since h is arbitrary, it follows immediately that

$$\operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) + \ker\left(\mathbf{T}_{\tilde{\mathbf{K}},\tau}\right) = \boldsymbol{L}^2_{[\tau,\infty)}.$$

Now we show that $\mathbf{\Pi}_{\mathscr{G}^\tau_\mathbf{P} \| \mathscr{G}'^\tau_\mathbf{C}} = \mathbf{T}_{\mathbf{G},\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau}$ when (3.10) holds. To this end, note that (3.10) implies any $h \in \boldsymbol{L}^2_{[\tau,\infty)}$ can be uniquely decomposed into $h = g + k$ with

$$g = \boldsymbol{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}} \| \mathscr{G}'^\tau_{\mathbf{C}}} h \in \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} \quad \text{and} \quad k = \boldsymbol{\Pi}_{\mathscr{G}'^\tau_{\mathbf{C}} \| \mathscr{G}^\tau_{\mathbf{P}}} h \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^2_{[\tau,\infty)}.$$

Now since $\mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau})$ and $\mathbf{T}_{\mathbf{G},\tau}$ has a left causal inverse, there exists a unique $q \in \boldsymbol{L}^2_{[\tau,\infty)}$ such that $g = \mathbf{T}_{\mathbf{G},\tau} q$. Correspondingly, for all $h \in \boldsymbol{L}^2_{[\tau,\infty)}$,

$$(\boldsymbol{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}} \| \mathscr{G}'^\tau_{\mathbf{C}}} - \mathbf{T}_{\mathbf{G},\tau} \mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau} \mathbf{T}_{\tilde{\mathbf{K}},\tau}) h = g - \mathbf{T}_{\mathbf{G},\tau} \mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau} \mathbf{T}_{\tilde{\mathbf{K}},\tau} (\mathbf{T}_{\mathbf{G},\tau} q + k) = g - \mathbf{T}_{\mathbf{G},\tau} q = 0,$$

whereby the claim is proved. The expression for $\boldsymbol{\Pi}_{\mathscr{G}'^\tau_{\mathbf{C}} \| \mathscr{G}^\tau_{\mathbf{P}}}$ can be established similarly. □

By the equivalence of (i) and (ii) in Lemma 3.2.15, $[\mathbf{P}, \mathbf{C}]$ is stable if, and only if, $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ is boundedly invertible for all $\tau \in \mathbb{R}$ and $\sup_{\tau \in \mathbb{R}} \bar{\gamma}(\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau}) < \infty$. In turn, this is equivalent to $\inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}) = \underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) > 0$, where the equality follows from Lemma 3.1.3(v). Analogously, by the equivalence of Lemma 3.2.15(i) and (iii), $[\mathbf{P}, \mathbf{C}]$ is stable if, and only if, $\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}$ is boundedly invertible for all $\tau \in \mathbb{R}$ and $\inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) = \underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0$.

Lemma 3.2.16. *The following are equivalent:*

1. $[\mathbf{P}, \mathbf{C}]$ *is stable;*
2. $\underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) = \inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}) > 0$ *and* $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ *is Fredholm with* $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}) = 0 \, \forall \tau \in \mathbb{R}$*;*
3. $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) = \inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) > 0$ *and* $\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}$ *is Fredholm with* $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) = 0 \, \forall \tau \in \mathbb{R}$.

Proof. By combining Lemmas 3.1.6 and 3.2.15, and noting the immediately preceding observation as in [JC10, Lem. 8]. □

When graph symbols have the property of being normalised as in Assumption 2.2.2, several useful identities may be derived.

Using the properties of *normalised* graph symbols in Assumptions 2.2.1 and 2.2.2, one obtains

$$\begin{bmatrix} \tilde{\mathbf{G}} \\ \mathbf{G}^* \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{G}}^* & \mathbf{G} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\mathbf{K}} \\ \mathbf{K}^* \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{K}}^* & \mathbf{K} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{I} \end{bmatrix},$$

i.e. $\begin{bmatrix} \tilde{\mathbf{G}} \\ \mathbf{G}^* \end{bmatrix}$ and $\begin{bmatrix} \tilde{\mathbf{K}} \\ \mathbf{K}^* \end{bmatrix}$ are right-invertible with bounded right inverses $\begin{bmatrix} \tilde{\mathbf{G}}^* & \mathbf{G} \end{bmatrix}$ and $\begin{bmatrix} \tilde{\mathbf{K}}^* & \mathbf{K} \end{bmatrix}$, respectively. Mimicking the argument in the proof of [Can06, Prop. 4], observe that

$$\ker \left(\begin{bmatrix} \tilde{\mathbf{G}} \\ \mathbf{G}^* \end{bmatrix} \right) = \ker \left(\tilde{\mathbf{G}} \right) \cap \ker (\mathbf{G}^*) = \ker \left(\tilde{\mathbf{G}} \right) \cap \operatorname{img} (\mathbf{G})^\perp = \{0\},$$

whereby $\begin{bmatrix} \tilde{\mathbf{G}} \\ \mathbf{G}^* \end{bmatrix}$ is left-invertible and the left inverse is equal to the right inverse $\begin{bmatrix} \tilde{\mathbf{G}}^* & \mathbf{G} \end{bmatrix}$. A similar argument applies to $\mathbf{C}$, that is, $\begin{bmatrix} \tilde{\mathbf{K}} \\ \mathbf{K}^* \end{bmatrix}$ has the left inverse $\begin{bmatrix} \tilde{\mathbf{K}}^* & \mathbf{K} \end{bmatrix}$. These imply

$$\tilde{\mathbf{G}}^*\tilde{\mathbf{G}} + \mathbf{G}\mathbf{G}^* = \mathbf{I} \text{ and } \tilde{\mathbf{K}}^*\tilde{\mathbf{K}} + \mathbf{K}\mathbf{K}^* = \mathbf{I}. \tag{3.11}$$

It follows that for two causal operators $\mathbf{P}_1$ and $\mathbf{P}_2$ with respective right and left normalised graph symbols $\mathbf{G}_i$ and $\tilde{\mathbf{G}}_i$, $\begin{bmatrix} \mathbf{G}_2^*\mathbf{G}_1 \\ \tilde{\mathbf{G}}_2\mathbf{G}_1 \end{bmatrix}$ and $\begin{bmatrix} \mathbf{G}_2^*\tilde{\mathbf{G}}_1^* \\ \tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^* \end{bmatrix}$ are isometries, while $\begin{bmatrix} \mathbf{G}_2^*\mathbf{G}_1 & \mathbf{G}_2^*\tilde{\mathbf{G}}_1^* \end{bmatrix}$ and $\begin{bmatrix} \tilde{\mathbf{G}}_2\mathbf{G}_1 & \tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^* \end{bmatrix}$ are co-isometries. These lead to

$$\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) = \sqrt{1 - \underline{\gamma}^2(\mathbf{G}_2^*\mathbf{G}_1)} = \bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_2) \tag{3.12}$$

$$\underline{\gamma}(\mathbf{G}_2^*\mathbf{G}_1) = \sqrt{1 - \bar{\gamma}^2(\tilde{\mathbf{G}}_2\mathbf{G}_1)} = \underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*). \tag{3.13}$$

Furthermore, notice that $\begin{bmatrix} \mathbf{K}^*\tilde{\mathbf{G}}^* \\ \tilde{\mathbf{K}}\tilde{\mathbf{G}}^* \end{bmatrix}$ is an isometry while $\begin{bmatrix} \tilde{\mathbf{K}}\mathbf{G} & \tilde{\mathbf{K}}\tilde{\mathbf{G}}^* \end{bmatrix}$ and $\begin{bmatrix} \tilde{\mathbf{G}}\mathbf{K} & \tilde{\mathbf{G}}\tilde{\mathbf{K}}^* \end{bmatrix}$ are a co-isometries, whereby

$$\underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) = \sqrt{1 - \bar{\gamma}^2(\tilde{\mathbf{K}}\tilde{\mathbf{G}}^*)} = \underline{\gamma}(\mathbf{K}^*\tilde{\mathbf{G}}^*) = \sqrt{1 - \bar{\gamma}^2(\tilde{\mathbf{G}}\tilde{\mathbf{K}}^*)} = \underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}). \tag{3.14}$$

Similarly, $\begin{bmatrix} \mathbf{K}^*\tilde{\mathbf{G}}^* & \mathbf{K}^*\mathbf{G} \end{bmatrix}$ is a co-isometry, whereby

$$\bar{\gamma}(\mathbf{G}^*\mathbf{K}) = \bar{\gamma}(\mathbf{K}^*\mathbf{G}) = \sqrt{1 - \underline{\gamma}^2(\mathbf{K}^*\tilde{\mathbf{G}}^*)} = \sqrt{1 - \underline{\gamma}^2(\tilde{\mathbf{G}}\mathbf{K})}. \tag{3.15}$$

Lemma 3.2.17. *If $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) = \underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) > 0$, then $\tilde{\mathbf{G}}\mathbf{K}$ and $\tilde{\mathbf{K}}\mathbf{G}$ are boundedly invertible. Similarly, if $\underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*) > 0$, then $\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*$ is boundedly invertible.*

Proof. The proof is based on [JC10, Lem. 10]. First we remark that $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) = \underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G})$ from (3.14). We prove the result only for $\tilde{\mathbf{G}}\mathbf{K}$ as the case for $\tilde{\mathbf{K}}\mathbf{G}$ follows a similar proof. Notice that $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0$ implies $\bar{\gamma}(\tilde{\mathbf{K}}\tilde{\mathbf{G}}^*) < 1$ and $\bar{\gamma}(\mathbf{G}^*\mathbf{K}) < 1$ by (3.14) and (3.15) respectively. Consequently, it holds that $\bar{\gamma}(\tilde{\mathbf{G}}\tilde{\mathbf{K}}^*\tilde{\mathbf{K}}\tilde{\mathbf{G}}^*) = \bar{\gamma}(\tilde{\mathbf{K}}\tilde{\mathbf{G}}^*)^2 < 1$ and $\bar{\gamma}(\mathbf{K}^*\mathbf{G}\mathbf{G}^*\mathbf{K}) = \bar{\gamma}(\mathbf{G}^*\mathbf{K})^2 < 1$. Moreover, we have from (3.11) that

$$\begin{aligned} \mathbf{I} &= \tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \tilde{\mathbf{G}}\mathbf{K}\mathbf{K}^*\tilde{\mathbf{G}}^* + \tilde{\mathbf{G}}\tilde{\mathbf{K}}^*\tilde{\mathbf{K}}\tilde{\mathbf{G}}^* \\ \text{and} \quad \mathbf{I} &= \mathbf{K}^*\mathbf{K} = \mathbf{K}^*\tilde{\mathbf{G}}^*\tilde{\mathbf{G}}\mathbf{K} + \mathbf{K}^*\mathbf{G}\mathbf{G}^*\mathbf{K}. \end{aligned}$$

As such, $\tilde{\mathbf{G}}\mathbf{K}\mathbf{K}^*\tilde{\mathbf{G}}^* = \mathbf{I} - \tilde{\mathbf{G}}\tilde{\mathbf{K}}^*\tilde{\mathbf{K}}\tilde{\mathbf{G}}^*$ and $\mathbf{K}^*\tilde{\mathbf{G}}^*\tilde{\mathbf{G}}\mathbf{K} = \mathbf{I} - \mathbf{K}^*\mathbf{G}\mathbf{G}^*\mathbf{K}$ are boundedly invertible by [Kre89, Thm. 7.3-1], implying respectively that $\mathrm{img}\left(\tilde{\mathbf{G}}\mathbf{K}\right) = \boldsymbol{L}^2_{\mathbb{R}}$ and $\ker\left(\tilde{\mathbf{G}}\mathbf{K}\right) = \{0\}$, i.e. $\tilde{\mathbf{G}}\mathbf{K}$ is bijective. That $(\tilde{\mathbf{G}}\mathbf{K})^{-1}$ is bounded follows from the open mapping theorem [Kre89, Thm. 4.12-2]. The second part of the lemma can be established using the same line of arguments along with (3.12), (3.13), and (3.11); see [JC10, Lem. 10]. $\square$

Note that if $\tilde{\mathbf{K}}\mathbf{G}$ or $\tilde{\mathbf{G}}\mathbf{K}$ has a bounded *causal* inverse, then $[\mathbf{P}, \mathbf{C}]$ is stable by Lemma 3.2.16 and Remark 3.1.5. That the converse also holds is shown below.

Observe that when $[\mathbf{P}, \mathbf{C}]$ is stable, $\underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) > 0$ by Lemma 3.2.16, and hence it follows from Lemma 3.2.17 that $(\tilde{\mathbf{K}}\mathbf{G})^{-1} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$. Furthermore, by (3.5) and Lemma 3.2.15, $\mathbf{\Pi}_{\mathscr{G}_\mathbf{P}\|\mathscr{G}'_\mathbf{C},\tau} = \mathbf{T}_{\mathbf{G},\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau}$ is causal for all $\tau \in \mathbb{R}$. Thus, $\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ is causal by the fact that $\mathbf{T}_{\mathbf{G},\tau}$ has a left causal inverse and $\mathbf{T}_{\tilde{\mathbf{K}},\tau}$ has a right causal inverse; recall Remark 3.2.10. Now, using Lemma 3.1.3(iv), we have that $\mathbf{T}_{(\tilde{\mathbf{K}}\mathbf{G})^{-1},\tau}\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau} = \mathbf{T}_{(\tilde{\mathbf{K}}\mathbf{G})^{-1}\tilde{\mathbf{K}}\mathbf{G},\tau} = \mathbf{I}$, i.e. $\mathbf{T}_{(\tilde{\mathbf{K}}\mathbf{G})^{-1},\tau}$ is the left inverse of $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$. Since $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ is invertible, its left inverse is equal to its inverse, i.e. $\mathbf{T}_{(\tilde{\mathbf{K}}\mathbf{G})^{-1},\tau} = \mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$, which is causal for all $\tau \in \mathbb{R}$. It follows by Lemma 3.1.3(iii) that $(\tilde{\mathbf{K}}\mathbf{G})^{-1}$ is causal.

All of the above arguments hold analogously for $\mathbf{\Pi}_{\mathscr{G}'_\mathbf{C}\|\mathscr{G}_\mathbf{P}}$. In other words, $[\mathbf{P}, \mathbf{C}]$ is stable if, and only if, $\tilde{\mathbf{G}}\mathbf{K}$ has a bounded causal inverse.

Remark 3.2.18. It is shown above that stability of $[\mathbf{P}, \mathbf{C}]$ is equivalent to either $\tilde{\mathbf{K}}\mathbf{G}$ or $\tilde{\mathbf{G}}\mathbf{K}$ being boundedly and causally invertible, where $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are respectively right and left *normalised* graph symbols for $\mathbf{P}$ whereas $\mathbf{K}$ and $\tilde{\mathbf{K}}$ are those for $\mathbf{C}$. Now recall by Lemma 3.2.11 that graph symbols are related by composition with a bounded causal operator that has a bounded causal inverse. As such, given another (not necessarily normalised) left graph symbol $\tilde{\mathbf{K}}_1$ for $\mathbf{C}$ and a right graph symbol $\mathbf{G}_1$ for $\mathbf{P}$, bounded causal invertibility of $\tilde{\mathbf{K}}\mathbf{G}$ is equivalent of that of $\tilde{\mathbf{K}}_1\mathbf{G}_1$. A similar argument applies to $\tilde{\mathbf{G}}\mathbf{K}$.

When $[\mathbf{P}, \mathbf{C}]$ is stable, we have by Lemma 3.1.3(iv) that $\mathbf{\Pi}_{\mathscr{G}^\tau_\mathbf{P}\|\mathscr{G}'^\tau_\mathbf{C}} = \mathbf{T}_{\mathbf{G}(\tilde{\mathbf{K}}\mathbf{G})^{-1}\tilde{\mathbf{K}},\tau}$, and thus

$$\begin{aligned}\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}^\tau_\mathbf{P}\|\mathscr{G}'^\tau_\mathbf{C}}\right) &= \sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathbf{T}_{\mathbf{G}(\tilde{\mathbf{K}}\mathbf{G})^{-1}\tilde{\mathbf{K}},\tau}\right)\\ &= \bar{\gamma}(\mathbf{G}(\tilde{\mathbf{K}}\mathbf{G})^{-1}\tilde{\mathbf{K}}) = \bar{\gamma}((\tilde{\mathbf{K}}\mathbf{G})^{-1}) = 1/\underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}),\end{aligned}$$

where the second equality holds by Lemma 3.1.3(v) and the third by Assumption 2.2.2, whereby the graph symbols are normalised. Therefore,

$$b_{\mathbf{P},\mathbf{C}} := \left(\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}^\tau_\mathbf{P}\|\mathscr{G}'^\tau_\mathbf{C}}\right)\right)^{-1} = \underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) > 0.$$

Similarly, $b_{\mathbf{C},\mathbf{P}} := \left(\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}'^\tau_\mathbf{C}\|\mathscr{G}^\tau_\mathbf{P}}\right)\right)^{-1} = \underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0$. Using (3.14), it holds that $b_{\mathbf{P},\mathbf{C}} = b_{\mathbf{C},\mathbf{P}}$, which is consistent with [DGS93, Prop. 6] since $\mathbf{P}$ and $\mathbf{C}$ are linear.

We summarise all the preceding results in the following main theorem of this section.

Theorem 3.2.19. *Given a causal* $\mathbf{P} : \operatorname{dom}(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$*, for which* $\mathbf{G}$ *and* $\tilde{\mathbf{G}}$ *are respectively right and left graph symbols, and a causal* $\mathbf{C} : \operatorname{dom}(\mathbf{C}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$*, for which* $\mathbf{K}$ *and* $\tilde{\mathbf{K}}$ *are respectively right and left graph symbols, the following are equivalent:*

1. $[\mathbf{P}, \mathbf{C}]$ *is stable in the sense of Definition 3.2.1;*
2. $\tilde{\mathbf{K}}\mathbf{G}$ *has a bounded causal inverse;*
3. $\tilde{\mathbf{G}}\mathbf{K}$ *has a bounded causal inverse;*
4. $\underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) = \inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}) > 0$ *and* $\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}$ *is Fredholm with* $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G},\tau}) = 0 \, \forall \tau \in \mathbb{R}$*;*
5. $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) = \inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) > 0$ *and* $\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}$ *is Fredholm with* $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) = 0 \, \forall \tau \in \mathbb{R}$.

Suppose further that Assumption 2.2.2 holds, i.e. these graph symbols are normalised, then when $[\mathbf{P}, \mathbf{C}]$ *is stable, we have* $b_{\mathbf{P},\mathbf{C}} = b_{\mathbf{C},\mathbf{P}} = \underline{\gamma}(\tilde{\mathbf{K}}\mathbf{G}) = \underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0$.

3.3 Robust stability properties of the ν-gap metric

The ν-gap metric for linear time-varying systems is formally defined in this section. We present sufficient conditions for robust feedback stability and properties of the ν-gap metric, as delineated in the introduction to this chapter. A robust stability result that motivates the definition of the ν-gap is derived in Section 3.3.1. That the ν-gap is a metric is shown in Section 3.3.2, together with several of its robustness and topological properties.

3.3.1 The ν-gap metric

We begin with the following important robustness result, which generalises the time-invariant case considered in [Vin01, Lem. 3.6]. The proof is constructed by combining aspects of [JC10, Lem. 10 and Thm. 1]. First recall the following definition from Section 2.2.

Definition 2.2.4. We denote by $\mathbb{S}$ the set of causal operators for which all of Assumptions 2.2.1, 2.2.2, and 2.2.3 are satisfied.

Theorem 3.3.1. *Given* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathbb{S}$*, suppose that* $\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) < b_{\mathbf{P}_1,\mathbf{C}} = \underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K})$*. Then*

(i) $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ *is Fredholm for all* $\tau \in \mathbb{R}$*; and*

(ii) $[\mathbf{P}_2, \mathbf{C}]$ *is stable if, and only if,* $\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}) = 0\,\forall\tau \in \mathbb{R}$.

Proof. (i): First, by (3.12) and hypothesis, $\bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) < \underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) \le \bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) \le 1$, where the last inequality holds since the graph symbols are normalised. As such, for all $\tau \in \mathbb{R}$,

$$\bar{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{G}_2\mathbf{G}_2^*\tilde{\mathbf{G}}_1^*,\tau}) \le \bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_2\mathbf{G}_2^*\tilde{\mathbf{G}}_1^*) < 1 \quad \text{and} \quad \bar{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\tilde{\mathbf{G}}_2^*,\tau}) \le \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\tilde{\mathbf{G}}_2^*) < 1. \tag{3.16}$$

Now using (3.11), note that

$$\mathbf{T}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau} = \mathbf{I} - \mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{G}_2\mathbf{G}_2^*\tilde{\mathbf{G}}_1^*,\tau} \quad \text{and} \quad \mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau} = \mathbf{I} - \mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\tilde{\mathbf{G}}_2^*,\tau}. \tag{3.17}$$

As such, with (3.16) it follows by Lemma 3.1.2(iii) and (3.17) that $\mathbf{T}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ and $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau}$ are Fredholm with

$$\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}) = \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau}) = \mathrm{ind}(\mathbf{I}) = 0.$$

Now note that by Lemma 3.1.3(iv) and Definition 2.1.9,

$$\begin{aligned}\mathbf{T}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau} &= (\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau} + \mathbf{H}^{+-}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau} \quad \text{and} \\ \mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau} &= \mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^* + \mathbf{H}^{+-}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau},\end{aligned}$$

in which the composition of Hankel operators $\mathbf{H}^{+-}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ and $\mathbf{H}^{+-}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_1\tilde{\mathbf{G}}_2^*,\tau}$ are compact by Assumption 2.2.3, Lemma 3.2.14, and Lemma 2.1.1. Consequently, applying Lemma 3.1.2(iv) yields $(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ and $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*$ are Fredholm with $\mathrm{ind}\left((\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\right) = \mathrm{ind}\left(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*\right) = 0$. That $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm then follows from $\ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\right) = \ker\left((\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\right)$ and $\ker\left((\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*\right) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})^*\right)$.

(ii): Our objective is to show that

1. $\underline{\gamma}(\tilde{\mathbf{G}}_2\mathbf{K}) > 0$; and

2. $\mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{K},\tau}$ is Fredholm with $\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{K},\tau}) = \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})$ for all $\tau \in \mathbb{R}$,

from which the claimed result follows by Theorem 3.2.19.

To begin with, note that when $\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) < 1$, we have by hypothesis

$$\frac{\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)}{\sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_2\mathbf{G}_1)}} < \frac{\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K})}{\sqrt{1-\underline{\gamma}^2(\tilde{\mathbf{G}}_1\mathbf{K})}},$$

since $x \mapsto x/\sqrt{1-x^2}$ is monotonic on $[0,1)$. This implies

$$\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)\sqrt{1-\underline{\gamma}^2(\tilde{\mathbf{G}}_1\mathbf{K})} < \underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K})\sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_2\mathbf{G}_1)}, \tag{3.18}$$

which also holds trivially when $\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) = 1$. Combining (3.13), (3.18), and (3.15) yields

$$\begin{aligned}\underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\mathbf{K}) \geq \underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*)\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) &= \sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_2\mathbf{G}_1)}\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) \\ &> \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)\sqrt{1-\underline{\gamma}^2(\tilde{\mathbf{G}}_1\mathbf{K})} \\ &= \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)\bar{\gamma}(\mathbf{G}_1^*\mathbf{K}) \geq \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\mathbf{K}).\end{aligned} \tag{3.19}$$

Now using (3.11), we arrive at the identity

$$\tilde{\mathbf{G}}_2\mathbf{K} = \tilde{\mathbf{G}}_2\left(\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1 + \mathbf{G}_1\mathbf{G}_1^*\right)\mathbf{K} = \tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\mathbf{K} + \tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\mathbf{K}, \tag{3.20}$$

whereby $\underline{\gamma}(\tilde{\mathbf{G}}_2\mathbf{K}) \geq \underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\mathbf{K}) - \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\mathbf{K}) > 0$, as required by the first objective identified above.

Because $\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) > 0$ by hypothesis and $\underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*) > 0$ by (3.19), both $\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*$ and $\tilde{\mathbf{G}}_1\mathbf{K}$ are boundedly invertible by Lemma 3.2.17. As a result, (3.20) can be rewritten as

$$\tilde{\mathbf{G}}_2\mathbf{K} = (\mathbf{I}+\mathbf{X})\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\mathbf{K} \quad \text{with} \quad \mathbf{X} := \tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\mathbf{K}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\mathbf{K})^{-1}.$$

Application of both parts of Lemma 3.1.3(iv) then yields

$$\mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{K},\tau} = \mathbf{T}_{\mathbf{I}+\mathbf{X},\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau} + \mathbf{H}^{+-}_{\mathbf{I}+\mathbf{X},\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}, \forall\tau\in\mathbb{R}. \tag{3.21}$$

Note that $\mathbf{H}^{+-}_{\mathbf{I}+\mathbf{X},\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}$ is compact by Lemma 2.1.1, since $\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is compact by Lemma 3.2.14. Furthermore, by (3.19),

$$\bar{\gamma}(\mathbf{T}_{\mathbf{X},\tau}) \leq \bar{\gamma}(\mathbf{X}) \leq \frac{\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{G}_1^*\mathbf{K})}{\underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*\tilde{\mathbf{G}}_1\mathbf{K})} < 1.$$

Hence by Lemma 3.1.2(iii), $\mathbf{T}_{\mathbf{I}+\mathbf{X},\tau}$ is Fredholm and $\operatorname{ind}(\mathbf{T}_{\mathbf{I}+\mathbf{X},\tau}) = \operatorname{ind}(\mathbf{I}+\mathbf{T}_{\mathbf{X},\tau}) = \operatorname{ind}(\mathbf{I}) = 0$ for all $\tau\in\mathbb{R}$. Also, since $[\mathbf{P}_1,\mathbf{C}]$ is stable by the hypothesis that $b_{\mathbf{P}_1,\mathbf{C}} > 0$, the use of Theorem 3.2.19 implies $\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}$ is Fredholm and $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}) = 0$ for all $\tau\in\mathbb{R}$.

All in all, applying Lemma 3.1.2, parts (iv) and (ii), to (3.21) yields $\mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{K},\tau}$ is Fredholm for all $\tau \in \mathbb{R}$ with

$$\begin{aligned}\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{K},\tau}) &= \operatorname{ind}(\mathbf{T}_{\mathbf{I}+\mathbf{X},\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}) \\ &= \operatorname{ind}(\mathbf{T}_{\mathbf{I}+\mathbf{X},\tau}) + \operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}) + \operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}) \\ &= \operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}),\end{aligned}$$

as required by the second objective identified above. □

Remark 3.3.2. The robustness result above is developed in terms of (normalised) left graph symbols. On the contrary, right graph symbols are used to develop the analogous time-invariant result in [Vin01, Lem. 3.6]. The next lemma shows the equivalence between the two. It is of note that the derivation in [Vin01, Lem. 3.6] does not directly carry over to the time-varying case as above, which explains the rationale behind using left graph symbols.

Lemma 3.3.3 ([JC10, Prop. 1])**.** *Given $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{S}$ and any $\tau \in \mathbb{R}$, $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm if, and only if, $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ is Fredholm, in which case $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}) = -\operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau})$.*

Proof. First we identify a few useful identities and properties of the graph symbols. Recall from Assumption 2.2.1 that for any $\mathbf{P} \in \mathbb{S}$, $\mathscr{G}_{\mathbf{P}}^{\tau} = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau}) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right)$ for all $\tau \in \mathbb{R}$. It follows that

$$\mathscr{G}_{\mathbf{P}}^{\tau\,\perp} = \operatorname{img}(\mathbf{T}_{\mathbf{G},\tau})^{\perp} = \ker\left(\mathbf{T}_{\mathbf{G}^*,\tau}\right) \quad \text{and} \quad \mathscr{G}_{\mathbf{P}}^{\tau\,\perp} = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}},\tau}\right)^{\perp} = \operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau}\right),$$

where the last equality holds because $\operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau}\right)$ is closed. To see this, note that by Assumption 2.2.1 and Lemma 3.1.3(iv), $\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau}$ has a bounded left (anti-causal) inverse $\mathbf{T}_{[\tilde{\mathbf{X}}^*\ -\tilde{\mathbf{Y}}^*],\tau}$, implying that $\underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau}) > 0$, and hence $\operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau}\right)$ is closed [GGK90, Thm. XI.2.1]. Also recall that $\underline{\gamma}(\mathbf{T}_{\mathbf{G},\tau}) > 0$ since the operator has a bounded left (causal) inverse.

Now application of Lemma 3.1.3(iv) gives

$$\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau} = \mathbf{T}_{\tilde{\mathbf{G}}_2,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1^*,\tau} + \mathbf{H}^{+-}_{\tilde{\mathbf{G}}_2,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_1^*,\tau},$$

where the second term is compact by Assumption 2.2.3 and Lemma 2.1.1. Suppose that

$\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm. It follows from Lemma 3.1.2(iv) and the development above that

$$\begin{aligned}
\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}) &= \operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1^*,\tau}) \\
&:= \dim\ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_2,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1^*,\tau}\right) - \dim\ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_1,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2^*,\tau}\right) \\
&= \dim\left(\ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_2,\tau}\right)\cap\operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{G}}_1^*,\tau}\right)\right) - \dim\left(\ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_1,\tau}\right)\cap\operatorname{img}\left(\mathbf{T}_{\tilde{\mathbf{G}}_2^*,\tau}\right)\right) \\
&= \dim\left(\mathscr{G}^{\tau}_{\mathbf{P}_2}\cap\mathscr{G}^{\tau\,\perp}_{\mathbf{P}_1}\right) - \dim\left(\mathscr{G}^{\tau}_{\mathbf{P}_1}\cap\mathscr{G}^{\tau\,\perp}_{\mathbf{P}_2}\right) \\
&= \dim\left(\ker\left(\mathbf{T}_{\mathbf{G}_1^*,\tau}\right)\cap\operatorname{img}\left(\mathbf{T}_{\mathbf{G}_2,\tau}\right)\right) - \dim\left(\ker\left(\mathbf{T}_{\mathbf{G}_2^*,\tau}\right)\cap\operatorname{img}\left(\mathbf{T}_{\mathbf{G}_1,\tau}\right)\right) \\
&= \dim\ker\left(\mathbf{T}_{\mathbf{G}_1^*,\tau}\mathbf{T}_{\mathbf{G}_2,\tau}\right) - \dim\ker\left(\mathbf{T}_{\mathbf{G}_2^*,\tau}\mathbf{T}_{\mathbf{G}_1,\tau}\right) \\
&=: -\operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*,\tau}\mathbf{T}_{\mathbf{G}_1,\tau}) = -\operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}),
\end{aligned}$$

where causality of $\mathbf{G}_1$ here and Lemma 3.1.3(iv) have been exploited in the last equality. It is clear that $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ is Fredholm, as required.

Conversely, starting with the assumption that $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ is Fredholm and reversing the line of argument above, we have $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm and $\operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = -\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})$. □

Motivated by Theorem 3.3.1 and Lemma 3.3.3 , we have the following definition.

Definition 3.3.4. The ν-gap function $\delta_\nu : \mathbb{S}\times\mathbb{S}\to[0,1]$ is defined as

$$\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) := \begin{cases} \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) & \text{if } \mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau} \text{ is Fredholm and } \operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = 0 \text{ for all } \tau\in\mathbb{R} \\ \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 3.3.5. *The ν-gap δ_ν is well-defined in the sense that it is independent of the choice of* normalised *graph symbols.*

Proof. Suppose we have two sets of normalised graph symbols for $\mathbf{P}_i$, denoted $\{\mathbf{G}_i, \tilde{\mathbf{G}}_i\}$ and $\{\mathbf{\Gamma}_i, \tilde{\mathbf{\Gamma}}_i\}$, where $i\in\{1,2\}$. Note that by Remark 3.2.13 there exist unitary memoryless bounded operators $\mathbf{Q}_i$ and $\tilde{\mathbf{Q}}_i$ such that

$$\mathbf{\Gamma}_i = \mathbf{G}_i\mathbf{Q}_i \quad \text{and} \quad \tilde{\mathbf{\Gamma}}_i = \tilde{\mathbf{Q}}_i\tilde{\mathbf{G}}_i,$$

from which it follows that

$$\bar{\gamma}(\tilde{\mathbf{\Gamma}}_2\mathbf{\Gamma}_1) = \bar{\gamma}(\tilde{\mathbf{Q}}_2\tilde{\mathbf{G}}_2\mathbf{G}_1\mathbf{Q}_1) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1).$$

It remains to show that for any $\tau\in\mathbb{R}$, $\mathbf{T}_{\tilde{\mathbf{\Gamma}}_2\tilde{\mathbf{\Gamma}}_1^*,\tau}$ is Fredholm if, and only if, $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$

is, and when this is the case, that $\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{\Gamma}}_2\tilde{\mathbf{\Gamma}}_1^*,\tau}) = \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau})$. First recall from Remark 3.2.13 that $\tilde{\mathbf{Q}}_i$ are memoryless, whereby use of Lemma 3.1.3(iv) implies $\mathbf{T}^{-1}_{\tilde{\mathbf{Q}}_i,\tau} = \mathbf{T}_{\tilde{\mathbf{Q}}_i^*,\tau}$, and hence $\mathbf{T}_{\tilde{\mathbf{Q}}_i,\tau}$ is Fredholm with $\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{Q}}_i,\tau}) = \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{Q}}_i^*,\tau}) = 0$. Furthermore, $\mathbf{T}_{\tilde{\mathbf{\Gamma}}_2\tilde{\mathbf{\Gamma}}_1^*,\tau} = \mathbf{T}_{\tilde{\mathbf{Q}}_2,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{T}_{\tilde{\mathbf{Q}}_1^*,\tau}$ by the second part of Lemma 3.1.3(iv). Thus, when $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm, we have by Lemma 3.1.2(ii) that $\mathbf{T}_{\tilde{\mathbf{\Gamma}}_2\tilde{\mathbf{\Gamma}}_1^*,\tau}$ is Fredholm and

$$\begin{aligned}
\mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{\Gamma}}_2\tilde{\mathbf{\Gamma}}_1^*,\tau}) &= \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{Q}}_2,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}\mathbf{T}_{\tilde{\mathbf{Q}}_1^*,\tau}) \\
&= \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{Q}}_2,\tau}) + \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}) + \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{Q}}_1^*,\tau}) \\
&= \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}).
\end{aligned}$$

The converse can be established by assuming $\mathbf{T}_{\tilde{\mathbf{\Gamma}}_2\tilde{\mathbf{\Gamma}}_1^*,\tau}$ is Fredholm and reversing the line of argument above. □

It follows from Theorem 3.3.1 and Lemma 3.3.3 that for any $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathbb{S}$ such that $b_{\mathbf{P}_1,\mathbf{C}} > \delta_\nu(\mathbf{P}_1, \mathbf{P}_2)$, $[\mathbf{P}_2, \mathbf{C}]$ is stable. Moreover, when $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < 1$, it is necessarily true that $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)$. It is illustrated in Section 4.6.2 of the next chapter that in the *time-invariant* case, the above definition reduces to the well-known Vinnicombe's ν-gap metric for frequency-domain objects [Vin93, Vin01, CJK12, CJK10, CJK09].

The following corollary provides a lower bound on the residual robust performance margin that can be guaranteed when a $\mathbf{C}$ that achieves a certain $b_{\mathbf{P}_1,\mathbf{C}}$ is connected in feedback with $\mathbf{P}_2$, which may represent a perturbed version of $\mathbf{P}_1$. Its time-invariant version can be found in [Vin93, Thm. 4.2].

Corollary 3.3.6. *For any* $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathbb{S}$,

$$\arcsin b_{\mathbf{P}_2,\mathbf{C}} \geq \arcsin b_{\mathbf{P}_1,\mathbf{C}} - \arcsin \delta_\nu(\mathbf{P}_1, \mathbf{P}_2),$$

which implies the weaker inequality $b_{\mathbf{P}_2,\mathbf{C}} \geq b_{\mathbf{P}_1,\mathbf{C}} - \delta_\nu(\mathbf{P}_1, \mathbf{P}_2)$.

Proof. The result is trivially true when $[\mathbf{P}_1, \mathbf{C}]$ is unstable or $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) \geq b_{\mathbf{P}_1,\mathbf{C}}$. Therefore, suppose that the converse is true, by which $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < b_{\mathbf{P}_1,\mathbf{C}} \leq 1$, and therefore $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)$. It follows by Definition 3.3.4, Theorem 3.3.1, and Lemma 3.3.3 that $[\mathbf{P}_2, \mathbf{C}]$ is stable, and as a consequence the only step left now is to establish the bound on $b_{\mathbf{P}_2,\mathbf{C}}$. Towards this end, we make use of (3.20), which yields

$$\begin{aligned}
b_{\mathbf{P}_2,\mathbf{C}} = \underline{\gamma}(\tilde{\mathbf{G}}_2\mathbf{K}) &\geq \underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*)\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) - \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)\bar{\gamma}(\mathbf{G}_1^*\mathbf{K}) \\
&= \sqrt{1-\delta_\nu(\mathbf{P}_1,\mathbf{P}_2)^2}\, b_{\mathbf{P}_1,\mathbf{C}} - \delta_\nu(\mathbf{P}_1,\mathbf{P}_2)\sqrt{1-b^2_{\mathbf{P}_2,\mathbf{C}}} \\
&= \cos(\arcsin\delta_\nu(\mathbf{P}_1,\mathbf{P}_2))\sin(\arcsin b_{\mathbf{P}_1,\mathbf{C}}) \\
&\qquad - \sin(\arcsin\delta_\nu(\mathbf{P}_1,\mathbf{P}_2))\cos(\arcsin b_{\mathbf{P}_2,\mathbf{C}}) \\
&= \sin(\arcsin b_{\mathbf{P}_1,\mathbf{C}} - \arcsin\delta_\nu(\mathbf{P}_1,\mathbf{P}_2)),
\end{aligned}$$

where the second equality follows from (3.13) and (3.15). The proof is completed by taking arcsin of both sides and noting that $\arcsin(\cdot)$ is monotonically increasing. $\square$

3.3.2 Properties of the generalised ν-gap metric

Here we establish various properties of the ν-gap, including proofs for the claim that it is a metric and that it induces the coarsest topology under which closed-loop stability and performance are robust properties.

We first prove the metric property of $\delta_\nu(\cdot,\cdot)$ following the LTI development in [Vin93, Vin01]. Recall from Definition 3.2.1 that stability of a feedback interconnection is determined only by the behaviour of its subsystems on $\boldsymbol{L}^{2+}$. Furthermore, recall from Sections 2.3, 2.4, and 2.5 that given time-varying state-space realisations or frequency domain symbols, the construction of $\mathbf{P} : \mathrm{dom}\,(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ is such that $\mathrm{dom}\,(\mathbf{P}) \subset \boldsymbol{L}^{2+}$ and so $\mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^{2+} = \mathscr{G}_{\mathbf{P}}$. By virtue of these, we identify operators having the same behaviour on $\boldsymbol{L}^{2+}$:

$$\mathbf{P}_1 = \mathbf{P}_2 \text{ if } \mathscr{G}_{\mathbf{P}_1} \cap \boldsymbol{L}^{2+} = \mathscr{G}_{\mathbf{P}_2} \cap \boldsymbol{L}^{2+}. \tag{3.22}$$

This identification is used in the proof of part (b) of the theorem below.

Theorem 3.3.7. *$\delta_\nu(\cdot,\cdot)$ is a metric on $\mathbb{S}$.*

Proof. We need to show the following for any $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3 \in \mathbb{S}$:

(a) $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) = \delta_\nu(\mathbf{P}_2,\mathbf{P}_1)$ (symmetry);

(b) $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) \geq 0$, with $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) = 0$ if, and only if, $\mathbf{P}_1 = \mathbf{P}_2$ (positive definiteness);

(c) $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) \leq \delta_\nu(\mathbf{P}_1,\mathbf{P}_3) + \delta_\nu(\mathbf{P}_2,\mathbf{P}_3)$ (triangle inequality).

Throughout, let $\mathbf{G}_i$ and $\tilde{\mathbf{G}}_i$ denote right and left normalised graph symbols for $\mathbf{P}_i$, respectively.

(a): From (3.12), we have $\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) = \bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_2)$. Furthermore, by Lemma 3.1.2(i), $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ is Fredholm if, and only if, $\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau}$ is, with $\mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = -\mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau})$ for all $\tau \in \mathbb{R}$. Symmetry therefore holds by the definition of $\delta_\nu(\cdot,\cdot)$.

(b): That $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) \geq 0$ is obvious. When $\mathbf{P}_1 = \mathbf{P}_2$, $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) = 0$ since $\mathbf{G}_1^*\mathbf{G}_1 = \mathbf{I}$ and $\tilde{\mathbf{G}}_1\mathbf{G}_1 = 0$ by Assumptions 2.2.1 and 2.2.2. Now suppose $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) = 0$. Note that this implies $\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) = \bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_2) = 0$ and hence $\tilde{\mathbf{G}}_2\mathbf{G}_1 = \tilde{\mathbf{G}}_1\mathbf{G}_2 = 0$. As such, by Lemma 3.1.3(iv), $\mathbf{T}_{\tilde{\mathbf{G}}_2,\tau}\mathbf{T}_{\mathbf{G}_1,\tau} = \mathbf{T}_{\tilde{\mathbf{G}}_2\mathbf{G}_1,\tau} = 0$ and $\mathbf{T}_{\tilde{\mathbf{G}}_1,\tau}\mathbf{T}_{\mathbf{G}_2,\tau} = \mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{G}_2,\tau} = 0$ for all $\tau \in \mathbb{R}$, and thus

$$\mathrm{img}\left(\mathbf{T}_{\mathbf{G}_1,\tau}\right) \subset \ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_2,\tau}\right) \quad \text{and} \quad \mathrm{img}\left(\mathbf{T}_{\mathbf{G}_2,\tau}\right) \subset \ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_1,\tau}\right).$$

But $\mathscr{G}_{\mathbf{P}_1} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \mathrm{img}\left(\mathbf{T}_{\mathbf{G}_1,\tau}\right) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_1,\tau}\right)$ and $\mathscr{G}_{\mathbf{P}_2} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \mathrm{img}\left(\mathbf{T}_{\mathbf{G}_2,\tau}\right) = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_2,\tau}\right)$, implying that $\mathscr{G}_{\mathbf{P}_1} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \mathscr{G}_{\mathbf{P}_2} \cap \boldsymbol{L}^2_{[\tau,\infty)}$ for all $\tau \in \mathbb{R}$. Consequently, $\mathbf{P}_1 = \mathbf{P}_2$ by the identification in (3.22).

(c): First note that by Definition 3.3.4, it is always the case that $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) \leq 1$, hence the triangle inequality holds vacuously if $\delta_\nu(\mathbf{P}_1,\mathbf{P}_3)+\delta_\nu(\mathbf{P}_2,\mathbf{P}_3) \geq 1$. Thus, suppose that

$$\delta_\nu(\mathbf{P}_1,\mathbf{P}_3) + \delta_\nu(\mathbf{P}_2,\mathbf{P}_3) < 1. \tag{3.23}$$

This implies $\delta_\nu(\mathbf{P}_1,\mathbf{P}_3) < 1$ and $\delta_\nu(\mathbf{P}_2,\mathbf{P}_3) < 1$, whereby $\delta_\nu(\mathbf{P}_1,\mathbf{P}_3) = \bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1)$ and $\delta_\nu(\mathbf{P}_2,\mathbf{P}_3) = \bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_2)$, i.e.

$$\mathbf{T}_{\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*,\tau} \text{ and } \mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*,\tau} \text{ are Fredholm with } \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*,\tau}) = \mathrm{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*,\tau}) = 0\,\forall\tau \in \mathbb{R} \tag{3.24}$$

by Definition 3.3.4 and Lemma 3.3.3.

Now notice that (3.11) gives

$$\mathbf{G}_2^*\mathbf{G}_1 = \mathbf{G}_2^*\left(\mathbf{G}_3\mathbf{G}_3^* + \tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3\right)\mathbf{G}_1 = \mathbf{G}_2^*\mathbf{G}_3\mathbf{G}_3^*\mathbf{G}_1 + \mathbf{G}_2^*\tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3\mathbf{G}_1,$$

whereby

$$\begin{aligned}
\underline{\gamma}(\mathbf{G}_2^*\mathbf{G}_1) &\geq \underline{\gamma}(\mathbf{G}_2^*\mathbf{G}_3\mathbf{G}_3^*\mathbf{G}_1) - \bar{\gamma}(\mathbf{G}_2^*\tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3\mathbf{G}_1) \\
&\geq \underline{\gamma}(\mathbf{G}_3^*\mathbf{G}_2)\underline{\gamma}(\mathbf{G}_3^*\mathbf{G}_1) - \bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_2)\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1) \\
\iff \sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_2\mathbf{G}_1)} &\geq \sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_2)}\sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_1)} - \bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_2)\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1) \\
\iff \bar{\gamma}^2(\tilde{\mathbf{G}}_2\mathbf{G}_1) &\leq \bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_2) + \bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_1) + 2\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_2)\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1)\sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_2\mathbf{G}_1)} \\
\implies \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) &\leq \bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1) + \bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_2) = \delta_\nu(\mathbf{P}_1,\mathbf{P}_3) + \delta_\nu(\mathbf{P}_2,\mathbf{P}_3).
\end{aligned}$$

where (3.13) has been used to arrive at the first equivalence relation. To complete the proof, we show below that $\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm with $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau}) = 0$ for all $\tau \in \mathbb{R}$, whereby $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)$.

First observe that the condition (3.23) implies $\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1) + \bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_2) < 1$, whereby

$$\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1)\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_2) < \sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_1)}\sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_2)}.$$

Combining the above with (3.12) and (3.13) yields

$$\begin{aligned}
\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_3\mathbf{G}_3^*\tilde{\mathbf{G}}_1^*) \leq \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_3)\bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_3) &= \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_3)\bar{\gamma}(\tilde{\mathbf{G}}_3\mathbf{G}_1) \\
&< \sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_2)}\sqrt{1-\bar{\gamma}^2(\tilde{\mathbf{G}}_3\mathbf{G}_1)} \\
&= \underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*)\underline{\gamma}(\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*) \leq \underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*).
\end{aligned} \tag{3.25}$$

Now using (3.11), one obtains

$$\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^* = \tilde{\mathbf{G}}_2\left(\tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3 + \mathbf{G}_3\mathbf{G}_3^*\right)\tilde{\mathbf{G}}_1^* = \tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^* + \tilde{\mathbf{G}}_2\mathbf{G}_3\mathbf{G}_3^*\tilde{\mathbf{G}}_1^*. \tag{3.26}$$

Since according to (3.25), $\underline{\gamma}(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*) > 0$ and $\underline{\gamma}(\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*) > 0$, we have by Lemma 3.2.17 that $\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*$ and $\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*$ are boundedly invertible. As such, (3.26) becomes

$$\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^* = (\mathbf{I}+\mathbf{X})\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^* \quad \text{with} \quad \mathbf{X} := \tilde{\mathbf{G}}_2\mathbf{G}_3\mathbf{G}_3^*\tilde{\mathbf{G}}_1^*(\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*)^{-1}.$$

Applying Lemma 3.1.3(iv) twice yields for any $\tau \in \mathbb{R}$,

$$\begin{aligned}
\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_1^*,\tau} &= \mathbf{T}_{(\mathbf{I}+\mathbf{X})\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*,\tau} + \mathbf{H}^{+-}_{(\mathbf{I}+\mathbf{X})\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*,\tau} \\
&= \mathbf{T}_{(\mathbf{I}+\mathbf{X}),\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*,\tau} + \mathbf{H}^{+-}_{(\mathbf{I}+\mathbf{X}),\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*,\tau} + \mathbf{H}^{+-}_{(\mathbf{I}+\mathbf{X})\tilde{\mathbf{G}}_2\tilde{\mathbf{G}}_3^*,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_3\tilde{\mathbf{G}}_1^*,\tau}.
\end{aligned} \tag{3.27}$$

Note that through (3.25),

$$\bar{\gamma}(\mathbf{T}_{\mathbf{X},\tau}) \leq \bar{\gamma}(\mathbf{X}) \leq \frac{\bar{\gamma}(\tilde{\mathbf{G}}_2 \mathbf{G}_3 \mathbf{G}_3^* \tilde{\mathbf{G}}_1^*)}{\underline{\gamma}(\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^* \tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*} < 1.$$

Thus by Lemma 3.1.2(iii), $\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm with $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_1^*,\tau}) = \operatorname{ind}(\mathbf{T}_{\mathbf{I}+\mathbf{X},\tau}) = \operatorname{ind}(\mathbf{I} + \mathbf{T}_{\mathbf{X},\tau}) = \operatorname{ind}(\mathbf{I}) = 0$. By (3.24) and Lemma 3.1.2(ii), this means the operator $\mathbf{T}_{(\mathbf{I}+\mathbf{X}),\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm with

$$\operatorname{ind}(\mathbf{T}_{(\mathbf{I}+\mathbf{X}),\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*,\tau}) = \operatorname{ind}(\mathbf{T}_{(\mathbf{I}+\mathbf{X}),\tau}) + \operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^*,\tau}) + \operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*,\tau}) = 0.$$

Note that $\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^*,\tau}$ and $\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*,\tau}$ are compact by Assumption 2.2.3 and Lemma 3.2.14. Thus, by Lemma 2.1.1, $\mathbf{H}^{+-}_{(\mathbf{I}+\mathbf{X}),\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*,\tau}$ and $\mathbf{H}^{+-}_{(\mathbf{I}+\mathbf{X})\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^*,\tau}\mathbf{H}^{-+}_{\tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*,\tau}$ are compact. Since the sum of two compact operators is compact [Kat80, Thm. III.4.7], applying Lemma 3.1.2(iv) to (3.27) then yields $\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_1^*,\tau}$ is Fredholm with

$$\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_1^*,\tau}) = \operatorname{ind}(\mathbf{T}_{(\mathbf{I}+\mathbf{X}),\tau}\mathbf{T}_{\tilde{\mathbf{G}}_2 \tilde{\mathbf{G}}_3^*,\tau}\mathbf{T}_{\tilde{\mathbf{G}}_3 \tilde{\mathbf{G}}_1^*,\tau}) = 0,$$

as required. □

In the following theorem, we bound the errors in the closed-loop mappings due to perturbations of $\mathbf{P}$ in terms of the ν-gap measure of distance between the nominal and perturbed systems; see [Vin01, Thm. 3.19] for a time-invariant equivalent. This leads to useful topological properties of the ν-gap metric. A similar result in terms of the standard gap metric can be found in [CV02, Thm. III.2].

Theorem 3.3.8. *For any $\mathbf{P}_1, \mathbf{P}_2, \mathbf{C} \in \mathbb{S}$ such that $[\mathbf{P}_1, \mathbf{C}]$ and $[\mathbf{P}_2, \mathbf{C}]$ are stable,*

$$\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) \leq \sup_{\tau \in \mathbb{R}} \bar{\gamma}\left(\mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_1} \| \mathscr{G}'^\tau_{\mathbf{C}}} - \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_2} \| \mathscr{G}'^\tau_{\mathbf{C}}}\right) \leq \frac{\delta_\nu(\mathbf{P}_1, \mathbf{P}_2)}{b_{\mathbf{P}_1,\mathbf{C}} b_{\mathbf{P}_2,\mathbf{C}}}. \tag{3.28}$$

Proof. Let $\mathbf{\Delta}_\tau := \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_2} \| \mathscr{G}'^\tau_{\mathbf{C}}} - \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_1} \| \mathscr{G}'^\tau_{\mathbf{C}}}$. We show below that $\mathbf{\Delta}_\tau = \mathbf{\Pi}_{\mathscr{G}'^\tau_{\mathbf{C}} \| \mathscr{G}^\tau_{\mathbf{P}_1}} \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_2} \| \mathscr{G}'^\tau_{\mathbf{C}}}$ as in the proof for [CV02, Thm III.2]. Given any $u \in \boldsymbol{L}^2_{[\tau,\infty)}$, let

$$v_i := \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}_i} \| \mathscr{G}'^\tau_{\mathbf{C}}} u \quad \text{and} \quad w_i := \mathbf{\Pi}_{\mathscr{G}'^\tau_{\mathbf{C}} \| \mathscr{G}^\tau_{\mathbf{P}_i}} u \quad (i = 1, 2),$$

so that $u = v_1 + w_1 = v_2 + w_2$. It follows that

$$\mathbf{\Delta}_\tau u = v_1 - v_2 = w_1 - w_2 =: w \in \mathscr{G}'^\tau_{\mathbf{C}}.$$

Consequently, as $\mathbf{\Pi}_{\mathscr{G}'^{\tau}_{\mathbf{C}}\|\mathscr{G}^{\tau}_{\mathbf{P}_1}} w = w$ and u was arbitrary, we have that

$$\mathbf{\Delta}_\tau = \mathbf{\Pi}_{\mathscr{G}'^{\tau}_{\mathbf{C}}\|\mathscr{G}^{\tau}_{\mathbf{P}_1}} \mathbf{\Delta}_\tau = \mathbf{\Pi}_{\mathscr{G}'^{\tau}_{\mathbf{C}}\|\mathscr{G}^{\tau}_{\mathbf{P}_1}} \mathbf{\Pi}_{\mathscr{G}^{\tau}_{\mathbf{P}_2}\|\mathscr{G}'^{\tau}_{\mathbf{C}}},$$

where the last equality holds because $\mathbf{\Pi}_{\mathscr{G}'^{\tau}_{\mathbf{C}}\|\mathscr{G}^{\tau}_{\mathbf{P}_1}} \mathbf{\Pi}_{\mathscr{G}^{\tau}_{\mathbf{P}_1}\|\mathscr{G}'^{\tau}_{\mathbf{C}}} = 0$. From Lemma 3.2.15, we then have for any $\tau \in \mathbb{R}$,

$$\begin{aligned}
\mathbf{\Delta}_\tau &= \mathbf{T}_{\mathbf{G}_2,\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G}_2,\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau} - \mathbf{T}_{\mathbf{G}_1,\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G}_1,\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau} \\
&= \mathbf{T}_{\mathbf{K},\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1,\tau}\mathbf{T}_{\mathbf{G}_2,\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G}_2,\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau} \\
&= \mathbf{T}_{\mathbf{K},\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau}\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{G}_2,\tau}\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G}_2,\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau},
\end{aligned} \tag{3.29}$$

where the last equality holds by Lemma 3.1.3(iv) and causality of $\mathbf{G}_2$. It follows immediately that

$$\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{\Delta}_\tau) \leq \sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{T}^{-1}_{\tilde{\mathbf{G}}_1\mathbf{K},\tau})\bar{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{G}_2,\tau})\bar{\gamma}(\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G}_2,\tau}) \leq \frac{\delta_\nu(\mathbf{P}_1,\mathbf{P}_2)}{b_{\mathbf{P}_1,\mathbf{C}}b_{\mathbf{P}_2,\mathbf{C}}},$$

i.e. the upper bound in (3.28) holds.

To establish the lower bound, first define

$$\mathbf{Q} := (\tilde{\mathbf{K}}\mathbf{G}_2)^{-1}\tilde{\mathbf{K}}\mathbf{G}_1,$$

so that $\mathbf{Q}, \mathbf{Q}^{-1} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ are both causal; see Theorem 3.2.19. Then it follows from (3.29) that

$$\mathbf{\Delta}_\tau = (\mathbf{T}_{\mathbf{G}_2,\tau}\mathbf{T}_{\mathbf{Q},\tau} - \mathbf{T}_{\mathbf{G}_1,\tau})\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G}_1,\tau}\mathbf{T}_{\tilde{\mathbf{K}},\tau}.$$

Noting that $\bar{\gamma}(\mathbf{T}_{\tilde{\mathbf{K}}\mathbf{G}_1,\tau}) \leq \bar{\gamma}(\tilde{\mathbf{K}}\mathbf{G}_1) \leq 1$, and so $\underline{\gamma}(\mathbf{T}^{-1}_{\tilde{\mathbf{K}}\mathbf{G}_1,\tau}) \geq 1$, one gets

$$\begin{aligned}
\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{\Delta}_\tau) &\geq \sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{T}_{\mathbf{G}_1,\tau} - \mathbf{T}_{\mathbf{G}_2,\tau}\mathbf{T}_{\mathbf{Q},\tau}) \\
&= \bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) \\
&= \bar{\gamma}\left(\begin{bmatrix} \mathbf{G}_2^*\mathbf{G}_1 - \mathbf{Q} \\ \tilde{\mathbf{G}}_2\mathbf{G}_1 \end{bmatrix}\right) \geq \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1),
\end{aligned} \tag{3.30}$$

where the first equality follows from Lemma 3.1.3(v) and the second from the fact that $\begin{bmatrix} \mathbf{G}_2^* \\ \tilde{\mathbf{G}}_2 \end{bmatrix}$ is an isometry; see (3.11). It remains to confirm the Fredholm index condition in the definition of the ν-gap metric. First observe that if $\bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) \geq 1$, then from (3.30) we have $\sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{\Delta}_\tau) \geq 1$ and as a result the lower bound in (3.28) holds trivially.

So we consider the case where $\bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) < 1$, which in turn implies

$$\bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q}) = \bar{\gamma}(\mathbf{G}_1^*(\mathbf{G}_1 - \mathbf{G}_2\mathbf{Q})) = \bar{\gamma}(\mathbf{I} - \mathbf{G}_1^*\mathbf{G}_2\mathbf{Q}) < 1.$$

Consequently, as $\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q} = \mathbf{I} - (\mathbf{I} - \mathbf{G}_1^*\mathbf{G}_2\mathbf{Q})$, we have

$$\underline{\gamma}(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q}) > 0 \quad \text{and} \quad \mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q},\tau} \text{ is Fredholm with } \operatorname{ind}(\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q},\tau}) = 0\,\forall\tau \in \mathbb{R},$$

where the latter holds by Lemma 3.1.2(iii). Because $\mathbf{Q}$ has a bounded causal inverse, $\underline{\gamma}(\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q}) > 0$ implies $\underline{\gamma}(\mathbf{G}_1^*\mathbf{G}_2) > 0$. Moreover, $\mathbf{T}_{\mathbf{Q},\tau}$ is boundedly invertible, hence for all $\tau \in \mathbb{R}$, $\operatorname{ind}(\mathbf{T}_{\mathbf{Q},\tau}) = 0$ (cf. Remark 3.1.5) and Fredholmness of $\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q},\tau} = \mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau}\mathbf{T}_{\mathbf{Q},\tau}$ implies that of $\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau}$. Applying Lemma 3.1.2(ii) again, it follows that

$$0 = \operatorname{ind}(\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2\mathbf{Q},\tau}) = \operatorname{ind}(\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau}) + \operatorname{ind}(\mathbf{T}_{\mathbf{Q},\tau}) = \operatorname{ind}(\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau})\,\forall\tau \in \mathbb{R}.$$

Application of Lemma 3.1.2(i) then yields $\operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = -\operatorname{ind}(\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau}) = 0\,\forall\tau \in \mathbb{R}$, whereby $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1)$, as required for the lower bound in (3.28) to hold via (3.30). □

In the sequel, the topology generated by the ν-gap metric is termed the *graph topology*, in line with the existing literature [Vin93, Vin01, You98, Vid84, FGS93, CV02]. The bounds in Theorem 3.3.8 facilitate simple and direct proofs of the following properties of the graph topology. Specifically, the graph topology is the weakest topology with respect to which both feedback stability and performance are robust properties. The previous work in [CV02] establishes a similar result for the standard gap metric (i.e. only semi-infinite signal space with a fixed 'initial' time) in a potentially LTV setting, while [Vin01, Chapter 7] does so for the ν-gap metric in the time-invariant setting. The proofs of the following corollaries are based on these references.

Corollary 3.3.9. *Given $\mathbf{P}$ and a sequence $\{\mathbf{P}_i\}$ in $\mathbb{S}$, the following are equivalent:*

(i) $\delta_\nu(\mathbf{P}, \mathbf{P}_i) \to 0$*;*

(ii) *For any $\mathbf{C}$ such that $[\mathbf{P}, \mathbf{C}]$ is stable, $\mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}_i}^\tau \| \mathscr{G}_{\mathbf{C}}'^\tau} \to \mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}}^\tau \| \mathscr{G}_{\mathbf{C}}'^\tau}$ uniformly in $\tau \in \mathbb{R}$.*

Proof. That (ii) implies (i) follows from the lower bound in Theorem 3.3.8 while the converse implication from the upper bound and the robustness result Corollary 3.3.6. □

Corollary 3.3.10. *When $\mathbb{S}$ is equipped with the graph topology, for any $\mathbf{C} \in \mathbb{S}$, the mapping $\mathbf{X} := \mathbf{P} \in \mathbb{S} \mapsto \mathbf{\Pi}_{\mathscr{G}_{\mathbf{P}} \| \mathscr{G}_{\mathbf{C}}'}$ is continuous at all points $\mathbf{P}$ in $\mathscr{S}(\mathbf{C}) := \{\mathbf{P} \in \mathbb{S} : [\mathbf{P}, \mathbf{C}]$ is stable$\}$. Furthermore, the graph topology is the weakest topology on $\mathbb{S}$ for which this holds.*

Proof. Corollary 3.3.6 and the upper bound in Theorem 3.3.8 guarantee that given any $\mathbf{P} \in \mathscr{S}(\mathbf{C})$ and $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\delta_\nu(\mathbf{P}, \tilde{\mathbf{P}}) < \delta \implies [\tilde{\mathbf{P}}, \mathbf{C}] \text{ is stable and } \sup_{\tau \in \mathbb{R}} \bar{\gamma} \left(\mathbf{\Pi}_{\mathscr{G}^\tau_{\tilde{\mathbf{P}}} \| \mathscr{G}'^\tau_{\mathbf{C}}} - \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}} \| \mathscr{G}'^\tau_{\mathbf{C}}} \right) < \epsilon,$$

from which continuity of $\mathbf{X}$ follows. For the second part, suppose $\mathscr{T}$ is a topology having the property under consideration. This implies for any $\mathbf{P} \in \mathscr{S}(\mathbf{C})$ and $\epsilon > 0$, there exists an open neighbourhood of $\mathbf{P}$, denoted $\mathscr{N}(\mathbf{P}, \epsilon) \in \mathscr{T}$, such that

$$[\tilde{\mathbf{P}}, \mathbf{C}] \text{ is stable and } \sup_{\tau \in \mathbb{R}} \bar{\gamma} \left(\mathbf{\Pi}_{\mathscr{G}^\tau_{\tilde{\mathbf{P}}} \| \mathscr{G}'^\tau_{\mathbf{C}}} - \mathbf{\Pi}_{\mathscr{G}^\tau_{\mathbf{P}} \| \mathscr{G}'^\tau_{\mathbf{C}}} \right) < \epsilon, \forall \tilde{\mathbf{P}} \in \mathscr{N}(\mathbf{P}, \epsilon).$$

By the lower bound in Theorem 3.3.8, it follows that

$$\delta_\nu(\mathbf{P}, \tilde{\mathbf{P}}) < \epsilon \, \forall \tilde{\mathbf{P}} \in \mathscr{N}(\mathbf{P}, \epsilon),$$

i.e. $\mathscr{N}(\mathbf{P}, \epsilon) \subset \mathbb{B}_\epsilon(\mathbf{P}) := \{\tilde{\mathbf{P}} \in \mathbb{S} : \delta_\nu(\mathbf{P}, \tilde{\mathbf{P}}) < \epsilon\}$. In other words, any set that is open with respect to the graph topology is also open with respect to the topology $\mathscr{T}$. $\square$

3.4 Robustness analysis for periodic systems

The previous section presented sufficient conditions for robust feedback stability in terms of the ν-gap metric. Here we derive a necessary condition which is analogous to the first part of [Vin01, Thm. 3.10]. Since the proof of this result relies on explicit construction of a perturbed system, we depart from the purely abstract setting and focus in this section on the class of linear periodically time-varying (LPTV) systems $\mathbb{P}_{HS} \subset \mathbb{P}_{CF}$ introduced in Section 2.5.

3.4.1 The ν-gap metric for periodic systems

The same notation from Section 2.5 is used here. Given $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{P}_{CF}$, by the h-periodicity property of the operators, $\mathbf{S}_{kh}\mathbf{G}_2^*\mathbf{G}_1 = \mathbf{G}_2^*\mathbf{G}_1\mathbf{S}_{kh} \, \forall k \in \mathbb{Z}$, where $\mathbf{S}_\tau$ denotes the continuous-time shift operator and $\mathbf{G}_1, \mathbf{G}_2 \in \mathbb{P}$ are right graph symbols for $\mathbf{P}_1$ and

$\mathbf{P}_2$, respectively. Consequently, for all $\tau \in \mathbb{R}$ and $k \in \mathbb{Z}$,

$$\begin{aligned}\mathbf{S}_{-kh}\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau+kh}\mathbf{S}_{kh}|_{\boldsymbol{L}^2_{[\tau,\infty)}} &= \mathbf{S}_{-kh}(\mathbf{I}-\mathbf{\Pi}_{\tau+kh})\mathbf{G}_2^*\mathbf{G}_1|_{\boldsymbol{L}^2[\tau+kh,\infty)}\mathbf{S}_{kh}|_{\boldsymbol{L}^2_{[\tau,\infty)}} \\ &= (\mathbf{I}-\mathbf{\Pi}_\tau)\mathbf{S}_{-kh}\mathbf{G}_2^*\mathbf{G}_1\mathbf{S}_{kh}|_{\boldsymbol{L}^2_{[\tau,\infty)}} \\ &= (\mathbf{I}-\mathbf{\Pi}_\tau)\mathbf{G}_2^*\mathbf{G}_1|_{\boldsymbol{L}^2_{[\tau,\infty)}} \\ &= \mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}.\end{aligned} \tag{3.31}$$

Furthermore, note that as bijective mappings the restricted shift operators $\mathbf{S}_{-kh}|_{\boldsymbol{L}^2[\tau+kh,\infty)}$: $\boldsymbol{L}^2[\tau+kh,\infty) \to \boldsymbol{L}^2_{[\tau,\infty)}$ and $\mathbf{S}_{kh}|_{\boldsymbol{L}^2_{[\tau,\infty)}} : \boldsymbol{L}^2_{[\tau,\infty)} \to \boldsymbol{L}^2[\tau+kh)$ are both Fredholm with zero index. Thus, it follows by Lemma 3.1.2(ii) that $\mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = \mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau+kh})\,\forall\tau \in \mathbb{R}, k \in \mathbb{Z}$. Thus, for the system class $\mathbb{P}_{CF}$, Definition 3.3.4 of the ν-gap metric then simplifies to

$$\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) := \begin{cases} \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) = \|\tilde{G}_2 G_1\|_\infty & \text{if } \mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau} \text{ is Fredholm} \\ & \text{and } \mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = 0 \text{ for all } \tau \in [0,h) \\ 1 & \text{otherwise,} \end{cases} \tag{3.32}$$

where $\tilde{G}_2, \tilde{G}_1 \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ are respectively normalised left and right graph symbols for $P_2, P_1 \in \boldsymbol{\mathcal{L}}_{CF}$ as in Proposition 2.5.4.

3.4.2 A necessary condition for robust stability

The main result here characterises the maximal ν-gap metric ball of perturbations a nominal stable feedback system can tolerate before becoming unstable. It is analogous to the time-invariant case of [Vin01, Rem 3.11(i)]. The proof presented below borrows ideas from [CG00, Thm 4.2], in which the standard gap metric is studied within a largely different setting.

Theorem 3.4.1. *Given $\mathbf{P}_1, \mathbf{C} \in \mathbb{P}_{HS}$ and $\beta < b_{\mathrm{opt}}(\mathbf{P}_1;\mathbb{P}_{HS}) := \sup_{\mathbf{C}\in\mathbb{P}_{HS}\,:\,[\mathbf{P},\mathbf{C}] \text{ is stable}} b_{\mathbf{P},\mathbf{C}}$, $[\mathbf{P}_2,\mathbf{C}]$ is stable for all $\mathbf{P}_2 \in \mathbb{P}_{HS}$ satisfying $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) < \beta$ if, and only if, $b_{\mathbf{P}_1,\mathbf{C}} \geq \beta$.*

Proof. Sufficiency is immediate from Corollary 3.3.6. For the necessity proof, suppose to the contrary that $b_{\mathbf{P}_1,\mathbf{C}} < \beta$. We show below that it is possible to then construct a system $\mathbf{P}_2 \in \mathbb{P}_{HS}$ such that $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) < \beta$ and $[\mathbf{P}_2,\mathbf{C}]$ is unstable.

First note that the stability of $[\mathbf{P}_1,\mathbf{C}]$ is equivalent to $\tilde{\mathbf{K}}\mathbf{G}_1$ having a bounded causal inverse by Theorem 3.2.19, which in turn is equivalent to $(\tilde{K}G_1)^{-1} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$. In addition, we have $b_{\mathbf{P}_1,\mathbf{C}}^{-1} = \bar{\gamma}((\tilde{\mathbf{K}}\mathbf{G}_1)^{-1}) = \|(\tilde{K}G_1)^{-1}\|_\infty = \|(\tilde{K}G_1)^{-1}\tilde{K}\|_\infty$, where the last equality holds since $\tilde{K}$ is normalised, i.e. $\tilde{K}\tilde{K}^* = I$. Let

$$\Gamma := (\tilde{K}G_1)^{-1}\tilde{K} \in \boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}.$$

Since Γ is analytic in $\mathbb{D}$, for any $\epsilon_0 > 0$ and $\epsilon_1 > 0$, there exists by the maximum modulus principle a $z_0 \in \{z \in \mathbb{C} : (1 - \epsilon_0) \le z < 1\}$ such that

$$(b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1)^{-1} < \bar{\gamma}(\Gamma(z_0)) \le b_{\mathbf{P}_1,\mathbf{C}}^{-1}.$$

It follows by the definition of the induced norm that there exists $u \in \boldsymbol{L}_{[0,h)}^2$ such that $\|u\|_2 = 1$ and

$$\bar{\gamma}(\Gamma(z_0)) \ge \|\Gamma(z_0)u\|_2 \ge (b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1)^{-1}.$$

We define $\Delta_0 := \hat{C}B : \boldsymbol{L}_{[0,h)}^2 \to \boldsymbol{L}_{[0,h)}^2$, where $B \in \mathscr{L}(\boldsymbol{L}_{[0,h)}^2, \mathbb{C}^n)$ maps $\alpha\Gamma(z_0)u$ to $(-\alpha, 0, \cdots, 0)^T$ for all $\alpha \in \mathbb{C}$ and every element in $\{x \in \boldsymbol{L}_{[0,h)}^2 : \langle x, \alpha\Gamma(z_0)u\rangle_2 = 0; \alpha \in \mathbb{C}\}$ to 0, while $\hat{C} := (v_1, v_2, \cdots, v_n)^T \mapsto v_1 u \in \mathscr{L}(\mathbb{C}^n, \boldsymbol{L}_{[0,h)}^2)$, so that

$$\bar{\gamma}(\Delta_0) \le (b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1) \quad \text{and} \quad \Gamma(z_0)u \in \ker\left(I + \Gamma(z_0)\Delta_0\right).$$

As such, $I + \Gamma(z_0)\Delta_0$ is not invertible. Define

$$\Delta(z) := \frac{z}{z_0}\Delta_0 = (0, B, \hat{C}/z_0, 0) \in \boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty},$$

so that $\Delta(z_0) = \Delta_0$, whereby it is clear that $I + \Gamma\Delta$ is not invertible in $\boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}$. We set $\epsilon_1 := (\beta - b_{\mathbf{P}_1,\mathbf{C}})/2$ and $\epsilon_0 := \epsilon_1/2\beta$, so that

$$\|\Delta\|_\infty = \frac{1}{|z_0|}\bar{\gamma}(\Delta_0) \le \frac{b_{\mathbf{P}_1,\mathbf{C}} + \epsilon_1}{1 - \epsilon_0} < \beta.$$

Now define

$$\hat{G}_2 := G_1 + \Delta \in \boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}.$$

Since G_1 is normalised, i.e. $\left(G_1(\frac{1}{\bar{z}})\right)^* G_1(z) = I \,\forall z \in \mathbb{D} \cup \mathbb{T}$ and $\|\Delta\|_\infty < \beta < 1$, we have $\inf_{z\in\mathbb{T}} \underline{\gamma}(\hat{G}_2(z)) > 0$. Partition conformably $G_1 = \begin{bmatrix} N_1 \\ M_1 \end{bmatrix}$ and $\hat{G}_2 = \begin{bmatrix} \hat{N}_2 \\ \hat{M}_2 \end{bmatrix}$ and recall that by Definition 2.5.1, G_1 is left-invertible in $\boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}$. Given that Δ does not have a feedthrough 'D' term in its realisation, $\hat{G}_2$ and $\hat{M}_2$ have the same feedthrough terms as those of G_1 and M_1, respectively. Consequently, $\hat{M}_2$ is invertible in $\boldsymbol{\mathcal{L}}$ since M_1 is, and hence $\hat{G}_2$ is left-invertible in $\boldsymbol{\mathcal{L}}$. Let the Hardy space

$$\boldsymbol{H}_{\mathbb{D}}^{\infty} := \left\{ \Phi : \mathbb{D} \to \mathscr{L}(\boldsymbol{L}_{[0,h)}^2, \boldsymbol{L}_{[0,h)}^2) \;\middle|\; \begin{array}{l} \Phi \text{ is analytic in } \mathbb{D} \text{ and} \\ \|\Phi\|_\infty := \sup_{z\in\mathbb{D}} \bar{\gamma}(\Phi(z)) < \infty \end{array} \right\}.$$

We now show that there exists a $\hat{G}_2^L \in \boldsymbol{H}_{\mathbb{D}}^{\infty}$ such that $\hat{G}_2^L \hat{G}_2 = I$ on $\mathbb{D}$ (i.e. $\hat{G}_2$ is a legitimate graph symbol), by following the argument employed at the end of [Vin01, Pg. 144]. To this end, let $\mathbf{C}_1 \in \mathbb{P}_{HS}$ be such that $b_{\mathbf{P}_1,\mathbf{C}_1} > \beta$, which exists by hypothesis. Then

$$\begin{aligned}\inf_{z\in\mathbb{D}} \underline{\gamma}\left(\hat{G}_2(z)\right) \geq \frac{\inf_{z\in\mathbb{D}} \underline{\gamma}\left(\tilde{K}_1(z)\hat{G}_2(z)\right)}{\|\tilde{K}_1\|_\infty} &\geq \inf_{z\in\mathbb{D}} \underline{\gamma}\left(\tilde{K}_1(z)\hat{G}_2(z)\right) \\ &\geq \inf_{z\in\mathbb{D}} \underline{\gamma}\left(\tilde{K}_1(z)G_1(z)\right) - \|\tilde{K}_1\Delta\|_\infty \\ &= b_{\mathbf{P}_1,\mathbf{C}_1} - \beta > 0,\end{aligned}$$

where the fact that $\tilde{K}_1\tilde{K}_1^* = I$ has been used. As such, $\hat{G}(z)$ has a left bounded inverse for all $z \in \mathbb{D}$, and hence $\hat{G}_2$ has a left inverse in $\boldsymbol{H}_{\mathbb{D}}^{\infty}$, which we denote by $\hat{G}^L$. It follows that $\bigcup_{k\in\mathbb{Z}} \operatorname{img}\left(\boldsymbol{M}_{\hat{G}_2}|_{z^k \boldsymbol{H}_{\mathbb{D}}^2}\right)$ is the graph of a multiplication operator with symbol $P_2 := \hat{N}_2\hat{M}_2^{-1} \in \boldsymbol{\mathcal{L}}$; see Proposition 2.5.4. Note that P_2 has the same feedthrough term as that of $P_1 \in \boldsymbol{\mathcal{L}}_{HS}$, and hence $P_2 \in \boldsymbol{\mathcal{L}}_{HS}$.

We now proceed to show that $[\mathbf{P}_2, \mathbf{C}]$, where $\mathbf{P}_2 \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{P_2}$, is not a stable feedback interconnection. Let G_2 be a normalised right graph symbol for P_2 with a left inverse $G_2^L \in \boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}$, which exists by Proposition 2.5.2. Define $\hat{Q} := \hat{M}_2^{-1}M_2 \in \boldsymbol{\mathcal{L}}$, so that $\hat{Q}^{-1} = M_2^{-1}\hat{M}_2 \in \boldsymbol{\mathcal{L}}$ and $G_2 = \hat{G}_2\hat{Q}$. Furthermore, note that $\hat{Q} = \hat{G}_2^L G_2 \in \boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}$ and $\hat{Q}^{-1} = G_2^L\hat{G}_2 \in \boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}$. Now observe that

$$(\tilde{K}G_1)^{-1}\tilde{K}G_2 = (\tilde{K}G_1)^{-1}\tilde{K}\hat{G}_2\hat{Q} = (I + \Gamma\Delta)\hat{Q},$$

from which it follows that $\tilde{K}G_2$ is not invertible in $\boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}$. By Theorem 3.2.19, this implies that $[\mathbf{P}_2, \mathbf{C}]$ is not stable. To complete the proof, we establish below that $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < \beta$.

First we see that $G_1^*G_2 = G_1^*(G_1 + \Delta)\hat{Q} = (I + G_1^*\Delta)\hat{Q}$. By defining $\mathbf{G}_1 \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{G_1}$, $\mathbf{G}_2 \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{G_2}$, $\boldsymbol{\Delta} \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{\Delta}$, $\hat{\mathbf{Q}} \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{\hat{Q}}$ and noting that G_1 is normalised, we have

$$\bar{\gamma}(\mathbf{G}_1^*\boldsymbol{\Delta}) = \|G_1^*\Delta\|_\infty \leq \|\Delta\|_\infty < \beta < 1.$$

Now by Lemma 3.1.2(i), (ii), and (iii) in the order they are stated, for all $\tau \in [0, h)$,

$$\begin{aligned}-\operatorname{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = \operatorname{ind}(\mathbf{T}_{\mathbf{G}_1^*\mathbf{G}_2,\tau}) = \operatorname{ind}(\mathbf{T}_{(I+\mathbf{G}_1^*\boldsymbol{\Delta})\hat{\mathbf{Q}},\tau}) &= \operatorname{ind}(\mathbf{T}_{I+\mathbf{G}_1^*\boldsymbol{\Delta},\tau}) + \operatorname{ind}(\mathbf{T}_{\hat{\mathbf{Q}},\tau}) \\ &= \operatorname{ind}(\mathbf{T}_{\hat{\mathbf{Q}},\tau}) = 0,\end{aligned}$$

where the last equality holds by Remark 3.1.5 because $\hat{Q}^{-1} \in \boldsymbol{\mathcal{L}H}_{\mathbb{D}}^{\infty}$, $\hat{\mathbf{Q}}^{-1} \overset{\mathscr{Z}\mathscr{W}_h}{\sim} \boldsymbol{M}_{\hat{Q}^{-1}}$ is a bounded causal operator. Finally, with

$$\hat{\mathbb{Q}} := \left\{ \mathbf{Q} \overset{\mathscr{Z}\mathscr{W}}{\hookleftarrow}{}^{h} \boldsymbol{M}_Q : Q \in \boldsymbol{\mathcal{L}H}^{\infty}_{\mathbb{D}} \right\},$$

we have that

$$\begin{aligned}
\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2 \mathbf{G}_1) \leq \inf_{\mathbf{Q} \in \hat{\mathbb{Q}}} \bar{\gamma}\left(\begin{bmatrix} \mathbf{G}_2^* \mathbf{G}_1 - \mathbf{Q} \\ \tilde{\mathbf{G}}_2 \mathbf{G}_1 \end{bmatrix} \right) &= \inf_{\mathbf{Q} \in \hat{\mathbb{Q}}} \bar{\gamma}\left(\begin{bmatrix} \mathbf{G}_2^* \\ \tilde{\mathbf{G}}_2 \end{bmatrix} (\mathbf{G}_1 - \mathbf{G}_2 \mathbf{Q}) \right) \\
&= \inf_{\mathbf{Q} \in \hat{\mathbb{Q}}} \bar{\gamma}(\mathbf{G}_1 - \mathbf{G}_2 \mathbf{Q}) \\
&= \inf_{Q \in \boldsymbol{\mathcal{L}H}^{\infty}_{\mathbb{D}}} \| G_1 - G_2 Q \|_\infty \\
&= \inf_{Q \in \boldsymbol{\mathcal{L}H}^{\infty}_{\mathbb{D}}} \| G_1 - (G_1 + \Delta) \hat{Q} Q \|_\infty \\
&\leq \| \Delta \|_\infty < \beta,
\end{aligned}$$

where the third equality follows from the fact that $\begin{bmatrix} \mathbf{G}_2^* \\ \tilde{\mathbf{G}}_2 \end{bmatrix}$ is an isometry as in (3.11). $\square$

3.5 Summary

This chapter motivates a generalised ν-gap from [JC10, JC11] as a measure of distance between open-loop causal *linear time-varying* systems from the perspective of capturing their 'closed-loop' difference. The development relies only on assumptions regarding the existence of normalised strong graph symbols/representations satisfying a corresponding Hankel operator compactness property as in [JC10, JC11], as defined in Chapter 2. Several notions including feedback stability and robustness margin are also generalised from the *time-invariant* case [Vin93, Vin01] and characterised in terms of system graph symbols building upon the initial development in [JC10, JC11]. The ν-gap metric is shown to possess various useful closed-loop robustness characteristics. These lay the foundations for the development of more advanced stability analysis and model approximation methods in the forthcoming chapters.

Chapter 4

Stability analysis via the integral quadratic constraints

The theory of integral quadratic constraints (IQCs) [MR97, Jön01] presents a general tool for characterising the input-output behaviour of systems. It is well-documented in the literature [MR97, RM97] that IQC based robustness analysis generalises the small-gain, passivity, and circle/Popov criterion type stability arguments [Kha02]. The main goal of this chapter is to corroborate the benefits brought about by incorporating IQCs into the ν-gap metric based analysis along the lines of [JC10, JC11], where IQCs are used to characterise input/output behaviour and continuous ν-gap homotopies to characterise the robustness properties of uncertain feedback interconnections. Towards this end, the theories are consolidated within a unified robustness analysis framework, from which a pure ν-gap metric ball type robust stability result is then shown to be recoverable. Several intermediate results are derived along the way, each of which is interesting in its own right. In particular, a useful characterisation of the ν-gap distance in terms of a linear fractional transformation (LFT) being stable and contractive is developed in an operator-theoretic setting, in the spirit of [Can06, BC07]. The LFT provides a bijective and continuous map between a ν-gap ball and a corresponding norm-ball. This is exploited to establish that sufficiently small ν-gap metric balls are pathwise connected in the graph topology. A ν-gap ball based robustness result then follows from this result within the IQC/ν-gap framework of [JC10, JC11].

In the literature, LFT characterisation of similar kind has been used to examine various model approximation and validation problems; see, for example, [Dav95, Vin01, Can01, CV04]. These *shift-invariant* studies are closely tied to the so-called DGKF solution to $\boldsymbol{H}^\infty$ optimal control problems [DGKF89, ZDG96]. A different and more direct route is adopted here, based on a J-spectral factorisation approach. Earlier works in this direction include [Can06, BC07] and [BC08], where the ν-gap metric for finite-dimensional linear time-invariant (LTI) systems is characterised in terms of an LFT and a step-wise procedure for reduced-order approximation proposed, respectively.

The LFT characterisation requires the existence of a certain J-spectral factorisation of a normalised graph symbol expression. This is established for two generic classes of linear systems from Chapter 2, namely time-varying systems with finite-dimensional stabilisable and detectable state-space realisations [IS04, MC10] and multiplications by distributed-parameter transfer functions in the constantly proper subclass of the Callier-Desoer algebra [CD78, CZ95]. In the latter case, the definition of the generalised ν-gap metric is also shown to reduce to the more familiar winding number expression; see [CJK12, CJK10, CJK09, Vin93, Vin01].

This chapter has the following structure. The IQC/ν-gap robust stability analysis from [JC10, JC11] is reviewed in the next section. The aforementioned linear fractional characterisation of the metric is developed in Section 4.2 under the assumption that a certain J-spectral factorisation exists. A pathwise connectedness result on ν-gap balls is established in Section 4.3. The flexibility of the unified framework over that of the ν-gap alone is then demonstrated in Section 4.4. Section 4.5 contains a characterisation of the robust stability margin. The existence of the required J-spectral factorisation is established for specific classes of systems in Section 4.6.

4.1 A unified IQC and ν-gap based robustness result

In this section, a mixed integral quadratic constraint (IQC) and ν-gap metric based feedback robustness result is presented in a general setting. It is based on the initial developments in [JC10, JC11, CJK12, CJK10, CJK09, JCK08].

Recall that the set of causal operators which satisfy all of Assumptions 2.2.1, 2.2.2, and 2.2.3 is denoted by $\mathbb{S}$ as in Definition 2.2.4.

Definition 4.1.1 (Integral Quadratic Constraints [MR97])**.** An operator $\mathbf{X} \in \mathbb{S}$ is said to satisfy the IQC defined by the multiplier $\mathbf{\Psi} = \mathbf{\Psi}^* \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ if

$$\langle v, \mathbf{\Psi} v\rangle_2 \geq 0, \forall v \in \mathscr{G}_{\mathbf{X}} \cap \boldsymbol{L}^{2+}.$$

This is denoted $\mathbf{X} \in \mathrm{IQC}(\mathbf{\Psi})$. On the other hand, $\mathbf{X}$ is said to satisfy the strict complementary IQC, denoted $\mathbf{X} \in \mathrm{IQC}^c(\mathbf{\Psi})$, if there exists an $\epsilon > 0$ such that

$$\langle v, \mathbf{\Psi} v\rangle_2 \leq -\epsilon\|v\|_2^2, \forall v \in \mathscr{G}'_{\mathbf{X}} \cap \boldsymbol{L}^{2+}.$$

Lemma 4.1.2. *Given $\mathbf{P}, \mathbf{C} \in \mathbb{S}$, suppose there exists a $\mathbf{\Psi} = \mathbf{\Psi}^* \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ such that*

$$\mathbf{P} \in \mathrm{IQC}(\mathbf{\Psi}) \quad \text{and} \quad \mathbf{C} \in \mathrm{IQC}^c(\mathbf{\Psi}).$$

Then there exists an $\eta > 0$ such that

$$\|v + w\|_2^2 \geq \eta^2 \|v\|_2^2 \quad \text{for all } v \in \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^{2+} \text{ and } w \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^{2+}.$$

Note that if in addition $[\mathbf{P}, \mathbf{C}]$ is stable, Lemma 3.2.8 yields $b_{\mathbf{P},\mathbf{C}} \geq \eta$.

Proof. The proof follows from an argument employed in the proof of [JC11, Thm. 1], which is included here for completeness. From hypothesis, we have

$$\exists \epsilon > 0 \text{ such that } \langle w, \mathbf{\Psi} w\rangle_2 \leq -\epsilon \|w\|_2^2, \forall w \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^{2+} \quad \text{and} \quad \langle v, \mathbf{\Psi} v\rangle_2 \geq 0, \forall v \in \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^{2+},$$

for any $\mathbf{P} \in \mathscr{P}$. With $\mathbf{\Upsilon} := 2\mathbf{\Psi} + \epsilon I$, these conditions become

$$\langle v, \mathbf{\Upsilon} v\rangle_2 \geq \epsilon \|v\|_2^2, \forall v \in \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^{2+} \quad \text{and} \quad \langle w, \mathbf{\Upsilon} w\rangle_2 \leq -\epsilon \|w\|_2^2, \forall w \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^{2+}.$$

It follows that for any $v \in \mathscr{G}_{\mathbf{P}} \cap \boldsymbol{L}^{2+}$ and $w \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^{2+}$,

$$\begin{aligned} \epsilon(\|v\|_2^2 + \|w\|_2^2) \leq \langle v, \mathbf{\Upsilon} v\rangle_2 - \langle w, \mathbf{\Upsilon} w\rangle_2 &= \langle v + w, \mathbf{\Upsilon}(v + w)\rangle_2 - 2\langle w, \mathbf{\Upsilon}(v + w)\rangle_2 \\ &\leq \|\mathbf{\Upsilon}\| \|v + w\|_2^2 + 2\|\mathbf{\Upsilon}\| \|w\|_2 \|v + w\|_2 \\ &\leq \|\mathbf{\Upsilon}\| \|v + w\|_2^2 + \frac{2\|\mathbf{\Upsilon}\|^2 \|v + w\|_2^2}{\epsilon} + \frac{\epsilon}{2}\|w\|_2^2, \end{aligned}$$

where the last inequality holds since $2xy \leq \frac{x^2}{\gamma} + \gamma y^2$ for any $x, y, \gamma \in \mathbb{R}$. This implies

$$\begin{aligned} \left(1 + \frac{2}{\epsilon}\|\mathbf{\Upsilon}\|\right) \|\mathbf{\Upsilon}\| \|v + w\|_2^2 &\geq \epsilon \|v\|_2^2 + \frac{\epsilon}{2}\|w\|_2^2 \geq \epsilon \|v\|_2^2 \\ \implies \|v + w\|_2^2 &\geq \eta^2 \|v\|_2^2, \end{aligned}$$

for any positive $\eta \leq \frac{\epsilon}{\sqrt{\|\mathbf{\Upsilon}\|(\epsilon + 2\|\mathbf{\Upsilon}\|)}}$. □

In IQC-based feedback robustness analysis framework of [JC10, JC11, CJK12, CJK10, CJK09, JCK08], homotopies of systems in the ν-gap metric play a crucial role.

Definition 4.1.3. We say that two operators $\mathbf{P}_a$ and $\mathbf{P}_b$ in $\mathbb{S}$ are joined by a *path* if there exists a homotopy of operators $\{\mathbf{P}_\lambda \in \mathbb{S} : \lambda \in (0, 1)\}$ such that the mapping $\lambda \in [0, 1] \mapsto \mathbf{P}_\lambda$ with $\mathbf{P}_0 := \mathbf{P}_a$ and $\mathbf{P}_1 := \mathbf{P}_b$ is continuous with respect to $\delta_\nu(\cdot, \cdot)$. $\mathbf{P}_a$ and $\mathbf{P}_b$ are said to be connected by a *weak path* if for every $\eta > 0$, there exists

$\mathscr{P}(\eta) := \{\mathbf{P}_i \in \mathbb{S} : i = 1, 2, \ldots, N(\eta)\}$ such that $\mathbf{P}_1 = \mathbf{P}_a$, $\mathbf{P}_N = \mathbf{P}_b$ and $\delta_\nu(\mathbf{P}_i, \mathbf{P}_{i+1}) < \eta, \forall i = 1, 2, \ldots, N-1$. When this is the case, the weak path is denoted by $\mathscr{P}(\mathbf{P}_a, \mathbf{P}_b) := \bigcup_{\eta>0} \mathscr{P}(\eta)$. An uncertainty set of operators $\mathscr{W}$ is said to be (weakly) path-connected in the graph topology if for any two elements in $\mathscr{W}$, there exists a (weak) path connecting them in $\mathscr{W}$.

Theorem 4.1.4. *Suppose $[\mathbf{P}_a, \mathbf{C}]$ is stable, then $[\mathbf{P}_b, \mathbf{C}]$ is stable if*

1. *$\mathbf{P}_a$ and $\mathbf{P}_b$ are connected by a weak path $\mathscr{P}(\mathbf{P}_a, \mathbf{P}_b)$;*
2. *there exists a $\mathbf{\Psi} = \mathbf{\Psi}^* \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ such that*

$$\mathbf{P} \in \mathrm{IQC}(\mathbf{\Psi}), \forall \mathbf{P} \in \mathscr{P}(\mathbf{P}_a, \mathbf{P}_b) \quad \textit{and} \quad \mathbf{C} \in \mathrm{IQC}^c(\mathbf{\Psi}).$$

Proof. By hypothesis and Lemma 4.1.2, $b_{\mathbf{P}_a,\mathbf{C}} \geq \eta$ for some $\eta > 0$. Given this η, it follows by hypothesis that there exists $\mathscr{P}(\eta) := \{\mathbf{P}_i \in \mathbb{S} : i = 1, 2, \ldots, N\}$ such that $\mathbf{P}_1 = \mathbf{P}_a$, $\mathbf{P}_N = \mathbf{P}_b$ and $\delta_\nu(\mathbf{P}_i, \mathbf{P}_{i+1}) < \eta, \forall i = 1, 2, \ldots, N-1$. Noting that $[\mathbf{P}_1, \mathbf{C}]$ is stable by hypothesis, the result follows by a simple inductive argument which establishes that $[\mathbf{P}_i, \mathbf{C}]$ is stable for all $i = 2, \ldots, N$. Specifically, suppose that $[\mathbf{P}_k, \mathbf{C}]$ is stable, whereby $b(\mathbf{P}_k, \mathbf{C}) \geq \eta$ by hypothesis and Lemma 4.1.2 again. Since $\delta_\nu(\mathbf{P}_k, \mathbf{P}_{k+1}) < \eta$, it follows by Corollary 3.3.6 that $[\mathbf{P}_{k+1}, \mathbf{C}]$ is stable. ☐

Corollary 4.1.5. *Given a $\mathbf{C} \in \mathbb{S}$ and a weakly path-connected set $\mathscr{W} \subset \mathbb{S}$, suppose there exists a $\mathbf{P}_0 \in \mathscr{W}$ such that $[\mathbf{P}_0, \mathbf{C}]$ is stable. If in addition there exists a $\mathbf{\Psi} = \mathbf{\Psi}^* \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ such that $\mathbf{P} \in \mathrm{IQC}(\mathbf{\Psi}), \forall \mathbf{P} \in \mathscr{W}$ and $\mathbf{C} \in \mathrm{IQC}^c(\mathbf{\Psi})$, then $[\mathbf{P}, \mathbf{C}]$ is stable for all $\mathbf{P} \in \mathscr{W}$.*

Proof. By hypothesis, for every $\mathbf{P} \in \mathscr{W}$, there exists a weak path connecting it with $\mathbf{P}_0$. The claim that $[\mathbf{P}, \mathbf{C}]$ is stable then holds by Theorem 4.1.4. ☐

For analogous IQC/ν-gap based results to the above, see [JC10, JC11, CJK12, CJK10, CJK09, JCK08]. Moreover, [RM97] contains a similar result for nonlinear operators developed within a different setting using the generalised gap metric from [GS97]. Other works which exploit gap-homotopies in a similar fashion include [Vin99, JSV05].

4.2 A linear fractional characterisation of the ν-gap metric

In this section, we establish a characterisation of the generalised ν-gap metric based on [Can06, BC07]. To be specific, given two $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{S}$, it is shown that $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < r$

if, and only if, the linear fractional transformation (LFT) of a certain bounded causal operator $\mathbf{R}$ on $\mathbf{P}_2$ is bounded, causal, and strictly contractive. The $\mathbf{R}$ involved has a bounded causal inverse and is dependent on $\mathbf{P}_1$ and $r \in (0,1)$. The characterisation is of crucial importance in this thesis, forasmuch as it underlies the path-connectedness proof in the succeeding section and is useful for tackling problems in model approximation with respect to the ν-gap measure of error as delineated in Chapter 5.

We begin with two preliminary lemmata from [Can06] stated in the form of complementary integral quadratic constraints (IQCs). They clearly demonstrate the close relationship between feedback performance margin and the ν-gap metric. Recall the notation introduced for operator graph symbols in Section 3.2.3, which is also used below.

Lemma 4.2.1. *Given $\mathbf{P}, \mathbf{C} \in \mathbb{S}$, the following are equivalent for any $r \in (0,1)$:*

1. $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > r$;
2. *for all* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^{2+} = \bigcup_{\tau \in \mathbb{R}} \operatorname{img}(\mathbf{T}_{\mathbf{K},\tau})$, *there exists an $\epsilon > 0$ such that*

$$\left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left(r^2 \mathbf{I} - \tilde{\mathbf{G}}^* \tilde{\mathbf{G}}\right) \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \le -\epsilon \left\| \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\|_2^2 .$$

Likewise, $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) \ge r$ if, and only if, $\left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left(r^2 \mathbf{I} - \tilde{\mathbf{G}}^* \tilde{\mathbf{G}}\right) \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \le 0$ for all $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^{2+}$.

Proof. The proof is straightforward from the definition of induced gains. In particular, since graph symbols are causal by definition, application of Lemma 3.1.3(i) yields

$$\underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau})^2 = \inf_{q \in \boldsymbol{L}^2_{[\tau,\infty)} : \|q\|_2 = 1} \left\langle \tilde{\mathbf{G}}\mathbf{K}q, \tilde{\mathbf{G}}\mathbf{K}q \right\rangle_2 = \inf_{\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \operatorname{img}(\mathbf{T}_{\mathbf{K},\tau}) : \left\| \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\|_2 = 1} \left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \tilde{\mathbf{G}}^* \tilde{\mathbf{G}} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 ,$$

where the last equality holds since $\mathbf{T}_{\mathbf{K},\tau}$ is an isometry. It follows immediately that

$$\inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) \ge r \iff \left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left(r^2 \mathbf{I} - \tilde{\mathbf{G}}^* \tilde{\mathbf{G}}\right) \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \le 0, \forall \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \bigcup_{\tau \in \mathbb{R}} \operatorname{img}(\mathbf{T}_{\mathbf{K},\tau}) \quad \text{and}$$

$$\inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) > r \iff \left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left(r^2 \mathbf{I} - \tilde{\mathbf{G}}^* \tilde{\mathbf{G}}\right) \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \le -\epsilon \left\| \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\|_2^2, \forall \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \bigcup_{\tau \in \mathbb{R}} \operatorname{img}(\mathbf{T}_{\mathbf{K},\tau}),$$

where ϵ is some positive real number. The claimed result then follows from Lemma 3.1.3(v), whereby $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) = \inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau})$. □

Lemma 4.2.2. *Given $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{S}$, the following are equivalent for any $r \in (0,1)$ and any $i, j \in \{1,2\}$ satisfying $i \ne j$:*

1. $\bar{\gamma}(\tilde{\mathbf{G}}_i \mathbf{G}_j) < r$;

2. *for all* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}_j} \cap \boldsymbol{L}^{2+} = \bigcup_{\tau \in \mathbb{R}} \operatorname{img}\left(\mathbf{T}_{\mathbf{G}_j,\tau}\right)$, *there exists* $\epsilon > 0$ *such that*

$$\left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left(r^2\mathbf{I} - \tilde{\mathbf{G}}_i^* \tilde{\mathbf{G}}_i\right) \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \geq \epsilon \left\| \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\|_2^2.$$

Likewise, $\bar{\gamma}(\tilde{\mathbf{G}}_i \mathbf{G}_j) \leq r$ *if, and only if,* $\left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left(r^2\mathbf{I} - \tilde{\mathbf{G}}_i^* \tilde{\mathbf{G}}_i\right) \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \geq 0$ *for all* $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}_j} \cap \boldsymbol{L}^{2+}$.

Proof. First note that $\bar{\gamma}(\tilde{\mathbf{G}}_2 \mathbf{G}) = \bar{\gamma}(\tilde{\mathbf{G}}_1 \mathbf{G}_2)$ by (3.12), which explains the subscripts i and j. The proof for this lemma is largely similar to that for the previous one. Specifically, because graph symbols are causal by definition and $\mathbf{T}_{\mathbf{G}_j,\tau}$ is an isometry,

$$\bar{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}_i \mathbf{G}_j,\tau})^2 = \sup_{q \in \boldsymbol{L}^2_{[\tau,\infty)} : \|q\|_2 = 1} \left\langle \tilde{\mathbf{G}}_i \mathbf{G}_j q, \tilde{\mathbf{G}}_i \mathbf{G}_j q \right\rangle_2 = \sup_{\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \operatorname{img}\left(\mathbf{T}_{\mathbf{G}_j,\tau}\right) : \left\|\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right\|_2 = 1} \left\langle \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \tilde{\mathbf{G}}_i^* \tilde{\mathbf{G}}_i \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2.$$

The result follows immediately from this equality and Lemma 3.1.3(v). □

We are now ready to establish the LFT characterisation of the ν-gap metric. First define the maximal robustness margin of any $\mathbf{P} \in \mathbb{S}$ by

$$b_{\text{opt}}(\mathbf{P}; \mathbb{S}) := \sup_{\mathbf{C} \in \mathbb{S} : [\mathbf{P}, \mathbf{C}] \text{ is stable}} b_{\mathbf{P},\mathbf{C}}. \tag{4.1}$$

Theorem 4.2.3. *Given* m*-input* p*-output operators* $\mathbf{P}_1$, $\mathbf{P}_2 \in \mathbb{S}$ *and a positive number* $r < b_{\text{opt}}(\mathbf{P}_1; \mathbb{S})$, *suppose there exists a boundedly invertile causal*

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$$

such that $\mathbf{R}^{-1} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ *is causal and*

$$r^2\mathbf{I}_{p+m} - \tilde{\mathbf{G}}_1^* \tilde{\mathbf{G}}_1 = \mathbf{R}^* \begin{bmatrix} -\mathbf{I}_p & 0 \\ 0 & \mathbf{I}_m \end{bmatrix} \mathbf{R}. \tag{4.2}$$

Then the following are equivalent:

1. $\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) < r$;

2. *the LFT operator illustrated in Figure 4.1,*

$$\mathcal{F}(\mathbf{R}, \mathbf{P}_2) : \operatorname{dom}(\mathcal{F}(\mathbf{R}, \mathbf{P}_2)) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}},$$

   for which $\mathbf{RG}$ *is a (not necessarily normalised) right graph symbol, is such that*

$\mathcal{F}(\mathbf{R}, \mathbf{P}_2)$ *is causal,* $\boldsymbol{L}^{2+} \subset \operatorname{dom}(\mathcal{F}(\mathbf{R}, \mathbf{P}_2))$*, and*

$$\bar{\gamma}(\mathcal{F}(\mathbf{R}, \mathbf{P}_2)|_{\boldsymbol{L}^{2+}}) := \sup_{u \in \boldsymbol{L}^{2+} : \|u\|_2 = 1} \|\mathcal{F}(\mathbf{R}, \mathbf{P}_2) u\|_2 < 1.$$

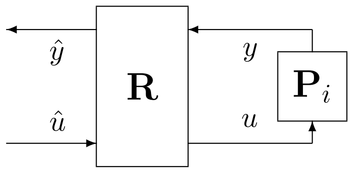

Figure 4.1: Graphical representation of the LFT $\mathcal{F}(\mathbf{R}, \mathbf{P}_i)$

Remark 4.2.4. J-spectral factorisation in (4.2) is not unique. Specifically, given any $\hat{\mathbf{R}}$ that solves (4.2) and any causal $\mathbf{Q} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ such that $\mathbf{Q}^{-1} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ is causal and $\mathbf{Q}^* \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \mathbf{Q} = \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right]$, $\mathbf{R} := \mathbf{Q}\hat{\mathbf{R}}$ is also a solution to (4.2). Indeed, $\mathbf{Q}$ can be constructed in such a way that $\mathbf{R}_{21}\mathbf{N}_2 + \mathbf{R}_{22}\mathbf{M}_2$ has non-singular instantaneous gain (cf. Definition 2.1.10), so that by Lemma 2.1.13

$$\mathbf{R}\mathbf{G}_2 \boldsymbol{L}^{2+} = \begin{bmatrix} \mathbf{R}_{11}\mathbf{N}_2 + \mathbf{R}_{12}\mathbf{M}_2 \\ \mathbf{R}_{21}\mathbf{N}_2 + \mathbf{R}_{22}\mathbf{M}_2 \end{bmatrix} \boldsymbol{L}^{2+}$$

is the graph of a causal linear operator $\mathcal{F}(\mathbf{R}, \mathbf{P}_2) : \operatorname{img}(\mathbf{R}_{21}\mathbf{N}_2 + \mathbf{R}_{22}\mathbf{M}_2|_{\boldsymbol{L}^{2+}}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ satisfying

$$\mathcal{F}(\mathbf{R}, \mathbf{P}_2)\Big|_{\operatorname{img}\left(\mathbf{R}_{21}\mathbf{N}_2 + \mathbf{R}_{22}\mathbf{M}_2|_{\boldsymbol{L}^2_{[\tau,\infty)}}\right)} = \mathbf{T}_{\mathbf{R}_{11}\mathbf{N}_2 + \mathbf{R}_{12}\mathbf{M}_2, \tau} \mathbf{T}^{-1}_{\mathbf{R}_{21}\mathbf{N}_2 + \mathbf{R}_{22}\mathbf{M}_2, \tau} \; \forall \tau \in \mathbb{R}.$$

This linear fractional transformation (LFT) of the pair $(\mathbf{R}, \mathbf{P}_2)$ is consistent with chain-scattering formalism of [Kim97]. As an example, see [BC08, Lem. 2], which considers the class of finite-dimensional LTI systems with sufficiently small instantaneous gains. The $\mathbf{R}$ constructed there is such that $\mathbf{R}_{21} = 0$ and $\mathbf{R}_{22}$ has non-singular instantaneous gain. Combining this with the non-singular instantaneous gain of $\mathbf{M}_2$, it follows that $\mathbf{R}_{21}\mathbf{N}_2 + \mathbf{R}_{22}\mathbf{M}_2$ has non-singular instantaneous gain. In the special case where $\mathbf{P}_2$ that has zero instantaneous gain, i.e. strictly causal, see Proposition 5.3.1 for a formula of such an $\mathbf{R}$.

Proof of Theorem 4.2.3. The line of proof follows the arguments underlying [Can06, Thm. 1] and [BC07, Thm. 3]. We first note a consequence of the hypothesis $r < b_{\mathrm{opt}}(\mathbf{P}_1; \mathbb{S})$, whereby there exists a $\mathbf{C} \in \mathbb{S}$ such that $[\mathbf{P}_1, \mathbf{C}]$ is stable and $b_{\mathbf{P}_1, \mathbf{C}} = \underline{\gamma}(\tilde{\mathbf{G}}_1 \mathbf{K}) \geq r$.

Combining (4.2) with Lemmas 4.2.1 and 4.2.2 yields respectively

$$\underline{\gamma}(\tilde{\mathbf{G}}_1\mathbf{K}) \geq r \iff \left\langle \mathbf{R}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \mathbf{R}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \leq 0, \forall \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}'_{\mathbf{C}} \cap \boldsymbol{L}^{2+} \quad \text{and} \tag{4.3}$$

$$\bar{\gamma}(\tilde{\mathbf{G}}_i\mathbf{G}_1) < r \iff \left\langle \mathbf{R}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \mathbf{R}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \geq \epsilon \left\|\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right\|_2^2, \forall \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}_i} \cap \boldsymbol{L}^{2+}. \tag{4.4}$$

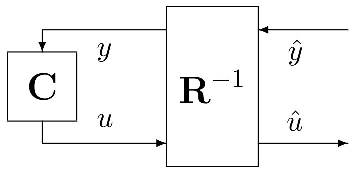

Figure 4.2: Graphical representation of the LFT $\mathcal{X}(\mathbf{R}, \mathbf{C})$

Let $\mathcal{X}(\mathbf{R}, \mathbf{C}) : \mathrm{dom}\,(\mathcal{X}(\mathbf{R}, \mathbf{C})) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ be the causal LFT operator illustrated in Figure 4.2, for which $\mathbf{RK}$ is a (not necessarily normalised) right graph symbol $\mathcal{X}(\mathbf{R}, \mathbf{C})$, i.e.

$$\mathscr{G}'_{\mathcal{X}(\mathbf{R},\mathbf{C})} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \mathrm{img}\,(\mathbf{T}_{\mathbf{RK},\tau}) \ \forall \tau \in \mathbb{R};$$

see Remark 4.2.4. By (4.3), it follows that

$$\left\langle \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \right\rangle_2 \leq 0 \ \forall \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \in \mathscr{G}'_{\mathcal{X}(\mathbf{R},\mathbf{C})} \cap \boldsymbol{L}^{2+}. \tag{4.5}$$

Define

$$\mathcal{C}(\tau) := \left\{ \hat{y} \in \mathrm{dom}\,(\mathcal{X}(\mathbf{R}, \mathbf{C})) \cap \boldsymbol{L}^2_{[\tau,\infty)} \mid \hat{u} = \mathcal{X}(\mathbf{R}, \mathbf{C})\hat{y} \in \boldsymbol{L}^2_{[\tau,\infty)} \right\}.$$

As such, we have from (4.5) that $\mathcal{X}(\mathbf{R}, \mathbf{C})|_{\mathcal{C}(\tau)}$ is contractive, i.e. $\bar{\gamma}\left(\mathcal{X}(\mathbf{R}, \mathbf{C})|_{\mathcal{C}(\tau)}\right) \leq 1$, for all $\tau \in \mathbb{R}$. In fact, it can be shown that $\mathcal{C}(\tau) = \boldsymbol{L}^2_{[\tau,\infty)}$ for all $\tau \in \mathbb{R}$, from which it follows that $\bar{\gamma}(\mathcal{X}(\mathbf{R}, \mathbf{C})|_{\boldsymbol{L}^{2+}}) \leq 1$. To this end, let

$$\mathcal{F}(\mathbf{R}, \mathbf{P}_1) : \mathrm{dom}\,(\mathcal{F}(\mathbf{R}, \mathbf{P}_1)) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$$

be the causal LFT operator for which $\mathbf{RG}_1$ is a (not necessarily normalised) right graph symbol; see Figure 4.1 and Remark 4.2.4. It follows that $\tilde{\mathbf{G}}_1\mathbf{R}^{-1}$ is a (not necessarily normalised) left graph symbol for $\mathcal{F}(\mathbf{R}, \mathbf{P}_1)$, satisfying $\mathscr{G}_{\mathcal{F}(\mathbf{R},\mathbf{P}_1)} \cap \boldsymbol{L}^2_{[\tau,\infty)} = \ker\left(\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{R}^{-1},\tau}\right)$ for all $\tau \in \mathbb{R}$. Since $\bar{\gamma}(\tilde{\mathbf{G}}_1\mathbf{G}_1) = 0 < r$, and in light of (4.4), we have

$$\left\langle \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \right\rangle_2 \geq \epsilon \left\|\left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right]\right\|_2^2, \forall \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \in \mathscr{G}_{\mathcal{F}(\mathbf{R},\mathbf{P}_1)} \cap \boldsymbol{L}^{2+}. \tag{4.6}$$

Define

$$\mathcal{P}_1(\tau) := \left\{ \hat{u} \in \mathrm{dom}\,(\mathcal{F}(\mathbf{R}, \mathbf{P}_1)) \cap \boldsymbol{L}^2_{[\tau,\infty)} \mid \hat{y} = \mathcal{F}(\mathbf{R}, \mathbf{P}_1)\hat{u} \in \boldsymbol{L}^2_{[\tau,\infty)} \right\}.$$

It follows from (4.6) that $\mathcal{F}(\mathbf{R},\mathbf{P}_1)|_{\mathcal{P}_1(\tau)}$ is uniformly *strictly* contractive in $\tau \in \mathbb{R}$, i.e. $\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathcal{F}(\mathbf{R},\mathbf{P}_1)|_{\mathcal{P}_1(\tau)}\right) < 1$. Also, since $[\mathbf{P}_1,\mathbf{C}]$ is stable, it follows by Theorem 3.2.19 that

$$\tilde{\mathbf{G}}_1\mathbf{K} = \tilde{\mathbf{G}}_1\mathbf{R}^{-1}\mathbf{R}\mathbf{K}$$

has a bounded and causal inverse, whereby $\mathbf{T}_{\tilde{\mathbf{G}}_1\mathbf{R}^{-1}\mathbf{R}\mathbf{K},\tau}$ is boundedly invertible for all $\tau \in \mathbb{R}$; see Remark 3.1.5. As such, it can be shown as in the proof of Lemma 3.2.15 that

$$\mathbf{F}_{\mathcal{F}(\mathbf{R},\mathbf{P}_1),\mathcal{X}(\mathbf{R},\mathbf{C}),\tau} := \left.\begin{bmatrix} \mathbf{I} & \mathcal{F}(\mathbf{R},\mathbf{P}_1) \\ \mathcal{X}(\mathbf{R},\mathbf{C}) & \mathbf{I} \end{bmatrix}\right|_{\mathcal{C}(\tau)\times\mathcal{P}_1(\tau)}$$

is boundedly invertible for all $\tau \in \mathbb{R}$. Now suppose, to the contrapositive, that there exists a $\tau \in \mathbb{R}$ such that $\mathcal{C}(\tau) \neq \boldsymbol{L}^2_{[\tau,\infty)}$ or $\mathcal{P}_1(\tau) \neq \boldsymbol{L}^2_{[\tau,\infty)}$. Then by the large-gain theorem [GKM97], it would follow that

$$\bar{\gamma}(\mathcal{F}(\mathbf{R},\mathbf{P}_1)\mathcal{X}(\mathbf{R},\mathbf{C})|_{\mathcal{P}_1\mathcal{C}(\tau)}) \geq 1,$$

where

$$\mathcal{P}_1\mathcal{C}(\tau) := \{x \in \mathcal{C}(\tau) \mid \mathcal{X}(\mathbf{R},\mathbf{C})x \in \mathcal{P}_1(\tau)\},$$

contradicting $\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathcal{F}(\mathbf{R},\mathbf{P}_1)|_{\mathcal{P}_1(\tau)}\right) < 1$ and $\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathcal{X}(\mathbf{R},\mathbf{C})|_{\mathcal{C}(\tau)}\right) \leq 1$. As such, one may conclude that $\mathcal{C}(\tau) = \boldsymbol{L}^2_{[\tau,\infty)}$ and $\mathcal{P}_1(\tau) = \boldsymbol{L}^2_{[\tau,\infty)}$ for all $\tau \in \mathbb{R}$, whereby $\bar{\gamma}(\mathcal{X}(\mathbf{R},\mathbf{C})|_{\boldsymbol{L}^{2+}}) \leq 1$ and $\bar{\gamma}(\mathcal{F}(\mathbf{R},\mathbf{P}_1)|_{\boldsymbol{L}^{2+}}) < 1$. Using the preceding development, we are now ready to establish the equivalence of 1) and 2).

2) $\implies$ 1): First note that $\tilde{\mathbf{G}}_2\mathbf{R}^{-1}$ is a left graph symbol for the causal LFT $\mathbf{F}(\mathbf{R},\mathbf{P}_2)$, as in the analogous case for $\mathbf{F}(\mathbf{R},\mathbf{P}_1)$ before. Furthermore, since $\bar{\gamma}(\mathcal{F}(\mathbf{R},\mathbf{P}_2)|_{\boldsymbol{L}^{2+}}) < 1$, or equivalently, for some $\epsilon > 0$,

$$\left\langle \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \right\rangle_2 \geq \epsilon \left\| \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \right\|_2^2 \ \forall \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \in \mathscr{G}_{\mathcal{F}(\mathbf{R},\mathbf{P}_2)} \cap \boldsymbol{L}^{2+},$$

we have that

$$\left\langle \mathbf{R} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \mathbf{R} \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \geq \epsilon \left\| \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\|_2^2 \ \forall \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}_2} \cap \boldsymbol{L}^{2+} = \bigcup_{\tau\in\mathbb{R}} \mathrm{img}\,(\mathbf{T}_{\mathbf{G}_2,\tau}).$$

Therefore from (4.4), $\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) < r$. Moreover, the small-gain argument in Lemma 3.2.9 can be used to conclude that $[\mathcal{F}(\mathbf{R},\mathbf{P}_2),\mathcal{X}(\mathbf{R},\mathbf{C})]$ is stable, which by Theorem 3.2.19 is equivalent to $\tilde{\mathbf{G}}_2\mathbf{R}^{-1}\mathbf{R}\mathbf{K} = \tilde{\mathbf{G}}_2\mathbf{K}$ having a bounded causal inverse. This in turn is equivalent to $[\mathbf{P}_2,\mathbf{C}]$ being stable. Using Theorem 3.3.1, we may thus conclude that for all $\tau \in \mathbb{R}$, $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ is Fredholm and $\mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = 0$, whereby $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) = \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) < r$.

1) $\implies$ 2): Since $\bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) < r$, it follows by (4.4) that

$$\left\langle \mathbf{R}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \mathbf{R}\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \right\rangle_2 \geq \epsilon \left\|\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]\right\|_2^2, \forall \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}_2} \cap \boldsymbol{L}^{2+},$$

and hence

$$\left\langle \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right], \left[\begin{smallmatrix} -\mathbf{I} & 0 \\ 0 & \mathbf{I} \end{smallmatrix}\right] \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \right\rangle_2 \geq \epsilon \left\|\left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right]\right\|_2^2, \forall \left[\begin{smallmatrix} \hat{y} \\ \hat{u} \end{smallmatrix}\right] \in \mathscr{G}_{\mathcal{F}(\mathbf{R},\mathbf{P}_2)} \cap \boldsymbol{L}^{2+}. \tag{4.7}$$

Define

$$\mathcal{P}_2(\tau) := \left\{ \hat{u} \in \operatorname{dom}(\mathcal{F}(\mathbf{R},\mathbf{P}_2)) \cap \boldsymbol{L}^2_{[\tau,\infty)} \mid \hat{y} = \mathcal{F}(\mathbf{R},\mathbf{P}_2)\hat{u} \in \boldsymbol{L}^2_{[\tau,\infty)} \right\}.$$

It follows from (4.7) that $\mathcal{F}(\mathbf{R},\mathbf{P}_2)|_{\mathcal{P}_2(\tau)}$ is uniformly *strictly* contractive in $\tau \in \mathbb{R}$, i.e. $\sup_{\tau\in\mathbb{R}} \bar{\gamma}\left(\mathcal{F}(\mathbf{R},\mathbf{P}_2)|_{\mathcal{P}_2(\tau)}\right) < 1$. Thus, it remains to show that $\mathcal{P}_2(\tau) = \boldsymbol{L}^2_{[\tau,\infty)}$ for all $\tau \in \mathbb{R}$. To this end, observe that by Corollary 3.3.6, $[\mathbf{P}_2, \mathbf{C}]$ is stable, since $\delta_\nu(\mathbf{P}_1,\mathbf{P}_2) < r \leq b_{\mathbf{P}_1,\mathbf{C}}$. By Theorem 3.2.19, this implies that $\tilde{\mathbf{G}}_2\mathbf{K} = \tilde{\mathbf{G}}_2\mathbf{R}^{-1}\mathbf{R}\mathbf{K}$ has a bounded causal inverse. The remainder of the proof is the same as the above case of showing $\mathcal{F}(\mathbf{R},\mathbf{P}_1)$ is bounded on $\boldsymbol{L}^{2+}$ via the large-gain theorem [GKM97]. $\square$

4.3 Path-connectedness of ν-gap metric balls

We establish that any ν-gap ball of radius less than the maximal robust stability margin of its centre, is pathwise connected in the graph topology. This is exploited in Section 4.4 to reconcile the IQCs based framework discussed in Section 4.1 with the ν-gap robustness results of Chapter 3. The proof makes use of the LFT characterisation developed in the previous section. More specifically, the LFT is shown to provide an invertible and continuous mapping from a unit ball of stable causal operators to a ν-gap ball of interest, under which path-connectedness is preserved. The proof closely follows the original ideas developed in [CJK10, CJK12] for the time-invariant case.

Theorem 4.3.1. *Given $\mathbf{P}_c \in \mathbb{S}$ for which $\tilde{\mathbf{G}}_c$ is a normalised left graph symbol, an $r < b_{\mathrm{opt}}(\mathbf{P}_c;\mathbb{S})$ and any $\mathbf{P}_0, \mathbf{P}_1 \in \mathbb{G}_r(\mathbf{P}_c) := \{\mathbf{P} \in \mathbb{S} \mid \delta_\nu(\mathbf{P}_c,\mathbf{P}) < r\}$, the following path is continuous with respect to the ν-gap metric:*

$$\theta \in [0,1] \mapsto \mathbf{P}_\theta := \mathcal{F}(\mathbf{R}_c^{-1}, \mathbf{Q}_\theta) \in \mathbb{G}_r(\mathbf{P}_c),$$

where $\mathbf{Q}_\theta := (1-\theta)\mathbf{Q}_0 + \theta\mathbf{Q}_1$, with

$$\mathbf{Q}_i := \mathcal{F}(\mathbf{R}_c, \mathbf{P}_i) \text{ for } i = 0, 1,$$

and $\mathbf{R}_c$ is a J-spectral factor for $\tilde{\mathbf{G}}_c^*\tilde{\mathbf{G}}_c - r^2\mathbf{I}$; see (4.2). As such, $\mathbb{G}_r(\mathbf{P}_c)$ is path-connected in the graph topology. Moreover, $\mathcal{F}(\mathbf{R}_c^{-1}, \mathbb{Q}) \subset \mathbb{G}_r(\mathbf{P}_c)$, where $\mathbb{Q}$ is any convex subset of $\mathbb{B}_1 := \{\mathbf{Q} \in \mathbb{S} : \mathbf{Q}$ is causal and bounded on $\boldsymbol{L}^{2+}$ with $\bar{\gamma}(\mathbf{Q}) < 1\}$, is path-connected.

Proof. We first show that $\mathbf{P}_\theta \in \mathbb{G}_r(\mathbf{P}_c)\,\forall\theta \in [0,1]$. Towards this end, note that because $\mathbf{P}_i \in \mathbb{G}_r(\mathbf{P}_c)$ for $i \in \{1,2\}$, whereby $\delta_\nu(\mathbf{P}_c, \mathbf{P}_i) < r < b_{\mathrm{opt}}(\mathbf{P}_c; \mathbb{S})$, $\mathbf{Q}_i$ is bounded and causal on $\boldsymbol{L}^{2+}$ with $\bar{\gamma}(\mathbf{Q}_i|_{\boldsymbol{L}^{2+}}) < 1$ by Theorem 4.2.3. Correspondingly, for all $\theta \in [0,1]$, $\mathbf{Q}_\theta$ is bounded and causal on $\boldsymbol{L}^{2+}$ with

$$\bar{\gamma}(\mathbf{Q}_\theta|_{\boldsymbol{L}^{2+}}) \le (1-\theta)\bar{\gamma}(\mathbf{Q}_0|_{\boldsymbol{L}^{2+}}) + \theta\bar{\gamma}(\mathbf{Q}_1|_{\boldsymbol{L}^{2+}}) < 1.$$

Furthermore, since $\mathbf{R}_c$ has a bounded causal inverse, it follows that

$$\mathcal{F}(\mathbf{R}_c, \mathbf{P}_\theta) = \mathcal{F}(\mathbf{R}_c, \mathcal{F}(\mathbf{R}_c^{-1}, \mathbf{Q}_\theta)) = \mathbf{Q}_\theta.$$

By Theorem 4.2.3, we thus have $\delta_\nu(\mathbf{P}_c, \mathbf{P}_\theta) < r$, whereby $\mathbf{P}_\theta \in \mathbb{G}_r(\mathbf{P}_c)$ as claimed.

Now note that since $r < b_{\mathrm{opt}}(\mathbf{P}_c; \mathbb{S})$, there exists by definition a $\mathbf{C} \in \mathbb{S}$ such that $[\mathbf{P}_c, \mathbf{C}]$ is stable and $b_{\mathbf{P}_c,\mathbf{C}} > r$. By Corollary 3.3.6, we then have that $[\mathbf{P}, \mathbf{C}]$ is stable for every $\mathbf{P} \in \mathbb{G}_r(\mathbf{P}_c)$ with $b_{\mathbf{P},\mathbf{C}} \ge b_{\mathbf{P}_c,\mathbf{C}} - r =: \Delta > 0$. Furthermore, for any $\theta_1, \theta_2 \in [0,1]$ such that $\bar{\gamma}(\tilde{\mathbf{G}}_{\theta_2}\mathbf{G}_{\theta_1}) < \Delta$, application of Theorem 3.3.1 yields that for all $\tau \in \mathbb{R}$, $\mathbf{T}_{\mathbf{G}_{\theta_2}^*\mathbf{G}_{\theta_1},\tau}$ is Fredholm and $\mathrm{ind}(\mathbf{T}_{\mathbf{G}_{\theta_2}^*\mathbf{G}_{\theta_1},\tau}) = 0$. As such, provided that $\bar{\gamma}(\tilde{\mathbf{G}}_{\theta_2}\mathbf{G}_{\theta_1})$ can be made arbitrarily small via the choice of $|\theta_2 - \theta_1|$, continuity of the path $\theta \mapsto \mathbf{P}_\theta$ in the graph topology follows. This is now established.

To begin with, recall that for any $\theta \in [0,1]$, the causal operator $\mathbf{Q}_\theta|_{\boldsymbol{L}^{2+}} : \boldsymbol{L}^{2+} \to \boldsymbol{L}^2_{\mathbb{R}}$ is bounded. Therefore, given that $\boldsymbol{L}^2_{\mathbb{R}}$ is the closure of $\boldsymbol{L}^{2+}$, $\mathbf{Q}_\theta|_{\boldsymbol{L}^{2+}}$ has a bounded linear extension $\hat{\mathbf{Q}}_\theta : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ satisfying $\hat{\mathbf{Q}}_\theta|_{\boldsymbol{L}^{2+}} = \mathbf{Q}_\theta|_{\boldsymbol{L}^{2+}}$ and $\bar{\gamma}(\hat{\mathbf{Q}}_\theta) = \bar{\gamma}(\mathbf{Q}_\theta|_{\boldsymbol{L}^{2+}})$ [Kre89, Thm. 2.7-11]. The causality of $\hat{\mathbf{Q}}_\theta$ follows from that of $\mathbf{Q}_\theta$ by Lemma 3.1.3(iii). Now observe that by definition

$$\mathbf{Z}_\theta := \mathbf{R}_c^{-1}\begin{bmatrix}\hat{\mathbf{Q}}_\theta \\ \mathbf{I}\end{bmatrix} \quad \text{and} \quad \tilde{\mathbf{Z}}_\theta := \begin{bmatrix}-\mathbf{I} & \hat{\mathbf{Q}}_\theta\end{bmatrix}\mathbf{R}_c$$

are respectively right and left graph symbols for $\mathbf{P}_\theta$ that are not necessarily normalised. Their (non-unique) respective left and right causal bounded inverses can be taken as

$$\mathbf{Z}_\theta^L = \begin{bmatrix}0 & \mathbf{I}\end{bmatrix}\mathbf{R}_c \quad \text{and} \quad \tilde{\mathbf{Z}}_\theta^R = \mathbf{R}_c^{-1}\begin{bmatrix}-\mathbf{I} \\ 0\end{bmatrix}.$$

Given normalised right and left graph symbols $\mathbf{G}_\theta$ and $\tilde{\mathbf{G}}_\theta$ for $\mathbf{P}_\theta \in \mathbb{S}$, there exist by

Lemma 3.2.11 bounded causal $\mathbf{X}_\theta$ and $\tilde{\mathbf{X}}_\theta$ such that $\mathbf{G}_\theta = \mathbf{Z}_\theta \mathbf{X}_\theta$ and $\tilde{\mathbf{G}}_\theta = \tilde{\mathbf{X}}_\theta \tilde{\mathbf{Z}}_\theta$. Indeed, $\mathbf{X}_\theta = \mathbf{Z}_\theta^L \mathbf{G}_\theta$ and $\tilde{\mathbf{X}}_\theta = \tilde{\mathbf{G}}_\theta \tilde{\mathbf{Z}}_\theta^R$, whereby $\bar{\gamma}(\mathbf{X}_\theta) \leq \bar{\gamma}(\mathbf{R}_c)$ and $\bar{\gamma}(\tilde{\mathbf{X}}_\theta) \leq \bar{\gamma}(\mathbf{R}_c^{-1})$. Finally, notice that for any $\theta_1, \theta_2 \in [0,1]$,

$$\begin{aligned}\bar{\gamma}(\tilde{\mathbf{G}}_{\theta_2}\mathbf{G}_{\theta_1}) = \bar{\gamma}(\tilde{\mathbf{X}}_{\theta_2}\tilde{\mathbf{Z}}_{\theta_2}\mathbf{Z}_{\theta_1}\mathbf{X}_{\theta_1}) &\leq \bar{\gamma}(\tilde{\mathbf{X}}_{\theta_2})\bar{\gamma}(\hat{\mathbf{Q}}_{\theta_2} - \hat{\mathbf{Q}}_{\theta_1})\bar{\gamma}(\mathbf{X}_{\theta_1}) \\ &\leq 2|\theta_2 - \theta_1|\bar{\gamma}(\mathbf{R}_c)\bar{\gamma}(\mathbf{R}_c^{-1}),\end{aligned}$$

thereby establishing continuity of the path $\theta \mapsto \mathbf{P}_\theta$. □

4.4 Reconciling IQC and ν-gap based robust stability results

We argue in this section that ν-gap homotopy-type robustness results are more general and powerful than the ball-type ones when it is possible to exploit via IQCs known structure of a feedback system in stability analysis following [MR97, JC10, JC11, CJK12, CJK09]. The development relies on the pathwise connectedness result on ν-gap metric balls in Section 4.3 to recover a sufficient ball-type robust stability condition within the IQC setting of Section 4.1.

Suppose that we are given $\mathbf{P}_c, \mathbf{C} \in \mathbb{S}$ such that $b_{\mathbf{P}_c,\mathbf{C}} > r$ for some $r < b_{\mathrm{opt}}(\mathbf{P}_c; \mathbb{S})$. Define the ν-gap metric ball of causal operators centred at $\mathbf{P}_c$:

$$\mathbb{G}_r(\mathbf{P}_c) := \{\mathbf{P} \in \mathbb{S} \mid \delta_\nu(\mathbf{P}_c, \mathbf{P}) < r\}.$$

Application of Lemmas 4.2.1 and 4.2.2 yields that

$$\mathbf{P} \in \mathrm{IQC}(\mathbf{\Psi}), \forall \mathbf{P} \in \mathbb{G}_r(\mathbf{P}_c) \text{ and } \mathbf{C} \in \mathrm{IQC}^c(\mathbf{\Psi}),$$

where

$$\mathbf{\Psi} = \mathbf{\Psi}^* := r^2\mathbf{I} - \tilde{\mathbf{G}}_c^*\tilde{\mathbf{G}}_c$$

and $\tilde{\mathbf{G}}_c$ is a normalised left graph symbol for $\mathbf{P}_c$. This is originally noted in the time-invariant case in [CJK12, CJK10, JCK08]. Moreover, $\mathbb{G}_r(\mathbf{P}_c)$ is path-connected (and hence weakly path-connected) in the graph topology by Theorem 4.3.1. Therefore, it is true that $[\mathbf{P}, \mathbf{C}]$ is stable for all $\mathbf{P} \in \mathbb{G}_r(\mathbf{P}_c)$ by Corollary 4.1.5. Notice that this also follows directly by the ν-gap ball based results, Theorem 3.3.1 or Corollary 3.3.6. Nevertheless, Corollary 4.1.5 offers greater flexibility in robust stability analysis since it can be applied to any path-connected subset of $\mathbb{G}_r(\mathbf{P}_c)$, and $\mathbf{C}$ is only required a priori to stabilise any element of the set. Note that IQCs may be found by exploiting known structure of the systems of interest [MR97, JC10, JC11, CJK12, CJK09, JCK08].

4.5 The optimal stability margin

A characterisation of the feedback robust stability/performance margin is derived in this section. The proof is based, in large part, on the the time-invariant case [Vin01, Prop. 4.1]. It relates the performance margin to the solution of a Nehari extension problem, whose optimal value can be expressed as the norm of a Hankel operator for certain classes of systems. The result is used in the ensuing section to relate the existence of the J-spectral factorisation underpinning the LFT characterisation developed in Section 4.2 to maximal robustness margin of the system involved. See [GM89, Thm. 4.1], and [CZ95, Thm. 9.4.3] for similar results in the linear time-invariant case.

Theorem 4.5.1. *Given causal operators $\mathbf{P}$ and $\mathbf{C}$ that satisfy Assumptions 2.2.1 and 2.2.2, the following are equivalent for any $r \in (0,1)$:*

(i) *$[\mathbf{P}, \mathbf{C}]$ is stable and $b_{\mathbf{P},\mathbf{C}} > r$;*

(ii) *Given a normalised left graph symbol $\tilde{\mathbf{G}}$ for $\mathbf{P}$, the operator $\mathbf{C}$ has a not necessarily normalised right graph symbol $\mathbf{K}_u$ satisfying*

$$\bar{\gamma}(\tilde{\mathbf{G}}^* - \mathbf{K}_u) < \sqrt{1-r^2};$$

(iii) *Given $\mathbf{K}$, a normalised right graph symbol for $\mathbf{C}$,*

$$\inf_{\mathbf{Q}\in\mathbb{Q}} \bar{\gamma}(\tilde{\mathbf{G}}^* - \mathbf{KQ}) < \sqrt{1-r^2}, \tag{4.8}$$

where $\mathbb{Q} := \{\mathbf{Q} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}}) \mid \mathbf{Q}$ is boundedly invertible and $\mathbf{Q}, \mathbf{Q}^{-1}$ are causal$\}$.

Proof. We establish that both (i) and (ii) are equivalent to (iii). We begin with the proof for (i) $\implies$ (iii). First recall $\begin{bmatrix} \mathbf{K}^* \\ \tilde{\mathbf{K}} \end{bmatrix}$ is an isometry from (3.11). Now note that

$$\begin{aligned}
\inf_{\mathbf{Q}\in\mathbb{Q}} \bar{\gamma}(\tilde{\mathbf{G}}^* - \mathbf{KQ}) &\geq \inf_{\mathbf{Q}\in\mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})} \bar{\gamma}(\tilde{\mathbf{G}}^* - \mathbf{KQ}) \\
&= \inf_{\mathbf{Q}\in\mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})} \bar{\gamma}\left(\begin{bmatrix} \mathbf{K}^*\tilde{\mathbf{G}}^* - \mathbf{Q} \\ \tilde{\mathbf{K}}\tilde{\mathbf{G}}^* \end{bmatrix}\right) \\
&= \bar{\gamma}(\tilde{\mathbf{K}}\tilde{\mathbf{G}}^*) = \sqrt{1-\underline{\gamma}^2(\tilde{\mathbf{K}}\mathbf{G})} = \sqrt{1-b^2_{\mathbf{P},\mathbf{C}}},
\end{aligned} \tag{4.9}$$

where (3.14) and Theorem 3.2.19 have been used. Recall from (3.11) that $\begin{bmatrix} \tilde{\mathbf{G}} \\ \mathbf{G}^* \end{bmatrix}$ is an isometry. Set $\mathbf{Q} := b^2_{\mathbf{P},\mathbf{C}}(\tilde{\mathbf{G}}\mathbf{K})^{-1} \in \mathbb{Q}$, it follows that

$$\begin{aligned}\bar{\gamma}(\tilde{\mathbf{G}}^* - \mathbf{K}\mathbf{Q}) = \bar{\gamma}\left(\begin{bmatrix} \mathbf{I} - \tilde{\mathbf{G}}\mathbf{K}\mathbf{Q} \\ \mathbf{G}^*\mathbf{K}\mathbf{Q} \end{bmatrix}\right) &= \bar{\gamma}\left(\begin{bmatrix} (1 - b^2_{\mathbf{P},\mathbf{C}})\mathbf{I} \\ b^2_{\mathbf{P},\mathbf{C}}\mathbf{G}^*\mathbf{K}(\tilde{\mathbf{G}}\mathbf{K})^{-1} \end{bmatrix}\right) \\ &= \sqrt{(1 - b^2_{\mathbf{P},\mathbf{C}})^2 + \bar{\gamma}^2(\mathbf{G}^*\mathbf{K}(\tilde{\mathbf{G}}\mathbf{K})^{-1})b^4_{\mathbf{P},\mathbf{C}}} \\ &= \sqrt{(1 - b^2_{\mathbf{P},\mathbf{C}})^2 + \frac{1 - b^2_{\mathbf{P},\mathbf{C}}}{b^2_{\mathbf{P},\mathbf{C}}} b^4_{\mathbf{P},\mathbf{C}}} = \sqrt{1 - b^2_{\mathbf{P},\mathbf{C}}},\end{aligned}$$

where the second last equality holds since $\begin{bmatrix} \mathbf{G}^*\mathbf{K} \\ \tilde{\mathbf{G}}\mathbf{K} \end{bmatrix}$ is an isometry, whereby

$$\frac{\left\langle \begin{bmatrix} \mathbf{G}^*\mathbf{K} \\ \tilde{\mathbf{G}}\mathbf{K} \end{bmatrix} u, \begin{bmatrix} \mathbf{G}^*\mathbf{K} \\ \tilde{\mathbf{G}}\mathbf{K} \end{bmatrix} u \right\rangle_2}{\langle u, u \rangle_2} = 1 \implies \frac{\|\mathbf{G}^*\mathbf{K}u\|_2^2}{\|u\|_2^2} = 1 - \frac{\|\tilde{\mathbf{G}}\mathbf{K}u\|_2^2}{\|u\|_2^2}, \forall u \in \boldsymbol{L}^2_{\mathbb{R}}.$$

To summarise, the lower bound in (4.9) is achievable when $[\mathbf{P}, \mathbf{C}]$ is stable, from which we have (i) $\implies$ (iii).

We now show that (iii) $\implies$ (i). Note that since $\tilde{\mathbf{G}}$ is normalised,

$$\bar{\gamma}(\tilde{\mathbf{G}}^* - \mathbf{K}\mathbf{Q}) < 1 \implies \bar{\gamma}\left(\tilde{\mathbf{G}}(\tilde{\mathbf{G}}^* - \mathbf{K}\mathbf{Q})\right) = \bar{\gamma}(\mathbf{I} - \tilde{\mathbf{G}}\mathbf{K}\mathbf{Q}) < 1. \tag{4.10}$$

Thus we have $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q}) > 0$, because $\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q} = \mathbf{I} - (\mathbf{I} - \tilde{\mathbf{G}}\mathbf{K}\mathbf{Q})$. Since $\mathbf{Q}$ is boundedly invertible, it then also follows that $\underline{\gamma}(\tilde{\mathbf{G}}\mathbf{K}) > 0$. Using (4.10), we see that for all $\tau \in \mathbb{R}$,

$$\bar{\gamma}(\mathbf{I} - \mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q},\tau}) \le \bar{\gamma}(\mathbf{I} - \tilde{\mathbf{G}}\mathbf{K}\mathbf{Q}) < 1,$$

which implies by Lemma 3.1.2(iii) that $\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q},\tau} = \mathbf{I} - (\mathbf{I} - \mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q},\tau})$ is Fredholm with

$$\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q},\tau}) = \operatorname{ind}(\mathbf{I}) = 0. \tag{4.11}$$

Since $\mathbf{T}^{-1}_{\mathbf{Q},\tau} = \mathbf{T}_{\mathbf{Q}^{-1},\tau}$ is Fredholm as in Remark 3.1.5 and $\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q},\tau} = \mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q},\tau}\mathbf{T}^{-1}_{\mathbf{Q},\tau}$ by Lemma 3.1.3(iv), it follows by Lemma 3.1.2(ii) that $\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}$ is Fredholm. Moreover, applying Lemma 3.1.2(ii) to (4.11) yields

$$\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) + \operatorname{ind}(\mathbf{T}_{\mathbf{Q},\tau}) = \operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K}\mathbf{Q},\tau}) = 0, \forall \tau \in \mathbb{R}.$$

However, $\operatorname{ind}(\mathbf{T}_{\mathbf{Q},\tau}) = 0\, \forall \tau \in \mathbb{R}$; see Remark 3.1.5. Therefore $\operatorname{ind}(\mathbf{T}_{\tilde{\mathbf{G}}\mathbf{K},\tau}) = 0\, \forall \tau \in \mathbb{R}$, and hence $[\mathbf{P}, \mathbf{C}]$ is stable by Theorem 3.2.19. Also, we have $b_{\mathbf{P},\mathbf{C}} > r$ by (4.8) and (4.9).

Clearly (iii) $\implies$ (ii) by Lemma 3.2.11. To show that (ii) $\implies$ (iii), we simply need to find a $\mathbf{Q} \in \mathbb{Q}$ such that $\mathbf{K}_u = \mathbf{K}\mathbf{Q}$. Such a $\mathbf{Q}$ exists by Lemma 3.2.11. $\square$

4.6 Existence of J-spectral factorisation

We consider two general classes of linear systems in this section and show existence of the J-spectral factorisation upon which the linear fractional characterisation of the ν-gap metric (Theorem 4.2.3) and the path-connectedness of ν-gap balls (Theorem 4.3.1) are dependent to be valid. The first of these is the class of finite-dimensional linear time-varying (LTV) systems with stabilisable and detectable state-space realisations in Section 2.3, while the second is the class of systems associated with multiplication by an infinite-dimensional transfer function in the constantly proper Callier-Desoer algebra in Section 2.4.

4.6.1 Finite-dimensional time-varying systems

A simplified ν-gap metric formula

Recall the class of LTV systems $\mathbb{V}$ considered in Section 2.3 and the notion of exponential dichotomy in Definition 2.3.1. Given any $\mathbf{Z} \in \mathbb{V}$ with a state-space realisation (A, B, C, D) that is exponentially dichotomous with respect to the projection P, suppose D has a uniformly bounded inverse and $A^\times := A - BD^{-1}C$ defines an exponentially dichotomous evolution with respect to the projection $P^\times$, it follows by [GKS84, Thm. II.5.2] that $\mathbf{T}_{\mathbf{Z},\tau}$ is Fredholm for all $\tau \in \mathbb{R}$ with

$$\begin{aligned}\operatorname{ind}(\mathbf{T}_{\mathbf{Z},\tau}) &= \operatorname{rank}(P) - \operatorname{rank}(P^\times) \\ &= \dim \operatorname{img}(P) - \dim \operatorname{img}\left(P^\times\right) \\ &= \dim \ker(P) - \dim \ker\left(P^\times\right).\end{aligned}$$

Now given $\mathbf{P}_1, \mathbf{P}_2 \in \mathbb{V}$, let $\mathbf{G}_1$ be a normalised right graph symbol for $\mathbf{P}_1$ and $\mathbf{G}_2, \tilde{\mathbf{G}}_2$ be respectively normalised left and right graph symbols for $\mathbf{P}_2$. Suppose a state-space realisation for $\mathbf{G}_2^*\mathbf{G}_1$ is given by (A, B, C, D), in which D has a bounded inverse and $A^\times := A - BD^{-1}C$ defines an exponential dichotomy, then Definition 3.3.4 of the ν-gap distance between $\mathbf{P}_1$ and $\mathbf{P}_2$ can be written as

$$\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) := \begin{cases} \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) & \text{if } \operatorname{rank}(P_{\mathbf{G}_2^*\mathbf{G}_1}) - \operatorname{rank}(P^\times_{\mathbf{G}_2^*\mathbf{G}_1}) \\ & \\ 1 & \text{otherwise,} \end{cases}$$

where $P_{\mathbf{G}_2^*\mathbf{G}_1}$ and $P^{\times}_{\mathbf{G}_2^*\mathbf{G}_1}$ are the projections defining the exponential dichotomies of A and $A^{\times}$, respectively.

J-spectral factorisation

The objective of this subsection is to show that the J-spectral factor in (4.2) of Theorem 4.2.3 exists for $\mathbb{V}$. To this end, the following result, which is a consolidation of Remark 3.16, Corollary 3.26, and Theorem 4.9 from [IS04], is essential.

Proposition 4.6.1. *Given an exponentially dichotomic system $\mathbf{\Sigma} \in \mathbb{V}$ of the form*

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} \begin{bmatrix} A & 0 \\ Q & -A^T \end{bmatrix} & \begin{bmatrix} B \\ -L \end{bmatrix} \\ \hline \begin{bmatrix} L^T & B^T \end{bmatrix} & S \end{array} \right),$$

in which A defines an exponentially stable evolution, $Q^T = Q$, and $S^T = S$, suppose

1. $\inf_{t\in\mathbb{R}} |\det(S(t))| > 0$*, where* det *denotes the determinant of a square matrix;*
2. *S is of a constant Jordan structure, i.e. the number of its (real) distinct eigenvalues does not depend on time $t \in \mathbb{R}$;*
3. *the family of algebraic inverses of Wiener-Hopf operators $\left\{(\mathbf{T}_{\mathbf{\Sigma},\tau})^{-1}\right\}_{\tau\in\mathbb{R}}$ is well-defined and uniformly bounded with respect to τ, i.e. there exists an $\eta > 0$ such that*
$$\|\mathbf{T}_{\mathbf{\Sigma},\tau} u\|_2 \geq \eta \|u\|_2, \forall u \in \boldsymbol{L}^2_{[\tau,\infty)}, \forall \tau \in \mathbb{R}.$$

Then the Kalman-Popov-Yakubovich system of equations in J-form

$$\begin{aligned} S &= V^T J_{p,m} V; \\ L + XB &= W^T J_{p,m} V; \\ \dot{X} + A^T X + XA + Q &= W^T J_{p,m} W, \end{aligned}$$

has a stabilising solution V, W, and $X = X^T$ that is continuous and bounded on $\mathbb{R}$ such that X is continuously differentiable on $\mathbb{R}$ and $A - BV^{-1}W$ defines an exponentially stable evolution, where $J_{p,m} := \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ denotes the sign matrix of S. Furthermore, with $\mathbf{R} := (A, B, W, V) \in \mathbb{V}$, $\mathbf{R}^{-1} \in \mathbb{V}$ is exponentially stable (or equivalently, bounded and causal on $\boldsymbol{L}^2_{\mathbb{R}}$) and

$$\mathbf{\Sigma} = \mathbf{R}^* \mathbf{J}_{p,m} \mathbf{R},$$

where $\mathbf{J}_{p,m} := \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & -\mathbf{I}_m \end{bmatrix}$. $\mathbf{\Sigma}$ *is said to admit a J-spectral factorisation when the above is true.*

Now suppose we are given an m-input p-output system $\mathbf{P} \in \mathbb{V}$, for which $\tilde{\mathbf{G}} \in \mathbb{V}$ is a normalised left graph symbol. Consider

$$\mathbf{\Sigma}_r := \tilde{\mathbf{G}}^* \tilde{\mathbf{G}} - r^2 \mathbf{I}_{p+m} \quad r \in (0,1).$$

Let $\tilde{\mathbf{G}} = (A, B, C, D)$. Using the composition of systems formula (2.5), one may write

$$\mathbf{\Sigma}_r = \left(\begin{array}{c|c} \begin{bmatrix} A & 0 \\ -C^T C & -A^T \end{bmatrix} & \begin{bmatrix} B \\ -C^T D \end{bmatrix} \\ \hline \begin{bmatrix} D^T C & B^T \end{bmatrix} & D^T D - r^2 I \end{array} \right).$$

Theorem 4.6.2. *For any* $r \in \left(0, \sqrt{1 - \sup_{\tau \in \mathbb{R}} \bar{\gamma}^2(\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau})}\right)$, $\mathbf{\Sigma}_r$ *admits a J-spectral factorisation:*

$$\mathbf{\Sigma}_r = \mathbf{R}^* \mathbf{J}_{p,m} \mathbf{R}.$$

Proof. Define $S_r := D^T D - r^2 I_{p+m}$. We first show that $\inf_{t \in \mathbb{R}} |\det(S_r(t))| > 0$ and S_r is of a constant Jordan structure, for any $r \in (0,1)$. By the normalisation property of the left graph symbol, we have that $\tilde{\mathbf{G}}\tilde{\mathbf{G}}^* = \mathbf{I}$, whereby $D(t)D^T(t) = I_p$ for all $t \in \mathbb{R}$. This implies $D(t)$ has only p non-zero singular values, and these are all equal to unity [HJ85, Thm. 7.3.5]. Consequently, as discussed in [HJ85, Section 0.4.6],

$$\operatorname{rank}(D(t)) = \operatorname{rank}\left(D^T(t)D(t)\right) = \operatorname{rank}\left(D(t)D^T(t)\right) = p.$$

Putting these together, we conclude that $D^T(t)D(t)$ has p eigenvalues equal to 1 and m zero-valued eigenvalues. It follows that $S_r(t)$ has p (positive) eigenvalues equal to $1 - r^2$ and m (negative) eigenvalues equal to $-r^2$, for any $r \in (0,1)$. As such,

$$\inf_{t \in \mathbb{R}} |\det(S_r(t))| = (1 - r^2)^p r^{2m} > 0$$

and the number of distinct eigenvalues of $S_r(t)$ is 2 for all $t \in \mathbb{R}$, i.e. S_r is of a constant Jordan structure.

Now we show that $\inf_{\tau \in \mathbb{R}} \underline{\gamma}(\mathbf{T}_{\mathbf{\Sigma}_r,\tau}) > 0$, from which the claimed result then follows by Proposition 4.6.1. To this end, fix any $\tau \in \mathbb{R}$. By Lemma 3.1.3(iv),

$$(\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau})^* \mathbf{T}_{\tilde{\mathbf{G}}^*,\tau} + (\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau})^* \mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau} = \mathbf{T}_{\tilde{\mathbf{G}}\tilde{\mathbf{G}}^*,\tau} = \mathbf{I},$$

i.e. $\begin{bmatrix} \mathbf{T}_{\tilde{\mathbf{G}}^*,\tau} \\ \mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau} \end{bmatrix}$ is an isometry, and thus we may conclude that

$$\underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau})^2 = 1 - \bar{\gamma}^2(\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau}).$$

Define the operator

$$\mathbf{B}_\tau := \tilde{\mathbf{G}}|_{\boldsymbol{L}^2_{[\tau,\infty)} \ominus \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau})}.$$

Recall that by Assumption 2.2.1, there exists a causal $\mathbf{W} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}})$ such that $\tilde{\mathbf{G}}\mathbf{W} = \mathbf{I}$ on $\boldsymbol{L}^2_{\mathbb{R}}$. Define

$$\mathbf{A}_\tau := \mathbf{\Pi}_{\boldsymbol{L}^2_{[\tau,\infty)} \ominus \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau})} \mathbf{W}|_{\boldsymbol{L}^2_{[\tau,\infty)}},$$

from which it follows that $\mathbf{B}_\tau \mathbf{A}_\tau = \mathbf{I}$ on $\boldsymbol{L}^2_{[\tau,\infty)}$. Furthermore, note that $\ker(\mathbf{B}_\tau) = \{0\}$, and hence $\mathbf{B}_\tau$ has a left inverse, which is equal to its right inverse $\mathbf{A}_\tau$, i.e. $\mathbf{A}_\tau \mathbf{B}_\tau = \mathbf{I}$ on $\boldsymbol{L}^2_{[\tau,\infty)} \ominus \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau})$. As such, $\mathbf{B}_\tau$ is boundedly invertible, and hence $\underline{\gamma}(\mathbf{B}_\tau) = \underline{\gamma}(\mathbf{B}^*_\tau)$. Now note that

$$\mathbf{B}^*_\tau = \mathbf{\Pi}_{\boldsymbol{L}^2_{[\tau,\infty)} \ominus \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau})} \tilde{\mathbf{G}}^*|_{\boldsymbol{L}^2_{[\tau,\infty)}} = \mathbf{\Pi}_{\boldsymbol{L}^2_{[\tau,\infty)}} \tilde{\mathbf{G}}^*|_{\boldsymbol{L}^2_{[\tau,\infty)}},$$

where $\mathbf{\Pi}_{\mathcal{X}}$ denotes the orthogonal projection onto $\mathcal{X}$. Note that $\mathbf{B}^*_\tau = \mathbf{T}_{\tilde{\mathbf{G}}^*,\tau}$, and hence

$$\underline{\gamma}(\mathbf{B}^*_\tau) = \underline{\gamma}(\mathbf{T}_{\tilde{\mathbf{G}}^*,\tau}) = \sqrt{1 - \bar{\gamma}^2(\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau})}.$$

Consequently,

$$\underline{\gamma}(\mathbf{B}^*_\tau \mathbf{B}_\tau) \geq \underline{\gamma}(\mathbf{B}^*_\tau)\underline{\gamma}(\mathbf{B}_\tau) = 1 - \bar{\gamma}^2(\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau}).$$

Thus for any $r \in \left(0, \sqrt{1 - \sup_{\tau\in\mathbb{R}} \bar{\gamma}^2(\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau})}\right)$, we have

$$\inf_{\tau\in\mathbb{R}} \underline{\gamma}(\mathbf{T}_{\mathbf{\Sigma}_r,\tau}|_{\boldsymbol{L}^2_{[\tau,\infty)} \ominus \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau})}) = \inf_{\tau\in\mathbb{R}} \underline{\gamma}(\mathbf{B}^*_\tau \mathbf{B}_\tau - r^2\mathbf{I}) > 0.$$

Combining the above with the observation that

$$\mathbf{T}_{\mathbf{\Sigma}_r,\tau} u = -r^2 u, \forall u \in \ker(\mathbf{T}_{\tilde{\mathbf{G}},\tau}), \tau \in \mathbb{R}$$

yields the following uniform non-zero lower bound

$$\inf_{\tau\in\mathbb{R}} \underline{\gamma}(\mathbf{T}_{\mathbf{\Sigma}_r,\tau}) > 0, \forall r \in \left(0, \sqrt{1 - \sup_{\tau\in\mathbb{R}} \bar{\gamma}^2(\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau})}\right),$$

as required. □

Remark 4.6.3. Here, we restrict to systems (A, B, C, D) that admit uniformly bounded, positive semidefinite solutions to the differential Riccati equation pair:

$$\begin{aligned}\dot{P} + PA + A^T P - PBB^T P &= 0;\\ \dot{Q} - AQ - QA^T + QC^T CQ &= 0,\end{aligned}$$

such that $A - BB^T P$ and $A - QC^T C$ define exponentially stable evolutions and the inverses of the positive definite blocks of P and Q are uniformly bounded; see [TV92, (1.10)]. By the Nehari Theorem for finite-dimensional LTV systems [TV92, Section 3], given an exponentially dichotomous state-space realisation (2.1) and the corresponding convolution operator $\mathbf{Z} : \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}$ of the form (2.3),

$$\inf_{\substack{\mathbf{Q}\text{ has an exponentially}\\ \text{stable realisation (2.1)}}} \bar{\gamma}(\mathbf{Z} - \mathbf{Q}) = \sup_{\tau\in\mathbb{R}} \bar{\gamma}(\mathbf{H}^{-+}_{\mathbf{Z},\tau}).$$

An immediate consequence of this and the equivalence between (i) and (ii) of Theorem 4.5.1 is the following bound on the maximal robustness margin of $\mathbf{P} \in \mathbb{V}$:

$$b_{\mathrm{opt}}(\mathbf{P};\mathbb{V}) := \sup_{\mathbf{C}\in\mathbb{V}\,:\,[\mathbf{P},\mathbf{C}]\text{ is stable}} b_{\mathbf{P},\mathbf{C}} \leq \sqrt{1 - \sup_{\tau\in\mathbb{R}} \bar{\gamma}^2(\mathbf{H}^{-+}_{\tilde{\mathbf{G}}^*,\tau})}, \tag{4.12}$$

where $\tilde{\mathbf{G}}$ is a normalised left graph symbol for $\mathbf{P}$. Analogous results in the linear time-invariant case can be found in [GM89, Thm. 4.2], [GS90, Thm. 2], [Vin01, Section 4.1], and [CZ95, Lem. 9.4.7]. Using (4.12), Theorem 4.6.2 implies that $\boldsymbol{\Sigma}_r$ admits a J-spectral factorisation for any $r \in (0, b_{\mathrm{opt}}(\mathbf{P};\mathbb{V}))$.

4.6.2 Infinite-dimensional time-invariant systems

The ν-gap metric for time-invariant systems

The same notation from Section 2.4 is used here. The ν-gap metric is defined on $\mathbb{W} \subset \mathbb{S}$, or directly on the frequency-domain space $\hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$, as

$$\delta_\nu(P_1, P_2) := \delta_\nu(\mathbf{P}_1, \mathbf{P}_2) = \begin{cases} \bar{\gamma}(\tilde{\mathbf{G}}_2\mathbf{G}_1) = \|\tilde{G}_2 G_1\|_\infty & \text{if for all } \tau \in \mathbb{R}, \mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau} \text{ is Fredholm} \\ & \text{and } \mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = 0 \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}$ denotes the *time-domain* Wiener-Hopf operator given by Definition 2.1.9 in Section 3.1. The definition above can be reformulated in terms of more familiar conditions in the literature [Vin93, CJK12, CJK10, CJK09], as we demonstrate below.

Following [IZ01], the Wiener algebra is defined as

$$\hat{\mathcal{W}} := \left\{ \hat{\phi} = \hat{\nu} + \hat{\mu} \;\middle|\; \hat{\nu}, \hat{\mu}^* \in \hat{\mathcal{A}}^{\mathrm{cp}} \right\}$$

and the sub-algebra

$$\hat{\mathcal{W}}_- := \left\{ \hat{\phi} = \hat{\nu} + \hat{\mu} \in \hat{\mathcal{W}} \;\middle|\; \hat{\nu}, \hat{\mu}^* \in \hat{\mathcal{A}}^{\mathrm{cp}}_- \right\}.$$

Let $\hat{\Phi} \in \hat{\mathcal{W}}$. The *frequency-domain* Wiener-Hopf (a.k.a. Toeplitz) operator with symbol $\hat{\Phi}$ is defined by

$$\boldsymbol{T}_{\hat{\Phi}} := \hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_+} \mapsto \boldsymbol{\Pi}_+ \boldsymbol{M}_{\hat{\Phi}} \hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_+},$$

where $\boldsymbol{\Pi}_+ : \boldsymbol{L}^2_{j\mathbb{R}} \to \boldsymbol{H}^2_{\mathbb{C}_+}$ denotes the orthogonal projection. By contrast, the forward and backward Hankel operators with symbol $\hat{\Phi}$ are defined respectively by

$$\boldsymbol{H}^{+-}_{\hat{\Phi}} := \hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_-} \mapsto \boldsymbol{\Pi}_+ \boldsymbol{M}_{\hat{\Phi}} \hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_+},$$

and

$$\boldsymbol{H}^{-+}_{\hat{\Phi}} := \hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_+} \mapsto \boldsymbol{\Pi}_- \boldsymbol{M}_{\hat{\Phi}} \hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_-},$$

where $\boldsymbol{\Pi}_-$ denotes orthogonal projection onto $\boldsymbol{H}^2_{\mathbb{C}_-}$. Note the Hilbert adjoints $\boldsymbol{T}^*_{\hat{\Phi}} = \boldsymbol{T}_{\hat{\Phi}^*}$ and $(\boldsymbol{H}^{-+}_{\hat{\Phi}})^* = \boldsymbol{H}^{+-}_{\hat{\Phi}^*}$.

First observe that following a similar argument to (3.31) in Section 2.5, one may show by exploiting the time-invariance property that for all $\tau \in \mathbb{R}$, $\mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,\tau}) = \mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,0})$, the latter of which is equal to $\mathrm{ind}(\boldsymbol{T}_{G_2^*G_1})$ by Lemma 3.1.2(ii), since $\mathbf{T}_{\mathbf{G}_2^*\mathbf{G}_1,0} = \mathscr{F}^{-1}\boldsymbol{T}_{G_2^*G_1}\mathscr{F}$. Indeed, given any square $\Phi \in \hat{\mathcal{W}}$, $\det(\Phi(j\omega)) \neq 0 \,\forall \omega \in \mathbb{R} \cup \{\infty\}$ if, and only if, $\boldsymbol{T}_\Phi$ is Fredholm [GGK90, Thm. XII.3.1]. In this case, the Fredholm index

$$\mathrm{ind}(\boldsymbol{T}_\Phi) := \dim\ker(\boldsymbol{T}_\Phi) - \dim\mathrm{img}(\boldsymbol{T}_\Phi)^\perp = -\mathrm{wno}\det(\Phi),$$

where wno denotes the winding number around the origin of the curve parameterised by $\omega \mapsto \det(\Phi(j\omega))$, as ω decreases from ∞ to $-\infty$; i.e. $\frac{1}{2\pi}\times$ the net increase in the argument $\angle \det \Phi(j\omega)$ as the curve is traversed. Therefore, the ν-gap metric can be written more familiarly [Vin93, Vin01, CJK12, CJK10, CJK09] as

$$\delta_\nu(P_1, P_2) := \begin{cases} \|\tilde{G}_2 G_1\|_\infty & \text{if } \det(G_2^* G_1(j\omega)) \neq 0 \,\forall \omega \in \mathbb{R} \cup \{\infty\} \\ & \text{and } \mathrm{wno}\det(G_2^* G_1) = 0 \\ & \\ 1 & \text{otherwise.} \end{cases}$$

The robust stability margin of a feedback configuration, has also an equivalent frequency-domain representation. The feedback interconnection of Figure 3.1 in Section 3.2 is reproduced by Figure 4.3 in the frequency domain with $P, C \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$. Notice that by Definition 3.2.1 and (3.5), feedback stability is equivalent to

$$[P, C] := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \in \hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}},$$

in which $\boldsymbol{M}_{[P,C]} = \begin{bmatrix} d_y \\ d_u \end{bmatrix} \mapsto \begin{bmatrix} y_p \\ u_p \end{bmatrix}$. When this is the case, the robust stability margin of $[P, C]$ is $b_{P,C} := b_{\mathbf{P},\mathbf{C}} = \|[P, C]\|_{\infty}^{-1}$. Moreover, the maximal robustness margin

$$b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}) := \sup_{C \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}} : [P,C] \in \hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}}} b_{P,C} = \sup_{\mathbf{C} \in \mathbb{W} : [\mathbf{P},\mathbf{C}] \text{ is stable}} b_{\mathbf{P},\mathbf{C}} =: b_{\mathrm{opt}}(\mathbf{P}; \mathbb{W}).$$

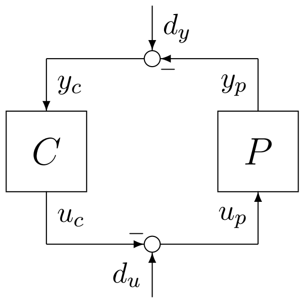

Figure 4.3: Feedback interconnection of transfer functions in $\hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$

J-spectral factorisation

Given any m-input p-output $P \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$ and its associated time-domain operator $\mathbf{P} \overset{\mathscr{F}}{\leftrightarrow} \boldsymbol{M}_P$, we show here that for any $r \in (0, b_{\mathrm{opt}}(\mathbf{P}; \mathbb{W}))$, there exists a causal $\mathbf{R} \in \mathscr{L}(\boldsymbol{L}_{\mathbb{R}}^2, \boldsymbol{L}_{\mathbb{R}}^2)$ such that $\mathbf{R}^{-1} \in \mathscr{L}(\boldsymbol{L}_{\mathbb{R}}^2, \boldsymbol{L}_{\mathbb{R}}^2)$ is causal and

$$\boldsymbol{\Sigma}_r := \tilde{\mathbf{G}}^* \tilde{\mathbf{G}} - r^2 \mathbf{I}_{p+m} = \mathbf{R}^* \mathbf{J}_{p,m} \mathbf{R},$$

where $\mathbf{J}_{p,m} := \begin{bmatrix} \mathbf{I}_p & 0 \\ 0 & -\mathbf{I}_m \end{bmatrix}$ and $\tilde{\mathbf{G}}$ is a normalised left graph symbol for $\mathbf{P}$. The J-spectral factorisation problem can be equivalently formulated in the frequency domain by exploiting the equivalence of operators via the Fourier transform isomorphism as

$$\Sigma_r := \tilde{G}^* \tilde{G} - r^2 I_{p+m} = R^* J_{p,m} R \quad \text{on } j\mathbb{R},$$

where $r \in (0, b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}))$, $J_{p,m} := \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix}$, and $R, R^{-1} \in \hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}}$. Towards resolving the existence issue, some preliminary results are needed.

In general, we say that a $\Phi = \Phi^* \in \hat{\boldsymbol{\mathcal{W}}}$ admits a $J_{p,m}$-spectral factorisation over $\hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}$ (resp. $\hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}}$) if there exists a spectral factor $R, R^{-1} \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}$ (resp. $\hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}}$) such that

$$\Phi(j\omega) = R^*(j\omega) J_{p,m} R(j\omega) \quad \forall \omega \in \mathbb{R};$$

this is possible only if $\Phi(j\omega)$ has p positive and m negative eigenvalues for all $\omega \in \mathbb{R}$.

Proposition 4.6.4. *[IZ01, Thm. 3.2] Let $\Phi = \Phi^* \in \hat{\boldsymbol{\mathcal{W}}}$ be such that* $\det(\Phi(j\omega)) \neq 0$ *for all $\omega \in \mathbb{R} \cup \{\infty\}$. The following statements are equivalent:*

(i) *Φ admits a $J_{p,m}$-spectral factorisation over $\hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}$;*

(ii) *Φ has no equalising vector — there exists no non-zero $\hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_+}$ such that $\boldsymbol{M}_\Phi \hat{u} \in \boldsymbol{H}^2_{\mathbb{C}_-}$;*

(iii) *$\boldsymbol{T}_\Phi$ is boundedly invertible.*

Lemma 4.6.5. *Let $\Phi = \Phi^* \in \hat{\boldsymbol{\mathcal{W}}}_-$. If Φ admits a $J_{p,m}$-spectral factorisation over $\hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}$, then it admits a $J_{p,m}$-spectral factorisation over $\hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}}$.*

Proof. This holds via the argument employed in Step 2 of the proof for [CW90, Thm. 1], where a special type of J-spectral factorisation (the standard spectral factorisation) is considered; specifically, with $J_{p,m} = J_{p,0} = I_p$. The proof continues to hold for the general J-spectral factorisation considered here, since multiplication by $J = J^{-1}$ does not change the holomorphicity or continuity properties of a transfer function. $\square$

We are now ready to establish the main result of this subsection.

Theorem 4.6.6. *Given an m-input p-output $P \in \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$, for any $r \in (0, b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}))$, the transfer function $\Sigma_r := \tilde{G}^*\tilde{G} - r^2 I_{p+m} \in \hat{\boldsymbol{\mathcal{W}}}_-$ admits a $J_{p,m}$-spectral factorisation over $\hat{\boldsymbol{\mathcal{A}}}_{-}^{\mathrm{cp}}$.*

Proof. Below we proceed to show that

i) $\det(\Sigma_r(j\omega)) \neq 0 \, \forall \, \omega \in \mathbb{R} \cup \{\infty\}$; and

ii) $\ker(\boldsymbol{T}_{\Sigma_r}) = \{0\}$, which is equivalent to Σ_r having no equalising vector.

The assertion then follows by Proposition 4.6.4 and Lemma 4.6.5.

i). Since $\tilde{G}$ is normalised, $\tilde{G}(j\omega)\tilde{G}^*(j\omega) = I_p$ for all $\omega \in \mathbb{R} \cup \{\infty\}$; see Proposition 2.4.1. This implies $\tilde{G}(j\omega)$ has only p non-zero singular values, which are all 1's [HJ85, Thm.

7.3.5]. As such, by [HJ85, Sec. 0.4.6],

$$\operatorname{rank}(\tilde{G}(j\omega)) = \operatorname{rank}(\tilde{G}^*(j\omega)\tilde{G}(j\omega)) = \operatorname{rank}(\tilde{G}(j\omega)\tilde{G}^*(j\omega)) = p.$$

Together, one may conclude that $\tilde{G}^*(j\omega)\tilde{G}(j\omega)$ has p unity eigenvalues and m zero-valued eigenvalues. It follows that $\Sigma_r(j\omega)$ has p positive and m negative eigenvalues, for any $r \in (0,1)$ and $\omega \in \mathbb{R} \cup \{\infty\}$. Finally, note that $\det(\Sigma_r(j\omega)) = 0$ if, and only if, zero is an eigenvalue of $\Sigma_r(j\omega)$, by which $i)$ holds as claimed.

ii). Observe that for any $u \in \ker(\boldsymbol{T}_{\tilde{G}})$, $\boldsymbol{T}_{\Sigma_r} u = -r^2 u$. Thus, $\ker(\boldsymbol{T}_{\Sigma_r}) \cap \ker(\boldsymbol{T}_{\tilde{G}}) = \{0\}$. Moreover, as established below, $\ker(\boldsymbol{T}_{\Sigma_r}) \cap (\boldsymbol{H}^2_{\mathbb{C}_+} \ominus \ker(\boldsymbol{T}_{\tilde{G}})) = \{0\}$, from which the result follows as claimed. Specifically, by [CZ95, Lem. 9.4.7] and Proposition 2.4.1 we have

$$b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}})^2 = 1 - \bar{\gamma}(\boldsymbol{H}^{-+}_{\tilde{G}^*})^2 > 0,$$

and because $\boldsymbol{T}^*_{\tilde{G}^*}\boldsymbol{T}_{\tilde{G}^*} + (\boldsymbol{H}^{-+}_{\tilde{G}^*})^*\boldsymbol{H}^{-+}_{\tilde{G}^*} = I$, we conclude that

$$\underline{\gamma}(\boldsymbol{T}_{\tilde{G}^*})^2 := \inf_{x \in \boldsymbol{H}^2_{\mathbb{C}_+} : \|x\|_2 = 1} \|\boldsymbol{T}_{\tilde{G}^*} x\|_2^2 = 1 - \bar{\gamma}(\boldsymbol{H}^{-+}_{\tilde{G}^*})^2 = b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}})^2.$$

Now define the operator $\boldsymbol{B} := \boldsymbol{M}_{\tilde{G}}|_{\boldsymbol{H}^2_{\mathbb{C}_+} \ominus \ker(\boldsymbol{T}_{\tilde{G}})}$, whereby

$$\boldsymbol{B}^* = \boldsymbol{\Pi}_{\boldsymbol{H}^2_{\mathbb{C}_+} \ominus \ker(\boldsymbol{T}_{\tilde{G}})} \boldsymbol{M}_{\tilde{G}^*}|_{\boldsymbol{H}^2_{\mathbb{C}_+}} = \boldsymbol{\Pi}_{\boldsymbol{H}^2_{\mathbb{C}_+}} \boldsymbol{M}_{\tilde{G}^*}|_{\boldsymbol{H}^2_{\mathbb{C}_+}}.$$

Note that $\boldsymbol{B}^* = \boldsymbol{T}_{\tilde{G}^*}$, and hence

$$\underline{\gamma}(\boldsymbol{B}^*) = \underline{\gamma}(\boldsymbol{T}_{\tilde{G}^*}) = b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}) > 0.$$

Since $\boldsymbol{B}$ has a bounded inverse $\boldsymbol{A} := \boldsymbol{\Pi}_{\boldsymbol{H}^2_{\mathbb{C}_+} \ominus \ker(\boldsymbol{T}_{\tilde{G}})} \boldsymbol{M}_W|_{\boldsymbol{H}^2_{\mathbb{C}_+}}$, where $W \in \hat{\boldsymbol{\mathcal{A}}}^{\mathrm{cp}}$ is such that $\tilde{G}W = I$, which exists by Proposition 2.4.1, we have $\underline{\gamma}(\boldsymbol{B}) = \underline{\gamma}(\boldsymbol{B}^*)$. As such, it follows that $\underline{\gamma}(\boldsymbol{B}^*\boldsymbol{B}) \geq \underline{\gamma}(\boldsymbol{B}^*)\underline{\gamma}(\boldsymbol{B}) = b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}})^2$. Thus for any $r < b_{\mathrm{opt}}(P; \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}})$, we have

$$\underline{\gamma}(\boldsymbol{T}_{\Sigma_r}|_{\boldsymbol{H}^2_{\mathbb{C}_+} \ominus \ker(\boldsymbol{T}_{\tilde{G}})}) = \underline{\gamma}(\boldsymbol{B}^*\boldsymbol{B} - r^2 I) > 0,$$

implying $\ker\left(\boldsymbol{T}_{\Sigma_r}|_{\boldsymbol{H}^2_{\mathbb{C}_+} \ominus \ker(\boldsymbol{T}_{\tilde{G}})}\right) = \{0\}$, as required. □

4.7 Summary

The integration of integral quadratic constraint (IQC) based system analysis with the ν-gap metric from [JC10, JC11] is reviewed in this chapter. The main significance of

IQC conditions lies in providing a uniform lower bound on the robust stability margin of an uncertain feedback interconnection. A powerful stability analysis method results when this is unified with ν-gap homotopies, allowing exploitation of system structure in an explicit fashion. Reconciliation with the original ν-gap framework in Chapter 3 is carried out via a ν-gap ball path-connectedness result, shown using a linear fractional characterisation of the ν-gap metric. An underlying assumption is the existence of a certain J-spectral factorisation, which is also developed herein for two standard classes of linear systems from Chapter 2: time-varying systems with finite-dimensional stabilisable and detectable state-space realisations and distributed-parameter time-invariant systems in the constantly proper Callier-Desoer algebra.

Chapter 5

Sampled-data approximation in the ν-gap metric

Approximation involves the construction of a model satisfying *structural constraints*, while remaining *close* to another model that is perhaps less tractable. Such problems arise in the simulation and/or design of systems from broad array of application areas [Ant05]. Error is inevitably incurred in an approximation process. Within the context of feedback modelling or compensator design, it is more important to measure approximation error in terms of the difference in closed-loop behaviour. Endowed with the feedback robustness properties delineated in Chapter 3, the ν-gap metric is a natural candidate to serve this purpose. In addition, the linear-fractional characterisation developed in Chapter 4 presents an elegant way through which to approach model approximation problems in the ν-gap due to the fact that linear fractional transformations (LFTs) are well-studied in the literature.

This chapter revisits the problem of sampled-data approximation of a continuous linear time-invariant (LTI) system with respect to a *weighted* ν-gap measure; see [CV04]. This is particularly important in robust digital implementation of feedback control strategies for continuous-time processes. A main concern in this chapter is to establish that the class of linear periodically time-varying (LPTV) systems introduced in Section 2.5 is sufficiently rich to equivalently represent both types of systems involved in the problem via the time-lifting technique [BPFT91, BP92, Yam94, CF95], whereby they satisfy the assumptions for the ν-gap distance to be well-defined. Using the LFT characterisation of the ν-gap metric developed in Chapter 4, a linear matrix inequality (LMI) [GA94] based stepwise algorithm is proposed for numerically synthesising a sampled-data approximation which lies within the minimal ν-gap distance from a nominal LTI system. The difference between the work in this chapter, which relies on a single J-spectral factorisation underpinning the

LFT characterisation, and [CV04], which employes a less-direct three-term factorisation to establish the result, is of note.

The chapter evolves along the following lines. First in the next section model approximation in the ν-gap metric is formulated. Using the LFT characterisation, an equivalent more tractable optimisation problem is proposed. Section 5.2 develops shift-invariant representations for LTI and sampled-data systems. In Section 5.3, the sampled-data approximation problem is formulated and solved, and a system of LMI conditions is explicitly provided to characterise the convex LFT synthesis problem arising from the proposed procedure. Finally, examples are provided to illustrate the algorithm's numerical tractability.

5.1 General approximation problem formulation

First recall from Section 3.3 that the set of causal systems on which the ν-gap metric is defined is denoted by $\mathbb{S}$. Now given subsets $\mathbb{S}_{\mathrm{T}}$ and $\mathbb{S}_{\mathrm{A}}$, which reflect the structural constraints on the targets of approximation and approximations, respectively, the problem may be posed as a constrained optimisation:

$$\text{For some } \mathbf{P} \in \mathbb{S}_{\mathrm{T}},\ \beta_{opt} := \inf \left\{\beta \in (0, b_{\mathrm{opt}}(\mathbf{P};\mathbb{S})) \mid \exists \mathbf{P}_a \in \mathbb{S}_{\mathrm{A}} \text{ such that } \delta_\nu(\mathbf{P},\mathbf{P}_a) < \beta \right\}. \tag{5.1}$$

Let $\mathbb{Q} := \{\mathbf{Q} \in \mathscr{L}(\boldsymbol{L}^2_{\mathbb{R}}, \boldsymbol{L}^2_{\mathbb{R}}) \mid \mathbf{Q} \text{ is boundedly invertible and } \mathbf{Q}, \mathbf{Q}^{-1} \text{ are causal}\}$. As in Theorem 4.2.3 of Chapter 4, suppose there exists a ν-gap J-spectral factor $\mathbf{R}(\mathbf{P},\beta) \in \mathbb{Q}$ such that

$$\tilde{\mathbf{G}}^*\tilde{\mathbf{G}} - \beta^2\mathbf{I} = \mathbf{R}^* \left[\begin{smallmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{smallmatrix}\right] \mathbf{R}, \quad \beta \in (0, b_{\mathrm{opt}}(\mathbf{P};\mathbb{S})), \tag{5.2}$$

where $\tilde{\mathbf{G}}$ is a normalised left graph symbol for $\mathbf{P}$. Then given any $\mathbf{P}_a \in \mathbb{S}$ for which $\tilde{\mathbf{G}}_a$ is a left graph symbol, $\delta_\nu(\mathbf{P},\mathbf{P}_a) < \beta$ if, and only if, the linear fractional transformation (LFT) depicted in Figure 4.1

$$\mathcal{F}(\mathbf{R},\mathbf{P}_a) : \mathrm{dom}\,(\mathcal{F}(\mathbf{R},\mathbf{P}_a)) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}},$$

for which $\tilde{\mathbf{G}}_a\mathbf{R}^{-1}$ is a left graph symbol, is causal and strictly contractive on $\boldsymbol{L}^{2+}$, that is, $\bar{\gamma}(\mathcal{F}(\mathbf{R},\mathbf{P}_a)|_{\boldsymbol{L}^{2+}}) < 1$. Correspondingly, the problem (5.1) may be recast as: Given a

$\mathbf{P} \in \mathbb{S}_{\mathrm{T}}$,

$$\beta_{opt} := \inf \left\{ \beta \in (0, b_{\mathrm{opt}}(\mathbf{P};\mathbb{S})) \;\middle|\; \begin{array}{l} \exists \mathbf{P}_a \in \mathbb{S}_{\mathrm{A}} \text{ such that } \mathcal{F}(\mathbf{R},\mathbf{P}_a) \text{ is causal and} \\ \bar{\gamma}(\mathcal{F}(\mathbf{R},\mathbf{P}_a)|_{\boldsymbol{L}^{2+}}) < 1, \text{where } \mathbf{R}(\mathbf{P},\beta) \in \mathbb{Q} \text{ solves (5.2)} \end{array} \right\}. \tag{5.3}$$

Observe that tractability of the problem (5.3) is tied with that of finding a structurally constrained operator $\mathbf{P}_a \in \mathbb{S}_{\mathrm{A}}$ such that an LFT $\mathcal{F}(\mathbf{R},\mathbf{P}_a)$ is causal and its gain/norm on $\boldsymbol{L}^{2+}$ is strictly bounded above by 1, where $\mathbf{R}$ is a ν-gap J-spectral factor for $\mathbf{P} \in \mathbb{S}_{\mathrm{T}}$.

In the problem of sampled-data approximation, $\mathbb{S}_{\mathrm{T}}$ corresponds to the set of finite-dimensional LTI systems with real-rational transfer function representations and $\mathbb{S}_{\mathrm{A}}$ consists systems with a sampled-data structure, composed of a stable finite-dimensional anti-aliasing LTI filter, a periodic sampler, a discrete-time finite-dimensional linear shift-invariant (LSI) system, and a synchronised zero-order hold. These are further detailed in the next section.

5.2 Shift-invariant representations for periodic systems

This section constructs shift-invariant representations for two classes of linear periodically time-varying (LPTV) systems which appear in the problem of sampled-data approximation. Specifically, we consider the class of finite-dimensional linear time-invariant (LTI) systems, characterised by a rational frequency-domain symbol, and a class of LPTV systems with sampled-data structure. We study them here by working with the system graphs and establish appropriate representations of these systems in line with the ν-gap metric based robustness analysis from the previous chapters.

5.2.1 Preliminaries on system classes

Let $\mathcal{X}$ and $\mathcal{Y}$ denote arbitrary separable Hilbert spaces. Define the following transfer function classes with 'rational' realisations:

$$\begin{aligned}
\boldsymbol{\mathcal{R}}^{p\times m} &:= \left\{ \begin{array}{l} s \mapsto C(sI-A)^{-1}B + D \\ \in \mathscr{L}(\mathbb{C}, \mathscr{L}(\mathbb{C}^m,\mathbb{C}^p)) \text{ (a.e. on } \mathbb{C}) \end{array} \;\middle|\; \begin{array}{l} A \in \mathbb{R}^{n\times n};\ B \in \mathbb{R}^{n\times m}; \\ C \in \mathbb{R}^{p\times n};\ D \in \mathbb{R}^{p\times m} \end{array} \right\}; \\
\boldsymbol{\mathcal{D}}(\mathcal{X},\mathcal{Y}) &:= \left\{ \begin{array}{l} z \mapsto zC(I-zA)^{-1}B + D \\ \in \mathscr{L}(\mathbb{C}, \mathscr{L}(\mathcal{X},\mathcal{Y})) \text{ (a.e. on } \mathbb{C}) \end{array} \;\middle|\; \begin{array}{l} A \in \mathbb{R}^{n\times n};\ B \in \mathscr{L}(\mathcal{X},\mathbb{R}^n); \\ C \in \mathscr{L}(\mathbb{R}^n,\mathcal{Y});\ D \in \mathscr{L}(\mathcal{X},\mathcal{Y}) \end{array} \right\}; \\
\boldsymbol{\mathcal{D}}^{p\times m} &:= \boldsymbol{\mathcal{D}}(\mathbb{C}^m,\mathbb{C}^p).
\end{aligned}$$

Also recall the notation $\boldsymbol{\mathcal{L}}$, $\boldsymbol{\mathcal{L}}_{CF}$, and $\boldsymbol{\mathcal{L}}_{HS}$ defined in Section 2.5.

In the sequel, we may suppress the dimensions of the classes for notational simplicity. If the operator 'D' in a realisation of a transfer function is 0, we say the transfer function is *strictly proper*. The *order/complexity* of a realisation of a transfer function is defined to be the dimension of its 'A' matrix.

Remark 5.2.1. A realisation of a transfer function is not unique. A minimal realisation is one of minimal order. Given any realisation of a transfer function in $\boldsymbol{\mathcal{R}}$, $\boldsymbol{\mathcal{D}}$ or $\boldsymbol{\mathcal{L}}$, a minimal realisation can be constructed via a Kalman canonical decomposition; see [IOW99, Thm. 1.13.2] for $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{\mathcal{D}}$, and [Can98] for $\boldsymbol{\mathcal{L}}$.

We define the following sub-classes with an additional condition on the spectrum of the 'A' matrix in any minimal realisation of a transfer function P:

$$\begin{aligned} \boldsymbol{\mathcal{R}H}^{\infty}_{\mathbb{C}_+} &:= \{P \in \boldsymbol{\mathcal{R}} \ \mid \operatorname{spec}(A) \subset \mathbb{C}_- \}\,; \\ \boldsymbol{\mathcal{D}H}^{\infty}_{\mathbb{D}}(\mathcal{X},\mathcal{Y}) &:= \{P \in \boldsymbol{\mathcal{D}}(\mathcal{X},\mathcal{Y}) \ \mid \operatorname{spec}(A) \subset \mathbb{D}\}\,, \end{aligned} \tag{5.4}$$

where $\mathbb{C}_-$ denotes the open left-half complex plane and $\mathbb{D}$ the unit disk. For a $P \in \boldsymbol{\mathcal{R}H}^{\infty}_{\mathbb{C}_+}$, $\|P\|_\infty := \sup_{s\in\mathbb{C}_{0+}} \bar{\sigma}(P(s)) < \infty$, where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. On the other hand, for a $P \in \boldsymbol{\mathcal{D}H}^{\infty}_{\mathbb{D}}(\mathcal{X},\mathcal{Y})$, $\|P\|_\infty := \sup_{z\in\mathbb{D}} \bar{\gamma}(P(z)) < \infty$.

Multiplication operators with symbols in $\boldsymbol{\mathcal{L}}$ are as defined in Section 2.5. They can be generalised to symbols in $\boldsymbol{\mathcal{D}}(\mathcal{X},\mathcal{Y})$ as follows. Given a $P \in \boldsymbol{\mathcal{D}}(\mathcal{X},\mathcal{Y})$, define the associated multiplication $\boldsymbol{M}_P : \operatorname{dom}(\boldsymbol{M}_P) : \boldsymbol{L}^2_{\mathbb{T}}(\mathcal{X}) \to \boldsymbol{L}^2_{\mathbb{T}}(\mathcal{Y})$ by $(\boldsymbol{M}_P u)(z) := P(z)u(z)$, where

$$\operatorname{dom}(\boldsymbol{M}_P) := \bigcup_{k\in\mathbb{Z}} z^k \left\{u \in \boldsymbol{H}^2_{\mathbb{D}}(\mathcal{X}) \ \middle|\ \boldsymbol{M}_P u \in \boldsymbol{H}^2_{\mathbb{D}}(\mathcal{Y}) \right\},$$

and $\boldsymbol{L}^2_{\mathbb{T}}(\mathcal{Z})$ and $\boldsymbol{H}^2_{\mathbb{D}}(\mathcal{Z})$ are respectively discrete-time Fourier transforms[1] $\mathscr{Z}$ of signals in

$$\begin{aligned} \boldsymbol{\ell}^2_{\mathbb{Z}}(\mathcal{Z}) &:= \left\{ f : \mathbb{Z} \to \mathcal{Z} \ \middle|\ \|f\|^2_{\boldsymbol{\ell}^2_{\mathbb{Z}}} := \sum_{i=-\infty}^{\infty} \|f_i\|^2_{\mathcal{X}} < \infty \right\} \quad \text{and} \\ \boldsymbol{\ell}^2_{\mathbb{Z}_+}(\mathcal{Z}) &:= \left\{ f \in \boldsymbol{\ell}^2_{\mathbb{Z}}(\mathcal{Z}) \,|\, f_i = 0, \forall i < 0 \right\}. \end{aligned}$$

When $P \in \boldsymbol{\mathcal{D}H}^{\infty}_{\mathbb{D}}$, $\operatorname{dom}(\boldsymbol{M}_P) := \boldsymbol{L}^2_{\mathbb{T}}(\mathcal{X})$. Note that given a $P \in \boldsymbol{\mathcal{D}}$, $P \in \boldsymbol{\mathcal{D}H}^{\infty}_{\mathbb{D}}$ if, and only if, $\bar{\gamma}(\boldsymbol{M}_P) = \|P\|_\infty < \infty$ and $\boldsymbol{M}_P \boldsymbol{H}^2_{\mathbb{D}}(\mathcal{X}) \subset \boldsymbol{H}^2_{\mathbb{D}}(\mathcal{Y})$; see [SNF70, Chapter 5]. On the other hand, multiplication operators with symbols in $\boldsymbol{\mathcal{R}} \subset \hat{\boldsymbol{\mathcal{B}}}^{\mathrm{cp}}$ are defined as in Section 2.4.

Fundamental to the subsequent analysis is the result that all transfer functions in $\boldsymbol{\mathcal{R}}$ admit normalised doubly coprime factorisations over $\boldsymbol{\mathcal{R}H}^{\infty}_{\mathbb{C}_+}$. Recall that a similar result holds for $\boldsymbol{\mathcal{L}}_{CF}$ and $\boldsymbol{\mathcal{L}}_{HS}$; see Definition 2.5.1 and Proposition 2.5.2.

[1] $\mathscr{Z} : \boldsymbol{\ell}^2_{\mathbb{Z}}(\mathcal{Z}) \to \boldsymbol{L}^2_{\mathbb{T}}(\mathcal{Z})$; $\quad (\mathscr{Z} f)(z) := \sum_{i\in\mathbb{Z}} z^i f_i$.

Proposition 5.2.2 ([ZDG96, Thm. 13.37]). *Given any $P \in \boldsymbol{\mathcal{R}}$, there exist*

$$N, M, \tilde{M}, \tilde{N}, X, \tilde{X}, Y, \tilde{Y} \in \boldsymbol{\mathcal{R}H}^{\infty}_{\mathbb{C}_+}$$

such that

$$\begin{aligned} \begin{bmatrix} Y & X \\ \tilde{M} & -\tilde{N} \end{bmatrix} \begin{bmatrix} N & \tilde{X} \\ M & -\tilde{Y} \end{bmatrix} = I; \quad NM^{-1} = \tilde{M}^{-1}\tilde{N} = P; \\ M^*M + N^*N = I; \quad \tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I. \end{aligned} \tag{5.5}$$

5.2.2 Time-lifting results

The same notation from Section 2.1.1 on equivalence of operators is used in this section.

Time-invariant systems

We show that multiplication by any element in $\boldsymbol{\mathcal{R}}$ is equivalent to multiplication by a corresponding element in $\boldsymbol{\mathcal{L}}_{CF}$ via a composition of isomorphisms: the continuous-time Fourier transform $\mathscr{F}$, the time-lifting $\mathscr{W}_h$, and the discrete-time Fourier transform $\mathscr{Z}$ (cf. Section 2.5). Note that the converse is not true. In particular, $\boldsymbol{\mathcal{L}}_{CF}$ is larger than $\boldsymbol{\mathcal{R}}$ in the sense that it can include symbols that correspond to (non-stationary) periodic realisations in the time-domain. Our derivation makes use of an *exponentially stable* normalised characterisation of the system graph. Such a characterisation naturally admits a convolution/integral operator realisation in the time domain, as required to directly apply the lifting isomorphism from [BPFT91, BP92, Yam94, CF95].

The following theorem demonstrates the lifting procedure of a *stable* transfer function. It includes a condition for lifted transfer function invertibility. The fact that multiplication by a stable transfer function defines an exponentially stable evolution over the doubly infinite time is exploited in the proof. The more general case is considered later in Theorem 5.2.5.

Theorem 5.2.3. *Given $P = (A, B, C, D) \in \boldsymbol{\mathcal{R}H}^{\infty}_{\mathbb{C}_+}$, let $\underline{P} := (\grave{A}, \grave{B}, \grave{C}, \grave{D}) \in \boldsymbol{\mathcal{L}}$, where for any $x \in \mathbb{R}^n$ and $w \in \boldsymbol{L}^2_{[0,h)}$,*

$$\begin{aligned} \grave{A}x = e^{Ah}x; \quad \grave{B}w = \int_0^h e^{A(h-\tau)}Bw(\tau)\, d\tau; \\ (\grave{C}x)(\theta) = Ce^{A\theta}x, \ \forall \theta \in [0, h); \\ (\grave{D}w)(\theta) = Dw(\theta) + \int_0^\theta Ce^{A(\theta-\tau)}Bw(\tau)\, d\tau, \ \forall \theta \in [0, h). \end{aligned}$$

Then $\boldsymbol{M}_P \overset{\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}}{\sim} \boldsymbol{M}_{\underline{P}}$. *Moreover,* $\underline{P}^{-1} \in \boldsymbol{\mathcal{L}}$ *if* $P^{-1} \in \boldsymbol{\mathcal{R}}$.

Proof. Clearly $\grave{D}$ is causal. Note that $\operatorname{spec}(\grave{A}) \in \mathbb{D}$ since $\operatorname{spec}(A) \in \mathbb{C}_-$ and thus, $\underline{P} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$; see (5.4). Since $P \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$, i.e. A defines an exponentially stable dichotomy, by [IOW99, Thm. 2.2.1], any $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}}$, where $\mathbf{P} \overset{\mathscr{F}}{\leftrightarrow} \boldsymbol{M}_P$, satisfies

$$y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t), \forall t \in \mathbb{R}. \tag{5.6}$$

Likewise, because $\underline{P} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$, by [IOW99, Thm. 2.6.1], any $\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] \in \mathscr{G}_{\underline{\mathbf{P}}}$, where $\underline{\mathbf{P}} \overset{\mathscr{Z}}{\leftrightarrow} \boldsymbol{M}_{\underline{P}}$, satisfies

$$\underline{y}_k = \sum_{i=-\infty}^{k-1} \grave{C}\grave{A}^{k-i-1}\grave{B}\underline{u}_i + \grave{D}\underline{u}_k, \forall k \in \mathbb{Z}. \tag{5.7}$$

We now show that $\mathbf{P} \overset{\mathscr{W}_h}{\sim} \underline{\mathbf{P}}$, which, together with $\mathbf{P} \overset{\mathscr{F}}{\sim} \boldsymbol{M}_P$ and $\underline{\mathbf{P}} \overset{\mathscr{Z}}{\sim} \boldsymbol{M}_{\underline{P}}$, implies that $\boldsymbol{M}_P \overset{\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}}{\sim} \boldsymbol{M}_{\underline{P}}$.

Given any $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}}$ satisfying (5.6), for any $h > 0$, $k \in \mathbb{Z}$ and $\theta \in [0, h)$,

$$\begin{aligned}
& y(kh+\theta) \\
&= \int_{-\infty}^{kh+\theta} Ce^{A(kh+\theta-\tau)}Bu(\tau)\,d\tau + Du(kh+\theta) \\
&= \int_{-\infty}^{kh} Ce^{A(kh+\theta-\tau)}Bu(\tau)\,d\tau + \int_{kh}^{kh+\theta} Ce^{A(kh+\theta-\tau)}Bu(\tau)\,d\tau + Du(kh+\theta) \\
&= Ce^{A\theta} \int_{-\infty}^{kh} e^{A(kh-\tau)}Bu(\tau)\,d\tau + \int_0^\theta Ce^{A(\theta-\tau)}Bu(kh+\tau)\,d\tau + Du(kh+\theta) \\
&= Ce^{A\theta} \sum_{i=-\infty}^{k-1} \int_0^h e^{A(kh-ih-\tau)}Bu(ih+\tau)\,d\tau + \int_0^\theta Ce^{A(\theta-\tau)}Bu(kh+\tau)\,d\tau + Du(kh+\theta) \\
&= Ce^{A\theta} \sum_{i=-\infty}^{k-1} e^{Ah(k-i-1)} \int_0^h e^{A(h-\tau)}Bu(ih+\tau)\,d\tau \\
&\qquad\qquad + \int_0^\theta Ce^{A(\theta-\tau)}Bu(kh+\tau)\,d\tau + Du(kh+\theta).
\end{aligned}$$

Letting $\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] := \mathscr{W}_h \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$ yields

$$\underline{y}_k(\theta) = \sum_{i=-\infty}^{k-1} Ce^{A\theta}(e^{Ah})^{k-i-1} \int_0^h e^{A(h-\tau)}B\underline{u}_i(\tau)\,d\tau + \int_0^\theta Ce^{A(\theta-\tau)}B\underline{u}_k(\tau)\,d\tau + D\underline{u}_k(\theta),$$

which is (5.7). In particular, $\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] \in \mathscr{G}_{\underline{\mathbf{P}}}$. Conversely, given any $\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] \in \mathscr{G}_{\underline{\mathbf{P}}}$, by reversing the line of argument above, it follows that $\mathscr{W}_h^{-1}\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}}$. As such, $\mathscr{G}_{\mathbf{P}} \overset{\mathscr{W}_h}{\sim} \mathscr{G}_{\underline{\mathbf{P}}}$ and hence, $\mathbf{P} \overset{\mathscr{W}_h}{\sim} \underline{\mathbf{P}}$.

For the final part of the lemma, observe that if D is invertible,

$$\begin{aligned}(\grave{D}w)(\theta) &= Dw(\theta) + \int_0^\theta Ce^{A(\theta-\tau)}Bw(\tau)\,d\tau \\ &= D\left(w(\theta) + \int_0^\theta D^{-1}Ce^{A(\theta-\tau)}Bw(\tau)\,d\tau\right).\end{aligned}$$

Now note that the kernel of the integral operator above is norm-quadratically Lebesgue-integrable in the following sense:

$$\int_0^h \int_0^h \bar{\gamma}(D^{-1}Ce^{A(\theta-\tau)}B)^2\,d\theta\,d\tau < \infty.$$

As such, the equation

$$v(\theta) = (\check{D}w)(\theta) := w(\theta) + \int_0^\theta D^{-1}Ce^{A(\theta-\tau)}Bw(\tau)\,d\tau$$

is a Volterra Integral equation of the second kind, which has a unique (up to a set of measure zero) solution $w \in \boldsymbol{L}^2_{[0,h)}$ given any $v \in \boldsymbol{L}^2_{[0,h)}$ [Tri85, Section 1.5]. Specifically, the solution is given by the formula

$$w(\theta) = (\check{D}^{-1}v)(\theta) = v(\theta) - \int_0^\theta H(\theta,\tau)v(\tau)\,d\tau,$$

where the resolvent kernel H satisfies

$$H(\theta,\tau) + D^{-1}Ce^{A(\theta-\tau)}B = \int_\tau^\theta H(\theta,z)D^{-1}Ce^{A(z-\tau)}B\,dz.$$

This implies that $\grave{D}^{-1} = \check{D}^{-1}D^{-1}$ exists as a causal map and that $\grave{D}^{-1} \in \mathscr{L}(\boldsymbol{L}^2_{[0,h)}, \boldsymbol{L}^2_{[0,h)})$ by the open mapping theorem [Kre89, Thm. 4.12-2], which in turn implies the existence of $\underline{P}^{-1} \in \boldsymbol{\mathcal{L}}$, for which one realisation is given by (2.11) in Section 2.5. □

Throughout, we use an underline to denote the lifted equivalent of a transfer function.

Lemma 5.2.4. *Given $P_1, P_2 \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$, define $P_3 := P_1P_2 \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$. Then we have that $\underline{P}_3 = \underline{P}_1\underline{P}_2 \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$.*

Proof. Given any $X \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$, the corresponding $\underline{X} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$, as defined in Theorem 5.2.3, satisfies

$$\mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\boldsymbol{M}_{\underline{X}}\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}u = \boldsymbol{M}_X u, \tag{5.8}$$

for all $u \in \operatorname{dom}(\boldsymbol{M}_X) = \boldsymbol{L}^2_{j\mathbb{R}}$; see Remark 2.1.3. In particular, for all $u \in \boldsymbol{L}^2_{j\mathbb{R}}$, repetitive application of (5.8) yields

$$\begin{aligned}
\mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\boldsymbol{M}_{\underline{P}_3}\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}u &= \boldsymbol{M}_{P_3}u \\
&= \boldsymbol{M}_{P_1}(\boldsymbol{M}_{P_2}u) \\
&= \mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\boldsymbol{M}_{\underline{P}_1}\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}(\boldsymbol{M}_{P_2}u) \\
&= \mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\boldsymbol{M}_{\underline{P}_1}\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}\mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\boldsymbol{M}_{\underline{P}_2}\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}u \\
&= \mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\boldsymbol{M}_{\underline{P}_1}\boldsymbol{M}_{\underline{P}_2}\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}u.
\end{aligned}$$

Therefore, $\boldsymbol{M}_{\underline{P}_3} = \boldsymbol{M}_{\underline{P}_1}\boldsymbol{M}_{\underline{P}_2} = \boldsymbol{M}_{\underline{P}_1\underline{P}_2}$ and hence, $\underline{P}_3 = \underline{P}_1\underline{P}_2 \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$. $\square$

Theorem 5.2.5. *Given any $P \in \boldsymbol{\mathcal{R}}$, there exists $\underline{P} \in \boldsymbol{\mathcal{L}}_{CF}$ such that $\boldsymbol{M}_P \overset{\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}}{\sim} \boldsymbol{M}_{\underline{P}}$.*

Proof. By Proposition 5.2.2, there exist $N, M, \tilde{M}, \tilde{N}, X, \tilde{X}, Y, \tilde{Y} \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$ such that

$$\begin{gathered}
\begin{bmatrix} Y & X \\ \tilde{M} & -\tilde{N} \end{bmatrix}\begin{bmatrix} N & \tilde{X} \\ M & -\tilde{Y} \end{bmatrix} = I; \quad NM^{-1} = \tilde{M}^{-1}\tilde{N} = P; \\
M^*M + N^*N = I; \quad \tilde{M}\tilde{M}^* + \tilde{N}\tilde{N}^* = I.
\end{gathered} \tag{5.9}$$

Applying Lemma 5.2.4 to (5.9) yields

$$\begin{bmatrix} \underline{Y} & \underline{X} \\ \underline{\tilde{M}} & -\underline{\tilde{N}} \end{bmatrix}\begin{bmatrix} \underline{N} & \underline{\tilde{X}} \\ \underline{M} & -\underline{\tilde{Y}} \end{bmatrix} = I$$

with $\underline{N}, \underline{M}, \underline{\tilde{M}}, \underline{\tilde{N}}, \underline{X}, \underline{\tilde{X}}, \underline{Y}, \underline{\tilde{Y}} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$. Thus, $\underline{N}$ and $\underline{M}$ define a right coprime factorisation for a $\underline{P} := \underline{N}\,\underline{M}^{-1} \in \boldsymbol{\mathcal{L}}$, where the invertibility of $\underline{M}$ in $\boldsymbol{\mathcal{L}}$ follows from Theorem 5.2.3 since $M^{-1} \in \boldsymbol{\mathcal{R}}$. Similarly, $\{\underline{\tilde{N}}, \underline{\tilde{M}}\}$ is a left coprime factor pair for $\underline{P} = \underline{\tilde{M}}^{-1}\underline{\tilde{N}}$.

We now show that $\mathscr{G}_{\boldsymbol{M}_P} \overset{\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}}{\sim} \mathscr{G}_{\boldsymbol{M}_{\underline{P}}}$. First recall as in Sections 2.4 and 2.5 that $\mathscr{G}_{\boldsymbol{M}_P} = \operatorname{img}(\boldsymbol{M}_G|_{\mathscr{F}\boldsymbol{L}^{2+}})$ and $\mathscr{G}_{\boldsymbol{M}_{\underline{P}}} = \operatorname{img}\big(\boldsymbol{M}_{\underline{G}}|_{\mathscr{Z}\mathscr{W}_h\boldsymbol{L}^{2+}}\big)$, where

$$G := \begin{bmatrix} N \\ M \end{bmatrix} \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+} \quad \text{and} \quad \underline{G} := \begin{bmatrix} \underline{N} \\ \underline{M} \end{bmatrix} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}.$$

right graph symbols for P and $\underline{P}$, respectively. By Theorem 5.2.3, $\boldsymbol{M}_G \overset{\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}}{\sim} \boldsymbol{M}_{\underline{G}}$.

Therefore, by Definition 2.1.2 and Remark 2.1.3,

$$\operatorname{img}\left(\boldsymbol{M}_G|_{\mathscr{F}\boldsymbol{L}^{2+}}\right) \overset{\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}}{\sim} \operatorname{img}\left(\boldsymbol{M}_{\underline{G}}|_{\mathscr{Z}\mathscr{W}_h\boldsymbol{L}^{2+}}\right).$$

All in all, $\underline{P} := \underline{NM}^{-1} \in \boldsymbol{\mathcal{L}}$ satisfies $\boldsymbol{M}_P \overset{\mathscr{Z}\mathscr{W}_h\mathscr{F}^{-1}}{\sim} \boldsymbol{M}_{\underline{P}}$.

Finally, let

$$\tilde{G} := \begin{bmatrix} -\tilde{M} & \tilde{N} \end{bmatrix} \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+} \quad \text{and} \quad \underline{\tilde{G}} := \begin{bmatrix} -\underline{\tilde{M}} & \underline{\tilde{N}} \end{bmatrix} \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$$

be left graph symbols for P and $\underline{P}$, respectively. By (5.9), note that $G^*G = I$ and $\tilde{G}\tilde{G}^* = I$. It can actually be shown using these identities that $\underline{G}^*\underline{G} = I$ and $\underline{\tilde{G}}\underline{\tilde{G}}^* = I$, i.e. $\underline{G}$ and $\underline{\tilde{G}}$ are respectively *normalised* right and left graph symbols for $\underline{P}$, whereby $\underline{P} \in \boldsymbol{\mathcal{L}}_{CF}$. To this end, note that for any $\underline{u}, \underline{v} \in \boldsymbol{L}^2_{\mathbb{T}}$,

$$\langle \underline{u}, \underline{G}^*\underline{Gv}\rangle_{\boldsymbol{L}^2_{\mathbb{T}}} = \langle \underline{Gu}, \underline{Gv}\rangle_{\boldsymbol{L}^2_{\mathbb{T}}} = \langle Gu, Gv\rangle_{\boldsymbol{L}^2_{j\mathbb{R}}} = \langle u, G^*Gv\rangle_{\boldsymbol{L}^2_{j\mathbb{R}}} = \langle u, v\rangle_{\boldsymbol{L}^2_{j\mathbb{R}}} = \langle \underline{u}, \underline{v}\rangle_{\boldsymbol{L}^2_{\mathbb{T}}},$$

where $u := \mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\underline{u}$ and $v := \mathscr{F}\mathscr{W}_h^{-1}\mathscr{Z}^{-1}\underline{v}$. A similar argument holds for $\tilde{G}$. $\square$

Sampled-data systems

In what follows, time-lifting of a sampled-data system is performed to obtain an equivalent multiplication operator with a transfer function in the class $\boldsymbol{\mathcal{L}}_{CF}$. A sampled-data system is one which processes (pre-filtered) information discretely in time, via sampling and hold operations.

Consider a sampled-data system:

$$\mathbf{P} = \mathcal{H}_h\mathbf{P}_d\mathcal{S}_h\mathbf{F} : \operatorname{dom}(\mathbf{P}) \subset \boldsymbol{L}^2_{\mathbb{R}} \to \boldsymbol{L}^2_{\mathbb{R}}, \tag{5.10}$$

where

1. $\mathbf{F} \overset{\mathscr{F}}{\looparrowright} \boldsymbol{M}_F$, with strictly proper $F \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$, is a continuous-time LTI anti-aliasing filter, and we assume $(A_F, B_F, C_F, 0) = F$ is a minimal realisation;
2. $\mathbf{P}_d \overset{\mathscr{Z}}{\looparrowright} \boldsymbol{M}_{P_d}$, with $P_d \in \boldsymbol{\mathcal{D}}^{p\times m}$, is a discrete-time LSI system, and we assume a minimal realisation for P_d is (A_d, B_d, C_d, D_d);
3. $\mathcal{S}_h$ is an ideal sampler of period $h > 0$, i.e. $(\mathcal{S}_h u)[\cdot] := u(\cdot h)$; and
4. $\mathcal{H}_h$ is a zero-order-hold of the same period h and synchronised with $\mathcal{S}_h$, that is $(\mathcal{H}_h u)(\cdot) := u(\lfloor \cdot/h \rfloor)$, where $\lfloor\cdot\rfloor$ denotes the floor function.

Note that $\mathcal{H}_h \in \mathscr{L}(\boldsymbol{\ell}^2_{\mathbb{Z}}(\mathbb{R}^p), \boldsymbol{L}^2_{\mathbb{R}})$ with $\bar{\gamma}(\mathcal{H}_h) = \sqrt{h}$. We now show that a sampled-data system is equivalent to multiplication by a transfer function in $\boldsymbol{\mathcal{L}}_{CF}$.

Theorem 5.2.6. *Let* $\underline{\mathcal{H}_h P_d \mathcal{S}_h F} := \Omega\Psi$, *where*

$$\Psi := \left(\begin{array}{c|c} e^{A_F h} & \grave{B}_F \\ \hline C_F & 0 \end{array}\right) \in \boldsymbol{\mathcal{D}H}^\infty_{\mathbb{D}}(\boldsymbol{L}^2_{[0,h)}, \mathbb{C}^m);$$
$$\Omega := \left(\begin{array}{c|c} A_d & B_d \\ \hline \grave{C}_d & \grave{D}_d \end{array}\right) \in \boldsymbol{\mathcal{D}}(\mathbb{C}^m, \boldsymbol{L}^2_{[0,h)});$$
$$\grave{B}_F w := \int_0^h e^{A_F(h-\tau)} B_F w(\tau)\, d\tau;$$
$$(\grave{C}_d x)(\theta) := C_d x, \quad (\grave{D}_d u)(\theta) := D_d u, \quad \forall \theta \in [0,h).$$

Then $\mathbf{P} = \mathcal{H}_h \mathbf{P}_d \mathcal{S}_h \mathbf{F} \overset{\mathscr{Z}\mathscr{W}_h}{\sim} \boldsymbol{M}_{\underline{\mathcal{H}_h P_d \mathcal{S}_h F}}$. *Given the transfer functions* Ω *and* Ψ *above, a possibly non-minimal realisation of* $\underline{\mathcal{H}_h P_d \mathcal{S}_h F} \in \boldsymbol{\mathcal{L}}^{p\times m}_{HS} \subset \boldsymbol{\mathcal{L}}^{p\times m}_{CF}$ *is* $(A, B, C, 0)$, *where*

$$A := \begin{bmatrix} e^{A_F h} & 0 \\ B_d C_F & A_d \end{bmatrix} \in \mathbb{R}^{n\times n}; \quad B := \begin{bmatrix} \grave{B}_F \\ 0 \end{bmatrix};$$
$$C\left(\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]\right)(\theta) := D_d C_F x_1 + C_d x_2, \forall \theta \in [0,h).$$

Proof. Exploiting the fact that $F \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$ and following the proof of Theorem 5.2.3, it can be shown that $\mathcal{S}_h \mathbf{F} \overset{\mathscr{Z}\oplus\mathscr{Z}\mathscr{W}_h}{\sim} \boldsymbol{M}_\Psi$. Note the only difference here is that the additional sampling operation effectively discards the anti-aliasing filter's output between sampling instants. For now, suppose that $\operatorname{dom}(\boldsymbol{M}_\Omega) = \operatorname{dom}(\boldsymbol{M}_{P_d})$ and $\mathcal{H}_h \mathbf{P}_d \overset{\mathscr{Z}\mathscr{W}_h\oplus\mathscr{Z}}{\sim} \boldsymbol{M}_\Omega$, which is established later. In particular, see (5.13) and (5.15).

The graph of $\mathbf{P} = \mathcal{H}_h \mathbf{P}_d \mathcal{S}_h \mathbf{F}$ can be characterised as

$$\mathscr{G}_{\mathbf{P}} = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \boldsymbol{L}^{2+} \;\middle|\; \begin{array}{l} \exists u_S \in \operatorname{dom}(\mathbf{P}_d) \text{ satisfying} \\ u_S = \mathcal{S}_h \mathbf{F} u \text{ and } y = \mathcal{H}_h \mathbf{P}_d u_S \end{array} \right\}. \tag{5.11}$$

Given any $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}}$, let $u_S \in \operatorname{dom}(\mathbf{P}_d)$ be such that $u_S = \mathcal{S}_h \mathbf{F} u$ and $y = \mathcal{H}_h \mathbf{P}_d u_S$. Define $\hat{u}_S := \mathscr{Z} u_S$ and $\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] := \mathscr{Z}\mathscr{W}_h \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$. It follows by the equivalences noted above that

$$\underline{y} = \boldsymbol{M}_\Omega \hat{u}_S = \boldsymbol{M}_\Omega \boldsymbol{M}_\Psi \underline{u} = \boldsymbol{M}_{\Omega\Psi} \underline{u} = \boldsymbol{M}_{\underline{\mathcal{H}_h P_d \mathcal{S}_h F}} \underline{u}.$$

Clearly $\underline{u} \in \operatorname{dom}\left(\boldsymbol{M}_{\underline{\mathcal{H}_h P_d \mathcal{S}_h F}}\right)$, since $\underline{y} \in \mathscr{Z}\mathscr{W}_h \boldsymbol{L}^{2+}$. Thus, $\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{M}_{\underline{\mathcal{H}_h P_d \mathcal{S}_h F}}}$. Conversely, given any $\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right] \in \mathscr{G}_{\boldsymbol{M}_{\underline{\mathcal{H}_h P_d \mathcal{S}_h F}}}$,

$$\underline{y}(\cdot) = \left(\boldsymbol{M}_{\underline{\mathcal{H}_h P_d \mathcal{S}_h F}} \underline{u}\right)(\cdot) = \Omega(\cdot)\Psi(\cdot)\underline{u}(\cdot) = \Omega(\cdot)\hat{u}_S(\cdot),$$

where $\hat{u}_S := \boldsymbol{M}_\Psi \underline{u}$. Since $\Psi \in \boldsymbol{\mathcal{D}H}^\infty_{\mathbb{D}}(\boldsymbol{L}^2_{[0,h)}, \mathbb{C}^m)$, observe that $\underline{u} \in \mathrm{dom}\,(\boldsymbol{M}_\Psi) = \mathscr{Z}\mathscr{W}_h \boldsymbol{L}^{2+}$ and $\hat{u}_S \in \bigcup_{k\in\mathbb{Z}} z^k \boldsymbol{H}^2_{\mathbb{D}}(\mathbb{R}^m)$. Also,

$$\underline{y}(\cdot) = \Omega(\cdot)\hat{u}_S(\cdot) \in \mathscr{Z}\mathscr{W}_h \boldsymbol{L}^{2+}$$

implies $\hat{u}_S \in \mathrm{dom}\,(\boldsymbol{M}_\Omega) = \mathrm{dom}\,(\boldsymbol{M}_{P_d})$, where the last equality holds by our starting assumption. Moreover, noting that with $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] := \mathscr{W}_h^{-1}\mathscr{Z}^{-1}\left[\begin{smallmatrix} \underline{y} \\ \underline{u} \end{smallmatrix}\right]$ and $u_S := \mathscr{Z}^{-1}\hat{u}_S \in \mathrm{dom}\,(\mathbf{P}_d) = \mathrm{dom}\,(\mathcal{H}_h\mathbf{P}_d)$ (see Remark 2.1.5 for the last equality), we have $y = \mathcal{H}_h\mathbf{P}_d u_S$ and $u_S = \mathcal{S}_h\mathbf{F}u$. In particular, $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{G}_{\mathbf{P}}$ by (5.11). Thus, $\mathbf{P} \overset{\mathscr{Z}\mathscr{W}_h}{\sim} \boldsymbol{M}_{\underline{\mathcal{H}_h P_d \mathcal{S}_h F}}$, as required.

To complete the proof we show below $\mathrm{dom}\,(\boldsymbol{M}_\Omega) = \mathrm{dom}\,(\boldsymbol{M}_{P_d})$ and $\mathcal{H}_h\mathbf{P}_d \overset{\mathscr{Z}\mathscr{W}_h \oplus \mathscr{Z}}{\sim} \boldsymbol{M}_\Omega$. Towards this end, note that since $\mathcal{H}_h \in \mathscr{L}(\boldsymbol{\ell}^2_{\mathbb{Z}}(\mathbb{R}^p), \boldsymbol{L}^2_{\mathbb{R}})$, we have $\mathcal{H}_h \overset{\mathscr{Z}\mathscr{W}_h \oplus \mathscr{Z}}{\sim} \boldsymbol{M}_H$, where

$$\begin{aligned} H &:= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & D_H \end{array}\right) \in \boldsymbol{\mathcal{D}H}^\infty_{\mathbb{D}}(\mathbb{C}^p, \boldsymbol{L}^2_{[0,h)}); \quad \text{and} \\ (D_H u)(\theta) &:= u, \forall \theta \in [0,h). \end{aligned} \tag{5.12}$$

Also note that $D_H D_d = \grave{D}_d$, $D_H C_d = \grave{C}_d$, and consequently $HP_d = \Omega$.

We first show that $\mathrm{dom}\,(\boldsymbol{M}_\Omega) = \mathrm{dom}\,(\boldsymbol{M}_{P_d})$. Suppose $u \in \mathrm{dom}\,(\boldsymbol{M}_{P_d})$, then

$$y(\cdot) := P_d(\cdot)u(\cdot) \in \bigcup_{k\in\mathbb{Z}} z^k \boldsymbol{H}^2_{\mathbb{D}}(\mathbb{R}^p) \subset \mathrm{dom}\,(\boldsymbol{M}_H)\,,$$

and hence $H(\cdot)y(\cdot) = H(\cdot)P_d(\cdot)u(\cdot) = \Omega(\cdot)u(\cdot) \in \mathscr{Z}\mathscr{W}_h\boldsymbol{L}^{2+}$. That is, $u \in \mathrm{dom}\,(\boldsymbol{M}_\Omega)$ and thus $\mathrm{dom}\,(\boldsymbol{M}_{P_d}) \subset \mathrm{dom}\,(\boldsymbol{M}_\Omega)$. To see the converse inclusion, suppose that $u \in \mathrm{dom}\,(\boldsymbol{M}_\Omega)$. That is,

$$\begin{aligned} \|\boldsymbol{M}_\Omega u\|^2_{\boldsymbol{L}^2_{\mathbb{T}}(\boldsymbol{L}^2_{[0,h)})} &:= \int_0^{2\pi} \|H(e^{j\omega})P_d(e^{j\omega})u(e^{j\omega})\|_2^2 \, d\omega < \infty \\ &\Longrightarrow \int_0^{2\pi} \|D_H P_d(e^{j\omega})u(e^{j\omega})\|_2^2 \, d\omega < \infty \\ &\Longrightarrow \int_0^{2\pi} h|P_d(e^{j\omega})u(e^{j\omega})|^2 \, d\omega < \infty, \end{aligned}$$

where the last implication follows from the definition of D_H in (5.12), whereby

$$\|D_H x\|_2^2 = h|x|^2, \forall x \in \mathbb{R}^p.$$

Thus, we have that

$$\int_0^{2\pi} |P_d(e^{j\omega})u(e^{j\omega})|^2 \, d\omega =: \|\boldsymbol{M}_{P_d} u\|^2_{\boldsymbol{L}^2_{\mathbb{T}}(\mathbb{R}^p)} < \infty,$$

i.e. $u \in \mathrm{dom}\,(\boldsymbol{M}_{P_d})$, and hence $\mathrm{dom}\,(\boldsymbol{M}_\Omega) \subset \mathrm{dom}\,(\boldsymbol{M}_{P_d})$. Altogether,

$$\mathrm{dom}\,(\boldsymbol{M}_\Omega) = \mathrm{dom}\,(\boldsymbol{M}_{P_d})\,. \tag{5.13}$$

We now establish that $\mathcal{H}_h \mathbf{P}_d \overset{\mathscr{L}\mathscr{W}_h \oplus \mathscr{Z}}{\approx} \boldsymbol{M}_H \boldsymbol{M}_{P_d}$ and that $\boldsymbol{M}_H \boldsymbol{M}_{P_d} = \boldsymbol{M}_\Omega$, which follows by $HP_d = \Omega$ provided $\mathrm{dom}\,(\boldsymbol{M}_H \boldsymbol{M}_{P_d}) = \mathrm{dom}\,(\boldsymbol{M}_\Omega)$ (see (5.14) below). Recall that $\mathcal{H}_h \overset{\mathscr{L}\mathscr{W}_h \oplus \mathscr{Z}}{\approx} \boldsymbol{M}_H$ and $\mathbf{P}_d \overset{\mathscr{Z}}{\approx} \boldsymbol{M}_{P_d}$, and note that $\mathrm{img}\,(\mathbf{P}_d) \subset \boldsymbol{\ell}^2_{\mathbb{Z}}(\mathbb{R}^p) = \mathrm{dom}\,(\mathcal{H}_h)$. So, by Lemma 2.1.6, $\mathcal{H}_h \mathbf{P}_d \overset{\mathscr{L}\mathscr{W}_h \oplus \mathscr{Z}}{\approx} \boldsymbol{M}_H \boldsymbol{M}_{P_d}$. Finally, note that $\mathrm{dom}\,(\boldsymbol{M}_H \boldsymbol{M}_{P_d}) = \mathrm{dom}\,(\boldsymbol{M}_{P_d})$, which together with (5.13), implies

$$\mathrm{dom}\,(\boldsymbol{M}_H \boldsymbol{M}_{P_d}) = \mathrm{dom}\,(\boldsymbol{M}_\Omega)\,. \tag{5.14}$$

Specifically, the fact that $\mathrm{dom}\,(\boldsymbol{M}_H) = \boldsymbol{L}^2_{\mathbb{T}}(\mathbb{R}^p)$ implies that $\mathrm{dom}\,(\boldsymbol{M}_{P_d}) \subset \mathrm{dom}\,(\boldsymbol{M}_H \boldsymbol{M}_{P_d})$. Furthermore, we have $\mathrm{dom}\,(\boldsymbol{M}_H \boldsymbol{M}_{P_d}) \subset \mathrm{dom}\,(\boldsymbol{M}_{P_d})$ by definition; see Remark 2.1.5. Thus, $\mathrm{dom}\,(\boldsymbol{M}_{P_d}) = \mathrm{dom}\,(\boldsymbol{M}_H \boldsymbol{M}_{P_d})$. Putting everything together yields

$$\mathcal{H}_h \mathbf{P}_d \overset{\mathscr{L}\mathscr{W}_h \oplus \mathscr{Z}}{\approx} \boldsymbol{M}_H \boldsymbol{M}_{P_d} = \boldsymbol{M}_\Omega, \tag{5.15}$$

as required. $\square$

5.3 Sampled-data approximation and proposed solution

The results in the preceding section establish that the transfer function class $\boldsymbol{\mathcal{L}}_{CF}$ is sufficiently rich to equivalently represent both the LTI and SD system classes. Recall the definition in Section 3.4 of the ν-gap metric (3.32) on the set $\mathbb{P}_{CF}$, which is reproduced below:

$$\delta_\nu(\mathbf{P}_1, \mathbf{P}_2) := \begin{cases} \bar{\gamma}(\tilde{\mathbf{G}}_2 \mathbf{G}_1) = \|\tilde{G}_2 G_1\|_\infty & \text{if for all } \tau \in [0,h), \mathbf{T}_{\mathbf{G}_2^* \mathbf{G}_1, \tau} \text{ is Fredholm} \\ & \text{and } \mathrm{ind}(\mathbf{T}_{\mathbf{G}_2^* \mathbf{G}_1, \tau}) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Following Section 5.1, sampled-data approximation can be formulated as in (5.1):

$$\text{For some } \mathbf{P} \in \mathbb{S}_\mathrm{T},\ \beta_{opt} := \inf\left\{\beta \in (0, b_{\mathrm{opt}}(\mathbf{P}; \mathbb{E}))\,|\,\exists \mathbf{P}_a \in \mathbb{S}_\mathrm{A} \text{ such that } \delta_\nu(\mathbf{P}, \mathbf{P}_a) < \beta\right\}.$$

where

$$b_{\mathrm{opt}}(\mathbf{P}; \mathbb{E}) := \sup_{\mathbf{C} \in \mathbb{E}\,:\,[\mathbf{P}, \mathbf{C}] \text{ is stable}} b_{\mathbf{P},\mathbf{C}} \quad \text{with} \quad \mathbb{E} := \left\{\mathbf{P} \overset{\mathscr{F}}{\hookleftarrow} \boldsymbol{M}_P : P \in \boldsymbol{\mathcal{R}}\right\},$$

$$\mathbb{S}_{\mathrm{T}} := \left\{ \mathbf{P} \overset{\mathscr{F}}{\hookleftarrow} \boldsymbol{M}_P : P \in \boldsymbol{\mathcal{R}} \text{ is strictly proper} \right\} \subset \mathbb{P}_{CF},$$

and

$$\mathbb{S}_{\mathrm{A}} := \left\{ \mathbf{P}_a \overset{\mathscr{L}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{P_a} : P_a \in \mathbb{S}_{sd}(F, h) \right\} \subset \mathbb{P}_{CF},$$

where the sampled-data structure set

$$\mathbb{S}_{sd}(F, h) := \left\{ P_a := \underline{\mathcal{H}_h P_d \mathcal{S}_h F} \in \boldsymbol{\mathcal{L}}_{CF} \mid P_d \in \boldsymbol{\mathcal{D}} \right\}$$

and the notation $\underline{\mathcal{H}_h P_d \mathcal{S}_h F}$ is defined in Theorem 5.2.6 given a strictly proper $F \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$ and sampling period $h > 0$. Let $\tilde{G} \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$ be a normalised left graph symbol for P, suppose there exists $R, R^{-1} \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$ such that

$$\tilde{G}^* \tilde{G} - \beta^2 I_{p+m} = R^* \begin{bmatrix} I_p & 0 \\ 0 & -I_m \end{bmatrix} R,$$

then as demonstrated in Section 5.1, sampled-data approximation can be recast into the following equivalent problem:

$$\beta_{opt} := \inf \left\{ \beta \in (0, b_{\mathrm{opt}}(\mathbf{P}; \mathbb{E})) \;\middle|\; \begin{array}{l} \exists \mathbf{P}_a \in \mathbb{S}_{\mathrm{A}} \text{ such that } \mathcal{F}(\mathbf{R}, \mathbf{P}_a) \text{ is causal} \\ \text{and } \bar{\gamma}(\mathcal{F}(\mathbf{R}, \mathbf{P}_a)|_{\boldsymbol{L}^{2+}}) < 1 \end{array} \right\},$$

where $\mathbf{R} \overset{\mathscr{F}}{\hookleftarrow} \boldsymbol{M}_R$. Recall that $\mathcal{F}(\mathbf{R}, \mathbf{P}_a)$ denotes an LFT for which $\mathbf{G}_a \mathbf{R}^{-1}$ is a left graph symbol, where $\mathbf{G}_a$ is a normalised left graph symbol for $\mathbf{P}_a$. Let $\underline{R} \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ denote the lifted equivalent of R and $\mathcal{F}_c(\underline{R}, P_a) \in \boldsymbol{\mathcal{L}}$ the transfer function representation of $\mathcal{F}(\mathbf{R}, \mathbf{P}_a)$ satisfying $\mathcal{F}(\mathbf{R}, \mathbf{P}_a) \overset{\mathscr{L}\mathscr{W}_h}{\sim} \boldsymbol{M}_{\mathcal{F}_c(\underline{R}, P_a)}$. Denoting by $G_a \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ a normalised left graph symbol for $P_a \in \mathbb{S}_{sd}(F, h) \subset \boldsymbol{\mathcal{L}}$, it follows that $G_a \underline{R}^{-1}$ is a left graph symbol for $\mathcal{F}_c(\underline{R}, P_a)$. Indeed, according to the LFT setup in Figure 4.1, we have

$$\mathcal{F}_c(\underline{R}, P_a) = (\underline{R}_{11} P_a + \underline{R}_{12})(\underline{R}_{21} P_a + \underline{R}_{22})^{-1} \in \boldsymbol{\mathcal{L}},$$

provided the feedthrough 'D'-term of $\underline{R}_{21} P_a + \underline{R}_{22}$ has a bounded causal inverse, which is the case when that of $\underline{R}_{22}$ has a bounded causal inverse and that of P_a is zero; see Theorem 5.2.6. Note that $\mathcal{F}(\mathbf{R}, \mathbf{P}_a)$ is causal and $\bar{\gamma}(\mathcal{F}(\mathbf{R}, \mathbf{P}_a)|_{\boldsymbol{L}^{2+}}) < 1$ is equivalent to $\mathcal{F}_c(\underline{R}, P_a) \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ and $\|\mathcal{F}_c(\underline{R}, P_a)\|_\infty < 1$.

Weighted sampled-data approximation

In sampled-data approximation, it is more convenient to formulate the problem involving objects in $\boldsymbol{\mathcal{L}}_{CF}$ directly. To this end, define the inherited ν-gap metric on $\boldsymbol{\mathcal{L}}_{CF}$ as

$$\delta_\nu(P_1, P_2) := \delta_\nu(\mathbf{P}_1, \mathbf{P}_2),$$

where $\mathbf{P}_1 \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{P_1}$ and $\mathbf{P}_2 \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_{P_2}$. As is similar to Section 4.6.2, given $P, C \in \boldsymbol{\mathcal{L}}_{CF}$ (resp. $\boldsymbol{\mathcal{R}}$), define $\mathbf{P} \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_P$ and $\mathbf{C} \overset{\mathscr{Z}\mathscr{W}_h}{\hookleftarrow} \boldsymbol{M}_C$ (resp. $\mathbf{P} \overset{\mathscr{F}}{\hookleftarrow} \boldsymbol{M}_P$ and $\mathbf{C} \overset{\mathscr{F}}{\hookleftarrow} \boldsymbol{M}_C$), it follows that closed-loop stability of $[\mathbf{P}, \mathbf{C}]$ is equivalent to

$$[P, C] := \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}} \text{ (resp. } \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}),$$

When this is the case, the robust stability margin $b_{P,C} := b_{\mathbf{P},\mathbf{C}} = \|[P, C]\|^{-1}_\infty$.

Given $P \in \boldsymbol{\mathcal{R}}$, input-output weights $W_i, W_o \in \boldsymbol{\mathcal{R}}$ for which $W_o P W_i$ is strictly proper, sampling period $h > 0$, and strictly proper $F \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}$, the problem of optimal weighted ν-gap metric based sampled-data approximation may be formulated as the following structurally constrained optimisation:

$$\beta_{opt} := \inf\left\{\beta \in (0, b_{\text{opt}}(P; \boldsymbol{\mathcal{R}})) \mid \exists P_a \in \mathbb{S}_{sd}(F, h) \text{ such that } \delta_\nu(\underline{W_o P W_i}, \underline{W_o} P_a \underline{W_i}) < \beta\right\}, \tag{5.16}$$

where the maximal robustness margin

$$b_{\text{opt}}(P; \boldsymbol{\mathcal{R}}) := \sup_{C \in \boldsymbol{\mathcal{R}} \,:\, [P, C] \in \boldsymbol{\mathcal{R}H}^\infty_{\mathbb{C}_+}} b_{P,C},$$

the sampled-data structure set

$$\mathbb{S}_{sd}(F, h) := \left\{P_a := \underline{\mathcal{H}_h P_d \mathcal{S}_h F} \in \boldsymbol{\mathcal{L}}_{CF} \mid P_d \in \boldsymbol{\mathcal{D}}\right\},$$

and the notation $\underline{\mathcal{H}_h P_d \mathcal{S}_h F}$ is defined in Theorem 5.2.6. As explained before, sampled-data approximation (5.16) may be transformed into the equivalent problem:

$$\inf\{\beta \in (0, b_{\text{opt}}(P; \boldsymbol{\mathcal{R}})) \mid \mathbb{Y}(R(W_o P W_i, \beta), W_i, W_o, F, h) \neq \emptyset\}, \tag{5.17}$$

where

$$\mathbb{Y}(R, W_i, W_o, F, h) := \left\{ P_a \in \mathbb{S}_{sd}(F, h) \;\middle|\; \begin{array}{l} \mathcal{F}_c(\underline{R}, \underline{W_o} P_a \underline{W_i}) \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}} \text{ and} \\ \|\mathcal{F}_c(\underline{R}, \underline{W_o} P_a \underline{W_i})\|_\infty < 1 \end{array} \right\}, \tag{5.18}$$

where the frequency-domain LFT is defined by $\mathcal{F}_c(\Lambda, \Delta) := (\Lambda_{11}\Delta + \Lambda_{12})(\Lambda_{21}\Delta + \Lambda_{22})^{-1}$, and $R(W_oPW_i, \beta) \in \boldsymbol{\mathcal{RH}}^\infty_{\mathbb{C}_+}$ is a ν-gap J-spectral factor for $W_oPW_i \in \boldsymbol{\mathcal{R}}$ satisfying $R^{-1} \in \boldsymbol{\mathcal{RH}}^\infty_{\mathbb{C}_+}$ and

$$\tilde{G}_w^*\tilde{G}_w - \beta^2 I = R^* \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right] R,$$

with $\tilde{G}_w$ a normalised left graph symbol for W_oPW_i. An R always exists in this case, as illustrated by the next result from [BC08, Lem. 2], which makes use of the J-spectral factorisation result in [GGLD90].

Proposition 5.3.1. *Given a strictly proper $\Phi \in \boldsymbol{\mathcal{R}}^{p\times m}$ and a $\beta \in (0, b_{\mathrm{opt}}(\Phi; \boldsymbol{\mathcal{R}}))$, suppose a minimal realisation for Φ is $(A, B, C, 0)$. Define the following:*

$$L := -YC^T;$$
$$Z := \frac{\beta^2}{1-\beta^2} X \left(I - \frac{\beta^2}{1-\beta^2} YX \right)^{-1},$$

where X and Y are respectively the stabilising solutions to the generalised control algebraic Riccati equation (GCARE)

$$A^T X + XA + C^T C - XBB^T X = 0$$

and the generalised filtering algebraic Riccati equation (GFARE)

$$AY + YA^T - YC^T CY + BB^T = 0.$$

Then the transfer function matrix $R := (A_R, B_R, C_R, D_R)$ with

$$A_R := A + LC; \qquad B_R := \begin{bmatrix} -L & B \end{bmatrix};$$
$$C_R := \begin{bmatrix} \frac{1}{\sqrt{1-\beta^2}}(L^T Z - C) \\ \frac{1}{\beta} B^T Z \end{bmatrix}; \quad D_R := \begin{bmatrix} \sqrt{1-\beta^2} I_p & 0 \\ 0 & \beta I_m \end{bmatrix},$$

is such that $R, R^{-1} \in \boldsymbol{\mathcal{RH}}^\infty_{\mathbb{C}_+}$ and

$$\tilde{G}_\Phi^*\tilde{G}_\Phi - \beta^2 I_{p+m} = R^* \left[\begin{smallmatrix} I_p & 0 \\ 0 & -I_m \end{smallmatrix}\right] R,$$

where $\tilde{G}_\Phi \in \boldsymbol{\mathcal{RH}}^\infty_{\mathbb{C}_+}$ denotes a normalised left graph symbol of Φ.

The formulation (5.17) gives rise to the following algorithm for solving problem (5.16), which is optimal to within a pre-determined tolerance.

Algorithm 5.3.2. Suppose we are given $P, W_i, W_o \in \boldsymbol{\mathcal{R}}$ for which W_oPW_i is strictly proper, a strictly proper $F \in \boldsymbol{\mathcal{R}H}^{\infty}_{\mathbb{C}_+}$, and a fixed sampling period $h > 0$. The following bisection search yields β_{opt} in problem (5.16) to within a specified tolerance $\epsilon \in (0, b_{\rm opt}(P; \boldsymbol{\mathcal{R}}))$, as well as a transfer function $P_a \in \mathbb{S}_{sd}(F, h)$ achieving β_{opt}:

1. Set $\beta_{opt} := b_{\rm opt}(P; \boldsymbol{\mathcal{R}})$, $\beta_{min} := 0$, $\beta_{max} := b_{\rm opt}(P; \boldsymbol{\mathcal{R}}) - \epsilon$, and $\beta := \beta_{max}$.

2. Construct $R(W_oPW_i, \beta)$ as per Proposition 5.3.1, with $\Phi := W_oPW_i$.

3. If $\mathbb{Y}(R(W_oPW_i, \beta), W_i, W_o, F, h)$ is nonempty, set $\beta_{max} := \beta$ and $\beta_{opt} := \beta$. Otherwise, set $\beta_{min} := \beta$.

4. If $|\beta - (\beta_{max} + \beta_{min})/2| > \epsilon$, loop from Step 2 with $\beta := (\beta_{max} + \beta_{min})/2$.

5. If $\beta_{opt} < b_{\rm opt}(P; \boldsymbol{\mathcal{R}})$, choose a P_a from the set $\mathbb{Y}(R(W_oPW_i, \beta_{opt}), W_i, W_o, F, h)$ which is nonempty. Otherwise, problem (5.16) is unsolvable and the fixed pre-filter F and/or sampling period h should be redesigned.

The set $\mathbb{Y}(R, W_i, W_o, F, h)$ is convex, as shown in the sequel using standard $\boldsymbol{H}^{\infty}$ sampled-data synthesis techniques described by [BP92, Yam94, CF95, CG97]. As such, Step 3 of Algorithm 5.3.2 is a convex feasibility problem and Step 5 involves locating a feasibility point in a convex set.

A linear matrix inequality based solution

Here we demonstrate that the set $\mathbb{Y}(R, W_i, W_o, F, h) \subset \mathbb{S}_{sd}(F, h)$ in Step 3 of Algorithm 5.3.2, as defined in (5.18), is convex via an LMI approach [GA94] and that semidefinite programming methods [BV04] can be used for determining if the set is empty or not, as well as constructing a feasible point in the latter case. We begin with a preliminary lemma.

Lemma 5.3.3. *Given* $R = \left(\begin{array}{c|c} A_R & \begin{bmatrix} B_{R1} & B_{R2} \end{bmatrix} \\ \hline \begin{bmatrix} C_{R1} \\ C_{R2} \end{bmatrix} & \begin{bmatrix} D_{R11} & 0 \\ 0 & D_{R22} \end{bmatrix} \end{array}\right) \in \boldsymbol{\mathcal{R}}$ *(resp.* $\boldsymbol{\mathcal{L}}$*), suppose* $R^{-1} \in \boldsymbol{\mathcal{R}}$ *(resp.* $\boldsymbol{\mathcal{L}}$*) and let*

$$\tilde{R} = \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{21} & \tilde{R}_{22} \end{bmatrix} := \left(\begin{array}{c|c} A & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} & \begin{bmatrix} 0 & D_{12} \\ D_{21} & 0 \end{bmatrix} \end{array}\right),$$

where

$$A := A_R - B_{R2}D_{R22}^{-1}C_{R2}; \quad B_1 := -B_{R2}D_{R22}^{-1}; \quad B_2 := B_{R1};$$
$$C_1 := -C_{R1}; \quad C_2 := -D_{R22}^{-1}C_{R2}; \quad D_{12} := -D_{R11}; \quad D_{21} := -D_{R22}^{-1}.$$

Then, for any $P_1 \in \boldsymbol{\mathcal{R}}$ *(resp.* $\boldsymbol{\mathcal{L}}$*), the chain-scattering LFT*

$$\mathcal{F}_c(R, P_1) := (R_{11}P_1 + R_{12})(R_{21}P_1 + R_{22})^{-1}$$

is equal to the Redheffer's lower LFT

$$\mathcal{F}_l(\tilde{R}, P_1) := \tilde{R}_{11} + \tilde{R}_{12}P_1(I - \tilde{R}_{22}P_1)^{-1}\tilde{R}_{21},$$

i.e. $\mathcal{F}_c(R, P_1) = \mathcal{F}_l(\tilde{R}, P_1)$.

Proof. By noting that $(A_{cl}, B_{cl}, C_{cl}, D_{cl})$ is a realisation for both $\mathcal{F}_c(R, P_1)$ and $\mathcal{F}_l(\tilde{R}, P_1)$:

$$A_{cl} = \bar{A} + \underline{B}\Phi\underline{C}; \quad B_{cl} = \bar{B} + \underline{B}\Phi\underline{D}_{22}; \quad C_{cl} = \bar{C} + \underline{D}_{11}\Phi\underline{C}; \quad D_{cl} = \underline{D}_{12}\Phi\underline{D}_{21},$$

where

$$\begin{aligned} &\bar{A} := \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}; \quad \bar{B} := \begin{bmatrix} B_1 \\ 0 \end{bmatrix}; \quad \underline{B} := \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix}; \\ &\bar{C} := \begin{bmatrix} C_1 & 0 \end{bmatrix}; \quad \underline{C} := \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}; \quad \underline{D}_{12} := \begin{bmatrix} 0 & D_{12} \end{bmatrix}; \quad \underline{D}_{21} := \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}, \end{aligned} \tag{5.19}$$

and $\Phi := \left[\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right]$ with (A_1, B_1, C_1, D_1) being a realisation for P_1. In particular, see [BC08, Lem. 2] and [GA94, Section 2]. □

Definition 5.3.4. Given a $P \in \boldsymbol{\mathcal{R}}^{p\times m}$, weights $W_i \in \boldsymbol{\mathcal{R}}^{m\times m}$ and $W_o \in \boldsymbol{\mathcal{R}}^{p\times p}$ for which $W_oPW_i \in \boldsymbol{\mathcal{R}}^{p\times m}$ is strictly proper, a fixed sampling period $h > 0$, and a strictly proper $F \in \boldsymbol{\mathcal{RH}}_{\mathbb{C}_+}^{\infty\ m\times m}$, suppose a ν-gap J-spectral factor for W_oPW_i, constructed following the procedure in Proposition 5.3.1, is $R \in \boldsymbol{\mathcal{RH}}_{\mathbb{C}_+}^{\infty\ (p+m)\times(p+m)}$. Obtain $\tilde{R} \in \boldsymbol{\mathcal{R}}^{(p+m)\times(m+p)}$ from R using Lemma 5.3.3 and then define

$$G := \begin{bmatrix} \tilde{R}_{11} & \tilde{R}_{12}W_o \\ FW_i\tilde{R}_{21} & FW_i\tilde{R}_{22}W_o \end{bmatrix} \in \boldsymbol{\mathcal{R}}^{(p+m)\times(m+p)}.$$

Let a minimal realisation of G be given by $\left(\begin{array}{c|c} A & \left[\begin{smallmatrix} B_1 & B_2 \end{smallmatrix}\right] \\ \hline \left[\begin{smallmatrix} C_1 \\ C_2 \end{smallmatrix}\right] & \left[\begin{smallmatrix} 0 & D_{12} \\ 0 & 0 \end{smallmatrix}\right] \end{array}\right)$, with $A \in \mathbb{R}^{n\times n}$. Note that the term D_{21} in the realisation of G are zero since F is assumed strictly proper. Now calculate the matrix exponential

$$\hat{Q}(h) = \begin{bmatrix} \hat{Q}_{11}(h) & \hat{Q}_{12}(h) \\ \hat{Q}_{21}(h) & \hat{Q}_{22}(h) \end{bmatrix} := \exp\left\{ h \begin{bmatrix} -\hat{A}^T & -\hat{C}^T\hat{C} \\ \hat{B}\hat{B}^T & \hat{A} \end{bmatrix} \right\} \in \mathbb{R}^{2(n+p)\times 2(n+p)}$$

where

$$\hat{A} := \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{(n+p)\times(n+p)};$$
$$\hat{C}^T\hat{C} := \begin{bmatrix} C_1^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} C_1 & D_{12} \end{bmatrix}; \quad \hat{B}\hat{B}^T := \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{bmatrix} B_1^T & 0 \end{bmatrix}.$$

Let $\grave{G} := \left(\begin{array}{c|c} \underline{A} & \begin{bmatrix} \grave{B}_1 & \underline{B}_2 \end{bmatrix} \\ \hline \begin{bmatrix} \grave{C}_1 \\ C_2 \end{bmatrix} & \begin{bmatrix} 0 & \grave{D}_{12} \\ 0 & 0 \end{bmatrix} \end{array} \right) \in \boldsymbol{\mathcal{D}}^{(\bar{p}+m)\times(\bar{m}+p)}$, where

$$\underline{A} := \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{Q}_{11}(h)^{-T} \begin{bmatrix} I_n \\ 0 \end{bmatrix};$$
$$\underline{B}_2 := \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{Q}_{11}(h)^{-T} \begin{bmatrix} 0 \\ I_p \end{bmatrix};$$
$$\grave{B}_1\grave{B}_1^T := \begin{bmatrix} I_n & 0 \end{bmatrix} \hat{Q}_{21}(h)\hat{Q}_{11}(h)^{-1} \begin{bmatrix} I_n \\ 0 \end{bmatrix};$$
$$\begin{bmatrix} \grave{C}_1^T \\ \grave{D}_{12}^T \end{bmatrix} \begin{bmatrix} \grave{C}_1 & \grave{D}_{12} \end{bmatrix} := -\hat{Q}_{11}(h)^{-1}\hat{Q}_{12}(h).$$

Note $\grave{B}_1 \in \mathbb{R}^{n\times\bar{m}}$, $\grave{C}_1 \in \mathbb{R}^{\bar{p}\times n}$, and $\grave{D}_{12} \in \mathbb{R}^{\bar{p}\times p}$.

Theorem 5.3.5. *Using the notation developed in Definition 5.3.4, let* $\tilde{D}_{11} : \boldsymbol{L}^2_{[0,h)} \to \boldsymbol{L}^2_{[0,h)}$ *be defined as*

$$(\tilde{D}_{11}w)(t) := \int_0^t C_1 e^{A(t-\tau)} B_1 w(\tau)\, d\tau,$$

and suppose that[2] $\bar{\gamma}(\tilde{D}_{11}) := \|\tilde{D}_{11}\|_{\boldsymbol{L}^2_{[0,h)}} < 1$. *Then we have* $P_d \in \mathbb{Y}_d\left(\grave{G}\,(R, W_i, W_o, F, h)\right)$ *if, and only if,* $P_a := \underline{\mathcal{H}_h P_d \mathcal{S}_h F} \in \mathbb{Y}(R, W_i, W_o, F, h)$, *where* $\mathbb{Y}$ *is as defined in (5.18) and*

$$\mathbb{Y}_d(\grave{G}) := \{P_d \in \boldsymbol{\mathcal{D}} \mid \mathcal{F}_l(\grave{G}, P_d) \in \boldsymbol{\mathcal{D}H}^\infty_{\mathbb{D}} \text{ and } \|\mathcal{F}_l(\grave{G}, P_d)\|_\infty < 1\}.$$

Proof. As in Lemma 5.3.3, we have $\mathcal{F}_c(\underline{R}, P_a) = \mathcal{F}_l(\underline{\tilde{R}}, P_a)$. Furthermore, the equivalence between

1. $\mathcal{F}_l(\underline{\tilde{R}}, P_a) \in \boldsymbol{\mathcal{L}H}^\infty_{\mathbb{D}}$ and $\|\mathcal{F}_l(\underline{\tilde{R}}, P_a)\|_\infty < 1$ and

2. $\mathcal{F}_l(\grave{G}, P_d) \in \boldsymbol{\mathcal{D}H}^\infty_{\mathbb{D}}$ and $\|\mathcal{F}_l(\grave{G}, P_d)\|_\infty < 1$,

is proved in [CG97, Thm 4.1] and [BP92, Thm. 6]. In particular, the realisation formulae

[2]One way of computing the operator norm $\|\cdot\|_{\boldsymbol{L}^2_{[0,h)}}$ can be found in [Dul99].

in the statement of this theorem are from [CG97]. The result then follows from the above equivalence. □

Proposition 5.3.6 ([GA94, Section 6]). *There exists $P_d \in \boldsymbol{\mathcal{D}}^{p\times m}$ such that*

$$\mathcal{F}_l(\grave{G}, P_d) \in \boldsymbol{\mathcal{D}H}^{\infty}_{\mathbb{D}} \text{ and } \|\mathcal{F}_l(\grave{G}, P_d)\|_\infty < 1$$

if, and only if, there exist symmetric matrices $R, S \in \mathbb{R}^{n\times n}$ satisfying the following LMI system:

$$\left[\begin{array}{c|c} N_R & 0_{(n+\bar{p})\times\bar{m}} \\ \hline 0 & I_{\bar{m}} \end{array}\right]^T \left[\begin{array}{cc|c} \underline{A}R\underline{A}^T - R & \underline{A}R\grave{C}_1^T & \grave{B}_1 \\ \grave{C}_1 R\underline{A}^T & \grave{C}_1 R\grave{C}_1^T - I_{\bar{p}} & 0_{\bar{p}\times\bar{m}} \\ \hline \grave{B}_1^T & 0_{\bar{m}\times\bar{p}} & -I_{\bar{m}} \end{array}\right] \left[\begin{array}{c|c} N_R & 0_{(n+\bar{p})\times\bar{m}} \\ \hline 0 & I_{\bar{m}} \end{array}\right] < 0;$$

$$\left[\begin{array}{c|c} N_S & 0_{(n+\bar{m})\times\bar{p}} \\ \hline 0 & I_{\bar{p}} \end{array}\right]^T \left[\begin{array}{cc|c} \underline{A}^T S\underline{A} - S & \underline{A}^T S\grave{B}_1 & \grave{C}_1^T \\ \grave{B}_1^T S\underline{A} & \grave{B}_1^T S\grave{B}_1 - I_{\bar{m}} & 0_{\bar{m}\times\bar{p}} \\ \hline \grave{C}_1 & 0_{\bar{p}\times\bar{m}} & -I_{\bar{p}} \end{array}\right] \left[\begin{array}{c|c} N_S & 0_{(n+\bar{m})\times\bar{p}} \\ \hline 0 & I_{\bar{p}} \end{array}\right] < 0;$$

$$\begin{bmatrix} R & I_n \\ I_n & S \end{bmatrix} \geq 0, \tag{5.20}$$

where $\operatorname{img}(N_R) = \ker\left(\begin{bmatrix} \underline{B}_2^T & \grave{D}_{12}^T \end{bmatrix}\right)$ *and* $\operatorname{img}(N_S) = \ker\left(\begin{bmatrix} C_2 & 0_{m\times\bar{m}} \end{bmatrix}\right)$.

Suppose that the solutions R and S to the LMIs in (5.20) are found and that

$$\operatorname{rank}(I_n - RS) = k \leq n.$$

Let $M, N \in \mathbb{R}^{n\times k}$ be any two *full-column-rank* matrices satisfying

$$MN^T = I_n - RS$$

and $0 < X_{cl} \in \mathbb{R}^{(n+k)\times(n+k)}$ be the unique solution of the linear equation

$$\begin{bmatrix} S & I_n \\ N^T & 0_{k\times n} \end{bmatrix} = X_{cl} \begin{bmatrix} I_n & R \\ 0_{k\times n} & M^T \end{bmatrix}. \tag{5.21}$$

Proposition 5.3.7 ([GA94, Section 7]). *Suppose R and S have been found satisfying the LMIs in Proposition 5.3.6, construct X_{cl} as in (5.21). Then there exists $\left[\begin{smallmatrix} A_d & B_d \\ C_d & D_d \end{smallmatrix}\right] \in \mathbb{R}^{(k+p)\times(k+m)}$ satisfying the LMI*

$$H_{X_{cl}} + Q^T \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix}^T P + P^T \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} Q < 0,$$

where

$$P := \begin{bmatrix} 0_{k\times n} & I_k & 0_{k\times n} & 0_{k\times k} & 0_{k\times\bar{m}} & 0_{k\times\bar{p}} \\ \underline{B}_2^T & 0_{p\times k} & 0_{p\times n} & 0_{p\times k} & 0_{p\times\bar{m}} & \grave{D}_{12}^T \end{bmatrix};$$

$$Q := \begin{bmatrix} 0_{k\times n} & 0_{k\times k} & 0_{k\times n} & I_k & 0_{k\times\bar{m}} & 0_{k\times\bar{p}} \\ 0_{m\times n} & 0_{m\times k} & C_2 & 0_{m\times k} & 0_{m\times\bar{m}} & 0_{m\times\bar{p}} \end{bmatrix};$$

$$H_{X_{cl}} := \begin{bmatrix} -X_{cl}^{-1} & A_0 & B_0 & 0_{(n+k)\times\bar{p}} \\ A_0^T & -X_{cl} & 0_{(n+k)\times\bar{m}} & C_0^T \\ B_0^T & 0_{\bar{m}\times(n+k)} & -I_{\bar{m}} & 0_{\bar{m}\times\bar{p}} \\ 0_{\bar{p}\times(n+k)} & C_0 & 0_{\bar{p}\times\bar{m}} & -I_{\bar{p}} \end{bmatrix};$$

$$A_0 := \begin{bmatrix} \underline{A} & 0_{n\times k} \\ 0_{k\times n} & 0_{k\times k} \end{bmatrix}; \quad B_0 := \begin{bmatrix} \grave{B}_1 \\ 0_{k\times\bar{m}} \end{bmatrix}; \quad C_0 := \begin{bmatrix} \grave{C}_1 & 0_{\bar{p}\times k} \end{bmatrix}.$$

Moreover, $P_d := (A_d, B_d, C_d, D_d) \in \boldsymbol{\mathcal{D}}^{p\times m}$ *satisfies* $\mathcal{F}_l(\grave{G}, P_d) \in \boldsymbol{\mathcal{D}H}_{\mathbb{D}}^{\infty}$ *and* $\|\mathcal{F}_l(\grave{G}, P_d)\|_\infty < 1$*. Note that the order or degree of the constructed* P_d *satisfies*

$$\deg(P_d) \leq \deg(P) + \deg(F) + 2(\deg(W_i) + \deg(W_o)). \tag{5.22}$$

The set of solutions of a system of LMIs is convex [BV04]. Thus, $\mathbb{Y}_d\left(\grave{G}\left(R, W_i, W_o, F, h\right)\right)$ is convex by Proposition 5.3.6. This in turn means that $\mathbb{Y}(R, W_i, W_o, F, h)$ is also convex by Theorem 5.3.5. In the case where $\mathbb{Y}_d(\grave{G})$ is non-empty, Proposition 5.3.7 can be used to identify an element of the set and hence the corresponding element in $\mathbb{Y}(R, W_i, W_o, F, h)$. In summary, Theorem 5.3.5 and Proposition 5.3.6 can together be employed to address the convex feasibility problem in Step 3 of Algorithm 5.3.2, while Proposition 5.3.7 can be used subsequently to construct a feasible point in Step 5.

5.4 Numerical examples

The numerical examples in this section are generated by making use of the MATLAB toolboxes YALMIP [Löf04] and SDPT3 [TTT99] for defining and solving LMIs, respectively.

First consider a strictly proper $P \in \boldsymbol{\mathcal{R}}$ defined by $P(s) = \frac{0.5s+0.5}{(0.05s+1)(0.2s+1)}$, whose 3-dB bandwidth is approximately 140 rad/sec and $b_{\text{opt}}(P; \boldsymbol{\mathcal{R}}) = 0.849$. We use unity weights and set $F(s) = \frac{b^2}{(s+b)^2}$, where b denotes the bandwidth of the anti-aliasing filter. Figure 5.1 illustrates the value of $\delta_\nu(\underline{P}, \underline{\mathcal{H}_h P_d \mathcal{S}_h F})$, where $\mathcal{H}_h \mathbf{P}_d \mathcal{S}_h \mathbf{F}$ denotes the optimal sampled-data approximation for P, obtained by applying Algorithm 5.3.2 while varying b and h. It can be observed that a higher sampling frequency and an anti-aliasing filter's bandwidth matching that of the nominal transfer function P, around 140 rad/sec, results in better approximations with respect to the ν-gap metric.

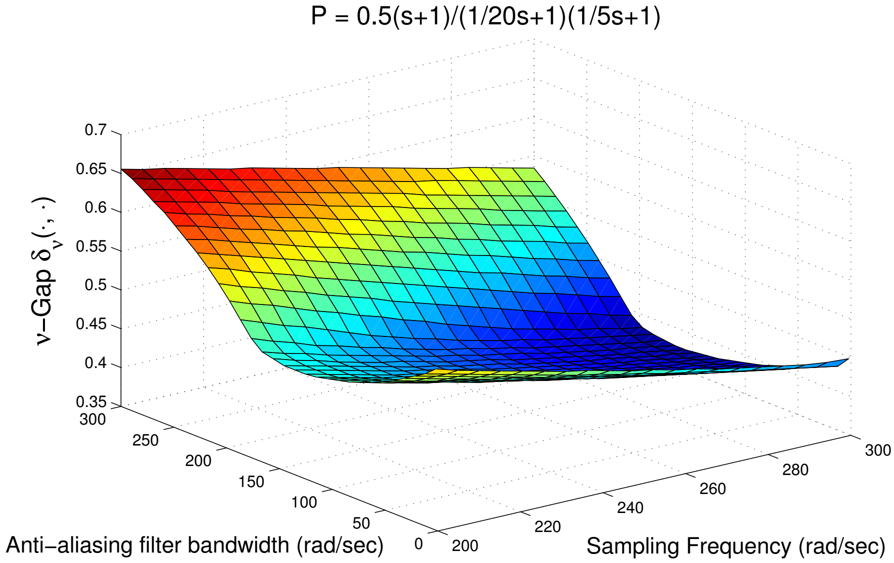

Figure 5.1: ν-gap between P and its optimal SD approximation

Consider now a plant

$$P(s) = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)},$$

where $K = 2$, $\tau_1 = 0.05$, and $\tau_2 = 0.001$. A transfer function of such form may model, for instance, the dynamics of a DC servo motor from its input voltage to the output angular velocity. Setting $W_i(s) := \frac{20}{s+3}$ and $W_o(s) := 1$, a $\boldsymbol{H}^\infty$ loop-shaping controller [MG90] for P may be designed to be $C(s) = W_o(s)C_s(s)W_i(s)$, where $C_s(s) = \frac{-1.469s^2-1507s-37850}{s^2+1064s+65680}$. The achieved robustness margin is $b(P, C) = 0.562$. Selecting $F(s) := \frac{5}{s+5}$, we discretise $C_s(s)$ to obtain $\hat{C}_s \in \boldsymbol{\mathcal{L}}$ by: (i) Applying Algorithm 5.3.2; and (ii) Taking the Tustin's bilinear transformation ($z = \frac{1+sh/2}{1-sh/2}$ with h the sampling period). The resulting closed-loop characteristics are summarised in Table 5.1, in which $\hat{C} := \underline{W_o}\hat{C}_s\underline{W_i}$ and $\Delta := [\underline{P}, \underline{C}] - [\underline{P}, \hat{C}]$.

Observe that approach (i) outperforms (ii) significantly based on the achieved closed-loop behaviours of their respective approximations. This owes to Algorithm 5.3.2 automatically taking into account the anti-aliasing filter F and the input and output weights W_i, W_o when executing sampled-data approximation, in such a way that is optimal in the weighted ν-gap measure of distance, with respect to which the feedback robustness results in Chapter 3 hold true. Nevertheless, note that the more accurate approximations are acquired at the expense of a higher controller order. Specifically, the order of $C_d \in \boldsymbol{\mathcal{D}}$ in $\hat{C}_s = \underline{\mathcal{H}_h C_d \mathcal{S}_h F} \in \boldsymbol{\mathcal{L}}$ for approach (ii) is 2 while that for approach (i) can be as high as 5; see (5.22).

Alg.	Samp. freq.	$\delta_\nu(\underline{C}, \hat{C})$	$b(\underline{P}, \hat{C})$	$\|\Delta\|_\infty$
(i)	100 rad/s	0.521	0.335	2.491
	250 rad/s	0.342	0.403	1.504
(ii)	100 rad/s	0.707	0.084	11.239
	250 rad/s	0.652	0.146	6.099

Table 5.1: Closed-loop performances of sampled-data control systems

5.5 Summary

This chapter reconsiders the problem of optimal sampled-data approximation from [CV04] for robust feedback design, in which the error incurred is measured by the generalised ν-gap metric from Chapter 3. An appropriate mathematical framework is developed to rigorously define the ν-gap measure of distance between an LTI and a sampled-data system. Through the previously established linear fractional characterisation of the ν-gap metric in Section 4.2, it is shown that the approximation problem is solvable by a line search involving at each step a convex feasibility check equivalent to the existence of a solution to a set of LMIs. Numerical results demonstrate that the proposed algorithm is well-suited for approximating controllers designed using robust $\boldsymbol{H}^\infty$-loop shaping.

Chapter 6

Conclusions

This dissertation generalises the ν-gap metric based framework of Vinnicombe [Vin93, Vin01] for analysing the robustness of LTI feedback interconnections, to a general time-varying setting based on initial developments made in [JC10, JC11]. It is rigorously established that the incorporation of integral quadratic constraints (IQCs) with the analysis framework, as originally suggested in [JC10, JC11], provides additional flexibility over purely ν-gap ball based results. On the synthesis side, sampled-data approximation problem [CV04] in the ν-gap metric is also considered and optimally solved. Below, the main contributions are summarised and future research directions identified.

6.1 Contributions

- In Chapter 3, a framework for closed-loop stability which implicitly incorporates an arrow of time [GS10] is developed. This extends the initial development in [JC10, JC11], where a redundant instantaneous gain condition was included in the definition of closed-loop stability.

- A generalised ν-gap metric from [JC10, JC11] for causal linear time-varying systems having normalised strong graph symbol representations is studied in Chapter 3. Several ν-gap ball based sufficient conditions for robust closed-loop stability are derived in a system-theoretic setting, leading to conclusions on the nice properties of the graph topology.

- A necessary and sufficient ν-gap ball based robustness condition is established by exploiting the periodically time-varying structure of a class of systems. This result, together with the aforementioned ones, substantiate the fact that the ν-gap metric has been generalised in the right way.

- A useful characterisation of the ν-gap metric in terms of a gain bound on a linear fractional transformation (LFT) is presented in Chapter 4. Based on this characterisation, pathwise connectedness of ν-gap balls in the graph topology is established, allowing for reconciliation of the more flexible IQC based system analysis of [JC10, JC11] with the ν-gap ball based results of Chapter 3.
- J-spectral factorisation underpinning the LFT characterisation is shown to exist for two generic classes of linear systems: finite-dimensional time-varying systems and distributed-parameter time-invariant systems.
- In Chapter 5, appropriate representations of finite-dimensional linear time-invariant and sampled-data systems are developed, resulting in a well-defined ν-gap measure of distance between them.
- The problem of sampled-data approximation from [CV04] in the weighted ν-gap metric is reformulated using the LFT characterisation and an iterative linear matrix inequality based approach is proposed for optimally solving it.

6.2 Directions for further work

- Development of a ν-gap analysis framework similar to that in Chapter 3 to accommodate nonlinear systems. Unifying this with the IQC based stability analysis would be accomplishable using the arguments in Section 4.1, which exploit neither the linearity property nor any function-theoretic representations.
- Recall from Chapter 4 that IQC based analysis relies on homotopies which are continuous with respect to the ν-gap metric to conclude feedback stability. However, unlike the ν-gap ball type robustness results, the analysis does not explicitly account for closed-loop stability/performance margin of the perturbed feedback systems. An interesting research direction is to examine if this is possible, perhaps by imposing stronger condition on the ν-gap homotopies such as differentiability.
- Pathwise connectedness of different types of uncertainty sets may be verified. For instance, intersections of ν-gap metric balls and a manifold of systems with dimensions no larger than a fixed integer or sets which are structured in a particular way.
- In view of the solvability of sampled-data approximation in the ν-gap metric in Chapter 5, a characterisation of which structural constraints in model approximation give rise to convex optimisation problems may be investigated via an LFT based approach using ideas from, for instance, [RL06].

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