TOPICS IN STABILIZATION UNDER CONSTRAINTS AND SYNCHRONIZATION PROBLEMS

By

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TOPICS IN STABILIZATION UNDER CONSTRAINTS AND SYNCHRONIZATION PROBLEMS

ABSTRACT

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All physical systems are operating under a variety of constraints. Control designs without taking these constraints into account will result in performance deterioration and even instability. A major part of the thesis is devoted to the constrained stabilization problems. The goal is to develop various controller design methodologies to achieve stabilization under different constraints. In the case where a linear system is subject to hard constrains on state and input, the notions of semi-global stabilization in the admissible set and recoverable region are explicitly related to certain structural properties of the systems. For a class of sandwich nonlinear systems consisting of cascaded linear systems and saturation elements, necessary and sufficient conditions for semi-global and global stabilization are presented. Under these conditions, a generalized low-gain design methodology is proposed to solve the stabilization problems. For linear system with input saturation and multiple time delays, upper bounds on the delays are found and corresponding controllers can be designed to achieve semi-global stabilization under input saturation and tolerable delays.

When the issues related to internal stabilization are resolved, the research is directed to simultaneous stabilization problems. The focus here is linear system subject to input saturation. In the case where disturbances that are additive to the input, we complement the existing results in the literature by solving the simultaneous stabilization problems for discrete-time linear systems subject to input saturation. Then attention is paid to non-input-additive disturbances. This research is carried out using a progressive approach. The results are obtained and generalized from a simple double integrator to the most general linear systems. It is found that the simultaneous stabilization problems are solvable if the disturbances

do not contained large frequency component corresponding to the open-loop eigenvalues on the stability margin. For those disturbances that meet this criterion, dynamical feedback controller can be constructed to achieve simultaneous stabilization.

The rest of the thesis studies synchronization problems in multi-agent networks with uniform constant communication delays. Both homogeneous and heterogenous networks are considered. In the homogenous case, we assume that agents that are at most critically unstable. An achievable upper bound of delay tolerance is obtained which explicitly depends on agent dynamics and network topology. For any delay satisfying the proposed upper bounds, a controller design methodology without exact knowledge of the network topology is proposed so that the multi-agent synchronization in a set of networks can be achieved. For heterogeneous networks, under the assumption that the agents are introspective and right-invertible, synchronization problem can be solved for an arbitrarily given delay via decentralized dynamical controllers. The synchronization with output regulation problem in a heterogeneous network is also investigated and solved under mild conditions. The proposed design method can be applied to the output formation problem.

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Dedication

This dissertation is dedicated to my parents.

CHAPTER 1

Introduction

1.1. Low-gain feedback

The research demonstrated in this dissertation is centered around the development, extension and application of a core theory, that is,the low-gain theory. The low-gain feedback was originally developed to solve the semi-global stabilization problem for linear system subject to input saturation. Four different design methods are independently proposed in the literature, which are based on direct eigenstructure assignment, the solution of an H_2/H_∞ algebraic Riccati equation, or the solution of an parametric Laypunov equation. We shall show that these four method can be unified under one theoretic framework, that is, H_2/H_∞ control theory, and extended into a more general form. It will be shown in future chapters that this H_2/H_∞ low-gain feedback design can be generalized or combined with other tools and techniques, such as adaptive gain scheduling and/or Model Predictive Control, to address a variety of issues in stabilization under constraints and synchronization problems.

1.2. Internal stabilization of dynamical systems under constraints

All physical system are operating under a variety of constraints. Most constraints can be roughly characterized into two categories, namely soft constraints and hard constraints. Soft constraints are intrinsic property of the system, which generally result from limited capacity of physical devices. In this case, if a quantity is subject to soft constraints, the outcome is forced to take only certain allowed values, even though the original quantity may not. One ubiquitous type of soft constraints is saturation nonlinearity whose output always lies in a limited range. The hard constraints are referred to those restrictions imposed on the system by its operator or designer for reasons such as security, linearization accuracy and etc. In contract to soft constrains like a saturation which may be overloaded, the hard constraints can never be violated. The stabilization problem under soft and hard constraints may have different solvability condition and require different methodologies.

Physical quantities such as speed, acceleration, pressure, flow, current, voltage, and so on, are always limited to a finite range, and saturation non-linearities are therefore a ubiquitous feature of physical systems. One class of such systems is the class of linear systems subject to actuator saturation. Early work on internal stabilization of linear systems subject to actuator saturation started with the seminal work of [24] which established that a chain of integrators with order higher than two cannot be globally asymptotically stabilized by any saturating linear control law. Continuing the theme of Fuller, [110, 120, 154, 155] established that, in general, global asymptotic stabilization of linear systems with bounded inputs can only be achieved using non-linear feedback laws. Moreover, this stabilization can be achieved if and only if the given system in the absence of saturation is stabilizable and critically unstable (equivalently, *asymptotically null controllable with bounded control* (ANCBC)). We note that critically unstable systems are those systems that have all their open-loop poles within the closed left half plane (continuous-time systems) or within the closed unit disc (discrete-time systems). The works of Sontag, Sussman, and Yang unleashed a flurry of activity in internally stabilizing linear systems subject to actuator saturation. Along one direction, [128, 129] proposed certain design methodologies to design appropriate controllers for global stabilization. [68] came up with a gain scheduling based non-linear control law utilizing Riccati equations. Along another direction, Saberi and his students queried as to what can be achieved by utilizing only linear feedback control laws. In this respect, [51, 53, 50] proposed and emphasized a semi-global rather than global framework for stabilization using bounded controls. All this early work of these authors and others is surveyed in [5, 94, 123, 95, 32, 35], and the references therein.

Following the early phase work outlined above, during the last decade and a half, among others, our research team probed intensely into a number of problems concerning the stabilization of linear systems subject to input and state constraints, where the controller must guarantee that the output which is a linear combination of inputs and states of a linear system remains in a given set (see, e.g., [87, 88, 97] and references therein). This is a general type of hard constraints. Another progressive generalization of linear systems subject to input saturation is so-called sandwich nonlinear systems which consist of cascaded linear systems with static nonlinear elements, such as deadbeat and saturation, embedded in between. The stabilization of linear system under input and state constraints is somehow related to that of the sandwich nonlinear systems. However, there is an obvious bifurcation in controller design and analysis owing to the inherent difference between hard and soft constraints.

In the last few decades, time-delayed system has been greeted with great enthusiasm from researchers in recognition of its theoretical and applied importance, see [84]. Many control problems have been extensively studied, among which stability and stabilization are of particular interest (see, for instance, [73, 28, 72, 37, 23, 15] and references therein). When both saturation and time-delay are present in the system, controller design can be challenging, especially when the delay is not known.

1.2.1. Simultaneous internal and external stabilization of linear systems subject to input saturation and disturbances

Once the issues related to internal stabilization were resolved, the research was directed towards simultaneous external and internal stabilization. Such simultaneous stabilization also has a long history. A well-known result in linear system theory states that asymptotically stable systems have very good external stability properties. Thus, for linear systems the notions of internal stability and external stability in any sense are highly coupled. However, for general non-linear systems, these two notions of stability are vastly different. The relation between external stability and internal stability of a nonlinear systems has been of interest to researchers, which is also one subject of this part because its theoretical importance. A more complete review of work along this direction is given in Chapter 10 where we give a detail a study into this problem.

For the class of linear systems subject to actuator saturations, the external disturbances may be roughly classified into two categories.

1. Input-additive disturbances. This case can be described by the following model:

$$\dot{x} = Ax + B\sigma(u + d);$$

2. Non-additive disturbances, which can be written as

$$\dot{x} = Ax + B\sigma(u) + Ed.$$

Historically, the non-input-additive disturbances are further classified as *matched* disturbances if $E = B$ or *mismatched* disturbances if $E \neq B$.

The concept of simultaneous external and internal stabilization for a linear system with actuator saturation was first studied in [30, 31] and [47, 2] mainly for input-additive disturbances. Subsequent to this work, there exist numerous other works on simultaneous external and internal stabilization (see e.g. [89, 13]). The picture that emerges from all these works is that, for the case when external disturbance is additive to the control input, all the issues associated with simultaneous external and internal stabilization are more or less resolved, but only for continuous-time systems. In this part, we shall fill the blank by solving the simultaneous stabilization problem for discrete-time linear with actuator saturation and input-additive disturbances.

On the other hand, [115] studies the non-input-additive case and finds that $\mathcal{L}_p$ and ℓ_p stabilization with finite gain are impossible, but $\mathcal{L}_p$ and ℓ_p stabilization without finite gain are always attainable via a dynamic low-gain feedback. Moreover, for an open-loop neutrally stable system, it is attainable via a linear static state feedback (see [107]). Nevertheless, these results only apply to $\mathcal{L}_p$ and ℓ_p disturbances for $p \in [1, \infty)$ (i.e., disturbances whose "energy" vanishes asymptotically), and not to sustained signals belonging to $\mathcal{L}_\infty$ and ℓ_∞.

For sustained signals that are non-input-additive, clearly not all disturbances can be managed appropriately as, for instance, a large constant disturbance aligned (matched) with the input could overpower the saturated control and lead to unbounded states. In view of this, a natural starting point for the study of non-additive disturbances is the matched case where disturbances should have magnitude smaller than the level of saturation by a known margin. As we move further, in an effort of dealing with more general non-additive sustained disturbances, of particular interest is the study on identifying classes of disturbances for which a controller can be designed to yield bounded closed-loop state trajectories. In Part III, substantial attention will be paid to address this problem.

1.2.2. Synchronization in the networks

The synchronization analysis and design in networks have received substantial attention in recent years, partly due to the wide applications in areas such as sensor networks and autonomous vehicle control. A relatively complete coverage of earlier work can be found in the survey paper [74], the recent books [150, 83] and references therein.

The research can be generally divided into two categories: one studies *homogeneous* networks, that is–network consist of identical agents–and the other studies *heterogeneous* networks in which non-identical agents are interconnected. The study on state synchronization in homogeneous network has been quite fruitful. Depending on what information the agents collect from the network, synchronization in homogenous networks can be classified into two categories. In some networks, each agent measures its own state relative to that of neighbors, which is referred to as *full-state coupling* [75, 76, 79, 80, 82, 138]; In other networks, the agents may collect information of its output relative to that of its neighboring agents, which we refer to as *partial-state coupling* [77, 139, 45, 104]. Although the work is primarily focused on continuous-time case, synchronization in homogeneous networks of discrete-time agents has also been studied in [76, 44, 140] (also see the references therein). A distributed observer-based synchronization controller was developed in [44] which communicates information over the same network. In [140], the author considers a very special case of neutrally stable agent with full actuation ($B = I$). A network of first-order agents with Laplacian communication topology is studied in [76] where the topology can be switching. All the aforementioned work only considers identical agents, in another word, homogeneous networks.

In contrast to the flourishing research on synchronization in homogenous networks, relatively limited results have been obtained for heterogeneous networks. For heterogenous networks, the notion of state synchronization may no longer make sense as each agent possesses a set of state information which may be inherently different from others. In this case, it is more natural to study an alternative problem of *output synchronization*, that is, all the agents should agree on a set of pre-selected outputs (see, for example, [19, 38, 152]). In this body of work, it is commonly assumed that each agent has a local measurement of its own states, which we refer to as introspective agents. Moreover, the result on discrete-time heterogeneous network is even sparse.

Due to the ubiquity of communication delay during the transmission of information, the research has also been directed to synchronization in networks with time delay. Most results in the literature consider the agent model as described by single-integrator dynamics ([7, 134, 76]), or double-integrator dynamics ([135, 46, 9]). Specifically, it is shown by [76] that a network of single-integrator agents subject to uniform constant communication delay can achieve consensus with a particular linear local control protocol if and only if the delay is bounded by a maximum that is inversely proportional to the largest eigenvalue of the graph Laplacian associated with the network. This result was later on generalized in [7] to non-uniform constant or time-varying delays. Sufficient conditions for consensus among agents with first order dynamics were also obtained in [134]. The results in [76] were extended in [46, 9] to double integrator dynamics. An upper bound on the maximum network delay tolerance for second-order consensus of multi-agent systems with any given linear control protocol was obtained. Despite aforementioned advances, this research is still largely situated in a limited framework – that is, homogenous networks of simple agents mostly with first order or second order dynamics. To the best of the authors' knowledge, the results that explicitly consider heterogenous networks of higher-order agents and time-delay are [40, 43]. The single-output synchronization is studied in [40]. A frequency-domain approach based on Geršgorin's theorem and spectral radius stability theorem is proposed to design a decentralized linear consensus controller. However, the consensus condition obtained in [40] is very conservative (see [134]). [43] studies single-input single-output agents and undirected communication topologies. A consensus condition is derived based on the notion of *S-hull*. However, the results on synchronization in a homogenous or heterogenous network of complex agents under communication delay remains largely unknown.

1.3. Organization

The dissertation is written as a collection of published works. There is repetitiveness, to some extent, among chapters so that each chapter presents self-contained technical results and can be read independently. Also, each chapter may have different structure and even call upon incoherent notations. However, this only happens when no confusion is incurred.

The rest of the dissertation is divided into four parts.

1. Part I comprises Chapter 4 - 6, where we develop H_2/H_∞ low-gain feedback. This is essentially the basis of the entire work.

   (a) Chapter 2: continuous-time case.

   (b) Chapter 3: discrete-time case.

2. Part II comprises Chapter 4-9, which is devoted to internal stabilization of dynamical systems under constraints.

(a) Chapter 4 and 5 consider linear systems with hard constraints on the inputs and states. Semi-global stabilization in admissible set and recoverable region problems are solved based on the taxonomy of constrains and a special coordinate basis.

(b) Chapter 6 studies a class of sandwich nonlinear systems. Necessary and sufficient solvability conditions for global and semi-global stabilization problems are obtained. Whenever these conditions are satisfied, a generalized low-gain design is explicitly constructed to solve the stabilization problems.

(c) Chapter 7 and 8 solve the semi-global stabilization problem for linear systems subject to input saturation and multiple input delays. A upper bound on the delay tolerance is found for which a low-gain state or measurement feedback can be design to achieve semi-global stabilization.

(d) Chapter 9 propose a new low-and-high gain design methodology based on the low-gain feedback and a ultra-short-horizon MPC.

3. Part III comprises Chapter 10 - 16, in which we deal with simultaneous external and internal stabilization of linear system subject to input saturation and disturbances.

   (a) The first objective is to establish the relation between $\mathcal{L}_p$ stability and internal stability of general nonlinear systems. This is done in Chapter 10;

   (b) The second objective is to solve the remaining simultaneous external and internal stabilization problem related to input-additive disturbances for discrete-time linear systems subject to actuator saturation. This problem is solved in Chapter 11;

   (c) The third objective is to address two issues related to non-additive sustained disturbances. One issue is to identify classes of non-additive sustained disturbances that are manageable, while the other is to design feedback controller to solve the simultaneous stabilization problem for these classes of disturbances. Chapter 12-16 are devoted into achievement of this objective. Specifically, Chapter 12 marks our first attempt to deal with non-additive disturbances in which we consider a matched disturbance whose magnitude is restricted within the range of saturation. Then we move to general cases in a progressive manner from simple double integrators (Chapter 13), chain of integrators (Chapter 14), neutrally stable systems (Chapter 15) to eventually general linear systems (Chapter 16).

4. Part IV comprises Chapter 17 - 19, which is concerned with synchronization in the networks.

   (a) Chapter 17 studies studies output synchronization and regulation problem for a heterogenous network of discrete-time introspective right-invertible agents.

   (b) Chapter 18 is concerned with synchronization in a homogeneous network of continuous-time agents that are at most critically unstable and coupled through networks with uniform constant communication delay.

   (c) Chapter 19 will further investigate the heterogenous network of introspective right-invertible agents. Both output synchronization and output regulation problems are studied.

Part I

Low-gain feedback theory

CHAPTER 2

H_2 and H_∞ low-gain feedback – Continuous-time systems

2.1. Introduction

The low-gain feedback design methodology was first developed in [51, 52] to achieve semi-global stabilization of linear systems subject to input saturation. Since then, it has been widely employed in various control problems, such as output regulation with constraints, H_2 and H_∞ optimal control etc [61, 93]. The low-gain feedback can be constructed using four different approaches, namely direct eigenstructure assignment [51, 52], H_2 and H_∞ algebraic Riccati equation (ARE) based methods [61, 130], and parametric Lyapunov equation based method [159, 161]. Although these four methodologies were independently proposed in literature, we shall show that they are all rooted in and can be unified under two fundamental control theories, H_2 and H_∞ theory.

Moreover, all these designs for low-gain feedbacks only consider the case where a low gain is required in all input channels, and consequently require the asymptotic null controllability with bounded input (ANCBC) of the given system. In this note, we introduce the concept of H_2 and H_∞ low-gains in a general setting where either some or all input channels have constraints of low-gain imposed upon them. We provide explicit existence conditions and design methods which yield the classical ANCBC condition and the four design methods as special cases.

Standard notations are used in this chapter. $\mathbb{C}^-$, $\mathbb{C}^\circ$ and $\mathbb{C}^+$ denote open left half complex plane, the imaginary axis and open right half complex plane respectively. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x. For $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X. For a vector-valued continuous-time signal y, $\|y\|_{\mathcal{L}_p}$ denotes the $\mathcal{L}_p$ norm of y. For a continuous-time system Σ having a $q \times \ell$ stable transfer function G, $\|G\|_2$ and $\|G\|_\infty$ denote respectively the standard H_2 and H_∞ norm of G.

2.2. Definition of H_2 and H_∞ low-gain sequences

Consider the linear time invariant continuous-time system,

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ z = Du \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $z \in \mathbb{R}^{m_0}$. Without loss of generality, we assume that $D = \begin{pmatrix} I_{m_0} & 0 \end{pmatrix}$.

In what follows, a state feedback gain such as F_ε parameterized in a parameter ε is called a gain sequence since as ε changes one obtains a sequence of gains. We define below formally what we mean by H_2 and H_∞ low-gain sequences.

Definition 2.1 For the system Σ in (2.1), the ***H*₂ low-gain sequence** is a sequence of parameterized static state feedback gains F_ε for which there exists an ε^* such that the following properties hold:

1. There exists a constant M such that $\|F_\varepsilon\| \leq M$ for any $\varepsilon \in (0, \varepsilon^*]$;
2. $A + BF_\varepsilon$ is Hurwitz stable for any $\varepsilon \in (0, \varepsilon^*]$;
3. For any $x(0) \in \mathbb{R}^n$, the closed-loop system with $u = F_\varepsilon x$ satisfies $\lim_{\varepsilon \to 0} \|z\|_{\mathcal{L}_2} = 0$.

The H_∞ low-gain sequence will depend on an, a priori given, constant γ. Therefore we will call it a γ-level H_∞ low-gain sequence. Whenever we refer to the H_∞ low-gain sequence, we always imply the γ-level H_∞ low-gain sequence.

Definition 2.2 For Σ in (2.1) and for an arbitrary $E \in \mathbb{R}^{n \times p}$, define an auxiliary system

$$\Sigma_\infty : \begin{cases} \dot{x} = Ax + Bu + E\omega \\ z = Du, \end{cases} \tag{2.2}$$

and the infimum

$$\gamma^* = \inf_F \left\{ \|DF(sI - A - BF)^{-1}E\|_\infty \mid \lambda(A + BF) \in C^- \right\}. \tag{2.3}$$

For a given $\gamma > \gamma^*$, the **γ-level H_∞ low-gain sequence** is a sequence of parameterized static state feedback gains $F_\varepsilon(E, \gamma)$ for which there exists an ε^* such that

1. There exists a constant M such that $\|F_\varepsilon(E, \gamma)\| \leq M$ for any $\varepsilon \in (0, \varepsilon^*]$;

2. $A + BF_\varepsilon(E,\gamma)$ is Hurwitz stable for any $\varepsilon \in (0, \varepsilon^*]$;

3. For system Σ_∞ with $u = F_\varepsilon(E,\gamma)$ and any $x(0) \in \mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} \left\{ \sup_{\omega \in \mathcal{L}_2} (\|z\|^2_{\mathcal{L}_2} - \gamma \|\omega\|^2_{\mathcal{L}_2}) \right\} = 0.$$

2.3. Properties of H_2 and H_∞ low-gain sequences

Theorem 2.1 For the system Σ in (2.1) with a given $E \in \mathbb{R}^{n\times p}$ and a $\gamma > \gamma^*$ where γ^* is defined in (2.3), a sequence of feedback gains $F_\varepsilon(E,\gamma)$ is a γ-level H_∞ low-gain sequence only if it is an H_2 low-gain sequence.

Proof : By setting $\omega = 0$ in the definition of H_∞-γ-level low-gain sequence, we immediately conclude this result. ■

The next theorem shows that for the closed-loop system Σ in (2.1) with either an H_2 low-gain controller $u = F_\varepsilon x$ or an H_∞ low-gain controller $u = F_\varepsilon(E,\gamma)x$, the magnitude of z and DF_ε or $DF_\varepsilon(E,\gamma)$ can be made arbitrarily small.

Theorem 2.2 The closed-loop system (2.1) with either $u = F_\varepsilon x$ or $u = F_\varepsilon(E,\gamma)x$ satisfies the following properties:

1. $\lim_{\varepsilon\to 0} \|z\|_{\mathcal{L}_\infty} = 0$,

2. $\lim_{\varepsilon\to 0} DF_\varepsilon = 0$ and $\lim_{\varepsilon\to 0} DF_\varepsilon(E,\gamma) = 0$.

Proof : Owing to Theorem 2.1, we only need to prove these two properties for an H_2 low-gain sequence. The fact that $\|z\|_{\mathcal{L}_2} \to 0$ as $\varepsilon \to 0$ for any $x(0)$ implies that

$$\lim_{\varepsilon\to 0} \|F_\varepsilon e^{(A+BF_\varepsilon)t}\|_{\mathcal{L}_2} = 0.$$

Since $\|F_\varepsilon\|$ is bounded for all $\varepsilon \in (0,\varepsilon^*]$, $\|A + BF_\varepsilon\|$ is also bounded for all $\varepsilon \in (0,\varepsilon^*]$. This yields,

$$\lim_{\varepsilon\to 0} \|F_\varepsilon e^{(A+BF_\varepsilon)t}(A + BF_\varepsilon)\|_{\mathcal{L}_2} = 0.$$

This implies that $\dot{z} \in \mathcal{L}_2$ and moreover $\lim_{\varepsilon\to 0} \|\dot{z}\|_{\mathcal{L}_2} = 0$. Applying Cauchy-Schwartz inequality, we can show that

$$\big| \|z(t)\|^2 - \|z(0)\|^2 \big| \le 2\|\dot{z}\|_{\mathcal{L}_2}^{[0,t]} \|z\|_{\mathcal{L}_2}^{[0,t]}. \tag{2.4}$$

Let ε be fixed and $t \to \infty$. Since $A + BF_\varepsilon$ is Hurwitz, $\|z(t)\| \to 0$. This yields that $\|z(0)\|^2 \le 2\|\dot{z}\|_{\mathcal{L}_2}\|z\|_{\mathcal{L}_2}$. Next, let $\varepsilon \to 0$. We conclude that for any $x(0) \in \mathbb{R}^n$,

$$\lim_{\varepsilon\to 0} \|z(0)\|^2 = \lim_{\varepsilon\to 0} \|DF_\varepsilon x(0)\|^2 = 2 \lim_{\varepsilon\to 0} \|\dot{z}\|_{\mathcal{L}_2}\|z\|_{\mathcal{L}_2} = 0,$$

and hence $\lim_{\varepsilon\to 0} DF_\varepsilon = 0$. On the other hand, (2.4) also yields

$$\|z(t)\|^2 \le 2\|\dot{z}\|_{\mathcal{L}_2}^{[0,t]} \|z\|_{\mathcal{L}_2}^{[0,t]} + \|z(0)\|^2 \le 2\|\dot{z}\|_{\mathcal{L}_2}\|z\|_{\mathcal{L}_2} + \|z(0)\|^2.$$

Therefore, $\lim_{\varepsilon\to 0} \|z\|_{\mathcal{L}_\infty} = 0$. ∎

We emphasize that if F_ε is not bounded, the above theorem is not true in general.

Remark 2.1 *If F_ε is not bounded, the above theorem is not true in general. As an example, consider the system,*

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_0 \\ \quad z \quad = u_0. \end{cases}$$

Choosing $u_0 = \varepsilon x_2$ yields

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1+\varepsilon \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_1 \\ \quad z \quad = \varepsilon x_2. \end{cases}$$

Choose $u_1 = -\frac{1}{(1+\varepsilon)\varepsilon^2} x_1 - \frac{1}{\varepsilon} x_2$. Define $y_1 = \varepsilon x_1$, $y_2 = (1+\varepsilon)\varepsilon^2 x_2$ and $\tilde{t} = \frac{1}{\varepsilon} t$. The closed-loop system in the new coordinates and in the new time scale is given by

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{2.5}$$

We first verify that the controller $(u_0, u_1)'$ is H_2 low-gain. Note that $\|y(0)\| \le \varepsilon\|x(0)\|$ provided ε is small. There exists a γ_2 independent of ε such that $\|y_2\|_{\mathcal{L}_2} \le \gamma_2\|y(0)\| \le \varepsilon\gamma_2\|x(0)\|$. Then

$$\|z\|_{\mathcal{L}_2}^2 = \varepsilon^2\|x_2\|_{\mathcal{L}_2}^2 = \varepsilon^2 \int_0^\infty \frac{y_2^2}{(1+\varepsilon)^2\varepsilon^4} \varepsilon d\tilde{t} \le \frac{1}{\varepsilon}\|y_2\|_{\mathcal{L}_2}^2 \le \varepsilon\gamma_2\|x(0)\|.$$

Therefore for any $x(0)$, we have $\lim_{\varepsilon\to 0}\|z\|_{\mathcal{L}_2} = 0$.

However, if we fix initial condition at $(x_1(0), x_2(0))' = (1,0)'$, we get $(y_1(0), y_2(0))' = (\varepsilon, 0)$ and $\|y_2\|_{\mathcal{L}_\infty} = \varepsilon\gamma_\infty$ where

$$\gamma_\infty := \|y_2\|_{\mathcal{L}_\infty} \text{ with } \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Note that γ_∞ is independent of ε. Then

$$\|z\|_{\mathcal{L}_\infty} = \varepsilon\tfrac{1}{(1+\varepsilon)\varepsilon^2}\varepsilon\gamma_\infty = \tfrac{\gamma_\infty}{1+\varepsilon},$$

and $\|z\|_{\mathcal{L}_\infty}$ cannot be reduced to zero.

Theorem 2.2 enables us to connect to the literature and explain why the H_2 and γ-level H_∞ sequences as defined in Definitions 2.1 and 2.2 are termed as '*low-gain*' sequences. As we alluded to in the introduction, the name *low-gain* sequence arose and has roots in one of the classical problems, namely the problem of semi-globally stabilizing a linear system subject to actuator saturation. (For readers not familiar with the saturation literature, we refer to [5, 32, 35, 98, 123] for more details.) To be precise, let

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\sigma(\bar{u}) \tag{2.6}$$

where the function $\sigma(\cdot)$ denotes a standard saturation; that is, $\sigma(\bar{u}) = \operatorname{sign}(\bar{u})\min\{1, |\bar{u}|\}$. Let the pair $(\bar{A}, \bar{B})$ be stabilizable and $\bar{A}$ has all its eigenvalues in the closed left half plane.

Consider a state feedback controller, $\bar{u} = \bar{F}_\varepsilon\bar{x}$ where $\bar{F}_\varepsilon$ is a parameterized sequence with the parameter as ε. If the feedback sequence $\bar{F}_\varepsilon$ satisfies all the three conditions posed in Theorem 3.1 of [52], it is known as a '*low-gain*' feedback in the context of stabilization of linear systems subject to saturation (see also [49]). In fact, the state feedback controller $\bar{u} = \bar{F}_\varepsilon\bar{x}$ where $\bar{F}_\varepsilon$ is such a *low-gain* sequence semi-globally stabilizes (2.6) for a small enough value of ε. That is, there exists an ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, the closed-loop system comprising (2.6) and $\bar{u} = \bar{F}_\varepsilon\bar{x}$ is semi-globally stable with *a priori* given (arbitrarily large) bounded set Ω being in the region of attraction, and moreover the smaller the value of ε the larger can be the *a priori* prescribed set Ω.

Having recalled above the classical semi-global stabilization problem of a linear system with saturating linear feedbacks, we can now emphasize its connection to Theorem 2.2. As is done in classical semi-global stabilization problem, let us first assume that all the control channels are subject to saturation. Then, to see the connection between such a semi-global stabilization problem and Theorem 2.2, set

$D = I_m$ and thus take $z = u$ as the constrained variable subject to saturation. In that case, Theorem 2.2 shows that the H_2 and γ-level H_∞ sequences as defined in Definitions 2.1 and 2.2 satisfy all the three conditions posed in Theorem 3.1 of [52], and hence they can appropriately be termed as *low-gain* sequences. Furthermore, as is evident from Theorem 2.2, they can readily achieve semi-global stabilization of a continuous-time linear system where all control inputs are subject to saturation whenever it is achievable.

For the general setting when $D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$ for some $m_0 < m$, in the scenario of a linear system subject to input saturation, not all the input channels are constrained. To be precise, let

$$\dot{\xi} = A\xi + B_0\sigma(u_0) + B_1 u_1 \tag{2.7}$$

where $\xi \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^{m_0}$, $u_1 \in \mathbb{R}^{m-m_0}$ and $B = \begin{bmatrix} B_0 & B_1 \end{bmatrix}$. Part of the inputs, as represented by u_0, are subject to saturation. In other words, we have the constrained variable $z = Du = u_0$. In this case, property 1 of Theorem 2.2 implies that for an initial condition x_0 in a given set and a pre-specified saturation level Δ, there exists an ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$ the closed-loop system satisfies $\|z(t)\| = \|u_0(t)\| = \|DF_\varepsilon e^{(A+BF_\varepsilon)t} x_0\| \leq \Delta$ for all $t \geq 0$. This implies that the saturation can be made inactive for all time, and hence the closed-loop system can in fact be linear. Therefore, the stability of the closed-loop system directly follows from Definitions 2.1 and 2.2.

2.4. Existence of H_2 and H_∞ low-gain sequences

Theorem 2.3 For the system Σ in (2.1) with an arbitrarily given $E \in \mathbb{R}^{n\times p}$ and $\gamma > \gamma^*$ where γ^* is defined in (2.3), the H_2 and γ-level H_∞ low-gain sequences exist if and only if

1. (A, B) is stabilizable;
2. $(A, B, 0, D)$ is at most weakly non-minimum phase, i.e it has all its invariant zeros in $\mathbb{C}^- \cup \mathbb{C}^\circ$.

Remark 2.2 *In the special case of $D = I_m$, the invariant zeros of $(A, B, 0, I)$ coincide with the eigenvalues of A. Hence Condition 2 requires all the eigenvalues of A are in closed left half plane. In this case, a system that satisfies Conditions 1 and 2 is said to be asymptotically null controllable with bounded control (ANCBC), see [119].*

Proof : For the case of H_2 low-gain sequence, let $\gamma_2^* = \sqrt{\text{trace}(P)}$ where P is the unique semi-stabilizing solution to the continuous-time linear matrix inequality (CLMI),

$$\begin{pmatrix} A'P + PA & PB \\ B'P & D'D \end{pmatrix} \geq 0. \tag{2.8}$$

It is shown in [93] that designing an H_2 low-gain sequence as formulated in Definition 1 can be converted into a singular H_2 suboptimal control problem (see Section 2.2.2). Hence it is evident from [93] that the H_2 low-gain sequence or the corresponding H_2-suboptimal controller exists if and only if $\gamma_2^* = 0$, i.e. $P = 0$. This is equivalent to the conditions that (A, B) is stabilizable and

$$\text{rank} \begin{pmatrix} sI - A & -B \\ 0 & D \end{pmatrix} = \text{normrank} \begin{pmatrix} sI - A & -B \\ 0 & D \end{pmatrix}$$

for any $s \in \mathbb{C}^+$, i.e. $(A, B, 0, D)$ is at most weakly non-minimum phase.

For the case of H_∞ low-gain sequence, note that designing H_∞ low-gain sequence is equivalent to solving a singular H_∞ control problem. We can easily verify [113] that given $\gamma > \gamma^*$ the γ-level H_∞ low-gain sequences exist if and only if, $P = 0$ is a semi-stabilizing solution to the continuous-time quadratic matrix inequality (CQMI),

$$\begin{pmatrix} A'P + PA + \gamma^{-2} PEE'P & PB \\ B'P & D'D \end{pmatrix} \geq 0,$$

which is equivalent to the conditions that (A, B) is stabilizable and that the matrix pencil

$$\begin{pmatrix} sI - A & -B \\ 0 & D \end{pmatrix}$$

does not have any zeros on the open right half plane, i.e. the system is at most weakly non-minimum phase. ■

Remark 2.3 *As shown in the foregoing discussion, the low-gain sequences achieve semi-global stabilization of linear systems subject to input saturation. In order to design a low-gain sequence for the system* (2.6)*, one can choose $D = I_m$ in* (2.1)*. The above theorem then shows that the necessary and sufficient conditions for semi-global stabilization are that (A, B) is stabilizable and all the invariant zeros of $(A, B, 0, I_m)$ are in the closed left half plane. It is known that the invariant zeros of $(A, B, 0, I_m)$ coincide with eigenvalues of A. Hence Conditions 2 implies that all the eigenvalues of A are in the closed* *left half plane. Note that in this case of $D = I_m$, conditions 1 and 2 are well known to the saturation community as classical ANCBC conditions, see [119].*

However, in general not all of the system inputs may be subject to saturation as shown in (2.7)*. To design a low-gain feedback sequence for this type of system, we can choose $D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$ in* (2.1)*. Then the necessary and sufficient conditions as required in Theorem 2.3 are that (A, B) is stabilizable and the invariant zeros of $(A, B, 0, D)$ are in the closed left half plane. It can be shown that the invariant zeros of $(A, B, 0, D)$ in this case are a subset of eigenvalues of A (see [99]). Therefore, only some eigenvalues of A have to be constrained while the others can be completely free. Moreover, Theorem 2 identifies those eigenvalues that need to be restricted. To illustrate this, consider a linear system with a partial input subject to saturation,*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \sigma(u_0) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1.$$

Clearly (A, B) *is stabilizable. Matrix A has eigenvalues* $(j, -j, 2, 3)$*. It can be identified that* $(j, -j)$ *are the invariant zeros of* $(A, B, 0, D)$*, which are on the imaginary axis. Hence the two conditions in Theorem 3 are still satisfied while the two eigenvalues* $(2, 3)$ *are in the right half plane.*

2.5. Design of H_2 low-gain sequences

The H_2 low-gain design procedures developed here yield the classical low-gain design methods as special cases. We note that the H_2 low-gain sequence as defined in Definition 2.1 for the system Σ in (2.1) is equivalent to a bounded H_2 sub-optimal sequence of controllers for the following auxiliary system,

$$\Sigma_2 \begin{cases} \dot{x} = Ax + Bu + \omega \\ z = Du. \end{cases}$$

Such an H_2 sub-optimal controller for Σ_2 can be constructed using either direct eigenstructure assignment method or perturbation method, see [56, 93].

2.5.1. Direct eigenstructure assignment method

The design basically follows the "H_2 suboptimal state feedback gain sequence" (H_2-SOSFGS) algorithm developed in [55, 56]. There exists a nonsingular state transformation $[x_a', x_c']' = T_1 x$ such that the system Σ_2 can be transformed into a compact Special Coordinate Basis (SCB) form:

$$\bar{\Sigma}_2 : \begin{cases} \begin{pmatrix} \dot{x}_a \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} \bar{A}_a & 0 \\ \star & A_c \end{bmatrix} \begin{pmatrix} x_a \\ x_c \end{pmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_1 + \begin{bmatrix} \bar{B}_a \\ B_{ac} \end{bmatrix} u_0 + E\omega \\ \quad z \quad = u_0, \end{cases} \tag{2.9}$$

where $x_a \in \mathbb{R}^{n_a}$, $x_c \in \mathbb{R}^{n_c}$, $u_0 \in \mathbb{R}^{m_0}$, $u_c \in \mathbb{R}^{m_c}$, $n_a + n_c = n$ and $m_0 + m_c = m$, and $\star$ denotes matrix of less interest. The eigenvalues of A_a are the invariant zeros of system Σ. Theorem 2.3 implies that (A_a, B_a) is stabilizable and A_a has all its eigenvalues in the closed left half plane. Moreover, (A_c, B_c) is controllable. Details of SCB can be found in [99].

In order to use the eigenstructure assignment method, we need to perform another transformation $[\bar{x}_a', x_c']' = T_2[x_a', x_c']'$ such that the system can be further converted into:

$$\bar{A}_a = \begin{pmatrix} A_1 & A_{12} & \cdots & A_{1\ell} & 0 \\ 0 & A_2 & \cdots & A_{2\ell} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_\ell & 0 \\ 0 & 0 & 0 & 0 & A_o \end{pmatrix}, \qquad \bar{B}_a = \begin{pmatrix} B_1 & 0 & \cdots & 0 & B_{1,o} \\ 0 & B_2 & \cdots & 0 & B_{2,o} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\ell & B_{\ell,o} \\ B_{o,1} & B_{o,2} & \cdots & B_{o,\ell} & B_o \end{pmatrix},$$

and where A_o is Hurwitz stable, (A_i, B_i) is controllable, and A_i has all its eigenvalues on the imaginary axis. Moreover, (A_i, B_i) is in the controllability canonical form as given by

$$A_i = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_{i,0} & -\alpha_{i,1} & \cdots & -\alpha_{i,n_i-2} & -\alpha_{i,n_i-1} \end{pmatrix}, \quad B_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

For each pair (A_i, B_i), let the feedback gain $F_i(\varepsilon)$ be such that $\lambda(A_i + B_i F_i(\varepsilon)) = -\varepsilon - \lambda(A_i)$. Define

$$F_{a,\varepsilon} = \begin{bmatrix} \text{blkdiag}\{\bar{F}_i(\varepsilon)\}_{i=1}^{\ell} & \\ & 0 \end{bmatrix}, \quad \bar{F}_i(\varepsilon) = F_i(\varepsilon^{2^{\ell-i}(r_{i+1}+1)\dots(r_\ell+1)})$$

where r_i is the largest algebraic multiplicity of eigenvalues of A_i. Since (A_c, B_c) is controllable, we can choose a bounded F_c such that $A_c + B_c F_c$ is stable and has a desired set of eigenvalues. The sequence of feedback gains for the system Σ_2 can then be constructed as

$$F_\varepsilon = \begin{pmatrix} F_{a,\varepsilon} & 0 \\ 0 & F_c \end{pmatrix} T_2 T_1.$$

Clearly, F_ε is bounded and $A + BF_\varepsilon$ is Hurwitz. It follows from [56] that F_ε also satisfies Property 3 in Definition 2.1. Therefore, F_ε is an H_2 low-gain sequence.

Remark 2.4 *For $D = I_m$, the above design procedure recovers the direct eigenstructure assignment method in the classical low-gain design of [51] for linear systems subject to input saturation.*

2.5.2. Perturbation methods

The philosophy of perturbation methods used in H_2 low-gain design is the same as in classical H_2 sub-optimal controller design, that is to perturb the data of the system so that an H_2 optimal controller exists for the perturbed system and then based on continuity argument, we can obtain a sequence of H_2 low-gains for the original system utilizing H_2 optimal control design techniques developed in [93].

For a given quadruple (A, B, C, D), let a sequence of perturbed data $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ be such that $A_\varepsilon \to A$, $B_\varepsilon \to B$, $\bar{Q}_\varepsilon \to \bar{Q}_0$ as $\varepsilon \to 0$ and $\bar{Q}_\varepsilon$ is continuous at $\varepsilon = 0$ where

$$\bar{Q}_0 = \begin{bmatrix} C & D \end{bmatrix}' \begin{bmatrix} C & D \end{bmatrix}, \quad \bar{Q}_\varepsilon = \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}' \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}. \tag{2.10}$$

For this perturbation $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ to be admissible for H_2 low-gain design, it has to satisfy the following conditions:

1. The positive semi-definite semi-stabilizing solution P_ε to the CLMI ,

   $$\begin{bmatrix} A_\varepsilon' P_\varepsilon + P_\varepsilon A_\varepsilon & P_\varepsilon B_\varepsilon + C_\varepsilon' D_\varepsilon \\ B_\varepsilon' P_\varepsilon + D_\varepsilon' C_\varepsilon & D_\varepsilon' D_\varepsilon \end{bmatrix} \geq 0, \tag{2.11}$$

   converges to 0.

2. An H_2 optimal state feedback controller F_ε exists for the perturbed system characterized by $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, I)$ and can, for instance, be constructed using the so-called "Continuous-time optimal gains, fixed modes and decoupling zeros" (($COGFMDZ$) for left invertible system or $(COGFMDZ)_{nli}$ for non-left-invertible system) algorithm in [93]. (See Section 7.2)

Moreover, the obtained F_ε should satisfy:

3. F_ε is bounded.

4. F_ε is such that $A + BF_\varepsilon$ is Hurwitz.

5. $\lim_{\varepsilon\to 0} \|(C + DF_\varepsilon)(sI - A - BF_\varepsilon)^{-1}\|_2 = 0$.

If $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ and the corresponding F_ε satisfy the 5 conditions stated above, then F_ε is an H_2 low-gain sequence. Specifically in our problem, for the system Σ in (2.1) characterized by (A, B, C, D) with $C = 0$, we can use two perturbation methods to design an H_2 low-gain sequence.

Perturbation method I The classical perturbation that is used in H_2 sub-optimal control is in the form of $(A, B, C_\varepsilon, D_\varepsilon)$ where C_ε and D_ε are such that $(A, B, C_\varepsilon, D_\varepsilon)$ has neither invariant zeros nor infinite zeros, and

$$\bar{Q}_\varepsilon \to \bar{Q}_0 \text{ as } \varepsilon \to 0, \quad \bar{Q}_{\varepsilon_1} \le \bar{Q}_{\varepsilon_2} \text{ with } 0 \le \varepsilon_1 \le \varepsilon_2 \le \beta, \tag{2.12}$$

for some $\beta > 0$ and $\bar{Q}_\varepsilon$ and $\bar{Q}_0$ are defined in (2.10). This leads to a perturbed system

$$\Sigma_2^\varepsilon : \begin{cases} \dot{x} &= Ax + Bu + w \\ z_\varepsilon &= C_\varepsilon x + D_\varepsilon u \end{cases}$$

For this perturbation, we have:

- since C_ε and D_ε satisfy (2.12), condition 1 follows from Theorem 2.6 in Appendix.

- since the quadruple $(A, B, C_\varepsilon, D_\varepsilon)$ has neither finite invariant zeros nor infinite zeros, condition 2 follows from Lemma 5.6.3 in [93].

- since we do not perturb A and B, condition 4 is obvious.

- since $u = F_\varepsilon x$ is an H_2 optimal state feedback for the perturbed system and $P_\varepsilon \to 0$, we have that $\|(C_\varepsilon + D_\varepsilon F_\varepsilon)(sI - A - BF_\varepsilon)^{-1}\|_2 \to 0$. Then (2.12) implies that

  $$\|(C + DF_\varepsilon)(sI - A - BF_\varepsilon)^{-1}\|_2 \le \|(C_\varepsilon + D_\varepsilon F_\varepsilon)(sI - A - BF_\varepsilon)^{-1}\|_2.$$

  Therefore, $\|(C + DF_\varepsilon)(sI - A - BF_\varepsilon)^{-1}\|_2 \to 0$ as $\varepsilon \to 0$.

We find that conditions 1, 2, 4, 5 are always satisfied by this type of perturbation. It remains to verify condition 3. We note that since $C = 0$ in our problem, we can always find a $(C_\varepsilon, D_\varepsilon)$ such that a bounded F_ε can be constructed following $(COGFMDZ)$ or $(COGFMDZ)_{nli}$ algorithm in [93] (see Section 7.2). In what follows, we give an example for this type of perturbation.

Example: We can perturb the auxiliary system $\bar{\Sigma}_2$ in its compact SCB form (2.9) as:

$$\bar{\Sigma}^{\varepsilon}_{2,I}: \begin{cases} \begin{bmatrix} \dot{x}_a \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + T_1 w \\ \begin{bmatrix} z \\ z_{\varepsilon,1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_c \end{bmatrix}, \end{cases}$$

where

$$Q_\varepsilon > 0 \ \text{ and } \ \lim_{\varepsilon \to 0} Q_\varepsilon = 0. \tag{2.13}$$

In this case,

$$C_\varepsilon = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

The perturbed system does not have zero structure (that is, neither invariant zeros nor infinite zeros) and $(C_\varepsilon, D_\varepsilon)$ satisfies (2.12). We proceed to check condition 3.

Let X_ε be the positive definite solution of H_2 ARE,

$$A_a' X_\varepsilon + X_\varepsilon A_a + Q_\varepsilon - X_{\varepsilon,1} B_a B_a' X_\varepsilon = 0,$$

and choose a bounded F_c such that $A_c + B_c F_c$ is Hurwitz. An H_2 optimal static state feedback gain F_ε for the perturbed system can be constructed as

$$F_\varepsilon = \begin{bmatrix} -B_a' X_\varepsilon & 0 \\ 0 & F_c \end{bmatrix} T_1.$$

F_ε is bounded for any $m_0 \leq m$ and $\varepsilon \in [0, 1]$. Therefore, it is an H_2 low-gain sequence.

Remark 2.5 *For a special case of $D = I_m$, the above design procedure recovers the standard H_2-ARE low-gain design of [61].*

Perturbation method II In perturbation method I, we add fictitious outputs to completely remove zero dynamics. However, we can also directly perturb system dynamics to move those invariant zeros on the imaginary axis without adding outputs. Consider a perturbation $(A_\varepsilon, B, 0, D)$ which leads to the following perturbed system

$$\bar{\Sigma}^{\varepsilon}_{2,II} : \begin{cases} \dot{\bar{x}} = A_\varepsilon \bar{x} + Bu + \omega \\ \bar{z} = Du \end{cases}$$

where $A_\varepsilon = A + \frac{\varepsilon}{2} I$ and ε small enough such that $(A + \frac{\varepsilon}{2} I, B)$ is stabilizable. For the sake of clarity, we focus on this particular choice of perturbation. The conditions required for perturbation can be verified as follows:

- Since both $(A_\varepsilon, B, 0, D)$ and $(A, B, 0, D)$ have the same normal rank, condition 1 follows from Theorem 2.5 in Appendix.

- since $(A_\varepsilon, B, 0, D)$ does not have any invariant zeros on the imaginary axis and has no infinite zeros, condition 2 follows from Lemma 5.6.3 in [93].

- Note that $DF_\varepsilon e^{(A+BF_\varepsilon+\frac{\varepsilon}{2}I)t} = e^{\frac{\varepsilon}{2}t} DF_\varepsilon e^{(A+BF_\varepsilon)t}$. This implies that $\|DF_\varepsilon(sI - A - BF_\varepsilon)\|_2 \leq \|DF_\varepsilon(sI - A - \frac{\varepsilon}{2}I - BF_\varepsilon)\|_2$. Therefore, $\|DF_\varepsilon(sI - A - BF_\varepsilon)\|_2 \to 0$ if $\|DF_\varepsilon(sI - A - \frac{\varepsilon}{2}I - BF_\varepsilon)\|_2 \to 0$. We find that conditions 5 is satisfied.

- Obviously, $A + BF$ is Hurwitz stable if $A + BF + \frac{\varepsilon}{2} I$ is Hurwitz stable. Therefore, condition 4 is satisfied.

Therefore, the conditions 1, 2, 3 and 4 can be satisfied. For this perturbation, we can always construct a bounded H_2 optimal controller following $(COGFMDZ)_{nli}$ algorithm. This can be done as follows. We first find a nonsingular state transformation independent of ε, $\begin{pmatrix} x_a^{-'} & x_a^{\bigcirc'} & x_c' \end{pmatrix} = T_2 x'$, such that the perturbed system can be transformed into its SCB form,

$$\bar{\Sigma}^{\varepsilon}_{2,II} : \begin{cases} \begin{pmatrix} \dot{x}_a^- \\ \dot{x}_a^{\bigcirc} \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} A_a^- + \frac{\varepsilon}{2} I & 0 & 0 \\ 0 & A_a^{\bigcirc} + \frac{\varepsilon}{2} I & 0 \\ \star & \star & A_c + \frac{\varepsilon}{2} I \end{bmatrix} \begin{pmatrix} x_a^- \\ x_a^{\bigcirc} \\ x_c \end{pmatrix} \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a^- \\ B_a^{\bigcirc} \\ B_{ac} \end{bmatrix} u_0 + E\omega \\ z \quad = u_0, \end{cases} \tag{2.14}$$

where A_a^- is Hurwitz stable, the pairs $(A_a^{\bigcirc}, B_a^{\bigcirc})$ and (A_c, B_c) are controllable and the eigenvalues of $A_a^{\bigcirc}$ are on the imaginary axis. The eigenvalues of $(1 + \varepsilon)A_a^{\bigcirc}$ and $(1 + \varepsilon)A_a^-$ are the invariant zeros of the perturbed system. For a small ε, $(1+\varepsilon)A_a^-$ is also Hurwitz stable. Let X_ε be the positive definite solution of ARE,

$$(A_a^{\bigcirc} + \tfrac{\varepsilon}{2}I)'X_\varepsilon + X_\varepsilon(A_a^{\bigcirc} + \tfrac{\varepsilon}{2}I) - X_\varepsilon B_a^{\bigcirc} B_a^{\bigcirc\prime} X_\varepsilon = 0. \tag{2.15}$$

It is shown in [159] that X_ε strictly decreases to zero as ε goes to zero and hence X_ε is bounded. Choose a bounded F_c such that $A_c + B_c F_c$ is Hurwitz. The H_2 low-gain sequence F_ε can be constructed as

$$F_\varepsilon = \begin{bmatrix} 0 & -B_a^{\bigcirc\prime} X_\varepsilon & 0 \\ 0 & 0 & F_c \end{bmatrix} T_2.$$

Remark 2.6 *In the special case when $D = I_m$, this method recovers the parametric Lyapunov approach to low-gain design as in [159] for linear systems subject to input saturation.*

2.6. Design of H_∞ low-gain sequences

Different alternate design procedures for γ-level H_∞ low-gain sequences we develop here recover the classical H_∞-ARE low-gain design methods in [130] as a special case.

2.6.1. Direct eigenstructure assignment method

The direct eigenstructure assignment method of γ-level H_∞ low-gain design can be found in [10]. In this chapter, we focus on designing γ-level H_∞ low-gain sequences using perturbation methods.

2.6.2. Perturbation methods

The philosophy of the perturbation methods is similar to that in H_2 low-gain design. However, the conditions imposed on perturbations are more restrictive. For a given quintuple (A, B, C, D, E), let a sequence of perturbations $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ be such that $A_\varepsilon \to A$, $B_\varepsilon \to B$, $E_\varepsilon \to E$ and $\bar{Q}_\varepsilon \to Q$ where Q and $\bar{Q}_\varepsilon$ are defined in (2.10). $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ is admissible for γ-level H_∞ low-gain design if it satisfies the following conditions:

1. Define

$$\gamma_\varepsilon^* = \inf_F \left\{ \|(C_\varepsilon + D_\varepsilon F)(zI - A_\varepsilon - B_\varepsilon F)^{-1} E_\varepsilon\|_\infty \mid \lambda(A_\varepsilon + B_\varepsilon F) \in C^- \right\}. \tag{2.16}$$

For a sufficiently small ε, we have $\gamma_\varepsilon^* < \gamma$.

2. The positive semi-definite semi-stabilizing solution P_ε to CQMI,

$$\begin{bmatrix} A'_\varepsilon P_\varepsilon + P_\varepsilon A_\varepsilon + C'_\varepsilon C_\varepsilon + \gamma^{-2} P_\varepsilon E_\varepsilon E'_\varepsilon P_\varepsilon & P_\varepsilon B_\varepsilon + C'_\varepsilon D_\varepsilon \\ B'_\varepsilon P_\varepsilon + D'_\varepsilon C_\varepsilon & D'_\varepsilon D_\varepsilon \end{bmatrix} \geq 0,$$

satisfies $P_\varepsilon \to 0$ as $\varepsilon \to 0$.

3. $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ has neither invariant zeros on the imaginary axis nor any infinite zeros.

Using the above, a γ-level H_∞ sub-optimal state feedback $F_\varepsilon(E,\gamma)$ with $\gamma > \gamma^*(\varepsilon)$ for the perturbed system can be easily constructed following [112]. Moreover, such an $F_\varepsilon(E,\gamma)$ should satisfy the next three conditions:

4. For ε sufficiently small, $\|(C + DF_\varepsilon(E,\gamma))(sI - A - BF_\varepsilon(E,\gamma))^{-1}E\|_\infty < \gamma$,

5. The $F_\varepsilon(E,\gamma)$ is bounded,

6. The $F_\varepsilon(E,\gamma)$ is such that $A + BF_\varepsilon(E,\gamma)$ is Hurwitz.

If $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ and a constructed $F_\varepsilon(E,\gamma)$ satisfy all 6 conditions, this $F_\varepsilon(E,\gamma)$ is a γ-level H_∞ low-gain sequence.

In our problem, for a given 5-tuple (A, B, C, D, E) with $C = 0$ and the given $\gamma > 0$ satisfying $\gamma > \gamma^*$, two perturbation methods can be used for γ-level H_∞ low-gain design.

Perturbation method I Similar to that in H_2 low-gain design, the first perturbation is in the form of $(A, B, C_\varepsilon, D_\varepsilon, E)$ where C_ε and D_ε satisfy (2.12). We give an example.

Example: First, we can transfer the system into the SCB form (2.9) with transformation $(x'_a, x'_c)' = T_1 x$. Then consider a perturbed system based on (2.9) as

$$\Sigma^{\varepsilon}_{\infty,I} : \begin{cases} \begin{pmatrix} \dot{x}_a \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{pmatrix} x_a \\ x_c \end{pmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + \begin{bmatrix} E_a \\ E_c \end{bmatrix} \omega \\ \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix} \begin{pmatrix} x_a \\ x_c \end{pmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_0 \\ u_c \end{pmatrix}, \end{cases}$$

where Q_ε satisfies (2.13). We first verify below that this perturbation is admissible for H_∞ low-gain design.

- Suppose we apply any bounded F to the system (2.1) characterized by $(A, B, 0, D, E)$ such that $A + BF$ is Hurwitz. Let $\gamma_F = \|DF(sI - A - BF)^{-1}E\|_\infty$. It is easy to verify that $\|(C_\varepsilon + D_\varepsilon F)(sI - A - BF)^{-1}E\|_\infty \to \gamma_F$ as $\varepsilon \to 0$. This together with (2.13) implies that for a given γ, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$ conditions 1 and 4 are satisfied.

- $(A, B, C_\varepsilon, D_\varepsilon)$ has neither invariant zeros nor infinite zeros. One can then design a γ-level H_∞ sub-optimal feedback $F_\varepsilon(E, \gamma)$ using the techniques from [112].

- It is easy to see that C_ε and D_ε satisfy (2.12). Then condition 2 follows from Theorem 2.6 in Appendix.

- Since we only perturb C and D and $F_\varepsilon(E, \gamma)$ is obtained using H_∞ control techniques, condition 6 is obvious.

It remains to check condition 5. Next we construct a γ-level H_∞ sub-optimal feedback F_ε for the perturbed system following [112]. Let X_ε be the positive definite solution of H_∞ ARE,

$$A_a' X_\varepsilon + X_\varepsilon A_a + Q_\varepsilon - X_\varepsilon B_a B_a' X_\varepsilon + \gamma^{-2} X_\varepsilon E_a E_a' X_\varepsilon = 0,$$

and choose a bounded F_c such that $A_c + B_c F_c$ is Hurwitz. The $F_\varepsilon(E, \gamma)$ can be constructed as

$$F_\varepsilon(E, \gamma) = \begin{bmatrix} -B_a' X_\varepsilon & 0 \\ 0 & F_c \end{bmatrix} T_1.$$

Clearly, $F_\varepsilon(E, \gamma)$ is bounded for $\varepsilon \in (0, \varepsilon^*]$. Therefore, $F_\varepsilon(E, \gamma)$ is a γ-level low-gain sequence.

Remark 2.7 *When $D = I_m$, the above method recovers the H_∞-ARE based low-gain design for semi-global stabilization of linear systems subject to input saturation [130].*

Perturbation method II We can also directly perturb the system dynamics to move those invariant zeros on the imaginary axis. Consider the perturbation $(A + \frac{\varepsilon}{2} I, B, 0, D, E)$ with ε small enough such that $(A + \frac{\varepsilon}{2} I, B)$ is stabilizable. We shall focus on this particular choice of perturbation.

- Given $A + \frac{\varepsilon}{2} I + BF$ Hurwitz stable, we have $\|DF(sI - A - BF)^{-1}E\|_\infty \leq \|DF(sI - A - \frac{\varepsilon}{2} I - BF)^{-1}E\|_\infty$. This implies that conditions 1 and 4 are satisfied.

- Since $(A + \frac{\varepsilon}{2}I, B, 0, D)$ always have the same normal rank as that of $(A, B, 0, D)$, the condition 2 follows from Theorem 2.5 in Appendix.

- Since $(A + \frac{\varepsilon}{2}I, B, 0, D)$ does not have any invariant zeros on the imaginary axis, the condition 3 is satisfied.

- $A + BF$ is Hurwitz if $A + \frac{\varepsilon}{2}I + BF$ is Hurwitz.

Therefore, Conditions 1, 2, 3, 4 and 6 are satisfied for sufficiently small ε. Moreover, one can always design a bounded γ-level H_∞ state feedback as in [112] as follows:

The perturbed system can be transformed into its compact SCB form using a nonsingular state transformation: $\begin{bmatrix} x_a' & x_a^{\bigcirc\prime} & x_c' \end{bmatrix}' = T_2 x$ as:

$$\bar{\Sigma}^{\varepsilon}_{\infty,II} : \begin{cases} \begin{pmatrix} \dot{x}_a^- \\ \dot{x}_a^\bigcirc \\ \dot{x}_c \end{pmatrix} = \begin{bmatrix} A_a^- + \frac{\varepsilon}{2}I & 0 & 0 \\ 0 & A_a^\bigcirc + \frac{\varepsilon}{2}I & 0 \\ \star & \star & A_c + \frac{\varepsilon}{2}I \end{bmatrix} \begin{pmatrix} x_a^- \\ x_a^\bigcirc \\ x_c \end{pmatrix} \\ \qquad + \begin{bmatrix} 0 \\ 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a^- \\ B_a^\bigcirc \\ B_{ac} \end{bmatrix} u_0 + \begin{bmatrix} E_a^- \\ E_a^\bigcirc \\ E_c \end{bmatrix} \omega \\ z \quad = u_0, \end{cases} \tag{2.17}$$

where A_a^- is Hurwitz, (A_c, B_c) is controllable and $(A_a^\bigcirc, B_a^\bigcirc)$ is controllable. For a sufficiently small ε, $A_a^- + \frac{\varepsilon}{2}I$ is Hurwitz as well. Let X_ε be the positive definite solution of H_∞ ARE,

$$(A_a^\bigcirc + \tfrac{\varepsilon}{2}I)'X_\varepsilon + X_\varepsilon(A_a^\bigcirc + \tfrac{\varepsilon}{2}I) - X_\varepsilon B_a^\bigcirc B_a^{\bigcirc\prime} X_\varepsilon + \gamma^{-2} X_\varepsilon E_a^\bigcirc E_a^{\bigcirc\prime} X_\varepsilon = 0.$$

Let F_c be bounded and such that $A_c + B_c F_c$ is Hurwitz, and the γ-level H_∞ sub-optimal controller is given by

$$F_\varepsilon(E, \gamma) = \begin{bmatrix} 0 & -B_a^{\bigcirc\prime} X_\varepsilon & 0 \\ 0 & 0 & F_c \end{bmatrix} T_2.$$

Since X_ε is bounded, $F_\varepsilon(E, \gamma)$ is bounded. Therefore, $F_\varepsilon(E, \gamma)$ is a γ-level H_∞ low-gain sequence.

Appendix

Existence of H_2 optimal controller

Consider the system,

$$\Sigma_a : \begin{cases} \dot{x} = Ax + Bu + \omega \\ z = Cx + Du, \end{cases} \tag{2.18}$$

We recall the following existence conditions of H_2 optimal static state feedback controller for system Σ_a:

Theorem 2.4 For the system Σ_a in (2.18), H_2 optimal state feedback controller of a static type exists if and only if

1. (A, B) is stabilizable;
2. Σ_a does not have any invariant zero on the imaginary axis;
3. Σ_a does not have any infinite zero of order greater or equal to 1.

Proof : The results follow from Lemma 6.6.5 and Lemma 6.6.1 in [93]. ■

Continuity of solution of CQMI

Here our concern is the continuity of semi-stabilizing solution of the following CQMI associated with the 5-tuple (A, B, C, D, E) and $\gamma > \gamma^*$

$$\begin{bmatrix} A'P + PA + C'C + \gamma^{-2}PEE'P & PB + C'D \\ B'P + D'C & D'D \end{bmatrix} \geq 0, \tag{2.19}$$

where

$$\gamma^* := \inf_F \left\{ \|(C + DF)(sI - A - BF)^{-1}E\|_\infty \mid \lambda(A + BF) \in \mathbb{C}^- \right\} \tag{2.20}$$

We recall the following theorem from [114]:

Theorem 2.5 Consider a 5-tuple (A, B, C, D, E). Suppose (A, B) is stabilizable, (A, B, C, D) does not have any invariant zeros in $\mathbb{C}^+$, and $\gamma > \gamma^*$. Let a sequence of perturbed data $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ converges to (A, B, C, D, E). Moreover, assume that the normal rank of $C_\varepsilon(sI - A_\varepsilon)^{-1}B_\varepsilon + D_\varepsilon$ is equal to the normal rank of $C(sI - A)^{-1}B + D$ for all ε. Then, the smallest positive semi-definite semi-stabilizing solution of CQMI (2.19) associated with $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ converges to the smallest positive semi-definite semi-stabilizing solution of CQMI associated with (A, B, C, D, E).

In the perturbation method I of both H_2 and H_∞ low-gain design, we use perturbations which do not necessarily preserve the normal rank. In this case, we use the following:

Theorem 2.6 Consider a 5-tuple (A, B, C, D, E) and $\gamma > \gamma^*$. Suppose a sequence of perturbations $(C_\varepsilon, D_\varepsilon)$ converges to (C, D), and satisfies the following conditions:

1. $\bar{Q}_\varepsilon$ is continuous at $\varepsilon = 0$;
2. there exists a β such that for $0 \le \varepsilon_1 \le \varepsilon_2 \le \beta$, we have $\bar{Q}_{\varepsilon_1} \le \bar{Q}_{\varepsilon_2}$,

where $\bar{Q}_\varepsilon$ is defined in (2.10). Then the semi-stabilizing solution to CQMI (2.19) associated with $(A, B, C_\varepsilon, D_\varepsilon, E)$ converges to the semi-stabilizing solution of CQMI (2.19) associated with (A, B, C, D, E).

Proof : The case $\gamma = \infty$ was proved in [136]. We shall prove this result for a finite γ.

First, we need to show that given $\gamma > \gamma^*$, for sufficiently small ε, we have $\gamma > \gamma_\varepsilon^*$ where γ_ε^* is defined in (2.20) with (A, B, C, D, E) replaced by $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$. This follows from the fact that there exists a stabilizing state feedback $u = Fx$ such that the H_∞ norm from w to z equals $\gamma_0 < \gamma$. The transfer matrix $G_{\varepsilon,cl}$ from w to z_ε satisfies,

$$G_{\varepsilon,cl}(s)' G_{\varepsilon,cl}(s) = G_{cl}(s)' G_{cl}(s) + G_0(s)'(\bar{Q}_\varepsilon - \bar{Q}_0) G_0(s)$$

where G_{cl} is the transfer matrix from w to z while G_0 is defined by

$$G_0(s) = \begin{bmatrix} I \\ F \end{bmatrix} (sI - A - BF)^{-1} E.$$

Since G_0 has a finite H_∞ norm and $\bar{Q}_\varepsilon \to \bar{Q}_0$ we find that

$$\lim_{\varepsilon \downarrow 0} \|G_{\varepsilon,cl}\|_\infty \to \|G_{cl}\|_\infty = \gamma_0 < \gamma$$

Next we investigate

$$x_0' P_\varepsilon x_0 = \sup_{w \in \mathcal{L}_2} \inf_u \left\{ \|z_\varepsilon\|_{\mathcal{L}_2}^2 - \gamma^2 \|w\|_{\mathcal{L}_2}^2 \mid x \in \mathcal{L}_2 \right\},$$

where $x(0) = x_0$. Since $\bar{Q}_\varepsilon > \bar{Q}_0$ for small ε, we find that

$$x_0' P_\varepsilon x_0 \ge x_0' P_0 x_0$$

for small ε. If we choose $u = Fx$, we obtain

$$0 \le x_0' P_\varepsilon x_0 \le \left\{ \sup_{w \in \mathcal{L}_2} \|z_\varepsilon\|^2_{\mathcal{L}_2} - \gamma^2 \|w\|^2_{\mathcal{L}_2} \mid u = Fx \right\}.$$

We always have for any η,

$$\|a+b\|^2 \le (1+\tfrac{1}{\eta})\|a\|^2 + (1+\eta)\|b\|^2.$$

Let γ_1 be such that $\gamma_0 < \gamma_1 < \gamma$ such that for all sufficiently small ε we know the H_∞ norm from w to z_ε is less than γ_1 for the feedback $u = Fx$. With u fixed by $u = Fx$, we can write $z_\varepsilon = z_{x_0} + z_w$ where z_{x_0} is the output for initial condition x_0 and $w = 0$ and z_w is the output for initial condition $x_0 = 0$ and disturbance w. Let L be such that

$$\|x_{x_0}\|^2_{\mathcal{L}_2} = x_0' L x_0$$

where x_{x_0} is the state for initial condition x_0 and $w = 0$. Choose

$$\eta = \frac{\gamma^2 - \gamma_1^2}{2\gamma_1^2}.$$

We find

$$\|z_\varepsilon\|^2_{\mathcal{L}_2} \le \tfrac{\gamma^2+\gamma_1^2}{\gamma^2-\gamma_1^2} \|\bar{Q}_\varepsilon\| x_0' L x_0 + \tfrac{\gamma^2+\gamma_1^2}{2} \|w\|^2_{\mathcal{L}_2}.$$

But then if w is such that

$$\|w\|^2 > \beta x_0' L x_0 \tag{2.21}$$

where $\beta > \beta_\varepsilon$ for all sufficiently small ε with

$$\beta_\varepsilon = \frac{2(\gamma^2 + \gamma_1^2)}{2(\gamma^2 - \gamma_1^2)^2} \|\bar{Q}_\varepsilon\|$$

we have

$$\|z_\varepsilon\|^2_{\mathcal{L}_2} - \gamma^2 \|w\|^2_{\mathcal{L}_2} < 0$$

for $u = Fx$. We find that for w for which (2.21) is satisfied we obtain for a suboptimal u already a negative cost. Since

$$\sup_{w \in \mathcal{L}_2} \inf_u \left\{ \|z_\varepsilon\|^2_{\mathcal{L}_2} - \gamma^2 \|w\|^2_{\mathcal{L}_2} \mid x \in \mathcal{L}_2 \right\} > 0,$$

we can without loss of generality assume that w satisfies

$$\|w\|^2 < \beta x_0' L x_0 \tag{2.22}$$

provided ε is small enough. By setting $u = Fx + v$ the above inf-sup problem is equivalent to

$$\sup_{w \in \mathscr{L}_2} \inf_{v \in \mathscr{L}_2} \left\{ \|z_\varepsilon\|^2_{\mathscr{L}_2} - \gamma^2 \|w\|^2_{\mathscr{L}_2} \mid u = Fx + v \right\} > 0$$

Since we showed that w can be, without loss of generality, assumed to be bounded, it is clear that in the above optimization for $\varepsilon = 0$ we can also assume without loss of generality that v is bounded as well, i.e.

$$\|v\|_2 \leq N \|x_0\|$$

If the system is left-invertible from v to z then as v gets sufficiently large in $\mathscr{L}_2$ norm then the cost can be made arbitrarily large. If the system is not left-invertible we can split the input in a part which has no effect on the output and a part which has a left-invertible effect on the output. The latter has to be bounded for a bounded cost. The first can be set to zero without loss of generality. But with v and w bounded we can find M such that

$$\|x\|_2 \leq M \|x_0\|^2,$$

but then

$$\begin{aligned}
&\sup_{w \in \mathscr{L}_2} \inf_{v \in \mathscr{L}_2} \left\{ \|z_\varepsilon\|^2_{\mathscr{L}_2} - \gamma^2 \|w\|^2_{\mathscr{L}_2} \right\} \\
&\qquad \leq \sup_{w \in \mathscr{L}_2} \inf_{v \in \mathscr{L}_2} \left\{ \|z_\varepsilon\|^2_{\mathscr{L}_2} - \gamma^2 \|w\|^2_{\mathscr{L}_2} \mid \|v\|_2 \leq N \|x_0\| \right\} \\
&\qquad \leq \sup_{w \in \mathscr{L}_2} \inf_{v \in \mathscr{L}_2} \left\{ \|z_0\|^2_{\mathscr{L}_2} - \gamma^2 \|w\|^2_{\mathscr{L}_2} \mid \|v\|_2 \leq N \|x_0\| \right\} + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2 \\
&\qquad = \sup_{w \in \mathscr{L}_2} \inf_{v \in \mathscr{L}_2} \left\{ \|z_0\|^2_{\mathscr{L}_2} - \gamma^2 \|w\|^2_{\mathscr{L}_2} \right\} + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2 \\
&\qquad = x_0' P_0 x_0 + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2
\end{aligned}$$

where in each case $u = Fx + v$. In conclusion, we find

$$x_0' P_0 x_0 \leq x_0' P_\varepsilon x_0 \leq x_0' P_0 x_0 + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2,$$

which implies that $P_\varepsilon \to P_0$ as $\varepsilon \downarrow 0$. ■

CHAPTER 3

H_2 and H_∞ low-gain feedback–Discrete-time systems

3.1. Introduction

This chapter is a discrete-time counterpart of Chapter 2. We define the concept of H_2 and H_∞ low-gains in a general setting where part of or the complete input is restricted to a low gain feedback. Although the philosophy and structure of this chapter are essentially the same as in previous one, many technical aspects of the design and analysis are very different from continuous-time case.

3.2. Definitions of discrete-time H_2 and H_∞ low-gain sequences

We first recall some standard notations. $\mathbb{C}^{\odot}$, $\mathbb{C}^{\bigcirc}$ and $\mathbb{C}^{\otimes}$ denote open unit disk, unit circle and outside of closed unit disk. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes its transpose. For $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes its transpose. For a vector-valued discrete-time signal y, we define the ℓ_p norm for $p \in [1,\infty)$, and ℓ_∞ norm as

$$\|y\|_{\ell_p} := \left(\sum_{k=0}^{\infty}\sum_{i=1}^{n} y_i^p(k)\right)^{\frac{1}{p}},$$
$$\|y\|_{\ell_\infty} := \sup_{k\geq 0}\ \max_{1\leq i\leq n} |y_i(k)|.$$

Consider a discrete-time system Σ having a $q \times \ell$ stable transfer function G. Then the H_2 norm of the discrete-time system Σ or, equivalently, of the transfer matrix G is defined as

$$\|G\|_2 = \Big(\frac{1}{2\pi}\text{trace}\left[\int_0^{2\pi} G(e^{j\omega})G^*(e^{j\omega})\mathrm{d}\omega\right]\Big)^{1/2}, \tag{3.1}$$

where * denotes complex conjugate transpose, or equivalently as

$$\|G\|_2 = \left(\text{trace}\left[\sum_{k=0}^{\infty} g(k)g(k)'\right]\right)^{1/2}, \tag{3.2}$$

where $g(k)$ is the unit impulse response matrix of $G(s)$.

The H_∞ norm of G is defined as

$$\|G\|_\infty := \sup_{-\pi\leq\omega\leq\pi} \sigma_{\max}[G(e^{j\omega})]. \tag{3.3}$$

Consider next the following linear time invariant discrete-time system:

$$\Sigma : \begin{cases} \rho x = Ax + Bu \\ z \ = Du \end{cases} \tag{3.4}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $z \in \mathbb{R}^{m_0}$ and ρx represents $x(k+1)$. Here the variable z represents a desired variable that can be constrained as required. Without loss of generality, we assume that

$$D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}.$$

This is because for a general D there always exist non-singular matrices U and V such that $UDV = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$.

In what follows, a state feedback gain such as F_ε parameterized in a parameter ε is called a gain sequence since as ε changes one obtains a sequence of gains. We define below formally what we mean by discrete-time H_2 and H_∞ low-gain sequences.

Definition 3.1 Discrete-time $\mathbf{H_2}$ low-gain sequence: For the system Σ in (3.4), the discrete-time H_2 low-gain sequence is a sequence of parameterized static state feedback gains F_ε for which there exists an ε^* such that the following properties hold:

1. $A + BF_\varepsilon$ is Schur stable for any $\varepsilon \in (0, \varepsilon^*]$;
2. For any $x(0) \in \mathbb{R}^n$, the closed-loop system with $u = F_\varepsilon x$ satisfies

$$\lim_{\varepsilon \to 0} \|z\|_{\ell_2} = 0.$$

The discrete-time H_∞ low-gain sequence will depend on an a priori given data γ, hence we define it as discrete-time γ-level H_∞ low-gain sequence to explicitly indicate such a dependence. When we refer to a discrete-time H_∞ low-gain sequence, we always imply discrete-time γ-level H_∞ low-gain sequence.

Definition 3.2 Discrete-time ,-level $\mathbf{H_\infty}$ low-gain sequence: For the system Σ in (3.4) and an arbitrary given $E \in \mathbb{R}^{n\times p}$, define an auxiliary system

$$\Sigma_\infty : \begin{cases} \rho x = Ax + Bu + E\omega \\ z \ = Du \end{cases} \tag{3.5}$$

and infimum

$$\gamma^* = \inf_F \left\{ \| DF(zI - A - BF)^{-1} E \|_\infty \mid \lambda(A + BF) \in C^{\odot} \right\}. \tag{3.6}$$

Let a $\gamma > \gamma^*$ be given. The γ-level H_∞ low-gain sequence is a sequence of parameterized static state feedback gains $F_\varepsilon(E, \gamma)$ for which there exists an ε^* such that the following properties hold:

1. $A + BF_\varepsilon(E, \gamma)$ is Schur stable for any $\varepsilon \in (0, \varepsilon^*]$;

2. For the system Σ_∞ with $u = F_\varepsilon(E, \gamma)x$ and any $x(0) \in \mathbb{R}^n$,

$$\lim_{\varepsilon \to 0} \left\{ \sup_{w \in \ell_2} (\|z\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) \right\} = 0.$$

Remark 3.1 *Unlike in continuous-time case, we do not require boundedness of F_ε or $F_\varepsilon(E, \gamma)$ as a function of ε. However, from the perspective of applications, a bounded F_ε or $F_\varepsilon(E, \gamma)$ is desirable and in fact can always be constructed.*

3.3. Properties of discrete-time H_2 and H_∞ low-gain sequences

The first theorem shows the relationship between the discrete-time H_2 and γ-level H_∞ low-gain sequences.

Theorem 3.1 For the system Σ in (3.4) with a given $E \in \mathbb{R}^{n \times p}$ and a $\gamma > \gamma^*$ where γ^* is defined in (3.6), a sequence of feedback gains $F_\varepsilon(E, \gamma)$ is a γ-level H_∞ low-gain sequence only if it is an H_2 low-gain sequence.

Proof : By setting $w = 0$ in the definition of γ-level H_∞ low-gain sequence, we immediately conclude this result. ■

For the closed-loop system consisting of system Σ in (3.4) and either a discrete-time H_2 low-gain controller $u = F_\varepsilon x$ or a discrete-time H_∞ low-gain controller $u = F_\varepsilon(E, \gamma)x$, the next theorem shows that the magnitude of z and DF_ε or $DF_\varepsilon(E, \gamma)$ can be made arbitrarily small.

Theorem 3.2 The closed-loop system comprising of (3.4) and either $u = F_\varepsilon x$ or $u = F_\varepsilon(E, \gamma)x$ satisfies the following properties:

1. $\lim_{\varepsilon\to 0} \|z\|_{\ell_\infty} = 0$,

2. $\lim_{\varepsilon\to 0} DF_\varepsilon = 0$ or $\lim_{\varepsilon\to 0} DF_\varepsilon(E,\gamma) = 0$.

Proof : Owing to Theorem 3.1, we only need to prove these two properties for a discrete-time H_2 low-gain sequence. Since

$$\|z\|_{\ell_\infty} \le \|z\|_{\ell_2},$$

the fact that $\|z\|_{\ell_2} \to 0$ as $\varepsilon \to 0$ for any $x(0)$ immediately yields 1. Moreover,

$$\|z(0)\| = \|DF_\varepsilon x(0)\| \le \|z\|_{\ell_\infty}.$$

Therefore, $\|z\|_{\ell_\infty} \to 0$ as $\varepsilon \to 0$ for any $x(0)$ implies that $\|DF_\varepsilon\| \to 0$ as $\varepsilon \to 0$. ■

Like in continuous-time case, Theorem 3.2 enables us to connect to the literature and explain why the discrete H_2 and γ-level H_∞ sequences as defined in Definitions 3.1 and 3.2 are termed as '*low-gain*' sequences. For the sake of completeness, we borrow the following remark from previous chapter. As we alluded to in introduction, the name *low-gain* sequence arose or has roots in one of the classical problems, namely the problem of semi-globally stabilizing a linear system subject to actuator saturation. (For readers not familiar with saturation literature, we refer to [5, 32, 35, 98, 123] for more details.) To be precise, let us consider a linear system

$$\rho\bar{x} = \bar{A}\bar{x} + \bar{B}\sigma(\bar{u}) \tag{3.7}$$

where the function $\sigma(\cdot)$ denotes a standard saturation; that is, $\sigma(\bar{u}) = \operatorname{sign}(\bar{u})\min\{1, |\bar{u}|\}$. Let the pair $(\bar{A}, \bar{B})$ be stabilizable and $\bar{A}$ has all its eigenvalues within or on the unit disc. Consider a state feedback controller,

$$\bar{u} = \bar{F}_\varepsilon\bar{x}, \tag{3.8}$$

where $\bar{F}_\varepsilon$ is a parameterized sequence with the parameter as ε. If the feedback sequence $\bar{F}_\varepsilon$ satisfies all the three conditions posed in Theorem 3.1 of [52], it is known as a '*low-gain*' feedback in the context of stabilization of linear systems subject to saturation (see also [49]). In fact, the state feedback controller $\bar{u} = \bar{F}_\varepsilon\bar{x}$ where $\bar{F}_\varepsilon$ is such a *low-gain* sequence semi-globally stabilizes (3.7) for a small enough value of ε. That is, there exists an ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, the closed-loop system comprising (3.7) and (3.8) is semi-globally stable with *a priori* given (arbitrarily large) bounded set Ω being in the region of attraction, and moreover the smaller the value of ε the larger can be the *a priori* prescribed set Ω.

Having recalled above the classical semi-global stabilization problem of a linear system with saturating linear feedbacks, we can now emphasize its connection to Theorem 3.2. As is done in classical semi-global stabilization problem, let us first assume that all the control channels are subject to saturation. Then, to see the connection between such a semi-global stabilization problem and Theorem 3.2, set $D = I_m$ and thus take $z = u$ as the constrained variable subject to saturation. Then, Theorem 3.2 shows that the discrete H_2 and γ-level H_∞ sequences as defined in Definitions 3.1 and 3.2 satisfy all the three conditions posed in Theorem 3.1 of [52], and hence they can appropriately be termed as *low-gain* sequences. Furthermore, as is evident from Theorem 3.2, they can readily achieve semi-global stabilization of a discrete-time linear system where all control inputs are subject to saturation whenever it is achievable.

We now proceed with the general setting, where we assume without loss of generality $D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$ for some $m_0 < m$. This means, in the scenario of a linear system subject to input saturation, all the input channels are not necessarily constrained, that is some are constrained and others are not. To be precise, we can assume the following system configuration

$$\rho\xi = A\xi + B_0\sigma(u_0) + B_1u_1 \tag{3.9}$$

where $\xi \in \mathbb{R}^n$, $u_0 \in \mathbb{R}^{m_0}$, $u_1 \in \mathbb{R}^{m-m_0}$ and $B = \begin{bmatrix} B_0 & B_1 \end{bmatrix}$. Partial inputs as represented by u_0 are subject to saturation. In another word, we have the constrained variable $z = Du = u_0$. In this case, property 1 of Theorem 3.2 implies that for an initial condition x_0 in a given set and a pre-specified saturation level Δ, there exists an ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$ the closed-loop system satisfies $\|z(k)\| = \|u_0(k)\| = \|DF_\varepsilon(A + BF_\varepsilon)^k x_0\| \leq \Delta$ for all $k \geq 0$. This implies that the saturation can be made inactive for all the time, and hence the closed-loop system can in fact be linear. Therefore, the stability of the closed-loop system directly follows from Definitions 3.1 and 3.2.

3.4. Existence of discrete-time H_2 and H_∞ low-gain sequences

We have the following theorem regarding the existence of discrete-time H_2 and γ-level H_∞ low-gain sequences.

Theorem 3.3 For the system Σ in (3.4) with an arbitrarily given $E \in \mathbb{R}^{n\times p}$ and $\gamma > \gamma^*$ where γ^* is defined in (3.6), the discrete-time H_2 and γ-level H_∞ low-gain sequences exist if and only if

1. (A, B) is stabilizable;
2. $(A, B, 0, D)$ is at most weakly non-minimum phase[1].

Proof : Consider the case of discrete-time H_2 low-gain sequence. Define $\gamma_2^* = \sqrt{\text{trace}(P)}$ where P is the semi-stabilizing solution of the discrete-time linear matrix inequality (DLMI) [2],

$$\begin{bmatrix} A'PA - P & A'PB \\ B'PA & D'D + B'PB \end{bmatrix} \geq 0. \tag{3.10}$$

According to Definition 2.4.1 in [93], a discrete-time H_2 low gain sequence is equivalent to a H_2 sub-optimal control sequence with $\gamma_2^* = 0$, i.e. $P = 0$. Then it follows from Theorem 4.4.8 in [93] that $P = 0$ is a unique positive semi-definite semi-stabilizing solution if and only if (A, B) is stabilizable and $(A, B, 0, D)$ has no invariant zeros in $\mathcal{C}^{\otimes}$. Note that a system characterized by a quadruple $(A, B, 0, D)$ is always right invertible and does not have any infinite zero structure.

Consider next the case of discrete-time γ-level H_∞ low-gain sequence. Following [113], we can easily verify that given $\gamma > \gamma^*$ the discrete-time γ-level H_∞- low-gain sequences exist if and only if, $P = 0$ is a semi-stabilizing solution to the Discrete Algebraic Riccati Equation (DARE)[3],

$$P = A'PA - \begin{bmatrix} B'PA \\ E'PA \end{bmatrix}' G(P)^\dagger \begin{bmatrix} B'PA \\ E'PA \end{bmatrix}$$

where

$$G(P) = \begin{bmatrix} D'D & 0 \\ 0 & -\gamma^{-2} I \end{bmatrix} + \begin{bmatrix} B' \\ E' \end{bmatrix} P \begin{bmatrix} B & E \end{bmatrix}.$$

[1] A system characterized by a quadruple (A, B, C, D) is said to be *at most weakly non-minimum phase* if all its invariant zeros are in $\mathbb{C}^{\odot} \cup \mathbb{C}^{\bigcirc}$. The detailed definitions can be found in [145].

[2] The definition of semi-stabilizing solution of DLMI is given in [93] as Definition 4.4.4 on page 130.

[3] For the definition of semi-stabilizing solution of DARE, see [114].

This is equivalent to the conditions that (A, B) is stabilizable and that the matrix pencil

$$\begin{bmatrix} zI - A & -B \\ 0 & D \end{bmatrix}$$

does not have any zeros in $\mathbb{C}^{\otimes}$, i.e. the system is at most weakly non-minimum phase. ■

Remark 3.2 *As shown in the foregoing discussion, the low-gain sequences achieve semi-global stabilization of linear systems subject to input saturation. Consider the system* (3.7). *In order to design a low-gain sequence for this system, one can choose* $D = I_m$ *in* (3.4). *The above theorem then shows that the necessary and sufficient conditions for semi-global stabilization are that* (A, B) *is stabilizable and all the invariant zeros of* $(A, B, 0, I_m)$ *are in the closed unit disc. It is known that the invariant zeros of* $(A, B, 0, I_m)$ *coincide with eigenvalues of A. Hence Conditions 2 implies that all the eigenvalues of A are in the closed unit disc. Note that in this particular case of* $D = I_m$*, Conditions 1 and 2 are well known to the saturation community as classical ANCBC conditions, see [119].*

However, in general all the system inputs may not have to be subject to saturation as shown in (3.9). *To design a low-gain feedback sequence for this type of system, we can choose* $D = \begin{bmatrix} I_{m_0} & 0 \end{bmatrix}$ *in* (3.4). *Then the necessary and sufficient conditions as required in Theorem 3.3 are that* (A, B) *is stabilizable and the invariant zeros of* $(A, B, 0, D)$ *are in the closed unit disc. It can be shown that the invariant zeros of* $(A, B, 0, D)$ *in this case are a subset of eigenvalues of A (see [99]). Therefore, only some eigenvalues of A have to be constrained while the others can be completely free. Moreover, Theorem 2 identifies those eigenvalues that need to be restricted. This can be illustrated by the following example. Consider a linear system with a partial input subject to saturation,*

$$\begin{bmatrix} \rho x_1 \\ \rho x_2 \\ \rho x_3 \\ \rho x_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \sigma(u_0) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_1. \tag{3.11}$$

Clearly (A, B) *is stabilizable. Matrix A has eigenvalues* $(j, -j, 2, 3)$. *It can be identified that* $(j, -j)$ *are the invariant zeros of* $(A, B, 0, D)$*, which are on the unit circle. Hence the two conditions in Theorem 3 are still satisfied while the two eigenvalues* $(2, 4)$ *are strictly outside unit circle.*

3.5. Design of discrete-time H_2 low-gain sequences

In this section, we present the design of discrete-time H_2 low-gain sequences. We emphasize that we present different alternate procedures. Thus the designer has a choice of choosing one method or the other. The design procedures we develop here yield the classical low-gain design methods as special cases.

The discrete-time H_2 low-gain sequence as defined in Definition 3.1 for the system Σ in (3.4) is equivalent to the H_2 sub-optimal sequence of controllers for the following auxiliary system,

$$\Sigma_2 : \begin{cases} \rho x = Ax + Bu + w \\ \ z \ = Du \end{cases}$$

provided that both conditions in Theorem 3.3 are satisfied. (This can be seen from Definition 2.4.1 in [93] with $\gamma_p^* = 0$.) Such an H_2 sub-optimal sequence of controllers for Σ_2 can be constructed using either direct eigenstructure assignment method or perturbation method, see [57] and Chapter 11.3.2 in [93].

3.5.1. Direct eigenstructure assignment

The design basically follows the SOSFGS algorithm developed in [57]. There exists a nonsingular state transformation $[x_a', x_c']' = T_1 x$ such that the system Σ_2 can be transformed into a compact Special Coordinate Basis(SCB) Form:

$$\bar{\Sigma}_2 : \begin{cases} \begin{bmatrix} \rho x_a \\ \rho x_c \end{bmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_1 + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + T_1 \omega \\ \quad z \quad = u_0, \end{cases} \tag{3.12}$$

where $x_a \in \mathbb{R}^{n_a}$, $x_c \in \mathbb{R}^{n_c}$, $u_0 \in \mathbb{R}^{m_0}$, $u_c \in \mathbb{R}^{m_c}$, $n_a + n_c = n$ and $m_0 + m_c = m$, and $\star$ denotes a matrix of not much interest. The eigenvalues of A_a are the invariant zeros of system Σ. In view of the properties of SCB, Theorem 3.3 implies that (A_a, B_a) is stabilizable and A_a has all its eigenvalues in the closed unit disc. Moreover, (A_c, B_c) is controllable. Detailed definitions and properties of SCB can be found in [99].

In order to use the eigenstructure assignment method, we need to perform another transformation

$[\bar{x}_a', x_c']' = T_2[x_a', x_c']'$ such that the system can be further converted into:

$$\tilde{\Sigma}_2 : \begin{cases} \begin{bmatrix} \rho\bar{x}_a \\ \rho x_c \end{bmatrix} = \begin{bmatrix} \bar{A}_a & 0 \\ \star & A_c \end{bmatrix} \begin{bmatrix} \bar{x}_a \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_1 + \begin{bmatrix} \bar{B}_a \\ B_{ac} \end{bmatrix} u_0 + T\omega \\ \quad z \quad = u_0, \end{cases}$$

where $T = T_2 T_1$, $\bar{A}_a$ and $\bar{B}_a$ are in the following form:

$$\bar{A}_a = \begin{bmatrix} A_1 & A_{12} & \cdots & A_{1\ell} & 0 \\ 0 & A_2 & \cdots & A_{2\ell} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_\ell & 0 \\ 0 & 0 & 0 & 0 & A_o \end{bmatrix}, \quad \bar{B}_a = \begin{bmatrix} B_1 & 0 & \cdots & 0 & B_{1,o} \\ 0 & B_2 & \cdots & 0 & B_{2,o} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & B_\ell & B_{\ell,o} \\ B_{o,1} & B_{o,2} & \cdots & B_{o,\ell} & B_o \end{bmatrix}, \tag{3.13}$$

and where A_o is Schur stable, (A_i, B_i) is controllable, and A_i has all its eigenvalues on the unit circle. Moreover, (A_i, B_i) is in the controllability canonical form as given by

$$A_i = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \ddots & \vdots & 0 \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -\alpha_{i,0} & -\alpha_{i,1} & \cdots & -\alpha_{i,n_i-2} & -\alpha_{i,n_i-1} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{3.14}$$

For each pair (A_i, B_i), let the feedback gain $F_i(\varepsilon)$ be such that

$$\lambda(A_i + B_i F_i(\varepsilon)) = (1 - \varepsilon)\lambda(A_i).$$

Define

$$F_{a,\varepsilon} = \begin{bmatrix} \bar{F}_1(\varepsilon) & 0 & \cdots & 0 & 0 & 0 \\ 0 & \bar{F}_2(\varepsilon) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \bar{F}_\ell(\varepsilon) & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

where

$$\bar{F}_i(\varepsilon) = F_i(\varepsilon^{2^{\ell-i}(r_{i+1}+1)\cdots(r_\ell+1)}),$$

and r_i is the largest algebraic multiplicity of eigenvalues of A_i.

Since (A_c, B_c) is controllable, we can choose F_c such that $A_c + B_c F_c$ is stable and has a desired set of eigenvalues.

The sequence of feedback gains for the system Σ_2 can then be constructed as

$$F_\varepsilon = \begin{bmatrix} F_{a,\varepsilon} & 0 \\ 0 & F_c \end{bmatrix} T_2 T_1.$$

Clearly, $A + BF_\varepsilon$ is Schur stable. It follows from [57] that, for any $x(0) \in \mathbb{R}^n$, F_ε also renders,

$$\lim_{\varepsilon \to 0} \|z\|_{\ell_2} = 0.$$

Therefore, F_ε is a discrete-time H_2 low-gain sequence.

Remark 3.3 *In a special case of $D = I_m$, the above design procedure recovers the direct eigenstructure assignment method in the classical low-gain design developed in [52] for stabilization of discrete-time linear systems subject to input saturation.*

To highlight the explicit nature of the above method and to illustrate the constructive procedures, we design a discrete-H_2 low-gain sequence for the example given in (3.11). *Note that for this system, A and B are already in the form of* (3.13) *and* (3.14) *where*

$$A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_c = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

With a bit of algebra, we find

$$F_1(\varepsilon) = \begin{bmatrix} 1 - (1-\varepsilon)^2 & 0 \end{bmatrix}.$$

It is easy to verify that $A_1 + B_1 F_1(\varepsilon)$ has eigenvalues at $((1-\varepsilon)j, -(1-\varepsilon)j)$. *Choose $F_c = \begin{bmatrix} -3.75 & -5 \end{bmatrix}$ so that $A_c + B_c F_c$ has eigenvalues at* $(0.5, -0.5)$. *The discrete H_2 low-gain sequence can then be constructed as*

$$F_\varepsilon = \begin{bmatrix} 1-(1-\varepsilon)^2 & 0 & 0 & 0 \\ 0 & 0 & -3.75 & -5 \end{bmatrix}.$$

3.5.2. Perturbation methods

There exists a classical perturbation method that has long been used in discrete-time H_2 suboptimal controller sequence design, see for instance chapter 11.3.2 in [93]. The philosophy of perturbation methods used in discrete-time H_2 low-gain design is the same as for H_2 sub-optimal controller sequence design, that is to perturb the data of the system so that an H_2 optimal controller exists for the perturbed system and then based on continuity argument, we can obtain a sequence of discrete-time H_2 low-gains for the original system utilizing H_2 optimal control design techniques developed in [93].

In the discrete-time H_2 low gain design, we consider a perturbation of the auxiliary system Σ_2 as follows:

$$\Sigma_2^\varepsilon : \begin{cases} \rho x = A_\varepsilon x + B_\varepsilon u + I\omega \\ z_\varepsilon = C_\varepsilon x + D_\varepsilon u \end{cases} \tag{3.15}$$

where $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ are such that $A_\varepsilon \to A$, $B_\varepsilon \to B$, $\bar{Q}_\varepsilon \to \bar{Q}_0$ as $\varepsilon \to 0$ where

$$\bar{Q}_0 = \begin{bmatrix} 0 & D \end{bmatrix}' \begin{bmatrix} 0 & D \end{bmatrix}, \quad \bar{Q}_\varepsilon = \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}' \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}. \tag{3.16}$$

In order for this perturbation $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ to be admissible for discrete-time H_2 low-gain design, it has to satisfy the following conditions:

1. The smallest positive semi-definite semi-stabilizing solution P_ε to the DLMI ,

$$\begin{bmatrix} C_\varepsilon' C_\varepsilon + A_\varepsilon{}' P_\varepsilon A_\varepsilon - P_\varepsilon & A_\varepsilon' P_\varepsilon B_\varepsilon + C_\varepsilon' D_\varepsilon \\ B_\varepsilon' P_\varepsilon A_\varepsilon + D_\varepsilon' C_\varepsilon & D_\varepsilon' D_\varepsilon + B_\varepsilon' P_\varepsilon B_\varepsilon \end{bmatrix} \geq 0, \tag{3.17}$$

   converges to 0.

2. H_2 optimal state feedback controller of a static type $u = F_\varepsilon x$ exists for the perturbed system Σ_2^ε. Moreover, such an F_ε should satisfy the next two conditions:

3. F_ε is such that $A + BF_\varepsilon$ is Schur stable.

4. F_ε satisfies that $\|(C + DF_\varepsilon)(zI - A - BF_\varepsilon)^{-1}\|_2 \to 0$ as $\varepsilon \to 0$.

If $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ and one constructed sequence of F_ε satisfy all the 4 conditions stated above, such a sequence of F_ε is a discrete-time H_2 low-gain sequence.

Remark 3.4 *Since F_ε is obtained from H_2 optimal controller design, we immediately see that $A_\varepsilon + B_\varepsilon F_\varepsilon$ is Schur stable and $\|(C_\varepsilon + D_\varepsilon F_\varepsilon)(zI - A_\varepsilon - B_\varepsilon F_\varepsilon)^{-1}\|_2 \to 0$. But these do not necessarily imply that $A + BF_\varepsilon$ is Schur stable and $\|DF_\varepsilon(zI - A - BF_\varepsilon)^{-1}\|_2 \to 0$ as $\varepsilon \to 0$ even though the condition 1 is satisfied.*

Remark 3.5 *One can construct the desired H_2 optimal static state feedback controller $u = F_\varepsilon x$ using, for instance, the* ((DOGFMDZ)$_{li}$ *or* (DOGFMDZ)$_{nli}$ *algorithm in [93] (Chapter 8.2, page 309).*

We shall present two perturbation methods to design a discrete-time H_2 low-gain sequence.

Perturbation method I:

The classical perturbation that is used in H_2 sub-optimal controller sequence design is in the form $(A, B, C_\varepsilon, D_\varepsilon)$ where C_ε and D_ε are such that $(A, B, C_\varepsilon, D_\varepsilon)$ has no invariant zero, and

$$\begin{gathered} \bar{Q}_\varepsilon \to \bar{Q}_0 \text{ as } \varepsilon \to 0, \\ \bar{Q}_{\varepsilon_1} \le \bar{Q}_{\varepsilon_2} \text{ with } \; 0 \le \varepsilon_1 \le \varepsilon_2 \le \beta \end{gathered} \tag{3.18}$$

for some $\beta > 0$ and $\bar{Q}_\varepsilon$ and $\bar{Q}_0$ are defined in (3.16). For this perturbation, we have:

- since C_ε and D_ε satisfy (3.18), condition 1 follows from Theorem 3.6 in Appendix.
- since the quadruple $(A, B, C_\varepsilon, D_\varepsilon)$ does not have invariant zeros, condition 2 follows from Theorem 3.4 in Appendix.
- since we do not perturb A and B, condition 3 is obvious.
- since $u = F_\varepsilon x$ is an H_2 optimal state feedback for the perturbed system and $P_{\varepsilon,2} \to 0$, we note that $\|(C_\varepsilon + D_\varepsilon F_\varepsilon)(zI - A - BF_\varepsilon)^{-1}\|_2 \to 0$. Then (3.18) implies that

$$\|DF_\varepsilon(zI - A - BF_\varepsilon)^{-1}\|_2 \le \|(C_\varepsilon + D_\varepsilon F_\varepsilon)(zI - A - BF_\varepsilon)^{-1}\|_2.$$

  Therefore, $\|DF_\varepsilon(zI - A - BF_\varepsilon)^{-1}\|_2 \to 0$ as $\varepsilon \to 0$.

We find that conditions 1, 2, 3, and 4 are always satisfied by this type of perturbation. The discrete H_2 low gain F_ε can be constructed following the *((DOGFMDZ)* or *(DOGFMDZ)*$_{\text{nli}}$ algorithm in [93] (Chapter 8.2, page 309). In what follows, we give two examples for this type of perturbation which recover, in a special case of $D = I_m$, the standard H_2-ARE low-gain design for a discrete-time linear system subject to input saturation.

Example 1: One choice of perturbation for system Σ_2 is $(A, B, C_\varepsilon, D_\varepsilon)$ where

$$C_\varepsilon = \begin{bmatrix} 0 \\ 0 \\ \sqrt{Q_\varepsilon} \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} D \\ \varepsilon I \\ 0 \end{bmatrix},$$

and $Q_\varepsilon \in \mathbb{R}^{n \times n}$ is such that

$$Q_\varepsilon > 0 \;\text{ and }\; \lim_{\varepsilon \to 0} Q_\varepsilon = 0. \tag{3.19}$$

Clearly, $(A, B, C_\varepsilon, D_\varepsilon)$ does not have invariant zeros, and $(C_\varepsilon, D_\varepsilon)$ satisfies (3.18). Let X_ε be the positive definite solution of the H_2 DARE,

$$X_\varepsilon = A' X_\varepsilon A + Q_\varepsilon - A' B X_\varepsilon (B' X_\varepsilon B + D'_\varepsilon D_\varepsilon)^{-1} X_\varepsilon B' A. \tag{3.20}$$

The H_2 optimal static state feedback for the perturbed system can then be constructed as

$$F_\varepsilon = -(B' X_\varepsilon B + D'_\varepsilon D_\varepsilon)^{-1} B' X_\varepsilon A.$$

F_ε satisfies all the required conditions and hence is a discrete-time H_2 low-gain sequence for the original system. Moreover, when $m_0 = m$, it recovers the standard H_2-ARE based low-gain design for a linear system subject to input saturation [61].

Example 2: We can also perturb the auxiliary system $\bar{\Sigma}_2$ in its compact SCB form (3.12) as:

$$\bar{\Sigma}^\varepsilon_{2,I} : \begin{cases} \begin{bmatrix} \rho x_a \\ \rho x_c \end{bmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + T_1 w \\ \begin{bmatrix} z \\ z_{\varepsilon,1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_c \end{bmatrix}, \end{cases}$$

where Q_ε satisfies (3.19). In this case,

$$C_\varepsilon = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix}.$$

The perturbed system does not have invariant zeros and $(C_\varepsilon, D_\varepsilon)$ satisfies (3.18). Let X_ε be the positive definite solution of H_2 DARE,

$$X_\varepsilon = A'_a X_\varepsilon A_a + Q_\varepsilon - A'_a X_\varepsilon B_a (I + B'_a X_\varepsilon B_a)^{-1} B'_a X_\varepsilon A_a,$$

and choose F_c such that $A_c + B_c F_c$ is Schur stable. An H_2 optimal static state feedback gain F_ε for the perturbed system can be constructed as

$$F_\varepsilon = \begin{bmatrix} -(I + B'_a X_\varepsilon B_a)^{-1} B'_a X_\varepsilon A_a & 0 \\ 0 & F_c \end{bmatrix} T_1.$$

F_ε satisfies all 4 conditions and hence is a discrete-time H_2 low-gain sequence.

When $m_0 = m$, i.e. $D = I_m$, the above perturbation method also recovers the standard H_2-ARE low-gain design developed in [61].

Perturbation method II:

In perturbation method I, we add fictitious outputs to completely remove zero dynamics. However, we can also directly perturb system dynamics to move those invariant zeros on the unit circle without adding outputs. Consider a class of perturbation $(A_\varepsilon, B_\varepsilon, 0, D_\varepsilon)$ where

$$A_\varepsilon = (1+\varepsilon)A, \quad B_\varepsilon = (1+\varepsilon)B, \quad D_\varepsilon = (1+\varepsilon)D$$

and ε small enough such that $((1+\varepsilon)A, (1+\varepsilon)B)$ is stabilizable and $(1+\varepsilon)A$ does not have eigenvalues on the unit circle. For the sake of clarity, we focus on this particular choice of perturbation. The conditions required for perturbation can be verified as follows:

- since $((1+\varepsilon)A, (1+\varepsilon)B, 0, (1+\varepsilon)D)$ has the same normal rank as that of $(A, B, 0, D)$, condition 1 follows from Theorem 3.5 in Appendix.

- since $((1+\varepsilon)A, (1+\varepsilon)B, 0, (1+\varepsilon)D)$ does not have any invariant zeros on the unit circle, condition 2 follows from Theorem 3.4 in Appendix.

- Obviously, any F_ε for which $(1+\varepsilon)A + (1+\varepsilon)BF_\varepsilon$ is Schur stable also yields that $A + BF_\varepsilon$ is Schur stable. Therefore, condition 3 is satisfied.
- Note that
$$(1+\varepsilon)DF_\varepsilon((1+\varepsilon)A + (1+\varepsilon)BF_\varepsilon)^k = (1+\varepsilon)^{k+1}DF_\varepsilon(A+BF_\varepsilon)^k.$$
  This together with (3.2) implies that $\|DF_\varepsilon(zI - A - BF_\varepsilon)\|_2 \leq \|(1+\varepsilon)DF_\varepsilon(zI - (1+\varepsilon)A - (1+\varepsilon)BF_\varepsilon)\|_2$. Therefore, $\|DF_\varepsilon(zI - A - BF_\varepsilon)^{-1}\|_2 \to 0$ if $\|(1+\varepsilon)DF_\varepsilon(zI - (1+\varepsilon)A - (1+\varepsilon)BF_\varepsilon)^{-1}\|_2 \to 0$. We find that condition 4 is satisfied.

Hence there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, all 4 conditions are satisfied. For this specific perturbation, we can directly construct a discrete-time H_2 optimal controller following the *(DOGFMDZ)*$_{\text{nli}}$ algorithm. This can be done as follows: the perturbed system can be transformed into its compact SCB form using a nonsingular state transformation: $\begin{bmatrix} x_a^{\odot\prime} & x_a^{\bigcirc\prime} & x_c' \end{bmatrix} = T_3 x$ as such:

$$\bar{\Sigma}_{2,II}^{\varepsilon} : \begin{cases} \begin{bmatrix} \rho\bar{x}_a^{\odot} \\ \rho\bar{x}_a^{\bigcirc} \\ \rho\bar{x}_c \end{bmatrix} = (1+\varepsilon)\begin{bmatrix} A_a^{\odot} & 0 & 0 \\ 0 & A_a^{\bigcirc} & 0 \\ \star & \star & A_c \end{bmatrix}\begin{bmatrix} \bar{x}_a^{\odot} \\ \bar{x}_a^{\bigcirc} \\ \bar{x}_c \end{bmatrix} + (1+\varepsilon)\begin{bmatrix} 0 \\ 0 \\ B_c \end{bmatrix} u_c + (1+\varepsilon)\begin{bmatrix} B_a^{\odot} \\ B_a^{\bigcirc} \\ B_{ac} \end{bmatrix} u_0 + T_3\omega \\ \quad\bar{z} \quad = (1+\varepsilon)u_0, \end{cases} \tag{3.21}$$

where $A_a^{\odot}$ is Schur stable, the pairs $(A_a^{\bigcirc}, B_a^{\bigcirc})$ and (A_c, B_c) are controllable and the eigenvalues of $A_a^{\bigcirc}$ are on the unit circle. The eigenvalues of $(1+\varepsilon)A_a^{\bigcirc}$ and $(1+\varepsilon)A_a^{\odot}$ are the invariant zeros of the perturbed system. For a small ε, $(1+\varepsilon)A_a^{\odot}$ is also Schur stable. Moreover, T_3 is independent of ε. Let X_ε be the positive definite solution of DARE,

$$\frac{1}{(1+\varepsilon)^2} X_\varepsilon = A_a^{\bigcirc\prime} X_\varepsilon A_a^{\bigcirc} - A_a^{\bigcirc\prime} X_\varepsilon B_a^{\bigcirc} (I + B_a^{\bigcirc\prime} X_\varepsilon B_a^{\bigcirc})^{-1} B_a^{\bigcirc\prime} X_\varepsilon A_a^{\bigcirc}, \tag{3.22}$$

and choose F_c such that $A_c + B_c F_c$ is Schur stable. The discrete-time H_2 suboptimal controller sequence F_ε can be constructed as

$$F_\varepsilon = \begin{bmatrix} 0 & -(I + B_a^{\bigcirc\prime} X_\varepsilon B_a^{\bigcirc})^{-1} B_a^{\bigcirc\prime} X_\varepsilon A_a^{\bigcirc} & 0 \\ 0 & 0 & F_c \end{bmatrix} T_3.$$

Since all the conditions are satisfied, we conclude that F_ε is a discrete-time H_2 low gain sequence.

Remark 3.6 *In the special case when $D = I_m$, the above perturbation method recovers the parametric Lyapunov approach to low-gain design developed in [161] for linear systems subject to input saturation.*

3.6. Design of discrete-time H_∞ low-gain sequences

We present the design of discrete-time γ-level H_∞ low-gain sequences in this section. Similar to that in the preceding section, we give different alternate procedures. These design procedures we develop here recover the classical H_∞-ARE low-gain design methods in [130] as a special case.

3.6.1. Direct eigenstructure assignment

The direct eigenstructure assignment method of discrete-time γ-level H_∞ low-gain design can be found in [10]. In this chapter, we focus on designing discrete-time γ-level H_∞ low-gain sequences using perturbation methods.

3.6.2. Perturbation methods

We now proceed to design discrete-time H_∞ low-gain sequences using perturbation methods. The philosophy of the perturbation methods is similar to that in discrete-time H_2 low-gain design. However, the conditions imposed on perturbations are more restrictive. For the auxiliary system Σ_∞, consider a sequence of perturbations $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ which leads to the perturbed system: Let

$$\Sigma_\infty^\varepsilon : \begin{cases} \rho x = A_\varepsilon x + B_\varepsilon u + E_\varepsilon w \\ z_\varepsilon = C_\varepsilon x + D_\varepsilon u \end{cases} \tag{3.23}$$

be such that $A_\varepsilon \to A$, $B_\varepsilon \to B$, $E_\varepsilon \to E$ and $\bar{Q}_\varepsilon \to \bar{Q}_0$ where $\bar{Q}_\varepsilon$ and $\bar{Q}_0$ are defined in (3.16). $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ is admissible for discrete-time γ-level H_∞ low-gain design if it satisfies the following conditions:

1. Define

$$\gamma_\varepsilon^* \quad = \quad \inf_F \left\{ \|(C_\varepsilon + D_\varepsilon F)(zI - A_\varepsilon - B_\varepsilon F)^{-1} E_\varepsilon\|_\infty \mid \lambda(A_\varepsilon + B_\varepsilon F) \in C^{\odot} \right\}. \tag{3.24}$$

   Given $\gamma > \gamma^*$ where γ^* is defined in (3.6), for all sufficiently small ε, we have $\gamma > \gamma_\varepsilon^*$.

2. Provided that $\gamma > \gamma_\varepsilon^*$, consider the H_∞ DARE,

$$P_\varepsilon = A_\varepsilon' P_\varepsilon A_\varepsilon + C_\varepsilon' C_\varepsilon - \begin{bmatrix} B_\varepsilon' P_\varepsilon A_\varepsilon + D_\varepsilon' C_\varepsilon \\ E_\varepsilon' P_\varepsilon A_\varepsilon \end{bmatrix}' G(P_\varepsilon)^\dagger \begin{bmatrix} B_\varepsilon' P_\varepsilon A_\varepsilon + D_\varepsilon' C_\varepsilon \\ E_\varepsilon' P_\varepsilon A_\varepsilon \end{bmatrix} \tag{3.25}$$

   with

$$G(P_\varepsilon) = \begin{bmatrix} D_\varepsilon' D_\varepsilon & 0 \\ 0 & -\gamma^{-2} I \end{bmatrix} + \begin{bmatrix} B_\varepsilon' \\ E_\varepsilon' \end{bmatrix} P_\varepsilon \begin{bmatrix} B_\varepsilon & E_\varepsilon \end{bmatrix} \tag{3.26}$$

   subject to

$$E_\varepsilon' P_\varepsilon E_\varepsilon + E_\varepsilon' P_\varepsilon B_\varepsilon (D_\varepsilon' D_\varepsilon + B_\varepsilon' P B_\varepsilon)^\dagger B_\varepsilon' P_\varepsilon E_\varepsilon < \gamma^2 I.$$

   The smallest positive semi-definite semi-stabilizing solution P_ε satisfies $P_\varepsilon \to 0$ as $\varepsilon \to 0$.

3. $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon)$ does not have invariant zeros on the unit circle. Under this condition, a discrete-time γ-level H_∞ controller $F_\varepsilon(E, \gamma)$ with $\gamma > \gamma_\varepsilon^*$ for the perturbed system $\Sigma_\infty^\varepsilon$ can be easily constructed. However, such an $F_\varepsilon(E, \gamma)$ should also satisfy the next two conditions:

4. $F_\varepsilon(E, \gamma)$ is such that $A + BF_\varepsilon(E, \gamma)$ is Schur stable.

5. The closed-loop system comprising Σ_∞ and $u = F_\varepsilon(E, \gamma)x$ satisfies

$$\lim_{\varepsilon \to 0} \left\{ \sup_{w \in \ell_2} (\|z\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) \right\} = 0.$$

If $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ and a constructed sequence of $F_\varepsilon(E, \gamma)$ satisfy all 5 conditions, this $F_\varepsilon(E, \gamma)$ is a discrete-time γ-level H_∞ low-gain sequence.

Remark 3.7 *Given $\gamma > \gamma_\varepsilon^*$, a γ-level H_∞ state feedback can be constructed for the perturbed system Σ_∞ using the techniques developed in [112].*

Perturbation method I:

Similar to the discrete-time H_2 low-gain design, the first class of perturbations for system Σ_∞ in (3.5) is in the form of $(A, B, C_\varepsilon, D_\varepsilon, E)$ where C_ε and D_ε satisfy (3.18). We give two examples.

Example 1: One classical perturbation for system Σ_∞ which is widely used in the literature is $(A, B, C_\varepsilon, D_\varepsilon, E)$ where

$$C_\varepsilon = \begin{bmatrix} 0 \\ 0 \\ \sqrt{Q_\varepsilon} \end{bmatrix}, \quad D_\varepsilon = \begin{bmatrix} D \\ \varepsilon I \\ 0 \end{bmatrix},$$

and Q_ε satisfies (3.19). We first verify below that this perturbation is admissible for discrete-time H_∞ low-gain design.

- Condition 1 is proved in the proof of Theorem 3.6 in Appendix.
- It is easy to see that C_ε and D_ε satisfy (3.18). Then, the condition 2 follows from Theorem 3.6 in Appendix.
- Clearly $(A, B, C_\varepsilon, D_\varepsilon)$ does not have invariant zeros. One can then design a discrete-time γ-level H_∞ sub-optimal feedback $F_\varepsilon(E, \gamma)$ using the techniques developed in [112].
- Since we only perturb C and D and $F_\varepsilon(E, \gamma)$ is obtained using H_∞ optimal control techniques, condition 4 is obvious.

Therefore, for $\varepsilon \in (0, \varepsilon^*]$, conditions 1, 2, 3 and 4 are all satisfied. Next, we construct a discrete-time γ-level H_∞ suboptimal controller using the techniques developed in [112]. Let P_ε be the unique positive semi-definite semi stabilizing solution of H_∞ DARE,

$$P_\varepsilon = A' P_\varepsilon A + Q_\varepsilon - \begin{bmatrix} B' P_\varepsilon A \\ E' P_\varepsilon A \end{bmatrix}' G(P_\varepsilon)^{-1} \begin{bmatrix} B' P_\varepsilon A \\ E' P_\varepsilon A \end{bmatrix} \tag{3.27}$$

where

$$G(P_\varepsilon) = \begin{bmatrix} D_\varepsilon' D_\varepsilon & 0 \\ 0 & -\gamma^{-2} I \end{bmatrix} + \begin{bmatrix} B' \\ E' \end{bmatrix} P_\varepsilon \begin{bmatrix} B & E \end{bmatrix} \tag{3.28}$$

subject to

$$E_\varepsilon' P_\varepsilon E_\varepsilon + E_\varepsilon' P_\varepsilon B_\varepsilon (D_\varepsilon' D_\varepsilon + B_\varepsilon' P_\varepsilon B_\varepsilon)^{-1} B_\varepsilon' P_\varepsilon E_\varepsilon < \gamma^2 I.$$

Then a discrete-time γ-level H_∞ sub-optimal static state feedback can be constructed as

$$F_\varepsilon(E,\gamma) = (B' P_\varepsilon B + D_\varepsilon' D_\varepsilon + B' P_\varepsilon E(\gamma^2 I - E' P_\varepsilon E)^{-1} E' P_\varepsilon B)^{-1} \\ (B' P_\varepsilon A + B' P_\varepsilon E(\gamma^2 I - E' P_\varepsilon E)^{-1} E' P_\varepsilon A).$$

If we apply this $u = F_\varepsilon(E,\gamma)x$ to the original system Σ_∞ and the perturbed system $\Sigma_\infty^\varepsilon$ with this class of perturbation, since our perturbation satisfies (3.18), we have $\|z\|_{\ell_2} \leq \|z_\varepsilon\|_{\ell_2}$ for the same initial condition x_0 and w. This implies that

$$\sup_{w \in \ell_2} (\|z\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) \leq \sup_{w \in \ell_2} (\|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) = x_0' P_\varepsilon x_0.$$

The last equality follows from [112]. Since $P_\varepsilon \to 0$ as $\varepsilon \to 0$ according to Theorem 3.6 of the Appendix, condition 5 is satisfied. Therefore, $F_\varepsilon(E,\gamma)$ is a discrete-time γ-level H_∞ low-gain sequence. Moreover, it recovers the H_∞-ARE based low-gain design for semi-global stabilization of linear system subject to input saturation introduced in [130].

Example 2: Similar to that in H_2 low-gain sequence design, we can first transform the system into its SCB form with transformation $(x_a', x_c')' = T_1 x$:

$$\Sigma_{\infty,I}: \begin{cases} \begin{bmatrix} \rho x_a \\ \rho x_c \end{bmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + \begin{bmatrix} E_a \\ E_c \end{bmatrix} w \\ \quad z \quad = u_0. \end{cases}$$

Then we perturb the above transformed system. After doing so, we get

$$\Sigma_{\infty,I}^\varepsilon: \begin{cases} \begin{bmatrix} \rho x_a \\ \rho x_c \end{bmatrix} = \begin{bmatrix} A_a & 0 \\ \star & A_c \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} 0 \\ B_c \end{bmatrix} u_c + \begin{bmatrix} B_a \\ B_{ac} \end{bmatrix} u_0 + \begin{bmatrix} E_a \\ E_c \end{bmatrix} w \\ \begin{bmatrix} z \\ z_{\varepsilon,1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \sqrt{Q_\varepsilon} & 0 \end{bmatrix} \begin{bmatrix} x_a \\ x_c \end{bmatrix} + \begin{bmatrix} I_{m_0} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_0 \\ u_c \end{bmatrix}, \end{cases}$$

where Q_ε satisfies (3.19). For the same reasons as argued in the previous example, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, conditions 1, 2, 3 and 4 are all satisfied.

Next we construct a discrete-time γ-level H_∞ sub-optimal feedback F_ε for the perturbed system following the design procedure in [112]. Let P_ε be the positive semi-definite semi stabilizing solution of

H_∞ DARE,

$$P_\varepsilon = A_a' P_\varepsilon A_a + Q_\varepsilon - \begin{bmatrix} B_a' P_\varepsilon A_a \\ E_a' P_\varepsilon A_a \end{bmatrix}' G(P_\varepsilon)^{-1} \begin{bmatrix} B_a' P_\varepsilon A_a \\ E_a' P_\varepsilon A_a \end{bmatrix}$$

where

$$G(P_\varepsilon) = \begin{bmatrix} I & 0 \\ 0 & -\gamma^{-2} I \end{bmatrix} + \begin{bmatrix} B_a' \\ E_a' \end{bmatrix} P_\varepsilon \begin{bmatrix} B_a & E_a \end{bmatrix},$$

subject to

$$E_a' P_\varepsilon E_a + E_a' P_\varepsilon B_a (I + B_a' P_\varepsilon B_a)^{-1} B_a' P_\varepsilon E_a < \gamma^2 I,$$

and choose F_c such that $A_c + B_c F_c$ is Schur stable. The $F_\varepsilon(E, \gamma)$ can be constructed as

$$F_\varepsilon(E, \gamma) = \begin{bmatrix} \bar{F}_\varepsilon & 0 \\ 0 & F_c \end{bmatrix} T_1,$$

where

$$\bar{F}_\varepsilon = (B_a' P_\varepsilon B_a + I + B_a' P_\varepsilon E_a (\gamma^2 I - E_a' P_\varepsilon E_a)^{-1} E_a' P_\varepsilon B_a)^{-1}$$
$$(B_a' P_\varepsilon A + B_a' P_\varepsilon E_a (\gamma^2 I - E_a' P_\varepsilon E_a)^{-1} E_a' P_\varepsilon A_a).$$

If we apply this constructed feedback $u = F_\varepsilon(E, \gamma) x$ to the original system Σ_∞ and perturbed system $\Sigma_\infty^\varepsilon$ with this class of perturbation, since our perturbation satisfies (3.18), we have $\|z\|_{\ell_2} \leq \|z_\varepsilon\|_{\ell_2}$ for the same initial condition x_0 and w . This implies that

$$\sup_{w \in \ell_2} (\|z\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) \leq \sup_{w \in \ell_2} (\|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) = x_a(0)' P_\varepsilon x_a(0)$$

where $x_a(0)$ is the initial condition of x_a dynamics. The last equality follows from [112]. Since $P_\varepsilon \to 0$ as $\varepsilon \to 0$, according to Theorem 3.6 of the Appendix, we find that condition 5 is satisfied. Therefore, $F_\varepsilon(E, \gamma)$ is a γ-level low-gain sequence.

Perturbation method II:

We can also directly perturb the system dynamics to move those invariant zeros on the unit circle. Consider the perturbation $(A_\varepsilon, B_\varepsilon, 0, D, E_\varepsilon)$ where

$$A_\varepsilon = (1 + \varepsilon) A, \;\; B_\varepsilon = (1 + \varepsilon) B, \;\; E_\varepsilon = (1 + \varepsilon) E$$

and ε small enough such that $((1+\varepsilon)A, (1+\varepsilon)B)$ is stabilizable and $(1+\varepsilon)A$ does not have eigenvalues on the unit circle. We shall focus on this particular choice of perturbation.

- Given $(1+\varepsilon)A + (1+\varepsilon)BF$ Schur stable, we note that $\|DF(zI - A - BF)^{-1}E\|_\infty \leq \|(1+\varepsilon)DF(zI - (1+\varepsilon)A - (1+\varepsilon)BF)^{-1}E\|_\infty$. This implies that conditions 1 is satisfied.

- Since $((1+\varepsilon)A, (1+\varepsilon)B, 0, D)$ always have the same normal rank as that of $(A, B, 0, D)$, the condition 2 follows from Theorem 3.5 in Appendix.

- Since $((1+\varepsilon)A, (1+\varepsilon)B, 0, D)$ does not have any invariant zeros on the unit circle, the condition 3 is satisfied.

- $A + BF$ is Schur stable if $(1+\varepsilon)A + (1+\varepsilon)BF$ is Schur Stable.

Therefore, conditions 1, 2, 3 and 4 are satisfied for a sufficiently small ε. One can design a discrete-time γ-level H_∞ state feedback according to [112] as follows:

The perturbed system can be transformed into its compact SCB form using a nonsingular state transformation: $\begin{bmatrix} x_a^{\odot\prime} & x_a^{\bigcirc\prime} & x_c' \end{bmatrix}' = T_3 x$ as:

$$\bar{\Sigma}^{\varepsilon}_{\infty,II} : \begin{cases} \begin{bmatrix} \rho\bar{x}_a^{\odot} \\ \rho\bar{x}_a^{\bigcirc} \\ \rho\bar{x}_c \end{bmatrix} = (1+\varepsilon)\begin{bmatrix} A_a^{\odot} & 0 & 0 \\ 0 & A_a^{\bigcirc} & 0 \\ \star & \star & A_c \end{bmatrix}\begin{bmatrix} \bar{x}_a^{\odot} \\ \bar{x}_a^{\bigcirc} \\ \bar{x}_c \end{bmatrix} + (1+\varepsilon)\begin{bmatrix} 0 \\ 0 \\ B_c \end{bmatrix} u_c + (1+\varepsilon)\begin{bmatrix} B_a^{\odot} \\ B_a^{\bigcirc} \\ B_{ac} \end{bmatrix} u_0 + (1+\varepsilon)\begin{bmatrix} E_a^{\odot} \\ E_a^{\bigcirc} \\ E_c \end{bmatrix} w \\ \bar{z} = u_0, \end{cases} \tag{3.29}$$

where $A_a^{\odot}$ is Schur stable, (A_c, B_c) is controllable, $(A_a^{\bigcirc}, B_a^{\bigcirc})$ is controllable, and $A_a^{\bigcirc}$ has all its eigenvalues on the unit circle. The eigenvalues of $(1+\varepsilon)A_a^{\bigcirc}$ and $(1+\varepsilon)A_a^{\odot}$ are the invariant zeros of the perturbed system. For a sufficiently small ε, $(1+\varepsilon)A_a^{\odot}$ is also Schur stable. Moreover, T_3 is independent of ε. Let P_ε be the positive semi-definite semi-stabilizing solution of H_∞ DARE,

$$\tfrac{1}{(1+\varepsilon)^2} P_\varepsilon = A_a^{\bigcirc\prime} P_\varepsilon A_a^{\bigcirc} - \begin{bmatrix} B_a^{\bigcirc\prime} P_\varepsilon A_a^{\bigcirc} \\ E_a^{\bigcirc\prime} P_\varepsilon A_a^{\bigcirc} \end{bmatrix}' G(P_\varepsilon)^{-1} \begin{bmatrix} B_a^{\bigcirc\prime} P_\varepsilon A_a^{\bigcirc} \\ E_a^{\bigcirc\prime} P_\varepsilon A_a^{\bigcirc} \end{bmatrix},$$

where

$$G(P_\varepsilon) = \begin{bmatrix} \frac{1}{(1+\varepsilon)^2} I & 0 \\ 0 & -\frac{\gamma^{-2}}{(1+\varepsilon)^2} I \end{bmatrix} + \begin{bmatrix} B_a^{\bigcirc\prime} \\ E_a^{\bigcirc\prime} \end{bmatrix} P_\varepsilon \begin{bmatrix} B_a^{\bigcirc} & E_a^{\bigcirc} \end{bmatrix}.$$

Let F_c be such that $A_c + B_c F_c$ is Schur stable, and the γ-level H_∞ sub-optimal controller is given by

$$F_\varepsilon(E, \gamma) = \begin{bmatrix} 0 & \bar{F}_\varepsilon & 0 \\ 0 & 0 & F_c \end{bmatrix} T_3$$

where

$$\bar{F}_\varepsilon = H_\varepsilon^{-1} \left[B_a^{\bigcirc\prime} P_\varepsilon A_a^{\bigcirc} + B_a^{\bigcirc\prime} P_\varepsilon E_a^{\bigcirc} (\gamma^2 I - E_a^{\bigcirc\prime} P_\varepsilon E_a^{\bigcirc})^{-1} E_a^{\bigcirc\prime} P_\varepsilon A_a^{\bigcirc} \right]$$

and

$$H_\varepsilon = B_a^{\bigcirc'} P_\varepsilon B_a^{\bigcirc} + I + B_a^{\bigcirc'} P_\varepsilon E_a^{\bigcirc} (\gamma^2 I - E_a^{\bigcirc'} P_\varepsilon E_a^{\bigcirc})^{-1} E_a^{\bigcirc'} P_\varepsilon B_a^{\bigcirc}.$$

Note that if we apply $u = F_\varepsilon(E, \gamma)$ to the original system Σ_∞ and perturbed system $\Sigma_\infty^\varepsilon$ with this perturbation data, we have, for the same initial condition and w, $z_\varepsilon(k) = (1+\varepsilon)^k z(k)$ and hence $\|z\|_{\ell_2} \leq \|z_\varepsilon\|_{\ell_2}$. Therefore,

$$\sup_{w \in \ell_2} (\|z\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) \leq \sup_{w \in \ell_2} (\|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2) = x_a^{\bigcirc'}(0) P_\varepsilon x_a^{\bigcirc}(0).$$

The last inequality follows from [112]. Since $P_\varepsilon \to 0$ as $\varepsilon \to 0$, according to Theorem 3.5 of the Appendix, we find that condition 5 is satisfied. Therefore, $F_\varepsilon(E, \gamma)$ is a discrete-time γ-level H_∞ low-gain sequence.

3.7. Conclusion

These two chapters can be summarized as follows:

- Four different existing approaches to low-gain design, namely direct eigenstructure assignment, H_2 and H_∞ algebraic Riccati equation (ARE) based methods, and parametric Lyapunov equation based method, are shown to be rooted in two fundamental control theories, H_2 and H_∞ theory. This unification not only brings the existing methods together but also reveals the interconnections between them.

- The second is that by making explicit the connection between the proposed low-gain design and the stabilization of linear systems subject to saturation, the paper also gives the necessary and sufficient conditions for semi-global stabilization of linear systems when only some but not necessarily all the input channels are subject to saturation; this aspect has never been considered by the saturation community.

Appendix

Existence of H_2 optimal controller

Consider the system,

$$\Sigma_a : \begin{cases} \rho x = Ax + Bu + I\omega \\ z \ = Cx + Du. \end{cases} \tag{3.30}$$

We recall the following existence conditions of H_2 optimal static state feedback controller for system Σ:

Theorem 3.4 For the system Σ_a in (3.30), H_2 optimal state feedback controller of a static type exists if and only if

1. (A, B) is stabilizable;
2. Σ_a does not have any invariant zero on the unit circle.

Proof : The results follow from Lemma 6.6.5 and Lemma 6.6.1 in [93]. ■

Continuity of solution of discrete-time H_∞ Riccati Equation

In this paper, our concern is about the continuity of solutions of the DARE associated with the 5-tuple (A, B, C, D, E) and $\gamma > \gamma^*$,

$$P = A'PA + C'C - \begin{bmatrix} B'PA + D'C \\ E'PA \end{bmatrix}' G(P)^\dagger \begin{bmatrix} B'PA + D'C \\ E'PA \end{bmatrix} \tag{3.31}$$

where

$$G(P) = \begin{bmatrix} D'D & 0 \\ 0 & -\gamma^{-2}I \end{bmatrix} + \begin{bmatrix} B' \\ E' \end{bmatrix} P \begin{bmatrix} B & E \end{bmatrix},$$

and

$$\gamma^* = \inf_F \left\{ \|(C + DF)(zI - A - BF)^{-1}E\|_\infty \mid \lambda(A + BF) \in C^{\odot} \right\}. \tag{3.32}$$

We recall the following theorem from [114]:

Theorem 3.5 Consider a 5-tuple (A, B, C, D, E). Suppose (A, B) is stabilizable, (A, B, C, D) does not have any invariant zeros outside the unit disc, and $\gamma > \gamma^*$ where γ^* is defined in (3.32). Let a sequence of perturbed data $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ converge to (A, B, C, D, E). Moreover, assume that the normal rank of $C_\varepsilon(zI - A_\varepsilon)^{-1}B_\varepsilon + D_\varepsilon$ is equal to the normal rank of $C(zI - A)^{-1}B + D$ for all ε. Then, the smallest positive semi-definite semi-stabilizing solution of DARE (3.31) associated with $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$ converges to the smallest positive semi-definite semi-stabilizing solution of DARE associated with (A, B, C, D, E).

In the perturbation method I of both H_2 and H_∞ low-gain design, we use perturbations which do not necessarily preserve the normal rank. In this case, we need the following result. Define

$$\bar{Q}_\varepsilon = \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}' \begin{bmatrix} C_\varepsilon & D_\varepsilon \end{bmatrix}.$$

Theorem 3.6 Consider a 5-tuple (A, B, C, D, E) and $\gamma > \gamma^*$ where γ^* is defined in (3.6). Suppose a sequence of perturbations $(C_\varepsilon, D_\varepsilon)$ converge to (C, D), and satisfies the following conditions:

1. $\bar{Q}_\varepsilon$ is continuous at $\varepsilon = 0$;
2. there exists a β such that for $0 \le \varepsilon_1 \le \varepsilon_2 \le \beta$, we have $\bar{Q}_{\varepsilon_1} \le \bar{Q}_{\varepsilon_2}$.

Then the semi-stabilizing solution to DARE (3.31) associated with $(A, B, C_\varepsilon, D_\varepsilon, E)$ converges to the semi-stabilizing solution of DARE (3.31) associated with (A, B, C, D, E).

Proof : The case $\gamma = \infty$ was proved in [137]. We shall prove this result for a finite γ.

First, we need to show that given $\gamma > \gamma^*$, for sufficiently small ε, we have $\gamma > \gamma_\varepsilon^*$ where γ_ε^* is defined in (3.32) with (A, B, C, D, E) replaced by $(A_\varepsilon, B_\varepsilon, C_\varepsilon, D_\varepsilon, E_\varepsilon)$. This follows from the fact that there exists a stabilizing state feedback $u = Fx$ such that the H_∞ norm from w to z equals $\gamma_0 < \gamma$. The transfer matrix $G_{\varepsilon,cl}$ from w to z_ε satisfies,

$$G_{\varepsilon,cl}(z)'G_{\varepsilon,cl}(z) = G_{cl}(z)'G_{cl}(z) + G_0(z)'(\bar{Q}_\varepsilon - \bar{Q}_0)G_0(z)$$

where G_{cl} is the transfer matrix from w to z while G_0 is defined by

$$G_0(z) = \begin{bmatrix} I \\ F \end{bmatrix} (zI - A - BF)^{-1}E.$$

Since G_0 has a finite H_∞ norm and $\bar{Q}_\varepsilon \to \bar{Q}_0$ we find that

$$\lim_{\varepsilon \downarrow 0} \|G_{\varepsilon,cl}\|_\infty \to \|G_{cl}\|_\infty = \gamma_0 < \gamma$$

Next we investigate

$$x_0' P_\varepsilon x_0 = \sup_{w \in \ell_2} \inf_u \left\{ \|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \mid x \in \ell_2 \right\},$$

where $x(0) = x_0$. Since $\bar{Q}_\varepsilon > \bar{Q}_0$ for small ε, we find that

$$x_0' P_\varepsilon x_0 \geq x_0' P_0 x_0$$

for small ε. If we choose $u = Fx$, we obtain

$$0 \leq x_0' P_\varepsilon x_0 \leq \left\{ \sup_{w \in \ell_2} \|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \mid u = Fx \right\}.$$

We always have for any η,

$$\|a + b\|^2 \leq (1 + \tfrac{1}{\eta})\|a\|^2 + (1 + \eta)\|b\|^2.$$

Let γ_1 be such that $\gamma_0 < \gamma_1 < \gamma$ and that, for all sufficiently small ε, H_∞ norm from w to z_ε be less than γ_1 for the feedback $u = Fx$. With u fixed by $u = Fx$, we can write $z_\varepsilon = z_{x_0} + z_w$ where z_{x_0} is the output for initial condition x_0 and $w = 0$ and z_w is the output for initial condition $x_0 = 0$ and disturbance w. Let L be such that

$$\|x_{x_0}\|_{\ell_2}^2 = x_0' L x_0$$

where x_{x_0} is the state for initial condition x_0 and $w = 0$. Choose

$$\eta = \frac{\gamma^2 - \gamma_1^2}{2\gamma_1^2}.$$

We find

$$\|z_\varepsilon\|_{\ell_2}^2 \leq \tfrac{\gamma^2 + \gamma_1^2}{\gamma^2 - \gamma_1^2} \|\bar{Q}_\varepsilon\| x_0' L x_0 + \tfrac{\gamma^2 + \gamma_1^2}{2} \|w\|_{\ell_2}^2.$$

But then if w is such that

$$\|w\|^2 > \beta x_0' L x_0 \tag{3.33}$$

where $\beta > \beta_\varepsilon$ for all sufficiently small ε with

$$\beta_\varepsilon = \frac{2(\gamma^2 + \gamma_1^2)}{2(\gamma^2 - \gamma_1^2)^2} \|\bar{Q}_\varepsilon\|$$

we have

$$\|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 < 0$$

for $u = Fx$. We find that for w for which (3.33) is satisfied we obtain for a suboptimal u already a negative cost. Since

$$\sup_{w \in \ell_2} \inf_{u} \left\{ \|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \mid x \in \ell_2 \right\} > 0,$$

we can without loss of generality assume that w satisfies

$$\|w\|^2 < \beta x_0' L x_0 \tag{3.34}$$

provided ε is small enough. By setting $u = Fx + v$ the above inf-sup problem is equivalent to

$$\sup_{w \in \ell_2} \inf_{v \in \ell_2} \left\{ \|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \mid u = Fx + v \right\} > 0.$$

Since we showed that w can be, without loss of generality, assumed to be bounded, it is clear that in the above optimization for $\varepsilon = 0$ we can also assume without loss of generality that v is bounded as well, i.e.

$$\|v\|_2 \leq N \|x_0\|.$$

If the system is left-invertible from v to z then as v gets sufficiently large in ℓ_2 norm then the cost can be made arbitrarily large. If the system is not left-invertible we can split the input in a part which has no effect on the output and a part which has a left-invertible effect on the output. The latter has to be bounded for a bounded cost. The first can be set to zero without loss of generality. But with v and w bounded we can find M such that

$$\|x\|_2 \leq M \|x_0\|^2,$$

but then

$$\begin{aligned}
&\sup_{w \in \ell_2} \inf_{v \in \ell_2} \left\{ \|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \right\} \\
&\quad \leq \sup_{w \in \ell_2} \inf_{v \in \ell_2} \left\{ \|z_\varepsilon\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \mid \|v\|_2 \leq N \|x_0\| \right\} \\
&\quad \leq \sup_{w \in \ell_2} \inf_{v \in \ell_2} \left\{ \|z_0\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \mid \|v\|_2 \leq N \|x_0\| \right\} + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2 \\
&\quad = \sup_{w \in \ell_2} \inf_{v \in \ell_2} \left\{ \|z_0\|_{\ell_2}^2 - \gamma^2 \|w\|_{\ell_2}^2 \right\} + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2 \\
&\quad = x_0' P_0 x_0 + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2
\end{aligned}$$

where in each case $u = Fx + v$. In conclusion, we find

$$x_0' P_0 x_0 \le x_0' P_\varepsilon x_0 \le x_0' P_0 x_0 + \|\bar{Q}_\varepsilon - \bar{Q}_0\| M \|x_0\|^2,$$

which implies that $P_\varepsilon \to P_0$ as $\varepsilon \downarrow 0$. ■

Part II

Issues related to internal stabilization of complex dynamical systems subject to constraints

CHAPTER 4

Semi-global stabilization of discrete-time systems subject to non-right invertible constraints

4.1. Introduction

In this chapter we revisit the problem of stabilization of general linear time-invariant discrete-time systems subject to constraints. Over the past decade there has been rather strong interest in this problem, in part due to a wide recognition of the inherent constraints on the input and state in most practical control systems. Consequently, several important results have appeared in the open literature.

It is becoming evident that the taxonomy of constraints developed in [87] plays dominant roles in every type of control design problem, not only for continuous-time systems but also for discrete-time systems. The taxonomy of constraints is developed by appropriately modeling the constraints in terms of what is called a constrained output (of the given system) with its magnitude subject to some prescribed constraint sets. It turns out that structural properties of the mapping from the input to the constrained output vector play dominant roles in dictating what is feasible and what is not feasible. Such structural properties have been categorized in three directions. The first direction of categorization is based on the right invertibility of the mapping from the input to the constrained output vector. This direction of categorization delineates the constraints into two mutually exclusive categories, 1) right invertible constraints representing the case when the mapping from the input to the constrained output vector is right invertible, and 2) non-right invertible constraints representing the case when the mapping from the input to the constrained output vector is not right invertible. The second direction of categorization is based on the so called constrained invariant zeros of the plant, i.e. the invariant zeros of the mapping from the input to the constrained output vector. Like in the first categorization, this second categorization also delineates the constraints into two mutually exclusive main categories, 1) at most weakly non-minimum phase constraints representing the case when the constrained invariant zeros are in the closed left-half complex plane for continuous-time systems or in the closed unit disc for discrete-time systems, and 2) strongly non-minimum phase constraints representing the case when one or more of the constrained invariant zeros are in the open right half complex plane for continuous-time systems or outside the unit disc for discrete-time systems. The third direction of categorization is based on the order of constraint infinite zeros, i.e. the infinite zeros of the mapping from the input to the constrained output.

Based on such a taxonomy of constraints, two main features emerge:

- Neither the constrained semi-global nor the constrained global stabilization problem is solvable whenever the constraints are strongly non-minimum phase.

- There exists a perceptible demarcation line between the right and non-right invertible constraints. In particular, the solvability conditions for the constrained semi-global and global stabilization problems via state feedback do not depend on the shape of the constraint sets for right invertible constraints, whereas for non-right invertible constraints they indeed do so.

The initial work on stabilization of linear systems subject to constraints can be found in two special issues of IJRNC, [5] and [98], and the references cited there. During the last decade several aspects of control design problems for linear systems with magnitude and rate constraints on control variables have been studied among others by the second and third authors and their students and collaborators. A number of powerful analysis and design methods such as low gain, low-high gain, scheduled low gain, scheduled low-high gain and many variations of them have been developed for several core control design problems including global and semi-global internal stabilization, external stabilization, output regulation, and disturbance rejection. They have studied stabilization (continuous-time in [89] and discrete-time in [58]) and output regulation problems (continuous-time in [61] and discrete-time in [65]) associated with magnitude and rate constraints on control variables. Many of these issues have also been addressed in the book [95]. The research thrust of the second and third authors and their students has broadened to include additionally magnitude and rate constraints on state variables. In connection with stabilization, whenever amplitude and rate constraints on both state as well as input variables exist, a taxonomy of all possible constraints is introduced in [87] as discussed above. The work of [87], while considering mainly right-invertible constraints, focuses on continuous-time systems and generalizes, extends, and covers all the existing results by then including those developed in the seminal works of Fuller [24, 25], Sontag and Sussmann [110], Sussmann and Yang [120], as well as Sussmann, Sontag, and Yang [119], all of which dwell only with control magnitude saturation. The discrete-time version of results analogous to [87], once again mainly for right-invertible constraints, are developed in [88]. Output regulation for systems with both input and state constraints has in the mean time also been studied in the papers [96] (continuous-time) and [108] (discrete-time).

As said above, both the works of [87] and [88] are mainly concerned with right invertible constraints. The work of [97] continues the theme of semi-global stabilization with respect to the admissible set by considering linear continuous-time systems however with non-right invertible constraints. We remark here that non-right invertible constraints arise inherently due to state constraints, while the magnitude and rate constraints only on the input belong to the right invertible constraints. The focus of this chapter is to address the same issues as in [97] by considering discrete-time systems with non-right invertible constraints. Although the development for discrete-time systems parallels somewhat that in continuous-time systems, there are several fundamental differences between continuous- and discrete-time systems:

1. the solvability conditions for semi-global stabilization, unlike in continuous-time systems, requires that the order of the constraint infinite zeros be less than or equal to one,

2. the methods of constructing appropriate controllers need to be revised as needed, and

3. some new issues arise in proofs of the results obtained, which do not exist in continuous-time systems.

We emphasize that semi-global stabilization with respect to the admissible set requires that the constraints are at most weakly non-minimum phase. When constraints are strongly non-minimum phase, semi-global stabilization can only be accomplished with respect to recoverable sets. Such stabilization issues and construction of recoverable sets for continuous-time systems have been addressed in [116]. The discrete-time counterpart of these results will be the topic of the next chapter.

We must mention that, besides the authors and their collaborators, there have been other efforts on dealing with state and input constraints utilizing the concept of positive invariant sets [6] and techniques of model predictive control [8, 26, 64]. However, the available tools along these lines are computationally very demanding and it is very difficult to guarantee feasibility in the case of state constraints using those numerical tools.

This chapter is organized as follows. Following the problem formulation in the next section, a taxonomy of constraints is reviewed in Section 4.3. This taxonomy facilitates the statements of main results of semi-global stabilization for non-right invertible constraints in Section 4.5. Proofs of the main results for the state feedback case are presented in Section 4.6, while Section 4.7 presents proofs for the measurement feedback case. The issues involved when adding rate constraints in the non-right-invertible case are addressed in Section 4.9. Conclusions are drawn in Section 4.10.

4.2. Problem formulation

Consider a linear discrete-time system,

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad y(k) \quad = C_y x(k) + D_y u(k) \\ \quad z(k) \quad = C_z x(k) + D_z u(k), \end{cases} \tag{4.1}$$

where $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the input, $y(k) \in \mathbb{R}^\ell$ is the measured output, $z(k) \in \mathbb{R}^p$ is the constrained output, which is subject to the constraint $z(k) \in \mathcal{S}$ for all $k \geq 0$. The issues involved when adding rate constraints are dealt with in Section 4.9.

The set $\mathcal{S}$ is a subset in $\mathbb{R}^p$ satisfying the following assumptions. These assumptions are identical to those in the constrained stabilization problem subject to right-invertible constraints. We refer to [87, 88] for a full motivation.

Assumption 4.1

1. The set $\mathcal{S}$ is compact, convex and contains 0 as interior point.

2. $C_z' D_z = 0$ and

$$\mathcal{S} = (\mathcal{S} \cap \operatorname{im} C_z) \oplus (\mathcal{S} \cap \operatorname{im} D_z).$$

In order to guarantee that the constrained output remains in $\mathcal{S}$, the initial conditions must be restricted. We have the following definition regarding the admissible set of initial conditions:

Definition 4.1 Define the admissible set of initial conditions $\mathcal{V}(\mathcal{S})$ as

$$\mathcal{V}(\mathcal{S}) := \{\, x \in \mathbb{R}^n \mid \exists u \text{ such that } C_z x + D_z u \in \mathcal{S} \,\}.$$

Definition 4.2 The constrained recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ is defined to be the set of all initial states $x(0) \in \mathcal{V}(\mathcal{S})$ for which there exists a control input such that $x(k) \to 0$ as $k \to \infty$ while $z(k) \in \mathcal{S}$ for all k.

Definition 4.3 Assume that $u(k) = f(x(k), k)$ is a static control law for the system Σ. The constrained domain of attraction $\mathcal{R}_A^f(\Sigma, \mathcal{S})$ is defined to be the set of all $x(0) \in \mathbb{R}^n$ such that the state trajectory of the closed-loop system satisfies $x(k) \to 0$ as $t \to \infty$ while $z(k) \in \mathcal{S}$ for all $k > 0$. In this case we say that the controller f achieves the constrained domain of attraction $\mathcal{R}_A^f(\Sigma, \mathcal{S})$.

We will address two problems for this system as outlined in the following problem formulations:

Problem 4.1 The **constrained semi-global stabilization via state feedback problem** is to find, if possible, for any a priori given compact set $\mathcal{W} \subset \operatorname{int} \mathcal{V}(\mathcal{S})$, a static state feedback $u(k) = f(x(k), k)$ such that the equilibrium point $x = 0$ of the closed-loop system is asymptotically stable with $\mathcal{W} \subset \mathcal{R}_A^f(\Sigma, \mathcal{S})$.

Problem 4.2 The **constrained semi-global stabilization via measurement feedback problem** is to find (if possible) a parameterized family of measurement feedbacks of the form,

$$\begin{cases} v(k+1) = g_\varepsilon(v(k), y(k), k), & v \in \mathbb{R}^q \\ u(k) \quad = h_\varepsilon(v(k), y(k), k), & \end{cases} \tag{4.2}$$

such that for any compact set $\mathcal{W} \subset \operatorname{int} \mathcal{V}(\mathcal{S})$ and any compact set $\mathcal{V} \subset \mathbb{R}^q$ there exist a measurement feedback in this family such that the following conditions hold:

1. The equilibrium point $(x, v) = (0, 0)$ of the closed-loop system is asymptotically stable with $\mathcal{W} \times \mathcal{V}$ contained in its region of attraction.

2. For any $(x(0), v(0)) \in \mathcal{W} \times \mathcal{V}$, we have $z(t) \in \mathcal{S}$ for all $t \geq 0$.

4.3. Taxonomy of constraints

In this section, we review the taxonomy of constraints as developed in [87]. To do so, we let $\mathbb{C}$, $\mathbb{C}^{\oplus}$, $\mathbb{C}^{\ominus}$ and $\mathbb{C}^{\bigcirc}$ denote respectively the set of complex numbers in the entire complex plane, outside the unit circle, inside the unit circle, and on the unit circle.

The following notions are fundamental to the taxonomy of constraints given below.

Definition 4.4 A subsystem Σ_{zu} characterized by the quadruple (A, B, C_z, D_z),

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad z(k) \quad = C_z x(k) + D_z u(k), \end{cases}$$

is said to be **right invertible** if for any sequence $z_{\text{ref}}(k)$ defined for $k \geq 0$ there exists an input u and a choice of $x(0)$ such that $z(k) = z_{\text{ref}}(k)$ for all $k \geq 0$.

Definition 4.5 The **invariant zeros** of a linear system with a realization $(A,\ B,\ C,\ D)$ are those points $\lambda \in \mathbb{C}$ for which

$$\text{rank} \begin{pmatrix} \lambda I - A & -B \\ C & D \end{pmatrix} < \text{normrank} \begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}$$

where "normrank" denotes the *normal rank*.

The first categorization is based on whether the subsystem Σ_{zu} is right invertible or not. We have the following definition:

Definition 4.6 The constraints are said to be

- **right invertible constraints** if the subsystem Σ_{zu} is right invertible.
- **non-right invertible constraints** if the subsystem Σ_{zu} is not right invertible.

It turns out that the location of the invariant zeros of the subsystem Σ_{zu} is also important in characterizing the solvability of stabilization problems. We refer to these invariant zeros as constraint invariant zeros:

Definition 4.7 The invariant zeros of the system characterized by the quadruple (A, B, C_z, D_z) are called the **constraint invariant zeros** of the given system Σ.

The second categorization of constraints is based on the location of the constrained invariant zeros. We have the following definition:

Definition 4.8 The constraints are said to be

- **minimum phase constraints** if all the constraint invariant zeros are in $\mathbb{C}^{\ominus}$.

- **weakly minimum phase constraints** if all the constrained invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$ with the restriction that any invariant zero in $\mathbb{C}^{\bigcirc}$ is simple,
- **weakly non-minimum phase constraints** if all the constrained invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$ with at least one non-simple invariant zero in $\mathbb{C}^{\bigcirc}$.
- **at most weakly non-minimum phase constraints** if all the constrained invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$.
- **strongly non-minimum phase constraints** if at least one constrained invariant zeros is in $\mathbb{C}^{\oplus}$.

The third categorization is based on the order of the infinite zeros of the subsystem Σ_{zu} (see [99] for a definition of infinite zeros of a system). Because of their importance, we specifically label the infinite zeros of the subsystem Σ_{zu} as the constraint infinite zeros of the plant.

Definition 4.9 The infinite zeros of the subsystem Σ_{zu} are called the **constraint infinite zeros** of the plant associated with the constrained output z.

We have the following definition regarding the third categorization of constraints.

Definition 4.10 The constraints are said to be **type one constraints** if the order of all constraint infinite zeros is less than or equal to one.

4.4. Review of results for semi-global stabilization in admissible set for right-invertible constraints

In this section we review the necessary and/or sufficient conditions for the solvability of Problems 4.1 and 4.2, under the assumption that the subsystem Σ_{zu} characterized by (A, B, C_z, D_z) is right invertible, i.e. the system (4.1) has right invertible constraints. These results are extracted from [88].

It is worth pointing out that for the discrete-time systems the solvability conditions for the global and semi-global constrained stabilization are the same. This is in contrast to the continuous-time case [87].

The first result is about the solvability conditions for the constrained global or semi-global stabilization via state feedback.

Theorem 4.1 Consider the plant Σ as given by (4.1) with the constraint set $\mathcal{S}$ satisfying Assumption 4.1. Assume that the constraints are right-invertible and the set $\mathcal{S}$ is bounded. Then the global or semi-global constrained stabilization problem via state feedback as defined in Problem 4.1 is solvable if and only if:

1. (A, B) is stabilizable.

2. The constraints are at most weakly non-minimum phase.

3. The constraints are of type one.

Remark 4.1 *A fundamental consequence of Theorem 4.1 is that the conditions are independent of any specific shapes of the given constraint set. That is, for the case of a right invertible system Σ, if the semi-global constrained stabilization problem is solvable for some given constraint set satisfying Assumption 4.1, then it is also solvable for any other constraint sets satisfying Assumption 4.1.*

Note that the controller needed can be chosen either as a time-invariant nonlinear controller or as a time-varying linear controller.

For the case of measurement feedback, we have the following theorem.

Theorem 4.2 Consider the plant Σ as given by (4.1) with the constraint set $\mathcal{S}$ satisfying Assumption 4.1. Assume that the constraints are right-invertible and the set $\mathcal{S}$ is bounded. Then, the semi-global constrained stabilization problem via measurement feedback as defined in Problem 4.2 is solvable if the following conditions hold:

1. (A, B) is stabilizable.

2. The constraints are at most weakly non-minimum phase.

3. The constraints are of type one.

4. The pair (C_y, A) is observable.

5. $\ker C_y \subset \ker C_z A$.

6. $\ker \begin{pmatrix} C_y & D_y \end{pmatrix} \subset \ker \begin{pmatrix} C_z & D_z \end{pmatrix}$.

Moreover, conditions (1) to (3) are necessary.

Remark 4.2 *Note that in the above theorem, conditions* (1)*,* (2) *and* (3) *are necessary. Condition* (4) *can be weakened by assuming only detectability. However, clearly some additional assumptions would then be needed if undetectable states can affect the constrained output* z*. However, this is excluded by condition* (6)*. Regarding condition* (5) *we know that a necessary condition for solvability equals*

$$\ker \begin{pmatrix} C_y \\ C_z \end{pmatrix} \subseteq \ker C_z A$$

which is equal to condition (5) *given condition* (6)*. Condition* (6) *is not necessary but it is a natural condition to impose that the constrained variables* z *are part of the observations variables* y*, which is another way to express condition* (6)*.*

Remark 4.3 *The sufficient conditions as given by Theorems 4.2 are independent of any specific features of the given constraint sets. But the solvability of the semi-global constrained stabilization problem in the measurement feedback case is in general dependent on the shape of the constraint sets even for the case of right-invertible constraints (An example of [88] demonstrates this). Note that this is not in contradiction with Theorems 4.2 since we only presented there sufficient conditions for solvability. Also, we point out that the above is in contrast with continuous-time where the solvability is always independent of the constraint set for right-invertible systems.*

4.5. Main results

Although the goal of this section is to present the main results when the constraints are non-right invertible, at first we recall a basic result from [88] to clarify a certain issue related to non-right invertible constraints. As mentioned in the introduction, unlike in the case of right invertible constraints, in general the solvability conditions for semi-global stabilization in the presence of non-right invertible constraints depend on the shape of constraint set $\mathcal{S}$. We can however present one additional assumption which must be satisfied independent of the shape of the constraint set. In this regard, we recall a result from [88].

In order to analyze these problems, we first investigate the structure of the system. By using a suitable basis transformation, the given system can be transformed via a state space transformation T, input basis transformation T_u, and output basis transformation T_z into its SCB [99, 92] with the state described in the new coordinates by

$$Tx = \bar{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \qquad T_u u = \bar{u} = \begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix},$$

and

$$T_z z = \bar{z}(k) = \begin{pmatrix} z_b(k) \\ z_0(k) \\ z_d(k) \end{pmatrix} = \begin{pmatrix} C_b x_b(k) \\ u_0(k) \\ C_d x_d(k) \end{pmatrix}.$$

We have that $\bar{z}(k)$ is subject to the constraint $\bar{z}(k) \in \bar{\mathcal{S}}$ for all $k \geq 0$, where $\bar{\mathcal{S}} = T_z^{-1}\mathcal{S}$. Since $C_z' D_z = 0$, it is guaranteed that the new constraint set still satisfies Assumption 4.1. The given system (4.1) can be written in the following form:

$$\begin{cases} x_a(k+1) = A_{aa}x_a(k) + K_a\tilde{z}(k) \\ x_b(k+1) = A_{bb}x_b(k) + K_b\tilde{z}(k) \\ x_c(k+1) = A_{cc}x_c(k) + K_c\tilde{z}(k) + B_c[u_c(k) + J_a x_a(k)] \\ x_d(k+1) = u_d(k) + G_a x_a(k) + G_b x_b(k) + G_c x_c(k) + G_d x_d(k) \\ \quad y(k) \quad = C_{ya}x_a(k) + C_{yb}x_b(k) + C_{yc}x_c(k) + C_{yd}x_d(k) + \tilde{D}_y\tilde{u}(k) \\ \quad z_0(k) \quad = u_0(k) \\ \quad z_b(k) \quad = C_b x_b(k) \\ \quad z_d(k) \quad = C_d x_d(k). \end{cases}$$

Defining

$$\tilde{A}_1 = \begin{pmatrix} A_{aa} & 0 \\ 0 & A_{bb} \end{pmatrix}, \quad \tilde{B}_1 = \begin{pmatrix} K_a \\ K_b \end{pmatrix}, \quad \tilde{x}_1 = \begin{pmatrix} x_a \\ x_b \end{pmatrix},$$

$\tilde{C}_1 = \begin{pmatrix} 0 & C_b \end{pmatrix}$, $\tilde{v}_1 = \tilde{z}$, and $\tilde{z}_1 = z_b$, we obtain for $i = 1$ the following reduced system:

$$\tilde{\Sigma}_i : \begin{cases} \tilde{x}_i(k+1) = \tilde{A}_i\tilde{x}_i(k) + \tilde{B}_i\tilde{v}_i(k) \\ \quad \tilde{z}_i(k) \quad = \tilde{C}_i\tilde{x}_i(k), \end{cases} \tag{4.3}$$

where both $\tilde{v}_1$ and $\tilde{z}_1$ are constrained. Temporarily dropping the constraint on $\tilde{v}_1$, we can repeat the same procedure based on finding the SCB of $\tilde{\Sigma}_1$ to obtain $\tilde{\Sigma}_2$ from $\tilde{\Sigma}_1$ and so on. At each step of the construction we should make sure that the matrix $\tilde{B}_i$ has full column rank and the matrix $\tilde{C}_i$ has full row rank. This can be done without loss of generality. This chain ends if we obtain a subsystem $\tilde{\Sigma}_i$ which is right invertible in the sense that $\tilde{\Sigma}_{i+1}$ satisfies $\tilde{C}_{i+1} = 0$. Another possibility of termination is that at some step we get $\tilde{B}_i = 0$, which obviously implies that we can end the chain. It can be shown easily that if the pair (A, B) of the given system Σ is stabilizable, then all the systems $\tilde{\Sigma}_i$ as defined in (4.3) are stabilizable.

The following theorem contains some necessary conditions for constrained semi-global stabilization when the system is not right invertible.

Theorem 4.3 Consider the system Σ given by (4.1) while the constraint set $\mathcal{S}$ satisfies Assumption 4.1. Moreover, let the chain of systems $\tilde{\Sigma}_i$ $(i = 1, \ldots s)$ be as described above. Then the semi-global constrained stabilization problem formulated in Problem 4.1 is solvable **only if** the following conditions are satisfied:

1. (A, B) is stabilizable.

2. The constraints of system Σ are at most weakly non-minimum phase.

3. The constraints of system Σ are of type one.

4. All the subsystems $\tilde{\Sigma}_i$ $(i = 1, \ldots, s)$ have at most weakly non-minimum phase constraints.

5. The subsystems $\tilde{\Sigma}_i$ $(i = 1, \ldots, s)$ with realization (4.3) satisfy,

$$\ker \tilde{C}_i \subset \ker \tilde{C}_i \tilde{A}_i . \tag{4.4}$$

We emphasize that the above theorem presents only necessary conditions for the solvability of the semi-global constrained stabilization problem formulated in Problem 4.1. However, it is important to observe that the necessary conditions of Theorem 4.3 do not depend on the shape of the constraint set $\mathcal{S}$.

We proceed now to present the main results, namely the necessary and sufficient conditions for the solvability of the semi-global constrained stabilization problems formulated in Problems 4.1 and 4.2. We first consider the state feedback case pertaining to Problem 4.1.

Theorem 4.4 Consider the constrained system Σ given by (4.1) while the constraint set $\mathcal{S}$ satisfies Assumption 4.1. The constrained semi-global stabilization via state feedback problem (Problem 4.1) is solvable if and only if

1. (A, B) is stabilizable,

2. The constraints are at most weakly non-minimum phase,

3. For any $x \in \mathcal{V}(\mathcal{S})$ there exists u such that $Ax + Bu \in \mathcal{V}(\mathcal{S})$ while $C_z x + D_z u \in \mathcal{S}$.

We consider next the measurement feedback case pertaining to Problem 4.2.

Theorem 4.5 Consider the constrained system Σ given by (4.1) while the constraint set $\mathcal{S}$ satisfies Assumption 4.1. The constrained semi-global stabilization via measurement feedback problem (Problem 4.2) is solvable if

1. (A, B) is stabilizable,
2. The constraints are at most weakly non-minimum phase,
3. For any $x \in \mathcal{V}(\mathcal{S})$ there exists u such that $Ax + Bu \in \mathcal{V}(\mathcal{S})$ while $C_z x + D_z u \in \mathcal{S}$,
4. The pair (C_y, A) is observable.
5. We have
$$\ker C_y \subseteq \ker C_z A,$$
6. We have
$$\ker \begin{pmatrix} C_y & D_y \end{pmatrix} \subseteq \ker \begin{pmatrix} C_z & D_z \end{pmatrix}.$$

Note that in the above theorem, conditions (1), (2) and (3) are necessary. Condition (4) can be weakened by assuming only detectability. However, clearly some additional assumptions would then be needed if undetectable states can affect the constrained output z. However, this is excluded by condition (6). Regarding condition (5) we know that a necessary condition for solvability equals

$$\ker \begin{pmatrix} C_y \\ C_z \end{pmatrix} \subseteq \ker C_z A,$$

which is equal to condition (5) given condition (6). Condition (6) is not necessary but it is a natural condition to impose that the constrained variables z are part of the observations variables y, which is another way to express condition (6).

4.6. Proofs for the state feedback case

Using the SCB as introduced in the previous section, we can decompose the original system into two subsystems:

$$\Sigma_1 : \begin{cases} x_a(k+1) = A_{aa}x_a(k) + K_{ab}C_b x_b(k) + K_{a2}\zeta(k) \\ x_b(k+1) = A_{bb}x_b(k) + K_{bb}C_b x_b(k) + K_{b2}\zeta(k) \\ x_d(k+1) = A_{dd}x_d(k) + B_d[u_d(k) + G\bar{x}(k)] + K_d\bar{z}(k) \\ \quad \zeta(k) \quad = \begin{pmatrix} 0 \\ C_d \end{pmatrix} x_d(k) + \begin{pmatrix} I \\ 0 \end{pmatrix} u_0(k) \\ \quad \bar{z}(k) \quad = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta(k), \end{cases} \tag{4.5}$$

and

$$\Sigma_2 : \{ \; x_c(k+1) = A_{cc}x_c(k) + B_c[u_c(k) + J_a x_a(k)] + K_c\bar{z}(k). \tag{4.6}$$

Our design methodology will amount to designing a controller for the system Σ_1. Finding a controller for Σ_2 does not affect Σ_1 nor the constraints. We set

$$u_c = -J_a x_a(k) + F_c x_c(k),$$

where F_c is any matrix for which $A_{cc} + B_c F_c$ is asymptotically stable.

We first establish the necessity of the conditions in Theorem 4.4.

Lemma 4.1 Consider the constrained system (4.1). The constrained semi-global stabilization via state feedback problem is solvable only if:

1. The constraints are weakly non-minimum phase.

2. For any $x \in \mathcal{V}(\mathcal{S})$ there exists u such that $Ax + Bu \in \mathcal{V}(\mathcal{S})$ while $C_z x + D_z u \in \mathcal{S}$.

Moreover, condition (2) implies the following:

3. The constraints are weakly non-right invertible, i.e. the matrix C_b is injective.

4. The constraints are of type one, i.e. the matrix C_d is injective.

Proof : We first note that the first equation of Σ_1 reads as

$$x_a(k+1) = A_{aa}x_a(k) + K_{ab}C_b x_b(k) + K_{a2}\zeta(k).$$

Clearly $\bar{x}(0) \in \mathcal{V}(\mathcal{S})$ still allows for an arbitrary initial condition $x_a(0)$ for this system (provided we choose the other initial conditions for x_b, x_c and x_d appropriately). On the other hand, the inputs $C_b x_b$ and ζ to this system are bounded. It is well known from the theory for linear systems subject to input constraints that if the system is exponentially unstable there exist initial conditions $x_a(0)$ for which there exists no $C_b x_b$ and ζ such that x_a converges to zero. This is clearly in contradiction with the requirements for the semi-global constrained stabilization problem. Therefore A_{aa} must have its eigenvalues in the closed unit disc or, equivalently, the constraints are at most weakly non-minimum phase.

Condition (2) is clearly necessary since the existence of $x_0 \in \mathcal{V}(\mathcal{S})$ for which there does not exist any u such that $Ax_0 + Bu \in \mathcal{V}(\mathcal{S})$ implies the existence of a $\tilde{x}_0 \in \operatorname{int} \mathcal{V}(\mathcal{S})$ for which there does not exist any u such that $A\tilde{x} + Bu \in \mathcal{V}(\mathcal{S})$ because the set $\mathcal{V}(\mathcal{S})$ is closed. But clearly semi-global stabilization requires that for all initial conditions in the interior of $\mathcal{V}(\mathcal{S})$ we must be able to avoid constraint violation. The fact that we cannot guarantee for initial condition $\tilde{x}$ that $x(1) \in \mathcal{V}(\mathcal{S})$ implies that we will get a constraint violation which yields a contradiction.

Assume that the constraints are not weakly non-right invertible or, in other words, the matrix C_b is not injective. In that case, we can find a x_b such that $C_b x_b = 0$. However, since (C_b, A_{bb}) is observable there always exists some $k < n$ such that $C_b A_{bb}^k x_b(0) \neq 0$. Without loss of generality, choose k such that $C_b A_{bb}^{k-1} x_b = 0$. But then the initial condition,

$$\bar{x}_0 = \begin{pmatrix} 0 \\ \lambda A_{bb}^{k-1} x_b \\ 0 \\ 0 \end{pmatrix},$$

is in $\mathcal{V}(\bar{\mathcal{S}})$ for all λ. However,

$$\bar{z}(1) = \begin{pmatrix} \lambda C_b A_{bb}^k x_b + C_b K_{bb} C_b x_b(0) + C_b K_{b2} \zeta(0) \\ \zeta(1) \end{pmatrix}$$

will not be in $\bar{\mathcal{S}}$ for sufficiently large λ since $\mathcal{S}$ is bounded. After all $\lambda C_b A_{bb}^k x_b$ can be made arbitrarily large by choosing λ large while all other terms are bounded since we know that $\bar{z}(0) \in \mathcal{S}$. Therefore $\bar{z}(1)$ is not in $\bar{\mathcal{S}}$ for any input even though $\bar{x}(0) \in \mathcal{V}(\bar{\mathcal{S}})$. This violates condition (2).

In order to establish (4), we first assume that there exists an $\tilde{x}_d$ such that $C_d \tilde{x}_d = 0$ while $C_d A_{dd} \tilde{x}_d + C_d B_d u \neq 0$ for any u. For any λ there exists $\bar{x}(0)$ in $\mathcal{V}(\bar{\mathcal{S}})$ which yields initial condition $x_d(0) = \lambda \tilde{x}_d$

for this system (provided we choose the other initial conditions for x_b, x_c and x_d appropriately). But then

$$C_d x_d(1) = \lambda C_d A_{dd} \tilde{x}_d + C_d B_d[u_d(0) + G\bar{x}(0)] + C_d K_d \bar{z}(0)$$

can be made arbitrary large independent of our choice for $u_d(0)$ since the second term on the right cannot cancel the first term while the third term on the right is bounded. This violates condition (2). On other hand, if for all x_d satisfying $C_d x_d = 0$ there exists a u such that $C_d A_{dd} x_d + C_d B_d u = 0$, then there exists a matrix F such that for all x_d such that $C_d x_d = 0$ we have $C_d(A_{dd} + B_d F)x_d = 0$. But this in turn implies that $C_d B_d v = 0$ for some $v \neq 0$ yields that $C_d(A_{dd} + BF)^k Bv = 0$ for all k which is in contradiction with the left-invertibility of $(A_{dd}, B_d, C_d, 0)$. Hence $C_d B_d$ is injective (which implies the infinite zeros are at most of order 1). The structure of the SCB then also guarantees that C_d is injective.

■

Since the eigenvalues of A_{aa} must be in the closed unit disc, it can be established that the critical part of the system is actually the x_b and x_d dynamics presented in the following subsystem:

$$\Sigma_{bd} : \begin{cases} x_b(k+1) = A_{bb}x_b(k) + K_{bb}C_b x_b(k) + K_{b2}\zeta(k) \\ x_d(k+1) = A_{dd}x_d(k) + B_d[u_d(k) + G\bar{x}(k)] + K_d\bar{z}(k) \\ \zeta \quad = \begin{pmatrix} 0 \\ C_d \end{pmatrix} x_d(k) + \begin{pmatrix} I \\ 0 \end{pmatrix} u_0(k) \\ \bar{z}(k) \quad = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta(k). \end{cases} \tag{4.7}$$

We define the admissible set for subsystem Σ_{bd} as

$$\mathcal{V}_{bd}(\bar{\mathcal{S}}) := \left\{ x_b \in \mathbb{R}^{n_b}, x_d \in \mathbb{R}^{n_d} \mid \exists u_0 \quad \text{such that} \quad \begin{pmatrix} C_b x_b \\ u_0 \\ C_d x_d \end{pmatrix} \in \bar{\mathcal{S}} \right\}. \tag{4.8}$$

In order to establish sufficiency of the conditions of Theorem 4.4, we will construct an appropriate controller for the system Σ. We will start the construction of this controller by determining an appropriate controller for the system Σ_{bd}. Using condition (3), it is not difficult to construct a controller such that the state cannot leave the set $\mathcal{V}_{bd}(\bar{\mathcal{S}})$. However, to establish the convergence to the origin, we need to do some extra work which is presented in the following lemma.

Lemma 4.2 The constrained semi-global stabilization problem for Σ_{bd} is solvable by a static state feedback,

$$u_0 = \bar{f}_1(x_b, x_d) \ \text{ and } \ u_d = \bar{f}_2(x_b, x_d) - G\bar{x},$$

if the conditions of Theorem 4.4 are satisfied.

Proof : We first define a modified system:

$$\Sigma_{bd}^{\ell} : \begin{cases} x_b^{\ell}(k+1) = (1+\ell)(A_{bb} + K_{bb}C_b)x_b^{\ell}(k) + K_{b2}\zeta^{\ell}(k) \\ x_d^{\ell}(k+1) = (1+\ell)A_{dd}x_d^{\ell}(k) + B_d\tilde{u}_d^{\ell}(k) + K_d\bar{z}^{\ell}(k) \\ \quad \zeta(k) \quad = \begin{pmatrix} 0 \\ C_d \end{pmatrix} x_d^{\ell}(k) + \begin{pmatrix} I \\ 0 \end{pmatrix} u_0^{\ell}(k) \\ \quad \bar{z}^{\ell}(k) \quad = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b^{\ell}(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta^{\ell}(k). \end{cases}$$

Let $\mathcal{R}_{c,bd}^{\ell}(\bar{\mathcal{S}})$ be the largest set of initial conditions for the system Σ_{bd}^{ℓ} for which there exists an input such that the constraints are satisfied while we stay inside the set for all k. Condition (2) of Theorem 4.4 implies that for all

$$x_{bd}^0(0) = \begin{pmatrix} x_b^0(0) \\ x_d^0(0) \end{pmatrix} \in \mathcal{V}_{bd}(\bar{\mathcal{S}}),$$

there exist $u_0^0(0)$ and $\tilde{u}_d^0(0)$ such that

$$x_{bd}^0(1) = \begin{pmatrix} x_b^0(1) \\ x_d^0(1) \end{pmatrix} \in \mathcal{V}_{bd}(\bar{\mathcal{S}}) \text{ and } \bar{z}^0(0) \in \bar{\mathcal{S}}.$$

This might no longer be the case for $\ell > 0$, but we do claim that for any ρ there exists an $\ell > 0$ sufficiently small such that

$$\rho\mathcal{V}_{bd}(\bar{\mathcal{S}}) \subset \mathcal{R}_{c,bd}^{\ell}(\bar{\mathcal{S}}) \subset \mathcal{V}_{bd}(\bar{\mathcal{S}}). \tag{4.9}$$

It is trivial to see that

$$\mathcal{R}_{c,bd}^{\ell}(\bar{\mathcal{S}}) \subset \mathcal{V}_{bd}(\bar{\mathcal{S}}).$$

It remains to establish that

$$\rho\mathcal{V}_{bd}(\bar{\mathcal{S}}) \subset \mathcal{R}_{c,bd}^{\ell}(\bar{\mathcal{S}}).$$

Let

$$x_{bd}^{\ell}(k) = \begin{pmatrix} x_1^{\ell}(k) \\ x_2^{\ell}(k) \end{pmatrix} \quad \text{and} \quad \bar{u}^{\ell}(k) = \begin{pmatrix} u_0^{\ell}(k) \\ u_d^{\ell}(k) \end{pmatrix},$$

where x_1^{ℓ} is controllable and x_2^{ℓ} is uncontrollable.

Choose $R > n$. Consider time $r > R$ for the system Σ_{bd}^{ℓ}. Since the system is controllable there exists a $\delta > 0$ such that for any x_1 for which there exists x_2 such that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \delta\mathcal{V}_{bd}(\bar{\mathcal{S}})$$

there exists, for $x^\ell_{bd}(0) = 0$, an input $\bar{u}^\ell := (u^\ell_0, \tilde{u}^\ell_d)$ such that $x^\ell_1(r) = -x_1$ and $x^\ell_2(r) = 0$ while

$$\bar{z}^\ell(k) \in \tfrac{1-\rho}{2}\mathcal{S}, \qquad k = 0, 1, \dots, r-1.$$

Moreover, we can choose δ independent of ℓ and r provided ℓ is small enough.

Let $r > n$ be such that for any $x^\ell_{bd}(0) \in \mathcal{V}_{bd}(\bar{\mathcal{S}})$ we have that

$$\begin{pmatrix} 0 \\ x^\ell_2(r) \end{pmatrix} \in \delta\rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$$

for all ℓ sufficiently small. This is clearly possible due to the fact that the system is stabilizable and hence the uncontrollable dynamics of x^ℓ_2 must be asymptotically stable.

Consider any initial condition $x_{bd}(0) \in \mathcal{V}_{bd}(\bar{\mathcal{S}})$. We have an input $\bar{u}^0$ for the system Σ^0_{bd} such that $\bar{z}^0(k) \in \bar{\mathcal{S}}$. Hence for any $\rho < 1$ we can find, for any initial condition $x^0_{bd}(0) \in \rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$, an input $\bar{u}^0$ for the system Σ^0_{bd} such that $\bar{z}^0(k) \in \rho\bar{\mathcal{S}}$ for all k. But then for ℓ small enough we find that there exists $\bar{u}_1$ for which we have $\bar{x}^\ell_{bd}(k) \in (1+\delta)\rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$ and $\bar{z}^\ell(k) \in (1+\delta)\rho\bar{\mathcal{S}}$ for $k = 0, \dots, r$. Choose

$$x_1 = \delta x^\ell_1(r).$$

Choose input $\bar{u}_2$ such that, for $x^\ell_{bd}(0) = 0$, we have $x^\ell_1(r) = -x_1$ and $x^\ell_2(r) = 0$ while $\bar{z}^\ell(k) \in \delta\mathcal{S}$. But then for initial condition $x^\ell_{bd}(0) \in \rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$ and input $\bar{u}_1 + \bar{u}_2$ we obtain that

$$\bar{z}^\ell(k) \in \bar{\mathcal{S}} \text{ for } k = 0, \dots, r-1,$$

and

$$\begin{aligned} x^\ell_{bd}(r) = (1-\delta)\begin{pmatrix} x^\ell_1(r) \\ x^\ell_2(r) \end{pmatrix} + \delta\begin{pmatrix} 0 \\ x^\ell_2(r) \end{pmatrix} &\in \left[(1-\delta)(1+\delta)\rho + \delta^2\rho\right]\mathcal{V}_{bd}(\bar{\mathcal{S}}) \\ &\in \rho\mathcal{V}_{bd}(\bar{\mathcal{S}}). \end{aligned}$$

If we repeat this construction between $k = r$ and $k = 2r$ and so forth it becomes clear that we can find for any initial condition

$$\bar{x}^\ell_{bd}(0) \in \rho\mathcal{V}_{bd}(\bar{\mathcal{S}}),$$

an input such that

$$\bar{z}^\ell(k) \in \mathcal{S}$$

for all k. Hence $\bar{x}^{\ell}_{bd}(0) \in \rho\mathcal{R}^{\ell}_{c,bd}(\bar{\mathcal{S}})$. This clearly implies that (4.9) is satisfied.

For semi-global constrained stabilization we take any compact set $\mathcal{H}_{bd}$ contained in the interior of $\mathcal{V}_{bd}(\bar{\mathcal{S}})$ and we construct a static controller which will stabilize the system and the constrained domain of attraction contains $\mathcal{H}$. But then clearly, using (4.9) we can find ℓ such that $\mathcal{H}_{bd} \subset \mathcal{V}^{\ell}_{bd}(\bar{\mathcal{S}})$. Next, we choose a feedback f on the boundary of $\mathcal{V}^{\ell}_{b}(\bar{\mathcal{S}})$ such that, for any $x^{\ell}_{bd}(k) \in \partial\mathcal{V}^{\ell}_{bd}(\bar{\mathcal{S}})$, we have $x^{\ell}_{bd}(k+1) \in \mathcal{V}^{\ell}_{b}(\bar{\mathcal{S}})$. We expand this feedback f to the whole state space. Define $g : \mathbb{R}^n \to \mathbb{R}^+$ such that for any x

$$g(x)x \in \partial\mathcal{V}^{\ell}_{bd}(\bar{\mathcal{S}}).$$

Since $\mathcal{V}^{\ell}_{bd}(\bar{\mathcal{S}})$ is a convex set containing 0 in its interior, this mapping is well-defined. Then we expand f to the whole state space by

$$\bar{f}(x) = \frac{f(g(x)x)}{g(x)}.$$

This expansion has the property that for any $\eta > 0$ we have $x^{\ell}_{bd}(k+1) \in \eta\mathcal{V}^{\ell}_{b}(\bar{\mathcal{S}})$ for all $x^{\ell}_{bd}(k) \in \eta\mathcal{V}^{\ell}_{b}(\bar{\mathcal{S}})$. Note that $\bar{f}$ is positively homogeneous, that is,

$$\bar{f}(\alpha x) = \alpha\bar{f}(x),$$

for any $\alpha > 0$.

Clearly for the system Σ^{ℓ}_{bd} we then have for the feedback

$$u^{\ell}_{0}(k) = \bar{f}_1(x^{\ell}_{bd}(k)) \text{ and } \tilde{u}^{\ell}_{d}(k) = \bar{f}_2(x^{\ell}_{bd}(k)),$$

that for all initial conditions in the set $\mathcal{V}^{\ell}_{bd}(\bar{\mathcal{S}})$ we have $x^{\ell}_{bd}(k) \in \mathcal{V}^{\ell}_{bd}(\bar{\mathcal{S}})$ for all k.

But then the feedback,

$$u_0(k) = \bar{f}_1(x_{bd}(k)) \text{ and } u_d(k) = \bar{f}_2(x_{bd}(k)) - G\bar{x}(k),$$

for the original system with $x_{bd}(0) = x^{\ell}_{bd}(0)$, results in a state

$$x_{bd}(k) = \frac{1}{(1+\ell)^k}x^{\ell}_{bd}(k).$$

Hence, we obviously have $x_{bd}(k) \in \mathcal{V}^{\ell}_{bd}(\bar{\mathcal{S}})$ for all k but also $x_{bd}(k) \to 0$ as $k \to \infty$. ■

Lemma 4.3 The constrained semi-global stabilization problem for Σ_1 is solvable by a static state feedback if the conditions of Theorem 4.4 are satisfied.

Proof : It is easy to verify that the admissible set of initial conditions $\mathcal{V}_1(\bar{\mathcal{S}})$ and $\mathcal{V}_{bd}(\bar{\mathcal{S}})$ for Σ_1 and Σ_{bd} respectively have the relationship,

$$\mathcal{V}_1(\bar{\mathcal{S}}) = \mathbb{R}^{n_a} \oplus \mathcal{V}_{bd}(\bar{\mathcal{S}}).$$

For any compact set $\mathcal{H}$ in $\mathcal{V}_1(\bar{\mathcal{S}})$ we choose a compact set $\mathcal{H}_1$ and $\rho < 1$ such that

$$\mathcal{H} \subset \mathcal{H}_1 \oplus \rho\mathcal{V}_{bd}(\bar{\mathcal{S}}).$$

The controller $u_0 = \bar{f}_1(x_{bd})$ and $u_d = \bar{f}_2(x_{bd}) - G\bar{x}$ is such that for all initial conditions in $\rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$, the origin of the closed-loop system is exponentially stable. Hence there exist $M > 0$ and λ with $|\lambda| < 1$ such that

$$\|x_{bd}(k)\| \leq M\lambda^k \tag{4.10}$$

for all k and for all $x_{bd}(0) \in \rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$.

Next, let P_0 be the semi-stabilizing solution of the discrete-time algebraic Riccati equation,

$$P_0 = A_0' P_0 A_0 + C_0' C_0 - A_0' P_0 B_0 (B_0' P_0 B_0 + D_0' D_0)^{\dagger} B_0' P_0 A_0,$$

where

$$A_0 = \begin{pmatrix} A_{aa} & K_{ab}C_b & K_{ad}C_d \\ 0 & A_{bb} + K_{bb}C_b & K_{bd}C_d \\ 0 & K_{db}C_b & A_{dd} \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_{a0} & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & C_b & 0 \\ 0 & 0 & C_d \\ 0 & 0 & 0 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{pmatrix}.$$

We have

$$P_0 \begin{pmatrix} x_a \\ 0 \\ 0 \end{pmatrix} = 0 \tag{4.11}$$

for all $x_a \in R_{na}$ since the eigenvalue of A_{aa} are in the closed unit disc. Choose a level set,

$$V_0(c) := \{\, \xi \in \mathbb{R}^{n_1} \mid \xi(k)' P_0 \xi(k) \leq c \,\},$$

such that we have

$$(C_0 + D_0(B_0' P_0 B_0 + D_0' D_0)^{\dagger} B_0' P_0 A_0)\xi \in \bar{S}/3 \tag{4.12}$$

for all $\xi \in V_0(c)$. Then with the controller,

$$u_0 = \bar{f}_1(x_{bd}) \text{ and } u_d = \bar{f}_2(x_{bd}) - G\bar{x},$$

which we can abbreviate as

$$\begin{pmatrix} u_0 \\ u_d \end{pmatrix} = \bar{f}(\bar{x}),$$

there exists a T such that, for any initial state in

$$\mathcal{H}_a \oplus \rho \mathcal{V}_{bd}(\bar{\mathcal{S}}),$$

we have

$$\begin{pmatrix} x_a(T) \\ x_{bd}(T) \end{pmatrix} \in V_0(c). \tag{4.13}$$

Let P_ε be the stabilizing solution of the algebraic equation,

$$P_\varepsilon = A_0' P_\varepsilon A_0 + C_0' C_0 + \varepsilon I - A_0' P_\varepsilon B_0 (B_0' P_\varepsilon B_0 + I)^{-1} B_0' P_\varepsilon A_0.$$

We have $P_\varepsilon \to P_0$ as ε approach zero. Define the level set,

$$V_\varepsilon(c) := \{\, \xi \in \mathbb{R}^{n_1} \mid \xi' P_\varepsilon \xi \le c \,\},$$

such that for ε small enough we have

$$\begin{pmatrix} x_a(T) \\ x_{bd}(T) \end{pmatrix} \in 2V_\varepsilon(c),$$

and

$$(C_0 - D_0(B_0' P_\epsilon B_0 + D_0' D_0)^{\dagger} B_0' P_\epsilon A_0)\xi \in \bar{S}$$

for any initial condition $\xi \in 2V_\varepsilon(c)$. Hence the feedback

$$\begin{pmatrix} u_0 \\ u_d \end{pmatrix} = -(B_0' P_\varepsilon B_0 + D_0' D_0)^{\dagger} B_0' P_\varepsilon A_0 \begin{pmatrix} x_a \\ x_{bd} \end{pmatrix}$$

is an asymptotically stabilizing controller for Σ_1, and achieves a domain of attraction containing $2V_\varepsilon(c)$.

Next, consider the controller,

$$\begin{pmatrix} u_0 \\ u_d \end{pmatrix} = \begin{cases} \bar{f}(\bar{x}), & x_{abd} \notin 2V_\varepsilon(c) \\ -(B_0' P_\varepsilon B_0 + D_0' D_0)^{\dagger} B_0' P_\varepsilon A_0 x_{abd}, & x_{abd} \in 2V_\varepsilon(c). \end{cases}$$

It is easily verified that this controller asymptotically stabilizes the system. ■

The above lemma yields an appropriate controller for the subsystem Σ_2. Finally, we need to construct a controller for the original system Σ which will complete our proof of sufficiency for Theorem 4.4.

Proof of Theorem 4.4 : The necessity was already established in Lemma 4.1. For sufficiency, it is easily seen that the controllers designed in Lemma 4.3 combined with a controller,

$$u_c(k) = -J_a x_a(k) + F_c x_c(k),$$

where F_c is such that $A_{cc} + B_c F_c$ is asymptotically stable, solve the semi-global constrained stabilization problem via state feedback. ■

4.7. Proofs for the measurement feedback case

Note that the assumptions of Theorem 4.5 imply that x_b and x_d can be directly deduced from the measurements. In other words, we have (in a suitable basis):

$$y(k) = \begin{pmatrix} y_1(k) \\ y_2(k) \\ y_3(k) \end{pmatrix} = \begin{pmatrix} C_{ya} & 0 & C_{yc} & 0 \\ 0 & C_b & 0 & 0 \\ 0 & 0 & 0 & C_d \end{pmatrix} \begin{pmatrix} x_a(k) \\ x_b(k) \\ x_c(k) \\ x_d(k) \end{pmatrix} + \begin{pmatrix} D_y \\ 0 \\ 0 \end{pmatrix} u_0,$$

where C_b and C_d are injective. However, we need an observer to estimate x_a and x_c:

$$\begin{aligned} \begin{pmatrix} \hat{x}_a(k+1) \\ \hat{x}_c(k+1) \end{pmatrix} = \begin{pmatrix} A_{aa} & 0 \\ B_c J_a & A_{cc} \end{pmatrix} \begin{pmatrix} \hat{x}_a(k) \\ \hat{x}_c(k) \end{pmatrix} + \begin{pmatrix} K_{ab} & K_{a2} & 0 \\ K_{c1} & K_{c2} & B_c \end{pmatrix} \begin{pmatrix} C_b x_b(k) \\ \zeta(k) \\ u_c(k) \end{pmatrix} \\ + \begin{pmatrix} L_a \\ L_c \end{pmatrix} \left[y_1(k) - C_{ya}\hat{x}_a(k) - C_{yc}\hat{x}_b(k) - D_y u_0(k) \right]. \end{aligned}$$

Clearly,

$$\begin{aligned} \begin{pmatrix} \hat{x}_a - x_a \\ \hat{x}_c - x_c \end{pmatrix}(k+1) &= \begin{pmatrix} A_{aa} - L_a C_{ya} & -L_a C_{yc} \\ B_c J_a - L_c C_{ya} & A_{cc} - L_c C_{yc} \end{pmatrix} \begin{pmatrix} \hat{x}_a - x_a \\ \hat{x}_c - x_c \end{pmatrix}(k) \\ &= \tilde{A}_{ac} \begin{pmatrix} \hat{x}_a - x_a \\ \hat{x}_c - x_c \end{pmatrix}(k), \end{aligned}$$

where L_a and L_c are chosen such that $\tilde{A}_{ac}$ is asymptotically stable.

The feedback $\bar{f}$ can be directly implemented even in the measurement feedback case since the condition (5) of Theorem 4.5 guarantees that

$$G = \begin{pmatrix}0 & G_b & 0 & G_d\end{pmatrix} + L_G \begin{pmatrix}C_{ya} & 0 & C_{yc} & 0\end{pmatrix}.$$

Hence $u_d = \bar{f}_2(x_{bd}) - G\bar{x}$ is equivalent to

$$u_d = \bar{f}_1(x_{bd}) - G_b x_b - G_b x_d - L_G y_1 + L_G D_y u_0.$$

Next, we follow the same arguments as in Lemma 4.3 with small modifications such as the inclusion of x_c since the observer does not allow a separate controller design for x_c and x_a.

It is easy to verify that the admissible set of initial conditions $\mathcal{V}(\bar{\mathcal{S}})$ and $\mathcal{V}_{bd}(\bar{\mathcal{S}})$ for $\bar{\Sigma}$ and Σ_{bd} respectively have the relationship,

$$\mathcal{V}(\bar{\mathcal{S}}) = \mathbb{R}^{n_a} \oplus \mathbb{R}^{n_c} \oplus \mathcal{V}_{bd}(\bar{\mathcal{S}}).$$

For any compact set $\mathcal{H}$ in $\mathcal{V}(\bar{\mathcal{S}})$ we choose a compact set $\mathcal{H}_1$ and $\rho < 1$ such that

$$\mathcal{H} \subset \mathcal{H}_1 \oplus \rho\mathcal{V}_{bd}(\bar{\mathcal{S}}).$$

The controller $u_0 = \bar{f}_1(x_{bd})$ and $u_d = \bar{f}_2(x_{bd}) - G\bar{x}$ is such that for all initial conditions in $\rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$, the origin of the closed-loop system is exponentially stable. Hence there exist $M > 0$ and λ with $|\lambda| < 1$ such that

$$\|x_{bd}(k)\| \le M\lambda^k \tag{4.14}$$

for all k and for all $x_{bd}(0) \in \rho\mathcal{V}_{bd}(\bar{\mathcal{S}})$.

Next, let P_0 be the semi-stabilizing solution of the discrete-time algebraic Riccati equation,

$$P_0 = A_0' P_0 A_0 + C_0' C_0 - A_0' P_0 B_0 (B_0' P_0 B_0 + D_0' D_0)^{\dagger} B_0' P_0 A_0,$$

where

$$A_0 = \begin{pmatrix} A_{aa} & K_{ab}C_b & 0 & K_{ad}C_d \\ 0 & A_{bb} + K_{bb}C_b & 0 & K_{bd}C_d \\ B_c J_a & K_{cb}C_b & A_{cc} & K_{cd}C_d \\ 0 & K_{db}C_b & 0 & A_{dd} \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_{a0} & 0 & 0 \\ B_{b0} & 0 & 0 \\ B_{c0} & B_c & 0 \\ B_{d0} & 0 & B_d \end{pmatrix},$$

$$C_0 = \begin{pmatrix} 0 & C_b & 0 & 0 \\ 0 & 0 & C_d & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad D_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}.$$

We have

$$P_0 \begin{pmatrix} x_a \\ 0 \\ x_c \\ 0 \end{pmatrix} = 0 \tag{4.15}$$

for all $x_a \in \mathbb{R}^{n_a}$ and $x_c \in \mathbb{R}^{n_c}$ since the eigenvalue of A_{aa} are in the closed unit disc while u_c can stabilize the x_c dynamics without incurring cost. Choose a level set,

$$V_0(c) := \{ \bar{x} \in \mathbb{R}^n \mid \bar{x}(k)' P_0 \bar{x}(k) \le c \},$$

such that we have

$$(C_0 + D_0(B_0' P_0 B_0 + D_0' D_0)^\dagger B_0' P_0 A_0)\bar{x} \in \bar{S}/3 \tag{4.16}$$

for all $\bar{x} \in V_0(c)$. Then with the controller,

$$u_0 = \bar{f}_1(x_{bd}), \quad u_c = 0, \quad u_d = \bar{f}_2(x_{bd}) - G\bar{x},$$

which we can abbreviate as

$$\begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix} = \bar{f}(\bar{x}),$$

there exists a T such that, for any initial state in

$$\mathcal{H}_{ac} \oplus \rho \mathcal{V}_{bd}(\bar{\mathcal{S}}),$$

we have

$$\bar{x}(T) \in V_0(c). \tag{4.17}$$

Let P_ε be the stabilizing solution of the algebraic equation,

$$P_\varepsilon = A_0' P_\varepsilon A_0 + C_0' C_0 + \varepsilon I - A_0' P_\varepsilon B_0 (B_0' P_\varepsilon B_0 + D_0' D_0)^\dagger B_0' P_\varepsilon A_0.$$

We have $P_\varepsilon \to P_0$ as ε approach zero. Defining the level set,

$$V_\varepsilon(c) := \{ \bar{x} \in \mathbb{R}^n \mid \bar{x}' P_\varepsilon \bar{x} \le c \},$$

there exists an ε such that

$$\bar{x}(T) \in \tfrac{4}{3} V_\varepsilon(c).$$

Moreover, we can guarantee that for ε small enough

$$\begin{pmatrix} x_a - \hat{x}_a \\ 0 \\ x_c - \hat{x}_c \\ 0 \end{pmatrix} (k) \in \tfrac{1}{3} V_\varepsilon(c)$$

for all $k > 0$ given initial conditions for the system and the observer in the compact sets $\mathscr{H}$ and $\mathscr{H}_{\text{obs}}$ respectively. For ε small enough, we have

$$(C_0 + D_0(B_0' P_\epsilon B_0 + D_0' D_0)^\dagger B_0' P_\epsilon A_0)\tilde{x} \in \bar{S}$$

for any initial condition $\tilde{x} \in 2V_\varepsilon(c)$. Note that

$$\hat{\hat{x}} = \begin{pmatrix} \hat{x}_a \\ x_b \\ \hat{x}_c \\ x_d \end{pmatrix} \in \frac{4}{3} V_\varepsilon(c)$$

implies that

$$\bar{x} = \begin{pmatrix} \hat{x}_a \\ x_b \\ \hat{x}_c \\ x_d \end{pmatrix} + \begin{pmatrix} x_a - \hat{x}_a \\ 0 \\ x_c - \hat{x}_c \\ 0 \end{pmatrix} (k) \in 2V_\varepsilon(c),$$

and hence the feedback,

$$\begin{aligned} \begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix} &= -(B_0' P_\varepsilon B_0 + D_0' D_0)^\dagger B_0' P_\varepsilon A_0 \hat{\hat{x}} - \begin{pmatrix} 0 \\ 0 \\ G \end{pmatrix} \bar{x} \\ &= F_\varepsilon \hat{\hat{x}} + Ny, \end{aligned}$$

with the associated observer is an asymptotically stabilizing controller for Σ_1 and achieves a domain of attraction containing $\mathscr{H}_{\text{obs}} \oplus 2V_\varepsilon(c)$. Next, consider the following controller,

$$\begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix} = \begin{cases} \bar{f}(\hat{\hat{x}}), & \hat{\hat{x}} \notin \frac{4}{3} V_\varepsilon(c), \\ F_\varepsilon \hat{\hat{x}} + Ny, & \hat{\hat{x}} \in \frac{4}{3} V_\varepsilon(c), \end{cases}$$

together with our observer. It is easily verified that this controller asymptotically stabilizes the given system.

4.8. Example

Consider the following system:

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = C_y x(k) + D_y u(k) \\ z(k) = C_z x(k) + D_z u(k) \end{cases} \tag{4.18}$$

where $x(k) \in \mathbb{R}^4$, $u(k) \in \mathbb{R}^3$, $y(k) \in \mathbb{R}^3$, $z(k) \in \mathbb{R}^3$ and

$$A = \begin{pmatrix} -1 & -1 & 1 & 3 \\ -2 & -1 & 1 & 2 \\ -5.5 & -3 & 2.5 & 5.5 \\ -4.5 & -2 & 1.5 & 6.5 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 5 & 3 \\ 2 & 3 & 3 \\ 7 & 8 & 8 \\ 8 & 11 & 9 \end{pmatrix},$$

$$C_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -0.5 & -1 & 0.5 & 0.5 \\ -0.5 & 1 & -0.5 & 0.5 \end{pmatrix}, \quad D_y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$C_z = \begin{pmatrix} -0.5 & 0 & 0 & 0.5 \\ 0 & 1 & -0.5 & 0 \\ 0 & -1 & 0.5 & 0 \end{pmatrix}, \quad D_z = \begin{pmatrix} -0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \end{pmatrix}.$$

The system is subject to the constraints $z(k) \in \mathcal{S}$ where $\mathcal{S}$ is given by:

$$\mathcal{S} = \left\{ \gamma \in \mathbb{R}^3 \mid \begin{pmatrix} 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \end{pmatrix} \gamma \in [-1,1] \times [-1,1] \times [-1,1] \right\}.$$

The problem is to stabilize the system with a priori given set $\mathcal{W}$ contained in its domain of attarction. Where

$$\mathcal{W} = \left\{ \gamma \in \mathbb{R}^4 \mid \begin{pmatrix} 0.5 & 0 & 0.5 & -0.5 \\ -0.5 & -1.0 & 0.5 & 0.5 \\ 0.5 & 0 & -0.5 & 0.5 \\ -0.5 & 1.0 & -0.5 & 0.5 \end{pmatrix} \gamma \in [-10,10] \times [-1,1] \times [-10,10] \times [-1,1] \right\}.$$

4.8.1. State feedback case

Step 1, it is easy to verify that (A, B) is stabilizable.

Step 2, there exist a state transformation $\bar{x} = T_x x$, a input basis transformation $\bar{u} = T_u u$ and output basis transformation $\bar{z} = T_z z$ that converts the original system into its SCB form. These transformations are given by:

$$\bar{x} = \begin{pmatrix} x_a(k) \\ x_b(k) \\ x_c(k) \\ x_d(k) \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & 0.5 & -0.5 \\ -0.5 & -1.0 & 0.5 & 0.5 \\ 0.5 & 0 & -0.5 & 0.5 \\ -0.5 & 1.0 & -0.5 & 0.5 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \end{pmatrix},$$

$$\bar{u}(k) = \begin{pmatrix} u_0(k) \\ u_c(k) \\ u_d(k) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_1(k) \\ u_2(k) \\ u_3(k) \end{pmatrix}, \quad \bar{z} = \begin{pmatrix} z_b(k) \\ z_d(k) \\ z_0(k) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} z_1(k) \\ z_2(k) \\ z_2(k) \end{pmatrix}.$$

The transformed system is as follows:

$$\tilde{\Sigma} : \begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k) \\ \bar{y}(k) = \bar{C}_y \bar{x}(k) + \bar{D}_y \bar{u}(k) \\ \bar{z}(k) = \bar{C}_z x(k) + \bar{D}_z \bar{u}(k) \end{cases} \tag{4.19}$$

where

$$\bar{A} = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 0 & 2 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$\bar{C}_y = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{D}_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\bar{C}_z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \bar{D}_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

and the system is subject to the constraints $\bar{z}(k) \in \bar{\mathcal{S}}$, where $\bar{\mathcal{S}}$ is given by

$$\bar{\mathcal{S}} = [-1, 1] \times [-1, 1] \times [-1, 1],$$

and $\bar{\mathcal{W}}$ is given by

$$\bar{\mathcal{W}} = [-10, 10] \times [-1, 1] \times [-10, 10] \times [-1, 1].$$

Extract subsystem Σ_1 composed of x_a, x_b and x_d dynamics.

$$\Sigma_1 : \begin{cases} x_{abd}(t) = A_0 x_{abd}(t) + B_0 \bar{u}(t) \\ \bar{z}(t) = C_0 x_{abd}(t) + D_0 \bar{u}(k), \end{cases} \tag{4.20}$$

where

$$A_0 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \quad B_0 = \begin{pmatrix} 1 & 0 \\ 4 & 0 \\ 1 & 1 \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then extract x_d and x_d dynamics from Σ_1 and form Σ_{bd},

$$\Sigma_{bd} : \begin{cases} x_b(k) = 3x_b(k) + 2x_d(k) + 4u_0(k) \\ x_d(k) = x_b(k) + x_d(k) + u_0(k) + u_d(k) \\ \bar{z}(t) = \begin{pmatrix} x_b(k) \\ x_d(k) \\ u_0(k) \end{pmatrix}. \end{cases} \tag{4.21}$$

Step 3, design a state feedback for subsystem Σ_{bd}. A suitable controller is given by

$$\begin{pmatrix} u_0(k) \\ u_d(k) \end{pmatrix} = \bar{f}(x_{bd}(k)) = \begin{pmatrix} -\frac{5}{8}x_b(x) - \frac{3}{8}x_d(k) \\ -\frac{3}{8}x_b(k) \end{pmatrix}.$$

Step 4, design state feedback controller for subsystem Σ_{abd}. Let P_0 be the sem-stabilizing solution of the discrete-time algebraic Riccati equation

$$P_0 = A_0' P_0 A_0 + C_0' C_0 - A_0' P_0 B_0 (B_0' P_0 B_0 + D_0' D_0)^{\dagger} B_0' P_0 A_0,$$

we have

$$P_0 = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.5390 & 0.3593 \\ 0.0000 & 0.3593 & 1.2396 \end{pmatrix}.$$

Then choose $c = 0.2$ and $\epsilon = 0.000004$. P_ϵ is the stabilizing solution of the algebraic Riccati equation,

$$P_\epsilon = A_0' P_\epsilon A_0 + C_0' C_0 + \epsilon I - A_0' P_\epsilon B_0 (B_0' P_\epsilon B_0 + D_0' D_0)^\dagger B_0' P_\epsilon A_0$$

We have

$$P_\epsilon = \begin{pmatrix} 0.0015 & 0.0021 & 0.0009 \\ 0.0021 & 1.5421 & 0.3607 \\ 0.0009 & 0.3607 & 1.2401 \end{pmatrix}.$$

Choose the level set

$$V_\epsilon(c) = \left\{ \xi \in \mathbb{R}^3 \mid \xi' P_\epsilon \xi < c \right\}.$$

Design low gain feedback $F_\epsilon = -(B_0' P_\epsilon B_0 + D_0' D_0)^\dagger B_0' P_\epsilon A_0$, that is

$$F_\epsilon = \begin{pmatrix} 0.0004 & 0.7192 & 0.4794 \\ -0.0001 & 0.3175 & 0.5450 \end{pmatrix}.$$

Hence, the state feedback for Σ_1 is given by

$$\begin{pmatrix} u_0 \\ u_d \end{pmatrix} = \begin{cases} \bar{f}(x_{bd}), & x_{abd} \notin 2V_\epsilon(c) \\ F_\epsilon x_{abd}, & x_{abd} \in 2V_\epsilon(c). \end{cases}$$

Step 5, design the state feedback controller for entire system. Let

$$u_c(k) = -u_0(k) - x_a(k) - x_b(k) - \tfrac{3}{2} x_c(k) - x_d(k).$$

Then the controller designed in step 4 combined with this controller solves the semi-global constraint stabilization problem for this system and we denote this controller as

$$\begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix} = \hat{f}(\bar{x}).$$

The simulation data for state feedback case is shown in Figure 1.

4.8.2. Measurement feedback case

Steps 1 and 2 are identical to state feedback case.

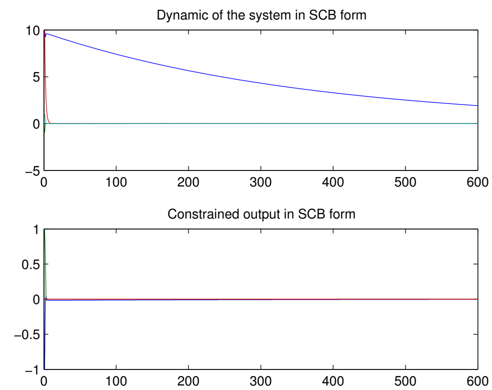


Figure 4.1: State feedback case

Step 3, we have

$$\bar{y}(k) = \bar{C}_y \bar{x} + \bar{D}_y \bar{u}.$$

Replace the x_b and x_d in the controller designed for state feedback with the observation $\hat{x}_b$ and $\hat{x}_d$ which can be directly determined from the measurement.

Step 4, design an observer for x_a and x_c. The observer is given by

$$\begin{pmatrix} \hat{x}_a(k+1) \\ \hat{x}_c(k+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \hat{x}_a(k) \\ \hat{x}_c(k) \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_b(k) \\ x_d(k) \\ u_0(k) \\ u_c(k) \end{pmatrix} + \begin{pmatrix} \frac{3}{4} \\ \frac{9}{4} \end{pmatrix} \left[\bar{y}_1(k) - \hat{x}_c(k) - u_0(k) \right].$$

Step 5, design measurement feedback for entire system. Denote

$$\hat{\bar{x}} = \begin{pmatrix} \hat{x}_a \\ x_b \\ \hat{x}_c \\ x_d \end{pmatrix}.$$

Let P_0 be the solution of algebraic Riccati equation,

$$P_0 = \bar{A}' P_0 \bar{A} + \bar{C}_z' \bar{C}_z - \bar{A}' P_0 \bar{B} (\bar{B}' P_0 \bar{B} + \bar{D}_z' \bar{D}_z)^{\dagger} \bar{B}' P_0 \bar{A}.$$

We have

$$P_0 = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 1.5390 & 0.0000 & 0.3593 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.3593 & 0.0000 & 1.2396 \end{pmatrix}.$$

Choose $c = 0.2$ and $\epsilon = 0.000004$. P_ϵ is stabilizing solution of the algebraic Riccati equation,

$$P_\epsilon = \bar{A}' P_\epsilon \bar{A} + \bar{C}_z' \bar{C}_z + \epsilon I - \bar{A}' P_\epsilon \bar{B}(\bar{B}' P_\epsilon \bar{B} + \bar{D}_z' \bar{D}_z)^\dagger \bar{B}' P_\epsilon \bar{A}.$$

We have

$$P_\epsilon = \begin{pmatrix} 0.0015 & 0.0021 & 0.0000 & 0.0009 \\ 0.0021 & 1.5421 & 0.0000 & 0.3608 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.0009 & 0.3608 & 0.0000 & 1.2401 \end{pmatrix}.$$

Choose the level set

$$V_\epsilon(c) = \left\{ \xi \in \mathbb{R}^4 \mid \xi' P_\epsilon \xi < c \right\}.$$

Design low gain feedback $F_\epsilon = -(\bar{B}' P_\epsilon \bar{B} + \bar{D}_z' \bar{D}_z)^\dagger \bar{B}' P_\epsilon \bar{A}$, that is

$$F_\epsilon = \begin{pmatrix} 0.0004 & 0.7192 & 0.0000 & 0.4794 \\ 0.9996 & 0.2808 & 1.0000 & 0.5206 \\ -0.0001 & 0.3175 & 0.0000 & 0.5450 \end{pmatrix}.$$

Then the following measurement feedback controller solves the semi-global stabilization problem:

$$\begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix} = \begin{cases} \hat{f}(\hat{\hat{x}}), & \hat{\hat{x}} \notin \frac{4}{3} V_\epsilon(c) \\ F_\epsilon \hat{\hat{x}}, & \hat{\hat{x}} \in \frac{4}{3} V_\epsilon(c) \end{cases}$$

where $\hat{f}$ is the controller designed in statefeedback case. The simulation data for measurement feedback case is shown in Figure 2.

4.9. Rate constraints

In the papers [87] (continuous-time) and [88] (discrete-time), we considered systems where the constrained output was subject to both amplitude and rate constraints. Many papers do not consider rate constraints separately since one can convert rate to amplitude constraints through a system expansion. Consider the system,

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad y(k) \quad = C_y x(k) + D_y u(k) \\ \quad z(k) \quad = C_z x(k) + D_z u(k), \end{cases} \tag{4.22}$$

subject to the constraint $z(k) \in \mathcal{S}$ and $z(k+1) - z(k) \in \mathcal{T}$ for all $k \geq 0$. This is clearly equivalent to

$$\Sigma : \begin{cases} x(k+1) &= Ax(k) + Bx_1(k) \\ x_1(k+1) &= x_1(k) + v(k) \\ y(k) &= C_y x(k) + D_y x_1(k) \\ z(k) &= C_z x(k) + D_z x_1(k) \\ z_1(k) &= C_z(A - I)x(k) + C_z B x_1(k) + D_z v(k), \end{cases} \tag{4.23}$$

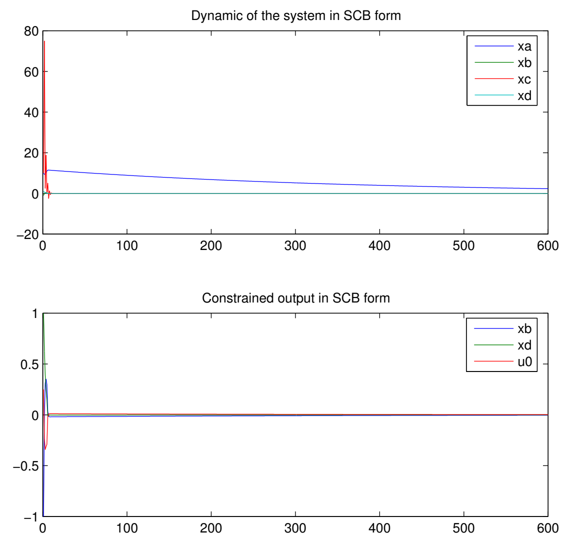


Figure 4.2: Measurement feedback case

provided $x_1(0) = u(0)$ and $v(k) = u(k+1) - u(k)$. Moreover the constraints of the original system now convert to amplitude constraints: $z(k) \in \mathcal{S}$ and $z_1(k) \in \mathcal{T}$ for all $k \geq 0$.

There are however two main drawbacks to this approach. First of all, if we have right-invertible constraints then this expansion will result in a system with non-right-invertible constraints. Hence, for systems with right-invertible constraints the direct approach of [87, 88] is to be preferred.

The second drawback of this expansion is the choice of the initial condition. If we can avoid constraint violations for a certain initial condition $x(0)$ for the system (4.22) then there exists an initial condition $x_1(0) = u(0)$ for the system (4.23) such that constraint violations can be avoided. It yields conservative results if we impose semi-global stabilization in the admissible set for the expanded system since we then suddenly need to guarantee that we can avoid constraint violations for *all* initial conditions for $x_1(0)$ in the admissible set.

The following example illustrates the difficulties with the direct approach in the non-right-invertible case. It also illustrates the difficulties in choosing $u(0)$ appropriately which shows up in the expanded system.

Example 4.1 Consider the system:

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \alpha & \beta \\ 0 & \beta & \alpha \end{pmatrix} x(k) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u$$

where $\beta = \sqrt{1-\alpha^2}$ with the constraints:

$$x_1(k) \in [-1, 1],$$

$$u_2(k) \in [-\varepsilon, \varepsilon],$$

$$u_1(k+1) - u_1(k) \in [-\delta, \delta],$$

and

$$x_2^2(k) + x_3^2(k) \leq 7.$$

We claim that for any $r > 0$ there exists ε, α and δ such that for all initial conditions in the admissible set there exists an input which does not violate constraints in the first r steps even though there does exist an initial condition in the admissible set for which we will get a constraint violation at some time $k > r$ for any input.

Choose α so close to 1 that $x_2(k) - x_2(0) \in [-1, 1]$ for all initial conditions with $k < r$. On the other hand choose α such that for $x_2(0) = 7$ and $x_3(0) = 0$ we have $x(5r) = 3$ for $u_2(k) = 0$. This is clearly possible.

If $u_1(k) = -x_2(0)$ for $k = 0, \ldots, r$ and $u_2(0) = 0$ then we will not have any constraint violation in the first r steps.

On the other hand for initial condition $x_2(0) = \sqrt{7}$ and $x_3(0) = 0$ we need to choose $u_2(0) \in [-8, -6]$. By choosing δ small enough we must have that $u_1(5r) \in [-\frac{17}{2}, -\frac{11}{2}]$. On the other hand for ε small enough we will have $x_2(5r) \in [\frac{5}{2}, \frac{7}{2}]$ for any choice of u_2 satisfying the constraints. But then

$$x_1(4n+1) \in [-6, -2]$$

and hence we have a constraint violation.

In our main results of Theorems 4.4 and 4.5, we established that guaranteeing the fact that, for all initial conditions in the admissible set, we will still be in the admissible set *one* time step later, implies that we can stay in the admissible set forever without any constraint violations. In contrast, in the case of rate constraints a guarantee of the fact that, for all initial conditions in the admissible set, we will still be in the admissible set after r steps (with r being arbitrarily large), does *not* imply that we can stay in the admissible set forever without any constraint violations. This shows that for rate constraints we need a different approach which will be dependent on a suitable choice for the input at time 0.

4.10. Conclusions

For discrete-time time-invariant linear systems subject to non-right-invertible constraints, necessary and sufficient conditions are developed under which semi-global stabilization with respect to the admissible set can be achieved by state feedback. Sufficient conditions are also developed for such a stabilization however by utilizing measurement feedback. Such sufficient conditions are almost necessary. Controllers for both state feedback and measurement feedback are constructed as well.

CHAPTER 5

Computation of the recoverable region and stabilization problem in the recoverable region for discrete-time systems

5.1. Introduction

For over a decade, Saberi and Stoorvogel as well as their coworkers and students, have focused heavily in resolving most of the issues related to global and semi-global stabilization of linear systems subject to both input and state constraints (see for initial work [51] in the presence of input constraints, and the recent work in the presence of both input and state constraints that culminated in [87, 88, 97, 145] and the references there in). For the most part, their effort was concentrated on global and semi-global stabilization in the admissible set. The admissible set is defined as the set of initial conditions that do not violate the constraints at time 0. Necessary and sufficient conditions as well as appropriate controllers for global and semi-global stabilization in the admissible set have been developed. It turns out that invariant zeros, infinite zeros and right-invertibility properties of the subsystem from control input to the constrained output play a crucial role. In [87] these invariant zeros and infinite zeros are labeled as constraint invariant zeros and constraint infinite zeros. More specifically, the taxonomy of constraints presented in [87, 88] delineates the constraints into several categories, such as right and non-right invertible constraints, minimum phase, at most weakly non-minimum phase, strongly non-minimum phase constraints, and type one constraints, etc. For systems with right invertible constraints (the subsystem from control input to the constrained output is right invertible), it is shown that the necessary conditions for global and semi-global stabilization are that the system is stabilizable and the constraints are at most weakly non-minimum phase (i.e. the constraint invariant zeros are in the closed left-half plane (continuous-time) or in the closed unit disc (discrete-time)). Moreover for global stabilization one needs an additional condition that the constraints be of type one (the order of all constraint infinite zeros is less than or equal to one). For constraints that are right invertible and at most weakly non-minimum phase, it is possible to achieve semi-global stabilization by a linear control; however, in general one has to use nonlinear control laws for global stabilization [87, 88]. For the case of non-right invertible constraints, the complete development of necessary and sufficient conditions for global and semi-global stabilization turns out to be very complex and challenging. Nevertheless, such conditions have been developed in [97] for continuous-time and in [145] for discrete-time.

Out of the work that is described above, one can establish two general properties:

- There exists a perceptible demarcation line between the right and non-right invertible constraints. In particular, the solvability conditions for the semi-global and global stabilization problems in the admissible set via state feedback do not depend on the shape of the constraint sets for right invertible constraints, whereas for non-right invertible constraints they indeed do so.
- Neither the semi-global nor the global stabilization problem in the admissible set is solvable whenever the constraints are strongly non-minimum phase (at least one of the constraint invariant zeros is in the right half complex plane or outside the unit disc).

In view of the above properties, especially the later one, the notion of *recoverable region (set)*, sometimes also called as the domain of null controllability or null controllable region, arises in stabilizing linear systems subject to constraints. Generally speaking, for a system with constraints, an initial state is said to be *recoverable* if it can be driven to zero by some control without violating the constraints on the state and input. The set of all recoverable initial conditions denoted by $\mathcal{R}_C$ is said to be the recoverable region. The recoverable region is thus indeed the maximum achievable domain of attraction in stabilizing linear systems subject to non-minimum phase constraints.

Paper [116] considers continuous-time systems which do not satisfy the solvability conditions for semi-global stabilization in the admissible set, for instance, because the constraints are strongly non-minimum phase. For such systems,

- it develops methods for constructing the recoverable region $\mathcal{R}_C$, and then
- develops methods of constructing controllers that achieve semi-global stabilization in the recoverable region.

Also, let us emphasize that [116] provide a reduction in computation and removal of some of the computational complexity involved in obtaining the recoverable regions.

The focus of this chapter is to address the above stated issues however for discrete-time systems. Although the development for discrete-time systems parallels somewhat that in continuous-time systems, there are two fundamental differences between continuous- and discrete-time systems:

1. The methods of constructing recoverable regions as well as appropriate controllers need to be greatly revised as needed, and
2. Some new issues arise, which do not exist in continuous-time systems.

Regarding the construction of recoverable regions, the earliest literature can be traced back to 1960's. For the case of input constraints, J. L. LeMay in 1964 first studied the conditions for characterizing the *maximal region of recoverability* and the *maximal region of reachability* [42]. LeMay also derived a method for calculation of recoverable regions based on optimal control techniques. It is known that for any state in the recoverable region there exists a time-optimal control law that drives the state to zero. This fact builds a direct connection between the characterization of the recoverable region and time-optimal control. There exists a vast literature in the 60's and 70's that were devoted to time-optimal control. Pertinent literature is briefly reviewed in [116]. To be explicit, in connection with discrete-time systems, we mention first the work of Choi [17, 18]. For semi-global stabilization problem, Choi [17] showed that for exponentially unstable *discrete-time* linear systems subject to input constraints any compact subset of the maximal recoverable region can be exponentially stabilized via periodic linear variable structure controller. Also, Choi [18] showed that in general linear feedback cannot achieve global stabilization for discrete-time unstable systems. It should be emphasized that Choi's work deals only with the case when the constraints are posed on the inputs. On the other hand, Cwikel and Gutman [20] developed an algorithm to find maximal state constraint sets for discrete-time linear dynamical systems with bounded controls and states.

To distinguish our work with that of Choi [17] and Cwikel and Gutman [20], let us emphasize that [17] only considers input constraints, and [20] uses a simple algorithm without exploiting structure. It is exactly the exploitation of the structure in computing the recoverable region that is the first objective of this chapter. Our second objective is to design a state feedback controller that achieves semi-global stabilization in the recoverable region.

This chapter is organized as follows. After the introduction we present some preliminary results in Section 5.2. In Section 5.3 we discuss the issues related to computing the recoverable region and present a reduction technique which allows us to reduce the computational effort by developing an explicit relationship between the recoverable region of the full system and the recoverable region of a subsystem of lower order. In Section 5.4 we establish that for any compact set contained in the interior of the recoverable region, there exists a state feedback controller that stabilizes the system and contains the chosen compact set in its domain of attraction while satisfying the constraints.

5.2. Preliminaries and problem statement

This section discusses some preliminaries and statement of problems that will be resolved. Consider a linear discrete-time system,

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \;\; y(k) \;\;\; = C_y x(k) + D_y u(k) \quad k \geq 0 \\ \;\; z(k) \;\;\; = C_z x(k) + D_z u(k), \end{cases} \tag{5.1}$$

where (A, B) is stabilizable, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m$ is the control input, $y(k) \in \mathbb{R}^\ell$ is the measured output, and $z(k) \in \mathbb{R}^p$ is the constrained output, which is subject to the constraint $z(k) \in \mathcal{S}$ for all $k \geq 0$, where $\mathcal{S}$ is a given subset in $\mathbb{R}^p$. Note that the case of input constraints is included as a special case in this general setup by letting $C_z = 0$ and $D_z = I$ in the constrained output equation. However, one should note the difference of input saturation from the input constraint: a saturation can be overloaded, whereas a constraint can never be violated.

We make some general assumption on the constraint set $\mathcal{S}$:

Assumption 5.1 The set $\mathcal{S}$ is compact, convex and contains 0 as interior point. Furthermore, we assume that $C_z' D_z = 0$ and

$$\mathcal{S} = (\mathcal{S} \cap \operatorname{im} C_z) + (\mathcal{S} \cap \operatorname{im} D_z), \tag{5.2}$$

where $\operatorname{im} C_z$ and $\operatorname{im} D_z$ are image spaces of C_z and D_z.

Remark 5.1 *Note that this assumption is not restrictive. In fact, it is a general reflection of the separability of input constraints and state constraints.* (5.2) *is actually the reflection of assumption on plant data, namely* $C_z' D_z = 0$*. We observe that* $\operatorname{im} C_z$ *reflects the state constraints while* $\operatorname{im} D_z$ *reflects the* *input constraints. Therefore the decomposition of $\mathcal{S}$ as required in (5.2) only implies that we have constraints on states and/or inputs. We have no mixed constraints where allowable inputs depend on the current state and conversely.*

Given the constraint on the output, obviously the initial states of the system must be restricted, since, if the initial state of the system is arbitrary then constraint violation can never be avoided. For this reason, we need to define an admissible set of initial conditions. It is straightforward to see that if an initial state is not in this set, then no controller can avoid constraint violation.

Definition 5.1 Given the system Σ in (5.1) and a constraint set $\mathcal{S} \subset \mathbb{R}^p$ satisfying Assumption 5.1, define the admissible set of initial conditions $\mathcal{V}(\mathcal{S})$ as

$$\mathcal{V}(\mathcal{S}) := \{\, x \in \mathbb{R}^n \mid \exists u \text{ such that } z(k) = C_z x(k) + D_z u(k) \in \mathcal{S} \text{ for all } k \geq 0 \,\}.$$

In the papers [88, 145] we established under what conditions, for all compact sets $\mathcal{H}$ in the interior of $\mathcal{V}(\mathcal{S})$, we can find a controller which avoids constraint violation for all time and for all initial conditions in $\mathcal{H}$ while, additionally, guaranteeing that the state converges to zero. This is clearly not always possible. Hence, we define the recoverable region as the largest set of initial conditions for which we can avoid constraint violation while steering the state to the origin.

Definition 5.2 Consider the system given by (5.1) and a constraint set $\mathcal{S} \subset \mathbb{R}^p$ satisfying Assumption 5.1. The set

$$\mathcal{R}_c(\Sigma, \mathcal{S}) = \Big\{ x(0) \in \mathcal{V}(\mathcal{S}) \mid \exists u \text{ such that } \lim_{k\to\infty} x(k) = 0 \\ \text{and } z(k) = C_z x(k) + D_z u(k) \in S \text{ for all } k \geq 0 \Big\}$$

is called the **recoverable region** with constraint set $\mathcal{S}$.

Our goals in this chapter are two fold. At first we would like to explore the properties and computational issues in constructing the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ for a given system Σ and for a given constraint set $\mathcal{S}$ by exploiting the structure of Σ. Then, we would like to solve the problem of semi-global stabilization in the recoverable region via state feedback. Before we proceed to examine these goals and to determine ways of achieving them, it is appropriate to mention that certain structural properties of a system play important roles in determining the properties of the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ and the solvability of certain stabilization problems. Specifically, so called right invertibility, the location of invariant zeros, and the order of infinite zeros of the subsystem Σ_{zu} characterized by the quadruple (A, B, C_z, D_z) play prominent roles. This implies that we need to digress here to recall certain terminology pertaining to a taxonomy of constraints. To do so, we use the standard notation that $\mathbb{C}$, $\mathbb{C}^{\oplus}$, $\mathbb{C}^{\ominus}$ and $\mathbb{C}^{\bigcirc}$ denote respectively the set of complex numbers in the entire complex plane, outside the unit circle, inside the unit circle, and on the unit circle. We recall the following definitions.

Definition 5.3 A subsystem Σ_{zu} characterized by the quadruple (A, B, C_z, D_z),

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad z(k) \quad = C_z x(k) + D_z u(k), \end{cases}$$

is said to be **right invertible** if for any sequence $z_{\text{ref}}(k)$ defined for $k \geq 0$ there exists an input u and a choice of $x(0)$ such that $z(k) = z_{\text{ref}}(k)$ for all $k \geq 0$.

Definition 5.4 The **invariant zeros** of a linear system with a realization (A, B, C, D) are those points $\lambda \in \mathbb{C}$ for which

$$\operatorname{rank}\begin{pmatrix} \lambda I - A & -B \\ C & D \end{pmatrix} < \operatorname{normrank}\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix}$$

where “normrank” denotes *normal rank*.

The first categorization is based on whether the subsystem Σ_{zu} is right invertible or not. We have the following definition:

Definition 5.5 The constraints are said to be

- **right invertible constraints** if the subsystem Σ_{zu} is right invertible.
- **non-right invertible constraints** if the subsystem Σ_{zu} is not right invertible.

It turns out that the location of the invariant zeros of the subsystem Σ_{zu} is also important in characterizing the solvability of stabilization problems. We refer to these invariant zeros as constraint invariant zeros:

Definition 5.6 The invariant zeros of the system characterized by the quadruple (A, B, C_z, D_z) are called the **constraint invariant zeros** of the given system Σ.

The second categorization of constraints is based on the location of the constraint invariant zeros. We have the following definition:

Definition 5.7 The constraints are said to be

- **minimum phase constraints** if all the constraint invariant zeros are in $\mathbb{C}^{\ominus}$.
- **weakly minimum phase constraints** if all the constraint invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$ with the restriction that any invariant zero in $\mathbb{C}^{\bigcirc}$ is simple,
- **weakly non-minimum phase constraints** if all the constraint invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$ with at least one non-simple invariant zero in $\mathbb{C}^{\bigcirc}$.
- **at most weakly non-minimum phase constraints** if all the constraint invariant zeros are in $\mathbb{C}^{\ominus} \cup \mathbb{C}^{\bigcirc}$.
- **strongly non-minimum phase constraints** if at least one constraint invariant zero is in $\mathbb{C}^{\oplus}$.

The third categorization is based on the order of the infinite zeros of the subsystem Σ_{zu} (see [99] for a definition of infinite zeros of a system). Because of their importance, we specifically label the infinite zeros of the subsystem Σ_{zu} as the constraint infinite zeros of the plant.

Definition 5.8 The infinite zeros of the subsystem Σ_{zu} are called the **constraint infinite zeros** of the plant associated with the constrained output z.

We have the following definition regarding the third categorization of constraints.

Definition 5.9 The constraints are said to be **type one constraints** if the order of all constraint infinite zeros is less than or equal to one.

Having recalled above the taxonomy of constraints, we are now in a good position to examine our first goal of exploring the properties and computational issues in constructing the recoverable region

$\mathcal{R}_c(\Sigma, \mathcal{S})$. Since the computation of the admissible set of initial conditions $\mathcal{V}(\mathcal{S})$ is relatively trivial, we can enquire under what conditions the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ coincides with the admissible set of initial conditions $\mathcal{V}(\mathcal{S})$. It is known that $\mathcal{R}_c(\Sigma, \mathcal{S})$ coincides with $\mathcal{V}(\mathcal{S})$ whenever the constraints are at most weakly non-minimum phase and right invertible. Also, it is so for non-right invertible constraints under certain conditions (see [97] for continuous-time and [145] for discrete-time). However, whenever the constraints are strongly non-minimum phase, irrespective of whether they are right or non-right invertible constraints, the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ is always a proper subset of the admissible set of initial conditions $\mathcal{V}(\mathcal{S})$. As such, our main interest is indeed to reduce the complexity in computing the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ whenever the constraints are strongly non-minimum phase. We emphasize that such a construction is very involved where as the construction of the admissible set of initial conditions $\mathcal{V}(\mathcal{S})$ is somewhat trivial. In order to reduce the complexities involved in the computation of $\mathcal{R}_c(\Sigma, \mathcal{S})$, we exploit here the structural properties of the given system. In fact, by exploiting the structural properties, the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ for a given system is constructed by constructing the same however for a reduced order subsystem of the given system. Such a reduction in the order or dimension of the system generally leads to a considerable simplification in the computational effort. One appreciates the reduction in the order of a system, when we note that in the literature so far the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ is constructed at the most for fourth order systems.

We have the following problem statement which expresses formally the first goal.

Problem 5.1 For a given system Σ as in (5.1) along with the constraint set $\mathcal{S}$ satisfying Assumption 5.1, examine the properties of $\mathcal{R}_c(\Sigma, \mathcal{S})$ and then explore the computational issues in constructing the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$.

Next, our second goal of solving via state feedback the semi-global stabilization problem in the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ can be formally stated as follows:

Problem 5.2 (**Semi-global stabilization problem in the recoverable region** $\mathcal{R}_c(\Sigma, \mathcal{S})$) Consider the system (5.1) with the constraint set $\mathcal{S}$ satisfying Assumption 5.1. For any a prior given compact set $\mathcal{W}$ contained in the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ find, if possible, a static state feedback $u(k) = f(x(k), k)$ such that the closed-loop system is asymptotically stable with a domain of attraction containing $\mathcal{W}$ in its interior and that all the constraints are satisfied, i.e. $z(k) \in \mathcal{S}$ for all $k \geq 0$.

Regarding Problem 5.1, the properties and computational issues of the recoverable region are explored in the following section. As required in Problem 5.2, the determination of a state feedback controller that achieves the semi-global stabilization in the recoverable region is done in Section 5.4.

5.3. Properties and computational issues of the recoverable region

Our goal in this section is to show how to reduce the complexities involved in computing the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$ by utilizing the structural properties of the given system. For this purpose we first express the system in terms of the Special Coordinate Basis (SCB) [99, 92] (See also, [11]) which displays explicitly both the finite as well as the infinite zero structure of a given system. A compact form of SCB is given in Appendix. Using suitable basis transformations for the state, input, and constrained output spaces, we can re-write (5.1) as

$$\bar{\Sigma} : \begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}\bar{u}(k) \\ \quad \bar{y}(k) \quad = \bar{C}_y\bar{x}(k) + \bar{D}_y\bar{u}(k) \\ \quad \bar{z}(k) \quad = \bar{C}_z\bar{x}(k) + \bar{D}_z\bar{u}(k). \end{cases} \tag{5.3}$$

The complete form of such a discrete-time system in terms of SCB and some properties of it are given in Appendix.

In this new basis we can decompose the state into several components:

$$\bar{x}(k) = T_x x(k) = \begin{pmatrix} x_a^{0-}(k) \\ x_a^{+}(k) \\ x_b(k) \\ x_c(k) \\ x_d(k) \end{pmatrix},$$

where n_a^{0-}, n_a^{+}, n_b, n_c and n_d are the dimensions of x_a^{0-}, x_a^{+}, x_b, x_c and x_d respectively. We have that $\bar{z}(k)$ is subject to the constraint $\bar{z}(k) \in \bar{\mathcal{S}}$ for all $k \geq 0$, where $\bar{\mathcal{S}} = T_z\mathcal{S}$. In view of the fact that $C_z' D_z = 0$ as given in Assumption 5.1, we can re-write the constrained output $z(k)$ in the new coordinate system as $\bar{z}(k)$,

$$\bar{z}(k) = \begin{pmatrix} C_b x_b(k) \\ u_0(k) \\ C_d x_d(k) \end{pmatrix}.$$

The decomposition of the state allows us to decompose the system $\bar{\Sigma}$ into certain subsystems. In order to characterize the recoverable region efficiently, we can extract the first subsystem Σ_{a+b} from $\bar{\Sigma}$,

$$\Sigma_{a+b}: \begin{cases} x_a^+(k+1) = A_{aa}^+ x_a^+(k) + K_{ab}^+ C_b x_b(k) + K_{a2}^+ \zeta(k) \\ x_b(k+1) = (A_{bb} + K_{bb} C_b) x_b(k) + K_{b2} \zeta(k) \\ \bar{z}(k) \quad = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta(k). \end{cases} \tag{5.4}$$

where A_{aa}^+ has all its eigenvalues inside the closed unit disc. $\zeta(k)$ is defined in (5.5).

We also make extensive use of the following subsystem of the original system:

$$\Sigma_{a+bd}: \begin{cases} x_a^+(k+1) = A_{aa}^+ x_a^+(k) + K_{ab}^+ C_b x_b(k) + K_{a2}^+ \zeta(k) \\ x_b(k+1) = (A_{bb} + K_{bb} C_b) x_b(k) + K_{b2} \zeta(k) \\ x_d(k+1) = A_{dd} x_d(k) + B_d [u_d(k) + G \bar{x}(k)] + K_d \bar{z}(k) \\ \zeta(k) \quad = \begin{pmatrix} 0 \\ C_d \end{pmatrix} x_d(k) + \begin{pmatrix} I \\ 0 \end{pmatrix} u_0(k) \\ \bar{z}(k) \quad = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta(k). \end{cases} \tag{5.5}$$

In Σ_{a+bd}, we define

$$\tilde{u}_d(k) = u_d(k) + G\bar{x}(k) \text{ and } \tilde{u}(k) = \begin{pmatrix} u_0(k) \\ \tilde{u}_d(k) \end{pmatrix}$$

Let $\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}})$ denote the recoverable region of subsystem Σ_{a+bd} with input $\tilde{u}$.

We claim that we can compute the recoverable region for the full system from the recoverable region for the subsystem Σ_{a+b}. We define,

$$\mathcal{V}_q(\bar{\Sigma}, \bar{\mathcal{S}}) = \left\{ \bar{x} \in \mathbb{R}^n \mid x_{a+b} \in \mathcal{R}_c(\Sigma_{a+b}, \bar{\mathcal{S}}) \right\},$$

where q is an integer chosen larger than the maximum order of infinite zeros of the system. Next, we define the following recursion:

$$\mathcal{V}_k(\bar{\Sigma}, \bar{\mathcal{S}}) = \left\{ \bar{x} \in \mathbb{R}^n \mid \exists \bar{u} \text{ such that } \bar{A}\bar{x} + \bar{B}\bar{u} \in \mathcal{V}_{k+1}(\bar{\Sigma}, \bar{\mathcal{S}}) \text{ and } \bar{C}_z \bar{x} + \bar{D}_z \bar{u} \in \bar{\mathcal{S}} \right\} \tag{5.6}$$

for $k = q-1, \ldots, 0$. Our first main result claims that V_0 leads to the recoverable region for the original system.

Theorem 5.1 Consider the system Σ of (5.1) along with the constraint set $\mathcal{S}$ satisfying Assumption 5.1. We have

$$\mathcal{R}_c(\Sigma, \mathcal{S}) = T_x^{-1} \mathcal{R}_c(\bar{\Sigma}, \bar{\mathcal{S}}) = T_x^{-1} \mathcal{V}_0(\bar{\Sigma}, \bar{\mathcal{S}}).$$

Remark 5.2 *A special case of the above theorem is obtained when the system Σ is right-invertible and at most weakly non-minimum-phase since in that case the system Σ_{a+b} is empty and we obtain,*

$$\mathcal{V}_q(\bar{\Sigma}, \bar{\mathcal{S}}) = \mathbb{R}^n,$$

in which case we can obtain the recoverable region through a finite recursion. In general, however, the above only results in a reduction of complexity since we only need to obtain the recoverable region for a system of lower dimension. However, note this is crucial since the classical results for computation of the recoverable region, such as the book [86], only consider the cases $n = 2$ and $n = 3$; This is primarily because with growing dimension the required computational effort grows dramatically.

Proof : In order to prove Theorem 5.1, we need some preparatory work. Consider the recursive definition of $\mathcal{V}_k(\bar{\mathcal{S}})$ for $0 \le k \le q$. We define,

$$\tilde{\mathcal{V}}_q(\Sigma_{a+bd}, \bar{\mathcal{S}}) := \left\{ x_{a+bd} \mid x_{a+b} \in \mathcal{R}_c(\Sigma_{a+b}, \bar{\mathcal{S}}) \right\},$$

and

$$\tilde{\mathcal{V}}_k(\Sigma_{a+bd}, \bar{\mathcal{S}}) = \left\{ x_{a+bd} \mid \exists \tilde{u} \text{ such that } \tilde{A}_0 x_{a+bd} + \tilde{B}_0 \tilde{u} \in \mathcal{V}_{k+1}(\Sigma_{a+bd}, \bar{\mathcal{S}}) \right.$$
$$\left. \text{and } \tilde{C}_0 x_{a+bd} + \tilde{D}_0 \tilde{u} \in \bar{\mathcal{S}} \right\},$$

where

$$\tilde{A}_0 = \begin{pmatrix} A_{aa}^{+} & K_{ab}^{+} C_b & K_{ad}^{+} C_d \\ 0 & A_{bb} + K_{bb} C_b & K_{bd} C_d \\ 0 & K_{db} C_b & A_{dd} \end{pmatrix}, \quad \tilde{B}_0 = \begin{pmatrix} B_{a0}^{+} & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{pmatrix},$$
$$\tilde{C}_0 = \begin{pmatrix} 0 & C_b & 0 \\ 0 & 0 & C_d \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{D}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_0 \\ \tilde{u}_d \end{pmatrix}.$$

We have the relationship,

$$\mathcal{V}_k(\bar{\Sigma}, \bar{\mathcal{S}}) = \mathbb{R}^{n_a^{0-}} \times \tilde{\mathcal{V}}_k(\Sigma_{a+bd}, \bar{\mathcal{S}}) \times \mathbb{R}^{n_c}, \tag{5.7}$$

for $k = 0, \ldots, q$. This relationship between $\mathcal{V}_k(\bar{\Sigma}, \bar{\mathcal{S}})$ and $\tilde{\mathcal{V}}_k(\Sigma_{a+bd}, \bar{\mathcal{S}})$ for $0 \le k \le q$ results from the fact that the relationship is obviously true for $k = q$ while for $k < q$ we note that the dynamics of x_a^{0-} and x_c do not impact the constraints directly as shown by the structure of the SCB.

It can be easily verified that $\mathcal{V}_k(\bar{\Sigma}, \bar{\mathcal{S}})$ defined in equation (5.6) can be characterized as

$$\mathcal{V}_k(\bar{\Sigma}, \bar{\mathcal{S}}) = \big\{ \bar{x}(k) \mid \exists \bar{u} \text{ such that } \bar{z}(i) \in \bar{\mathcal{S}} \text{ for } i = k, \dots, q \\ \text{and } x(q) \in \mathcal{V}_q(\bar{\Sigma}, \bar{\mathcal{S}}) \big\}. \quad (5.8)$$

Similarly, $\tilde{\mathcal{V}}_k(\Sigma_{a+bd}, \bar{\mathcal{S}})$ is given by

$$\tilde{\mathcal{V}}_k(\Sigma_{a+bd}, \bar{\mathcal{S}}) := \Big\{ x_{a+bd}(k) \mid \exists \tilde{u} \text{ such that } \bar{z}(i) \in \bar{\mathcal{S}} \\ \text{for } i = k, \dots, q \text{ and } x_{a+bd}(q) \in \tilde{\mathcal{V}}_q(\Sigma_{a+bd}, \bar{\mathcal{S}}) \Big\}. \quad (5.9)$$

As a first step in the proof of Theorem 5.1, the next lemma shows that $\tilde{\mathcal{V}}_0(\bar{\mathcal{S}})$ is the recoverable region of Σ_{a+bd}.

Lemma 5.1 Consider the system given by (5.5) with constraint set $\bar{\mathcal{S}}$ satisfying Assumption 5.1. We have

$$\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}) = \tilde{\mathcal{V}}_0(\Sigma_{a+bd}, \bar{\mathcal{S}}).$$

Proof : First we will show that

$$\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}) \subseteq \tilde{\mathcal{V}}_0(\Sigma_{a+bd}, \bar{\mathcal{S}}). \quad (5.10)$$

Suppose $x_{a+bd}(0) \in \mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}})$. By definition, we know there exist a $\tilde{u}(k)$ such that $x_{a+bd}(k) \to 0$ as $k \to \infty$ and $\bar{z}(k) \in \bar{\mathcal{S}}$ for all $k \geq 0$, which implies that, for subsystem Σ_{a+b}, there exists a sequence $\zeta(k)$ such that $x_{a+b}(k)$ approaches zero as k goes to infinity. This means that $x_{a+b}(q) \in R_c(\Sigma_{a+b}, \bar{S})$. Hence we have $x_{a+bd}(q) \in \tilde{\mathcal{V}}_q(\Sigma_{a+bd}, \bar{\mathcal{S}})$. The fact that $x_{a+bd}(k) \in \tilde{\mathcal{V}}_k(\Sigma_{a+bd}, \bar{\mathcal{S}})$ for $0 \leq k \leq q-1$ follows then directly from (5.9) since we have no constraint violation in the interval $[0, q]$. This concludes $x_{a+bd}(0) \in \tilde{\mathcal{V}}_0(\Sigma_{a+bd}, \bar{\mathcal{S}})$.

The next step is to show that

$$\tilde{\mathcal{V}}_0(\Sigma_{a+bd}, \bar{\mathcal{S}}) \subseteq \mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}). \quad (5.11)$$

Suppose $x_{a+bd}(0) \in \tilde{\mathcal{V}}_0(\Sigma_{a+bd}, \bar{\mathcal{S}})$ and define $x_{d0} = x_d(0)$. From the definition of $\tilde{\mathcal{V}}_0(\Sigma_{a+bd}, \bar{\mathcal{S}})$, there exists a control input $\tilde{u}(k)$, say $\tilde{u}_1(k)$, for $0 \leq k \leq q$, such that $x_{a+bd}(q) \in \tilde{\mathcal{V}}_q(\Sigma_{a+bd}, \bar{\mathcal{S}})$ and no constraints violation occurs within the first $q-1$ steps. This implies that there exists a $\zeta(k)$ from 0 to q such that $x_{a+b}(q) \in \mathcal{R}_c(\Sigma_{a+b}, \bar{\mathcal{S}})$ and the constraints are not violated. From time q onward, since $x_{a+b}(q) \in R_c(\Sigma_{a+b}, \bar{S})$, there exists an input $\zeta(k)$ for $k \geq q$ that steers $x_{a+b}(k)$ to zero and causes no constraints violation.

In this way we find a signal $\zeta(k)$ for all k. Clearly, we can generate this $\zeta(k)$ via a suitable input $\tilde{u}(k)$, say $\tilde{u}_2(k)$, for $k \geq 0$ together with an appropriate initial condition $\tilde{x}_{d0}$ because the x_d dynamics with inputs u_0 and $\tilde{u}_d$ and output ζ is right-invertible by construction.

Next we note that x_{d0} and inputs $\tilde{u}_1(k)$ also generate the same $\zeta(k)$ for $k = 0, \ldots, q$. The special structure of x_d dynamics guarantees that the initial conditions x_{d0} and $\tilde{x}_{d0}$ must be the same since they result in the same output $\zeta(k)$ for an interval at least as long as the order of the infinite zeros. The structure guarantees this even though the associated inputs might be different. We conclude that for our initial conditions there exists inputs which generate the signal $\zeta(k)$ for all k.

We have noted before that this signal ζ is such that no constraint violations will occur. It remains to show that $x_{a+bd}(k) \to 0$ from time q onward. As noted earlier this signal $\zeta(k)$ is such that $x_{a+b}(k) \to 0$ as $k \to \infty$, which also implies $\zeta(k) \to 0$ as $k \to \infty$. Again the structure of x_d dynamics guarantees that x_d also approaches zero as $k \to \infty$.

Hence we can find a input $\tilde{u}(k)$ for all $k \geq 0$ such that

$$\lim_{k \to 0} x_{a+bd}(k) = 0,$$

and no constraints violation occurs. This completes the proof. ∎

Now we proceed to prove Theorem 5.1. It is obvious from Lemma 5.1 and (5.7) that

$$\mathcal{V}_0(\bar{\Sigma}, \bar{\mathcal{S}}) = \mathbb{R}^{n_a^{0-}} \times \mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}) \times \mathbb{R}^{n_c}.$$

It is obvious that this implies that

$$\mathcal{R}_c(\bar{\Sigma}, \bar{\mathcal{S}}) \subseteq \mathcal{V}_0(\bar{\Sigma}, \bar{\mathcal{S}}).$$

It remains to prove the converse inclusion. This will be shown through an explicit controller design as presented later in the proof of Theorem 5.2. ∎

5.4. Semi-global stabilization in the recoverable region by state feedback controllers

The first objective of this chapter is the reduction in the computation of the recoverable region as outlined in the previous section. The second objective of this chapter is to solve the semi-global stabilization problem (as stated in Problem 5.2) in the recoverable region by state feedback controllers. That is, our intention here is to show that semi-global stabilization can be achieved by a state feedback controller without violating the constraints for any compact subset $\mathcal{W}$ contained in the interior of $\mathcal{R}_c(\Sigma, \mathcal{S})$.

The following theorem establishes the solvability conditions for the semi-global stabilization problem as stated in Problem 5.2.

Theorem 5.2 : Consider the system Σ of (5.1) along with the constraint set $\mathcal{S}$ satisfying Assumption 5.1. The semi-global stabilization described in Problem 5.2 is solvable. More specifically, for any prior given compact set $\mathcal{W}$ contained in the recoverable region $\mathcal{R}_c(\Sigma, \mathcal{S})$, there exists a time-invariant static state feedback $u(k) = f(x(k))$ such that the closed-loop system is asymptotically stable with a domain of attraction containing $\mathcal{W}$ in its interior and that all the constraints are satisfied, i.e. $z(k) \in \mathcal{S}$ for all $k \geq 0$.

Before we start proving this theorem, it is necessary to define the following subsystem:

$$\Sigma_{abd} : \begin{cases} x_a(k+1) = A_{aa}x_a(k) + K_{ab}C_b x_b(k) + K_{a2}\zeta(k) \\ x_b(k+1) = (A_{bb} + K_{bb}C_b)x_b(k) + K_{b2}\zeta(k) \\ x_d(k+1) = A_{dd}x_d(k) + B_d[u_d(k) + G\bar{x}(k)] + K_d\bar{z}(k) \\ \quad \zeta(k) \quad = \begin{pmatrix} 0 \\ C_d \end{pmatrix} x_d(k) + \begin{pmatrix} I \\ 0 \end{pmatrix} u_0(k) \\ \quad \bar{z}(k) \quad = \begin{pmatrix} C_b \\ 0 \end{pmatrix} x_b(k) + \begin{pmatrix} 0 \\ I \end{pmatrix} \zeta(k). \end{cases} \tag{5.12}$$

Similarly,

$$\tilde{u}_d(k) = u_d(k) + G\bar{x}(k) \text{ and } \tilde{u}(k) = \begin{pmatrix} u_0(k) \\ \tilde{u}_d(k) \end{pmatrix}.$$

Let $\mathcal{R}_c(\Sigma_{abd}, \bar{\mathcal{S}})$ denote the recoverable region of system Σ_{abd}.

Then let us give a brief road-map of how we prove Theorem 5.2, that is how we construct a semi-globally stabilizing controller for the given system Σ. Lemma 5.2 that follows considers the semi-global stabilization of the subsystem Σ_{a+bd} as given by (5.5). Based on the result of Lemma 5.2, we proceed to construct in Lemma 5.3 a semi-globally stabilizing controller for the newly defined subsystem Σ_{abd}.

Finally, the controller constructed in Lemma 5.3 for the subsystem Σ_{abd} is augmented to form a semi-globally stabilizing controller for the given system Σ.

We proceed now to construct a semi-globally stabilizing controller for the subsystem Σ_{a+bd}.

Lemma 5.2 The semi-global stabilization problem in recoverable region for Σ_{a+bd} is solvable by a non-linear static state feedback of the form,

$$u_0 = f_1(x_{a+bd}) \text{ and } \tilde{u}_d = f_2(x_{a+bd}).$$

Proof : To start with, we transform the subsystem Σ_{a+bd} to its controllable canonical form. That is, we utilize a nonsingular state transformation T

$$\tilde{x} = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = T x_{a+bd}$$

such that the system Σ_{a+bd} given by (5.5) is transformed to the form,

$$\tilde{\Sigma}_{a+bd} : \begin{cases} \begin{pmatrix} x_1(k+1) \\ x_2(k+1) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \tilde{u}(k) \\ \bar{z}(k) = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \tilde{u}(k), \end{cases}$$

where the dynamics of x_1 is controllable, the dynamics of x_2 is uncontrollable, and

$$\tilde{u}(k) = \begin{pmatrix} u_0(k) \\ \tilde{u}_d(k) \end{pmatrix}.$$

We observe that the recoverable region of system $\tilde{\Sigma}_{a+bd}$ is given by

$$\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}) = T\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}).$$

In order to construct a controller for $\tilde{\Sigma}_{a+bd}$, we define a slightly modified form of $\tilde{\Sigma}_{a+bd}$. That is, we define the modified system,

$$\tilde{\Sigma}^{\ell}_{a+bd} : \begin{cases} \begin{pmatrix} x^{\ell}_1(k+1) \\ x^{\ell}_2(k+1) \end{pmatrix} = (1+\ell) \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x^{\ell}_1(k) \\ x^{\ell}_2(k) \end{pmatrix} + (1+\ell) \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \tilde{u}^{\ell}(k) \\ \bar{z}(k) = \begin{pmatrix} C_1 & C_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^{\ell}_1(k) \\ x^{\ell}_2(k) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \tilde{u}^{\ell}(k), \end{cases}$$

where $\ell > 0$ is small enough that $\tilde{\Sigma}^{\ell}_{a+bd}$ is still stabilizable.

Let $\tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}})$ be the largest set of initial conditions for the system $\tilde{\Sigma}^{\ell}_{a+bd}$ for which there exists an input such that the constraints are satisfied while we stay inside the set for all k (and where we do **not** impose convergence to zero). We do claim that for any $\rho \in (0,1)$ there exists an $\ell > 0$ sufficiently small such that

$$\rho \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}) \subset \tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}}) \subset \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}). \tag{5.13}$$

It is trivial to see that

$$\tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}}) \subset \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}).$$

It remains to establish that

$$\rho \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}) \subset \tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}}).$$

Consider any $r > n$ for the system $\tilde{\Sigma}^{\ell}_{a+bd}$. Since the $x_1^{\ell}(k)$ dynamics is controllable there exists a $\delta^* > 0$ such that for any $\delta \in (0, \delta^*)$ and for any

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \delta \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$$

there exist, an input $\tilde{u}^{\ell} := \begin{pmatrix} u_0^{\ell} \\ u_d^{\ell} \end{pmatrix}$ and initial condition $\tilde{x}^{\ell}(0) = 0$ such that $x_1^{\ell}(r) = -x_1$ and $x_2^{\ell}(r) = 0$ while

$$\bar{z}^{\ell}(k) \in \tfrac{1-\rho}{2} \mathcal{S}, \qquad k = 0, 1, \ldots, r-1.$$

Moreover, δ^* is independent of ℓ and r provided ℓ is small enough.

Let $r > n$ be such that for any $\tilde{x}^{\ell}(0) \in \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$ we have that

$$\begin{pmatrix} 0 \\ x_2^{\ell}(r) \end{pmatrix} \in \delta \rho \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$$

for all ℓ sufficiently small. This is clearly possible due to the fact that the system is stabilizable and hence the uncontrollable dynamics of x_2^{ℓ} must be asymptotically stable.

Consider any initial condition $\tilde{x}(0) \in \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$. We have an input $\tilde{u}$ for the system $\tilde{\Sigma}_{a+bd}$ such that $\bar{z}(k) \in \bar{\mathcal{S}}$. Hence for any $\rho < 1$ we can find, for any initial condition $\tilde{x}(0) \in \rho \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$, an input $\rho \tilde{u}$ for the system $\tilde{\Sigma}_{a+bd}$ such that $\bar{z}(k) \in \rho \bar{\mathcal{S}}$ for all k. But then for $\tilde{x}^{\ell}(0) \in \rho \mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$ and ℓ small enough we find that there exists a control input, say $\tilde{u}_1^{\ell}$, for which we have $\tilde{x}^{\ell}(k) \in (1 +$

$\delta)\rho\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$ for $k = 0, \dots, r$ and $\bar{z}^\ell(k) \in (1+\delta)\rho\bar{\mathcal{S}}$ for $k = 0, \dots, r-1$. Also we observe that if we choose $\delta < \frac{1-\rho}{2}$, we have

$$\delta\tilde{x}^\ell(r) = \begin{pmatrix} \delta x_1^\ell(r) \\ \delta x_2^\ell(r) \end{pmatrix} \in \delta\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}).$$

Then let

$$x_1 = \delta x_1^\ell(r).$$

Hence we can choose an input, say $\tilde{u}_2^\ell$ such that, for $\tilde{x}^\ell(0) = 0$, we have $x_1^\ell(r) = -x_1$ and $x_2^\ell(r) = 0$ while $\bar{z}^\ell(k) \in \frac{1-\rho}{2}\bar{\mathcal{S}}$. But then for initial condition $\tilde{x}^\ell(0) \in \rho\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$, the input $\tilde{u}_1^\ell + \tilde{u}_2^\ell$ and $\delta < \min\{\delta^*, \frac{1-\rho}{2}\}$ we obtain that

$$\bar{z}^\ell(k) \in (1+\delta)\rho\bar{\mathcal{S}} + \tfrac{1-\rho}{2}\bar{\mathcal{S}} \in \bar{\mathcal{S}} \text{ for } k = 0, \dots, r-1,$$

and

$$\begin{aligned} \tilde{x}^\ell(r) = (1-\delta)\begin{pmatrix} x_1^\ell(r) \\ x_2^\ell(r) \end{pmatrix} + \delta\begin{pmatrix} 0 \\ x_2^\ell(r) \end{pmatrix} &\in \left[(1-\delta)(1+\delta)\rho + \delta^2\rho\right]\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}) \\ &\in \rho\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}) \text{ for } k = 0, \dots, r. \end{aligned}$$

This is true due to the fact that $\bar{\mathcal{S}}$ and $\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$ are convex and contain 0 as interior point.

If we repeat this construction between $k = r$ and $k = 2r$ and so forth it becomes clear that we can find for any initial condition,

$$\tilde{x}^\ell(0) \in \rho\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}}),$$

an input such that

$$\bar{z}^\ell(k) \in \mathcal{S}$$

for all k. Hence $\tilde{x}^\ell(0) \in \tilde{\mathcal{R}}(\tilde{\Sigma}_{a+bd}^\ell, \bar{\mathcal{S}})$. This clearly implies that (5.13) is satisfied.

For semi-global stabilization we take any compact set $\tilde{\mathcal{H}}_{a+bd}$ contained in the interior of $\mathcal{R}_c(\tilde{\Sigma}_{a+bd}, \bar{\mathcal{S}})$ and we construct a static controller which will stabilize the system and the constrained domain of attraction contains $\tilde{\mathcal{H}}_{a+bd}$. But then clearly, using (5.13) we can find ℓ such that $\mathcal{H}_{a+bd} \subset \tilde{\mathcal{R}}(\tilde{\Sigma}_{a+bd}^\ell, \bar{\mathcal{S}})$. Next, we choose a feedback $\tilde{f} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix}$ on the boundary of $\tilde{\mathcal{R}}(\tilde{\Sigma}_{a+bd}^\ell, \bar{\mathcal{S}})$ such that for any $\tilde{x}^\ell(k) \in \partial\tilde{\mathcal{R}}(\tilde{\Sigma}_{a+bd}^\ell, \bar{\mathcal{S}})$ we have $\tilde{x}^\ell(k+1) \in \tilde{\mathcal{R}}(\tilde{\Sigma}_{a+bd}^\ell, \bar{\mathcal{S}})$. We expand this feedback $\tilde{f}$ to the whole state space. Define $g : \mathbb{R}^n \to \mathbb{R}^+$ such that for any x,

$$g(x)x \in \partial\tilde{\mathcal{R}}(\tilde{\Sigma}_{a+bd}^\ell, \bar{\mathcal{S}}).$$

Since $\tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}})$ is a convex set containing 0 in its interior this mapping is well-defined. Then we expand f to the whole state space by

$$\bar{f}(x) = \begin{pmatrix} \bar{f}_1(x) \\ \bar{f}_2(x) \end{pmatrix} = \frac{\tilde{f}(g(x)x)}{g(x)}.$$

This expansion has the property that for any $\eta > 0$ we have $x^{\ell}_{a+bd}(k+1) \in \eta\tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}})$ for all $\tilde{x}^{\ell}(k) \in \eta\tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}})$. Note that $\bar{f}$ is positively homogeneous, that is

$$\bar{f}(\alpha x) = \alpha \bar{f}(x)$$

for any $\alpha > 0$.

Clearly, for the system $\tilde{\Sigma}^{\ell}_{a+bd}$ we then have, given the feedback

$$u^{\ell}_0(k) = \bar{f}_1(\tilde{x}^{\ell}(k)) \text{ and } \tilde{u}^{\ell}_d(k) = \bar{f}_2(\tilde{x}^{\ell}(k)),$$

that for all initial conditions in the set $\tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}})$ we have $\tilde{x}^{\ell}(k) \in \tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}})$ for all k.

But then the feedback,

$$u_0(k) = \bar{f}_1(\tilde{x}(k)) \text{ and } \tilde{u}_d(k) = \bar{f}_2(\tilde{x}(k))$$

for the system $\tilde{\Sigma}_{a+bd}$ with $\tilde{x}(0) = \tilde{x}^{\ell}(0)$ results in a state,

$$\tilde{x}(k) = \frac{1}{(1+\ell)^k}\tilde{x}^{\ell}(k).$$

Hence, we obviously have $\tilde{x} \in \tilde{\mathcal{R}}(\tilde{\Sigma}^{\ell}_{a+bd}, \bar{\mathcal{S}})$ for all k but also $\tilde{x}(k) \to 0$ as $k \to \infty$.

Finally, the controller for the original system Σ_{a+bd} is given by:

$$u_0(k) = \bar{f}_1(T^{-1}\tilde{x}(k)) = f_1(x_{a^+}, x_b, x_d) \text{ and } \tilde{u}_d(k) = \bar{f}_2(T^{-1}\tilde{x}(k)) = f_2(x_{a^+}, x_b, x_d).$$

■

Next we design a controller for the subsystem Σ_{abd} based on the state feedback established for Σ_{a+bd}. We have the following lemma.

Lemma 5.3 We have

$$\mathcal{R}_c(\Sigma_{abd}, \bar{\mathcal{S}}) = \mathbb{R}^{n^{0-}_a} \times \mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}), \tag{5.14}$$

where $\mathcal{R}_c(\Sigma_{abd}, \bar{\mathcal{S}})$ and $\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}})$ are the recoverable regions of Σ_{abd} and Σ_{a+bd} respectively. Moreover, the semi-global stabilization problem for Σ_{abd} is solvable.

Proof : It is easy to verify that $\mathcal{R}_c(\Sigma_{abd}, \bar{\mathcal{S}})$ and $\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}})$ have the relationship,

$$\mathcal{R}_{abd}(\Sigma, \bar{\mathcal{S}}) \subseteq \mathbb{R}^{n_a^{0-}} \times \mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}).$$

We will prove the reverse implication and establish the solvability of the stabilization problem for system Σ_{abd} by constructing a state feedback controller for it.

For any compact set $\mathcal{K}$ in $\mathbb{R}^{n_a^{0-}} \times \mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}})$ we choose a compact set $\mathcal{K}_1$ and $\rho < 1$ such that

$$\mathcal{K} \subset \mathcal{K}_1 \times \rho\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}).$$

The controllers $u_0 = f_1(x_{a+bd})$ and $\tilde{u}_d = f_2(x_{a+bd})$ are such that, for all initial conditions in $\rho\mathcal{R}(\Sigma_{a+bd}, \bar{\mathcal{S}})$, the origin of the closed-loop system is exponentially stable. Hence there exists $M > 0$ and λ with $|\lambda| < 1$ such that

$$\|x_{a+bd}(k)\| \leq M\lambda^k \tag{5.15}$$

for all k and for all $x_{a+bd}(0) \in \rho\mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}})$.

Next, let P_0 be the semi-stabilizing solution of the discrete time algebraic Riccati equation,

$$P_0 = A_0' P_0 A_0 + C_0' C_0 - A_0' P_0 B_0 (B_0' P_0 B_0 + D_0' D_0)^{\dagger} B_0' P_0 A_0,$$

where

$$A_0 = \begin{pmatrix} A_{aa} & K_{ab}C_b & K_{ad}C_d \\ 0 & A_{bb} + K_{bb}C_b & K_{bd}C_d \\ 0 & K_{db}C_b & A_{dd} \end{pmatrix}, \quad B_0 = \begin{pmatrix} B_{a0} & 0 \\ B_{b0} & 0 \\ B_{d0} & B_d \end{pmatrix},$$
$$C_0 = \begin{pmatrix} 0 & C_b & 0 \\ 0 & 0 & C_d \\ 0 & 0 & 0 \end{pmatrix}, \qquad D_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{pmatrix}.$$

We have

$$P_0 \begin{pmatrix} x_a^{0-} \\ 0 \\ 0 \end{pmatrix} = 0 \tag{5.16}$$

for all $x_a^{0-} \in \mathbb{R}^{n_a^{0-}}$ since the eigenvalue of A_{aa}^{0-} are in the closed unit disc. Choose a level set,

$$V_0(c) := \{\, \xi \in \mathbb{R}^{n_{abd}} \mid \xi' P_0 \xi \leq c \,\},$$

such that we have

$$(C_0 + D_0(B_0' P_0 B_0 + D_0' D_0)^{\dagger} B_0' P_0 A_0)\xi \in \bar{S}/3 \tag{5.17}$$

for all $\xi \in V_0(c)$. Then with the controllers $u_0 = f_1(x_{a+bd})$ and $\tilde{u}_d = f_2(x_{a+bd})$ there exists a T such that, for any initial state in

$$\mathcal{H}_1 \times \rho\mathcal{R}(\Sigma_{a+bd}, \bar{\mathcal{S}}),$$

we have

$$\begin{pmatrix} x_{a^{0-}}(T) \\ x_{a+bd}(T) \end{pmatrix} \in V_0(c). \tag{5.18}$$

Let P_ε be the stabilizing solution of the algebraic equation,

$$P_\varepsilon = A_0' P_\varepsilon A_0 + C_0' C_0 + \varepsilon I - A_0' P_\varepsilon B_0 (B_0' P_\varepsilon B_0 + D_0' D_0)^\dagger B_0' P_\varepsilon A_0.$$

We have $P_\varepsilon \to P_0$ as ε approaches zero. Define the level set,

$$V_\varepsilon(c) := \{ \xi \in R^{n_{abd}} \mid \xi' P_\varepsilon \xi \le c \}.$$

Then, there exists an ε such that

$$\begin{pmatrix} x_{a^{0-}}(T) \\ x_{a+bd}(T) \end{pmatrix} \in 2V_\varepsilon(c).$$

For ε small enough, we have

$$\left[C_0 - D_0 (B_0' P_\varepsilon B_0 + D_0' D_0)^\dagger B_0' P_\varepsilon A_0 \right] \xi \in \bar{S}$$

for any $\xi \in 2V_\varepsilon(c)$. Hence the feedback

$$\begin{pmatrix} u_0 \\ \tilde{u}_d \end{pmatrix} = -(B_0' P_\varepsilon B_0 + D_0' D_0)^\dagger B_0' P_\varepsilon A_0 \begin{pmatrix} x_a^{0-} \\ x_{a+bd} \end{pmatrix}$$

is an asymptotically stabilizing controller for Σ_{abd} and achieves a domain of attraction containing $2V_\varepsilon(c)$. Next, consider the controller,

$$\begin{pmatrix} u_0 \\ \tilde{u}_d \end{pmatrix} = \begin{cases} f(x_{a+bd}), & x_{abd} \notin 2V_\varepsilon(c) \\ -(B_0' P_\varepsilon B_0 + D_0' D_0)^\dagger B_0' P_\varepsilon A_0 x_{abd}, & x_{abd} \in 2V_\varepsilon(c). \end{cases}$$

It is easily verified that this controller asymptotically stabilizes the system. Hence we have

$$\mathbb{R}^{n_a^{0-}} \times \mathcal{R}_c(\Sigma_{a+bd}, \bar{\mathcal{S}}) \subset \mathcal{R}(\Sigma_{abd}, \bar{\mathcal{S}}).$$

This completes the proof. ■

The above lemma yields an appropriate controller for the subsystem Σ_{abd}. Finally we need to construct a controller for the original system Σ which will complete our proof of Theorem 5.2 but will also complete our proof of Theorem 5.1.

Proof of Theorem 5.2 : It is easily seen that the controllers designed in Lemma 5.3 combined with a controller

$$u_c(k) = -J_a x_a(k) + F_c x_c(k)$$

solve the semi-global constraint stabilization problem in recoverable region via state feedback, where F_c is such that $A_{cc} + B_c F_c$ is asymptotically stable and A_{cc},J_a and B_c are given in the SCB form of the system (see Appendix). ■

Let us next comment on measurement feedback controllers. For discrete-time system with measurement feedback, it is not sensible to consider the recoverable region. After all, the recoverable region is intrinsically an open-loop concept relying on our knowledge of the state which in the measurement feedback case is clearly not available. Moreover, in the discrete time, in contrast with the continuous time, semi-global stabilization with measurement feedback is in general not possible. In continuous-time a fast observer could guarantee a highly accurate estimate of the state in a short period of time in which we do not leave the recoverable region. However, in discrete time, it might take up to n time steps before we get a good estimate of the state and in this period of time we might leave the recoverable region.

An alternative is to use a concept such as maximum domain of attraction. Basically we look for a measurement feedback controller with the largest constrained domain of attraction, i.e. the largest set of initial conditions for which we can guarantee convergence to the origin without constraint violation. However, this concept is also problematic in discrete-time system. If two controllers achieve constrained domain of attractions $\mathcal{R}_1$ and $\mathcal{R}_2$ respectively, then there might not exist a controller for which $\mathcal{R}_1 \cup \mathcal{R}_2$ is contained in its constrained domain of attraction. This is established in the following example. The fact that this is not possible makes it impossible to decide which measurement feedback controller we should use.

Example 5.1 We consider the system,

$$\Sigma_l : \begin{cases} x(k+1) = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \\ y(k) \quad = \begin{pmatrix} 1 & 0 \end{pmatrix} x(k) \\ z(k) \quad = x(k), \end{cases}$$

with a constraint set $\mathcal{S} = \{ x \in \mathbb{R}^2 \mid x \in [-1,1] \times [-1,1] \}$. Note that there is one step delay from the input to the output. Consider the time $k = 0$. Suppose $x(0) \in [-1,1] \times \left[-\frac{3}{4}, \frac{1}{4}\right]$, we can choose $u(0) = \frac{1}{2}$ so that no constraint violation occurs at $k = 1$. Similarly, if $x(0) \in [-1,1] \times \left[-\frac{1}{4}, \frac{3}{4}\right]$, we can choose $u(0) = -\frac{1}{2}$ to avoid constraint violation. However if $x(0) \in [-1,1] \times \left[-\frac{3}{4}, \frac{3}{4}\right]$ which is the union of these two regions, it is impossible to find a $u(0)$ that guarantees no constraint violation. After all, the measurement at time 0 does not yield any information about $x_2(0)$ and hence we can never guarantee that the state $x_2(1)$ will be in $[-1,1]$. Hence, we cannot avoid constraint violation.

5.5. Appendix – Special coordinate basis

Consider a linear discrete-time system,

$$\Sigma : \begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) \quad = C_y x(k) + D_y u(k) \quad k \geq 0 \\ z(k) \quad = C_z x(k) + D_z u(k). \end{cases} \tag{5.19}$$

In what follows we give a very compact version of Special Coordinate Basis (SCB) [99, 92]. Various subsystems used in the body of the chapter are extracted from this compact version. An expanded version of SCB can be found in [99, 92].

By using a suitable basis transformation, the above given system Σ can be transformed via a state space transformation T_x, input basis transformation T_u, and output basis transformation T_z into its SCB with the state $\bar{x}$, control $\bar{u}$, and constrained output $\bar{z}$ described in the new coordinates by

$$T_x x = \bar{x} = \begin{pmatrix} x_a \\ x_b \\ x_c \\ x_d \end{pmatrix}, \qquad T_u u = \bar{u} = \begin{pmatrix} u_0 \\ u_c \\ u_d \end{pmatrix},$$

and

$$T_z z = \bar{z}(k) = \begin{pmatrix} z_b(k) \\ z_0(k) \\ z_d(k) \end{pmatrix} = \begin{pmatrix} C_b x_b(k) \\ u_0(k) \\ C_d x_d(k) \end{pmatrix}.$$

The constrained output $\bar{z}(k)$ is subject to the constraint $\bar{z}(k) \in \bar{\mathcal{S}}$ for all $k \geq 0$, where $\bar{\mathcal{S}} = T_z \mathcal{S}$. Since $C_z' D_z = 0$, it is guaranteed that the new constraint set still satisfies Assumption 5.1. The given system (5.1) can be written in the following form:

$$\bar{\Sigma} : \begin{cases} x_a(k+1) = A_{aa} x_a(k) + K_a \bar{z}(k) \\ x_b(k+1) = A_{bb} x_b(k) + K_b \bar{z}(k) \\ x_c(k+1) = A_{cc} x_c(k) + K_c \bar{z}(k) + B_c[u_c(k) + J_a x_a(k)] \\ x_d(k+1) = A_{dd} x_d(k) + K_d \bar{z}(k) + B_d[u_d(k) + G\bar{x}(k)] \\ \quad y(k) \quad = C_{ya} x_a(k) + C_{yb} x_b(k) + C_{yc} x_c(k) + C_{yd} x_d(k) + \tilde{D}_y \tilde{u}(k) \\ \quad z_0(k) \quad = u_0(k) \\ \quad z_b(k) \quad = C_b x_b(k) \\ \quad z_d(k) \quad = C_d x_d(k). \end{cases}$$

Furthermore, the state x_a can be decomposed into three parts, x_a^-, x_a^0, and x_a^+ such that

$$\begin{aligned} x_a^-(k+1) &= A_{aa}^- x_a^-(k) + K_{a0}^- z_0(k) + K_{ab}^- z_b(k) + K_{ad}^- z_d(k), \\ x_a^0(k+1) &= A_{aa}^0 x_a^0(k) + K_{a0}^0 z_0(k) + K_{ab}^0 z_b(k) + K_{ad}^0 z_d(k), \\ x_a^+(k+1) &= A_{aa}^+ x_a^+(k) + K_{a0}^+ z_0(k) + K_{ab}^+ z_b(k) + K_{ad}^+ z_d(k), \end{aligned}$$

where the eigenvalues of A_{aa}^-, A_{aa}^0, and A_{aa}^+ are respectively inside, on, and outside the unit circle.

SCB has many properties as it displays both the finite and infinite zero structure of a given system Σ. Specific properties of SCB used in the context of this chapter are highlighted below:

- The dynamics of x_a represents the finite zero dynamics of Σ. Thus, the constraint invariant zeros of Σ are given by the eigenvalues of A_{aa}. Hence, the constraints are at most weakly non-minimum phase if and only if the state x_a^+ and thus A_{aa}^+ are non-existent.

- The constraints are right invertible if and only if the state x_b and thus y_b are nonexistent.

- Infinite zero structure is a set of integers which coincides with the set $\mathcal{I}_4$ of Morse [70]. SCB displays such a set, to do so however one needs an expanded form of SCB as given in [99, 92].

CHAPTER 6

Stabilization of a class of sandwich nonlinear systems

6.1. Introduction

Many physical systems can be modeled as interconnections of several distinct subsystems, some of which are linear and some of which are nonlinear. One common type of structure consists of two linear systems connected in cascade via a static nonlinearity. We refer to such systems as *sandwich systems*, because the static nonlinearity is *sandwiched* between the two linear systems.

We focus on sandwich systems where the sandwiched nonlinearity is a saturation. Saturations can occur due to the limited capacity of an actuator, limited range of a sensor, or physical limitations within a system. Physical quantities such as speed, acceleration, pressure, flow, current, voltage, and so on, are always limited to a finite range, and saturations are therefore a ubiquitous feature of physical systems. Our primary goal is to develop design methodologies for semi-global and global stabilization of such systems via feedback controller. We observe that the resulting sandwiched non-linear systems, as shown in Figure 6.1 are extensive generalizations of linear systems subject to actuator saturation.

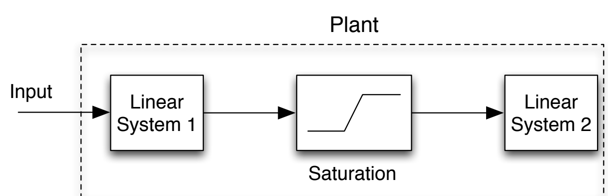


Figure 6.1: Single-layer sandwich system

Figure 6.1 depicts a single-layer sandwich system, where the single layer refers to the saturation element that is sandwiched between two linear systems. A natural extension of this class of systems is depicted in Figure 6.2, which shows a single-layer sandwich system subject to actuator saturation. These types of systems can be further extended to multi-layer sandwich systems, and multi-layer sandwich systems subject to actuator saturation, shown in Figures 6.3 and 6.4.

Sandwiched systems such as those depicted in Figure 6.1 are a special case of so-called cascaded

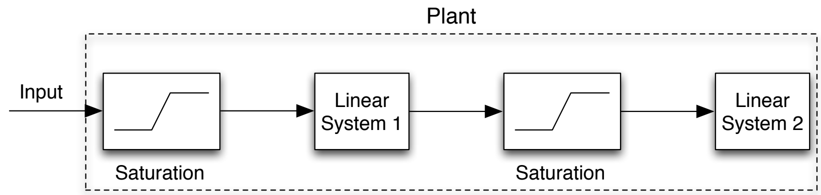


Figure 6.2: Single-layer sandwich system subject to actuator saturation

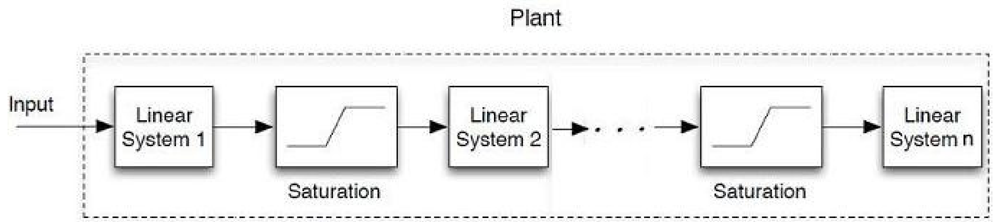


Figure 6.3: Multi-layer sandwich system

systems which are linear systems whose output affects a nonlinear system. This research was initiated in [90] but has also been studied for instance in [102, 103]. Note that in our case the nonlinear system has a very special structure of an interconnection of a static nonlinearity with a linear system. Moreover, in these references the nonlinear system is assumed to be stable and the goal was to see whether the output of a stable linear system can affect this stability.

Some researchers have previously studied linear systems with sandwiched nonlinearities. The most recent activity in this area is the work of Tao et al. [126, 127, 124, 125]. The main technique used in these papers is based on an approximate inversion of the nonlinearities. An example studied in these references is a deadzone, which is a right-invertible nonlinearity. By contrast, a saturation has a very limited range and cannot be inverted even approximately, except in a local region. The work of Tao et al. is therefore not applicable to the case of a saturation nonlinearity. To achieve our goal of semi-global and global stabilization, we need to deal with the saturation directly, by exploiting the structural properties of the given linear systems.

The systems illustrated by Figures 6.1–6.4 are progressive generalizations of the class of systems consisting of a single linear system with an actuator saturation. Over the past years there has been a strong interest in stabilization of this class of systems. Several important results have appeared in the literature, starting with the works of Fuller [24, 25], Sontag and Sussmann [110], Sussmann and Yang

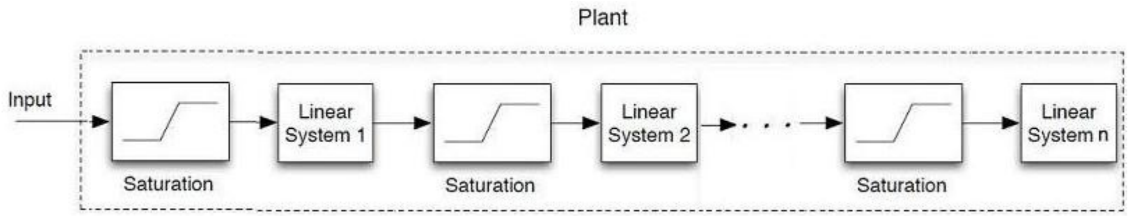


Figure 6.4: Multi-layer sandwich system subject to actuator saturation

[120], as well as Sussmann, Sontag, and Yang [119]. (See also two special issues of the International Journal of Robust and Nonlinear Control [5, 94].) These works led to the development of *low-gain* design methodologies for semi-global stabilization, and *scheduled low-gain* design methodologies for global stabilization of linear systems subject to actuator saturation [51, 52, 49]. The scheduled low-gain design methodology is based on the concept of scheduling, developed by Megretski [68]. Since being developed in the 1990's, low-gain and scheduled low-gain design methodologies have formed an integral part of several related design methodologies, such as low-and-high-gain design methodologies.

Recent research has also focused on linear systems subject to state constraints, where the controller must guarantee that the output of a linear system remains in a given set (see, for instance, [87, 88, 97, 145] and references therein). The approach developed in these works can be used for the class of nonlinear sandwich systems, albeit with some drawbacks. First, the approach does not allow arbitrary initial conditions. Instead, the initial conditions must belong to a constrained set known as the *admissible set of initial conditions* to avoid constraints violation at time 0. Second, the approach is based on limiting the input to avoid activation of *all* the saturations for *all* time, so that the closed-loop system only operates in the linear region. This requires either further restrictions on the initial conditions or constraints on the zero structure to be imposed. In our problem formulations, we allow that the saturation is activated and in this way we avoid the above restrictions.

In the sequel, we first establish conditions for semi-global and global stabilizability of single-layer sandwich systems, portrayed in Figure 6.1, and we construct appropriate control laws by state feedback. We then extend the stabilization results to single-layer sandwich systems subject to actuator saturation, portrayed in Figure 6.2. The design methodologies that emerge from this extension are generalizations of the classical low-gain and scheduled low-gain design methodologies, developed for semi-global and global stabilization of linear systems subject only to actuator saturation. Indeed, when the first linear system is a static and invertible, the new design methodologies reduce to their classical counterparts, and we therefore refer to the new design methodologies as *generalized low-gain* design (for semi-global stabilization) and *generalized scheduled low-gain* design (for global stabilization). We furthermore discuss the natural extension of the results to the multi-layer sandwich systems portrayed in Figures 6.3 and 6.4. We illustrate the results with an example.

6.2. Problem formulations and preliminaries

We first consider the single-layer sandwich systems, portrayed in Figure 6.1. We then formulate the semi-global and global stabilization problems for this class of systems.

Single-layer sandwich systems consist of two interconnected systems, L_1 and L_2, given by

$$L_1 : \begin{cases} \rho x = Ax + Bu \\ \;\; z \; = Cx, \end{cases} \tag{6.1}$$

and

$$L_2 : \quad \rho\omega = M\omega + N\sigma(z) \tag{6.2}$$

where $x \in \mathbb{R}^{n_1}$, $\omega \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^{m_1}$ and $z \in \mathbb{R}^{m_2}$. ρx represents $\dot{x}$ for continuous-time systems and $x(k+1)$ for discrete-time systems.

As will become clear in the design procedure, different saturation levels do not cause any intrinsic differences in controller design methodology except for some changes on ranges of certain design parameters. Therefore, without loss of generality, we assume that all the saturation elements studied in this paper are indeed the same and equal to the standard saturation function defined as $\sigma(z) = [\sigma_1(z_1), \ldots, \sigma_1(z_{m_2})]'$ where $\sigma_1(s) = \operatorname{sgn}(s) \min\{|s|, \Delta\}$ for some $\Delta > 0$.

The dynamics of system L_1 can be modified to include an actuator saturation, and we refer to the resulting system as $\bar{L}_1$. Single-layer sandwich systems subject to actuator saturation therefore consist of two systems, $\bar{L}_1$ and L_2, given by

$$\bar{L}_1 : \begin{cases} \rho x = Ax + B\sigma(u), \\ \;\; z \; = Cx, \end{cases} \tag{6.3}$$

and

$$L_2 : \quad \rho\omega = M\omega + N\sigma(z), \tag{6.4}$$

where $x \in \mathbb{R}^{n_1}$, $\omega \in \mathbb{R}^{n_2}$, $u \in \mathbb{R}^{m_1}$ and $z \in \mathbb{R}^{m_2}$.

This type of system configuration can be generalized to an interconnection of n linear systems, namely the multi-layer nonlinear sandwich systems. Consider the following interconnection of n systems:

$$L_i : \begin{cases} \rho x_i = A_i x_i + B_i \sigma(u_i), & i = 1, \dots, n \\ z_i = C_i x_i, & i = 1, \dots, n-1 \\ u_i = z_{i-1}, & i = 2, \dots, n \end{cases} \tag{6.5}$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$ for $i = 1, \dots, n$, $z_i \in \mathbb{R}^{m_{i+1}}$ for $i = 1, \dots, n-1$.

Let $\bar{x}$ and u denote the state and input of the over-all sandwich systems. The semi-global and global stabilization problems for the three sandwich systems as defined above can be formulated as follows:

Problem 6.1 Consider the (single layer, single layer with input saturation and multilayer) sandwich nonlinear systems as defined above. The semi-global stabilization problem for sandwich nonlinear systems is said to be solvable if for any compact subset $\mathcal{W}$ of whole state space, there exists a state feedback control law $u = f(\bar{x})$ such that the origin of the closed-loop system is asymptotically stable with $\mathcal{W}$ contained in its domain of attraction.

Problem 6.2 Consider the (single layer, single layer with input saturation and multilayer) sandwich nonlinear systems as defined above. The global stabilization problem for the sandwich systems is said to be solvable if there exists a state feedback control law $u = f(\bar{x})$ such that the origin of the closed-loop system is globally asymptotically stable.

6.3. Necessary and sufficient conditions for stabilization of nonlinear sandwich systems

6.3.1. Single layer sandwich system

In this subsection, we present two theorems that give necessary and sufficient conditions for the solvability of semi-global and global stabilization problems as defined in Problems 6.1 and 6.2 for a single layer nonlinear sandwich system.

Theorem 6.1 Consider the interconnection of the two systems given by (6.1) and (6.2). Both the semi-global and the global stabilization problems, as formulated in Problems 6.1 and 6.2 respectively, are solvable if and only if,

1. The linearized cascaded system is stabilizable, i.e. $(\mathscr{A}, \mathscr{B})$ is stabilizable, where

$$\mathscr{A} = \begin{pmatrix} A & 0 \\ NC & M \end{pmatrix} \text{ and } \mathscr{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{6.6}$$

2. All the eigenvalues of M are in the closed left half plane for continuous-time systems and in the closed unit disc for discrete-time systems.

6.3.2. Single layer nonlinear sandwich systems with input saturation

In this subsection, we present two theorems that give necessary and sufficient conditions for solving Problems 6.1 and 6.2 for a single layer sandwich system with input saturation.

Theorem 6.2 Consider the interconnection of the two systems given by (6.3) and (6.4). Both the semi-global and the global stabilization problems, as formulated in Problems 6.1 and 6.2, are solvable if and only if,

1. The linearized cascaded system is stabilizable, i.e. $(\mathscr{A}, \mathscr{B})$ is stabilizable where $\mathscr{A}$ and $\mathscr{B}$ are given by (6.6).

2. All the eigenvalues of A are in the closed left half plane for continuous-time systems and in the closed unit disc for discrete-time systems.

3. All the eigenvalues of M are in the closed left half plane for continuous-time systems and in the closed unit disc for discrete-time systems..

6.3.3. Multi-layer nonlinear sandwich systems

In this subsection, we establish the necessary and sufficient conditions for solving Problems 6.1 and 6.2 for a multi-layer sandwich system.

Theorem 6.3 Consider the interconnection of L_i, $i = 1, \ldots, n$ as given by (6.5). The semi-global and global stabilization problems defined in 6.1 and 6.2 are solvable if and only if

1. $(\mathcal{A}_0, \mathcal{B}_0)$ is stabilizable, where

$$\mathcal{A}_0 = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ B_2C_1 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & B_nC_{n-1} & A_n \end{pmatrix}, \quad \mathcal{B}_0 = \begin{pmatrix} B_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{6.7}$$

2. All A_i have their eigenvalues in the closed left half plane for continuous-time systems and in the closed unit disc for discrete-time systems.

Remark 6.1 *Note that for all three types of sandwich systems, the solvability conditions for semi-global and global stabilization are the same. The intrinsic difference is that global stabilization, unlike the semi-global stabilization, in general requires a nonlinear state feedback law.*

Proof of Theorem 6.1, 6.2 and 6.3 : Necessity of these conditions is quite immediate. Whenever a system needs to be stabilized through a saturated signal and it is well known, see for instance [119], that this can only be done if the eigenvalues of the system are in the closed left half plane for continuous-time systems and in the closed unit disc for discrete-time systems. The cascaded system is linear in a small neighborhood around $(0,0)$ and hence the stabilizability of the nonlinear cascaded system clearly requires the stabilizability of the local linear system, which is equivalent to the stabilizability of the linear cascade systems in the absence of saturation.

Sufficiency is established in the next section by an explicit construction of a stabilizing controller. ■

Remark 6.2 *Note that the existence conditions are the same but semi-global stabilization allows for a linear controller at the expense of a compact but arbitrarily large domain of attraction.*

6.4. Generalized low-gain design for single layer sandwich systems

The design methodologies utilized here are generalizations of classical low-gain design methodologies for linear systems subject to input saturation. The principle behind classical low-gain design is to create a control law with a sufficiently low gain to keep the input saturation inactive for all time. In the semiglobal case, the gain is fixed, based on an *a priori* given set of admissible initial conditions; in the global case, the gain is scheduled to be sufficiently low regardless of the initial conditions.

For the systems considered in this paper, the principle is similar. However, there are now multiple saturations, and the problem is more complex because the sandwiched saturations cannot be made inactive from the start by using low gain. Instead, the sandwiched saturation must be deactivated by controlling the states of the preceding subsystem toward the origin. Conceptually, the control task can therefore be viewed as consisting of a sequence of subtasks. The ith subtask is to control the states of the L_i subsystem toward the origin, in order to deactivate the following sandwiched saturation. Once all the sandwiched saturations have been deactivated, the last subtask consists of controlling the state of the whole system to the origin without reactivating the sandwiched saturations. All of this should be accomplished without activating the input saturation if it exists.

Let's first consider a single layer sandwich systems. To accomplish the two subtasks, the control law is divided into two terms, referred to as the L_1 *term* and the L_1/L_2 *term*. The L_1 term is a function of x, and the purpose of this term is to control the state of the L_1 subsystem toward the origin, in order to permanently deactivate the sandwiched saturation. The gain used in this term is chosen sufficiently low to avoid activating the input saturation, by adjusting a low-gain parameter $\varepsilon_1 > 0$. The L_1/L_2 term is a function of x and ω, and the purpose of this term is to control the states of both subsystems to the origin once the sandwiched saturation becomes inactive. The gain of this term is chosen sufficiently low that it does not interfere with the L_1 term's ability to permanently deactivate the sandwiched saturation, by adjusting a low-gain parameter $\varepsilon_2 > 0$.

6.4.1. Continuous-time systems

Semi-global stabilization

We first choose F such that $A + BF$ is asymptotically stable and consider the system:

$$\begin{cases} \dot{x} = (A + BF)x + Bv \\ z = Cx \end{cases} \tag{6.8}$$

We have

$$\begin{aligned} z(t) &= Ce^{(A+BF)t}x(0) + \int_0^t Ce^{(A+BF)(t-\tau)}Bv(\tau)\,\mathrm{d}\tau \\ &= Ce^{(A+BF)t}x(0) + z_0(t) \end{aligned}$$

Since $A + BF$ is asymptotically stable, we know that there exists δ such that

$$\|v(\tau)\| < \delta \quad \forall \tau > 0 \tag{6.9}$$

implies that $\|z_0(t)\| < \frac{1}{2}$. Next we consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} A + BF & 0 \\ NC & M \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} v \tag{6.10}$$

As is defined earlier, $\bar{x}$ denotes the state of system (6.10). Our initial objective is, for any a priori given compact set $\mathcal{W}$, to find a stabilizing controller for the system (6.10) such that $\mathcal{W}$ is contained in its domain of attraction and $\|v(\tau)\| < \delta$ for all $\tau > 0$.

There exists $P_\varepsilon > 0$ satisfying

$$\begin{pmatrix} A + BF & 0 \\ NC & M \end{pmatrix}' P_\varepsilon + P_\varepsilon \begin{pmatrix} A + BF & 0 \\ NC & M \end{pmatrix} - P_\varepsilon \begin{pmatrix} BB' & 0 \\ 0 & 0 \end{pmatrix} P_\varepsilon + \varepsilon I = 0 \tag{6.11}$$

The following lemma is already obtained in [51].

Lemma 6.1 Consider the system (6.10) with constraint $\|v(t)\| < \delta$ and let conditions 1 and 2 of Theorem 6.1 hold. For any a priori given compact set $\bar{\mathcal{W}} \in \mathbb{R}^{n_1+n_2}$, there exists ε^* such that for any $0 < \varepsilon < \varepsilon^*$, the feedback:

$$v = -\begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \bar{x} \tag{6.12}$$

achieves asymptotic stability of the equilibrium $\bar{x} = 0$. Moreover, for any initial condition in $\bar{\mathcal{W}}$, the constraint does not get violated for any $t > 0$.

Theorem 6.4 Consider the interconnection of the two systems given by (6.1) and (6.2) satisfying conditions 1 and 2 of Theorem 6.1. Let F be such that $A + BF$ is asymptotically stable while $P_\varepsilon > 0$ is the solution of (6.11). We define a state feedback by

$$u = Fx - \begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \begin{pmatrix} x \\ \omega \end{pmatrix} = F_{1,\varepsilon} x + F_{2,\varepsilon} \omega. \tag{6.13}$$

For any compact set of initial conditions $\mathcal{W} \in \mathbb{R}^{n_1+n_2}$ there exists $\varepsilon^* > 0$ such that for all ε with $0 < \varepsilon < \varepsilon^*$ the controller (6.13) asymptotically stabilizes the equilibrium $(0, 0)$ with a domain of attraction containing $\mathcal{W}$.

Proof : Condition 2 of Theorem 6.1 immediately implies the existence of $P_\varepsilon > 0$ satisfying (6.11). Moreover, condition 1 immediately implies

$$P_\varepsilon \to 0 \tag{6.14}$$

as $\varepsilon \to 0$. This implies that

$$F_{1,\varepsilon} \to F, \qquad F_{2,\varepsilon} \to 0$$

Note that the initial conditions are in some compact set $\mathcal{W}$ and hence there exists compact sets $\mathcal{X}$ and Ω such that $x(0) \in \mathcal{X}$ and $\omega(0) \in \Omega$.

Note that for $u = Fx$, there exists $T > 0$ such that for any $x(0) \in \mathcal{X}$ we have

$$\|Ce^{(A+BF)t}x(0)\| < \tfrac{1}{2}.$$

for all $t > T$ and there exists a compact set $\bar{\mathcal{X}}$ such that $x(t) \in \bar{\mathcal{X}}$ for all $t \in [0, T]$. This immediately follows from the asymptotic stability of $A + BF$.

Since $\omega(0) \in \Omega$ which is a compact set and $\sigma(z(t))$ is bounded we find that, independent of ε, there exists a compact set $\bar{\Omega}$ such that $\omega(t) \in \bar{\Omega}$ for all $t \in [0, T]$.

Next, there exists $\varepsilon^* > 0$ such that for

$$u = F_{1,\varepsilon}x + F_{2,\varepsilon}\omega$$

and $\varepsilon < \varepsilon^*$ we have

$$x(t) \in 2\bar{\mathcal{X}}$$

for all $t \in [0, T]$. This follows from the fact that $F_{1,\varepsilon} \to F$ and $F_{2,\varepsilon} \to 0$ while $\omega(t)$ is bounded.

We also note that, from Lemma 6.1, there exists $\varepsilon_2^* < \varepsilon^*$ such that, for $\varepsilon < \varepsilon_2^*$, the controller:

$$v = -\begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \begin{pmatrix} x(t) \\ \omega(t) \end{pmatrix}$$

stabilizes system (6.10) and satisfies $\|v(t)\| < \delta$ for all $t > 0$ given $x(t) \in 2\bar{\mathcal{X}}$ and $\omega(t) \in \bar{\Omega}$ over $[0, T]$. However, this implies $z(t)$ generated by (6.8) satisfies $\|z(t)\| < 1$ for $t > T$. Then the interconnection of (6.1) and (6.2) with controller (6.13) for $t > T$ is equivalent to the interconnection of (6.10) with controller (6.12) for $t > T$. The asymptotic stability of the latter system follows from Lemma 6.1. Hence we have

$$x(t) \to 0, \quad \omega(t) \to 0.$$

Since this follows for any $(x(0), \omega(0)) \in \mathcal{W}$, we find that $\mathcal{W}$ is contained in the domain of attraction as required. ■

Global stabilization

We claim that the same controller given in (6.13) with scheduled low gain parameter $\varepsilon_s(\bar{x})$ solves the global stabilization problem.

First, we are looking for a scheduling parameter satisfying:

1. $\varepsilon_s(\bar{x}) \in C^1$.
2. $\varepsilon_s(0) = 1$.
3. For any $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^{n+m}$ such that

   $$\bar{x}_1' P_{\varepsilon_s(\bar{x}_2)} \bar{x}_1 \leq \bar{x}_2' P_{\varepsilon_s(\bar{x}_2)} \bar{x}_2,$$

   we have

   $$\| B' P_{\varepsilon_s(\bar{x}_2)} \bar{x}_1 \|_\infty \leq \delta$$

4. $\varepsilon_s(\bar{x}) \to 0$ as $\|\bar{x}\|_\infty \to \infty$.
5. $\{ \bar{x} \in \mathbb{R}^{n+m} \mid \bar{x}' P_{\varepsilon_s(\bar{x})} \bar{x} \leq c \}$ is a bounded set for all $c > 0$.
6. $\varepsilon_s(\bar{x})$ is uniquely determined given that $x' P_{\varepsilon_s(\bar{x})} \bar{x} = c$ for some $c > 0$.

A particular choice satisfying the above criteria is given by:

$$\varepsilon_s(\bar{x}) = \max \{ r \in (0, 1] \mid (\bar{x}' P(r) \bar{x}) \operatorname{trace} \left[\begin{pmatrix} B \\ 0 \end{pmatrix}' P(r) \begin{pmatrix} B \\ 0 \end{pmatrix} \right] \leq \delta^2 \}. \tag{6.15}$$

Then the following result has already been obtained in [68]:

Lemma 6.2 Consider the system (6.10) and assume $(\mathcal{A}, \mathcal{B})$ as given by (6.6) is stabilizable and the eigenvalues of M are in the closed left half plane. The feedback:

$$v = -\begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})}\bar{x} \tag{6.16}$$

then achieves global stability of the equilibrium $\bar{x} = 0$.

Theorem 6.5 Consider the interconnection of the two systems given by (6.1) and (6.2) satisfying conditions 1 and 2 of Theorem 6.1. Choose F such that $A + BF$ is asymptotically stable. Let P_ε and ε_s be as defined by (6.11) and (6.15) respectively. In that case, the feedback

$$u = Fx - \begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})}\bar{x} \tag{6.17}$$

achieves global asymptotic stability.

Proof : If we consider the interconnection of (6.1) and (6.2), then we note that close to the origin the saturation does not get activated. Moreover, close to the origin the feedback (6.17) is given by:

$$u = Fx - \begin{pmatrix} B \\ 0 \end{pmatrix}' P_1\bar{x}$$

which immediately yields that the interconnection of (6.1), (6.2) and (6.17) is locally asymptotically stable. It remains to show that we have global asymptotic stability.

Consider an arbitrary initial condition $x(0)$ and $\omega(0)$. Then there exists $T > 0$ such that

$$\|Ce^{(A+BF)t}x(0)\| < \tfrac{1}{2}.$$

for $t > T$. Moreover, by construction

$$v = -\begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})}\bar{x}$$

yields $\|v(t)\| \leq \delta$ for all $t > 0$. However, this implies that $z(t)$ generated by (6.8) satisfies $\|z(t)\| < 1$ for all $t > T$. But this yields that the interconnection of (6.1) and (6.2) with controller (6.17) behaves for $t > T$ like the interconnection of (6.10) with controller (6.16). From Lemma 6.2, global asymptotic stability of the latter system then implies that $\bar{x}(t) \to 0$ as $t \to \infty$. Since this property holds for any initial condition and we have local asymptotic stability we can conclude that the controller yields global asymptotic stability. This completes the proof. ■

6.4.2. Discrete-time systems

Semi-global stabilization

Next, we present a generalized low-gain design to solve semi-global stabilization problem for the discrete-time single-layer sandwich system described by (6.1), (6.2). The design is in a strict parallel with that of continuous-time case. We start by applying a preliminary state feedback $u = Fx + v$ where F is such that $A + BF$ is asymptotically stable. Consider the resulting L_1 system:

$$\begin{aligned} x(k+1) &= (A+BF)x(k) + Bv(k) \\ z(k) &= Cx(k). \end{aligned} \tag{6.18}$$

We have

$$z(k) = C(A+BF)^k x(0) + \sum_{i=0}^{k-1} C(A+BF)^{k-i-1} Bv(i) \tag{6.19}$$

$$= C(A+BF)^k x(0) + z_0(k). \tag{6.20}$$

Define

$$\delta_1 = \frac{1}{2\sum_{k=0}^{\infty} \left\| C(A+BF)^k B \right\|}. \tag{6.21}$$

Since $A + BF$ is asymptotically stable, the above summation is well defined. We know that if

$$\|v(k)\| < \delta_1 \quad \forall k > 0, \tag{6.22}$$

then $\|z_0(k)\| < \frac{1}{2}$. Next we consider the system

$$\bar{x}(k+1) = \tilde{\mathcal{A}}\bar{x}(k) + \mathcal{B}v(k) \tag{6.23}$$

where

$$\bar{x}(k) = \begin{pmatrix} x(k) \\ \omega(k) \end{pmatrix}, \quad \tilde{\mathcal{A}} = \begin{pmatrix} A+BF & 0 \\ NC & M \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}. \tag{6.24}$$

Note that Conditions 1 and 2 of Theorem 6.1 and asymptotic stability of $A + BF$ together imply that $(\tilde{\mathcal{A}}, \mathcal{B})$ is stabilizable and $\tilde{\mathcal{A}}$ has all its eigenvalues in the closed unit disc.

Our next objective is, for any a priori given compact set $\mathcal{W}$, to find a stabilizing controller for the system (6.23) such that $\mathcal{W}$ is contained in its domain of attraction and $\|v(k)\| < \delta_1$ for all $k > 0$.

We note that there exists a unique $P_\varepsilon > 0$ satisfying

$$P_\varepsilon = \tilde{\mathcal{A}}' P_\varepsilon \tilde{\mathcal{A}} + \varepsilon I - \tilde{\mathcal{A}}' P_\varepsilon \mathcal{B}(\mathcal{B}' P_\varepsilon \mathcal{B} + I)^{-1} \mathcal{B}' P_\varepsilon \tilde{\mathcal{A}}. \tag{6.25}$$

The following lemma is already obtained in [52].

Lemma 6.3 Consider the system (6.23) with constraint $\|v(k)\| < \delta_1$, and assume that $(\tilde{\mathcal{A}}, \mathcal{B})$ is stabilizable and $\tilde{\mathcal{A}}$ has all its eigenvalues in closed unit disc. For any a priori given compact set $\bar{\mathcal{W}} \in \mathbb{R}^{n_1+n_2}$, there exists an ε^* such that for any $0 < \varepsilon < \varepsilon^*$, the feedback:

$$v = -(\mathcal{B}' P_\varepsilon \mathcal{B} + I)^{-1} \mathcal{B}' P_\varepsilon \tilde{\mathcal{A}} \bar{x} \tag{6.26}$$

achieves asymptotic stability of the equilibrium $\bar{x} = 0$ with $\bar{\mathcal{W}}$ contained in its domain of attraction. Moreover, for any initial condition in $\bar{\mathcal{W}}$, the constraint does not get violated for any $k > 0$.

We can now use Lemma 6.3 to prove that a particular family of control laws achieves semi-global stability of the single-layer nonlinear sandwich system.

Theorem 6.6 Consider the interconnection of the two systems given by (6.1) and (6.2) satisfying Conditions 1 and 2 of Theorem 6.1. Let F be an arbitrary matrix such that $A + BF$ is asymptotically stable while $P_\varepsilon > 0$ is the solution of (6.25). We define a low-gain state feedback by

$$u = Fx - (\mathcal{B}' P_\varepsilon \mathcal{B} + I)^{-1} \mathcal{B}' P_\varepsilon \tilde{\mathcal{A}} \bar{x} = F_{1,\varepsilon} x + F_{2,\varepsilon} \omega. \tag{6.27}$$

For any compact set of initial conditions $\mathcal{W} \in \mathbb{R}^{n_1+n_2}$ there exists an $\varepsilon^* > 0$ such that for all ε with $0 < \varepsilon < \varepsilon^*$ the controller (6.27) asymptotically stabilizes the equilibrium $(0, 0)$ with a domain of attraction containing $\mathcal{W}$.

Proof : Condition 2 of Theorem 6.1 immediately implies the existence and uniqueness of $P_\varepsilon > 0$ satisfying (6.25). Moreover, Condition 1 immediately implies $P_\varepsilon \to 0$ as $\varepsilon \to 0$. This immediately implies that $F_{1,\varepsilon} \to F$ and $F_{2,\varepsilon} \to 0$ as $\varepsilon \to 0$. Note that the initial conditions are in some compact set $\mathcal{W}$ and hence there exist compact sets $\mathcal{X}$ and Ω such that $x(0) \in \mathcal{X}$ and $\omega(0) \in \Omega$. Define a family of sets $\mathcal{V}(c_1) = \{\bar{x} \in \mathbb{R}^{n_1+n_2} \mid \bar{x}' P_\varepsilon \bar{x} \leq c_1\}$.

Note that if we apply $u = Fx$, there exists a $K > 0$ such that for any $x(0) \in \mathcal{X}$ we have

$$\|C(A + BF)^k x(0)\| < \tfrac{1}{2}$$

for all $k > K$ and there exists a compact set $\bar{\mathcal{X}}$ such that $x(k) \in \bar{\mathcal{X}}$ for all $0 \leq k \leq K$. This immediately follows from the asymptotic stability of $A + BF$.

Since $\omega(0) \in \Omega$ which is a compact set and $\sigma(z(k))$ is bounded we find that, independent of ε, there exists a compact set $\bar{\Omega}$ such that $\omega(k) \in \bar{\Omega}$ for all $0 \leq k \leq K$.

Next, there exists an $\varepsilon^{\#} > 0$ such that for $u(k) = F_{1,\varepsilon}x(k) + F_{2,\varepsilon}\omega(k)$ and $\varepsilon < \varepsilon^{\#}$ we have $x(k) \in 2\bar{\mathcal{X}}$ for all $0 \leq k \leq K$. This follows from the fact that $F_{1,\varepsilon} \to F$ and $F_{2,\varepsilon} \to 0$ while $\omega(k)$ is bounded in $\bar{\Omega}$. Let c_1 be such that

$$c_1 = \sup_{\substack{\varepsilon \in (0,1] \\ \bar{x} \in 2\bar{\mathcal{X}} \times \bar{\Omega}}} \bar{x}' P_\varepsilon \bar{x}.$$

Define a family of level sets $\mathcal{V}(c_1) = \{\bar{x} \in \mathbb{R}^{n_1+n_2} \mid \bar{x}' P_\varepsilon \bar{x} \leq c_1\}$.

From Lemma 6.3, we also note that there exists an $\varepsilon^* < \varepsilon^{\#}$ such that, for $\varepsilon < \varepsilon^*$, the controller

$$v = -(\mathcal{B}' P_\varepsilon \mathcal{B} + I)^{-1} \mathcal{B}' P_\varepsilon \tilde{\mathcal{A}} \bar{x}$$

stabilizes the system (6.23), and satisfies $\|v\| < \delta_1$ for all $k > 0$ given $\bar{x}(K) \in \mathcal{V}(c_1)$. This implies that $z(k)$ generated by (6.18) satisfies $\|z(k)\| < 1$ for $k > K$. Then the interconnection of (6.1) and (6.2) with controller (6.27) for $k > K$ is equivalent to the interconnection of (6.23) with controller (6.26) for $k > K$. The asymptotic stability of the latter system follows from Lemma 6.3. Hence we have $x(k) \to 0, \quad \omega(k) \to 0$. Since this follows for any $(x(0), \omega(0)) \in \mathcal{W}$, we find that $\mathcal{W}$ is contained in the domain of attraction as required. ■

Remark 6.3 *For semi-global stabilization, we can enlarge the domain of attraction by choosing a sufficiently small low-gain parameter. However, this incurs a deterioration of closed-loop performance near the origin since a small low-gain parameter results in conservativeness in feedback gain and hence does not allow full utilization of control capacity when the state is close to the origin. In order to rectify this problem, a generalized low-and-high gain feedback design methodology for continuous-time sandwich nonlinear systems is recently introduced in [118]. It was shown that a refined performance can* *be achieved with the so-called low-and-high-gain feedback controller. Because of some inherent differences between continuous- and discrete-time systems, development of a low-and-high-gain design for discrete-time counter-part remains an open research problem.*

Remark 6.4 *To implement the semi-globally stabilizing controller, it is necessary to find appropriate low-gain parameters ε. It is difficult to derive tight upper bounds on ε analytically, and thus the parameters are typically found experimentally, by gradually decreasing them until the desired stability is achieved.*

Global stabilization

In what follows, we show that the family of controllers defined by (6.27), with ε replaced by a scheduled low-gain parameter $\varepsilon(\bar{x})$, solves Problem 6.2 for single-layer sandwich system. We consider the scheduling introduced in Section 6.2 as given by:

$$\varepsilon(\bar{x}) = \max\{\, r \in (0,1] \mid (\bar{x}' P_r \bar{x}) \|\mathcal{B}' P_r \mathcal{B}\| \leq \tfrac{\delta_1^2}{M_p} \,\} \tag{6.28}$$

where P_r is the unique positive definite solution of algebraic Riccati equation (6.25) with $\varepsilon = r$, δ_1 is defined by (6.21),

$$M_p = \sigma_{\max}(P_1^{\frac{1}{2}} \mathcal{B}\mathcal{B}' P_1^{\frac{1}{2}}) + 1$$

and P_1 is the solution of (6.25) with $\varepsilon = 1$. It has been shown in Section 6.2 that this scheduling guarantees that

$$\|(\mathcal{B}' P_{\varepsilon(x)} \mathcal{B} + I)^{-1} \mathcal{B}' P_{\varepsilon(x)} \tilde{\mathcal{A}} x\| \leq \delta_1.$$

To prove Theorem 6.1, we need the following lemma from [68], which defines a control law that stabilizes the linear system (6.23).

Lemma 6.4 Consider the system (6.23) and assume that $(\tilde{\mathcal{A}}, \mathcal{B})$ as given by (6.24) is stabilizable, and that the eigenvalues of $\tilde{\mathcal{A}}$ are within the closed unit disc. The control law

$$v = -(\mathcal{B}' P_{\varepsilon(\bar{x})} \mathcal{B} + I)^{-1} \mathcal{B}' P_{\varepsilon(\bar{x})} \tilde{\mathcal{A}} \bar{x} \tag{6.29}$$

achieves global stability of the equilibrium $\bar{x} = 0$.

We can now use Lemma 6.4 to prove that a particular family of control laws achieves global stability of the single-layer nonlinear sandwich system.

Theorem 6.7 Consider the systems given by (6.1) and (6.2), satisfying Conditions 1 and 2 of Theorem 6.1. Choose an arbitrary matrix F such that $A + BF$ is asymptotically stable. Let $P_{\varepsilon(\bar{x})}$ be the unique positive definite solution of ARE (6.25), with ε replaced by the scheduled low-gain parameter $\varepsilon(\bar{x})$ defined by (6.28). Then, the control law

$$u = Fx - (\mathcal{B}' P_{\varepsilon(\bar{x})} \mathcal{B} + I)^{-1} \mathcal{B}' P_{\varepsilon(\bar{x})} \tilde{\mathcal{A}} \bar{x} \tag{6.30}$$

achieves global asymptotic stability of the origin where $\tilde{\mathcal{A}}$ and $\mathcal{B}$ are given by (6.24).

Proof :

If we consider the interconnection of (6.1) and (6.2), then we note that close to the origin the saturation does not get activated. Moreover, close to the origin the feedback (6.30) is given by:

$$u = Fx - (\mathcal{B}' P_1 \mathcal{B} + I)^{-1} \mathcal{B}' P_1 \tilde{\mathcal{A}} \bar{x}$$

where P_1 is the solution of (6.25) with $\varepsilon = 1$. This immediately yields that the interconnection of (6.1), (6.2) and (6.30) is locally asymptotically stable. It remains to show that we have global asymptotic stability.

Consider an arbitrary initial condition $x(0)$ and $\omega(0)$. Then there exists a $K > 0$ such that

$$\|C(A + BF)^k x(0)\| < \tfrac{1}{2}$$

for $k > K$. Moreover, by construction

$$v = -(\mathcal{B}' P_{\varepsilon(\bar{x})} \mathcal{B} + I)^{-1} \mathcal{B}' P_{\varepsilon(\bar{x})} \tilde{\mathcal{A}} \bar{x}$$

yields $\|v(k)\| \leq \delta_1$ for all $k > 0$. However, this implies that $z(k)$ generated by (6.18) satisfies $\|z(k)\| < 1$ for all $k > K$. But this yields that the interconnection of (6.1) and (6.2) with controller (6.30) behaves for $k > K$ like the interconnection of (6.23) with controller (6.29). From Lemma 6.4, global asymptotic stability of the latter system then implies that $\bar{x}(k) \to 0$ as $k \to \infty$. Since this property holds for any initial condition and we have local asymptotic stability we can conclude that the controller yields global asymptotic stability. This completes the proof. ∎

6.5. Generalized low-gain design for single layer sandwich systems with input saturation

6.5.1. Continuous-time systems

Semi-global stabilization

To construct a semi-globally stabilizing class of controllers, we begin by letting P_{ε_1} denote the unique symmetric positive-definite solution of the algebraic Riccati equation (ARE)

$$A'P_{\varepsilon_1} + P_{\varepsilon_1}A - P_{\varepsilon_1}BB'P_{\varepsilon_1} + \varepsilon_1 I_n = 0. \tag{6.31}$$

Define $F_{\varepsilon_1} := -B'P_{\varepsilon_1}$ and $\bar{F} := [F_{\varepsilon_1}, 0] \in \mathbb{R}^{p\times(n_1+n_2)}$. We continue by letting $\mathcal{P}_{\varepsilon_2}$ denote the unique symmetric positive-definite solution of the ARE

$$(\mathcal{A} + \mathcal{B}\bar{F})'\mathcal{P}_{\varepsilon_2} + \mathcal{P}_{\varepsilon_2}(\mathcal{A} + \mathcal{B}\bar{F}) - \mathcal{P}_{\varepsilon_2}\mathcal{B}\mathcal{B}'\mathcal{P}_{\varepsilon_2} + \varepsilon_2 I_{n+m} = 0. \tag{6.32}$$

Define $\mathcal{F}_{\varepsilon_2} := -\mathcal{B}'\mathcal{P}_{\varepsilon_2}$. The interconnection of (6.3) and (6.4) is now semi globally stabilized by the control law

$$u = F_{\varepsilon_1}x + \mathcal{F}_{\varepsilon_2}\chi. \tag{6.33}$$

The low-gain parameters $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ must be chosen sufficiently small depending on the size of the set of admissible initial conditions, as shown by the following theorem:

Theorem 6.8 Let $\mathcal{W} \subset \mathbb{R}^{n_1+n_2}$ be a compact set, and suppose that conditions 1-3 of Theorem 6.2 are satisfied. Then there exists an $\varepsilon_1^* > 0$ such that for each $0 < \varepsilon_1 < \varepsilon_1^*$, there exists an $\varepsilon_2^*(\varepsilon_1) > 0$ such that for all $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$, the controller described by (6.33) renders the origin of (6.3) and (6.4) asymptotically stable with $\mathcal{W}$ contained in the region of attraction.

Proof : Consider first the system

$$\dot{\chi} = \mathcal{A}\chi + \mathcal{B}, \quad \chi = \begin{bmatrix} x \\ \omega \end{bmatrix}, \tag{6.34}$$

with $u = F_{\varepsilon_1}x + \mathcal{F}_{\varepsilon_2}\chi$, which is valid locally around the origin where both saturations are inactive. Defining the Lyapunov function candidate $V(\chi) = x'P_{\varepsilon_1}x + \chi'\mathcal{P}_{\varepsilon_2}\chi$, it is easily confirmed that we obtain the time derivative

$$\begin{aligned}\dot{V}(\chi) =& -\varepsilon_1 x'x - x'P_{\varepsilon_1}BB'P_{\varepsilon_1}x - 2x'P_{\varepsilon_1}B\mathcal{B}'\mathcal{P}_{\varepsilon_2}\chi - \varepsilon_2\chi'\chi \\ & - \chi'\mathcal{P}_{\varepsilon_2}\mathcal{B}\mathcal{B}'\mathcal{P}_{\varepsilon_2}\chi \\ =& -\varepsilon_1 x'x - \varepsilon_2\chi'\chi - (B'P_{\varepsilon_1}x + \mathcal{B}'\mathcal{P}_{\varepsilon_2}\chi)'(B'P_{\varepsilon_1}x + \mathcal{B}'\mathcal{P}_{\varepsilon_2}\chi).\end{aligned}$$

Thus, we know that the system is locally exponentially stable. Since $\chi(0)$ belongs to the compact set $\mathcal{W}$, there exist compact sets X and Ω such that $x(0) \in X$ and $\omega(0) \in \Omega$.

Because the eigenvalues of $\mathcal{A}$, and therefore the eigenvalues of A, are in the closed left-half plane, the solutions of (6.31) are such that $P_{\varepsilon_1} \to 0$ as $\varepsilon_1 \to 0$ [49, Lemma 2.2.6]. Furthermore, the matrix $F_{\varepsilon_1} = -B'P_{\varepsilon_1}$ is such that the matrix $A + BF_{\varepsilon_1}$ is Hurwitz, and it follows that the eigenvalues of the matrix $\mathcal{A} + \mathcal{B}\bar{F}$ are in the closed left-half plane. This in turn implies that for each $\varepsilon_1 > 0$, the solutions of (6.32) are such that $\mathcal{P}_{\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$. From these considerations, we may conclude that $\lim_{\varepsilon_1 \to 0} F_{\varepsilon_1} = 0$, and for each $\varepsilon_1 > 0$, $\lim_{\varepsilon_2 \to 0} \mathcal{F}_{\varepsilon_2} = 0$.

We first investigate the effect of the L_1 term alone; that is, the feedback matrix F_{ε_1}. Since the matrix $A + BF_{\varepsilon_1}$ is Hurwitz and $F_{\varepsilon_1} \to 0$ as $\varepsilon_1 \to 0$, there exists an $\varepsilon_1^* > 0$ such that for all $0 < \varepsilon_1 < \varepsilon_1^*$ and for all $x(0) \in X$, the input saturation remains inactive in the sense that $\|F_{\varepsilon_1}x(t)\| = \|F_{\varepsilon_1}\exp^{(A+BF_{\varepsilon_1})t}x(0)\| \le \frac{1}{4}$ (see [61, Theorem 2.8]). Let ε_1 be fixed such that this inequality is satisfied, and define $\gamma > 0$ such that $x'P_{\varepsilon_1}x \le \gamma$ implies $\|Cx\| \le \frac{1}{4}$ and $\|F_{\varepsilon_1}x\| \le \frac{1}{4}$. Define $K = \{x \in \mathbb{R}^n \mid x'P_{\varepsilon_1}x \le \gamma\}$, and let $T > 0$ be chosen large enough that for all $x(0) \in X$, $x(T) = \exp^{(A+BF_{\varepsilon_1})T}x(0) \in K$.

Next, consider the complete control law, with both the L_1 and the L_1/L_2 terms; that is, $u = F_{\varepsilon_1}x + \mathcal{F}_{\varepsilon_2}\chi$. The L_1/L_2 term can be partitioned as $\mathcal{F}_{\varepsilon_2}\chi = \mathcal{F}_{1,\varepsilon_2}x + \mathcal{F}_{2,\varepsilon_2}\omega$, where $\mathcal{F}_{1,\varepsilon_2} \to 0$ and $\mathcal{F}_{2,\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$. Since $\omega(0) \in \Omega$ and the input $\sigma(z)$ to the L_2 subsystem is bounded, we know that there exists a compact set $\bar{\Omega} \supset \Omega$ such that for all $t \in [0, T]$, $\omega(t) \in \bar{\Omega}$. Using the property that $\mathcal{F}_{2,\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$, we therefore see that the term $\mathcal{F}_{2,\varepsilon_2}\omega$ can be made arbitrarily small on the time interval $[0, T]$ by decreasing ε_2. This, combined with the property that $\mathcal{F}_{1,\varepsilon_2} \to 0$ as $\varepsilon_2 \to 0$, shows that for small ε_2, the control law on the interval $[0, T]$ can be viewed as a small perturbation of the control law $u = F_{\varepsilon_1}x$. Thus, we know that for all sufficiently small ε_2, $x(T) \in 2K$ is satisfied for all $\chi(0) \in \mathcal{W}$.

Accordingly, let $\varepsilon_2^*(\varepsilon_1)$ be chosen small enough that, for all $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$ and all $\chi(0) \in \mathcal{W}$, we have $x(T) \in 2K$. Furthermore, let $\varepsilon_2^*(\varepsilon_1)$ be chosen small enough that the following two properties hold for all $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$: (i) $x' P_{\varepsilon_1} x \leq 4\gamma$ and $\omega \in \bar{\Omega}$ implies $V(\chi) \leq 9\gamma$; and (ii) $V(\chi) \leq 9\gamma$ implies $\|\mathcal{F}_{\varepsilon_2}\chi\| \leq \frac{1}{4}$.

We can now make several observations. At time T, we know that $x(T) \in 2K$ and $\omega(T) \in \bar{\Omega}$, which means that $x'(T) P_{\varepsilon_1} x(T) \leq 4\gamma$, and thus we can conclude that $V(\chi(T)) \leq 9\gamma$. Furthermore, for all χ such that $V(\chi) \leq 9\gamma$, we have $x' P_{\varepsilon_1} x \leq 9\gamma$, which means that $x \in 3K$. This in turn implies that $\|F_{\varepsilon_1} x\| \leq \frac{3}{4}$ and $\|Cx\| \leq \frac{3}{4}$. Combined with the expression $\|\mathcal{F}_{\varepsilon_2}\chi\| \leq \frac{1}{4}$, this implies that $\|u\| = \|F_{\varepsilon_1} x + \mathcal{F}_{\varepsilon_2}\chi\| \leq 1$. Thus, for all χ such that $V(\chi) \leq 9\gamma$, both the input saturation and the sandwiched saturation are inactive. The proof is completed by noting that when both saturations are inactive, $V(\chi)$ is a Lyapunov function. Thus, χ never escapes from the level set defined by $V(\chi) \leq 9\gamma$, and the system therefore behaves like a linear, exponentially stable system for all $t \geq T$. ■

Remark 6.5 *To implement the semiglobally stabilizing controller, it is necessary to find appropriate low-gain parameters ε_1 and ε_2. It is difficult to derive tight upper bounds ε_1^* and $\varepsilon_2^*(\varepsilon_1)$ analytically, and thus the parameters are typically found experimentally, by gradually decreasing them until the desired stability is achieved.*

Global stabilization

To achieve global stabilization, we use a control law that is very similar to the semi global case. The main difference is that, instead of being fixed, the low-gain parameters are scheduled as functions of the state of the system.

Let $P_{\varepsilon_1(x)}$ be the unique symmetric positive-definite solution of the ARE (6.31) with $\varepsilon_1 = \varepsilon_1(x)$. Define $F_{\varepsilon_1(x)} := -B' P_{\varepsilon_1(x)}$ and $\bar{F} := [F_1, 0] \in \mathbb{R}^{p\times(n+m)}$ (where $F_1 = -B'P_1$ and P_1 is the solution of (6.31) with $\varepsilon_1 = 1$). Let $\mathcal{P}_{\varepsilon_2(\chi)}$ be the unique symmetric positive-definite solution of the ARE (6.32) with $\varepsilon_2 = \varepsilon_2(\chi)$. Define $\mathcal{F}_{\varepsilon_2(\chi)} = -\mathcal{B}'\mathcal{P}_{\varepsilon_2(\chi)}$. When the scheduled low-gain parameters $\varepsilon_1(x)$ and $\varepsilon_2(\chi)$ are properly defined, the interconnection of (6.3) and (6.4) is globally stabilized by the control law

$$u = F_{\varepsilon_1(x)} x + \varepsilon_1(x) \mathcal{F}_{\varepsilon_2(\chi)} \chi. \tag{6.35}$$

We now specify our requirements for the scheduled low-gain parameters $\varepsilon_1(x)$ and $\varepsilon_2(x)$. The function $\varepsilon_1\colon \mathbb{R}^n \to (0,1]$ must be continuous and satisfy the following properties:

1. There exists an open neighborhood O of the origin such that for all $x \in O$, $\varepsilon_1(x) = 1$.
2. For any $x \in \mathbb{R}^n$, $\|B' P_{\varepsilon_1(x)} x\| \le \frac{1}{2}$.
3. $\varepsilon_1(x) \to 0 \implies \|x\| \to \infty$.
4. For each $c > 0$, the set $\{x \in \mathbb{R}^n \mid x' P_{\varepsilon_1(x)} x \le c\}$ is bounded.
5. There is a function $g\colon \mathbb{R}_{>0} \to (0,1]$ such that for all $x \neq 0$, $\varepsilon_1(x) = g(x' P_{\varepsilon_1(x)} x)$.

A particular choice that satisfies the above conditions is

$$\varepsilon_1(x) = \max\left\{r \in (0,1] \mid x' P_r x \cdot \operatorname{trace}(B' P_r B) \le \frac{1}{4}\right\} \tag{6.36}$$

where P_r is the solution of (6.31) with $\varepsilon_1 = r$.

To define $\varepsilon_2(\chi)$, first define

$$\delta := \min\left\{\frac{1}{2}, \frac{\ell}{4\|F_1\|}, \frac{1}{2\rho}\right\}, \quad \ell := \frac{1}{2\sqrt{\|P_1\| \operatorname{trace}(B' P_1 B)}},$$
$$\rho := \int_0^\infty \|C \exp^{(A+BF_1)t} B\| \, \mathrm{d}t. \tag{6.37}$$

Note that ρ is well-defined because $A + BF_1$ is Hurwitz. The function $\varepsilon_2\colon \mathbb{R}^{n+m} \to (0,1]$ must be continuous and satisfy properties 1–4 above, with x replaced by χ, B replaced by $\mathcal{B}$, $P_{\varepsilon_1(x)}$ replaced by $\mathcal{P}_{\varepsilon_2(\chi)}$, and the number $\frac{1}{2}$ in Property 2 replaced by δ. A particular choice that satisfies these conditions is

$$\varepsilon_2(\chi) = \max\left\{r \in (0,1] \mid \chi' \mathcal{P}_r \chi \cdot \operatorname{trace}(\mathcal{B}' \mathcal{P}_r \mathcal{B}) \le \delta^2\right\} \tag{6.38}$$

where $\mathcal{P}_r$ is the solution of (6.32) with $\varepsilon_2 = r$.

Theorem 6.9 Suppose that conditions 1-3 of Theorem 6.2 are satisfied. Then the controller described by (6.35), with $\varepsilon_1(x)$ and $\varepsilon_2(\chi)$ defined by (6.36), (6.38), renders the origin of (6.3) and (6.4) globally asymptotically stable.

Proof : We start by noting that the properties of the scheduling guarantee that

$$\|F_{\varepsilon_1(x)}x\| = \|B'P_{\varepsilon_1(x)}x\| \leq \frac{1}{2}, \quad \|\varepsilon_1(x)\mathcal{F}_{\varepsilon_2(\chi)}\chi\| \leq \|\mathcal{B}'\mathcal{P}_{\varepsilon_2(\chi)}\chi\| \leq \delta \leq \frac{1}{2}.$$

It follows that $\|u\| \leq 1$, and hence the input saturation is always inactive.

For sufficiently small χ, both saturations are inactive, and we have $\varepsilon_1(x) = \varepsilon_2(\chi) = 1$. Thus, the system behaves like a linear system with a linear control law $u = F_1x + \mathcal{F}_1\chi$ in a region around the origin. As in the semiglobal case, it is easy to show that the origin of the resulting system is locally exponentially stable by using the Lyapunov function $V(\chi) = x'P_1x + \chi'\mathcal{P}_1\chi$.

Define $K = \{x \in \mathbb{R}^n \mid \varepsilon_1(x) = 1\}$. We wish to show that whenever $x \notin K$, $\varepsilon_1(x)$ is strictly increasing with respect to time. Suppose, for the sake of establishing a contradiction, that $\varepsilon_1(x)$ is not strictly increasing when $x \notin K$, that is, $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) \leq 0$. Then we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(x'P_{\varepsilon_1(x)}x) = &- \varepsilon_1(x)x'x - x'P_{\varepsilon_1(x)}BB'P_{\varepsilon_1(x)}x \\ &- 2\varepsilon_1(x)x'P_{\varepsilon_1(x)}B\mathcal{B}'\mathcal{P}_{\varepsilon_2(\chi)}\chi + x'\frac{\mathrm{d}}{\mathrm{d}t}\left(P_{\varepsilon_1(x)}\right)x.\end{aligned}$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) \leq 0$, the properties of the ARE imply that $\frac{\mathrm{d}}{\mathrm{d}t}P_{\varepsilon_1(x)} \leq 0$. Furthermore,

$$\|2\varepsilon_1(x)x'P_{\varepsilon_1(x)}B\mathcal{B}'\mathcal{P}_{\varepsilon_2(\chi)}\chi\| \leq 2\varepsilon_1(x)\|x\|\|F_{\varepsilon_1(x)}\|\delta < \frac{2\delta}{\ell}\varepsilon_1(x)\|x\|^2\|F_1\| \leq \frac{1}{2}\varepsilon_1(x)x'x,$$

where we have used the properties $\|\mathcal{B}'\mathcal{P}_{\varepsilon_2(\chi)}\chi\| \leq \delta \leq \frac{\ell}{4\|F_1\|}$, $\|P_{\varepsilon_1(x)}B\| = \|F_{\varepsilon_1(x)}\| \leq \|F_1\|$, and $x \notin K \implies \varepsilon_1(x) < 1 \implies \|x\| > \ell$. (The latter implication can be confirmed from (6.36) by noting that $\|x\| \leq \ell \implies x'P_1x \cdot \mathrm{trace}(B'P_1B) \leq \frac{1}{4}$.)

Combining the above expressions, we obtain $\frac{\mathrm{d}}{\mathrm{d}t}(x'P_{\varepsilon_1(x)}x) \leq -\frac{1}{2}\varepsilon_1(x)x'x < 0$. However, the properties of the scheduling then imply that $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) > 0$, which yields a contradiction with the assumption $\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_1(x) \leq 0$. We have therefore shown that $\varepsilon_1(x)$ is strictly increasing when $x \notin K$, which implies that x converges to, and remains in, K.

Let $t^* > 0$ be such that for all $t \geq t^*$, $x \in K$. Then for all $t \geq t^*$, $u = F_1x + v$, where $v = -\mathcal{B}\mathcal{P}_{\varepsilon_2(\chi)}\chi$. For all $t \geq t^*$, the output z of the L_1 subsystem is therefore described by

$$z(t) = C\exp^{(A+BF_1)(t-t^*)}x(t^*) + \int_{t^*}^{t} C\exp^{(A+BF_1)(t-\tau)}Bv(\tau)\,\mathrm{d}\tau.$$

The properties of the scheduling guarantee that $\|v\| \leq \delta \leq \frac{1}{2\rho}$. Let $T \geq t^*$ be such that for all $t \geq T$, $\|C \exp^{(A+BF_1)(t-t^*)} x(t^*)\| \leq \frac{1}{2}$. Then for all $t \geq T$,

$$\begin{aligned}
\|z(t)\| &\leq \|C \exp^{(A+BF_1)(t-t^*)} x(t^*)\| + \left\| \int_{t^*}^{t} C \exp^{(A+BF_1)(t-\tau)} Bv(\tau)\,\mathrm{d}\tau \right\| \\
&\leq \frac{1}{2} + \int_{t^*}^{t} \left\| C \exp^{(A+BF_1)(t-\tau)} B \right\| \|v(\tau)\|\,\mathrm{d}\tau \\
&\leq \frac{1}{2} + \int_{0}^{\infty} \left\| C \exp^{(A+BF_1)t} B \right\| \mathrm{d}\tau\, \frac{1}{2\rho} = 1.
\end{aligned}$$

Hence, for all $t \geq T$, the sandwiched saturation is inactive, and the system is therefore described by the equation $\dot{\chi} = (\mathcal{A} + \mathcal{B}\bar{F})\chi - \mathcal{B}\mathcal{P}_{\varepsilon_2(\chi)}\chi$. From [68] we know that the origin of this system is globally asymptotically stable. ∎

Remark 6.6 *To implement the globally stabilizing controller, one needs to calculate the parameter* δ*, which is used in the scheduling* (6.38)*. This, in turn, requires calculating* P_1*,* F_1*, and* ρ*.* P_1 *is found by solving* (6.31) *with* $\varepsilon_1 = 1$*, and* $F_1 = -B'P_1$*. After* F_1 *has been found,* ρ *can be calculated by numerical integration according to* (6.37)*.*

6.5.2. Discrete-time systems

Semi-global stabilization

We now present a generalized low-gain design for solving Problem 6.1 concerning the semi-global stabilization of the origin of the single-layer sandwich system subject to input saturation described by (6.3) and (6.4).

Let $P_{1,\varepsilon_1} = P'_{1,\varepsilon_1} > 0$ be the unique positive-definite solution of the algebraic Riccati equation

$$P_{1,\varepsilon_1} = A'P_{1,\varepsilon_1}A + \varepsilon_1 I - A'P_{1,\varepsilon_1}B(B'P_{1,\varepsilon_1}B + I)^{-1}B'P_{1,\varepsilon_1}A, \tag{6.39}$$

and define

$$F_{1,\varepsilon_1} = -(B'P_{1,\varepsilon_1}B + I)^{-1}B'P_{1,\varepsilon_1}A. \tag{6.40}$$

Next, let $P_{2,\varepsilon_2} = P'_{2,\varepsilon_2} > 0$ be the unique positive-definite solution of the algebraic Riccati equation

$$P_{2,\varepsilon_2} = \tilde{\mathcal{A}}' P_{2,\varepsilon_2} \tilde{\mathcal{A}} + \varepsilon_2 I - \tilde{\mathcal{A}}' P_{2,\varepsilon_2} \mathcal{B}(\mathcal{B}' P_{2,\varepsilon_2} \mathcal{B} + I)^{-1} \mathcal{B}' P_{2,\varepsilon_2} \tilde{\mathcal{A}}, \tag{6.41}$$

and define

$$F_{2,\varepsilon_2} = -(\mathcal{B}' P_{2,\varepsilon_2} \mathcal{B} + I)^{-1} \mathcal{B}' P_{2,\varepsilon_2} \tilde{\mathcal{A}}, \tag{6.42}$$

where $\tilde{\mathcal{A}}$ and $\mathcal{B}$ are given by

$$\tilde{\mathcal{A}} = \begin{pmatrix} A + BF_{1,\varepsilon_1} & 0 \\ NC & M \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}.$$

We define the following family of control laws:

$$u = F_{1,\varepsilon_1} x + F_{2,\varepsilon_2} \bar{x}. \tag{6.43}$$

The family of control laws is parameterized by the parameters $\varepsilon_1, \varepsilon_2 > 0$, and we show in the next theorem that semi-global stabilization is achieved for suitably chosen values of these parameters.

Theorem 6.10 Consider the systems given by (6.3) and (6.4), satisfying Conditions 2, 3, and 1 of Theorem 6.2. For any compact set of initial conditions $\mathcal{W} \in \mathbb{R}^{n_1+n_2}$, there exists an $\varepsilon_1^* > 0$ such that for any ε_1 with $0 < \varepsilon_1 < \varepsilon_1^*$, there exists an $\varepsilon_2^*(\varepsilon_1)$ such that for all $0 < \varepsilon_2 < \varepsilon_2^*(\varepsilon_1)$, the controller defined by (6.43) asymptotically stabilizes the origin with a domain of attraction containing $\mathcal{W}$.

Proof : By Conditions 2 and 3 of Theorem 6.2, we know that the eigenvalues of A and M are in the closed unit disc. This implies that $\lim_{\varepsilon_1 \to 0} P_{1,\varepsilon_1} = 0$ and $\lim_{\varepsilon_2 \to 0} P_{2,\varepsilon_2} = 0$, and hence we know that

$$\lim_{\varepsilon_1 \to 0} F_{1,\varepsilon_1} = 0, \qquad \lim_{\varepsilon_2 \to 0} F_{2,\varepsilon_2} = 0. \tag{6.44}$$

Note that the initial conditions belong to some compact set $\mathcal{W}$, and hence there exist compact sets $\mathcal{X} \subset \mathbb{R}^{n_1}$ and $\Omega \subset \mathbb{R}^{n_2}$ such that $x(0) \in \mathcal{X}$ and $\omega(0) \in \Omega$. Define a family of sets $\mathcal{V}_1(c) = \{ x \in \mathbb{R}^{n_1} \mid x' P_{1,\varepsilon_1} x \leq c \}$.

If we apply $u = F_{1,\varepsilon_1} x$, it is proved in [58] that there exists an $\varepsilon_1^* > 0$ such that for all $0 < \varepsilon_1 < \varepsilon_1^*$ and for all $x(0) \in \mathcal{X}$,

$$\| F_{1,\varepsilon_1} (A + BF_{1,\varepsilon_1})^k x(0) \| \leq \tfrac{1}{4}. \tag{6.45}$$

Moreover, there exists a $K > 0$, dependent on ε_1, such that $x(K) \in \mathcal{V}_1(c_1)$ for all $x(0) \in \mathcal{X}$. Here c_1 is such that $x \in \mathcal{V}_1(c_1)$ implies that $\|Cx\| \leq \frac{1}{4}$ and $\|F_{1,\varepsilon_1}x\| \leq \frac{1}{4}$. Since $\omega(0) \in \Omega$, where Ω is a compact set, and $\sigma(z(k))$ is bounded, it follows that there exists a compact set $\bar{\Omega}$, independent of ε_2, such that $\omega(k) \in \bar{\Omega}$ for all $0 \leq k \leq K$. Define a family of sets

$$\mathcal{V}_2(c) = \left\{ \bar{x} \in \mathbb{R}^{n_1+n_2} \;\mid\; x'P_{1,\varepsilon_1}x + \bar{x}'P_{2,\varepsilon_2}\bar{x} \leq c \right\}.$$

Next, we note that for $u = F_{1,\varepsilon_1}x$, we have $x(K) \in \mathcal{V}_1(c_1)$. From (6.44) and our earlier conclusion that $\omega(k)$ is bounded for $0 \leq k \leq K$, we see that if we apply $u = F_{1,\varepsilon_1}x + F_{2,\varepsilon_2}\bar{x}$ then there exists an ε_2^*, dependent on ε_1, such that for all $0 < \varepsilon_2 < \varepsilon_2^*$, the following properties hold:

- $x(K) \in 2\mathcal{V}_1(c_1)$.
- If $x \in 2\mathcal{V}_1(c_1)$ and $\omega \in \bar{\Omega}$, then $\bar{x} \in 3\mathcal{V}_2(c_1)$.
- For any $\bar{x}$ such that $\bar{x} \in 3\mathcal{V}_2(c_1)$, we have $\|F_{2,\varepsilon_2}\bar{x}\| < \frac{1}{4}$.

At time $k = K$, we have $\bar{x} \in 3\mathcal{V}_2(c_1)$. This immediately implies that $\|F_{2,\varepsilon_2}\bar{x}\| \leq \frac{1}{4}$. Note that for any $\bar{x} \in 3\mathcal{V}_2(c_1)$

$$x'P_{1,\varepsilon}x \leq x'P_{1,\varepsilon}x + \bar{x}'P_{2,\varepsilon_2}\bar{x} \leq 9c_1,$$

and hence $x \in 3\mathcal{V}_1(c_1)$. But this implies that $\|F_{1,\varepsilon_1}x\| \leq \frac{3}{4}$. Therefore, we have that $\|F_{1,\varepsilon_1}x + F_{2,\varepsilon_2}\bar{x}\| \leq 1$.

Similarly for any $\bar{x} \in 3\mathcal{V}_2(c_1)$, we have $x \in 3\mathcal{V}_1(c_1)$ and this implies that $\|Cx\| \leq \frac{3}{4}$. Therefore, for any $\bar{x} \in 3\mathcal{V}_2(c_1)$, both saturations are inactive.

We know at time K, the closed-loop system is linear and can be written as

$$\bar{x}(k+1) = (\tilde{\mathcal{A}} + \mathcal{B}F_{2,\varepsilon_2})\bar{x}(k). \tag{6.46}$$

It is straightforward to see that (6.46) is asymptotically stable and $3\mathcal{V}_2(c_1)$ is invariant. We know that the two saturations will remain inactive for all $k \geq K$. The asymptotic stability of (6.46) implies $\bar{x}(k) \to 0$ as $k \to \infty$. Since this holds for any $\bar{x}(0) \in \mathcal{W}$, it follows that $\mathcal{W}$ is contained in the domain of attraction. This completes the proof. ■

Global stabilization

We now present a generalized scheduled low-gain design for solving Problem 6.2 concerning the global stabilization of the single-layer sandwich system subject to input saturation described by (6.3), (6.4). As in previous section, this controller is formed by equipping semi-global controller (6.43) with scheduled parameters.

Let $P_{1,\varepsilon_1} = P'_{1,\varepsilon_1} > 0$ be the unique positive-definite solution of the algebraic Riccati equation (6.39) and F_{1,ε_1} be defined as (6.40) with scheduled parameter $\varepsilon_1 = \varepsilon_1(x)$.

Similar to that in the preceding section, a particular choice of scheduling is given by

$$\varepsilon_1(x) = \max\{\, r \in (0,1] \mid (x' P_{1,r} x)\|B' P_{1,r} B\| \le \tfrac{1}{4M_2} \,\} \tag{6.47}$$

where $P_{1,r}$ is the solution of ARE (6.39) with $\varepsilon_1 = r$, $M_2 = \sigma_{\max}(P_{1,1}^{\frac{1}{2}} B B' P_{1,1}^{\frac{1}{2}}) + 1$ and $P_{1,1}$ is the solution of ARE (6.39) with $\varepsilon_1 = 1$. It has been shown that above scheduling guarantees that

$$\left\|(B' P_{1,\varepsilon_1(x)} B + I)^{-1} B' P_{1,\varepsilon_1(x)} x\right\| \le \tfrac{1}{2}.$$

Let $\ell > 0$ be such that

$$(\lambda_{\max}(P_{1,1}) + \tfrac{1}{2})\ell^2 \le \tfrac{1}{4M_2\|B' P_{1,1} B\|},$$

and let $P_{2,\varepsilon_2} = P'_{2,\varepsilon_2} > 0$ be the unique positive-definite solution of the algebraic Riccati equation (6.41) and F_{2,ε_2} be defined by (6.42) where in both (6.41) and (6.42), we take

$$\tilde{\mathcal{A}} = \begin{pmatrix} A + BF_{1,1} & 0 \\ NC & M \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} B \\ 0 \end{pmatrix}, \tag{6.48}$$

and $\varepsilon_2 = \varepsilon_2(\bar{x})$ is a scheduled parameter. Choose

$$\delta_2 = \min\left\{\frac{1}{2}, \frac{\ell^2}{2(3\,\|B' P_{1,1} B\| + 1)}, \frac{1}{2\rho}\right\}, \tag{6.49}$$

where $\rho = \sum_{k=0}^{\infty} \|C(A + BF_{1,1})^k B\|$. Consider an associated scheduled parameter given by

$$\varepsilon_2(\bar{x}) = \max\{\, r \in (0,1] \mid (\bar{x}' P_{2,r} \bar{x})\|\mathcal{B}' P_{2,r} \mathcal{B}\| \le \tfrac{\delta_2^2}{M_3} \,\} \tag{6.50}$$

where $M_3 = \sigma_{\max}(P_{2,1}^{\frac{1}{2}} \mathcal{B}\mathcal{B}' P_{2,1}^{\frac{1}{2}}) + 1$. We have $\|F_{2,\varepsilon}\| \le \delta_2$. The following theorem shows that a particular control law achieves global stability of the single-layer nonlinear sandwich system subject to input saturation.

Theorem 6.11 Consider the two systems given by (6.3) and (6.4), satisfying Conditions 2, 3 and 1 of Theorem 6.2. Let $P_{1,\varepsilon_1(x)}$ be the solution of (6.39) with ε_1 replaced by the scheduled low-gain parameter $\varepsilon_1(x)$ defined by (6.47). Let $P_{2,\varepsilon_2(\bar{x})}$ be the solution of (6.41) with $\tilde{\mathcal{A}}$ and $\mathcal{B}$ given by (6.48) and ε_2 replaced by the scheduled low-gain parameter $\varepsilon_2(\bar{x})$ defined by (6.50). The control law

$$u = F_{1,\varepsilon_1(x)}x + \varepsilon_1(x)F_{2,\varepsilon_2(\bar{x})}\bar{x} \tag{6.51}$$

achieves global asymptotic stability of the origin where $F_{1,\varepsilon_1(x)}$ and $F_{2,\varepsilon_2(\bar{x})}$ are respectively defined by (6.40) and (6.42) with ε_1 and ε_2 replaced by $\varepsilon_1(x)$ and $\varepsilon_2(\bar{x})$.

Proof : Note that our scheduled parameter guarantees that $\|u(k)\| \leq 1$ for all $k \geq 0$. The input saturation is always inactive.

Considering the interconnection of (6.3) and (6.4), we note that the sandwiched saturation is not activated near the origin. Moreover, near the origin the control law (6.51) is given by $u = F_{1,1}x + F_{2,1}\bar{x}$. This means that state matrix of the interconnection of (6.3), (6.4), and (6.51) equals $\tilde{\mathcal{A}} + \mathcal{B}F_{2,1}$ which is asymptotically stable by the properties of the algebraic Riccati equation. We have therefore established local asymptotic stability. It remains to show that we have global asymptotic stability.

Define $V = x'P_{\varepsilon_1(x)}x$ and $\mathcal{V}_1 = \{x \in \mathbb{R}^{n_1} \mid \|x\| \leq \ell\}$ and $\mathcal{V}_2 = \{x \in \mathbb{R}^{n_1} \mid V(x) \leq (\lambda_{\max}(P_{1,1})+1/2)\ell^2\}$. Since $\|x(k)\| \leq \ell$ implies that $V(x) \leq \lambda_{\max}(P_{\varepsilon_1(x(k))})\|x(k)\|^2 \leq \lambda_{\max}(P_{1,1})\ell^2$, we have that $\mathcal{V}_2 \supset \mathcal{V}_1$. Moreover, from definition of ℓ, we have that $\varepsilon_1(x) = 1$ for $x \in \mathcal{V}_2$. We first want to establish that $V(k)$ is strictly decreasing in time when $x \notin \mathcal{V}_1$.

Assume that this is not the case and we can find $x(k) \notin \mathcal{V}_1$ such that $V(k+1) - V(k) \geq 0$. Denote $\varepsilon_1(x(k))$ and $P_{1,\varepsilon_1(x(k))}$ by $\varepsilon_1(k)$ and $P_1(k)$ respectively. We obtain

$$\begin{aligned}V(k+1) - V(k) \leq -\varepsilon_1(k)x(k)'x(k) - x(k+1)'P_1(k)x(k+1) + x(k+1)'P_1(k+1)x(k+1) - \\ 2x(k)'A'P_1(k)Bv_2(k) - 2v_1(k)'B'P_1(k)Bv_2(k) + v_2(k)'B'P_1(k)Bv_2(k)\end{aligned}$$

where $v_1(k) = F_{1,\varepsilon_1(k)}x(k)$ and $v_2(k) = -\varepsilon_1(k)F_{2,\varepsilon_2(k)}\bar{x}(k)$.

Our scheduling guarantees that $\|v_1(k)\| \leq \frac{1}{2}$ and $\|v_2(k)\| \leq \varepsilon_1(k)\delta_2$ and hence

$$\begin{aligned}
&\|x(k)'A'P_1(k)Bv_2(k)\| = \|v_1(k)'(B'P_1(k)B + I)v_2(k)\| \leq \tfrac{1}{2}\varepsilon_1(k)(\left\|B'P_{1,1}B\right\| + 1)\delta_2 \\
&\|v_1(k)'B'P_1(k)B'v_2(k)\| \leq \tfrac{1}{2}\varepsilon_1(k)\left\|B'P_{1,1}B\right\|\delta_2 \\
&\|v_2(k)'B'P_1(k)Bv_2(k)\| \leq \varepsilon_1(k)^2\left\|B'P_{1,1}B\right\|\delta_2^2 \leq \varepsilon_1(k)\left\|B'P_{1,1}B\right\|\delta_2.
\end{aligned} \tag{6.52}$$

Therefore

$$\begin{aligned}
&V(k+1) - V(k) \\
\leq &-\varepsilon_1(k)x'(k)x(k) + x(k+1)'(P_1(k+1) - P_1(k))x(k+1) + \varepsilon_1(k)(3\left\|B'P_{1,1}B\right\| + 1)\delta_2 \\
\leq &-\varepsilon_1(k)x'(k)x(k) + x(k+1)'(P_1(k+1) - P_1(k))x(k+1) + \tfrac{1}{2}\varepsilon_1(k)\ell^2 \\
\leq &-\tfrac{1}{2}\varepsilon_1(k)\|x(k)\|^2 + x(k+1)'(P_1(k+1) - P_1(k))x(k+1),
\end{aligned} \tag{6.53}$$

where we use that $x(k) \notin \mathcal{V}_1$ and hence $\|x(k)\| \geq \ell$. Since $V(k+1) - V(k) \geq 0$, the properties of our scheduling imply that $x(k+1)'(P_1(k+1) - P_1(k))x(k+1) \leq 0$. We get

$$V(k+1) - V(k) \leq -\tfrac{1}{2}\varepsilon_1(k)\|x(k)\|^2 < 0.$$

This yields a contradiction. Hence when $x(k) \notin \mathcal{V}_1$ we have that $V(k)$ is strictly decreasing, and it follows that $x(k)$ enters $\mathcal{V}_1$ within finite time, say K_1. When $x(k) \in \mathcal{V}_1$, we have either $V(k+1) - V(k) \leq 0$ or $x(k+1)'(P_1(k+1) - P_1(k))x(k+1) \leq 0$, and (6.53) yields that

$$V(k+1) - V(k) \leq \tfrac{1}{2}\varepsilon_1(k)\ell^2 \leq \tfrac{1}{2}\ell^2.$$

This implies that $V(k+1) \leq \lambda_{\max}(P_{1,1})\ell^2 + \frac{1}{2}\ell^2$ and hence $x(k+1) \in \mathcal{V}_2$. We find that if $x(k) \in \mathcal{V}_1$ then $x(k+1) \in \mathcal{V}_2$. On the other hand, if $x(k) \in \mathcal{V}_2 \backslash \mathcal{V}_1$ then $V(k)$ is strictly decreasing and hence $x(k+1) \in \mathcal{V}_2$. Therefore, $x(k)$ will enter $\mathcal{V}_2$ and it cannot escape from $\mathcal{V}_2$. On $\mathcal{V}_2$ we have $\varepsilon_1(k) = 1$. The L_1 system then becomes:

$$\begin{aligned}
x(k+1) &= (A + BF_{1,1})x(k) + Bv_2(k) \\
z(k) &= Cx(k),
\end{aligned} \tag{6.54}$$

where $\|v_2(k)\| \leq \delta_2$. We have for any $k > K_1$

$$z(k) = C(A + BF_{1,1})^{k-K_1}x(K_1) + z_0(k)$$

where

$$z_0(k) = \sum_{i=K_1}^{k-1} C(A + BF_{1,1})^{k-i-1} Bv_2(i). \tag{6.55}$$

Given that $\delta_2 \leq \frac{1}{2\rho}$ as given by (6.49), we have $\|v(k)\| < \frac{1}{2\rho}$ for all $k > K_1$. But this guarantees that $\|z_0(k)\| < \frac{1}{2}$ for all $k > K_1$, where $z_0(k)$ is defined by (6.55). Therefore there exists a K_2 such that for $k \geq K_2$

$$\|C(A + BF_{1,1})^{k-K_1} x(K_1)\| \leq \tfrac{1}{2}$$

and hence $\|z(k)\| \leq 1$ for $k \geq K_2$. We can then apply Lemma 6.4 as in the previous subsection, and we conclude that the system therefore behaves like a stable system after a finite amount of time, and it follows that $x(k) \to 0$ and $\omega(k) \to 0$ as $k \to \infty$. ■

6.6. Generalized low-gain design for multi-layer sandwich systems

6.6.1. Semi-global stabilization

Now we construct a linear semi-globally stabilizing controller for the multi-layer sandwich system which solves semi-global stabilization as formulated in Problem 6.1.

Consider the interconnection of L_i as defined in (6.5). Let P_i be the positive definite solution of Riccati equation

$$P_{\varepsilon_i} = \mathcal{A}_i' P_{\varepsilon_i} \mathcal{A}_i + \varepsilon_i I - \mathcal{A}_i' P_{\varepsilon_i} \mathcal{B}_i (\mathcal{B}_i' P_{\varepsilon_i} \mathcal{B}_i + I)^{-1} \mathcal{B}_i' P_{\varepsilon_i} \mathcal{A}_i \tag{6.56}$$

and define

$$F_{\varepsilon_i} = -(\mathcal{B}_i' P_{\varepsilon_i} \mathcal{B}_i + I)^{-1} \mathcal{B}_i' P_{\varepsilon_i} \mathcal{A}_i \tag{6.57}$$

where

$$\mathcal{A}_1 = A_1, \quad \mathcal{A}_i = \begin{pmatrix} \mathcal{A}_{i-1} + \mathcal{B}_{i-1} F_{\varepsilon_{i-1}} & 0 \\ B_i \mathcal{C}_{i-1} & A_i \end{pmatrix} \quad \text{for } i = 2, \dots, n$$

and

$$\mathcal{B}_i = \begin{pmatrix} B_1' & 0 & \cdots & 0 \end{pmatrix}', \quad \mathcal{C}_i = \begin{pmatrix} 0 & \dots & 0 & C_i \end{pmatrix} \tag{6.58}$$

are of appropriate dimensions. The parameters ε_i, $i = 1, \dots, n$ are to be determined appropriately shortly. We have the following theorem:

Theorem 6.12 Consider interconnection of n systems as given by (6.5), satisfying Conditions 1, 2 of Theorem 6.3. Let P_{ε_i} be the solution of Riccati equations in (6.56) with $\varepsilon_i \in (0, 1]$, $i = 1, \dots, n$. For any compact set $\mathcal{W} \subset \mathbb{R}^{\sum_{i=1}^{n} n_i}$, we can determine ε_i, $i = 1, \dots, n$ such that the controller

$$u = \sum_{i=1}^{n} F_{\varepsilon_i} \chi_i \tag{6.59}$$

renders the origin asymptotically stable with a domain of attraction containing $\mathcal{W}$ where

$$\chi_i = \begin{pmatrix} x_1' & \cdots & x_i' \end{pmatrix}'. \tag{6.60}$$

Proof : For simplicity of presentation, denote P_{ε_i} and F_{ε_i} by P_i and F_i.

Conditions (1) and (2) of Theorem 6.3 and the fact that $\mathcal{A}_i + \mathcal{B}_i F_i$ is asymptotically stable imply that

$$\lim_{\varepsilon_i \to 0} P_i = 0, \quad \lim_{\varepsilon_i \to 0} F_i = 0 \tag{6.61}$$

Define function $V_i(\chi_i) = \sum_{j=1}^{i} \chi_j' P_j \chi_j$ and set $\mathcal{V}_i(c) = \left\{ \chi_i \in \mathbb{R}^{\sum_{j=1}^{i} n_j} \mid V_i(\chi_i) \le c \right\}$.

Since $\mathcal{W}$ is compact, there exist for $i = 1, \dots, n$, compact sets $\mathcal{W}_i$ such that $\chi_n(0) \in \mathcal{W}$ implies that $x_i(0) \in \mathcal{W}_i$. Next we determine ε_i recursively.

Determine ε_1: Let's consider applying a controller $v_1 = F_1 \chi_1 = F_1 x_1$.

Note that (6.61) implies the existence of an ε_1^* such that for any $\varepsilon \in (0, \varepsilon_1^*]$ and $x_1(0) \in \mathcal{W}_1$, we have

$$\|F_1(A_1 + B_1 F_1)^k x_1(0)\| \le \tfrac{1}{4^{n-1}}$$

for all $k \ge 0$. Let c_1 be such that, $x_1 \in \mathcal{V}_1(c_1)$ implies $\|F_1 x_1\| \le \frac{1}{4^{n-1}}$ and $\|C_1 x_1\| \le \frac{1}{3^{n-1}}$. Since $A_1 + B_1 F_1$ is asymptotically stable, there exists a K_1 such that for all $x_1 \in \mathcal{W}_1$, we have $x_1(K_1) \in \mathcal{V}_1(c_1)$.

Determine ε_2: Since $x_2(0) \in \mathcal{W}_2$ and the input to L_2 is bounded, there exists a $\bar{\mathcal{W}}_2$ such that

$$x_2(k) \in \bar{\mathcal{W}}_2, \quad \text{for } k \le K_1.$$

Let ε_1 be fixed. Consider applying the controller $v_2 = F_1 x_1 + F_2 \chi_2$. Due to (6.61), given $x_2 \in \mathcal{W}_2$, there exists an $\varepsilon_2^*(\varepsilon_1)$ such that the following properties hold:

1. For any $\varepsilon_2 \in (0, \varepsilon_2^*(\varepsilon_1)]$, $x_1(K_1) \in 2\mathcal{V}_1(c_1)$.

2. For any $\varepsilon_2 \in (0, \varepsilon_2^*(\varepsilon_1)]$, $x_1 \in 2\mathcal{V}_1(c_1)$ and $x_2 \in \bar{\mathcal{W}}_2$ imply $\chi_2 \in 3\mathcal{V}_2(c_1)$.

3. For any $\varepsilon_2 \in (0, \varepsilon_2^*(\varepsilon_1)]$, $\chi_2 \in 3\mathcal{V}_2(c_1)$ implies $\|F_2\chi_2\| \le \frac{1}{4^{n-1}}$.

At time $k = K_1$, we know $\chi_2 \in 3\mathcal{V}_2(c_1)$. For any $\chi_2 \in 3\mathcal{V}_2(c_1)$, $\|F_2\chi_2\| \le \frac{1}{4^{n-1}}$. Also note that $\chi_2 \in 3\mathcal{V}_2(c_1)$ implies then $V_1(x_1) \le 9c_1$ and hence $F_1 x_1 \le \frac{3}{4^{n-1}}$ and $\|C_1 x_1\| \le \frac{1}{3^{n-2}}$. We have

$$\|u\| = \|F_1 x_1 + F_2\chi_2\| \le \frac{1}{4^{n-2}}.$$

Therefore two saturations are both inactive in $3\mathcal{V}_2(c_1)$, it is straightforward to see that with controller v_2, $\chi_2(k) \in 3\mathcal{V}_2(c_1)$ for all $k \ge K_1$ and moreover $\chi_2(k) \to 0$ as $k \to \infty$.

Let c_2 be such that $\chi_2' P_2 \chi_2 \le c_2$ implies $\|C_1 x_1\| \le \frac{1}{3^{n-2}}$, $\|\mathcal{C}_2\chi_2\| \le \frac{1}{3^{n-2}}$. There exists a K_2 such that for all $\chi_2(K_1) \in 3\mathcal{V}_2(c_1)$, we have $\chi_2(K_2) \in \mathcal{V}_2(c_2)$. At time K_2, we get

1. $\chi_2(K_2) \in \mathcal{V}_2(c_2)$.

2. $\|C_1 x_1(K_2)\| \le \frac{1}{3^{n-2}}$ and $\|\mathcal{C}_2\chi_2(K_2)\| \le \frac{1}{3^{n-2}}$.

3. $\|F_1 x_1 + F_2\chi_2\| \le \frac{1}{4^{n-2}}$ for all $\chi_2 \in \mathcal{V}_2(c_2)$ and $k \le K_2$.

Determine $\varepsilon_3, \ldots, \varepsilon_n$: Consider system L_i, $i \ge 3$. At this moment, ε_j, c_j and K_j for $j \le i-1$ have been determined in previous $i-1$ steps. The resulting controller $v_{i-1} = \sum_{j=1}^{i-1} F_j\chi_j$ yields

1. $\chi_{i-1}(K_{i-1}) \in \mathcal{V}_{i-1}(c_{i-1})$.

2. $\|\mathcal{C}_j\chi_j(K_{i-1})\| \le \frac{1}{3^{n-i+1}}$ for all $j \le i-1$.

3. $\|\sum_{j=1}^{i-1} F_j\chi_j\| \le \frac{1}{4^{n-i+1}}$ for all $\chi_{i-1} \in \mathcal{V}_{i-1}(c_{i-1})$.

Since the input to L_i is bounded and $x_i(0) \in \mathcal{W}_i$, we know that there exists a $\bar{\mathcal{W}}_i$ such that $x_i(k) \in \bar{\mathcal{W}}_i$ for all $k \le K_{i-1}$. Consider the controller $v_i = \sum_{j=1}^{i} F_j\chi_j$. Then (6.61) implies the existence of an $\varepsilon_i^*(\varepsilon_1, \ldots, \varepsilon_{i-1})$ such that the following properties hold:

1. $\chi_{i-1}(K_{i-1}) \in 2\mathcal{V}_{i-1}(c_{i-1})$.

2. $\chi_{i-1} \in 2\mathcal{V}_{i-1}(c_{i-2})$ and $x_i \in \bar{\mathcal{W}}_i$ imply that $\chi_i \in 3\mathcal{V}_i(c_{i-1})$.

3. $\chi_i \in 3\mathcal{V}_i(c_{i-1})$ implies $F_i\chi_i \leq \frac{1}{4^{n-i+1}}$.

Therefore, we get at $k = K_{i-1}$, $\chi_i(K_{i-1}) \in 3\mathcal{V}_i(c_{i-1})$, i.e $V_i(\chi) \leq 9c_{i-1}$. But this implies that $V_{i-1}(\chi) \leq 9c_{i-1}$. Hence we have $\|\mathcal{C}_j\chi_j\| \leq \frac{1}{3^{n-i}}$ for all $j = 1, \ldots, i-1$ and that $\|F_i\chi_i\| \leq \frac{3}{4^{n-i+1}}$. Moreover

$$\|v_i\| = \|F_j\chi_i\| + \|\sum_{j=1}^{i-1} F_j\chi_j\| \leq \frac{1}{4^{n-i+1}} + \frac{3}{4^{n-i+1}} = \frac{1}{4^{n-i}}.$$

In conclusion, the first i saturations are inactive for any $\chi_i \in 3\mathcal{V}_i(c_{i-1})$. It is easy to see that with controller v_i, $\chi(k) \in 3\mathcal{V}_i(c_{i-1})$ for all $k \geq K_{i-1}$ and moreover $\chi_i(k) \to 0$ as $k \to \infty$.

Let c_i be such that $V_i(\chi_i) \leq c_i$ implies that $\|\mathcal{C}_j\chi_j\| \leq \frac{1}{3^{n-i}}$ for all $j \leq i$. There exists a K_i such that $\chi_i(K_i) \in \mathcal{V}(c_i)$ for all $\chi_i(K_{i-1}) \in 3\mathcal{V}_i(c_{i-1})$. At time K_i, we have

1. $\chi_i(K_i) \in \mathcal{V}_i(c_i)$.

2. $\|\mathcal{C}_j\chi_j(K_{i-1})\| \leq \frac{1}{3^{n-i}}$ for all $j \leq i$.

3. $\|\sum_{j=1}^{i} F_j\chi_j\| \leq \frac{1}{4^{n-i}}$ for all $\chi_i \in \mathcal{V}_i(c_i)$.

Repeating this procedure, we can determine $\varepsilon_1, \ldots, \varepsilon_n$, c_n, K_n and a controller $u(\chi_n) = v_n(\chi_n) = \sum_{i=1}^{n} F_i\chi_i$ such that for $k \geq K_n$ we have:

1. $\chi_n(K_n) \in \mathcal{V}_n(c_n)$.

2. $\|\mathcal{C}_j\chi_j(K_n)\| \leq 1$ for all $j \leq n$.

3. $\|\sum_{j=1}^{n} F_j\chi_j\| \leq 1$ for all $\chi_{i-1} \in 3\mathcal{V}_{i-1}(c_0)$.

Then the interconnection of n systems is equivalent to

$$\dot{\chi} = (\mathcal{A}_n + \mathcal{B}_nF_n)\chi_n.$$

The stability of this system implies that $\chi_n(k) \to 0$ as $k \to \infty$. This completes the proof. ■

6.6.2. Global stabilization

In this section we construct global stabilizing controller for multi-layer systems to prove sufficiency of Conditions 1 and 2 in Theorem 6.3. This controller is formed by assembling semi-global stabilizing controller (6.59) with scheduled parameters.

Let $P_{\varepsilon_i(\chi_i)}$ be the positive definite solution of Riccati equation (6.56) and $F_{\varepsilon_i(\chi_i)}$ be defined by (6.57) where $\varepsilon_i = \varepsilon_i(\chi_i)$ is a scheduled parameter, $\mathcal{B}$ is given by (6.58) and

$$\mathcal{A}_1 = A_1, \quad \mathcal{A}_i = \begin{pmatrix} \mathcal{A}_{i-1} + \mathcal{B}_{i-1}F_{i-1,1} & 0 \\ B_i\mathcal{C}_{i-1} & A_i \end{pmatrix}, \quad i = 2, \ldots, n \tag{6.62}$$

where $F_{i,1} = -(\mathcal{B}_i' P_{\varepsilon_i}\mathcal{B}_i + I)^{-1}\mathcal{B}_i' P_{\varepsilon_i}\mathcal{A}_i$ with $\varepsilon_i = 1$.

We need n scheduled parameters which satisfy similar properties as given in Section 6.2. Choose

$$\delta_1 = \tfrac{1}{n}, \quad \delta_i = \min\left\{\frac{1}{n}, \delta_{i-1}, \frac{\ell_{i-1}^2}{2(n-i+1)^2(\frac{n+2}{n}\,\|\mathcal{B}_{i-1}' P_{i-1,1}\mathcal{B}_{i-1}\| + \frac{2}{n})}, \frac{1}{2(n-i+1)\rho_{i-1}}\right\} \tag{6.63}$$

for $i = 2, \ldots, n$ where ℓ_i is such that

$$(\lambda_{\max}(P_{i,1}) + \tfrac{1}{2})\ell_i^2 \leq \frac{\delta_i^2}{M_i\|B' P_{i,1} B\|}$$

and

$$\rho_i = \sum_{k=0}^{\infty} \|\mathcal{C}_i(\mathcal{A}_i + \mathcal{B}_i F_{i,1})^k \mathcal{B}_i\|\}.$$

Consider the following scheduled parameters

$$\varepsilon_i(\chi_i) = \max\{\, r \in (0,1] \mid (\chi_i' P_r \chi_i)\|\mathcal{B}_i' P_r \mathcal{B}_i\| \leq \tfrac{\delta_i^2}{M_i}\,\} \tag{6.64}$$

where χ_i is given by (6.60), P_r is the solution of (6.56) with $\varepsilon_i = r$, $M_i = \sigma_{\max}(P_{i,1}^{\frac{1}{2}}\mathcal{B}_i\mathcal{B}_i' P_{i,1}^{\frac{1}{2}}) + 1$ and $P_{i,1}$ is the solution of (6.56) with $\varepsilon_i = 1$. Consider the controller

$$u_1 = \sum_{i=1}^{n}\Big(\prod_{j=0}^{i-1}\varepsilon_j(\chi_j)\Big)F_{\varepsilon_i(\chi_i)}\chi_i \tag{6.65}$$

with $\varepsilon_0 = 1$. It has been shown that our scheduling with δ_i defined in (6.63) guarantees that $\|F_{\varepsilon_i(\chi_i)}\chi_i\| \leq \frac{1}{n}$ and hence $\|u_1\| \leq 1$. This implies that the input saturation to the first system never gets activated. The following theorem shows that the controller (6.65) with tuning parameters defined by (6.64) achieves global asymptotic stability of the origin for multi-layer nonlinear sandwich system.

Theorem 6.13 Consider interconnection of system L_i given in (6.5), satisfying Conditions 1, 2 of Theorem 6.3. The control (6.65) achieves global asymptotic stability of the origin.

Proof : For the simplicity of presentation, we denote $\varepsilon_i(\chi_i(k))$, $P_{\varepsilon_i(\chi_i(k))}$ and $F_{\varepsilon_i(\chi_i(k))}$ by $\varepsilon_i(k)$, $P_i(k)$ and $F_i(k)$ respectively. But we emphasize that they always depend on χ_i.

When the state is sufficiently close to the origin, all saturation elements are inactive and $\varepsilon_i(\chi_i) = 1$ for all $i = 1, \dots, n$. The state matrix of the closed-loop system is given by $\mathcal{A}_n + \mathcal{B}_n F_{n,1}$. From the property of ARE, we know that the above matrix is asymptotically stable. Then local stability follows.

We shall prove global attractivity using induction. We have argued that for all $k \geq 0$, the input saturation on L_1 remains inactive and by construction $\varepsilon_0 = 1$. Suppose there exists a K_i with $1 \leq i \leq n-1$ such that $\varepsilon_j = 1$ for $j \leq i-1$ and the first i saturations are inactive for all $k \geq K_i$. We shall show that there exists a K_{i+1} such that $\varepsilon_i = 1$ and saturation on L_{i+1} will be inactive for all $k \geq K_{i+1}$. By assumption, for $k \geq K_i$, the interconnection of first i systems is equivalent to the following linear system

$$\dot{\chi}_i = \mathcal{A}_i \chi_i + \mathcal{B}_i v_1 \tag{6.66}$$

where $\mathcal{A}_i$ is given by (6.62) and v_1 is given by

$$v_1 = v_{1,1} + v_{1,2} = F_i \chi_i + \sum_{j=i+1}^{n} \Big(\prod_{t=i}^{j-1} \varepsilon_t\Big) F_j \chi_j.$$

Define $V_i(k) = \chi_i' P_i \chi_i$ and the family of sets $\mathcal{V}_{i,1} = \{ \chi_i \in \mathbb{R}^{\sum_{j=1}^{i} n_j} \mid \|\chi_i\| \leq \ell_i \}$ and $\mathcal{V}_{i,2} = \{\chi_i \in \mathbb{R}^{\sum_{j=1}^{i} n_j} \mid V_i \leq (\lambda_{\max}(P_{i,1}) + 1/2)\ell_i^2 \}$. Since $x(k) \in \mathcal{V}_{i,1}$ implies

$$V_i(k) \leq \lambda_{\max}(P_i(k))\|\chi_i(k)\|^2 \leq \lambda_{\max}(P_{i,1})\ell_i^2,$$

we find that $\mathcal{V}_{i,1} \subset \mathcal{V}_{i,2}$. Moreover, the definition of ℓ_i implies that $\varepsilon_i(k) = 1$ for $\chi_i(k) \in \mathcal{V}_{i,2}$.

Evaluating $V_i(k+1) - V_i(k)$ along the trajectories yields:

$$\begin{aligned} V_i(k+1) - V_i(k) \leq\ & -\varepsilon_i(k)\chi_i(k)'\chi_i(k) - \chi_i(k+1)' P_i(k)\chi_i(k+1) + \chi_i(k+1)' P_i(k+1)\chi_i(k+1) - \\ & 2\chi_i(k)'\mathcal{A}_i' P_i(k)\mathcal{B}_i v_{1,2}(k) - 2v_{1,1}(k)'\mathcal{B}_i' P_i(k)\mathcal{B}_i v_{1,2}(k) + v_{1,2}(k)'\mathcal{B}_i' P_i(k)\mathcal{B}_i v_{1,2}(k) \end{aligned}$$

where

$$v_{1,1}(k) = F_i(k)\chi_i(k), \qquad v_{1,2}(k) = \sum_{j=i+1}^{n} (\prod_{t=i}^{j-1} \varepsilon_t(k)) F_j(k)\chi_j(k).$$

Our scheduling guarantees that $\|v_{1,1}(k)\| \leq \frac{1}{n}$ and $\|v_{1,2}(k)\| \leq \varepsilon_i(k)(n-i)\delta_{i+1}$ and hence

$$\|\chi_i(k)'\mathcal{A}_i' P_i(k)\mathcal{B}_i v_{1,2}(k)\| = \|v_{1,1}(k)'(\mathcal{B}_i' P_i(k)\mathcal{B}_i + I)v_{1,2}(k)\| \leq \varepsilon_i(k)\tfrac{(n-i)^2}{n}(\left\|\mathcal{B}_i' P_i(k)\mathcal{B}_i\right\| + 1)\delta_{i+1}$$

$$\|v_{1,1}(k)'\mathcal{B}_i' P_i(k)\mathcal{B}_i v_{1,2}(k)\| \leq \varepsilon_i(k)\tfrac{(n-i)^2}{n}\left\|\mathcal{B}_i' P_i(k)\mathcal{B}_i\right\|\delta_{i+1}$$

$$\|v_{1,2}(k)'\mathcal{B}_i' P_i(k)\mathcal{B}_i v_{1,2}(k)\| \leq \varepsilon_i(k)(n-i)^2\left\|\mathcal{B}_i' P_i(k)\mathcal{B}_i\right\|\delta_{i+1}.$$

With the above inequalities, we have

$$\begin{aligned}
& V_i(k+1) - V_i(k) \\
\leq & -\varepsilon_i(k)\|\chi_i(k)\|^2 + \chi_i(k+1)'(P_i(k+1) - P_i(k))\chi_i(k+1) + \varepsilon_i(k)(n-i)^2(\tfrac{n+4}{n}\left\|\mathcal{B}_i' P_i(k)\mathcal{B}_i\right\| + \tfrac{2}{n})\delta_{i+1} \\
\leq & -\varepsilon_i(k)\|\chi_i(k)\|^2 + \chi_i(k+1)'(P_i(k+1) - P_i(k))\chi_i(k+1) + \tfrac{1}{2}\varepsilon_i(k)\ell_i^2.
\end{aligned}$$

Using the same argument as in the proof of Theorem 6.11, we can show that if $\chi_i(k) \notin \mathcal{V}_{i,1}$ then $V_i(k)$ is strictly decreasing and hence χ_i will enter $\mathcal{V}_{i,1}$ within finite time. On the other hand, if $\chi_i(k) \in \mathcal{V}_{i,1}$ then $\chi_i(k+1) \in \mathcal{V}_{i,2}$. Since $\mathcal{V}_{i,1} \subset \mathcal{V}_{i,2}$, we conclude that χ_i will enter $\mathcal{V}_{i,2}$ within finite time, say $K_{i,1}$, and can not escape from it. On $\mathcal{V}_{i,2}$ we have $\varepsilon_i(k) = 1$.

Consider $z_i(k) = C_i x_i(k) = \mathcal{C}_i\chi_i(k)$ for $k \geq K_{i,1}$. Since $\varepsilon_i(k) = 1$, we have

$$z_i(k) = \mathcal{C}_i(\mathcal{A}_i + \mathcal{B}_i F_{i,1})^{k-K_{i,1}}\chi_i(K_{i,1}) + z_{i,0}(k)$$

where

$$z_{i,0}(k) = \sum_{j=K_{i,1}}^{k-1} \mathcal{C}_i(\mathcal{A}_i + \mathcal{B}_i F_{i,1})^{k-j-1}\mathcal{B}_i v_{1,2}(j).$$

Our scheduling guarantees that

$$v_{1,2} \leq (n-i)\delta_{i+1} \leq \tfrac{1}{2\rho_i} = \frac{1}{2\sum_{k=0}^{\infty}\|\mathcal{C}_i(\mathcal{A}_i - \mathcal{B}_i F_{i,1})^k\mathcal{B}_i\|}.$$

This implies that $\|z_{i,0}(k)\| \leq \frac{1}{2}$ for all $k \geq K_{i,1}$. Since $\mathcal{A}_i + \mathcal{B}_i F_{i,1}$ is asymptotically stable, there exists a $K_{i+1} > K_{i,1}$ such that for all $k \geq K_i$, we have $\|z_i(k)\| \leq 1$.

Therefore the input saturation on L_{i+1} will be inactive and $\varepsilon_i = 1$ for all $k \geq K_{i+1}$. By induction, there exists a K_n such that all the saturations are inactive for $k \geq K_n$ and $\varepsilon_i = 1$ for all $i = 0, \ldots, n-1$.

Then the interconnection of n systems (6.5) and controller (6.65) is equivalent to the interconnection of linear system

$$\chi_n(k+1) = \mathcal{A}_n \chi_n(k) + \mathcal{B}_n v_1$$

with controller $v_1 = F_{\varepsilon_n(\chi_n)}\chi_n = -(\mathcal{B}_n' P_{\varepsilon_n(\chi_n)} \mathcal{B}_n + I)^{-1} \mathcal{B}_n' P_{\varepsilon_n(\chi_n)} \mathcal{A}_n \chi_n$. It follows from Lemma 6.4 that the closed-loop system is globally asymptotically stable, i.e. $\chi_n(k) \to 0$ as $k \to \infty$. This shows global attractivity of the origin and completes the proof. ∎

6.7. Generalized low-and-high-gain design for single layer sandwich systems

We have developed the necessary and sufficient conditions under which sandwich non-linear systems and their generalizations can be stabilized either semi-globally or globally. We also developed low-gain and generalized scheduled low-gain design methodologies for constructing appropriate stabilizing controllers. The philosophy behind the design in the previous section can be briefly sketched as follows: we designed a controller such that the saturation does not get activated after some finite time. Thereafter, the design methodology reduces to a simple low gain or scheduled low gain design. However, such design methods based on standard low-gain or scheduled low gain design methods are conservative as they are constructed in such a way that the control forces do not exceed a certain level in an arbitrary, a priori given, region of the state space in the semi-global case or the whole state space in the global case. Hence the saturation remains inactive. Therefore, such generalized low-gain design methods do not allow full utilization of the available control capacity. Design methods based on low-and-high gain feedback design are conceived to rectify the drawbacks of low-gain design methods, and can utilize the available control capacity fully. As such, they have been successfully used for control problems beyond stabilization, to enhance transient performance and to achieve robust stability and disturbance rejection [53, 54, 91]. In this section, we shall develop a generalized low-and-high gain feedback design methodologies for semi-global and global stabilization of single layer sandwich systems. The application of such designs to more general type of sandwich systems is still subject to future study.

6.7.1. Continuous-time systems

Semi-global controller design

The first part of design concerned with choosing F for L_1 system is exactly the same as in the generalized low-gain design in Section 6.4.1. We first choose F such that $A + BF$ is asymptotically stable and consider the system (6.8), and then compute δ according to (6.9).

Next we consider the system (6.10). Our objective is, for any a priori given compact set $\mathcal{W}$, to find a stabilizing controller for the system (6.10) such that $\mathcal{W}$ is contained in its domain of attraction and $\|v(\tau)\| < \delta$ for all $\tau > 0$.

Let $Q_\varepsilon > 0$ be a parameterized family of matrices which satisfies $\frac{\mathrm{d}Q}{\mathrm{d}\varepsilon} > 0$ for $\varepsilon > 0$ with $\lim_{\varepsilon\to 0} Q_\varepsilon = 0$. In that case, there exists for any $\varepsilon > 0$ a $P_\varepsilon > 0$ satisfying

$$\begin{pmatrix} A+BF & 0 \\ NC & M \end{pmatrix}' P_\varepsilon + P_\varepsilon \begin{pmatrix} A+BF & 0 \\ NC & M \end{pmatrix} - P_\varepsilon \begin{pmatrix} BB' & 0 \\ 0 & 0 \end{pmatrix} P_\varepsilon + Q_\varepsilon = 0. \quad (6.67)$$

We first show the following lemma.

Lemma 6.5 Consider the system (6.10) and assume that the pair $(\tilde{A}, \tilde{B})$ as given by (6.6) is stabilizable and the eigenvalues of M are in the closed left half plane. Then, for any a priori given compact set $\mathcal{W} \in \mathbb{R}^{n+m}$, there exists an ε^* such that for any $0 < \varepsilon < \varepsilon^*$ and $\rho > 0$, the state feedback,

$$v = -\delta\sigma(\tfrac{1+\rho}{\delta} \begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \bar{x}), \quad (6.68)$$

achieves asymptotic stability of the equilibrium point $\bar{x} = 0$ where we denote by $\bar{x}$ the state of the system (6.10). Moreover, for any initial condition in $\mathcal{W}$, the constraint $\|v(t)\| \le \delta$ does not get violated for any $t > 0$.

Proof : Note that condition 2 of Theorem 6.1 immediately implies the existence of a $P_\varepsilon > 0$ satisfying (6.67). Moreover, condition 1 immediately implies that

$$P_\varepsilon \to 0 \quad (6.69)$$

as $\varepsilon \to 0$. Obviously, controller (6.68) satisfies $\|v\| < \delta$. It remains to show that such a controller achieves semi-global stabilization. Define $V(\bar{x}) = \bar{x}' P_\varepsilon \bar{x}$. Let c be defined as

$$c = \sup_{\substack{\varepsilon \in (0,1] \\ \bar{x} \in \mathcal{W}}} \{\bar{x}' P_\varepsilon \bar{x}\}.$$

There exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, we have that $\bar{x} \in \mathcal{L}_V(c) = \{\bar{x} \mid \bar{x}' P_\varepsilon \bar{x} \le c\}$, implies that

$$\|\tilde{v}\| \le \delta$$

where we denote

$$\tilde{v} = \begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \bar{x}.$$

Consider $\dot{V}$ along any trajectory,

$$\begin{aligned} \dot{V} &\le -\bar{x}' Q_\varepsilon \bar{x} - 2\delta \tilde{v}' [\sigma(\tfrac{1+\rho}{\delta}\tilde{v}) - \tfrac{1}{\delta}\tilde{v}] \\ &= -\bar{x}' Q_\varepsilon \bar{x} - 2\delta \tilde{v}' [\sigma(\tfrac{1+\rho}{\delta}\tilde{v}) - \sigma(\tfrac{1}{\delta}\tilde{v})]. \end{aligned}$$

We have $\dot{V} < 0$ for any $\rho > 0$. This completes the proof. ■

Theorem 6.14 Consider the interconnection of the two systems given by (6.1) and (6.2) satisfying conditions 1 and 2 of Theorem 6.1. Let F be such that $A + BF$ is asymptotically stable while $P_\varepsilon > 0$ is defined by (6.67). Define a state feedback law by

$$u = Fx - \delta\sigma(\tfrac{1+\rho}{\delta} \begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \begin{pmatrix} x \\ \omega \end{pmatrix}). \tag{6.70}$$

Then, for any compact set of initial conditions $\mathcal{W} \in \mathbb{R}^{n+m}$, there exists an $\varepsilon^* > 0$ such that for all ε with $0 < \varepsilon < \varepsilon^*$ and any $\rho > 0$ the controller (6.70) asymptotically stabilizes the equilibrium point $(0, 0)$ with a domain of attraction containing $\mathcal{W}$.

Proof : Consider any $(x(0)', \omega(0)')' \in \mathcal{W}$. Then there exists a $T > 0$ independent of particular initial condition such that

$$\|Ce^{(A+BF)t} x(0)\| < \tfrac{1}{2}$$

for $t > T$. Denote

$$v(t) = -\delta\sigma(\tfrac{1+\rho}{\delta} \begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \begin{pmatrix} x \\ \omega \end{pmatrix}).$$

By construction, we have $\|v(t)\| \leq \delta$ for $t > 0$. This together with (6.9) implies that $\|z(t)\| \leq 1$ for $t \geq T$.

Since $A + BF$ is Hurwitz stable and the input to the second system is bounded, there exists a $\bar{\mathcal{W}}$ such that for any $(x(0)', \omega(0)')' \in \mathcal{W}$, we have $(x(T)', \omega(T)')' \in \bar{\mathcal{W}}$.

Then the interconnection of (6.1) and (6.2) with controller (6.70) for $t > T$ is equivalent to the interconnection of (6.10) with controller (6.68) for $t > T$. From Lemma 6.5, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$ and any $\rho > 0$, the closed-loop system of (6.10) and controller (6.68) is asymptotically stable with $(x(T)', \omega(T)')' \in \bar{\mathcal{W}}$. Therefore we have

$$x(t) \to 0, \quad \omega(t) \to 0.$$

Since this follows for any $(x(0), \omega(0)) \in \mathcal{W}$, we find that $\mathcal{W}$ is contained in the domain of attraction as required. ■

Global controller design

We claim that the same controller given in (6.70) with ε being replaced by the scheduled low gain parameter $\varepsilon_s(\bar{x})$ as defined below solves the global stabilization problem.

At first, we look for a scheduling parameter satisfying the following:

1. $\varepsilon_s(x) : \mathbb{R}^{n+m} \to (0, 1]$ is continuous and piecewise continuously differentiable.
2. There exists an open neighborhood $\mathcal{O}$ of the origin such that $\varepsilon_s(x) = 1$ for all $x \in \mathcal{O}$.
3. For any $\bar{x} \in \mathbb{R}^{n+m}$, we have
$$\| \begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})} \bar{x} \|_\infty \leq \delta.$$
4. $\varepsilon_s(\bar{x}) \to 0$ as $\|\bar{x}\|_\infty \to \infty$.
5. $\{ \bar{x} \in \mathbb{R}^{n+m} \mid \bar{x}' P_{\varepsilon_s(\bar{x})} \bar{x} \leq c \}$ is a bounded set for all $c > 0$.
6. $\varepsilon_s(\bar{x})$ is uniquely determined given that $\bar{x}' P_{\varepsilon_s(\bar{x})} \bar{x} = c$ for some $c > 0$.

A particular choice satisfying the above criteria is given by

$$\varepsilon_s(\bar{x}) = \max\{\, r \in (0,1] \mid (\bar{x}' P_r \bar{x}) \operatorname{trace}\left[\begin{pmatrix} B \\ 0 \end{pmatrix}' P_r \begin{pmatrix} B \\ 0 \end{pmatrix} \right] \leq \delta^2 \}. \quad (6.71)$$

Then we first show the following result:

Lemma 6.6 Consider the system (6.10) and assume that the pair $(\tilde{A}, \tilde{B})$ as given by (6.6) is stabilizable and the eigenvalues of M are in the closed left half plane. Then, for any $\rho > 0$, the feedback,

$$v = -\delta\sigma(\tfrac{1+\rho}{\delta} \begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})} \bar{x}), \tag{6.72}$$

achieves global stability of the equilibrium point $\bar{x} = 0$.

Proof : Obviously, controller (6.72) satisfies $\|v\| < \delta$. It remains to show that such a controller achieves global stabilization. Define $V(\bar{x}) = \bar{x}' P_{\varepsilon_s(\bar{x})} \bar{x}$.

Denote

$$\tilde{v} = \begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})} \bar{x}.$$

Consider $\dot{V}$ along any trajectory,

$$\dot{V} \leq -\bar{x}' Q_{\varepsilon_s(\bar{x})} \bar{x} - 2\delta \tilde{v}' [\sigma(\tfrac{1+\rho}{\delta} \tilde{v}) - \tfrac{1}{\delta} \tilde{v}] + \bar{x}' \tfrac{dP_{\varepsilon_s(\bar{x})}}{dt} \bar{x}.$$

By construction, $\|\frac{1}{\delta} \tilde{v}\| < 1$. We get

$$\dot{V} \leq \bar{x}' Q_{\varepsilon_s(\bar{x})} \bar{x} - 2\delta \tilde{v}' [\sigma(\tfrac{1+\rho}{\delta} \tilde{v}) - \sigma(\tfrac{1}{\delta} \tilde{v})] + \bar{x}' \tfrac{dP_{\varepsilon_s(\bar{x})}}{dt} \bar{x}.$$

If $\rho > 0$, we have

$$\dot{V} < -\bar{x}' Q_{\varepsilon_s(\bar{x})} \bar{x} + \bar{x}' \tfrac{dP_{\varepsilon_s(\bar{x})}}{dt} \bar{x}.$$

The scheduling law (6.71) implies

$$V(x) \operatorname{trace}\left[\begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})} \begin{pmatrix} B \\ 0 \end{pmatrix} \right] = \delta^2$$

whenever $\varepsilon_s(\bar{x}) \neq 1$ or equivalently $P_{\varepsilon_s(\bar{x})}$ is not a constant locally. This implies that $\dot{V}$ and $\bar{x}' \frac{dP_{\varepsilon_s(\bar{x})}}{dt} \bar{x}$ are either both zero or of opposite signs. Hence for $x \neq 0$

$$\dot{V} < 0$$

If not, we know $\bar{x}'\frac{dP_{\varepsilon_s(\bar{x})}}{dt}\bar{x} \leq 0$. But this implies $\dot{V} < -\bar{x}' Q_{\varepsilon_s(\bar{x})}\bar{x}$ which yields a contradiction. Therefore, the global asymptotic stability follows. ■

Theorem 6.15 Consider the interconnection of the two systems given by (6.1) and (6.2) satisfying the conditions 1 and 2 of Theorem 6.1. Choose F such that $A + BF$ is asymptotically stable. Let P_ε and ε_s be as defined by (6.67) and (6.71) respectively. In that case, for any $\rho > 0$, the state feedback,

$$u = Fx - \delta\sigma\left(\tfrac{1+\rho}{\delta}\begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})}\bar{x}\right) \tag{6.73}$$

achieves global asymptotic stability.

Proof : If we consider the interconnection of (6.1) and (6.2), then we note that close to the origin the saturation does not get activated. Moreover, close to the origin the feedback (6.73) is given by

$$u = Fx - (1+\rho)\begin{pmatrix} B \\ 0 \end{pmatrix}' P_1\bar{x},$$

which immediately yields that the interconnection of (6.1), (6.2) and (6.73) is locally asymptotically stable. It remains to show that we have global asymptotic stability.

Consider an arbitrary initial condition $x(0)$ and $\omega(0)$. Then there exists a $T > 0$ such that

$$\|Ce^{(A+BF)t}x(0)\| < \tfrac{1}{2}$$

for $t > T$. Moreover, by construction, the control

$$v = -\delta\sigma\left(\tfrac{1+\rho}{\delta}\begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})}\bar{x}\right)$$

yields $\|v(t)\| \leq \delta$ for all $t > 0$. However, this implies that $z(t)$ generated by (6.8) satisfies $\|z(t)\| < 1$ for all $t > T$. But this yields that the interconnection of (6.1) and (6.2) with controller (6.73) behaves for $t > T$ like the interconnection of (6.10) with controller (6.72). Global asymptotic stability of the latter system then implies that $\bar{x}(t) \to 0$ as $t \to \infty$. Since this property holds for any initial condition and we have local asymptotic stability we can conclude that the controller yields global asymptotic stability. This completes the proof. ■

Similar to the results in Section 6.4.1, the construction of our controller guarantees the saturation does not get activated after some finite time T and the stabilization of sandwich non-linear systems becomes stabilization of a linear system subject to input saturation. It is clear from the proof that T is determined by the inital condition of L_1. Since $A + BF$ is Hurwitz stable with the preliminary feedback, this T can be fairly small. However, after time T the design meothodology presented above yields a regular low-and-high gain feedback controller, while in Section 6.4.1, it reduces to the classical low-gain feedback controller. Therefore, we expect an enhanced system performance from our design techinque.

We like to emphasize that an appropriate selection of the matrix Q_ε plays an important role in the design process. A judicious choice of Q_ε can tremendously improve the performance. This is also illustrated by an example given in next section.

6.7.2. Discrete-time systems

6.7.3. Semi-global controller design

In this subsection, we design a controller which solves the semi-global stabilization problem for discrete-time sandwich systems. For ease of presentation, denote $\bar{x} = (x', \omega)'$. Our design progresses in three steps:

Step 1 Choose preliminary feedback F such that $A + BF$ is Schur stable.

Step 2 Define δ as

$$\delta = \frac{\alpha}{\sum_{k=0}^{\infty} \|C(A + BF)^k B\|} \tag{6.74}$$

for arbitrary $\alpha \in (0, 1)$. Such a δ is well defined since $A + BF$ is Schur stable.

Step 3 Let $Q_\varepsilon > 0$ be a matrix function: $(0, 1] \to \mathbb{R}^{(n+m)\times(n+m)}$ which satisfies $\frac{\mathrm{d}Q_\varepsilon}{\mathrm{d}\varepsilon} > 0$ for $\varepsilon > 0$ and $\lim_{\varepsilon \to 0^+} Q_\varepsilon = 0$. Solve the ARE (6.25). Note that condition 2 of Theorem 6.1 guarantees existence of the positive definite solution P_ε.

The generalized low-and-high gain feedback can be constructed as

$$u = Fx - \delta\sigma\left(\tfrac{1}{\delta}(1 + \tfrac{\rho}{\|\bar{B}' P_\varepsilon \bar{B}\|})(I + \bar{B}' P_\varepsilon \bar{B})^{-1} \bar{B}' P_\varepsilon \tilde{A} \bar{x}\right).$$

We show in the next theorem that the above low-and-high gain controller solves the semi-global stabilization problem:

Theorem 6.16 Consider the interconnection of the two systems given by (6.1) and (6.2) satisfying conditions 1 and 2 of Theorem 6.1. Define a state feedback controller by

$$u = Fx - \delta\sigma\left(\tfrac{1}{\delta}(1 + \tfrac{\rho}{\|\bar{B}' P_\varepsilon \bar{B}\|})(I + \bar{B}' P_\varepsilon \bar{B})^{-1} \bar{B}' P_\varepsilon \tilde{A}\bar{x}\right). \tag{6.75}$$

Then, for any compact set of initial conditions $\mathcal{W} \in \mathbb{R}^{n+m}$, there exists an $\varepsilon^* > 0$ such that for all ε with $\varepsilon \in (0, \varepsilon^*]$ and any $\rho \in [0, 2]$, the controller (6.75) asymptotically stabilizes the equilibrium point $(0, 0)$ with a domain of attraction containing $\mathcal{W}$.

Proof : Define

$$v = -\delta\sigma\left(\tfrac{1}{\delta}(1 + \tfrac{\rho}{\|\bar{B}' P_\varepsilon \bar{B}\|})(I + \bar{B}' P_\varepsilon \bar{B})^{-1} \bar{B}' P_\varepsilon \tilde{A}\bar{x}\right) \tag{6.76}$$

Consider the system L_1 with the preliminary feedback $u = Fx + v$ as given by

$$\begin{aligned} x(k+1) &= (A + BF)x(k) + Bv(k) \\ z(k) &= Cx(k) \end{aligned} \tag{6.77}$$

We have

$$\begin{aligned} z(k) &= C(A + BF)^k x(0) + \sum_{i=0}^{k-1} C(A + BF)^{k-i-1} Bv(i) \\ &= C(A + BF)^k x(0) + z_0(k) \end{aligned}$$

where

$$z_0(k) = \sum_{i=0}^{k-1} C(A + BF)^{k-i-1} Bv(i) \tag{6.78}$$

For any a priori given set of initial conditions $\mathcal{W}$, there exists a $K > 0$ such that

$$\|C(A + BF)^k x(0)\| < 1 - \alpha$$

for $k > K$ and any $x(0) \in \mathcal{W}$.

By construction, $\|v\| \leq \delta$ for all $k \geq 0$. From the definition of δ, we get

$$\|z_0(k)\| = \sum_{i=0}^{k-1} \|C(A + BF)^{k-i-1}B\|\|v(i)\| \leq \alpha$$

This implies that for all $k \geq K$ we have $\|z(k)\| \leq 1$ i.e. the sandwiched saturation remains inactive after time K. Therefore, for all $k \geq K$, the closed-loop system is equivalent to the interconnection of the linear cascaded system

$$\bar{x}(k+1) = \begin{pmatrix} A + BF & 0 \\ NC & M \end{pmatrix} \bar{x}(k) + \begin{pmatrix} B \\ 0 \end{pmatrix} v(k). \tag{6.79}$$

with controller v given by (6.76).

There exists a compact set $\bar{\mathcal{W}}$ such that for any $\bar{x}(0) \in \mathcal{W}$, we have $\bar{x}(K) \in \bar{\mathcal{W}}$. This is due to the fact that $\mathcal{W}$ is compact, $A + BF$ is Schur stable and the input to L_2 is bounded.

In the next lemma, we shall show that the interconnection of (6.79) and (6.76) is asymptotically stable with $\bar{\mathcal{W}}$ contained in its domain of attraction.

Lemma 6.7 Consider the system (6.79) and assume that the pair $(\bar{A}, \bar{B})$ as given by (6.6) is stabilizable and the eigenvalues of M are in the closed unit disc. Then, for any a priori given compact set $\bar{\mathcal{W}} \in \mathbb{R}^{n+m}$, there exists an ε^* such that for any $0 < \varepsilon < \varepsilon^*$ and $\rho \in [0, 2]$, the state feedback,

$$v = -\delta\sigma(\tfrac{1}{\delta}(1 + \tfrac{\rho}{\|\bar{B}' P_\varepsilon \bar{B}\|})(I + \bar{B}' P_\varepsilon \bar{B})^{-1} \bar{B}' P_\varepsilon \tilde{A}\bar{x}), \tag{6.80}$$

achieves asymptotic stability of the equilibrium point $\bar{x} = 0$ with domain of attraction containing $\bar{\mathcal{W}}$.

Proof : First we introduce the following notation

$$\tilde{v} = (I + \bar{B}' P_\varepsilon \bar{B})^{-1} \bar{B}' P_\varepsilon \bar{A}\bar{x} \quad \text{and} \quad \mu = \|\bar{B}' P_\varepsilon \bar{B}\|.$$

Note that condition 1 of Theorem 6.1 immediately implies that

$$P_\varepsilon \to 0 \tag{6.81}$$

as $\varepsilon \to 0$. Define $V(\bar{x}) = \bar{x}' P_\varepsilon \bar{x}$. Let c be defined as

$$c = \sup_{\substack{\varepsilon \in (0,1] \\ \bar{x} \in \mathcal{W}}} \{\bar{x}' P_\varepsilon \bar{x}\}.$$

There exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, we have that $\bar{x} \in \mathcal{L}_v(c) = \{\bar{x} \mid \bar{x}' P_\varepsilon \bar{x} \leq c\}$, implies that

$$\|\tilde{v}\| \leq \delta$$

and hence

$$\|\tilde{v}\| \leq \|v\| \leq (1 + \tfrac{2}{\mu})\|\tilde{v}\| \tag{6.82}$$

Consider $V(k+1) - V(k)$ along any trajectory,

$$\begin{aligned}
&V(k+1) - V(k)\\
&\quad= \tilde{v}(k)'(I + \bar{B}' P_\varepsilon \bar{B})\tilde{v}(k) - \bar{x}(k)' Q_\varepsilon \bar{x}(k)\\
&\qquad - 2\tilde{v}(k)'(I + \bar{B}' P_\varepsilon \bar{B})v(k) + v(k)'\bar{B}' P_\varepsilon \bar{B} v(k)\\
&\quad= -\bar{x}(k)' Q_\varepsilon \bar{x}(k) - v(k)'v(k)\\
&\qquad + [v(k) - \tilde{v}(k)]'(I + \bar{B}' P_\varepsilon \bar{B})[v(k) - \tilde{v}(k)]\\
&\quad\leq (1+\mu)[v(k) - \tilde{v}(k)]'[v(k) - \tilde{v}(k)] - \bar{x}(k)' Q_\varepsilon \bar{x}(k)\\
&\qquad - v(k)'v(k)\\
&\quad\leq -\bar{x}(k)' Q_\varepsilon \bar{x}(k) - \tfrac{1}{\mu}\tilde{v}(k)'\tilde{v}(k) +\\
&\qquad \mu[v(k) - \tfrac{1+\mu}{\mu}\tilde{v}(k)]'[v(k) - \tfrac{1+\mu}{\mu}\tilde{v}(k)]
\end{aligned}$$

Note that (6.82) implies

$$\|v(k) - \tfrac{1+\mu}{\mu}\tilde{v}(k)\| \leq \tfrac{1}{\mu}\|\tilde{v}(k)\|$$

Therefore,

$$\mu[v(k) - \tfrac{1+\mu}{\mu}\tilde{v}(k)]'[v(k) - \tfrac{1+\mu}{\mu}\tilde{v}(k)] - \tfrac{1}{\mu}\tilde{v}(k)'\tilde{v}(k) \leq 0$$

We conclude $V(k+1) - V(k) < 0$ for any $\varepsilon \in (0, \varepsilon^*]$, $\rho \in [0, 2]$ and $x \neq 0$. This completes the proof.

■

The above lemma indicates

$$x(k) \to 0, \quad \omega(k) \to 0.$$

Since this follows for any $(x(0), \omega(0)) \in \mathcal{W}$, we find that $\mathcal{W}$ is contained in the domain of attraction as required. ■

6.7.4. Global controller design

We claim that the same controller given in (6.75) with ε being replaced by the scheduled low gain parameter $\varepsilon_s(\bar{x})$ as defined below solves the global stabilization problem. Specifically, the deign methodology follows in 4 steps:

Step 1, 2 and 3 This step is exactly the same as for semi-global stabilization given in the preceding subsections.

Step 4 We look for a scheduling parameter $\varepsilon = \varepsilon_s(\bar{x})$ satisfying the following:

1. $\varepsilon_s(\bar{x}) : \mathbb{R}^{n+m} \to (0, 1]$ is continuous and piecewise continuously differentiable.
2. There exists an open neighborhood $\mathcal{O}$ of the origin such that $\varepsilon_s(\bar{x}) = 1$ for all $\bar{x} \in \mathcal{O}$.
3. For any $\bar{x} \in \mathbb{R}^{n+m}$, we have $\|F_{\varepsilon_s(\bar{x})} x\| \leq \delta$.
4. $\varepsilon_s(\bar{x}) \to 0$ as $\|\bar{x}\| \to \infty$.
5. $\{ \bar{x} \in \mathbb{R}^n \mid \bar{x}' P_{\varepsilon_s(\bar{x})} \bar{x} \leq c \}$ is a bounded set for all $c > 0$.
6. $\varepsilon_s(\bar{x})$ is uniquely determined given that $\bar{x}' P_{\varepsilon_s(\bar{x})} \bar{x} = c$, for any $c > 0$.

A particular choice satisfying the above criteria is given by

$$\varepsilon_s(\bar{x}) = \max \{ r \in (0, 1] \mid (\bar{x}' P_r \bar{x}) trace \left[\bar{B}' P_r \bar{B} \right] \leq \delta^2 / M_p \}. \quad (6.83)$$

where $M_p = \sigma_{max}(P_1^{\frac{1}{2}} \bar{B} \bar{B}' P_1^{\frac{1}{2}}) + 1$ and P_r is the solution of (6.25) with $\varepsilon = r$.

We prove the following theorem:

Theorem 6.17 Consider the interconnection of the two systems given by (6.1) and (6.2) satisfying the conditions 1 and 2 of Theorem 6.1. Choose F such that $A + BF$ is Schur stable. Let P_ε and ε_s be as defined by (6.25) and (6.83) respectively. In that case, for any $\rho \in [0, 2]$, the state feedback,

$$u = Fx - \delta\sigma\left(\frac{1}{\delta}(1 + \frac{\rho}{\|\bar{B}' P_{\varepsilon_s(\bar{x})} \bar{B}\|})(I + \bar{B}' P_{\varepsilon_s(\bar{x})} \bar{B})^{-1} \bar{B}' P_{\varepsilon_s(\bar{x})} \tilde{A}\bar{x}\right) \quad (6.84)$$

achieves global asymptotic stability of the origin, where F, δ $P_{\varepsilon_s(\bar{x})}$ and $\varepsilon_s(\bar{x})$ are obtained in Step 1,2,3 and 4 respectively.

Proof : Define

$$v = \delta\sigma(\frac{1}{\delta}(1 + \frac{\rho}{\|\bar{B}' P_{\varepsilon_s(\bar{x})} \bar{B}\|})(I + \bar{B}' P_{\varepsilon_s(\bar{x})} \bar{B})^{-1} \bar{B}' P_{\varepsilon_s(\bar{x})} \tilde{A}\bar{x}) \quad (6.85)$$

If we consider the interconnection of (6.1) and (6.2), then we note that close to the origin the saturation does not get activated. Moreover, close to the origin the feedback (6.84) is given by

$$u = Fx - (1 + \frac{\rho}{\|\bar{B}' P_1 \bar{B}\|})(I + \bar{B}' P_1 \bar{B})^{-1} \bar{B}' P_1 \tilde{A}\bar{x},$$

which immediately yields that the interconnection of (6.1), (6.2) and (6.84) is locally asymptotically stable. It remains to show that we have global asymptotic stability.

Consider an arbitrary initial condition $x(0)$ and $\omega(0)$. Then there exists a $K > 0$ such that

$$\|C(A + BF)^k x(0)\| < 1 - \alpha$$

for $k > K$. The definition of δ implies $\|z_0(t)\| \leq \alpha$ for all $k \geq 0$ where z_0 is given by (6.78). However, this implies that $z(k)$ generated by (6.77) satisfies $\|z(k)\| < 1$ for all $k > K$. Then the closed-loop system becomes the interconnection of the linear cascade system (6.79) with controller (6.85).

We next show the following lemma

Lemma 6.8 Consider the system (6.79) and assume that the pair $(\bar{A}, \bar{B})$ as given by (6.6) is stabilizable and the eigenvalues of $\tilde{A}$ are in the closed unit disc. Then, for any $\rho \in [0, 2]$, the feedback (6.85) achieves global stability of the equilibrium point $\bar{x} = 0$.

Proof : Define $V(\bar{x}) = \bar{x}' P_{\varepsilon_s(\bar{x})} \bar{x}$. Denote

$$\tilde{v}(k) = (I + \bar{B}' P_{\varepsilon_s(\bar{x}(k))} \bar{B})^{-1} \bar{B}' P_{\varepsilon_s(\bar{x}(k))} \tilde{A}\bar{x}$$

and

$$\mu(k) = \|\bar{B}' P_{\varepsilon_s(\bar{x}(k))} \bar{B}\|.$$

Consider $V(k+1) - V(k)$ along any trajectory,

$$\begin{aligned}
&V(k+1) - V(k) \\
\le& -\bar{x}(k+1)' P_{\varepsilon_s(x(\bar{k}))} \bar{x}(k+1) + \bar{x}(k+1)' P_{\varepsilon_s(x(\bar{k+1}))} \bar{x}(k+1) \\
&- \bar{x}(k)' Q_{\varepsilon_s(x(\bar{k}))} \bar{x}(k) - \tfrac{1}{\mu(k)} \tilde{v}(k)' \tilde{v}(k) \\
&+ \mu(k)[v(k) - \tfrac{1+\mu(k)}{\mu(k)} \tilde{v}(k)]'[v(k) - \tfrac{1+\mu(k)}{\mu(k)} \tilde{v}(k)]
\end{aligned}$$

Since $\rho \le 2$ and by construction $\|\tilde{v}\| \le \delta$, we get

$$\|\tilde{v}(k)\| \le \|v(k)\| \le (1 + \tfrac{2}{\mu(k)}) \|\tilde{v}(k)\|,$$

This implies $\|v(k) - \frac{1+\mu(k)}{\mu(k)} \tilde{v}(k)\| \le \frac{1}{\mu(k)} \|\tilde{v}(k)\|$. Therefore

$$\begin{aligned}
V(k+1) - V(k) \le& -\bar{x}(k)' Q_{\varepsilon_s(x(\bar{k}))} \bar{x}(k) \\
&- \bar{x}(k+1)' P_{\varepsilon_s(x(\bar{k}))} \bar{x}(k+1) + \bar{x}(k+1)' P_{\varepsilon_s(x(\bar{k+1}))} \bar{x}(k+1)
\end{aligned}$$

The scheduling law (6.83) guarantees that $V(k+1) - V(k)$ and $\bar{x}(k+1)'(P_{\varepsilon_s(\bar{x}(k+1))} - P_{\varepsilon_s(\bar{x}(k))})\bar{x}(k+1)$ can not have the same signs. This implies that for $x \neq 0$ and $\rho \in [0, 2]$

$$V(k+1) - V(k) < 0.$$

The global asymptotic stability follows. ■ The above lemma yields that $\bar{x}(k) \to 0$ as $k \to \infty$. Since

this property holds for any initial condition and we have local asymptotic stability, we can conclude that the controller yields global asymptotic stability. This completes the proof. ■

6.8. Simulation example

The design methods are illustrated by several numerical examples.

6.8.1. Single layer sandwich systems

Continuous-time systems

Semi-global stabilization The two systems L_1 and L_2 in (6.1) and (6.2) are given by

$$L_1 : \begin{cases} \dot{x}(t) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t) \\ z(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} x(t) \end{cases}$$

and

$$L_2 : \quad \dot{\omega}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \omega(t) + \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \sigma(z(t))$$

We will design a controller to stabilize the systems with an a priori given compact set $\mathcal{W}$ contained in its domain of attraction, where

$$\mathcal{W} = \{\gamma \in \mathbb{R}^5 \mid \gamma \in [-1, 1]^6 \}$$

Step 1. Choose

$$F = \begin{pmatrix} -12 & -6 & -7 \end{pmatrix}$$

such that $A + BF$ is Hurwitz stable.

Step 2. Choose $\delta = 2.28$. Then for system (6.8), we have that

$$\|v(\tau)\| < \delta \quad \forall \tau > 0$$

implies $\|z_0(t)\| < \frac{1}{2}$ for all $t > 0$.

Step 3. We set the low gain parameter $\varepsilon = 0.0001$. After solving the associated algebraic Riccati equation, we obtain the following state feedback:

$$u = \begin{pmatrix} -15.2016 & -6.4139 & -7.2370 \end{pmatrix} x \\ + \begin{pmatrix} 0.0100 & 0.1869 & 1.7412 \end{pmatrix} \omega$$

The simulation data is shown in Figure 6.5.

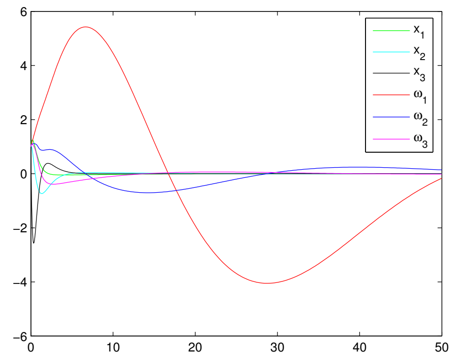


Figure 6.5: Semi-global stabilization via state feedback

Global stabilization The two systems L_1 and L_2 in (6.1) and (6.2) are the same as in the preceding example. We solve the global stabilization problem as follows:

Step 1. Choose

$$F = \begin{pmatrix} -12 & -6 & -7 \end{pmatrix}$$

such that $A + BF$ is Hurwitz stable.

Step 2. Choose the same $\delta = 2.28$ as preceding example.

Step 3. Design a controller

$$u = Fx - \begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})}\bar{x}$$

where $P_{\varepsilon_s(\bar{x})}$ is given by (6.11) and (6.15).

The resulting simulation is shown in Figure 6.6.

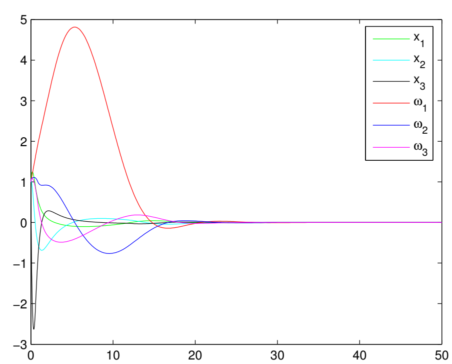


Figure 6.6: Global stabilization via state feedback

Discrete-time systems

Consider the following two systems:

$$L_1 : \begin{cases} x(k+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k), \\ z(k) \quad = \begin{pmatrix} 1 & 0 \end{pmatrix} x(k), \end{cases} \tag{6.86}$$

and

$$L_2 : \quad \omega(k+1) = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix} \omega(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma(z(k)), \tag{6.87}$$

and $\mathcal{W} = [-2, 2] \times [-2, 2] \times [-2, 2] \times [-2, 2]$. We shall design controllers for both semi-global and global stabilization of the origin of (6.86) and (6.87). The initial condition for simulations is $x(0) = (-2, 2)'$ and $\omega(0) = (2, -2)'$.

Semi-global stabilization

- Choose $F = \begin{pmatrix} -0.7321 & 0 \end{pmatrix}$.
- From (6.21), we calculate $\delta_1 = 0.366$.
- Determine ε according to $\mathcal{W}$ and δ_1. We choose $\varepsilon = 3 \times 10^{-3}$.
- The feedback controller is given by

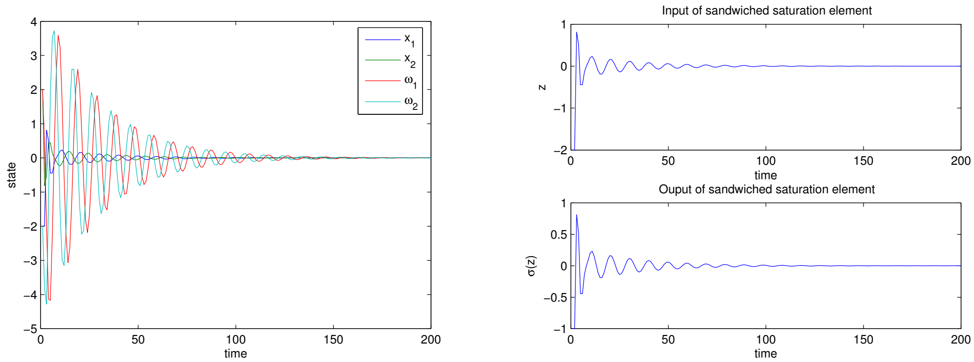
$$u = \begin{pmatrix} -0.7145 & -0.055 & -0.0740 & -0.0087 \end{pmatrix} \bar{x}.$$

The simulation data is shown in Figure 6.7.

Global stabilization

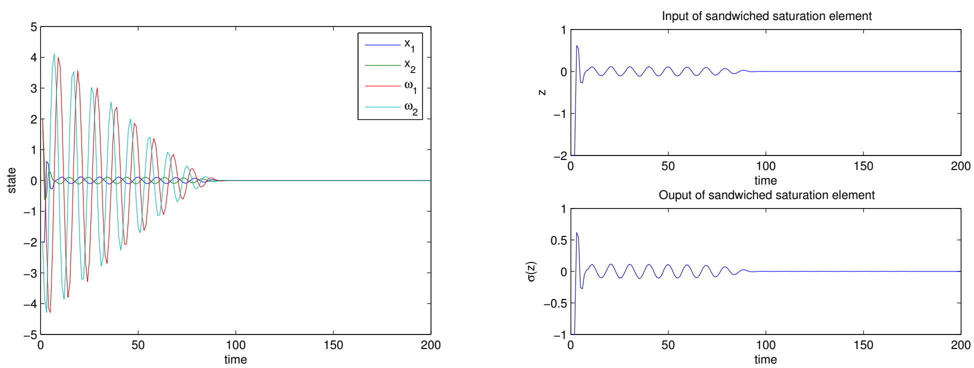
- Choose $F = \begin{pmatrix} -0.7321 & 0 \end{pmatrix}$.
- From (6.21), we compute $\delta_1 = 0.366$.
- The global stabilizing controller is formed by semi-global controller together with scheduled parameter.

The simulation data is shown in Figure 6.8 and 6.9.



State evolution I/O of sandwiched saturation

Figure 6.7: Semi-global stabilization of single layer sandwich system



State evolution I/O of saturation

Figure 6.8: Global stabilization of single layer sandwich system

6.8.2. Single layer sandwich systems with input saturation

Continuous-time systems

Consider the interconnection of (6.3) and (6.4) with

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The L_1 subsystem has an eigenvalue at the origin of multiplicity two; thus, it is open-loop unstable. The L_2 subsystem has imaginary eigenvalues at $\pm 1j$; thus, it is marginally stable. Following the procedure in Section 6.5, we design a semi-globally stabilizing controller for this system with $\varepsilon_1 = 10^{-4}$ and $\varepsilon_2 = 5 \cdot$

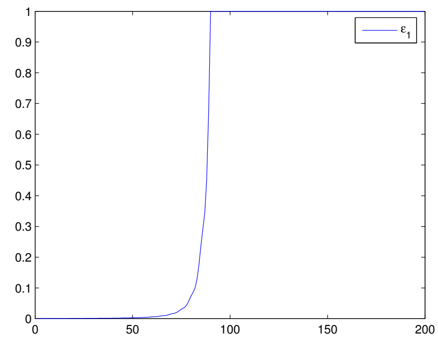
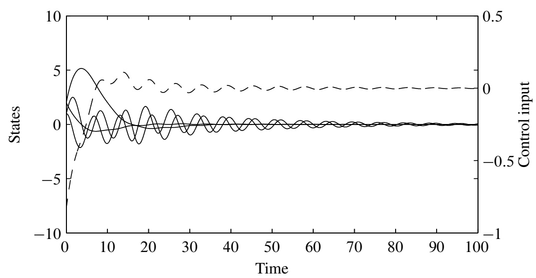
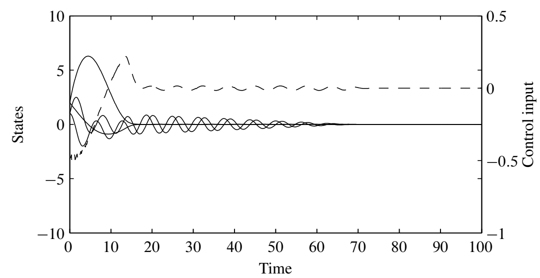


Figure 6.9: Time evolution of ε



(a) States (solid, left axis) and control input (dashed, right axis) for semiglobally stabilizing controller



(b) States (solid, left axis) and control input (dashed, right axis) for globally stabilizing controller

Figure 6.10: Simulation results

10^{-4}. Similarly, we design a globally stabilizing controller according the procedure in Section 6.5, which gives $\delta \approx 0.03$. Fig. 6.10 shows the simulation results with initial conditions $x(0) = [2, 2]'$ and $\omega(0) = [1, 1]'$. In this example, the globally stabilizing controller uses the available control input somewhat more efficiently than the semi-globally stabilizing controller, thereby ensuring a shorter settling time.

Discrete-time systems

Consider the systems as given by (6.88) and (6.89)

$$L_1 : \begin{cases} x(k+1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma(u(k)), \\ z(k) \quad = \begin{pmatrix} 1 & 0 \end{pmatrix} x(k), \end{cases} \tag{6.88}$$

and

$$L_2 : \quad \omega(k+1) = \begin{pmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{pmatrix} \omega(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sigma(z(k)), \tag{6.89}$$

and $\mathcal{W} = [-2, 2] \times [-2, 2] \times [-2, 2] \times [-2, 2]$. The initial condition for simulation is $x(0) = \begin{pmatrix} -2 & 2 \end{pmatrix}'$ and $\omega(0) = \begin{pmatrix} 2 & -2 \end{pmatrix}'$.

Semi-global stabilization

- According to $\mathcal{W}$, we choose $\varepsilon_1 = 0.05$.
- According to $\mathcal{W}$ and ε_1, we choose $\varepsilon_2 = 3 \times 10^{-3}$.
- The controller is given by $u = \begin{pmatrix} -0.2674 & -0.0442 & -0.0738 & 0.0119 \end{pmatrix} \bar{x}$

The simulation data and I/O of saturation elements are shown respectively in Figure 6.11 and 6.12:

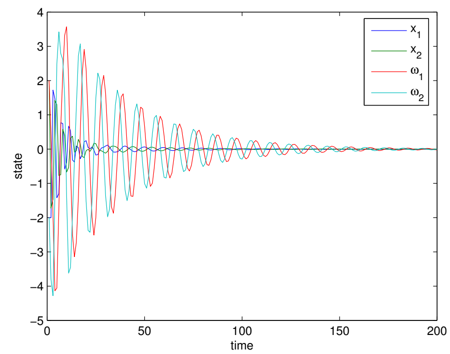
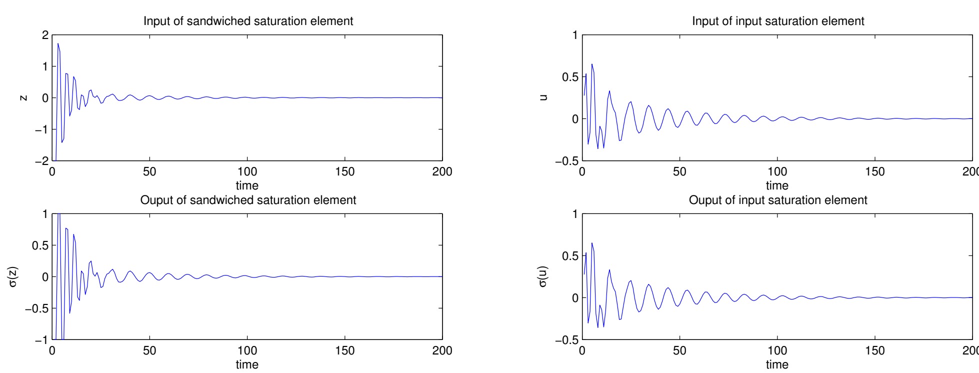


Figure 6.11: Semi-global stabilization of single layer sandwich system with input saturation



Sandwiched saturation　　　　Input saturation

Figure 6.12: I/O of saturation elements in semi-global stabilization of single layer sandwich system with input saturation

Global stabilization Note that the theoretical bounds on δ in Theorem 6.11 as given by (6.49) certainly suffice to prove solvability but it might be unnecessarily conservative in practice, since it is derived based on a worst-scenario estimation of v_1, v_2 and z_0 in (6.52) and (6.55). A proper δ can be obtained by relaxing one or more conservative bounds in (6.49) and reducing it again if necessary until stability is achieved as well as reasonable performance.

- We use $\delta = 0.2$. This choice is verified by a simulation of an 1296-point array of initial conditions without any observation of instability.
- $M_2 = 3.7321$ and $M_3 = 6.5474$.
- The controller is formed by the semi-global stabilizing controller together with scheduling (6.47) and (6.50).

The simulation data is shown in Figure 6.13.

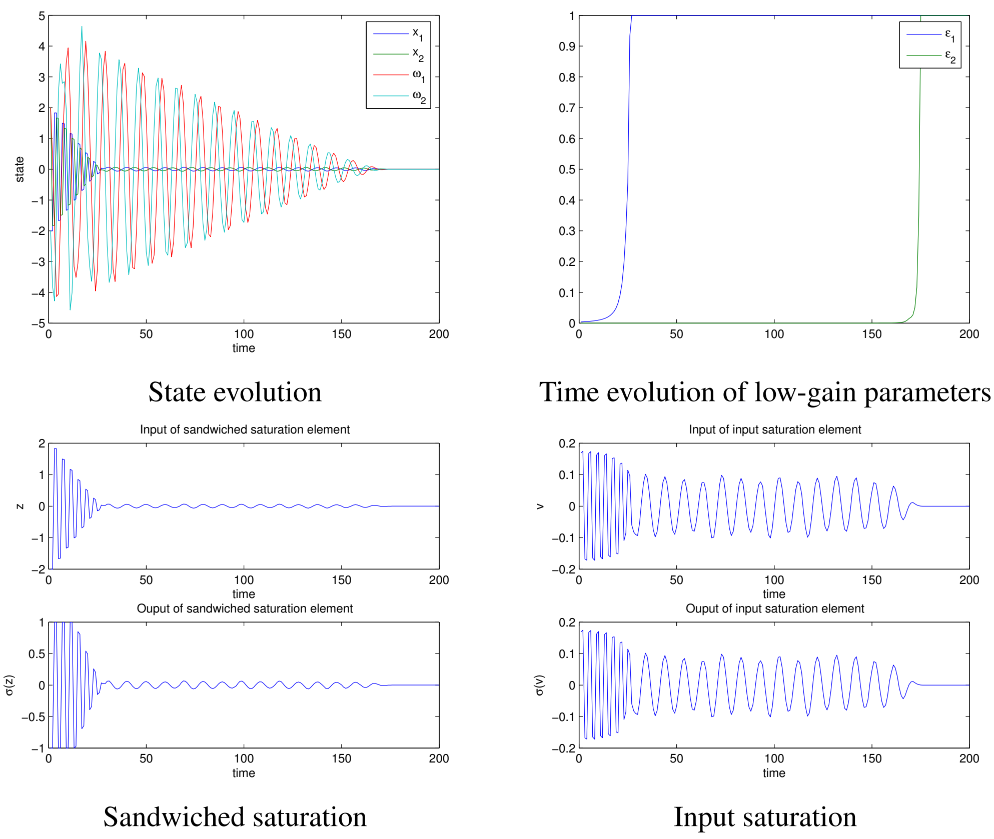


Figure 6.13: Global stabilization of single layer sandwich system with Input saturation

6.8.3. Generalized low-and-high-gain design

Continuous-time systems

Semi-global stabilization via state feedback Consider the two systems L_1 and L_2 given in (6.1) and (6.2),

$$L_1 : \begin{cases} \dot{x}(t) = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) \\ z(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t), \end{cases}$$

and

$$L_2 : \quad \dot{\omega}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \omega(t) + \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \sigma(z(t)).$$

We design below a controller that stabilizes the cascaded system of L_1 and L_2 with an a priori given compact set $\mathcal{W}$ to be contained in the domain of attraction of the closed-loop system, where

$$\mathcal{W} = \{\xi \in \mathbb{R}^4 \;\mid\; [-3, 3]^4 \}.$$

Step 1. Choose

$$F = \begin{pmatrix} -22.2474 & -8.4495 \end{pmatrix}$$

such that $A + BF$ is Hurwitz stable.

Step 2. Choose $\delta = 2.0772$ and $\rho = 1000$. Then for system (6.8), we have

$$\|v(\tau)\| < \delta \quad \forall \tau > 0,$$

implying that $\|z_0(t)\| < \frac{1}{2}$ for all $t > 0$.

Step 3. We set the low gain parameter $\varepsilon = 10^{-4}$. Choose $Q_\varepsilon = \varepsilon I$. After solving the associated algebraic Riccati equation, we obtain the following state feedback controller:

$$u = \begin{pmatrix} -22.2474 & -8.4495 \end{pmatrix} x - 1.0491\sigma\{\begin{pmatrix} 386.5181 & 25.1874 \end{pmatrix} x + \begin{pmatrix} -4.8190 & -198.2508 \end{pmatrix} \omega\}.$$

For comparison purpose, a low gain feedback controller of the form,

$$u = Fx + \begin{pmatrix} B \\ 0 \end{pmatrix}' P_\varepsilon \begin{pmatrix} x \\ \omega \end{pmatrix},$$

is also given as

$$u = \begin{pmatrix} -23.0495 & -8.5018 \end{pmatrix} x +$$
$$\begin{pmatrix} 0.0100 & 0.4114 \end{pmatrix} \omega.$$

The simulation data is shown in Figure 6.14. For comparison, the simulation data of low-gain controller is shown in Figure 6.15. As we can see, the low-high gain enhances the performance by incurring much lower overshoot and undershoot.

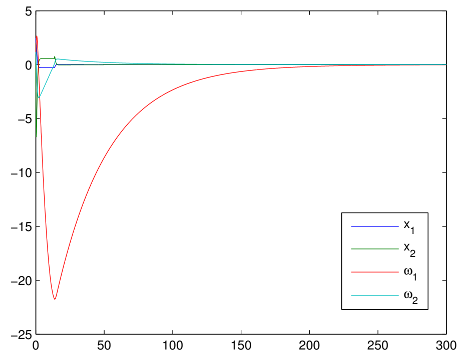


Figure 6.14: Semi-global stabilization via state feedback-low high gain approach

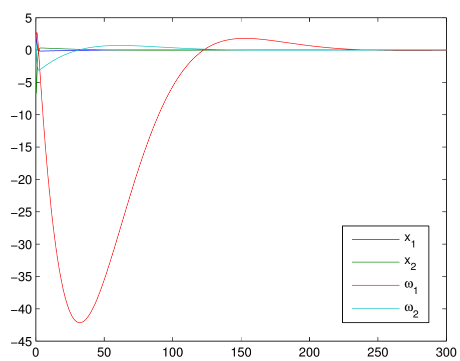


Figure 6.15: Semi-global stabilization via state feedback-low gain approach

Global stabilization via state feedback The two systems L_1 and L_2 in (6.1) and (6.2) are the same as in the preceding example. We solve the global stabilization problem as follows:

Step 1. Choose

$$F = \begin{pmatrix} -22.2474 & -8.4495 \end{pmatrix}$$

such that $A + BF$ is Hurwitz stable.

Step 2. Choose the same $\delta = 2.0772$ as in the preceding example and $\rho = 1000$.

Step 3. Design a controller

$$u = Fx - \delta\sigma\left(\frac{1+\rho}{\delta}\begin{pmatrix} B \\ 0 \end{pmatrix}' P_{\varepsilon_s(\bar{x})}\bar{x}\right)$$

where $P_{\varepsilon_s(\bar{x})}$ is given by (6.11) and (6.15).

The resulting simulation is shown in Figure 6.16. For comparison, the simulation data of a closed-loop system under a scheduled low gain feedback controller is shown in Figure 6.17. Clearly, the dynamics achieved by the low-and-high gain feedback has a lower overshoot.

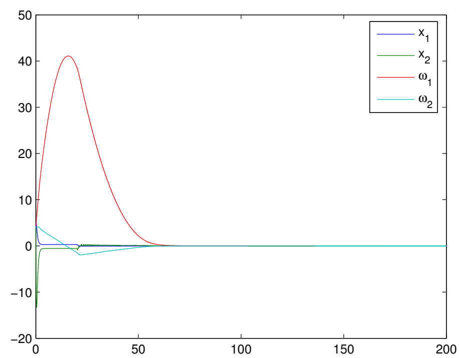


Figure 6.16: Global stabilization via low-and-high state feedback

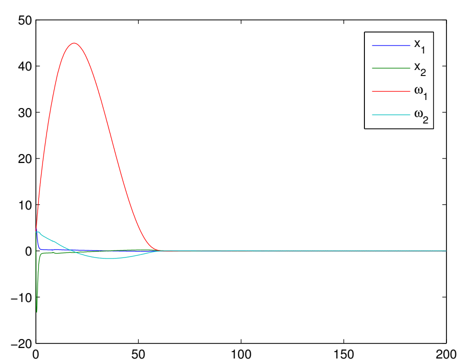


Figure 6.17: Global stabilization via low gain state feedback

6.9. Concluding remarks

In this chapter, we have considered a class of nonlinear sandwich systems, where the nonlinear element is a saturation. At first we dealt with single-layer sandwich systems, consisting of a single saturation sandwiched between two linear systems. We have established necessary and sufficient conditions for semi-global and global internal stabilization of such systems, and we have presented generalized low-gain and generalized scheduled low-gain design methodologies to achieve the prescribed stabilization. We have extended the design methodology to single-layer sandwich systems subject to input saturation, and further to multi-layer sandwich systems.

For ease of presentation, we have chosen to base the design methodologies in this paper on Riccati equations. It is also possible to generalize the classical eigenstructure assignment method from [52] to achieve the same results.

Current research is focused on constructing measurement feedback controllers to solve the semi-global and global internal stabilization problems, as well as external stabilization problems.

CHAPTER 7

Stabilization of linear system with input saturation and unknown delay–Continuous-time

7.1. Introduction

In the last few decades, time-delayed system has been greeted with great enthusiasm from researchers in recognition of its theoretical and applied importance, see [84]. Many control problems have been extensively studied, among which stability and stabilization are of particular interest (see, for instance, [73, 28, 72, 37, 23, 15] and references therein). Like time delay, actuator saturation is also ubiquitous in control application and is well known as the bane of closed-loop performance and stability. The study on stabilization subject to actuator saturation has a long history and still receives renewed attention. Numerous results have been reported in the literature. Some earlier work is surveyed in [5, 94, 123, 95, 32, 35].

When both actuator saturation and input time-delay are present, controller design can be challenging. What is worse, the precise knowledge of delay is not available in most circumstances while only an approximation, usually an upper bound, is known. In this case, [67] studied the global asymptotic stabilization for chains of integrators using nested-saturation type controller originally developed in [129]. This result was later on extended to a class of nonlinear feedforward systems in [66]. Chains of integrators were also studied in [69]. A linear low-gain state feedback was constructed to achieve the semi-global stabilization for integrator chains with input saturation and unknown input delay that has a known upper bound which can be arbitrarily large. A different low-gain design based on the parametric Lyapunov equation was used in [162] to prove a similar result for a broader class critically unstable systems with eigenvalues on the imaginary axis being zero. Both state and measurement feedback were developed. However, in the measurement feedback case, delays have to be known by the observer.

In this chapter, we investigate the stabilization of general linear critically unstable system subject to input saturation and multiple unknown constant input delays. We give nonconservative upper bounds on the delays which are inversely proportional to the maximal magnitude of the open-loop eigenvalues on the imaginary axis. This makes sense because when delay is unknown, a system with highly oscillatory behavior is obviously more difficult to stabilize than a system with dynamics that do not change "direction" so frequently. As the eigenvalues on imaginary axis move towards the origin, the upper bounds on delay turn to infinity. For unknown input delays satisfying these bounds, a linear low-gain state or finite dimensional measurement feedback controller can be designed to achieve semi-global stabilization. The design in this chapter only relies on the upper bounds. This chapter recovers and expands upon the results in [69] and [162].

The rest of the chapter is organized as follows: In Section 7.2, we formulate the stabilization problems and make necessary assumptions. Main results are presented in Section 7.3. We illustrate our designs with a numerical example in Section 7.4. Section 7.5 is conclusion. Proofs of some auxiliary lemmas are given in the Appendix.

The following standard notations will be used:
For a vector $x \in \mathbb{C}^n$,

$$x^* : \text{conjugate transpose of } x;$$
$$\|x\| : \text{2 norm of } x.$$

For a matrix $X \in \mathcal{C}^{n\times m}$,

$$X^* : \text{conjugate transpose of } X;$$
$$\sigma(X) : \text{singular value of } X;$$
$$\bar{\sigma}(X) : \text{maximal singular value of } X;$$
$$\underline{\sigma}(X) : \text{minimal singular value of } X;$$
$$\|X\| : \text{induced 2 norm};$$
$$\lambda(X) : \text{eigenvalue of } X \text{ if } X \text{ is square};$$
$$\det(X) : \text{determinant of } X \text{ if } X \text{ is square}.$$

For a continuous vector function $y : [0,\infty) \to \mathbb{R}^n$,

$$\|y\|_\infty : \mathscr{L}_\infty \text{ norm of } y;$$

$$\|y\|_2 : \mathscr{L}_2 \text{ norm of } y.$$

For a transfer function $H(s) : \mathbb{C} \to \mathbb{C}^{n\times m}$,

$$\|H(s)\|_\infty : \mathscr{H}_\infty \text{ norm of } H(s).$$

Let $\mathscr{C}_\tau^n := C([-\tau, 0], \mathbb{R}^n)$ denote the Banach space of all continuous functions from $[-\tau, 0] \to \mathbb{R}^n$ with norm $\|x\|_C = \sup_{t\in[-\tau,0]} \|x(t)\|$.

We denote a diagonal matrix as

$$\text{diag}\{A_i\}_{i=1}^m = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{bmatrix}$$

A standard saturation function $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined as

$$\sigma(s) = \begin{cases} 1, & s \geq 1; \\ s, & -1 < s < 1; \\ -1, & s \leq -1. \end{cases}$$

7.2. Problem formulation

Consider the following system:

$$\begin{cases} \quad \dot{x} & = Ax + \sum_{i=1}^m B_i \sigma[u_i(t-\tau_i)], \\ \quad y & = Cx, \\ x(\theta) = \phi(\theta), \theta \in [-\bar{\tau}, 0] \end{cases} \tag{7.1}$$

where $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $y \in \mathbb{R}^p$, $\phi \in \mathscr{C}_{\bar{\tau}}^n$. Each input u_i has delay $\tau_i \in [0, \bar{\tau}_i]$ and $\bar{\tau} = \max \bar{\tau}_i$.

We formulate two semi-global stabilization problems as follows:

Problem 7.1 The semi-global asymptotic stabilization via state feedback problem for system (7.1) is to find a set of $\bar{\tau}_i > 0$ and find, for any a priori given bounded set of initial conditions $\mathcal{W} \subset \mathscr{C}_{\bar{\tau}}^n$ with $\bar{\tau} = \max\{\bar{\tau}_i\}$, a linear state feedback controller $u = Fx$ independent of specific delay such that the zero solution of the closed-loop system is locally asymptotically stable for any $\tau_i \in [0, \bar{\tau}_i]$ with $\mathcal{W}$ contained in its domain of attraction, i.e. the following properties hold for all $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, ..., m$:

1. $\forall \varepsilon > 0, \exists \delta$ such that if $\|\phi\|_C \leq \delta$, $\|x(t)\| \leq \varepsilon$ for all $t \geq 0$;

2. $\forall \phi \in \mathcal{W}$, $x(t) \to 0$ as $t \to \infty$.

Problem 7.2 The semi-global asymptotic stabilization via measurement feedback problem for system (7.1) is to find a positive integer $q > 0$, a set of $\bar{\tau}_i > 0$ and for any a priori given bounded set $\mathcal{W} \subset \mathcal{C}_{\bar{\tau}}^{n+q}$ with $\bar{\tau} = \max\{\bar{\tau}_i\}$, a linear finite dimensional measurement feedback controller independent of delay

$$\begin{cases} \dot{\chi} = A_k \chi + B_k y, & \chi \in \mathbb{R}^q \\ u = C_k \chi + D_k y, \end{cases} \tag{7.2}$$

such that the zero solution of the closed-loop system is locally asymptotically stable for all $\tau_i \in [0, \tau_i]$ with $\mathcal{W}$ contained in its domain of attraction, i.e. the following properties hold for all $\tau_i \in [0, \bar{\tau}_i]$:

1. $\forall \varepsilon > 0, \exists \delta$ such that if $\|(\phi; \psi)\|_C \leq \delta$, $\|x(t)\| \leq \varepsilon$ for all $t \geq 0$;

2. $\forall (\phi; \psi) \in \mathcal{W}$, $(x(t), \chi(t)) \to 0$ as $t \to \infty$.

If $\tau_i = 0, i = 1, ..., m$, it is well known that the semi-global stabilization problem is solvable only if system (7.6) is Asymptotically Null Controllable with Bounded Control (ANCBC), i.e. the following assumption holds:

Assumption 7.1 (A, B) is stabilizable with $B = [B_1, \cdots, B_m]$ and A has all its eigenvalues in the closed left half plane.

Moreover, for stabilization via measurement feedback, the next assumption is also necessary.

Assumption 7.2 (A, C) is detectable.

7.3. Main result

We start from designing the state and measurement feedback controllers that will solve the stabilization problems. The methodology we use here are the classical $H_2 - ARE$ based low-gain feedback design (see [148]) which was originally developed in [61] in the context of semi-global stabilization of linear systems subject to input saturation.

Assume (A, B) is stabilizable and A has all its eigenvalues in the closed left half plane. For $\varepsilon \in (0, 1]$, let P_ε be the solution of Algebraic Riccati Equation

$$A' P_\varepsilon + P_\varepsilon A - P_\varepsilon B B' P_\varepsilon + \varepsilon I = 0. \tag{7.3}$$

The low-gain state feedback can be constructed as

$$u = F_\varepsilon = -B' P_\varepsilon x. \tag{7.4}$$

The low-gain state feedback (7.4) can be implemented into a dynamic compensator, which we refer to as a low-gain compensator

$$\begin{cases} \dot{\chi} = A\chi + BF_\varepsilon \chi - K(y - C\chi) \\ u = F_\varepsilon \chi, \end{cases} \tag{7.5}$$

where K is chosen such that $A + KC$ is Hurwitz stable.

In the design of (7.4) and (7.5), ε is called a low-gain parameter. With a properly chosen ε, the low-gain feedback (7.4) and low-gain compensator (7.5) solve Problem 1 and 2 respectively for suitably chosen $\bar{\tau}_i$. To prove this, we will proceed in two steps: first, we will show that our controllers globally asymptotically stabilize (7.1) without saturation and provide us with a non-conservative input-delay tolerance. Then, we will extend the result to the case where saturation is present by selecting the low-gain parameter differently.

7.3.1. Global stabilization of linear systems with input delay

Ignoring saturation, we can write (7.1) as follows:

$$\begin{cases} \dot{x} &= Ax + \sum_{i=1}^{m} B_i u_i(t - \tau_i) \\ y &= Cx \\ x(\theta) = \phi(\theta), & \forall \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{7.6}$$

Since the system is linear, it is possible to solve the global asymptotic stabilization problems for (7.6) using the low-gain feedback (7.4) and compensator (7.5), which means in Problem 1 and 2, the bounded set of initial condition $\mathcal{W}$ is actually the entire Banach space $\mathcal{C}^n_{\bar{\tau}}$ and $\mathcal{C}^{n+q}_{\bar{\tau}}$.

In order to present our result, we need the following notation. For each input u_i $i = 1, ..., m$, define the maximal controllable frequency as

$$\omega^i_{max} := \max\{\omega \in \mathbb{R} \mid \exists v \in \mathbb{C}^n, \text{ s.t. } A'v = j\omega v \text{ and } v^* B_i \neq 0\}. \tag{7.7}$$

It is clear that $j\omega_{\max}^i$ is the eigenvalue of A on the imaginary axis with the maximal magnitude which is controllable via input channel u_i. Now, we are ready to present the following theorem:

Theorem 7.1 For any $\bar{\tau}_i < \frac{\pi}{3\omega_{max}^i}$, $i = 1, ..., m$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop of (7.6) and the low-gain feedback (7.4) is globally asymptotically stable for any $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, ..., m$.

Proof : Consider the closed-loop system

$$\dot{x} = Ax + \sum_{i=1}^{m} B_i F_i x(t - \tau_i). \tag{7.8}$$

Define

$$G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1} B \text{ and } D(s) = \text{diag}\{e^{-\tau_i s}\}_{i=1}^{m}$$

The following result is classical:

Lemma 7.1 The system (7.8) is asymptotically stable if and only if

$$\det\Big[I - G_\varepsilon(j\omega)\Big(D(j\omega) - I\Big)\Big] \neq 0, \quad \forall\omega,\ \forall\tau_i \in [0, \bar{\tau}_i]. \tag{7.9}$$

(see [21, 157])

Note that in general (7.9) has to be satisfied for all $\omega \in \mathbb{R}$. However, due to the merit of low-gain feedback, we are only concerned with those ω's that are in a finite number of small intervals.

Assume A has r eigenvalues on the imaginary axis which are denoted by $j\omega_k$, $k = 1, ..., r$. Given $\bar{\tau}_i < \frac{\pi}{3\omega_{max}^i}$ for $i = 1, ..., m$, there exists a $\delta > 0$ such that

1. The neighborhoods $\mathcal{E}_k := (\omega_k - \delta, \omega_k + \delta)$, $k = 1, ..., r$ around these eigenfrequencies, are mutually disjoint;

2. $\omega\bar{\tau}_i < \pi/3$ for $\omega \in \mathcal{E}_k$ if ω_k is at least partially controllable through input i.

Lemma 7.2 The following properties hold:

1. If $j\omega_k$ is not controllable via input u_i for some i, then

$$\lim_{\varepsilon \downarrow 0} F_\varepsilon (j\omega I - A - BF_\varepsilon)^{-1} B e_i = 0,$$

uniformly in ω for $\omega \in \mathcal{E}_k$ where e_i is the standard basis of $\mathbb{R}^m$ and F_ε is given by (7.4).

2. There exists ε^* such that for $\varepsilon \in (0, \varepsilon^*]$,

$$\| F_\varepsilon (j\omega I - A - BF_\varepsilon)^{-1} B \| \leq \tfrac{1}{3}, \quad \forall \omega \in \Omega := \mathbb{R} \setminus \cup_{k=1}^{r} \mathcal{E}_k.$$

Thanks to Lemma 7.2, we find that there exists an ε_1 such that (7.9) is satisfied if for all $k = 1, ..., r$

$$\det[I - G_\varepsilon(j\omega) \left(\tilde{D}_k(j\omega) - I\right)] \neq 0, \quad \forall \omega \in \mathcal{E}_k, \forall \tau_i \in [0, \bar{\tau}_i] \tag{7.10}$$

provided $\varepsilon \leq \varepsilon_1$ where $\tilde{D}_k(s)$ equals $D(s)$ with $\tau_i = 0$ for all i's such that the eigenvalue $j\omega_k$ is not controllable via input channel i.

Let's consider (7.10),

$$\begin{aligned} I - G_\varepsilon(j\omega)(\tilde{D}_k(j\omega) - I) =& I - (I + G_\varepsilon(j\omega))(\tilde{D}_k(j\omega) - I) + (\tilde{D}_k(j\omega) - I) \\ =& \tilde{D}_k(j\omega) - (I + G_\varepsilon(j\omega))(\tilde{D}_k(j\omega) - I). \end{aligned}$$

First of all, we know that for all $\varepsilon > 0$

$$\underline{\sigma}[I - F_\varepsilon (j\omega I - A)^{-1} B] \geq 1, \quad \forall \omega$$

(see Section 5.4, p.122 in [1]), and this implies that

$$\bar{\sigma}[I + G_\varepsilon(j\omega)] \leq 1, \quad \forall \omega. \tag{7.11}$$

Since $\tilde{D}_k(j\omega)$ is unitary, it is easy to see that given (7.11), $\tilde{D}_k(j\omega) - (I + G_\varepsilon(j\omega))(\tilde{D}_k(j\omega) - I)$ is nonsingular if $\bar{\sigma}(\tilde{D}_k(j\omega) - I) < 1$. Therefore, we have the condition (7.10) holding for $\varepsilon \leq \varepsilon_1$ if for all $k = 1, ..., r$

$$\bar{\sigma}(\tilde{D}_k(j\omega) - I) < 1, \; \forall \omega \in \mathcal{E}_k, \forall \tau_i \in [0, \bar{\tau}_i]. \tag{7.12}$$

This is guaranteed by our choice of δ and $\mathcal{E}_k$. ■

In a special case where A has all its eigenvalues at the origin, the low-gain feedback can tolerate any bounded delay that can be arbitrarily large.

Corollary 7.1 Suppose A has only zero eigenvalues. For any $\bar{\tau}_i > 0$, $i = 1, ..., m$, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (7.6) and (7.4) is asymptotically stable for any $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, ..., m$.

The next theorem concerns stabilization of (7.6) via measurement feedback.

Theorem 7.2 For any $\bar{\tau}_i < \frac{\pi}{3\omega_{max}^i}$, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (7.6) and low-gain compensator (7.5) is asymptotically stable for $\tau_i \in [0, \bar{\tau}_i]$.

Proof : The closed-loop system is given by

$$\begin{cases} \dot{x} &= Ax + \sum_{i=1}^{m} B_i F_i \chi(t - \tau_i) \\ \dot{\chi} &= (A + BF_\varepsilon + KC)\chi - KCx \\ x(\theta) = \phi(\theta), & \forall \theta \in [-\bar{\tau}, 0] \\ \chi(\theta) = \psi(\theta), & \forall \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{7.13}$$

Define

$$G_\varepsilon^m(s) = -F_\varepsilon(sI - A - BF_\varepsilon)^{-1} KC(sI - A - KC)^{-1} B.$$

Obviously, $G_\varepsilon^m(s)$ is stable.

It follows from Lemma 7.1 that the closed-loop system of (7.6) and (7.5) is global asymptotically stable if and only if

$$\det[I - G_\varepsilon^m(j\omega)\,(D(j\omega) - I)] \neq 0, \quad \forall \omega \in \mathbb{R}, \forall \tau_i \in [0, \bar{\tau}_i], \tag{7.14}$$

We have the following lemma

Lemma 7.3 Let $G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1} B$. Then

$$\lim_{\varepsilon \downarrow 0} \left(G_\varepsilon^m(j\omega) - G_\varepsilon(j\omega) \right) = 0$$

uniformly in ω.

Proof of Lemma 7.3 : See Appendix. ■

If, by Theorem 1, there exists an $\varepsilon_2 \leq \varepsilon_1$ such that for all $\varepsilon \in (0, \varepsilon_2]$ we have (7.9) satisfied with $G_\varepsilon(j\omega)$, then we can find an $\varepsilon_3 \leq \varepsilon_2$ such that (7.14) holds for all $\varepsilon \in (0, \varepsilon_3]$. ■

7.3.2. Semi-global stabilization subject to input saturation

In this subsection, we shall extend the results for linear systems to the case where input saturation is considered and solve the semi-global stabilization problems as formulated in Problem 1 and 2.

Theorem 7.3 Consider the system (7.1). The semi-global asymptotic stabilization via state feedback problem can be solved by the low-gain feedback (7.4). Specifically, for a set of positive real numbers $\bar{\tau}_i < \frac{\pi}{3\omega_{max}^i}$, $i = 1, ..., m$ and any *a priori* given compact set of initial conditions $\mathcal{W} \subset \mathcal{C}_{\bar{\tau}}^n$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the low-gain feedback (7.4) achieves local asymptotic stability of the closed-loop system with the domain of attraction containing $\mathcal{W}$ for any $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, ..., m$.

Proof : The closed-loop system can be written as

$$\begin{cases} \dot{x} & = Ax + \sum_{i=1}^{m} B_i \sigma(F_i x(t - \tau_i)) \\ x(\theta) = \phi(\theta), \ \forall \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{7.15}$$

Suppose $\bar{\tau}_i$'s satisfy the bound $\bar{\tau}_i < \frac{\pi}{3\omega_{max}^i}$. Let ε_1 be such that the closed-loop system in the absence of saturation, i.e. (7.8), is asymptotically stable. Then the local stability of (7.15) for $\varepsilon \leq \varepsilon_1$ follows.

It remains to show the attractivity. It is sufficient to prove that for system (7.15), given $\mathcal{W}$, there exists an $\varepsilon^* \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$, we have

$$\|F_\varepsilon x(t - \bar{\tau})\| \leq 1, \forall t \geq 0.$$

Then we can avoid saturation for all $t \geq 0$. The closed-loop system becomes linear and the attractivity of zero solution is therefore guaranteed with $\varepsilon \leq \varepsilon_1$.

Let us define two linear time invariant operators g_ε and δ with the following transfer matrices:

$$G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1} B$$

$$\Delta(s) = D(s) - I = \text{diag}\{e^{-\tau_i s} - 1\}_{1=1}^{m}.$$

Note that the operators g_ε and δ have zero initial conditions. From the proof of Theorem 7.1, we know that (7.9) is satisfied which guarantees that there exists a μ such that

$$\underline{\sigma}(I - G_\varepsilon(j\omega)\Delta(j\omega)) > \mu, \forall \omega \in \mathbb{R}, \forall \tau_i \in [0, \bar{\tau}_i]$$

for all $\varepsilon \le \varepsilon_1$ and this μ only depends on $\bar{\tau}_i$ provided that $\varepsilon \le \varepsilon_1$. This implies that

$$\|(I - G_\varepsilon(s)\Delta(s))^{-1}\|_\infty \le \tfrac{1}{\mu}.$$

Moreover, we already have in (7.11)

$$\bar{\sigma}(I + G_\varepsilon(j\omega)) \le I, \ \forall \omega \in \mathbb{R}$$

which implies $\|G_\varepsilon(s)\|_\infty \le 2$. Note that for $t \ge 0$

$$\dot{x} = (A + BF_\varepsilon)x + B\delta(F_\varepsilon x) + Bv_\varepsilon,$$

where

$$v_\varepsilon(t) = \begin{bmatrix} v_1(t) \\ \vdots \\ v_m(t) \end{bmatrix}, \quad v_i(t) = \begin{cases} F_i\phi(t-\tau_i), & t < \tau_i, \\ 0, & t \ge \tau_i. \end{cases}$$

Since $v_\varepsilon(t)$ vanishes for $t \ge \bar{\tau}$, $\phi \in \mathcal{W}$ and $F_\varepsilon \to 0$, we have for any $\phi \in \mathcal{W}$, $\|v_\varepsilon\|_\infty \to 0$ and $\|v_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$.

We have

$$F_\varepsilon x(t) = F_\varepsilon e^{(A+BF_\varepsilon)t}x(0) + (g_\varepsilon \circ \delta)(F_\varepsilon x)(t) + g_\varepsilon(v_\varepsilon)(t)$$

and hence

$$F_\varepsilon x(t) = (1 - g_\varepsilon \circ \delta)^{-1}\left[F_\varepsilon e^{(A+BF_\varepsilon)t}x(0) + g_\varepsilon(v_\varepsilon)(t)\right]. \tag{7.16}$$

Let $w_\varepsilon(t) = g_\varepsilon(v_\varepsilon)(t)$. By the definition of g_ε, we have

$$\begin{cases} \dot{\xi} = (A + BF_\varepsilon)\xi + Bv_\varepsilon, \quad \xi(0) = 0 \\ w_\varepsilon = F_\varepsilon\xi. \end{cases}$$

Clearly, $\|w_\varepsilon\|_2 \le \|G_\varepsilon(s)\|_\infty\|v_\varepsilon\|_2 \le 2\|v_\varepsilon\|_2$. Hence for any given initial condition ϕ, $\|w_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$.

For $t \in [0, \bar{\tau}]$,

$$\begin{aligned} \dot{w}_\varepsilon(t) &= F_\varepsilon(A + BF_\varepsilon)\xi(t) + F_\varepsilon Bv_\varepsilon(t) \\ &= F_\varepsilon(A + BF_\varepsilon)\int_0^t e^{(A+BF_\varepsilon)(t-r)}Bv_\varepsilon(s)\mathrm{d}r + F_\varepsilon Bv_\varepsilon(t) \end{aligned}$$

Since $A + BF_\varepsilon$ is bounded for all $\varepsilon \in [0, 1]$ and $\|v_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$, we will have

$$\sup_{t\in[0,\bar{\tau}]} \|\dot{w}_\varepsilon(t)\| \to 0 \text{ as } \varepsilon \to 0. \tag{7.17}$$

This also implies

$$\int_0^{\bar{\tau}} \|\dot{w}(t)\|^2 \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0. \tag{7.18}$$

From $\bar{\tau}$ onward, $v_\varepsilon(t)$ vanishes and

$$\dot{w}(t) = F_\varepsilon e^{(A+BF_\varepsilon)t}(A + BF_\varepsilon)\xi(\bar{\tau})$$

It is shown by [148] that

$$\int_{\bar{\tau}}^{\infty} \|\dot{w}(t)\|^2 \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0. \tag{7.19}$$

provided that $\xi(\bar{\tau})$ is bounded which is obvious by noticing that

$$\xi(\bar{\tau}) = \int_0^{\bar{\tau}} e^{(A+BF)(\bar{\tau}-t)} B v_\varepsilon(t)\mathrm{d}t$$

and $\|v_\varepsilon\|_\infty \to 0$ as $\varepsilon \to 0$. Combining (7.18) and (7.19), we have shown that for any given $\phi \in \mathcal{W}$, $\|\dot{w}\|_2 \to 0$ as $\varepsilon \to 0$.

Now let us go back to (7.16). We get

$$\begin{aligned}\|F_\varepsilon x\|_2 \leq & \|(1 - G_\varepsilon(s)D(s))^{-1}\|_\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t} x(0)\|_2 + \|(1 - G_\varepsilon(s)D(s))^{-1}\|_\infty \|w_\varepsilon\|_2 \\ \leq & \tfrac{1}{\mu}\|F_\varepsilon e^{(A+BF_\varepsilon)t} x(0)\|_2 + \tfrac{1}{\mu}\|w_\varepsilon\|_2.\end{aligned}$$

Since for any ϕ, $\|F_\varepsilon e^{(A+BF_\varepsilon)t} x(0)\|_2 \to 0$ (see [148]) and $v_\varepsilon \to 0$ as $\varepsilon \to 0$ and μ is independent of ε (provided ε is smaller than ε_1), there exists an ε_3 such that for $\varepsilon \in (0, \varepsilon_3]$, we get

$$\|F_\varepsilon x\|_2 \leq \tfrac{1}{2},\ \forall \phi \in \mathcal{W}. \tag{7.20}$$

Note that (7.16) also yields

$$F\dot{x}(t) = (1 - g_\varepsilon \circ \delta)^{-1} \left[F_\varepsilon e^{(A+BF_\varepsilon)t}(A + BF_\varepsilon)x(0) \ + \dot{w}_\varepsilon(t)\right],$$

and thus

$$\begin{aligned}\|F_\varepsilon \dot{x}\|_2 \leq & \|(1-G_\varepsilon(s)D(s))^{-1}\|_\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}\tilde{x}\|_2 + \|(1-G_\varepsilon(s)D(s))^{-1}\|_\infty \|\dot{w}_\varepsilon\|_2 \\ \leq & \frac{1}{\mu}\|F_\varepsilon e^{(A+BF_\varepsilon)t}\tilde{x}\| + \frac{1}{\mu}\|\dot{w}_\varepsilon\|_2\end{aligned}$$

with $\tilde{x} = (A + BF_\varepsilon)x(0)$. There exists an ε_4 such that for $\varepsilon \in (0, \varepsilon_4]$, we have

$$\|F_\varepsilon \dot{x}\|_2 \leq \tfrac{1}{2},\ \forall \phi \in \mathcal{W}. \tag{7.21}$$

Applying Cauchy-Schwartz inequality, we can prove that for any $t \geq 0$,

$$\left|\|F_\varepsilon x(t)\|^2 - \|F_\varepsilon x(0)\|^2\right| \leq 2\|F_\varepsilon \dot{x}\|_2 \|F_\varepsilon x\|_2,$$

and

$$\|F_\varepsilon x(t)\|^2 \leq \|F_\varepsilon x(0)\|^2 + 2\|F_\varepsilon \dot{x}\|_2 \|F_\varepsilon x\|_2. \tag{7.22}$$

Finally, there exists an ε_5 such that for $\varepsilon \in (0, \varepsilon_5]$

$$\|F_\varepsilon x(0)\|^2 \leq \|F_\varepsilon \phi\|_C^2 \leq \tfrac{1}{2}, \quad \phi \in \mathcal{W}. \tag{7.23}$$

Let $\varepsilon^* = \min\{\varepsilon_1, \cdots, \varepsilon_5\}$. We conclude from (7.20), (7.21), (7.22) and (7.23) that for $\varepsilon \in (0, \varepsilon^*]$,

$$\|F_\varepsilon x(t - \bar{\tau})\| \leq 1, \forall t \geq 0.$$

■

The next theorem solves Problem 2.

Theorem 7.4 Consider the system (7.1). The semi-global asymptotic stabilization via measurement feedback problem can be solved by the low-gain compensator (7.5). Specifically, for any *a priori* given compact set of initial conditions $\mathcal{W} \subset \mathcal{C}_{\bar{\tau}}^{2n}$ and a set of positive real numbers $\bar{\tau}_i < \frac{\pi}{3\omega_{max}^i}$, $i = 1, ..., m$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the low-gain feedback (7.5) achieves local asymptotic stability of the closed-loop system for any $\tau_i \in [0, \bar{\tau}_i]$, $i = 1, ..., m$ with the domain of attraction containing $\mathcal{W}$.

Proof : The closed-loop system can be written as

$$\begin{cases} \dot{x} & = Ax + \sum_{i=1}^{m} B_i \sigma(F_i \chi(t-\tau_i)) \\ \dot{\chi} & = (A + BF_\varepsilon + KC)\chi - KCx \\ x(\theta) = \phi(\theta), \ \forall \theta \in [-\bar{\tau}, 0] \\ \chi(\theta) = \psi(\theta), \ \forall \theta \in [-\bar{\tau}, 0]. \end{cases} \tag{7.24}$$

Suppose $\bar{\tau}_i$'s satisfy the bound $\bar{\tau}_i < \frac{\pi}{3\omega_{max}^i}$. Let ε_1 be given by Theorem 7.2 such that the closed-loop system without saturation is asymptotically stable. Then the local stability of (7.24) for $\varepsilon \leq \varepsilon_1$ follows.

Define two linear time invariant operators g_ε^m and δ with Laplacian transform

$$G_\varepsilon^m(s) = -F_\varepsilon(sI - A - BF_\varepsilon)^{-1} KC(sI - A - KC)^{-1} B$$

$$\Delta(s) = D(s) - I = \text{diag}\{e^{-\tau_i s} - 1\}_{i=1}^m.$$

From the proof of Theorem 2, we know that (7.14) holds for $\varepsilon \leq \varepsilon_1$. There exists a $\mu > 0$ such that

$$\underline{\sigma}(I - G_\varepsilon^m(j\omega)\Delta(j\omega)) > \mu, \forall \omega \in \mathbb{R}, \forall \tau_i \in [0, \bar{\tau}_i], \tag{7.25}$$

where μ is independent of ε provided that $\varepsilon \leq \varepsilon_1$. It follows from Lemma 7.3 that $G_\varepsilon^m(j\omega) \to G_\varepsilon(j\omega)$ uniformly in ω where $G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B$. Hence given $\bar{\sigma}(G_\varepsilon(j\omega)) \leq 2$ for any $\varepsilon > 0$ and $\omega \in \mathbb{R}$, there exists an ε_2 such that

$$\bar{\sigma}(G_\varepsilon^m(j\omega)) \leq 3, \ \forall \omega \in \mathbb{R}. \tag{7.26}$$

We also have that following lemma

Lemma 7.4 For any $\xi \in \mathbb{R}^{2n}$,

$$\lim_{\varepsilon \downarrow 0} \int_0^\infty \|\mathcal{F}_\varepsilon e^{(\mathcal{A} + \mathcal{B}\mathcal{F}_\varepsilon)t} \xi\|^2 \text{d}t = 0,$$

where

$$\mathcal{A} = \begin{bmatrix} A & BF_\varepsilon \\ -KC & A + BF_\varepsilon + KC \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ \mathcal{F} = \begin{bmatrix} 0 & F_\varepsilon \end{bmatrix}.$$

Proof of Lemma 7.4 : See Appendix. ■

Given (7.25), (7.26) and Lemma 7.4 hold, we can use exactly the same argument as in the proof of Theorem 7.3 to prove that there exists an $\varepsilon^* \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$,

$$\|F_\varepsilon \chi(t - \bar{\tau})\| \leq 1, \ \forall t \geq 0, \ (\phi, \psi) \in \mathcal{W}.$$

■

7.4. Example

Consider the following example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(t-\tau_1) \\ u_2(t-\tau_2) \end{bmatrix}$$

$$y_1 = x_1, \quad y_2 = x_2.$$

First, we have

$$\omega_{max}^1 = 0, \quad \omega_{max}^2 = 1.$$

The upper bounds on delay are given by

$$\bar{\tau}_1 < \infty, \quad \bar{\tau}_2 < \tfrac{\pi}{3}.$$

In this example, we choose $\bar{\tau}_1 = 1$ and $\bar{\tau}_2 = \frac{\pi}{4}$. The initial condition is given by

$$x(\theta) = \phi(\theta) = \begin{bmatrix} 2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \ \forall \theta \in [-1, 0].$$

7.4.1. State feedback

Choose $\varepsilon = 0.001$. The low-gain state feedback can constructed according to (7.4) which is

$$F_\varepsilon = \begin{bmatrix} -0.0281 & -0.2319 & 0.2262 & -0.0587 \\ -0.0145 & -0.0587 & 0.0512 & -0.1120 \end{bmatrix}.$$

The simulation data is shown in the following figures.

7.4.2. Measurement feedback

The low-gain compensator can be constructed as in (7.5) with

$$K = \begin{bmatrix} -7 & -2 \\ 1 & -7 \\ -11 & -7 \\ 7 & -9 \end{bmatrix},$$

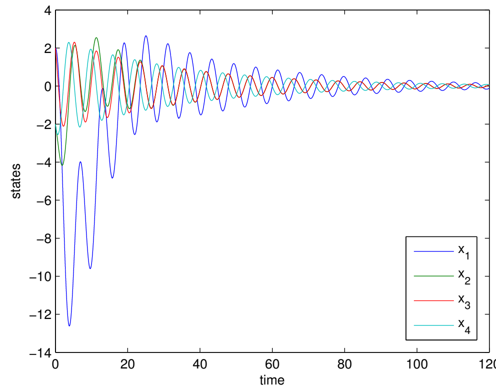


Figure 7.1: Evolution of states

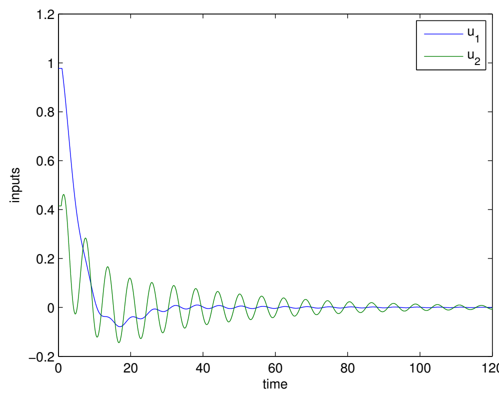


Figure 7.2: Inputs to the system

and the initial condition of the compensator is given by

$$\chi(\theta) = \psi(\theta) = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \ \forall \theta \in [-1, 0].$$

In this case, ε is chosen to be 0.0001. Simulation data is shown in Fig. 7.4.2 and 7.4.2.

7.5. Conclusion

In this chapter, the semi-global stabilization problems for general uncritically unstable systems subject to input saturation and multiple unknown input delays are solved. Nonconservative upper bounds on delays are found for which a low-gain state feedback or a low-gain compensator can be constructed to

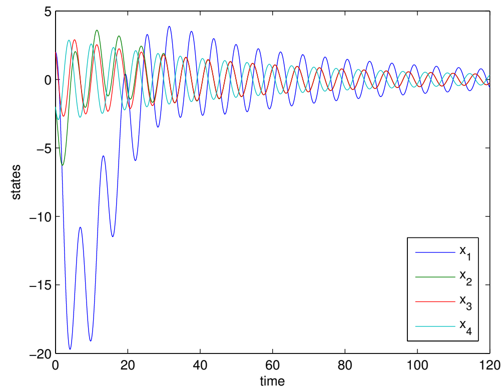


Figure 7.3: Evolution of states

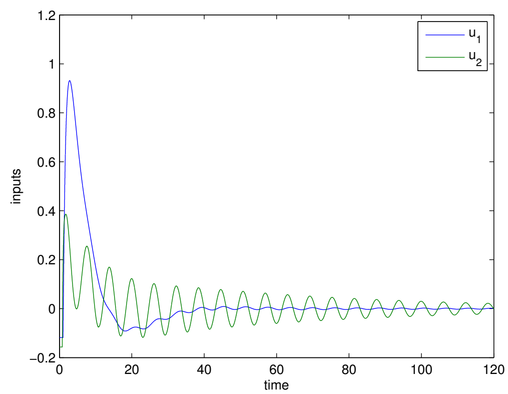


Figure 7.4: Inputs to the system

achieve the semi-global stabilization.

Appendix

Proof of Lemma 7.2 : To prove item (1), we first note that

$$\begin{aligned}
&F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}Be_i \\
=&F_\varepsilon(I - (j\omega I - A)^{-1}BF_\varepsilon)^{-1}(j\omega I - A)^{-1}Be_i \\
=&(I - F_\varepsilon(j\omega I - A)^{-1}B)^{-1}F_\varepsilon(j\omega I - A)^{-1}Be_i.
\end{aligned}$$

Next we note that:

$$\bar{\sigma}(I - F_\varepsilon(j\omega I - A)^{-1}B)^{-1} \leq 1,\ \forall \omega \in \mathbb{R}$$

(see [1]). Moreover, $\forall \omega \in \mathcal{E}_k$, $(j\omega I - A)^{-1}Be_i$ has no pole and therefore

$$\|(j\omega I - A)^{-1}Be_i\| \leq M, \forall \omega \in \mathcal{E}_k.$$

for $M > 0$ independent of ω.

But then

$$\|F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}Be_i\| \leq M\|F_\varepsilon\|, \forall \omega \in \mathcal{E}_k,$$

and since F_ε converges to zero we get

$$\|F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}Be_i\| \to 0$$

as $\varepsilon \to 0$ uniformly in $\mathcal{E}_k$.

It remain to show item (2). By definition, $\det(j\omega I - A) \neq 0$ for all $\omega \in \Omega$. There exists a μ such that

$$\underline{\sigma}(j\omega I - A) > \mu,\ \forall \omega \in \Omega.$$

After all assume this is not the case. Then there exists a sequence $\omega^i \in \Omega$ such that

$$\underline{\sigma}(j\omega^i I - A) \to 0$$

as $i \to \infty$. We can ensure that this sequence ω^i is bounded since for ω satisfying $|\omega| > \|A\| + 1$ we have:

$$\underline{\sigma}(j\omega I - A) > |\omega| - \|A\| > 1$$

But a bounded sequence ω^i has a convergent subsequence whose its limit, denoted by $\bar{\omega}$, is in Ω (since Ω is closed). The limit $\bar{\omega}$ would have the property

$$\underline{\sigma}(j\bar{\omega}I - A) = 0.$$

This implies $\bar{\omega}$ is an eigenvalue of A which is in contradiction with definition of Ω.

Choose ε^* such that $\|F_\varepsilon\| \leq \frac{\mu}{4}\|B\|^{-1}$ for $\varepsilon \leq \varepsilon^*$. In that case:

$$\underline{\sigma}(\omega I - A - BF) > \mu - \|B\|\|F_\varepsilon\| > \tfrac{3\mu}{4}, \forall \omega \in \Omega,$$

and hence

$$\|(j\omega I - A - BF_\varepsilon)^{-1}\| < \tfrac{4}{3\mu}, \forall \omega \in \Omega,$$

but then

$$\|F_\varepsilon(j\omega I - A - BF_\varepsilon)^{-1}B\| \leq \|F_\varepsilon\|\|(j\omega I - A - BF_\varepsilon)^{-1}\|\|B\| \leq \tfrac{1}{3}$$

for all $\omega \in \Omega$. ■

Proof of Lemma 7.3 :

The error between $G_\varepsilon^m(s)$ and $G_\varepsilon(s)$ is

$$\begin{aligned} G_\varepsilon(s) - G_\varepsilon^m(s) &= \left[I + F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B\right] F_\varepsilon(sI - A - KC)^{-1}B \\ &= [I + G_\varepsilon(s)]\, F_\varepsilon(sI - A - KC)^{-1}B \end{aligned}$$

From (7.11) we obtain

$$\bar{\sigma}(I + G_\varepsilon(j\omega)) \leq 1,\ \forall \varepsilon > 0,\ \omega \in \mathbb{R}.$$

Moreover,

$$\|F_\varepsilon(sI - A - KC)^{-1}B\|_\infty \leq \|F_\varepsilon\|\|(sI - A - KC)^{-1}B\|_\infty$$

Since $F_\varepsilon \to 0$ as $\varepsilon \to 0$, we immediately have that

$$\lim_{\varepsilon \downarrow 0} \left(G_\varepsilon^m(j\omega) - G_\varepsilon(j\omega)\right) = 0,$$

uniformly in ω. ■

Proof of Lemma 7.4 : Define a system as

$$\begin{cases} \dot{x}_1 = Ax_1 + BF_\varepsilon x_2 \\ \dot{x}_2 = (A + BF_\varepsilon + KC)x_2 - KCx_1 \\ \ z\ = F_\varepsilon x_2 \end{cases}, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \xi$$

It is obvious that for any ξ

$$\|z\|_2 = \int_0^\infty \|\mathcal{F}_\varepsilon e^{(\mathcal{A}+\mathcal{B}\mathcal{F})t}\xi\|^2 \mathrm{d}t.$$

Let $e = x_1 - x_2$. In the new coordinates of (x_1, e), the above system can be written as

$$\begin{cases} \dot{x}_1 = (A + BF_\varepsilon)x_1 - BF_\varepsilon e \\ \dot{e} = (A + KC)e \\ z = F_\varepsilon(x_1 - e) \end{cases},$$

with $e_1(0) = x_1(0) - x_2(0)$. We get $\|z\|_2 \leq \|F_\varepsilon e\|_2 + \|F_\varepsilon x_1\|_2$.

Since $A + KC$ is Hurwitz, there exists a γ such that $\|e\|_2 \leq \gamma\|e(0)\|$ for any $e(0) \in \mathbb{R}^n$. Then

$$\|F_\varepsilon e\|_2 \leq \gamma\|F_\varepsilon\|\|e(0)\| \to 0 \text{ as } \varepsilon \to 0.$$

But for x_1, we have

$$\begin{aligned} \|F_\varepsilon x_1\|_2 &\leq \|G_\varepsilon(s)\|_\infty\|F_\varepsilon e\|_2 + \int_0^\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}x_1(0)\|^2 \mathrm{d}t \\ &\leq 2\gamma\|F_\varepsilon\|\|e(0)\| + \int_0^\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}x_1(0)\|^2 \mathrm{d}t \end{aligned}$$

where $G_\varepsilon(s) = F_\varepsilon(sI - A - BF_\varepsilon)^{-1}B$. It was shown in [148] that

$$\lim_{\varepsilon\downarrow 0} \int_0^\infty \|F_\varepsilon e^{(A+BF_\varepsilon)t}x_1(0)\|^2 \mathrm{d}t = 0.$$

and thus $\lim_{\varepsilon\downarrow 0}\|F_\varepsilon x_1\|_2 = 0$. We conclude that $\lim_{\varepsilon\downarrow 0}\|z\|_2 = 0$.

■

CHAPTER 8

Stabilization of linear systems subject to input saturation and multiple unknown constant delays–discrete-time systems

8.1. Introduction

The aim of this chapter is to extend the results in previous Chapter to discrete-time case. The analysis and design here is also based on a simple frequency-domain stability criterion analogous to the one for continuous linear time-delay systems. It turns out that the results of discrete-time systems are in a strict parallel with those of continuous-time systems. The upper bound of tolerable delays found here is also inversely proportional to the argument of eigenvalues on the unit circle. If all the delays satisfy the proposed upper bounds, linear state and finite dimensional dynamic measurement feedback can be constructed using the H_2 low-gain design technique to achieve the semi-global stabilization.

This chapter is organized as follows. Two stabilization problems are formulated in Section 8.2. Some preliminary results, including a stability criterion for linear discrete time-delay systems and some key properties of H_2 low-gain feedback, are presented in Section 8.3. The main results of this paper are developed in Section 8.4. In this part, we first stabilize the linearized system without saturation using the low-gain feedback and then show that by proper selection of a tuning parameter, the same controller will solve the semi-global stabilization problems in the presence of saturation. Proofs of several technical lemmas used in the stability analysis are given in the Appendix.

8.1.1. Notations

The following notations are used. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$ and $\mathbb{N}$ denote respectively the sets of all complex numbers, real numbers, positive real numbers, integers and natural numbers. For any open set $\mathscr{G} \subset \mathbb{C}$, $\partial\mathscr{G}$ and $\overline{\mathscr{G}}$ denote its boundary and closure. For $z_0 \in \mathbb{C}$ and $r \in \mathbb{R}^+$, $\mathscr{D}(z_0, r)$ denotes an open disc centered at z_0 with radius r. Among all, the unit open disc centered at the origin is of particular importance and will be used very often, as such we denote specially

$$\mathscr{D}_0 := \mathscr{D}(0,1), \quad \mathbb{C}^{\odot} := \overline{\mathscr{D}(0,1)}, \quad \mathbb{C}^{\circ} := \partial\mathscr{D}(0,1).$$

For any $K_1, K_2 \in \mathbb{Z}$ and $K_1 \leq K_2$,

$$\overline{[K_1, K_2]} := \{k \in \mathbb{Z} \mid K_1 \leq k \leq K_2\}.$$

For a vector $x \in \mathbb{C}^n$,

$$x^* : \text{conjugate transpose of } x;$$

$$\|x\| : \text{2 norm of } x.$$

For a matrix $X \in \mathbb{C}^{n \times m}$,

$$X^* : \text{conjugate transpose of } X;$$

$$\sigma(X) : \text{singular value of } X;$$

$$\bar{\sigma}(X) : \text{maximal singular value of } X;$$

$$\underline{\sigma}(X) : \text{minimal singular value of } X;$$

$$\|X\| : \text{induced 2 norm;}$$

$$\lambda(X) : \text{eigenvalue of } X \text{ if } X \text{ is square;}$$

$$\det(X) : \text{determinant of } X \text{ if } X \text{ is square.}$$

For a sequence $\{y_n\} \subset \mathbb{R}^n$,

$$\|y\|_\infty : \ell_\infty \text{ norm of } y;$$

$$\|y\|_2 : \ell_2 \text{ norm of } y.$$

For a discrete-time transfer function $H(z) : \mathbb{C} \to \mathbb{C}^{n \times m}$,

$$\|H\|_\infty : \mathcal{H}_\infty \text{ norm of } H(z).$$

Let $\ell_\infty^n(K)$ denote the Banach space of finite sequences $\{y_1, ..., y_K\} \subset \mathbb{C}^n$ with norm $\|\cdot\|_\infty = \max_i\{\|y_i\|\}$.

We denote a diagonal or block diagonal matrix as

$$\text{diag}\{A_i\}_{i=1}^m = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_m \end{bmatrix},$$

where A_i can be scalar or matrix.

A standard saturation function $\sigma(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined as

$$\sigma(s) = \begin{cases} 1, & s \geq 1; \\ s, & -1 < s < 1; \\ -1, & s \leq -1. \end{cases}$$

8.2. Problem formulation

Consider a discrete-time linear system subject to input saturation and delay

$$\begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{m} B_i \sigma\left(u_i(k-\kappa_i)\right), \\ \quad y(k) \quad = Cx(k), \\ \quad x(\theta) \quad = \phi_{\theta+K}, \theta \in \overline{[-K, 0]} \end{cases} \tag{8.1}$$

where $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}$, $\kappa_i \in \overline{[0, K_i]}$, $K_i \in \mathbb{N}$ and $K = \max\{K_i\}$. The initial condition $\phi \in \ell_\infty^n(K)$. Let

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix}.$$

We can formulate two semi-global stabilization problems as follows:

Problem 8.1 The semi-global asymptotic stabilization via state feedback problem for system (8.1) is to find, for any set of positive integers $K_i > 0$ and a priori given bounded set of initial conditions $\mathcal{W} \subset \ell_\infty^n(K)$ with $K = \max\{K_i\}$, a delay-independent linear state feedback controller $u = Fx$ such that the zero solution of the closed-loop system is locally asymptotically stable for any $\kappa_i \in \overline{[0, K_i]}$ with $\mathcal{W}$ contained in its domain of attraction, i.e. the following properties hold for all $\kappa_i \in \overline{[0, K_i]}$, $i = 1, ..., m$:

1. $\forall \varepsilon > 0, \exists \delta$ such that if $\|\phi\|_\infty \leq \delta$, $\|x(k)\| \leq \varepsilon$ for all $k \geq 0$;

2. $\forall \phi \in \mathcal{W}$, $x(k) \to 0$ as $k \to \infty$.

Problem 8.2 The semi-global asymptotic stabilization via measurement feedback problem for system (8.1) is to find an integer q, and find, for any set of positive integers $K_i > 0$ and any a priori given bounded set $\mathcal{W} \subset \ell_\infty^{n+q}(K)$ with $K = \max\{K_i\}$, a delay-independent linear finite dimensional measurement feedback controller

$$\begin{cases} \chi(k+1) = A_k \chi(k) + B_k y(k), \quad \chi(k) \in \mathbb{R}^q, \\ \quad u(k) \quad = C_k \chi(k) + D_k y(k), \end{cases} \tag{8.2}$$

such that the zero solution of the closed-loop system is locally asymptotically stable for all $\kappa_i \in \overline{[0, K_i]}$ with $\mathcal{W}$ contained in its domain of attraction, i.e. the following properties hold for all $\tau_i \in \overline{[0, K_i]}$:

1. $\forall \varepsilon > 0, \exists \delta$ such that if $\|(\phi; \psi)\|_\infty \leq \delta$, $\|x(k)\| \leq \varepsilon$ for all $k \geq 0$;
2. $\forall (\phi; \psi) \in \mathcal{W}$, $(x(k), \chi(k)) \to 0$ as $k \to \infty$.

Since the input of (8.1) is bounded, it is well known that the following assumption is necessary for semi-global stabilization.

Assumption 8.1 (A, B) is stabilizable, (A, C) is detectable and A has all its eigenvalues in the closed unit disc $\mathbb{C}^{\odot}$.

8.3. Preliminaries

In this section, we shall present stability criteria for discrete time-delay system which are the basic of this paper and recall the standard low-gain feedback design and some of its properties.

8.3.1. Stability of discrete linear time-delay systems

Consider system

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} A_i x(k - \kappa_i), \tag{8.3}$$

where $x(k) \in \mathbb{R}^n$ and $\kappa_i \in \mathbb{N}$. Suppose $A + \sum_{i=1}^{m} A_i$ is Schur stable. The next lemma is a standard result.

Lemma 8.1 System (8.3) is asymptotically stable if and only if

$$\det\left[zI - A - \sum_{i=1}^{m} z^{-\kappa_i} A_i\right] \neq 0, \quad \forall z \notin \mathcal{D}_0, \quad \forall \kappa_i \in \overline{[0, K_i]}. \tag{8.4}$$

Define for $\alpha \in [0, 1]$

$$F_\alpha(z) = \det\left[zI - A - (1-\alpha)\sum_{i=1}^{m} A_i - \alpha \sum_{i=1}^{m} z^{-\kappa_i} A_i\right]. \tag{8.5}$$

The results of this paper are all based on the following lemma.

Lemma 8.2 The system (8.3) is asymptotically stable if

$$\det\left(F_\alpha(z)\right) \neq 0,\ \forall z \in \mathbb{C}^{\bigcirc},\ \forall \alpha \in [0, 1]. \tag{8.6}$$

Proof : Suppose (8.6) holds but (8.4) does not hold, that is, $F_1(z)$ has zeros in $\mathbb{C}/\mathcal{D}_0$. However, since $A + \sum_{i=1}^{m} B_i F_i$ is Schur stable, all the zeros of $F_0(z)$ must be in the open unit disc $\mathcal{D}_0$. Note that the zeros of $F_\alpha(z)$ move continuously as α varies. Hence, there exists $\alpha_0 \in (0, 1]$ such that $\det(F_{\alpha_0}(z_0)) = 0$ for some $z_0 \in \mathbb{C}^{\bigcirc}$, which contradicts (8.6). ∎

Consider a special case of (8.3) where $A_i = B_i F_i$, that is,

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} B_i F_i x(k - \kappa_i). \tag{8.7}$$

Assume that $\bar{A} = A + \sum_{i=1}^{m} B_i F_i$ is Schur stable. The following variation of Lemma 8.2 is more convenient to use.

Lemma 8.3 The system (8.7) is asymptotically stable if

$$\det\left[I + \alpha G(z)(I - D(z))\right] \neq 0, \quad \forall z \in \mathbb{C}^{\bigcirc},\ \forall \alpha \in [0, 1], \tag{8.8}$$

where $G(z) = F(zI - A - BF)^{-1}B$, $D(z) = \text{diag}\{z^{-\kappa_i}\}_{i=1}^{m}$ and

$$B = \begin{bmatrix} B_1 & \cdots & B_m \end{bmatrix}, \quad F = \begin{bmatrix} F_1 \\ \vdots \\ F_m \end{bmatrix}.$$

Proof : The proof is straightforward. Note that

$$\begin{aligned}
&\det\left[zI - A - (1-\alpha)\sum_{i=1}^{m} B_i F_i - \alpha \sum_{i=1}^{m} z^{-\kappa_i} B_i F_i\right] \\
&= \det\left[zI - A - BF + \alpha B(I - D(z))F\right] \\
&= \det\left[zI - A - BF\right] \det\left[I + \alpha (zI - A - BF)^{-1} B(I - D(z))F\right] \\
&= \det\left[zI - A - BF\right] \det\left[I + \alpha F(zI - A - BF)^{-1} B(I - D(z))\right].
\end{aligned}$$

Since $A + BF$ is Schur stable, (8.6) holds if and only if (8.8) holds. ∎

Remark 8.1 *Lemma 8.2 and 8.3 are discrete-time counterparts of Lemma 2 and 3 in [158] (see also the work in [21, 41]). However, the conditions* (8.6) *and* (8.8) *are only sufficient for discrete-time system.*

8.3.2. H_2 low-gain state feedback and compensator

Consider a discrete-time linear system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \\ y(k) \quad = Cx(k), \\ z(k) \quad = u(k). \end{cases} \tag{8.9}$$

Let Assumption 8.1 hold. Recall the following definition from [147]. An H_2 low-gain sequence is a family of parameterized matrices F_ε with $\varepsilon \in (0, 1]$ such that the following properties hold

1. $A + BF_\varepsilon$ is Schur stable for any $\varepsilon \in (0, 1]$;

2. the closed-loop system of (8.9) and $u = F_\varepsilon x$ satisfies

$$\lim_{\varepsilon \downarrow 0} \|z\|_2 = 0, \quad \forall x_0 \in \mathbb{R}^n. \tag{8.10}$$

The H_2 low-gain sequence can be constructed as

$$F_\varepsilon = -(B' P_\varepsilon B + I)^{-1} B' P_\varepsilon A \tag{8.11}$$

where for $\varepsilon \in (0, 1]$, P_ε is the positive definite solution of H_2 Algebraic Riccati Equation

$$P_\varepsilon = A' P_\varepsilon A + \varepsilon I - A' P_\varepsilon B(B' P_\varepsilon B + I)^{-1} B' P_\varepsilon A. \tag{8.12}$$

It is known that under Assumption 8.1, $P_\varepsilon \to 0$, and thus $F_\varepsilon \to 0$, as $\varepsilon \to 0$. Moreover, we also have the following lemma

Lemma 8.4 Define transfer function $G_\varepsilon(z) = F_\varepsilon(zI - A - BF_\varepsilon)^{-1} B$. We have

$$\|I + G_\varepsilon\|_\infty \le \sqrt{1 + \lambda_{\max}(B' P_\varepsilon B)}, \tag{8.13}$$

Proof : Define $R(z) = I - F_\varepsilon(zI - A)^{-1} B$. $R(z)$ satisfies the following return difference equality ([105]):

$$R(z^{-1})'(I + B' P_\varepsilon B) R(z) = I + \varepsilon B'(z^{-1} I - A')^{-1}(zI - A)^{-1} B.$$

This implies for $z \in \mathbb{C}^{\bigcirc}$,

$$\left(1 + \lambda_{\max}(B' P_\varepsilon B)\right) R(z)^* R(z) \ge I,$$

and hence

$$\underline{\sigma}\left(I - F_\varepsilon(zI - A)^{-1}B)\right) \geq \frac{1}{\sqrt{1+\lambda_{\max}(B'P_\varepsilon B)}}, \quad z \in \mathbb{C}^{\bigcirc}. \tag{8.14}$$

By matrix inversion lemma,

$$\bar{\sigma}\left(I + F_\varepsilon(zI - A - BF_\varepsilon)^{-1}B\right) \leq \sqrt{1+\lambda_{\max}(B'P_\varepsilon B)}, \quad z \in \mathbb{C}^{\bigcirc},$$

which yields (8.13). ■

Remark 8.2 *An immediate consequence of Lemma 8.4 is the following relations which will be useful in our analysis.*

$$\|I + G_\varepsilon\|_\infty \leq \eta := \sqrt{1+\lambda_{\max}(BP_1B)}, \quad \|G_\varepsilon\|_\infty \leq 1+\eta, \quad \forall \varepsilon \in (0,1] \tag{8.15}$$

The low-gain state feedback $u = F_\varepsilon x$ can be realized with an observer, which we refer to as low-gain compensator

$$\begin{cases} \chi(k+1) = A\chi(k) + BF_\varepsilon\chi(k) - K\left(y(k) - C\chi(k)\right), & \chi(0) = \chi_0, \\ \quad u(k) \quad = F_\varepsilon\chi(k). \end{cases} \tag{8.16}$$

where K is such that $A + KC$ is Schur stable. It can be shown that (8.16) is a generalized H_2 low-gain "sequence" as it satisfies the aforementioned two properties of an H_2 low-gain. First, it is easy to see $\mathcal{A} + \mathcal{B}\mathcal{F}_\varepsilon$ is Schur stable, where

$$\mathcal{A} = \begin{bmatrix} A & 0 \\ -KC & A+KC+BF_\varepsilon \end{bmatrix}, \ \mathcal{B} = \begin{bmatrix} 0 \\ B \end{bmatrix} \text{ and } \mathcal{F}_\varepsilon = \begin{bmatrix} 0 & F_\varepsilon \end{bmatrix}.$$

The next lemma proves property (8.10) for the closed-loop of (8.9) and (8.16).

Lemma 8.5 The closed-loop of (8.9) and (8.16) satisfies

$$\lim_{\varepsilon \downarrow 0} \|z\|_2 = 0, \quad \forall x_0, \chi_0 \in \mathbb{R}^n.$$

Proof : See Appendix. ■

8.4. Main result

Now we are in good position to solve the two stabilization problems formulated in Section 8.2. We will develop the results for a linearized system ignoring saturation utilizing the low-gain state feedback and compensator. Then it can be shown that the input of the resulting closed-loop systems can be made sufficient small to avoid saturation for a compact set of initial conditions, which will lead to the solution of Problem 8.1 and 8.2.

8.4.1. Global stabilization of linear discrete-time system with input delay

We first consider the stabilization problem for system (8.1) in the absence of saturation

$$\begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{m} B_i u(k-\kappa_i), \\ \quad y(k) \quad = Cx(k) \end{cases} \tag{8.17}$$

Since the system (8.17) is linear, it is possible to achieve the global stabilization via linear feedback. We shall show that this in fact can be achieved by a low-gain feedback $u = F_\varepsilon x$ with F_ε given by (8.11).

Define

$$\omega_{\max}^{i} = \max\{\omega \in [0,\pi] \mid \exists v \in \mathbb{C}^n,\ A'v = e^{j\omega}v,\ v'B_i \neq 0\}.$$

Clearly, $\omega_{\max}^{i}$ is the largest argument of eigenvalues that are, at least partially, controllable via input u_i. It will be made clear in the following theorem and its proof that this $\omega_{\max}^{i}$ dictates the delay tolerance in the channel u_i.

Theorem 8.1 Consider system (8.17). Let Assumption 8.1 hold and $F = F_\varepsilon$ be given by (8.11) and (8.12) with $\varepsilon \in (0,1]$. For any $K_i < \frac{\pi}{3\omega_{\max}^{i}}$, there exists $\varepsilon^* \in (0,1]$ such that the system(8.17) where $u = F_\varepsilon x$ with F_ε given by (8.11) is asymptotically stable for $\varepsilon \in (0,\varepsilon^*]$ and $\kappa_i \in \overline{[0, K_i]}$.

In a special case where A has all the eigenvalues equal to 1, Theorem 8.1 immediately implies that any bounded delay can be tolerated with using low-gain feedback $u = F_\varepsilon x$. This is stated in the following corollary.

Corollary 8.1 Consider system (8.17). Let Assumption 8.1 hold and $F = F_\varepsilon$ be given by (8.11) and (8.12) with $\varepsilon \in (0,1]$. Suppose A has all the eigenvalues equal to 1. For any given positive integers

K_i, there exists $\varepsilon^* \in (0, 1]$ such that the system(8.17) where $u = F_\varepsilon x$ with F_ε given by (8.11) is asymptotically stable for $\varepsilon \in (0, \varepsilon^*]$ and $\kappa_i \in \overline{[0, K_i]}$.

Proof of Theorem 8.1 : Consider the closed-loop system

$$x(k+1) = Ax(k) + \sum_{i=1}^{m} B_i F_{\varepsilon,i} x(k - \kappa_i), \tag{8.18}$$

where $F_{\varepsilon,i}$ is the ith row of F_ε. Let $G_\varepsilon(z) = F_\varepsilon(zI - A - BF_\varepsilon)^{-1}B$. It follows from Lemma 8.3 that the system (8.18) is asymptotically stable if

$$\det\Big[I + \alpha G_\varepsilon(e^{j\omega})(I - D(e^{j\omega})\Big] \neq 0, \quad \forall \omega \in [-\pi, \pi],\ \forall \alpha \in [0, 1], \tag{8.19}$$

where $D(z) = \operatorname{diag}\{z^{-\kappa_i}\}$. Due to symmetry, we only need to consider the $\omega \in [0, \pi]$. Assume A has r eigenvalues on the unit circle which are denoted by $e^{j\omega_q}$, $q = 1, ..., r$ with $\omega_q \in [0, \pi]$. Given $K_i < \frac{\pi}{3\omega^i_{\max}}$ for $i = 1, ..., m$, there exists a $\delta > 0$ such that

1. The neighborhoods $\mathcal{E}_q := [\omega_q - \delta, \omega_q + \delta] \cap [0, \pi]$, $q = 1, ..., r$, around these eigenfrequencies are mutually disjoint;

2. If $e^{j\omega_q}$ is at least partially controllable through input i,

$$\omega K_i < \frac{\pi}{3} - \frac{1}{2}\left(\frac{\pi}{3} - K_i \omega^i_{\max}\right), \quad \forall \omega \in \mathcal{E}_q. \tag{8.20}$$

Lemma 8.6 The following properties hold:

1. If $e^{j\omega_q}$ is not controllable via input u_i for some i, then

$$\lim_{\varepsilon \downarrow 0} F_\varepsilon(e^{j\omega}I - A - BF_\varepsilon)^{-1}B_i = 0,$$

   uniformly in ω for $\omega \in \mathcal{E}_q$.

2. There exists ε^* such that for $\varepsilon \in (0, \varepsilon^*]$,

$$\|F_\varepsilon(e^{j\omega}I - A - BF_\varepsilon)^{-1}B\| \leq \tfrac{1}{3},\ \forall \omega \in \overline{\Omega},$$

   where $\Omega := [0, \pi] \backslash \cup_{k=1}^{r} \mathcal{E}_q$

Proof : See Appendix. ■

Owing to Lemma 8.6, we find that there exists an ε_1 such that (8.19) is satisfied if for all $q = 1, ..., r$

$$\det[I + \alpha G_\varepsilon(e^{j\omega})\left(I - \tilde{D}_q(e^{j\omega})\right)] \neq 0, \ \forall \omega \in \mathcal{E}_q, \forall \kappa_i \in \overline{[0, K_i]}, \ \alpha \in [0, 1] \tag{8.21}$$

provided $\varepsilon \leq \varepsilon_1$ where $\tilde{D}_q(e^{j\omega})$ equals $D(e^{j\omega})$ with $\kappa_i = 0$ for all i's such that the eigenvalue $e^{j\omega_q}$ is not controllable via input channel i. Clearly, $\tilde{D}_q(e^{j\omega})$ is still unitary. Moreover, by (8.20), we find that

$$\mathrm{Re}(\tilde{D}_q(e^{j\omega})) > \tfrac{1}{2}I, \quad \forall \omega \in \mathcal{E}_q.$$

Let's consider (8.21). We can write

$$I + \alpha G_\varepsilon(e^{j\omega})\left(I - \tilde{D}_q(e^{j\omega})\right) = (1-\alpha)I + \alpha\tilde{D}_q(e^{j\omega}) + \alpha(I + G_\varepsilon(e^{j\omega}))\left(I - \tilde{D}_q(e^{j\omega})\right). \tag{8.22}$$

Note that,

$$\left[(1-\alpha)I + \alpha\tilde{D}_q(e^{j\omega})^*\right]\left[(1-\alpha)I + \alpha\tilde{D}_q(e^{j\omega})\right] = \left((1-\alpha)^2 + \alpha^2\right)I + 2\alpha(1-\alpha)\,\mathrm{Re}(\tilde{D}_q(e^{j\omega})) \geq \alpha I.$$

Therefore

$$\underline{\sigma}\left((1-\alpha)I + \alpha\tilde{D}_q(e^{j\omega})\right) \geq \sqrt{\alpha} \geq \alpha.$$

This together with (8.22) imply that (8.21) holds if

$$\bar{\sigma}\left((I + G_\varepsilon(e^{j\omega}))\left(I - \tilde{D}_q(e^{j\omega})\right)\right) < 1 \tag{8.23}$$

By (8.20), for any $q = 1, ..., r$, there exists $\xi_q \in (0, 1)$ solely depending on K_i such that we get $\bar{\sigma}(I - \tilde{D}_q(e^{j\omega})) < 1 - \xi_q$ for $\omega \in \mathcal{E}_q$. According to (8.13), there exists a $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, $\bar{\sigma}(I + G_\varepsilon(e^{j\omega})) < 1/(1 - \xi_q)$ for any $q = 1, ..., r$. Therefore, condition (8.19) is satisfied. ■

The next theorem is concerned with measurement feedback.

Theorem 8.2 Consider system (8.17). Let Assumption 8.1 hold. For any positive integers $K_i < \frac{\pi}{3\omega^i_{max}}$, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (8.17) and low-gain compensator (8.16) is asymptotically stable for $\kappa_i \in \overline{[0, K_i]}$.

Corollary 8.2 Consider system (8.17). Let Assumption 8.1 hold and A has all its eigenvalues equal to 1. For any positive integers K_i, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (8.17) and low-gain compensator (8.16) is asymptotically stable for $\kappa_i \in \overline{[0, K_i]}$.

Proof of Theorem 8.2 : The closed-loop system is given by

$$\begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^{m} B_i F_i \chi(k-\kappa_i) \\ \chi(k+1) = (A + BF_\varepsilon + KC)\chi(k) - KCx(k). \end{cases} \tag{8.24}$$

It is well known that system (8.24) without delay is asymptotically stable. Define

$$G_\varepsilon^m(z) = -F_\varepsilon(zI - A - BF_\varepsilon)^{-1} KC(zI - A - KC)^{-1} B.$$

Obviously, $G_\varepsilon^m(z)$ is stable. It follows from Lemma 8.3 that (8.24) is global asymptotically stable if

$$\det[I + \alpha G_\varepsilon^m(e^{j\omega})\left(I - D(e^{j\omega})\right)] \neq 0,\ \forall\omega \in [-\pi, \pi], \forall\kappa_i \in \overline{[0, K_i]}\ \forall\alpha \in [0, 1], \tag{8.25}$$

where $D(z) = \operatorname{diag}\{z^{-\kappa_i}\}_{i=1}^{m}$. We have the following lemma

Lemma 8.7 Let $G_\varepsilon(z) = F_\varepsilon(zI - A - BF_\varepsilon)^{-1} B$. Then

$$\lim_{\varepsilon \downarrow 0} \left(G_\varepsilon^m(e^{j\omega}) - G_\varepsilon(e^{j\omega}) \right) = 0$$

uniformly ω.

If, by Theorem 1, there exists an ε_1 such that for all $\varepsilon \in (0, \varepsilon_1]$ we have (8.19) satisfied with $G_\varepsilon(j\omega)$, then we can find an $\varepsilon_2 \leq \varepsilon_1$ such that (8.25) holds for all $\varepsilon \in (0, \varepsilon_2]$. ■

8.4.2. Semi-global stabilization subject to input saturation

In this subsection, we shall show that the low-gain state feedback and compensator which stabilize the linearized systems (8.17) also solve the semi-global stabilization problem for the same linear system with input saturation (8.1) by a proper selection of the tuning parameter ε with respect to a set of initial conditions.

Theorem 8.3 Consider the system (8.1). Let Assumption 8.1 hold. The semi-global asymptotic stabilization via state feedback problem can be solved by the low-gain feedback (8.11). Specifically, for a set of non-negative integers $K_i < \frac{\pi}{3\omega_{max}^i}$, $i = 1, ..., m$ and any *a priori* given compact set of initial conditions $\mathcal{W} \subset \ell_\infty^n(K)$ where $K = \max\{K_i\}$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the low-gain feedback (8.11) achieves local asymptotic stability of the closed-loop system with the domain of attraction containing $\mathcal{W}$ for any $\kappa_i \in \overline{[0, K_i]}$, $i = 1, ..., m$.

In the special case where all the eigenvalues of A are 1, the low-gain feedback allows any bounded but arbitrarily large input delays. This recovers the partial results in [160].

Corollary 8.3 Consider the system (8.1). Let Assumption 8.1 hold and A has all its eigenvalues equal to 1. For any given set of non-negative integers K_i, $i = 1, ..., m$ and any *a priori* given compact set of initial conditions $\mathcal{W} \subset \ell_\infty^n(K)$ where $K = \max\{K_i\}$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the low-gain feedback (8.11) achieves local asymptotic stability of the closed-loop system with the domain of attraction containing $\mathcal{W}$ for any $\kappa_i \in \overline{[0, K_i]}$, $i = 1, ..., m$.

Proof of Theorem 8.3 : The closed-loop system can be written as

$$x(k) = Ax(k) + \sum_{i=1}^{m} B_i \sigma(F_i x(k - \kappa_i)) \tag{8.26}$$

Since $K_i < \frac{\pi}{3}\omega_{\max}^i$, the local Lyapunov stability of the origin for sufficiently small ε follows from Theorem 8.1, that is, there exists $\varepsilon_1 \in (0, 1]$ such that for $\varepsilon \in (0, \varepsilon_1]$, the origin of (8.26) is locally stable.

It remains to show the attractivity. It suffices to prove that for system (8.18) with initial condition in $\mathcal{W}$, there exists $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, we shall have that

$$\|F_\varepsilon x(k - K)\| \leq 1, \ \forall k \geq 0.$$

This will imply that for system (8.26) no saturation will be active for all $k \geq 0$, and hence, the system is linear and stable for $\varepsilon \leq \varepsilon_1$. This will complete the proof.

Define two linear time invariant operators g_ε and δ with the following transfer matrices:

$$G_\varepsilon(z) = F_\varepsilon(zI - A - BF_\varepsilon)^{-1} B$$
$$\Delta(z) = I - D(z) = \text{diag}\{1 - z^{-\kappa_i}\}_{1=1}^m.$$

Note that the operators g_ε and δ have zero initial conditions. From the proof of Theorem 8.1, we know that (8.19) is satisfied which guarantees that there exists a $\mu > 0$ such that

$$\underline{\sigma}(I + G_\varepsilon(z)\Delta(z)) > \mu, \quad \forall z \in \mathbb{C}^{\bigcirc},\ \forall \kappa_i \in \overline{[0, K_i]},$$

for all $\varepsilon \le \varepsilon_1$ and this μ only depends on K_i provided that $\varepsilon \le \varepsilon_1$. This implies that

$$\|(I + G_\varepsilon(z)\Delta(z))^{-1}\|_\infty \le \tfrac{1}{\mu}.$$

Note that for $k \ge 0$

$$x(k+1) = (A + BF_\varepsilon)x(k) - B\delta(F_\varepsilon x)(k) + Bv_\varepsilon(k),$$

where

$$v_\varepsilon(k) = \begin{bmatrix} v_1(k) \\ \vdots \\ v_m(k) \end{bmatrix}, \quad v_i(k) = \begin{cases} F_i\phi(k-\tau_i), & k < \kappa_i, \\ 0, & k \ge \kappa_i. \end{cases}$$

Since $v_\varepsilon(k)$ vanishes for $k \ge K$, $\phi \in \mathcal{W}$ is bounded and $F_\varepsilon \to 0$, we have for any $\phi \in \mathcal{W}$, $\|v_\varepsilon\|_\infty \to 0$ and $\|v_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$.

We have

$$F_\varepsilon x(k) = F_\varepsilon(A + BF_\varepsilon)^k x(0) - (g_\varepsilon \circ \delta)(F_\varepsilon x)(k) + g_\varepsilon(v_\varepsilon)(k)$$

and hence

$$F_\varepsilon x(k) = (1 + g_\varepsilon \circ \delta)^{-1}\Big[F_\varepsilon(A + BF_\varepsilon)^k x(0) + g_\varepsilon(v_\varepsilon)(k)\Big]. \tag{8.27}$$

Let $w_\varepsilon(k) = g_\varepsilon(v_\varepsilon)(k)$. By the definition of g_ε, we have $\|w_\varepsilon\|_2 \le \|G_\varepsilon(z)\|_\infty\|v_\varepsilon\|_2 \le (1+\eta)\|v_\varepsilon\|_2$ where η is given by (8.15). Hence for any given initial condition ϕ, $\|w_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$. Then from (8.27), we get

$$\begin{aligned}\|F_\varepsilon x\|_2 \le& \|(1 + G_\varepsilon(z)\Delta(z))^{-1}\|_\infty \|F_\varepsilon(A + BF_\varepsilon)^k x(0)\|_2 + \|(1 + G_\varepsilon(z)\Delta(z))^{-1}\|_\infty\|w_\varepsilon\|_2 \\ \le& \tfrac{1}{\mu}\|F_\varepsilon(A + BF_\varepsilon)^k x(0)\|_2 + \tfrac{1}{\mu}\|w_\varepsilon\|_2.\end{aligned}$$

Since, by (8.10), $\|F_\varepsilon(A+BF_\varepsilon)^k x(0)\|_2 \to 0$ and $v_\varepsilon \to 0$ as $\varepsilon \to 0$ and μ is independent of ε (provided ε is smaller than ε_1), there exists an ε_2 such that $\|F_\varepsilon x\|_2 \le 1$ for $\varepsilon \in (0, \varepsilon_1]$ and $\phi \in \mathcal{W}$. This implies that

$\|F_\varepsilon x(k)\| \leq \|F_\varepsilon x\|_2 \leq 1$ for $k \geq 0$. At last, since $\phi \in \mathcal{W}$, there exists $\varepsilon^* \leq \varepsilon_2$ such that $\|F_\varepsilon x(k)\| \leq 1$ for $k \geq -K$. ■

The next theorem solves Problem 2.

Theorem 8.4 Consider the system (8.1). Let Assumption 8.1 hold. The semi-global asymptotic stabilization via measurement feedback problem can be solved by the low-gain compensator (8.16). Specifically, for any *a priori* given compact set of initial conditions $\mathcal{W} \subset \ell_\infty^{2n}(K)$ and a set of positive integers $K_i < \frac{\pi}{3\omega_{\max}^i}$, $i = 1, ..., m$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the origin of the closed-loop system of (8.1) and (8.16) is local asymptotic stable for any $\kappa_i \in \overline{[0, K_i]}$, $i = 1, ..., m$ with the domain of attraction containing $\mathcal{W}$.

Corollary 8.4 Consider the system (8.1). Let Assumption 8.1 hold and A has all its eigenvalues equal to 1. For any *a priori* given compact set of initial conditions $\mathcal{W} \subset \ell_\infty^{2n}(K)$ and any given set of positive integers K_i, $i = 1, ..., m$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the origin of the closed-loop system of (8.1) and (8.16) is local asymptotic stable for any $\kappa_i \in \overline{[0, K_i]}$, $i = 1, ..., m$ with the domain of attraction containing $\mathcal{W}$.

Proof of Theorem 8.4 : The closed-loop system can be written as

$$\begin{cases} x(k+1) = Ax(k) + \sum_{i=1}^m B_i \sigma(F_i \chi(k-\kappa_i)) \\ \chi(k+1) = (A + BF_\varepsilon + KC)\chi(k) - KCx(k) \\ \quad x(\theta) \quad = \phi(\theta),\ \forall \theta \in \overline{[-K_i, 0]} \\ \quad \chi(\theta) \quad = \psi(\theta),\ \forall \theta \in \overline{[-K_i, 0]}. \end{cases} \tag{8.28}$$

Suppose K_i's satisfy the bound $K_i < \frac{\pi}{3\omega_{\max}^i}$. By Theorem 8.2, there exists an ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, the closed-loop system without saturation is asymptotically stable. Then the local stability of (8.28) for $\varepsilon \leq \varepsilon_1$ follows.

Define two linear time invariant operators g_ε^m and δ with z transform

$$G_\varepsilon^m(z) = -F_\varepsilon(zI - A - BF_\varepsilon)^{-1} KC(zI - A - KC)^{-1} B$$

$$\Delta(z) = I - D(z) = \mathrm{diag}\{1 - z^{-\kappa_i}\}_{i=1}^m.$$

From the proof of Theorem 2, we know that (8.25) holds for $\varepsilon \le \varepsilon_1$. There exists a $\mu > 0$ such that

$$\underline{\sigma}(I + G_\varepsilon^m(z)\Delta(z)) > \mu, \forall z \in \mathbb{C}^{\bigcirc}, \forall \kappa_i \in \overline{[0, K_i]}, \tag{8.29}$$

where μ is independent of ε provided that $\varepsilon \le \varepsilon_1$. It follows from Lemma 8.7 that $G_\varepsilon^m(z) \to G_\varepsilon(z)$ uniformly on $\mathbb{C}^{\bigcirc}$ where $G_\varepsilon(z) = F_\varepsilon(zI - A - BF_\varepsilon)^{-1}B$. Since $\|G_\varepsilon\|_\infty \le 1 + \eta$ for any $\varepsilon \in (0, 1]$ with η given by (8.15), there exists an ε_2 such that

$$\|G_\varepsilon^m\|_\infty \le 2(1 + \eta). \tag{8.30}$$

Given (8.29), (8.30) and Lemma 8.5 hold, we can use exactly the same argument as in the proof of Theorem 8.3 to prove that there exists an $\varepsilon^* \le \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$,

$$\|F_\varepsilon \chi(k - K)\| \le 1, \ \forall k \ge 0, \ (\phi, \psi) \in \mathcal{W}.$$

■

8.5. Conclusion

In this paper, the semi-global stabilization problems for general uncritically unstable systems subject to input saturation and multiple unknown delays are solved. We propose upper bounds on delays based on a frequency-domain stability criterion for linear discrete time-delay system and constructed a low-gain state feedback and compensator to achieve the semi-global stabilization with feasible delays.

8.6. Appendix

Proof of Lemma 8.5 : The closed-loop of (8.9) and (8.16) is given by

$$\begin{cases} x(k+1) = Ax(k) + BF_\varepsilon\chi(k) \\ \chi(k+1) = (A + BF_\varepsilon + KC)\chi(k) - KCx(k) \\ \quad z(k) \quad = F_\varepsilon\chi(k) \end{cases}, \quad \begin{bmatrix} x(0) \\ \chi(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ \chi_0 \end{bmatrix}.$$

Let $e = x - \chi$. In the new coordinates of (x, e), the above system can be written as

$$\begin{cases} x(k+1) = (A + BF_\varepsilon)x(k) - BF_\varepsilon e(k) \\ e(k+1) = (A + KC)e(k) \\ \quad z(k) \quad = F_\varepsilon\,(x(k) - e(k)) \end{cases},$$

with $e_1(0) = x_0 - \chi_0$. We get $\|z\|_2 \leq \|F_\varepsilon e\|_2 + \|F_\varepsilon x\|_2$.

Since $A + KC$ is Schur stable, there exists a γ such that $\|e\|_2 \leq \gamma \|e(0)\|$ for any $e(0) \in \mathbb{R}^n$. Then

$$\|F_\varepsilon e\|_2 \leq \gamma \|F_\varepsilon\| \|e(0)\| \to 0 \text{ as } \varepsilon \to 0.$$

But for x, we have

$$\begin{aligned}\|F_\varepsilon x\|_2 &\leq \|G_\varepsilon(z)\|_\infty \|F_\varepsilon e\|_2 + \sum_{k=0}^{\infty} \|F_\varepsilon (A + BF_\varepsilon)^k x_0\|^2 \\ &\leq 2(1+\eta)\|F_\varepsilon\| \|e(0)\| + \sum_{k=0}^{\infty} \|F_\varepsilon (A + BF_\varepsilon)^k x_0\|^2\end{aligned}$$

where $G_\varepsilon(z) = F_\varepsilon(zI - A - BF_\varepsilon)^{-1}B$ and we use (8.15). It was shown in (8.10) that

$$\lim_{\varepsilon \downarrow 0} \sum_{k=0}^{\infty} \|F_\varepsilon (A + BF_\varepsilon)^k x_0\|^2 = 0.$$

and thus $\lim_{\varepsilon \downarrow 0} \|F_\varepsilon x\|_2 = 0$. We conclude that $\lim_{\varepsilon \downarrow 0} \|z\|_2 = 0$. ∎

Proof of Lemma 8.6 : To prove item (1), we first note that

$$\begin{aligned}&F_\varepsilon(e^{j\omega} I - A - BF_\varepsilon)^{-1} B e_i \\ =&F_\varepsilon(I - (e^{j\omega} I - A)^{-1} BF_\varepsilon)^{-1} (e^{j\omega} I - A)^{-1} B e_i \\ =&(I - F_\varepsilon(e^{j\omega} I - A)^{-1} B)^{-1} F_\varepsilon (e^{j\omega} I - A)^{-1} B e_i.\end{aligned}$$

By (8.13),

$$\bar{\sigma}(I - F_\varepsilon(e^{j\omega} I - A)^{-1} B)^{-1} \leq \sqrt{1 + \lambda_{\max}(B' P_1 B)}, \ \forall \omega \in \mathbb{R}$$

Moreover, $\forall \omega \in \mathcal{E}_q$, $(j\omega I - A)^{-1} B e_i$ has no pole and therefore

$$\|(e^{j\omega} I - A)^{-1} B e_i\| \leq M, \forall \omega \in \mathcal{E}_q.$$

for $M > 0$ independent of ω. But then

$$\|F_\varepsilon(e^{j\omega} I - A - BF_\varepsilon)^{-1} B e_i\| \leq M \|F_\varepsilon\|, \forall \omega \in \mathcal{E}_q,$$

and since F_ε converges to zero we get

$$\lim_{\varepsilon \downarrow 0} \|F_\varepsilon(e^{j\omega} I - A - BF_\varepsilon)^{-1} B e_i\| = 0, \text{ uniformly in } \mathcal{E}_q.$$

It remain to show item (2). By definition, $\det(e^{j\omega}I - A) \neq 0$ for all $\omega \in \overline{\Omega}$, which implies $\underline{\sigma}(e^{j\omega I - A}) > 0$ for $\omega \in \overline{\Omega}$. Note that $\underline{\sigma}(e^{j\omega}I - A)$ depends continuously on $\overline{\Omega}$ and $\overline{\Omega}$ is closed and bounded. There exists a μ such that

$$\underline{\sigma}(e^{j\omega}I - A) > \mu, \; \forall \omega \in \overline{\Omega}.$$

Choose ε^* such that $\|F_\varepsilon\| \leq \frac{\mu}{4}\|B\|^{-1}$ for $\varepsilon \leq \varepsilon^*$. In that case:

$$\underline{\sigma}(e^{j\omega I} - A - BF) > \mu - \|B\|\|F_\varepsilon\| > \tfrac{3\mu}{4}, \forall \omega \in \overline{\Omega},$$

and hence

$$\|(e^{j\omega}I - A - BF_\varepsilon)^{-1}\| < \tfrac{4}{3\mu}, \forall \omega \in \overline{\Omega},$$

but then

$$\|F_\varepsilon(e^{j\omega}I - A - BF_\varepsilon)^{-1}B\| \leq \|F_\varepsilon\|\|(e^{j\omega}I - A - BF_\varepsilon)^{-1}\|\|B\| \leq \tfrac{1}{3}$$

for all $\omega \in \overline{\Omega}$. ■

Proof of Lemma 8.7 : The error between $G_\varepsilon^m(z)$ and $G_\varepsilon(z)$ is

$$\begin{aligned} G_\varepsilon(z) - G_\varepsilon^m(z) &= \left[I + F_\varepsilon(zI - A - BF_\varepsilon)^{-1}B\right] F_\varepsilon(zI - A - KC)^{-1}B \\ &= \left[I + G_\varepsilon(z)\right] F_\varepsilon(zI - A - KC)^{-1}B \end{aligned}$$

From (8.13) we obtain

$$\bar{\sigma}(I + G_\varepsilon(e^{j\omega})) \leq \sqrt{1 + \lambda_{\max}(B'P_1B)}, \; \forall \varepsilon \in (0,1], \; \omega \in \mathbb{R},$$

where P_1 is the positive definite solution of (8.12) for $\varepsilon = 1$. Moreover,

$$\|F_\varepsilon(zI - A - KC)^{-1}B\|_\infty \leq \|F_\varepsilon\|\|(zI - A - KC)^{-1}B\|_\infty$$

Since $F_\varepsilon \to 0$ as $\varepsilon \to 0$ and $A + KC$ is Schur stable, we immediately have that

$$\lim_{\varepsilon \downarrow 0} G_\varepsilon^m(e^{j\omega}) - G_\varepsilon(e^{j\omega}) = 0,$$

uniformly in ω. ■

CHAPTER 9

A new low-and-high gain feedback design using MPC for global stabilization of linear systems subject to input saturation

9.1. Introduction

Stabilization of linear systems subject to actuator saturation has been extensively studied during the past two decades and is still drawing renewed attention, largely because saturation is widely recognized as ubiquitous in engineering applications and inherent constraints in control system designs. Many significant results have already been obtained in the literature. Some early works in this area are summarized in [5, 94, 123, 95, 32, 35] and references therein.

The low-gain method, proposed in [51, 53, 50], was originally developed as a linear feedback design methodology in the context of semi-global stabilization under actuator saturation and later on extended to the global framework with a gain scheduling [68, 31]. The low-gain feedback is parameterized by a so-called low-gain parameter, which is determined a priori in the semi-global setting according to a pre-selected compact set or adaptively with respect to states in the global setting. By properly selecting this parameter, we are able to limit the input magnitude to a sufficiently small level and avoid saturation for all time so as to stabilize the system.

On one hand the low-gain proves to be successful in solving stabilization problems, on the other hand it does not utilize the full actuation level and hence is conservative and less capable regarding performance. Low-and-high gain feedback designs are conceived to rectify the drawbacks of low-gain design methods, and can make better use of available control capacity. As such, they have been used for control problems beyond stabilization, to enhance transient performance and to achieve robust stability and disturbance rejection, see, for instance, [53, 54, 91, 31, 89] and also Chapter 2 and 3. However, as will be shown in this chapter, there is still a room for improvement.

Model predictive controllers (MPC) have a reputation of dealing with constraints and achieving good closed-loop performance. It numerically solves a finite-horizon constrained optimal control problem at each sample. Hence, a MPC may choose to operate exactly at the constraints, while a low-gain strategy would be to avoid the constraints. The more aggressive approach of the MPC complements the more conservative low-gain strategy, and is an interesting approach to include in a low-and-high gain feedback design in order to improve performance.

A drawback of MPC is the computational complexity of solving online numerically a constrained optimization problem (usually a quadratic program) at each sample. Guarantees of MPC stability requires particular formulations of the finite-horizon optimal control problem, such as sufficiently long prediction horizon and the use of a terminal cost, [121], or terminal constraints, [36]. Reduction of computational complexity typically requires that the prediction horizon is made shorter, which comes at the cost of more complex terminal costs and constraints, as well as sub-optimality compared to an infinite horizon constrained optimal control formulation, see e.g. [101, 122] for examples of such reformulations.

Explicit MPC of constrained linear systems admits a piecewise affine state feedback solution to be pre-computed using multi-parametric quadratic programming, [4]. Although online computational complexity can be reduced by orders of magnitude, the approach is still limited by available computer memory and the cost of off-line pre-computations, [156, 3]. Consequently, low-complexity sub-optimal strategies are also of interest in explicit MPC in order to manage complexity due to long prediction horizon, high system order, or many constraints, while preserving stability (see e.g. [34, 39, 27, 33]).

The key idea pursued in this chapter is to use an ultra-short-horizon MPC as the high gain strategy in a low-high-gain feedback design, where simple constraints resulting from the low-gain design are imposed on the MPC in order to guarantee stability.

9.2. Classical low-gain design and MPC

Consider a discrete-time system

$$x_{k+1} = Ax_k + B\sigma(u_k) \tag{9.1}$$

where $\sigma(\cdot)$ is standard saturation, i.e. for $u \in \mathbb{R}^m$, $\sigma(u) = [\sigma_0(u_1); \cdots ; \sigma_0(u_m)]$, $\sigma_0(u_i) = \text{sign}(u_i) \min\{1, |u_i|\}$ and $\text{sign}(s)$ is defined as

$$\text{sign}(s) = \begin{cases} 1, & s \geq 0; \\ -1, & s < 0. \end{cases} \tag{9.2}$$

It is well-known that the global stabilization problem is solvable if and only if the following assumption holds

Assumption 9.1 (A, B) is stabilizable and A has all its eigenvalues in the closed unit circle.

9.2.1. Classical ARE-based low-gain feedback design

The low-gain feedback is a sequence of parameterized feedback gains F_ε satisfying the following properties:

1. $A + BF_\varepsilon$ is Schur stable;
2. $\lim_{\varepsilon\to 0} F_\varepsilon = 0$;
3. $\lim_{\varepsilon\to 0} \|F_\varepsilon(A + BF_\varepsilon)^k x_0\|_{\ell_\infty} = 0$ for any x_0.

The parameter ε is called low-gain parameter. Low-gain feedback can be design using different methods (see Chapter 3 and references therein). One way of designing low-gain feedback based on the solution of an H_2 algebraic Riccati equation (ARE) is as follows [58]:

$$u_L = F_\varepsilon x = -(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax$$

where P_ε is the positive definite solution of ARE:

$$P_\varepsilon = A'P_\varepsilon A + \varepsilon I - A'P_\varepsilon B(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon A.$$

Following the argument in [58], it is straightforward to show that the control formulation can be generalized by selecting a matrix $R > 0$ and a parameterized matrix $Q_\varepsilon > 0$ such that $\lim_{\varepsilon\to 0} Q_\varepsilon = 0$ and $\frac{\mathrm{d}Q_\varepsilon}{\mathrm{d}\varepsilon} > 0$, and choosing

$$u_L = F_\varepsilon x = -(R + B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax \tag{9.3}$$

where P_ε is the solution of ARE:

$$P_\varepsilon = A'P_\varepsilon A + Q_\varepsilon - A'P_\varepsilon B(R + B'P_\varepsilon B)^{-1}B'P_\varepsilon A, \tag{9.4}$$

which has the property that $P_\varepsilon \to 0$ as $\varepsilon \to 0$ provided that Assumption 1 holds. It is shown in [58] that (9.3) satisfies the three low-gain properties.

The low-gain feedback has been successfully employed to solve the semi-global stabilization of a linear system subject to input saturation given by (9.1). In this context, the low-gain parameter ε controls the domain of attraction of the closed-loop system. It is clear from the properties of low-gain feedback that with a smaller ε, we can shrink the control input to avoid saturation for a large set of initial conditions. Hence by tuning ε sufficiently small, the domain of attraction can be made arbitrarily large to contain any a priori given compact set. To be precise, for any a priori given compact set, say $\mathcal{W}$, there exists ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the origin of closed-loop system of (9.1) and (9.3) is locally asymptotically stable with $\mathcal{W}$ contained in the domain of attraction. In this case, the resulting low-gain feedback is linear.

In order to solve the global stabilization problem, the low-gain parameter ε can be scheduled adaptively with respect to the states. This has been done in the literature, see for instance [31]. In general, the scheduled parameter $\varepsilon(x)$ should satisfy the following properties:

1. $\varepsilon(x) : \mathbb{R}^n \to (0, 1]$ is continuous and piecewise continuously differentiable.
2. There exists an open neighborhood $\mathcal{O}_s$ of the origin such that $\varepsilon(x) = 1$ for all $x \in \mathcal{O}_s$.
3. For any $x \in \mathbb{R}^n$, we have $\|F_{\varepsilon(x)}x\| \leq 1$.
4. $\varepsilon(x) \to 0$ as $\|x\| \to \infty$.
5. $\{ x \in \mathbb{R}^n \mid x' P_{\varepsilon(x)} x \leq c \}$ is a bounded set for all $c > 0$.

One particular choice of scheduling ε, which satisfies the above conditions, is given in [31] as follows

$$\varepsilon(x) = \max\{r \in (0, 1] \mid (x' P_r x) \operatorname{trace}(P_r) \leq \tfrac{1}{b}\} \tag{9.5}$$

where $b = 2 \operatorname{trace}(BB')$ while P_r is the unique positive definite solution of ARE (9.4) with $\varepsilon = r$.

The scheduled version of low-gain feedback controllers for global stabilization is given by

$$u_L(x) = F_{\varepsilon(x)} x = -(B' P_{\varepsilon(x)} B + R)^{-1} B' P_{\varepsilon(x)} A x \tag{9.6}$$

where $P_{\varepsilon(x)}$ is the solution of (9.4) with ε replaced by $\varepsilon(x)$.

9.2.2. MPC

Let U denote the region $\{u \in \mathbb{R}^m \mid u_i \in [-1, 1],\ i = 1, ..., m\}$. An MPC problem with prediction horizon N can be formulated by solving the optimization problem

$$\begin{gathered}\min_{\{u_k\}_{k=0}^{N-1}} \sum_{k=0}^{N-1} x_k' Q x_k + u_k' R u_k + x_N' P x_N \\ \text{s.t.} \\ x_{k+1} = Ax_k + Bu_k,\ \forall k = 0, \dots, N-1 \\ u_k \in U,\ \forall k = 0, \dots, N-1, \\ R > 0, \quad Q > 0, \quad P \geq 0\end{gathered}$$

Under the Assumption 9.1, the optimization problem is feasible for every initial condition. By solving the above, we obtain an open-loop optimal control sequence $(u_0, \dots, u_{N-1})$. Only the first input is applied to the system. This process is repeated at the each sample time.

The optimization problem can be reformulated in the mp-QP

$$\begin{gathered}J(x_0) = x_0' Y x_0 + \min_{U_N} U_N' H U_N + x_0' F U_N \\ \text{s.t.} \\ GU_N \preceq W + Ex_0,\end{gathered}$$

where $H > 0$, $U_N = (u_0', \dots, u_{N-1}')$ is the optimal control sequence and Y, H, F, G, W, E can be obtained from system dynamics and Q, R and P (see [4]).

It is shown in [14] that, by selecting P as the unique positive definite solution of the ARE

$$P = (A - BL')P(A - BL) + L'RL + Q \tag{9.7}$$

where

$$L = (B'PB + R)^{-1} B'PA,$$

there is a positively invariant region $\mathcal{O}_\infty$ around the equilibrium for which the MPC controller corresponds to $u_k = -Lx_k$, and in which no constraints are activated (i.e., the controller becomes an unconstrained LQR controller).

Moreover, if N is chosen sufficiently large, then the MPC controller is sure to bring any initial condition within $\mathcal{W}$ to $\mathcal{O}_\infty$ within N steps (i.e., by the end of the prediction horizon). In this case, the MPC solution is equivalent to the solution of the infinite-horizon optimization problem

$$\begin{gathered}\min_{\{u_k\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} x_k' Q x_k + u_k' R u_k \\ \text{s.t.} \\ x_{k+1} = Ax_k + Bu_k,\ \forall k = 0, 1, \dots \\ u_k \in U,\ \forall k = 0, 1, \dots .\end{gathered}$$

and stability is therefore guaranteed.

9.2.3. Connection between scheduled low-gain and MPC

Suppose we choose in scheduled low-gain design that $Q_\varepsilon = \varepsilon Q$ and R to be the same as in MPC. Note that for $x_k \in \mathcal{O}_s$, we have $\varepsilon(x_k) = 1$ and $Q_\varepsilon = Q$. Hence for $x_k \in \mathcal{O}_s \cap \mathcal{O}_\infty$, the scheduled low-gain controller corresponds precisely to the MPC controller $u_k = -Lx_k$. It is also easily verified that the AREs (9.4) and (9.7) are the same, with $\varepsilon = 1$.

From the comparison above, we can conclude that the MPC and scheduled low-gain formulations produce an "inner region" $\mathcal{O}_s \cap \mathcal{O}_\infty$ around the equilibrium, where they share an unconstrained optimal linear controller. However, for x outside this region, they determine the control input differently.

In the semi-global case, if we formulate the MPC problem with a weighting matrix Q_ε such that $\lim_{\varepsilon \to 0} Q_\varepsilon = 0$, in the inner region $\mathcal{O}_\infty$, an unconstrained linear controller applies, which corresponds precisely to an ARE-based low-gain controller (9.3). As $\varepsilon \to 0$, $\mathcal{O}_\infty$ will expand to become arbitrarily large. Eventually, the MPC controller within $\mathcal{W}$ will simply be a linear controller corresponding to a stabilizing ARE-based low-gain controller.

9.3. Low-and-high gain design using MPC

9.3.1. Classical low-and-high-gain feedback design for discrete-time system

Although in the global framework, the low-gain parameter is adapted with respect to the states so that the control input gets as close to the admissible limits as possible while avoiding saturation, the low-gain feedback still does not fully utilize the actuation capability, especially in the MIMO case. To rectify this drawback, the so-called low-and-high gain feedback design method was developed in [31, 89] and also in Chapter 11.

The low-and-high-gain state feedback is composed of a low-gain state feedback and a high-gain state feedback as

$$u_k = u_L + u_H = F_{\varepsilon(x_k)} x_k + F_H x_k \tag{9.8}$$

where $F_{\varepsilon(x_k)} x_k$ is the scheduled low-gain feedback designed in previous section with $R = I$. The high-gain feedback is of the form,

$$F_H x_k = \rho F_{\varepsilon(x_k)} x_k$$

where $\rho > 0$ is called the high-gain parameter.

For continuous-time systems, the high gain parameter ρ does not affect the domain of attraction and can be any positive real number. It aims mainly at achieving control objectives beyond stability, such as robustness, disturbance rejection and performance. However, the high-gain parameter can not be arbitrarily large for discrete-time systems. In order to preserve local asymptotic stability, this high gain has to be bounded at least near the equilibrium. A suitable choice of such a high-gain parameter satisfies

$$\rho \in \left[0, \frac{2}{\|B' P_\varepsilon B\|}\right] \tag{9.9}$$

where P_ε is the solution of ARE (9.4) with $R = I$ (see Chapter 11). This high-gain can also be adapted respect to states and accompanied with the scheduled low-gain parameter to solve the global stabilization problem. This result is stated in the next lemma:

Lemma 9.1 Consider system (9.1). Suppose $R = I$ and $Q_\varepsilon > 0$ is such that

$$\lim_{\varepsilon \to 0} Q_\varepsilon = 0, \quad \frac{\mathrm{d}Q_\varepsilon}{\mathrm{d}\varepsilon} > 0 \text{ for } \varepsilon > 0.$$

Let P_ε be the solution of (9.4). The equilibrium of the interconnection of (9.1) with the low-and-high-gain feedback

$$u_k = -\left(1 + \frac{2}{\|B' P_{\varepsilon(x_k)} B\|}\right)(I + B' P_{\varepsilon(x_k)} B)^{-1} B' P_{\varepsilon(x_k)} A x_k \tag{9.10}$$

is globally asymptotically stable.

Proof : For simplicity, we denote $\varepsilon(x_k)$, $F_{\varepsilon(x_k)}$ and $P_{\varepsilon(x_k)}$ respectively by ε_k, F_k and P_k.

Define a Lyapunov function $V_k = x_k' P_k x_k$. The scheduling (9.5) guarantees that

$$\|(I + B' P_k B)^{-1} B' P_k A x_k\| \leq 1.$$

Define $\mu_k = \|B' P_k B\|$, $v_k = -(I + B' P_\varepsilon B)^{-1} B' P_\varepsilon A x_k$ and $\tilde{u}_k = \sigma(u_k)$. We evaluate $V_{k+1} - V_k$

along the trajectories as

$$\begin{aligned}
V_{k+1} - V_k &= -x_k' Q_k x_k - \tilde{u}_k' \tilde{u}_k + x_{k+1}' [P_{k+1} - P_k] x_{k+1} \\
&\qquad + [\tilde{u}_k - v_k]' (I + B' P_k B) [\tilde{u}_k - v_k] \\
&\le -x_k' Q_k x_k - \|\tilde{u}_k\|^2 + (1 + \mu_k) \|\tilde{u}_k - v_k\|^2 \\
&\qquad + x_{k+1}' [P_{k+1} - P_k] x_{k+1} \\
&= -x_k' Q_k x_k + \mu_k \|\tilde{u}_k - \tfrac{1+\mu_k}{\mu_k} v_k\|^2 \\
&\qquad - \tfrac{1+\mu_k}{\mu_k} \|v_k\|^2 + x_{k+1}' [P_{k+1} - P_k] x_{k+1}.
\end{aligned}$$

Since $\|v_k\| \le 1$, we have

$$\|v_k\| \le \|\tilde{u}_k\| \le (1 + \tfrac{2}{\mu_k}) \|v_k\|.$$

This implies that

$$\|\tilde{u}_k - \tfrac{1+\mu_k}{\mu_k} v_k\| \le \tfrac{1}{\mu_k} \|v_k\|,$$

and thus,

$$\mu_k \|\tilde{u}_k - \tfrac{1+\mu_k}{\mu_k} v_k\|^2 - \tfrac{1}{\mu_k} \|v_k\|^2 \le 0.$$

Combining the above, we get for any $x_k \neq 0$,

$$V_{k+1} - V_k \le -x_k' Q_k x_k - \|v_k\|^2 + x_{k+1}' [P_{k+1} - P_k] x_{k+1}.$$

Note that the scheduling (9.5) implies that $V_{k+1} - V_k$ and $x_{k+1}' [P_{k+1} - P_k] x_{k+1}$ can not have the same signs (see [31]). Consequently, we have for $x_k \neq 0$

$$V_{k+1} - V_k < 0$$

This proves global asymptotic stability of the origin. ■

9.3.2. A new low-and-high-gain feedback design using MPC

The underlying philosophy behind the above low-and-high gain design is, based on the low-gain design and its associated Lyapunov function V_k, to find a high gain part and form a composed controller which not only renders V_k to decay every step to ensure stability but also potentially accelerates the convergence.

With this in mind, we shall propose another low-and-high gain design methodology using MPC with a prediction horizon $N = 1$. For $Q > 0$ and $R > 0$, let $Q_\varepsilon = \varepsilon Q$, P_ε be the solution of (9.4) with Q_ε and R and $\varepsilon = \varepsilon(x_k)$ be determined by (9.5).

Consider the Lyapunov candidate $V(x_k) = x_k' P_{\varepsilon(x_k)} x_k$. We also take the same abbreviations as used in the proof of Lemma 1. Under the constraints

$$u_k \in U, \ \forall k, \tag{9.11}$$

we have that for arbitrary u_k along the trajectory of (9.1),

$$\begin{aligned} V_{k+1} - V_k &= -x_k' Q_k x_k - u_k' R u_k + x_{k+1}'[P_{k+1} - P_k]x_{k+1} \\ &\quad + [u_k - F_k x_k]'(R + B' P_k B)[u_k - F_k x_k] \\ &= -x_k' Q_k x_k - x_k' F_k' R F_k x_k \\ &\quad - 2x_k' F_k' R[u_k - F_k x_k] \\ &\quad + [u_k - F_k x_k]' B' P_k B[u_k - F_k x_k] \\ &\quad + x_{k+1}'[P_{k+1} - P_k]x_{k+1}. \end{aligned}$$

where $F_k = -(R + B' P_k B)^{-1} B' P_k A$. In view of the property that $V_{k+1} - V_k$ can not have the same sign with $x_{k+1}'[P_{k+1} - P_k]x_{k+1}$, to ensure $V_{k+1} - V_k < 0$ for $x_k \neq 0$, it is sufficient to restrict that

$$2x_k' F_k' R[u_k - F_k x_k] - [u_k - F_k x_k]' B' P_\varepsilon B[u_k - F_k x_k] \geq 0 \tag{9.12}$$

This can be satisfied by enforcing constraints for $i = 1, ...m$,

$$\text{sign}(D_{k,i} x_k)(u_{k,i} - F_{k,i} x_k) \geq 0, \tag{9.13}$$

$$[2D_{k,i} x_k - C_{k,i}(u_k - F_k x_k)](u_{k,i} - F_{k,i} x_k) \geq 0 \tag{9.14}$$

where $C_k = B' P_k B$, $D_k = R F_k$ and $C_{k,i}$, $D_{k,i}$, $F_{k,i}$ and $u_{k,i}$ denote the ith row of C_k, D_k, F_k and u_k. Function sign$(\cdot)$ is defined in (9.2). Observe that (9.13) and (9.14) hold if (9.13) and the following constraints are satisfied:

$$\text{sign}(D_{k,i} x_k)\left[2D_{k,i} x_k - C_{k,i}(u_k - F_k x_k)\right] \geq 0. \tag{9.15}$$

Note that (9.13) and (9.15) are a conservative reformulation of (9.12).

The constraints (9.13) and (9.15) are linear in u_k only for current step. We can obtain u_k by solving the following MPC problem with $N = 1$

$$\min_{u_k} \; J = u_k' R u_k + x_{k+1}' P x_{k+1} \tag{9.16}$$

subject to

$$x_{k+1} = Ax_k + Bu_k \tag{9.17}$$

$$-1 \leq u_{k,i} \leq 1, \; i = 1, ..., m, \tag{9.18}$$

$$\text{sign}(D_{k,i} x_k)(u_{k,i} - F_{k,i} x_k) \geq 0, \; i = 1, ..., m, \tag{9.19}$$

$$\text{sign}(D_{k,i} x_k) \left[2D_{k,i} x_k - C_{k,i}(u_k - F_k x_k) \right] \geq 0, \; i = 1, ..., m, \tag{9.20}$$

where P is P_ε with $\varepsilon = 1$.

This problem is always feasible since $u_k = F_k x_k$ is a feasible solution. The solution to the above MPC problem can be obtained by online solving the following convex Quadratic Programming problem

$$\min_{u_k} \; J = u_k'(R + B'PB)u_k + 2x_k' A' P B u_k \tag{9.21}$$

subject to

$$-1 \leq u_{k,i} \leq 1, \; i = 1, ..., m, \tag{9.22}$$

$$\text{sign}(D_{k,i} x_k)(u_{k,i} - F_{k,i} x_k) \geq 0, \; i = 1, ..., m, \tag{9.23}$$

$$\text{sign}(D_{k,i} x_k) \left[2D_{k,i} x_k - C_{k,i}(u_k - F_k x_k) \right] \geq 0, \; i = 1, ..., m, \tag{9.24}$$

The resulting u_k is a nonlinear function of x_k, which we can denote as $u_k = f(x_k)$.

Note that by a re-parametrization with introducing more parameters, an explicit solution of (9.21)-(9.24) with affine dependence on the parameters can be obtained using multi-parametric quadratic programming (mp-QP). This will increase complexity, but it is expected to contribute less to the added complexity than increasing the prediction horizon N.

We have the following theorem

Theorem 9.1 The equilibrium of the closed-loop system of (9.1) and the controller $u_k = f(x_k)$ constructed through (9.21)-(9.24) is globally asymptotically stable.

Proof : By construction, the obtained controller u_k guarantees that $V_{k+1} - V_k < 0$ for any $x_k \neq 0$. The result follows immediately. ■

Remark 9.1 *Compared with the classical low-and-high gain design, the proposed modified approach using MPC yields an LQ optimal controller in a local region around the equilibrium. Moreover, while preserving $V_{k+1} - V_k$, it allows more freedom in choosing u_k when states are large and hence will potentially improve the performance, however at the cost of more computational loads. On the other hand, compared to MPC with a long prediction horizon, the modified approach achieves a guaranteed global asymptotic stability of the closed-loop with very short prediction horizon $N = 1$ and it is more computationally efficient than MPC with large N.*

9.4. Example and Simulation

Consider the following system

$$x_{k+1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma(u_{k,1}) \\ \sigma(u_{k,2}) \end{bmatrix} \tag{9.25}$$

Choose $Q = I$ and $R = I$. We simulate the closed-loop systems of (9.25) with classical low-and-high gain feedback (9.10) and modified low-and-high gain feedback defined by (9.21)-(9.24). The simulations are initialized from 8 corner points of cubic $[-4, 4] \times [-4, 4] \times [-4, 4]$. The state evolutions are shown in the following figures. On average, we observe a $20\% - 25\%$ improvement on the settling time and overshoot.

9.5. Conclusion

In this chapter, we developed a new low-and-high gain feedback design methodology for global stabilization of discrete-time linear systems subject to input saturation using an ultra-short horizon MPC. Simulation on a triple-integrator type system shows improved performance compared with classical low-and-high gain method. The design can also be done in a semi-global framework, in which scheduling of ε is not needed and computational complexity can be further reduced.

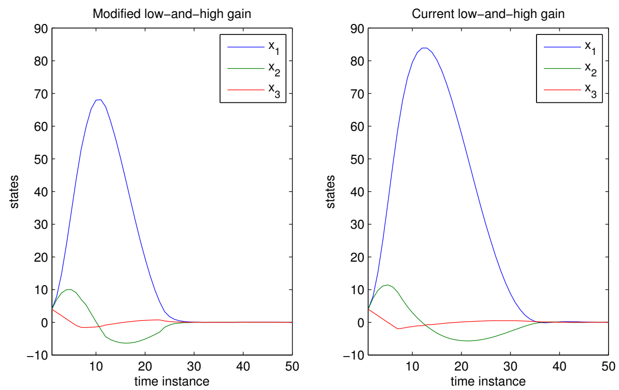


Figure 9.1: Initial condition $[4, 4, 4]'$

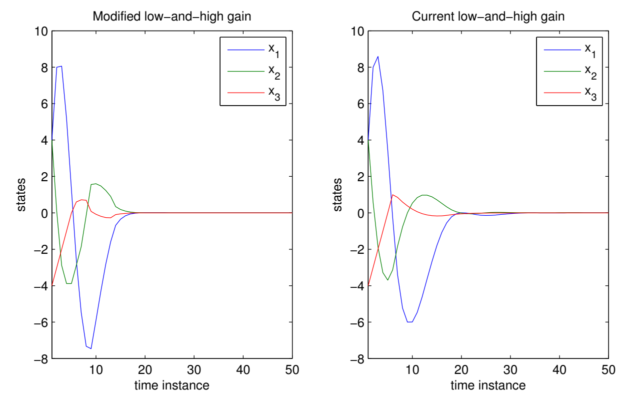


Figure 9.2: Initial condition $[4, 4, -4]'$

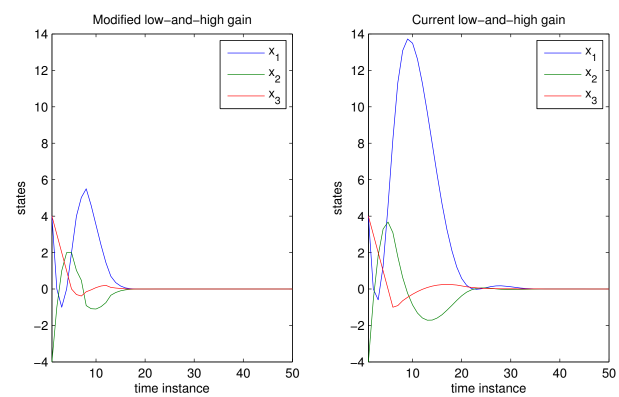


Figure 9.3: Initial condition $[4, -4, 4]'$

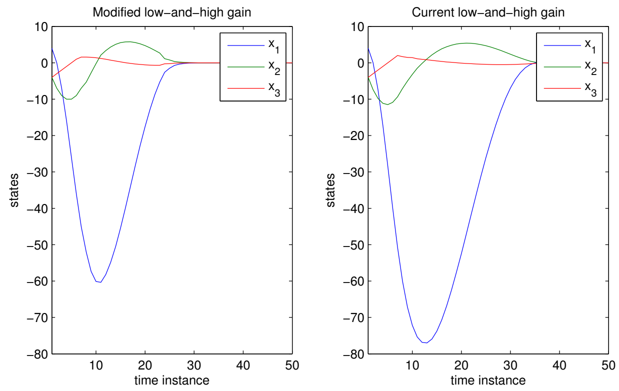


Figure 9.4: Initial condition $[4, -4, -4]'$

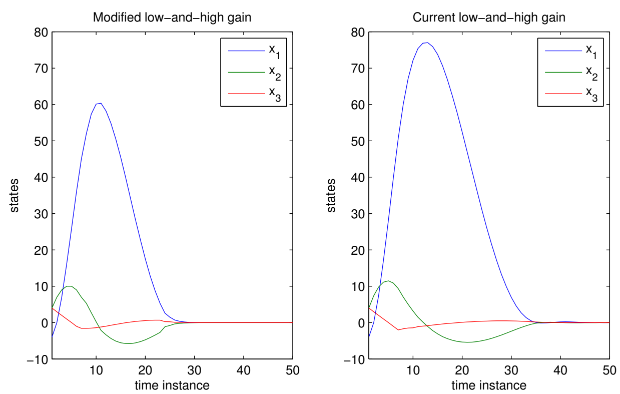


Figure 9.5: Initial condition $[-4, 4, 4]'$

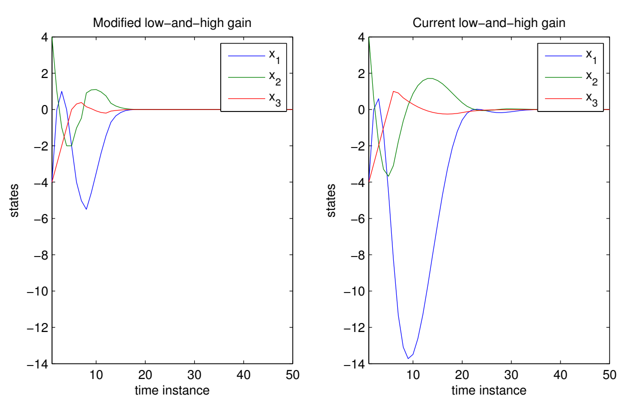


Figure 9.6: Initial condition $[-4, 4, -4]'$

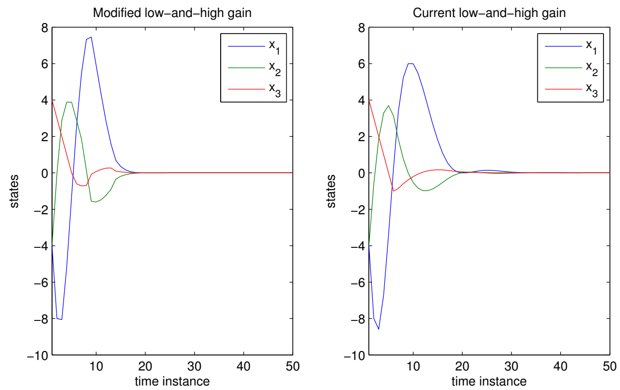


Figure 9.7: Initial condition $[-4, -4, 4]'$

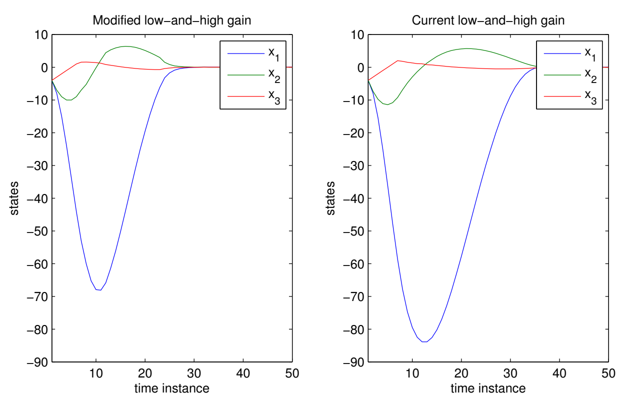


Figure 9.8: Initial condition $[-4, -4, -4]'$

Part III

Simultaneous external and internal stabilization of linear system with input saturation

Notation

Let $\mathbb{C}^-$ and $\mathbb{C}^\odot$ denote the open left-half complex plane and open unit disc. $\mathbb{C}^\circ$ denotes the imaginary axis for continuous-time system and unit circle for discrete-time system. For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x. For $X \in \mathbb{C}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X. For continuous-time (discrete-time) signal y, $\|y\|_\infty$ denotes it $\mathcal{L}_\infty$ (ℓ_∞) norm. $\mathcal{L}_\infty(\delta)$ ($\ell_\infty(\delta)$) represent a set of continuous-time (discrete-time) signals whose $\mathcal{L}_\infty$ (ℓ_∞) norm is less than δ.

For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x. For $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X. $\operatorname{trace}(X)$ denotes the trace of X. If X is symmetric, $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the smallest and largest eigenvalues of X respectively. For a subset $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{X}^c$ denotes the complement of $\mathcal{X}$. For $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \leq k_2$, $\overline{k_1, k_2}$ denotes the integer set $\{k_1, k_1+1, \ldots, k_2\}$.

A continuous function $\phi : [0,\infty) \to [0,\infty)$ is said to be a class $\mathcal{K}$ function if

1. $\phi(0) = 0$;
2. ϕ is strictly increasing.

The $\mathcal{L}_p$ space with $p \in [1,\infty)$ consists of all vector-valued continuous-time signals y from $\mathbb{R}^+$ to $\mathbb{R}^n$ for which

$$\int_{t=0}^{\infty} \|y(t)\|^p \mathrm{d}t < \infty.$$

For a signal $y \in \mathcal{L}_p$, the $\mathcal{L}_p$ norm of y is defined as

$$\|y\|_p = \left(\int_{t=0}^{\infty} \|y(t)\|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

The $\mathcal{L}_\infty$ space consists of all vector-valued continuous-time signals y from $\mathbb{R}^+$ to $\mathbb{R}^n$ for which

$$\sup_{t\geq 0} \|y(t)\| < \infty.$$

For a signal $y \in \mathcal{L}_\infty$, the $\mathcal{L}_\infty$ norm of y is defined as

$$\|y\|_\infty = \sup_{t\geq 0} \|y(t)\|.$$

The ℓ_p space with $p \in [1, \infty)$ consists of all vector-valued discrete-time signals y from $\mathbb{Z}^+ \cup \{0\}$ to $\mathbb{R}^n$ for which

$$\sum_{k=0}^{\infty} \|y(k)\|^p < \infty.$$

For a signal $y \in \ell_p$, the ℓ_p norm of y is defined as

$$\|y\|_p = \left(\sum_{k=0}^{\infty} \|y(k)\|^p \right)^{\frac{1}{p}}.$$

The ℓ_∞ space consists of all vector-valued discrete-time signals y from $\mathbb{Z}^+ \cup \{0\}$ to $\mathbb{R}^n$ for which

$$\sup_{k \geq 0} \|y(k)\| < \infty.$$

For a signal $y \in \ell_\infty$, the ℓ_∞ norm of y is defined as

$$\|y\|_\infty = \sup_{k \geq 0} \|y(k)\|.$$

The following relationship holds for all ℓ_p spaces: for $1 < p < q < \infty$

$$\ell_1 \subset \ell_p \subset \ell_q \subset \ell_\infty.$$

Moreover, for any $y \in \ell_p$ with $p \in [1, \infty)$, the following properties hold:

1. $\|y\|_\infty \leq \|y\|_p$;

2. $y(k) \to 0$ as $k \to \infty$.

The notations $\mathcal{L}_\infty(\delta)$ or $\ell_\infty(\delta)$ represent a set of continuous-time or discrete-time signals whose $\mathcal{L}_\infty$ or ℓ_∞ norm is less than δ.

CHAPTER 10

Relation between external and internal stability of nonlinear systems

10.1. Introduction

A well-known result in linear system theory states that asymptotically stable systems have very good external stability properties. Thus, for linear systems the notions of internal stability and external stability in any sense are highly coupled. However, for general non-linear systems, these two notions of stability are vastly different. By external stability, we always refer to $\mathcal{L}_p/\ell_p$ stability in this context. In this chapter, we study the relation between external stability and internal stability of nonlinear systems. Specifically, for a nonlinear system that is $\mathcal{L}_p/\ell_p$ stable for $p \in [1, \infty)$, we are interested in investigating the internal stability of the autonomous system when the input is zero. Although the focus of this chapter lies in continuous-time systems, as it will be remarked later, to obtain similar result for discrete-time system is generally in a parallel and actually much easier due to the different nature of ℓ_p function space.

The research on this topic evolves mainly along two lines. The first line starts with $\mathcal{L}_p$ stability. An important result that emerges in this direction is [62]. It is shown that under a fairly restrictive condition on the structural property of the system, $\mathcal{L}_p$ stability implies global attractivity of the equilibrium. In fact, it turns out that this conclusion can be attained under much weaker conditions than those in [62]. It will be shown in this chapter that under mild conditions, global $\mathcal{L}_p$ stability ensures attractivity of the equilibrium in the absence of input and attractivity of the origin with any $\mathcal{L}_p$ input.

The other line emanates from $\mathcal{L}_p$ stability with finite gain. There is a large body of work in the literature in this direction; see, for instance, [29, 142, 16, 62]. Along this line of research, the objective is to conclude local asymptotic stability of the equilibrium based on $\mathcal{L}_p$ stability with finite gain. It was shown in [29] that under a uniform reachability condition, global $\mathcal{L}_p$ stability with finite gain implies local asymptotic stability of the equilibrium. In [142], the notion of small-signal $\mathcal{L}_p$ stability with finite gain was introduced and its connection to attractivity of the equilibrium was established. This concept of small-signal $\mathcal{L}_p$ stability was extended in [16] by so-called gain-over-set stability, and it was shown that finite-gain $\mathcal{L}_p$ stability over a set in $\mathcal{L}_p$ space yields local asymptotic stability of the equilibrium.

In this chapter, we prove a result on the relationship between Lyapunov stability and local $\mathcal{L}_p$ stability with finite gain, which further extends, to some level, the result in [16].

10.2. Preliminaries

Consider a nonlinear system

$$\Sigma_1 : \quad \dot{x} = f(x, u), \quad x(0) = x_0, \tag{10.1}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. We assume that $f(\cdot, u)$ is continuous. Let $x(t, t_0, u, x_0)$ denote the trajectory of Σ_1 initialized at time t_0 with input u and initial condition x_0.

We shall investigate the internal stability of the unforced system

$$\Sigma_2 : \quad \dot{x} = f(x, 0), \quad x(0) = x_0, \tag{10.2}$$

under the assumption that Σ_1 is $\mathcal{L}_p$ stable in some sense.

We formally define the notions of $\mathcal{L}_p$ stability as follows:

Definition 10.1 Σ_1 is said to be globally $\mathcal{L}_p$ stable if for $x_0 = 0$ and any $u \in \mathcal{L}_p$, there exists a unique solution $x(\cdot, 0, u, 0) \in \mathcal{L}_p$. Σ_1 is said to be locally $\mathcal{L}_p$ stable with finite gain if there exists a δ and γ such that for $x_0 = 0$ and any u with $\|u\|_{\mathcal{L}_p} \leq \delta$, a unique solution exists and $\|x(\cdot, 0, u, 0)\|_{\mathcal{L}_p} \leq \gamma \|u\|_{\mathcal{L}_p}$.

The domain of attraction and the notion of an $\mathcal{L}_p$-reachable set are defined as follows:

Definition 10.2 The set

$$\mathcal{A}(\Sigma_2) = \{x_0 \in \mathbb{R}^n \mid x(t, 0, 0, x_0) \to 0 \text{ as } t \to \infty\}$$

is called the domain of attraction of the system Σ_2.

Definition 10.3 A point $\xi \in \mathbb{R}^n$ is an $\mathcal{L}_p$-reachable point of system Σ_1 if there exist finite T, M and a measurable input $u : [0, T] \to \mathbb{R}^m$ such that $x(T, 0, u, 0) = \xi$ and

$$\int_0^T \|u(t)\|^p \mathrm{d}t \leq M. \tag{10.3}$$

The set of all $\mathcal{L}_p$-reachable points of Σ_1 is called the $\mathcal{L}_p$-reachable set of Σ_1, which is denoted as $\mathcal{R}_p(\Sigma_1)$.

Remark 10.1 *The requirement* (10.3) *in Definition 10.3 is a weak condition that ensures that the integral of* $\|u(t)\|^p$ *over the interval* $[0, T]$ *is finite. For example, any* x_0 *that is reachable via a signal* $u(t)$ *that is essentially bounded on* $[0, T]$ *is* $\mathcal{L}_p$*-reachable for any* $p \in [1, \infty)$.

The following definition of small-signal local $\mathcal{L}_p$-reachability is adapted from [16]:

Definition 10.4 The system Σ_1 is said to be small-signal locally $\mathcal{L}_p$-reachable if for any $\varepsilon > 0$, there exists δ such that for any $\xi \in \mathbb{R}^n$ with $\|\xi\| \leq \delta$, we can find a finite time T and a measurable input $u : [0, T] \to \mathbb{R}^m$ such that $x(T, 0, u, 0) = \xi$ and $\|u\|_{\mathcal{L}_p} \leq \varepsilon$.

10.3. Main result

Theorem 10.1 Suppose system Σ_1 is globally $\mathcal{L}_p$ stable for some $p \in [1, \infty)$. Then $\mathcal{A}(\Sigma_2) \supseteq \mathcal{R}_p(\Sigma_1)$.

In order to prove Theorem 10.1, we need the following lemma:

Lemma 10.1 Consider system Σ_2. If $x(\cdot, 0, 0, x_0) \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then $x(t, 0, 0, x_0) \to 0$.

Proof : For simplicity, we denote $x(t, 0, 0, x_0)$ by $x(t)$ and $f(x(t), 0)$ by $f(x(t))$ in this proof. Suppose, for the sake of establishing a contradiction, that $x(t) \to 0$ does not hold. Then there exists a $\delta > 0$ such that, for any arbitrarily large $T \geq 0$, there is a $\tau \geq T$ such that $\|x(\tau)\| \geq 2\delta$. Let m be a bound on $\|f(x)\|$ on the closed ball $B(2\delta)$. This bound exists due to continuity of $f(x)$ with respect to x.

For some τ such that $\|x(\tau)\| \geq 2\delta$, let $t_2 > \tau$ be the smallest value such that $\|x(t_2)\| = \delta$, and let t_1 be the largest value such that $t_1 < t_2$ and $\|x(t_1)| = 2\delta$. Such t_1 and t_2 exist because $x(t)$ is absolutely continuous and $x \in \mathcal{L}_p$. Since $\|x(t)\| \in B(2\delta)$ for all $t \in [t_1, t_2]$, we have, due to the absolute continuity of the solution,

$$\|x(t_1)\| - \|x(t_2)\| \leq \|x(t_2) - x(t_1)\| = \left\| \int_{t_1}^{t_2} f(x(\tau))\, \mathrm{d}\tau \right\| \leq \int_{t_1}^{t_2} \|f(x(\tau))\|\, \mathrm{d}\tau \leq (t_2 - t_1)m.$$

Hence, $t_2 - t_1 \geq (\|x(t_1)\| - \|x(t_2)\|)/m = \delta/m$. Clearly $\|x(t)\| \geq \delta$ for all $t \in [\tau, t_2]$, and furthermore $t_2 - \tau \geq t_2 - t_1 \geq \delta/m$. It follows that for each τ such that $\|x(\tau)\| \geq 2\delta$, we have $\|x(t)\| \geq \delta$ for all $t \in [\tau, \tau + \delta/m]$.

Let T be chosen large enough that

$$\int_T^\infty \|x(t)\|^p \, \mathrm{d}\tau < \frac{\delta^{p+1}}{m}. \tag{10.4}$$

Such a T must exist, since $x(t) \in \mathcal{L}_p$. Let $\tau \geq T$ be chosen such that $\|x(\tau)\| \geq 2\delta$. We have

$$\int_T^\infty \|x(t)\|^p \, \mathrm{d}\tau \geq \int_\tau^{\tau+\delta/m} \|x(t)\|^p \, \mathrm{d}\tau \geq \frac{\delta^{p+1}}{m}.$$

This contradicts (10.4), which proves that $x(t) \to 0$. ■

Remark 10.2 *The result in Lemma 1 is closely connected to Theorem 1 in [132]. The proof given above also employs a similar computation technique as used in [132]. Here we are only concerned about attractivity of the equilibrium whereas in Theorem 1 of [132], a result of global asymptotic stability was proved. Therefore, only a weaker condition that* $x(\cdot, 0, 0, x_0) \in \mathcal{L}_p$ *is required compared with [132] where the* $\mathcal{L}_p$ *norm of the trajectory needs to be a class* $\mathcal{K}$ *function of the* $\|x_0\|$.

Proof of Theorem 10.1 : For any $x_0 \in \mathcal{R}_p(\Sigma_1)$, there exist finite T, M and an input $u_0(t)$ for $t \in [0, T]$ such that $x(T, 0, u_0, 0) = x_0$ and

$$\int_0^T \|u_0(t)\|^p \mathrm{d}t \leq M$$

Define

$$u(t) = \begin{cases} u_0(t), & t \in [0, T] \\ 0, & t > T \end{cases}$$

Clearly, $u \in \mathcal{L}_p$. Since Σ_1 is globally $\mathcal{L}_p$ stable, we have that $x(\cdot, 0, u, 0) \in \mathcal{L}_p$. On the other hand, $u(t) = 0$ for $t > T$ implies that after T the system Σ_1 is equivalent with system Σ_2 initialized at x_0, i.e. $x(t, 0, u, 0) = x(t - T, 0, 0, x_0)$ with $t > T$. Therefore, $x(t, 0, 0, x_0) \in \mathcal{L}_p$ over $[0, \infty)$. It follows from Lemma 10.1 that $x(t, 0, 0, x_0) \to 0$ as $t \to \infty$. This completes the proof. ■

Corollary 10.1 Suppose system Σ_1 is globally $\mathcal{L}_p$ stable for some $p \in [1, \infty)$. If $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$, then the origin of Σ_2 is globally attractive.

The next theorem shows that under a certain condition on the structure of $f(x, u)$, the origin of Σ_1 is attractive for any input $u \in \mathcal{L}_p$.

Theorem 10.2 Suppose that Σ_1 is globally $\mathcal{L}_p$ stable for some $p \in [1, \infty)$. If there exist $\delta > 0, m_1 \geq 0$, $m_2 \geq 0$ and $q \in [0, p]$ such that for any x with $\|x\| \leq \delta$

$$\|f(x, u)\| \leq m_1 + m_2\|u\|^q, \tag{10.5}$$

then for $x_0 = 0$ and any $u \in \mathcal{L}_p$, $x(t, 0, u, 0) \to 0$ as $t \to \infty$.

Proof : Define a generalized saturation function $\bar{\sigma}(\cdot) : \mathcal{R}^n \to \mathcal{R}^n \in C^1$ as

$$\bar{\sigma}(x) = \begin{bmatrix} \bar{\sigma}_1(x_1) \\ \vdots \\ \bar{\sigma}_n(x_n) \end{bmatrix}, \quad \bar{\sigma}_i(x_i) = \begin{cases} -\frac{2\delta}{\pi}, & x_i < -\delta \\ \frac{2\delta}{\pi} \sin(\frac{\pi}{2\delta} x_i), & |x_i| \leq \delta \\ \frac{2\delta}{\pi}, & x_i > \delta \end{cases}$$

Consider $\bar{x}(t) = \bar{\sigma}(x(t, 0, u, 0))$. Note that $\bar{x}(t)$ is still absolutely continuous on any compact interval. Let $\bar{x}_i$ and f_i denote the ith element of $\bar{x}$ and $f(x, u)$ respectively. We have

$$|\dot{\bar{x}}_i(t)| = \begin{cases} 0, & |x_i(t)| > \delta \\ |\cos(\frac{\pi}{2\delta} x_i) f_i(x(t), u(t))| \leq m_1 + m_2\|u(t)\|^q, & |x_i(t)| \leq \delta \end{cases}$$

Therefore, $\|\dot{\bar{x}}(t)\| \leq \sqrt{n}(m_1 + m_2\|u\|^q)$ for all $t > 0$. Note that $\|u(t)\|^q \leq 1 + \|u(t)\|^p$ and hence $\|u\|^q$ is locally uniformly integrable. Then it follows from [131] that $\bar{x}(t) \to 0$ as $t \to 0$. This implies that $x(t, 0, u, 0) \to 0$ as $t \to 0$. ■

Remark 10.3 *In [62], in order to prove the same result as in Theorem 10.2, the following condition was imposed on $f(x, u)$: there exists δ_1, K_1, K_2 and $\alpha \in [0, p]$ such that for $x \in \mathbb{R}^n$ with $\|x\| \leq \delta_1$,*

$$\|f(x, u)\| \leq K_1(\|x\| + \|u\|) + K_2(\|x\|^\alpha + \|u\|^\alpha)$$

Theorem 10.2 shows that the restrictions on x in the above condition are not necessary.

An immediate consequence of Theorem 10.2 is the next theorem.

Theorem 10.3 Suppose that Σ_1 is globally $\mathcal{L}_p$ stable and $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$ for some $p \in [1, \infty)$. If there exist $\delta > 0$, $m_1 \geq 0$, $m_2 \geq 0$ and $q \in [0, p]$ such that for any x with $\|x\| \leq \delta$

$$\|f(x,u)\| \leq m_1 + m_2\|u\|^q,$$

then Σ_1 is globally $\mathcal{L}_p$ stable with arbitrary initial condition.[1] Moreover, for any $x_0 \in \mathbb{R}^n$ and any $u \in \mathcal{L}_p$, $x(t, 0, u, x_0) \to 0$ as $t \to \infty$.

Proof : Since $\mathcal{R}_p(\Sigma_1) = \mathbb{R}^n$, for any $x_0 \in \mathbb{R}^n$, there exist finite T, M and a measurable input u_0 : $[0, T] \to \mathbb{R}^m$ such that $x(T, 0, u_0, 0) = x_0$ and

$$\int_0^T \|u_0(t)\|^p \, \mathrm{d}t \leq M.$$

For any $u \in \mathcal{L}_p$, define

$$\bar{u}(t) = \begin{cases} u_0(t), & t \in [0, T] \\ u(t - T), & t > T \end{cases}$$

Then we have $x(t, 0, u, x_0) = x(t + T, 0, \bar{u}, 0)$. Clearly $\bar{u} \in \mathcal{L}_p$. This implies that $x(\cdot, 0, \bar{u}, 0) \in \mathcal{L}_p$ and hence $x(\cdot, 0, u, x_0) \in \mathcal{L}_p$. This proves $\mathcal{L}_p$ stability with arbitrary initial condition and it follows from Theorem 10.2 that $x(t, 0, \bar{u}, 0) \to 0$ as $t \to \infty$ and therefore $x(t, 0, u, x_0) \to 0$ as $t \to \infty$. ■

In what follows, we prove a theorem that is a slight generalization of results in [16].

Theorem 10.4 Suppose that Σ_1 is locally $\mathcal{L}_p$ stable with finite gain and small-signal locally $\mathcal{L}_p$-reachable. Then the origin of Σ_2 is locally asymptotically stable.

Proof : Let ε be an arbitrary positive real number. We need to show that there exists a $\delta > 0$ such that $\|x_0\| \leq \delta$ implies $\|x(t, 0, 0, x_0)\| \leq \varepsilon$ for all $t \geq 0$. Toward this end, let $\delta \leq \frac{\varepsilon}{2}$ be chosen such that for any $x_0 \in \mathbb{R}^n$ with $\|x_0\| \leq \delta$, there exist a finite time T and measurable input $u : [0, T] \to \mathbb{R}^m$ such that

$$x(T, 0, u, 0) = x_0 \text{ and } \|u\|_{\mathcal{L}_p} < \frac{\varepsilon}{2\gamma}\left(\frac{\varepsilon}{2M(\varepsilon)}\right)^{\frac{1}{p}}.$$

This is possible due to $\mathcal{L}_p$ local reachability.

[1] Σ_1 is said to be global $\mathcal{L}_p$ stable with arbitrary initial condition if for any $x_0 \in \mathbb{R}^n$ and $u \in \mathcal{L}_p$, we have $x(\cdot, 0, u, x_0) \in \mathcal{L}_p$.

Set $u(t) = 0$ for $t > T$. Since Σ_1 is locally $\mathcal{L}_p$ stable with finite gain, from Definition 10.1, there exists γ such that

$$\int_T^\infty \|x(t,0,u,0)\|^p \,\mathrm{d}t \le \int_0^\infty \|x(t,0,u,0)\|^p \,\mathrm{d}t \le \gamma^p \|u\|_{\mathcal{L}_p}^p < \frac{\varepsilon^{p+1}}{2^{p+1}M(\varepsilon)}.$$

For $t > T$, the system Σ_1 is equivalent to Σ_2 initialized at $x(0) = x_0$, i.e. $x(t,0,u,0) = x(t - T,0,0,x_0)$. Hence we have

$$\int_0^\infty \|x(t,0,0,x_0)\|^p \,\mathrm{d}t < \frac{\varepsilon^{p+1}}{2^{p+1}M(\varepsilon)}. \tag{10.6}$$

It immediately follows from Lemma 10.1 that $x(t,0,0,x_0) \to 0$ as $t \to \infty$.

We proceed to show that $\|x(t,0,0,x_0)\| < \varepsilon$ for all $t \ge 0$. Suppose, for the sake of establishing a contradiction, that there exists a τ such that $\|x(\tau,0,0,x_0)\| \ge \varepsilon$. Let $t_1 < \tau$ be the largest value such that $\|x(t_1,0,0,x_0)\| = \varepsilon/2$, and let $t_2 \le \tau$ be the smallest value such that $t_2 > t_1$ and $\|x(t_2,0,0,x_0)\| = \varepsilon$. Such t_1 and t_2 exist because $\|x_0\| \le \frac{\varepsilon}{2}$. Then $\varepsilon/2 \le \|x(t,0,0,x_0)\| \le \varepsilon$ for all $t \in [t_1,t_2]$. Let $M(\varepsilon)$ be a bound on $f(x,0)$ for $\|x\| \le \varepsilon$. We have, owing to the absolute continuity of $x(\cdot,0,0,x_0)$,

$$\begin{aligned} \|x(t_2,0,0,x_0)\| - \|x(t_1,0,0,x_0)\| &\le \|x(t_2,0,0,x_0) - x(t_1,0,0,x_0)\| \\ &\le \left\| \int_{t_1}^{t_2} f(x(t),0)\,\mathrm{d}t \right\| \le \int_{t_1}^{t_2} M(\varepsilon)\,\mathrm{d}t \le M(\varepsilon)(t_2 - t_1) \end{aligned}$$

This gives that $t_2 - t_1 \ge \frac{\varepsilon}{2M(\varepsilon)}$ and hence that

$$\int_0^\infty \|x(t,0,0,x_0)\|^p \,\mathrm{d}t \ge \int_{t_1}^{t_2} \|x(t,0,0,x_0)\|^p \,\mathrm{d}t \ge \int_{t_1}^{t_2} \left(\frac{\varepsilon}{2}\right)^p \mathrm{d}t = \frac{\varepsilon^{p+1}}{2^{p+1}M(\varepsilon)},$$

which contradicts (10.6). Hence $\|x(t,0,0,x_0)\| < \varepsilon$ for all $t \ge 0$, which completes the proof. ■

Remark 10.4 *Compared with the result in [16], Theorem 10.4 requires a finite gain only within an arbitrary small neighborhood of the origin of $\mathcal{L}_p$ space.*

Remark 10.5 *We assume that $f(x,u)$ is continuous with respect to x, which covers a large class of dynamical systems. In fact, it can be seen from the proof that we only need continuity of $f(x,u)$ with respect to x at $x = 0$.*

Remark 10.6 *The reader is also referred to a related paper [132] which presents the integral characterizations of uniform asymptotic stability and uniform exponential stability for differential equations and inclusions.*

10.4. A remark for discrete-time systems and conclusion

Note that in the discrete-time case, $x \in \ell_p$ automatically means $x(k) \to 0$ as $k \to \infty$. Therefore, under similar standard assumptions and reachability conditions as in the continuous-time case, if applicable, all the results carry over to discrete-time systems straightforwardly.

In this chapter, we study the connection between two notions of stability of nonlinear systems, namely Lyapunov stability and external $\mathcal{L}_p/\ell_p$ stability. While no direct translation can be made between these notions of stability in general, it represents another effort in exposing the relationship between them by studying attractivity for $\mathcal{L}_p/\ell_p$ stable systems with additional structural properties, such as local reachability and bounds on the derivative.

CHAPTER 11

Simultaneous global external and internal stabilization of linear time-invariant discrete-time systems subject to actuator saturation

11.1. Introduction

Our focus in this chapter is on discrete-time linear systems subject to actuator saturation and input-additive disturbances. For *continuous-time* systems, a key result is given in [89]. This work, while pointing out all the complexities involved in simultaneous *global* external and *global* internal stabilization, resolves all such issues and develops certain scheduled low-and-high gain design methodologies to achieve the required simultaneous global-global stabilization. Analogous results for discrete-time systems do not exist so far in the literature. Discrete-time has its own peculiarities. High-gain cannot be as freely used as in continuous-time but also almost disturbance decoupling could always be achieved in continuous-time case while in discrete-time case, this is not possible in general. This chapter can be thought of as a companion to [89] as it resolves fully all the issues for discrete-time systems. In particular, we develop here the necessary and sufficient conditions for simultaneous *global* external and *global* internal stabilization, and furthermore develop also the required design methodologies to accomplish such a stabilization whenever it is feasible.

We organize this chapter as follows: In Section 11.2, we formulate precisely two problems studied in this chapter, namely simultaneous global ℓ_p stabilization without finite gain and internal global asymptotic stabilization (G_p/G), and (2) simultaneous global ℓ_p stabilization with finite gain and internal global asymptotic stabilization $(G_p/G)_{fg}$. In Section 11.3, we describe controller design methodologies, and in Section 11.4, we establish the solvability conditions for (G_p/G) and $(G_p/G)_{fg}$ and construct an adaptive-low-gain and high-gain controller that solves the two problems by using a parametric Lyapunov equation.

11.2. Preliminary notations and problem formulation

In this section, we recall the notions of external stability for a general discrete-time nonlinear system. Based on these notions, we formulate the simultaneous stabilization problems in this section.

Consider a system

$$\Sigma : \begin{cases} x(k+1) = f(x(k), d(k)), & x(0) = x_0 \\ \quad y(k) \quad = g(x(k), d(k)) \end{cases}$$

with $x(k) \in \mathbb{R}^n$ and $d(k) \in \mathbb{R}^m$. The two classical ℓ_p stabilities are defined as follows:

Definition 11.1 For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with fixed initial condition and without finite gain** if for $x(0) = 0$ and any $d \in \ell_p$, we have $y \in \ell_p$.

Definition 11.2 For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with fixed initial condition and with finite gain** if for $x(0) = 0$ and any $d \in \ell_p$, we have $y \in \ell_p$ and there exists a γ_p such that for any $d \in \ell_p$,

$$\|y\|_p \le \gamma_p \|d\|_p.$$

The infimum over all γ_p with this property is called the ℓ_p gain of the system Σ.

As observed in [107], the initial condition plays a dominant role in whether achieving ℓ_p stability is possible or not. Hence any definition of external stability must take into account the effect of initial condition. The notion of external stability with arbitrary initial condition was introduced in [107]. We recall these definitions below:

Definition 11.3 For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with arbitrary initial condition and without finite gain** if for any $x_0 \in \mathbb{R}^n$ and any $d \in \ell_p$, we have $y \in \ell_p$.

Definition 11.4 For any $p \in [1, \infty]$, the system Σ is said to be **ℓ_p stable with arbitrary initial condition with finite gain and with bias** if for any $x_0 \in \mathbb{R}^n$ and any $d \in \ell_p$, we have $y \in \ell_p$ and there exists a γ_p and a class $\mathcal{K}$ function $\phi(\cdot)$ such that for any $d \in \ell_p$

$$\|y\|_p \le \gamma_p \|d\|_p + \phi(\|x_0\|).$$

The infimum over all γ_p with this property is called the ℓ_p gain of the system Σ.

Now we are ready to formulate our control problems. Consider a linear discrete-time system subject to actuator saturation,

$$x(k+1) = Ax(k) + B\sigma(u(k) + d(k)), \tag{11.1}$$

where state $x \in \mathbb{R}^n$, the output $y = x$, control input $u \in \mathbb{R}^m$, and external input $d \in \mathbb{R}^m$. Here $\sigma(\cdot)$ denotes the standard saturation function defined as

$$\sigma(u) = [\sigma_1(u_1), \ldots, \sigma_1(u_m)]$$

where $\sigma_1(s) = \operatorname{sgn}(s) \min\{|s|, \Delta\}$ for some $\Delta > 0$.

The simultaneous global external and internal stabilization problems are formulated as follows:

Problem 11.1 For any $p \in [1, \infty]$, the system (11.1) is said to be simultaneously globally ℓ_p stabilizable with fixed initial condition and without finite gain and globally asymptotically stabilizable via static time invariant state feedback, which we refer to as (G_p/G), if there exists a static state feedback controller $u = f(x)$ such that the following properties hold:

1. the closed-loop system is ℓ_p stable with fixed initial condition and without finite gain where the output $y = x$;
2. In the absence of external input d, the equilibrium $x = 0$ is globally asymptotically stable.

Problem 11.2 For any $p \in [1, \infty]$, the system (11.1) is said to be simultaneously globally ℓ_p stabilizable with fixed initial condition with finite gain with zero bias and globally asymptotically stabilizable via state feedback, which we refer to as $(G_p/G)_{fg}$, if there exists a static time invariant state feedback controller $u = f(x)$ such that the following properties hold:

1. the closed-loop system is finite gain ℓ_p stable with fixed initial condition with finite gain and with zero bias where the output $y = x$;
2. In the absence of external input d, the equilibrium $x = 0$ is globally asymptotically stable.

Note that the notion of global ℓ_p stability with arbitrary initial condition embeds in it the internal stability in some sense. We also formulate below additional external stabilization problems with arbitrary initial conditions.

Problem 11.3 For any $p \in [1, \infty]$, the system (11.1) is said to be globally ℓ_p stabilizable with arbitrary initial condition and without finite gain via static time invariant state feedback, if there exists a static state feedback controller $u = f(x)$ such that the closed-loop system is ℓ_p stable with arbitrary initial condition and without finite gain where the output $y = x$.

Problem 11.4 For any $p \in [1, \infty]$, the system (11.1) is said to be globally ℓ_p stabilizable with arbitrary initial condition with finite gain and with bias via state feedback, if there exists a static time invariant state feedback controller $u = f(x)$ such that the closed-loop system is finite gain ℓ_p stable with arbitrary initial condition with finite gain and with bias where the output $y = x$.

Since global asymptotic stabilization is required in all the problems, it is well-known that the following assumption is ***necessary***.

Assumption 11.1

1. the pair (A, B) is stabilizable;
2. A has all its eigenvalues in the closed unit disc.

In fact, as will become clear in the sequel, Assumption 11.1 is also ***sufficient*** to solve Problem 11.1-11.4. To see this, we first note that under Assumption 11.1, the system (11.1) can be transformed into the form,

$$\begin{pmatrix} x_s(k+1) \\ x_u(k+1) \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix} \begin{pmatrix} x_s(k) \\ x_u(k) \end{pmatrix} + \begin{pmatrix} B_s \\ B_u \end{pmatrix} \sigma(u(k) + d(k)) \tag{11.2}$$

where A_s is Schur stable, A_u has all its eigenvalues on the unit circle and (A_u, B_u) is controllable. Suppose (G_p/G) and/or $(G_p/G)_{f.g}$ of the x_u dynamics can be achieved by a feedback controller $u = f(x_u)$. If B_u has full column rank, it is straight forward to show that $u = f(x_u)$ also achieves (G_p/G) and/or $(G_p/G)_{f.g}$ of the overall system. However, it takes some effort to reach the same conclusion in the general case. We show this in the appendix under a generic assumption on controller structure.

Therefore, to solve Problem 11.1 to 11.4 for system (11.1), it is sufficient to solve these problems only for the unstable sub-dynamics. In the rest of this chapter, we impose the following assumption

Assumption 11.2

1. the pair (A, B) is controllable;

2. A has all its eigenvalues on the unit circle.

11.3. Controller design

In this section, we would like to present the controller design methodologies which we shall employ to solve the problems formulated in Section 11.2. The controller design is based on the classical low-gain and low-and-high-gain feedback design methodologies. The low-gain feedback can be constructed using different approaches such as direct eigenstructure assignment [51], H_2 and H_∞ algebraic Riccati equation based methods [61, 130], and parametric Lyapunov equation based method [159, 161]. In our effort to solve the simultaneous stabilization problems, we choose parametric Lyapunov equation method to build the low-gain feedback because of its special properties; as will become clear later on, it greatly simplifies the expressions for our controllers and the subsequent analysis.

Since the low-gain feedback, as indicated by its name, does not allow complete utilization of control capacities, the low-and-high-gain feedback was developed to rectify this drawback and was intended to achieve control objectives beyond stability, such as performance enhancement, robustness and disturbances rejection. The low-and-high gain feedback is composed of a low-gain and a high-gain feedback. As shown in [31], the solvability of simultaneous global external and internal stabilization problem critically relies on a proper choice of high-gain. In this section, we shall first recall the low-gain feedback design and propose a new high-gain design methodology.

11.3.1. Low gain state feedback

In this subsection, we review the low-gain feedback design methodology recently introduced in [159, 161] which is based on the solution of a parametric Lyapunov equation. Five key properties of the parametric Lyapunov equation are summarized in the next lemma, where the first three properties are adopted from [161].

Lemma 11.1 Assume that (A, B) is controllable and A has all its eigenvalues on the unit circle. For any

$\varepsilon \in (0,1)$, the Parametric Algebraic Riccati Equation,

$$(1-\varepsilon)P_\varepsilon = A'P_\varepsilon A - A'P_\varepsilon B(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon A, \tag{11.3}$$

has a unique positive definite solution $P_\varepsilon = W_\varepsilon^{-1}$ where W_ε is the solution for W of

$$W - \frac{1}{1-\varepsilon}AWA' = -BB'.$$

Moreover, the following properties hold:

1. $A_c(\varepsilon) = A - B(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon A$ is Schur stable for any $\varepsilon \in (0,1)$;
2. $\frac{\mathrm{d}P_\varepsilon}{\mathrm{d}\varepsilon} > 0$ for any $\varepsilon \in (0,1)$;
3. $\lim_{\varepsilon \downarrow 0} P_\varepsilon = 0$;
4. There exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$,
$$\|[P_\varepsilon^{\frac{1}{2}} A P_\varepsilon^{-\frac{1}{2}}\| \leq \sqrt{2};$$
5. Let ε^* be given by property 4. There exists a M_{ε^*} such that $\|\frac{P_\varepsilon}{\varepsilon}\| \leq M_{\varepsilon^*}$ for all $\varepsilon \in (0, \varepsilon^*]$.

Proof : The existence of the positive definite solution $P_\varepsilon = W_\varepsilon^{-1}$ and properties 1, 2 and 3 were shown in [161]. Regarding property 4, multiplying by $P_\varepsilon^{-1/2}$ on both sides of (11.3) gives

$$V_\varepsilon'[I - P_\varepsilon^{\frac{1}{2}} B(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon^{\frac{1}{2}}]V_\varepsilon = (1-\varepsilon)I$$

where $V_\varepsilon = P_\varepsilon^{1/2}AP_\varepsilon^{-1/2}$. Since $P_\varepsilon \to 0$ as $\varepsilon \to 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$

$$I - P_\varepsilon^{\frac{1}{2}} B(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon^{\frac{1}{2}} \geq \tfrac{1}{2}I.$$

Hence

$$V_\varepsilon' V_\varepsilon < 2I,$$

or equivalently

$$\|V_\varepsilon\| \leq \sqrt{2}.$$

It remains to show property 5. Note that W_ε is a rational matrix in ε and thus P_ε is a rational matrix in ε. Property 3 implies that $P = \varepsilon \bar{P}_\varepsilon$ where $\bar{P}_\varepsilon$ is rational in ε and satisfies $\|\bar{P}_\varepsilon\| < M_{\varepsilon^*}$ for any $\varepsilon \in (0, \varepsilon^*]$. Hence, property 5 holds. This concludes the proof of Lemma 11.1. ■

We define the low-gain state feedback which is a family of parameterized state feedback laws given by

$$u_L(x) = F_L x = -(I + B' P_\varepsilon B)^{-1} B' P_\varepsilon A x, \tag{11.4}$$

where P_ε is the solution of (11.3). Here, as usual, ε is called the low-gain parameter. From the properties given by Lemma 11.1, it can be seen that the magnitude of the control input can be made arbitrarily small by choosing ε sufficiently small so that the input never saturates for any, a priori given, set of initial conditions.

11.3.2. Low-and-high-gain feedback

The low-and-high-gain state feedback is composed of a low-gain state feedback and a high-gain state feedback as

$$u_{LH}(x) = F_{LH} x = F_L x + F_H x \tag{11.5}$$

where $F_L x$ is given by (11.4). The high-gain feedback is of the form,

$$F_H x = \rho F_L x$$

where ρ is called the high-gain parameter.

For continuous-time systems, the high gain parameter ρ can be any positive real number. However, this is not the case for discrete-time systems. In order to preserve local asymptotic stability, this high gain has to be bounded at least near the origin. The existing results in literature on the choice of high-gain parameter for discrete-time system are really sparse. To the best of our knowledge, the only available result is in [59, 60] where the high-gain parameter is a nonlinear function of x as

$$\kappa(x) = \max\{z \in [0, 1] \mid \|F_L x + \alpha z K_H x\|_\infty \leq \Delta\}$$

where $K_H = -(B' P_\varepsilon B)^{-1} B' P_\varepsilon (A + B F_L)$ and $\alpha \in [0, 2]$ (assume without loss of generality that B has full rank). This high gain always yields a controller smaller than Δ in magnitude, which lacks the capability of dealing with disturbances. Furthermore, to solve the global external and internal stabilization problem, we need to schedule the high-gain parameter with respect to x. However, this nonlinear high-gain parameter is also not suitable for adaptation since it will make the analysis extremely complicated. Instead, we need a constant high-gain parameter so that the controller (11.5) remains linear. A suitable choice of such a high-gain parameter satisfies

$$\rho \in \left[0, \frac{2}{\|B'P_\varepsilon B\|}\right] \tag{11.6}$$

where P_ε is the solution of parametric Lyapunov equation (11.3).

Lemma 11.2 Consider system (11.1) satisfying Assumption 11.2. Let P_ε be the solution of (11.3). For any *a priori* given compact set $\mathcal{X}$, there exists an ε^* such that for any $\varepsilon \in [0, \varepsilon^*]$ and ρ satisfying (11.6), the origin of the interconnection of (11.1) with the low-and-high-gain feedback

$$u_{LH} = -(1+\rho)(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax$$

is locally asymptotically stable with domain of attraction containing $\mathcal{X}$.

Proof : Let c be such that

$$c = \sup_{\substack{\varepsilon\in(0,\varepsilon^*] \\ x\in\mathcal{X}}} x'P_\varepsilon x.$$

where ε^* is given by Property (4) and (5) in Lemma 11.1. Define a Lyapunov function $V(x) = x'P_\varepsilon x$ and a level set $\mathcal{V}(c) = \{x \mid V(x) \le c\}$. We have $\mathcal{X} \subset \mathcal{V}_c$. From Lemma 11.1, there exists an $\varepsilon_1 \le \varepsilon^*$ such that for any $\varepsilon \in (0, \varepsilon_1]$ and $x \in \mathcal{V}_c$,

$$\|(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax\| \le \Delta.$$

Define $\mu = \|B'P_\varepsilon B\|$. We evaluate $V(k+1) - V(k)$ along the trajectories as

$$\begin{aligned} V(k+1) - V(k) =& -\varepsilon V(k) - \sigma(u_{LH})'\sigma(u_{LH}) + [\sigma(u_{LH}) - u_L]'(I + B'PB)[\sigma(u_{LH}) - u_L] \\ \le& -\varepsilon V(k) - \sigma(u_{LH})'\sigma(u_{LH}) + (1+\mu)[\sigma(u_{LH}) - u_L]'[\sigma(u_{LH}) - u_L)] \\ =& -\varepsilon V(k) - \tfrac{1+\mu}{\mu}\|u_L\|^2 + \mu\|\sigma(u_{LH}) - \tfrac{1+\mu}{\mu}u_L\|^2. \end{aligned}$$

where we abbreviated $u_{LH}(k)$ and $u_L(k)$ by u_{LH} and u_L respectively. Since $\|u_L\| \le \Delta$ and ρ satisfies (11.6), we have

$$\|u_L\| \le \|\sigma(u_{LH})\| \le (1 + \tfrac{2}{\mu})\|u_L\|.$$

This implies that

$$\|\sigma(u_{LH}) - \tfrac{1+\mu}{\mu} u_L\| \le \tfrac{1}{\mu}\|u_L\|,$$

and thus,

$$\mu\|\sigma(u_{LH}) - \tfrac{1+\mu}{\mu} u_L\|^2 - \tfrac{1}{\mu}\|u_L\|^2 \le 0.$$

Combining the above, we get for any $x(k) \in \mathcal{V}(c)$,

$$V(k+1) - V(k) \le -\varepsilon V(k).$$

We conclude local asymptotic stability of the origin with a domain of attraction containing $\mathcal{X}$. ■

Remark 11.1 *We would like to explain the role played by the high-gain parameter ρ in the controller design. For semi-global asymptotic stabilization, the domain of attraction is basically determined by the low-gain parameter ε provided that ρ lies in a proper range. When ρ is too large, stabilization is not possible. This is different from continuous-time systems for which the high gain parameter ρ does not have any impact on internal stability. But like continuous-time systems, ρ plays a dominant role in issues other than internal stability such as external stabilization, robust stabilization and disturbance rejection.*

11.3.3. Scheduling of low-gain parameter

In the semi-global framework, with controller (11.4), the domain of attraction of the closed-loop system is determined by the low-gain parameter ε. In order to solve the global stabilization problem, this ε can be scheduled with respect to the state. This has been done in the literature, see for instance [31].

A family of scheduled low-gain feedback controllers for global stabilization is given by

$$u_L(x) = F_{\varepsilon(x)}x = -(B'P_{\varepsilon(x)}B + I)^{-1}B'P_{\varepsilon(x)}Ax \tag{11.7}$$

where $P_{\varepsilon(x)}$ is the solution of (11.3) with ε replaced by $\varepsilon(x)$. In general, the scheduled parameter $\varepsilon(x)$ should satisfy the following properties:

1. $\varepsilon(x) : \mathbb{R}^n \to (0, \varepsilon^*]$ is continuous and piecewise continuously differentiable where ε^* is a design parameter.

2. There exists an open neighborhood $\mathcal{O}$ of the origin such that $\varepsilon(x) = 1$ for all $x \in \mathcal{O}$.

3. For any $x \in \mathbb{R}^n$, we have $\|F_{\varepsilon(x)}x\| \leq \delta$.

4. $\varepsilon(x) \to 0$ as $\|x\| \to \infty$.

5. $\{ x \in \mathbb{R}^n \mid x' P_{\varepsilon(x)} x \leq c \}$ is a bounded set for all $c > 0$.

Because of the specific problem facing us, we use the scheduling given in [31] which not only satisfies the above conditions but also yields an adaptive low-gain parameter with certain properties that are fundamental to our design,

$$\varepsilon(x) = \max\{r \in (0, \varepsilon^*] \mid (x' P_r x)\,\mathrm{trace}(P_r) \leq \tfrac{\Delta^2}{b}\} \tag{11.8}$$

where $\varepsilon^* \in (0, 1)$ is any a priori given constant and $b = 2\,\mathrm{trace}(BB')$ while P_r is the unique positive definite solution of parametric Lyapunov equation (11.3) with $\varepsilon = r$.

Note that the scheduled low-gain controller (11.7) with (11.8) satisfies

$$\|(B' P_{\varepsilon(x)} B + I)^{-1} B' P_{\varepsilon(x)} A x\| \leq \Delta.$$

To see this, observe that

$$\begin{aligned}
\|(B' P_{\varepsilon(x)} B + I)^{-1} B' P_{\varepsilon(x)} A x\|^2 &\leq \|B' P_{\varepsilon(x)} A x\|^2 \\
&\leq \|B'\|^2 \|P_{\varepsilon(x)}\| \|P_{\varepsilon(x)}^{1/2} A P_{\varepsilon(x)}^{-1/2}\|^2 \|P_{\varepsilon(x)}^{1/2} x\|^2 \\
&\leq 2\|BB'\| \|P_{\varepsilon(x)}\| x' P_{\varepsilon(x)} x \\
&\quad \text{(where we use property 4 of Lemma 11.1)} \\
&\leq 2\,\mathrm{trace}(BB')\,\mathrm{trace}(P_{\varepsilon(x)})\, x' P_{\varepsilon(x)} x \\
&\leq \Delta^2.
\end{aligned}$$

11.3.4. Scheduling of high-gain parameter

As emphasized earlier, the high gain parameter plays a crucial role in dealing with external inputs/disturbances. In order to solve the simultaneous external and internal stabilization problems for continuous-time systems, different methods of schedulings of high-gain parameter have been developed in the literature [31, 47, 89]. Unfortunately, none of them carry over to discrete-time case because the high gain has to be restricted near the origin. In this subsection, we introduce a new scheduling of the high-gain parameter with which we shall solve the (G_p/G) and $(G_p/G)_{fg}$ problems as formulated in Section 11.2.

Our scheduling depends on the specific control objective. If one is not interested in finite gain, the following scheduled high gain suffices to solve (G_p/G) problem,

$$\rho_0(x) = \tfrac{1}{\|B' P_{\varepsilon(x)} B\|}. \tag{11.9}$$

Clearly, this high gain satisfies the constraints that

$$\rho_0(x) \le \tfrac{2}{\|B' P_{\varepsilon(x)} B\|}.$$

We observe that this high-gain parameter is radially unbounded. However, if we further pursue finite gain ℓ_p stabilization, the rate of growth of $\rho(x)$ with respect to $\|x\|$ as given in (11.9) is not sufficient for us. The scheduled high-gain parameter must rise quickly enough to overwhelm any disturbances in ℓ_p before the state is steered so large that it actually prevents finite gain. Therefore, we shall introduce a different scheduling of high-gain parameter. In order to do so, we need the following lemma:

Lemma 11.3 Assume that $2p \ge 1$. For any $\eta > 1$ there exists a $\beta > 0$ such that

$$(u+v)^p \le u^p + \eta u^p + \beta v^p \tag{11.10}$$

for all $u, v \ge 0$.

Proof : The lemma is a known result for $p \ge 1$; see, for instance, [107]. For $p \in [\frac{1}{2}, 1)$, we have $2p \ge 1$ and then

$$(\sqrt{u+v})^{2p} \le (\sqrt{u} + \sqrt{v})^{2p} \le u^{2p} + \eta u^{2p} + \beta v^{2p}$$

where we use the lemma with p replaced by $2p$ which is the known case. ■

Let ε^* and M_{ε^*} be given by Lemma 11.1 and let P^* be the solution of (11.3) with $\varepsilon = \varepsilon^*$. The scheduled high gain parameter is given by:

$$\rho_f(x) = \begin{cases} \rho_0(x) = \frac{1}{\|B' P_{\varepsilon(x)} B\|}, & x' P_\varepsilon(x) x \le c \\ \frac{8\rho_1(x)}{\varepsilon(x)\lambda_{\min} P_{\varepsilon(x)}}, & \text{otherwise} \end{cases} \tag{11.11}$$

with

$$\rho_1(x) = \frac{\lambda_{\max} P_{\varepsilon(x)}}{\lambda_{\min} P_{\varepsilon_1(x)}} \rho_2(x) \tag{11.12}$$

where

$$\rho_2(x) = \begin{cases} 1 & p = \infty \\ \left[\dfrac{\rho_p \beta(\varepsilon(x))}{1 - \left(1 - \frac{\varepsilon_1(x)}{4(1+L_{\varepsilon_1(x)})}\right)^{p/2}} + 1 \right]^{2/p}, & p \in [1, \infty) \end{cases}$$

where ρ_p is a positive constant to be determined later and c, $\varepsilon_1(x)$ and L_s are given by

$$\begin{aligned} c &= \Delta^2 \max\{4M_{\varepsilon^*} b, 4(1 + \|B' P^* B\|)\}, \\ \varepsilon_1(x) &= \max\{r \in (0, \varepsilon^*] \mid 2x' P_r x \,\text{trace}(P_r) \leq \tfrac{\Delta^2}{b}\}, \\ L_s &= \tfrac{\text{trace}(P^*)}{\lambda_{\min} P_s}. \end{aligned} \tag{11.13}$$

Finally, in order to define $\beta(\varepsilon) > 1$ we first define $\eta(\varepsilon)$ satisfying

$$\left[1 - \tfrac{\varepsilon}{4(1+L_\varepsilon)}\right]^{p/2} \leq (1 + \eta(\varepsilon)) \left[1 - \tfrac{\varepsilon}{2(1+L_\varepsilon)}\right]^{p/2} < 1.$$

Next we choose $\beta(\varepsilon) > 1$ such that Lemma 11.3 holds for $\eta = \eta(\varepsilon)$. In other words, $\beta(\varepsilon)$ is such that for a given $p > 1/2$, ε and $\eta(\varepsilon)$

$$(u + v)^p \leq (1 + \eta(\varepsilon)) u^p + \beta(\varepsilon) v^p$$

for all $u > 0, v > 0$.

11.4. Main results

In this section, we shall solve the simultaneous external and internal stabilization problems as formulated in Section 11.2 using the proposed low-and-high-gain controller in Section 11.3. We first study the simultaneous stabilization without finite gain as formulated in Problemds 11.1 and 11.3. Then we will solve Problems 11.2 and 11.4.

The theorem given below solves the global ℓ_p stabilization with arbitrary initial condition and without finite gain as formulated in Problem 11.3.

Theorem 11.1 Consider the system (11.1) satisfying Assumption 11.2. For any $p \in [1, \infty]$, the ℓ_p stabilization with arbitrary initial conditions and without finite gain as formulated in Problem 11.3 can be solved by the adaptive-low-gain and high-gain controller,

$$u = -(1 + \rho_0(x))(I + B' P_{\varepsilon(x)} B)^{-1} B' P_{\varepsilon(x)} A x, \tag{11.14}$$

where $P_{\varepsilon(x)}$ is the solution of (11.3), $\varepsilon(x)$ is determined adaptively by the scheduling (11.8) and $\rho_0(x)$ is determined by (11.9).

Theorem 11.1 immediately yields the following result.

Corollary 11.1 Consider the system (11.1) satisfying Assumption 11.2. For any $p \in [1, \infty]$, the (G_P/G) as formulated in Problem 11.1 can be solved by the same adaptive-low-gain and high-gain controller (11.14).

Proof of Theorem 11.1 : In this proof, we denote $\varepsilon(x(k))$, $\rho_0(x(k))$, and $P_{\varepsilon(x(k))}$ by $\varepsilon(k)$, $\rho_0(k)$, and $P(k)$ respectively. This abbreviation should not cause any notational confusions.

Define $v(k) = -(I + B'P(k)B)^{-1} B'P(k)Ax(k)$, $u(k) = v(k) + \rho_0(k)v(k)$ and $\mu(k) = \|B'P(k)B\|$. We have shown that (11.8) implies that $\|v(k)\|_\infty < \Delta$.

We proceed now to show global asymptotic stability. In the absence of d, we can evaluate the increment of $V(k)$ along the trajectory as:

$$\begin{aligned}
&V(k+1) - V(k)\\
=&x(k+1)'[P(k+1) - P(k)]x(k+1) - \varepsilon(k)V(k) - \|\sigma(u(k))\|^2\\
&+ [\sigma(u(k)) - v(k)]'(I + B'P(k)B)[\sigma(u(k)) - v(k)]\\
\leq&x(k+1)'[P(k+1) - P(k)]x(k+1) - \varepsilon(k)V(k) - \|\sigma(u(k))\|^2\\
&+ (1 + \mu(k))[\sigma(u(k)) - v(k)]'[\sigma(u(k)) - v(k)]\\
=&x(k+1)'[P(k+1) - P(k)]x(k+1) - \varepsilon(k)V(k)\\
&- \tfrac{1+\mu(k)}{\mu(k)}\|v(k)\|^2 + \mu(k)\|\sigma(u(k)) - \tfrac{1+\mu(k)}{\mu(k)}v(k)\|^2.
\end{aligned}$$

As noted before, $\|v(k)\| \leq \Delta$ for all $k \geq 0$, and therefore

$$\|v(k)\| \leq \|\sigma(u(k))\| \leq (1 + \tfrac{1}{\mu(k)})\|v(k)\|.$$

This implies that

$$\|\sigma(u(k)) - \tfrac{1+\mu(k)}{\mu(k)} v(k)\| \leq \tfrac{1}{\mu(k)} \|v(k)\|,$$

and thus,

$$\mu(k)\|\sigma(u(k)) - \tfrac{1+\mu(k)}{\mu(k)} v(k)\|^2 - \tfrac{1+\mu(k)}{\mu(k)} \|v(k)\|^2 \leq -\|v(k)\|^2.$$

Finally, we get

$$V(k+1) - V(k) \leq -\varepsilon(k)V(k) + x(k+1)'[P(k+1) - P(k)]x(k+1). \tag{11.15}$$

Our scheduling (11.8) implies that $V(k+1) - V(k)$ and $x(k+1)'[P(k+1) - P(k)]x(k+1)$ cannot have the same sign. To see this, assume that $V(k+1) > V(k)$ and $P(k+1) > P(k)$. This implies that

$$\varepsilon(k) < \varepsilon^*.$$

If $V(k)\,\mathrm{trace}(P(k)) < \frac{\Delta^2}{b}$, then (11.8) implies that $\varepsilon(k) = \varepsilon^*$, which yields a contradiction. If $V(k)\,\mathrm{trace}(P(k)) = \frac{\Delta^2}{b}$, then $V(k+1)\,\mathrm{trace}(P(k+1)) > \frac{\Delta^2}{b}$ since by assumption $V(k+1) > V(k)$ and $P(k+1) > P(k)$. But this is impossible by our scheduling (11.8). A similar argument can be used to establish that $V(k+1) - V(k) < 0$ and $P(k+1) - P(k) < 0$ cannot happen simultaneously either.

Using this property, (11.15) then implies that for all $x \neq 0$,

$$V(k+1) - V(k) < 0.$$

This concludes the global asymptotic stability.

What remains is to show ℓ_p stability. Similar to our earlier development, we have

$$\begin{aligned} &V(k+1) - V(k) \\ \leq &- x(k+1)'[P(k) - P(k+1)]x(k+1) - \varepsilon(k)V(k) - \tfrac{1}{\mu(k)}\|v(k)\|^2 \\ &+ \mu(k)\|\sigma(u(k) + d(k)) - \tfrac{1+\mu(k)}{\mu(k)} v(k)\|^2. \end{aligned}$$

Let $d_i(k)$, $v_i(k)$ and $u_i(k)$ denote the ith element of $d(k)$, $v(k)$ and $u(k)$ respectively.

If $|d_i(k)| \leq \frac{1}{\mu(k)}|v_i(k)|$, recalling that $|v_i(k)| \leq \Delta$, we have

$$|v_i(k)| \leq |\sigma_1(u_i(k) + d_i(k))| \leq (1 + \tfrac{2}{\mu(k)})|v_i(k)|.$$

Hence

$$\mu(k)|\sigma_1(u_i(k)+d_i(k)) - \tfrac{1+\mu(k)}{\mu(k)} v_i(k)|^2 - \tfrac{1}{\mu(k)}|v_i(k)|^2 \leq 0.$$

If $|d_i(k)| \geq \frac{1}{\mu(k)}|v_i(k)|$, we have

$$\begin{aligned}
&- \tfrac{1}{\mu(k)}|v_i(k)|^2 + \mu(k)|\sigma_1(u_i(k)+d_i(k)) - \tfrac{1+\mu(k)}{\mu(k)} v_i(k)|^2 \\
&\leq \mu(k)\,[(1+\mu(k))d_i(k) + d_i(k) + (1+\mu(k))d_i(k)]^2 + \mu(k)|d_i(k)|^2 \\
&\leq a\mu(k)|d_i(k)|^2,
\end{aligned}$$

where $a = (2\mu^* + 3)^2 + 1$, $\mu^* = \|B'P^*B\|$ and P^* is the solution of (11.3) with $\varepsilon = \varepsilon^*$. Therefore, we conclude that

$$V(k+1) - V(k) \leq x(k+1)'[P(k+1) - P(k)]x(k+1) - \varepsilon(k)V(k) + a\mu(k)\|d(k)\|^2. \quad (11.16)$$

Note that this implies that

$$V(k+1) - V(k) \leq \max\{-\varepsilon(k)V(k) + a\mu(k)\|d(k)\|^2, 0\} \quad (11.17)$$

since $V(k+1) - V(k)$ and $x(k+1)'[P(k+1) - P(k)]x(k+1)$ can not have the same sign. Let us first address the case of $p = \infty$. We will show that there exists a c_1 such that $V(k) \leq c_1$ for all $k \geq 0$ with $V(0) = 0$.

If

$$V(k) \geq a\tfrac{\mu(k)}{\varepsilon(k)}\|d(k)\|^2, \quad (11.18)$$

we have

$$V(k+1) - V(k) \leq 0. \quad (11.19)$$

Property 5 of Lemma 11.1 yields that there exists a M_{ε^*} independent of k and d such that $V(k) \geq abM_{\varepsilon^*}\|d\|_\infty^2$ implies that (11.18) is satisfied and therefore $V(k+1) - V(k) \leq 0$, where, as defined earlier, $b = 2\,\mathrm{trace}(BB')$.

On the other hand, we have

$$V(k+1) - V(k) \leq a\mu(k)\|d(k)\|^2 \leq a\mu^*\|d\|_\infty^2.$$

We conclude that

$$V(k) \leq V(0) + abM_{\varepsilon^*}\|d\|_\infty^2 + a\mu^*\|d\|_\infty^2. \tag{11.20}$$

Property 5 of our scheduling then implies that $x(k)$ is bounded for all $k \geq 0$. This shows ℓ_∞ stability of the closed-loop system with arbitrary initial condition.

We proceed now with the case of $p \in [1, \infty)$. First of all, due to the fact that $\|d\|_\infty \leq \|d\|_p$, (11.20) implies that $V(k)$ is bounded for all $k \geq 0$. Hence by our scheduling, there exists an ε_0 such that $\varepsilon(k) \geq \varepsilon_0$ for all $k \geq 0$.

Next, we consider two possible cases:

Case 1. For $V(k+1) - V(k) \geq 0$, (11.17) implies that

$$V(k+1) - V(k) \leq -\varepsilon(k)V(k) + a\mu(k)\|d(k)\|^2. \tag{11.21}$$

Case 2. For $V(k+1) - V(k) \leq 0$, our scheduling implies that

$$x(k+1)'[P(k+1) - P(k)]x(k+1) \geq 0.$$

But this implies that $\varepsilon(k) \leq \varepsilon(k+1) \leq \varepsilon^*$, and thus

$$V(k+1)\,\mathrm{trace}(P(k+1)) \leq V(k)\,\mathrm{trace}(P(k)).$$

Hence

$$[V(k+1) - V(k)]\,\mathrm{trace}(P(k+1)) \leq -V(k)\,\mathrm{trace}[P(k+1) - P(k)].$$

Then we have

$$\begin{aligned}
&|x(k+1)'[P(k+1) - P(k)]x(k+1)| \\
\leq& |\,\mathrm{trace}(P(k+1) - P(k))| \cdot \|x(k+1)\|^2 \\
\leq& \frac{\mathrm{trace}(P(k+1))}{V(k)} \cdot |V(k+1) - V(k)| \cdot \|x(k+1)\|^2 \\
\leq& \frac{V(k+1)\,\mathrm{trace}(P(k+1))}{V(k)\lambda_{\min}P(k+1)} \cdot |V(k+1) - V(k)| \\
\leq& \frac{\mathrm{trace}(P^*)}{\lambda_{\min}P(k)} \cdot |V(k+1) - V(k)| \\
\leq& L(k) \cdot |V(k+1) - V(k)|
\end{aligned}$$

where $L(k) = \frac{\text{trace}(P^*)}{\lambda_{\min}(P(k))}$. We have

$$V(k+1) - V(k) \leq \tfrac{-\varepsilon(k)}{1+L(k)} V(k) + a\mu(k)\|d(k)\|^2. \tag{11.22}$$

Given $\varepsilon(k) \in [\varepsilon_0, \varepsilon^*]$ for all $k \geq 0$, (11.21) in case 1 and (11.22) in case 2 ensure that

$$V(k+1) - V(k) \leq -\tfrac{\varepsilon_0}{1+L} V(k) + a\mu^*\|d(k)\|^2, \tag{11.23}$$

where $L = \frac{\text{trace}(P^*)}{\lambda_{\min} P_0}$ and P_0 is the solution of (11.3) with $\varepsilon = \varepsilon_0$. Also, $\varepsilon_0 < 1$ implies that $\varepsilon_0/(1+L) < 1$.

Applying Lemma 11.3 with η such that

$$(1+\eta)(1 - \tfrac{\varepsilon_0}{1+L})^{p/2} < 1,$$

we find that there exists a β such that

$$V(k+1)^{p/2} \leq (1+\eta)(1 - \tfrac{\varepsilon_0}{1+L})^{p/2} V(k)^{p/2} + \beta(a\mu^*)^{p/2}\|d(k)\|^p.$$

This yields

$$\left[1 - (1+\eta)(1 - \tfrac{\varepsilon_0}{1+L})^{p/2}\right] \sum_{k=0}^{\infty} V(k)^{p/2} \leq \beta(a\mu^*)^{p/2}\|d\|_p^p + V(0)^{p/2}.$$

Since $\varepsilon(k) \geq \varepsilon_0$ for all k,

$$\|x\|_p^p \leq \sum_{k=0}^{\infty} \frac{V(k)^{p/2}}{(\lambda_{\min} P_0)^{p/2}} \leq \frac{\beta(a\mu^*)^{p/2}}{(\lambda_{\min} P_0)^{p/2}\left[1-(1+\eta)(1-\frac{\varepsilon_0}{1+L})^{p/2}\right]}\|d\|_p^p + \frac{V(0)^{p/2}}{(\lambda_{\min} P_0)^{p/2}\left[1-(1+\eta)(1-\frac{\varepsilon_0}{1+L})^{p/2}\right]}, \tag{11.24}$$

we conclude that $d \in \ell_p$ implies that $x \in \ell_p$ for any $x(0) \in \mathbb{R}^n$. This concludes the proof of Theorem 11.1. ■

We observe from (11.20) and (11.24) that as $\|d\|_p$ and $x(0)$ become larger, the ε_0 becomes smaller and the ℓ_p gain becomes larger. In order to pursue finite gain ℓ_p stabilization, it is necessary to modify the high gain parameter. We first consider the case $p = \infty$.

Theorem 11.2 Consider the system (11.1) satisfying Assumption 11.2. For $p = \infty$, ℓ_p stabilization with arbitrary initial condition with finite gain and with bias, as formulated in Problem 11.4, can be achieved by the adaptive-low-gain and high-gain controller,

$$u = -(1 + \rho_f(x))(I + B' P_{\varepsilon(x)} B)^{-1} B' P_{\varepsilon(x)} A x, \tag{11.25}$$

where $P_{\varepsilon(x)}$ is the solution of (11.3) with $\varepsilon = \varepsilon(x)$, $\varepsilon(x)$ is determined adaptively by (11.8) and $\rho_f(x)$ is determined by (11.11) and (11.12).

Theorem 11.2 readily yields the following corollary:

Corollary 11.2 Consider the system (11.1) satisfying Assumption 11.2. For $p = \infty$, the $(G_p/G)_{fg}$ as formulated in Problem 11.2 can be solved by the same adaptive-low-gain and high-gain controller as (11.25).

Proof of Theorem 11.2 : For simplicity, we denote $P_{\varepsilon(x(k))}$, $P_{\varepsilon_1(x(k))}$ respectively by $P(k)$ and $P_1(k)$ whenever this does not cause any notational confusions.

Define $v(k) = -(I + B'P(k)B)^{-1}B'P(k)Ax(k)$ and $u(k) = v(k) + \rho_f(k)v(k)$. We have already shown that the controller (11.14) along with (11.8) satisfies $\|v\|_\infty < \Delta$.

Define the Lyapunov function $V(k) = x(k)'P(k)x(k)$ and a set $\mathcal{V}(c) = \{V(x) \le c\}$ where c is given by (11.13). Owing to Property 5 of Lemma 11.1, it is easy to verify that for $x(k) \in \mathcal{V}(c)^c$, the following inequality holds:

$$\varepsilon(k)V(k) \ge 4\varepsilon(k)M_{\varepsilon^*}b\Delta^2 \ge 8\|B'P(k)B\|\Delta^2. \tag{11.26}$$

In the absence of d, we can evaluate the increment of V along the trajectory as

$$\begin{aligned}
&V(k+1) - V(k) \\
=&x(k+1)'\left[P(k+1) - P(k)\right]x(k+1) - \varepsilon(k)V(k) - 2v(k)'[\sigma(u(k)) - v(k)] \\
&+ \left[\sigma(u(k)) - v(k)\right]' B'P(k)B \left[\sigma(u(k)) - v(k)\right].
\end{aligned}$$

Also, $\|v(k)\| \le \Delta$ implies that $-2v(k)'[\sigma(u(k)) - v(k)] \le 0$ for any $\rho(k) > 0$. Using this property, we find that for $x(k) \in \mathcal{V}(c)^c$,

$$\begin{aligned}
&V(k+1) - V(k) \\
\le&x(k+1)'\left[P(k+1) - P(k)\right]x(k+1) - \varepsilon(k)V(k) - 2v(k)'[\sigma(u(k)) - v(k)] + 4\|B'P(k)B\|\Delta^2 \\
\le&x(k+1)'\left[P(k+1) - P(k)\right]x(k+1) - \tfrac{\varepsilon(k)}{2}V(k).
\end{aligned}$$

The last inequality is owing to (11.26). If

$$x(k+1)' [P(k+1) - P(k)] x(k+1) < 0, \tag{11.27}$$

the last inequality implies that $V(k+1) - V(k) < 0$. But we have argued earlier that (11.27) and $V(k+1) - V(k) < 0$ cannot happen simultaneously by our scheduling (11.8). Therefore $x(k+1)' [P(k+1) - P(k)] x(k+1) \geq 0$. From the proof of Theorem 11.1,

$$x(k+1)' [P(k+1) - P(k)] x(k+1) \leq L(k)[V(k) - V(k+1)].$$

Hence, for $x(k) \in \mathcal{V}(c)^c$,

$$V(k+1) - V(k) < -\tfrac{\varepsilon(k)}{2(1+L(k))} V(k).$$

The trajectory will enter $\mathcal{V}(c)$ within finite time. However, for $x(k) \in \mathcal{V}(c)$, we have already proved in the proof of Theorem 11.1 that

$$V(k+1) - V(k) < 0$$

since in $\mathcal{V}(c)$, $\rho(k) = \rho_0(k) = \frac{1}{\|B'P(k)B\|}$. This proves global asymptotic stability of the origin.

We proceed to show ℓ_∞ stability with arbitrary initial conditions with finite gain with bias. In order to do so, we first find an upper bound of $\frac{V(k)}{\lambda_{\min} P(k)}$ in terms of $\|d\|_\infty$ and then conclude ℓ_∞ stability by observing that $\|x\|_\infty \leq \sqrt{\|\frac{V}{\lambda_{\min} P}\|_\infty}$. To this end, we note that the case $V(k+1) - V(k) \leq 0$ is not interesting since it is equivalent with

$$\tfrac{V(k+1)}{\lambda_{\min} P(k+1)} - \tfrac{V(k)}{\lambda_{\min} P(k)} \leq 0$$

due to the fact that $V(k+1) \leq V(k)$ implies $\lambda_{\min} P(k+1) \geq \lambda_{\min} P(k)$. Therefore, it will not affect the upper bound of $\frac{V(k)}{\lambda_{\min} P(k)}$. In view of this, throughout the remainder of the proof, we only consider $V(k+1) - V(k) > 0$.

Suppose $V(k+1) - V(k) > 0$, scheduling (11.8) implies that $x(k+1)' [P(k+1) - P(k)] x(k+1) \leq 0$. By construction, $\|v(k)\| \leq \Delta$. We get

$$\begin{aligned}
&V(k+1) - V(k) \\
\leq& -\varepsilon(k)V(k) - 2v(k)'[\sigma(u(k) + d(k)) - v(k)] + 4\|B'P^*B\|\Delta^2 \\
\leq& 4(1 + \|B'P^*B\|)\Delta^2.
\end{aligned}$$

Since $c > 4(1 + \|B'P^*B\|)\Delta^2$, we have

$$V(k+1) - V(k) \leq c. \tag{11.28}$$

The above inequality holds for any $x(k) \in \mathbb{R}^n$. Since different high-gains are applied in different regions, we have two possible cases:

Case 1: $x(k) \in \mathcal{V}(c)^c$. Then (11.28) implies that $V(k+1) \leq 2V(k)$. But this implies that $\varepsilon_1(k) \leq \varepsilon(k+1)$ and $P_1(k) \leq P(k+1)$. Let $v_i(k)$ and $d_i(k)$ denote the ith element of $v(k)$ and $d(k)$.

If $|d_i(k)| < \rho_f(k)|v_i(k)|$, then

$$-v_i(k)\left[\sigma(v_i(k) + \rho_f(k)v_i(k) + d_i(k)) - v_i(k)\right] \leq 0.$$

If $|d_i(k)| \leq |\rho_f(k)v_i(k)|$, we have

$$\begin{aligned}
&- v_i(k)\left[\sigma(v_i(k) + \rho_f(k)v_i(k) + d_i(k)) - v_i(k)\right] \\
=&- v_i(k)\left[\sigma(v_i(k) + \rho_f(k)v_i(k) + d_i(k)) - \sigma(v_i(k))\right] \\
\leq& \tfrac{|d_i(k)|}{\rho_f(k)} \cdot |2d_i(k)| \\
=& \tfrac{2d_i(k)^2}{\rho_f(k)}.
\end{aligned}$$

In summary, we find that

$$-2v(k)'[\sigma(u(k) + d(k)) - v(k)] \leq \tfrac{4\|d(k)\|^2}{\rho_f(k)}.$$

This yields

$$\begin{aligned}
&V(k+1) - V(k) \\
\leq& - \tfrac{\varepsilon(k)}{2} V(k) - 2v(k)'[\sigma(u(k) + d(k)) - v(k)] \\
\leq& - \tfrac{\varepsilon(k)}{2} V(k) + 4\tfrac{\|d(k)\|^2}{\rho_f(k)} \\
\leq& - \tfrac{\varepsilon(k)\lambda_{\min} P(k)}{2} (\|x(k)\|^2 - \tfrac{\|d(k)\|^2}{\rho_1(k)}).
\end{aligned}$$

Clearly, $V(k+1) - V(k) \geq 0$ requires that

$$\|x(k)\|^2 \leq \tfrac{\|d(k)\|^2}{\rho_1(k)}.$$

Then

$$\frac{V(k+1)}{\lambda_{\min}P(k+1)} \le \frac{2V(k)}{\lambda_{\min}P_1(k)} \le \frac{2\lambda_{\max}P(k)}{\lambda_{\min}P_1(k)}\|x(k)\|^2 = 2\rho_1\|x(k)\|^2 \le 2\|d(k)\|^2. \tag{11.29}$$

Case 2: $x(k) \in \mathcal{V}(c)$. We have $\rho(k) = \rho_0(k)$ and hence the same controller as in Theorem 11.1 is used. In the proof of Theorem 11.1, the following two properties have already been shown:

1. if $V(k) \ge abM_{\varepsilon^*}\|d(k)\|^2$, we have $V(k+1) - V(k) \le 0$;

2. $V(k+1) - V(k) \le a\mu^*\|d(k)\|^2$.

We can immediately draw the conclusion that for $V(k+1) - V(k) > 0$ and $x(k) \in \mathcal{V}(c)$,

$$V(k+1) \le (abM_{\varepsilon^*} + a\mu^*)\|d(k)\|^2.$$

On the other hand, (11.28) and the fact $V(k) \le c$ imply that $V(k+1) \le 2c$. But this implies that there exists a λ_1 independent of d such that

$$\frac{V(k+1)}{\lambda_{\min}P(k+1)} \le \frac{abM_{\varepsilon^*}+a\mu^*}{\lambda_1}\|d\|_\infty^2. \tag{11.30}$$

In summary, whenever $V(k)$ or, equivalently, $\frac{V(k)}{\lambda_{\min}P(k)}$ is increasing, we have either (11.30) or (11.29) holds depending on $x(k) \in \mathcal{V}(c)$ or not. Therefore,

$$\|\frac{V}{\lambda_{\min}P}\|_\infty \le \frac{V(0)}{\lambda_{\min}P(0)} + \max\{2, \frac{abM_{\varepsilon^*}+a\mu^*}{\lambda_1}\}\|d\|_\infty.$$

Using the fact that $\|x\|_\infty^2 \le \|\frac{V}{\lambda_{\min}P}\|_\infty$, we have

$$\|x\|_\infty \le \sqrt{\|\frac{V}{\lambda_{\min}P}\|_\infty} \le \sqrt{\frac{V(0)}{\lambda_{\min}P(0)}} + \max\{\sqrt{2}, \sqrt{\frac{abM_{\varepsilon^*}+a\mu^*}{\lambda_1}}\}\|d\|_\infty. \tag{11.31}$$

Note that $\sqrt{\frac{V(0)}{\lambda_{\min}P(0)}}$ is clearly a class $\mathcal{K}$ function of $\|x(0)\|$. The finite gain ℓ_∞ stability of closed-loop system with arbitrary initial condition and bias follows. ■

In Theorem 11.2, we only need to consider the case that $V(x(k))$ is increasing. However, this does not work when the external input d is in ℓ_p with $p \in [1,\infty)$. The decay rate of $V(x(k))$ when $V(x(k))$ is decreasing definitely has an impact on the ℓ_p norm of x. Therefore, we have to consider both cases and obtain bounds on $\|x\|_p$ in terms of $\|d\|_p$. As will be seen in the next theorem, it requires even more complicated high-gain design and involved analysis.

Theorem 11.3 Consider the system (11.1) satisfying Assumption 11.2. For any $p \in [1, \infty)$, the ℓ_p stabilization with arbitrary initial condition with finite gain with bias problem as formulated in Problem 11.4 can be solved by the adaptive-low-gain and high-gain controller,

$$u = -(1 + \rho_f(x))(I + B' P_{\varepsilon(x)} B)^{-1} B' P_{\varepsilon(x)} A x, \tag{11.32}$$

where $P_{\varepsilon(x)}$ is the solution of (11.3) with $\varepsilon = \varepsilon(x)$, $\varepsilon(x)$ is determined adaptively by (11.8) and $\rho_f(x)$ is determined by (11.11), (11.12) with ρ_p sufficiently large.

Theorem 11.3 also produces as a special case the solution to $(G_p/G)_{fg}$. This is stated in the following corollary.

Corollary 11.3 Consider the system (11.1) satisfying Assumption 11.2. For any $p \in [1, \infty)$, the $(G_p/G)_{fg}$ as formulated in Problem 11.2 can be solved by the adaptive-low-gain and high-gain controller (11.32).

Proof of Theorem 11.3 : For simplicity, we denote $\varepsilon(x(k))$, $\varepsilon_1(x(k))$, $\beta(\varepsilon(x(k)))$, $\rho_f(x(k))$ and $\rho_1(x(k))$ by $\varepsilon(k)$, $\varepsilon_1(k)$, $\beta(k)$, $\rho_f(k)$ and $\rho_1(k)$ respectively and denote $P_{\varepsilon(x(k))}$, $P_{\varepsilon_1(x(k))}$, $L_{\varepsilon_1(x(k))}$ respectively by $P(k)$, $P_1(k)$ and $L_1(k)$. This does not cause any notational confusions.

Define $v(k) = -(I + B'P(k)B)^{-1}B'P(k)Ax(k)$ and $u(k) = v(k) + \rho_f(k)v(k)$. We have already shown that $v(k)$ along with (11.8) satisfies $\|v\|_\infty < \Delta$.

Define the Lyapunov function $V(k) = x(k)'P(k)x(k)$ and a set $\mathcal{V}(c) = \{x \mid V(x) \leq c\}$ with c given by (11.13). As in the proof of Theorem 11.2, for $x \in \mathcal{V}(c)^c$, the following inequality holds:

$$\varepsilon(k)V(k) \geq 4\varepsilon(k)M_{\varepsilon^*}b\Delta^2 \geq 8\|B'P(k)B\|\Delta^2. \tag{11.33}$$

Using exactly the same argument as used in Theorem 11.2, we conclude the global asymptotic stability of the origin of the closed-loop system.

It remains to prove global ℓ_p stability with finite gain. The proof proceeds in several steps:

Step 1. Define a function

$$\alpha(s) = \frac{s^{p/2}}{(\lambda_{\min} P_s)^{p/2}\left[1 - \left(1 - \frac{\varepsilon_s}{4(1+L_s)}\right)^{p/2}\right]},$$

where ε_s is a function of s as given by

$$\varepsilon_s = \max\{r \in [0, \varepsilon^*] \mid s\,\mathrm{trace}(P_r) \le \tfrac{\Delta^2}{b}\},$$

and P_s is the solution of (11.3) with $\varepsilon = \varepsilon_s$, $L_s = \frac{\mathrm{trace}(P^*)}{\lambda_{\min} P_s}$. Note that if s is strictly increasing, by the property of our scheduling, ε_s is decreasing and hence $\lambda_{\min} P_s$ is decreasing and L_s is increasing. This implies that $\alpha(s)$ is strictly increasing and is a class $\mathcal{K}$ function.

Define

$$\kappa = \frac{(\lambda_{\min} P^*)^{p/2}\left[1-\left(1-\frac{\varepsilon^*}{4(1+L^*)}\right)^{p/2}\right]}{(\lambda_{\min} P_{2c})^{p/2}\left[1-\left(1-\frac{\varepsilon_{2c}}{4(1+L_{2c})}\right)^{p/2}\right]},$$

where P^* is the solution of (11.3) with $\varepsilon = \varepsilon^*$ and $L^* = \frac{\mathrm{trace}(P^*)}{\lambda_{\min} P^*}$. Since c is given, ε_{2c}, P_{2c}, L_{2c} and κ are fixed constants. Choose $\rho_p > \max\{1+\kappa, (\lambda_{\min} P^*)^{p/2}\}$. We have $\rho_f(k) \ge 1$ for any $x(k)$.

We can always divide the whole time horizon into a sequence of successive intervals $\{I_i\}_{i\ge 1}$ with $I_i = \overline{[k_i, k_{i+1}-1]}$ such that for each I_i, one of the following cases holds:

1. For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)^c$ and $V(k+1) - V(k) > 0$;

2. For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)^c$ and $V(k+1) - V(k) \le 0$;

3. For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)$ with $k_{i+1} < \infty$;

4. For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)$ with $k_{i+1} = \infty$.

Step 2. For case 1, since $V(k+1)-V(k) > 0$, the adaptation (11.8) implies that $x(k+1)'\,[P(k+1) - P(k)]\,x(k+1) \le 0$. As in the proof of Theorem 11.2, we find

$$V(k+1) - V(k) \le -\frac{\varepsilon(k)\lambda_{\min} P(k)}{2}\left[\|x(k)\|^2 - \frac{\|d(k)\|^2}{\rho_1(k)}\right].$$

Then, $V(k+1) - V(k) > 0$ implies that

$$\|d(k)\|^2 \ge \rho_1(k)\|x(k)\|^2 \ge \|x(k)\|^2 \tag{11.34}$$

since $\rho_1(k) \ge 1$ by construction.

Furthermore, we have already shown that for all $x(k)$, $V(k+1) - V(k) \le c$. Hence

$$V(k+1) \le 2V(k).$$

From the definition of $\varepsilon_1(k)$ and $L_1(k)$, this implies that

$$\varepsilon_1(k) \leq \varepsilon(k+1),\ L_1(k) \geq L(k+1) \text{ and } \lambda_{\min} P_1(k) \leq \lambda_{\min} P(k+1). \tag{11.35}$$

Consider specifically $k = k_{i+1} - 1$. We have

$$\begin{aligned}
&\|d(k_{i+1}-1)\|^p - \|x(k_{i+1}-1)\|^p \\
\geq &\left(\rho_1(k_{i+1}-1)^{p/2} - 1\right)\|x(k_{i+1}-1)\|^p \\
\geq &\frac{\rho_p \lambda_{\max} P(k_{i+1}-1)^{p/2} \|x(k_{i+1}-1)\|^p}{\lambda_{\min} P_1(k_{i+1}-1)^{p/2}\left[1-\left(1-\frac{\varepsilon_1(k_{i+1}-1)}{4(1+L_1(k_{i+1}-1))}\right)^{p/2}\right]} \\
\geq &\frac{\rho_p V(k_{i+1}-1)^{p/2}}{\lambda_{\min} P_1(k_{i+1}-1)^{p/2}\left[1-\left(1-\frac{\varepsilon(k_{i+1}-1)}{4(1+L(k_{i+1}-1))}\right)^{p/2}\right]} \\
\geq &\frac{(1+\kappa) V(k_{i+1})^{p/2}}{\lambda_{\min} P(k_{i+1})^{p/2}\left[1-\left(1-\frac{\varepsilon(k_{i+1})}{4(1+L(k_{i+1}))}\right)^{p/2}\right]}
\end{aligned}$$

where we use (11.35), $\rho_p > 1 + \kappa$ and $V(k_{i+1} - 1) > V(k_{i+1})$ in the derivation of the last inequality. We get

$$\|d(k_{i+1}-1)\|^p \geq \|x(k_{i+1}-1)\|^p + (1+\kappa)\alpha(V(k_{i+1})). \tag{11.36}$$

then (11.34) and (11.36) yield

$$\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p \leq \sum_{k=k_i}^{k_{i+1}-1} \|d(k)\|^p - (1+\kappa)\alpha(V(k_{i+1})).$$

Step 3. For case 2, the following relationship has been established in the proof of Theorem 11.1,

$$\begin{aligned}
0 &\leq x(k+1)'\left[P(k+1) - P(k))x(k+1)\right] \\
&\leq L(k)(V(k) - V(k+1))
\end{aligned}$$

where $L(k) = \frac{\text{trace}(P^*)}{\lambda_{\min} P(k)}$. Therefore,

$$\begin{aligned}
V(k+1) - V(k) &\leq -\frac{\varepsilon(k)}{2(1+L(k))} V(k) + \frac{\varepsilon(k)\lambda_{\min} P(k)}{\rho_1(k)(1+L(k))}\|d(k)\|^2 \\
&\leq -\frac{\varepsilon(k)}{2(1+L(k))} V(k) + \frac{\lambda_{\min} P(k)}{\rho_1(k)}\|d(k)\|^2,
\end{aligned}$$

and hence

$$V(k+1) \leq \left[1 - \frac{\varepsilon(k)}{2(1+L(k))}\right] V(k) + \frac{\lambda_{\min} P(k)}{\rho_1(k)}\|d(k)\|^2.$$

Since $V(k)$ is decreasing, we have $\lambda_{\min} P(k+1) \geq \lambda_{\min} P(k)$ and

$$\frac{V(k+1)}{\lambda_{\min} P(k+1)} \leq \left[1 - \frac{\varepsilon(k)}{2(1+L(k))}\right] \frac{V(k)}{\lambda_{\min} P(k)} + \frac{1}{\rho_1(k)} \|d(k)\|^2.$$

By definition of $\beta(k)$,

$$\left(\frac{V(k+1)}{\lambda_{\min} P(k+1)}\right)^{p/2} \leq \left[1 - \frac{\varepsilon(k)}{4(1+L(k))}\right]^{p/2} \left(\frac{V(k)}{\lambda_{\min} P(k)}\right)^{p/2} + \beta(k) \frac{\|d(k)\|^p}{\rho_1(k)^{p/2}}.$$

Using standard comparison principle, we get for $k \geq k_i$,

$$\begin{aligned}\left(\frac{V(k)}{\lambda_{\min} P(k)}\right)^{p/2} \leq \prod_{j=k_i}^{k} \left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2} \left(\frac{V(k_i)}{\lambda_{\min} P(k_i)}\right)^{p/2} \\ + \sum_{j=k_i}^{k-1} \left(\prod_{s=j}^{k-1} \left[1 - \frac{\varepsilon(s)}{4(1+L(s))}\right]^{p/2}\right) \frac{\beta(j)}{\rho_1(j)^{p/2}} \|d(j)\|^p.\end{aligned}$$

Since $V(k)$ is decreasing, $\left[1 - \frac{\varepsilon(k)}{4(1+L(k))}\right]^{p/2}$ is decreasing. Hence,

$$\begin{aligned}\left(\frac{V(k)}{\lambda_{\min} P(k)}\right)^{p/2} \leq \left\{\left[1 - \frac{\varepsilon(k_i)}{4(1+L(k_i))}\right]^{p/2}\right\}^{k-k_i} \left(\frac{V(k_i)}{\lambda_{\min} P(k_i)}\right)^{p/2} \\ + \sum_{j=k_i}^{k-1} \left\{\left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2}\right\}^{k-1-j} \frac{\beta(j)}{\rho_1(j)^{p/2}} \|d(j)\|^p.\end{aligned}$$

We have

$$\sum_{k=k_i}^{k_{i+1}-1} \left(\frac{V(k)}{\lambda_{\min} P(k)}\right)^{p/2} \leq \frac{1}{1 - \left[1 - \frac{\varepsilon(k_i)}{4(1+L(k_i))}\right]^{p/2}} \left(\frac{V(k_i)}{\lambda_{\min} P(k_i)}\right)^{p/2} + \sum_{j=k_i}^{k_{i+1}-2} \frac{\beta(j)}{1 - \left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2}} \frac{\|d(j)\|^p}{\rho_1(j)^{p/2}}.$$

By definition, for any $x(k)$

$$\varepsilon_1(k) \leq \varepsilon(k) \text{ and } L_1(k) \geq L(k),$$

and from (11.12)

$$\rho_1(j)^{p/2} \geq \frac{\beta(j)}{1 - \left[1 - \frac{\varepsilon_1(j)}{4(1+L_1(j))}\right]^{p/2}} \geq \frac{\beta(j)}{1 - \left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2}}.$$

We conclude that

$$\begin{aligned}\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p &\leq \sum_{k=k_i}^{k_{i+1}-2} \|d(j)\|^p + \alpha(V(k_i)) \\ &\leq \sum_{k=k_i}^{k_{i+1}-1} \|d(j)\|^p + \alpha(V(k_i)).\end{aligned}$$

Note that $\alpha(V(k_i))$ is increasing. Therefore $\alpha(V(k_i)) \geq \alpha(V(k_{i+1}))$. We can rewrite the above inequality as

$$\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p \leq \sum_{k=k_i}^{k_{i+1}-1} \|d(j)\|^p + (1+\kappa)\alpha(V(k_i)) - \kappa\alpha(V(k_{i+1})).$$

Step 4. For case 3 and 4, if $x(k) \in \mathcal{V}(c)$, from (11.21) and (11.22), we have

$$V(k+1) - V(k) \leq -\tfrac{\varepsilon(k)}{1+L(k)} V(k) + a\mu^* \|d(k)\|^2.$$

If $x(k) \in \mathcal{V}(c)^c \cap \mathcal{V}(2c)$ and $V(k+1) - V(k) > 0$, then $x(k+1)'[P(k+1) - P(k)]x(k+1) \leq 0$, we have

$$\begin{aligned} V(k+1) - V(k) \leq & -\tfrac{\varepsilon(k)}{2} V(k) \\ & \quad - 2v(k)'[\sigma(u(k) + d(k)) - v(k)] \\ \leq & -\tfrac{\varepsilon(k)}{2} V(k) + 4\tfrac{\|d(k)\|^2}{\rho_f(k)} \\ \leq & -\tfrac{\varepsilon(k)}{2} V(k) + 4\|d(k)\|^2. \end{aligned}$$

If $x(k) \in \mathcal{V}(c)^c \cap \mathcal{V}(2c)$ and $V(k+1) - V(k) \leq 0$, then $x(k+1)'[P(k+1) - P(k)]x(k+1) \leq L(k)(V(k) - V(k+1))$. We have

$$\begin{aligned} V(k+1) - V(k) &\leq -\tfrac{\varepsilon(k)}{2(1+L(k))} V(k) + 4\tfrac{\|d(k)\|^2}{\rho_f(k)(1+L(k))} \\ &\leq -\tfrac{\varepsilon(k)}{2(1+L(k))} V(k) + 4\|d(k)\|^2. \end{aligned}$$

Hence there exists a $\zeta = \max\{4, a\mu^*\}$ such that for all $x(k) \in \mathcal{V}(2c)$, we have

$$V(k+1) - V(k) \leq -\tfrac{\varepsilon(k)}{2(1+L(k))} V(k) + \zeta\|d(k)\|^2.$$

Note that our adaptation (11.8) and the fact that $V(x) \leq 2c$ for $k = k_i, \ldots, k_{i+1} - 1$ imply that for $k = k_i, \ldots, k_{i+1} - 1$, $\varepsilon(k) \geq \varepsilon_{2c}$ and hence

$$-\tfrac{\varepsilon(k)}{2(1+L(k))} \leq -\tfrac{\varepsilon_{2c}}{2(1+L_{2c})}, \quad \lambda_{\min} P(k) \geq \lambda_{\min} P_{2c}.$$

Choose η_{2c} such that

$$\left[1 - \tfrac{\varepsilon_{2c}}{4(1+L_{2c})}\right]^{p/2} \leq (1+\eta_{2c}) \left[1 - \tfrac{\varepsilon_{2c}}{2(1+L_{2c})}\right]^{p/2} < 1.$$

Applying Lemma 11.3, there exists a β_{2c} independent of d and k such that

$$V(k+1)^{p/2} \leq \left[1 - \frac{\varepsilon_{2c}}{4(1+L_{2c})}\right]^{p/2} V(k)^{p/2} + \beta_{2c}\zeta^{p/2}\|d(k)\|^p.$$

Using the same comparison principle as used in case 2, we can find a constant γ_1 dependent on ε_{2c}, L_{2c}, β_{2c} and ζ such that

$$\begin{aligned}
\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p &\leq \sum_{k=k_i}^{k_{i+1}-1} \frac{V(k)^{p/2}}{(\lambda_{\min} P_{2c})^{p/2}} \\
&\leq \gamma_1 \sum_{k=k_i}^{k_{i+1}-2} \|d(k)\|^p + \frac{V(k_i)^{p/2}}{(\lambda_{\min} P_{2c})^{p/2}\left\{1-\left[1-\frac{\varepsilon_{2c}}{4(1+L_{2c})}\right]^{p/2}\right\}} \\
&\leq \gamma_1 \sum_{k=k_i}^{k_{i+1}-2} \|d(k)\|^p + \frac{\kappa V(k_i)^{p/2}}{(\lambda_{\min} P(k_i))^{p/2}\left\{1-\left[1-\frac{\varepsilon(k_i)}{4(1+L(k_i))}\right]^{p/2}\right\}} \\
&\leq \gamma_1 \sum_{k=k_i}^{k_{i+1}-2} \|d(k)\|^p + \kappa\alpha(V(k_i)).
\end{aligned}$$

For case 3 where $k_{i+1} < \infty$, consider specifically $k = k_{i+1} - 1$. Since the states are leaving $\mathcal{V}(2c)$, we have $V(k_{i+1}) - V(k_{i+1} - 1) > 0$. Moreover, we have argued that the increment of $V(k)$ for any $x(k)$ is at most c. This implies that $x(k_{i+1} - 1) \in \mathcal{V}(c)^c \cap \mathcal{V}(2c)$. Following the same argument as used in case 1, we have

$$\|d(k_{i+1} - 1)\|^p \geq \|x(k_{i+1} - 1)\|^p + (1+\kappa)\alpha(V(k_{i+1})).$$

Finally, we conclude for $k \in \overline{[k_i, k_{i+1} - 1]}$,

$$\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p \leq \gamma_1 \sum_{k=k_i}^{k_{i+1}-1} \|d(k)\|^p + \kappa\alpha(V(k_i)) - (1+\kappa)\alpha(V(k_{i+1})).$$

For case 4 where $k_{i+1} = \infty$, we only have

$$\sum_{k=k_i}^{k_{i+1}} \|x(k)\|^p \leq \gamma_1 \sum_{k=k_i}^{k_{i+1}} \|d(k)\|^p + \kappa\alpha(V(k_i)).$$

Step 5. In summary of previous steps, we find the following results:

- if I_i belongs to case 1,

$$\sum_{k_i}^{k_{i+1}-1} \|x(k)\|^p \leq \sum_{k=k_i}^{k_{i+1}-1} \|d(k)\|^p - (1+\kappa)\alpha(V(k_{i+1})).$$

- if I_i belongs to case 2,

$$\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p \leq \sum_{k=k_i}^{k_{i+1}-1} \|d(j)\|^p + (1+\kappa)\alpha(V(k_i)) - \kappa\alpha(V(k_{i+1})).$$

- if I_i belongs to case 3,

$$\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p \leq \gamma_1 \sum_{k=k_i}^{k_{i+1}-1} \|d(k)\|^p + \kappa\alpha(V(k_i)) - (1+\kappa)\alpha(V(k_{i+1})).$$

- if I_i belongs to case 4,

$$\sum_{k=k_i}^{k_{i+1}} \|x(k)\|^p \leq \gamma_1 \sum_{k=k_i}^{k_{i+1}} \|d(k)\|^p + \kappa\alpha(V(k_i)).$$

Note that if I_i belongs to cases 1, 3 and 4, we have either $i = 1$ or I_{i-1} belongs to cases 1, 2 or 3. Then the positive term $\kappa\alpha(V(k_i))$ of I_i can always be canceled by the corresponding negative term of I_{i-1} for $i > 1$.

Similarly, if I_i belongs to case 2, we have either $i = 1$ or I_{i-1} belongs to case 1 or 3. The positive term $(1+\kappa)\alpha(V(k_i))$ can also be canceled by the negative term of I_{i-1} for $i > 1$.

In conclusion, we find that for any $x(0)$ and k,

$$\sum_{k=0}^{k} \|x(k)\|^p \leq \max\{1, \gamma_1\} \sum_{k=0}^{k} \|d(k)\|^p + (1+\kappa)\alpha(V(0)).$$

This completes the proof. ■

11.5. Conclusions

It is shown that (G_p/G) and $(G_p/G)_{fg}$ problems for discrete-time linear systems subject to actuator saturation are solvable if and only if the given linear system is stabilizable and it has all its poles within the unit disc, i.e. if it is ANCBC. We also develop here an adaptive-low-gain and high-gain controller design methodology by using a parametric Lyapunov equation. By utilizing the developed methodology, one can explicitly construct the required state feedback controllers that solve the (G_p/G) and $(G_p/G)_{fg}$ problems whenever they are solvable.

Appendix

We show in this section that for system (11.2) if a feedback controller of the form $u = B' f(x_u)$ achieves (G/G_p) and/or $(G/G_p)_{f.g}$ for the unstable dynamics x_u, it also achieves (G/G_p) and/or $(G/G_p)_{f.g}$ for the overall system.

Let us consider the unstable part of the input-additive case.

$$x_u^+ = Ax_u + B_u\sigma(u + d)$$

Assume we have a feedback $u = B_u' f(x_u)$ such that $x_u \in \ell_p$ and, if possible, with finite gain:

$$\|x_u\|_{\ell_p} \leq c_1\|d\|_{\ell_p}$$

Note that we impose a bit of special structure on the feedback. Namely $u = B_u' f(x_u)$ instead of $u = f(x_u)$ but all our standard controllers satisfy this property which is easily seen if we recall that:

$$-(I + B'P_\varepsilon B)^{-1}B'P_\varepsilon A = B'P_\varepsilon(I + BB'P_\varepsilon)^{-1}A$$

If we achieve (G/G_p) for the unstable dynamics then it is easily verified that we must have that

$$B_u\sigma(B_u' f(x_u) + d) \in \ell_p$$

while achieving $(G/G_p)_{f.g}$ for the unstable dynamics implies:

$$\|B_u\sigma(B_u' f(x_u) + d)\|_{\ell_p} \leq c_2\|d\|_{\ell_p} \tag{11.37}$$

Now in order to incorporate the stable dynamics we want to establish that:

$$\sigma(B_u' f(x_u) + d) \in \ell_p$$

and ideally with a finite gain:

$$\|\sigma(B_u' f(x_u) + d)\|_{\ell_p} \leq c_3\|d\|_{\ell_p}$$

This implies that for stable dynamics, we shall have

$$\|x_s\|_{\ell_p} \leq \gamma\|\sigma(B_u' f(x_u) + d)\|_{\ell_p} \leq c_3\gamma\|d\|_{\ell_p}$$

where γ is ℓ_p gain of the pair (A_s, B_s).

We first note that

$$B_u\sigma(B_u' f(x_u) + d) = B_u\sigma(B_u' f(x_u)) + B_u d_1$$

with $\|d_1\|_{\ell_p} \leq \|d\|_{\ell_p}$. But this implies that

$$\|B_u\sigma(B_u' f(x_u))\|_{\ell_p} \leq \|B_u\sigma(B_u' f(x_u) + d)\|_{\ell_p} + \|B_u\|\|d\|_{\ell_p}$$

In other words it is sufficient to prove that

$$\|\sigma(B_u' f(x_u))\|_{\ell_p} \leq c_4\|B_u\sigma(B_u' f(x_u))\|_{\ell_p} \tag{11.38}$$

to obtain that:

$$\begin{aligned}
\|\sigma(B_u' f(x_u) + d)\|_{\ell_p} &\leq \|\sigma(B_u' f(x_u))\|_{\ell_p} + \|d\|_{\ell_p} \\
&\leq c_4\|B_u\sigma(B_u' f(x_u))\|_{\ell_p} + \|d\|_{\ell_p} \\
&\leq c_4\|B_u\sigma(B_u' f(x_u) + d)\|_{\ell_p} \\
&\qquad + (1 + c_4\|B_u\|)\,\|d\|_{\ell_p} \\
&\leq (c_4c_2 + 1 + c_4\|B_u\|)\,\|d\|_{\ell_p}
\end{aligned}$$

where we used (11.37).

Remains to verify (11.38) which is implied by the following static inequality:

$$\|\sigma(B_u' v)\|_p \leq c_4\|B_u\sigma(B_u' v)\|_p \tag{11.39}$$

Since this is a static finite-dimensional problem and all finite-dimensional norms are equivalent, it suffices to prove (11.39) for $p = 2$.

Note that we can find a matrix S such that:

$$B_u = S\begin{pmatrix} B_{u1} \\ 0 \end{pmatrix}$$

with B_{u1} surjective. Next, we note that it is sufficient to prove that:

$$\|\sigma(B_{u1}' w)\|_2 \leq c_5\|B_{u1}\sigma(B_{u1}' w)\|_2 \tag{11.40}$$

for some suitably chosen c_5 since for $w = Sv$ we get:

$$\begin{aligned}
\|\sigma(B'_u v)\|_2 &\le c_5 \|B_{u1}\sigma(B'_u v)\|_2 \\
&\le \frac{c_5}{\sigma_{\min}(S)} \left\| S \begin{pmatrix} B_{u1} \\ 0 \end{pmatrix} \sigma(B'_u v) \right\|_2 \\
&\le \frac{c_5}{\sigma_{\min}(S)} \|B_u \sigma(B'_u v)\|_2
\end{aligned}$$

which yields (11.39) for suitable chosen c_4. Remains to show (11.40). We consider two cases. If $B'_{u1}w$ saturates at least one channel then

$$\begin{aligned}
\|B_{u1}\sigma(B'_{u1}w)\|_2 &\ge \langle B'_{u1}w_n, \sigma(B'_{u1}w) \rangle \\
&\ge \|B'_{u1}w_n\|_\infty \\
&\ge \frac{1}{\sqrt{m}} \sigma_{\min}(B'_{u1})
\end{aligned}$$

where $w_n = \frac{w}{\|w\|}$ is the normalized vector of w.

In that case:

$$\begin{aligned}
\|\sigma(B'_{u1}w)\|_2 &\le \sqrt{m}\, \|\sigma(B'_{u1}w)\|_\infty \\
&= \sqrt{m} \\
&\le \frac{m}{\sigma_{\min}(B'_{u1})} \|B_{u1}\sigma(B'_{u1}w)\|_2
\end{aligned}$$

On the other hand without saturation:

$$\begin{aligned}
\|B'_{u1}w\|_2 &\le \|B'_{u1}(B'_{u1}B'_{u1})^{-1} B_{u1} B'_{u1} w\|_2 \\
&\le \|B'_{u1}(B'_{u1}B'_{u1})^{-1}\|_2 \; \|B_{u1}B'_{u1}w\|_2
\end{aligned}$$

Combining the two cases with and without saturation yields (11.40) for suitable chosen c_5, i.e.

$$c_5 \ge \max\left\{ \frac{m}{\sigma_{\min}(B'_{u1})} \,,\; \|B'_{u1}(B'_{u1}B'_{u1})^{-1}\|_2 \right\}$$

CHAPTER 12

Simultaneous external and internal stabilization of linear system subject to input saturation and matched disturbances

12.1. Introduction

After we obtain the results in previous chapter, all the issues associated with input-additive disturbances have been more or less resolved. The research has moved to the non-input-additive case. In particular, we are concerned with non-input-additive disturbances which do not vanish over time, that is, belonging to $\mathcal{L}_\infty/\ell_\infty$ space. In this case, clearly not all disturbances can be managed appropriately as, for instance, a large constant disturbance aligned with the input could overpower the saturated control and lead to unbounded states.

This chapter marks our first attempt to solve the simultaneous stabilization problem for linear system with input saturation and non-input-additive disturbances. Specifically, we investigate the semi-global and global robust stabilization and disturbance rejection in the presence of input saturation and matched uncertainty and disturbance. That is, we consider a system of the form,

$$\dot{x} = Ax + B\sigma(u(t)) + Bf(x,t).$$

In this chapter, we assume the magnitude of $f(x,t)$ is restricted. This is motivated by a simple observation that the states of a double integrator with a constant disturbance $|f(x,t)| = 1$ will diverge to infinity regardless of the controller we use. Therefore, we assume that $|f(x,t)| < 1-\delta$, for any a priori given $\delta \in (0,1)$.

In the previous study of stabilization problem related to input-additive disturbances, a low-and-high-gain and scheduled low-and-high-gain feedback design methodologies have been developed in the literature (see [91, 48]). In this chapter, we expand and generalize these low-and-high gain design and scheduled low-and-high gain design methodologies from the input-additive case to the matched case. This chapter is organized as follows: In Section 12.2, we formulate formally the control problems. A low-and-high gain design is introduced in Section 12.3 which solves the semi-global robust stabilization and disturbance rejection problem by state feedback. In Section 12.4, a scheduled low-and-high gain controller is constructed to solve the global counterpart by state feedback. Section 12.5 considers semi-global observer based measurement feedback designs.

12.2. Problem formulation

Consider a linear system:

$$\Sigma : \begin{cases} \dot{x} = Ax + B\sigma(u(t)) + Bf(x,t), \\ y = Cx, \end{cases} \tag{12.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $y \in \mathbb{R}^p$ are the state, control input, and measured output respectively, and $\sigma(\cdot) : \mathbb{R}^m \to \mathbb{R}^m$ is a standard saturation defined as $\sigma(u) = [\sigma_1(u_1), \ldots, \sigma_m(u_m)]$ where $\sigma_i(s) = \operatorname{sgn}(s) \min\{|s|, 1\}$. Moreover, the term $f(x,t)$ represents an unknown uncertainty or disturbance. Without loss of generality, we assume here that B and C have full rank.

We make the following assumptions:

Assumption 12.1 The given system (12.1) is *asymptotically null controllable with bounded control* (ANCBC), or equivalently the given system (12.1) in the absence of saturation is stabilizable and has all its open-loop poles in the closed left-half plane.

Assumption 12.2 The given system (12.1) in the absence of saturation and uncertain element $f(x,t)$, which is then characterized by the triple (A, B, C), is left invertible and minimum phase. Moreover, we assume that the matrix pair (A, C) is detectable.

Assumption 12.3 The uncertainty and disturbance $f(x,t)$ is piecewise continuous in t and locally Lipschitz in x, and satisfies

$$\|f(x,t)\| \le 1 - \delta \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

for some $\delta \in (0,1)$.

Assumption 12.4 The norm of $f(x,t)$ is bounded by a known function

$$\|f(x,t)\| \le f_0(\|x\|) \quad \forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}^n$$

where $f_0 : \mathbb{R}^+ \to \mathbb{R}^+$ is locally Lipschitz and satisfies $f_0(0) = 0$.

We present two formal problem statements, first one for the semi-global case and the second one for the global case, utilize state feedback.

Problem 12.1 Consider the given system (12.1), and let Assumptions 12.1 and 12.3 be satisfied. The semi-global stabilization problem is to find, if possible, for any arbitrary large bounded subset $\mathcal{W} \subset \mathbb{R}^n$ and arbitrary small bounded subset $\mathcal{W}_0 \subset \mathbb{R}^n$ containing the origin, a state feedback law $u = Fx$, such that the closed-loop system satisfies the following conditons:

1. Any trajectory starting in $\mathcal{W}$ will enter $\mathcal{W}_0$ and remain in $\mathcal{W}_0$ thereafter.

2. If $f(x,t)$ satisfies Assumption 12.4 for a certain f_0, then the equalibrium point $x = 0$ is locally asympotically stable with $\mathcal{W}$ contained in its domain of attraction.

Problem 12.2 Consider the system (12.1) satisfying Assumption 12.1 and 12.3. The global stabilization problem is to find, if possible, for any arbitrary small bounded subset $\mathcal{W}_0 \subset \mathbb{R}^n$ containing the origin, a state feedback law $u = s(x,t)$, such that the closed-loop system satisfies the following conditons:

1. For all initial conditions in $\mathbb{R}^n$, the trajectories will enter $\mathcal{W}_0$ and remain in $\mathcal{W}_0$ thereafter.

2. If $f(x,t)$ satisfies Assumption 12.4 for a certain f_0, then the equalibrium point $x = 0$ is globally asympotically stable.

Next, we present a formal problem statement for the semi-global case, utilizes measurement feedback.

Problem 12.3 Consider the given system (12.1), and let Assumptions 12.1, 12.2, and 12.3 be satisfied. The semi-global stabilization problem is to find, if possible, for any arbitrary large bounded subset $\mathcal{W} \subset \mathbb{R}^{2n}$ and arbitrary small bounded subset $\mathcal{W}_0 \subset \mathbb{R}^{2n}$ containing the origin, a measurement feedback law,

$$\begin{cases} \dot{\hat{x}} = g(\hat{x}, y, t), \hat{x} \in \mathbb{R}^n \\ u = h(\hat{x}, t), \end{cases}$$

such that the closed-loop system satisfies the following conditons:

1. Any trajectory starting in $\mathcal{W}$ will enter $\mathcal{W}_0$ and remain in $\mathcal{W}_0$ thereafter.

2. if $f(x,t)$ satisfies Assumption 12.4 along with a given f_0, the equalibrium point $x = 0$ is locally asympotically stable with $\mathcal{W}$ contained in its domain of attraction.

12.3. Semi-global state feedback designs

In this section, we construct a low-high-gain feedback control law which can solve Problem 12.1.

Let $P_\varepsilon > 0$ be the solution of the continuous-time algebraic Riccati equation,

$$A'P_\varepsilon + P_\varepsilon A - P_\varepsilon BB'P_\varepsilon + \varepsilon I = 0. \tag{12.2}$$

Since the system is stabilizable, such a P_ε always exists. Moreover, since all eigenvalues of A are in the closed left half plane, $P_\varepsilon \to 0$ as $\varepsilon \to 0$. The low-gain controller is then given by

$$u_L = -B'P_\varepsilon x$$

We choose a high gain state feedback law of the form, $u_H = -\rho B'P_\varepsilon x$ and P_ε is the same as in the low-gain feedback design while $\rho \geq 0$ is to be determined.

The low-high-gain state feedback control law is formed by adding together a low and high gain feedback control. We have

$$u = F_{LH}(\varepsilon, \rho)x = u_L + u_H = -(1+\rho)B'P_\varepsilon x. \tag{12.3}$$

We claim that the controller (12.3) solves Problem 12.1 for appropriately chosen ε and ρ, as stated formally in the following theorem:

Theorem 12.1 Consider the given system (12.1) that satisfies Assumption 12.1. For any bounded subsets $\mathcal{W} \subset \mathbb{R}^n$ and $\mathcal{W}_0 \subset \mathbb{R}^n$ containing the origin, there exists an ε^* such that for each $\varepsilon \in (0, \varepsilon^*]$ there exists a ρ^* with the property that for $\rho > \rho^*$, the low-high-gain feedback $u = F_{LH}x$ solves Problem 12.1.

Proof : Let c be such that

$$c = \sup\{x'P_\varepsilon x \mid \varepsilon \in (0,1],\ x \in \mathcal{W}\}.$$

Define $V(x) = x'P_\varepsilon x$ and $\mathcal{L}_v(c) = \{x|V(x) < c\}$. There exists an $\varepsilon^* \in (0,1]$ such that for all $x \in \mathcal{L}_v(c)$ we have $\|B'P_\varepsilon x\| \leq \delta$.

Consider the derivative of $V = x'P_\varepsilon x$ along the trajectory of the closed-loop system. We have

$$\dot{V}(x) \leq -x'Q_\varepsilon x - 2x'P_\varepsilon B[\sigma((1 + \rho)B'P_\varepsilon x) - B'P_\varepsilon x - f(x,t)].$$

Denote $B'P_\varepsilon x$ by v and denote the ith component of v and $f(x,t)$ by v_i and f_i respectively. We have

$$\dot{V} \leq -\tfrac{\lambda_{\min}(Q_\varepsilon)}{\lambda_{\max}(P_\varepsilon)}V(x) - 2v'[\sigma(v + \rho v) - v - f(x,t)].$$

We know that

$$|v_i| < \delta, \quad \text{and} \quad |f_i| < 1 - \delta. \tag{12.4}$$

This implies that $|v_i + f_i| < 1$.

If $|\rho v_i| > |f_i|$, then

$$|v_i + \rho v_i| = |v_i| + |\rho v_i| \geq |v_i| + |f_i| \geq |v_i + f_i|.$$

Together with (12.4), we get

$$-v_i[\sigma(v_i + \rho v_i) - (v_i + f_i)] < 0.$$

If $|\rho v_i| < |f_i|$, we have

$$|v_i + \rho v_1| = |v_i| + |\rho v_i| < |v_i| + |f_i|.$$

Then (12.4) implies that $|v_i + \rho v_1| < 1$. Therefore, we get

$$-2v_i[\sigma(v_i + \rho v_i) - (v_i + f_i)] \leq -2v_i[\rho v_i - f_i] \leq \tfrac{f_i^2}{2\rho}.$$

Hence,

$$\dot{V}(x) \leq -\tfrac{\lambda_{\min}(Q_\varepsilon)}{\lambda_{\max}(P_\varepsilon)}V(x) + \sum_{i=1}^{m} \tfrac{f_i^2}{2\rho}. \tag{12.5}$$

Since $|f_i| < 1$, we get

$$\dot{V}(x) \leq -\tfrac{\lambda_{\min}(Q_\varepsilon)}{\lambda_{\max}(P_\varepsilon)}V(x) + \tfrac{m}{2\rho}.$$

Choose ν such that $\mathcal{L}_v(\nu) \subset \mathcal{W}_0$. Define

$$\rho_1^* = \tfrac{m\lambda_{\max}(P_\varepsilon)}{2\nu\lambda_{\min}(Q_\varepsilon)}.$$

If $\rho > \rho_1^*$, we have $\dot{V} < 0$ for all $x \in \mathcal{L}_v(c)$ for which $x \notin \mathcal{L}_v(\nu)$. This implies that any trajectory starting from $\mathcal{W}$ will enter and remain in $\mathcal{W}_0$ within finite time.

If $f(x,t)$ satisfies Assumption 12.4, we can define

$$M = \sup\{\tfrac{f_0(s)}{s} \mid s \in (0, c/\sqrt{\lambda_{\min}(P_\varepsilon)}]\}.$$

Such a M exists because f_0 is locally Lipschitz.

Therefore, from (12.5), we can conclude that for $x \in L_v(c)$,

$$\dot{V}(x) \leq [-\lambda_{\min}(Q_\varepsilon) + \tfrac{mM^2}{2\rho}]\|x\|^2.$$

Define

$$\rho_2^* = \tfrac{mM^2}{2\lambda_{\min}(Q_\varepsilon)}.$$

If $\rho > \rho_2^*$, we have $\dot{V} < 0$ for all $x \in \mathcal{L}_v(c)$. Hence the origin is asymptotically stable with $\mathcal{W}$ contained in its domain of attraction. ■

12.4. Global low-high-gain state feedback designs

In this chapter, we use the same scheduling of low-gain parameter as in [48] which is developed in [68]. Consider

$$\varepsilon_s(x) = \max\{ r \in (0,1] \mid (x' P_r x)\, \text{trace}\left[B' P_r B\right] \leq \delta^2 \}. \tag{12.6}$$

Choose the scheduled high-gain parameter ρ_s as

$$\rho_s(x) = \tfrac{\rho_0[g^2(\|x\|)+1]\lambda_{\max}(P_{\varepsilon_S(x)})}{2\lambda_{\min}(Q_{\varepsilon_S(x)})} \tag{12.7}$$

where ρ_0 is to be determined and $g(x)$ is defined as follows: if Assumption 12.4 is not satified, $g(x) \equiv 0$; if Assumption 12.4 is satisfied, $g(x)$ is any locally Lipschitz function such that $g(x) \geq \frac{f_0(x)}{x}$. Such a $g(x)$ exists since $f_0(x)$ is locally Lipschitz and $f_0(0) = 0$.

We claim that the controller constructed in the preceding section together with the scheduling low-gain and high-gain parameters solves Problem 12.2.

Theorem 12.2 Consider the given system (12.1) that satisfies Assumptions 12.1 and 12.3. For any bounded subset $\mathcal{W}_0$ there exists ρ_0^* such that the low-high-gain feedback controller (12.3) when ε and ρ are replaced with the scheduling parameters $\varepsilon_s(x)$ as in (12.6) and $\rho_s(x)$ as in (12.7) with $\rho_0 > \rho_0^*$ solves Problem 12.2.

Proof : Consider the derivative of $V = x' P(\varepsilon_s) x$ along any trajectory,

$$\dot{V}(x) \leq -\lambda_{\min}(Q)\|x\|^2 - 2v'[\sigma(v + \rho v) - v - f(x,t)] + x' \frac{dP_{\varepsilon_s(x)}}{dt} x.$$

As shown in the previous section,

$$\begin{aligned} \dot{V}(x) &\leq -\frac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})} V(x) + \sum_{i=1}^{m} \frac{f_i^2}{2\rho_s} + x' \frac{dP_{\varepsilon_s(x)}}{dt} x \\ &\leq -\frac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})} [V(x) - \frac{m}{\rho_0}] + x' \frac{dP_{\varepsilon_s(x)}}{dt} x. \end{aligned}$$

Let $\nu \leq 1$ be such that $\mathcal{L}_v(\nu) \subset \mathcal{W}_0$. Define $\rho_0^* = \frac{m}{\nu}$. We have

$$\dot{V} < x' \frac{dP_{\varepsilon_s(x)}}{dt} x, \quad \forall \rho_0 \geq \rho_0^* \text{ and } x \notin \mathcal{L}_V(\nu). \tag{12.8}$$

Assume that $\dot{V} \geq 0$ for some $x(t) \notin \mathcal{L}_V(\nu)$. We have two possible cases:

1. *Case I:* $\varepsilon_s(x) = 1$. We have $\frac{dP_{\varepsilon_s}}{dt} = 0$. But then (12.8) implies that $\dot{V} < 0$. This yields a contradiction.

2. *Case II:* $\varepsilon_s(x) \neq 1$. Note that V trace$\left[B' P_{\varepsilon_s(x)} B\right] = \delta^2$ whenever $\varepsilon_s \neq 1$. Hence $\dot{V} \geq 0$ implies that $\frac{dP_{\varepsilon_s}}{dt} \leq 0$. But (12.8) gives $\dot{V} < 0$. This yield a contradiction.

Therefore, we conclude that $\dot{V} < 0$ for all $x \notin \mathcal{L}_V(\nu)$. Any trajectory will enter and remain in $\mathcal{W}_0$ after finite time.

If $f(x,t)$ satisfies Assumption 12.4, we have

$$\begin{aligned} \dot{V}(x) &\leq -\frac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})} V(x) + \sum_{i=1}^{m} \frac{f_i^2}{2\rho_s} + x' \frac{dP_{\varepsilon_s(x)}}{dt} x \\ &\leq -\frac{\lambda_{\min}(Q_{\varepsilon_s(x)})}{\lambda_{\max}(P_{\varepsilon_s(x)})} V(x)(1 - \frac{m}{\rho_0}) + x' \frac{dP_{\varepsilon_s(x)}}{dt} x. \end{aligned}$$

we get

$$\dot{V} < x' \frac{dP_{\varepsilon_s(x)}}{dt} x, \quad \forall\, x \neq 0 \text{ and } \rho_0 > \rho_0^*.$$

We have $\dot{V} < 0$ for all $x \neq 0$. Therefore the origin is globally asymptotically stable. ∎

12.5. Semi-global observer based measurement feedback designs

Before we proceed with our design, it is necessary to introduce a Special Coordinate Basis (SCB) of the given system (12.1) in the absence of saturation and uncertain element $f(x,t)$. Consider

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases} \tag{12.9}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. Without loss of generality, we assume that B and C have full rank. Then, there exist nonsingular transformation matrices Γ_x, Γ_y and Γ_u such that

$$\bar{x} = \Gamma_x^{-1} x \quad \bar{y} = \Gamma_y^{-1} y \quad \bar{u} = \Gamma_u^{-1} u,$$

where

$$\begin{gathered} \bar{x} = \begin{pmatrix} x_a' & x_b' & x_c' & x_d' \end{pmatrix}', \ \bar{y} = \begin{pmatrix} y_b' & y_d' \end{pmatrix}', \bar{u} = \begin{pmatrix} u_c' & u_d' \end{pmatrix}', \\ x_b = \begin{pmatrix} x_{b,1}' & \dots & x_{b,m_b}' \end{pmatrix}', \quad x_{b,i} = \begin{pmatrix} x_{b,i_1} & \dots & x_{b,i_{r_i}} \end{pmatrix}', \\ x_d = \begin{pmatrix} x_{d,1}' & \dots & x_{d,m_d}' \end{pmatrix} \quad x_{d,i} = \begin{pmatrix} x_{d,i_1} & \dots & x_{d,i_{q_i}} \end{pmatrix}', \\ y_b = \begin{pmatrix} y_{b,1} & \dots & y_{b,m_b} \end{pmatrix}', \quad y_d = \begin{pmatrix} y_{d,1} & \dots & y_{d,m_d} \end{pmatrix}', \\ u_d = \begin{pmatrix} u_{d,1} & \dots & u_{d,m_d} \end{pmatrix}', \end{gathered}$$

and where x_a, x_b, x_c and x_d are of dimension n_a, n_b, n_c and n_d respectively, y_b and y_d are of dimension m_b and m_d respectively, u_c and u_d are of dimension $m - m_d$ and m_d respectively,

$$\begin{gathered} n = n_a + n_b + n_c + n_d, \quad n_b = \textstyle\sum_{i=1}^{m_b} r_i, \quad n_d = \textstyle\sum_{i=1}^{m_d} q_i, \\ m_b + m_d = p. \end{gathered}$$

In the new coordinate basis, we have

$$\begin{aligned} \dot{x}_a &= A_{aa} x_a + L_{ab} y_b + L_{ad} y_d; \\ \dot{x}_c &= A_{cc} x_c + L_{cb} y_b + L_{cd} x_d + B_c [E_{ca} x_a + u_c]. \end{aligned}$$

For $i = 1, \dots, m_b$,

$$\begin{aligned} \dot{x}_{b,i} &= A_{r_i} x_{b,i} + L_{b_i b} y_b + L_{b_i d} y_d \\ y_{b,i} &= C_{r_i} x_{b,i} = x_{b,i_1}. \end{aligned}$$

For $i = 1, \dots, m_d$,

$$\begin{aligned} \dot{x}_{d,i} &= A_{q_i} x_{d,i} + L_i y_d + B_{qi} [u_{d,i} + E_{ia} x_a + E_{ib} x_b + E_{ic} x_c \ + E_{id} x_d] \\ y_{d,i} &= C_{q_i} x_{d,i} = x_{d,i_1}, \end{aligned}$$

where

$$A_r = \begin{pmatrix} 0 & I_{r-1} \\ 0 & 0 \end{pmatrix}, \qquad \begin{aligned} B_r &= \begin{pmatrix} 0 & \dots & 0 & 1 \end{pmatrix}', \\ C_r &= \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}. \end{aligned}$$

The SCB given above displays explicitly both finite and infinite zero structure of the system given in (12.9), and has a number of important properties (see [99, 92]). We need to stress that in view of these SCB properties, Assumption 12.2 implies that A_{aa} is Hurwitz, and x_c and u_c do not exist. Hence we have $m_d = m$ and $m_b = p - m$. Moreover, the input transformation $\Gamma_u = I$, in another word, we don't need to transform the input.

We now proceed to implement the low-and-high gain controller designed in Section 12.3 using a high-gain observer under Assumption 12.2.

The measurement feedback is of the form:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + Bu + L(\ell)(y - C\hat{x}) \\ u = F_{LH}(\varepsilon, \rho)\hat{x} \end{cases} \tag{12.10}$$

where $F_{LH}(\varepsilon, \rho)$ is given by (12.3) and where parameterized observer gain $L(\ell)$ is designed shortly as given in (12.11).

We construct the high gain observer in following steps:

Step 1: Transform the system into the Special coordinate basis. Given Assumption 12.2 satisfied, we have

$$\begin{aligned} \bar{x} &= \begin{pmatrix} x_a' & x_b' & x_d' \end{pmatrix}', \ \bar{y} = \begin{pmatrix} y_b' & y_d' \end{pmatrix}', \ \bar{u} = \begin{pmatrix} u_{d,1} & \dots & u_{d,m} \end{pmatrix}' \\ x_b &= \begin{pmatrix} x_{b,1}' & \dots & x_{b,p-m}' \end{pmatrix}', \quad x_{b_i} = \begin{pmatrix} x_{b,i_1} & \dots & x_{b,i_{r_i}} \end{pmatrix}' \\ x_d &= \begin{pmatrix} x_{d,1}' & \dots & x_{d,m}' \end{pmatrix}', \quad x_{d_i} = \begin{pmatrix} x_{d,i_1} & \dots & x_{d,i_{q_i}} \end{pmatrix}' \\ y_b &= \begin{pmatrix} y_{b,1} & \dots & y_{b,p-m} \end{pmatrix}', \quad y_d = \begin{pmatrix} y_{d,1} & \dots & y_{d,m} \end{pmatrix}' \end{aligned}$$

$$\dot{x}_a = A_{aa}x_a + L_{ab}y_b + L_{ad}y_d,$$

and for $i = 1, \dots, p - m$,

$$\begin{aligned} \dot{x}_{b,i} &= A_{r_i}x_{b,i} + L_{b_i b}y_b + L_{b_i d}y_d \\ y_{b,i} &= C_{r_i}x_{b,i} = x_{b,i_1} \end{aligned}$$

For $i = 1, \dots, m$,

$$\dot{x}_{d,i} = A_{q_i}x_{d,i} + L_i y_d + B_{qi}[u_{d,i} + E_{ia}x_a + E_{ib}x_b + E_{id}x_d]$$

$$y_{d,i} = C_{q_i}x_{d,i} = x_{d,i_1},$$

Step 2, Since (A_{r_i}, C_{r_i}) is observable, for $i = 1$ to $p - m$, choose $L_{b,i} \in \mathbb{R}^{r_i \times 1}$ such that $A_{r_i}^c = A_{r_i} - L_{b,i}C_{r_i}$ is Hurwitz.

Similarly, (A_{q_i}, C_{q_i}) is observable. For $i = 1$ to m, choose $L_{d,i} \in \mathbb{R}^{q_i \times 1}$ such that $A^c_{q_i} = A_{q_i} - L_{d,i}C_{q_i}$ is Hurwitz

Step 3, For any $\ell \in (0, 1]$, define a matrix $L(\ell) \in \mathbb{R}^{n\times p}$ as

$$L(\ell) = \Gamma_x \begin{pmatrix} L_{ab} & L_{ad} & \\ L_{bb} + L_b(\ell) & L_{bd} & \\ 0 & L_{dd} & L_d(\ell) \end{pmatrix} \Gamma_y^{-1}, \tag{12.11}$$

where

$$L_{bb} = \begin{pmatrix} L_{b_1 b} \\ L_{b_2 b} \\ \vdots \\ L_{b_{p-m} b} \end{pmatrix}, \; L_{bd} = \begin{pmatrix} L_{b_1 d} \\ L_{b_2 d} \\ \vdots \\ L_{b_m d} \end{pmatrix}, \; L_{dd} = \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_m \end{pmatrix},$$
$$L_b(\ell) = \text{blkdiag}\left\{S_{r_i}(\ell)L_{b,i}\right\}_{i=1}^{p-m},$$
$$L_d(\ell) = \text{blkdiag}\left\{S_{q_i}(\ell)L_{d,i}\right\}_{i=1}^{m}$$

and $S_r(\ell) = \text{blkdiag}\left\{\ell^i\right\}_{i=1}^{r}$ for any integer $r \geq 1$. We have following theorem

Theorem 12.3 Consider the system (12.1). Let Assumptions 12.1, 12.2, 12.3 be satisfied. There exist ε^*, $\rho*$ and ℓ^* such that for any $\varepsilon \in (0, \varepsilon^*]$, $\rho > \rho^*$ and $\ell > \ell^*$, the measurement feedback controller (12.10) solves the Problem 12.3.

In order to prove this theorem, we need to establish the following lemmas. Let $\tilde{\Sigma}$ denote the system,

$$\tilde{\Sigma} : \begin{cases} \dot{x} = Ax + B[\sigma(u) + f(x + Te, t) + Ee] \\ \dot{e} = A_o e, \end{cases} \tag{12.12}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $e \in \mathbb{R}^k$ and A_o is Hurwitz stable. Let P_o be the solution of the Lyapunov equation, i.e.

$$A_o' P_o + P_o A_o = -I.$$

Define $\tau = \sqrt{\lambda(E'E)}$ and $\kappa = \sqrt{\lambda(T'T)}$.

Lemma 12.1 Given $\delta, \epsilon \in (0, 1)$. Let $c > 0$ be such that

$$\|B' P_\varepsilon x\| < 1 \quad \forall x \in \{x \in \mathbb{R}^n : x' P_\varepsilon x < c^2 + 1\},$$

where P_ε is as in (12.2). Define

$$\gamma = \frac{\max\{1, (\tau^2+1)\lambda_{\max}(P_o)\}}{\min\{1, \frac{\lambda_{\max}(P_\varepsilon)}{\lambda_{\min}(Q_\varepsilon)}\}}, \quad M = \sup_{s\in(0,F)}\left\{\frac{g_0(s)}{s}\right\},$$
$$F = \sqrt{c^2+1}\left(\sqrt{\lambda_{\min}(P_\varepsilon)^{-1}} + \kappa\sqrt{[(\tau^2+1)\lambda_{\min}(P_o)]^{-1}}\right),$$
$$\rho_1^* = \frac{2m}{(c^2+1)\gamma}, \quad \rho_2^* = \frac{2mM^2}{\lambda_{\min}(Q_\varepsilon)}, \quad \rho_3^* = 2mM^2\kappa^2.$$

Assume $\rho > \max\{\rho_1^*, \rho_2^*, \rho_3^*\}$. For the system $\tilde{\Sigma}$ that satisfies Assumption 12.1 and 12.3, and with controller (12.3), there exists a continuous function $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^+$ such that the function

$$V(x,e) = x' P_\varepsilon x + (\tau^2 + 1) e' P_o e$$

satisfies $\dot{V} \le -\psi(x,e)$.

If Assumption 12.4 is not satisfied, then

$$(x', e')' \in L_V(c^2+1) \Rightarrow \psi(x,e) \ge \gamma(V - \tfrac{\rho_1^*}{\rho}\tfrac{c^2+1}{2}).$$

If Assumption 12.4 is satisfied, then

$$(x', e')' \in L_V(c^2+1) \Rightarrow \psi(x,e) \ge 0.5\gamma V.$$

Proof : Note that $u = -(1+\rho)B' P_\varepsilon x$. We denote $B' P_\varepsilon x$ by v and denote the ith component of v and $f(x+Te,t)$ by v_i and f_i respectively. Consider the derivative of V along the trajectory in the set $L_V(c^2+1)$,

$$\dot{V} = -x' Q_\varepsilon x - 2v'[\sigma((1+\rho)v) - f(x+Te,t) - v] - v'v + 2v' E e - (\tau^2+1) e'e.$$

Similar with the proof in Section 12.3, we have

$$2v'[\sigma((1+\rho)v) - f(x+Te,t) - v] \le \sum_{i=1}^{m} \tfrac{f_i^2}{2\rho} \le \tfrac{m}{2\rho}.$$

Hence

$$\dot{V} \le -x' Q_\varepsilon x + \sum_{i=1}^{m} \tfrac{f_i^2}{2\rho} - e'e \le -\gamma[V(x) - \tfrac{\rho_1^*}{\rho}\tfrac{c^2+1}{2}].$$

Moreover, if Assumption 12.4 is satisfied and $\rho > \max\{\rho_1^*, \rho_2^*\}$, we have

$$\begin{aligned}
\dot{V}(x) =& - x' Q_\varepsilon x + \tfrac{mM^2\|x+Te\|^2}{2\rho} - e'e \\
\le& - x' Q_\varepsilon x + \tfrac{mM^2(\|x\|^2 + \kappa^2\|e\|^2)}{\rho} - e'e \\
\le& - [0.5\lambda_{\min}(Q_\varepsilon) - \tfrac{mM^2}{\rho}]\|x\|^2 - [0.5 - \tfrac{mM^2\kappa^2}{\rho}]\|e\|^2 \\
& - 0.5\gamma V(x) \\
\le& - 0.5\gamma V(x).
\end{aligned}$$

■

The following Lemma is the same as Lemma 4 in [91], which is adapted from [133].

Lemma 12.2 Consider the nonlinear system

$$\begin{cases} \dot{z} = f(z, e, t), & z \in \mathbb{R}^n; \\ \dot{e} = \ell A e + g(z, e, t), & e \in \mathbb{R}^m \end{cases}$$

where $\ell > 0$ and A is Hurwitz matrix. Assume that for the system $\dot{z} = f(z, 0, t)$, there exists a neighborhood $\mathcal{W}_1$ of the origin in $\mathbb{R}^n$ and a $\mathcal{C}_1$ function $V_1 : \mathcal{W}_1 \to R^+$ which is positive definte on $\mathcal{W}_1 \setminus \{0\}$ and proper on $\mathcal{W}_1$ and satisfies

$$\frac{\partial V_1}{\partial z} f(z, 0, t) \leq -\psi_1(z),$$

where $\psi_1(z)$ is continuous on $\mathcal{W}_1$ and positive definite on $\{z : \nu_1 < V_1(z) \leq c_1 + 1\}$ for some nonnegative real number $\nu_1 < 1$ and some real number $c_1 \geq 1$. Also assume that there exist positive real numbers α and β and a bounded function γ with $\gamma(0) = 0$ satisfying

$$\left.\begin{array}{r} \|f(z, e, t) - f(z, 0, t)\| \leq \gamma(\|e\|) \\ \|g(z, e, t)\| \leq \alpha\|e\| + \beta \end{array}\right\} \forall (z, e, t) \in \{z \in \mathbb{R}^n : V_1(z) \leq c_1 + 1\} \times \mathbb{R}^m \times \mathbb{R}^+$$

Let $c_2(\ell)$ be a class $\mathcal{K}_\infty$ function satisfying $\lim_{\ell \to \infty} \frac{\ell}{c_2^4(\ell)} = \infty$ and P solves the Lyapunov equation $A'P + PA = -I$. Define the function

$$V(z, e) = c_1 \frac{V_1(z)}{c_1 + 1 - V_1(z)} + c_2(\ell) \frac{\ln(1 + e'Pe)}{c_2(\ell) + 1 - \ln(1 + e'Pe)},$$

and the set

$$\mathcal{W} := \{z : V_1(z) < c_1 + 1\} \times \{e : \ln(1 + e'Pe) < c_2(\ell) + 1\}.$$

Then, for $\ell > 0$, $V : \mathcal{W} \to R^+$ is positive definite on $\mathcal{W}$ $\{0\}$ and proper on $\mathcal{W}$. Furthermore, for any $\nu_2 \in (0, 1)$, there exists an $\ell^*(\nu_2) > 0$ such that, for all $\ell \in [\ell^*(\nu_2), \infty)$, the derivative of V along the trajectories of systems satisfies $\dot{V} \leq -\psi_2(z, e)$ where $\psi_2(z, e)$ is positive definite on $\{(z, e) : \nu_1 + \nu_2 \leq V(z, e) \leq c_1^2 + c_2^2(\ell) + 1\}$.

Next, we proceed to prove theorem 12.3.

Proof of theorem 12.3 : Consider the closed-loop system of (12.1) and (12.10),

$$\begin{cases} \dot{x} = Ax + B[\sigma(u) + f(x, t)] \\ \dot{\hat{x}} = A\hat{x} + Bu + L(\ell)(y - C\hat{x}) \\ u = F_{LH}(\varepsilon, \rho)\hat{x}. \end{cases} \tag{12.13}$$

Using the state and output transformation Γ_x and Γ_y, we transform the system into its SCB form,

$$\bar{x} = \Gamma_x^{-1} x = \begin{pmatrix} x_a' & x_b' & x_d' \end{pmatrix}', \ \bar{\hat{x}} = \Gamma_x^{-1} x = \begin{pmatrix} \hat{x}_a' & \hat{x}_b' & \hat{x}_d' \end{pmatrix}'.$$

We construct a new state as

$$\tilde{x} = \Gamma_x \begin{pmatrix} \hat{x}_a' & x_b' & x_d' \end{pmatrix}, \quad \tilde{e} = \begin{pmatrix} e_a' & e_b' & e_d' \end{pmatrix}',$$

where $e_a = x_a - \hat{x}_a$, $e_b = S_b(\ell)(x_b - \hat{x}_b)$, $e_d = S_d(\ell)(x_d - \hat{x}_d)$,

$$S_b(\ell) = \text{blkdiag}\left\{\ell^{r_i} S_{r_i}^{-1}(\ell)\right\}_{i=1}^{p-m},$$
$$S_d(\ell) = \text{blkdiag}\left\{\ell^{q_i} S_{q_i}^{-1}(\ell)\right\}_{i=1}^{m}.$$

We denote $e_{bd} = (e_b', e_d')'$. Then the closed-loop system in the new basis is

$$\dot{\tilde{x}} = A\tilde{x} + B[\sigma(u) + f(\tilde{x} + \Gamma_{xa} e_a, t) + E_a e_a], \tag{12.14}$$

$$\dot{e}_a = A e_a, \tag{12.15}$$

$$\dot{\tilde{e}}_{bd} = \ell A_{bd} e_{bd} + B_{bf}[\sigma(u) + f(\tilde{x} + \Gamma_{xa} e_a, t) - u + E_{bd} S_{bd}^{-1}(\ell) e_{bd}], \tag{12.16}$$

$$u = F_{LH}(\varepsilon, \rho)[\tilde{x} - \Gamma_{xbd} S_{bd}(\ell)^{-1} e_{bd}], \tag{12.17}$$

where

$$\begin{gathered} E_a = \begin{bmatrix} E_{1a} & \dots & Ema \end{bmatrix} \\ A_{bd} = diag\{A_{r_1}^c, A_{r_2}^c, \dots, A_{r_{p-m}}^c, A_{q_1}^c, A_{q_2}^c, \dots, A_{q_m}^c\} \\ B_{bd} = [0, diag\{B_{q_1}, B_{q_2}, \dots, B_{q_m}\}']' \\ E_{bd} = \begin{pmatrix} E_{1b} & E_{1d} \\ E_{2b} & E_{2d} \\ \vdots & \vdots \\ E_{mb} & E_{md} \end{pmatrix} \\ S_{bd}(\ell) = \text{blkdiag}\{S_b(\ell), S_d(\ell)\} \end{gathered}$$

and $\Gamma_x = (\Gamma_{xa}, \Gamma_{xbd})$ with $\Gamma_{xa} \in \mathbb{R}^{n \times n_a}$ and $\Gamma_{xbd} \in \mathbb{R}^{n \times (n_b + n_d)}$.

Consider the dynamic of $\tilde{x}$ and e_a. We will apply Lemma 12.1. Set $e_{bd} = 0$ in the closed-loop equations (12.14), (12.16) and (12.17). Then, we have

$$\begin{cases} \dot{\tilde{x}} = A\tilde{x} + B[\sigma(u) + f(\tilde{x} + \Gamma_{xa} e_a, t) + E_a e_a] \\ \dot{e}_a = A_{aa} e_a \\ u = F_{LH}(\varepsilon, \rho)\tilde{x}. \end{cases}$$

By Assumption 12.2, A_{aa} is Hurwitz stable. Let $P_a > 0$ be the solution of

$$A'_{aa} P_a + P_a A_{aa} = -I.$$

Following Lemma 12.1, we define

$$V_1(\tilde{x}, e_a) = \tilde{x}' P_\varepsilon \tilde{x} + (\tau^2 + 1) e'_a P_a e_a,$$

where $\tau = \sqrt{\lambda_{\max} E'_a E_a}$. Let $c_1 > 1$ be such that

$$c_1^2 > \sup\{V_1(\tilde{x}, e_a) \mid (x, \hat{x}) \in \mathcal{W}, \varepsilon \in (0, 1)\}.$$

There exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*)$

$$\|B' P_\varepsilon \tilde{x}\| < 1, \quad \forall (x, \hat{x}) \in L_{V_1}(c_1) = \{V_1(\tilde{x}, e_a) < c_1^2 + 1\}.$$

Fix $\varepsilon \in (0, \varepsilon^*]$. Let P_{bd} satisfy the Lyapunov equation

$$A'_{bd} P_{bd} + P_{bd} A_{bd} = -I,$$

and let $V_3 = e'_{bd} P_{bd} e_{bd}$. Observe that from the definition of $S_r(\ell)$, if we assume $\ell > 1$, there exists a $k > 0$ such that, for any $r \geq 0$, we have

$$\|(\tilde{x}', e'_a, e'_{bd})'\| \leq r \Rightarrow \|(x', \hat{x}')'\| \leq kr. \tag{12.18}$$

Moreover, this k is independent of ℓ provided that $\ell > 1$.

We can choose $\nu \in (0, 1)$, a strictly positive real number such that, for all $\ell > 1$, we have

$$L_{V_1}(\nu) \times L_{V_3}(\exp(\nu) - 1) \subset \mathcal{W}_0.$$

Such a ν exsits since $\mathcal{W}_0$ contains zero in its interior, P_ε, P_a and P_{bd} are positive definite and (12.18) holds for all $\ell > 1$. It follows from Lemma 12.1 that if

$$\rho > \max\{\rho_1^*, \tfrac{\rho_1^*(c_1^2+1)}{2\nu}\},$$

we get $\dot{V}_1 \leq -\psi_1(\tilde{x}, e_a)$, where

$$(\tilde{x}', e'_a)' \in \{\tfrac{\nu}{4} < V_1(\tilde{x}, e_a) \leq c_1^2 + 1\} \Rightarrow -\psi_1(\tilde{x}, e_a) < 0. \tag{12.19}$$

Let ρ be fixed. Choose

$$c_2(\ell) = \ln(1 + \lambda_{\max}(P_{bf} R^2 \ell^{2(n_b+n_d)})),$$

where R is such that $(x, \hat{x}) \in \mathcal{W}$ implies that $\|x_b - \hat{x}_b\| < R/2$ and $\|x_d - \hat{x}_d\| < R/2$. Obviously $c_2(\ell)$ is of class $\mathcal{K}_\infty$ and satisfies

$$\lim_{\ell \to \infty} \frac{\ell}{c_2^4(\ell)} = \infty.$$

We then define the Lyapunov function

$$V_2(\tilde{x}, e_a.e_{bd}) = \frac{c_1^2 V_1}{c_1^2 + 1 - V_1} + \frac{c_2(\ell) \ln(1 + e_{bd}' P_{bd} e_{bd})}{c_2(\ell) + 1 - \ln(1 + e_{bd} P_{bd} e_{bd})},$$

and the set

$$\mathcal{W}_2 = \{(\tilde{x} \times e_a) : V_1(\tilde{x}, e_a) < c_1^2 + 1\} \times \{e_{bd} : \ln(1 + e_{bd}' P_{bd} e_{bd}) < c_2(\ell) + 1\}. \quad (12.20)$$

It then follows from Lemma 12.1 that for all $\ell > 0$, V_2 is positive definite on $\mathcal{W}_2$ $\{0\}$ and proper on $\mathcal{W}_2$. Furthermore, there exists an $\ell^*(\varepsilon, \rho, \nu)$ such that, for all $\ell > \ell^*(\varepsilon, \rho, \nu)$, we have

$$\dot{V}_2 \leq -\psi_2(\tilde{x}, e_a, e_{bd}),$$

where $\psi_2(\tilde{x}, e_a.e_{bd})$ is positive definite on

$$\mathcal{W}_3 := \{(\tilde{x}, e_a, e_{bd}), \nu/2 < V_2 < c_1^4 + c_2(\ell)^2 + 1\}.$$

It is clear that $(x, \hat{x}) \in \mathcal{W}$ implies $V_2 < c_1^4 + c_2^2(\ell)$ and $V_2 < \nu/2$ implies $(x, \hat{x}) \in \mathcal{W}_0$. This completes the proof of item 1 in Problem 12.3.

If Assumption 12.4 is satisfied, it follows from Lemma 12.1 that for $\rho \geq \max\{\rho_2^*, \rho_3^*\}$ and for any $\nu \in (0, 1)$, we have

$$(\tilde{x}', e_a')' \in \{\nu/4 < V_1(\tilde{x}, e_a) \leq c_1^2 + 1\} \Rightarrow \psi_1(\tilde{x}, e_a) > 0.5\gamma V_1.$$

This implies that the origin of $(\tilde{x}, e_a)$ is locally exponentially stable. Then for any a priori given neighborhood $\mathcal{H}$ of the origin, the local asymptotic stability of the origin of $(\tilde{x}, e_a, e_{bd})$ with a domain of attraction containning $\mathcal{H}$ follows from the standard singular perturbation result. ∎

So far we have discussed semi-global stabilization along with disturbance rejection while utilizing measurement feedback. Along the same lines, a similar result for global stabilization can be developed.

12.6. Conclusions

Low-and-high gain and scheduled low-and-high gain state and measurement feedback design methodologies are expanded and generalized to solve semi-global and global internal stabilization along with disturbance rejection for the case of matched disturbances and uncertainties for linear systems subject to actuator saturation.

CHAPTER 13

Further results on the disturbance response of a double integrator controlled by saturating linear static state feedback

13.1. Introduction

In the previous chapter, we proved that for disturbances that are exactly matched to the input and have magnitude smaller than the level of saturation by a known margin, a nonlinear static state or dynamic measurement feedback can be constructed to ensure a bounded closed-loop state. The assumption that the disturbances magnitude are strictly below the saturation level is, although natural, too restrictive. From this chapter to the end of Part II, we shall consider the general non-input additive case which does have to be matched and can have large magnitude. Of particular interest in dealing with such disturbances, is the study on identifying classes of disturbances for which a controller can be designed to yield bounded closed-loop state trajectories. We begin this study with a simple double-integrator system in chapter.

The double integrator system is commonly seen in control applications including low-friction, free rigid-body motion, such as single-axis spacecraft rotation and rotary crane motion (see [78] and references therein). An interesting and widely studied problem is the control of double integrators subject to input saturation. A classical result is that a double integrator with a saturating linear static feedback provides global asymptotic stability of the origin. This result has been extended to mixed-type systems in [141] and [153]. Many other control methods have also been proposed. A brief summary and comparison of various methods is given in [78].

Compared with the relatively mature study of internal stabilization, the dynamic response of a double integrator with saturating feedback to external disturbances is still not fully understood. In this chapter, we study the disturbance response of a double integrator controlled by a saturating linear static state feedback, as given below:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sigma(-k_1 x_1 - k_2 x_2) + d. \end{aligned} \tag{13.1}$$

where σ represents the standard saturation function $\sigma(s) = \operatorname{sign}(s)\min\{1, |s|\}$. The goal is to identify a class of disturbances for which the states of the above controlled system remain bounded. [12] and [117] have previously studied this problem in the context of $\mathcal{L}_p$ stability. They showed that for any $k_1 > 0$ and $k_2 > 0$, (13.1) is $\mathcal{L}_p$ stable for $p \in [1, 2]$, and, moreover, the trajectories remain bounded for any $d \in \mathcal{L}_p$ with $p \in [1, 2]$. However, for any $p > 2$ there exists $d \in \mathcal{L}_p$ that cause the states to grow unbounded from certain initial conditions. [106] studied the notion of input-to-state stability (ISS) (see [111]) for system (13.1), and proved that no choice of k_1, k_2 can achieve ISS. Specifically, there exist bounded disturbances with arbitrarily small $\mathcal{L}_\infty$ norm that cause the states to grow unbounded from certain initial conditions. Even more dramatically, unbounded growth can be achieved by *vanishing* disturbances with arbitrarily small $\mathcal{L}_\infty$ norm.

[149] extended the negative result from [106] to classes of small $\mathcal{L}_\infty$ signals with further restrictions. However, [149] also showed that boundedness of the states of (13.1) is preserved for a particular class of small disturbances that have bounded integrals over all intervals. To be precise, let a family of parameterized sets be defined by

$$\tilde{\Omega}_M = \left\{ d \in \mathcal{L}_\infty(1) \;\middle|\; \forall\, t_1, t_2 \geq 0, \left| \int_{t_1}^{t_2} d(t)\, \mathrm{d}t \right| \leq M \right\},$$

where $\mathcal{L}_\infty(1)$ denotes the set of $\mathcal{L}_\infty$ signals of magnitude less than 1. For a given linear state feedback, it was proven that for any $M > 0$ there exists a $q^* > 0$ such that the states of (13.1) remain bounded for any disturbance $d(t) = qs(t)$, where $s(t) \in \tilde{\Omega}_M$ and $q \leq q^*$. In other words, signals with bounded integrals can be tolerated if they are scaled down by a sufficient amount. It was furthermore shown that a small bias in $d(t)$ can be tolerated while still achieving boundedness of the states. The class of disturbances considered by [149] covers a broad class of signals, such as periodic, quasi-periodic, and $\mathcal{L}_1$ signals.

This chapter is an extension of the work in [149]. Our focus is also on disturbances with bounded integrals; however, we strengthen the results from [149] by dispensing with the magnitude restriction on d and removing the attenuation factor q. Specifically, we consider the family of parameterized sets

$$\Omega_M = \left\{ d \in \mathcal{L}_\infty \;\middle|\; \forall\, t_1, t_2 \geq 0, \left| \int_{t_1}^{t_2} d(t)\, \mathrm{d}t \right| \leq M \right\}. \tag{13.2}$$

If a signal d belongs to Ω_M for some $M > 0$, we refer to it as an *integral-bounded signal*, and we refer to M as an *integral bound* on d.

We first prove a new negative result, namely, that for a given linear static feedback law, there always exist integral-bounded signals that cause the trajectories of the system to grow unbounded from some initial conditions. Our next result, however, shows that if an integral bound M is known *a priori*, then k_1 and k_2 can always be designed to ensure boundedness of the states, regardless of initial conditions. Moreover, boundedness can be ensured also if the integral-bounded disturbance is biased by a DC signal of magnitude less than 1. Finally, we prove an even stronger result for disturbances consisting of a finite number of sinusoids plus a DC offset of magnitude less than 1. In this case, *any* internally stabilizing linear static feedback law ensures boundedness of the states.

13.2. Main result

The first theorem shows that not every internally stabilizing static linear law can maintain boundedness of the trajectories in the face of integral-bounded disturbances.

Theorem 13.1 Consider the system (13.1) with $k_1 > 0$ and $k_2 > 0$. There exists an integral-bounded signal d and an initial condition such that x_1 and x_2 grow unbounded.

Proof : Define $y_1 = k_1 x_1 + k_2 x_2$, $y_2 = k_2 x_2$ and $\tilde{t} = \frac{k_1}{k_2} t$. Then the closed-loop system in the new coordinates and with $\tilde{t}$ as the time variable becomes

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}\tilde{t}}(\tilde{t}) &= y_2(\tilde{t}) - \lambda\left[\sigma(y_1(\tilde{t})) - d(\tilde{t})\right], \\ \frac{\mathrm{d}y_2}{\mathrm{d}\tilde{t}}(\tilde{t}) &= -\lambda\left[\sigma(y_1(\tilde{t})) - d(\tilde{t})\right], \end{aligned} \tag{13.3}$$

where $\lambda = k_2^2/k_1 > 0$. We shall construct an integral-bounded disturbance d that causes the states to grow unbounded from a particular initial condition.

Step 1: Suppose the trajectory of (13.3) starts from $A = (1, N\lambda)$, for some large integer N, at time $\tilde{t}_A = 0$. We will construct a d to drive the states from point A to $B = (1, -(N+2)\lambda)$ at time $\tilde{t}_{\boldsymbol{B}}$. Choose

$$d(\tilde{t}) = 2\pi \sin(\pi\tilde{t}).$$

Let

$$d_1(\tilde{t}) = \int_0^{\tilde{t}} d(\tau)\,\mathrm{d}\tau = 2(1-\cos(\pi\tilde{t})), \quad d_2(\tilde{t}) = \int_0^{\tilde{t}} d_1(\tau)\,\mathrm{d}\tau = 2\tilde{t} - \tfrac{2}{\pi}\sin(\pi\tilde{t}).$$

Since $\frac{\mathrm{d}y_1}{\mathrm{d}\tilde{t}}(0) = y_2(0) - \lambda$, we see that for large N, the trajectory will initially move to the right. If $y_1(\tilde{t}) > 1$, we have

$$y_2(\tilde{t}) = y_2(0) - \lambda\tilde{t} + \lambda d_1(\tilde{t}) = N\lambda - \lambda\tilde{t} + 2\lambda(1-\cos(\pi\tilde{t}))$$

and

$$\begin{aligned} y_1(\tilde{t}) &= y_1(0) + \int_0^{\tilde{t}} y_2(\tau)\,\mathrm{d}\tau - \lambda\tilde{t} + \lambda d_1(\tilde{t}) \\ &= y_1(0) + N\lambda\tilde{t} - \tfrac{\lambda}{2}\tilde{t}^2 + \lambda d_2(\tilde{t}) - \lambda\tilde{t} + \lambda d_1(\tilde{t}) \\ &= y_1(0) + N\lambda\tilde{t} - \tfrac{\lambda}{2}\tilde{t}^2 + 2\lambda\tilde{t} - \tfrac{2\lambda}{\pi}\sin(\pi\tilde{t}) - \lambda\tilde{t} + 2\lambda(1-\cos(\pi\tilde{t})) \\ &= -\tfrac{\lambda}{2}\tilde{t}^2 + (N\lambda+\lambda)\tilde{t} + 2\lambda(1-\cos(\pi\tilde{t})) - \tfrac{2\lambda}{\pi}\sin(\pi\tilde{t}) + 1. \end{aligned}$$

Given a sufficiently large N, $y_1(\tilde{t})$ only has one intersection with $y_1 = 1$ for $\tilde{t} > 0$. This is shown in the Appendix. For $\tilde{t} = 2N+2$, we have

$$y_1 = -\lambda\tfrac{(2N+2)^2}{2} + (N\lambda+\lambda)\times(2N+2) + 1 = 1$$

and

$$y_2 = N\lambda - \lambda(2N+2) + \lambda\int_0^{2N+2} d(\tilde{t})\,\mathrm{d}\tilde{t} = -(N+2)\lambda.$$

This shows that the trajectory will cross $y_1 = 1$ at $B = (1, -(N+2)\lambda)$ at time $\tilde{t}_B = 2N+2$. We have

$$\int_0^{\tilde{t}_B} d(\tilde{t})\,\mathrm{d}\tilde{t} = 0, \quad \left|\int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t})\,\mathrm{d}\tilde{t}\right| \le 4 \text{ with } \tilde{t}_A \le \tilde{t}_1 \le \tilde{t}_2 \le \tilde{t}_B.$$

Step 2: From $\tilde{t}_B$, set $d(\tilde{t}) = \sigma(y_1(\tilde{t}))$. Then

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}\tilde{t}}(\tilde{t}) &= y_2, \\ \frac{\mathrm{d}y_2}{\mathrm{d}\tilde{t}}(\tilde{t}) &= 0. \end{aligned}$$

The trajectory will move directly toward the left to $C = (-1, -(N+2)\lambda)$ at time $\tilde{t}_c = \tilde{t}_B + \frac{2}{(N+2)\lambda}$. Clearly, for N sufficiently large,

$$\int_{\tilde{t}_B}^{\tilde{t}_C} d(\tilde{t})\, \mathrm{d}\tilde{t} = 0, \quad \left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t})\, \mathrm{d}\tilde{t} \right| \leq 1 \text{ with } \tilde{t}_B \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \tilde{t}_C.$$

Step 3: From $\tilde{t}_C$, choose

$$d(\tilde{t}) = -2\pi \sin(\pi \tilde{t}).$$

Following the same argument as in Step 1 with N replaced by $\tilde{N} = N+2$, we find that the trajectory will re-cross $y_1 = -1$ at $D = (-1, (\tilde{N}+2)\lambda) = (-1, (N+4)\lambda)$ at time $\tilde{t}_D = \tilde{t}_C + 2\tilde{N} + 2 = \tilde{t}_C + 2N + 6$. Similarly

$$\int_{\tilde{t}_C}^{\tilde{t}_D} d(\tilde{t})\, \mathrm{d}\tilde{t} = 0, \quad \left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t})\, \mathrm{d}\tilde{t} \right| \leq 4 \text{ with } \tilde{t}_C \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \tilde{t}_D.$$

Step 4: From $\tilde{t}_D$, choose

$$d(\tilde{t}) = \sigma(y_1(\tilde{t})).$$

The trajectory will move directly to the right and cross $y_1 = 1$ at $E = (1, (N+4)\lambda)$ at time $\tilde{t}_E = \tilde{t}_D + \frac{2}{N+4}$. We have

$$\int_{\tilde{t}_D}^{\tilde{t}_E} d(\tilde{t})\, \mathrm{d}\tilde{t} = 0, \quad \left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t})\, \mathrm{d}\tilde{t} \right| \leq 1 \text{ with } \tilde{t}_D \leq \tilde{t}_1 \leq \tilde{t}_2 \leq \tilde{t}_E.$$

The system trajectory resulting from Steps 1 through 4 is visualized in Fig. 1. By repeating these steps, the state grows unbounded, and we can check that the constructed disturbance signal satisfies

$$\left| \int_{\tilde{t}_1}^{\tilde{t}_2} d(\tilde{t})\, \mathrm{d}\tilde{t} \right| \leq 4 + 1 + 1 + 4 = 10 = \tilde{M}$$

for any $0 = \tilde{t}_A \leq \tilde{t}_1 \leq \tilde{t}_2$. In the original time variable t, we calculate the integral bound as

$$\left| \int_{t_1}^{t_2} d(\tilde{t})\, \mathrm{d}t \right| = \left| \frac{k_2}{k_1} \int_{\frac{k_1}{k_2} t_1}^{\frac{k_1}{k_2} t_2} d(\tilde{t})\, \mathrm{d}\tilde{t} \right| \leq \frac{k_2}{k_1} \tilde{M}.$$

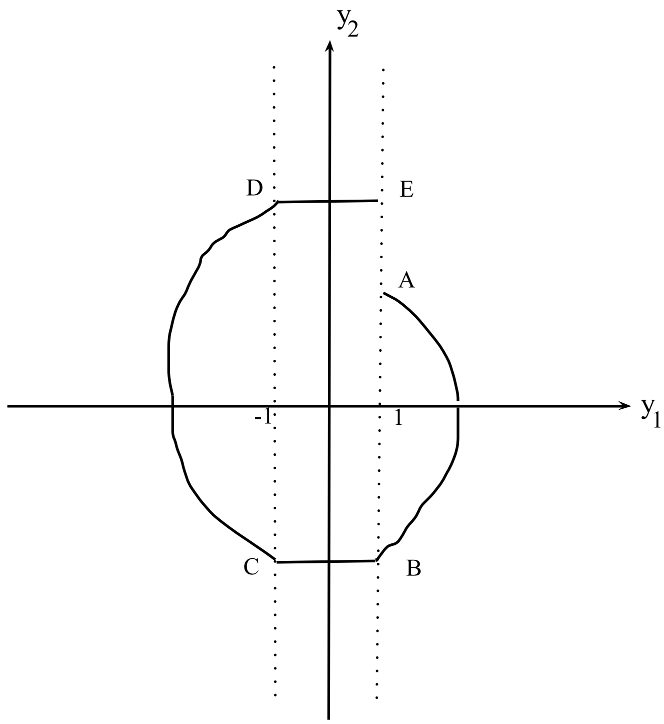


Figure 13.1: State trajectories of double integrator with $\lambda = 1$

Hence, we have shown that there exists an integral-bounded signal with integral bound $M = \frac{k_2}{k_1}\tilde{M} = 10\frac{k_2}{k_1}$ that causes the states to grow unbounded for a particular initial condition. ■

Next, we show that if an integral bound M is given *a priori*, then we can always design a static stabilizing linear feedback to ensure boundedness of the trajectories.

Theorem 13.2 Let M be given. If k_1 and k_2 satisfy $\frac{k_2}{k_1} > 16M$, then for any $d \in \Omega_M$ and any initial condition, we have $x_1, x_2 \in \mathcal{L}_\infty$.

Proof : The proof of Theorem 13.2 is a consequence of Lemmas 13.1 and 13.2 which are stated and proved below.

Lemma 13.1 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + y, \\ \dot{x}_2 &= \sigma(-k_1 x_1 - k_2 x_2), \end{aligned} \tag{13.4}$$

where $|y(t)| < 2M$ for all $t \geq 0$ and $\frac{k_2}{k_1} > 16M$. In that case, we have $x_1, x_2 \in \mathcal{L}_\infty$ for any initial condition.

Proof of Lemma 13.1 : Define a positive definite function V as

$$V = \int_0^{k_1 x_1} \sigma(s)\,\mathrm{d}s + \int_0^{k_1 x_1 + k_2 x_2} \sigma(s)\,\mathrm{d}s + k_1 x_2^2.$$

This function was first introduced in [12]. Differentiating V along the trajectories yields

$$\begin{aligned}
\dot{V} &= (k_1 x_2 + k_1 y)\sigma(k_1 x_1) - 2k_1 x_2 \sigma(k_1 x_1 + k_2 x_2) \\
&\quad + [k_1 x_2 + k_1 y - k_2 \sigma(k_1 x_1 + k_2 x_2)]\,\sigma(k_1 x_1 + k_2 x_2) \\
&= k_1 x_2\,[\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] - k_2 \sigma^2(k_1 x_1 + k_2 x_2) k_1 y[\sigma(k_1 x_1 + k_2 x_2) + \sigma(k_1 x_1)] \\
&\leq k_1 x_2\,[\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] - k_2 \sigma^2(k_1 x_1 + k_2 x_2) + 2k_1 |y|.
\end{aligned}$$

If $|k_1 x_1 + k_2 x_2| > \frac{1}{2}$, then

$$-k_2 \sigma^2(k_1 x_1 + k_2 x_2) + 2k_1 |y| \leq -16 M k_1 \times \tfrac{1}{4} + 4 k_1 M \leq 0.$$

Hence

$$\dot{V} \leq k_1 x_2\,[\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] \leq 0.$$

If $|k_1 x_1 + k_2 x_2| \leq \frac{1}{2}$, then by using Lemma B.2 in [107], we get

$$k_1 x_2\,[\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] \leq -\tfrac{k_1}{2} x_2 \sigma(k_2 x_2).$$

If we also have that $|x_2| \geq \max\{8M, \frac{1}{k_2}\}$, then

$$k_1 x_2\,[\sigma(k_1 x_1) - \sigma(k_1 x_1 + k_2 x_2)] \leq -\tfrac{k_1}{2} x_2 \sigma(k_2 x_2) \leq -4 k_1 M,$$

which yields $\dot{V} \leq 0$. We therefore conclude that $\dot{V} \leq 0$ outside the region defined by $|k_1 x_1 + k_2 x_2| \leq \frac{1}{2}$ and $|x_2| \leq \max\{8M, \frac{1}{k_2}\}$. Hence, V remains bounded, which implies that $x_1, x_2 \in \mathcal{L}_\infty$. ■

Now consider the double integrator system (13.1). We construct a fictitious state

$$\dot{y} = \sigma(-k_1 x_1 - k_2 x_2) - \sigma(-k_1 x_1 - k_2 x_2 + k_2 y) + d, \quad y(0) = 0.$$

By defining $z = x_2 - y$, we obtain the augmented system

$$\begin{aligned}\dot{x}_1 &= z + y,\\ \dot{z} &= \sigma(-k_1 x_1 - k_2 z),\\ \dot{y} &= \sigma(-k_1 x_1 - k_2 x_2) - \sigma(-k_1 x_1 - k_2 x_2 + k_2 y) + d,\end{aligned}$$

with $y(0) = 0$, $z(0) = x_2(0)$. From Lemma 13.1, we know that given $\frac{k_2}{k_1} > 16M$, x_1 and z remain bounded provided $|y| \leq 2M$. The latter statement is proven by the following lemma.

Lemma 13.2 Consider the system

$$\dot{y} = \sigma(v) - \sigma(v + k_2 y) + d, \quad y(0) = 0, \tag{13.5}$$

where $k_2 > 0$, $d \in \Omega_M$ and v is continuous. We have $|y(t)| \leq 2M$ for all $t \geq 0$.

Proof of Lemma 13.2 : Define

$$\dot{\bar{y}} = d, \quad \bar{y}(0) = 0.$$

Since $d \in \Omega_M$, the solution satisfies $|\bar{y}(t)| \leq M$ for all $t \geq 0$. Define $\tilde{y} = y - \bar{y}$. We have

$$\dot{\tilde{y}} = \sigma(v) - \sigma(v + k_2(\tilde{y} + \bar{y})), \quad \tilde{y}(0) = 0.$$

Define a positive definite function $\tilde{V} = \tilde{y}^2$. Taking the derivative of $\tilde{V}$ with respect to t, we get

$$\dot{\tilde{V}} = \tilde{y}\left[\sigma(v) - \sigma(v + k_2(\tilde{y} + \bar{y}))\right].$$

If $\tilde{V} \geq M^2$, then $|\tilde{y}(t)| \geq M \geq |\bar{y}(t)|$, which implies that $k_2(\tilde{y} + \bar{y})$ has the same sign as $\tilde{y}$. It then follows that $\dot{\tilde{V}} \leq 0$. Since $\tilde{V}(0) = 0$, we can conclude that $\tilde{V} \leq M^2$ and thus $|\tilde{y}(t)| \leq M$ for all $t \geq 0$, and it follows that $|y(t)| \leq |\bar{y}(t)| + |\tilde{y}(t)| \leq 2M$ for all $t \geq 0$. ∎

From Lemmas 13.1 and 13.2, we know that x_1 and z are bounded. Since y is bounded as shown in Lemma 13.2, we conclude that $x_1, x_2 \in \mathcal{L}_\infty$. ∎

An immediate consequence of Theorems 13.1 and 13.2 is that if k_1 and k_2 are arbitrary positive real numbers, then boundedness is guaranteed if the integral bound M is sufficiently small. This is formally stated in the following corollary.

Corollary 13.1 For any given $k_1 > 0$ and $k_2 > 0$, we have $x_1, x_2 \in \mathcal{L}_\infty$ if $d \in \Omega_M$ with $M \le \frac{k_2}{16k_1}$.

In the next theorem we consider integral-bounded disturbances that are biased by a DC signal. We show that, if the magnitude of the bias is less than 1 by a known margin, and an integral bound M is known *a priori*, then k_1, k_2 can be chosen to ensure boundedness of x_1, x_2.

Theorem 13.3 Let $M > 0$ and $\delta \in (0, 1]$ be given, and suppose that $d = d_1 + d_2$ where d_1 is a constant with $|d_1| \le 1 - \delta$ and $d_2 \in \Omega_M$. If k_1, k_2 satisfy $k_2 \ge \max\{\frac{1-\delta}{M}, \frac{48k_1 M}{\delta^2}\}$, then $x_1, x_2 \in \mathcal{L}_\infty$.

Proof : The closed-loop system is given by

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = \sigma(-k_1x_1 - k_2x_2) + d_1 + d_2.$$

We construct a fictitious state

$$\dot{y} = \sigma(-k_1x_1 - k_2x_2) - \sigma(-k_1x_1 - k_2x_2 + k_2y) + d_2, \quad y(0) = 0.$$

Lemma 13.1 shows that $|y(t)| \le 2M$ for all $t \ge 0$. Similar to the proof of Theorem 13.2, we define $z = x_2 - y$ and convert the closed-loop system to the form

$$\dot{x}_1 = z + y,$$

$$\dot{z} = \sigma(-k_1x_1 - k_2z) + d_1,$$

with $z(0) = x_2(0)$ and $y \in \mathcal{L}_\infty(2M)$.

We also introduce another fictitious state

$$\dot{w} = \sigma(-k_1x_1 - k_2z) - \sigma(-k_1x_1 - k_2z + k_2w - d_1)$$

with $w(0) = 0$. Following the same argument as in the proof of Lemma 13.2, we can show that $|w(t)| \le \frac{1-\delta}{k_2} \le M$ for all $t \ge 0$. Define $\xi_1 = x_1$, $\xi_2 = z - w = x_2 - y - w$. Then (13.1) can be transformed into

$$\dot{\xi}_1 = \xi_2 + w + y,$$

$$\dot{\xi}_2 = \sigma(-k_1\xi_1 - k_2\xi_2 - d_1) + d_1,$$

where $\xi_1(0) = x_1(0)$, $\xi_2(0) = x_2(0)$ and $|w(t) + y(t)| \leq M + 2M = 3M$ for all $t \geq 0$. Since $w(t)$ and $y(t)$ are bounded, we know that x_1 and x_2 are bounded if ξ_1 and ξ_2 are bounded.

Define $\tilde{\sigma}_{d_1}(s) = \sigma(s - d_1) + d_1$ with $|d_1| \leq 1 - \delta$. Then

$$\tilde{\sigma}_{d_1}(s) = \begin{cases} 1 + d_1, & s \geq 1 + d_1, \\ s, & -1 + d_1 \leq s < 1 + d_1, \\ -1 + d_1, & s \leq -1 + d_1. \end{cases} \tag{13.6}$$

This function can be viewed as a generalized saturation function, which is visualized in Fig. 13.2. It is

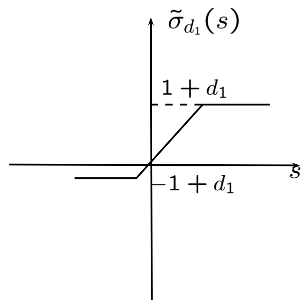


Figure 13.2: Generalized saturation function $\tilde{\sigma}_{d_1}(s)$

easy to verify that $\tilde{\sigma}_{d_1}$ satisfies the following properties:

1. $|\tilde{\sigma}_{d_1}(s)| \leq 2$
2. $s\tilde{\sigma}_{d_1}(s) \geq 0$ and $s\tilde{\sigma}_{d_1}(s) = 0$ iff $s = 0$
3. $s\left[\tilde{\sigma}_{d_1}(v + s) - \tilde{\sigma}_{d_1}(v)\right] \geq 0$

Moreover, it is shown in Lemma 13.3 in the Appendix that if $|v| \leq \frac{\delta}{2}$, then

$$s\left[\tilde{\sigma}_{d_1}(v + s) - \tilde{\sigma}_{d_1}(v)\right] \geq s\sigma_{\delta/2}(s),$$

where $\sigma_{\delta/2}(s)$ is the standard saturation function with saturation level $\delta/2$, which is defined by $\sigma_{\delta/2}(s) = \mathrm{sign}(s)\min\{\delta/2, |s|\}$.

With this generalized saturation function, the closed-loop system can be rewritten as

$$\dot{\xi}_1 = \xi_2 + w + y,$$
$$\dot{\xi}_2 = \tilde{\sigma}_{d_1}(-k_1\xi_1 - k_2\xi_2).$$

Define a positive definite function

$$V = \int_0^{k_1\xi_1} \tilde{\sigma}_{d_1}(s)\,\mathrm{d}s + \int_0^{k_1\xi_1+k_2\xi_2} \tilde{\sigma}_{d_1}(s)\,\mathrm{d}s + k_1\xi_2^2.$$

Differentiating V along the trajectory yields

$$\begin{aligned}
\dot{V} &= (k_1\xi_2 + k_1 w + k_1 y)\tilde{\sigma}_{d_1}(k_1\xi_1) - 2k_1\xi_2\tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2) + \\
&\qquad \left[k_1\xi_2 + k_1 w + k_1 y - k_2\tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)\right]\tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2) \\
&= k_1\xi_2\left[\tilde{\sigma}_{d_1}(k_1\xi_1) - \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)\right] - k_2\tilde{\sigma}_{d_1}^2(k_1\xi_1 + k_2\xi_2) \\
&\qquad + k_1(w+y)\left[\tilde{\sigma}_{d_1}(k_1\xi_1) + \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)\right] \\
&\le k_1\xi_2\left[\tilde{\sigma}_{d_1}(k_1\xi_1) - \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)\right] - k_2\tilde{\sigma}_{d_1}^2(k_1\xi_1 + k_2\xi_2) + 12k_1 M.
\end{aligned}$$

If $|k_1\xi_1 + k_2\xi_2| \ge \frac{\delta}{2}$, then

$$-k_2\tilde{\sigma}_{d_1}^2(k_1\xi_1 + k_2\xi_2) + 12k_1 M \le -\tfrac{48k_1 M}{\delta^2}\tfrac{\delta^2}{4} + 12k_1 M = 0,$$

and hence $\dot{V} \le 0$. If $|k_1\xi_1 + k_2\xi_2| \le \frac{\delta}{2}$ and $|\xi_2| \ge \max\{\frac{\delta}{2k_2}, \frac{24M}{\delta}\}$, then by using Lemma 13.3 we have

$$k_1\xi_2\left[\tilde{\sigma}_{d_1}(k_1\xi_1) - \tilde{\sigma}_{d_1}(k_1\xi_1 + k_2\xi_2)\right] \le -k_1\xi_2\sigma_{\delta/2}(k_2\xi_2) \le -k_1\tfrac{24M}{\delta}\tfrac{\delta}{2} \le -12k_1 M,$$

and hence $\dot{V} \le 0$. We therefore find that $\dot{V} \le 0$ outside the region defined by $|k_1\xi_1 + k_2\xi_2| \le \frac{1}{2}$ and $|\xi_2| \le \max\{\frac{\delta}{2k_2}, \frac{24M}{\delta}\}$. It follows that V remains bounded, which implies that ξ_1 and ξ_2 remain bounded. ■

Our final result concerns a special case where the disturbance consists of a finite number of sinusoids together with a DC bias of magnitude less than 1. In this case, any internally stabilizing linear static feedback controller guarantees that the states of the system (13.1) remain bounded.

Theorem 13.4 Consider the system (13.1) with $k_1 > 0$ and $k_2 > 0$. Suppose that $d = d_1 + d_2$, where d_1 is a constant satisfying $|d_1| < 1$ and d_2 is generated by an exogenous system

$$\begin{aligned}
\dot{w} &= Aw, \quad w(0) = w_0, \\
d &= Cw,
\end{aligned}$$

where A is non-singular and satisfies $A + A' = 0$. We have $x_1, x_2 \in \mathcal{L}_\infty$ for any initial condition.

Proof : We can rewrite the closed-loop system in a compact form:

$$\begin{bmatrix} \dot{w} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [\sigma(-k_1 x_1 - k_2 x_2) + d_1].$$

Consider the state transformation

$$\begin{bmatrix} w \\ \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -CA^{-2} & 1 & 0 \\ -CA^{-1} & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ x_1 \\ x_2 \end{bmatrix}.$$

This transformation results in the system

$$\begin{bmatrix} \dot{w} \\ \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \left[\sigma\left(-(k_1 CA^{-2} + k_2 CA^{-1})w - k_1 \bar{x}_1 - k_2 \bar{x}_2\right) + d_1\right].$$

Define $v = -(k_1 CA^{-2} + k_2 CA^{-1})w + d_1$. Then

$$\begin{aligned} \sigma(-(k_1 CA^{-2} + k_2 CA^{-1})w - k_1 \bar{x}_1 - k_2 \bar{x}_2) + d_1 &= \sigma(v - k_1 \bar{x}_1 - k_2 \bar{x}_2 - d_1) + d_1 \\ &= \tilde{\sigma}_{d_1}(-k_1 \bar{x}_1 - k_2 \bar{x}_2 + v), \end{aligned}$$

where $\tilde{\sigma}_{d_1}$ is the generalized saturation function defined in the proof of Theorem 13.3. The dynamics of $\bar{x}_1$ and $\bar{x}_2$ can now be written as

$$\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2, \\ \dot{\bar{x}}_2 &= \tilde{\sigma}_{d_1}(-k_1 \bar{x}_1 - k_2 \bar{x}_2 + v). \end{aligned}$$

Clearly $v \in \mathcal{L}_\infty$. It was shown by [12] that the $(\bar{x}_1, \bar{x}_2)$ dynamics is $\mathcal{L}_\infty$ stable from v to $\bar{x}_1$ and $\bar{x}_2$ for any $k_1 > 0$ and $k_2 > 0$. ■

Remark 13.1 *For ease of presentation, we use a standard saturation function with saturation level* 1*, but all the results obtained in this chapter can easily be extended to the case where a saturation function with arbitrary saturation level* Δ *is used.*

Appendix

Intersection problem in the proof of Theorem 1

We shall show that

$$y_1(\tilde{t}) = -\frac{\lambda}{2}\tilde{t}^2 + (N\lambda + \lambda)\tilde{t} + 2\lambda(1 - \cos(\pi\tilde{t})) - \frac{2\lambda}{\pi}\sin(\pi\tilde{t}) + 1$$

has only one intersection with $y_1 = 1$ for $\tilde{t} > 0$ and sufficiently large N. Let

$$\tilde{t}_1 = \min\{\tilde{t} > 0 : y_1(\tilde{t}) = 1\}, \quad \tilde{t}_0 = \min\left\{\tilde{t} > 0 : \frac{\mathrm{d}y_1}{\mathrm{d}\tilde{t}}(\tilde{t}) = 0\right\}.$$

Given $y_1(0) = 1$ and $\frac{\mathrm{d}y_1}{\mathrm{d}\tilde{t}} > 0$, we must have $\tilde{t}_1 > \tilde{t}_0$. Note that

$$\frac{\mathrm{d}y_1}{\mathrm{d}\tilde{t}}(\tilde{t}) = -\lambda\tilde{t} + N\lambda + \lambda + 2\lambda\pi\sin(\pi\tilde{t}) - 2\lambda\cos(\pi\tilde{t}).$$

Hence $\tilde{t}_0 \geq N + 1 - 2\pi - 2 > \frac{N}{2}$ for sufficiently large N. However,

$$y_1(\tilde{t}) \geq -\frac{\lambda}{2}\tilde{t}^2 + (N\lambda + \lambda)\tilde{t} - 4\lambda - 2\lambda + 1$$

Hence we have

$$-\frac{\lambda}{2}\tilde{t}_1^2 + (N\lambda + \lambda)\tilde{t}_1 - 4\lambda - 2\lambda \leq 0$$

or equivalently

$$\frac{1}{2}\tilde{t}_1^2 - (N+1)\tilde{t}_1 + 6 \geq 0$$

This implies

$$\tilde{t}_1 < r_1, \text{ or } \tilde{t}_1 > r_2$$

where r_1 and r_2 are two roots of $\frac{1}{2}\tilde{t}^2 - (N+1)\tilde{t} + 6$.

$$r_{1,2} = N + 1 \mp \sqrt{(N+1)^2 - 12}$$

Note that

$$r_1 = N + 1 - \sqrt{(N+1)^2 - 12} = \frac{12}{N + 1 + \sqrt{(N+1)^2 - 12}} \leq \frac{N}{2}$$

for sufficiently large N. Since we already know $\tilde{t}_1 > \tilde{t}_0 > \frac{N}{2}$, we must have $\tilde{t}_1 > r_2$. But then

$$r_2 = N + 1 + \sqrt{(N+1)^2 - 12} \geq N + 1 + \sqrt{(N+1)^2/4} = \frac{3}{2}(N+1)$$

for large N. We find that $\tilde{t}_1 > \frac{3}{2}(N+1)$. But for $\tilde{t} > \frac{3}{2}(N+1)$, we have

$$\begin{aligned}\frac{\mathrm{d}y_1}{\mathrm{d}\tilde{t}}(\tilde{t}) &< -\frac{3}{2}\lambda(N+1) + N\lambda + \lambda + 2\lambda\pi\sin(\pi\tilde{t}) - 2\lambda\cos(\pi\tilde{t}) \\ &< -\frac{1}{2}\lambda(N+1) + 2\lambda\pi\sin(\pi\tilde{t}) - 2\lambda\cos(\pi\tilde{t}) < 0\end{aligned}$$

for sufficiently large N. This shows that $y_1(\tilde{t}) < 1$ for all $\tilde{t} > \tilde{t}_1$, and hence, the only intersection with $y_1 = 1$ is at $\tilde{t} = \tilde{t}_1$.

Property of $\tilde{\sigma}_{d_1}(s)$

Lemma 13.3 The generalized saturation function $\tilde{\sigma}_{d_1}$ defined in (13.6) with $|d_1| \leq 1 - \delta$ satisfies

$$s\left[\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v)\right] \geq s\sigma_{\delta/2}(s)$$

for $|v| \leq \frac{\delta}{2}$ where $\sigma_{\delta/2}(s)$ denotes the standard saturation function with saturation level $\delta/2$ defined as $\sigma_{\delta/2}(s) = \text{sign}(s)\min\{\delta/2, |s|\}$.

Proof : If $|s| < \frac{\delta}{2}$, we have $|v+s| \leq \delta \leq 1 - |d_1|$. By definition (13.6), we have $\tilde{\sigma}_{d_1}(s+v) = s+v$. Hence

$$\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v) = s + v - v = s.$$

If $|s| \geq \frac{\delta}{2}$, it can be seen from Fig. 13.2 that

$$|\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v)| \geq |\,\text{sign}(s)\tfrac{\delta}{2} + v - v| = \tfrac{\delta}{2}.$$

Hence $s\left[\tilde{\sigma}_{d_1}(s+v) - \tilde{\sigma}_{d_1}(v)\right] \geq s\sigma_{\delta/2}(s)$. ∎

CHAPTER 14

Control of a chain of integrators subject to actuator saturation and disturbances

14.1. Introduction

This chapter is a continuation of previous chapter onto the general case where integrator-chain has length greater than 2. Specifically, we are interested in the following system:

$$\dot{x} = Ax + B\sigma(u) + Ed, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad d \in \mathbb{R}^m, \tag{14.1}$$

where x, A and B are given by

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \tag{14.2}$$

and u is a control input. The function $\sigma(\cdot)$ denotes a standard saturation; that is, $\sigma(u) = \text{sign}(u)\min\{1, |u|\}$. Since there are more places where disturbances can affect the system, the situation becomes more complicated compared with double integrator. Hence, it is prudent to first classify the disturbances according to the convention. If $E = B$, the disturbance is said to be *matched* with the control input. Otherwise, the disturbance is said to be *unmatched*. We will also deal specifically with the situation $B'E = 0$, in which case we say that the disturbance is *misaligned* with the input. The goal is to identify a class of disturbances for which the boundedness of the state can be ensured by a static or dynamic feedback controller.

This chapter is a further extension of the results in [106, 149] and previous chapter for $n > 2$. We shall show that a result similar to the double-integrator case holds for the case $n > 2$ as well; namely, that by the proper choice of feedback law, boundedness of the states can be ensured for both (i) matched, integral-bounded disturbances; (ii) misaligned, magnitude-bounded disturbances; and (iii) a combination of the two.

The chapter is organized as follows: In Section 2, we recall some standard notations and present the main results of the paper. In Section 3, we first recall the classical low-gain feedback design, which we use to develop a nonlinear dynamic low-gain feedback. In Section 4, we prove our main results based on the feedback laws developed in Section 3.

14.2. Main result

We first recall some standard notations used in this chapter. Let the vectors $e_1, \ldots, e_n$ denote the standard basis for $\mathbb{R}^n$; that is, e_i is a unit vector with the ith entry equal to 1. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes its transpose. For a matrix $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes its transpose. For a positive-definite matrix $X \in \mathbb{R}^{n\times n}$, $C = X^{1/2} \in \mathbb{R}^{n\times n}$ is a non-singular matrix such that $X = C'C$.

We define a class of integral bounded signal as in previous chapter

$$\Omega_\infty = \left\{ d \in \mathcal{L}_\infty \mid \text{ there exsits } M > 0 \text{ such that } \left| \int_{t_1}^{t_2} d(t)\, \mathrm{d}t \right| \leq M, \quad \forall\, t_2 \geq t_1 \geq 0 \right\}.$$

We now present the main results. The first theorem shows that if the disturbance is misaligned with the input (i.e., $B'E = 0$), then boundedness of the state can be ensured for any bounded disturbance by using a nonlinear static state feedback.

Theorem 14.1 Consider the system (14.1) with $B'E = 0$. There exists a nonlinear state feedback such that the closed-loop system satisfies the following properties:

1. In the absence of d, the origin is globally asymptotically stable.
2. If $d \in \mathcal{L}_\infty$, then $x \in \mathcal{L}_\infty$ for any $x(0) \in \mathbb{R}^n$.

If the disturbance is matched with the input (i.e. $E = B$), then the boundedness of the state trajectories can be preserved if the disturbance is integral-bounded.

Theorem 14.2 Consider the system (14.1) with $B = E$. There exists a nonlinear state feedback such that the closed-loop system satisfies the following properties:

1. In the absence of d, the origin is globally asymptotically stable.

2. If $d \in \Omega_\infty$, then $x \in \mathcal{L}_\infty$ for any $x(0) \in \mathbb{R}^n$.

We can combine the matched and the misaligned cases to obtain a more general case of mismatched disturbances. Specifically, consider the system

$$\dot{x} = Ax + B\sigma(u) + E_1 d_1 + E_2 d_2, \tag{14.3}$$

where $B' E_1 = 0$ and E_2 is given by

$$E_2 = \begin{bmatrix} \bar{E}_2 \\ \alpha \end{bmatrix}, \tag{14.4}$$

where α is either a non-zero real number or a row vector with only non-zero elements and $\bar{E}_2$ can be an arbitrary matrix with appropriate dimension. Based on the previous theorems, we can prove the following result.

Theorem 14.3 Consider the system (14.3). There exists a nonlinear state feedback such that the closed-loop system satisfies the following properties:

1. In the absence of d, the origin is globally asymptotically stable.

2. If $d_1 \in \mathcal{L}_\infty$ and $d_2 \in \Omega_\infty$, then $x \in \mathcal{L}_\infty$ for any $x(0) \in \mathbb{R}^n$.

14.3. Controller design

In this section we shall construct controllers that will be used to prove our main results. We start with a brief review of classical low-gain state feedback design.

14.3.1. Classical low-gain state feedback design

Classical low-gain feedback provides a family of parameterized stabilizing static feedback gains that vanish asymptotically as the parameter approaches zero. The philosophy behind classical low-gain design is that, by choosing the parameter small enough, the feedback gain can be made sufficiently small so that the saturation remains inactive in the whole state space or within any pre-specified compact subset. Classical low-gain design can be carried out using one of three approaches, namely, the method of direct eigen-structure assignment [51]; the H_2 and H_∞ ARE-based method [61, 130]; or the parametric

Lyapunov-based method [159]. In this paper, we choose the parametric Lyapunov-based method because of its convenient properties when applied to a chain of integrators.

Consider the system (14.1) and let P_ε be the unique positive-definite solution of the parametric Riccati equation

$$A'P_\varepsilon + P_\varepsilon A - P_\varepsilon BB'P_\varepsilon + \varepsilon P_\varepsilon = 0. \tag{14.5}$$

The classical low-gain state feedback is given by

$$u = -B'P_\varepsilon x. \tag{14.6}$$

It is shown in [159] that (14.6) solves the semi-global stabilization problem for the system (14.1). In the global setting, the feedback takes the same form as in (14.6), but the low-gain parameter ε, instead of being fixed, is scheduled as a function of the state of the system. Such a scheduling has to satisfy the following properties for some design parameter $\delta \leq 1$.

1. There exists an open neighborhood O of the origin such that for all $x \in O$, $\varepsilon_a(x) = 1$.
2. For any $x \in \mathbb{R}^n$, $|B'P_{\varepsilon_a(x)}x| \leq \delta$.
3. $\varepsilon_a(x) \to 0 \implies \|x\| \to \infty$.
4. For each $c > 0$, the set $\{x \in \mathbb{R}^n \mid x'P_{\varepsilon_a(x)}x \leq c\}$ is bounded.
5. There is a function $g\colon [0,\infty) \to (0,1]$ such that for all $x \neq 0$, $\varepsilon_a(x) = g(x'P_{\varepsilon_a(x)}x)$.

A particular choice of $\varepsilon_a(x)$, given in [68], is

$$\varepsilon_a(x) = \max\left\{r \in (0,1] \mid (x'P_r x) \times (B'P_r B) \leq \delta^2\right\}, \tag{14.7}$$

where P_r is the solution of (14.5) with $\varepsilon = r$. Based on this scheduling, the feedback law is given by

$$u = -B'P_{\varepsilon_a(x)}x. \tag{14.8}$$

14.3.2. Dynamic low-gain state feedback design

We now consider the chain of integrators and construct controllers that will be used to prove all the theorems of Section 2. For the case of misaligned disturbances, which is treated in Theorem 1, we can simply apply the classical scheduled low-gain state feedback (14.8) and (14.7) with $\delta = 1$. However, for the matched case, treated in Theorem 2, and the combined case, treated in Theorem 3, we construct a dynamic controller as follows:

$$\begin{cases} \dot{y} = \sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x}), \\ u = -B' P_{\varepsilon_a(\bar{x})} x, \end{cases} \tag{14.9}$$

where $P_{\varepsilon_a(\bar{x})}$ is the solution of (14.5) with

$$\varepsilon = \varepsilon_a(\bar{x}) := \max \left\{ r \in [0, 1] \mid (\bar{x}' P_r \bar{x}) \times (B' P_r B) \leq \tfrac{1}{4} \right\} \tag{14.10}$$

and

$$\bar{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y \end{bmatrix}.$$

14.4. Proofs of main results

We first prove Theorem 14.1 for the misaligned case $B'E = 0$.

Proof of Theorem 14.1 : Consider the scheduled static low-gain state feedback (14.8) and (14.7) with $\delta = 1$.

Define a Lyapunov function $V(x) = x' P_{\varepsilon_a(x)} x$. Differentiating $V(x)$ along the trajectories yields

$$\begin{aligned} \dot{V} &= x' A' P_{\varepsilon_a(x)} x + x' P_{\varepsilon_a(x)} A x - 2x' P_{\varepsilon_a(x)} B B' P_{\varepsilon_a(x)} x + 2x' P_{\varepsilon_a(x)} E d + x' \tfrac{\mathrm{d} P_{\varepsilon_a(x)}}{\mathrm{d}t} x \\ &\leq -\varepsilon V + 2x' P_{\varepsilon_a(x)} E d + x' \tfrac{\mathrm{d} P_{\varepsilon_a(x)}}{\mathrm{d}t} x. \end{aligned}$$

In absence of d, we have that

$$\dot{V} \leq -\varepsilon V + x' \tfrac{\mathrm{d} P_{\varepsilon_a(x)}}{\mathrm{d}t} x$$

It was shown in [48] that (14.7) implies that $\dot{V}$ and $x' \frac{\mathrm{d} P_{\varepsilon_a(x)}}{\mathrm{d}t} x$ can not have the same sign. Therefore, we find that

$$\dot{V} < 0$$

for all $x \in \mathbb{R}^n$. This shows global asymptotic stability. We proceed to prove Property 2. Lemma 14.1 given in the appendix implies that if $B'E = 0$, then there exists an $M > 0$ depending on system data such that

$$\|P_\varepsilon^{1/2} E\| \leq \varepsilon M$$

for $\varepsilon \in [0, 1]$.

For $d \in \mathcal{L}_\infty$, we have

$$\begin{aligned}
\dot{V} &\leq -\varepsilon V + 2\|x' P_{\varepsilon_a(x)}^{1/2}\| \|P_{\varepsilon_a(x)}^{1/2} E\| \|d\|_\infty + x' \tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t} x \\
&\leq -\varepsilon V + 2\varepsilon M \sqrt{V} \|d\|_\infty + x' \tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t} x \\
&= -\varepsilon \sqrt{V} (\sqrt{V} - 2M \|d\|_\infty) + x' \tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t} x.
\end{aligned}$$

For $V \geq 4M^2 \|d\|_\infty^2$, we have

$$\dot{V} \leq x' \tfrac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t} x.$$

The scheduling (14.7) guarantees that $\dot{V}$ and $x' \frac{\mathrm{d}P_{\varepsilon_a(x)}}{\mathrm{d}t} x$ cannot have the same sign. This implies that $\dot{V} < 0$ for $V \geq 4M^2 \|d\|_\infty^2$. Hence, V is bounded for all $t \geq 0$. Boundedness of x follows from Property 4 of the scheduling. ■

Next, we proceed to the matched case $E = B$.

Proof of Theorem 14.2 : Consider the nonlinear dynamic low-gain state feedback controller (14.9) and (14.10). Define $\bar{y} = x_n - y$. We have

$$\dot{\bar{y}} = \sigma(-B' P_{\varepsilon_a(\bar{x})} x) - \sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x}) + d.$$

Note that $\bar{x} = x - B x_n + B y = x - B\bar{y}$. Hence

$$\dot{\bar{y}} = \sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x} - B' P_{\varepsilon_a(\bar{x})} B \bar{y}) - \sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x}) + d.$$

We therefore have

$$\dot{\bar{x}} = \dot{x} - B\dot{\bar{y}} = A\bar{x} + B\sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x}) + e_{n-1} \bar{y}.$$

In the new coordinates $(\bar{x}, \bar{y})$, the closed-loop system is given by

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x}) + e_{n-1} \bar{y}, \\ \dot{\bar{y}} = \sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x} - B' P_{\varepsilon_a(\bar{x})} B \bar{y}) - \sigma(-B' P_{\varepsilon_a(\bar{x})} \bar{x}) + d. \end{cases} \tag{14.11}$$

We first show global asymptotic stability in the absence of disturbances. Close to the origin, we have $\varepsilon_a(\bar{x}) = 1$, and all the saturations are inactive. Equation (14.11) then reduces to a linear system

$$\begin{cases} \dot{\bar{x}} = A\bar{x} - BB'P_1\bar{x} + e_{n-1}\bar{y}, \\ \dot{\bar{y}} = -B'P_1B\bar{y}, \end{cases}$$

where P_1 is the solution of (14.5) with $\varepsilon = 1$. Local stability is therefore obvious. To prove global attractivity, consider the dynamics of $\bar{y}$. Define a Lyapunov function $V_1 = \bar{y}^2$. We then have

$$\dot{V}_1 = 2\bar{y}\left[\sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x} - B'P_{\varepsilon_a(\bar{x})}B\bar{y}) - \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x})\right].$$

The scheduling (14.10) guarantees that $|B'P_{\varepsilon_a(\bar{x})}\bar{x}| \leq \frac{1}{2}$. Therefore, owing to Lemma 14.3 in the appendix, we find that

$$\dot{V}_1 \leq -\bar{y}\sigma(B'P_{\varepsilon_a(\bar{x})}B\bar{y}). \tag{14.12}$$

This shows that $\bar{y}$ is bounded. Since $B'e_{n-1} = 0$, Theorem 14.1 implies that $\bar{x}$ is bounded for all $t \geq 0$. Hence $\varepsilon_a(x)$ is bounded away from zero, which, together with (14.12) implies $\bar{y} \to 0$ as $t \to 0$.

Next consider the dynamics of $\bar{x}$. Define another Lyapunov function $V_2(\bar{x}) = \bar{x}'P_{\varepsilon_a(\bar{x})}\bar{x}$ and a set

$$\mathcal{K} = \{\bar{x} \mid V_2(\bar{x}) \leq \tfrac{1}{2B'P_1B}\}.$$

It can be easily seen from (14.10) that for $\bar{x} \in \mathcal{K}$, $\varepsilon_a(\bar{x}) = 1$. Differentiating V_2 along the trajectory, we have

$$\begin{aligned} \dot{V}_2 &\leq -\varepsilon_a(\bar{x})V_2 + 2\bar{x}'P_{\varepsilon_a(x)}e_{n-1}\bar{y} + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x} \\ &\leq -\varepsilon_a(\bar{x})V_2 + 2|\bar{y}|\sqrt{V_2}\|P^{1/2}_{\varepsilon_a(\bar{x})}e_{n-1}\| + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x} \\ &\leq -\varepsilon_a(\bar{x})V_2 + 2M_2\varepsilon_a(\bar{x})|\bar{y}|\sqrt{V_2} + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x} \\ &= -\varepsilon_a(\bar{x})\sqrt{V_2}(\sqrt{V_2} - 2M_2|\bar{y}|) + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x}. \end{aligned}$$

Since $\bar{y} \to 0$, for given $\bar{y}(0)$ and $\bar{x}(0)$, there exists a T such that $|y(t)| \leq \min\{\frac{1}{2}, \frac{1}{4M_2\sqrt{B'P_1B}}\}$ for $t \geq T$. Therefore, for $t \geq T$ and $\bar{x} \notin \mathcal{K}$, $\sqrt{V_2} - 2M_2|\bar{y}| \geq \frac{\sqrt{V_2}}{2}$, and thus

$$\dot{V}_2 \leq -\tfrac{\varepsilon_a(\bar{x})}{2}V_2 + \bar{x}'\tfrac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x}.$$

Since $\dot{V}_2$ cannot have the same sign as $\bar{x}'\frac{\mathrm{d}P_{\varepsilon_a(\bar{x})}}{\mathrm{d}t}\bar{x}$, we conclude that $\dot{V}_2 < 0$ for $\bar{x} \notin \mathcal{K}$. This implies that $\bar{x}$ will enter $\mathcal{K}$ within finite time after $t = T$ and remain in $\mathcal{K}$ thereafter. For $t > T$ and $\bar{x} \in \mathcal{K}$, we have $\varepsilon_a(\bar{x}) = 1$ and $|y| \leq \frac{1}{2}$. All saturations are inactive and the system becomes linear. It therefore follows that $\bar{x} \to 0$, which shows that the origin is globally attractive.

When disturbances are present, Lemma 14.2 shows that $|\bar{y}| \in \mathcal{L}_\infty$ given $d \in \Omega_\infty$. Boundedness of $\bar{x}$ therefore follows from Theorem 14.1. ■

Finally, we prove Theorem 14.3 for the combined case by using Theorems 14.1 and 14.2.

Proof of Theorem 14.3 : This proof is basically a combination of those of Theorems 14.1 and 14.2. Consider the dynamic low-gain state feedback (14.9) and the scheduling (14.10). Define $\bar{y} = x_n - y$. We have

$$\dot{\bar{y}} = \sigma(-B'P_{\varepsilon_a(\bar{x})}x) - \sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + \alpha d_2.$$

Note that $\bar{x} = x - Bx_n + By = x - B\bar{y}$. Hence

$$\dot{\bar{y}} = \sigma(-B'P_{\varepsilon_a(\bar{x})}x) - \sigma(-B'P_{\varepsilon_a(\bar{x})}x + B'P_{\varepsilon_a(\bar{x})}B\bar{y}) + \alpha d_2.$$

Lemma 14.2 shows that $|\bar{y}| \in \mathcal{L}_\infty$ given $d \in \Omega_\infty$ for any $\bar{y}(0)$. We have

$$\dot{\bar{x}} = \dot{x} - B\dot{\bar{y}} = A\bar{x} + B\sigma(-B'P_{\varepsilon_a(\bar{x})}\bar{x}) + \begin{bmatrix} E_1 & \tilde{E}_2 & e_{n-1} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ \bar{y} \end{bmatrix},$$

where

$$\tilde{E}_2 = \begin{bmatrix} \bar{E}_2 \\ 0 \end{bmatrix}.$$

Note that

$$B'\begin{bmatrix} E_1 & \tilde{E}_2 & e_{n-1} \end{bmatrix} = 0.$$

The rest of the proof now proceeds in the same way as the proof of Theorem 14.1. ■

14.5. Example

We conclude the paper with an example. Consider the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sigma(u) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} d_2$$

where $d_1 = \sin t$ and $d_2 = 2$. The controller u is given by (14.9) and (14.10). The simulation data is shown in the following figure:

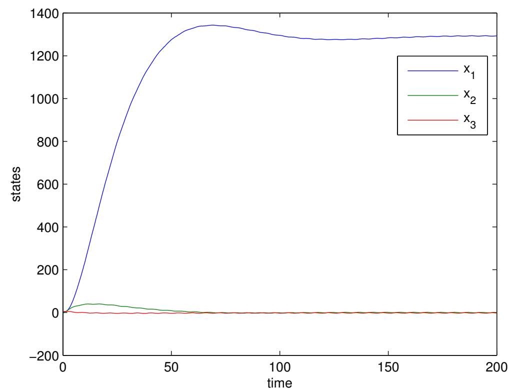


Figure 14.1: Triple integrator with actuator saturation and disturbances

Appendix

The following lemma regarding the properties of P_ε is adapted from [159].

Lemma 14.1 The parametric Riccati equation (14.5) associated with data A, B given by (14.2) has a unique positive-definite solution P_ε with the following properties:

1. P_ε is a polynomial matrix in ε.

2. $P_\varepsilon \to 0$ as $\varepsilon \to 0$.

3. $\frac{\mathrm{d}P_\varepsilon}{\mathrm{d}\varepsilon} > 0$ for all $\varepsilon \in [1, 0)$.

4. There exists an $M > 0$ such that for any $\varepsilon \in [0, 1]$,

$$e_i' P_\varepsilon e_i \le M\varepsilon^2,$$

   where $i < n$ and e_i is a unit vector whose ith entry is 1.

Proof : The first three properties were proven in [159]. Regarding Property 4, it was shown in [159] (see Lemma 1) that the unique positive-definite solution $P_\varepsilon = \left[p_{i,j}\right]_{n\times n}$ to the parametric Riccati equation associated with A, B given by (14.2) can be computed using the following recursion: for $i = n-1, \ldots, 0$

$$p_{i+1,n} = p_{n,i+1} = (-1)^{i+1}\left[\sum_{i+1}^{n}(-1)^k p_{n,k+1} C_k^{k-i} \varepsilon^{k-i} + (-1)^n C_n^{n-i} \varepsilon^{n-i}\right]$$

with $C_k^{k-i} = \frac{k!}{i!(k-i)!}$ and $p_{n,n+1} = 0$. For $k = j, j-1, \dots, 1$ and $j = n-1, \dots, 1$

$$p_{k,j} = p_{k,n}p_{n,j+1} - p_{j+1,k-1} - \varepsilon p_{k,j+1}$$

with $p_{i,0} = p_{0,i} = 0$.

This shows that P_ε is a polynomial matrix in ε and for $i < n$ and $j < n$, $p_{i,j}$ is at least of order ε^2. Therefore, for $i < n$, $e_i' P_\varepsilon e_i$ is at least of order ε^2. ■

Lemma 14.2 Consider the system

$$\dot{y}(t) = \sigma(v(t)) - \sigma(v(t) + k(t)y) + d, \tag{14.13}$$

where $d \in \Omega_\infty$ and $k(t) > 0$ and $v(t)$ are continuous. We have then $y \in \mathcal{L}_\infty$ for all $y(0)$.

Proof : Define

$$\dot{\bar{y}} = d, \quad \bar{y}(0) = y(0).$$

Since $d \in \Omega_\infty$, there exists a $M > 0$ such that $|\bar{y}(t)| \leq |y(0)| + M$ for all $t > 0$. Define $\tilde{y} = y - \bar{y}$. We have

$$\dot{\tilde{y}} = \sigma(v) - \sigma(v + k(\tilde{y} + \bar{y})), \quad \tilde{y} = 0.$$

Let $\tilde{V} = \tilde{y}^2$. Taking the derivative of $\tilde{V}$ with respect to t, we get

$$\dot{\tilde{V}} = \tilde{y}\left[\sigma(v) - \sigma(v + k(\tilde{y} + \bar{y}))\right].$$

If $\tilde{V} \geq (|y(0)| + M)^2$, then $|\tilde{y}| \geq M + |y(0)| \geq |\bar{y}|$. But this implies that $k(\tilde{y} + \bar{y})$ has the same sign as $\tilde{y}$. Thus

$$\dot{\tilde{V}} = \tilde{y}\left[\sigma(v) - \sigma(v + k(\tilde{y} + \bar{y}))\right] \leq 0.$$

Since $\tilde{V}(0) = 0$, we have $\tilde{V} \leq (|y(0)| + M)^2$ and $|\tilde{y}| \leq |y(0)| + M$ for all $t > 0$. Therefore, $|y| \leq |\bar{y}| + |\tilde{y}| \leq 2M + 2y(0)$. ■

The following lemma was shown in [107]:

Lemma 14.3 For any $w \in \mathbb{R}^m$ satisfying $\|w\| \leq \frac{1}{2}$ we have

$$2u'[\sigma(w) - \sigma(w - u)] \geq u'\sigma(u)$$

CHAPTER 15

Control of open-loop neutrally stable systems subject to actuator saturation and external disturbances

15.1. Introduction

In previous two chapters, we focused on systems that have all the eigenvalues at zero. Now we shall consider systems with oscillatory behavior, that is, the open-loop neutrally stable systems. A linear system $\rho x = Ax + Bu$ is said to be neutrally stable if A has all its eigenvalues in the closed left half plane (closed unit disc for discrete-time systems) and at least one eigenvalue on the imaginary axis (unit circle for discrete-time systems); and all the eigenvalues on the imaginary axis (unit circle for discrete-time systems) have Jordan block size 1.

In continuous-time case, systems consisting only of single integrators (i.e., eigenvalues at the origin with Jordan block size 1) can be viewed as neutrally stable systems. It has been shown in previous chapter that the state trajectories remain bounded for all initial conditions and all integral-bounded disturbances. Moreover, this result also holds if we add a sufficiently small DC signal to the disturbances.

We shall extend the results for single-integrator systems to neutrally stable systems. Although a similar result for discrete-time integrator-chain type system as obtained in previous two chapters is not available yet, we do observe a substantial similarity between continuous- and discrete-time neutrally stable systems. The extension made to continuous-time system carries over to its discrete-time counter part. Roughly speaking, we shall show that for disturbances that do not have large sustained frequency components corresponding to the system's eigenvalues on the stability margin, a linear static state feedback can be employed to achieve boundedness of the trajectories for any initial condition and at the same time yield a globally asymptotically stable equilibrium.

15.2. Problem formulation

Consider the following system

$$\rho x = Ax + B\sigma(u) + Ed, \quad x(0) = x_0, \tag{15.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $d \in \mathbb{R}^p$ and ρx represents $\dot{x}$ for continuous-time systems and $x(k+1)$ for discrete-time systems. $\sigma(\cdot)$ denotes the standard saturation function defined as

$$\sigma(\xi) = \begin{bmatrix} \operatorname{sign}(\xi_1)\min\{1, |\xi_1|\} \\ \vdots \\ \operatorname{sign}(\xi_m)\min\{1, |\xi_m|\} \end{bmatrix} \tag{15.2}$$

The pair (A, B) is stabilizable and A has all its eigenvalues in $\mathbb{C}^- \cup \mathbb{C}^\circ$ for continuous-time system and $\mathbb{C}^\odot \cup \mathbb{C}^\bigcirc$ for discrete-time system, with those on $\mathbb{C}^\bigcirc$ simple. We also assume $d \in \mathcal{L}_\infty$ in the continuous-time case and $d \in \ell_\infty$ in the discrete-time case.

In the sequel, we shall identify a class of disturbances for which a properly chosen linear state feedback $u = Fx$ can be found such that the states of closed-loop system remain bounded for any initial condition and that in the absence of d the origin is globally asymptotically stable. Note that system (15.1) can be decomposed into the following form:

$$\begin{bmatrix} \rho x_s \\ \rho x_u \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} \sigma(u) + \begin{bmatrix} E_s \\ E_u \end{bmatrix} d,$$

where A_s is asymptotically stable, (A_u, B_u) is controllable and A_u only has eigenvalues on $\mathbb{C}^\bigcirc$ with Jordan size 1. Since A_s is asymptotically stable, $d \in \mathcal{L}_\infty$ or $d \in \ell_\infty$ and $\sigma(\cdot)$ is uniformly bounded, it follows that the x_s dynamics will remain bounded no matter what controller is used. Therefore, without loss of generality, we can ignore the asymptotically stable dynamics and assume in (15.1) that A has eigenvalues on $\mathbb{C}^\bigcirc$ with Jordan size 1. Equivalently, we can assume that $A + A' = 0$ for continuous-time systems or $A'A = I$ for discrete-time systems.

To establish the results in this paper, we shall need two fundamental lemmas.

Lemma 15.1 Suppose $A + A' = 0$ and (A, B) is controllable. Consider the system

$$\dot{x} = Ax - B\sigma(B'x + v_1) + Bv_2, \quad x(0) = x_0$$

We have that

1. In the absence of v_1 and v_2, the origin is globally asymptotically stable;

2. $x \in \mathcal{L}_\infty$ for all $v_1 \in \mathcal{L}_\infty$, $v_2 \in \mathcal{L}_\infty(1/2)$ and any initial condition.

Lemma 15.2 Suppose $A'A = I$ and (A, B) is controllable. Consider

$$x(k+1) = Ax(k) - B\sigma(\kappa B'Ax(k) + v_1(k)) + Bv_2(k), \quad x(0) = x_0.$$

For κ such that $4\kappa B'B \le I$, we have

1. In the absence of v_1 and v_2, the origin is globally asymptotically stable;

2. $x \in \ell_\infty$ for all $v_1 \in \ell_\infty$, $v_2 \in \ell_\infty(1/2)$ and any initial condition.

Lemma 15.1 is similar to Lemma 2 in [62] and Proposition 1 in [151]. Lemma 15.2 basically follows from the same argument as used in proof of Proposition 2.3 in [13]. The detailed proofs are appended at the end of the paper.

15.3. Continuous-time systems

We first study a continuous-time system

$$\dot{x} = Ax + B\sigma(u) + Ed$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$. Also assume that (A, B) is controllable, $A + A' = 0$ and $d \in \mathcal{L}_\infty$.

We employ a linear static state feedback $u = -B'x$, which results in a closed-loop system

$$\dot{x} = Ax - B\sigma(B'x) + Ed, \quad x(0) = x_0. \tag{15.3}$$

Global asymptotic stability follows from Lemma 1. We focus here only on the boundedness of the closed-loop states.

15.3.1. Extended class of disturbances

To present our results, we extend the definition of integral-bounded disturbances introduced in [149, 144, 143] by defining a new set

$$\Omega_\infty = \{d \in \mathcal{L}_\infty \mid \forall i \in 1, \ldots, q,\ d(t)\sin\omega_i t \in S_\infty \text{ and } d(t)\cos\omega_i t \in S_\infty\}, \tag{15.4}$$

where $\pm j\omega_i$, $i \in 1, \ldots, q$, represents the eigenvalues of A. The set Ω_∞ consists of those signals that remain integral-bounded when multiplied by $\sin\omega_i t$ and $\cos\omega_i t$. This definition is a natural generalization of S_∞, since $\Omega_\infty = S_\infty$ for $\omega_i = 0$ in a chain of integrators.

In practical terms, a signal that belongs to Ω_∞ is a signal that has no sustained frequency component at any of the frequencies ω_i, $i \in 1, \ldots, q$. To see this, note that we can equivalently write

$$\Omega_\infty = \left\{ d \in \mathcal{L}_\infty \mid \exists M \ \ s.\,t. \ \forall i \in 1, \ldots, q, \forall t_2 \geq t_1 \geq 0, \left\| \int_{t_1}^{t_2} d(t) e^{j\omega_i t} \,\mathrm{d}t \right\| \leq M \right\}. \tag{15.5}$$

The integral $\int_{t_1}^{t_2} d(t) e^{j\omega_i t}\,\mathrm{d}t$ is easily recognized as the value at ω_i of the Fourier transform of the signal $d(t)$ truncated to the interval $[t_1, t_2]$. The definition of Ω_∞ implies that this value must be uniformly bounded regardless of the choice of t_1 and t_2.

In tune with the results for the single-integrator case, we shall show in the following sections that the trajectories of the controlled system (15.3) remain bounded for all disturbances belonging to Ω_∞. Moreover, this result also holds if we add a sufficiently small signal that does not belong to Ω_∞.

15.3.2. Second order single-frequency system

We start by considering an example system with a pair of complex eigenvalues at $\pm j$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(x_2) + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d, \quad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = x_0. \tag{15.6}$$

Theorem 15.1 The trajectories of (15.6) remain bounded for any $d \in \Omega_\infty$ and any x_0.

Proof : To analyze the system, we start by introducing a rotation matrix

$$R = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix},$$

which represents a counterclockwise rotation by an angle t. The dynamics of the rotation matrix is given by

$$\dot{R} = -R \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We shall study the dynamics of x from a rotated coordinate frame, and toward this end we define the rotated state $y = Rx$. The dynamics of y is given by

$$\begin{aligned} \dot{y} &= \dot{R}x + R\dot{x} \\ &= R\left(\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(x_2) \right) \\ &= R\left(\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\begin{bmatrix} 0 & 1 \end{bmatrix} R' y) \right), \quad y(0) = x(0) = x_0. \end{aligned}$$

Next, define a fictitious system

$$\dot{\tilde{y}} = R \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} d, \quad \tilde{y}(0) = x_0. \tag{15.7}$$

We know from the definition of Ω_∞ that the signal $d(t)$ is integral-bounded when multiplied by $\sin t$ and $\cos t$. It therefore follows that the right-hand side of (15.7) is integral-bounded, and hence $\tilde{y} \in \mathcal{L}_\infty$.

Consider the difference between y and the fictitious state $\tilde{y}$, given by $z = y - \tilde{y}$, with dynamics

$$\begin{aligned} \dot{z} &= -R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\begin{bmatrix} 0 & 1 \end{bmatrix} R' y) \\ &= -R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\begin{bmatrix} 0 & 1 \end{bmatrix} R' z + \delta), \quad z(0) = 0 \end{aligned}$$

where $\delta = [0, 1] R' \tilde{y} \in \mathcal{L}_\infty$. We rotate z back to the original coordinate frame by introducing $w = R'z$, thereby obtaining the dynamics

$$\begin{aligned} \dot{w} &= \dot{R}' z + R' \dot{z} \\ &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} w - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\begin{bmatrix} 0 & 1 \end{bmatrix} w + \delta), \quad w(0) = 0. \end{aligned}$$

It follows from Lemma 15.1 that $w \in \mathcal{L}_\infty$. Finally, we have $x = w + R' \tilde{y}$, and hence $x \in \mathcal{L}_\infty$. ■

To demonstrate the importance of the disturbance belonging to Ω_∞, we shall now show that if d contains a large frequency component at $\pm j$, the states of (15.6) will diverge toward infinity for any initial condition. Suppose therefore that $d(t) = a \sin(t + \theta)$, where a is an amplitude yet to be chosen. For ease of presentation, we assume that $[e_1, e_2]' = [0, 1]'$. Consider the dynamics of the rotated state y from the proof of Theorem 1. We have

$$\begin{aligned} \dot{y} &= R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left(d - \sigma(\begin{bmatrix} 0 & 1 \end{bmatrix} R' y)\right) \\ &= a \begin{bmatrix} -\sin t \sin(t + \theta) \\ \cos t \sin(t + \theta) \end{bmatrix} + \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \sigma(\cdot). \end{aligned}$$

Using trigonometric identities, the dynamics can be rewritten as

$$\dot{y} = \frac{a}{2} \begin{bmatrix} \cos(2t + \theta) - \cos(\theta) \\ \sin(2t + \theta) + \sin(\theta) \end{bmatrix} + \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \sigma(\cdot).$$

We have that either $|\sin(\theta)| \geq \sqrt{2}/2$ or $|\cos(\theta)| \geq \sqrt{2}/2$. Without loss of generality, we assume $|\sin(\theta)| \geq \sqrt{2}/2$. Let a be chosen such that $a \geq 4/\sqrt{2}(1 + \varepsilon)$, where ε is a positive number. For the trajectory $y_2(t)$, we have

$$|y_2(t)| = \left| y_2(0) + \int_0^t \frac{a}{2} [\sin(2\tau + \theta) + \sin(\theta)] - \cos\tau\sigma(\cdot)\,\mathrm{d}\tau \right|.$$

Noting that the last term of the integrand is bounded by ± 1, and using the bound $|a/2\sin(\theta)| \geq \sqrt{2}a/4 \geq 1 + \varepsilon$, we therefore have

$$\begin{aligned} |y_2(t)| &\geq \int_0^t \varepsilon\,\mathrm{d}\tau - |y_2(0)| - \frac{a}{2}\left| \int_0^t \sin(2\tau + \theta)\,\mathrm{d}\tau \right| \\ &\geq \varepsilon t - |y_2(0)| - \frac{a}{2}. \end{aligned}$$

This shows that $y_2(t)$ diverges toward infinity.

15.3.3. Connection to single-integrator case

Before moving on to the case of general multi-frequency systems, it is instructive to compare some aspects of the above example with previous results for single-integrator systems. A single-integrator system with a saturated control input and an external disturbance has the form

$$\dot{x} = \sigma(\cdot) + ed.$$

In the absence of disturbances, the open-loop response of this system is stationary. It is intuitively easy to see that a large DC bias in d would constitute a problem, because it would tend to dominate the bounded control term $\sigma(\cdot)$, thus leading to unboundedness. The absence of such a DC bias is guaranteed by d belonging to S_∞.

The example system above has the form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\cdot) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d.$$

In the absence of disturbances, the open-loop response of this system is oscillatory rather than stationary, and it is less obvious why a disturbance that does not belong to Ω_∞ may be problematic. By introducing a rotated state $y = Rx$, however, we obtain the dynamics

$$\dot{y} = R \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(\cdot) + R \begin{bmatrix} 0 \\ 1 \end{bmatrix} d.$$

In the absence of disturbances, the open-loop response of y is stationary, and the dynamics of y are similar to the single-integrator case. In particular, it is easy to see that a large DC bias in the term $R[\begin{smallmatrix}0\\1\end{smallmatrix}]d$ would constitute a problem, because it would tend to dominate the bounded control term. Analogous to the single-integrator case, the absence of such a bias is guaranteed if $R[\begin{smallmatrix}0\\1\end{smallmatrix}]d$ belongs to S_∞, which is equivalent to d belonging to Ω_∞.

In the single-integrator case, a DC bias in d can be tolerated if it is sufficiently small. Similarly, a small signal that does not belong to Ω_∞ can be tolerated for systems with complex eigenvalues. This is demonstrated in the next section, which deals with general multi-frequency systems.

15.3.4. Multi-frequency systems

We first extend Theorem 15.1 to a multi-frequency neutrally stable system. Consider

$$\dot{x} = Ax - B\sigma(B'x) + Ed, \quad x(0) = x_0 \tag{15.8}$$

where $A + A' = 0$ and (A, B) is controllable. Without loss of generality, we assume that

$$A = \begin{bmatrix} A_1 & & & \\ & \ddots & & \\ & & A_s & \\ & & & 0 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_s \\ x_o \end{bmatrix} \tag{15.9}$$

where $x_i \in \mathbb{R}^2$, $i = 1, \ldots, s$, $x_o \in \mathbb{R}^{n-2s}$ and

$$A_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix}, \quad i = 1, \ldots, s$$

with $2s \le n$. We have the following theorem

Theorem 15.2 The states of (15.8) remain bounded for any $d \in \Omega_\infty$ and any x_0.

Proof : Consider the matrix

$$R = \begin{bmatrix} R_1 & & & \\ & \ddots & & \\ & & R_s & \\ & & & I \end{bmatrix}$$

where

$$R_i = \begin{bmatrix} \cos\omega_i t & -\sin\omega_i t \\ \sin\omega_i t & \cos\omega_i t \end{bmatrix}.$$

Note that R is unitary, i.e. $RR' = I$ and moreover

$$\dot{R} = -RA.$$

Define a transformed state $y = Rx$. As a result,

$$\dot{y} = -RB'\sigma(B'R'y) + REd, \quad y(0) = x_0.$$

Introduce a fictitious system

$$\dot{\tilde{y}} = REd, \quad \tilde{y}(0) = x_0.$$

It follows from the definition of Ω_∞ that $\tilde{y} \in \mathcal{L}_\infty$. Next, define the difference between y and $\tilde{y}$ by $z = y - \tilde{y}$. We get

$$\dot{z} = -RB\sigma(B'R'z + B'R'\tilde{y}), \quad z(0) = 0.$$

Finally transform z back to the original coordinates by defining $w = R'z$. Note that

$$\dot{R}' = AR'.$$

Hence

$$\dot{w} = Aw - B\sigma(B'w + B'R'\tilde{y}), \quad \tilde{w}(0) = 0.$$

Lemma 15.1 shows that $w \in \mathcal{L}_\infty$. Since $x = w + R'\tilde{y}$ and $\tilde{y} \in \mathcal{L}_\infty$ is bounded for all t, we conclude that $x \in \mathcal{L}_\infty$. ■

Next, we shall prove that the states of (15.8) also remain bounded if a small signal that does not belong to Ω_∞ is added on top of the original signal in Ω_∞. Consider the system

$$\dot{x} = Ax - B\sigma(B'x) + E_1d_1 + E_2d_2, \quad x(0) = x_0 \tag{15.10}$$

where $A + A' = 0$ and (A, B) is controllable. Without loss of generality we assume that A is in the form of (15.9).

Theorem 15.3 The states of system (15.10) remain bounded for any x_0, any $d_1 \in \Omega_\infty$ and $d_2 \in \mathcal{L}_\infty(\delta)$ with δ sufficiently small..

Proof : Using the same sequence of transformations as introduced in the proof of Theorem 2, we get the following transformed system

$$\dot{w} = Aw - B\sigma(B'w + B'R'\tilde{y}) + E_2 d_2, \quad w(0) = 0$$

where $w = x - R'\tilde{y}$ and

$$\dot{\tilde{y}} = RE_1 d_1, \quad \tilde{y} = x_0.$$

The fact that $d_1 \in \Omega_\infty$ implies that $\tilde{y} \in \mathcal{L}_\infty$. Introduce another fictitious system

$$\dot{\bar{w}} = (A - BB')\bar{w} + E_2 d_2, \quad \bar{w}(0) = 0.$$

Since $A - BB'$ is Hurwitz stable and $d_2 \in \mathcal{L}_\infty(\delta)$, we have that $\bar{w} \in \mathcal{L}_\infty$ and moreover $\|B'\bar{w}\|_\infty \leq \frac{1}{2}$ for sufficiently small δ.

Define $\tilde{w} = w - \bar{w}$. Then $\tilde{w}$ has the following dynamics

$$\dot{\tilde{w}} = A\tilde{w} - B\sigma(B'\tilde{w} + v_1) + Bv_2, \quad \tilde{w} = 0$$

where $v_1 = B'\bar{w} + B'R'\tilde{y}$ and $v_2 = B'\bar{w}$. It follows from Lemma 15.1 that $\tilde{w} \in \mathcal{L}_\infty$. Since $x = \bar{w} + \tilde{w} + R'\tilde{y}$ and $\bar{w}, R'\tilde{y} \in \mathcal{L}_\infty$, we conclude that $x \in \mathcal{L}_\infty$. ■

15.4. Discrete-time systems

In this section, we deal with discrete-time systems. Consider the following system

$$x(k+1) = Ax(k) + B\sigma(u(k)) + Ed(k), \quad x(0) = x_0. \tag{15.11}$$

We assume that (A, B) is controllable and $A'A = I$.

We use a linear state feedback controller $u = -\kappa B'Ax$ which gives a closed-loop system as

$$x(k+1) = Ax(k) + B\sigma(-\kappa B'Ax(k)) + Ed(k), \quad x(0) = x_0.$$

For κ such that $4\kappa B'B \leq I$, the global asymptotic stability of the origin in the absence of d follows from Lemma 15.2. As such, as in continuous-time case, we focus here only on the boundedness of closed-loop states with disturbances.

15.4.1. Extended class of disturbances

As in continuous-time case, we define a set of discrete disturbances

$$\Omega_\infty = \left\{ d \in \ell_\infty \mid \exists M > 0, \;\; s.\,t.\, \forall i \in 1, \ldots, q, \;\; \forall k_2 \geq k_1 \geq 0, \right.$$
$$\left. \left\| \sum_{k=k_1}^{k_2} d(k)\cos(\theta_i k) \right\| \leq M, \quad \left\| \sum_{k=k_1}^{k_2} d(k)\sin(\theta_i k) \right\| \leq M \right\}, \quad (15.12)$$

where $e^{j\theta_i}$, $i \in 1, \ldots, q$, represents the eigenvalues of A.

Ω_∞ contains signals which do not have sustained component at discrete frequency θ_i. Like in the continuous-time case, we can also rewrite the above definition as

$$\Omega_\infty = \left\{ d \in \ell_\infty \mid \exists M > 0, \;\; s.\,t.\, \forall i \in 1, \ldots, q, \;\; \forall k_2 \geq k_1 \geq 0, \quad \left\| \sum_{k=k_1}^{k_2} d(k) z_i^k \right\| \leq M \right\}, \quad (15.13)$$

where $z_i = e^{j\theta_i}$, $i = 1, ..., q$, denotes the eigenvalues of A. Since $d \in \ell_\infty$, the power series $\sum_0^\infty d(z)z^k$ or the z-transform of $d(k)$ always has a radius of convergence 1. On $|z| = 1$, definition (15.12) implies all partial sums of the power series is bounded at $z = z_i$.

Note that the set Ω_∞ in (15.12) and (15.13) are a discrete equivalent of (15.4) and (15.5).

15.4.2. Multi-frequency systems

Next we shall prove the boundedness of closed-loop trajectories with disturbances that belong to Ω_∞ as defined in (15.12). The philosophy of the proof is basically the same as in continuous-time case. We apply a sequence of successive rotations to state coordinates and eventually convert the non-input-additive disturbances to input-additive disturbances using the property of Ω_∞. Since this procedure has been made clear in the preceding section, we shall skip the proof for second-order single-frequency systems and only work on the general case.

Theorem 15.4 Consider the system

$$x(k+1) = Ax(k) - B\sigma(\kappa B' Ax(k)) + Ed(k), \quad x(0) = x_0 \quad (15.14)$$

where (A, B) is controllable, $A'A = I$ and $d \in \Omega_\infty$. Then for κ such that $4\kappa B'B \leq I$, we have $x(k)$ bounded for all $k \geq 0$ and for any initial condition.

Proof : Define $R(k) = (A')^k$. Since $A'A = I$, $R(k)$ represents a time-varying rotation matrix with difference equation

$$R(k+1) = R(k)A'$$

Also define

$$y(k) = R(k)x(k)$$

The transformed system becomes

$$y(k+1) = y(k) - R(k)A'B\sigma(\kappa B'AR'(k)y(k)) + R(k)A'Ed(k), \quad y(0) = x_0.$$

Introduce a fictitious system

$$\tilde{y}(k+1) = \tilde{y}(k) + R(k)A'Ed(k), \quad \tilde{y}(0) = x_0.$$

Note that $d \in \Omega_\infty$ implies that there exists a $M > 0$ such that

$$\forall k_2 > k_1 > 0, \left\| \sum_{k_1}^{k_2} (A')^k Ed(k) \right\| \leq M.$$

Therefore, we find $\tilde{y} \in \ell_\infty$. Let $z = y - \tilde{y}$. We get

$$z(k+1) = z(k) - R(k)A'B\sigma(\kappa B'AR'(k)z(k) + \kappa B'AR'(k)\tilde{y}(k)), \quad z(0) = 0.$$

Finally, define $w(k) = R'(k)z(k)$. The dynamics of w is given by

$$w(k+1) = Aw(k) - B\sigma(\kappa B'Aw(k) + v(k)), \quad w(0) = 0.$$

where $v(k) = \kappa B'A^{k+1}\tilde{y}(k)$. It follows from Lemma 15.2 that the above system is ℓ_∞ stable with respect to v given $4\kappa B'B \leq I$. Thus $\tilde{y}_\infty$ implies $w \in \ell_\infty$. Note that $x(k) = w(k) + R'(k)\tilde{y}(k)$. Therefore, we conclude $x \in \ell_\infty$. ∎

The next theorem shows that a small disturbance that does not belong to Ω_∞ can also be tolerated.

Theorem 15.5 Consider the discrete-time system

$$x(k+1) = Ax(k) - B\sigma(\kappa B'Ax(k)) + E_1d_1(k) + E_2d_2(k), \quad x(0) = x_0. \tag{15.15}$$

For κ such that $4\kappa B'B \leq I$, we have $x(k)$ bounded for all $k \geq 0$ and for any x_0, $d_1 \in \Omega_\infty$ and $d_2 \in \ell_\infty(\delta)$ with δ sufficiently small.

Proof : Following the same lines as in the proof of Theorem 15.4, we shall get a transformed system

$$w(k+1) = Aw(k) - B\sigma(\kappa B'Aw(k) + \kappa B'AR'(k)\tilde{y}(k)) + E_2 d_2(k), w(k) = 0$$

where $w(k) = x(k) - R'(k)\tilde{y}(k)$ and $\tilde{y}$ satisfies

$$\tilde{y}(k+1) = \tilde{y}(k) + R(k)A'E_1 d_1(k), \quad \tilde{y}(0) = x_0,$$

and hence $\tilde{y} \in \ell_\infty$. Introduce an auxiliary system

$$\bar{w}(k+1) = (A + BF)\bar{w}(k) + E_2 d_2(k), \quad \bar{w}(0) = 0,$$

where F is such that $A + BF$ is asymptotically stable. Since $d_2 \in \ell_\infty(\delta)$, we find that $F\bar{w} \in \ell_\infty$. Moreover, we have $\|F\bar{w}\|_\infty \leq 1/2$ with sufficiently small δ.

Define $\tilde{w} = w - \bar{w}$. Then we get

$$\tilde{w}(k+1) = A\tilde{w}(k) - B\sigma(\kappa B'A\tilde{w}(k) + v_1(k)) + Bv_2(k), \quad \tilde{w}(0) = 0.$$

where $v_1 = \kappa B'A\bar{w} + \kappa B'A^{k+1}\tilde{y}$ and $v_2 = F\bar{w}$.

Lemma 15.2 shows that $\tilde{w} \in \ell_\infty$. Since $x = \tilde{w} + \bar{w} + R'(k)\tilde{y}$, we conclude that $x \in \ell_\infty$ for any initial condition. ■

15.5. Conclusion

In this paper, we study the dynamics of an open-loop neutrally stable linear system controlled by a saturating linear feedback controller in the presence of external disturbances. Two classes of disturbances have been identified for which we can achieve bounded states of the closed-loop system. This paper extends the results for a single-integrator system as reported in [143] to a neutrally stable system. It is evident that the class of disturbances identified in this paper is a natural extension of the class of integral-bounded disturbances for a chain of integrator. The more general case of critically unstable systems which have complex eigenvalues with Jordan block size great than 1 is subject to current research.

Appendix

We shall develop proof for Lemma 15.1 and 15.2. In order to do so, we need the following inequalities, which were proven in [107]:

Lemma 15.3 For two vectors $u, w \in \mathbb{R}^m$, the following statements hold:

$$|u'[\sigma(u+w) - \sigma(u)]| \leq 2\sqrt{m}\|w\|; \tag{15.16}$$

$$2u'[\sigma(w) - \sigma(w-u)] \geq u'\sigma(u), \quad \|w\| \leq \tfrac{1}{2}; \tag{15.17}$$

$$\|u - \sigma(u)\| \leq u'\sigma(u); \tag{15.18}$$

$$-u'[\sigma(u) + w] \leq \tfrac{\|w\|^2}{4}, \quad \|w\| \leq 1, \tag{15.19}$$

where $\sigma(\cdot)$ is the standard saturation function defined in (15.2).

Proof of Lemma 15.1 : Item 1 has been proven in [62]. We only prove item 2. Denote $u = B'x$ and define $V_1 = \frac{1}{3}\|x\|^3$. Differentiating V_1 along the trajectories yields

$$\begin{aligned}
\dot{V}_1 &= \|x\|u'\left[\sigma(-u+v_1) + v_2\right] \\
&\leq \|x\|(u-v_1)'\left[-\sigma(u-v_1) + v_2\right] + 2\|x\|\|v_1\|_\infty \\
&= \|x\|\left\{(u-v_1)'[-\sigma(u-v_1) + \sigma(u-v_1+v_2)]\right. \\
&\quad \left. +(u-v_1)'[-\sigma(u-v_1+v_2) + \sigma(v_2)]\right\} + 2\|x\|\|v_1\|_\infty \\
&\leq -\tfrac{1}{2}\|x\|(u-v_1)\sigma(u-v_1) + 2\sqrt{m}\|x\|\|v_2\|_\infty + 2\|x\|\|v_1\|_\infty.
\end{aligned}$$

The last inequality results from (15.16), (15.17) and the condition $\|v_2\| \leq \frac{1}{2}$.

Next, since $A - BB'$ is Hurwitz stable, there exists a $P > 0$ satisfying

$$(A - BB')'P + P(A - BB') = -I.$$

Define $V_2 = x'Px$. There exists an α such that

$$\begin{aligned}
\dot{V}_2 &= -\|x\|^2 + 2x'P[B\sigma(-u+v_1) + Bu + Bv_2] \\
&= -\|x\|^2 + 2x'P[B(\sigma(-u+v_1) + u - v_1) + Bv_2 + Bv] \\
&\leq -\|x\|^2 + 2\alpha\|x\|(u-v_1)\sigma(u-v_1) + 2\alpha\|x\|\|v_2\|_\infty + 2\alpha\|x\|\|v_1\|_\infty,
\end{aligned}$$

where inequality (15.18) is used to derive the last inequality.

Finally, define a Lyapunov candidate $V = 4\alpha V_1 + V_2$. We find that

$$\begin{aligned}
\dot{V} &\leq -\|x\|^2 + (8\alpha\sqrt{m} + 2\alpha)\|x\|\|v_2\|_\infty + 10\alpha\|x\|\|v_1\|_\infty \\
&= -\|x\|\left[\|x\| - (8\alpha\sqrt{m} + 2\alpha)\|v_2\|_\infty - 10\alpha\|v_1\|_\infty\right].
\end{aligned}$$

Hence $\dot{V} \leq 0$ for $\|x\| \geq (8\alpha\sqrt{m} + 2\alpha)\|v_2\|_\infty + 10\alpha\|v_1\|_\infty$. Let c be such that

$$\{x \mid V(x) \leq c\} \supset \{x \mid \|x\| \leq (8\alpha\sqrt{m} + 2\alpha)\|v_2\|_\infty + 10\alpha\|v_1\|_\infty\}.$$

We have $\dot{V} \leq 0$ for $x \notin \{x \mid V(x) \leq c\}$. This implies that $x(t) \in \{x \mid V(x) \leq c\}$ for all $t \geq 0$. ∎

To prove Lemma 15.2, we borrow the next lemma from [107]

Lemma 15.4 Assume that $A'A = I$ and $\kappa B'B \leq 2I$ for some $\kappa > 0$ Then $\tilde{A} = A - \kappa BB'A$ is Schur stable if and only if (A, B) is controllable.

Proof of Lemma 15.2 : Denote $\kappa B'Ax$ by u. Define $V_1 = \|x\|^2$. We have that

$$\begin{aligned} V_1(k+1) - V_1(k) &= \|Ax + B\sigma(-u+v_1) + Bv_2\|^2 - \|x\|^2 \\ &= \tfrac{2}{\kappa}u'[\sigma(-u+v_1) + v_2] + \left[\sigma(-u+v_1)' + v_2'\right] B'B \left[\sigma(-u+v_1)' + v_2'\right] \\ &\leq \tfrac{2}{\kappa}[u - v_1]'[\sigma(-u+v_1) + v_2] + \tfrac{2}{\kappa}v_1'[\sigma(-u+v_1) + v_2] + \tfrac{1}{4\kappa}\|\sigma(-u+v_1) + v_2\|^2 \end{aligned}$$

where we use condition $4\kappa BB' \leq I$. Since $\|v_2\| \leq \frac{1}{2}$ and $\sigma(\cdot)$ is bounded by ± 1, we find that $v_1'[\sigma(-u + v_1) + v_2] \leq 2\|v_1\|$. This yields that

$$\begin{aligned} V_1(k+1) - V_1(k) &\leq \tfrac{2}{\kappa}[u - v_1]'[\sigma(-u+v_1) + v_2] + \tfrac{1}{2\kappa}\|\sigma(-u+v_1)\|^2 + \tfrac{1}{2\kappa}\|v_2\|^2 + \tfrac{4}{\kappa}\|v_1\| \\ &\leq \tfrac{2}{\kappa}[u - v_1]'[\sigma(-u+v_1) + v_2] + \tfrac{1}{2\kappa}\|\sigma(-u+v_1)\|^2 + \tfrac{1}{2\kappa}\|v_2\| + \tfrac{4}{\kappa}\|v_1\|. \end{aligned}$$

Note that

$$\begin{aligned} \tfrac{2}{\kappa}[u - v_1]'[\sigma(-u+v_1) + v_2] &= \tfrac{1}{\kappa}[u - v_1]'\sigma(-u+v_1) + \tfrac{1}{\kappa}[u - v_1]'[\sigma(-u+v_1) + 2v_2] \\ &\leq \tfrac{1}{\kappa}[u - v_1]'\sigma(-u+v_1) + \tfrac{1}{\kappa}\|v_2\|^2 \\ &\leq \tfrac{1}{\kappa}[u - v_1]'\sigma(-u+v_1) + \tfrac{1}{\kappa}\|v_2\| \end{aligned}$$

where we use (15.19) and $\|v_2\| \leq \frac{1}{2}$. Therefore,

$$\begin{aligned} V_1(k+1) - V_1(k) &\leq \tfrac{1}{\kappa}[u - v_1]'\sigma(-u+v_1) + \tfrac{1}{2\kappa}\|\sigma(-u+v_1)\|^2 + \tfrac{3}{2\kappa}\|v_2\| + \tfrac{4}{\kappa}\|v_1\| \\ &\leq \tfrac{1}{2\kappa}[u - v_1]'\sigma(-u+v_1) + \tfrac{3}{2\kappa}\|v_2\| + \tfrac{4}{\kappa}\|v_1\|. \end{aligned} \tag{15.20}$$

Since $4\kappa B'B \leq I$, $\tilde{A} = A - \kappa BB'A$ is Schur stable. Let P be the solution to the Lyapunov equation

$$\tilde{A}'P\tilde{A} - P + I = 0.$$

Define $V_2 = \|P^{1/2}x\|$. We have

$$\begin{aligned} V_2(k+1) - V_2(k) &= \|P^{1/2}\tilde{A}x + P^{1/2}B[u - v_1 - \sigma(u - v_1) + (v_2 + v_1)]\| - \|P^{1/2}x\| \\ &\leq \|P^{1/2}\tilde{A}x\| + \|P^{1/2}B[u - v_1 - \sigma(u - v_1) + (v_2 + v_1)]\| - \|P^{1/2}x\|. \end{aligned}$$

For $x \neq 0$, there exists a $\beta > 0$ such that

$$\|P^{1/2}\tilde{A}x\| - \|P^{1/2}x\| = \frac{\|P^{1/2}\tilde{A}x\|^2 - \|P^{1/2}x\|^2}{\|P^{1/2}\tilde{A}x\| + \|P^{1/2}x\|} = \frac{-\|x\|^2}{\|P^{1/2}\tilde{A}x\| + \|P^{1/2}x\|} \leq -\beta\|x\|.$$

Obviously, the above also holds for $x = 0$. Hence

$$\begin{aligned} V_2(k+1) - V_2(k) &\leq -\beta\|x\| + \|P^{1/2}B\|\|(u - v_1) - \sigma(u - v_1)\| + \|P^{1/2}B\|(\|v_2\| + \|v_1\|) \\ &\leq -\beta\|x\| + \|P^{1/2}B\|(u - v_1)'\sigma(u - v_1) + \|P^{1/2}B\|(\|v_2\| + \|v_1\|) \qquad (15.21) \end{aligned}$$

where we use (15.18) of Lemma 15.3.

Define $V = 2\kappa\|P^{1/2}B\|V_1 + V_2$. We obtain from (15.20) and (15.21) that

$$V(k+1) - V(k) \leq -\beta\|x\| + 9\|P^{1/2}B\|\|v_1\| + 4\|P^{1/2}B\|\|v_2\|. \qquad (15.22)$$

This immediately implies that $x \in \ell_\infty$ for any initial condition. ■

CHAPTER 16

Simultaneous external and internal stabilization of linear systems with input saturation and non-input-additive sustained disturbances

16.1. Introduction

Based on the construction for a chain of integrator and for neutrally stable systems in previous chapters, we shall extend the results to general ANCBC systems which may have non-zero degenerate eigenvalues on the imaginary axis (continuous-time) or on the unit circle (discrete-time); in other words, systems that are at most critically unstable. It will be shown that the same class of disturbances identified in Chapter 15 can be tackled by a properly designed feedback controller. At the same time, the resulting closed-loop system in the absence of disturbances is globally asymptotically stable.

The chapter is organized as follows: In the formulation section, we shall define the system and the problem to be studied and make several necessary assumptions. Next, a special Jordan decomposition is introduced which is instrumental in establishing our results. A special class of disturbances is introduced in Section 16.3. After these preparations, we present the main results of this chapter and its proof in Section 16.4. Finally, some technical results used in this chapter are given in the appendix.

16.2. Formulation

Consider the following system

$$\rho x = Ax + B\sigma(u) + Ed, \quad x(0) = x_0, \tag{16.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $d \in \mathbb{R}^p$. ρx denotes the derivative $\rho x = \dot{x}$ for continuous-time systems and the shift operator $(\rho x)(t) = x(t+1)$ for discrete-time systems. $\sigma(\cdot)$ is the standard saturation function, i.e. for $s \in \mathbb{R}^m$

$$\sigma(s) = \begin{bmatrix} \operatorname{sign}(s_1)\min\{1, |s_1|\} \\ \vdots \\ \operatorname{sign}(s_m)\min\{1, |s_m|\}. \end{bmatrix}$$

We are interested in sustained disturbances, for which we assume in the first place that $d \in \mathcal{L}_\infty$ for continuous-time systems and $d \in \ell_\infty$ for discrete-time systems.

The problem we will study is to find a class of disturbances, say Ω, for which the simultaneous global $\mathcal{L}_\infty$ or ℓ_∞ and global asymptotic stabilization problem is solvable, i.e. there exists a controller $u = f(x, t)$ possibly nonlinear and dynamic such that

1. in the absence of disturbances, the origin is globally asymptotically stable;

2. for $d \in \Omega$, the states of the closed-loop system remain bounded for $t \geq 0$.

Since the global asymptotic stabilization without disturbances is required, it is a classical result that the system needs to be asymptotically null controllable with bounded control, i.e.

1. (A, B) is stabilizable;

2. A has all its eigenvalues in $\mathbb{C}^s$.

Such a system can be decomposed into the following form:

$$\begin{bmatrix} \rho x_s \\ \rho x_u \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_u \end{bmatrix} \begin{bmatrix} x_s \\ x_u \end{bmatrix} + \begin{bmatrix} B_s \\ B_u \end{bmatrix} \sigma(u) + \begin{bmatrix} E_s \\ E_u \end{bmatrix} d,$$

where A_s is asymptotically stable, A_u has all its eigenvalues on $\mathbb{C}^b$ and (A_u, B_u) is controllable. Since A_s is stable and $\sigma(\cdot)$ and d are bounded, it follows that the x_s dynamics will remain bounded no matter what controller is used. Therefore, without loss of generality, we can ignore the asymptotically stable dynamics and assume in (16.1) that (A, B) is controllable and all the eigenvalues of A are on $\mathbb{C}^b$.

Under the above assumption, we consider a linear system with input saturation and disturbances:

$$\rho x = Ax + B\sigma(u) + Ed, \quad x(0) = x_0 \tag{16.2}$$

where (A, B) is controllable and A has all its eigenvalues on $\mathbb{C}^b$. Suppose the eigenvalues of A have q different Jordan block sizes denoted by $n_1, ..., n_q$. Without loss of generality, we can assume $x = (x_1', x_2', \ldots, x_q')'$, and A, B are in the following form

$$A = \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \bar{A}_q \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}, \; E = \begin{bmatrix} E_1 \\ E_2 \\ \vdots \\ E_q \end{bmatrix} \tag{16.3}$$

where

$$x_i = \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,n_i-1} \\ x_{i,n_i} \end{bmatrix}, \quad \bar{A}_i = \underbrace{\begin{bmatrix} A_i & I & 0 & \dots & 0 \\ 0 & A_i & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_i & I \\ 0 & \cdots & \cdots & 0 & A_i \end{bmatrix}}_{n_i \times n_i \text{ blocks}},$$
$$B_i = \begin{bmatrix} B_{i,1} \\ B_{i,2} \\ \vdots \\ B_{i,n_i-1} \\ B_{i,n_i} \end{bmatrix}, \quad E_i = \begin{bmatrix} E_{i,1} \\ E_{i,2} \\ \vdots \\ E_{i,n_i-1} \\ E_{i,n_i} \end{bmatrix}, \tag{16.4}$$

$x_{i,j} \in \mathbb{R}^{p_i}$ with $n = \sum_{i=1}^{q} n_i p_i$ and $A_i' + A_i = 0$ for continuous-time systems and $A_i' A_i = I$ for discrete-time systems. Note that the above form can be obtained by assembling together in the real Jordan canonical form those blocks corresponding to eigenvalues with the same Jordan block size.

We say the disturbance d is *aligned* if $E_{i,n_i} \neq 0$ for some $i = 1, \dots, q$ and *misaligned* if $E_{i,n_i} = 0$ for all $i = 1, \dots, q$.

16.3. A special class of disturbances

We consider the following class of disturbances as defined in Chapter 15:

$$\Omega_\infty = \left\{ d \in \mathscr{L}_\infty \mid \exists M > 0, \ s.\ t.\ \forall\, i \in 1, \dots, \ell, \ \forall\, t_2 > t_1 > 0, \ \left\| \int_{t_1}^{t_2} d(t) e^{j\omega_i t}\, \mathrm{d}t \right\| \leq M \right\}, \tag{16.5}$$

in continuous-time case and

$$\Omega_\infty = \left\{ d \in \ell_\infty \mid \exists M > 0, \ \ s.\ t.\ \forall i \in 1, \dots, \ell, \ \forall t_2 \geq t_1 \geq 0, \ \ \left\| \sum_{t=t_1}^{t_2} d(t) e^{j\omega_i t} \right\| \leq M \right\}, \tag{16.6}$$

in discrete-time case, where $j\omega_i$ (continuous-time) or $e^{j\omega_i}$ (discrete-time), $i \in 1, \dots, \ell$, represents the eigenvalues of A. Here we assume that the system has ℓ different eigenvalues (repeated eigenvalues are counted once).

The integral $\int_{t_1}^{t_2} d(t) e^{j\omega_i t}\, \mathrm{d}t$ or summation $\sum_{t=t_1}^{t_2} d(t) e^{j\omega_i t}$ is easily recognized as the value at ω_i of the Fourier transform of the signal $d(t)$ truncated to $[t_1, t_2]$. The definition of Ω_∞ implies that this value must be uniformly bounded regardless of the choice of time interval. In practical terms, a signal that belongs to Ω_∞ is a signal that has no sustained frequency component at any of the frequencies ω_i, $i \in 1, \ldots, \ell$.

To better motivate the definition of Ω_∞ and demonstrate its importance, we recall the following example in Chapter 15

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sigma(u) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d, \tag{16.7}$$

where $d(t) = a \sin(t + \theta)$. This system is clearly in the form of (16.2), (16.3) and (16.4) and d contains a frequency component corresponding to the system's eigenvalues at $\pm j$. It was shown that for a relatively large a, states diverge to infinity for any initial condition and any controller. A similar example for discrete-time systems can also be constructed.

16.4. Main results

In this section, we shall show that for aligned disturbances which either belong to Ω_∞ and/or misaligned disturbances which belong to $\mathcal{L}_\infty$ or ℓ_∞, a controller can be designed such that the states of the closed-loop system remain bounded for any initial condition, at the same time the origin in the absence of disturbances is globally asymptotically stable. Moreover, we shall show that a small aligned disturbances which does not belong to Ω_∞ can also be tolerated.

Without loss of generality, for any critically unstable system with input saturation and non-input-additive disturbances as given by (16.2), (16.3) and (16.4), we can equivalently rewrite the system in the following form

$$\rho x = Ax + B\sigma(u) + \bar{E}_1 d_1 + \bar{E}_2 d_2 + \bar{E}_3 d_3, \tag{16.8}$$

with $x(0) = x_0$. In the above system, d_1 is misaligned and contain arbitrary disturbances that belong to $\mathcal{L}_\infty$ (continuous-time) or ℓ_∞ (discrete-time), d_2 contains all aligned disturbances belonging to Ω_∞ and d_3 contains aligned disturbances which do not belong to Ω_∞ but are sufficiently small. The system data A and B are given by (16.3) and (16.4). The $\bar{E}_1$, $\bar{E}_2$ and $\bar{E}_3$ are in the form

$$\bar{E}_1 = \begin{bmatrix} \bar{E}_{1,1} \\ \vdots \\ \bar{E}_{1,q-1} \\ \bar{E}_{1,q} \end{bmatrix}, \ \bar{E}_{1,i} = \begin{bmatrix} E_{i,1} \\ \vdots \\ E_{i,n_i-1} \\ 0 \end{bmatrix} \tag{16.9}$$

and

$$\bar{E}_j = \begin{bmatrix} \bar{E}_{j,1} \\ \vdots \\ \bar{E}_{j,q-1} \\ \bar{E}_{j,q} \end{bmatrix}, \ \bar{E}_{j,i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E^{j}_{i,n_i} \end{bmatrix}, \quad j = 2, 3. \tag{16.10}$$

Next we shall design a controller which solves the simultaneous external and internal stabilization problem. Let (A, B) satisfy the assumptions made in the preceding section and $P(\varepsilon) > 0$ be the solution to a Continuous Parametric Lyapunov Equation (CPLE)

$$A'P(\varepsilon) + P(\varepsilon)A - P(\varepsilon)BB'P(\varepsilon) + \varepsilon P(\varepsilon) = 0. \tag{16.11}$$

or a Discrete Parametric Lyapunov Equation (DPLE)

$$(1-\varepsilon)P(\varepsilon) = A'P(\varepsilon)A - A'P(\varepsilon)B(B'P(\varepsilon)B + I)^{-1}B'P(\varepsilon)A. \tag{16.12}$$

with $\varepsilon \in (0, 0.9]$. The existence of the positive definite $P(\varepsilon)$ and its following properties were shown in [159, 161].

1. $P(\varepsilon) \to 0$ as $\varepsilon \to 0$;
2. $\frac{\mathrm{d}P(\varepsilon)}{\mathrm{d}\varepsilon} > 0$ for $\varepsilon > 0$;
3. $P(\varepsilon)$ is rational in ε.

The special structure of $\bar{E}_1$ yields the following crucial technical lemma.

Lemma 16.1 Let $P(\varepsilon)$ be the solution to CPLE (16.11) or DPLE (16.12) associated with A and B given by (16.3) and (16.4). For any matrix $\bar{E}_1$ in the form of (16.9), there exists M such that for $\varepsilon \in (0, 1]$

$$\bar{E}_1'P(\varepsilon)\bar{E}_1 \leq M\varepsilon^2 I$$

Proof : See Appendix. ■

We will construct a low-gain dynamic state feedback controller. The controller as given below has q states that will transiently replace the evolution of the bottom states of each Jordan block $\bar{A}_i$ in generating feedback input into the system.

$$\begin{cases} \rho \hat{x}_i = A_i \hat{x}_i + B_{i,n_i} \sigma(F(\varepsilon_a(\bar{x}))\bar{x}), \\ u = K(x_b - \hat{x}) + F(\varepsilon_a(\bar{x}))\bar{x}, \end{cases} \tag{16.13}$$

for $i = 1, \ldots, q$ where $\hat{x} = [\hat{x}_1', \hat{x}_2,' \cdots, \hat{x}_q']'$ and

$$x_b = \begin{bmatrix} x_{1,n_1} \\ x_{2,n_2} \\ \vdots \\ x_{q,n_q} \end{bmatrix}, \bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_{q-1} \\ \bar{x}_q \end{bmatrix}, \bar{x}_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,n_i-1} \\ \hat{x}_i \end{bmatrix}.$$

Note that $\bar{x}$ is the system state x with bottom state segment x_{i,n_i} of each Jordan block $\bar{A}_i$ replaced by controller states $\hat{x}_i$. The feedback input is generated based on $\bar{x}$ instead of x. As will become clear in the proof, the underlying idea behind (16.13) is that by utilizing the states of controller and the property of Ω_∞, we will be able to convert some aligned disturbances affecting the bottom states into misaligned disturbances which turns out to be less restricted.

The parameter K can be chosen as

$$K = \begin{cases} -\hat{B}', & \text{continuous-time;} \\ -\kappa \hat{B}' \hat{A}, & \text{discrete-time.} \end{cases}$$

where κ satisfies $8\kappa B'B \leq I$ and

$$\hat{B} = \begin{bmatrix} B_{1,n_1} \\ B_{2,n_2} \\ \vdots \\ B_{q,n_q} \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_q \end{bmatrix}.$$

The other feedback gain $F(\varepsilon_a(\bar{x}))$ can be designed as follows

$$F(\varepsilon) = \begin{cases} -B'P(\varepsilon), & \text{(continuous);} \\ -(B'P(\varepsilon)B + I)^{-1}B'P(\varepsilon)A, & \text{(discrete)} \end{cases}$$

where $P(\varepsilon)$ is the solution to CPLE (16.11) or DPLE (16.12). The parameter ε is determined by

$$\varepsilon = \varepsilon_a(\bar{x}) := \max\{r \in (0, 0.9] \mid (\bar{x}'P(r)\bar{x}) \operatorname{trace}(P(r)) \leq \tfrac{\delta^2}{b}\} \tag{16.14}$$

where $b = 2\operatorname{trace}(BB')$, $\delta = \frac{1}{4}$ and $P(r)$ is the solution of (16.11) and (16.12) with $\varepsilon = r$. The scheduling (16.14) satisfies the following properties:

1. There exists an open neighborhood $\mathcal{O}$ of the origin such that for all $\bar{x} \in \mathcal{O}$, $\varepsilon_a(\bar{x}) = 0.9$.

2. For any $\bar{x} \in \mathbb{R}^n$, $\|F(\varepsilon_a(\bar{x}))\bar{x}\| \le \delta$.

3. $\varepsilon_a(\bar{x}) \to 0 \iff \|\bar{x}\| \to \infty$.

4. For each $c > 0$, the set $\{\bar{x} \in \mathbb{R}^n \mid \bar{x}' P(\varepsilon_a(\bar{x}))\bar{x} \le c\}$ is bounded.

5. For any x_1 and x_2, $x_1 P(\varepsilon_a(x_1))x_1 \le x_2 P(\varepsilon_a(x_2))x_2 \Rightarrow \varepsilon_a(x_1) \ge \varepsilon_a(x_2)$.

(see [68, 48, 31, 146]). The main result is stated in the following theorem:

Theorem 16.1 Consider the system (16.8) with controller (16.13). We have that

1. in the absence of d_1, d_2 and d_3, the origin is globally asymptotically stable;

2. there exists a $\delta_1 > 0$ such that the state remains bounded for any initial condition x_0 and disturbances $d_1 \in \mathcal{L}_\infty$, $d_2 \in \Omega_\infty$, $d_3 \in \mathcal{L}_\infty(\delta_1)$ (continuous time) or $d_1 \in \ell_\infty$, $d_2 \in \Omega_\infty$, $d_3 \in \ell_\infty(\delta_1)$ (discrete time).

Proof : We shall only prove the results for continuous-time systems. The discrete-time counterpart can be shown using a very similar argument. For continuous-time system, define

$$\tilde{x} = x_b - \hat{x} = \begin{bmatrix} x_{1,n_1} - \hat{x}_1 \\ x_{2,n_2} - \hat{x}_2 \\ \vdots \\ x_{q,n_q} - \hat{x}_q \end{bmatrix}.$$

We have that

$$\dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} - B'P(\varepsilon_a(\bar{x}))\bar{x}) - \hat{B}\sigma(-B'P(\varepsilon_a(\bar{x}))\bar{x}) + \hat{E}_2 d_2 + \hat{E}_3 d_3,$$

where

$$\hat{E}_j = \begin{bmatrix} E^j_{1,n_1} \\ E^j_{2,n_2} \\ \vdots \\ E^j_{q,n_q} \end{bmatrix}, \quad j = 2, 3. \tag{16.15}$$

$\hat{E}_2 d_2$ and $\hat{E}_3 d_3$ contain all the aligned disturbances that affect the bottom states of each Jordan block $\bar{A}_i$. Note that (A, B) is controllable implies that $(\hat{A}, \hat{B})$ is controllable. Moreover, $\hat{A} + \hat{A}' = 0$. To simplify our presentation, we will denote $P(\varepsilon_a(\bar{x}))$ by P since the dependency on the scaling parameter should be clear from the context. The closed-loop system can be written in terms of $\tilde{x}, \bar{x}$ as

$$\begin{cases} \dot{\bar{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{E}_1 d_1 + \mathcal{I}\tilde{x} \\ \qquad +\bar{B}\left[\sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) - \sigma(-B'P\bar{x})\right] \\ \dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) \\ \qquad -\hat{B}\sigma(-B'P\bar{x}) + \hat{E}_2 d_2 + \hat{E}_3 d_3, \end{cases} \tag{16.16}$$

where $\bar{B}$ is the same as B in (16.3) and (16.4) with B_{i,n_i} blocks set to zero and

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_1 \\ \mathcal{I}_2 \\ \vdots \\ \mathcal{I}_q \end{bmatrix}, \quad \mathcal{I}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{I}_i \\ 0 \end{bmatrix}, \quad \bar{I}_i = [0 \quad \cdots \quad \underset{\substack{\uparrow \\ i\text{th block}}}{I} \quad \cdots \quad 0]$$

It should be noted that $\bar{B}$, $\bar{E}_1$ and $\mathcal{I}$ are all in the form of (16.9). We first prove global asymptotic stability without disturbances. Consider the dynamics of $\tilde{x}$. Let $v = -B'P\bar{x}$. Our scheduling (16.14) guarantees that $\|v\| \leq \delta \leq \frac{1}{2}$ for any $\bar{x}$. Then,

$$\dot{\tilde{x}} = \hat{A}\tilde{x} + \hat{B}\sigma(-\hat{B}'\tilde{x} + v) - \hat{B}\sigma(v).$$

and define a Lyapunov function as $V_1 = \tilde{x}'\tilde{x}$. Differentiating V_1 along the trajectories yields

$$\dot{V}_1 = 2\tilde{x}'\hat{B}[\sigma(-\hat{B}'\tilde{x} + v) - \sigma(v)].$$

Since $\|v\| \leq \frac{1}{2}$, (16.18) yields that

$$\dot{V}_1 \leq -\tilde{x}'\hat{B}\sigma(\hat{B}'\tilde{x}).$$

Since $\tilde{x}$ has a bounded derivative, by Barbalat's Lemma, this yields that $\lim_{t\to\infty} \hat{B}'\tilde{x}(t) = 0$ which implies that there exists T_0 such that we have $\|\hat{B}'\tilde{x}(t)\| \leq \frac{1}{2}$ for $t \geq T_0$ and hence

$$\dot{\tilde{x}} = (\hat{A} - \hat{B}\hat{B}')\tilde{x}$$

and since this system matrix is Hurwitz stable, we have $\tilde{x}(t) \to 0$ as $t \to \infty$. For $t > T_0$, we have that

$$\dot{\bar{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{\mathcal{I}}\tilde{x}$$

where $\bar{\mathcal{I}} = \mathcal{I} - \bar{B}\hat{B}'$. Define $V_2 = \bar{x}'P\bar{x}$ and a set

$$\mathcal{K} = \left\{\bar{x} \mid V_2(\bar{x}) \leq \frac{\delta^2}{b\,\mathrm{trace}(P(0.9))}\right\}.$$

It can be easily seen from (16.14) that for $\bar{x} \in \mathcal{K}$, $\varepsilon_a(\bar{x}) = 0.9$. Next, consider the derivative of V_2,

$$\begin{aligned}\dot{V}_2 &= -\varepsilon V_2 - \tilde{x}' P B B' P \tilde{x} + 2\bar{x}' P \bar{\mathcal{J}} \tilde{x} + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x} \\ &\leq -\varepsilon V_2 + 2\bar{x}' P \bar{\mathcal{J}} \tilde{x} + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x} \\ &\leq -\varepsilon V_2 + 2\sqrt{V_2} \| P^{1/2} \bar{\mathcal{J}} \tilde{x} \| + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x}.\end{aligned}$$

Note that $\mathcal{J}$, $\bar{B}$ and hence $\bar{\mathcal{J}}$ are in the form of (16.9). Lemma 16.1 shows that there exists an M such that

$$\| P^{1/2} \bar{\mathcal{J}} \tilde{x} \| = \sqrt{\tilde{x}' \bar{\mathcal{J}}' P \bar{\mathcal{J}} \tilde{x}} \leq \varepsilon \sqrt{M} \|\tilde{x}\|.$$

We use here that Lemma 16.1 holds for any matrix of the form (16.9) so it also holds for $\bar{E}_1$ replaced by $\bar{\mathcal{J}}$. Hence

$$\begin{aligned}\dot{V}_2 &\leq -\varepsilon V_2 + 2\varepsilon \sqrt{M} \|\tilde{x}\| \sqrt{V_2} + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x} \\ &\leq -\varepsilon \sqrt{V_2} \left[\sqrt{V_2} - 2\sqrt{M} \|\tilde{x}\| \right] + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x}.\end{aligned}$$

Since $\tilde{x} \to 0$, there exists a $T_1 > T_0$ such that for $t \geq T_1$,

$$\|\tilde{x}\| \leq \frac{\delta}{4\sqrt{M}\sqrt{b\, \mathrm{trace}(P(0.9))}}.$$

Therefore, for $t \geq T_1$ and $\bar{x} \notin \mathcal{K}$ we have

$$\sqrt{V_2} - 2\sqrt{M}\|\tilde{x}\| \geq \frac{\sqrt{V_2}}{2}$$

and thus

$$\dot{V}_2 \leq -\tfrac{\varepsilon}{2} V_2 + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x}.$$

Since $\dot{V}_2$ cannot have the same sign as $\bar{x}' \frac{\mathrm{d}P}{\mathrm{d}t} \bar{x}$ (see [31]), we conclude that $\dot{V}_2 < 0$ for $\bar{x} \notin \mathcal{K}$ and $t > T_1$. This implies that $\bar{x}$ will enter $\mathcal{K}$ within finite time, say $T_2 > T_1$, and remain in $\mathcal{K}$ thereafter. For $t > T_2$ and $\bar{x} \in \mathcal{K}$, we have $\varepsilon_a(\bar{x}) = 0.9$ and $\|\hat{B}'\tilde{x}\| \leq \frac{1}{2}$. All saturations are inactive and the system becomes

$$\begin{cases} \dot{\bar{x}} = (A - BB'P(0.9))\bar{x} + \bar{\mathcal{J}}\tilde{x}, \\ \dot{\tilde{x}} = (\hat{A} - \hat{B}\hat{B}')\tilde{x}. \end{cases}$$

The global asymptotic stability follows from the properties that $\hat{A} - \hat{B}\hat{B}'$ and $A - BB'P(0.9)$ are Hurwitz stable. We proceed to show the boundedness of trajectories in presence of d_1 and d_2. Define

$$R = e^{\hat{A}'t} \quad \text{and} \quad y = R\tilde{x}.$$

Note that since $\hat{A} + \hat{A}' = 0$, R defines a rotation matrix. Moreover, we have that $\dot{R} = -R\hat{A}$. We obtain that

$$\dot{y} = R\hat{B}\sigma(-\hat{B}'R'y + v) - R\hat{B}\sigma(v) + R\hat{E}_2 d_2 + R\hat{E}_3 d_3$$

with $y(0) = \tilde{x}_0$ where $v = -B'P\bar{x}$. Let $\bar{y}$ satisfy

$$\dot{\bar{y}} = R\hat{E}_2 d_2, \quad \bar{y}(0) = \tilde{x}_0.$$

Since $d_2 \in \Omega_\infty$, we find that $\bar{y} \in \mathcal{L}_\infty$ (see Chapter 15). Define $\tilde{y} = y - \bar{y}$. Then

$$\dot{\tilde{y}} = R\hat{B}\sigma(-\hat{B}'R'\tilde{y} - \hat{B}'R'\bar{y} + v) - R\hat{B}\sigma(v) + R\hat{E}_3 d_3$$

with $\tilde{y}(0) = 0$. Again define $z = R'\tilde{y}$. We get

$$\dot{z} = \hat{A}z + \hat{B}\sigma(-\hat{B}'z - \hat{B}'R'\bar{y} + v) - \hat{B}\sigma(v) + \hat{E}_3 d_3, \quad z(0) = 0,$$

Consider an auxiliary system

$$\dot{w} = (\hat{A} + \hat{B}\hat{F})w + \hat{E}_3 d_3, \quad w(0) = 0,$$

where $\hat{F}$ is such that $\hat{A} + \hat{B}\hat{F}$ is Hurwitz stable. For a selected $\hat{F}$, let δ_1 be sufficiently small such that $\|d_3\|_{\mathcal{L}_\infty} \leq \delta_1$ implies that $\|\hat{F}w\|_{\mathcal{L}_\infty} \leq 1/4$.

Let $\xi = z - w$. We have that

$$\dot{\xi} = \hat{A}\xi + \hat{B}\sigma(-\hat{B}'\xi + u) - \hat{B}\sigma(v) - \hat{B}\hat{F}w, \quad \xi(0) = 0,$$

where $u = -\hat{B}'w - \hat{B}'R'\bar{y} + v$. Since $u \in \mathcal{L}_\infty$ and $\|\sigma(v) + \hat{F}w\|_{\mathcal{L}_\infty} \leq 1/4 + 1/4 = 1/2$, it follows from Lemma 16.3 in the appendix that $\xi \in \mathcal{L}_\infty$. This implies that $\tilde{x} \in \mathcal{L}_\infty$.

Consider the dynamics of $\bar{x}$

$$\dot{\bar{x}} = A\bar{x} + B\sigma(-B'P\bar{x}) + \bar{B}\zeta + \bar{E}_1 d_1 + \mathcal{J}\tilde{x}$$

where $\zeta = \sigma(-\hat{B}'\tilde{x} - B'P\bar{x}) - \sigma(-B'P\bar{x})$. Since $\sigma(\cdot)$ is globally Lipschitz with Lipschitz constant 1, we have that $\|\zeta\| \leq \|\hat{B}'\tilde{x}\|$ and thus $\zeta \in \mathcal{L}_\infty$.

By differentiating $V_2 = \bar{x}' P \bar{x}$, we obtain

$$\begin{aligned}\dot{V}_2 &\leq -\varepsilon V_2 + 2x' P \bar{E}_1 d_1 + 2x' P \mathcal{J} \tilde{x} + 2\bar{x}' P \bar{B} \zeta + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x} \\ &\leq -\varepsilon V_2 + 2\sqrt{V_2} \| P^{1/2} \bar{E}_1 d_1 \| + 2\sqrt{V_2} \| P^{1/2} \bar{B} \zeta \| \\ &\quad + 2\sqrt{V_2} \| P^{1/2} \mathcal{J} \tilde{x} \| + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x}.\end{aligned}$$

We have already shown in Lemma 16.1 that there exist M_1, M_2 and M_3 such that

$$\| P^{1/2} \bar{E}_1 d_1 \| \leq \varepsilon \sqrt{M_1} \| d_1 \|, \quad \| P^{1/2} \bar{B} \zeta \| \leq \varepsilon \sqrt{M_2} \| \zeta \| \text{ and } \| P^{1/2} \mathcal{J} \tilde{x} \| \leq \varepsilon \sqrt{M_3} \| \tilde{x} \|.$$

We obtain,

$$\dot{V}_2 \leq -\varepsilon \sqrt{V_2} \Big[\sqrt{V}_2 - 2\sqrt{M_1} \| d_1 \|_\infty - 2\sqrt{M_3} \| \tilde{x} \|_\infty - 2\sqrt{M_2} \| \zeta \|_\infty \Big] + \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x}.$$

If $\sqrt{V_2} \geq 2\sqrt{M_1} \| d_1 \|_\infty + 2\sqrt{M_3} \| \tilde{x} \|_\infty + 2\sqrt{M_2} \| \zeta \|_\infty$, we have

$$\dot{V}_2 \leq \bar{x}' \tfrac{\mathrm{d}P}{\mathrm{d}t} \bar{x}.$$

Since $\dot{V}_2$ and $\bar{x}' \frac{\mathrm{d}P}{\mathrm{d}t} \bar{x}$ can not have the same sign, we find that $\dot{V}_2 \leq 0$ for

$$\bar{x} \in \Big\{ \bar{x} \mid \sqrt{V_2} \geq 2\sqrt{M_1} \| d_1 \|_\infty + 2\sqrt{M_3} \| \tilde{x} \|_\infty + 2\sqrt{M_2} \| \zeta \|_\infty \Big\},$$

which, from the property (4) of scheduling (16.14), implies that $\bar{x} \in \mathcal{L}_\infty$ and hence $x \in \mathcal{L}_\infty$. ■

16.5. Computational issues

The proposed controller design relies on scheduling of the parameter $\varepsilon_a(\bar{x})$ which is a convex optimization problem but requires online solving CPLE (16.11) or DPLE (16.12) and can be computationally demanding for large systems. However, compared with normal Riccati equation, (16.11) and (16.12) still have some numerical merit, for example the solution $P(\varepsilon)$ is a rational matrix in general (see [159, 161]). Moreover, in a special case where the system has a single input, $P(\varepsilon)$ is a polynomial matrix and can be solved easily and explicitly in a finite recursion. In such a case, $\varepsilon_a(\bar{x})$ is not difficult to obtain.

Appendix

We shall need the following bounds from [107]:

Lemma 16.2 For two vectors $s, t \in \mathbb{R}^m$, the following statements hold:

$$|s'[\sigma(s+t)-\sigma(s)]| \leq 2\sqrt{m}\|t\|, \tag{16.17}$$

$$2s'[\sigma(t)-\sigma(t-s)] \geq s'\sigma(s), \qquad \|t\| \leq \tfrac{1}{2}, \tag{16.18}$$

$$\|s-\sigma(s)\| \leq s'\sigma(s). \tag{16.19}$$

The following lemma is a core result for neutrally stable systems:

Lemma 16.3 Suppose (A, B) is controllable and $A'+A=0$ for continuous-time systems and $A'A=I$ for discrete-time systems. Consider the system

$$\rho x = Ax + B\sigma(Kx+v_1) + Bv_2, \quad x(0) = x_0$$

where

$$K = \begin{cases} -B', & \text{continuous-time;} \\ -\kappa B'A, & \text{discrete-time.} \end{cases}$$

and κ satisfies $8\kappa B'B \leq I$. We have

1. In absence of v_1 and v_2, the origin is globally asymptotically stable;
2. $x \in \mathcal{L}_\infty$ for any initial condition and for $v_1 \in \mathcal{L}_\infty$, $v_2 \in \mathcal{L}_\infty(1/2)$ (continuous time) or $v_1 \in \ell_\infty$, $v_2 \in \ell_\infty(1/2)$ (discrete time).

Proof : The result for continuous-time systems can be found in [62] (Lemma 2) and [151] (Proposition 1). The discrete-time counterpart was proved in Chapter 15. ■

Proof of Lemma 16.1 : We only prove the result for the continuous-time case. The corresponding discrete-time result follows from exactly the same argument. It is shown in [159] that $P(0) = 0$ and $P(\varepsilon)$ is rational in ε. Therefore, we can write

$$P(\varepsilon) = \varepsilon P_1 + \varepsilon^2 P_2 + \ldots + \varepsilon^i P_i + \ldots.$$

Substituting $P(\varepsilon)$ in (16.11), we find P_1 satisfies that

$$P_1 A + A' P_1 = 0, \tag{16.20}$$

where A is given by (16.3). Consider the diagonal block of P_1, say $P_{1,i}$, corresponding to $\bar{A}_i$ block. $P_{1,i}$ must satisfy

$$\bar{A}_i' P_{1,i} + P_{1,i} \bar{A}_i = 0 \tag{16.21}$$

where $\bar{A}_i$ is given by (16.4). Suppose

$$P_{1,i} = \begin{bmatrix} \bar{P}_{11} & \bar{P}_{12}' \\ \bar{P}_{12} & \bar{P}_{22} \end{bmatrix}$$

where $\bar{P}_{11} \in \mathbb{R}^{p_i \times p_i}$, $\bar{P}_{12}$ and $\bar{P}_{22}$ are of appropriate dimension. Define

$$\xi_{i,j} = \begin{bmatrix} x_{i,j}' & 0 & \cdots & 0 \end{bmatrix}'.$$

where the eigenvectors of A_i are $x_{i,j}$ with associated eigenvalues λ_j, $j = 1, \ldots, p_i$. Clearly we have $\bar{A}_i \xi_{i,j} = \lambda_j \xi_{i,j}$ and thus $\xi_{i,j}$ is an eigenvector of $\bar{A}_i$. Note that $\bar{A}_i$ has p_i linearly independent eigenvectors. We shall have that

$$(\bar{A}_i' P_{1,i} + \bar{P}_{1,i} \bar{A}_i)\xi_{i,j} = 0.$$

This implies that

$$\bar{A}_i' P_{1,i} \xi_{i,j} = -\lambda_j P_{1,i} \xi_{i,j}.$$

In other words, $P_{1,i}\xi_{i,j}$ is an eigenvector of $\bar{A}_i'$ associated with eigenvalue $-\lambda_j$ for $j = 1, \ldots, p_i$. On the other hand, we have a set of eigenvectors of $\bar{A}'$ in the form of

$$\nu_{i,j} = \begin{bmatrix} 0 & \cdots & 0 & v_{i,j}' \end{bmatrix}', i = 1, \ldots, p_i$$

where $v_{i,j}$ are the eigenvectors of A_i' associated with eigenvalue λ_j. Note that $\bar{A}_i'$ also has only p_i linearly independent eigenvectors. Therefore,

$$P_1 \xi_{i,j} = \begin{bmatrix} \bar{P}_{11} x_i \\ \bar{P}_{12} x_i \end{bmatrix} \in \text{span}\{\nu_{i,1}, \ldots, \nu_{i,p_i}\}.$$

This implies that $\bar{P}_{11} x_{i,j} = 0$, $j = 1, \ldots, p_i$. Since $x_{i,j}$ forms a basis of $\mathbb{R}^{p_i}$, we must have that $\bar{P}_{11} = 0$ and hence $\bar{P}_{12} = 0$ due to the fact that $P_{1,i}$ is positive semi-definite. Recursively, applying the above argument to $\bar{P}_{22}$, we shall eventually find that

$$P_{1,i} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & \bar{P}_{n_i n_i} \end{bmatrix}.$$

Since P_1 is positive semi-definite, we note that for any given matrix $\bar{E}_1$ in the form of (16.9), we must have $P_1 \bar{E}_1 = 0$. This implies that $\bar{E}_1' P(\varepsilon) \bar{E}_1$ must be of order ε^2. ∎

Part IV

Synchronization in multi-agent systems

Notation and preliminaries

The following notations are used in this Chapter. For a matrix $X \in \mathbb{C}^{n\times m}$,

$$X' : \text{transpose of } X;$$

$$X^* : \text{conjugate transpose of } X;$$

$$X^{-1} : \text{inverse of } X \text{ if it exists}$$

$$\bar{\sigma}(X) : \text{maximal singular value of } X;$$

$$\underline{\sigma}(X) : \text{minimal singular value of } X;$$

$$\|X\| : \text{induced 2 norm};$$

$$\det(X) : \text{determinant of } X.$$

For a continuous-time transfer function $H(s) : \mathbb{C} \to \mathbb{C}^{n\times m}$,

$$\|H(s)\|_\infty : \mathcal{H}_\infty \text{ norm of } H(s).$$

For a vector d, we denote a diagonal matrix by D=diag$\{d\}$ whose diagonal is specified by d. For column vectors $x_1, \dots, x_n$, the stacking column vector of $x_1, \dots, x_n$ is denoted by $[x_1; \dots; x_n]$.

For $A \in \mathbb{C}^{n\times m}$ and $B \in \mathbb{C}^{p\times q}$, the Kronecker product of A and B is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & \vdots & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}$$

The following property of the Kronecker product will be used:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

A matrix $D = \{d_{ij}\}_{n\times n}$ is called a row stochastic matrix if

1. $d_{ij} \geq 0$ for any i, j;
2. $\sum_{j=1}^{n} d_{ij} = 1$ for $i = 1, ..., n$.

A row stochastic matrix D has at least one eigenvalue at 1 with right eigenvector $\mathbf{1}$. D can be associated with a graph $G = (\mathcal{N}, \mathcal{E})$. The number of nodes in $\mathcal{N}$ is the dimension of D and an arc $(j, i) \in \mathcal{E}$

if $d_{ij} > 0$. It is shown in [81] that 1 is a simple eigenvalue of D if and only if G contains a directed spanning tree. Moreover, the other eigenvalues are in the open unit disk if $d_{ii} > 0$ for all i.

A graph G is defined by a pair $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, N\}$ is a vertex set and $\mathcal{E}$ is a set of pairs of vertices (i, j). Each pair in $\mathcal{E}$ is called an *arc*. G is *undirected* if $(i, j) \in \mathcal{E} \Rightarrow (j, i) \in \mathcal{E}$. Otherwise, G is *directed*. A *directed path* from vertex i_1 to i_k is a sequence of vertices $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, ..., k-1$. A directed graph G contains a *directed spanning tree* if there is a node r such that a directed path exists between r and every other node.

The graph G is *weighted* if each arc (i, j) is assigned with a real number a_{ij}. For a weighted graph G, a matrix $L = \{\ell_{ij}\}$ with

$$\ell_{ij} = \begin{cases} \sum_{j=1}^{N} a_{ij}, & i = j \\ -a_{ij}, & i \neq j, \end{cases}$$

is called *Laplacian matrix* associated with graph G. In the case where G has non-negative weights, L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$ (see [85]). If G has a directed spanning tree, L has a simple eigenvalue at zero and all the other eigenvalues have strictly positive real parts (see for example [81]).

CHAPTER 17

Synchronization in a heterogeneous network of discrete-time introspective right-invertible agents

17.1. Introduction

In this chapter, we consider a heterogeneous network of non-identical introspective right-invertible agents[1]. Both output synchronization and regulation problems are studied. We show that exchange of information among controllers is not needed. Depending on the desired frequencies in synchronization trajectories, different decentralized control schemes are proposed to achieve synchronization for a set of communication topologies. The paper is organized as follows: The network structure and preliminary assumptions and definitions are given in Section 17.2. The output synchronization and regulation problems are solved in Section 17.3 and 17.4 respectively. Technical development is left in the Appendix.

17.2. Network structure

Consider a heterogeneous network of N introspective agents

$$\begin{cases} x^i(k+1) = A^i x^i(k) + B^i u^i(k), \\ y^i(k) \quad = C_y^i x^i(k), \\ z^i(k) \quad = C_z^i x^i(k), \\ \zeta^i(k) \quad = \sum_{j=1}^{N} d_{ij}(y^i(k) - y^j(k)) \end{cases} \tag{17.1}$$

where $x^i \in \mathbb{R}^{n_i}$, $y^i \in \mathbb{R}^p$, $z^i \in \mathbb{R}^{q_i}$ and $u^i \in \mathbb{R}^{m_i}$. The matrix $D = \{d_{ij}\} \in \mathbb{R}^{N\times N}$ is a row-stochastic matrix that satisfies $d_{ii} > 0$, $d_{ij} \geq 0$ and $\sum_j d_{ij} = 1$. This D matrix defines a communication topology that can be captured by a directed graph $G = (\mathcal{N}, \mathcal{E})$. The set $\mathcal{N}$ contains all the node and $\mathcal{E}$ is the edge set such that an arc $(j,i) \in \mathcal{E}$ if $d_{ij} > 0$.

Assumption 17.1 The communication topology G contains a directed spanning tree.

Under Assumption 17.1, D has a simple eigenvalue at 1 associated with right eigenvector $\mathbf{1}$ and the other eigenvalues strictly inside the unit disk. Let $\lambda_1, ..., \lambda_N$ denote the eigenvalues of D such that $\lambda_1 = 1$ and $|\lambda_i| < 1$, $i = 2, ..., N$. We can define a set of communication topology as follows:

[1]The definition of right-invertibility of a linear system can be found in [71]

Definition 17.1 For $\delta \in (0, 1]$, let $\mathcal{G}_\delta$ denote a set of communication topologies such that

1. Assumption 1 holds;

2. $|\lambda_i| < \delta, i = 2, ..., N$.

Remark 17.1 *For $\delta = 1$, $\mathcal{G}_1$ is the set of all communication topologies that satisfies Assumption 17.1. In this case, we shall drop the subscription* 1 *and simply denote it as $\mathcal{G}$ but it implies $\delta = 1$.*

In the network (17.1), each agent collects two measurements:

1. a network measurement $\zeta^i \in \mathbb{R}^p$ which is a combination of its own output relative to that of neighboring agents;

2. a local measurement $z^i \in \mathbb{R}^{q_i}$ of its internal dynamics.

For each agent, we make the following standard assumption.

Assumption 17.2 The agents possesse the following properties:

1. (A^i, B^i) is stabilizable;

2. (A^i, C_z^i) is detectable;

3. (A^i, C_y^i) is detectable;

4. (A^i, B^i, C_y^i) is right-invertible.

17.3. Output synchronization

The first problem studied in this paper is the output synchronization problem. The output synchronization in a heterogeneous network of the form (17.1) is defined as follows:

Definition 17.2 *The agents in the network achieve output synchronization if*

$$\lim_{k \to \infty} (y^i(k) - y^j(k)) = 0, \quad \forall i, j \in \{1, \dots, N\}.$$

The output synchronization problem is formulated below:

Problem 17.1 Consider a heterogenous network of the form (17.1). For $\delta \in (0, 1]$ and a given set $\mathcal{G}_\delta$, the output synchronization problem with a set of communication topologies $\mathcal{G}_\delta$ is to design a local linear dynamical controller

$$\begin{cases} \hat{x}^i(k+1) = A_c^i \hat{x}^i(k) + B_c^i \zeta^i(k) + E_c^i z^i(k) \\ u^i(k) \quad = C_c^i \hat{x}^i(k) + D_c^i \zeta^i(k) + M_c^i z^i(k), \end{cases} \tag{17.2}$$

such that the output synchronization can be achieved in the network with any communication topology belonging to $\mathcal{G}_\delta$.

Remark 17.2 *Since (A^i, C_z^i) is detectable, one can always design a local stabilizing measurement feedback controller so that the network achieves output synchronization in the sense that $y^i(k) \to 0$ as $k \to \infty$. Such a case is not interested in this paper. We are aiming to reach synchronization with a non-trivial and possibly desirable synchronization trajectory.*

The synchronization trajectories considered in most applications are either bounded or polynomially increasing. We shall also present the main results respectively for these two cases. The first theorem is concerned with bounded synchronization trajectories.

Theorem 17.1 For the set $\mathcal{G}$, Problem 17.1 with bounded synchronization trajectories is always solvable via a decentralized dynamic controller (17.2).

Remark 17.3 *Theorem 17.1 indicates that in the case of bounded synchronization trajectories, a universal synchronization controller can be constructed which solve Problem 17.1 for any communication topology satisfying Assumption 17.1.*

If unbounded synchronization trajectories are demanded, the admissible set of communication topologies has to be more restricted. This is stated in the next theorem

Theorem 17.2 For $\delta \in (0, 1)$ and a given set $\mathcal{G}_\delta$, Problem 17.1 with unbounded increasing synchronization trajectories is solvable via a decentralized dynamic consensus controller (17.2).

We shall prove Theorem 17.1 and 17.2 by explicitly constructing the synchronization controllers. The design and analysis is done in the next three subsections. First, by exploiting the self-measurement of each agent, we design a local pre-compensator such that the agent model can be re-shaped as asymptotically identical, which we refer to *network homogenizing*. Next, in the resulting (asymptotically) homogeneous network, solvability of the output synchronization problem can be connected to that of a robust stabilization problem. Finally, the last step is to solve this robust stabilization problem by designing a compensator using a low-gain approach. In this stage, depending on different types of synchronization trajectories, two controllers can be constructed.

17.3.1. Homogenization of the network

For introspective agents, their self-reflection of internal dynamics provides us with additional freedom to manipulate the agent models so as to disguise them as being almost identical to the rest of the network viewed from their output. This is shown in the next lemma.

Lemma 17.1 Consider a heterogeneous network of the form (17.1). Let n_d denote the maximum order of infinite zeros of (A^i, B^i, C^i). Suppose a triple (A, B, C) is given such that

1. $\text{rank}(C) = p$.
2. (A, B, C) is invertible, of uniform rank $n_q \geq n_d$ and has no invariant zeros.

There exists a compensator

$$\begin{cases} \xi^i(k+1) = A_h^i \xi^i(k) + B_h^i z^i(k) + E_h^i v^i(k) \\ u^i(k) \quad = C_h^i \xi^i(k) + D_h^i v^i(k), \end{cases} \tag{17.3}$$

such that the closed-loop system of (17.1) and (17.3) can be written in the following form:

$$\begin{cases} \bar{x}^i(k+1) = A\bar{x}^i(k) + B\left(v^i(k) + d^i(k)\right) \\ y^i(k) \quad = C\bar{x}^i(k), \\ \zeta^i(k) \quad = \sum_{j=1}^N d_{ij}(y^i(k) - y^j(k)), \end{cases} \tag{17.4}$$

where d^i are generated by

$$\begin{cases} e^i(k+1) = A_s^i e^i(k), \quad i = 1, ..., N, \\ d^i(k) \quad = C_s^i e^i(k). \end{cases} \tag{17.5}$$

and A_s^i are Schur stable.

Proof : Proof and detailed design procedure can found in Appendix 17.6.1. ■

Remark 17.4 *We have the following observations*

1. *The first condition of Lemma 17.1 is natural in the sense that the new model much maintain the same interface with the network.*

2. *The condition that (A, B, C) is invertible and has no invariant zero implies that (A, B) is controllable and (A, C) is observable.*

3. *We have a substantial freedom in choosing the eigenvalues of A which, as will be seen, determine the modes in the synchronization trajectories.*

Remark 17.5 *It should also be noted that such a triple (A, B, C) always exists and, without loss of generality, takes the following form:*

$$A = A_0 + BF, \quad A_0 = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C = \begin{bmatrix} I_p & 0 \end{bmatrix}, \tag{17.6}$$

and F is such that $A_0 + B_0 F$ has desired eigenvalues. Such an F exists due to the fact that (A_0, B_0) is controllable.

17.3.2. Connection to robust simultaneous stabilization problem

In this subsection, we shall show that the output synchronization in an (asymptotically) homogeneous network (17.4) and (17.5) can be solved by equivalently solving a robust stabilization problem.

Suppose the synchronization problem for network (17.4) and (17.5) with any communication topology in $\mathcal{G}_\delta$ can be solved by a compensator

$$\begin{cases} \chi^i(k+1) = A_c \chi^i(k) + B_c \zeta^i(k) \\ v^i(k) \quad\;\; = C_c \chi^i(k). \end{cases} \tag{17.7}$$

Let $\tilde{x}^i = [\bar{x}^i; \chi^i]$. Then the closed-loop of each agent can be written as

$$\begin{cases} \tilde{x}^i(k+1) = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix} \tilde{x}^i(k) + \begin{bmatrix} 0 \\ B_c \end{bmatrix} \zeta^i(k) + \begin{bmatrix} B \\ 0 \end{bmatrix} d^i(k) \\ y^i(k) \quad\;\; = \begin{bmatrix} C & 0 \end{bmatrix} \tilde{x}^i(k) \\ \zeta^i(k) \quad\;\; = y^i(k) - \sum_{j=1}^N d_{ij} y^j(k). \end{cases} \tag{17.8}$$

Define $\tilde{x} = [\tilde{x}^1; \cdots; \tilde{x}^N]$,

$$\bar{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \text{ and } \bar{E} = \begin{bmatrix} B \\ 0 \end{bmatrix}.$$

The overall dynamics of the N agents can be written as

$$\tilde{x}(k+1) = \left[I_N \otimes \bar{A} + (I_N - D) \otimes \bar{B}\bar{C}\right]\tilde{x}(k) + (I_N \otimes \bar{E})d(k).$$

Define $\eta = [\eta^1; \cdots; \eta^N] = (T \otimes I_n)\tilde{x}$ where $\eta^i \in \mathbb{C}^n$ and T is such that $J_L = T(I_N - D)T^{-1}$ is in the Jordan canonical form and $J_L(1,1) = 0$. In the new coordinates, the dynamics of η can be written as

$$\eta(k+1) = \left[I_N \otimes \bar{A} + J_L \otimes \bar{B}\bar{C}\right]\eta(k) + (T \otimes \bar{E})d(k).$$

Lemma 17.2 The network of the form (17.8) achieves output synchronization if and only if $\eta^i(k) \to 0$ as $k \to \infty$ for $i = 2, ..., N$.

Proof : Let

$$\pi(k) = \begin{bmatrix} \eta^1(k) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \otimes \eta^1(k)$$

If $\eta(k) \to \pi(k)$, then $\tilde{x}(k) \to (T^{-1} \otimes I_n)\pi(k)$. Note that the columns of T^{-1} comprise all the right eigenvectors and generalized eigenvectors of $I - D$. The first column of T^{-1} is vector $\mathbf{1}$. Hence the fact that $\tilde{x}(k) \to (T^{-1} \otimes I_n)\pi(k)$ implies that

$$\tilde{x}(k) \to \mathbf{1} \otimes \eta^1(k).$$

On the other hand, suppose the network (17.8) reaches synchronization. In this case, we shall have

$$\tilde{x}(k) \to \mathbf{1} \otimes \tilde{x}^1(k).$$

But then $\eta(k) \to (T\mathbf{1}) \otimes \tilde{x}^1(k)$. Since $\mathbf{1}$ is the first column of T^{-1}, we have

$$T\mathbf{1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Therefore, $\eta(k) \to (T\mathbf{1}) \otimes \tilde{x}^1(t)$ implies that $\eta^1(k) \to \tilde{x}^1(k)$ and $\eta^i(k) \to 0$ for $i = 2, ..., N$. ■

Remark 17.6 *It also becomes clear from Lemma 17.2 that the synchronization trajectory is given by $\eta^1(k)$ which is governed by*

$$\eta^1(k+1) = A\eta^1(k) + (w \otimes \bar{E})d(k), \quad \eta^1(0) = (w \otimes I_n)\tilde{x}(0),$$

where w is the first row of T, i.e. the left eigenvector associated with eigenvalue 1. Note that $d(k) \to 0$ as $k \to \infty$. This shows that the modes of the synchronization trajectory are determined by the eigenvalues of A and the complete dynamics depends on both A and a weighted average of the agents' initial conditions.

Define $\bar{\eta} = [\eta^2; \cdots; \eta^N]$. Taking the dynamics of d into account, we can write

$$\begin{bmatrix} \bar{\eta}(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \bar{A} + \bar{J}_L \otimes \bar{B}\bar{C} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} \begin{bmatrix} \bar{\eta}(k) \\ e(k) \end{bmatrix}, \tag{17.9}$$

where $e = [e^1; \ldots; e^N]$,

$$\bar{C}_s = \text{blkdiag}\{C_s^i\}_{i=1}^N, \quad \bar{I} = [0, I_{N-1}], \quad \bar{A}_s = \text{blkdiag}\{A_s^i\}_{i=1}^N,$$

and $\bar{J}_L$ is such that

$$J_L = \begin{bmatrix} 0 & \\ & \bar{J}_L \end{bmatrix}.$$

Clearly $\bar{\eta} \to 0$ for any initial condition if the system (17.9) is globally asymptotically stable. Since $\bar{A}_s$ is Schur stable, the next lemma is straightforward:

Lemma 17.3 The network of the form (17.8) achieves output synchronization if the system

$$\tilde{\eta}(k+1) = (I_{N-1} \otimes \bar{A} + \bar{J}_L \otimes \bar{B}\bar{C})\tilde{\eta}(k), \tag{17.10}$$

is globally asymptotically stable.

Due to upper-triangular structure of $I_{N-1} \otimes \bar{A}$ and $(\bar{J}_L \otimes \bar{B}\bar{C})$, the system (17.10) is essentially a family of $N-1$ subsystems:

$$\tilde{\eta}^i(k+1) = (\bar{A} + (1-\lambda_i)\bar{B}\bar{C})\tilde{\eta}^i(k), \quad i = 2, ..., N, \tag{17.11}$$

where λ_i, $i = 2, ..., N$ are those eigenvalues of D that are not equal to 1.

Lemma 17.4 The network (17.8) achieves output synchronization if (17.11) is globally asymptotically stable for λ_i, $i = 2, ..., N$.

Note that (17.11) can be viewed as the closed-loop of

$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ \quad z(k) \quad = (1-\lambda)Cx(k). \end{cases} \tag{17.12}$$

and a compensator

$$\begin{cases} \chi(k+1) = A_c\chi(k) + B_c z(k) \\ \quad u(k) \quad = C_c\chi(k) \end{cases} \tag{17.13}$$

with unknown λ satisfying $|\lambda| < \delta$. It is easy to see that owing to linearity, (17.13) stabilizes (17.12) if it stabilizes

$$\begin{cases} x(k+1) = Ax(k) + (1-\lambda)Bu(k), \\ \quad z(k) \quad = Cx(k). \end{cases} \tag{17.14}$$

Therefore, we arrive at the following conclusion by the end of this subsection.

Lemma 17.5 Problem 17.1 is solved via a composite controller of (17.3) and (17.7) if the closed-loop of (17.14) and (17.13) is globally asymptotically stable for all $|\lambda_i| < \delta$.

Proof : By establishing Lemma 17.2-17.4, we have shown that if the closed-loop of (17.14) and (17.13) is globally asymptotically stable for all $|\lambda| < \delta$, then the interconnections of the closed-loop of compensator (17.7) and (17.4), which is the network (17.8), will reach synchronization. This implies that the composite controller of (17.3) and (17.7) solves Problem 17.1. ■

So far, we have converted the output synchronization problem to a simultaneous stabilization problem. Next, depending on different types of synchronization trajectories, the design bifurcates into two approaches.

17.3.3. Bounded synchronization trajectories

It has been shown that the eigenvalues of A dictate the modes in the synchronization trajectories. If the trajectories are required to be bounded, we can choose A matrix in Lemma 17.1 to have only semi-simple eigenvalues on the unit circle. This can be done by choosing proper F matrix in (17.6). Note that in this case, we can always assume without loss of generality that $A'A = I$. The controller designed based on this type of A matrix can be easily modified by a state transformation so as to be applicable to the agents with a more general form.

Based on the analysis in the preceding subsection, to prove Theorem 17.1, we need to design data (A_c, B_c, C_c) in the compensator (17.7) for which the closed-loop of (17.14) and (17.13) is globally asymptotically stable with any $|\lambda| < 1$.

We can construct (17.7) in the following form:

$$\begin{cases} \chi^i(k+1) = (A+KC)\chi^i(k) - K\zeta^i(k) \\ v^i(k) \quad = -\varepsilon B'A\chi^i(k), \end{cases} \tag{17.15}$$

where K is such that $A+KC$ is Schur stable and $\varepsilon > 0$ is a design parameter to be chosen later. In other words, we choose $A_c = A + KC$, $B_c = -K$ and $C_c = -\varepsilon B'A$. With this set of data, the closed-loop (17.14) and (17.13) can be written as

$$\begin{cases} x(k+1) = Ax(k) - (1-\lambda)\varepsilon BB'A\chi(k) \\ \chi(k+1) = (A+KC)\chi(k) - KCx(k). \end{cases} \tag{17.16}$$

Lemma 17.6 There exists an $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$, (17.16) is globally asymptotically stable for $|\lambda| \in (0, 1)$.

Proof : Define $e(k) = x(k) - \chi(k)$. The system (17.16) can be rewritten in terms of x and e as follows:

$$\begin{cases} x(k+1) = (A - (1-\lambda)\varepsilon BB'A)x(k) + (1-\lambda)\varepsilon BB'Ae(k) \\ e(k+1) = (A + KC + (1-\lambda)\varepsilon BB'A)e(k) - (1-\lambda)\varepsilon BB'Ax(k). \end{cases} \tag{17.17}$$

Let Q be the positive definite solution of Lyapunov equation

$$(A+KC)'Q(A+KC) - Q + 4I = 0.$$

Since $|\lambda| \in (0, \delta)$, there exists an ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$

$$(A + KC + (1-\lambda)\varepsilon BB'A)^* Q(A + KC + (1-\lambda)\varepsilon BB'A) - Q + 3I \leq 0.$$

Consider $V_1(k) = e(k)^* Q e(k)$. Let $\mu(k) = \varepsilon B'Ax(k)$. To ease our presentation, we shall omit the time label (k) whenever this causes no confusion.

$$\begin{aligned} &V_1(k+1) - V_1(k) \\ \leq& -3\|e\|^2 + 2\left|\mathrm{Re}\Big((1-\lambda)^*\mu^* B'Q[A + KC + (1-\lambda)BB'A]e\Big)\right| + |1-\lambda|^2\mu^* B'QB\mu \\ \leq& -3\|e\|^2 + |1-\lambda| M_1\|\mu\|\|e\| + |1-\lambda|^2 M_2\|\mu\|^2, \end{aligned}$$

where

$$M_1 = 2\|B'Q\|\|A + KC\| + 4\|B'Q\|\|BB'A\|, \quad M_2 = \|B'QB\|.$$

It should be noted that M_1 and M_2 are independent of ε and λ.

Consider $V_2(k) = \|x(k)\|^2$. Note that

$$\begin{aligned}&[A - (1-\lambda)\varepsilon BB'A]'[A - (1-\lambda)\varepsilon BB'A] - I\\ =& -2\operatorname{Re}(1-\lambda)\varepsilon A'BB'A + |1-\lambda|^2\varepsilon^2 A'BB'BB'A\end{aligned}$$

There exists an ε_2 such that for $\varepsilon \in (0, \varepsilon_2]$, $\varepsilon B'B \leq \frac{1}{2}I$. Since $|1-\lambda|^2 \leq 2\operatorname{Re}(1-\lambda)$ for $|\lambda| < 1$, we get for $\varepsilon \in (0, \varepsilon_2]$,

$$[A - (1-\lambda)\varepsilon BB'A]^*[A - (1-\lambda)\varepsilon BB'A] - A'A \leq -\tfrac{1}{2}|1-\lambda|^2\varepsilon A'BB'A.$$

Hence

$$\begin{aligned}&V_2(k+1) - V_2(k)\\ \leq& -\tfrac{1}{2\varepsilon}|1-\lambda|^2\|\mu\|^2 + 2\operatorname{Re}\left((1-\lambda^*)e^*A'B\mu - |1-\lambda|^2\varepsilon e^*A'BB'B\mu\right) + |1-\lambda|^2\varepsilon^2 e^*A'BB'BB'Ae\\ \leq& -\tfrac{1}{2\varepsilon}|1-\lambda|^2\|\mu\|^2 + \theta_1|1-\lambda|\|e\|\|\mu\| + \theta_3|1-\lambda|^2\|e\|\|\mu\| + \theta_2\|e\|^2,\end{aligned}$$

where

$$\theta_1 = 2\|A'B\|, \quad \theta_3 = 2\|A'BB'B\|, \quad \theta_2 = 4\|A'BB'B'A\|.$$

Define a Lyapunov candidate $V(k) = V_1(k) + \varepsilon\kappa V_2(k)$ with

$$\kappa = 4 + 2M_2 + 2M_1^2.$$

We get that

$$\begin{aligned}V(k+1) - V(k) \leq& -(3 - \varepsilon\theta_2\kappa)\|e\|^2 - (2 + M_1^2)|1-\lambda|^2\|\mu\|^2\\ &+ (M_1 + \varepsilon\theta_1\kappa)|1-\lambda|\|\mu\|\|e\| + \varepsilon\theta_3\kappa|1-\lambda|^2\|\mu\|\|e\|.\end{aligned}$$

There exists an ε_3 such that for $\varepsilon \in (0, \varepsilon_3]$,

$$3 - \varepsilon\theta_2\kappa \geq 2.5, \quad M_1 + \varepsilon\theta_1\kappa \leq 2M_1 \text{ and } \varepsilon\theta_3\kappa \leq 1.$$

This yields that

$$
\begin{aligned}
&V(k+1)-V(k)\\
&\leq -2.5\|e\|^2-(2+M_1^2)|1-\lambda|^2\|\mu\|^2+(2M_1|1-\lambda|+|1-\lambda|^2)\|\mu\|\|e\|\\
&\leq -0.5\|e\|^2-|1-\lambda|^2\|\mu\|^2-(\|e\|-M_1|1-\lambda|\|\mu\|)^2-|1-\lambda|^2(\tfrac{1}{2}\|e\|-\|\mu\|)^2\\
&\leq -0.5\|e\|^2-|1-\lambda|^2\|\mu\|^2.
\end{aligned}
$$

Since (A, B) is controllable, it follows from LaSalle's invariance principle that system (17.17) is globally asymptotically stable. ■

17.3.4. Unbounded synchronization trajectories

We proceed to consider synchronization trajectories that are possibly unbounded. In most applications, those unbounded synchronization trajectories are normally polynomially increasing. This can be achieved by choosing an A matrix that has all the eigenvalues on the unit circle, some of which may be degenerate. It should be pointed out we can not only assign the eigenvalues of A to arbitrary locations, but we are also able to assign the multiplicity structures of the eigenvalues as long as they are compatible, that is, the summation of all algebraic multiplicities equals to the dimension of A.

Our design is built upon the solution of the following discrete algebraic Riccati equation (DARE) which is also used in [44]:

$$P_\varepsilon = A'P_\varepsilon A+\varepsilon I-(1-\delta^2)A'P_\varepsilon B(B'P_\varepsilon B+I)^{-1}B'P_\varepsilon A. \tag{17.18}$$

The next lemma can be proved following the work in [109, 100] (see also [44])).

Lemma 17.7 For any $\varepsilon > 0$ and $\delta \in (0, 1)$, the DARE (17.18) has a unique positive definite solution P_ε and moreover $A-(1-\lambda)B(B'P_\varepsilon B+I)^{-1}B^*P_\varepsilon A$ is Schur stable for $|\lambda| < \delta$.

The compensator (17.7) can be designed as follows

$$\begin{cases} \chi^i(k+1) = (A+KC)\chi^i(k)-K\zeta^i(k)\\ v^i(k) \quad = F_\varepsilon\chi^i(k). \end{cases} \tag{17.19}$$

where K is such that $A+KC$ is Schur stable and

$$F_\varepsilon = -(B'P_\varepsilon B+I)^{-1}B'P_\varepsilon A. \tag{17.20}$$

In this case, $A_c = A + KC$, $B_c = -K$ and $C_c = F_\varepsilon$. We shall prove that with this set of data, the closed-loop of (17.14) and (17.13) is globally asymptotically stable for $|\lambda| < \delta$. The closed-loop system can be written as:

$$\begin{cases} x(k+1) = Ax(k) + (1-\lambda)BF_\varepsilon\chi(k) \\ \chi(k+1) = (A+KC)\chi(k) - KCx(k). \end{cases} \tag{17.21}$$

Lemma 17.8 Let $\delta \in (0,1)$ be given. There exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the system (17.21) is globally asymptotically stable for $|\lambda| < \delta$.

Proof : Define $e = x - \chi$. The (17.21) can be written in terms of x and e as follows:

$$\begin{cases} x(k+1) = [A + (1-\lambda)BF_\varepsilon]x(k) - (1-\lambda)BF_\varepsilon e(k) \\ e(k+1) = [A + KC - (1-\lambda)BF_\varepsilon]e(k) + (1-\lambda)BF_\varepsilon x(k) \end{cases}$$

Let Q be the positive definite solution of Lyapunov equation

$$(A+KC)'Q(A+KC) - Q + 4I = 0.$$

Since $F_\varepsilon \to 0$ as $\varepsilon \to 0$, there exists an ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$

$$(A + KC - (1-\lambda)BF_\varepsilon)'Q(A + KC - (1-\lambda)BF_\varepsilon) - Q + 3I \le 0.$$

Consider $V_1(k) = e(k)^*Qe(k)$. Let $\mu(k) = F_\varepsilon x(k)$. To ease our presentation, we shall omit the time label (k) whenever this causes no confusion.

$$\begin{aligned} &V_1(k+1) - V_1(k) \\ &\le -3\|e\|^2 + 2\,\mathrm{Re}\Big((1-\lambda)^*\mu^*B'Q[A + KC - (1-\lambda)BF_\varepsilon]e\Big) + |1-\lambda|^2\mu^*B'QB\mu \\ &\le -3\|e\|^2 + M_1\|\mu\|\|e\| + M_2\|\mu\|^2, \end{aligned} \tag{17.22}$$

where

$$M_1 = 4\|B'Q\|\|A+KC\| + 8\|B'Q\| \max_{\varepsilon\in[0,1]}\{\|BF_\varepsilon\|)\}, \quad M_2 = 4\|B'QB\|.$$

It should be noted that M_1 and M_2 are independent of ε and λ provided that $\|\lambda\| < \delta$.

Consider $V_2(k) = x(k)'P_\varepsilon x(k)$. We have that

$$\begin{aligned} V_2(k+1) - V_2(k) \le -\varepsilon\|x\|^2 - (1-\delta)^2\|\mu\|^2 + 2\,\mathrm{Re}\Big((1-\lambda)^*e^*F_\varepsilon'B'P_\varepsilon[A + (1-\lambda)BF_\varepsilon]x\Big) \\ + |1-\lambda|^2e^*F_\varepsilon'BP_\varepsilon BF_\varepsilon e. \end{aligned}$$

Note that

$$e^* F_\varepsilon' B' P_\varepsilon [A + (1-\lambda) B F_\varepsilon] x = e^* F_\varepsilon' B' P_\varepsilon A x + (1-\lambda) e^* F_\varepsilon' B' P_\varepsilon B \mu$$
$$= -e^* F_\varepsilon' (B' P_\varepsilon B + I)\mu + (1-\lambda) e^* F_\varepsilon' B' P_\varepsilon B \mu$$
$$= e^* [F_\varepsilon' + -\lambda F_\varepsilon' B' P_\varepsilon B]\mu,$$

and hence

$$V_2(k+1) - V_2(k) \leq -\varepsilon \|x\|^2 - (1-\delta)^2 \|\mu\|^2 + \theta_1(\varepsilon)\|e\|\|\mu\| + \theta_2(\varepsilon)\|e\|^2, \tag{17.23}$$

where

$$\theta_1(\varepsilon) = 4(\|F_\varepsilon'\| + 3\|F_\varepsilon' B' P_\varepsilon B\|), \quad \theta_2(\varepsilon) = 4\|F_\varepsilon' B P_\varepsilon B F_\varepsilon\|.$$

Consider a Lyapunov candidate $V(k) = V_1(k) + \kappa V_2(k)$ with

$$\kappa = \frac{M_2 + M_1^2}{1 - \delta^2}.$$

In view of (17.22) and (17.23), we get

$$V(k+1) - V(k) \leq -\varepsilon\kappa \|x\|^2 - M_1^2 \|\mu\|^2 - [3 - \kappa\theta_2(\varepsilon)]\|e\|^2 + [M_1 + \kappa\theta_1(\varepsilon)]\|\mu\|\|e\|.$$

There exists an $\varepsilon^* \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$,

$$3 - \kappa\theta_2(\varepsilon) \geq 2, \quad M_1 + \kappa\theta_1(\varepsilon) \leq 2M_1.$$

This yields that

$$V(k+1) - V(k) \leq -\varepsilon\kappa\|x\|^2 - \|e\|^2 - (\|e\| - M_1\|\mu\|)^2.$$

Therefore, for $\varepsilon \in (0, \varepsilon^*]$, the system (17.21) is globally asymptotically stable.

■

17.3.5. Application to output formation

The formation problem is closely related to synchronization. The design procedure in preceding subsections can be easily modified to solve the output formation problem.

Definition 17.3 An output formation is a family of vectors $\{h_1, \dots, h_N\}$, $h_i \in \mathbb{R}^p$. The network of the form (17.1) is said to achieve output formation if

$$\lim_{k\to\infty} \left[(y_i(k) - h_i) - (y_j(k) - h_j)\right] = 0.$$

Theorem 17.3 Consider a heterogeneous network of the form (17.1). For any $\delta \in (0, 1]$, a given set of communication topologies $\mathcal{G}_\delta$ and a formation vector $\{h_1, ..., h_N\}$, the output formation is always achievable via a local compensator in the form (17.2) in the network with any communication topology in $\mathcal{G}_\delta$.

The proof and controller design follows a similar procedure as in the output synchronization problem. First, we design a pre-compensator in the form of (17.3) for each agent to homogenize the network utilizing its local measurements so that the agents are asymptotically identical to a new model characterized by (A, B, C) for which the output formation is always achievable. The existence of such a triple (A, B, C) is shown in the next lemma.

Lemma 17.9 For a given family of vectors $\{h_1, ..., h_N\}$ and integer $n_q > 0$, $h_i \in \mathbb{R}^p$, there exists a triple (A, B, C) and another set of vectors $\{\bar{h}_1, ..., \bar{h}_N\}$ of appropriate dimensions such that

1. $\operatorname{rank}(C) = p$,
2. (A, B, C) is invertible, of uniform rank n_q and has no invariant zero,
3. $C\bar{h}_i = h_i$, $i = 1, ..., N$,
4. $A\bar{h}_i = \bar{h}_i$, $i = 1, ..., N$, i.e. A has some semi-simple eigenvalues at 1 with eigenvectors $\bar{h}_i$,
5. the other eigenvalues of A are at desired locations on the unit circle.

Proof : Choose $\bar{h}_i = h_i \otimes \mathbf{1}$ with $\mathbf{1} \in \mathbb{R}^{n_q}$ and let

$$A_0 = \operatorname{blkdiag}\{A_{ii}\}_{i=1}^p, \quad B = \operatorname{blkdiag}\{B_{ii}\}_{i=1}^p, \quad C = \operatorname{blkdiag}\{C_{ii}\}_{i=1}^p,$$

and

$$A_{ii} = \begin{bmatrix} 0 & I_{(n_q-1)} \\ 0 & 0 \end{bmatrix}, \quad B_{ii} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{ii} = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and $A = A_0 + BF$ for some matrix F of appropriate dimension. Obviously, Conditions 1,2 and 3 are satisfied. What remains is to choose an F such that conditions 4 and 5 can be satisfied.

Let $F = \text{blkdiag}\{F_{11}, \ldots, F_{pp}\}$ where F_{ii} is such that $A_i + B_i F_{ii}$ has all its eigenvalues on the unit circle and at least one eigenvalue at 1. Then we get

$$A = A_0 + B_0 F = \text{blkdiag}\{A_{ii} + B_{ii} F_{ii}\}_{i=1}^{p}. \tag{17.24}$$

and hence Condition 5 is satisfied.

It remains to show Condition 4. In view of the structure of $\bar{h}_i$ and A, we find that $A\bar{h}_i = \bar{h}_i$ if $\mathbf{1}$ is an eigenvector of $A_{ii} + B_{ii} F_{ii}$ associated with eigenvalue 1. Suppose $F_{ii} = [f_{n_q}, ..., f_1]$. We observe that

$$(A_{ii} + B_{ii} F_{ii})\mathbf{1} = \mathbf{1} \Leftrightarrow \sum_{i=1}^{n_q} f_i = 1.$$

On the other hand, the characteristic polynomial of $A_{ii} + B_{ii} F_{ii}$ is given by

$$C(\lambda) = \det(\lambda I - A_{ii} - B_{ii} F_{ii}) = \lambda^{n_q} - f_1 \lambda^{n_q - 1} - \cdots - f_{n_q - 1}\lambda - f_{n_q}$$

Since $A_{ii} + B_{ii} F_{ii}$ has at least one eigenvalue at 1, we get $C(1) = 1 - \sum_{i=1}^{n_q} f_i = 0$ and $\sum_{i=1}^{n_q} f_i = 1$. Therefore, $\mathbf{1}$ is an eigenvector of $A_{ii} + B_{ii} F_{ii}$ for $i = 1, ..., p$ and Condition 4 is satisfied. ■

Based on Lemma 17.9 and its proof, we can place p semi-simple eigenvalues of A at 1 with eigenvectors $\{\bar{h}_1, ..., \bar{h}_N\}$ (at most p of them can be linearly independent). We have have a complete freedom to choose the locations of the other eigenvalues and a relative freedom to assign their multiplicity structures. Similarly as in preceding subsections, we can put only semi-simple eigenvalues on the unit circle to ensure bounded synchronization trajectories or allow degenerate eigenvalues to have unbounded synchronization trajectories. Next, depending on the type of synchronization trajectories and the resulting choice of A, a local formation controller can be constructed for the new model as follows:

$$\begin{cases} \chi^i(k+1) = (A + KC)\chi^i(k) - K\left[\sum_{j=1}^{N} d_{ij}[(y^i(k) - h_i) - (y^j(k) - h_j)]\right] \\ v^i(k) \quad = C_c \chi^i(k), \end{cases} \tag{17.25}$$

where

$$C_c = \begin{cases} -\varepsilon B' A, & A \text{ only has semi-simple eigenvalues on the unit circle;} \\ -(B' P_\varepsilon B + I)^{-1} B' P_\varepsilon A, & A \text{ has degenerate eigenvalues on the unit circle,} \end{cases}$$

where P_ε is the positive definite solution of (17.18).

Proof of Theorem 17.3 : For any triple (A, B, C) satisfying the conditions in Lemma 17.9, there exists a shaping pre-compensator in the form of (17.3) such that the interconnection of the agents and shaping pre-compensator can be written in the following form:

$$\begin{cases} \bar{x}^i(k+1) = A\bar{x}^i(k) + B\left(v^i(k) + d^i(k)\right) \\ \quad y^i(k) \quad = C\bar{x}^i(k). \end{cases} \tag{17.26}$$

Let $\bar{x}_s^i = \bar{x}^i - \bar{h}_i$. In view of Condition 3 and 4 in Lemma 17.9, the closed-loop system of (17.26) and controller (17.25) can be written in terms of $\bar{x}_s^i$ and χ^i as

$$\begin{cases} \bar{x}_s^i(k+1) = A\bar{x}_s^i(k) + BC_c\chi^i(k) + Bd^i(k), \\ \quad \chi^i(k) \quad = A_c\chi^i(k) + B_cC\left[\sum_{j=1}^N d_{ij}(\bar{x}_s^i(k) - \bar{x}_s^j(k))\right] \end{cases}$$

We have already proved that the above network synchronizes. Hence

$$\lim_{k\to\infty}\left[C\bar{x}_s^i(k) - C\bar{x}_s^j(k)\right] = \lim_{k\to\infty}\left[(y_i(k) - h_i) - (y_j(k) - h_j)\right] = 0.$$

■

Remark 17.7 *We would like to emphasize that thanks to the freedom we have in choosing appropriate (A, B, C), no restriction on formation vector needs to be imposed.*

17.4. Output regulation

Despite a freedom in choosing the mode or frequencies of the synchronization trajectories, we can not plan the trajectories arbitrarily because they are partially determined by the weighted average of initial conditions. On the other hand, it is important in some scenario to regulate the output of the agents to desired trajectories when the output synchronization is reached. Suppose the objective trajectories are generated by an exo-system

$$\begin{cases} x^0(k+1) = A^0x^0(k), \quad x^0(0) = x_r, \\ \quad y^0(k) \quad = C^0x^0(k), \end{cases} \tag{17.27}$$

where A^0 has all its eigenvalues in the closed unit disc and (A^0, C^0) is observable. It is reasonable to assume that the synchronization trajectories are not geometrically increasing.

We want to regulate each agent's output to y^0. Instead of disseminating the information of exo-system to every agent, we assume that only the *root*, which is agent 1, receives such information. In this case, the root measures its output relative to y^0 besides what it originally receives from the network. To be precise, agent 1 takes the following form:

$$\begin{cases} x^1(k+1) = A^1 x^1(k) + B^1 u^1(k), \\ z^1(k) \quad = C_z^1 x^1(k), \\ y^1(k) \quad = C_y^1 x^1(k), \\ \zeta^1(k) \quad = \sum_{j=1}^N d_{1j}(y^1(k) - y^j(k)) + \delta(y^1(k) - y^0(k)), \end{cases} \tag{17.28}$$

where $\delta = \frac{d_{11}}{2} > 0$.

Definition 17.4 *The agents in the network achieve output regulation if*

$$\lim_{k\to\infty}(y^i(k) - y^0(k)) = 0, \quad \forall i \in \{1, \dots, N\}.$$

We can formulate the regulation problem as follows:

Problem 17.2 Consider a heterogeneous network of the form (17.1). For a given exo-system (17.27), a set $\mathcal{G}_\delta$, the output regulation problem with exo-system (17.27) and a set of communication topologies $\mathcal{G}_\delta$ is to design a local linear dynamical controller (17.2) such that the output regulation can be achieved in the network with any communication topology belonging to $\mathcal{G}_\delta$.

Before we present the result for output regulation problem, some preparatory work needs to be done. First, we augment the network by including the exo-system as agent 0. In this augmented network, the agent 0 does not have any network measurement. We can write network measurement of all the agents uniformly as

$$\tilde{\zeta}^i(k) = \sum_{j=0}^N \tilde{d}_{ij}(y^i(k) - y^j(k)), i = 0, ..., N,$$

where

$$\bar{D} = \{\bar{d}_{ij}\} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \frac{d_{11}}{2} & \frac{d_{11}}{2} & d_{12} & \cdots & d_{1N} \\ 0 & d_{21} & d_{22} & \cdots & d_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & d_{N1} & d_{N2} & \cdots & d_{NN} \end{bmatrix}. \tag{17.29}$$

This $\bar{D}$ is also an row stochastic matrix and defines an augmented topology $\bar{G}$. Moreover, since agent 0 (exo-system) is connected to the root of the original network via an out-coming arc $(0, 1)$, this $\bar{G}$ also has a directed spanning tree with a new root at agent 0 (exo-system). Suppose eigenvalues of $\bar{D}$ are denoted by $\bar{\lambda}_i,\ i = 0, ..., N$ with $\bar{\lambda}_0 = 1$ and $\bar{\lambda}_i,\ i = 1, ..., N$ are in open unit disc. For a given set $\mathcal{G}_\delta$, the set of augmented topologies by including agent 0 and the arc $(0, 1)$ can be denoted by $\bar{\mathcal{G}}_{\bar{\delta}}$ such that for any $\bar{G} \in \bar{\mathcal{G}}_{\bar{\delta}}$,

$$|\bar{\lambda}_i| < \bar{\delta},\ i = 1, ..., N.$$

We have the following theorem:

Theorem 17.4 Consider a heterogeneous network of the form (17.1) and an exo-system (17.27). For a given set $\mathcal{G}_\delta$, Problem 17.2 is solvable via a decentralized dynamic consensus controller (17.2).

Proof : For a given exo-system (17.27), it is shown in Appendix 17.6.2 that there exists another system and initial condition:

$$\begin{cases} \tilde{x}^0(k+1) = \tilde{A}^0 \tilde{x}^0(k), \quad \tilde{x}^0(0) = \tilde{x}^0_0 \\ \quad y^0(k) \quad = \tilde{C}^0 \tilde{x}^0(k) \end{cases} \tag{17.30}$$

that produces the same output as the original exo-system (17.27). Moreover, we can find a $\tilde{B}^0$ matrix such that $(\tilde{A}^0, \tilde{B}^0, \tilde{C}^0)$ is invertible, of uniform rank n_q and has no invariant zero where n_q is an integer greater than the maximal order of infinite zeros of all the agent and the observability index of (C^0, A^0). We can view this system as the exo-system.

It is shown in Lemma 17.1 that a pre-compensator of the form (17.3) can be designed for agent $1, ..., N$ such that their internal model can be shaped as asymptotically identical to the exo-system (17.30).

Then, depending on the eigenvalues of $\tilde{A}^0$, a synchronization controller (17.15) or (17.19) can be designed for all the agents in the homogenized augmented network with communication topologies belonging to $\bar{\mathcal{G}}_{\bar{\delta}}$. If such a controller were applied to all the agents, output synchronization could be achieved. However, the exo-system does not have any input. In fact, this is not a problem. We can assume the controller that should be applied to exo-system has zero initial condition and the network should still synchronize regardless. Moreover, it should be noted that the exo-system is not associated with any network measurement either. Consequently, the controller would produce zero input if it were applied to the exo-system. This implies that we actually need not apply controller to the exo-system but only to the agents $1, ..., N$ in order for the network to synchronize. Since all the agent output y^i, $i = 1, ..., N$ will synchronize with the output of exo-system, the output regulation is achieved. ■

17.5. Conclusion

In this paper, a decentralized control scheme is developed to solve the output synchronization and output regulation problems in a heterogeneous network of discrete-time introspective right-invertible agents. The essence of the proposed design is two-folds: first, by exploiting the introspection and right-invertibility property of each agent, we design a local shaping pre-compensator to manipulate the agent's internal dynamics as being asymptotically identical to a new model in which we enjoy a substantial freedom in assigning its eigenstructures. Then, different synchronization controllers depending on the two types of synchronization trajectories can be constructed on top of the new model so that the output synchronization can be achieved for a set of communication topologies.

17.6. Appendix

17.6.1. Shaping pre-compensator design

In the appendix, we develop the proof of Lemma 17.1 and present a detailed procedure for the design of the shaping pre-compensator. In the venture to achieve this, a Special Coordinate Basis (SCB) of linear system developed in [99] plays a fundamental role. We shall first review this canonical decomposition form and based on that develop some technical results that are instrumental to our proof.

Review of Special Coordinate Basis (SCB)

Consider a discrete-time strictly proper system

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ \quad y(k) \quad = Cx(k). \end{cases} \tag{17.31}$$

There exist state, input and output transformation

$$\Gamma_s x = \tilde{x} = \begin{bmatrix} x_a \\ x_b \\ x_c \\ x_d \end{bmatrix}, \quad \Gamma_o y = \tilde{y} = \begin{bmatrix} y_b \\ y_d \end{bmatrix}, \quad \Gamma_i u = \tilde{u} = \begin{bmatrix} u_c \\ u_d \end{bmatrix} \tag{17.32}$$

such that in the new coordinate, (17.31) can be rewritten as

$$\begin{aligned} & x_a(k+1) = A_{aa}x_a(k) + L_{ab}y_b(k) + L_{ad}y_d(k) \\ & x_b(k+1) = A_{bb}x_b(k) + L_{bd}y_d(k) \\ & x_c(k+1) = A_{cc}x_c(k) + B_c u_c(k) + E_{ca}x_a(k) + L_{cb}y_b(k) + L_{cd}y_d(k) \\ & x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y_d(k) + B_{d,j}[u_{d,j}(k) + G_j\tilde{x}(k)], \\ & y_{d,j}(k) = C_{d,j}x_{d,j}(k), \quad j = 1, ..., r, \\ & y_b = C_b x_b \end{aligned} \tag{17.33}$$

and

$$x_d = \begin{bmatrix} x_{d,1} \\ \vdots \\ x^i_{d,r} \end{bmatrix}, \quad u_d = \begin{bmatrix} u_{d,1} \\ \vdots \\ u^i_{d,r} \end{bmatrix}, y_d = \begin{bmatrix} y_{d,1} \\ \vdots \\ y^i_{d,r} \end{bmatrix}$$

where the dimension of x_a, x_b, x_c, x_d, $x_{d,j}$, y_b, y_d, u_c and u_d have dimensions n_a, n_b. n_c, n_d, $n_{d,j}$, $p-r$, r, $m-r$ and r ($n_d = \sum_{j=1}^{r} n_{d,j}$) respectively and

$$A_{d,j} = \begin{bmatrix} 0 & I_{n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_{d,j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{d,j} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Some important properties of SCB are summarized as follows:

1. The invariant zeros of system (17.31) are given by the eigenvalues of A_{aa}.
2. x_b is nonexistent and $\Gamma_o = I$ if (17.31) is right-invertible.
3. x_c is nonexistent and $\Gamma_i = I$ if (17.31) is left-invertible.
4. (A_{cc}, B_c) is controllable and (A_{bb}, C_b) is observable.
5. system (17.31) has r zeros at infinity with order $n_{d,j}$, $j = 1, ..., r$.

Technical Lemmas

Lemma 17.10 (Squaring down of right-invertible systems) Suppose system (17.31) is right-invertible. There exists a pre-compensator

$$\begin{cases} \chi_1(k+1) = A_1\chi_1(k) + B_1u_1(k) \\ \quad u(k) \quad = C_1\chi_1(k) + D_1u_1(k) \end{cases} \tag{17.34}$$

such that the interconnection of (17.31) and (17.34) is invertible.

Proof : Since (17.31) is right invertible, there exist state and input transformations Γ_s and Γ_i such that (17.31) can be transformed into its SCB form (17.33) while x_b is nonexistent. To make this explicit, we write its SCB form as follows

$$\begin{aligned} x_a(k+1) &= A_{aa}x_a(k) + L_a y(k) \\ x_c(k+1) &= A_{cc}x_c(k) + B_c u_c(k) + E_a x_a(k) + L_c y(k) \\ x_{d,j}(k+1) &= A_{d,j}x_{d,j}(k) + L_{d,j}y(k) \\ &\quad + B_{d,j}[u_{d,j}(k) + G_{a,j}x_a(k) + G_{c,j}x_c(k) + G_{d,j}x_d(k)], \\ y_j(k) &= C_{d,j}x_{d,j}(k), \quad j = 1, ..., p, \end{aligned} \tag{17.35}$$

Let $G_c = [G_{c,1}; \cdots ; G_{c,p}]$. Since (A, C) is observable, we find that (A_{cc}, G_c) is observable. Let F and K be such that $A_{cc} + B_c F$ and $A_{cc} + KG_c$ are Schur stable. A pre-compensator can be constructed as

$$\begin{cases} \chi_1(k+1) = (A_{cc} + B_c F)\chi_1(k) + K\nu(k) \\ \begin{bmatrix} u_c(k) \\ u_d(k) \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}\nu(k) + \begin{bmatrix} F \\ -G_c \end{bmatrix}\chi_1(k) \end{cases} \tag{17.36}$$

We proceed to show that the interconnection (17.36) and (17.35) is invertible. Suppose $\nu = [\nu_1; \cdots ; \nu_p]$ and define $e_1 = x_c - \chi_1$. Then

$$x_a(k+1) = A_{aa}x_a(k) + L_a y(k) \tag{17.37}$$

$$x_c(k+1) = (A_{cc} + B_c F)x_c(k) - B_c F e_1(k) + E_a x_a(k) + L_c y(k) \tag{17.38}$$

$$e_1(k+1) = A_{cc}e_1(k) + E_a x_a(k) + L_c y(k) - K\nu(k) \tag{17.39}$$

and

$$x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y(k) + B_{d,j}[\nu_j^i(k) + G_{a,j}x_a(k) + G_{c,j}e_1(k) + G_{d,j}x_d(k)] \tag{17.40}$$

while

$$y_j(k) = C_{d,j}x_{d,j}(k). \tag{17.41}$$

Note that (17.37), (17.38), (17.40) and (17.41) are already in SCB form, but (17.39) is not. We need to eliminate ν from (17.39). Define $\tilde{x}_d = [x_{d,1,n_{d,1}}; \cdots ; x_{d,p,n_{d,p}}]$. We get

$$\tilde{x}_d(k+1) = \nu(k) + L_1 y(k) + G_a x_a^i(k) + G_c e_1^i(k) + G_d x_d(k),$$

where $G_a = [G_{a,1}; \cdots ; G_{a,p}]$, $G_d = [G_{d,1}; \cdots ; G_{d,p}]$ and L_1 is of appropriate dimension. Let $e_2 = e_1 + K\tilde{x}_d$ whose dynamics are given by

$$e_2(k+1) = (A_{cc} + KG_c)e_2(k) + L_2 y(k) + KG_a x_a(k) + \tilde{G}_d x_d(k),$$

where $\tilde{G}$ is an appropriate matrix. Define $\bar{x}_a = [x_c; e_2; x_a]$. We get

$$\bar{x}_a(k+1) = \bar{A}_{aa}\bar{x}_a(k) + \bar{L}_a y(k) + \bar{G}_d x_d(k) \tag{17.42}$$

where

$$\bar{A}_{aa} = \begin{bmatrix} A_{cc} + B_c F & -B_c F & E_a \\ 0 & A_{cc} + KG_c & KG_a \\ 0 & 0 & A_{aa} \end{bmatrix}, \quad \bar{L}_a = \begin{bmatrix} L_c \\ L_c \\ L_2 \end{bmatrix}$$

and $\bar{G}_d$ is an appropriate matrices. Finally, we need to eliminate x_d from (17.42). According to [99], there exists a matrix M_d such that

$$\tilde{x}_a = \bar{x}_a + M_d x_d$$

and x_d satisfy

$$\begin{aligned} &\tilde{x}_a(k+1) = \bar{A}_{aa}\tilde{x}_a(k) + \bar{L}_a y(k) \\ &x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y(k) + B_{d,j}[\nu_j(k) + \tilde{G}_a\tilde{x}_a(k) + G_{d,j}x_d(k)] \\ &y_j(k) = C_{d,j}x_{d,j}(k), \end{aligned} \tag{17.43}$$

with appropriate $\tilde{G}_a$. Clearly, (17.43) is SCB form and is square invertible. Note that in the original coordinate, the pre-compensator takes the form

$$\begin{cases} \chi_1(k+1) = (A_{cc} + B_c F)\chi_1(k) + K\nu(k) \\ \quad u(k) \quad = \Gamma_i^{-1}\begin{bmatrix} 0 \\ I \end{bmatrix}\nu(k) + \Gamma_i^{-1}\begin{bmatrix} F \\ -G_c \end{bmatrix}\chi_1(k) \end{cases} \tag{17.44}$$

■

Lemma 17.11 (Rank equalizing of invertible system) Suppose system (17.31) is invertible. There exists a pre-compensator

$$\begin{cases} \chi_2(k+1) = A_2\chi_2(k) + B_2 v(k) \\ \quad w(k) \quad = C_2\chi_2(k) \end{cases} \tag{17.45}$$

such that the interconnection of (17.45) and (17.31) is of uniform rank.

Proof : Since (17.31) is invertible, with only a nonsingular state transformation Γ_s, we can put it in the following form:

$$\begin{aligned} &x_a(k+1) = A_{aa}x_a(k) + L_a y(k) \\ &x_{d,j}(k+1) = A_{d,j}x_{d,j}(k) + L_{d,j}y(k) + B_{d,j}[u_j(k) + G_{a,j}x_a(k) + G_{d,j}x_d(k)] \\ &y_j(k) = C_{d,j}x_{d,j}(k), \quad j = 1, ..., p \end{aligned} \tag{17.46}$$

We can add more delays to u_j so that all the infinite zeros have the same order. Let $r > \max_j n_{d,j}$. For $x_{d,j}$ with $n_{d,j} < r$, a pre-compensator can be constructed as

$$\begin{cases} \chi_{2,j}(k+1) = A_{2,j}\chi_{2,j}(k) + B_{2,j}v_j(k) \\ \quad u_j(k) \quad\;\; = C_{2,j}\chi_{2,j}, \end{cases}$$

where

$$A_{2,j} = \begin{bmatrix} 0 & I_{r-n_{d,j}-1} \\ 0 & 0 \end{bmatrix}, \quad B_{2,j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_{2,j} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

By adding these pre-compensators to u_j, all the infinite zeros will have the same order of r. We can write them together as

$$\chi_2(k+1) = A_2\chi_2(k) + B_2 v(k),$$

where $\chi_2 = [\chi_{2,1}; \cdots ; \chi_{2,p}]$, $v = [v_1; \cdots ; v_p]$, $u = [u_1; \cdots ; u_j]$ and

$$A_2 = \text{blkdiag}\{A_{2,j}\}, \quad B_2 = \text{blkdiag}\{B_{2,j}\}, \quad C_2 = \Gamma_i \,\text{blkdiag}\{C_{2,j}\}.$$

■

Proof of Lemma 1

Proof : Given Assumption 17.1, there exist non-singular state transformation Γ_s^i and input transformation Γ_c^i such that (A^i, B^i, C_y^i) can be transformed into SCB. Without loss of generality, we assume in the first place that the agent i is of the following form

The design of compensator (17.3) can be accomplished in three steps.

Step 1: Squaring down. It follows from Lemma 17.10 that for each agent i, there exists a pre-compensator

$$\begin{cases} \chi_1^i(k+1) = A_1^i\chi_1^i(k) + B_1^i u_1^i(k) \\ \quad u^i(k) \quad\;\; = C_1^i\chi_1^i(k) + D_1^i u_1^i(k) \end{cases} \tag{17.47}$$

such that the interconnection of (17.1) and (17.47) is invertible.

Step 2: Rank equalizing. Lemma 17.11 shows that a rank-equalizing compensator can be constructed as

$$\begin{cases} \chi_2^i(k+1) = A_2^i \chi_2^i(k) + B_2^i u_2^i(k) \\ u_1^i(k) \quad = C_2^i \chi_2^i(k) \end{cases} \tag{17.48}$$

such that the interconnection of (17.1), (17.47) and (17.48) is invertible and of uniform rank $n_q > n_d$.

Step 3: Zero decoupling and Pole placement. Using a non-singular state transformation

$$\Gamma_s^i \begin{bmatrix} x^i \\ \chi_1^i \\ \chi_2^i \end{bmatrix} = \tilde{\chi}^i = \begin{bmatrix} \tilde{\chi}_0^i \\ \tilde{\chi}_d^i \end{bmatrix},$$

the interconnection of (17.1), (17.47) and (17.48) can be written in the following form:

$$\begin{cases} \tilde{\chi}_0^i(k+1) = \tilde{A}_0^i \tilde{\chi}_0^i(k) + \tilde{L}_0^i y^i(k) \\ \tilde{\chi}_d^i(k+1) = \tilde{A}_d^i \tilde{\chi}_d^i(k) + \tilde{B}_d^i [u_2^i(k) + D_0^i \tilde{\chi}_0^i(k) + D_d^i \tilde{\chi}_d^i(k)] \\ y^i(k) \quad = \tilde{C}_d^i \tilde{\chi}_d^i(k), \end{cases}$$

where

$$\tilde{A}_d^i = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_d^i = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad \tilde{C}_d^i = \begin{bmatrix} I_p & 0 \end{bmatrix}.$$

Note that each agent also has a local measurement $\tilde{z}^i$ which consists of the original z^i and compensator states χ_1^i and χ_2^i. This $\tilde{z}^i$ can be written in terms of $\tilde{\chi}_0$ and $\tilde{\chi}_d$ as

$$\tilde{z}^i(k) = \tilde{C}_1^i \tilde{\chi}_0^i(k) + \tilde{C}_2^i \tilde{\chi}_d^i(k).$$

Let

$$\tilde{A}^i = \begin{bmatrix} \tilde{A}_0^i & \tilde{C}^i \tilde{L}_0^i \\ \tilde{B}_d^i D_0^i & \tilde{A}_d^i + \tilde{B}_d^i D_d^i \end{bmatrix}, \quad \tilde{B}^i = \begin{bmatrix} 0 \\ \tilde{B}_d^i \end{bmatrix}, \quad \tilde{C}^i = \begin{bmatrix} \tilde{C}_1^i & \tilde{C}_2^i \end{bmatrix}.$$

Clearly, $(\tilde{A}^i, \tilde{C}^i)$ is detectable. Then an observer based pre-feedback is designed as follows

$$\begin{cases} \hat{\chi}^i(k+1) = \tilde{A}^i \hat{\chi}^i(k) + \tilde{B}^i v^i(k) - \tilde{K}^i(\tilde{z}^i(k) - \tilde{C}^i \hat{\chi}^i(k)) \\ u_2^i(k) \quad = -\tilde{D}^i \hat{\chi}^i(k) + \tilde{F}^i \hat{\chi}^i(k) + v^i(k), \end{cases} \tag{17.49}$$

where $\tilde{A}^i + \tilde{K}^i \tilde{C}^i$ is Schur stable, $\tilde{D}^i = [D_0^i \quad D_d^i]$, $\tilde{F}^i = [0 \quad \tilde{F}_d^i]$ and $\tilde{A}_d^i + \tilde{B}_d^i \tilde{F}_d^i$ has a set of pre-specified eigenvalues. It is easy to see that the error dynamics $e^i = \tilde{\chi}^i - \hat{\chi}^i$ is asymptotically stable. Therefore, the coupling between $\tilde{\chi}_0$ and $\tilde{\chi}_d$ is canceled asymptotically. The mapping from v^i to y^i is described by the following dynamics:

$$\begin{cases} \tilde{\chi}_d^i(k+1) = (\tilde{A}_d^i + \tilde{B}_d^i \tilde{F}_d^i)\tilde{\chi}_d^i(k) + \tilde{B}_d^i v^i(k) + \tilde{B}_d^i d^i(k) \\ y^i(k) \quad = \tilde{C}_d^i \tilde{\chi}(k) \end{cases}$$

and

$$\begin{cases} e^i(k+1) = (\tilde{A}^i + \tilde{K}^i\tilde{C}^i)e^i(k) \\ d^i(k) \quad = (\tilde{D}^i - \tilde{F}^i)e^i(k). \end{cases}$$

Note that $(\tilde{A}^i_d + \tilde{B}^i\tilde{F}^i, \tilde{B}^i_d, \tilde{C}^i_d)$ is invertible, of uniform rank n_q and has no invariant zeros. Moreover, $\tilde{A}^i_d + \tilde{B}^i_d\tilde{F}^i_d$ has pre-selected eigenvalues.

In the original coordinate, (17.49) can be written as

$$\begin{cases} \hat{\chi}^i(k+1) = \tilde{A}^i\hat{\chi}^i(k) + \tilde{B}^i v^i(k) - \tilde{K}^i(\tilde{z}^i(k) - \tilde{C}^i\hat{\chi}^(k)) \\ u^i_2(k) \quad = -\tilde{D}^i\Gamma_s\hat{\chi}^i(k) + \tilde{F}^i\Gamma_s\hat{\chi}^i(k) + v^i(k), \end{cases} \tag{17.50}$$

■

17.6.2. Manipulation of exo-system

Consider an arbitrary exo-system

$$\begin{cases} x(k+1) = Ax(k), \quad x(0) = x_0, \\ y(k) \quad = Cx(k), \end{cases} \tag{17.51}$$

There exists a non-singular state transformation $x = T_s\tilde{x}$ and $y = T_o\tilde{y}$ where

$$\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_p \end{bmatrix}, \quad \tilde{x}_i = \begin{bmatrix} \tilde{x}_{i,1} \\ \vdots \\ \tilde{x}_{i,n_i} \end{bmatrix}, \quad \tilde{y} = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_p \end{bmatrix},$$

we have

$$\tilde{x}_i(k+1) = A_i\tilde{x}_i(k) + L_i\tilde{y}, \quad \tilde{y}_i(k) = \begin{bmatrix} 1 & 0 \end{bmatrix}\tilde{x}_i(k), \tag{17.52}$$

where

$$A_i = \begin{bmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{bmatrix}$$

and L_i is of appropriate dimension. The set of integers $\{n_1, ..., n_p\}$ is the observability index of (C, A) (see [11]).

Note that we can equalize the size of A_i to n_q by adding shift registers to the bottom of each chain $\tilde{x}_i$ with zero initial conditions. Define

$$\bar{x} = \begin{bmatrix} \bar{x}_1 \\ \vdots \\ \bar{x}_p \end{bmatrix}, \quad \bar{x}_i = \begin{bmatrix} \tilde{x}_i \\ s_i \end{bmatrix} \in \mathbb{R}^{n_q}, \quad s_i(0) = 0,$$

and

$$\bar{x}_i(k+1) = \bar{A}_i \bar{x}_i(k) + \bar{L}_i \tilde{y}, \quad \tilde{y}_i(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}_i(k), \tag{17.53}$$

with

$$\bar{A}_i = \begin{bmatrix} 0 & I_{n_q-1} \\ 0 & 0 \end{bmatrix}, \quad \bar{L}_i = \begin{bmatrix} L_i \\ 0 \end{bmatrix}.$$

By adding s_i, we introduce several zero eigenvalues to the system. It is easy to see that (17.52) and (17.53) generate exactly the same output $\tilde{y}$. We can write system (17.53) in a compact form as

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) \\ \tilde{y}(k) \quad = \bar{C}\bar{x}(k), \end{cases} \quad \bar{x}(0) = \begin{bmatrix} \bar{x}_1(0) \\ \vdots \\ \bar{x}_p(0) \end{bmatrix} \tag{17.54}$$

where

$$\bar{A} = \begin{bmatrix} \star & I_{n_q-1} & \cdots & \star & 0 \\ \star & 0 & \cdots & \star & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \star & 0 & \cdots & \star & I_{n_q-1} \\ \star & 0 & \cdots & \star & 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Choose

$$\bar{B} = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 1 \end{bmatrix}.$$

Then $(\bar{A}, \bar{B}, \bar{C})$ is invertible, of uniform rank n_q and has no invariant zero.

Finally we restore the output transformation in system (17.54) as

$$\begin{cases} \bar{x}(k+1) = \bar{A}\bar{x}(k) \\ y(k) \quad = T_o\bar{C}\bar{x}(k), \end{cases} \quad \bar{x}(0) = \begin{bmatrix} \bar{x}_1(0) \\ \vdots \\ \bar{x}_p(0) \end{bmatrix} \tag{17.55}$$

This system produces the same output as (17.52). Since non-singular output transformation does not change invertibility and zero structure. Therefore, the triple $(\bar{A}, \bar{B}, T_o\bar{C})$ is still invertible, of uniform rank n_q and has no invariant zero. According to the property of SCB, there exists a state transformation that put it in the form of (17.6).

CHAPTER 18

Synchronization in a homogenous network with uniform constant communication delay

18.1. Introduction

In this chapter, we study the multi-agent consensus problem with uniform constant communication delay. The agents are assumed to be at most critically unstable, i.e. each agent has all its eigenvalues in the closed left half plane. The contribution of this chapter with respect to [76, 7, 46, 9] is twofold: first, we find a sufficient condition on the tolerable communication delay for agents with high-order dynamics, which has an explicit dependence on the agent dynamics and network topology. For undirected network, this upper bound can be independent of network topology provided that the network is connected. Moreover, in a special case where the agents only have zero eigenvalues, such as single- and double-integrator dynamics, arbitrarily large but bounded delay can be tolerated. Another layer of contribution is that for delay satisfying the proposed upper bound, we present a controller design methodology without exact knowledge of network topology so that the multi-agent consensus in a set of unknown networks can be achieved. When the network topology is precisely known, the controller design can be modified to be topology-dependent and a larger delay tolerance is attainable.

The rest of this chapter is organized as follows: System and network configuration and consensus problem formulations are given in Section 18.2. The consensus problem with full-state coupling is solved in Section 18.3. The corresponding problem with partial-state coupling is dealt with in Section 18.4. In Section 18.5, we discuss the special of neutrally stable systems. Some illustrative examples are given in Section 18.6. Some technical lemmas are appended at the end of this chapter.

Finally, it should be emphasized that the results in this chapter are obtained using the low-gain techniques developed in Chapter 7. Therefore, an overlap in some proofs of these two chapters is not surprising. However, some technical aspects are distinguished due to the different nature of the problems studied in the two chapters. For the sake of completeness and clarity, all the proofs are presented independently from those in Chapter 7.

18.2. Problem formulation

Consider a network of N identical agents

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) + Bu^i(t), \quad i = 1, \dots, N, \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} x^j(t-\tau). \end{cases} \tag{18.1}$$

where $x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ and $z^i \in \mathbb{R}^n$, $\tau > 0$ is an unknown constant satisfying $\tau \in [0, \bar{\tau}]$. The coefficients ℓ_{ij} are such that $\ell_{ij} \leq 0$ for $i \neq j$ and $\ell_{ii} = -\sum_{j \neq i}^{N} \ell_{ij}$. In (18.1), each agent collects a delayed measurement z^i of the state of neighboring agents through the network, which we refer to as *full-state coupling*.

It is also common that z_i may consist of the output of neighboring agents instead of the complete state which can be formulated as follows:

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) + Bu^i(t), \\ y^i(t) = Cx^i(t), \qquad i = 1, \dots, N, \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} y^j(t-\tau), \end{cases} \tag{18.2}$$

where $x^i \in \mathbb{R}^n$, $u^i \in \mathbb{R}^m$ and $y^i, z^i \in \mathbb{R}^p$. We refer to the agents in this case as having *partial-state coupling*.

The matrix $L = \{\ell_{ij}\} \in \mathbb{R}^{N \times N}$ defines the *communication topology* which can be captured by a weighted graph $G = (\mathcal{N}, \mathcal{E})$ where $(j, i) \in \mathcal{E} \Leftrightarrow \ell_{ij} < 0$ and $a_{ii} = 0$ and $a_{ij} = -\ell_{ij}$ for $i \neq j$. The G is directed in general. However, in a special case where L is symmetric, G is undirected. This L is the the Laplacian matrix associated with G.

Assumption 18.1 The following assumptions are made throughout the chapter:

1. The agents are at most critically unstable, that is, A has all its eigenvalues in the closed left half plane;
2. (A, B) is stabilizable and (A, C) is detectable;
3. The communication topology described by the graph G contains a directed spanning tree.

Remark 18.1 *It should be noted that under Assumption 18.1, L has one simple eigenvalue in zero and the others lie in the open left half plane.*

It should be noted that in practice, perfect information of the communication topology is usually not available for controller design and that only some rough characterization of the network can be obtained. Using the non-zero eigenvalues of L as a "measure" for the graph, we can introduce the following definition to characterize a set of unknown communication topologies. Let $\lambda_1, \cdots, \lambda_N$ denoted the eigenvalues of L and assume $\lambda_1 = 0$.

Definition 18.1 For any $\gamma \geq \beta \geq 0$ and $\frac{\pi}{2} > \varphi \geq 0$, $\mathcal{G}_{\beta,\gamma,\varphi}$ is the set of graphs satisfying Assumption 18.1 and whose associated Laplacian satisfies

$$|\lambda_i| \in (\beta, \gamma) \text{ and } \arg \lambda_i \in [-\varphi, \varphi]$$

for $i = 2, \ldots, N$.

Definition 18.2 The agents in the network achieve consensus if

$$\lim_{t \to \infty} (x^i(t) - x^j(t)) = 0, \quad \forall i, j \in \{1, \ldots, N\}.$$

Two consensus problems for agents with full-state coupling (18.1) and partial-state coupling (18.2) respectively can be formulated for this set of networks as follows:

Problem 18.1 Consider a network of agents (18.1) with full state coupling. The consensus problem given a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$ is to design linear static controllers $u^i = F z^i$ for $i = 1, \ldots, N$ such that the agents (18.1) with $u^i = F z^i$ achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.

Problem 18.2 Consider a network of agents (18.2) with partial state coupling. The consensus problem with a set of possible communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ and a delay upper bound $\bar{\tau}$ is to design linear dynamic control protocols of the form:

$$\begin{cases} \dot{\chi}^i = A_c \chi^i + B_c z^i \\ u^i = C_c \chi^i, \end{cases} \tag{18.3}$$

for $i = 1, \ldots, N$ such that the agents (18.2) with controller (18.3) achieve consensus with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.

18.3. Consensus with full-state coupling

In this section, we consider agents with full-state coupling as given in (18.1) and solve Problems 18.1.

For a given set of networks $\mathscr{G}_{\beta,\gamma,\varphi}$, we design a decentralized local consensus controller for any network in $\mathscr{G}_{\beta,\gamma,\varphi}$ as follows:

$$u^i = \alpha F_\varepsilon z^i, \tag{18.4}$$

where $F_\varepsilon = B' P_\varepsilon$. Here P_ε is the positive definite solution of the Algebraic Riccati Equation (ARE)

$$A' P_\varepsilon + P_\varepsilon A - P_\varepsilon B B' P_\varepsilon + \varepsilon I = 0. \tag{18.5}$$

and ε, as well as α, are design parameters which will be chosen according to β, γ and φ so that the multi-agent consensus can be achieved with any communication topology belonging to $\mathscr{G}_{\beta,\gamma,\varphi}$. Let

$$\omega_{\max} = \begin{cases} 0, & A \text{ is Hurwitz.} \\ \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A) = 0\}, & \text{otherwise.} \end{cases}$$

The first main result of this chapter is stated in the next theorem which solves the network consensus problem with respect to $\mathscr{G}_{\beta,\gamma,\varphi}$.

Theorem 18.1 For a given set $\mathscr{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (18.1) and any coupling network belonging to the set $\mathscr{G}_{\beta,\gamma,\varphi}$. In that case Problem 18.1 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{18.6}$$

Moreover, it can be solved by the consensus controller (18.4) if (18.6) holds. Specifically, for given $\mathscr{G}_{\beta,\gamma,\varphi}$ and given $\bar{\tau}$ satisfying (18.6), there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for this α and any $\varepsilon \in (0, \varepsilon^*]$, the agents (18.1) with controller (18.4) achieve consensus for any communication topologies in $\mathscr{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof : It follows from Lemma 18.1 in the Appendix that Theorem 18.1 holds if for any $\gamma \geq \beta > 0$, $\bar{\tau} > 0$ and φ satisfying (18.6), there exist $\alpha > 0$ and ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\dot{x} = Ax - \lambda \alpha e^{j\psi} B F_\varepsilon x(t - \tau). \tag{18.7}$$

is asymptotically stable for all $\tau \in [0, \bar{\tau}]$, $\lambda \in (\beta, \gamma)$ and $\psi \in [-\varphi, \varphi]$.

Since $\bar{\tau}$ and φ satisfy condition (18.6), there exists an $\alpha > 0$ such that

$$\arccos(\tfrac{1}{\alpha\beta}) > \varphi + \omega_{\max}\bar{\tau}. \tag{18.8}$$

Let this α be fixed. By Lemma 18.4, (18.8) implies that $\alpha\lambda\cos(\varphi) > 1$, and hence, $A - \alpha\lambda e^{j\psi} BF_\varepsilon$ is Hurwitz stable for $\psi \in [-\varphi, \varphi]$. Then it follows from Lemma 18.3 that system (18.7) is asymptotically stable if

$$\det\left[j\omega I - A + \alpha\lambda e^{j(\psi-\omega\tau)} BF_\varepsilon\right] \neq 0, \tag{18.9}$$

for $\omega \in \mathbb{R}$, $\tau \in [0, \bar{\tau}]$ and $\psi \in [-\varphi, \varphi]$.

First, we note that given (18.8), there exists a $\delta > 0$ such that

$$\arccos(\tfrac{1}{2\alpha\beta}) > \varphi + \omega\bar{\tau}, \quad \forall |\omega| < \omega_{\max} + \delta. \tag{18.10}$$

Next we will split the proof of (18.9) in two cases where $|\omega| < \omega_{\max} + \delta$ and $|\omega| \geq \omega_{\max} + \delta$ respectively.

If $|\omega| \geq \omega_{\max} + \delta$, we have $\det(j\omega I - A) \neq 0$, in another word, $\underline{\sigma}(j\omega I - A) > 0$. Hence, there exists $\mu > 0$ such that

$$\underline{\sigma}(j\omega I - A) > \mu, \quad \forall \omega, \text{ s.t. } |\omega| \geq \omega_{\max} + \delta.$$

To see this, note that for ω satisfying $|\omega| > \bar{\omega} := \max\{\|A\| + 1, \omega_{\max} + \delta\}$,

$$\underline{\sigma}(j\omega I - A) > |\omega| - \|A\| > 1.$$

But for ω with $|\omega| \in [\omega_{\max} + \delta, \bar{\omega}]$, there exists $\mu \in (0, 1]$ such that

$$\underline{\sigma}(j\omega I - A) \geq \mu,$$

which is due to the fact that $\underline{\sigma}(j\omega I - A)$ depends continuously on ω. Given α and $\lambda \in (\beta, \gamma)$, there exists $\varepsilon^* > 0$ such that $\|\lambda\alpha BF_\varepsilon\| \leq \mu/2$ for $\varepsilon < \varepsilon^*$. Then

$$\underline{\sigma}(j\omega I - A + \alpha\lambda e^{j(\psi-\omega\tau)} BF_\varepsilon) \geq \mu - \mu/2 \geq \mu/2.$$

Therefore, condition (18.9) holds for $|\omega| \geq \omega_{\max} + \delta$.

It remains to verify (18.9) with $|\omega| < \omega_{\max} + \delta$. By the definition of δ, we find that

$$\alpha\lambda \cos(\psi - \omega\tau) > \alpha\beta \cos(\varphi + |\omega|\bar{\tau}) > \tfrac{1}{2},$$

and hence by Lemma 18.4, $A - \alpha\lambda e^{j(\psi-\omega\tau)}BF_\varepsilon$ is Hurwitz stable, for $\omega \in (-\omega_{\max} - \delta, \omega_{\max} + \delta)$, $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. Therefore, (18.9) also holds with $|\omega| < \omega_{\max} + \delta$. ■

Remark 18.2 *Some comments on implementation of the consensus controller* (18.4) *are worthwhile. Four parameters are chosen sequentially in the consensus design and analysis, namely α, δ, μ and ε. First, we select the scaling parameter α in* (18.8) *using the given data β, φ and $\omega_{\max}$. Then, δ is chosen based on network data and the choice of α and such a δ will yield corresponding value of μ. Eventually, ε is determined by μ and γ.*

Remark 18.3 *The consensus controller design depends only on the agent model and parameters $\bar{\tau}, \beta, \gamma$ and φ and is independent of specific network topology provided that the network satisfies Assumption 18.1.*

In the special case where $\omega_{\max} = 0$, i.e. the eigenvalues of A are either zero or in the open left half plane, then arbitrarily bounded communication delay can be tolerated as formulated in the following corollary:

Corollary 18.1 For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (18.1) and any communication topology belonging to the set $\mathcal{G}_{\beta,\gamma,\varphi}$. Suppose the eigenvalues of A are either zero or in the open left half plane. In that case, Problem 18.1 is always solvable via the consensus controller (18.4). Specifically, for given $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, there exist α and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (18.1) with controller (18.4) achieve consensus for any communication topologies in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Remark 18.4 *In the previous study of network consensus problem, agents are normally assumed to have single- or double-integrator type dynamics. Based on Corollary 18.1, we find that the delay tolerance in such cases is independent of network topology and can be made arbitrarily large. This result in no way contradicts that in [76, 46, 9] since the goal here is to find the maximal achievable delay tolerance by controller design whereas obtained in [76, 46, 9] are the conditions on delay for which the consensus with certain given controller is not spoiled.*

Remark 18.5 *Note that for undirected and connected networks, the Laplacian associated with G is symmetric and hence has only real eigenvalues, i.e. we can set $\varphi = 0$. In this case, the upper bound of tolerable delay is independent of network topology. However, in directed networks, we have to sacrifice some robustness in the delay tolerance in order to cope with the complex part of Laplacian eigenvalues.*

18.4. Consensus with partial-state coupling

Next, we proceed to the case of partial-state coupling and design a dynamic consensus controller (18.3) which solves Problem 18.2.

For $\varepsilon > 0$, let P_ε be the positive definite solution of the ARE (18.5). A dynamic low-gain consensus controller can be constructed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - Kz^i \\ u^i = \alpha B' P_\varepsilon \chi^i, \end{cases} \tag{18.11}$$

where K is such that $A + KC$ is Hurwitz stable. α and ε are design parameters to be chosen later. We shall prove that the consensus controller solves Problem 18.2.

Theorem 18.2 For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (18.2) with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$. In that case, Problem 18.2 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{18.12}$$

Moreover, it can be solved by the consensus controller (18.11) if (18.12) holds. Specifically, for given β and γ and given φ and $\bar{\tau}$ satisfying (18.12), there exist $\alpha > 0$ and ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (18.2) with controller (18.11) achieve consensus for any communication topology in $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof : It follows from Lemma 18.2 in the Appendix that Theorem 18.2 holds if there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\begin{cases} \dot{x}(t) = Ax(t) - \alpha\lambda e^{j\psi} BB' P_\varepsilon \chi(t-\tau) \\ \dot{\chi}(t) = (A + KC)\chi(t) - KCx(t) \end{cases} \tag{18.13}$$

is asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

Define

$$\bar{\mathscr{A}} = \begin{bmatrix} A & 0 \\ -KC & A+KC \end{bmatrix}, \quad \bar{\mathscr{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{\mathscr{F}}_\varepsilon = \begin{bmatrix} 0 & -B'P_\varepsilon \end{bmatrix}.$$

First of all, for $\lambda \in (\beta, \gamma)$ and $\psi \in (-\varphi, \varphi)$, there exists α such that

$$\lambda\alpha\cos(\varphi + \omega_{\max}\bar{\tau}) > 2. \tag{18.14}$$

Let this α be fixed. By Lemma 18.5 in the Appendix, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, $\bar{\mathscr{A}} + \alpha\lambda e^{j\psi}\bar{\mathscr{B}}\bar{\mathscr{F}}_\varepsilon$ is Hurwitz stable for $\lambda \in (\beta, \gamma)$ and $\psi \in (-\varphi, \varphi)$. It follows from Lemma 18.3 that (18.13) is asymptotically stable if

$$\det\left[j\omega I - \bar{\mathscr{A}} - \alpha\lambda e^{j(\psi-\omega\tau)}\bar{\mathscr{B}}\bar{\mathscr{F}}_\varepsilon\right] \neq 0, \ \forall\omega \in \mathbb{R},$$
$$\forall\lambda \in (\beta,\gamma), \ \forall\psi \in (-\varphi,\varphi), \ \forall\tau \in [0,\bar{\tau}]. \tag{18.15}$$

Given (18.14), there exists $\delta > 0$ such that

$$\lambda\alpha\cos(\varphi + \omega\bar{\tau}) > 1, \quad \forall|\omega| < \omega_{\max} + \delta. \tag{18.16}$$

We can show, as in the proof of Theorem 18.1, that there exists $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, condition (18.15) holds for $|\omega| \geq \omega_{\max} + \delta$.

For $|\omega| < \omega_{\max} + \delta$, it follows from (18.16) and Lemma 18.5 that $\bar{\mathscr{A}} + \alpha\lambda e^{j(\psi-\omega\tau)}\bar{\mathscr{B}}\bar{\mathscr{F}}_\varepsilon$ is Hurwitz stable. Therefore, condition (18.15) also holds with $|\omega| < \omega_{\max} + \delta$. ∎

Remark 18.6 *The low-gain compensator* (18.11) *is constructed based on the agent model and the network characteristics β, γ and φ. The four parameters α, δ, μ and ε used in the design of controller* (18.11) *are chosen with the same order and relation as in the proof of Theorem 18.1.*

The next corollary is concerned with the case $\omega_{\max}$ where the eigenvalues of A are either zero or in the open left half plane.

Corollary 18.2 For a given set $\mathscr{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (18.2) with any communication topology belonging to $\mathscr{G}_{\beta,\gamma,\pi}$. Suppose the eigenvalues of A are either zero or in the open left half plane. In that case, Problem 18.2 is solvable by the consensus controller (18.11). Specifically, for given β, γ, φ and $\bar{\tau} > 0$, there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (18.2) with controller (18.11) achieve consensus for any communication topology in $\mathscr{G}_{\beta,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

18.5. Special case: Neutrally stable agents

We observe that the consensus controller design in Theorem 18.1 and Theorem 18.2 for general critically unstable agents depends on β which is related to the algebraic connectivity of the graph. We next consider a special case where the agent dynamics are neutrally stable, that is, the eigenvalues of A on the imaginary axis, if any, are semi-simple. Without loss of generality, we assume that $A' + A \leq 0$ which can be obtained after a suitable basis transformation. In this case, we shall show that the consensus controller design no longer requires the knowledge of β and hence allows us to deal with a larger set of unknown communication topologies that can be denoted as $\mathscr{G}_{0,\gamma,\varphi}$.

Consider the agents (18.1). Assume $A' + A \leq 0$. A local consensus controller can be constructed as

$$u^i = \varepsilon B' z^i. \tag{18.17}$$

We have the following theorem:

Theorem 18.3 For a given set $\mathscr{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (18.1) and any communication topology belonging to the set $\mathscr{G}_{0,\gamma,\varphi}$. Suppose $A' + A \leq 0$. In that case, Problem 18.1 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}, \tag{18.18}$$

Moreover, it can be solved by the consensus controller (18.17) if (18.18) holds. Specifically, for given γ and given φ and $\bar{\tau}$ satisfying (18.18), there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (18.1) with controller (18.17) achieve consensus for any communication topology in $\mathscr{G}_{0,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof : It follows from Lemma 18.1 that Theorem 18.3 holds if the system

$$\dot{x} = Ax - \lambda \varepsilon e^{j\psi} BB' x(t - \tau) \tag{18.19}$$

is asymptotically stable for $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$, which, by Lemma 18.3, is true if and only if

$$\det\left[j\omega I - A + \varepsilon\lambda e^{j\psi - j\omega\tau} BB'\right] \neq 0,\ \forall \omega \in \mathbb{R},\ \lambda \in (0, \gamma),\ \psi \in [-\varphi, \varphi],\ \tau \in [0, \bar{\tau}]. \tag{18.20}$$

There exists $\delta > 0$ such that

$$\omega\bar{\tau} + \varphi < \tfrac{\pi}{2}, \quad \forall\omega \text{ s.t. } |\omega| < \omega_{\max} + \delta.$$

For given $\lambda \in (0, \gamma)$, we can show with a similar argument as in the proof of Theorem 18.1 that there exists a $\mu > 0$ and a ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$ and $\lambda \in (0, \gamma)$

$$\underline{\sigma}(j\omega I - A + \varepsilon\lambda e^{j\psi - j\omega\tau} BB') > \mu, \forall\omega \text{ s.t. } |\omega| \geq \omega_{\max} + \delta.$$

Hence, (18.20) is satisfied with $|\omega| \geq \omega_{\max} + \delta$.

It remains to show (18.20) for $|\omega| < \omega_{\max} + \delta$. Note that $\psi - \omega\tau \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by definition of δ and hence $\cos(\psi - \omega\tau) > 0$. Then

$$[A - \varepsilon\lambda e^{j\psi - j\omega\tau} BB']^* + [A - \lambda\varepsilon e^{j\psi - j\omega\tau} BB'] = -2\lambda\varepsilon\cos(\psi - \omega\tau)BB' \leq 0$$

Since (A, B) is controllable, we conclude that $A - \lambda\varepsilon e^{j\psi - j\omega\tau} BB'$ is Hurwitz, and hence (18.20) also holds, with $|\omega| < \omega_{\max} + \delta$. ■

The next theorem addresses the consensus problem for networks with partial state coupling. In this case, a low-gain consensus controller can be designed as

$$\begin{cases} \dot{\chi}^i = (A + KC)\chi^i - Kz^i \\ u^i = \varepsilon B'\chi^i, \end{cases} \tag{18.21}$$

where K is such that $A + KC$ is Hurwitz.

Theorem 18.4 For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$, consider the agents (18.2) with any communication topology belonging to $\mathcal{G}_{0,\gamma,\varphi}$. Suppose $A + A' \leq 0$. In that case, Problem 18.2 is solvable if,

$$\bar{\tau} < \frac{\frac{\pi}{2} - \varphi}{\omega_{\max}}. \tag{18.22}$$

Moreover, it can be solved by the consensus controller (18.21) if (18.22) holds. Specifically, for given γ and given φ and $\bar{\tau}$ satisfying (18.22), there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the agents (18.2) with controller (18.21) achieve consensus for any communication topology in $\mathcal{G}_{0,\gamma,\varphi}$ and $\tau \in [0, \bar{\tau}]$.

Proof : It follows from Lemma 18.2 in the Appendix that Theorem 18.2 holds if there exist $\alpha > 0$ and $\varepsilon^* > 0$ such that for $\varepsilon \in (0, \varepsilon^*]$, the system

$$\begin{cases} \dot{x}(t) = Ax(t) - \varepsilon\lambda e^{j\psi} BB'\chi(t-\tau) \\ \dot{\chi}(t) = (A+KC)\chi(t) - KCx(t) \end{cases} \tag{18.23}$$

is asymptotically stable for any $\lambda \in (0, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

Define

$$\bar{\mathcal{A}} = \begin{bmatrix} A & 0 \\ -KC & A+KC \end{bmatrix}, \quad \bar{\mathcal{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{\mathcal{F}}_\varepsilon = \begin{bmatrix} 0 & -\varepsilon B' \end{bmatrix}.$$

By Lemma 18.6 in the Appendix, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$, $\bar{\mathcal{A}} + \alpha\lambda e^{j\psi}\bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. It follows from Lemma 18.3 that (18.23) is asymptotically stable if

$$\det\left[j\omega I - \bar{\mathcal{A}} - \alpha\lambda e^{j(\psi-\omega\tau)}\bar{\mathcal{B}}\bar{\mathcal{F}}_\varepsilon\right] \neq 0, \ \forall \omega \in \mathbb{R},$$
$$\forall\lambda \in (\beta, \gamma), \ \forall\psi \in (-\varphi, \varphi), \ \forall\tau \in [0, \bar{\tau}]. \tag{18.24}$$

Similarly as before, there exist $\delta > 0$ and $\varepsilon_2 \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon_2]$, condition (18.24) holds for $|\omega| \geq \omega_{\max} + \delta$.

On the other hand, $|\omega| < \omega_{\max} + \delta$, it follows from Lemma 18.6 that $\bar{\mathcal{A}} + \alpha\lambda e^{j(\psi-\omega\tau)}\bar{\mathcal{F}}_\varepsilon$ is Hurwitz stable. Therefore, condition (18.24) also holds with $|\omega| < \omega_{\max} + \delta$. ∎

18.6. Simulation

18.6.1. Consensus with full-state coupling with a set of communication topologies

Consider the 4 identical agents

$$\begin{cases} \dot{x}^i(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x^i(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u^i(t), \\ z^i(t) = -\sum_{j=1}^N \ell_{ij} x^j(t-\tau) \end{cases} \tag{18.25}$$

for $i = 1, \ldots, 4$ with full-state coupling as given in (18.1). The communication topologies defined by $L = \{\ell_{ij}\}$ belong to the set $\mathcal{G}_{3,5,\pi/6}$. We have $\omega_{\max} = 1$ in this case. For this given set of data, we can choose the parameter ε to be 2×10^{-4} and design a consensus controller according to (18.4) and (18.5)

$$u^i = \frac{1}{\beta} F_\varepsilon z^i = \begin{bmatrix} 0.0045 & 0.0540 & -0.0536 & 0.0091 \\ 0.0015 & 0.0091 & -0.0087 & 0.0155 \end{bmatrix} z^i$$

We apply this u^i to two networks in the set $\mathcal{G}_{3,5,\pi/6}$ as shown in Fig. 18.6.1 and 18.6.1. The communication delay in these two networks is $\tau = 0.5$. The corresponding simulation data are shown in Fig 18.3 and Fig 18.4.

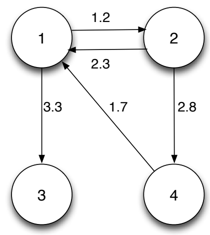


Figure 18.1: Network topology 1

18.6.2. Consensus with partial-state coupling with a set of communication topologies

Consider a network of 4 identical agents

$$\begin{cases} \dot{x}^i(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} x^i(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u^i(t), \\ y^i(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x^i(t) \\ z^i(t) = -\sum_{j=1}^{N} \ell_{ij} y^j(t-\tau) \end{cases} \tag{18.26}$$

with $\tau = 0.5$ and the same set of communication topology $\mathcal{G}_{3,5,\pi/6}$ as in Section 18.6.1.

We can choose $\varepsilon = 10^{-6}$ and

$$K = \begin{bmatrix} -10 & -29 & -5 & -20 \end{bmatrix}', \tag{18.27}$$

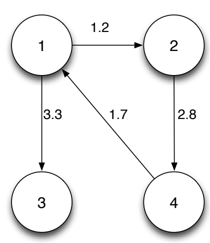


Figure 18.2: Network topology 2

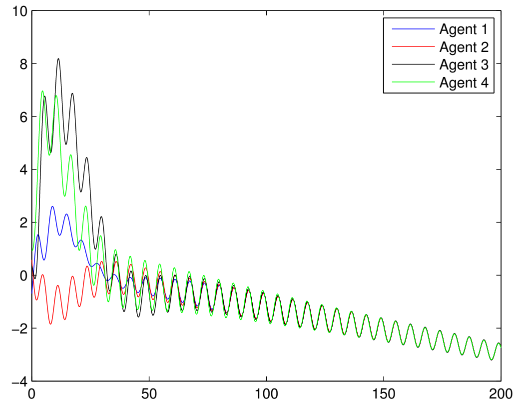


Figure 18.3: Evolution of the first state element of all four agents in Network 1

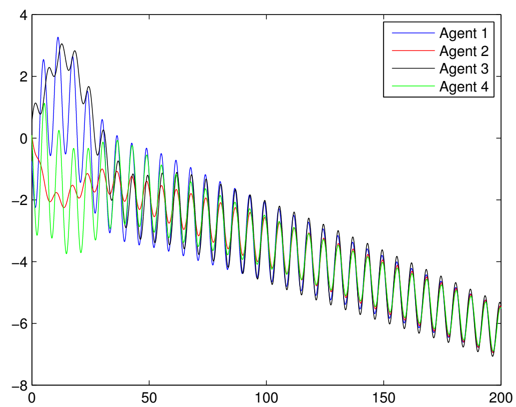


Figure 18.4: Evolution of the first state element of all four agents in Network 2

and design the dynamic low-gain consensus controller as follows:

$$\begin{cases} \dot{\chi}^i(t) = \begin{bmatrix} -10 & 1 & 1 & 0 \\ -29 & 0 & 0 & 1 \\ -5 & 0 & 0 & 1 \\ -20 & 0 & -1 & 0 \end{bmatrix} \chi^i(t) + \begin{bmatrix} -10 \\ -29 \\ -5 \\ -20 \end{bmatrix} z^i(t) \\ u^i(t) = \begin{bmatrix} 0.0003 & 0.0149 & -0.0149 & 0.0007 \\ 0.0000 & 0.0007 & -0.0007 & 0.0009 \end{bmatrix} \chi^i(t). \end{cases}$$

Consider the same two communication topologies depicted in Fig. 18.6.1 and 18.6.1. The respective simulation data are shown in Fig. 18.5 and Fig. 18.6.

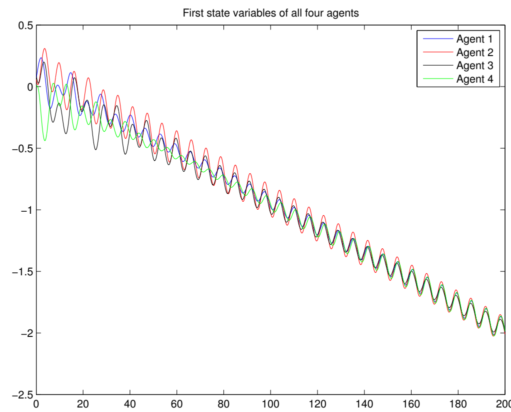


Figure 18.5: Evolution of x_1^i in Network 1

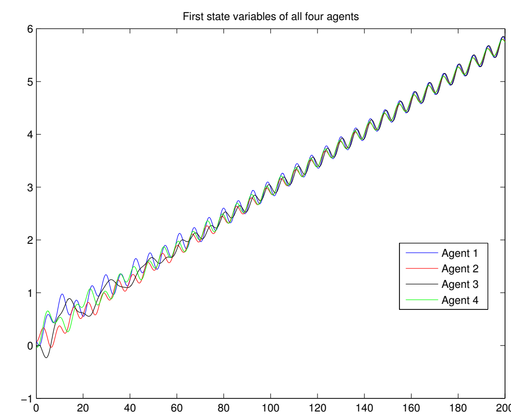


Figure 18.6: Evolution of x_1^i in Network 2

18.7. Concluding remarks

In this chapter, we study the multi-agent consensus with uniform constant communication delay for agents with high-order dynamics. A sufficient condition on delay is derived under which the multi-agent consensus is attainable. Whenever this condition is satisfied, a controller without the exact knowledge of network topology can be constructed such that consensus can be achieved in a set of networks.

Although this chapter focuses on unknown communication topologies, when the perfect information about the topology is in fact available, the design procedure can be easily modified to achieve a stronger result. In this case, input u_i to each agents can be first scaled as $u_i = d_i \bar{u}_i$ where these d_i are such that diag$\{d_i\}L$ has a simple eigenvalue at zero and the rest are real and strictly positive. The existence of such d_is is proved by [22]. Then we can design $\bar{u}_i$ following the procedure proposed in this chapter.

Future research will continuous in two directions: 1. extend the results to non-identical agents; 2. consider non-uniform and time-varying delay.

18.8. Appendix

18.8.1. Connection of network consensus to robust stabilization

Lemma 18.1 Problem 18.1 is solvable via consensus controller $u^i = Fz^i$ if the following $N-1$ systems

$$\dot{\xi}^i(t) = A\xi^i(t) - \lambda_i BF\xi^i(t-\tau) \tag{18.28}$$

are asymptotically stable where $\lambda_i, i = 2, ..., N$ are the non-zero eigenvalues of the Laplacian associated with the communication topology.

Proof : Define $\tilde{x} = [x^1; \cdots; x^N]$. The overall dynamics of N agents can be written as

$$\dot{\tilde{x}}(t) = (I_N \otimes A)\tilde{x}(t) - (L \otimes BF)\tilde{x}(t-\tau).$$

Define $\xi = [\xi^1; \cdots; \xi^N] = (T \otimes I_n)\tilde{x}$ where $\xi^i \in \mathbb{C}^n$ and T is such that $J_L = TLT^{-1}$ is in the Jordan canonical form and $J_L(1,1) = 0$. In the new coordinates, the dynamics of ξ can be written as

$$\dot{\xi}(t) = (I_N \otimes A)\xi(t) - (J_L \otimes BF)\xi(t-\tau).$$

We claim that the network consensus problem is solved if $\xi^i \to 0$ as $t \to \infty$ for $i = 2, \ldots, N$. This can be seen as follows. Let $\eta(t) = [\xi^1(t); 0; \cdots; 0]$. If $\xi(t) \to \eta(t)$, then $\tilde{x}(t) \to (T^{-1} \otimes I_n)\eta(t)$. Note that the columns of T^{-1} comprise all the right eigenvectors and generalized eigenvectors of L. The first column of T^{-1} is vector $\mathbf{1}$. This implies that for $i = 1, \ldots, N$

$$x^i(t) \to \xi^1(t),$$

i.e. the network achieves consensus.

The sub-dynamics of $\bar{\xi}(t) = [\xi^2(t); \cdots; \xi^N(t)]$ are

$$\dot{\bar{\xi}}(t) = (I_{N-1} \otimes A)\bar{\xi}(t) - (\bar{J}_L \otimes BF)\bar{\xi}(t-\tau) \tag{18.29}$$

where $\bar{J}_L$ is such that

$$J_L = \begin{bmatrix} 0 & \\ & \bar{J}_L \end{bmatrix}.$$

The eigenvalues of system (18.29) are given by the roots of its characteristic equation

$$H(s) = \det\left\{sI - (I_{N-1} \otimes A) + e^{-s\tau}(\bar{J}_L \otimes BF)\right\} = 0,$$

which, due to the upper-triangular structure of $I_{N-1} \otimes A$ and $\bar{J}_L \otimes BF$, are the union of the eigenvalues of the $N-1$ systems:

$$\dot{\xi}^i(t) = A\xi^i(t) - \lambda_i BF\xi^i(t-\tau), \quad i = 2, \ldots, N.$$

Then the result in Lemma 18.1 follows. ■

Lemma 18.2 Problem 18.2 is solvable via consensus controller (18.3) if the following $N-1$ systems

$$\begin{cases} \dot{x}^i(t) = Ax^i(t) - \lambda_i BC_c\xi^i(t-\tau) \\ \dot{\chi}^i(t) = A_c\chi^i(t) + B_c z^i(t) \end{cases} \tag{18.30}$$

are asymptotically stable where λ_i for $i = 2, \ldots, N$ are the non-zero eigenvalues of the Laplacian matrix L.

Proof : Let $\bar{x}^i = [x^i; \chi^i]$. Then for each agent, the closed-loop dynamics are

$$\begin{cases} \dot{\bar{x}}^i(t) = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}\bar{x}^i(t) + \begin{bmatrix} 0 \\ B_c \end{bmatrix} z^i(t) \\ y^i(t) = \begin{bmatrix} C & 0 \end{bmatrix}\bar{x}^i \\ z^i(t) = -\sum_{j=1}^N \ell_{ij} y^j(t-\tau). \end{cases}$$

Define $\tilde{x} = [\bar{x}^1; \cdots; \bar{x}^N]$, $\tilde{y} = [y^1; \cdots; y^N]$,

$$\mathcal{A} = \begin{bmatrix} A & BC_c \\ 0 & A_c \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ B_c \end{bmatrix} \text{ and } \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}.$$

The overall dynamics of the N agents can be written as

$$\dot{\tilde{x}}(t) = (I_N \otimes \mathcal{A})\tilde{x}(t) - (L \otimes \mathcal{B}\mathcal{C})\tilde{x}(t-\tau).$$

It follows from Lemma 18.1 that for any given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, Problem 18.2 is solvable via consensus controller (18.3) if the system

$$\dot{\xi}(t) = \mathcal{A}\xi(t) - \lambda e^{j\psi}\mathcal{B}\mathcal{C}\xi(t-\tau) \tag{18.31}$$

is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

The system (18.31) has a set of eigenvalues determined by

$$\det \begin{bmatrix} sI - A & -BC_c \\ \lambda e^{s\tau} B_c C & sI - A_c \end{bmatrix} = 0. \tag{18.32}$$

On the other hand, the eigenvalues of system (18.30) are given by

$$\det \begin{bmatrix} sI - A & \lambda e^{s\tau} BC_c \\ -B_c C & sI - A_c \end{bmatrix} = 0. \tag{18.33}$$

It is easily verified that $\lambda \in \mathbb{C}$ satisfies (18.32) if and only if λ satisfies (18.33).

We find that the two closed-loop systems have the same set of eigenvalues. Therefore, (18.31) is globally asymptotically stable if (18.30) is globally asymptotically stable for any $\lambda \in (\beta, \gamma)$, $\psi \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$. The result in Lemma 18.2 then follows. ■

18.8.2. Stability of linear time-delay system

The following lemma is adapted from [157].

Lemma 18.3 Consider a linear time-delay system

$$\dot{x} = Ax + A_d x(t - \tau). \tag{18.34}$$

Assume $A + A_d$ is Hurwitz. We have that (18.34) is globally asymptotically stable for $\tau \in [0, \bar{\tau}]$ if

$$\det \left[j\omega I - A - e^{-j\omega\tau} A_d \right] \neq 0, \quad \forall \omega \in \mathbb{R},\ \forall \tau \in [0, \bar{\tau}],$$

for all $\omega \in \mathbb{R}$ and $\tau \in [0, \bar{\tau}]$.

18.8.3. Robustness of low-gain state feedback and compensator

In this subsection, we recall some classical robust properties of low-gain feedback and compensator. Consider an uncertain system

$$\begin{cases} \dot{x} = Ax + \mu Bu \\ y = Cx, \end{cases} \tag{18.35}$$

where (A, B) is stabilizable, (A, C) is detectable and A has all its eigenvalues in the closed left half plane. The $\mu \in \mathbb{C}$ is input uncertainty. For $\varepsilon > 0$, let P_ε be the positive definite solution of ARE

$$A'P_\varepsilon + AP_\varepsilon - P_\varepsilon B'BP_\varepsilon + \varepsilon I = 0.$$

The robustness of a low-gain state feedback $u = -B'P_\varepsilon x$ is inherited from that of a classical LQR.

Lemma 18.4 $A - \mu BB'P_\varepsilon$ is Hurwitz stable for any $\mu \in \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \frac{1}{2}\}$.

Proof : We observe that for $\mu \in \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq \frac{1}{2}\}$,

$$(A - \mu BB'P_\varepsilon)^* P_\varepsilon + P_\varepsilon(A - \mu BB'P_\varepsilon) = -\varepsilon I - (2\,\mathrm{Re}(\mu) - 1)P_\varepsilon BB'P_\varepsilon < 0,$$

and hence, $A - \mu BB'P_\varepsilon$ is Hurwitz stable. ■

The next lemma proves similar property of a low-gain compensator.

Lemma 18.5 For any *a priori* given bounded set

$$\mathcal{W} \subseteq \{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\},$$

there exists ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (18.35) and the low-gain compensator

$$\begin{cases} \dot{\chi} = (A + KC)\chi - Ky, \\ u = -B'P_\varepsilon \chi \end{cases} \tag{18.36}$$

is asymptotically stable for any $\mu \in \mathcal{W}$.

Proof : Define $e = x - \chi$.The closed-loop of (18.35) and (18.36) can be rewritten in terms of x and e as

$$\begin{cases} \dot{x} = (A - \mu BB'P_\varepsilon)x + \mu BB'P_\varepsilon e \\ \dot{e} = (A + KC + \mu BB'P_\varepsilon)e - \mu BB'P_\varepsilon x. \end{cases} \tag{18.37}$$

Since $\mathrm{Re}(\mu) \geq 1$, we have

$$(A - \mu BB'P_\varepsilon)^* P_\varepsilon + P_\varepsilon(A - \mu BB'P_\varepsilon) \leq -\varepsilon I - P_\varepsilon BB'P_\varepsilon.$$

Define $V_1 = x^* P_\varepsilon x$ and $u = B' P_\varepsilon x$.

$$\begin{aligned}\dot{V}_1 &\le -\varepsilon\|x\|^2 - \|u\|^2 + 2\operatorname{Re}(\mu u^* B' P_\varepsilon e)\\ &\le -\varepsilon\|x\|^2 - \|u\|^2 + \theta(\varepsilon)\|e\|\|u\|,\end{aligned}$$

where $\theta(\varepsilon) = \|\mu B' P_\varepsilon\|$. Clearly, $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let Q be the positive definite solution of Lyapunov equation

$$(A + KC)'Q + Q(A + KC) = -2I.$$

Since $F_\varepsilon \to 0$ and μ is bounded in $\mathcal{W}$, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$,

$$(A + KC + \mu BF_\varepsilon)'Q + Q(A + KC + \mu BF_\varepsilon) \le -I.$$

Define $V_2 = e^* Q e$. We get

$$\begin{aligned}\dot{V}_2 &\le -\|e\|^2 - 2\operatorname{Re}(\mu e^* Q B u)\\ &\le -\|e\|^2 + M\|e\|\|u\|\end{aligned}$$

where

$$M = \max_{\mu \in \mathcal{W}} \{2\|\mu Q B\|\}.$$

Define $V = 4M^2 V_1 + 2V_2$. Then

$$\dot{V} \le -4M^2\varepsilon\|x\|^2 2 - 2\|e\|^2 - 4M^2\|u\|^2 + (4M^2\theta(\varepsilon) + 2M)\|e\|\|u\|$$

There exist $\varepsilon^* \le \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$,

$$4M^2\theta(\varepsilon) \le 2M.$$

Hence for $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} \le -4M^2\varepsilon\|x\|^2 - \|e\|^2 - (\|e\| - 2M\|u\|)^2.$$

We conclude that (18.36) is asymptotically stable for any $\mu \in \mathcal{W}$. ■

Lemma 18.6 Consider system (18.35). Suppose $A' + A \leq 0$. For any *a priori* given $\varphi \in (0, \frac{\pi}{2})$ and a bounded set

$$\mathcal{W} \subseteq \{s \in \mathbb{C} \mid s \neq 0, \ \ \arg(s) \in [-\varphi, \varphi]\},$$

there exists ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$, the closed-loop system of (18.35) and the low-gain compensator

$$\begin{cases} \dot{\chi} = (A + KC)\chi - Ky, \\ u = -\varepsilon B'\chi \end{cases} \tag{18.38}$$

is asymptotically stable for any $\mu \in \mathcal{W}$.

Proof : Define $e = x - \chi$. The closed-loop of (18.35) and (18.38) can be rewritten in terms of x and e as

$$\begin{cases} \dot{x} = (A - \varepsilon\mu BB')x + \varepsilon\mu BB'e \\ \dot{e} = (A + KC + \varepsilon\mu BB')e - \varepsilon\mu BB'x. \end{cases} \tag{18.39}$$

Define $V_1 = x^*x$ and $u = B'x$.

$$\begin{aligned} \dot{V}_1 &\leq -\varepsilon \operatorname{Re}(\mu)\|u\|^2 + 2\varepsilon \operatorname{Re}(\mu u^* B'e) \\ &\leq -\varepsilon \operatorname{Re}(\mu)\|u\|^2 + \varepsilon|\mu|\theta_1\|e\|\|u\|, \end{aligned}$$

where $\theta_1 = 2|B\|$.

Let Q be the positive definite solution of Lyapunov equation

$$(A + KC)'Q + Q(A + KC) = -2I.$$

Since μ is bounded in $\mathcal{W}$, there exists ε_1 such that for $\varepsilon \in (0, \varepsilon_1]$,

$$(A + KC + \varepsilon\mu BB')^*Q + Q(A + KC + \varepsilon\mu BB') \leq -I.$$

Define $V_2 = e^*Qe$. We get

$$\begin{aligned} \dot{V}_2 &\leq -\|e\|^2 - 2\operatorname{Re}(\varepsilon\mu e^* QBu) \\ &\leq -\|e\|^2 + \varepsilon|\mu|\theta_2\|e\|\|u\| \end{aligned}$$

where $\theta_2 = 2\|QB\|$.

Define $V = V_1 + V_2$. Then with $\theta_3 = \theta_1 + \theta_2$, we have

$$\begin{aligned}
\dot{V} &\leq -\|e\|^2 - \varepsilon \operatorname{Re}(\mu)\|u\|^2 + \varepsilon|\mu|\theta_3\|e\|\|u\| \\
&\leq -\|e\|^2 - \varepsilon \operatorname{Re}(\mu)\left[\|u\|^2 - \sec(\varphi)\theta_3\|e\|\|u\|\right] \\
&\leq -\left[1 - \varepsilon|\mu|\sec(\varphi)\right]\|e\|^2 - \frac{3}{4}\varepsilon \operatorname{Re}(\mu)\|u\|^2 \\
&\qquad - \varepsilon \operatorname{Re}(\mu)\left(\tfrac{1}{2}\|u\| - \sec(\varphi)\theta_3\|e\|\right)^2
\end{aligned}$$

Since $\mathcal{W}$ is bounded and φ is given, there exists $\varepsilon^* \leq \varepsilon_1$ such that for $\varepsilon \in (0, \varepsilon^*]$,

$$\varepsilon|\mu|\sec(\varphi) \leq \frac{1}{2}, \quad \forall \mu \in \mathcal{W}.$$

Hence for $\varepsilon \in (0, \varepsilon^*]$,

$$\dot{V} \leq -\frac{1}{2}\|e\|^2 - \frac{3}{4}\varepsilon \operatorname{Re}(\mu)\|u\|^2.$$

Since (A, B) is stabilizable, we conclude that (18.36) is asymptotically stable for any $\mu \in \mathcal{W}$. ∎

CHAPTER 19

Synchronization in a heterogenous network of introspective right-invertible agents with uniform constant communication delay–continuous-time case

19.1. Introduction

In this chapter, we consider heterogenous networks of introspective multi-input multi-output agents with uniform constant communication delay. We assume that the agents are right-invertible. Two problems are studied here, namely the output synchronization problem and output regulation problem. The underlying idea is to shape the agent dynamics into a particular form by exploiting the self-knowledge and right-invertibility property of the agents. Specifically, in the output synchronization problem, the agents are manipulated to imitate a neutrally stable system and as such can tolerate *arbitrary* bounded delay and accommodate more network uncertainties under standard assumption on the communication topology. However, when one is more concerned with the synchronization trajectories as in the output regulation problem, we can re-shape the agent to be the same with exo-system and regulate the agents' outputs by providing relative output measurement of the exo-system only to one particular agent. Moreover, we propose a *decentralized* controller design methodology that does not require exact knowledge of communication topologies so that these two problems can be solved for a set of unknown networks. Finally, we show that the design proposed in this chapter also applies to a formation control problem.

19.2. Problem formulation

Consider a heterogenous network of N introspective agents

$$\begin{cases} \dot{x}^i(t) = A^i x^i(t) + B^i u^i(t), \\ y^i(t) = C_y^i x^i(t), \\ z^i(t) = C_z^i x^i(t), \\ \zeta^i(t) = \sum_{j=1}^N \ell_{ij} y^j(t-\tau) \end{cases} \tag{19.1}$$

where $x^i \in \mathbb{R}^{n_i}$, $u^i \in \mathbb{R}^{m_i}$, $y^i, \zeta^i \in \mathbb{R}^p$, $z^i \in \mathbb{R}^{q_i}$ and $\tau > 0$ is an unknown constant satisfying $\tau \in [0, \bar{\tau}]$. The coefficients ℓ_{ij} are such that $\ell_{ij} \leq 0$ for $i \neq j$ and $\ell_{ii} = -\sum_{j \neq i}^N \ell_{ij}$.

The matrix $L = \{\ell_{ij}\} \in \mathbb{R}^{N\times N}$ defines the *communication topology* which can be captured by a weighted graph $G = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ where $(j,i) \in \mathcal{E} \Leftrightarrow \ell_{ij} < 0$ and $a_{ii} = 0$ and $a_{ij} = -\ell_{ij}$ for $i \neq j$.

Assumption 19.1 *The communication topology G contains a directed spanning tree whose root (without loss of generality) is agent N.*

In this case, L has a simple eigenvalue at zero and the rest are located in the open right half plane. Let $\lambda_1, \cdots, \lambda_N$ denote the eigenvalues of L and assume $\lambda_1 = 0$. When the perfect information of the communication topology is not available, we can use the non-zero eigenvalues of L as a rough "metric" of the graph and introduce the following definition to characterize a set of unknown communication topologies.

Definition 19.1 *For any $\gamma \geq \beta > 0$ and $\frac{\pi}{2} > \varphi \geq 0$, $\mathcal{G}_{\beta,\gamma,\varphi}$ is the set of networks whose Laplacian eigenvalues satisfy that*

$$|\lambda_i| \in (\beta, \gamma),\ \arg \lambda_i \in [-\varphi, \varphi] \ \text{ for } i = 2, \ldots, N.$$

In this network, each agent collects two measurements:

1. a network measurement $\zeta^i \in \mathbb{R}^p$ which is a combination of its own output relative to that of neighboring agents and is subject to a uniform constant communication delay;
2. a local measurement $z^i \in \mathbb{R}^{q_i}$ of its internal dynamics to which the agent has an instantaneous access.

Assumption 19.2 *The agents satisfy the following properties:*

1. (A^i, B^i) *is stabilizable;*
2. (A^i, C_y^i) *is detectable;*
3. (A^i, B^i, C_y^i) *is right-invertible;*
4. (A^i, C_z^i) *is detectable.*

The output synchronization in a heterogeneous network of agents (19.1) can be defined as follows:

Definition 19.2 *The agents in the network achieve output synchronization if*

$$\lim_{t\to\infty}(y^i(t) - y^j(t)) = 0, \quad \forall i, j \in \{1, \ldots, N\}.$$

With the above defined notations, the first problem studied in this chapter is formally stated below:

Problem 19.1 Consider a heterogenous network of the form (19.1). For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, the output synchronization problem with a set of communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ for all $\tau \leq \bar{\tau}$ is to design a local linear dynamical controller

$$\begin{cases} \dot{\chi}^i = A_c^i \chi^i + B_c^i \zeta^i + E_c^i z^i \\ u^i = C_c^i \chi^i + D_c^i \zeta^i + M_c^i z^i, \end{cases} \tag{19.2}$$

such that the synchronization can be achieved in the network with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.

Note that the above synchronization problem does not impose any restriction on the synchronization trajectories. The focus here is to solve this problem for as a large set of communication topologies and delay as possible. On the other hand, it is important in some scenario to regulate the output of the agents to desired trajectories when the output synchronization is reached. Let an exo-system be given as

$$\begin{cases} \dot{x}_r = A_r x_r, \quad x_r(0) = x_r, \\ y_r = C_r x_r, \end{cases} \tag{19.3}$$

where A_r has all its eigenvalues in the closed left half complex plane and (A_r, C_r) is observable. We want to regulate each agent's output to y_r. It is reasonable to assume that the synchronization trajectories are not exponentially increasing. In this case, we assume *the root* of network also measures its own output relative to y_r of the exo-system. To be precise, the root agent, which is the agent N, takes the following form:

$$\begin{cases} \dot{x}^N = A^N x^N(t) + B^N u^N(t), \\ z^N = C_z^N x^N(t), \\ y^N = C_y^N x^N(t), \\ \zeta^N = \sum_{j=1}^N \ell_{Nj} y^j(t-\tau) + \delta\left[y^N(t-\tau) - y_r(t-\tau)\right], \end{cases} \tag{19.4}$$

with $\delta > 0$.

Definition 19.3 *The agents in the network achieve output regulation if*

$$\lim_{t\to\infty}(y^i(t) - y_r(t)) = 0, \quad \forall i \in \{1, \ldots, N\}.$$

We can formulate the regulation problem as follows:

Problem 19.2 Consider a heterogenous network of the form (19.1). For a given exo-system (19.3), a set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, the output regulation problem with exo-system (19.3) and a set of communication topologies $\mathcal{G}_{\beta,\gamma,\varphi}$ for all $\tau \leq \bar{\tau}$ is to design a local linear dynamical controller (19.2) such that the output regulation can be achieved in the network with any communication topology belonging to $\mathcal{G}_{\beta,\gamma,\varphi}$ for $\tau \leq \bar{\tau}$.

19.3. Main result

The first main result of this chapter is stated in the following theorem:

Theorem 19.1 For a given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, the Problem 19.1 is always solvable via a decentralized dynamic consensus controller (19.2).

Before we present the result for output regulation problem, some preparatory work needs to be done. For any communication topology G, an augmented graph $\bar{G}$ can be defined by including the exo-system denoted by e and an arc (e, N) with weight δ into the topology. The Laplacian associated with $\bar{G}$ is

$$\bar{L} = \{\bar{\ell}_{ij}\} = \begin{bmatrix} \ell_{11} & \ell_{12} & \cdots & \ell_{1N} & 0 \\ \ell_{21} & \ell_{22} & \cdots & \ell_{2N} & 0 \\ \vdots & \vdots & \cdots & \vdots & 0 \\ \ell_{N1} & \ell_{N2} & \cdots & \ell_{NN} + \delta & -\delta \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{19.5}$$

whose eigenvalues are denoted by $\bar{\lambda}_i$, $i = 1, ..., N+1$ with $\bar{\lambda}_1 = 0$. Obviously, this $\bar{G}$ also has a directed spanning tree and thus $\bar{\lambda}_i$, $i = 2, ..., N+1$ are in the open right half plane. For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$, the set of augmented topologies can be denoted by $\bar{\mathcal{G}}_{\bar{\alpha},\bar{\beta},\bar{\varphi}}$ such that for any $\bar{G} \in \bar{\mathcal{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$,

$$|\bar{\lambda}_i| \in (\bar{\beta}, \bar{\gamma}),\ \arg(\bar{\lambda}_i) \in [-\bar{\varphi}, \bar{\varphi}],\ i = 2, ..., N.$$

We have the following theorem:

Theorem 19.2 For a given set $\mathcal{G}_{\beta,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, the Problem 19.2 is solvable via a decentralized dynamic consensus controller (19.2) if the set of augmented topologies $\bar{\mathcal{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ satisfies:

1. $\bar{\varphi} < \frac{\pi}{3}$;

2. $\bar{\tau} < \frac{\frac{\pi}{3}-\bar{\varphi}}{\omega_{\max}}$,

where $\omega_{\max} = \max\{\omega \in \mathbb{R} \mid \det(j\omega I - A_r) = 0\}$.

We shall prove Theorem 19.1 and 19.2 by explicitly constructing a synchronization or regulation controller in the form of (19.2) via a progressive design approach. First, we design a local pre-compensator to make the agents quasi-identical to a new common model, which we refer to as *homogenization of network*; Next, we show that for this new network, both problems can be reduced to a robust stabilization problem. Finally, we shall design a controller that solves the reformulated stabilization problem so that synchronization or output regulation can be achieved in the homogenized network.

19.3.1. Homogenization of the network

For introspective agents, their self-reflection of internal dynamics provides us with additional freedom to manipulate the agent models so as to disguise them as being almost identical to the rest of the network viewed from their output. This is shown in the next lemma.

Lemma 19.1 Consider a heterogenous network of the form (19.1) with communication topologies given by $\mathcal{G}_{\beta,\gamma,\varphi}$ and communication delay $\tau \leq \bar{\tau}$. Let n_d denote the maximum order of infinite zeros of (A^i, B^i, C^i). Suppose a triple (A, B, C) is given such that

1. $\operatorname{rank}(C) = p$.

2. (A, B, C) is invertible, of uniform rank $n_q \geq n_d$ and has no invariant zero.

There exists a compensator

$$\begin{cases} \dot{\xi}^i(t) = A_H^i \xi^i(t) + B_H^i z^i(t) + E_H^i v^i(t) \\ u^i(t) = C_H^i \xi^i(t) + D_H^i v^i(t), \end{cases} \tag{19.6}$$

such that the closed-loop system of (19.1) and (19.6) can be written in the following form:

$$\begin{cases} \dot{\bar{x}}^i(t) = A\bar{x}^i(t) + B\left(v^i(t) + d^i(t)\right) \\ y^i(t) = C\bar{x}^i(t), \\ \zeta^i(t) = \sum_{j=1}^N \ell_{ij} y^j(t-\tau), \end{cases} \tag{19.7}$$

where d^i are generated by

$$\begin{cases} \dot{\omega}^i(t) = A_s^i \omega^i(t), \quad i = 1, ..., N, \\ d^i(t) = C_s^i \omega^i(t). \end{cases} \tag{19.8}$$

and A_s^i are Hurwitz stable.

Proof : See [152]. ∎

Remark 19.1 *Lemma 1 shows that we can design a compensator* (19.6) *to make the agent identical to a new common model characterized by* a priori *given triple* (A, B, C) *except for an exponentially decaying exogenous signal injected in the range space of* B*. Moreover, we have a complete freedom to choose the modes of* A *which is fundamental in proving Theorems 19.1 and 19.2.*

The resulting network (19.7) can be viewed as a homogenous network affected by the exponentially decaying disturbances d_i generated by (19.8). The injection of such exponentially decaying d^i turns out to be irrelevant and the output synchronization problem in the original heterogenous network of agents (19.1) can be reduced to the output synchronization problem in a homogeneous network with the same communication topology.

19.3.2. Synchronization in homogeneous networks

Next, we consider the synchronization problem for the agents (19.7) as formulated in Problem 19.1. We can choose in Lemma 19.1 the triple (A, B, C) satisfying additional properties

$$A + A' = 0, \quad |\lambda(A)| < \frac{\frac{\pi}{2} - \varphi}{\bar{\tau}}. \tag{19.9}$$

Such a triple (A, B, C) always exists and in fact can be chosen in the following form:

$$A = \Gamma(A_0 + B_0 H)\Gamma^{-1}, \quad B = \Gamma B_0, \quad C = C_0 \Gamma^{-1}$$

and

$$A_0 = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_0 = \begin{bmatrix} I_p & 0 \end{bmatrix},$$

where H is such that $A_0 + B_0 H$ only has semi-simple eigenvalues on the imaginary axis satisfying (19.9). H exists due to the fact that (A_0, B_0) is controllable. Then a transformation Γ can be found such that $\Gamma(A_0 + B_0 H)\Gamma^{-1}$ is in the real Jordan canonical form and thus $A + A' = 0$.

For the above (A, B, C), a low-gain compensator can be constructed as

$$\begin{cases} \dot{\chi}^i(t) = (A + KC)\chi^i(t) - K\zeta^i(t) \\ v^i(t) = -\varepsilon B'\chi^i(t), \end{cases} \tag{19.10}$$

where K is such that $A + KC$ is Hurwitz stable. The existence of K is due to the fact that (A, C) is observable.

Define $\tilde{x}^i = [\bar{x}^i; \chi^i]$. Then for each agent, the closed-loop dynamics of (19.7) and (19.10) are

$$\begin{cases} \dot{\tilde{x}}^i(t) = \bar{A}\tilde{x}^i(t) + \bar{B}\zeta^i(t) + \bar{E}d^i(t) \\ y^i(t) = \bar{C}\tilde{x}^i(t) \\ \zeta^i(t) = \sum_{j\in\mathcal{N}} \ell_{ij} y^j(t-\tau), \end{cases}$$

where

$$\bar{A} = \begin{bmatrix} A & -\varepsilon BB' \\ 0 & A + KC \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ -K \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} B \\ 0 \end{bmatrix}. \tag{19.11}$$

Define $\tilde{x} = [\tilde{x}^1; \cdots; \tilde{x}^N]$ and $d = [d^1; \cdots; d^N]$. The overall dynamics of N agents can be written as

$$\dot{\tilde{x}}(t) = (I_N \otimes \bar{A})\tilde{x}(t) + (L \otimes \bar{B}\bar{C})\tilde{x}(t-\tau) + (I_N \otimes \bar{E})d.$$

Let T be a non-singular matrix such that $J = TLT^{-1}$ is in the Jordan Canonical Form with $J(1,1) = \lambda_1 = 0$ and $\eta = [\eta^1; \cdots; \eta^N] = (T \otimes I_n)\tilde{x}$ where n is the dimension of A. The dynamics of η are governed by

$$\dot{\eta}(t) = (I_N \otimes \bar{A})\eta(t) + (J \otimes \bar{B}\bar{C})\eta(t-\tau) + (T \otimes \bar{E})d.$$

Lemma 19.2 The interconnections of (19.7) and (19.10) reach output synchronization if $\eta^i \to 0$ as $t \to \infty$ for $i = 2, ..., N$.

Proof of Lemma 19.2 : Let $\pi(t) = [\eta^1(t); 0; \cdots; 0]$. If $\eta(t) \to \pi(t)$, then $\tilde{x}(t) \to (T^{-1} \otimes I_n)\pi(t)$ where n is the dimension of A. Note that the columns of T^{-1} comprise all the right eigenvectors and generalized eigenvectors of L. The first column of T^{-1} is vector $\mathbf{1}$. This implies that for $i = 1, ..., N$

$$\tilde{x}^i(t) \to \eta^1(t).$$

■

Define $\bar{\eta} = [\eta^2; \cdots; \eta^N]$ and take the dynamics of d into account. We will get

$$\begin{bmatrix} \dot{\bar{\eta}}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} I_{N-1} \otimes \bar{A} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} \begin{bmatrix} \bar{\eta}(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} \bar{J} \otimes \bar{B}\bar{C} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\eta}(t-\tau) \\ \omega(t-\tau) \end{bmatrix}, \tag{19.12}$$

where $\omega = [\omega_1; \ldots; \omega_N]$, $\bar{C}_s = \text{blkdiag}\{C_s^i\}_{i=1}^N$, $\bar{I} = [0, I_{N-1}]$ and $\bar{A}_s = \text{blkdiag}\{A_s^i\}_{i=1}^N$ is Hurwitz. Clearly $\bar{\eta} \to 0$ for any initial condition if the system (19.12) is globally asymptotically stable. Note that the system (19.12) is globally asymptotically stable if and only if

$$\det\left[sI - \begin{bmatrix} I_{N-1} \otimes \bar{A} & (\bar{I}T \otimes \bar{E})\bar{C}_s \\ 0 & \bar{A}_s \end{bmatrix} - \begin{bmatrix} \bar{J} \otimes \bar{B}\bar{C} & 0 \\ 0 & 0 \end{bmatrix} e^{-s\tau} \right] \neq 0, \quad \forall s \in \mathbb{C}^+. \tag{19.13}$$

Due to the upper triangular structure of both matrices in (19.13) and the fact that $\bar{A}_s$ is Hurwitz, it is easy to see that (19.13) holds if and only if

$$\det\left[sI - (I_{N-1} \otimes \bar{A}) - (\bar{J} \otimes \bar{B}\bar{C})e^{-s\tau} \right] \neq 0, \quad \forall s \in \mathbb{C}^+. \tag{19.14}$$

Therefore, we have the following lemma.

Lemma 19.3 The interconnections of agents (19.7) and (19.10) achieve output synchronization if the system

$$\dot{\tilde{\eta}}(t) = \bar{A}\tilde{\eta}(t) + \lambda \bar{B}\bar{C}\tilde{\eta}(t-\tau), \tag{19.15}$$

is globally asymptotically stable for $|\lambda| \in (0, \gamma)$, $\arg \lambda \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

The next lemma is proved in Chapter 18.

Lemma 19.4 Let $\bar{A}$, $\bar{B}$ and $\bar{C}$ be given in (19.11). For $\gamma \geq \beta > 0, \varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\bar{\tau} > 0$, there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the systems (19.15) are globally asymptotically stable for $|\lambda| \in (0, \gamma)$, $\arg \lambda \in [-\varphi, \varphi]$ and $\tau \in [0, \bar{\tau}]$.

Proof of Theorem 19.1 : For given set $\mathcal{G}_{0,\gamma,\varphi}$ and $\bar{\tau} \geq 0$, it follows from Lemma 19.1, 19.3 and 19.4 that there exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the composition of (19.6) and (19.10) will solve Problem 1. ■

19.3.3. Output regulation in homogenous network

Now we consider the output regulation problem. It is shown in the Appendix that without loss of generality, we can always manipulate the internal dynamics of exo-system (19.3) and find a matrix B_r such that (A_r, B_r, C_r) is invertible, of uniform rank $n_q > n_d$ and has no invariant zero. Therefore, according to Lemma 19.1, there exists a pre-compensator (19.6) such that the interconnection of (19.6) and agent (19.1) can be written in the form of (19.7) with A, B and C replaced by A_r, B_r and C_r.

Next, we design a controller for the homogenized network (19.7). By the definition of $\bar{L}$ in (19.5), we can define an augmented homogenized network by including the exo-system into (19.7) as follows:

$$\begin{cases} \dot{\bar{x}}^i(t) = A_r\bar{x}^i(t) + B_r\left(v^i(t) + d^i(t)\right), i = 1, ..., N+1 \\ y^i(t) = C_r\bar{x}^i(t), \\ \zeta^i(t) = \sum_{j=1}^{N+1} \bar{\ell}_{ij} y^j(t-\tau), \end{cases} \tag{19.16}$$

where agent $N+1$ is the exo-system and $d_{N+1}(t) = 0$. We can not control the exo-system, i.e. $v^{N+1}(t) = 0$. Obviously, the output regulation problem is solved if this augmented network reaches synchronization for any communication topology in $\bar{\mathscr{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ and $\tau \leq \bar{\tau}$. We shall design a controller to achieve this goal.

For $\varepsilon > 0$, let P_ε be the positive definition solution of Algebraic Riccati Equation (ARE)

$$A_r' P_\varepsilon + P_\varepsilon A_r - B_\varepsilon B_r B_r' P_\varepsilon + \varepsilon I = 0 \tag{19.17}$$

and K be such that $A_r + KC_r$ is Hurwitz stable. A low-gain compensator can be constructed for agent $i = 1, ..., N$ as

$$\begin{cases} \dot{\chi}^i(t) = (A_r + KC_r)\chi^i(t) - K\zeta^i(t), i = 1, ..., N. \\ v^i(t) = -\frac{1}{\beta} B_r' P_\varepsilon \chi^i(t). \end{cases} \tag{19.18}$$

We can imagine that (19.18) also apply to agent $N+1$ (exo-system) but with initial condition $\chi^{N+1}(0) = 0$. Since $\zeta^{N+1}(t) = 0$, we shall have $v^{N+1}(t) = 0$. In view of this, we can write the dynamics of the whole augmented network as

$$\dot{\tilde{x}}(t) = (I_N \otimes \bar{A})\tilde{x}(t) + (\bar{L} \otimes \bar{B}\bar{C})\tilde{x}(t-\tau) + (I_N \otimes \bar{E})d.$$

where

$$\bar{A} = \begin{bmatrix} A_r & -\frac{1}{\beta} B_r B_r' P_\varepsilon \\ 0 & A_r + KC_r \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ -K \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_r & 0 \end{bmatrix}, \quad \bar{E} = \begin{bmatrix} B_r \\ 0 \end{bmatrix}. \tag{19.19}$$

Similarly as in preceding subsection, we can prove

Lemma 19.5 The interconnections of agents (19.16) and controller (19.18) achieve synchronization for any communication topology in $\bar{\mathscr{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ and $\tau \leq \bar{\tau}$ if the following system

$$\dot{\tilde{\eta}}(t) = \bar{A}\tilde{\eta}(t) + \bar{\lambda}\bar{B}\bar{C}\tilde{\eta}(t-\tau) \tag{19.20}$$

is globally asymptotically stable for $|\bar{\lambda}| \in (\bar{\beta}, \bar{\gamma})$, $\arg(\bar{\lambda}) \in [-\bar{\varphi}, \bar{\varphi}]$ and $\tau \in [0, \bar{\tau}]$, where $\bar{A}$, $\bar{B}$ and $\bar{C}$ is given by (19.19).

The next lemma is shown in Chapter 18.

Lemma 19.6 For a given set $\bar{\mathscr{G}}_{\bar{\beta},\bar{\gamma},\bar{\varphi}}$ and $\bar{\tau} > 0$, let the conditions in Theorem 19.2 be satisfied. There exists an ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the system (19.20) with (19.19) is globally asymptotically stable for $|\bar{\lambda}| \in (\bar{\beta}, \bar{\gamma})$, $\arg(\bar{\lambda}) \in [-\bar{\varphi}, \bar{\varphi}]$ and $\tau \in [0, \bar{\tau}]$.

19.4. Application to formation

In this section, we show that the design method presented in preceding sections is also applicable to formation problem.

Definition 19.4 A *formation* is a family of vectors $\{h_1, \ldots, h_N\}$, $h_i \in \mathbb{R}^p$. The agents are said to achieve output formation if

$$\lim_{t\to\infty} \left[(y_i(t) - h_i) - (y_j(t) - h_j)\right] = 0.$$

Suppose a set of communication topologies $\mathscr{G}_{0,\gamma,\varphi}$ and $\bar{\tau} > 0$ are given. Let n_q be the maximum order of infinite zeros of all the agents. The controller design follows a similar procedure as in the synchronization problem. First, we design a pre-compensator (19.6) for each agent to homogenize the network utilizing its local measurement so that the agents are quasi-identical to a new common model characterized by a given trip (A, B, C) which satisfies the following conditions:

1. $\text{rank}(C) = p$.
2. (A, B, C) is invertible, of uniform rank n_q and has no invariant zero,
3. $A + A' = 0$,

4. The eigenvalues of A satisfy

$$|\lambda(A)| < \frac{\frac{\pi}{2} - \varphi}{\bar{\tau}}.$$

Moreover, there exists a family of vectors $\{\bar{h}_1, \ldots, \bar{h}_N\}$ of appropriate dimension such that for $i = 1, ..., N$,

5. $C\bar{h}_i = h_i$,

6. $A\bar{h}_i = 0$.

Remark 19.2 *For arbitrary given vectors $\{h_1, \ldots, h_N\}$, such a triple (A, B, C) always exists. One particular choice which satisfies the above conditions is the following*

$$A = T(A_0 + B_0 H)T^{-1}, \quad B = TB_0, \quad C = C_0 T^{-1}$$

and

$$A_0 = \begin{bmatrix} 0 & I_{(n_q-1)p} \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_0 = \begin{bmatrix} I_p & 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & H_0 \end{bmatrix}$$

where H_0 is such that

$$\bar{A}_0 + \bar{B}_0 H_0 = \begin{bmatrix} 0 & I_{(n_q-2)p} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_p \end{bmatrix} H_0$$

is non-singular and only has semi-simple eigenvalues on the imaginary axis. H_0 exists due to the fact that $(\bar{A}_0, \bar{B}_0)$ is controllable. It is easy to see that $A_0 + B_0 H$ has $(n_q - 1)p$ semi-simple non-zero eigenvalues on the imaginary axis and p semi-simple eigenvalues at zero. Then a transformation T can be found such that $T(A_0 + B_0 H)T^{-1}$ is in the real Jordan canonical form and thus $A + A' = 0$. For this triple (A, B, C), a family of vector $\{\bar{h}_1, \ldots, \bar{h}_N\}$ can be found as

$$\bar{h}_i = T \begin{bmatrix} h_i \\ 0 \end{bmatrix}$$

so that

$$C\bar{h}_i = \begin{bmatrix} I_p & 0 \end{bmatrix} \begin{bmatrix} h_i \\ 0 \end{bmatrix} = h_i.$$

Next, a local formation controller can be designed as follows:

$$\begin{cases} \dot{\chi}^i(t) = (A + KC)\chi^i(t) - K\left[\sum_{j=1}^{N} \ell_{ij}(y_j(t-\tau) - h_j)\right] \\ v^i(t) = -\varepsilon B'\chi^i(t), \end{cases} \tag{19.21}$$

We can prove the following result:

Theorem 19.3 For a given set $\mathcal{G}_{0,\gamma,\varphi}$, a formation $\{h_1, \ldots, h_N\}$ and $\bar{\tau} \geq 0$, there exists ε^* such that for $\varepsilon \in (0, \varepsilon^*]$, the agents (19.1) with controller (19.6) and (19.21) achieve formation for any communication topology belonging to $\mathcal{G}_{0,\gamma,\varphi}$ and $\tau \leq \bar{\tau}$.

Proof : It follows from Lemma 19.1 that the interconnection of the agents and (19.6) can be written in the following form:

$$\begin{cases} \dot{\bar{x}}^i(t) = A\bar{x}^i(t) + B\left(v^i(t) + d^i(t)\right) \\ y^i(t) = C\bar{x}^i(t). \end{cases} \tag{19.22}$$

Let $\bar{x}_s^i = \bar{x}^i - \bar{h}_i$. Then the closed-loop system of (19.22) and controller (19.21) can be written in terms of $\bar{x}_s^i$ and χ^i as

$$\begin{cases} \dot{\bar{x}}_s^i(t) = A\bar{x}_s^i(t) + B\left(v^i(t) + d^i(t)\right) + A\bar{h}_i, \\ \chi^i(t) = (A + KC)\chi^i(t) - K\left[\sum_{j=1}^N \ell_{ij}(C\bar{x}_s^j(t-\tau))\right] \end{cases}$$

Since $A\bar{h}_i = 0$, $i = 1, ..., N$, the rest of the proof is exactly the same as in the preceding section. ■

Remark 19.3 *One thing that should be noted is that owing to the freedom we have in choosing appropriate (A, B, C), no restriction on formation vector needs to be imposed.*

Appendix

Manipulation of exo-system

For a given exo-system (19.3), there exists a non-singular transformation $\tilde{x}_r = Tx_r$ such that (19.3) can be transformed in the following canonical form [63]:

$$\begin{cases} \dot{\tilde{x}}_r = \tilde{A}_r\tilde{x}_r \\ y_r = \tilde{C}_r\tilde{x}_r \end{cases} \tag{19.23}$$

where

$$\tilde{A}_r = TA_rT^{-1} = \begin{bmatrix} \tilde{A}_1 & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & \star & \star & \cdots & \star \\ 0 & \tilde{A}_2 & 0 & 0 & \cdots & 0 \\ \star & \star & \star & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \tilde{A}_p \\ \star & \star & \star & \star & \cdots & \star \end{bmatrix}, \tag{19.24}$$

$$\tilde{C}_r = C_rT^{-1} = \begin{bmatrix} \tilde{C}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{C}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{C}_p \end{bmatrix}, \tag{19.25}$$

and

$$\tilde{A}_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

$$\tilde{C}_i = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Here $\star$ denotes a possible non-zero row. Note that for the original system (19.3), $\tilde{A}_i$ may not have the same size. However, by adding integrators to the bottom of each block and setting the initial conditions of those extended states to zero, we can extend the dimension of $\tilde{A}_i$ to $n_q > n_d$ while system (19.23) still produces the same output as the original exo-system (19.3).

Eventually, we can choose

$$\tilde{B}_r = \begin{bmatrix} \tilde{B}_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{B}_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \tilde{B}_p \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

We find that $(\tilde{A}_r, \tilde{B}_r, \tilde{C}_r)$ is invertible, of uniform rank $n_q > n_d$ and has no invariant zero.

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