Electronic Theses and Dissertations

8-2011

Cantor set approximations and dimension computations in hyperspaces.

Matthew G. Zapf
University of Louisville

Follow this and additional works at: https://ir.library.louisville.edu/etd

Recommended Citation

Zapf, Matthew G., "Cantor set approximations and dimension computations in hyperspaces." (2011). *Electronic Theses and Dissertations.* Paper 1632.
https://doi.org/10.18297/etd/1632

CANTOR SET APPROXIMATIONS
AND
DIMENSION COMPUTATIONS IN HYPERSPACES

By

Matthew G. Zapf
B.A., Saint Louis University, 2004
M.A., Saint Louis University, 2006

A Dissertation
Submitted to the Faculty of the
College of Arts and Sciences of the University of Louisville
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy

Department of Mathematics
University of Louisville
Louisville, KY

August 2011

CANTOR SET APPROXIMATIONS
AND
DIMENSION COMPUTATIONS IN HYPERSPACES

Submitted by

Matthew G. Zapf

A Dissertation Approved on

June 24, 2011

by the Following Reading and Examination Committee:

Manabendra Das, Dissertation Director

Udayan Darji

Shudun Liu

Thomas Riedel

David Swanson

ACKNOWLEDGEMENTS

To my advisor, Professor Manav Das. Thank you for your skill, unyielding patience, and sound guidance during this long and arduous process. I am eternally grateful.

To Professors Udayan Darji and David Swanson for the many interesting and informative discussions of real analysis over the past five years.

To Professors Thomas Riedel and André Kézdy for the diligent guidance and advice you have given me.

To Professor Shudun Liu for the time you spent reading and understanding my thesis.

To Rebekah, Kim, Marshall, Adam, Max, Finley and all of the other mathematics doctoral students at the University of Louisville for your inspiration and friendship.

To the Puno family. I will never forget the kindness and generosity you have shown me at every turn during my time in Louisville. Thank you.

To Nate, Drew, Lance, Jeremy, Frank, Ronnie, Jasper, Phil, Jeff, Bru, Brad and Katie, Charles and Heend, Jake and Christen, and many others. Thank you all for being such dear friends.

To my sister Rachel, brother-in-law Ryan, niece Reese, and all the rest of my family for making life a joy. I love each and every one of you very much.

To my parents. If only I had the words to articulate my gratitude and love for you. Thank you.

ABSTRACT

CANTOR SET APPROXIMATIONS
AND
DIMENSION COMPUTATIONS IN HYPERSPACES

Matthew G. Zapf

June 24, 2011

Given a metric space (K, d), the hyperspace of K is defined by

$$\mathbb{H}(K) = \{F \subseteq K : F \text{ is compact}, F \neq \emptyset\}.$$

$\mathbb{H}(K)$ is itself a metric space under the Hausdorff metric d_H. Hyperspaces have been extensively studied by topologists since the 1970's, but the measure-theoretical study of hyperspaces has lagged. Boardman and Goodey concurrently provided a characterization of a one-parameter family of Hausdorff gauge functions that determine the dimension of $\mathbb{H}([0, 1])$, and this result was extended by McClure to $\mathbb{H}(X)$ where X is a self-similar fractal satisfying the Open Set Condition.

This dissertation further generalizes these results to include graph-self-similar and self-conformal fractals satisfying the Open Set Condition in $\mathbb{R}^d$. In Chapter 2 it is shown that the dimensions of the underlying fractals may be approximated by the dimensions of sets invariant under particularly constructed sub-iterated function systems that satisfy the Strong Separation Condition. In Chapter 3, a one-parameter family of gauge functions is constructed which computes the dimensions of the hyperspaces of graph-self-similar sets that satisfy the Strong

Separation Condition, after which the approximations of Chapter 2 are applied to extend the result to graph-self-similar sets which satisfy the Open Set Condition. The analogous results for self-conformal sets that satisfy the Open Set Condition are developed in Chapter 4.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Given a metric space (K, d), the hyperspace of K is defined by

$$\mathbb{H}(K) = \{F \subseteq K : F \text{ is compact}, \ F \neq \emptyset\}.$$

$\mathbb{H}(K)$ is itself a metric space under the Hausdorff metric d_H, and is complete (resp. compact) given that K is complete (resp. compact). It is an interesting problem to study the relationship of a hyperspace $\mathbb{H}(K)$ to its underlying metric space K, and much topological study of hyperspaces has taken place since the 1970's as well as some dimensional study (see the discussion at the beginning of Chapter 3). For our purposes, we wish to address the following question: Given a fractal K, can one derive the dimension of $\mathbb{H}(K)$ from the dimension of K? In particular, can we do this if K is the attractor of a suitable iterated function system?

We will address this question for the special cases of graph-directed IFS consisting of similitudes and IFS consisting of conformal maps. We now outline the arguments of this dissertation. The reader may wish to consult the flow chart in Figure 1.1 as well as the theorem mapping in Figure 1.2 to aid in clarifying the flow of our arguments.

Chapter 1 gives an overview of fractal geometry, including classical theorems that we will use in our arguments. Section 1.1 reviews the necessary classical notation and results from fractal geometry, Section 1.2 reviews the necessary measure theory and fractal dimension theory, and Section 1.3 reviews the particular
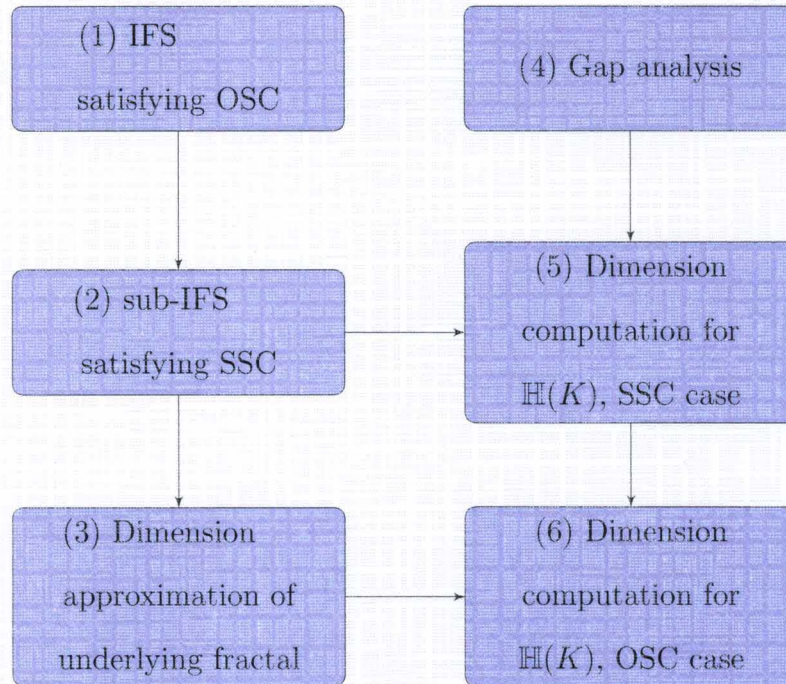
FIGURE 1.1 – Outline of thesis arguments.

Box	Pertinent definitions / lemmas / theorems
(1)	Definitions 5, 9
(2)	§2.1.1, §2.2, §2.3 all definitions
(3)	§2.1.1, §2.2, §2.3 all lemmas / theorems
(4)	Lemma 3.1, Propositions 3.1, 3.2, 4.1, 4.2
(5)	Lemmas 3.2, 3.3, 4.1, 4.2, Theorems 3.1, 4.1
(6)	Theorems 3.2, 4.2

FIGURE 1.2 – Mapping of theorems to the flow chart in Figure 1.1.

dimension computations for various types of fractals.

In Chapters 2 through 4, we perform an analysis of the gaps between the cylinders in the geometric constructions of the fractals. We give a general result that gives conditions for which the diameters of the sets in a covering of the fractal are small enough with respect to the sets relative distances in order to insure some nice geometric properties (Lemma 3.1). We then perform case specific analyses on the graph-directed and self-conformal fractals, respectively, in order to apply the general lemma (Propositions 3.1, 3.2, 4.1, 4.2).

With the appropriate geometric properties in hand, we proceed to construct measures on the fractals. We do this by constructing coverings that were consistent with the aforementioned gap analyses, and then by constructing measures relative to these coverings which satisfied a particular boundedness condition (Lemmas 3.3, 4.2). The properties of these measures allow us to apply a known density lemma to compute the dimensions of the hyperspaces (Theorems 3.1, 4.1).

Finally, we construct approximations of more general fractals by choosing sub-IFS that satisfy some appropriately chosen geometric conditions. We show that the dimensions of the sub-attractors given by the sub-IFS in fact approximate the dimensions of the big attractors (Theorems 2.2, 4.1) which allows us to generalize these results further (Theorems 3.2, 4.1).

1.1   Review of Fractal Geometry

The techniques and arguments of fractal geometry have a very distinctive flavor, as well as a distinctive notation. We review here the notation and results that will be used in the arguments in Chapters 2, 3 and 4.

1.1.1 Iterated Function Systems

We introduce here the notion of an *iterated function system* (or IFS) and discuss some examples. There is a vast literature on the study of IFSs with the first systematic account being Hutchinson's seminal paper [30].

Let (X, d) be a complete metric space and let $\{w_e\}_{e \in E}$ (for some finite set E) be a collection of contractive maps from X into itself. To say that f is **contractive** is to say there exists a constant $r \in (0, 1)$ such that

$$d(f(x), f(y)) \leq r d(x, y)$$

for all $x, y \in X$. In the case of the collection $\{w_e\}_{e \in E}$, we define the ratio r_e associated to the map w_e to be the minimum of all constants satisfying the above inequality. We call $\mathcal{F} = \{X, w_e\}_{e \in E}$ an **iterated function system** (IFS). Along with a metric space and an IFS, we wish to consider the solution to the self-referential equation

$$K = \bigcup_{e \in E} w_e(K).$$

Such a solution, which tends to be the fractal object of study, will be called the **invariant set** for the IFS. It is easy to see that finding the solution to the self-referential equation $K = \bigcup w_e(K)$ may be phrased as a fixed point problem on $\mathbb{H}(X)$. Given the maps $w_e : X \to X$, we may define $W : \mathbb{H}(X) \to \mathbb{H}(X)$ by

$$W(A) = \bigcup_{e \in E} w_e(A).$$

Each $w_e(A)$ is nonempty and compact by continuity of w_e, and since a finite union of compact sets is compact, W is well-defined. It is a simple exercise to show that W is a contractive map on $\mathbb{H}(X)$ under d_H given that each w_e is contractive on X. The existence and uniqueness of the invariant set then follows from the

Contraction Mapping Principle.[1]

A very broad class of fractal sets may be defined using IFSs, the full extent of which is far from understood. We will take the time now to introduce some of the widely studied subclasses and give some examples. We assume throughout this dissertation (unless otherwise noted) that $X \subset \mathbb{R}^d$ is compact and $X = \overline{\text{int}(X)}$, where $\text{int}(\cdot)$ denotes the interior.[2]

The simplest situation arises when the IFS maps are similitudes whose images satisfy a disjointness condition known as the *strong separation condition* (SSC). We formally define these two concepts with the following definitions.

DEFINITION 1. Let (X, d) be a metric space and let $f : X \to X$. We call f a **similitude** if there exists a constant $c > 0$ such that

$$d(f(x), f(y)) = cd(x, y)$$

for each $x, y \in X$.

DEFINITION 2. When an IFS $\mathcal{F}$ consists of similitudes we will refer to it as a **self-similar IFS**. Let (X, d) be a complete metric space and let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be an IFS with unique invariant set $K \subseteq X$. We say that $\mathcal{F}$ satisfies the **strong separation condition** if $w_{e_1}(K) \cap w_{e_2}(K) = \emptyset$ whenever $e_1 \neq e_2$.

The IFS that constructs the classical Cantor Middle-Third Set is the canonical example of an IFS satisfying the SSC (see Figure 1.3). Consider the following maps from $[0, 1]$ into itself:

$$w_1(x) = \frac{1}{3}x \qquad \text{and} \qquad w_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

---

[1]The use of the Contraction Mapping Principle to show existence and uniqueness of invariant sets and measures was first introduced in [30].

[2]The assumption that $X = \overline{\text{int}(X)}$ is necessary for conformal dimension computations (see [44]).
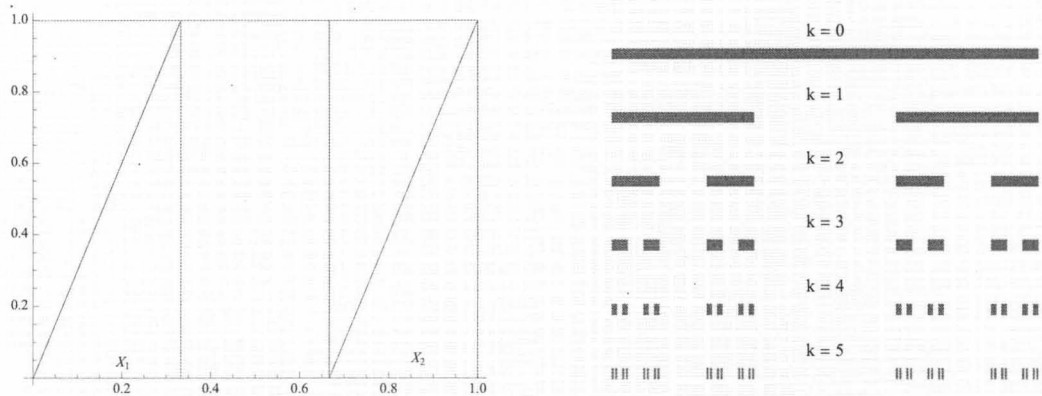
FIGURE 1.3 – The first five iterations in the construction of $\mathcal{C}$.

It is easily checked that the unique compact set satisfying the self-referential equation

tion

$$\mathcal{C} = w_1(\mathcal{C}) \cup w_2(\mathcal{C})$$

is the Cantor Middle-Third Set. It is also clear that

$$w_1(\mathcal{C}) \cap w_2(\mathcal{C}) = (\mathcal{C} \cap [0, 1/3]) \cap (\mathcal{C} \cap [2/3, 1]) = \emptyset$$

so that the IFS satisfies SSC.

A more complicated situation arises when the IFS maps are still similitudes, but the images of the maps are no longer assumed to be disjoint. If it is assumed that the images don't have too much overlap, we say the IFS satisfies the *open set condition* (OSC). This description is made precise with the following definition first given in [30].

DEFINITION 3. Let (X, d) be a complete metric space and let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be an IFS. If there exists an open set $U \subseteq X$ such that

1. $\bigcup_{e \in E} w_e(U) \subseteq U$, and

2. $w_{e_1}(U) \cap w_{e_2}(U) = \emptyset$

whenever $e_1 \neq e_2$, then we say that $\mathcal{F}$ satisfies the **open set condition**.

6
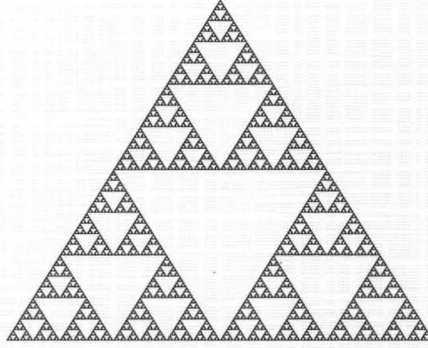
FIGURE 1.4 – Sierpiński Triangle

The canonical example in this situation is the famous Sierpiński Triangle. Define the triangle with side length 1 to be the following region

$$T = \overline{\text{conv}} \left\{ (0,0), (1/2, \sqrt{3}/2), (1,0) \right\} \subset \mathbb{R}^2$$

where $\overline{\text{conv}}$ denotes the closed convex hull. For $i = 1, 2, 3$ consider the following maps from T into itself:

$$w_i(\vec{x}) = (1/2)\vec{x} + t_i$$

where

$$t_1 = \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad t_3 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

There is a unique invariant set $\mathcal{T} \subset T$ which we call the *Sierpiński Triangle* that satisfies

$$\mathcal{T} = w_1(\mathcal{T}) \cup w_2(\mathcal{T}) \cup w_3(\mathcal{T}).$$

One can easily see that $w_i(\mathcal{T}) \cap w_j(\mathcal{T}) \neq \emptyset$ for $i \neq j$, since the smaller triangles intersect at the corners, hence the Sierpiński IFS does not satisfy the SSC. However, it is likewise easy to see that the IFS $\{T, w_1, w_2, w_3\}$ satisfies the OSC with $U = \text{int}(T)$.

7

It is often important for technical reasons to have the open set U intersect the invariant set K non-trivially. This added condition is important enough to warrant the naming of an autonomous separation condition, namely the *strong open set condition* (SOSC), which was first discussed in [32].

DEFINITION 4. Let (X, d) be a complete metric space and let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be an IFS satisfying the OSC with unique invariant set K. If the open set U from the OSC satisfies $U \cap K \neq \emptyset$, then we say that $\mathcal{F}$ satisfies the **strong open set condition**.

It may seem counterintuitive that an IFS could satisfy the OSC but not the SOSC. As one's intuition tends to work in $\mathbb{R}^d$, this intuition is warranted given the following result of A. Schief, which may be found in [47].

THEOREM 1.1. *Let* $\mathcal{F} = \{\mathbb{R}^d, w_i\}_{e \in E}$ *be an IFS consisting of similitudes. Then* $\mathcal{F}$ *satisfies the OSC if and only if it satisfies the SOSC.*

This theorem was extended to strongly connected GDIFS (see Definition 5) in [50] and to conformal maps in $\mathbb{R}^d$ in [44], facts that will play crucial roles in the results of this dissertation. As might be expected, it is not necessarily the case that the OSC and the SOSC are equivalent when we move to more general metric spaces.[3] A precise characterization of when the OSC and the SOSC are equivalent is still an open question, however.

Numerous other separation conditions have been studied, including the *Finite Type Condition*, the *Measure Separation Condition* and the *Weak Separation Condition*. We will not discuss these conditions here, but the interested reader may consult [24], [33], [42], [47].

---

[3]An example of a GDIFS which is not strongly connected and satisfies the OSC, but not the SOSC, is given in [50].

8
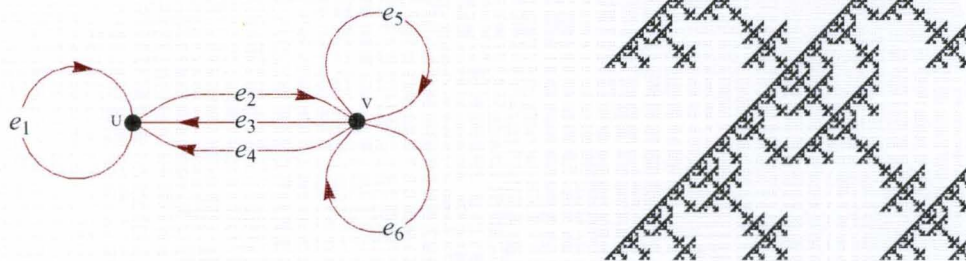
FIGURE 1.5 – A Mauldin-Williams Graph and Invariant Set List

We now consider a different generalization of IFSs, namely *graph directed iterated function systems*. Graph directed constructions were introduced in [36] and have been the subject of much study since. We give here the description presented in [16].

Consider a collection of vertices V and a collection of directed edges E, and suppose we have functions $i : E \to V$ and $t : E \to V$ where $i(e)$ and $t(e)$ are the initial and terminal vertices of e, respectively. Furthermore suppose we have a function $r : E \to (0, 1)$. We call the graph $G = (V, E, i, t, r)$ a **Mauldin-Williams Graph** (see for example Figure 1.5).

DEFINITION 5. Let $(J_v, d_v)_{v \in V}$ be a collection of metric spaces (sometimes called *seed sets*) and let

$$w_e : J_{t(e)} \to J_{i(e)}$$

be contractive maps with contraction ratios $r(e)$. We call the system $\mathcal{F}_G = \{J_v, w_e\}_{v \in V, e \in E}$ a **graph directed iterated function system** (GDIFS).

We will sometimes say that $\mathcal{F}_G$ "realizes" the Mauldin-Williams graph, and given that $\mathcal{F}_G$ realizes a strongly connected Mauldin-Williams graph, we call

$\mathcal{F}_G$ a *strongly connected GDIFS*. This setup yields a collection of self-referential equations

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} w_e(K_v)$$

for which there is a vector of solutions $(K_v)_{v \in V}$ that we call the **invariant set list**.

Graph directed systems allow for more mixing of shapes than standard IFSs, and as such require their own definitions for the various separation conditions.

DEFINITION 6. Let (V, E, i, t, r) be a Mauldin-Williams graph and let $\mathcal{F}_G = \{J_v, w_e\}_{v \in V, e \in E}$ be a realization of the graph.

1. $\mathcal{F}_G$ satisfies the graph strong separation condition if $w_{e_1}(J_{t(e_1)}) \cap w_{e_2}(J_{t(e_2)}) = \emptyset$ for each $e_1, e_2 \in E_v$, $e_1 \neq e_2$, $v \in V$.

2. $\mathcal{F}_G$ satisfies the graph open set condition if there exist open sets $(U_v)_{v \in V}$ such that $\bigcup_{e \in E_v} w_e(U_{t(e)}) \subset U_v$ for each $v \in V$, where $w_{e_1}(U_{t(e_1)}) \cap w_{e_2}(U_{t(e_2)}) = \emptyset$ for $e_1 \neq e_2$.

3. $\mathcal{F}_G$ satisfies the strong graph open set condition if it satisfies the graph open set condition and $K_v \cap U_v \neq \emptyset$ for each $v \in V$.

We will henceforth refer to the SSC, OSC, and SOSC, where it is understood that if we are discussing a graph directed system we are referring to the above definitions.

We may also consider the situation of IFSs where the maps being iterated are not similitudes, but rather are *conformal*. Let $x \in \mathbb{R}^d$ and let $L : \mathbb{R}^d \to \mathbb{R}^d$ denote a linear operator. We will here and throughout let $|x|$ denote the euclidean norm of x and $|L|$ denote the operator norm of L. Furthermore, if $U \subseteq \mathbb{R}^d$ is open

10

and $f \in C^1(U)$, we will use the notation

$$\|Df\| = \sup_{x \in U} |Df(x)|$$

where $Df(x)$ denotes the Jacobian of f at x.

DEFINITION 7. Let $V \subseteq \mathbb{R}^d$ be open. A function $f \in C^1(V)$ is said to be **conformal** if for each $x \in V$, $Df(x) : \mathbb{R}^d \to \mathbb{R}^d$ satisfies $|Df(x) \cdot y| = |Df(x)||y| \neq 0$ for every nonzero $y \in \mathbb{R}^d$.[4]

Naturally conformal IFS will consist of conformal maps. We will in fact need a little bit more: we will need the notion of a $C^{1+\epsilon}$ map.

DEFINITION 8. Let $V \subseteq \mathbb{R}^d$ be open and let $f \in C^1(V)$. We say that f is of class $C^{1+\epsilon}(V)$ if there exists a constant $c > 0$ such that

$$||Df(x)| - |Df(y)|| \leq c \cdot |x - y|^\epsilon$$

for each $x, y \in V$.

We are now ready to give a precise definition of a conformal IFS in $\mathbb{R}^d$. The theory of conformal IFS has been extended to much more general settings than discussed here, in particular conformal IFSs on V, where V is a connected subset of a Riemannian Manifold, are discussed in [43].

DEFINITION 9. Let $X \subset \mathbb{R}^d$ be compact and let $\mathcal{F}_c = \{X, w_e\}_{e \in E}$ be an IFS on X. We call $\mathcal{F}_c$ a **conformal IFS** if all of the following hold:

(a) each w_e extends to a conformal map on an open, convex set V with $X \subset V$

---

[4]The collection of maps given by this definition technically contains both conformal and anti-conformal maps. While we are slightly imprecise in our use of the term "conformal," this use is the standard in the fractal literature (see [44]).

(b) $w_e \in C^{1+\epsilon}(V)$ for each $e \in E$

(c) There exists $0 < r_{\min} < r_{\max} < 1$ such that $r_{\min} \leq |Dw_e(x)| \leq r_{\max}$ for each $e \in E$, $x \in V$

(d) $\mathcal{F}_c$ satisfies the OSC with $U = \text{int}(X)$.

Condition (c) will ensure that the maps are contractive, which insures the existence and uniqueness of the self-conformal invariant set.

1.1.2 Symbol Spaces

It is necessary in the construction and analysis of invariant sets to have some notational means of keeping track of the iterates. The object that serves this purpose is known as a symbol space.

Consider a finite collection of letters (or symbols) $E = \{e_1, e_2, \ldots, e_N\}$. We define the **symbol space over** E by

$$E^\infty = \{e_1, e_2, \ldots, e_N\}^{\mathbb{N}}.$$

This is the collection of infinite sequences whose elements are taken from E. We will refer to $\sigma \in E^\infty$ as a *string* (E^∞ is sometimes referred to as a string space, see for example [16]). We also wish to consider strings of finite length. For each $n \in \mathbb{N}$ we define *level-n* to be

$$E^n = \{\sigma_1 \sigma_2 \cdots \sigma_n | \sigma_i \in E \ \text{ each } \ 1 \leq i \leq n\}$$

and we let

$$E^* = \bigcup_{n=0}^{\infty} E^n.$$

The number of elements in a finite string will be called the *length* of the string and will be denoted by $|\sigma|$. It will be important to consider the restriction of strings to a given initial length, so we introduce the notation $\sigma|_n = \sigma_1\sigma_2 \cdots \sigma_n$ for $\sigma \in E^*$ and $1 \leq n \leq |\sigma|$ (with a similar definition holding for $\sigma \in E^\infty$). We will also let $\sigma- = \sigma|_{|\sigma|-1}$ for $\sigma \in E^*$. Just as important as restricting the length of a string is the expansion of the length of a string by another finite, or infinite, string. Given $\sigma \in E^m$ and $\tau \in E^n$, we define the *concatenation of σ and τ* to be the unique string $\eta \in E^*$ such that

$$\eta = \sigma_1\sigma_2 \cdots \sigma_m\tau_1\tau_2 \cdots \tau_n.$$

We will simply write $\sigma\tau$ to denote the concatenation of σ and τ. Note that a similar notion of the concatenation $\sigma\tau$ holds for $\sigma \in E^*$, $\tau \in E^\infty$, but that this notion is not well defined for $\sigma \in E^\infty$.

The notion of length induces a partial ordering on E^*, namely $\sigma \preceq \tau$ if there exists $n \in \mathbb{N}$ such that $\tau|_n = \sigma$, and in this case we refer to σ as the *ancestor* and τ as the *descendant*. In the special case that $\tau- = \sigma$, we refer to σ as the *parent* and τ as the *child*. Lastly, we introduce the notion of a **cylinder set**. Let $\sigma \in E^*$ and define

$$[\sigma] = \{\sigma\tau : \tau \in E^\infty\}.$$

Now that we have a multitude of definitions and notation out of the way, we can introduce some metric structure onto the symbol space. There are many ways to introduce a distance function to the symbol space, but for our purposes we would like the metric to mimic a geometric construction in a more general metric space. For this reason we introduce a function $\alpha : E^* \to \mathbb{R}^+$ that is decreasing on each totally ordered subset of E^*. As is shown in [16], so long as α is decreasing on totally ordered subsets it induces a metric ρ_α on E^* that satisfies $\rho_\alpha(\sigma, \tau) = |\eta|$

where η is the greatest common ancestor of σ and τ.

The function α will allow us to associate symbol spaces with many Cantor spaces in a Lipschitz way. We still need a candidate for the Lipschitz map, however. Let (X, d) be a compact metric space and let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be an IFS with unique invariant set K. For $\sigma \in E^n$ define

$$X_\sigma = (w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n})(X).$$

For $\sigma \in E^\infty$, the collection $\{X_{\sigma|_n}\}_{n \geq 1}$ is a nested collection of closed sets, hence its intersection is nonempty. Let $|X|$ denote the **diameter of X**, i.e.

$$|X| = \sup \{d(x, y) : x, y \in X\}.$$

Since each w_e is uniformly contractive, we have $|X_{\sigma|_n}| \to 0$. Thus

$$\{x_\sigma\} = \bigcap_{n=1}^{\infty} X_{\sigma|_n}.$$

We define $h : E^\infty \to K$ by $h(\sigma) = x_\sigma$ where x_σ is the unique element of the countable intersection of $\{X_{\sigma|_n}\}_{n \geq 1}$. Notice that we can equivalently define h by

$$h(\sigma) = \lim_{n \to \infty} w_{\sigma|_n}(z)$$

where the limit is independent of $z \in X$.

We need a bit more notation to appropriately define h for a graph directed system. In particular, if E denotes the collection of edges of the Mauldin-Williams graph, then the full symbol space E^∞ will contain erroneous strings. In particular if $e_1, e_2 \in E$ are such that $t(e_1) \neq i(e_2)$, then

$$w_{e_1} : J_{t(e_1)} \to J_{i(e_1)}$$

$$w_{e_2} : J_{t(e_2)} \to J_{i(e_2)}$$

but $J_{i(e_2)}$ and $J_{t(e_1)}$ are distinct metric spaces, hence $w_{e_1} \circ w_{e_2}$ is not defined. It follows that no string $\sigma \in E^\infty$ containing "$e_1 e_2$" has an associated point in any of the invariant sets K_v. We can make this idea notationally precise by defining

$$E_u = \{e \in E : i(e) = u\}$$

$$E_{uv} = \{e \in E : i(e) = u, t(e) = v\}$$

and considering the adjacency matrix

$$A = (a_{uv})_{u,v \in V}$$

of the Mauldin-Williams graph, where $a_{uv} = 1$ if $E_{uv} \neq \emptyset$ and is zero otherwise. We define the modified string space E_A^∞ by

$$E_A^\infty = \{\gamma \in E^\infty : a_{t(\gamma_j)i(\gamma_{j+1})} = 1, \forall j \geq 1\}.$$

For each $v \in V$, $h : (E_v)_A^\infty \to K_v$ is defined by

$$h(\gamma) = \lim_{n \to \infty} w_{\gamma|_n}(z_n)$$

where the limit is independent of the $z_n \in J_{t(\gamma_n)}$ chosen. We define cylinder sets analogous to the IFS case by

$$J_\gamma = (w_{\gamma_1} \circ \cdots \circ w_{\gamma_n})(J_{t(\gamma_n)})$$

for $\sigma \in E_A^n$. Also, we will henceforth let $i(\gamma) = i(\gamma_1)$ and $t(\gamma) = t(\gamma_{|\gamma|})$ for every $\gamma \in E_A^*$ (Recall that γ is a path in the Mauldin-Williams graph and so γ_1 and $\gamma_{|\gamma|}$ are edges). The function α generates a metric on E_A^* in the same way that it did for E^*, and we again refer the reader to [16] for this fact.

1.2 Measures and Dimensions

In all of fractal geometry, there are a few particular notions of dimension that are the most general and are not focused on specific applications. Among these there are two notions of dimension that are the most generally accepted, and most often used notions of dimension in fractal geometry. These dimensions are the *Hausdorff dimension* and the *box-counting dimension* (and the closely related *entropy index*), and to these notions of dimension we now turn our attention.[5]

1.2.1 Box Dimension and Entropy Index

We begin by motivating our definitions with an example: the Cantor set. We would like to find the correct "units" with which to measure $\mathcal{C}$ as positive and finite. In contrast to, say, the unit square, the Cantor set is totally disconnected, hence we will be unable to nicely place "units" in a finite way. We will therefore have to cover $\mathcal{C}$ with ever smaller pieces of units and see what their "measures" approach as we take a limit.

We have perfect candidates for these "smaller pieces" in the cylinder sets. For $k \geq 1$ there are 2^k cylinders, each of which has length $(1/3)^k$, which becomes $(1/3^s)^k = 3^{-sk}$ when we use (units)^s as our "units". This means our s-"measure" at level k is $2^k \cdot 3^{-sk}$. Our hope, then, would be that this s-"measure" would converge in k to a finite constant $c > 0$, which we may, without loss of generality, take to be 1. We should then have for large $k \in \mathbb{N}$,

$$2^k \cdot 3^{-sk} \approx 1.$$

---

[5]One could argue that the *packing dimension* is as important as the dimensions mentioned here, but as it does not factor into our arguments in the coming chapters, we omit it.

For this reason we might simply define s by

$$s = \lim_{k \to \infty} -\frac{\log 2^k}{\log 3^{-k}} = \frac{\log 2}{\log 3}$$

which we well know to be the fractal dimension of the Cantor Set (see section 2.2.1). This motivates the following definition of the most well-known fractal dimension, the *box-counting dimension* (or more simply the *box dimension*). The following definition may be found in [22].

DEFINITION 10. Let (X, d) be a metric space and let $K \subseteq X$. Let $N_{2\delta}(K)$ denote the smallest number of open balls of radius δ that cover K. We define the upper and lower box-counting dimensions of K by

$$\overline{\dim}_B(K) = \limsup_{\delta \to 0^+} -\frac{\log N_{2\delta}(K)}{\log \delta}$$

and

$$\underline{\dim}_B(K) = \liminf_{\delta \to 0^+} -\frac{\log N_{2\delta}(K)}{\log \delta}$$

respectively. If the limits are finite and equal, then we call this the **box dimension** of K and denote it by $\dim_B(K)$.

Notice, however, that in our covering of $\mathcal{C}$ with cylinders we were not covering $\mathcal{C}$ with open balls of radius 3^{-k}, but rather closed ones. For this reason we would actually need 2^k open balls of radius $c_k 3^{-k}$ to cover $\mathcal{C}$ (where $c_k \to 1^+$ as $k \to \infty$), but this doesn't affect our estimate as

$$\lim_{k \to \infty} -\frac{\log 2^k}{\log c_k 3^{-k}} = \lim_{k \to \infty} -\frac{\log 2^k}{\log c_k + \log 3^{-k}} = \lim_{k \to \infty} -\frac{\log 2^k}{\log 3^{-k}} = \frac{\log 2}{\log 3}.$$

There is a less often used notion of dimension that will nevertheless be important in our discussion, and that has form very similar to that of box dimension.

17

DEFINITION 11. Let (X, d) be a metric space and let $K \subseteq X$. Let $\hat{N}_{2\delta}(K)$ denote the largest number of disjoint closed balls of radius δ with centers in K. We define the upper and lower entropy indices of K by

$$\overline{\dim}_{\mathcal{E}}(K) = \limsup_{\delta \to 0^+} -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta}$$

and

$$\underline{\dim}_{\mathcal{E}}(K) = \liminf_{\delta \to 0^+} -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta}$$

respectively. If the limits are finite and equal, then we call this the **entropy index** of K and denote it by $\dim_{\mathcal{E}}(K)$ (see [16]).

We refer to the collection of closed balls with radius δ and centers in K as a δ-*packing* of K. As one might expect, the box dimension and entropy index can take different values in an arbitrary metric space. A mildly surprising fact, however, is that they always take the same value in $\mathbb{R}^d$. The following lemma is given for $\mathbb{R}^2$ as Proposition 6.8.7 in [16], and its proof is only a trivial modification of the proof in [16].

LEMMA 1.1. *Let $K \subseteq \mathbb{R}^d$, then $\overline{\dim}_B K = \overline{\dim}_{\mathcal{E}} K$ and $\underline{\dim}_B K = \underline{\dim}_{\mathcal{E}} K$.*

As will be shown, the box-dimension and the Hausdorff dimension take the same values on certain types of fractal subsets of $\mathbb{R}^d$, in particular the ones whose hyperspaces are the subject of this study. Lemma 1.1 implies that the entropy index is also equal to the Hausdorff dimension of these types of fractal sets, a fact that will play a crucial role in the arguments of chapter 2.

1.2.2 Invariant Measures

Before we delve into the definition of Hausdorff measure that will lead us to the definition of Hausdorff dimension, we will consider a natural type of measure

on the invariant set that will play a central role in our dimension computations, namely the *invariant measure*. In order to define the invariant measure we introduce the idea of an *IFS with probabilities*. There is a fully developed theory of IFS with probabilities (see [17]), and we will develop the main results for IFS in order to get a flavor of invariant measures, before stating the analogous results for GDIFS.

DEFINITION 12. Let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be a self-similar IFS and let $p_e \in (0, 1)$ be such that

$$\sum_{e \in E} p_e = 1.$$

We call $\mathcal{F} = \{X, w_e, p_e\}_{e \in E}$ an **IFS with probabilities**.

Given this definition we can consider what we would like an invariant measure to be. The idea is simple: just as the invariant set is the unique set that is invariant under the action of the IFS, we seek a unique measure that is invariant under the action of the IFS with probabilities.

What is the action on a measure by the IFS with probabilities, however? As one might assume given the notation in definition 12, an IFS with probabilities applies map w_e with probability p_e. We can think level by level of the probability distribution that this action generates. Initially, we give X a measure of one, which we may write $\mu_0(X) = 1$. At level-1, we have applied map w_e with probability p_e, and we can think of this as distributing μ_0 among the X_e with weights p_e, i.e. $\mu_1(X_e) = p_e \mu_0(X)$ for each $e \in E$. Since $w_{e_1}^{-1}(X_{e_2}) = \emptyset$ for $e_1 \neq e_2$, and $w_{e_1}^{-1}(X_{e_2}) = X$ for $e_1 = e_2$, we can similarly write

$$\mu_1(X_{e'}) = \sum_{e \in E} p_e \mu_0(w_e^{-1}(X_{e'})).$$

19

Rather than specifying any particular cylinder or set in this statement, we can write

$$\mu_1 = \sum_{e \in E} p_e(\mu_0 \circ w_e^{-1}).$$

We want to think of the above statement as follows: the measure μ_1 is the result of the action of the IFS with probabilities on the measure μ_0. This is phrased in a way that leads to the definition of an invariant measure.

DEFINITION 13. Let $\mathcal{F} = \{X, w_e, p_e\}_{e \in E}$ be an IFS with probabilities. Suppose there exists a Borel measure μ on X such that

$$\mu = \sum_{e \in E} p_e(\mu \circ w_e^{-1}).$$

We call μ an **invariant measure** for the IFS.

The existence and uniqueness of invariant measures follows from the Contraction Mapping Principle just as the existence and uniqueness of invariant sets. Whereas the IFS defined a contraction on the hyperspace $\mathbb{H}(X)$, the IFS with probabilities will define a contraction on $\mathfrak{B}(X)$, the space of Borel probability measures on X. The space $\mathfrak{B}(X)$ is a metric space under the Hutchinson metric as defined in [30], and, as was the case with $\mathbb{H}(X)$, $\mathfrak{B}(X)$ is complete (resp. compact) whenever X is complete (resp. compact, see [4]). Hence the CMP applies (see [30]).

The analogous result holds for GDIFS (see [17]). We first define a GDIFS with probabilities.

DEFINITION 14. Let $\mathcal{F}_G = \{J_v, w_e\}_{v \in V, e \in V}$ be a self-similar GDIFS. Let $p_e \in (0, 1)$ satisfy

$$\sum_{e \in E_u} p_e = 1$$

for each $u \in V$. We call $\mathcal{F}_G = \{J_v, w_e, p_e\}_{v \in V, e \in E}$ a **GDIFS with probabilities**.

THEOREM 1.2. *Let* $\mathcal{F}_G = \{J_v, w_e, p_e\}_{v \in V, e \in E}$ *be a GDIFS with probabilities. There exists a unique list of measures* $(\mu_v)_{v \in V}$ *such that* $\mu_v \in \mathfrak{B}(J_v)$ *and*

$$\mu_u = \sum_{v \in V} \sum_{e \in E_{uv}} p_e(\mu_v \circ w_e^{-1})$$

for each $u \in V$. *The list* $(\mu_v)_{v \in V}$ *is known as the* **invariant measure list**.

There is an important property of the invariant measure that we will need for our arguments in Chapter 3. This property has to do with how the invariant measure measures cylinders (see Lemma 8.4 in [21]).

PROPOSITION 1.1. *Let* $\mathcal{F} = \{X, w_e, p_e\}_{e \in E}$ *be an IFS with probabilities and let* μ *be the unique invariant measure. Suppose* $\mathcal{F}$ *satisfies the SSC, then*

$$\mu(X_\sigma) = p_\sigma := p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n}$$

for each $\sigma \in E^*$.

The analogous result holds for graph-directed systems (see [19]).

PROPOSITION 1.2. *Let* $\mathcal{F}_G$ *be a GDIFS with probabilities and let* $(\mu_v)_{v \in V}$ *be the unique invariant measure list. Suppose* $\mathcal{F}_G$ *satisfies the graph SSC, then*

$$\mu_{i(\gamma)}(J_\gamma) = p_\gamma = p_{\gamma_1} p_{\gamma_2} \cdots p_{\gamma_n}$$

for each $\gamma \in E_A^*$.

Propositions 1.1 and 1.2 are key tools in the Hausdorff dimension lower bound computations for self-similar and graph-self-similar sets, respectively. In addition, they will be our primary tools in the counting arguments of chapters 3 and 4.

1.2.3 Hausdorff Dimension

The box-dimension and the entropy index provide intuitive geometric ways to view dimension. In terms of measurement, however, they have a major drawback: neither defines a measure. A more subtle construction is necessary in order to define a measure, which leads us to the Hausdorff measure. The Hausdorff measure is the cornerstone tool in the study of Fractal Geometry, and has been studied extensively for a century (see [16], [17], [21], [22], [23], [29], [30], [45]).

We will first describe a general construction known as **Method I** which we will use to construct measures in Chapter 3. Then, as is done in [16], we construct Hausdorff measure as a generalization of Method I.

THEOREM 1.3 (Method I Theorem). *Let X be any set, let $\mathcal{A}$ be a collection of sets that covers X and let $c : \mathcal{A} \to [0, \infty]$ be any function. There exists an outer measure $\overline{\mathcal{M}}$ on X such that*

1. *$\overline{\mathcal{M}}(A) \leq c(A)$ for every $A \in \mathcal{A}$;*

2. *If $\overline{\mathcal{N}}$ is any other outer measure with $\overline{\mathcal{N}}(A) \leq c(A)$ for every $A \in \mathcal{A}$, then $\overline{\mathcal{N}}(U) \leq \overline{\mathcal{M}}(U)$ for every $U \subseteq X$.*

Furthermore, this outer measure is unique.

We note that if X is a metric space, the outer measure given by the Method I theorem is not necessarily a Borel measure. This in particular is what will lead us to the Hausdorff measure. For now, however, we will give some conditions under which a Method I measure is, in fact, a Borel measure. These conditions will allow us to define Borel measures on hyperspaces in Chapter 3. The following is standard measure theory that has been tailored to our purposes.

DEFINITION 15. Suppose (X, d) is a compact metric space and $\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_k$ is a collection of compact subsets of X with the following properties:

- $\mathcal{A}_0 = \{X\}$

- $\mathcal{A}_k$ is finite for each $k \geq 1$

- For every $k \geq 0$, $A \cap B = \emptyset$ for every $A, B \in \mathcal{A}_k$ with $A \neq B$

- For every $A \in \mathcal{A}_k$ there exists $B \in \mathcal{A}_{k+1}$ such that $B \subset A$

- For every $B \in \mathcal{A}_{k+1}$ there exists $A \in \mathcal{A}_k$ such that $B \subset A$

- $\max_{A \in \mathcal{A}_k} |A| \to 0$ as $k \to \infty$

We call such a collection $\mathcal{A}$ a **Cantor Net**.

It is easy to see that in the construction of the Cantor Set using the Cantor IFS, the collection of cylinders forms a Cantor net as described in Definition 15. We will not want to consider the "full" collection of (hyper)cylinders when constructing our measure on the hyperspace, but we will want to consider a Cantor net subcollection nonetheless.

Let $\mathcal{A}$ be a Cantor net. For $A \in \mathcal{A}_{k-1}$ define $\mathcal{A}_{k,A} = \{B \in \mathcal{A}_k : B \subset A\}$. Let κ be a **mass distributing function**, i.e. κ is such that $\kappa(X) = 1$ and

$$\kappa(B) = \frac{\kappa(A)}{\#\mathcal{A}_{k,A}}$$

for all $A \in \mathcal{A}_{k-1}$, $B \in \mathcal{A}_{k,A}$. Notice that κ is finitely additive on $\mathcal{A}$ since

$$\sum_{B \in \mathcal{A}_{k,A}} \kappa(B) = \#\mathcal{A}_{k,A} \left(\frac{\kappa(A)}{\#\mathcal{A}_{k,A}} \right) = \kappa(A)$$

for $A \in \mathcal{A}_{k-1}$. It follows by induction for $A \in \mathcal{A}_j$, $j \leq k - 1$, by simply grouping terms.

LEMMA 1.2. *Let $\mathcal{A}$ be a Cantor net and κ be a mass distributing function on $\mathcal{A}$. The Method I outer measure $\overline{\mathcal{M}}$ on X that is generated by κ is a metric outer measure.*

Proof. Define $\mathcal{A}_{\geq k}$ by

$$\mathcal{A}_{\geq k} = \bigcup_{j=k}^{\infty} \mathcal{A}_j.$$

Let $\mathcal{F} \subset \mathcal{A}$ be a countable cover of X. We claim that for every $k \geq 0$ there exists $\mathcal{G}_k \subset \mathcal{A}_{\geq k}$ such that

$$\sum_{A \in \mathcal{F}} \kappa(A) = \sum_{A \in \mathcal{G}_k} \kappa(A).$$

Let $k \geq 0$ be arbitrary. If $\mathcal{F} \subset \mathcal{A}_{\geq k}$ then the claim holds with $\mathcal{F} = \mathcal{G}_k$, so suppose there exists $A \in F$ such that $A \in \mathcal{A}_j$ some $j < k$. By definition of κ we have

$$\kappa(A) = \sum_{B \in \mathcal{A}_{k,A}} \kappa(B).$$

Let $\mathcal{G}_k$ be equal to $\mathcal{F}$, but remove every such A from $\mathcal{F}$ and replace it with the sets from $\mathcal{A}_{k,A}$. This will prove the claim.

Now let $U_1, U_2 \subset X$ be such that $\inf\{d(x,y) : x \in U_1, y \in U_2\} > 0$. Let $\{\mathcal{F}_k\}_{k \geq 0}$ be such that $\mathcal{F}_k \subset \mathcal{A}$ for each $k \geq 0$ and

$$\sum_{A \in \mathcal{F}_k} \kappa(A) \to \overline{\mathcal{M}}(U_1 \cup U_2).$$

By the above claim we may assume without loss of generality that $\mathcal{F}_k \subset \mathcal{A}_{\geq k}$ for each $k \geq 0$. Since $\max_{A \in \mathcal{A}_k} |A| \to 0$ as $k \to \infty$, we can choose $k_0 \in \mathbb{N}$ such that

$$\max_{A \in \mathcal{A}_k} |A| < \inf\{d(x,y) : x \in U_1, y \in U_2\}$$

for all $k \geq k_0$. It follows that for all $A \in \mathcal{F}_k$, $k \geq k_0$, we have either $A \cap U_1 \neq \emptyset, A \cap U_2 = \emptyset$ or $A \cap U_1 = \emptyset, A \cap U_2 \neq \emptyset$. We may then write

$$\mathcal{F}_k = \mathcal{F}_k^{U_1} \cup \mathcal{F}_k^{U_2}$$

such that $\mathcal{F}_k^{U_i}$ covers U_i and is disjoint from U_j. Finally let $\epsilon > 0$ be arbitrary. Choose $k_1 > k_0$ so that

$$\sum_{A \in \mathcal{F}_{k_1}} \kappa(A) \leq \overline{\mathcal{M}}(U_1 \cup U_2) + \epsilon.$$

We also have

$$\sum_{A \in \mathcal{F}_{k_1}} \kappa(A) = \sum_{A \in \mathcal{F}_{k_1}^{U_1}} \kappa(A) + \sum_{A \in \mathcal{F}_{k_1}^{U_2}} \kappa(A) \geq \overline{\mathcal{M}}(U_1) + \overline{\mathcal{M}}(U_2).$$

It follows that $\overline{\mathcal{M}}(U_1 \cup U_2) \geq \overline{\mathcal{M}}(U_1) + \overline{\mathcal{M}}(U_2)$, hence $\overline{\mathcal{M}}(U_1 \cup U_2) = \overline{\mathcal{M}}(U_1) + \overline{\mathcal{M}}(U_2)$, which completes the proof. $\square$

REMARK 1. It follows from Lemma 1.2 that $\overline{\mathcal{M}}$ is a Borel measure when restricted to its measurable sets (see [16]). Since the sets from $\mathcal{A}$ are all compact, it also follows that every set from $\mathcal{A}$ is $\mathcal{M}$-measurable. Also, if $A \in \mathcal{A}$ (i.e. $A \in \mathcal{A}_{k_0}$ some $k_0 \geq 1$) and $\mathcal{F} \subset \mathcal{A}$ covers A, we can, as in the proof of Lemma 1.2, assume without loss of generality that $\mathcal{F} \subset \mathcal{A}_{\geq k_0, A}$. By the construction of κ it follows that

$$\sum_{B \in \mathcal{F}} \kappa(B) = \kappa(A)$$

and hence $\mathcal{M}(A) = \kappa(A)$ for all $A \in \mathcal{A}$.

In chapter 3 we construct a measure $\mathcal{M}$ on $\mathbb{H}(K)$ using a Cantor net on $\mathbb{H}(K)$ and a mass distributing function κ, which is the same method used in [39]. The fact that $\mathcal{M}(A) = \kappa(A)$ for each $A \in \mathcal{A}$ is implicitly used in the dimension computation; this fact is left undiscussed in [39], and we have provided this discussion above.

Let us now turn our attention to the Hausdorff measure. Let (X, d) be a metric space and let $F \subseteq X$. For $s \geq 0$ and $\delta > 0$, let

$$\mathcal{H}_\delta^s(F) = \inf \left\{ \sum_{U \in \mathcal{G}} |U|^s : \mathcal{G} \text{ is a } \delta - \text{cover of } F \right\}$$

So $\mathcal{H}_\delta^s$ is the Method I measure on X generated by the set function $c : \mathcal{A}_\delta \to \mathbb{R}^+$ defined by $c(t) = t^s$, where

$$\mathcal{A}_\delta = \bigcup_{\mathcal{G}} \mathcal{G}$$

is taken over all δ-covers of F.

The first observation here is that $\mathcal{H}_\delta^s$ is not necessarily a Borel measure. The second is that $\mathcal{A}_{\delta'} \subseteq \mathcal{A}_\delta$ for $\delta' \leq \delta$, and hence $\mathcal{H}_{\delta'}^s \geq \mathcal{H}_\delta^s$. So if we want the "largest" measure defined in this way, we must take a limit. We thus define the **s-dimensional Hausdorff measure** of F to be

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^s(F) = \sup_{\delta > 0} \mathcal{H}_\delta^s(F)$$

Hausdorff measures were introduced by F. Hausdorff in [29], and one can find an extensive study of Hausdorff measures in [45]. It is an example of what is called a Method II measure in [16], hence it is a metric outer measure and a Borel measure.

The key reason that $\mathcal{H}^s$ is such an important measure is that it is metrically invariant. By definition, if $f : X \to X$ is an isometry, then $|X| = |f(X)|$. This necessarily means that there is a one-to-one correspondence of δ-coverings of $f(X)$ and X, hence $\mathcal{H}^s(f(X)) = \mathcal{H}^s(X)$. One hopes the size (dimension) of sets is independent of location and depends only on the geometry of the set. Metric invariance of $\mathcal{H}^s$ guarantees this and is the primary reason that $\mathcal{H}^s$ is more important than, say, the invariant measure (which is *not* metrically invariant).

Now that we have an appropriately, and rigorously, defined s-dimensional measure, we can realize a formal definition of dimension. It is easy to show (see [45]) that there exists a unique number $\dim_H F \geq 0$ such that $\mathcal{H}^s(F) = 0$ for $s > \dim_H F$ and $\mathcal{H}^s(F) > 0$ for $s < \dim_H F$. It is this unique number that we call the **Hausdorff dimension of F**.

While $\mathcal{H}^s$ preserves measure under isometries, it only needs to approximately preserve measure under a given type of map to preserve dimension under that map. The most general type of map that satisfies this approximate preservation of measure is a lipeomorphism.

DEFINITION 16. Let (X, d_x) and (Y, d_y) be metric spaces and let $f : X \to Y$. Suppose there exists a constant $c > 1$ such that

$$\frac{1}{c} d_x(a, b) \leq d_y(f(a), f(b)) \leq c d_x(a, b)$$

for all $a, b \in X$. Such a map f is called a **lipeomorphism**.

It is equivalent to say that for a map $f : X \to Y$, if f is Lipschitz and f^{-1} exists and is lipschitz, then f is a lipeomorphism. The following well-known proposition follows directly from the definitions of $\mathcal{H}^s$ and lipeomorphism (see [45]).

PROPOSITION 1.3. *Let (X, d_x) and (Y, d_y) be metric spaces. If X and Y are lipeomorphic, then $\dim_H X = \dim_H Y$.*

McClure uses Proposition 1.3 in his Cantor set approximations in [39], as we will see in §2.1.1. Having defined Hausdorff dimension and given sufficient conditions under which two sets have the same Hausdorff dimension, we still must ask how to directly compute the Hausdorff dimension of a set. This turns out to be quite a subtle issue.

To show $\mathcal{H}^s(F) < \infty$ for a certain $s \geq 0$, it suffices to find a particular $M > 0$ and particular δ-coverings of F for which $\mathcal{H}^s_\delta(F) \leq M$ for all $\delta > 0$. When dealing with IFSs, the collection of cylinder sets is usually an appropriate collection of δ-coverings for proving this upper bound. Showing that $\mathcal{H}^s(F) > 0$ turns out to be quite difficult, however, and there is no uniform method of argument in the

literature. One possible way to argue that $\mathcal{H}^s_\delta(F) > 0$ is to define a measure μ on F so that $\mu(U) \approx |U|^s$ for every set $U \subset F$ with $|U| \leq \delta$. This is known as the Mass Distribution Principle (MDP).

PROPOSITION 1.4 (Mass Distribution Principle). *Let μ be a Borel probability measure (or* mass distribution*) on F and suppose that for $s > 0$ there exist constants $c > 0$ and $\delta > 0$ such that*

$$\mu(U) \leq c|U|^s$$

for all sets U with $|U| \leq \delta$. Then $\mathcal{H}^s(F) \geq \frac{1}{c}\mu(U)$.

The MDP is sufficient for computing the lower bounds of $\mathcal{H}^s(F)$ for relatively "nice" sets F, e.g. self-similar and graph-self-similar sets. For sets with a more locally heterogeneous geometric structure, however, we need to construct a measure μ with properties that vary locally, and hence need a proposition to take the place of proposition 1.4. The following proposition which uses pointwise density properties of μ will serve this purpose (see [23]).

PROPOSITION 1.5. *Let $F \subset \mathbb{R}^d$ be a Borel set, let μ be a finite Borel measure on $\mathbb{R}^d$ and $0 < c < \infty$.*

(a) If $\limsup_{\epsilon \to 0} \mu(B_\epsilon(x))/r^s \leq c$ for every $x \in F$ then $\mathcal{H}^s(F) \geq \mu(F)/c$

(b) If $\limsup_{\epsilon \to 0} \mu(B_\epsilon(x))/r^s \geq c$ for every $x \in F$ then $\mathcal{H}^s(F) \leq 2^s\mu(F)/c$

Propositions 1.4 and 1.5 are sufficient for computing the lower bounds of self-similar, graph-self-similar, and self-conformal fractals in $\mathbb{R}^d$, which are the classes of fractals whose hyperspaces we concern ourselves with here. The computation of the Hausdorff dimension of an arbitrary subset of a metric space is an open question that is an ongoing topic in current research.

For many sets, however, it is known that it is not the case that $0 < \mathcal{H}^{s_0}(F) < \infty$ where $s_0 = \dim_H F$. Nor is it the case for many sets that $\mathcal{H}^{s_0}(F) > 0$ and F is $\mathcal{H}^{s_0}$-σ-finite.[6] This raises the question: is there a similar way to construct a Hausdorff measure that *can* measure these sets? While the answer is in general "no", we have a more general way of constructing measures for the $\mathcal{A}_\delta$ in Method I. In particular, theorem 1.3 yields a measure for *any* set function $c : \mathcal{A}_\delta \to \mathbb{R}^+$. In the setting of Hausdorff measures, however, we would like these set functions to satisfy a few conditions.

DEFINITION 17. Fix $\gamma > 0$. We define the space of **Hausdorff gauge functions**, Φ, to be the collection of all continuous, non-decreasing functions $\phi : [0, \gamma) \to \mathbb{R}^+$ such that $\phi(0) = 0$.

We can put a partial order on Φ by comparing the asymptotic behavior of two functions of Φ near zero. Let $\phi, \psi \in \Phi$, then

- $\phi \prec \psi$ if $\lim_{t \searrow 0} \frac{\psi(t)}{\phi(t)} = 0$

- $\phi \preceq \psi$ if $\limsup_{t \searrow 0} \frac{\psi(t)}{\phi(t)} < \infty$

- $\phi \asymp \psi$ if $0 < \liminf_{t \searrow 0} \frac{\psi(t)}{\phi(t)} < \limsup_{t \searrow 0} \frac{\psi(t)}{\phi(t)} < \infty$.

This is far from a total order as there exist many pairs $\phi, \psi \in \Phi$ for which

$$\liminf_{t \to 0+} \frac{\psi(t)}{\phi(t)} = 0$$

but

$$\limsup_{t \to 0+} \frac{\psi(t)}{\phi(t)} = \infty.$$

Such a pair is incomparable under $\prec$.

---

[6]In fact, it may be the case that "most" sets do not satisfy this property, see [20].

Now let (X, d) be a metric space and let $F \subseteq X$. For $\phi \in \Phi$ and $\delta > 0$, let

$$\mathcal{H}_\delta^\phi(F) = \inf \left\{ \sum_{U \in \mathcal{G}} \phi(|U|) : \mathcal{G} \text{ is a } \delta - \text{cover of } F \right\}.$$

As with the s-dimensional Hausdorff measure, we define the **ϕ-dimensional Hausdorff measure of F** to be

$$\mathcal{H}^\phi(F) = \lim_{\delta \to 0} \mathcal{H}_\delta^\phi(F) = \sup_{\delta > 0} \mathcal{H}_\delta^\phi(F).$$

While there exists a unique $s_0 \geq 0$ such that $\mathcal{H}^{s_0}(F) > 0$ for $s < s_0$ and $\mathcal{H}^{s_0}(F) < \infty$ for $s > s_0$, there does not exist such a gauge function $\phi_0 \in \Phi$. This is mainly because $\prec$ is not a total order on Φ, or as McClure puts it in [39], Φ is too "rich" of a set. The best we can do is the following.

LEMMA 1.3 (Rogers, 1970). *Let (X, d) be a metric space, $F \subseteq X$, and $\phi, \psi \in \Phi$. The following statements hold:*

(a) If $\mathcal{H}^\phi(F)$ is σ-finite and $\phi \prec \psi$, then $\mathcal{H}^\psi(F) = 0$.

(b) If $\mathcal{H}^\phi(F) > 0$ and $\phi \succ \psi$, then $\mathcal{H}^\psi(F)$ is non-σ-finite.

While Lemma 1.3 does not give a unique notion of ϕ-dimension, it does give us the following definition (see [45]).

DEFINITION 18. Let (X, d) be a metric space and let $F \subseteq X$. The **generalized Hausdorff dimension of F** is the partition of Φ given by

$$\Phi_\infty(F) = \left\{ \phi \in \Phi : F \text{ is of non} - \sigma - \text{finite } \mathcal{H}^\phi \text{ measure} \right\}$$

$$\Phi_+(F) = \left\{ \phi \in \Phi : \mathcal{H}^\phi(F) > 0 \text{ and } F \text{ is of } \sigma - \text{finite } \mathcal{H}^\phi \text{ measure} \right\}$$

$$\Phi_0(F) = \left\{ \phi \in \Phi : \mathcal{H}^\phi(F) = 0 \right\}.$$

We write $\dim_{GH} F = (\Phi_\infty(F), \Phi_+(F), \Phi_0(F))$ for the generalized Hausdorff dimension.

Explicitly determining this partition is a complicated proposition, and for this reason most researchers instead construct a one-parameter family in Φ, which hopefully has a unique ϕ_0 for which $\mathcal{H}^\phi(F) > 0$ for $\phi \prec \phi_0$ and $\mathcal{H}^\phi(F) = 0$ for $\phi \succ \phi_0$. In Chapter 3 we will be concerned with constructing such a one-parameter family for hyperspace of graph-self-similar and self-conformal fractals. The following corollary to Lemma 1.3, which is given in [39] in a slightly altered form, will aid us in constructing the upper half of this family.

COROLLARY 1.1. *Let (X, d) be a metric space, $F \subseteq X$ be a Borel set, and $\psi_{N,s}(t) = 2^{-N(1/t)^s}$. If there exists $N > 0$, $s_0 > 0$ such that $\mathcal{H}^{\psi_{N,s_0}}(F) < \infty$, then $\mathcal{H}^{\psi_{1,s}}(F) = 0$ for all $s > s_0$.*

We note (as in [39]) that $\psi_{N,s_1} \prec \psi_{N,s_2}$ for all $s_1 < s_2$ and all $N \in \mathbb{N}$, and $\psi_{N_1,s} \prec \psi_{N_2,s}$ for all $N_1 < N_2$ and all $s \geq 0$.

As the hyperspace of a fractal has very strong local heterogeneity, we will need the generalized dimension analogue of Proposition 1.5 in order to compute the lower half of our one-parameter family. We first need a notion of density with respect to dimension functions.

DEFINITION 19. Let (X, d) be a separable metric space and let μ be a Borel measure on X. For $x \in X$ and $\delta > 0$ let

$$\mu_\delta(x) = \sup\left\{\mu(U) : x \in U,\ U \text{ is a Borel set},\ |U| \leq \delta\right\}.$$

Let $\delta_k \searrow 0$. We define

$$\overline{D}_\mu^\phi(x) = \limsup_{k \to \infty} \frac{\mu_{\delta_k}(x)}{\phi(\delta_k)}$$

to be the **upper McClure ϕ-density of μ at x**.

The correct analogue of Proposition 1.5 will then be the following lemma given in [39].

LEMMA 1.4. *Let (X, d) be a separable metric space and let $F \subseteq X$ be a Borel set. Let $\delta_k \searrow 0$ and suppose $0 < N_1 < \infty$ and $\phi, \psi \in \Phi$ satisfy $\phi(\delta_k) \leq N_1 \psi(\delta_k)$ for every $k \in \mathbb{N}$. If there exists a Borel measure μ and a constant $0 < N_2 < \infty$ such that $\mu(F) > 0$ and*

$$\overline{D}_\mu^\phi(x) < N_2$$

for every $x \in F$, then $\mathcal{H}^\psi(F) > \frac{\mu(F)}{N_1 N_2} > 0$.

Combining Corollary 1.1 with Lemma 1.4 gives us the following corollary, which is given in [39] and which we will apply in chapter 3 to get the lower half of our one-parameter family of gauge functions.

COROLLARY 1.2. *Let (X, d) be a metric space, $F \subseteq X$ be a Borel set, $M > 0$ and $\phi_s(t) = 2^{-(1/t)^s}$. If $\overline{D}_\mu^{\phi_s}(x) < M < \infty$ for all $x \in F$, $s < s_0$, then $\mathcal{H}^{\phi_s}(F) > 0$ for all $s < s_0$.*

1.3 Dimension Computations for Invariant Sets

In this section we review the dimension computations for self-similar, graph-self-similar, and self-conformal sets. We do this both to give a flavor of the mathematics involved in dimension computations, and to set up some geometric propositions and lemmas that will come in to play in chapters 2 and 3. Recall the definition of the Open Set Condition (OSC) given in Definition 3, and note that the OSC is assumed throughout this section.

1.3.1 Self-Similar IFS

The use of invariant measures for the purpose of computing dimension was introduced in [30] and has been the primary method for computing lower bounds

of the Hausdorff dimensions of fractal sets. We present here the exact and concise presentation from [21], with the only difference being the notation.

In order to relate an invariant measure to a Hausdorff measure by way of Proposition 1.4, one needs two ingredients:

1. One needs precise controls over what values the invariant measure takes on the cylinder sets; and

2. One needs precise controls over the number of cylinder sets that a given set of diameter δ can intersect.

The first ingredient is taken care of (for self-similar sets) by Proposition 1.1. The second ingredient is taken care of by the following lemma, which can be found in [21].

LEMMA 1.5. *Let $\{V_i\}$ be a collection of open subsets of $\mathbb{R}^d$ such that each V_i contains a ball of radius $c_1\delta$ and is contained in a ball of radius $c_2\delta$. Then any ball B of radius δ intersects no more than $q(c_1, c_2) = (1 + 2c_2)^d c_1^{-d}$ sets from the collection $\{\overline{V}_i\}$.*

We now need a candidate for the Hausdorff dimension of a fractal set. The basic premise is this: the number of cylinders grows at one rate, and the diameters of the cylinders grow at a different rate such that the number of cylinders of a given diameter times the diameter (i.e. the measure) goes to zero asymptotically. The question is then: can we rescale the diameters of the cylinders in a way so as to have these rates offset one another?

In the case of self-similar sets, there is a simple way to do this rescaling. Recall that the ratio r_e associated to the map w_e is defined to be the minimum of all constants r satisfying $d(w_e(x), w_e(y)) \leq r d(x, y)$ for all $x, y \in X$. We then give the following definition:

33

DEFINITION 20. Let (X, d) be a metric space and let $\mathcal{F} = \{X, w_e\}_{e \in E}$. Define $s_0 > 0$ to be the unique solution to the equation

$$\sum_{e \in E} r_e^s = 1.$$

We call the value s_0 the **similarity dimension** of the IFS.

It turns out that the similarity dimension is precisely equal to the Hausdorff dimension, as is shown by the following theorem. We include the proof from [21] as it provides an archetypal example of a general dimension computation.

THEOREM 1.4. *Let $\mathcal{F}$ be a self-similar IFS in $\mathbb{R}^d$ with unique invariant set K. If $\mathcal{F}$ satisfies the OSC, then the similarity dimension and the Hausdorff dimension of K coincide. In particular $0 < \mathcal{H}^{s_0}(K) < \infty$ where s_0 is the similarity dimension.*

Proof. We write $r_{\max} = \max_{e \in E}\{r_e\}$ and $r_{\min} = \min_{e \in E}\{r_e\}$. Fix $\delta > 0$ and choose $n \geq 1$ so that $(r_{\max})^n < \delta$. It follows that $\{X_\sigma : \sigma \in E^n\}$ is a δ-cover of K, and so

$$\mathcal{H}_\delta^{s_0}(K) \leq \sum_{\sigma \in E^n} |X_\sigma|^{s_0} = \sum_{\sigma \in E^n} r_\sigma^{s_0} = \left(\sum_{e \in E} r_e^{s_0}\right)^n = 1.$$

Since δ was arbitrary, it follows that $\mathcal{H}^{s_0}(K) \leq 1$.

Now for the lower bound, we let μ be the invariant measure on K generated by the probabilities $p_e = r_e^{s_0}$. Let $V = \text{int}(X)$ be the open set satisfying the OSC. Since U is open, it contains a ball of radius c_1 and is contained in a ball of radius c_2. Fix $\delta > 0$ and let

$$L = \{\sigma \in E^* : r_\sigma \leq \delta < r_{\sigma-}\}$$

and consider the collection

$$\mathcal{L} = \{V_\sigma : \sigma \in L\}.$$

It follows from the OSC that $\mathcal{L}$ is a pairwise disjoint collection of sets. Also, each set in $\mathcal{L}$ contains a ball of radius $c_1 r_{\min}\delta$ and is contained in a ball of radius $c_2\delta$. By

Lemma 1.5, any ball B of radius δ intersects no more than $q = (1+2c_2)^d c_1^{-d} (r_{\min})^{-d}$ sets from the collection $\{\overline{V}_\sigma : \sigma \in L\} = \{X_\sigma : \sigma \in L\}$. Since $\operatorname{supp}(\mu) \subseteq K$, it follows that

$$\mu(B) \leq \sum_{\substack{\sigma \in L \\ X_\sigma \cap B \neq \emptyset}} \mu(X_\sigma) = \sum_{\substack{\sigma \in L \\ X_\sigma \cap B \neq \emptyset}} r_\sigma^{s_0} \leq q \delta^{s_0} = q 2^{-s_0} |B|^{s_0}.$$

Given any covering $\{U_i\}$ of K, we may cover K by balls $\{B_i\}$ with $|B_i| \leq 2|U_i|$, so

$$1 = \mu(K) \leq \sum \mu(B_i) \leq q 2^{-s_0} \sum |B_i|^{s_0} \leq q \sum |U_i|^{s_0}.$$

Hence $\mathcal{H}_\delta^{s_0}(K) \geq q^{-1}$, and as δ was arbitrary, we have $\mathcal{H}^{s_0}(K) > q^{-1} > 0$. $\qquad \square$

Before moving to GDIFS, we give a well-known corollary that may be found in [22] for example. This corollary will be crucial to our Cantor set approximations in chapter 2.

COROLLARY 1.3. $\dim_H K = \dim_{\mathcal{E}} K = \dim_B K$.

1.3.2 Self-Similar GDIFS

With our understanding of Theorem 1.4, we might expect that a GDIFS consisting of similitudes has a similarity dimension like that of a self-similar IFS, and that with this similarity dimension we can construct a finite and positive measure that is equivalent to the Hausdorff measure. This is the case to an extent, but certain modifications need to be made. We motivate these modifications by reconsidering the similarity dimension of self-similar IFSs.

Consider an IFS $\{X, w_e\}_{e \in E}$. We have been assuming "without loss of generality" that $|X| = 1$, but what happens if we don't assume $|X| = 1$? What would be our similarity dimension in this setting? The idea behind the similarity

dimension is to "preserve the diameter" when it is actually being decreased due to the action of the IFS. We want to find the correct magnification value so that the measure of the set remains positive and bounded. If the diameter is $|X| \neq 1$, we might expect this to mean we choose $s_0 \geq 0$ such that

$$\sum_{e \in E} |X_e|^{s_0} = |X|.$$

The problem with this definition is that we need the same rule to hold at an arbitrary level, but if the diameter of X_τ is measured as $|X_\tau|^{s_0}$ we then want

$$\sum_{\gamma \in E^n} |X_{\tau\gamma}|^{s_0} = |X_\tau|^{s_0}$$

for all $n \geq 1$ and all $\tau \in E^*$. As E^* includes the empty string, s_0 should in fact be defined by

$$\sum_{e \in E} |X_e|^{s_0} = |X|^{s_0}.$$

This, of course, gives the usual value of s_0 since $|w_e(X)|^{s_0} = r_e^{s_0} |X|^{s_0}$.

Suppose we want to generalize this idea to GDIFS. Given our definition of s_0 above, we should maybe define

$$\sum_{v \in V} \sum_{e \in E_{uv}} |w_e(J_v)|^{s_0} = |J_u|^{s_0}$$

or equivalently

$$\sum_{v \in V} \sum_{e \in E_{uv}} r_e^{s_0} |J_v|^{s_0} = |J_u|^{s_0}.$$

The problem here is that we get a different value of s_0 for each $u \in V$. However, suppose there exist unique numbers $\lambda_u > 0$ such that

$$\sum_{v \in V} \sum_{e \in E_{uv}} r_e^{s_0} \lambda_v^{s_0} = \lambda_u^{s_0}$$

for each $u \in V$.[7] As a constant rescaling of the metrics for the seed sets wouldn't change their Hausdorff dimensions, we could define a new metric on J_u by

$$\tilde{d}_u(x, y) = \frac{\lambda_u d_u(x, y)}{|J_u|}.$$

The metric $\tilde{d}_u$ differs from d_u only by a constant, and $|J_u|_{\tilde{d}_u} = \lambda_u$. Thus, upon existence of the numbers $\{\lambda_v\}_{v \in V}$ there would exist a single value $s_0 > 0$ such that

$$\sum_{v \in V} \sum_{e \in E_{uv}} |w_e(J_v)|^{s_0} = \sum_{v \in V} \sum_{e \in E_{uv}} r_e^{s_0} |J_v|^{s_0} = |J_u|^{s_0}$$

for each $u \in V$, where the diameters are computed under the metrics $\{\tilde{d}_v\}_{v \in V}$. **We will henceforth assume the seed sets are equipped with the rescaled metrics $\{\tilde{d}_v\}_{v \in V}$.**

The above property is precisely what we need to define a measure that is equivalent to the Hausdorff measure, but we first need some theory to assert the existence and uniqueness of the numbers $\{\lambda_v\}_{v \in V}$. Notice that an equivalent way of phrasing this is that there exists a positive eigenvector $\vec{\lambda} = (\lambda_v)_{v \in V}$ such that

$$M_{s_0} \vec{\lambda} = \vec{\lambda}$$

where $M_s = (m_{uv})_{u,v \in V}$ is defined by

$$m_{uv} = \sum_{e \in E_{uv}} r_e^s.$$

We may then rephrase the existence question in terms of finding $s_0 \geq 0$ such that M_{s_0} has an eigenvalue of 1. The theory that handles this issue is the Perron-Frobenius theory of non-negative matrices. We review this theory as presented in [18]. We first require a couple of definitions:

---

[7]Some texts will choose the λ_v such that the invariance statement holds with λ_v as opposed to $\lambda_v^{s_0}$. These are simply conventions of which we choose the latter so as to get equivalence of the Perron-Numbers to the diameters of the seed sets under the metrics $\{\tilde{d}_v\}_{v \in V}$.

DEFINITION 21. Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a non-negative $n \times n$ matrix (i.e. an $n \times n$ matrix with all non-negative entries).

1. The **Spectral Radius** of A is

$$\Phi = \sup\{|z| : z \in \mathbb{C} \text{ is an eigenvalue of } A\}.$$

2. A is called **reducible** if and only if $\{1, \ldots, n\}$ can be partitioned into two sets I and J such that $a_{ij} = 0$ for all $i \in I$, $j \in J$. A is **irreducible** if and only if it is not reducible.

The Perron-Frobenius theory of non-negative matrices has particularly strong results for matrices that are irreducible. The following proposition allow us to access the full strength of these results when dealing with GDIFS.

PROPOSITION 1.6. *Let (V, E, i, t, r) be a Mauldin-Williams graph and let M_s be defined as above. If (V, E, i, t, r) is strongly connected, then M_s is irreducible for all $s > 0$.*

We now state the needed results from Perron-Frobenius theory.

THEOREM 1.5. *Let $A = (a_{ij})$ be a non-negative, irreducible $n \times n$ matrix. All of the following statements hold:*

1. *The spectral radius, Φ, of A is also an eigenvalue of A. Furthermore, there exists a strictly positive eigenvector $\vec{x}$ with $A\vec{x} = \Phi\vec{x}$.*

2. *If $\vec{x} \geq 0$, $\vec{x} \neq 0$, and $A\vec{x} = \Phi'\vec{x}$ for some $\Phi' \in \mathbb{R}$, then $\Phi' = \Phi$.*

3. *If $B = (b_{ij})$ is a non-negative $n \times n$ matrix such that $a_{ij} \geq b_{ij}$ for all $1 \leq i, j \leq n$, then $\Phi_A \geq \Phi_B$.*

Part one of Theorem 1.5 is known by itself as the **Perron-Frobenius The-orem** and can be found in [41] (parts 2 and 3 may be found in [25] and [48], respectively). A simple application of Theorem 1.5 along with the Intermediate Value Theorem yields the following lemma, which gives us our candidate dimension for GDIFS.

LEMMA 1.6. *Let (V, E, i, t, r) be a Mauldin-Williams graph and consider M_s. Let $\Phi(s)$ be the spectral radius of M_s. There exists $s_0 > 0$ such that $\Phi(s_0) = 1$.*

REMARK 2. By Lemma 1.6 and Theorem 1.5(1) there exists a strictly positive eigenvector $\vec{\lambda} = (\lambda_v)_{v \in V}$ such that $M_{s_0} \vec{\lambda} = \vec{\lambda}$, i.e.

$$\sum_{v \in V} \sum_{e \in E_{uv}} r_e^{s_0} \lambda_v^{s_0} = \lambda_u^{s_0}.$$

We now define the probabilities p_e by

$$p_e = r_e^{s_0} \left(\frac{\lambda_{t(e)}}{\lambda_{i(e)}} \right)^{s_0}$$

and let $(\mu_v)_{v \in V}$ be the corresponding unique invariant measure list as given by Proposition 1.2. Notice that

$$p_\gamma = p_{\gamma_1} p_{\gamma_2} \cdots p_{\gamma_n} = r_{\gamma_1}^{s_0} \left(\frac{\lambda_{t(\gamma_1)}}{\lambda_{i(\gamma_1)}} \right)^{s_0} \cdot r_{\gamma_2}^{s_0} \left(\frac{\lambda_{t(\gamma_2)}}{\lambda_{i(\gamma_2)}} \right)^{s_0} \cdots r_{\gamma_n}^{s_0} \left(\frac{\lambda_{t(\gamma_n)}}{\lambda_{i(\gamma_n)}} \right)^{s_0}.$$

Since $t(\gamma_j) = i(\gamma_{j+1})$ for each j, this simplifies to

$$\mu_{i(\gamma)}(J_\gamma) = p_\gamma = r_\gamma^{s_0} \left(\frac{\lambda_{t(\gamma)}}{\lambda_{i(\gamma)}} \right)^{s_0}$$

for each $\gamma \in E_A^*$. The following may be found in [36].

THEOREM 1.6. *Let $s_0 > 0$ be the unique value such that $\Phi(s_0) = 1$, then $\dim_H K_v = s_0$ for each $v \in V$.*

We simply note that given the structure of the invariant measure, Theorem 1.6 will follow in much the same way as Theorem 1.4. The bulk of the extra work in the jump from IFS to GDIFS is in showing that one can construct the invariant measure list so as to satisfy the appropriate properties. We have the analogous corollary to Corollary 1.3 for GDIFS. Again, this result is well known (see [36]) and we state it without proof.

COROLLARY 1.4. *Let $\mathcal{F}_G$ be a self-similar GDIFS satisfying the graph OSC with invariant set list $(K_v)_{v \in v}$, then*

$$s_0 = \dim_H K_v = \dim_{\mathcal{E}} K_v = \dim_B K_v$$

for all $v \in V$.

1.3.3 Conformal IFS

While the dimension computations for IFS and GDIFS were somewhat similar, conformal IFS are much more difficult all around to deal with. Not only does the invariant set have more locally heterogeneous structure, the diameters of the cylinder sets must be estimated as opposed to explicitly computed. The techniques used to deal with these issues are collectively called the thermodynamic formalism and were introduced to symbol spaces in [9], and generalized in [46]. We use the concise presentation given in [23], but mix this presentation with the extra conditions and lemmas given in [44] which are necessary to put the theory into $\mathbb{R}^d$.

The two "ingredients" mentioned prior to Lemma 1.5 are much more subtle issues when dealing with conformal IFS. With IFS (and GDIFS), the cylinder sets had diameters which were explicitly computable, namely $|X_\sigma| = r_\sigma$ for each $\sigma \in E^*$. We would like to get something close to this with conformal IFS. Notice

in the case of a self-similar IFS on $\mathbb{R}$, that $r_\sigma = w'_\sigma(x)$ for each $\sigma \in E^*$, and so $|X_\sigma| = w'_\sigma(x)$ independently of $x \in X$. This clearly does not hold for conformal IFS, but there is an approximate result. Proposition 1.8, Lemma 1.8, Theorems 1.7 and 1.8, and Corollary 1.5 may all be found in [46].

PROPOSITION 1.7. *Let $\mathcal{F}_c$ be a conformal IFS, then*

$$\inf_{z \in V} |Dw_\sigma(z)||x - y| \leq |w_\sigma(x) - w_\sigma(y)| \leq \sup_{z \in V} |Dw_\sigma(z)||x - y|$$

for all $\sigma \in E^*$ and all $x, y \in V$.

Before proceeding with our study of the diameters of the cylinders, we note a product rule for conformal maps which can be found in any vector-valued calculus text.

PROPOSITION 1.8. *For each $\sigma \in E^*$, w_σ satisfies*

$$|Dw_\sigma(x)| = |Dw_{\sigma_n}(x)||Dw_{\sigma_{n-1}}(w_{\sigma_n}(x))| \cdots |Dw_{\sigma_1}(w_{\sigma_2 \cdots \sigma_n}(x))|$$

for all $x \in V$.

We can now give the key lemma that allows the thermodynamic formalism to work: the Bounded Distortion Principle (BDP). The idea behind the BDP is that while the diameters depend locally on the derivatives of the w_σ, these derivatives differ from one another by no more than a constant bound.

LEMMA 1.7 (Bounded Distortion Principle). *Let $\mathcal{F}_c$ be a conformal IFS. Then there exists a constant $C \geq 1$ such that*

$$|Dw_\sigma(x)| \leq C \cdot |Dw_\sigma(y)|$$

for each $x, y \in V$ and each $\sigma \in E^*$.[8]

---

[8]Lemma 1.7 is first given in [46], but was not given the name "Bounded Distortion Principle" until much later (see [23]).

The following corollary is a direct combination of Proposition 1.7 and Lemma 1.7. This is the result which we will apply to have the necessary control over the diameters of the cylinders.

COROLLARY 1.5. *There exists a constant $C \geq 1$ such that*

$$C^{-1}|Dw_\sigma(z)||x - y| \leq |w_\sigma(x) - w_\sigma(y)| \leq C|Dw_\sigma(z)||x - y|$$

for all $x, y, z \in V$ and $\sigma \in E^*$. In particular we have

$$C^{-1}|Dw_\sigma(z)| \leq |X_\sigma| \leq C|Dw_\sigma(z)|$$

for all $z \in X$ and all $\sigma \in E^*$.

In the case of self-similar IFS, we were able to compute $\dim_H K$ by choosing $s_0 \geq 0$ such that

$$\sum_{e \in E} r_e^{s_0} = 1.$$

One key reason that this led to an accurate dimension computation is that $|X_\sigma| = r_\sigma$ and

$$\sum_{\sigma \in E^n} |X_\sigma|^{s_0} = \sum_{\sigma \in E^n} r_\sigma^{s_0} = 1$$

for every $n \geq 1$. This statement no longer holds in the case of conformal IFS. In particular, if we choose s_n such that

$$\sum_{\sigma \in E^n} |X_\sigma|^{s_n} = 1$$

for every $n \geq 1$, then s_n will vary in n. The best we might hope for is some value $s_0 \geq 0$ for which $\sum |X_\sigma|^{s_0}$ remains positive and bounded in n. It was the remarkable insight of Bowen in [9] that allowed him to study the pressure of the system to compute such a value s_0.

DEFINITION 22. Let $\mathcal{F}_c$ be a conformal IFS. For $s \in \mathbb{R}$ and $x \in V$, we define the **pressure function** to be

$$P(s,x) = \lim_{k \to \infty} \frac{1}{k} \log \sum_{\sigma \in E^k} |Dw_\sigma(x)|^s$$

if this limit exists.

Given the definition of pressure, we can define the *Gibbs measure* which will ultimately play the role of our invariant measure in the dimension computation.

THEOREM 1.7. *Let $s \in \mathbb{R}$, then $P(s) = P(s,x)$ exists independently of the $x \in X$ chosen. Furthermore, there exists a Borel probability measure μ (called a **Gibbs measure**) supported by K and a constant $M > 0$ such that*

$$M^{-1} \leq \frac{\mu(X_\sigma)}{\exp(-kP(s))|Dw_\sigma(x)|^s} \leq M$$

for each $\sigma \in E^k$ and each $k \geq 1$.

It is clear from Theorem 1.7 that the Gibbs measure μ and the derivatives Dw_σ are intricately related. Hence by Corollary 1.5, $\mu(X_\sigma)$ and $|X_\sigma|$ are intricately related, which is precisely what we would like in order to make our lower bound estimate of the Hausdorff measure. The mitigating parameter in the inequality in Theorem 1.7, however, is the pressure. If we could somehow eliminate the pressure from this inequality, we would have what we need to argue the lower bound. We can achieve this by applying the Intermediate Value Theorem to $P(s)$. First we need a lemma which clarifies the continuity properties of the pressure $P(s)$.

LEMMA 1.8. *Let $s \in \mathbb{R}$ and $\delta > 0$, then*

$$\delta \log r_{\min} \leq P(s+\delta) - P(s) \leq \delta \log r_{\max}.$$

In particular, $P(s)$ is strictly decreasing and continuous in s, with $\lim_{s \to -\infty} P(s) = \infty$ and $\lim_{s \to \infty} P(s) = -\infty$.

REMARK 3. It follows from Lemma 1.8 and the Intermediate Value Theorem that there exists a unique number $s_0 \in \mathbb{R}$ such that $P(s_0) = 0$. This fact along with Theorem 1.7 gives us

$$M^{-1}|Dw_\sigma(x)|^{s_0} \leq \mu(X_\sigma) \leq M|Dw_\sigma(x)|^{s_0}$$

for each $\sigma \in E^*$.

Using this fact we have the following theorem which establishes $\dim_H K$.

THEOREM 1.8. *Let s_0 be the unique number satisfying $P(s_0) = 0$. Then $s_0 = \dim_H E$, and in particular there is a number $b > 0$ such that*

$$b^{-1}|X_\sigma|^{s_0} \leq \mathcal{H}^{s_0}(K \cap X_\sigma) \leq b|X_\sigma|^{s_0}$$

for all $\sigma \in E^*$.

The following corollary follows in much the same way as Corollary 1.3. As with GDIFS, we state the results without proof and refer the reader to [16].

COROLLARY 1.6. $\dim_H K = \dim_B K = \dim_\mathcal{E} K$.

CHAPTER 2
APPROXIMATIONS BY CANTOR SETS

It is a standard exercise in an introductory course in measure theory to construct a sequence of Cantor sets in the interval $[0, 1]$ whose (Lebesgue) measures approach 1. In addition to clarifying for the students that measure theory is not for the faint of heart, this exercise is meant to distinguish between topological "size" and measure theoretical "size". Cantor subsets of $[0, 1]$ are all topologically small, in particular they are first category, yet we can construct Cantor sets with as large of Lebesgue measure as we want, up to measure 1.

It is an interesting question whether one can approximate the dimension of a set with the dimensions of Cantor subsets in a way that admits a dimension estimate for the hyperspace. This question was answered in the positive for self-similar IFS satisfying the OSC in [39], and we answer this question in the positive for self-similar GDIFS, and self-conformal IFS in $\mathbb{R}^d$ satisfying the OSC. We use these results in Chapters 3 and 4 to extend the hyperspace dimension computations from IFS satisfying the SSC to IFS satisfying the OSC.

2.1 Self-Similar IFS

There is a more stringent question than the one posed at the beginning of this chapter: given an IFS satisfying the OSC, can one construct a subset which is *lipeomorphic* to a string space? It is in fact an affirmative answer to this question

that is given in [39], where McClure introduced the idea of an s-nested packing. We will first present this construction, and then answer the less restrictive question posed at the beginning of this chapter, but our answer will be given for GDIFS and conformal IFS as opposed to just self-similar IFS.

2.1.1 McClure's s-nested Packing

Fix $c, s > 0$, $0 < \epsilon < 1/4$ and $M > (1/\epsilon)^s + 1$. Let $E = \{e_1, \ldots, e_M\}$ and let E^∞ be a code space with length function $\alpha(\sigma) = c\epsilon^n$ where $|\sigma| = n$. Recall that α generates a metric ρ_α on E^∞ such that $\|[\sigma]\| = \alpha(\sigma)$ for each $\sigma \in E^*$.

Let $K \subseteq \mathbb{R}^d$. An s-**nested packing** of K is defined to be a collection of closed balls $\{\overline{B}_{c\epsilon^{|\sigma|}}(x_\sigma)\}_{\sigma \in E^*}$ satisfying

1. $x_\sigma \in K$ for every $\sigma \in E^*$

2. $\overline{B}_{c\epsilon^n}(x_\sigma) \cap \overline{B}_{c\epsilon^n}(x_\tau) = \emptyset$ for distinct $\sigma, \tau \in E^n$

3. $\overline{B}_{c\epsilon^{|\sigma|}}(x_\sigma) \subset \overline{B}_{c\epsilon^{|\sigma|-1}/4}(x_\sigma)$ for every $\sigma \in E^*$.

It is easily checked that an s-nested packing forms a particular type of Cantor Net (see §1.2.3) with $\mathcal{A}_k = \{\overline{B}_{c\epsilon^{|\sigma|}}(x_\sigma)\}_{\sigma \in E^k}$. If K admits an s-nested packing, then there exists a subset $K' \subset K$ defined by

$$K' = \bigcap_{n=1}^\infty \bigcup_{\sigma \in \Sigma^n} \overline{B}_{c\epsilon^n}(x_\sigma).$$

Define $h : E^\infty \to K'$ by

$$h(\sigma) = \bigcap_{n=1}^\infty \overline{B}_{c\epsilon^n}(x_{\sigma|_n}).$$

LEMMA 2.1. h *is a lipeomorphism.*

Proof. Let $\sigma, \tau \in E^\infty$ and suppose they have greatest common ancestor $\eta \in E^*$. Letting $n = |\eta|$ it follows that $h(\sigma), h(\tau) \in \overline{B}_{c\epsilon^n}(x_\eta)$, hence

$$d(h(\sigma), h(\tau)) \leq 2c\epsilon^n = 2\rho_\alpha(\sigma, \tau).$$

On the other hand, note that $h(\sigma) \in \overline{B}_{c\epsilon^{n+1}/4}(x_{\sigma|n+1})$ and $h(\tau) \in \overline{B}_{c\epsilon^{n+1}/4}(x_{\tau|n+1})$. Since $\overline{B}_{c\epsilon^{n+1}}(x_{\sigma|n+1}) \cap \overline{B}_{c\epsilon^{n+1}}(x_{\tau|n+1}) = \emptyset$ we have

$$c\epsilon^{n+1} \leq d(x_{\sigma|n+1}, x_{\tau|n+1})$$

$$\leq d(x_{\sigma|n+1}, h(\sigma)) + d(h(\sigma), h(\tau)) + d(h(\tau), x_{\tau|n+1})$$

$$\leq c\epsilon^{n+1}/4 + d(h(\sigma), h(\tau)) + c\epsilon^{n+1}/4.$$

Hence

$$d(h(\sigma), h(\tau)) \geq (1/2)c\epsilon^n = (1/2)\rho_\alpha(\sigma, \tau).$$

$\square$

It follows from Lemma 2.1 and Proposition 1.3 that

$$\dim_H K' = \dim_H E^\infty = -\frac{\log M}{\log c\epsilon}.$$

Now let $X \subset \mathbb{R}^d$ be compact and let $\mathcal{F} = \{X, w_e\}_{e \in E_1}$ be a self-similar IFS satisfying the OSC with contraction ratios $\{r_e\}_{e \in E_1}$ and unique invariant set K. Note by Corollary 1.3 we have $\dim_H K = \dim_{\mathcal{E}} K$. The existence of an s-nested packing for K is as follows. Define the following parameters:

$s > 0$

$r = \min\{r_i\}$

$c = \frac{8}{r} \max\{|K|, 1\}$

$\delta \in (0, \min\{\frac{1}{4}, \frac{1}{4}|K|\})$

$\epsilon = \frac{\delta}{c}.$

Fix $\gamma > 0$ and $\dim_{\mathcal{E}} K - \gamma < s < \dim_{\mathcal{E}} K$. Choose $\delta > 0$ so that

$$s < -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta/c}.$$

From this it follows that $\hat{N}_{2\delta}(K) > (\delta/c)^s$. Let $M = \hat{N}_{2\delta}(K)$ and $E_2 = \{e_1, \ldots, e_M\}$. Fix $x_\Lambda \in K$ arbitrarily and let $\overline{B}_c(x_\Lambda)$ be the first closed ball in the s-nested packing. Notice that $\frac{1}{r} > 1$ so $\frac{c}{8} > \max\{|K|, 1\}$ by our definition of c. Thus $K \subset \overline{B}_{c/8}(x_\Lambda)$ and so any ball $\overline{B}_\delta(x)$ with $x \in K$ satisfies $\overline{B}_\delta(x) \subset \overline{B}_{c/4}(x_\Lambda)$ since

$$|K| + \delta < \frac{1}{8}c + \delta = \frac{1}{8}c + c\epsilon = c(\frac{1}{8} + \epsilon) < \frac{1}{4}c$$

Choose a level 1 packing $\{\overline{B}_{c\epsilon}(x_\tau)\}_{\tau \in E_2^1}$ with each $x_\tau \in E_2$ that satisfies the three properties: (1) and (2) by the fact that $M = \hat{N}_{2\delta}(K)$ and (3) by the fact that $\overline{B}_\delta(x) \subset \overline{B}_{c/4}(x_\Lambda)$ for *every* $x \in K$.

We construct the rest of the s-nested packing by induction. Let $\tau \in E_2^n$ and assume we have the level-n packing $\{\overline{B}_{c\epsilon^n}(x_\tau)\}_{\tau \in E_2^n}$. For $\tau \in E_2^n$ choose $\sigma(\tau) \in E_1^*$ such that $x_\tau \in K_{\sigma(\tau)}$ and such that

$$|K_{\sigma(\tau)}| \leq \frac{1}{8}c\epsilon^n < |K_{\sigma(\tau)-}|.$$

The lower inequality tells us that $K_{\sigma(\tau)} \subset \overline{B}_{c\epsilon^n/8}(x_\tau)$ and so for *any* $x \in K_{\sigma(\tau)}$, we have $\overline{B}_{c\epsilon^{n+1}}(x) \subset \overline{B}_{c\epsilon^n/4}(x_\tau)$ by the same reasoning as above. At this point we may already choose a packing of $\hat{N}_{2c\epsilon^{n+1}}(K_{\sigma(\tau)})$ disjoint closed balls such that each ball is contained in $\overline{B}_{c\epsilon^n/4}(x_\tau)$ as required. We need only show that $\hat{N}_{2c\epsilon^{n+1}}(K_{\sigma(\tau)})$ is large enough. Notice by the right hand inequality of the above inequality and by our definition of c and r that

$$|K_{\sigma(\tau)}| \geq r|K_{\sigma(\tau)-}| > \frac{r}{8}c\epsilon^n \geq \epsilon^n|K|.$$

Since $|K_{\sigma(\tau)}| = r_{\sigma(\tau)}|K|$ this inequality simplifies to $r_{\sigma(\tau)} \geq \epsilon^n$. Finally we have

$$\hat{N}_{2c\epsilon^{n+1}}(K_{\sigma(\tau)}) = \hat{N}_{2c\epsilon\epsilon^n}(K_{\sigma(\tau)}) \geq \hat{N}_{2c\epsilon r(\sigma(\tau))}(K_{\sigma(\tau)})$$

$$\geq \hat{N}_{2c\epsilon}(K) = M.$$

The first inequality follows from the implication $a < b \implies \hat{N}_a(K) \geq \hat{N}_b(K)$. By self-similarity we have $w_\sigma(\overline{B}_\epsilon(x)) = \overline{B}_{r_\sigma \epsilon}(w_\sigma(x))$, so an M-packing of K can be mapped via w_σ to an M-packing of K_σ and the second inequality follows from this fact. Thus we may choose a packing of M disjoint closed balls centered in $K_{\sigma(\tau)}$ such that each ball is contained in $\overline{B}_{c\epsilon^n/4}(x_\tau)$ as required. This completes the induction and completes the construction of the s-nested packing. Note that it follows from the choice of δ that

$$\dim_H K - \gamma < \dim_H K' \leq \dim_H K.$$

This fact along with the fact that h is a lipeomorphism is exactly what was desired from K'.

2.1.2   The Sub-IFS Construction

While the s-nested packing is certainly valid, it is awkward to apply and relies heavily on the self-similarity of the maps of $\mathcal{F}$. The bulk of the effort put forth in the construction of the s-nested packing is to obtain Lemma 2.1, as the theorems concerning hyperspace dimension computations in [39] require h to be a lipeomorphism in order to use them in a more general setting. We have eliminated this need (see Theorems 3.1 and 4.1) and are thus able to present a new and simpler method of constructing a subset K_δ of a self-similar set K that still allows $\dim_H K_\delta$ to approximate $\dim_H K$. We will also show in the following sections that this construction generalizes to the case of graph-self-similar and self-conformal sets in a straightforward manner.

The idea to consider subsets of K by choosing pieces of $\mathcal{F}$ and looking at their invariant sets was motivated by the methods of M. Das in [13], [14] which considered sub-packings and sub-pseudo-packings for the purpose of analyzing mul-

tifractal structures. Similar notions were used by Kigami in [31] to construct metrics which retain the self-similar structure of sets and allow for volume doubling measures, and by Edgar and Golds in [18] to construct sub-attractors which geometrically approximate Julia sets.

Having reviewed the known theory on such approximations, let us turn to our construction which we will call the **sub-IFS construction**. Let $X \subset \mathbb{R}^d$ be compact and assume without loss of generality that $|X| = 1$. Let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be an IFS of similitudes satisfying the OSC, and suppose $\mathcal{F}$ has a unique invariant set K. Let

$$s_0 = \dim_H K = \dim_{\mathcal{E}} K.$$

Fix $\delta > 0$ and choose a packing $\{\overline{B}_\delta(x_i) | 1 \leq i \leq \hat{N}_{2\delta}(K)\}$ of closed balls centered in K. Note that for each $1 \leq i \leq \hat{N}_{2\delta}(K)$, $x_i = h(\sigma(i))$ for some $\sigma(i) \in E^\infty$. Choose $n_i \in \mathbb{N}$ such that

$$|X_{\sigma(i)|_{n_i}}| \leq \delta < |X_{\sigma(i)|_{n_i-1}}|.$$

Note that $x_i \in X_{\sigma(i)|_{n_i}}$. Define $E_\delta = \{\sigma(i)|_{n_i} : 1 \leq i \leq \hat{N}_{2\delta}(K)\} \subset E^*$ and consider the sub-IFS $\mathcal{F}_\delta = \{X, w_\tau\}_{\tau \in E_\delta}$. Since the balls $\{B_\delta(x_i) | 1 \leq i \leq \hat{N}_{2\delta}(K)\}$ are pairwise disjoint, $\mathcal{F}_\delta$ satisfies the SSC. Thus the unique invariant set $K_\delta \subset K$ is a Cantor Set for which $h_\delta : E_\delta^\infty \to K_\delta$ is bijective. The K_δ sets are self-similar with respect to $\mathcal{F}_\delta$ and sub-self-similar with respect to $\mathcal{F}$. The theory of sub-self-similar sets is discussed in [23]. This already completes the construction, and as it relies only on restricting the addresses of points in K based on the diameters of the cylinders which cover them, it is trivially non-vacuous and does not require the existence argument that was necessary for the s-nested packing. We note that in the current situation where $\mathcal{F}$ is self-similar, the diameter condition is equivalent

to

$$r_{\sigma(i)|_{n_i}} \leq \delta < r_{\sigma(i)|_{n_i-1}}.$$

We need only show the convergence of $\dim_H K_\delta$ in δ.

THEOREM 2.1. *Let $\mathcal{F}_\delta$ be as described above, then $\lim_{\delta \to 0+} \dim_H K_\delta = \dim_H K$.*

Proof. $\mathcal{F}_\delta$ still consists of similitudes having contraction ratios $\{r_\tau\}_{\tau \in E_\delta}$ where $r_\tau = r_{\tau_1} r_{\tau_2} \cdots r_{\tau_{|\tau|}}$. By Theorem 1.4, $\dim_H K_\delta$ is the unique solution to the equation

$$\sum_{\tau \in E_\delta} r_\tau^s = 1.$$

As this s is dependent upon δ, let us denote it by s_δ. Notice by our choice of the E_δ that $r_\tau \geq r_{\min} r_{\tau-} > r_{\min} \delta$ for each $\tau \in E_\delta$ ($\tau-$ is still defined in terms of $\tau \in E^*$). Thus

$$r_{\min} \delta < r_\tau \leq \delta$$

and so we must have

$$\hat{N}_{2\delta}(K)(r_{\min}\delta)^{s_\delta} < \sum_{\tau \in E_\delta} r_\tau^{s_\delta} \leq \hat{N}_{2\delta}(K)\delta^{s_\delta}.$$

Taking logarithms and solving for s_δ gives us

$$-\frac{\log \hat{N}_{2\delta}(K)}{\log \delta + \log r_{\min}} < s_\delta \leq -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta}$$

then taking limits in δ yields

$$\lim_{\delta \to 0} s_\delta > \lim_{\delta \to 0} -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta + \log r_{\min}} = \dim_{\mathcal{E}} K = \dim_H K$$

and

$$\lim_{\delta \to 0} s_\delta \leq \lim_{\delta \to 0} -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta} = \dim_{\mathcal{E}} K = \dim_H K$$

which completes the proof. $\qquad \square$

51

2.2 Self-Similar GDIFS

We can, without much new machinery, extend the above construction to the case of graph-self-similar sets in $\mathbb{R}^d$. Let $G = (V, E, i, t, r)$ be a strongly connected Mauldin-Williams graph and let $\mathcal{F}_G$ be a realization of this graph in $\mathbb{R}^d$. Suppose $\mathcal{F}_G$ satisfies the OSC as described in Definition 6 and note that by Theorem 1.1 the SOSC is also satisfied. Then, letting M_s denote the ratio matrix of G, we have independent of $v \in V$ that

$$s_0 = \dim_H K_v = \dim_B K_v = \dim_{\mathcal{E}} K_v$$

where $s_0 \geq 0$ is the unique positive real number such that $\Phi(s_0) = 1$ (see Lemma 1.6 and the remark that follows). Furthermore there exist positive numbers $(\lambda_v)_{v \in V}$ such that

$$\sum_{v \in V} \sum_{e \in (E_\delta)_{uv}} r_e^{s_0} \lambda_v^{s_0} = \lambda_u^{s_0}$$

for each $u \in V$. To begin constructing the sub-GDIFS, fix $\delta > 0$ and for each $v \in V$ choose a maximal packing of K_v by $\hat{N}_{2\delta}(K_v)$ balls of radius δ. Suppose these balls have centers

$$\left\{ x_v^j : 1 \leq j \leq \hat{N}_{2\delta}(K_v) \right\}.$$

Each $x_v^j = h(\gamma_v^j)$ for some infinite path $\gamma_v^j \in E_A^\infty$. Choose $n_v^j \in \mathbb{N}$ such that

$$|J_\gamma| \leq \delta < |J_{\gamma-}|$$

where $\gamma = \gamma_v^j|_{n_v^j}$ and the diameters are taken under the metrics $\{\tilde{d}_v\}_{v \in V}$. As the packing is by pairwise disjoint balls, this will ensure that the sub-GDIFS satisfies the graph SSC. Also, since $|J_\gamma| = r_\gamma \lambda_{t(\gamma)}$ for every $\gamma \in E_A^*$, this condition implies

$$r_{\min}\delta < r_\gamma \lambda_{t(\gamma)} \leq \delta$$

or, dividing by $\lambda_{t(\gamma)}$,

$$\frac{r_{\min}}{\lambda_{\max}}\delta < r_\gamma \leq \frac{1}{\lambda_{\min}}\delta.$$

We would hope at this point to be able to choose the sub-GDIFS using the γ's constructed above. There is a problem, however, in that the new Mauldin-Williams graph induced by these γ's is no longer strongly connected, thus causing problems with the dimension approximations. We correct this issue by concatenating paths with the γ's in order to make the induced graph strongly connected, while at the same time not changing the contraction ratios "too much." First we give a key lemma. Recall that $\overline{B}_\epsilon$ and B_ϵ denote closed and open balls of radius ϵ, respectively.

LEMMA 2.2. *For each $v \in V$ and $e \in E_v$ there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$ and any packing of K_v by $\hat{N}_{2\delta}(K_v)$ closed balls of radius δ, there exists at least one such ball that is centered in $w_e(U_{t(e)})$.*

Proof. The idea of the proof is simple. Since each open cylinder $w_e(U_{t(e)})$ intersects the invariant set at a point, say x, we can choose a small enough δ such that any ball of radius δ containing x must itself be contained in $w_e(U_{t(e)})$, and will in particular be centered in $w_e(U_{t(e)})$.

To this end, for each $v \in V$ we may choose $y_v \in K_v \cap U_v$ since the GDIFS satisfies the SOSC. Since U_v is open there exists $\epsilon_v > 0$ such that $B_{\epsilon_v}(y_v) \subset U_v \subset J_v$ for each $v \in V$. Since the maps are similitudes we have that

$$w_e\big(B_{\epsilon_{t(e)}}(y_{t(e)})\big) = B_{r_e \epsilon_{t(e)}}\big(w_e(y_{t(e)})\big)$$

for each $e \in E$, and by invariance of the invariant set list, $w_e(y_{t(e)}) \in K_{i(e)}$. Now let

$$\delta_0 < \frac{1}{2}\min_{e \in E}\left\{r_e \epsilon_{t(e)}\right\}.$$
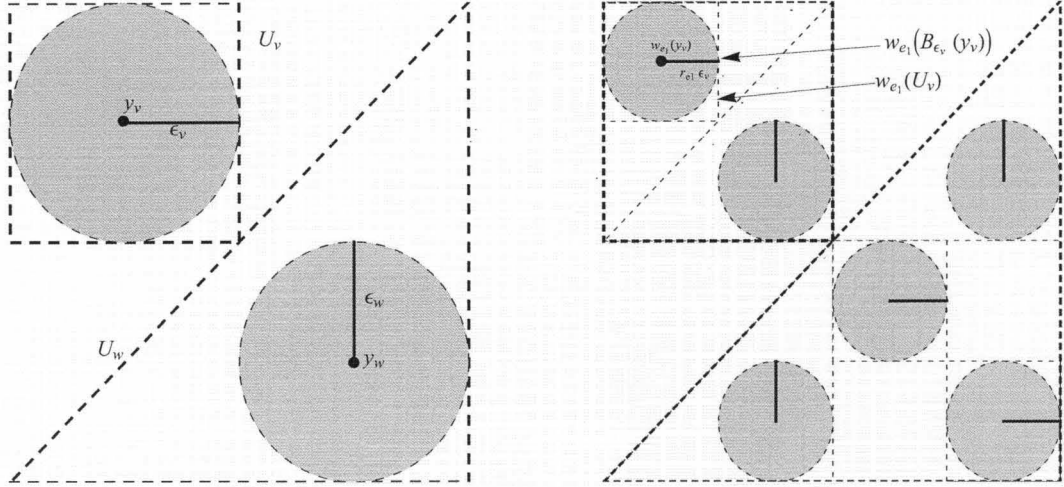
53

FIGURE 2.1 – Lemma 2.2 applied to the GDIFS from Figure 1.5.

Fix an arbitrary $\delta \in (0, \delta_0)$, $v \in V$ and consider a packing of K_v by $\hat{N}_{2\delta}(K_v)$ closed balls of radius δ, say

$$\left\{ \overline{B}_\delta(x_v^j) : 1 \le j \le \hat{N}_{2\delta}(K_v) \right\}.$$

Fix $e \in E_v$. If $x_v^j \notin B_{r_e \epsilon_{t(e)}}(w_e(y_{t(e)}))$ for any $1 \le j \le \hat{N}_{2\delta}(K_v)$, then $\overline{B}_\delta(x_v^j) \cap$ $\overline{B}_\delta(w_e(y_{t(e)})) = \emptyset$. But this would mean if we add $\overline{B}_\delta(w_e(y_{t(e)}))$ to the packing of K_v it is still a pairwise disjoint collection of closed balls centered in K_v, and this contradicts maximality of $\hat{N}_{2\delta}(K_v)$. It follows that if $\delta \in (0, \delta_0)$, then any maximal δ-packing of K_v must contain a closed ball centered in

$$B_{r_e \epsilon_{t(e)}}(w_e(y_{t(e)})) \subset w_e(U_{t(e)})$$

for each $e \in E_v$. This holds similarly for all $v \in V$. □

From this lemma we see that whenever we choose the γ's as described above, the collection $\{\gamma_1\}$ of first edges accounts for *all* of the $e \in E$. This means that for each $e \in E$ there exists a γ with $\gamma_1 = e$. So wherever the path γ ends, if we concatenate with it a path back to $t(\gamma_1)$, the resulting "edge" will be the same edge as γ_1. Since the γ_1's account for all of the $e \in E$, this will force the

induced Mauldin-Williams graph to be strongly connected, given that the original Mauldin-Williams graph was strongly connected.

For each pair of vertices $u, v \in V$ we can choose one (of the many) nonempty path from u to v since (V, E, i, t, r) is strongly connected. Call this collection of paths Ξ and define

$$\overline{r} = \max\{r_\xi : \xi \in \Xi\} \qquad \underline{r} = \min\{r_\xi : \xi \in \Xi\}.$$

Assume $\gamma = \gamma_v^j|_{n_v^j}$ is as constructed above. Since $t(\gamma_1), t(\gamma_n) \in V$, there exists a path $\xi = \xi(\gamma) \in \Xi$ from $t(\gamma_n)$ to $t(\gamma_1)$. Consider the concatenation $\gamma\xi$ and call this path $\zeta = \zeta_v^j$. Note that $i(\zeta) = i(\gamma_1)$ and $t(\zeta) = t(\gamma_1)$. This path satisfies $J_\zeta \subset J_\gamma$, and as

$$|J_\zeta| = r_\zeta \lambda_{t(\zeta)} = r_\gamma r_\xi \lambda_{t(\gamma_1)}$$

we also have

$$r_\zeta > \underline{r} \left(\frac{r_{\min}}{\lambda_{\max}}\right) \delta$$

and this inequality is independent of j and v. Finally let

$$\mathcal{S}_\delta = \{\zeta_v^j : v \in V, 1 \leq j \leq \hat{N}_{2\delta}(K_v)\}.$$

We define a new Mauldin-Williams graph using the same vertex set, but with edge set given by

$$E_\delta = \{i(\zeta)t(\zeta) : \zeta \in \mathcal{S}_\delta\}$$

and maps $i_\delta : E_\delta \rightarrow V$, $t_\delta : E_\delta \rightarrow V$ and $r_\delta : E_\delta \rightarrow (0, 1)$ given by

$$i_\delta(\zeta) = i(\zeta)$$

$$t_\delta(\zeta) = t(\zeta)$$

and

$$r_\delta(e) = r_\zeta$$

respectively.

LEMMA 2.3. $(V, E_\delta, i_\delta, t_\delta, r_\delta)$ *is strongly connected.*

Proof. Let $e \in E$ be arbitrary. By Lemma 2.2 there is a ball $\overline{B}_\delta(x^j_{i(e)})$ from the packing that is centered in $w_e(U_{t(e)}) \subset U_{i(e)} \subset J_{i(e)}$. By construction of E_δ there exists $\zeta \in E_\delta$ such that

$$J_\zeta \subset J_{\gamma^j_{i(e)}|_{n^j_{i(e)}}} \subset w_e(J_{t(e)})$$

from which it follows that $t(\zeta_1) = t(e)$. Since $t(\zeta_n) = t(\zeta_1)$ by construction, and since $i(\zeta_1) = i(e)$ trivially, it follows that the edge in E_δ given by $i(\zeta_1)t(\zeta_n)$ connects the same vertices as e. Since e was arbitrary, this shows that $(V, E_\delta, i_\delta, t_\delta, r_\delta)$ has the same adjacencies as (V, E, i, t, r), and at least as many edges. Hence $(V, E_\delta, i_\delta, t_\delta, r_\delta)$ is strongly connected. $\qquad\square$

We consider the sub-GDIFS $\mathcal{F}_{G,\delta} = \{(J_v)_{v \in V}, (w'_e)_{e \in E_\delta}\}$ where $w'_e = w_\zeta$. Clearly $\mathcal{F}_{G,\delta}$ satisfies the SSC as was the case with $\mathcal{F}_\delta$, and it yields an invariant set list $(K_{v,\delta})_{v \in V}$. We can also define the ratio matrix for $\mathcal{F}_{G,\delta}$ the same way we did for $\mathcal{F}_G$. Let $s \in \mathbb{R}$ and define

$$M^\delta_{uv}(s) = \sum_{e \in (E_\delta)_{uv}} r_e^s.$$

By Lemma 2.3 $(V, E_\delta, i_\delta, t_\delta, r_\delta)$ is strongly connected, hence Theorem 1.6 applies and gives us $s_\delta = \dim_H K_{v,\delta} > 0$ and positive $(\lambda_{v,\delta})_{v \in V}$ such that

$$\sum_{v \in V} \sum_{e \in (E_\delta)_{uv}} r_e^{s_\delta} \lambda_{v,\delta}^{s_\delta} = \lambda_{u,\delta}^{s_\delta}.$$

We will use this equation in a manner similar to the way we used the equation $\sum r_e^{s_\delta} = 1$ in the self-similar IFS case. In order to do this, however, we will need the ratio $\lambda_{u_1,\delta}^{s_\delta}/\lambda_{u_2,\delta}^{s_\delta}$ to be bounded below. Given the above observation we may

write

$$\frac{\lambda_{u_1,\delta}^{s\delta}}{\lambda_{u_2,\delta}^{s\delta}} = \frac{\sum_{v\in V}\sum_{e\in(E_\delta)_{u_1 v}} r_e^{s\delta}\lambda_{v,\delta}^{s\delta}}{\sum_{v\in V}\sum_{e\in(E_\delta)_{u_2 v}} r_e^{s\delta}\lambda_{v,\delta}^{s\delta}}.$$

Given our choice for the ratios r_e, we need only show that

$$\#(E_\delta)_{u_1 v} \approx \#(E_\delta)_{u_2 v}$$

in an appropriate manner. Since w_γ and w_τ^{-1} are both similarity maps for all $\gamma, \tau \in E_A^*$, it follows that $w_\gamma \circ w_\tau^{-1}$ is a similarity map. Since dimension is invariant under similitudes, we have $\dim_{\mathcal{E}} K_\gamma = \dim_H(w_\gamma \circ w_\tau^{-1})(K_\tau) = \dim_{\mathcal{E}} K_\tau$ where $\gamma, \tau \in E_A^*$. The above approximate equality will follow from this fact, as is shown in the following proposition.

PROPOSITION 2.1. *Fix $u^* \in V$. There exists $\delta_0 > 0$ and a constant $C \geq 1$ such that*

$$\frac{1}{C}\hat{N}_{2\delta}(K_{u^*}) \leq \#(E_\delta)_{uv} \leq C\hat{N}_{2\delta}(K_{u^*})$$

for all $u, v \in V$ and all $0 < \delta < \delta_0$.

Proof. For each $e \in E$, pick $x_e \in w_e(U_{t(e)}) \cap w_e(K_{t(e)})$ and define

$$\epsilon_0 = \min_{e\in E}\{d(x_e, \partial U_e)\}.$$

Fix $\delta_0 < \frac{2}{3}\epsilon_0$ and choose $n_0 \in \mathbb{N}$ so that $r_{\max}^{n_0} < \frac{1}{2\lambda_{\max}}\delta_0$. Let $\eta(e) \in E_A^{n_0}$ be the level-n_0 address of x_e, then it follows that

$$|K_{\eta(e)}| = r_{\eta(e)}\lambda_{t(\eta(e))} \leq r_{\max}^{n_0}\lambda_{\max} \leq \frac{1}{2}\delta_0.$$

Notice that since w_γ is a similitude for each $\gamma \in E_A^*$, we have $\dim_{\mathcal{E}} K_\gamma = \dim_{\mathcal{E}} K_\tau$ for each $\gamma, \tau \in E_A^*$. Thus there exists a constant $C \geq 1$ (independent of δ) such that

$$\frac{1}{C}\hat{N}_{2\delta}(K_\gamma) \leq \hat{N}_{2\delta}(K_\tau) \leq C\hat{N}_{2\delta}(K_\gamma)$$

for all $\gamma, \tau \in \bigcup_{k=0}^{n_0} E_A^k$.

Since $\#(E_\delta)_{uv}$ is precisely the number of balls from the collection $\hat{N}_{2\delta}(K_u)$ which are centered in $\bigcup_{e \in E_{uv}} w_e(K_{t(e)})$, we might hope that this sub-collection of balls forms a δ-packing of $\bigcup_{e \in E_{uv}} w_e(K_{t(e)})$. Were this true it would follow that

$$\#(E_\delta)_{uv} = \hat{N}_{2\delta} \left(\bigcup_{e \in E_{uv}} w_e(K_{t(e)}) \right)$$

and the result would be shown. Unfortunately this is untrue in general, so we must be a bit more careful with our estimate.

Fix $u, v \in V$, $\delta < \frac{1}{2}\delta_0$ and consider the collection $(E_\delta)_{uv}$. By construction, each $e \in (E_\delta)_{uv}$ corresponds to a closed ball $\overline{B}_\delta(y_e)$ with $y_e \in K_u$, and the collection $\{\overline{B}_\delta(y_e) : e \in (E_\delta)_{uv}\}$ is a pairwise disjoint collection. It follows that

$$\#(E_\delta)_{uv} \le \hat{N}_{2\delta}(K_u) \le C\hat{N}_{2\delta}(K_{u^*}).$$

Note that each $y_e \in w_{\tilde{e}}(K_{t(\tilde{e})})$ for some $\tilde{e} \in E_{uv}$. Fix one such $w_{\tilde{e}}(K_{t(\tilde{e})})$ and consider the sub-collection

$$\mathcal{G}_{\tilde{e}} = \{\overline{B}_\delta(y_e) : e \in (E_\delta)_{uv}, y_e \in w_{\tilde{e}}(K_{t(\tilde{e})})\}.$$

The collection $\mathcal{G}_{\tilde{e}}$ does not, unfortunately, define a δ-packing of $w_{\tilde{e}}(K_{t(\tilde{e})})$. However, consider the ball $\overline{B}_{\delta_0}(x_{\tilde{e}})$. If $\overline{B}_\delta(z)$ is any ball with $z \notin w_{\tilde{e}}(U_{t(\tilde{e})})$, then $\overline{B}_\delta(z) \cap \overline{B}_{\delta_0}(x_{\tilde{e}}) = \emptyset$ by our choices of $\delta, \delta_0, \epsilon_0$. Let $\mathcal{G}_{x_{\tilde{e}}}$ be the sub-collection of $\mathcal{G}_{\tilde{e}}$ consisting of balls which intersect $\overline{B}_{\delta_0}(x_{\tilde{e}})$, and note in particular that each $\overline{B}_\delta(z) \in \mathcal{G}_{x_{\tilde{e}}}$ satisfies $z \in w_{\tilde{e}}(K_{t(\tilde{e})})$. We also must have that any ball $\overline{B}_\delta(z)$ with $z \in K_{\eta(\tilde{e})}$ satisfies $\overline{B}_\delta(z) \subset \overline{B}_{\delta_0}(x_{\tilde{e}})$. Let $\mathcal{B}$ be a δ-packing of $K_{\eta(\tilde{e})}$ by $\hat{N}_{2\delta}(K_{\eta(\tilde{e})})$ closed balls and consider the collection

$$\left[\{\overline{B}_\delta(x_i) : 1 \le i \le \hat{N}_{2\delta}(K_u)\} \setminus \mathcal{G}_{x_{\tilde{e}}}\right] \cup \mathcal{B}.$$

This collection is pairwise disjoint since the only δ-balls from $\{\overline{B}_\delta(x_i) : 1 \leq i \leq \hat{N}_{2\delta}(K_u)\}$ which balls from $\mathcal{B}$ could have intersected were those from $\mathcal{G}_{x_{\tilde{e}}}$, and these have been removed.

If $\hat{N}_{2\delta}(K_{\eta(\tilde{e})}) > \#\mathcal{G}_{x_{\tilde{e}}}$, then we have contradicted the maximality of $\hat{N}_{2\delta}(K_u)$. It must then be the case that

$$\hat{N}_{2\delta}(K_{\eta(\tilde{e})}) \leq \#\mathcal{G}_{x_{\tilde{e}}} \leq \mathcal{G}_{\tilde{e}} \leq \#(E_\delta)_{uv}.$$

Since $\hat{N}_{2\delta}(K_{\eta(\tilde{e})}) \geq \frac{1}{C}\hat{N}_{2\delta}(K_{u^*})$, this completes the proof. $\qquad \square$

LEMMA 2.4. *Let δ_0 and C be as in Proposition 2.1, then*

$$\frac{\lambda_{u_1,\delta}^{s_\delta}}{\lambda_{u_2,\delta}^{s_\delta}} \geq \frac{1}{C^2}\left(\underline{r}\frac{\lambda_{\min}}{\lambda_{\max}}r_{\min}\right)^{s_\delta}$$

for all $u_1, u_2 \in V$ and $\delta < \delta_0$.

Proof. Recall by the construction of $\mathcal{F}_{G,\delta}$ that

$$\underline{r}\frac{r_{\min}}{\lambda_{\max}}\delta < r_e \leq \frac{1}{\lambda_{\min}}\delta$$

for each $e \in E_\delta$. Then by applying Proposition 2.1 we get

$$\frac{\lambda_{u_1,\delta}^{s_\delta}}{\lambda_{u_2,\delta}^{s_\delta}} = \frac{\sum_{v \in V}\sum_{e \in (E_\delta)_{u_1 v}} r_e^{s_\delta}\lambda_{v,\delta}^{s_\delta}}{\sum_{v \in V}\sum_{e \in (E_\delta)_{u_2 v}} r_e^{s_\delta}\lambda_{v,\delta}^{s_\delta}}$$

$$\geq \frac{(\underline{r}\frac{r_{\min}}{\lambda_{\max}})^{s_\delta}\delta^{s_\delta}\sum_{v \in V}\sum_{e \in (E_\delta)_{u_1 v}} \lambda_{v,\delta}^{s_\delta}}{(\frac{1}{\lambda_{\min}})^{s_\delta}\delta^{s_\delta}\sum_{v \in V}\sum_{e \in (E_\delta)_{u_2 v}} \lambda_{v,\delta}^{s_\delta}}$$

$$= \frac{(\underline{r}\frac{r_{\min}}{\lambda_{\max}})^{s_\delta}\sum_{v \in V}\lambda_{v,\delta}^{s_\delta}\sum_{e \in (E_\delta)_{u_1 v}} 1}{(\frac{1}{\lambda_{\min}})^{s_\delta}\sum_{v \in V}\lambda_{v,\delta}^{s_\delta}\sum_{e \in (E_\delta)_{u_2 v}} 1}$$

$$= \frac{(\underline{r}\frac{r_{\min}}{\lambda_{\max}})^{s_\delta}\sum_{v \in V}\lambda_{v,\delta}^{s_\delta}[\#(E_\delta)_{u_1 v}]}{(\frac{1}{\lambda_{\min}})^{s_\delta}\sum_{v \in V}\lambda_{v,\delta}^{s_\delta}[\#(E_\delta)_{u_2 v}]}$$

$$\geq \frac{[\min_{v \in V} \#(E_\delta)_{u_1 v}] \left(\underline{r} \frac{r_{\min}}{\lambda_{\max}}\right)^{s_\delta} \sum_{v \in V} \lambda_{v,\delta}^{s_\delta}}{[\max_{v \in V} \#(E_\delta)_{u_2 v}] \left(\frac{1}{\lambda_{\min}}\right)^{s_\delta} \sum_{v \in V} \lambda_{v,\delta}^{s_\delta}}$$

$$\geq \frac{\frac{1}{C} \hat{N}_{2\delta}(K_{u^*}) \left(\underline{r} \frac{r_{\min}}{\lambda_{\max}}\right)^{s_\delta}}{C \hat{N}_{2\delta}(K_{u^*}) \left(\frac{1}{\lambda_{\min}}\right)^{s_\delta}} = \frac{1}{C^2} \left(\underline{r} \frac{\lambda_{\min}}{\lambda_{\max}} r_{\min}\right)^{s_\delta} \cdot \cdot$$

which completes the proof. $\qquad\qquad$ $\square$

Now that we have the appropriate lower bound for the ratios of the Perron numbers, we can proceed with the dimension convergence argument as we did in the self-similar IFS case.

THEOREM 2.2. *Let $\mathcal{F}_{G,\delta}$ be as described above, then*

$$\lim_{\delta \to 0^+} \dim_H K_{v,\delta} = \dim_H K_v$$

for each $v \in V$.

Proof. By applying Lemma 2.4 we have that for each $u \in V$

$$\sum_{v \in V} \sum_{e \in (E_\delta)_{uv}} r_e^{s_\delta} \left(\frac{\lambda_{v,\delta}}{\lambda_{u,\delta}}\right)^{s_\delta} \geq \sum_{v \in V} \sum_{e \in (E_\delta)_{uv}} \left[\underline{r} \left(\frac{r_{\min}}{\lambda_{\max}}\right) \delta\right]^{s_\delta} \frac{1}{C^2} \left[\underline{r} \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right) r_{\min}\right]^{s_\delta}$$

$$\geq \hat{N}_{2\delta}(K_u) \left[\left(\underline{r} \frac{r_{\min}}{\lambda_{\max}} \lambda_{\min}^{1/2}\right)^2 \delta\right]^{s_\delta} \frac{1}{C^2}.$$

Taking logarithms and solving for s_δ we see that

$$s_\delta \geq -\frac{\log \hat{N}_{2\delta}(K_u) - \log C^2}{\log \delta + \log \left(\underline{r} \frac{r_{\min}}{\lambda_{\max}} \lambda_{\min}^{1/2}\right)^2}.$$

Then taking limits in δ we have

$$\lim_{\delta \to 0} s_\delta \geq \lim_{\delta \to 0} -\frac{\log \hat{N}_{2\delta}(K_u) - \log C^2}{\log \delta + \log \left(\underline{r} \frac{r_{\min}}{\lambda_{\max}} \lambda_{\min}^{1/2}\right)^2} = \dim_{\mathcal{E}} K_u = \dim_H K_u.$$

Since $K_{\delta,u} \subseteq K_u$ we also have $s_\delta \leq \dim_H K_u$ and so $\lim_{\delta \to 0} s_\delta \leq \dim_H K_u$. This completes the proof. $\qquad\qquad$ $\square$

2.3 Self-Conformal IFS

Let $\mathcal{F}_c = \{X, w_e\}_{e \in E}$ be a conformal IFS as described in Definition 9, and suppose in addition that $\mathcal{F}_c$ satisfies the OSC and note that by Theorem 1.1 in [44] the SOSC is also satisfied. The sub-IFS $\mathcal{F}_{c,\delta}$ is chosen in a similar way as in §2.1, but with a small adjustment. Recall from Corollary 1.5 that there exists a constant $c > 0$ such that

$$\frac{1}{c}|Dw_\sigma(z)||x - y| \leq |w_\sigma(x) - w_\sigma(y)| \leq c|Dw_\sigma(z)||x - y|$$

and

$$\frac{1}{c}|Dw_\sigma(z)| \leq |X_\sigma| \leq c|Dw_\sigma(z)|$$

for all $x, y, z \in V$ and $\sigma \in E^*$. For $\delta > 0$ choose

$$\frac{1}{(1 + c^2\delta)^2} \leq \alpha_\delta < \frac{1}{1 + c^2\delta}$$

and note that $\alpha_\delta \leq 1$, $\alpha_\delta \to 1$ as $\delta \to 0$. We define $E_\delta \subset E^*$ to be those strings σ which are addresses of the centers of closed balls from a δ-packing of K and whose lengths are chosen such that

$$|X_\sigma| \leq \alpha_\delta\delta < |X_{\sigma-}|.$$

It follows from Corollary 1.5 that

$$\frac{1}{c}r_{\min}\alpha_\delta\delta < |X_\sigma| \leq \alpha_\delta\delta$$

for each $\sigma \in E_\delta$.

The sub-IFS construction is relatively simple with self-similar IFS because once we have fixed the contraction ratios, we have all of the information necessary to compute the Hausdorff dimension of K_δ. In order to compute the Hausdorff dimension of K_δ for a conformal IFS, however, we will resort to taking an entropy

index estimate of K, and using this to construct an entropy index estimate of K_δ. We will then show that $\dim_{\mathcal{E}} K_\delta \to \dim_{\mathcal{E}} K$ and apply Corollary 1.6.

There are two main issues to deal with in this line of argument. First of all, one would hope that the δ-packing of K would be "like" a δ-packing of K_δ, but this isn't the case. The closed balls from the packing of K are centered in K, but they are not necessarily centered in K_δ. In order to ensure that we can center balls of the right size within K_δ, we need to be able to show the existence of points from K_δ that are close enough to the centers of the original closed balls. The following two lemmas show that we can, in fact, choose points of K_δ close to the points in K and still retain the size of the packing.

LEMMA 2.5. *Let $n(\delta)$ be the largest integer for which $\{\tau|_{n(\delta)} : \tau \in E_\delta\} = E^{n(\delta)}$, then*

$$\lim_{\delta \to 0} n(\delta) = \infty$$

independently of the packings chosen.

Proof. Recall that $X = \overline{U}$ and so also $X_\sigma = \overline{(U_\sigma)}$ for each $\sigma \in E^*$. Suppose $n(\delta) \not\to \infty$, then there exists $N \in \mathbb{N}$ and a sequence $\{\delta_k\}_{k \geq 1}$ with $\delta_k \to 0$ such that $n(\delta_k) < N$ for each $k \geq 1$.

Consider the level-N cylinders $\{X_\sigma : \sigma \in E^N\}$. Since the SOSC is satisfied, we may for each $\sigma \in E^N$ choose points $x_\sigma \in K \cap U_\sigma$, then choose $\epsilon > 0$ such that $B_\epsilon(x_\sigma) \subset U_\sigma \subset X_\sigma$ for all $\sigma \in E^N$. Let $0 < \epsilon_0 < \frac{2}{3}\epsilon$ and choose $k_0 \geq 1$ such that $\delta_{k_0} < \frac{1}{2}\epsilon_0$. Let

$$\{\overline{B}_{\delta_{k_0}}(x_i) : 1 \leq i \leq \hat{N}_{2\delta_{k_0}}(K)\}$$

be an arbitrary δ_{k_0}-packing of K.

We claim that each ball $B_{\epsilon_0}(x_\sigma)$ contains at least one x_i for $1 \leq i \leq \hat{N}_{2\delta_{k_0}}(K)$. To see this, suppose it is not the case, then there is a string $\sigma \in E^N$

such that $x_i \notin B_{\epsilon_0}(x_\sigma)$ for each $1 \leq i \leq \hat{N}_{2\delta_{k_0}}(K)$. However, this means that

$$\min \left\{ d(x_\sigma, x_i) : 1 \leq i \leq \hat{N}_{2\delta_{k_0}}(K) \right\} \geq \epsilon_0 > 2\delta_{k_0}$$

and in particular $\overline{B}_{\delta_{k_0}}(x_\sigma) \cap \overline{B}_{\delta_{k_0}}(x_i) = \emptyset$. It follows that

$$\{\overline{B}_{\delta_{k_0}}(x_i) : 1 \leq i \leq \hat{N}_{2\delta_{k_0}}(K)\} \cup \{\overline{B}_{\delta_{k_0}}(x_\sigma)\}$$

is a disjoint collection of closed balls with radius δ_{k_0} and centers in K. This contradicts maximality of $\hat{N}_{2\delta_{k_0}}(K)$ and completes the proof of the claim.

Now, for each $\sigma \in E^N$, choose one of the $x_i \in B_{\epsilon_0}(x_\sigma)$. By construction of $E_{\delta_{k_0}}$, there is a string $\tau \in E_{\delta_{k_0}}$ such that $X_\tau \subset \overline{B}_{\delta_{k_0}}(x_i)$. Also, since $\delta_{k_0} < \frac{1}{2}\epsilon_0 < \frac{1}{3}\epsilon$ we have that $\epsilon_0 + \delta_{k_0} < \epsilon$, and so

$$\overline{B}_{\delta_{k_0}}(x_i) \subset B_\epsilon(x_\sigma) \subset X_\sigma.$$

Thus $X_\tau \subset X_\sigma$ and $\tau|_N = \sigma$. Since $\sigma \in E^N$ was chosen arbitrarily, we have

$$\{\tau|_N : \tau \in E_\delta\} = E^N$$

and hence $n(\delta_{k_0}) \geq N$. This contradicts the original assumption about $\{\delta_k\}_{k \geq 1}$ and completes the proof of the lemma. $\qquad \square$

The reader will notice the similarity of Lemma 2.5 to Lemma 2.2, the difference being that Lemma 2.2 applies only to level-1 cylinders in the GDIFS setting, while Lemma 2.5 applies to *any* cylinder in the conformal IFS setting. If the present work is to be extended to conformal GDIFS, these two lemmas will have to be combined in an appropriate way.

In Lemma 2.5 it does not matter how large an n we choose, there exists a small enough δ such that any δ-packing of K must contain points centered in *each* level-n cylinder. This will provide us with control over the deviation of points of

K_δ from the centers of the closed balls from the original δ-packing of K. This in turn will allow us to define large enough closed balls centered in K_δ so as to mimic the δ-packing of K. This idea is clarified in the next lemma.

For the proof of the following lemma we will use slightly non-standard notation. Normally for $\sigma \in E^* \cup E^\infty$ we define

$$\sigma|_k = \sigma_1 \sigma_2 \cdots \sigma_k.$$

For the proof of Lemma 2.6, however, we will use the following notation:

$$\sigma|_1^k = \sigma_1 \sigma_2 \cdots \sigma_k$$

$$\sigma|_{k_1}^{k_2} = \sigma_{k_1} \sigma_{k_1+1} \cdots \sigma_{k_2}.$$

We will revert back to the standard notation after completion of the proof of Lemma 2.6.

LEMMA 2.6. *There exists $\delta_0 > 0$ such that for every $0 < \delta < \delta_0$ there exists $\delta^* > 0$ such that $\frac{1}{2}\alpha_\delta \delta < \delta^* < \delta$ and $\hat{N}_{2\delta^*}(K_\delta) = \hat{N}_{2\delta}(K)$.*

Proof. Choose $N \in \mathbb{N}$ such that $r_{\max}^N < \frac{1}{2c}$. Using Lemma 2.5, choose $\delta_0 > 0$ so that $n(\delta) > N$ for each $0 < \delta < \delta_0$.

Now fix $0 < \delta < \delta_0$ and let E_δ be as previously defined. Consider a δ-packing of K

$$\left\{ \overline{B}_\delta(x_i) : 1 \leq i \leq \hat{N}_{2\delta}(K) \right\}.$$

Fix $1 \leq i \leq \hat{N}_{2\delta}(K)$ and $\sigma = \sigma(i) \in E_\delta$ so that $X_\sigma \subset \overline{B}_\delta(x_i)$. Recalling that $h : E^\infty \to K$ is defined by $h(\sigma) = \lim_{n \to \infty} w_{\sigma|_n}(z)$, let $\eta \in h^{-1}(x_i)$ be an address of x_i. As before, we have that there exists $k \geq 1$ with $\eta|_1^k = \sigma$. Consider the string $\eta|_{k+1}^{k+n(\delta)} \in E^{n(\delta)}$. By definition of $n(\delta)$ there exists $\tau \in E_\delta$ such that

$$\tau|_1^{n(\delta)} = \eta|_{k+1}^{k+n(\delta)}$$

64

and hence

$$\sigma\tau|_1^{n(\delta)} = \eta|_1^{k+n(\delta)}.$$

Consider the cylinder $X_{\sigma\tau}$. We must have the following:

(a) $x_i \in X_{\sigma\tau|_1^{n(\delta)}}$

(b) $X_{\sigma\tau} \subset X_{\sigma\tau|_1^{n(\delta)}} \subset X_\sigma \subset \overline{B}_\delta(x_i)$

(c) $|X_{\sigma\tau}| \le cr_{\max}^{|\tau|}|X_\sigma| \le cr_{\max}^{n(\delta)}|X_\sigma| < \frac{1}{2}\alpha_\delta\delta.$

Note that (a) follows from the fact that $|\tau| \ge n(\delta) > N$. Facts (a)-(c) implicitly let $\sigma = \sigma(i)$ and $\tau = \tau(i)$, we will now make this explicit by referring to $\sigma(i)$ and $\tau(i)$. From fact (c) it follows that

$$|X_{\sigma(i)\tau(i)}| \le \frac{1}{2}\alpha_\delta\delta$$

$$2|X_{\sigma(i)\tau(i)}| \le \alpha_\delta\delta$$

$$|X_{\sigma(i)\tau(i)}| \le \alpha_\delta\delta - |X_{\sigma(i)\tau(i)}|.$$

Let

$$\delta^* = \min_{1 \le i \le \hat{N}_{2\delta}(E)} \left(\alpha_\delta\delta - |X_{\sigma(i)\tau(i)}|\right) = \alpha_\delta\delta - \max_{1 \le i \le \hat{N}_{2\delta}(E)} |X_{\sigma(i)\tau(i)}|.$$

Then for each $1 \le j \le \hat{N}_{2\delta}(E)$ we have

$$\delta^* = \alpha_\delta\delta - \max_{1 \le i \le \hat{N}_{2\delta}(E)} |X_{\sigma(i)\tau(i)}| \ge \max_{1 \le i \le \hat{N}_{2\delta}(E)} |X_{\sigma(i)\tau(i)}| \ge |X_{\sigma(j)\tau(j)}|.$$

So if $y \in X_{\sigma(j)\tau(j)}$ is any point, we have $X_{\sigma(j)\tau(j)} \subset \overline{B}_{\delta^*}(y)$.

Since the SOSC is satisfied, $U_{\sigma(i)\tau(i)} \cap K_\delta \ne \emptyset$ for each $1 \le i \le \hat{N}_{2\delta}(K)$ and we may choose $y_i \in X_{\sigma(i)\tau(i)} \cap K_\delta$ such that $d(x_i, y_i) < |X_{\sigma(i)\tau(i)}|$. Consider the collection of closed balls

$$\left\{\overline{B}_{\delta^*}(y_i) : 1 \le i \le \hat{N}_{2\delta}(K)\right\}.$$

We claim that this collection of balls defines a δ^*-packing of K_δ. To see this we first note that by construction $y_i \in K_\delta$ for each $1 \le i \le \hat{N}_{2\delta}(K)$. Suppose $\overline{B}_{\delta^*}(y_i) \not\subset \overline{B}_\delta(x_i)$ for some $1 \le i \le \hat{N}_{2\delta}(K)$. Then there exists $z \notin \overline{B}_\delta(x_i)$ such that $d(y_i, z) \le \delta^*$, but from this it follows that

$$\delta - |X_{\sigma(i)\tau(i)}| > \delta^* \ge d(z, y_i) \ge d(z, x_i) - d(x_i, y_i) > d(z, x_i) - |X_{\sigma(i)\tau(i)}|.$$

This shows that $d(x_i, z) < \delta$ which contradicts the assumption that $z \notin \overline{B}_\delta(x_i)$. Thus $\overline{B}_{\delta^*}(y_i) \subset \overline{B}_\delta(x_i)$ for each $1 \le i \le \hat{N}_{2\delta}(K)$. As

$$\left\{ \overline{B}_\delta(x_i) : 1 \le i \le \hat{N}_{2\delta}(K) \right\}$$

is a pairwise disjoint collection, it follows that

$$\left\{ \overline{B}_{\delta^*}(y_i) : 1 \le i \le \hat{N}_{2\delta}(K) \right\}$$

is a pairwise disjoint collection.

Finally, let $\overline{B}_{r_1}(a_1), \overline{B}_{r_2}(a_2)$ be closed balls with $a_1, a_2 \in X_\sigma$ and $r_1, r_2 > \frac{1}{2}\alpha_\delta\delta$. If we assume these balls are disjoint, then

$$d(a_1, a_2) \ge r_1 + r_2 > \alpha_\delta\delta \ge |X_\sigma|$$

which is impossible. Thus any such balls must intersect one another. Assume for the moment that $\delta^* > \frac{1}{2}\alpha_\delta\delta$, then in particular, if $\overline{B}_{\delta^*}(z)$ is any other δ^*-ball with $z \in K_\delta \cap X_\sigma$, we have that $\overline{B}_{\delta^*}(y_i) \cap \overline{B}_{\delta^*}(z) \ne \emptyset$. This is true for each $1 \le i \le \hat{N}_{2\delta}(K)$, hence the collection

$$\left\{ \overline{B}_{\delta^*}(y_i) : 1 \le i \le \hat{N}_{2\delta}(K) \right\}$$

is maximal and is a δ^*-packing. This completes the proof of the claim.

To complete the proof we remark that since $\left\{ \overline{B}_{\delta^*}(y_i) : 1 \le i \le \hat{N}_{2\delta}(K) \right\}$ is a maximal δ^*-packing of K_δ and by maximality of $\hat{N}_{2\delta^*}(K_\delta)$, we have that

66

$\hat{N}_{2\delta^*}(K_\delta) = \hat{N}_{2\delta}(K)$. Finally we observe that

$$\delta^* = \min_{1 \le i \le \hat{N}_{2\delta}(K)} \left(\alpha_\delta \delta - |X_{\sigma(i)\tau(i)}| \right)$$

$$\ge \min_{1 \le i \le \hat{N}_{2\delta}(K)} \left(\alpha_\delta \delta - \frac{1}{2}\alpha_\delta \delta \right) = \frac{1}{2}\alpha_\delta \delta \quad .$$

from which it follows that $\frac{1}{2}\alpha_\delta \delta < \delta^* < \delta$. $\qquad\qquad \square$

The point behind Lemma 2.6 is that when we take a packing of K by closed ball centered in K, and then map X inside of these balls and create a sub-IFS, the set K_δ likely no longer contains the centers of the balls from the packing of K. If we want to say that the packing of K is "like" a packing of K_δ this creates an issue that Lemmas 2.5 and 2.6 solve. We can now give our main approximation theorem for conformal IFS.

THEOREM 2.3. *Let $\mathcal{F}_{c,\delta}$ be as described above, then*

$$\lim_{\delta \to 0^+} \dim_H K_\delta = \dim_H K.$$

Proof. Fix $\delta > 0$ and consider E_δ, K_δ. We first note that since $K_\delta \subset K$, we trivially have $\dim_H K_\delta \le \dim_H K$. We will construct a lower estimate of $\dim_\varepsilon K_\delta$.

Note by Lemma 2.6 that we may choose a δ^*-packing of K_δ by $\hat{N}_{2\delta^*}(K_\delta) = \hat{N}_{2\delta}(K)$ balls. Recall by Definition 9(c) there exist constants $r_{\min,\delta}$, $r_{\max,\delta}$ such that for each $\sigma \in E_\delta$

$$r_{\min,\delta} \le |Dw_\sigma(x)| \le r_{\max,\delta}.$$

In particular we have that

$$r_{\min,\delta} = \min_{\sigma \in E_\delta} \inf_{x \in X} |Dw_\sigma(x)| \qquad\qquad r_{\max,\delta} = \max_{\sigma \in E_\delta} \sup_{x \in X} |Dw_\sigma(x)|.$$

By Corollary 1.5 and the construction of E_δ there exists a constant $c \geq 1$ such that

$$c^{-2} r_{\min} \alpha_\delta \delta \leq c^{-1}|X_\sigma| \leq |Dw_\sigma(x)| \leq c|X_\sigma| \leq c\alpha_\delta \delta$$

for each $x \in X$ and $\sigma \in E_\delta$. Hence

$$r_{\min,\delta} \geq c^{-2} r_{\min} \alpha_\delta \delta.$$

Now consider the collection of sets

$$\left\{ w_\sigma(\overline{B}_{\delta^*}(y_i)) | \sigma \in E_\delta, 1 \leq i \leq \hat{N}_{2\delta}(K) \right\}.$$

If $i \neq j$, then clearly $w_\sigma(\overline{B}_{\delta^*}(y_i)) \cap w_\sigma(\overline{B}_{\delta^*}(y_j)) = \emptyset$ by injectivity of w_σ. Let $\sigma, \tau \in E_\delta$ such that $\sigma \neq \tau$. Suppose $X_\sigma \subset \overline{B}_\delta(x_\sigma)$ where $\overline{B}_\delta(x_\sigma)$ is the corresponding ball from the δ-packing of K, and suppose the analogous statement for X_τ. Since $\overline{B}_\delta(x_\sigma) \cap \overline{B}_\delta(x_\tau) = \emptyset$, if

$$|X_\sigma| + \sup \left\{ d(w_\sigma(y_i), w_\sigma(z)) : z \in \partial \overline{B}_{\delta^*}(y_i) \right\} \leq \delta$$

for each $1 \leq i \leq \hat{N}_{2\delta}(K)$ (with the analogous statement holding for τ), then it will follow that the above collection is disjoint. By Corollary 1.5 we have

$$\sup\{d(w_\sigma(y_i), w_\sigma(z)) : z \in \partial \overline{B}_{\delta^*}(y)\} \leq c|Dw_\sigma(x)| \sup\{d(y_i, z) : z \in \overline{B}_{\delta^*}(y)\}$$
$$\leq c r_{\max,\delta} \delta^* \leq c^2 \alpha_\delta \delta \delta^* \leq c^2 \alpha_\delta \delta^2$$

hence

$$|X_\sigma| + \sup \left\{ d(w_\sigma(y_i), w_\sigma(z)) : z \in \partial \overline{B}_{\delta^*}(y_i) \right\} \leq \alpha_\delta \delta + c^2 \alpha_\delta \delta^2$$
$$= \alpha_\delta \delta(1 + c^2 \delta) < \delta.$$

The inequality holds similarly for X_τ and so

$$\left\{ w_\sigma(\overline{B}_{\delta^*}(y_i)) | \sigma \in E_\delta, 1 \leq i \leq \hat{N}_{2\delta}(K) \right\}$$

is a disjoint collection. Also if $\sigma \in E_\delta^n$ then

$$\sup\{d(w_\sigma(y_i), w_\sigma(z)) : z \in \partial \overline{B}_{\delta^*}(y)\} \leq c|Dw_\sigma(x)|d(y_i, z)$$

$$\leq c(r_{\max,\delta})^n \delta^* \leq c r_{\max,\delta} \delta^* \leq c^2 \alpha_\delta \delta^2.$$

Since $w_\sigma(y_i) \in X_{\sigma_1}$, a similar argument as above gives that

$$\left\{ w_\sigma(\overline{B}_{\delta^*}(y_i)) | \sigma \in E_\delta^n, 1 \leq i \leq \hat{N}_{2\delta}(K) \right\}$$

is a disjoint collection for all $n \geq 1$.

Now as the maps are iterates of conformal maps, it need not be the case that these sets are balls. However, we have that

$$d(w_\tau(x), w_\tau(y_i)) \geq c^{-1}(r_{\min,\delta})^n d(x, y_i) = c^{-1}(r_{\min,\delta})^n \delta^*$$

for each $1 \leq i \leq \hat{N}_{2\delta}(K)$, each $\sigma, \tau \in E_\delta^n$ and each $x \in \partial \overline{B}_{\delta^*}(y_i)$. This necessarily means that

$$\overline{B}_{c^{-1}(r_{\min,\delta})^n \delta^*}(w_\tau(y_i)) \subseteq w_\tau(\overline{B}_{\delta^*}(y_i))$$

for each $1 \leq i \leq \hat{N}_{2\delta}(K)$. The collection

$$\left\{ \overline{B}_{c^{-1}(r_{\min,\delta})^n \delta^*}(w_\tau(y_i)) | \tau \in E_\delta^n, 1 \leq i \leq \hat{N}_{2\delta}(K) \right\}$$

is then a pairwise disjoint collection of $c^{-1}(r_{\min,\delta})^n \delta^*$-balls centered in K_δ. Hence

$$\hat{N}_{2c^{-1}(r_{\min,\delta})^n \delta^*}(K_\delta) \geq \# \left\{ \overline{B}_{c^{-1}(r_{\min,\delta})^n \delta^*}(w_\tau(y_i)) | \tau \in E_\delta^n, 1 \leq i \leq \hat{N}_{2\delta}(K) \right\}$$

$$= \hat{N}_{2\delta}(K)[\hat{N}_{2\delta}(K)]^n = [\hat{N}_{2\delta}(K)]^{n+1}.$$

We finally have our lower estimate:

$$\dim_{\mathcal{E}} K_\delta = \lim_{\gamma \to 0} -\frac{\log \hat{N}_{2\gamma}(K_\delta)}{\log \gamma} = \lim_{n \to \infty} -\frac{\log \hat{N}_{2c^{-1}(r_{\min,\delta})^n \delta^*}(K_\delta)}{\log c^{-1}(r_{\min,\delta})^n \delta^*}$$

$$\geq \lim_{n \to \infty} -\frac{\log [\hat{N}_{2\delta}(K)]^{n+1}}{\log c^{-1}(r_{\min,\delta})^n \delta^*}$$

$$= \lim_{n \to \infty} -\frac{\log [\hat{N}_{2\delta}(K)]^{n+1}}{\log (r_{\min,\delta})^n + \log c^{-1} \delta^*}$$

$$= \lim_{n \to \infty} -\frac{\log[\hat{N}_{2\delta}(K)]^{n+1}}{\log(r_{\min,\delta})^n}$$

$$= \lim_{n \to \infty} -\left(\frac{n+1}{n}\right) \frac{\log \hat{N}_{2\delta}(K)}{\log r_{\min,\delta}}$$

$$= -\frac{\log \hat{N}_{2\delta}(K)}{\log r_{\min,\delta}}$$

$$\geq -\frac{\log \hat{N}_{2\delta}(K)}{\log c^{-2}r_{\min}\delta}$$

$$\geq -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta + \log c^{-2}r_{\min}}.$$

Since

$$\lim_{\delta \to 0} -\frac{\log \hat{N}_{2\delta}(K)}{\log \delta + \log c^{-2}r_{\min}} = \dim_{\mathcal{E}} K$$

it follows that $\lim_{\delta \to 0} \dim_{\mathcal{E}} K_\delta = \dim_{\mathcal{E}} K$. Hence by Corollary 1.6

$$\lim_{\delta \to 0} \dim_H K_\delta = \dim_H K$$

and the result follows. $\qquad\qquad\square$

2.4 Example: The Sierpiński Triangle

We will now consider the example of the Sierpiński Triangle. We must note that this particular example falls within the auspices of McClure's s-nested packing theory, but is appropriately simple to exemplify our arguments from this chapter.

Recall we define the triangle with side length 1 to be the following region

$$T = \overline{\mathrm{conv}} \left\{ (0,0), (1/2, \sqrt{3}/2), (1,0) \right\} \subset \mathbb{R}^2$$

where $\overline{\mathrm{conv}}$ denotes the closed convex hull. For $i = 1, 2, 3$ consider the following maps from T into itself.

$$w_i(\vec{x}) = (1/2)\vec{x} + t_i$$

where

$$t_1 = \begin{pmatrix} 1/4 \\ \sqrt{3}/4 \end{pmatrix} \qquad t_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad t_3 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

There is a unique invariant set $\mathcal{T} \subset T$ which we call the *Sierpiński Triangle* that satisfies

$$\mathcal{T} = w_1(\mathcal{T}) \cup w_2(\mathcal{T}) \cup w_3(\mathcal{T}).$$

We wish to choose a sequence of sub-IFSs $\mathcal{F}_{\delta_k}$ such that

$$\dim_H \mathcal{T}_{\delta_k} \rightarrow \dim_H \mathcal{T} = \frac{\log 3}{\log 2}.$$

Let $\epsilon > 0$ be very small and define

$$\delta_k = 2^{-k} \left(\frac{1}{1 + \epsilon^k} \right).$$

Notice that $\delta_k \rightarrow 0$ as $k \rightarrow \infty$, but also since $2^{-k} = |T_\sigma|$

$$|T_{\sigma+}| < \delta_k < |T_\sigma|$$

for all $k \geq 1$, $\sigma \in E^k$ and $2^k \delta_k \rightarrow 1$ as $k \rightarrow \infty$. We can see that

$$\hat{N}_{2\delta_k}(\mathcal{T}) = \frac{3}{2}(3^{k-1} + 1)$$

is equal to the number of distinct vertices of $\{T_\sigma : \sigma \in E^{k-1}\}$ (see Figure 2.2). Since the cylinders of $\mathcal{T}$ have diameter 2^{-k} at level-k we must choose level-$(k+1)$ cylinders to fit entirely inside these balls, hence they have diameters $2^{-(k+1)}$. The sub-IFS $\mathcal{F}_{\delta_k}$ then has similarity (hence Hausdorff) dimension

$$s_{\delta_k} = -\frac{\log[\frac{3}{2}(3^{k-1} + 1)]}{\log[2^{-(k+1)}]} \rightarrow \frac{\log 3}{\log 2} = \dim_H \mathcal{T}$$
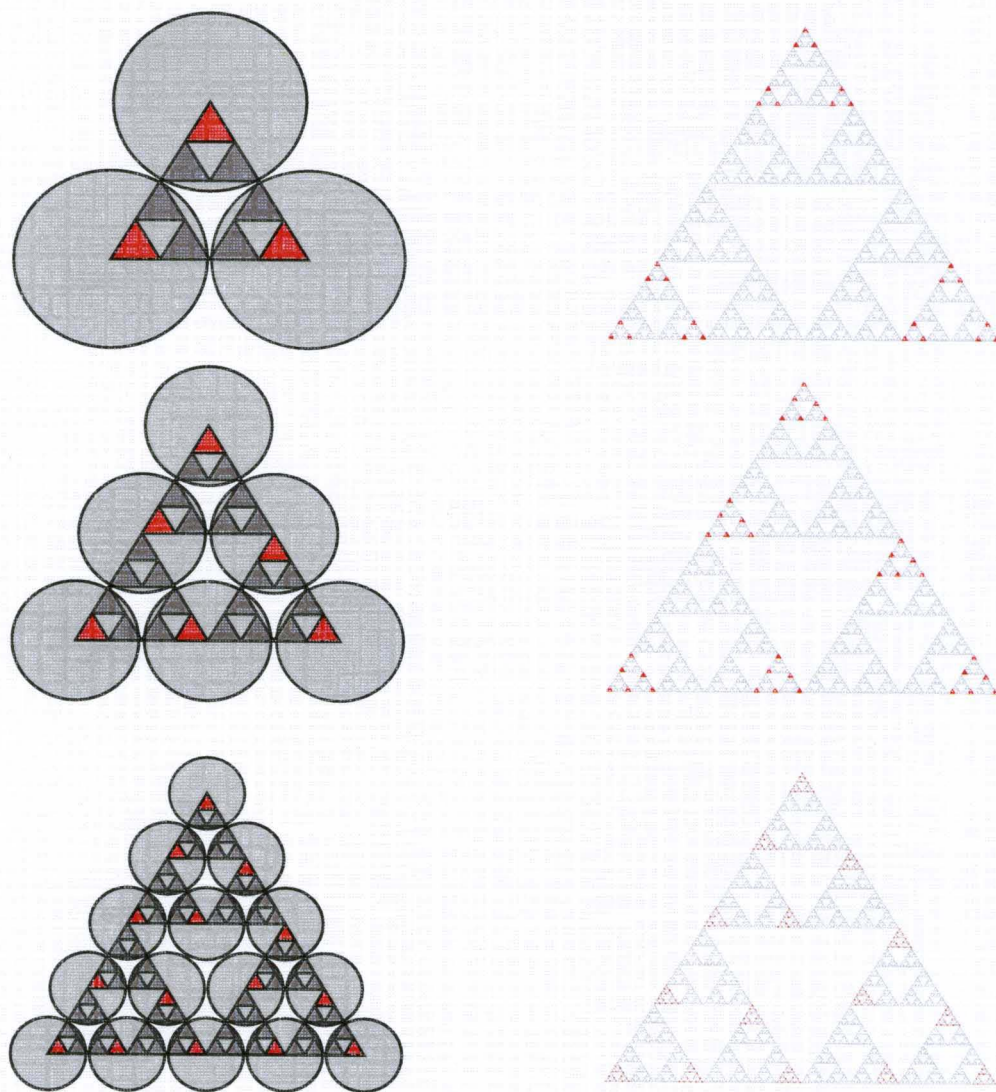
FIGURE 2.2 – The choice of $\mathcal{F}_{\delta_k}$ and the invariant set $\mathcal{T}_{\delta_k}$ for $k = 1, 2, 3$.

thus showing the required convergence. This unfortunately also demonstrates the numerical inefficiency of this algorithm, as one must compute all the way up to $s_{\delta_{2700}}$ in order to get within 3 significant digits of $\log 3/\log 2$.

CHAPTER 3
HYPERSPACE DIMENSIONS FOR GRAPH-SELF-SIMILAR SETS

Hyperspaces have been the objects of much topological study over the past 40 years, and more recently have entered the theory of fractals by way of the existence and uniqueness theory as described in §1.1.1. There has been surprisingly little dimensional study of hyperspaces, however. With the recent advent of the theory of *superfractals* (see [4]), it is worth revisiting the dimensional study and classification of hyperspaces as fractals in and of themselves. In this chapter we review the classical results concerning hyperspaces, and then extend the dimension computations of [39] to GDIFS in $\mathbb{R}^d$.

3.1   Historical Results Concerning Hyperspaces

Hyperspaces have a very rich topological structure and have been the subject of much topological study since the following famous result of Curtis and Schori in [11].

THEOREM. *Let $\mathcal{Q}$ denote the Hilbert cube. Then $\mathbb{H}(X) \cong_{\mathrm{Hom}} \mathcal{Q}$ if and only if (X, d) is a nondegenerate locally connected metric continuum.*

In contrast to the abundance of topological study, there has been for the most part a dearth of measure theoretical study of hyperspaces. Concurrently with Curtis and Schori developing their theory, Boardman was developing the following early measure theoretical results concerning hyperspaces (see [7], [8]).

THEOREM. *There exists no positive, σ-finite Hausdorff measure on $(\mathbb{H}([0,1]), d_H)$.*

THEOREM. *Let $\phi_s(t) = 2^{-(1/t)^s}$, then $\mathcal{H}^{\phi_s}(\mathbb{H}([0,1])) = 0$ for $s > 1$ and $\mathcal{H}^{\phi_s}(\mathbb{H}([0,1]))$ is non-σ-finite for $s < 1$.*

Despite this early progress, hyperspaces remained untouched in a measure-theoretical sense for over a decade, until Bandt and Baraki developed the following powerful theorem in [2].

THEOREM. *Let (X, d) be a locally compact separable metric space without isolated points. Then there exists no positive, σ-finite, metrically invariant Borel measure on $(\mathbb{H}(X), d_H)$.*

Bandt and Baraki's theorem decisively ended the hope of finding a measure on a nontrivial hyperspace that one could use to integrate or get other useful information out of. In an analysis sense, this settled the issue, and hyperspaces went untouched in the realm of measure theory for another decade, until McClure revisited the hyperspaces of self-similar fractals. Certainly Bandt and Baraki's theorem held for this class of hyperspaces, but when considering the *dimensional study*, as opposed to simply the measure theoretical study, of hyperspaces, one needs only some type of critical value where the measure switches from 0 to ∞. Building off of Boardman's work, this is precisely what McClure found in [39].

THEOREM. *Let (X, d) be a complete metric space and let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be an IFS of similitudes satisfying the OSC. Let K be the unique invariant set for $\mathcal{F}$ and suppose $s_0 = \dim_H K$. Let $\phi_s(t) = 2^{-(1/t)^s}$, then $\mathcal{H}^{\phi_s}(\mathbb{H}(K)) = 0$ for $s > s_0$ and $\mathcal{H}^{\phi_s}(\mathbb{H}(K))$ is non-σ-finite for $s < s_0$.*

More recently in [12], M. Das studies the converse problem of the effect of the Hausdorff dimension of $\mathbb{H}(K)$ on the underlying fractal K.

3.2 Dimension Computations for GDIFS

The arguments given in this section are generalizations of those given by McClure in [39]. In particular, the proof of Theorem 3.2 in [39] is done for $\mathbb{H}(E^\infty)$ where E^∞ is a self-similar code space, and we have split the various parts of this proof out into different propositions and lemmas, with innovations throughout, that have let us extend this Theorem to $\mathbb{H}(K)$ where K is a graph-self-similar set in $\mathbb{R}^d$.

We first need to introduce some notation. In order to construct a measure on a metric space in the manner described in §1.2.3 we require some type of efficient covering that plays the role of a Cantor Net. The cylinder sets serve this purpose for IFSs, and we need an analogous notion for hyperspaces.

For $\mathcal{F}$, the cylinders serve to partition K as we move down through the IFS. The cylinders at level-k, $\mathcal{L}_k = \{X_\sigma : \sigma \in E^k\}$, serve to approximate K in the sense that

$$K = \bigcap_{k=1}^{\infty} \bigcup_{\sigma \in E^k} X_\sigma$$

and $|X_{\sigma|_k}| \to 0$ as $k \to \infty$ for all $\sigma \in E^\infty$.

We want similar properties for sets in order for them to be considered "cylinders" for $\mathbb{H}(K)$. If we try to define $\mathbb{L}_k = \{\mathbb{H}(X_\sigma) : \sigma \in E^k\}$, however, we find that

$$\mathbb{H}(K) \cap \left(\bigcup_{\mathcal{K} \in \mathbb{L}_k} \mathcal{K} \right)^c \neq \emptyset$$

i.e. the sets of $\mathbb{L}_k$ do not cover $\mathbb{H}(K)$. In order to cover all of $\mathbb{H}(K)$, we need to consider collections of compacts which intersect all the possible non-empty subcollections of $\mathcal{L}_k$. We coin the term *hypercylinder* here, but an equivalent definition to the following is given in [39].

DEFINITION 23. Let $\mathcal{F} = \{X, w_e\}_{e \in E}$ be an IFS with a unique invariant set K. Let $L \subset E^*$ be such that no two elements of L are comparable under $\preceq$.[1] Define

$$\mathcal{K}_L = \left\{ F \in \mathbb{H}\left(\bigcup_{\sigma \in L} X_\sigma \right) : X_\sigma \cap F \neq \emptyset, \forall \sigma \in L \right\}.$$

We call $\mathcal{K}_L$ the **hypercylinder corresponding to L**.

It is easily checked that

$$\mathbb{H}(K) = \bigcap_{k=1}^{\infty} \bigcup_{\substack{L \subseteq E^k \\ L \neq \emptyset}} \mathcal{K}_L$$

and $|\mathcal{K}_L| \to 0$ as $k \to \infty$. Furthermore, if $\mathcal{F}$ satisfies the SSC, then $\{\mathcal{K}_L : L \subseteq E^k, L \neq \emptyset\}$ partitions $\mathbb{H}\left(\bigcup_{\sigma \in E^k} X_\sigma \right)$.

Consider for the moment the case of the Cantor Middle Third Set, $\mathcal{C}$. There are 2^k cylinders at level-k, each of which has diameter 3^{-k}. When we attempt to re-scale the diameters of the cylinders in order to estimate $\dim_H \mathcal{C}$, we choose $s = \log 2 / \log 3$ so that $3^{-sk} = 2^{-k}$. Now notice that there are $2^{2^k} - 1$ hypercylinders at level-k, each of which has diameter 3^{-k}. In order to estimate the dimension of $\mathbb{H}(\mathcal{C})$ we might then rescale these diameters by

$$\phi(3^{-k}) \approx 2^{-(1/3^{-k})^s} = 2^{-2^k}$$

so that $(2^{2^k} - 1)\phi(3^{-k}) \approx 1$. It is here that one can most easily see the origin of the form of the gauge functions in the theorems to follow.

The first lemma we give is the main innovation that has allowed us to directly extend the hyperspace arguments from the hyperspace of a symbol space to the hyperspace of a more general fractal. We apply it to a study of the relative distances, or *gaps*, between the cylinder sets.

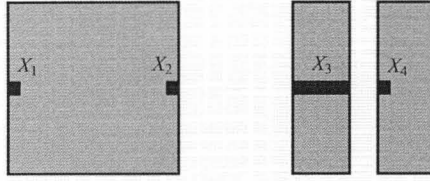[1] For the definition of $\preceq$, see §1.1.2.

77

FIGURE 3.1 – Situation motivating gap analysis.

Recall from §1.3.1 the following heuristic: in order to relate an invariant measure to a Hausdorff measure, one needs two ingredients:

1. One needs precise controls over what values the invariant measure takes on the cylinder sets; and

2. One needs precise controls over the number of cylinder sets that a given set of diameter δ can intersect.

The measure constructed in [39] satisfies the first property, but only satisfies the second property when restricted to hyperspaces of a symbol spaces, $\mathbb{H}(E^\infty)$.

To see this, suppose we have the compact sets $X_i \subset X$ depicted in Figure 3.1. Define $A = X_2 \cup X_3$, $B = X_1 \cup X_2 \cup X_3$, and $C = X_1 \cup X_2 \cup X_3 \cup X_4$. It is easy to see that

$$d_H(B, C) < d_H(A, B) = d_H(A, C).$$

Now suppose the sets containing the X_i are cylinders at some level of an IFS construction. If we consider the sets $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{H}(X)$ defined by $\mathcal{U}_1 = \{A, B\}$ and $\mathcal{U}_2 = \{A, B, C\}$, then $|\mathcal{U}_1| = |\mathcal{U}_2|$, but the constituent sets of $\mathcal{U}_2$ intersect more cylinders than that of $\mathcal{U}_1$. Since the diameter of an arbitrary collection of compact subsets of X is unaffected by the number of cylinders that its constituent sets intersect, we have no way of controlling how many hypercylinders such a collection can intersect. Thus any hope constructing a measure that satisfies the second of

the above properties is lost unless we can force some situation where $|\mathcal{U}_1| \neq |\mathcal{U}_2|$ if the constituent sets of the $\mathcal{U}_i$ intersect different numbers of cylinders. In order to force this situation, however, we need only have the cylinder diameters be small enough relative to the gaps. The following lemma makes precise this notion.

LEMMA 3.1. *Let* $\mathcal{F}_G$ *be a GDIFS consisting of conformal maps satisfying the SSC with invariant set list* $(K_v)_{v \in V}$. *Fix* $v \in V$ *and let* $L \subset (E_v)_A^*$ *be such that* $\{[\gamma] : \gamma \in L\}$ *partitions* $(E_v)_A^\infty$. *For* $F \in \mathbb{H}(K_v)$ *define*

$$L(F) = \{\gamma \in L : J_\gamma \cap F \neq \emptyset\}.$$

If $\mathcal{U} \subseteq \mathbb{H}(K_v)$ *satisfies* $|\mathcal{U}| < a$ *where*

$$a = \inf \{d(x, y) : x \in J_\gamma, y \in J_\tau,\ i(\gamma_1) = i(\tau_1) = v, \gamma \neq \tau,\ \gamma, \tau \in L(F)\}$$

and $F \in \mathcal{U}$, *then* $\mathcal{U} \subseteq \mathcal{K}_{L(F)}$.

Proof. Suppose $T \in \mathbb{H}(K_v)$ is such that $T \cap J_\gamma = \emptyset$ for some $\gamma \in L(F)$. Then since $F \cap J_\gamma \neq \emptyset$, $d_H(T, F) \geq a$. But this in turn means if $\mathcal{U} \subseteq \mathbb{H}(K_v)$ is such that $T, F \in \mathcal{U}$, then $|\mathcal{U}| \geq a$, a contradiction. So any set $\mathcal{U} \subseteq \mathbb{H}(K_v)$ satisfying $F \in \mathcal{U}$ and $|\mathcal{U}| < a$ satisfies $T \cap J_\gamma \neq \emptyset$ for every $T \in \mathcal{U}$, $\gamma \in L(F)$. A similar argument shows that any set $\mathcal{U} \subseteq \mathbb{H}(K_v)$ satisfying $F \in \mathcal{U}$ and $|\mathcal{U}| < a$ satisfies $T \cap J_\gamma = \emptyset$ for every $T \in \mathcal{U}$, $\gamma \in L \setminus L(F)$. It follows that $\mathcal{U} \subseteq \mathcal{K}_{L(F)}$ whenever $F \in \mathcal{U}$ and $|\mathcal{U}| < a$. $\qquad\square$

REMARK 4. Since similitudes are special cases of conformal maps, Lemma 3.1 covers the case of self-similar GDIFS. Also, since an IFS may be viewed as a GDIFS where the Mauldin-Williams graph has only one vertex, Lemma 3.1 covers the case of CIFS.

We now address the case of the hyperspace of a graph-self-similar set in $\mathbb{R}^d$. We first define the sizes of the gaps between cylinders, which will be critical in our analysis. Define

$$g_v = \inf \left\{ d(x,y) : x \in w_{e_1}(J_{t(e_1)}), y \in w_{e_2}(J_{t(e_2)}), e_1 \neq e_2, e_1, e_2 \in E_v \right\}$$

and

$$g = \min\{g_v : v \in V\}.$$

So g_v represents the closest that two points from distinct level-1 cylinders in J_v can get to one another, and g represents the closest that two points from any distinct level-1 cylinders can get to one another. Recall that we are considering the seed sets under the metrics $\{\tilde{d}_v\}_{v \in V}$, and that under these metrics each seed set has diameter less than 1, hence $g < 1$. We then choose the parameter u by letting

$$0 < u < r_{\min}g.$$

This parameter will play the analogous role to the parameter u in the proof of Theorem 3.2 in [39]. We also wish to define modified levels as is done in McClure's proof. To this end define

$$L_k^v = \{\gamma \in E_A^* : i(\gamma) = v, r_\gamma \leq u^k < r_{\gamma-}\}.$$

Note that it follows that $r_{\min}u^k < r_\gamma \leq u^k$ for each $\gamma \in L_k^v$. Finally we define the modified-level-k gap size by

$$g_k^v = \inf\{d(x,y) : x \in J_\gamma, y \in J_\tau, \gamma \neq \tau, \gamma, \tau \in L_k^v\}.$$

So g_v^k represents the closest that two points from distinct modified-level-k cylinders in J_v can get to one another. We proceed with some analysis of gap sizes.

PROPOSITION 3.1. *Let $\eta \in E_A^*$, then*

$$\inf\{d(x,y) : x \in J_{\eta e_1}, y \in J_{\eta e_2}\} \geq r_\eta g_v$$

for all $e_1, e_2 \in E_{t(\eta)}$ such that $e_1 \neq e_2$.

Proof. Fix $\eta \in E_A^*$. Applying the definition of g_v we see that

$$\inf\{d(x,y) : x \in J_{\eta e_1}, y \in J_{\eta e_2}\} = \inf\{d(w_\eta(x), w_\eta(y)) : x \in J_{e_1}, y \in J_{e_2}\}$$

$$= \inf\{r_\eta d(x,y) : x \in J_{e_1}, y \in J_{e_2}\}$$

$$= r_\eta \inf\{d(x,y) : x \in J_{e_1}, y \in J_{e_2}\}$$

$$\geq r_\eta g_v$$

for all $e_1, e_2 \in E_{t(\eta)}$ such that $e_1 \neq e_2$. $\qquad\square$

PROPOSITION 3.2. $g_k^v \geq u^{k+1}$ *for all $k \geq 1$.*

Proof. Fix $k \geq 1$ and let $\gamma, \tau \in L_k^v$ be such that $\gamma \neq \tau$. Let $\eta \in E_A^*$ be the greatest common ancestor of γ and τ. This means there exist $\gamma', \tau' \in E_A^*$ such that $\gamma = \eta\gamma'$ and $\tau = \eta\tau'$. Again we have $J_\gamma \subset J_{\eta\gamma_1'}$ and $J_\tau \subset J_{\eta\tau_1'}$. It follows from Proposition 3.1 and by our choice of u that

$$\inf\{d(x,y) : x \in J_\gamma, y \in J_\tau\} \geq \inf\{d(x,y) : x \in J_{\eta\gamma_1'}, y \in J_{\eta\tau_1'}\}$$

$$\geq r_\eta g \geq r_\gamma g \geq r_{\min} u^k g \geq u^{k+1}.$$

As $\gamma, \tau \in L_k^v$ were arbitrary (such that $\gamma \neq \tau$), it follows that $g_k^v \geq u^{k+1}$. $\qquad\square$

With Proposition 3.2 in mind, we make parameter choices and proceed to construct a measure on $\mathbb{H}(K_v)$. We make the following parameter choices: fix $\beta = 1/u^{s_0}$, $s < s_0$ and let $L_v = \#L_1^v$, $\beta_s' = 1/u^s$, and $\alpha_v = \max\{\frac{\log L_v}{\log \beta}, 1\}$. The parameters L_v and β_s' are the GDIFS analogues of parameters in McClure's proof, while α_v is introduced to correct a small gap in his induction argument. Recall that the Perron numbers $\{\lambda_v\}_{v \in V}$ and the invariant measure list $(\mu_v)_{v \in V}$ satisfy

$$\mu_{i(\gamma)}(J_\gamma) = r_\gamma^{s_0} \left(\frac{\lambda_{t(\gamma)}}{\lambda_{i(\gamma)}}\right)^{s_0}$$

81

for every $\gamma \in E_A^*$. McClure uses the very strong property that the Hausdorff measure of a cylinder for a self-similar symbol space satisfies $\mathcal{H}^{s_0}([\sigma]) = ||\sigma||^{s_0}$, but as this property is only used for counting purposes, the weaker property stated above in terms of the invariant measures will suffice. Define

$$c_* = \left(\frac{\lambda_{\max}}{\lambda_{\min}} \right)^{2s_0}$$

and choose $p_s \in (0,1)$ such that $\beta_s' < p_s \beta$, choose $j_* \in \mathbb{N}$ such that $p_s^{j_*} < 1/c_*$, and choose $j > j_*$ such that

$$\left(\frac{\beta_s'}{p_s \beta} \right)^j < \frac{1}{\beta}.$$

Since $u < 1$ it follows that $p_s^j \beta^{j-1} > (\beta_s')^j > 1$. In addition we define for $\gamma \in L_m^v$ ($m \le k$),

$$L_{k,\gamma}^v = \{ \tau \in L_k^v : \tau \text{ is a descendant of } \gamma \}.$$

Again, p_s and j are chosen analogously to parameters in McClure's proof, while j_* and c_* are chosen to allow the graph-directed setting. **Throughout this section, $L = L_v$, $\alpha = \alpha_v$, and $L_k = L_k^v$ will be understood to depend on $v \in V$, and $\beta' = \beta_s'$, $p = p_s$ will be understood to depend on $s < s_0$.**

The point in defining the set L_k is that the levels are much easier to deal with when their diameters satisfy some approximate uniformity. This nice property comes at a cost, however, in the form of lost knowledge of the cardinalities of the levels. We have the following lemma to help us overcome this issue. Note that the method of argument is the same as that in [39], but the proof uses the invariant measure instead of the Hausdorff measure and has been adjusted to allow the extra parameters.

LEMMA 3.2. *The following inequalities hold*

1. $(L/c_*)\beta^{k-2} < \#L_k < Lc_*\beta^k$

2. $(1/c_*)\beta^{j-1} < \#L_{(k+1)j,\gamma} < c_*\beta^{j+1}$ for each $\gamma \in L_{kj}$.

Proof. Using the definitions of L_k and u we have

$$\mu_{i(\gamma)}(J_\gamma) = r_\gamma^{s_0}\left(\frac{\lambda_{t(\gamma)}}{\lambda_{i(\gamma)}}\right)^{s_0} \leq (u^{s_0})^k \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{s_0} = \frac{1}{\beta^k}\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{s_0}$$

and

$$\mu_{i(\gamma)}(J_\gamma) = r_\gamma^{s_0}\left(\frac{\lambda_{t(\gamma)}}{\lambda_{i(\gamma)}}\right)^{s_0} \geq (u^{s_0})^{k+1}\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{s_0} = \frac{1}{\beta^{k+1}}\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{s_0}$$

for every $\gamma \in L_k$. In particular this means that

$$\frac{1}{\beta^2}\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{s_0} < \mu_{i(\gamma)}(J_\gamma) < \frac{1}{\beta}\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{s_0}$$

for each $\gamma \in L_1$. Now let $\gamma \in L_1$ and assume $\#L_{k,\gamma} \geq (\lambda_{\max}/\lambda_{\min})^{2s_0}\beta^k$. Then

$$\mu_{i(\gamma)}(J_\gamma) = \sum_{\tau \in L_{k,\gamma}} \mu_{i(\gamma)}(J_\tau) \geq \#L_{k,\gamma}\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{s_0}\frac{1}{\beta^{k+1}} \geq \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{s_0}\frac{1}{\beta}$$

which is a contradiction. Hence $\#L_{k,\gamma} < (\lambda_{\max}/\lambda_{\min})^{2s_0}\beta^k$ and it follows that

$$\#L_k = \sum_{\gamma \in L_1} \#L_{k,\gamma} < L\left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{2s_0}\beta^k = Lc_*\beta^k.$$

A similar line of reasoning gives $\#L_k > (L/c_*)\beta^{k-2}$.

The proof of (2) follows a similar line of reasoning. The additivity of the invariant measures again gives us

$$\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{s_0}\frac{1}{\beta^{kj+1}} < \mu_{i(\gamma)}(J_\gamma) < \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{s_0}\frac{1}{\beta^{kj}}$$

for $\gamma \in L_{kj}$ and

$$\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{s_0}\frac{1}{\beta^{(k+1)j+1}} < \mu_{i(\gamma)}(J_\gamma) < \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{s_0}\frac{1}{\beta^{(k+1)j}}$$

for $\gamma \in L_{(k+1)j}$. If we assume $\#L_{(k+1)j,\gamma} \geq c_*\beta^{j+1}$ where $\gamma \in L_{kj}$, then

$$\mu_{i(\gamma)}(J_\gamma) = \sum_{\tau \in L_{(k+1)j,\gamma}} \mu_{i(\gamma)}(J_\tau) \geq \#L_{(k+1)j,\gamma}\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)^{s_0}\frac{1}{\beta^{(k+1)j+1}} \geq \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{s_0}\frac{1}{\beta^{kj}}$$

which is a contradiction. Hence $\#L_{(k+1)j,\gamma} < c_*\beta^{j+1}$. A similar argument using the other inequality finishes the proof. $\square$

The following trivial proposition is omitted in [39], but in the interest of precision, we state it.

PROPOSITION 3.3. *Let $M \in \mathbb{N}$ and $x, y > 0$, then*

1. $M > \lfloor x \rfloor \implies M \geq \lceil x \rceil$

2. $x \lceil y \rceil \geq \lfloor xy \rfloor$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions, respectively.

We are finally ready to construct the necessary measure on the hyperspace that will allow us to compute the lower bound for the dimension. Again, the argument is essentially the same as that in [39], but with a few gaps and errors corrected, and with our extra parameters included to allow $\mathbb{H}(K_v)$ instead of simply $\mathbb{H}(E^\infty)$.

LEMMA 3.3. *Let $\mathcal{F}_g$ be a GDIFS satisfying the SSC that has unique invariant set list $(K_v)_{v \in V}$. For each $v \in V$ there exists a constant $L' = L'_v > 0$ and a Borel measure $\mathcal{M} = \mathcal{M}_v$ supported in $\mathbb{H}(K_v)$ such that*

$$\mathcal{M}(\mathcal{K}_A) \leq 2^{-L'(p^j \beta^{j-1})^k}$$

for every $k \geq 1$ and every nonempty $A \subseteq L_{kj}$.

Proof. Given $A \subseteq L_{kj}$, define

$$\pi(A) = \frac{\#A}{L\beta^{kj-\alpha}}.$$

We will construct a Cantor net, $\mathcal{A}$, that covers $\mathbb{H}(K)$ and that satisfies Definition 15. We then let κ be a mass distributing function on $\mathcal{A}$ and apply Lemmas 2.1,

2.2 to get the existence and uniqueness of $\mathcal{M}$. In addition to the properties listed in Definition 15, however, we will require that $A \subseteq L_{kj}$ and

$$\pi(A) \geq \frac{p^{kj}}{\beta^k}$$

for each $\mathcal{K}_A \in \mathcal{A}_k$ and $k \geq 0$.

First we let $\mathcal{A}_0 = \{\mathcal{K}_{\{\Lambda\}}\} = \{\mathbb{H}(K)\}$ where Λ denotes the empty string. We would then like $\{\Lambda\}$ to satisfy

$$\pi(\{\Lambda\}) \geq \frac{p^{0j}}{\beta^0} = 1.$$

It is easily checked that this is true by our choice of α. Now assume that $\mathcal{A}_k$ has been constructed for $k \geq 0$. We construct $\mathcal{A}_{k+1}$ by constructing $\mathcal{A}_{k+1,A}$ for each $A \in \mathcal{A}_k$. An arbitrary descendent $\mathcal{K}_B$ of $\mathcal{K}_A$ comes from a set B of the form

$$B = \bigcup_{\gamma \in A} B_\gamma$$

where $B_\gamma \subseteq L_{(k+1)j,\gamma}$ is nonempty for each $\gamma \in A$. Since $\pi(A) \geq p^{kj}/\beta^k$ by assumption, we have

$$\#A \geq \frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha}$$

which means

$$\#A \geq \left\lceil \frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha} \right\rceil \geq p^{j-j*} \left\lceil \frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha} \right\rceil.$$

So we may choose $\lceil p^{j-j*} \lceil (p^{kj}/\beta^k) L\beta^{kj-\alpha} \rceil \rceil$ of the γ's in A, and call this set A_1. Then we have

$$\#\left(\bigcup_{\gamma \in A_1} L_{(k+1)j,\gamma} \right) = \sum_{\gamma \in A_1} (\#L_{(k+1)j,\gamma})$$

$$\geq (\#A_1) \cdot \left(\min_{\gamma \in A_1} \{\#L_{(k+1)j,\gamma}\} \right)$$

$$> (p^{j-j*}) \left\lceil \frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha} \right\rceil (1/c_*)(\beta^{j-1})$$

$$\geq (p^{-j*})(1/c_*) \left\lfloor \frac{p^{(k+1)j}}{\beta^{k+1}} L\beta^{(k+1)j-\alpha} \right\rfloor$$

$$\geq \left\lfloor \frac{p^{(k+1)j}}{\beta^{k+1}} L\beta^{(k+1)j-\alpha} \right\rfloor.$$

It follows from Proposition 3.3 that

$$\#\left(\bigcup_{\gamma \in A_1} L_{(k+1)j,\gamma}\right) \geq \left\lceil \frac{p^{(k+1)j}}{\beta^{k+1}} L\beta^{(k+1)j-\alpha} \right\rceil \geq \frac{p^{(k+1)j}}{\beta^{k+1}} L\beta^{(k+1)j-\alpha}.$$

Let $A_2 = A \setminus A_1$, then for each $A \in \mathcal{A}_k$ we have shown the existence of at least one descendant of A of the form

$$B = \left(\bigcup_{\gamma \in A_1} L_{(k+1)j,\gamma}\right) \cup \left(\bigcup_{\gamma \in A_2} B_\gamma\right)$$

where $B_\gamma \subseteq L_{(k+1)j,\gamma}$ is nonempty for each $\gamma \in A_2$. By the above inequality any such B satisfies property $\pi(B) \geq p^{(k+1)j}/\beta^{k+1}$. This shows that $\mathcal{A}_{k+1}$ exists given the existence of $\mathcal{A}_k$, and the existence of all of $\mathcal{A}$ follows by induction. Let κ be a mass distributing function on $\mathcal{A}$, then by Theorem 1.3 and Lemma 1.2 we have that $\mathcal{M}$ exists, is unique, is Borel, and satisfies $\mathcal{M}(\mathcal{K}_A) = \kappa(\mathcal{K}_A)$ for every $\mathcal{K}_A \in \mathcal{A}$.

To finish the proof we must show the existence of $L' > 0$. We start by putting a lower bound on $\#\mathcal{A}_{k+1,A}$, i.e. the number of $B \subset L_{(k+1)j}$ per $A \subset L_{kj}$, so as to put an upper bound on $\mathcal{M}(\mathcal{K}_A)$. The part of a given B that we get from A_1 is fixed since B contains all of $L_{(k+1)j,\gamma}$ for each $\gamma \in A_1$, but the part we get from A_2 is arbitrary so long as $B_\gamma \neq \emptyset$ for each $\gamma \in A_2$. Thus

$$\#\{B : B \text{ descendent of } A\} \geq \prod_{\gamma \in A_2} (\# \text{ nonempty subsets of } L_{(k+1)j,\gamma}).$$

By Lemma 3.2(2) we have $\#L_{(k+1)j,\gamma} \geq (1/c_*)\beta^{j-1}$, and so

$$(\# \text{ nonempty subsets of } L_{(k+1)j,\gamma}) \geq 2^{(1/c_*)\beta^{j-1}} - 1$$

for each $\gamma \in A_2$. We also have

$$\#A_2 = \#A - \#A_1$$

86

so

$$\#A_2 \geq \left\lceil \frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha} \right\rceil - \left\lceil p^{j-j_*} \left\lceil \frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha} \right\rceil \right\rceil$$

$$\geq (1 - p^{j-j_*}) \left\lceil \frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha} \right\rceil - 1$$

$$\geq (1 - p^{j-j_*}) \cdot \left(\frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha} \right) - 1$$

thus

$$\#\{B : B \text{ descendent of } A\} \geq \left(2^{(1/c_*)\beta^{j-1}} - 1\right)^{(1-p^{j-j_*})(\frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha})-1}.$$

Since we distribute $\mathcal{M}(\mathcal{K}_A)$ evenly among the $\mathcal{K}_B \in \mathcal{A}_{k+1,A}$ we have

$$\mathcal{M}(\mathcal{K}_A) = (\#\{B : B \text{ descendent of } A\}) \cdot \mathcal{M}(\mathcal{K}_B)$$

$$\geq \left(2^{(1/c_*)\beta^{j-1}} - 1\right)^{(1-p^{j-j_*})(\frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha})-1} \cdot \mathcal{M}(\mathcal{K}_B)$$

or equivalently

$$\mathcal{M}(\mathcal{K}_B) \leq \left(2^{(1/c_*)\beta^{j-1}} - 1\right)^{-(1-p^{j-j_*})(\frac{p^{kj}}{\beta^k} L\beta^{kj-\alpha})+1} \cdot \mathcal{M}(\mathcal{K}_A).$$

We can make the same argument for $\mathcal{K}_A$ that we just made for $\mathcal{K}_B$ and continue iteratively to get

$$\mathcal{M}(\mathcal{K}_A) \leq \prod_{i=0}^{k-1} \left(\left(2^{(1/c_*)\beta^{j-1}} - 1\right)^{-(1-p^{j-j_*})(\frac{p^{ij}}{\beta^i} L\beta^{ij-\alpha})+1} \right).$$

Notice that

$$\sum_{i=0}^{k-1} \left(-(1-p^{j-j_*}) \left(\frac{p^{ij}}{\beta^i} L\beta^{ij-\alpha} \right) + 1 \right) = -(1-p^{j-j_*})L\beta^{-\alpha} \sum_{i=0}^{k-1} \left(\frac{p^{ij}}{\beta^i} \beta^{ij} \right) + k$$

$$= -(1-p^{j-j_*})L\beta^{-\alpha} \sum_{i=0}^{k-1} \left(p^j \beta^{j-1} \right)^i + k$$

$$= -(1-p^{j-j_*})L\beta^{-\alpha} \frac{(p^j \beta^{j-1})^k - 1}{p^j \beta^{j-1} - 1} + k.$$

Also $(1 - p^{j-j_*})L\beta^{-\alpha}$ is constant in k, and since $p^j\beta^{j-1} > 1$ we also have

$$\lim_{k\to\infty} \frac{1 - (p^j\beta^{j-1})^{-k}}{p^j\beta^{j-1} - 1} = \frac{1}{p^j\beta^{j-1} - 1}$$

and

$$\lim_{k\to\infty} \frac{k}{(p^j\beta^{j-1})^k} = 0.$$

It follows that

$$(1 - p^{j-j_*})L\beta^{-\alpha}\frac{1 - (p^j\beta^{j-1})^{-k}}{p^j\beta^{j-1} - 1} - \frac{k}{(p^j\beta^{j-1})^k} \geq L'$$

and hence

$$-(1 - p^{j-j_*})L\beta^{-\alpha}\frac{(p^j\beta^{j-1})^k - 1}{p^j\beta^{j-1} - 1} + k \leq -L'(p^j\beta^{j-1})^k$$

for some constant $L' > 0$. Thus we have that

$$\mathcal{M}(\mathcal{K}_A) \leq (2^{\beta^{j-1}} - 1)^{-L'(p^j\beta^{j-1})^k} \leq 2^{-L'(p^j\beta^{j-1})^k}$$

which completes the proof. $\qquad\square$

Given the measure $\mathcal{M}$ we can compute the lower bound for the dimension of $\mathbb{H}(K_v)$.

THEOREM 3.1. *Let $\mathcal{F}_g$ be a GDIFS in $\mathbb{R}^d$ satisfying the SSC. If $(K_v)_{v\in V}$ is the unique invariant set list for $\mathcal{F}_g$ and $\dim_H K_v = s_0$ for each $v \in V$, then $\mathcal{H}^{\phi_s}(\mathbb{H}(K_v)) > 0$ for all $s < s_0$.*

Proof. The proof will follow by applying Corollary 1.2 to the measure constructed in Lemma 3.3.

Let $v \in V$ be arbitrary and let $\mathcal{M} = \mathcal{M}_v$ be the measure on $\mathbb{H}(K_v)$ constructed in Lemma 3.3. Fix $F \in \mathbb{H}(K_v)$ and let $L(F)$ be as defined in Lemma 3.1

where $L = L_{kj}^v$ (note that this in turn means $a = g_{kj}^v$). Recall from the definition of $\mu_\delta(x)$ that

$$\mathcal{M}_{u^{kj+1}}(F) = \sup\{\mathcal{M}(\mathcal{U}) : F \in \mathcal{U},\ \mathcal{U} \text{ is Borel},\ |\mathcal{U}| \le u^{kj+1}\}$$

where u^{kj+1} plays the role of δ and F plays the role of x. By proposition 3.2 we have that $u^{kj+1} \le g_{kj}^v$ and it follows that $|\mathcal{U}| < a$ for each $\mathcal{U}$. Hence by Lemma 3.1 $\mathcal{U} \subset \mathcal{K}_{L(F)}$, whence it follows that $\mathcal{M}(\mathcal{U}) \le \mathcal{M}(\mathcal{K}_{L(F)})$ for every Borel set $\mathcal{U}$ with $F \in \mathcal{U}$ and $|\mathcal{U}| \le u^{kj+1}$. It follows that $\mathcal{M}_{u^{kj+1}}(F) \le \mathcal{M}(\mathcal{K}_{L(F)})$. Finally, since $\beta' = 1/u^s < \beta$ and $p^j \beta^{j-1} > (\beta')^j$, we have that

$$\frac{\mathcal{M}_{u^{kj+1}}(F)}{\phi_s(u^{kj+1})} \le \frac{\mathcal{M}(\mathcal{K}_{L(F)})}{\phi_s(u^{kj+1})} \le \frac{2^{-L'(p^j \beta^{j-1})^k}}{2^{-(\beta')^{kj+1}}}$$

$$= 2^{(\beta')^{kj+1} - L'(p^j \beta^{j-1})^k} \to 0$$

as $k \to \infty$. Thus $\overline{D}_{\mathcal{M}}^{\phi_s}(F, (u^{kj+1})_k) \le 1$ and the result follows from Corollary 1.2. $\qquad\square$

Combining Theorem 3.1 with the theory from Chapter 2, we can give our first main result of this chapter.

THEOREM 3.2. *Let* $\mathcal{F}_g = \{J_v, w_e\}_{v \in V, e \in E}$ *be a strongly connected graph-directed IFS with invariant set list* $(K_v)_{v \in V}$. *Suppose* $\mathcal{F}_g$ *satisfies the following:*

1. w_e is a similitude for each $e \in E$,

2. $J_v \subset \mathbb{R}^d$ is compact for each $v \in V$,

3. $\mathcal{F}_g$ satisfies the OSC.[2]

---

[2]Recall that by Theorem 1.1 the OSC and SOSC are equivalent in $\mathbb{R}^d$ for GDIFS consisting of similitudes. This allows us to assume OSC here, but it is very much the SOSC that is necessary for the argument to be valid.

Let $s_0 = \dim_H K_v$ and define $\phi_s(t) = 2^{-(1/t)^s}$. Then for all $v \in V$,

1. *$\mathcal{H}^{\phi_s}(\mathbb{H}(K_v)) = 0$ for all $s > s_0$, and*

2. *$\mathcal{H}^{\phi_s}(\mathbb{H}(K_v)) > 0$ for all $s < s_0$.*

Proof. Fix $v \in V$. We first show that $\mathcal{H}^{\phi_s}(\mathbb{H}(K_v)) = 0$ for all $s > s_0$. Recall that $|J_\gamma| = r_\gamma \lambda_{t(\gamma)}$ for each $\gamma \in E_A^*$, and that $r_\gamma < u^k$ for each $\gamma \in L_k^v$. This means that the collection

$$\{\mathcal{K}_A : A \subseteq L_k, A \neq \emptyset\}$$

forms a $(\lambda_{\max} u^k)$-cover of $\mathbb{H}(K_v)$. Let $N > \lambda_{\max}^{s_0} Lc_*$ and define

$$\psi_{N,s_0}(t) = 2^{-N(1/t)^{s_0}}.$$

Recalling from Lemma 3.2(1) that $\#L_k \leq Lc_* \beta^k$ we have

$$\begin{aligned}
\mathcal{H}^{\psi_{N,s_0}}_{\lambda_{\max} u^k}(\mathbb{H}(K_v)) &\leq (2^{\#L_k} - 1)\psi_{N,s_0}(\lambda_{\max} u^k) \\
&\leq (2^{Lc_* \beta^k} - 1)2^{-N\left(\frac{1}{\lambda_{\max} u^k}\right)^{s_0}} \\
&= (2^{Lc_* \beta^k} - 1)2^{-N\left(\frac{1}{\lambda_{\max}}\right)^{s_0} \beta^k} \\
&\leq 2^{Lc_* \beta^k - N\left(\frac{1}{\lambda_{\max}}\right)^{s_0} \beta^k} \leq 1.
\end{aligned}$$

It follows that $\mathcal{H}^{\psi_{N,s_0}}(\mathbb{H}(K_v)) \leq 1$. Thus by Corollary 1.1 we have that

$$\mathcal{H}^{\phi_s}(\mathbb{H}(K_v)) = \mathcal{H}^{\psi_{1,s}}(\mathbb{H}(K_v)) = 0$$

for all $s > s_0$, which completes the proof of the upper bound.

To prove the lower bound we will combine Theorem 2.2 with Theorem 3.1. First fix an arbitrary $s < s_0$ and choose a sub-GDIFS $\mathcal{F}_{G,\delta}$ that satisfies the SSC and satisfies $s < \dim_H K_{v,\delta} = s_\delta < s_0$ for each $v \in V$ (this is possible since $s_\delta \to s_0$ as $\delta \to 0$ by Theorem 2.2). Notice that $K_{v,\delta} \subseteq K_v$ implies $\mathbb{H}(K_{v,\delta}) \subseteq \mathbb{H}(K_v)$, hence

$$\mathcal{H}^{\phi_{s'}}(\mathbb{H}(K_v)) \geq \mathcal{H}^{\phi_{s'}}(\mathbb{H}(K_{v,\delta}))$$

for all $s' > 0$. Now applying Theorem 3.1 we have that

$$\mathcal{H}^{\phi_{s'}}(\mathbb{H}(K_{v,\delta})) > 0$$

for all $s' < s_\delta$. In particular this gives

$$\mathcal{H}^{\phi_s}(\mathbb{H}(K_v)) \geq \mathcal{H}^{\phi_s}(\mathbb{H}(K_{v,\delta})) > 0.$$

As $s < s_0$ was chosen arbitrarily, this completes the proof. $\qquad\square$

CHAPTER 4
HYPERSPACE DIMENSIONS FOR SELF-CONFORMAL SETS

4.1 Dimension Computations for Self-Conformal IFS

We proceed in much that same way for hyperspaces of self-conformal sets that we did for the hyperspaces of graph-self-similar sets. Let $c > 0$ be the constant from Corollary 1.5 so that

$$\frac{1}{c}|Dw_\sigma(z)||x - y| < |w_\sigma(x) - w_\sigma(y)| < c|Dw_\sigma(z)||x - y|$$

for all $\sigma \in E^*$, $x, y, z \in X$. Now define the gap size

$$g = \inf \left\{ d(x, y) : x \in w_{e_1}(X), y \in w_{e_2}(X), \ e_1 \neq e_2, \ e_1, e_2 \in E \right\}.$$

Since we are assuming $|X| = 1$ it follows that $g \leq 1$. Let $0 < u < (1/c)r_{\min}g$ such that $1/u^{s_0} = \beta \in \mathbb{N}$. Now fix $z_0 \in X$ and define

$$L_k = \left\{ \sigma \in E^* : |Dw_\sigma(z_0)| \leq u^k < |Dw_{\sigma-}(z_0)| \right\}.$$

By Proposition 1.8 and Definition 9 it follows that

$$r_{\min}u^k < |Dw_\sigma(z_0)| \leq u^k$$

for each $\sigma \in L_k$. Let

$$g_k = \inf \left\{ d(x, y) : x \in X_\sigma, y \in X_\tau, \ \sigma \neq \tau, \ \sigma, \tau \in L_k \right\}$$

and note that $g_1 \neq g$.

92

PROPOSITION 4.1. *Let $\eta \in E^*$ and $e_1, e_1 \in E$ such that $e_1 \neq e_2$. then*

$$\inf\{d(x,y) : x \in X_{\eta e_1}, y \in X_{\eta e_2}\} \geq \frac{g}{c}|Dw_\eta(z)|$$

for all $z \in X$.

Proof. Since $X_{\eta e_1}$ and $X_{\eta e_2}$ are compact, there exist $\hat{x}, \hat{y} \in X$ such that

$$\inf\{d(x,y) : x \in X_{\eta e_1}, y \in X_{\eta e_2}\} = d(w_{\eta e_1}(\hat{x}), w_{\eta e_2}(\hat{y})).$$

Then by Corollary 1.5 and by the definition of g we have

$$d(w_{\eta e_1}(\hat{x}), w_{\eta e_2}(\hat{y})) \geq \frac{1}{c}d(w_{e_1}(\hat{x}), w_{e_2}(\hat{y}))|Dw_\eta(z)| \geq \frac{g}{c}|Dw_\eta(z)|$$

for all $z \in X$. $\qquad\square$

PROPOSITION 4.2. $g_k \geq u^{k+1}$ *for all $k \geq 1$.*

Proof. Fix $k \geq 1$ and let $\sigma, \tau \in L_k$ be such that $\sigma \neq \tau$. If $\sigma_1 \neq \tau_1$ then since $X_\sigma \subset X_{\sigma_1}$ and $X_\tau \subset X_{\tau_1}$, it follows that

$$\inf\{d(x,y) : x \in X_\sigma, y \in X_\tau\} \geq \inf\{d(x,y) : x \in X_{\sigma_1}, y \in X_{\tau_1}\} \geq g \geq u^{k+1}.$$

If $\sigma_1 = \tau_1$ then let $\eta \in E^*$ be the greatest common ancestor of σ and τ. This means there exist $\sigma', \tau' \in E^*$ such that $\sigma = \eta\sigma'$ and $\tau = \eta\tau'$. Again we have $X_\sigma \subset X_{\eta\sigma_1'}$ and $X_\tau \subset X_{\eta\tau_1'}$. By choosing $u < (1/c)r_{\min}g$ we see that

$$u^{k+1} = u \cdot u^k < (1/c)r_{\min}g \cdot u^k.$$

It then follows from Proposition 4.1 that

$$\begin{aligned}
\inf\{d(x,y) : x \in X_\sigma, y \in X_\tau\} &\geq \inf\{d(x,y) : x \in X_{\eta\sigma_1'}, y \in X_{\eta\tau_1'}\} \\
&\geq \frac{g}{c}|Dw_\eta(z_0)| \\
&\geq \frac{g}{c}\left(\min_{\sigma \in L_k}|Dw_\sigma(z_0)|\right) \\
&\geq \frac{g}{c}r_{\min}u^k > u^{k+1}.
\end{aligned}$$

As $\sigma, \tau \in L_k$ were arbitrary (such that $\sigma \neq \tau$), it follows that $g_k \geq u^{k+1}$. $\qquad\square$

Similar to the case with GDIFS, we make the following parameter choices:

Let $L = \#L_1$, $\beta'_s = 1/u^s$, and $\alpha = \max\{\frac{\log L - \log M^2}{\log \beta}, 1\}$. Recall from Remark 3 (which followed Lemma 1.8) that there exists a constant $M \geq 1$ such that

$$M^{-1}|Dw_\sigma(x)|^{s_0} \leq \mu(X_\sigma) \leq M|Dw_\sigma(x)|^{s_0}$$

for each $\sigma \in E^*$, where μ is the invariant measure on K. Choose $p_s \in (0,1)$ such that $\beta'_s < p_s\beta$, choose $j_* \in \mathbb{N}$ such that $p_s^{j_*} < M^{-2}$, and choose $j > j_*$ such that

$$\left(\frac{\beta'_s}{p_s\beta}\right)^j < \frac{1}{\beta}.$$

In addition we define for $\sigma \in L_m$ $(m \leq k)$,

$$L_{k,\sigma} = \{\tau \in L_k : \tau \text{ is a descendant of } \sigma\}.$$

Again we will write $\beta' = \beta'_s$ and $p = p_s$ where it is understood that these values depend upon $s < s_0$. The point in defining the set L_k is that the levels are much easier to deal with when their diameters satisfy some approximate uniformity. This nice property comes at a cost, however, in the form of lost knowledge of the cardinalities of the levels. We have the following lemma to help us overcome this issue.

LEMMA 4.1. *Let all parameters be as defined above, then the following inequalities hold*

1. $LM^{-2}\beta^{k-2} < \#L_k < LM^2\beta^k$,

2. $M^{-2}\beta^{j-1} < \#L_{(k+1)j,\sigma} < M^2\beta^{j+1}$ *for each* $\sigma \in L_{kj}$.

Proof. Using the definitions of L_k and u we have

$$\mu(X_\sigma) < M|Dw_\sigma(z_0)|^{s_0} \leq M(u^{s_0})^k = \frac{M}{\beta^k}$$

94

and

$$\mu(X_\sigma) > M^{-1}|Dw_\sigma(z_0)|^{s_0} \geq M^{-1}(u^{s_0})^k = \frac{1}{M\beta^{k+1}}.$$

For every $\sigma \in L_k$. In particular this means that

$$\frac{1}{M\beta^2} < \mu(X_\sigma) < \frac{M}{\beta}$$

for each $\sigma \in L_1$. Now let $\sigma \in L_1$ and assume $\#L_{k,\sigma} \geq M^2\beta^k$. Then

$$\mu(X_\sigma) = \sum_{\tau \in L_{k,\sigma}} \mu(X_\tau) \geq \#L_{k,\sigma}\frac{1}{M\beta^{k+1}} \geq \frac{M}{\beta}$$

which is a contradiction. Hence $\#L_{k,\sigma} < M^2\beta^k$ and it follows that

$$\#L_k = \sum_{\sigma \in L_1} \#L_{k,\sigma} < LM^2\beta^k.$$

A similar line of reasoning gives $\#L_k > (L/M^2)\beta^{k-2}$ and completes the proof of (1).

The proof of (2) follows a similar line of reasoning. The additivity of the invariant measure again gives us

$$\frac{1}{M\beta^{kj+1}} < \mu(X_\sigma) < \frac{M}{\beta^{kj}}$$

for $\sigma \in L_{kj}$ and

$$\frac{1}{M\beta^{(k+1)j+1}} < \mu(X_\sigma) < \frac{M}{\beta^{(k+1)j}}$$

for $\sigma \in L_{(k+1)j}$. If we assume $\#L_{(k+1)j,\sigma} \geq M^2\beta^{j+1}$ where $\sigma \in L_{kj}$, then

$$\mu(X_\sigma) = \sum_{\tau \in L_{(k+1)j,\sigma}} \mu(X_\tau) \geq \#L_{(k+1)j,\sigma}\frac{1}{M\beta^{(k+1)j+1}} \geq \frac{M}{\beta^{kj}}$$

which is a contradiction. Hence $\#L_{(k+1)j,\sigma} < M^2\beta^{j+1}$. A similar argument using the other inequality finishes the proof. $\square$

LEMMA 4.2. *Let $\mathcal{F}_c$ be a conformal IFS satisfying the SSC that has unique invariant set K. There exists a constant $L' > 0$ and a Borel measure $\mathcal{M}$ supported in $\mathbb{H}(K)$ such that*

$$\mathcal{M}(\mathcal{K}_A) \leq 2^{-L'(p^j\beta^{j-1})^k}$$

for every $k \geq 1$ and every nonempty $A \subseteq L_{kj}$.

Proof. Given $A \subseteq L_{kj}$, define

$$\pi(A) = \frac{\#A}{LM^{-2}\beta^{kj-\alpha}}.$$

We will construct a Cantor net, $\mathcal{A}$, that covers $\mathbb{H}(K)$ and that satisfies Definition 15. We then let κ be a mass distributing function on $\mathcal{A}$ and apply Lemmas 2.1, 2.2 to get the existence and uniqueness of $\mathcal{M}$. In addition to the properties listed in Definition 15, however, we will require that $A \subseteq L_{kj}$ and

$$\pi(A) \geq \frac{p^{kj}}{\beta^k}$$

for each $\mathcal{K}_A \in \mathcal{A}_k$ and $k \geq 0$.

First we let $\mathcal{A}_0 = \{\mathcal{K}_{\{\Lambda\}}\} = \{\mathbb{H}(K)\}$ where Λ denotes the empty string. We would then like $\{\Lambda\}$ to satisfy

$$\pi(\{\Lambda\}) \geq \frac{p^{0j}}{\beta^0} = 1.$$

It is easily checked that this is true by our choice of α. Now assume that $\mathcal{A}_k$ has been constructed for $k \geq 0$. We construct $\mathcal{A}_{k+1}$ by constructing $\mathcal{A}_{k+1,A}$ for each $A \in \mathcal{A}_k$. An arbitrary descendent $\mathcal{K}_B$ of $\mathcal{K}_A$ comes from a set B of the form

$$B = \bigcup_{\sigma \in A} B_\sigma$$

where $B_\sigma \subseteq L_{(k+1)j,\sigma}$ is nonempty for each $\sigma \in A$. Since $\pi(A) \geq p^{kj}/\beta^k$ by assumption, we have

$$\#A \geq \frac{p^{kj}}{\beta^k}LM^{-2}\beta^{kj-\alpha}$$

which means

$$\#A \geq \left\lceil \frac{p^{kj}}{\beta^k} LM^{-2}\beta^{kj-\alpha} \right\rceil \geq p^{j-j*}\left\lceil \frac{p^{kj}}{\beta^k} LM^{-2}\beta^{kj-\alpha} \right\rceil .$$

So we may choose $\lceil p^{j-j*}\lceil (p^{kj}/\beta^k)LM^{-2}\beta^{kj-\alpha}\rceil\rceil$ of the σ's in A, and call this set A_1. Then we have

$$\#(\bigcup_{\sigma\in A_1} L_{(k+1)j,\sigma}) = \sum_{\sigma\in A_1}(\#L_{(k+1)j,\sigma})$$

$$\geq (\#A_1)\cdot\left(\min_{\sigma\in A_1}\{\#L_{(k+1)j,\sigma}\}\right)$$

$$> (p^{j-j*})\left\lceil \frac{p^{kj}}{\beta^k}LM^{-2}\beta^{kj-\alpha}\right\rceil (1/M^2)(\beta^{j-1})$$

$$\geq (p^{-j*})(1/M^2)\left\lfloor \frac{p^{(k+1)j}}{\beta^{k+1}}LM^{-2}\beta^{(k+1)j-\alpha}\right\rfloor$$

$$\geq \left\lfloor \frac{p^{(k+1)j}}{\beta^{k+1}}LM^{-2}\beta^{(k+1)j-\alpha}\right\rfloor .$$

It follows from Proposition 3.3 that

$$\#(\bigcup_{\sigma\in A_1} L_{(k+1)j,\sigma}) \geq \left\lceil \frac{p^{(k+1)j}}{\beta^{k+1}}LM^{-2}\beta^{(k+1)j-\alpha}\right\rceil \geq \frac{p^{(k+1)j}}{\beta^{k+1}}LM^{-2}\beta^{(k+1)j-\alpha}.$$

Let $A_2 = A \setminus A_1$, then for each $A \in \mathcal{A}_k$ we have shown the existence of at least one descendant of A of the form

$$B = \left(\bigcup_{\gamma\in A_1} L_{(k+1)j,\gamma}\right) \cup \left(\bigcup_{\gamma\in A_2} B_\gamma\right)$$

where $B_\gamma \subseteq L_{(k+1)j,\gamma}$ is nonempty for each $\gamma \in A_2$. By the above inequality any such B satisfies property $\pi(B) \geq p^{(k+1)j}/\beta^{k+1}$. This shows that $\mathcal{A}_{k+1}$ exists given the existence of $\mathcal{A}_k$, and the existence of all of $\mathcal{A}$ follows by induction. Let κ be a mass distributing function on $\mathcal{A}$, then by Theorem 1.3 and Lemma 1.2 we have that $\mathcal{M}$ exists, is unique, is Borel, and satisfies $\mathcal{M}(\mathcal{K}_A) = \kappa(\mathcal{K}_A)$ for every $\mathcal{K}_A \in \mathcal{A}$.

To finish the proof we must show the existence of $L' > 0$. We start by putting a lower bound on $\#\mathcal{A}_{k+1,A}$, i.e. the number of $B \subset L_{(k+1)j}$ per $A \subset L_{kj}$, so as to put an upper bound on $\mathcal{M}(\mathcal{K}_A)$. The part of a given B that we get from

A_1 is fixed since B contains all of $L_{(k+1)j,\gamma}$ for each $\gamma \in A_1$, but the part we get from A_2 is arbitrary so long as $B_\gamma \neq \emptyset$ for each $\gamma \in A_2$. Thus

$$\#\{B : B \text{ descendent of } A\} \geq \prod_{\gamma \in A_2} (\# \text{ nonempty subsets of } L_{(k+1)j,\gamma}).$$

By Lemma 4.1(2) we have

$$(\# \text{ nonempty subsets of } L_{(k+1)j,\gamma}) \geq 2^{(1/M^2)\beta^{j-1}} - 1$$

for each $\gamma \in A_2$. We also have

$$\#A_2 = \#A - \#A_1$$

so

$$
\begin{aligned}
\#A_2 &\geq \left\lceil \frac{p^{kj}}{\beta^k} L M^{-2} \beta^{kj-\alpha} \right\rceil - \left\lceil p^{j-j_*} \left\lceil \frac{p^{kj}}{\beta^k} L M^{-2} \beta^{kj-\alpha} \right\rceil \right\rceil \\
&\geq (1 - p^{j-j_*}) \left\lceil \frac{p^{kj}}{\beta^k} L M^{-2} \beta^{kj-\alpha} \right\rceil - 1 \\
&\geq (1 - p^{j-j_*}) \left(\frac{p^{kj}}{\beta^k} L M^{-2} \beta^{kj-\alpha} \right) - 1
\end{aligned}
$$

thus

$$\#\{B : B \text{ descendent of } A\} \geq \left(2^{(1/M^2)\beta^{j-1}} - 1 \right)^{(1-p^{j-j_*})(\frac{p^{kj}}{\beta^k} L M^{-2} \beta^{kj-\alpha})-1}.$$

Since we distribute $\mathcal{M}(\mathcal{K}_A)$ evenly among the $\mathcal{K}_B \in \mathcal{A}_{k+1,A}$ we have

$$
\begin{aligned}
\mathcal{M}(\mathcal{K}_A) &= (\#\{B : B \text{ descendent of } A\}) \cdot \mu(\mathcal{K}_B) \\
&\geq \left(2^{(1/M^2)\beta^{j-1}} - 1 \right)^{(1-p^{j-j_*})(\frac{p^{kj}}{\beta^k} L M^{-2} \beta^{kj-\alpha})-1} \cdot \mu(\mathcal{K}_B)
\end{aligned}
$$

or equivalently

$$\mathcal{M}(\mathcal{K}_B) \leq \left(2^{(1/M^2)\beta^{j-1}} - 1 \right)^{-(1-p^{j-j_*})(\frac{p^{kj}}{\beta^k} L M^{-2} \beta^{kj-\alpha})+1} \cdot \mathcal{M}(\mathcal{K}_A).$$

We can make the same argument for $\mathcal{K}_A$ that we just made for $\mathcal{K}_B$ and continue iteratively to get

$$\mathcal{M}(\mathcal{K}_A) \le \prod_{i=0}^{k-1} \left((2^{(1/M^2)\beta^{j-1}} - 1)^{-(1-p^{j-j_*})(\frac{p^{ij}}{\beta^i}LM^{-2}\beta^{ij-\alpha})+1} \right).$$

Notice that

$$\sum_{i=0}^{k-1} \left(-(1-p^{j-j_*}) \left(\frac{p^{ij}}{\beta^i}LM^{-2}\beta^{ij-\alpha} \right) + 1 \right) = -(1-p^{j-j_*})LM^{-2}\beta^{-\alpha} \sum_{i=0}^{k-1} \left(\frac{p^{ij}}{\beta^i}\beta^{ij} \right) + k$$

$$= -(1-p^{j-j_*})LM^{-2}\beta^{-\alpha} \sum_{i=0}^{k-1} \left(p^j\beta^{j-1} \right)^i + k$$

$$= -(1-p^{j-j_*})LM^{-2}\beta^{-\alpha} \frac{(p^j\beta^{j-1})^k - 1}{p^j\beta^{j-1} - 1} + k.$$

Notice that $(1-p^{j-j_*})LM^{-2}\beta^{-\alpha}$ is constant in k, and since $p^j\beta^{j-1} > 1$ we also have

$$\lim_{k\to\infty} \frac{1 - (p^j\beta^{j-1})^{-k}}{p^j\beta^{j-1} - 1} = \frac{1}{p^j\beta^{j-1} - 1}$$

and

$$\lim_{k\to\infty} \frac{k}{(p^j\beta^{j-1})^k} = 0.$$

It follows that

$$(1-p^{j-j_*})LM^{-2}\beta^{-\alpha} \frac{1 - (p^j\beta^{j-1})^{-k}}{p^j\beta^{j-1} - 1} - \frac{k}{(p^j\beta^{j-1})^k} \ge L'$$

and hence

$$-(1-p^{j-j_*})LM^{-2}\beta^{-\alpha} \frac{(p^j\beta^{j-1})^k - 1}{p^j\beta^{j-1} - 1} + k \le -L'(p^j\beta^{j-1})^k$$

for some constant $L' > 0$. Thus we have that

$$\mathcal{M}(\mathcal{K}_A) \le (2^{\beta^{j-1}} - 1)^{-L'(p^j\beta^{j-1})^k} \le 2^{-L'(p^j\beta^{j-1})^k}$$

which completes the proof. $\square$

THEOREM 4.1. *Let $\mathcal{F}_c$ be a conformal IFS in $\mathbb{R}^d$ satisfying the SSC. If K is the unique invariant set for $\mathcal{F}_c$ and $\dim_H K = s_0$, then $\mathcal{H}^{\phi_s}(\mathbb{H}(K)) > 0$ for all $s < s_0$.*

Proof. The proof will follow by applying Corollary 1.2 to the measure constructed in Lemma 4.2.

Let $\mathcal{M}$ be the measure on $\mathbb{H}(K)$ constructed in Lemma 4.2. Fix $F \in \mathbb{H}(K)$ and let $L(F)$ be as defined in Lemma 3.1 where $L = L_{kj}$ (note that this in turn means $a = g_{kj}$). Recall from the Definition of $\mu_\delta(x)$ that

$$\mathcal{M}_{u^{kj+1}}(F) = \sup\{\mathcal{M}(\mathcal{U}) : F \in \mathcal{U},\ \mathcal{U} \text{ is Borel},\ |\mathcal{U}| \le u^{kj+1}\}$$

where u^{kj+1} plays the role of δ and F plays the role of x. By Proposition 4.2 we have that $u^{kj+1} \le g_{kj}$ and it follows that $|\mathcal{U}| < a$ for each $\mathcal{U}$. Hence by Lemma 3.1 $\mathcal{U} \subset \mathcal{K}_{L(F)}$, whence it follows that $\mathcal{M}(\mathcal{U}) \le \mathcal{M}(\mathcal{K}_{L(F)})$ for every Borel set $\mathcal{U}$ with $F \in \mathcal{U}$ and $|\mathcal{U}| \le u^{kj+1}$. It follows that $\mathcal{M}_{u^{kj+1}}(F) \le \mathcal{M}(\mathcal{K}_{L(F)})$. Finally, since $\beta' = 1/u^s < \beta$ and $p^j \beta^{j-1} > (\beta')^j$, we have that

$$\frac{\mathcal{M}_{u^{kj+1}}(F)}{\phi_s(u^{kj+1})} \le \frac{\mathcal{M}(\mathcal{K}_{L(F)})}{\phi_s(u^{kj+1})} \le \frac{2^{-L'(p^j \beta^{j-1})^k}}{2^{-(\beta')^{kj+1}}}$$

$$= 2^{(\beta')^{kj+1} - L'(p^j \beta^{j-1})^k} \to 0$$

as $k \to \infty$. Thus $\overline{D}_{\mathcal{M}}^{\phi_s}(F, (u^{kj+1})_k) \le 1$ and the result follows from Corollary 1.2. $\qquad\square$

THEOREM 4.2. *Let $\mathcal{F}_c = \{X, w_e\}_{e \in E}$ be a conformal IFS satisfying the SOSC as described in Definition 9, and let K be the invariant set. Let $s_0 = \dim_H K$ and define $\phi_s(t) = 2^{-(1/t)^s}$. Then $\mathcal{H}^{\phi_s}(\mathbb{H}(K)) = 0$ for all $s > s_0$, and $\mathcal{H}^{\phi_s}(\mathbb{H}(K)) > 0$ for all $s < s_0$.*

Proof. We first show that $\mathcal{H}^{\phi_s}(\mathbb{H}(K)) = 0$ for all $s > s_0$. Recall by Corollary 1.5 that there exists a constant $c \geq 1$ such that

$$|X_\sigma| \leq c|Dw_\sigma(z)|$$

for all $z \in X$ and all $\sigma \in E^*$. By the construction of L_k, this says that $|X_\sigma| \leq cu^k$ for all $\sigma \in L_k$. This means that the collection

$$\{\mathcal{K}_A : A \subseteq L_k, A \neq \emptyset\}$$

forms a (cu^k)-cover of $\mathbb{H}(K)$. Let $N > c^{s_0} Lc_*$ and define

$$\psi_{N,s_0}(t) = 2^{-N(1/t)^{s_0}}.$$

Recalling from Lemma 4.1(1) that $\#L_k \leq Lc_* \beta^k$ we have

$$
\begin{aligned}
\mathcal{H}_{cu^k}^{\psi_{N,s_0}}(\mathbb{H}(K)) &\leq (2^{\#L_k} - 1)\psi_{N,s_0}(cu^k) \\
&\leq (2^{Lc_*\beta^k} - 1)2^{-N\left(\frac{1}{cu^k}\right)^{s_0}} \\
&= (2^{Lc_*\beta^k} - 1)2^{-Nc^{-s_0}\beta^k} \\
&\leq 2^{Lc_*\beta^k - Nc^{-s_0}\beta^k} \leq 1.
\end{aligned}
$$

It follows that $\mathcal{H}^{\psi_{N,s_0}}(\mathbb{H}(K)) \leq 1$. Thus by Corollary 1.1 we have that

$$\mathcal{H}^{\phi_s}(\mathbb{H}(K)) = \mathcal{H}^{\psi_{1.s}}(\mathbb{H}(K)) = 0$$

for all $s > s_0$, which completes the proof of the upper bound.

To prove the lower bound we will combine Theorem 2.3 with Theorem 4.1. First fix an arbitrary $s < s_0$ and choose a sub-IFS $\mathcal{F}_{c,\delta}$ that satisfies the SSC and satisfies $s < \dim_H K_\delta = s_\delta < s_0$ (this is possible since $s_\delta \to s_0$ as $\delta \to 0$ by Theorem 2.3). Notice that $K_\delta \subseteq K$ implies $\mathbb{H}(K_\delta) \subseteq \mathbb{H}(K)$, hence

$$\mathcal{H}^{\phi_{s'}}(\mathbb{H}(K)) \geq \mathcal{H}^{\phi_{s'}}(\mathbb{H}(K_\delta))$$

for all $s' > 0$. Now applying Theorem 4.1 we have that

$$\mathcal{H}^{\phi_{s'}}(\mathbb{H}(K_\delta)) > 0$$

for all $s' < s_\delta$. In particular this gives

$$\mathcal{H}^{\phi_s}(\mathbb{H}(K)) \geq \mathcal{H}^{\phi_s}(\mathbb{H}(K_\delta)) > 0.$$

As $s < s_0$ was chosen arbitrarily, this completes the proof. $\quad\square$

4.2 Example: A Conformal Cantor Set

In order to illustrate the dimension computations given in this chapter we will consider the case of a conformal Cantor set, i.e. a set akin to the classical Cantor Middle Third Set but with conformal maps and unequal contraction ratios. Let $X = [0, 1]$ and consider the IFS $\mathcal{F} = \{X, w_1, w_2\}$ where

$$w_1(x) = \frac{4}{25}x^2 + \frac{8}{25}x \qquad\qquad w_2(x) = \frac{4}{25}x^2 + \frac{8}{25}x + \frac{13}{25}$$

and

$$w_i'(x) = \frac{8}{25}x + \frac{8}{15}$$

for $i = 1, 2$.[1] In this situation, the contractions are a local (as opposed to a global) property. We can see however that

$$\frac{8}{25} \leq |w_i'(x)| \leq \frac{16}{25}$$

for all $x \in X$. So $r_{\min} = 8/25$ and $r_{\max} = 16/25$ and $\mathcal{F}$ has a unique invariant compact set $K \subset X$ which is a Cantor set. We also numerically compute the

---

[1]The coefficients in this example are chosen to exhibit the gap error described at the beginning of §3.2.

Level(k)	g_k	$\max_{\sigma \in L_{k+1}} |X_\sigma|$
1	0.04000000	0.48000000
2	0.01056807	0.19046400
3	0.00232528	0.06675273
4	0.00077564	0.02207382

FIGURE 4.1 – Table of gap sizes relative to maximum cylinder lengths.

constant $c \approx 2.13592$ for which

$$\frac{1}{c}|w_\sigma'(x)| \le |X_\sigma| \le c|w_\sigma'(x)|$$

for all $\sigma \in E^*$, $x \in X$.

We will first define the levels as in [39] and show where that proof breaks down, thus motivating our study of the gaps. Fix $z_0 = 0$ and define

$$L_k = \left\{ \sigma \in E^* : |w_\sigma'(z_0)| \le r_{\min}^k < |w_{\sigma-}'(z_0)| \right\}$$

where $E = \{1, 2\}$.[2] We can directly observe the first few levels with this formulation:

$L_1 = \{1, 2\}$

$L_2 = \{11, 21, 121, 122, 221, 222\}$

$L_3 = \{111, 211, 1211, 2211, 1121, 2121, 1221, 2221, 12121, 22121, 11221,$

$\qquad 12122, 22122, 11222, 21221, 21222, 12221, 12222, 22221, 22222\}.$

A numerical check using a VBA program yields the table in figure 4.1. Clearly it is possible that there are cylinders at level-$(k + 1)$ with diameters larger than

---

[2]In order to mimic [39] we should technically consider u^k where $u < r_{\min}$ instead of $r_{\min}^k$. For the purpose of highlighting the flaws in the direct application of the arguments from [39], however, we may simply consider $r_{\min}^k$.

the length of the smallest level-k gap g_k. This causes a breakdown in McClure's measure estimate as we will now see.

Recall in the proof of Theorems 3.1 and 4.1 that we defined $\mathcal{M}_{u^{kj+1}}(F)$ which in this example would be

$$\mathcal{M}_{r_{\min}^{kj+1}}(F) = \sup\{\mathcal{M}(\mathcal{U}) : F \in \mathcal{U},\ \mathcal{U} \text{ is Borel},\ |\mathcal{U}| \leq r_{\min}^{kj+1}\}.$$

If the hypotheses of these theorems had been followed in this example, we would have $\mathcal{U} \subseteq \mathcal{K}_{L(F)}$, but this is not the case. Consider two consecutive cylinders from L_{kj}, say X_σ and X_τ, separated by a single gap, say (x, y) (i.e. x is the right endpoint of X_σ and y is the left endpoint of X_τ). Suppose further that the largest cylinder from L_{kj+1} is an interval I that satisfies $|I| \geq g_{kj} = |x - y|$. Let $F = \{x, y\} \cup I$ and let $\mathcal{U} = \{\{x\} \cup I, F\}$. We then have

$$|\mathcal{U}| = |x - y| < |I| \leq r_{\min}^{kj+1}.$$

Hence $\mathcal{U}$ is in the collection of elements over which we take the supremum in the computation of $\mathcal{M}_{r_{\min}^{kj+1}}(F)$. But clearly $\{x\} \cap X_\tau = \emptyset$ and so $\mathcal{U} \not\subseteq \mathcal{K}_{L(F)}$. This shows that there are sets $\mathcal{U}$ allowable in the definition of $\mathcal{M}_{r_{\min}^{kj+1}}(F)$ for which $\mathcal{U} \not\subseteq \mathcal{K}_L$ for any $L \subset L_{kj+1}$. At this point we have very little information about $\mathcal{M}_{r_{\min}^{kj+1}}(F)$ and our estimate is lost.

Given this breakdown in the argument, we turn to our gap study to correct the issue. Now, instead of defining

$$L_k = \{\sigma \in E^* : |w_\sigma'(z_0)| \leq r_{\min}^k < |w_{\sigma-}'(z_0)|\}$$

we define

$$L_k = \{\sigma \in E^* : |w_\sigma'(z_0)| \leq u^k < |w_{\sigma-}'(z_0)|\}$$

where $u \leq (1/3) \cdot (8/25) \cdot (1/50) < (1/c) \cdot r_{\min} \cdot g$. We will choose $u = 1/500$. Now, if we consider any gap at level-kj with length g', it follows from Proposition 3.2

104

that

$$g' \geq g_{kj} \geq u^{kj+1} > |X_\sigma|$$

for all $\sigma \in L_{kj+1}$. Thus if $F \in \mathbb{H}(K)$ and $\mathcal{U} \subset \mathbb{H}(K)$ is Borel with $F \in \mathcal{U}$, it must be the case that either $\mathcal{U} \subset \mathcal{K}_{L(F)}$ or $|\mathcal{U}| > u^{kj+1}$.

Now that this issue is resolved, we continue with the dimension computation by choosing values for the parameters.[3] Define

$$L_k = \{\sigma \in E^* : r_\sigma \leq u^k < r_{\sigma-}\}.$$

The levels will grow much faster with this choice of u as the sets contract much more quickly. For example, even L_1 as defined here contains more strings than L_4 with the previous definition, since $u < r_{\min}^4$. A numerical computation in VBA gives $L = \#L_1 = 314$. We also numerically determine

$$s_0 = \dim_H K \approx 0.94383$$

so that

$$\beta = u^{-s_0} \approx 352.6703.$$

Choose $s < s_0$, say $s = 0.9$, then we also get the parameters $\beta_s' \approx 268.5796$, and $\alpha = 1$ (since $L < \beta$). From Remark 3 there exists a constant $M > 0$ such that

$$M^{-1}|w_\sigma'(x)|^{s_0} \leq \mu(X_\sigma) \leq M|w_\sigma'(x)|^{s_0}$$

where μ is the self-conformal measure. The parameter M is difficult to simulate, but its value should be somewhat close to that of c. For safety we will take $M = 5 > 2c$. Choose $p = 0.8$, $j_* = 15$ and $j = 120$ so that

$$p^{j_*} < M^{-2} \qquad \text{and} \qquad \left(\frac{\beta'}{p \cdot \beta}\right)^s < \frac{1}{\beta}.$$

---

[3]The upper bound computation is trivial and is omitted.

By Lemma 4.1 we have that

$$12.56 \cdot 352.6703^{k-2} < \#L_k < 7850 \cdot 352.6703^k$$

and

$$(1/25) \cdot 352.6703^{119} < \#L_{120(k+1),\gamma} < 25 \cdot 352.6703^{121}$$

for each $\gamma \in L_{120k}$.[4]

We will now construct the Cantor net, $\mathcal{A}$. Let $\mathcal{A}_0 = \{\mathcal{K}_{\{\Lambda\}}\} = \{\mathbb{H}(K)\}$ where Λ is the empty string. We want all levels to satisfy

$$\pi(A) \geq \frac{p^{kj}}{\beta^k} \geq \frac{0.8^{120k}}{352.6703^k}$$

where $\pi(A)$ is defined by

$$\pi(A) = \frac{\#A}{LM^{-2}\beta^{kj-\alpha}} = \frac{\#A}{314 \cdot 5^{-2} \cdot 352.6703^{120k-1}}.$$

Clearly at level-0 we have

$$\pi(\{\Lambda\}) = 314^{-1} \cdot 25 \cdot 352.6703 > \frac{p^{0j}}{\beta^0}.$$

Notice here that if $\alpha = 0$ as in the proof in [39],[5] we would have $\pi(\{\Lambda\}) < \frac{p^{0j}}{\beta^0}$, and we would be unable to begin the inductive construction of $\mathcal{A}$.

For the inductive step, suppose we have constructed $\mathcal{A}_k$ which satisfies $\pi(A) \geq \frac{0.8^{120k}}{352.6703^k}$ for each $A \in \mathcal{A}_k$. This means

$$\#A \geq \frac{p^{kj}}{\beta^k} LM^{-2}\beta^{kj-\alpha} \geq \frac{314}{25} \cdot 0.8^{120k} \cdot 352.6703^{119k-1}.$$

We may then choose

$$0.8^{105} \cdot \frac{314}{25} \cdot 0.8^{120k} \cdot 352.6703^{119k-1} = 0.8^{105} \cdot \frac{314}{25} \cdot \frac{0.8^{120k}}{352.6703^k} \cdot 352.6703^{120k-1}$$

---

[4]Unfortunately it is the nature of conformal systems that the values of the parameters involved quickly become very ugly.

[5]The parameter α is not considered in [39], which leaves it effectively as zero.

of these strings to take all descendants of when forming $B \in \mathcal{A}_{k+1}$. As each level-$120k$ string has at least $(1/25) \cdot 352.6703^{119}$ descendants at level-$(120(k+1))$, we have

$$\#B \geq \left(0.8^{105} \cdot \frac{314}{25} \cdot \frac{0.8^{120k}}{352.6703^k} \cdot 352.6703^{120k-1}\right)\left(\frac{1}{25} \cdot 352.6703^{119}\right)$$

$$\geq \frac{0.8^{-15}}{25} \cdot \frac{314}{25} \cdot \frac{0.8^{120(k+1)}}{352.6703^{k+1}} \cdot 352.6703^{120(k+1)-1}$$

$$\geq \frac{314}{25} \cdot \frac{0.8^{120(k+1)}}{352.6703^{k+1}} \cdot 352.6703^{120(k+1)-1}$$

which shows that $\pi(B) \geq \frac{0.8^{120(k+1)}}{352.6703^{k+1}}$. This shows the existence of $\mathcal{A}_{k+1}$ given the existence of $\mathcal{A}_k$, and the existence of all of $\mathcal{A}$ follows by induction.

Following the proof of Lemma 3.3 it is straighforward to see that

$$\mathcal{M}(\mathcal{K}_A) \leq 2^{-L'(0.8^{120}352.6703^{119})^k}$$

for some $L' > 0$. Since $\mathcal{U} \subset \mathcal{K}_{L(F)}$ as argued above, it then follows that

$$\frac{\mathcal{M}_{u^{120k+1}}(F)}{\phi_s(u^{120k+1})} \leq \frac{\mathcal{M}(\mathcal{K}_{L(F)})}{\phi_s(u^{120k+1})} \leq \frac{2^{-L'(0.8^{120}352.6703^{119})^k}}{2^{-(\beta')^{120k+1}}} \to 0$$

the convergence of which can be quickly checked numerically. The lower bound follows from Corollary 1.2.

CHAPTER 5
MOTIVATION AND FUTURE RESEARCH

In this dissertation we have computed the Hausdorff gauge functions (i.e. dimensions) of the hyperspaces of graph-self-similar and self-conformal sets in $\mathbb{R}^d$. Our motivation for addressing this problem was twofold. First, in recent work by Barnsley, Hutchinson, and Stenflo, the theory of *Superfractals* has been introduced (see [4], [5]). The idea behind a superfractal is to consider the hyperspace $\mathbb{H}(K)$ as a fractal in and of itself, and to define a *super-IFS* on the hyperspace where each "map" is itself and IFS on K. In this promising new field it is worthwhile to provide a method of dimension classification that is commensurate with treating a hyperspace as a fractal in and of itself. While the methods developed by McClure and generalized here do not fully provide this classification, they are a conceptual first step.

The second motivating factor has to do with Bandt's theorem in [2]. In recent work, Elekes and Keleti have begun to study so-called *dimensionless* or *immeasurable* compact sets in $\mathbb{R}$ (see [15], [20]). These are compact sets for which the μ-measure is either zero or non-σ-finite for *any* translation invariant measure μ. Bandt's theorem shows that hyperspaces are examples of compact dimensionless sets that do no lie in $\mathbb{R}$. While the results of this dissertation provide no direct insight into the study of the dimensionless properties of hyperspaces, the author put forth much effort into constructing nice measures on $\mathbb{H}(K)$ prior to finding

Bandt's result, and hopes that the insight gained in these pursuits will aid in his future study of hyperspaces using the methods of Elekes and Keleti.

Given the motivation for, and realization of, the results in this dissertation, we would now like to consider the further research that most naturally follows from this current work.

Firstly, the results here concern fractals in $\mathbb{R}^d$, and clearly one would hope to extend this discussion to more general metric spaces. The reader will notice that many of the arguments in this work seem to not rely on the particular properties of $\mathbb{R}^d$, and in fact there are only a few key properties that are crucially used, namely:

1. It is assumed that the SOSC and OSC are equivalent

2. It is assumed that for an arbitrary point x and an arbitrary $\epsilon > 0$, there necessarily exists a point y with $d(x, y) = \epsilon$

3. It is assumed that $\dim_H K = \dim_B K = \dim_{\mathcal{E}} K$ for any set K where on value is well-defined

These properties hold in particular in $\mathbb{R}^d$, with assumption 1 holding for the particular types of IFS under consideration. In a general metric space assumption 1 does not hold and SOSC would have to be assumed in order to extend our results using our presented arguments. Our arguments are still valid if we weaken assumption 2 to say that there exists $c > 0$ independent of x and ϵ such that for all x there exists y with $d(x, y) \geq c\epsilon$. In order to retain our arguments for conformal systems, at least this much would have to be assumed. Assumption 3, in particular lemma 1.1 may or may not hold in a general metric space, and this issue would need some resolution in order to facilitate the generalization of the current theory beyond $\mathbb{R}^d$. The author assumed this would have already been a resolved issue, but again the entropy index is not at all a commonly studied value.

Another natural extension of the results contained herein would involve considering GDIFSs consisting of conformal maps. This would clearly contain both the self-similar GDIFS and conformal IFS cases, and would allow for the approximations of Julia sets in a manner similar to the constructions of Edgar and Golds in [18]. The Edgar/Golds construction is an interesting one which we now outline.

Suppose $\mathcal{F}_{g,c} = \{J_v, w_e\}_{v \in V, e \in E}$ is a conformal GDIFS in $\mathbb{R}^d$ that satisfies the SOSC, then $\text{int}(J_v) \cap K_v \neq \emptyset$ for each $v \in V$. Fix $k \geq 1$ and pick points $x_\gamma \in \text{int}(J_\gamma) \cap K_{i(\gamma)}$ for each $\gamma \in E_A^k$. Choose $\xi(\gamma) \in E_{i(\gamma)}^*$ such that $K_{\xi(\gamma)} \subset \text{int}(J_\gamma)$. In addition choose $\zeta(\gamma) \in E_{t(\xi(\gamma))v}^*$ such that $\xi(\gamma)\zeta(\gamma) \in E_{i(\gamma)i(\gamma)}^*$. Then define the approximating IFS $\mathcal{F}_{c,g,k} = \{J_{\xi(\gamma)\zeta(\gamma)}, w_{e\xi(\gamma)\zeta(\gamma)}\}_{\gamma \in E_A^k, e \in E}$.

This construction is very similar to the sub-IFS construction developed in this dissertation, but with a few glaring differences. The clearest difference is that the invariant sets $K_{v,k} \subset K_v$ from the Edgar/Golds construction will never intersect the boundaries of the J_v's. This is in contrast to the sets $K_{v,\delta}$ from the sub-IFS construction which will likely intersect the boundaries of cylinders. The benefit of the Edgar/Golds method is that it uses only the GDIFS, hence it is computationally efficient. The sub-IFS construction in general employs the Axiom of Choice, which means in order for it to even be constructive there must be case-by-case and level-by-level arguments for optimality of the packings.

There is a penalty to be paid for this computational efficiency, however, in that there is no argument given in [18] and [26] for the convergence of the dimensions. The following theorem is given

THEOREM (Edgar and Golds, 1999). *Let r_e' be the lower lipschitz constants of the maps w_e and consider the Mauldin-Williams graph (V, E, r_e'). Let s_1 be the*
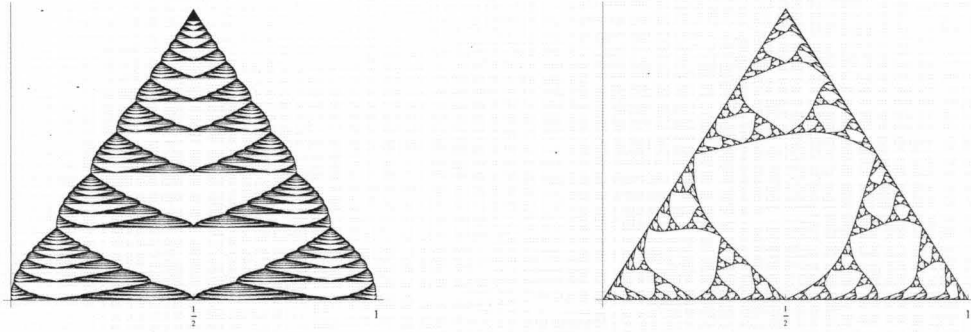
FIGURE 5.1 – Non-conformal Sierpiński Triangles

dimension of the graph and let $s_0 = \dim_H K_u$, then $s_0 \geq s_1$.

The values s_0 and s_1 are of course equal in the case that the w_e are similarities. This theorem provides an easy to compute lower bound for the dimension of the approximating Cantor sets, and numerical approximations of the dimensions of Julia sets are considered and compared to the methods of McMullen in [40] which, "provides evidence that [their] methods are correct." At no point is it argued that the values s_1 converge to the Hausdorff dimension of the larger fractal.

For the reasons mentioned above, it is implausible that the Edgar/Golds construction and the sub-IFS construction can realize the same sequence of Cantor sets for a particular system. It is possible, however, that the dimensions of the Cantor sets from the Edgar/Golds sequence might be bounded below by the dimensions of the Cantor sets from the sub-IFS sequence. Were this shown to be true and should the convergence arguments of this dissertation be extended to conformal GDIFS, the missing convergence argument in the Edgar/Golds construction would be provided. This would be a useful development, as Edgar and Golds use their construction to efficiently approximate the dimensions of particular Julia sets, which have proven notoriously difficult to compute.

In addition to extending the GDIFS arguments to allow conformal maps, it

111

would be interesting to extend the IFS arguments to allow non-conformal maps. In addition to providing very beautiful pictures such as non-conformal Sierpiński triangles (see Figure 5.1), non-conformal invariant sets are as of yet a wide-open research frontier. Very little is known about their dimension computations, and as such computations tend to be easier when the SSC is satisfied, a Cantor set approximation theorem for non-conformal invariant sets would provide a ready way to extend the SSC theorems to the OSC case as they arise.

CHAPTER 6
CONCLUSION

In this dissertation we have computed the Hausdorff dimensions of the hyperspaces of certain classes of fractals in $\mathbb{R}^d$, namely graph-self-similar and self-conformal fractals.

We first performed an analysis of the gaps between the cylinders in the geometric constructions of the fractals. We give a general result that gives conditions for which the diameters of the sets in a covering of the fractal are small enough with respect to the sets relative distances in order to insure some nice geometric properties (Lemma 3.1). We then performed case specific analyses on the graph-directed and self-conformal fractals, respectively, in order to apply the general lemma (Propositions 3.1, 3.2, 4.1, 4.2).

With the appropriate geometric properties in hand, we proceeded to construct measures on the fractals. We did this by constructing coverings that were consistent with the aforementioned gap analyses, and then by constructing measures relative to these coverings which satisfied a particular boundedness condition (Lemmas 3.3, 4.2). The properties of these measures allowed us to apply a known density lemma to compute the dimensions of the hyperspaces (Theorems 3.1, 4.1).

Finally, we constructed approximations of more general fractals by choosing sub-IFS that satisfied some appropriately chosen geometric conditions. We showed that the dimensions of the sub-attractors given by the sub-IFS in fact ap-

proximate the dimensions of the big attractors (Theorems 2.2, 4.1) which allowed us to generalize these results further (Theorems 3.2, 4.1).

REFERENCES

[1] T. Banakh and N. Mazurenko, *The topology of systems of hyperspaces determined by dimension functions*, Topology **48** (2009), no. 2-4, 43–53.

[2] C. Bandt and G. Baraki, *Metrically invariant measures on locally homogeneous spaces and hyperspaces*, Pacific J. Math. **121** (1986), no. 1, 13–28.

[3] M. F. Barnsley, *Fractals Everywhere*, Academic Press, Boston, MA (1988).

[4] —, *Superfractals*, Cambridge University Press, Cambridge, UK (2006).

[5] M. F. Barnsley et al., *V-variable fractals: fractals with partial self-similarity*, Adv. Math. **116** (2008), no. 6, 2051–2088.

[6] A. Besicovitch, *Sets of fractional dimensions. Part I.*, Math. Ann. **101** (1929), 161–193.

[7] E. Boardman, *Some Hausdorff measure properties of the space of compact subsets of* [0, 1], Quart. J. Math. Oxford Ser. (2) **24** (1973), 333–341.

[8] —, *The problem of subsets of finite positive measure in the space of compact subsets of* [0, 1], Quart. J. Math. Oxford Ser. (2) **26** (1975), no. 103, 329–332.

[9] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, vol 470 of *Lecture Notes in Mathematics*, Springer-Verlag, New York, NY (1975).

[10] R. Cawley and D. Mauldin, *Multifractal decompositions of Moran fractals*, Adv. Math. **92** (1992), no. 2, 196–236.

[11] D. W. Curtis and R. M. Schori, *Hyperspaces of Peano continua are Hilbert cubes*, Fund. Math. **101** (1978), 19–38.

[12] M. Das, *The upper entropy index of a set and the Hausdorff dimension of its hyperspace*, Monatsh Math DOI:10.1007/s00605-011-0304-6.

[13] —, *Pointwise local dimensions*, Ph.D. thesis, The Ohio State University (1996).

[14] —, *Hausdorff measures, dimensions, and mutual singularity*, Trans. Amer. Math. Soc. **357** (2005), 4249–4268.

[15] R. Davies, *Sets which are null or non-sigma-finite for every translation-invariant measure*, Mathematika **18** (1971), 161–162.

[16] G. A. Edgar, *Measure, topology, and fractal geometry*, Springer-Verlag, New York, NY (1990).

[17] —, *Integral, probability, and fractal measures*, Springer-Verlag, New York, NY (1998).

[18] G. A. Edgar and J. Golds, *A fractal dimension estimate for a graph-directed iterated function system of non-similarities*, Indiana Univ. Math. J. **48** (1999), no. 2, 429–447.

[19] G. A. Edgar and D. Mauldin, *Multifractal decompositions of digraph recursive fractals*, Proc. Lond. Math. Soc. **65** (1992), 604–628.

116

[20] M. Elekes and T. Keleti, *Borel sets which are null or non-σ-finite for every translation invariant measure*, Adv. Math. **201** (2006), no. 1, 102–115.

[21] K. J. Falconer, *The geometry of fractal sets*, Cambridge University Press, Cambridge, UK (1986).

[22] ──, *Fractal geometry. Mathematical foundations and applications*, John Wiley & Sons, Chichester, UK (1990).

[23] ──, *Techniques in fractal geometry*, John Wiley & Sons, Chichester, UK (1997).

[24] A. H. Fan and K.-S. Lau, *Iterated function system and ruelle operator*, J. Math. Anal. Appl. **231** (1999), no. 2, 319 – 344.

[25] F. R. Gantmacher, *The theory of matrices*, vol 2, AMS Chelsea Publishing, Providence, RI (1959).

[26] J. Golds, *Estimation of the fractal dimension of selected classes of Julia sets using spectral radius calculations*, Ph.D. thesis, The Ohio State University (1998).

[27] P. R. Goodey, *Some results concerning the dimension of the space of compact subsets of the unit interval*, Quart. J. Math. Oxford Ser. (2) **27** (1976), no. 108, 467–473.

[28] ──, *Hausdorff measure functions in the space of compact subsets of the unit interval*, Trans. Amer. Math. Soc. **226** (1977), 203–208.

[29] F. Hausdorff, *Dimension und ausseres Mass*, Math. Ann. **79** (1918), 157–179.

[30] J. E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math J. **30** (1981), 713 – 747.

[31] J. Kigami, *Volume doubling measures and heat kernel estimates on self-similar sets.*, Mem. Amer. Math. Soc. **199** (2009), no. 932, viii+94.

[32] S. P. Lalley, *The packing and covering functions of some self-similar fractals*, Indiana Univ. Math. J. **37** (1988), no. 3, 699–710.

[33] K.-S. Lau and S.-M. Ngai, *Multifractal measures and weak separation condition*, Adv. Math. **141** (1999), no. 1, 45 – 96.

[34] B. Mandelbrot, *The fractal geometry of nature*, W.H. Freeman and Co., San Francisco, CA (1982).

[35] D. Mauldin and M. Urbański, *Graph directed markov systems*, Cambridge University Press, Cambridge, UK (2003).

[36] D. Mauldin and S. C. Williams, *Hausdorff dimension in graph directed constructions*, Trans. Amer. Math. Soc. **309** (1988), no. 2, 811 – 829.

[37] M. McClure, *Fractal measures on infinite dimensional sets*, Ph.D. thesis, The Ohio State University (1994).

[38] —, *Entropy dimensions of the hyperspace of compact sets*, Real Anal. Exchange **21** (1995/96), no. 1, 194–202.

[39] —, *The Hausdorff dimension of the hyperspace of compact sets*, Real Anal. Exchange **22** (1996/97), no. 2, 611–625.

[40] C. McMullen, *Hausdorff dimension and conformal dynamics III: computation of dimensions*, Amer. J. Math. **120** (1998), 691–721.

[41] H. Minc, *Nonegative matrices*, John Wiley & Sons, New York, NY (1988).

[42] S.-M. Ngai and Y. Wang, *Hausdorff dimension of self-similar sets with overlaps*, J. London Math. Soc. (2) **63** (2001), no. 3, 655–672.

[43] N. Patzschke, *Self-conformal multifractal measures*, Adv. in Appl. Math. **19** (1997), no. 4, 486–513.

[44] Y. Peres et al., *Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2689–2699.

[45] C. A. Rogers, *Hausdorff measures*, Cambridge University Press, Cambridge, UK (1970).

[46] D. Ruelle, *Thermodynamic formalism: the mathematical structures of classical equilibrium statistical mechanics*, Addison-Wesley Publishing Co., Reading, MA (1978).

[47] A. Schief, *Separation properties for self-similar sets*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 111 – 115.

[48] E. Seneta, *Non-negative matrices. An introduction to theory and applications.*, John Wiley & Sons, New York, NY (1973).

[49] J. Tyson, *Bi-Lipschitz embeddings of hyperspaces of compact sets*, Fund. Math. **187** (2005), no. 3, 229–254.

[50] J. Wang, *The open set conditions for graph directed self-similar sets*, Random Compute Dynam. **5** (1997), no. 4, 283–305.

INDEX

CURRICULUM VITAE

Matthew Gilbert Zapf

Education

University of Louisville, Louisville, Kentucky
Ph.D., Applied and Industrial Mathematics, August 2011
Saint Louis University, St. Louis, MO
M.A., Mathematics, May 2006
B.A., Mathematics/German Studies, May 2004

Publications

Cantor Set Approximations and Dimension Computations in Hyperspaces. In Preparation.

Talks and Presentations

Dimension Computations in Hyperspaces *November 2010*
Kentucky Academy of Science Annual Meeting
The Existence and Uniqueness of Invariant Measures for Place-Dependent IFS *October 2009*
University of Louisville Analysis Seminar
The Hausdorff h-measure of Cantor Sets with Decreasing Gap Sequences
University of Louisville Analysis Seminar *March 2009*

Awards

Ken F. and Sandra S. Hohman Fellowship *August 2008-May 2009*