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PROPERTIES OF GENERIC AND ALMOST EVERY
MAPPINGS IN VARIOUS NONLOGICALLY COMPACT
POLISH ABELIAN GROUPS

By

Susan L. Calcote White
B.S., McNeese State University, 1992
M.A., University of Louisville, 2003

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Louisville, KY

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A Dissertation Approved on

March 26, 2007

By the following Dissertation Committee:

Udayan B. Darji, Dissertation Director

M. Elizabeth Bradley

Shudun Liu

Robert C. Powers

Kandi L. Walker

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ABSTRACT

PROPERTIES OF GENERIC AND ALMOST EVERY MAPPINGS IN VARIOUS NONLOCALLY COMPACT POLISH ABELIAN GROUPS

Susan L. Calcote White

May 12, 2007

In a nonlocally compact Polish abelian group G , we will consider two notions of smallness of subsets of G . Those subsets of G which are topologically small are said to be meager, and those which are measure-theoretically small are Haar null. We will say that a property P holds for a generic $g \in G$ if the property holds on the complement of a meager subset of G , and P holds for almost every $g \in G$ if the property holds on the complement of a Haar null set. Thus the phrase “a randomly chosen element of G is likely to have property P ” may be understood to have two different meanings in this paper.

The spaces $\mathbb{Z}^{\mathbb{Z}}$ and $\mathcal{C}(\mathbb{R}^n)$, $n \geq 1$, the continuous self-maps of \mathbb{Z} and \mathbb{R}^n , respectively, are both nonlocally compact Polish abelian groups. In this paper we will study properties of generic and almost every mappings in $\mathbb{Z}^{\mathbb{Z}}$ and $\mathcal{C}(\mathbb{R})$, and properties of generic mappings in $\mathcal{C}(\mathbb{R}^n)$. In the space $\mathbb{Z}^{\mathbb{Z}}$, we show that the behavior of a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ is quite different than the behavior of almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$. We will show that in the space $\mathcal{C}(\mathbb{R})$, the behavior of a generic $f \in \mathcal{C}(\mathbb{R})$ is analogous to the behavior of a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ in several ways, but the analogies between the spaces $\mathbb{Z}^{\mathbb{Z}}$ and $\mathcal{C}(\mathbb{R})$ seem to cease when the properties of almost every $f \in \mathcal{C}(\mathbb{R})$ are considered. In fact, many of the properties of functions in $\mathcal{C}(\mathbb{R})$ that

we consider in this paper are shown to be H-ambivalent; that is, the properties hold on a set which is neither Haar null nor the complement of a Haar null set. We will present preliminary results concerning the behavior of a generic $f \in \mathcal{C}(\mathbb{R}^n)$. We will show that several of the properties which hold for a generic $f \in \mathcal{C}(\mathbb{R})$ also hold in the more general setting of a generic $f \in \mathcal{C}(\mathbb{R}^n)$, although the proofs techniques differ. Finally, we will close with a discussion of future directions that this work may take.

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CHAPTER 1

INTRODUCTION

1.1 Introductory Remarks

Suppose that we are given a collection of objects, and we would like to know how likely it is that a randomly chosen object of the collection behaves in a certain way. Intuitively, we say that if the size of the set of objects which do not exhibit the prescribed behavior is negligible or small in some sense, then a randomly chosen object is likely to exhibit the behavior. In the space of real numbers \mathbb{R} , there are at least two natural notions of smallness, one topological and the other measure-theoretic. We say that a subset of \mathbb{R} is topologically negligible if it is meager in \mathbb{R} , and it is measure-theoretically negligible if its Lebesgue measure is zero. There are many examples of subsets of \mathbb{R} which are small in both senses (for example, the rational numbers \mathbb{Q} or the standard Cantor set), but it can be shown that \mathbb{R} can be written as the disjoint union of a meager set and a set of Lebesgue measure zero. Thus these two notions of smallness are not related; neither class of small sets includes the other.

These two notions of smallness hold in the more general setting of locally compact Polish abelian groups. Since all Polish groups are topological spaces, the topological notion of meagerness holds. Moreover, any locally compact Polish abelian group admits a unique (up to a multiplicative constant) translation-invariant σ -finite Borel measure which is finite on compact sets and positive on nonempty open sets. Hence, in a locally compact abelian Polish group, a set whose

Haar measure is zero is negligible in a measure-theoretic sense.

However, no nonlocally compact Polish abelian group admits a Haar measure. In 1972, J.P.R. Christensen generalized the concept of Haar measure zero sets to abelian Polish groups which may be nonlocally compact [12]. Christensen defined the Haar null set as a measure-theoretically negligible subset of a Polish abelian group, and he showed that the notions of Haar measure zero and Haar null are equivalent in a locally compact Polish abelian group. In 1992, the notion of Haar null sets was reintroduced by Hunt, Sauer, and Yorke in the setting of infinite dimensional Banach spaces [23]. Their terminology differed from Christensen's; they used the term "shy" rather than Haar null, and they referred to the complement of a Haar null set as a "prevalent" set. Mycielski provided a definition of Haar null sets for nonabelian Polish groups in [35].

With these definitions in mind, we see that the phrase "a randomly chosen element g of a Polish abelian group G is likely to have property P " may be interpreted in two different ways in this paper. We say that a property P holds for a *generic* $g \in G$ if the set on which P does not hold is meager in G , and we say that P holds for *almost every* $g \in G$ if the set on which G does not hold is Haar null. The complement of a meager (respectively, Haar null) set is said to be comeager (co-Haar null). In this paper we are interested in investigating which properties of various nonlocally compact Polish abelian groups are likely to hold on a randomly chosen element. In Chapter 2, we will provide the necessary background information concerning meager and Haar null subsets of Polish abelian groups.

The notion of Haar null is more recent than the notion of meagerness, so while there are many well-known results concerning meager subsets of various Polish groups, there is a growing body of literature concerning Haar null subsets. For example, consider the following pair of theorems concerning $C([0, 1], \mathbb{R})$, the space

of continuous real-valued functions on the closed unit interval. The first is a classical result of Banach's, and the second more recent result is due to Hunt [25] in 1994.

THEOREM (Banach). *A generic $f \in C([0, 1], \mathbb{R})$ is nowhere differentiable.*

THEOREM (Hunt). *Almost every $f \in C([0, 1], \mathbb{R})$ is nowhere differentiable.*

Of course it is not always the case that a subset of a group is both comeager and co-Haar null. As another example, consider the space consisting of all permutations on \mathbb{N} , denoted by S_∞ , which is a nonabelian Polish group under the group operation composition of functions. We have the following results, the first well-known, and the second due to Dougherty and Mycielski in 1994 [17]. Of particular interest is the fact that the properties of a generic $\sigma \in S_\infty$ and almost every $\sigma \in S_\infty$ are in some sense complementary.

THEOREM (Folklore). *A generic $\sigma \in S_\infty$ has no infinite cycle, and infinitely many cycles of length k for all $k \in \mathbb{N}$.*

THEOREM (Dougherty-Mycielski). *Almost every $\sigma \in S_\infty$ has infinitely many infinite cycles and only finitely many finite cycles.*

These two theorems concerning S_∞ are the motivating results for our work in the space $\mathbb{Z}^{\mathbb{Z}}$, which consists of all (continuous) functions from \mathbb{Z} to \mathbb{Z} . Although both S_∞ and $\mathbb{Z}^{\mathbb{Z}}$ are groups consisting of self-maps of a countably infinite set, there are important differences between them. $\mathbb{Z}^{\mathbb{Z}}$ is an abelian Polish group with group operation pointwise addition; moreover, $\mathbb{Z}^{\mathbb{Z}}$ includes any mapping from \mathbb{Z} to \mathbb{Z} , while S_∞ includes only bijections from \mathbb{N} to \mathbb{N} . In Chapter 3, we will state and prove our results concerning properties of generic and almost every mappings in $\mathbb{Z}^{\mathbb{Z}}$. By associating each $\phi \in \mathbb{Z}^{\mathbb{Z}}$ with a graph Γ_ϕ , we will establish a setting in

which the structure of elements of $\mathbb{Z}^{\mathbb{Z}}$ may be studied. We find it quite interesting that although the groups have very different structure, we have obtained results for $\mathbb{Z}^{\mathbb{Z}}$ which are analogous to the Folklore and Dougherty-Mycielski Theorems for S_{∞} stated above. We have also found that, with most of the properties that we considered for the elements of $\mathbb{Z}^{\mathbb{Z}}$, the behavior of a generic ϕ with respect to the property is either complementary or “almost” complementary to the behavior of almost every ϕ . For example, we will show that a generic ϕ is surjective and almost every ϕ is not surjective. We will also show that a generic ϕ is not injective and almost every ϕ is “almost” injective, in the sense that ϕ is injective on the complement of a finite set. Other properties of elements of $\mathbb{Z}^{\mathbb{Z}}$ will also be discussed in Chapter 3.

After completing the results of Chapter 3, we focused our attention on the space of continuous real-valued functions on \mathbb{R} , denoted by $\mathcal{C}(\mathbb{R})$. We anticipated that, just as some of our results in $\mathbb{Z}^{\mathbb{Z}}$ were analogous to the known results for S_{∞} , it would be the case that our results in $\mathcal{C}(\mathbb{R})$ would be analogous to those in $\mathbb{Z}^{\mathbb{Z}}$. For the properties of a generic $f \in \mathcal{C}(\mathbb{R})$, this was indeed the case. In Section 1 of Chapter 4, we will state and discuss our results concerning the properties of generic elements of $\mathcal{C}(\mathbb{R})$. The first main result of Chapter 4, which is stated as Theorem 4.1, shows that there are several ways in which the behavior of a generic $f \in \mathcal{C}(\mathbb{R})$ exhibits behavior which is similar to that of a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$. For example, a generic $f \in \mathcal{C}(\mathbb{R})$ is surjective and non-injective, as is a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$. A generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ has the property that every point has infinite (hence, unbounded) preimage under ϕ , while a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that every point has uncountable and unbounded preimage under f . Another analogous result between the two spaces is that a generic f has the property that the forward orbit of x under f , denoted by $orb(f, x) = \{x, f(x), f^2(x) \dots\}$, is bounded for all $x \in \mathbb{R}$,

while a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ has the property that $\text{orb}(\phi, n)$ is finite for all $n \in \mathbb{Z}$.

Our result concerning the boundedness of the (forward) orbits of all points for a generic $f \in \mathcal{C}(\mathbb{R})$ led us into the area of dynamical systems as we began to consider properties of the orbits and ω -limit sets of a generic f . (The ω -limit set of x under f , denoted $\omega(f, x)$, is defined as the set of all subsequential limits of $\text{orb}(f, x)$, when $\text{orb}(f, x)$ is viewed as a sequence.) If a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that every point has bounded orbit, what else might we say about these orbits? Are some or all of the orbits finite? If some orbit $\text{orb}(f, x)$ is infinite, then is its associated ω -limit set $\omega(f, x)$ finite? In Theorem 4.2, we will classify the structure of the orbits and ω -limit sets of a generic $f \in \mathcal{C}(\mathbb{R})$ and show which scenarios may occur.

The structure of ω -limit sets of elements in $C([0, 1])$, the space of continuous self-maps of the unit interval, have been well-studied. Agronsky, Bruckner, and Laczkovich proved the following in [2].

THEOREM (Agronsky-Bruckner-Laczkovich). *A generic $f \in C([0, 1])$ has the property that $\omega(f, x)$ is nowhere dense and perfect for a generic $x \in [0, 1]$.*

Lehning in [30] offered a simpler proof of this result which applied to a more general setting: Lehning showed that in the space of continuous self-maps of a compact N -dimensional manifold X , a generic f has the property that $\omega(f, x)$ is nowhere dense and perfect for a generic $x \in X$. In our setting, $\mathcal{C}(\mathbb{R})$, we are considering self-maps of a space which is σ -compact, but not compact. Nevertheless, in Theorem 4.2 of Chapter 4, we will show that the property of Agronsky, Bruckner, and Laczkovich is true of a generic $f \in \mathcal{C}(\mathbb{R})$ as well; i.e., a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $\omega(f, x)$ is nowhere dense and perfect for all x in a comeager subset of \mathbb{R} . The remainder of our results in Theorem 4.2 concern the meager subset of \mathbb{R} for which the ω -limit sets of a generic $f \in \mathcal{C}(\mathbb{R})$ are not perfect.

What types of orbits and ω -limit sets may occur on this meager subset of \mathbb{R} ? In [2], it is proven that a generic $f \in C([0, 1])$ has the property that the set of points with finite orbit (hence, finite ω -limit set) is dense in $[0, 1]$; we will show that for a generic $f \in C(\mathbb{R})$, the set of points with finite orbit is c -dense in \mathbb{R} . We will also show that the set of all points which have infinite orbit and finite ω -limit set is c -dense in \mathbb{R} , and the set of points with infinite orbit and non-perfect infinite ω -limit set is unbounded in \mathbb{R} .

In Section 2 of Chapter 4, we will state and prove our results concerning the properties of a generic $f \in C(\mathbb{R}^n), n \geq 1$, in Theorem 4.3. This theorem is the result of our efforts to generalize the results of Section 4.1 to the space $C(\mathbb{R}^n)$. We have found that, just as in $C(\mathbb{R})$, a generic $f \in C(\mathbb{R}^n)$ has the properties that f is a surjection, the preimage of every point is uncountable and unbounded, the (forward) orbit of every point is bounded, and the set of periodic points is unbounded. Thus some of the results in Theorem 4.1 are implied by Theorem 4.3; however, the proof techniques differ and so we include the statements and proofs of both theorems separately. We will close Chapter 4 with a brief discussion of the difficulties involved in the problem of finding which types of ω -limit sets might occur for a generic $f \in C(\mathbb{R}^n)$.

In Chapter 5, we will turn our attention to the properties of almost every $f \in C(\mathbb{R})$. Here is where the analogies between the spaces $\mathbb{Z}^{\mathbb{Z}}$ and $C(\mathbb{R})$ seem to cease. In the space $\mathbb{Z}^{\mathbb{Z}}$, the set of surjections is both comeager and Haar null. While the set of surjections in $C(\mathbb{R})$ is comeager in $C(\mathbb{R})$, we will prove in Chapter 5 that it is neither Haar null nor co-Haar null; i.e., it is H-ambivalent. We will show in Theorem 5.1 that many comeager subsets of $C(\mathbb{R})$ are H-ambivalent. For example, the set of all f such that every point of \mathbb{R} has unbounded preimage under f , the set of all f which are of monotonic type at no point, and the set of all f

which are monotone at no point, are all sets which are comeager in $\mathcal{C}(\mathbb{R})$ and are H -ambivalent.

While in Section 1 of Chapter 4, our work led us into the area of dynamical systems, in Chapter 5 our work leads to a study of the differentiability properties of functions in $\mathcal{C}(\mathbb{R})$. One of the results, namely, (6) of Theorem 5.1, is of particular interest. Recall that Hunt proved that the set of all $f \in C([0, 1], \mathbb{R})$ that have finite derivative at some point is Haar null. Hunt's result was improved by Kolář in [29].

THEOREM (Kolář). *The set of all $f \in C([0, 1], \mathbb{R})$ such that f has a finite one-sided approximate derivative (hence, a finite one-sided derivative) at some point is Haar null*

In [44], Zajíček proves the following.

THEOREM (Zajíček).

1. *The set of all $f \in C([0, 1], \mathbb{R})$ such that $f'(x) \in \mathbb{R} \cup \{\pm\infty\}$ for some $x \in (0, 1)$ is H -ambivalent.*
2. *For any fixed $a \in (0, 1)$, the set of all $f \in C([0, 1], \mathbb{R})$ such that f has derivative $+\infty$ at a is H -ambivalent.*

Zajíček notes that the sets in (1) and (2) are each comeager subsets of $C([0, 1], \mathbb{R})$, and so the “‘Haar null case’ differs from the ‘category case...’” (page 1144, [44]). The results of Hunt, Kolář, and Zajíček extend to $\mathcal{C}(\mathbb{R})$ by a simple argument. In (6) of Theorem 5.1, we will strengthen (2) of Zajíček's result by proving that, for any fixed $a \in \mathbb{R}$, the set of all $f \in \mathcal{C}(\mathbb{R})$ such that f has derivative $+\infty$ at a and f has a knot point at all $x \neq a$ is not Haar null. (The term *knot point* is defined in Chapter 5.) This result is interesting in its own right because, as a consequence, we are able to provide an explicit example of uncountably many pairwise

disjoint universally measurable non-Haar null subsets of $\mathcal{C}(\mathbb{R})$. Christensen asked in [12] whether any family of mutually disjoint universally measurable non-Haar null subsets of a Polish abelian group is at most countable. Dougherty in [16] showed that in many nonlocally compact Polish abelian groups, there exist such families of non-Haar null subsets which are uncountable. Solecki in [41] proved that such an uncountable family exists in every nonlocally compact Polish abelian group. With our result, we are able to provide an explicit example of such a family in the group $\mathcal{C}(\mathbb{R})$.

Although many of the properties of functions in $\mathcal{C}(\mathbb{R})$ under consideration in this paper are H-ambivalent, we will show in Theorem 5.2 that we have found several properties which hold for almost every $f \in \mathcal{C}(\mathbb{R})$. We will prove that the set of functions f which are of monotonic type on no interval is co-Haar null. It follows that the set of all f which are monotone on no interval is co-Haar null, and that the set of noninjective functions is co-Haar null; these sets are both comeager in $\mathcal{C}(\mathbb{R})$ as well. We will see in Chapter 3 that almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$ is injective on a co-finite set; as a contrast, we will show in Chapter 5 that almost every $f \in \mathcal{C}(\mathbb{R})$ has the property that given any bounded set $F \subseteq \mathbb{R}$, f is noninjective on the complement of F . We will show that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the preimage of every $x \in \mathbb{R}$ is uncountable; in Chapter 5, we will see that almost every $f \in \mathcal{C}(\mathbb{R})$ has the property that the preimage of every point is either empty or uncountable for a generic $x \in \mathbb{R}$. We will also show that $f(\mathbb{R})$ is either a line or a ray in \mathbb{R} for almost every $f \in \mathcal{C}(\mathbb{R})$.

Finally, in Chapter 6, we will close with a discussion of open questions and future directions that this work may take. Before we proceed to Chapter 2, we provide the reader with a table which defines the notation that will be used throughout this paper.

1.2 Table of Notation

\mathbb{N}	set of natural numbers $\{1, 2, 3, \dots\}$
\mathbb{Z}	set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$
\mathbb{Q}	set of rational numbers
\mathbb{R}	set of real numbers
ω	first infinite ordinal, cardinality of \mathbb{N}
\ni	such that
\exists	there exists
\forall	for all
\emptyset	empty set
\cap, \cup	intersection, union
S^c	complement of S
$S \setminus T$	set difference between the sets S and T , defined as $S \cap T^c$
\bar{S}	closure of the set S
$\text{Int}(S)$	interior of the set S
∂S	boundary of the set S , defined as $\bar{S} \setminus \text{Int}(S)$
$ S $	cardinality of the set S
$ I $	length of the interval I
$\mathbb{Z}^{\mathbb{Z}}$	space of all self-maps of \mathbb{Z}
$\text{dom}(\sigma)$	domain of the function σ
$\text{supp}(\mu)$	support of the measure μ
$C(X)$	space of all continuous self-maps of X
$C(X, Y)$	space of all continuous maps from X to Y
$\text{orb}(f, x)$	forward orbit of x under f , defined as $\bigcup_{n=0}^{\infty} f^n(x)$
$\omega(f, x)$	omega-limit set of x under f
$B_{\epsilon}(f)$	open ball of radius ϵ centered at f

CHAPTER 2 PRELIMINARIES

In the space of real numbers \mathbb{R} , there are at least two natural notions of smallness. We consider a set S to be small in the sense of category if S is a meager subset of \mathbb{R} , and we consider S to be small in the sense of measure if S is a set of Lebesgue measure zero. Neither notion of smallness implies the other. While some subsets of \mathbb{R} , such as the standard Cantor set, are small in both senses, it is not difficult to construct subsets of \mathbb{R} which are simultaneously meager in \mathbb{R} and of full Lebesgue measure. (For example, see Theorem 1.6 of [37] or Example 1.1 of [36].) Observe that these two classes of sets, the meager sets and the (Lebesgue) measure zero sets, share certain properties, listed below.

1. Every meager [measure zero] set has empty interior.
2. The translate of a meager [measure zero] set is also meager [measure zero].
3. Every subset of a meager [measure zero] set is meager [measure zero].
4. The countable union of meager [measure zero] sets is meager [measure zero].

Intuitively, we would hope that given any definition of “smallness” of sets in a topological space, the class of all such small sets would also satisfy these four properties.

In a more general setting, consider a locally compact abelian Polish group G . A Polish group G is a topological group whose topology is separable and completely metrizable; G is abelian if its group operation is abelian. Recall that a

set $K \subseteq G$ is compact if every open cover of K has a finite subcover; that is, given any collection \mathcal{U} of open sets whose union contains K , there is a finite collection $\{U_1, \dots, U_n\} \subseteq \mathcal{U}$ such that $K \subseteq \bigcup_{i=1}^n U_i$. We say that G is locally compact if every point in G has an open neighborhood with compact closure. Every locally compact abelian Polish group G admits a unique (up to a multiplicative constant) σ -finite Borel measure, called the Haar measure, which is finite on compact sets, positive on nonempty open sets, and translation invariant [27]. Thus for such a group G , there are topological and measure-theoretic notions of smallness which satisfy the four properties above. However, if G is not locally compact, then a translation-invariant measure with nice properties such as those of the Haar measure does not exist [31]. In 1972, J.P.R. Christensen showed in [12] that the concept of Haar measure zero could be generalized to nonlocally compact abelian Polish groups. He defined the notion of the “Haar zero” set for such groups. We will see that the class of all such sets satisfies the four properties listed above, and thus it is a reasonable definition of measure-theoretic smallness in the setting of nonlocally compact Polish abelian groups.

In this chapter, we will give the necessary definitions and background results which will be used in the following chapters. Throughout this chapter, G will be used to denote a Polish abelian group (unless stated otherwise), and X will be used to denote a complete metric space. In the first section, we state the topological definitions and theorems which will be used in this paper, and we discuss the topologies on the spaces $\mathbb{Z}^{\mathbb{Z}}$ and $\mathcal{C}(\mathbb{R}^n)$, $n \geq 1$. In the second section, we will introduce Christensen’s definition of “Haar zero” sets and provide some of the history, definitions, and theorems related to this idea.

2.1 The Categories of Baire

Many of the topological definitions and theorems of this section are due to R. Baire. These definitions and theorems are standard and may be found in texts such as [38] or [34]. Although our work in this paper will be in Polish groups, we state the results of this section for any complete metric space X so that the results may be presented in the most general possible setting. Since every Polish group is a complete metric space, the results of this section hold for Polish groups.

We say that $D \subseteq X$ is *dense* if $\overline{D} = X$. A set $E \subseteq X$ is *nowhere dense* if the interior of its closure is empty; equivalently, $E \subseteq X$ is nowhere dense if $(\overline{E})^c$ is dense. A subset A of X is said to be *dense in itself* if it contains no isolated points. If $A \subseteq X$ is closed and dense in itself, then A is *perfect*.

DEFINITION 1. *A set $E \subseteq X$ is **meager** (or, of the first category) in X if E is the countable union of nowhere dense sets. The complement of a meager set is said to be **comeager** (or, residual) in X . We say that a **generic** $x \in X$ has **property P** if*

$$\{x \in X : x \text{ does not have property } P\}$$

is meager in X .

The word “typical” is sometimes used in place of “generic.” In Baire’s original terminology, subsets of X are classified as belonging to one of two categories. The smallest sets, topologically speaking, are the meager sets, which are of the first category. All sets which are not meager are of the second category. In this paper we will not use the terms first category, second category, and residual; rather, we will use the terms meager, nonmeager, and comeager. We now state Baire’s Theorem.

THEOREM (Baire). *Let $\{U_n\}_{n \in \mathbb{N}}$ be a countable collection of dense open subsets of a complete metric space X . Then, $\bigcap_{n \in \mathbb{N}} U_n$ is dense.*

As a corollary, we obtain the following. Note that the corollary is significant in that it guarantees that any meager subset of X must have empty interior.

COROLLARY (Baire Category Theorem). *No nonempty open subset of a complete metric space X is meager in X .*

The following characterization of comeager subsets of X will be very useful when we prove results of a topological nature in the following chapters. Recall that a set is said to be a G_δ subset of X if it is the countable intersection of open subsets of X .

PROPOSITION 2.1. *A set W is comeager in X if and only if W contains a dense G_δ subset of X .*

The final proposition of this section will be used repeatedly in Chapters 4 and 5. The proof is not difficult and is not shown here.

PROPOSITION 2.2. *Let $W \subseteq X$. Suppose that for all $x \in X$ and $\epsilon > 0$, there exist $y \in B_\epsilon(x)$ and $\eta > 0$ such that $B_\eta(y) \subseteq W$. Then W contains a dense open subset of X .*

Before we proceed to the next section, we include some remarks concerning the topologies of the groups under consideration in this paper. In the group $\mathbb{Z}^{\mathbb{Z}}$, we will use the product topology, and in the group $\mathcal{C}(\mathbb{R}^n)$, $n \geq 1$, we will use the compact-open topology. We will give a basis for each of these topologies below. Recall that a collection \mathcal{B} of open subsets of X is a basis for the topology on X if for each open subset U of X and each $x \in U$, there exists $B_x \in \mathcal{B}$ satisfying $x \in B_x \subseteq U$. Note that the open subsets of X are the unions of sets in \mathcal{B} . X

is said to be separable if it contains a countable dense subset of X . Since X is a complete metric space, X is separable if and only if X has a countable basis.

2.1.1 Product Topology on $\mathbb{Z}^{\mathbb{Z}}$

$\mathbb{Z}^{\mathbb{Z}}$ is defined as the space of all continuous mappings $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$. Let \mathbb{Z} be given the discrete topology; i.e., every subset of \mathbb{Z} is open in \mathbb{Z} . Observe that, when \mathbb{Z} is given the discrete topology, every mapping from \mathbb{Z} to \mathbb{Z} is continuous. We endow $\mathbb{Z}^{\mathbb{Z}}$ with the product topology obtained from the discrete topology on \mathbb{Z} . A basis for the product topology on $\mathbb{Z}^{\mathbb{Z}}$ is defined as follows. Let $\mathbb{Z}^{<\mathbb{Z}}$ be the set of all functions σ such that, for some finite subset F of \mathbb{Z} (depending on σ), $\sigma : F \rightarrow \mathbb{Z}$. For each $\sigma \in \mathbb{Z}^{<\mathbb{Z}}$, let

$$[\sigma] = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(n) = \sigma(n) \forall n \in \text{dom}(\sigma)\},$$

where $\text{dom}(\sigma)$ denotes the domain of σ . Then the set $\{[\sigma] : \sigma \in \mathbb{Z}^{<\mathbb{Z}}\}$ is a basis of clopen sets for the product topology on $\mathbb{Z}^{\mathbb{Z}}$. A metric d on $\mathbb{Z}^{\mathbb{Z}}$ is defined by $d(\phi, \psi) = 2^{-n}$, where $\phi(n) \neq \psi(n)$ and $\phi(m) = \psi(m)$ for all $m \in \mathbb{Z}$ satisfying $|m| < |n|$. The metric d induces the product topology on $\mathbb{Z}^{\mathbb{Z}}$ obtained from the discrete topology on \mathbb{Z} [6].

2.1.2 Compact-Open Topology on $\mathcal{C}(\mathbb{R}^n), n \geq 1$

For any $n \in \mathbb{N}$, the space \mathbb{R}^n is defined as the set of all n -tuples of real numbers. Elements of \mathbb{R}^n are of the form

$$\mathbf{x} = (x_1, x_2, \dots, x_n),$$

where $x_1, \dots, x_n \in \mathbb{R}$. (When $n = 1$, we will write x rather than \mathbf{x} .) The Euclidean distance between two points \mathbf{x} and \mathbf{y} in \mathbb{R}^n is given by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}.$$

The metric topology on \mathbb{R}^n induced by d is compatible with the product topology on \mathbb{R}^n . We denote by $\mathcal{C}(\mathbb{R}^n)$ the space of all continuous mappings from \mathbb{R}^n into \mathbb{R}^n . The space $\mathcal{C}(\mathbb{R}^n)$ is endowed with the compact-open topology, which has as a subbasis all sets of the form

$$S(K, U) = \{f \in \mathcal{C}(\mathbb{R}^n) : f(K) \subseteq U\},$$

where $K \subseteq \mathbb{R}^n$ is compact and $U \subseteq \mathbb{R}^n$ is open [34]. The collection of all finite intersections of subbasis elements forms a basis for the compact-open topology in $\mathcal{C}(\mathbb{R}^n)$.

For $f, g \in \mathcal{C}(\mathbb{R}^n)$, we define $\|f - g\|_{[-N, N]^n} = \max_{\mathbf{x} \in [-N, N]^n} \{d(f(\mathbf{x}), g(\mathbf{x}))\}$.

Then a metric ρ on $\mathcal{C}(\mathbb{R}^n)$ is given by

$$\rho(f, g) = \sup_{N \in \mathbb{N}} \left\{ \min \left\{ \frac{1}{N}, \|f - g\|_{[-N, N]^n} \right\} \right\}.$$

The metric ρ metrizes the compact-open topology in $\mathcal{C}(\mathbb{R}^n)$ [18]. In Chapters 4 and 5, we will assume that an arbitrary basis element is of the form

$$B_\epsilon(f) = \{g \in \mathcal{C}(\mathbb{R}^n) : \rho(f, g) < \epsilon\},$$

where $f \in \mathcal{C}(\mathbb{R}^n)$ and $\epsilon > 0$.

2.2 Haar Null Sets in a Polish Abelian Group

A *topological group* is a group G endowed with a topology such that the mapping $(x, y) \mapsto xy^{-1}$ from $G \times G$ to G is continuous. G is said to be *locally*

compact if each point in G has an open neighborhood whose closure is compact. Any group, when endowed with the discrete topology, is a locally compact topological group [13]. We say that G is a *Polish abelian group* if G is a topological abelian group whose topology is separable and completely metrizable. For example, \mathbb{R} , with the usual topology and the group operation of addition, is a locally compact Polish abelian group. More generally, if X is a separable Banach space, then $(X, +)$ is a Polish group [27].

The spaces $\mathbb{Z}^{\mathbb{Z}}$ and $\mathcal{C}(\mathbb{R}^n)$ are nonlocally compact Polish abelian groups; each space has the group operation of pointwise addition. We will show that $\mathbb{Z}^{\mathbb{Z}}$ is nonlocally compact. Let $\phi \in \mathbb{Z}^{\mathbb{Z}}$ and let $U \subseteq \mathbb{Z}^{\mathbb{Z}}$ be a basis element containing ϕ . Since U is an element of the basis for the product topology on $\mathbb{Z}^{\mathbb{Z}}$, U is of the form $\prod_{n \in \mathbb{Z}} U_n$, where $U_n \subseteq \mathbb{Z}$ for all n , and $U_n \neq \mathbb{Z}$ for at most finitely many n . Note that every compact subset of $\mathbb{Z}^{\mathbb{Z}}$ is contained in a set of the form $\prod_{n \in \mathbb{Z}} K_n$, where K_n is a finite subset of \mathbb{Z} for all n . Since $U \subseteq \overline{U}$, the set \overline{U} cannot be compact.

Every locally compact Polish abelian group admits a Haar measure; however, no nonlocally compact Polish abelian group admits such a measure [31]. In the absence of a reasonable translation-invariant measure on a nonlocally compact Polish abelian group G , Christensen sought to define some notion of measure zero sets in G with properties which are analogous to the properties of Haar measure zero sets in a locally compact Polish abelian group. Christensen called such sets “Haar zero” sets [12]. Today the preferred term for such sets is “Haar null” (see [29, 41, 44], for example); we will follow this convention. One might wonder if the notions of Haar null and Haar measure zero are equivalent in a locally compact Polish abelian group. Christensen answered this question in the affirmative. In addition, he showed that the class of Haar null sets in a Polish abelian group is a σ -ideal.

Before we give Christensen's definition of Haar null, we include some remarks concerning the measurability of sets. Let G be a Polish space; i.e., G is a separable completely metrizable topological space. A set $A \subseteq G$ is said to be *universally measurable* if it is μ -measurable for any σ -finite Borel measure μ on G [27]. A subset A of G is *analytic* if there exists a Polish space Y and a continuous function $f : Y \rightarrow G$ such that $f(Y) = A$. The complement of an analytic set is said to be *co-analytic*. A subset of G is *Borel* if it belongs to the σ -algebra generated by the open subsets of G . Every Borel subset of a Polish space G is analytic, and every analytic subset of a Polish space G is universally measurable [13]. Every co-analytic subset of G is universally measurable as well.

DEFINITION 2. *A universally measurable subset A of a Polish abelian group G is Haar null if there exists a Borel probability measure μ on G with the property that $\mu(A + g) = 0$ for all $g \in G$. We call μ a test measure for A . More generally, a set is said to be Haar null if it is contained in a universally measurable Haar null set. A set is **co-Haar null** if its complement is Haar null. We say that **almost every (ae) $g \in G$ has property P** if the set*

$$\{g \in G : g \text{ does not have property } P\}$$

is Haar null.

In Chapters 3 and 5, we will prove that certain subsets of $\mathbb{Z}^{\mathbb{Z}}$ and $\mathcal{C}(\mathbb{R})$ are Haar null. Each of the sets under consideration in these chapters is either universally measurable set or the subset of a universally measurable Haar null set; we will not include a discussion of the measurability of each of the sets individually.

Twenty years later after Christensen's paper was published, Hunt, Sauer, and Yorke reintroduced the idea of Haar null sets in the setting of infinite dimensional Banach spaces [23]. Their terminology differed from Christensen's; they

referred to Haar null sets as “shy sets” and co-Haar null sets as “prevalent sets.” Hunt, Sauer, and Yorke were unaware of Christensen’s earlier work in the area and published an addendum to their paper in which they acknowledged the equivalence of the definitions of shy and Haar null sets [24]. Mycielski in [35] observes that the definition of Haar null is valid in a nonabelian Polish group if one replaces “ $\mu(A + g) = 0$ for all $g \in G$ ” with “ $\mu(g_1 A g_2) = 0$ for all $g_1, g_2 \in G$ ” in the definition.

Hunt, Sauer, and Yorke make several interesting observations in [23]. For example, if A is a Haar null subset of G and μ is its test measure, then it may be assumed without loss of generality that the support of μ , denoted by $\text{supp}(\mu)$, is contained in a compact subset of G . (The support of μ is defined as the smallest closed set whose complement has μ -measure zero [13].) They also note that in an infinite dimensional space, often a convenient choice for a test measure is the Lebesgue measure supported on a finite dimensional subspace. We will use this technique in Chapter 5. The following simple example shows how this technique might be used to show that for a fixed interval I , almost every $f \in \mathcal{C}(\mathbb{R})$ is not constant on I . Let S_I be the set of functions in $\mathcal{C}(\mathbb{R})$ which are constant on the fixed interval I . We will show that S_I is Haar null. For each $k \in [0, 1]$, let $\psi_k : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $\psi_k(x) = kx$. For all Borel subsets B of $\mathcal{C}(\mathbb{R})$, let $\mu(B) = \lambda(\{k : \psi_k \in B\})$, where λ is the Lebesgue measure on \mathbb{R} . Note that μ is a Borel probability measure supported on the one-dimensional subspace $K = \{\psi_k : k \in [0, 1]\}$ of $\mathcal{C}(\mathbb{R})$. Let $h \in \mathcal{C}(\mathbb{R})$. The claim is that $\mu(S_I + h) = 0$. Suppose that there exist $f_1, f_2 \in S_I$ such that $f_1 + h = \psi_{k_1}$ and $f_2 + h = \psi_{k_2}$. Then $f_1 - f_2$ is also constant on I , and $(f_1 - f_2)(x) = (k_1 - k_2)x$. If $k_1 \neq k_2$, then $f_1 - f_2$ is not constant on I . Thus $k_1 = k_2$, and so $f_1 = f_2$. It follows that the cardinality of the set $(S_I + h) \cap K$ is at most 1, and so $\mu(S_I + h) = 0$. Thus S_I is Haar null.

Hunt, Sauer, and Yorke also observe that all co-Haar null (or, prevalent) sets are dense. It immediately follows that every Haar null set has empty interior, and so the class of Haar null sets satisfies each of the four properties of small sets listed at the beginning of the chapter.

In [33], Matoušková gives the following characterization of Haar null sets in the setting of separable Banach spaces. The definition is useful in that it weakens the requirement for the test measure μ .

PROPOSITION. *Let X be a separable Banach space and let A be a Borel subset of X . A is Haar null if and only if for every $\delta > 0$ and $r > 0$, there exists a Borel probability measure μ on X with $\text{supp}(\mu) \subseteq \overline{B}_r(0)$ such that $\mu(A + x) \leq \delta$ for all $x \in X$.*

It is often the case that, given a Polish abelian group G and a property P , the set

$$\{g \in G : g \text{ has property } P\}$$

is neither Haar null nor co-Haar null. Using terminology by Zajíček in [44], we will say that such a property is *H-ambivalent*, and in addition, we will say that the set on which an H-ambivalent property holds is H-ambivalent. H-ambivalent subsets of G are analogous to subsets of G which are nonmeager but not comeager in G . In order to show that a set is not Haar null, we will use the following well-known lemma. (A proof is given in [44].)

LEMMA 2.1. *Let S be a subset of a Polish abelian group G . Suppose that given any compact subset K of G , there exists $g_K \in G$ such that $K + g_K \subseteq S$. Then S is not Haar null.*

CHAPTER 3

$\mathbb{Z}^{\mathbb{Z}}$, THE BAER-SPECKER GROUP

Let \mathbb{Z} denote the set of integers. We denote by $\mathbb{Z}^{\mathbb{Z}}$ the space of all mappings $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$. $\mathbb{Z}^{\mathbb{Z}}$ may be defined as the countably infinite direct product of copies of \mathbb{Z} , and may also be denoted by $\mathbb{Z}^{\mathbb{N}}$, \mathbb{Z}^{ω} , \mathbb{Z}^{\aleph_0} , or $\prod_{\aleph_0} \mathbb{Z}$. The group $\mathbb{Z}^{\mathbb{Z}}$ has the cardinality of the continuum.

The group $\mathbb{Z}^{\mathbb{Z}}$ is referred to as the Baer-Specker group. Two classical results concerning the Baer-Specker group were proven by R. Baer and E. Specker. Recall that an abelian group G is said to be a free abelian group if there exists a free set of generators $X = \{x_{\alpha}\}_{\alpha \in \mathcal{A}}$ such that every element of G can be written uniquely as a finite linear combination of elements of X . Any free abelian group is uniquely determined by the cardinality of the index set \mathcal{A} , up to isomorphism [20]. Baer proved in [4] that the group $\mathbb{Z}^{\mathbb{Z}}$ is not a free abelian group, and Specker proved in [42] that every countable subgroup of $\mathbb{Z}^{\mathbb{Z}}$ is a free abelian group. (See Theorem 19.2 of [20] for a proof of these results.) Other algebraic properties of $\mathbb{Z}^{\mathbb{Z}}$ have been well-studied. Coleman in [14] provides an overview of many of the known results concerning $\mathbb{Z}^{\mathbb{Z}}$.

In this chapter we are interested in identifying properties of elements of $\mathbb{Z}^{\mathbb{Z}}$ which are likely to hold on a randomly chosen element of $\mathbb{Z}^{\mathbb{Z}}$. Our work was inspired by the following pair of theorems.

THEOREM 3.1 (Folklore). *A generic $\sigma \in S_{\infty}$ has no infinite cycle, and infinitely many cycles of length k for all $k \in \mathbb{N}$.*

THEOREM 3.2 (Dougherty-Mycielski, [17]). *Almost every $\sigma \in S_\infty$ has infinitely many infinite cycles and only finitely many finite cycles.*

In these two theorems, we see that the structures of a generic $\sigma \in S_\infty$ and almost every $\sigma \in S_\infty$ are quite different. In fact, we see in these two theorems that S_∞ may be decomposed into two “small” sets: the set of functions which have at least one infinite cycle, which is meager in S_∞ , and the set of functions which have no infinite cycle, which is Haar null. If we now consider the set of all self-maps of \mathbb{Z} , what types of results might we expect? Is there a decomposition of $\mathbb{Z}^{\mathbb{Z}}$ into two small sets? We shall see that we will obtain results concerning the properties of a generic and almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$ which are analogous to the results seen in the theorems above. In addition, since the functions in $\mathbb{Z}^{\mathbb{Z}}$ include but are not limited to the permutations of \mathbb{Z} , we are able to investigate a broader range of properties of functions in $\mathbb{Z}^{\mathbb{Z}}$ than in S_∞ . With this in mind, one may ask whether a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ is, say, surjective. If so, does the opposite property hold for almost every ϕ ? Our results concerning the structure of a generic and almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$ are given in Theorems 3.3 and 3.4. Before we state those results, we will provide the reader with the necessary definitions and terminology regarding elements of $\mathbb{Z}^{\mathbb{Z}}$, which will be given in Section 3.1.

3.1 Directed Graphs of Mappings in $\mathbb{Z}^{\mathbb{Z}}$

$\mathbb{Z}^{\mathbb{Z}}$ is defined as the space of all mappings $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$. $\mathbb{Z}^{\mathbb{Z}}$ is a nonlocally compact Polish group, endowed with the product topology, with the group operation of pointwise addition. The sets $\phi^{-1}(n)$ and $P_{n,\phi}$ are the preimage and predecessor set of n under ϕ , respectively, and are defined as

$$\phi^{-1}(n) = \{m \in \mathbb{Z} : \phi(m) = n\}$$

and

$$P_{n,\phi} = \{m \in \mathbb{Z} : \phi^k(m) = n \text{ for some } k \in \mathbb{N}\}.$$

Observe that $\phi^{-1}(n) \subseteq P_{n,\phi}$, and if $\phi^k(m) = n$ for some $m \in \mathbb{Z}$ and $k > 1$, then the inclusion is strict.

Given any permutation σ of \mathbb{N} , it is well-known that σ may be written as the product of pairwise disjoint cycles. This representation of σ , known as its cyclic decomposition, is unique up to ordering of the cycles and inclusion or omission of 1-cycles [39]. Thus the cyclic decomposition of a function in S_∞ completely characterizes the function. Clearly we are not able to use cycle notation, as it is understood for permutations, to characterize elements of $\mathbb{Z}^{\mathbb{Z}}$. We must establish some other setting in which we are able to uniquely represent elements of $\mathbb{Z}^{\mathbb{Z}}$. In order to do so, we will borrow some ideas from graph theory.

To each ϕ in $\mathbb{Z}^{\mathbb{Z}}$, we will associate a graph Γ_ϕ . Using a graph to represent a function is not without precedent. For example, the authors of [22] describe a *functional digraph* as a directed graph in which the “out-degree” of each vertex is one, and in [8], *permutation graphs*, where graphs are used to represent permutations of a set, are considered. The graph Γ_ϕ is defined as follows.

DEFINITION 3. *Let $\phi \in \mathbb{Z}^{\mathbb{Z}}$. The **graph associated with ϕ** , denoted by Γ_ϕ , is the directed graph whose vertex set is \mathbb{Z} , and whose edge set consists of all directed edges (n_1, n_2) where $\phi(n_1) = n_2$.*

Throughout the remainder of the section, it is assumed that Γ_ϕ denotes a graph associated with some $\phi \in \mathbb{Z}^{\mathbb{Z}}$.

DEFINITION 4. [8] *Let n, m be vertices (not necessarily distinct) in Γ_ϕ . A **path of length k** from n to m is a sequence of vertices*

$$n = n_0, n_1, \dots, n_{k-1} = m$$

such that either (n_i, n_{i+1}) or (n_{i+1}, n_i) is an edge for $i = 0, 1, \dots, k - 2$. If n is a fixed point of ϕ , we say that n, n is a path of length 1 from n to n . We will also allow paths of length 0 by defining each single vertex of Γ_ϕ as a trivial path of length 0.

We now give the definition of a component of a graph Γ_ϕ . Roughly speaking, we may think of a component of a graph Γ_ϕ as a set of vertices which are somehow related or connected to each other under the mapping ϕ . This idea is made precise in the definition and proposition that follow.

DEFINITION 5. Let n, m be vertices of a graph Γ_ϕ . We say that $n \equiv m$ if there exists a finite path from n to m . Note that \equiv defines an equivalence relation on \mathbb{Z} . Let $[n]_\phi$ denote the equivalence class of n under the relation \equiv . We say that $[n]_\phi$ is a **component** of Γ_ϕ . The graph Γ_ϕ is **connected** if it has only one component.

PROPOSITION 3.1. Let n, m be vertices in Γ_ϕ . Then $n \equiv m$ if and only if one of the following occurs.

1. $\phi^r(n) = m$ for some $r \in \mathbb{N}$.
2. $\phi^s(m) = n$ for some $s \in \mathbb{N}$.
3. $\phi^r(n) = \phi^s(m)$ for some $r, s \in \mathbb{N}$.

Proof. Only the implication (\Rightarrow) needs to be proven. Let $n \equiv m$ and suppose that neither (1) nor (2) occurs. Let $n = n_0, n_1, \dots, n_{k-1} = m$ be the path between n and m . Observe that if (n_0, n_1) and (n_{k-2}, n_{k-1}) are directed edges, then it must be the case that for some $2 \leq i \leq k - 2$, both (n_i, n_{i+1}) and (n_i, n_{i-1}) are edges, contradicting that ϕ is a well-defined function. By the same argument, it cannot be the case that (n_1, n_0) and (n_{k-1}, n_{k-2}) are edges of Γ_ϕ . So either (n_1, n_0) and (n_{k-2}, n_{k-1}) are edges, or (n_0, n_1) and (n_{k-1}, n_{k-2}) are edges. Assume the former.

Then for some $1 \leq i \leq k - 2$, the vertex n_i has two edges leaving it, and ϕ is not well-defined. So, (n_0, n_1) and (n_{k-1}, n_{k-2}) are edges. Then it must also be the case that (n_1, n_2) and (n_{k-2}, n_{k-3}) are edges, and so on, until we have $\phi^r(n) = \phi^s(m)$ for some $r, s \in \mathbb{N}$. \square

Our final definition is for the cycle of a graph Γ_ϕ . The term *cycle* will be understood to have two different meanings in this paper – one for the permutations in the space S_∞ , and another for the digraphs of mappings in the space $\mathbb{Z}^{\mathbb{Z}}$. The usage of the term is standard in both settings. To avoid confusion, the reader should be aware of the space we are working in so that the correct meaning of the term will be clear. Note that the definition of cycle in the space S_∞ allows for infinite cycles, while all cycles in $\mathbb{Z}^{\mathbb{Z}}$ must be finite.

DEFINITION 6. *Suppose that $\{n_0, n_1, \dots, n_{k-1}\}$, where $k \geq 1$, is a set of distinct vertices in a graph Γ_ϕ such that $n_0, n_1, \dots, n_{k-1}, n_0$ is a path in Γ_ϕ . Then we say that $\{n_0, n_1, \dots, n_{k-1}\}$, together with the edges of the path, is a **cycle of length k** in the graph Γ_ϕ . A cycle of length 1 represents a fixed point of ϕ , and is said to be a **loop** in Γ_ϕ .*

In the next proposition, we show that, if C is a cycle in a graph Γ_ϕ , then no vertex of the cycle can have more than one edge of the cycle entering it. Intuitively, we may think of the edges of a cycle as being directed in either a clockwise or a counterclockwise direction.

PROPOSITION 3.2. *$\{n_0, n_1, \dots, n_{k-1}\}$ is a cycle in Γ_ϕ if and only if either $\phi(n_{i(\text{mod}k)}) = n_{i+1(\text{mod}k)}$ for all i , or $\phi(n_{i+1(\text{mod}k)}) = n_{i(\text{mod}k)}$ for all i .*

Proof. The direction (\Leftarrow) is true by definition of cycle. Now suppose that

$$\{n_0, n_1, \dots, n_{k-1}\}$$

is a cycle in Γ_ϕ . So there is a path from n_0 to n_0 , and one of the three cases given in Proposition 3.1 must occur. Observe that we cannot have $\phi^r(n_0) = \phi^s(n_0) = n_i$ for some $r, s \in \mathbb{N}$ and $1 \leq i \leq k-1$, for then the vertex n_0 would have two edges leaving it. So by Proposition 3.1, $\phi^r(n_0) = n_0$ for some $r \in \mathbb{N}$, in which case either $\phi(n_{i(\text{mod}k)}) = n_{i+1(\text{mod}k)}$ for all i , or $\phi(n_{i+1(\text{mod}k)}) = n_{i(\text{mod}k)}$ for all i . \square

Note that in a graph Γ_ϕ , any two distinct cycles can have no vertex in common. In our final proposition of the section, we show that any component of a graph Γ_ϕ has at most one cycle.

PROPOSITION 3.3. *Each component of a graph Γ_ϕ is either acyclic or unicyclic.*

Proof. Let $C_1 = \{n_0, \dots, n_{k-1}\}$ and $C_2 = \{m_0, \dots, m_{l-1}\}$ be distinct cycles in a graph Γ_ϕ . Suppose that C_1 and C_2 belong to the same component of Γ_ϕ . Note that C_1 and C_2 have no vertices in common, so for some i and j , there is a path from n_i to m_j . Let $n_i, v_1, \dots, v_s, m_j$ be the vertices in the path, where $v_1, \dots, v_s \notin C_1 \cup C_2$. There must be a directed edge from v_1 to n_i , for if not, n_i has two edges leaving it and ϕ is not well defined. Similarly, $(v_2, v_1), \dots, (m_j, v_s)$ must all be directed edges. But now m_j has two edges leaving it, which contradicts that ϕ is well-defined. \square

Now that we have provided the necessary background information, we proceed to the next section.

3.2 Structure of Generic and AE Mappings in $\mathbb{Z}^{\mathbb{Z}}$

We begin by presenting the main results of the chapter in Theorems 3.3 and 3.4. We posed the question earlier in the chapter: is a generic ϕ surjective? Is the opposite true of almost every ϕ ? Both questions are answered in the affirmative in Theorems 3.3 and 3.4. We find that there are other ways in which the behavior of a generic ϕ differs significantly from the behavior of almost every ϕ . For example,

not only is a generic ϕ not injective, but it has the property that every point has infinite preimage. By contrast, we cannot say that almost every ϕ is injective (see Proposition 3.6), but we can say that almost every ϕ is injective on a co-finite set. In some sense, we may consider a generic ϕ as “strongly” not injective and almost every ϕ as “almost” injective.

Theorems 3.1 and 3.2, above, were the inspiration for our work in $\mathbb{Z}^{\mathbb{Z}}$. In light of this, Properties (3)-(5) of Theorems 3.3 and 3.4 are of particular interest. By comparing Theorem 3.1 and Property (4) of Theorem 3.3, we see that a generic $\sigma \in S_{\infty}$ has infinitely many cycles of length k for all $k \in \mathbb{N}$, and a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ has the property that Γ_{ϕ} has infinitely many cycles of length k for all $k \in \mathbb{N}$. Consider the second property of Theorem 3.1, namely, that a generic $\sigma \in S_{\infty}$ has no infinite cycle. This means that for any $n \in \mathbb{N}$, $\sigma^r(n) = n$ for some $r \in \mathbb{N}$; i.e., the set

$$orb(\sigma, n) = \{n, \sigma(n), \sigma^2(n), \sigma^3(n), \dots\}$$

is finite. By Property (3) of Theorem 3.3, a generic ϕ has the property that every component of Γ_{ϕ} contains a cycle. Thus, given any $n \in \mathbb{Z}$, the forward orbit of n under ϕ eventually terminates in a cycle; i.e., $orb(\phi, n)$ is finite. Now compare the behavior of almost every $\sigma \in S_{\infty}$ with that of almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$. By the theorem of Dougherty and Mycielski, almost every σ has only finitely many finite cycles. By Properties (3) and (5) of Theorem 3.4, almost every ϕ has the property that only the finite components of Γ_{ϕ} contain cycles, and there are only finitely many finite components, so Γ_{ϕ} has only finitely many (finite) cycles. Dougherty and Mycielski also showed that almost every σ has infinitely many infinite cycles, so for infinitely many $n \in \mathbb{N}$, $orb(\sigma, n)$ is infinite; the same is true of almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$ by Properties (3) and (5) of Theorem 3.4.

THEOREM 3.3 (Structure of Generic Mapping in $\mathbb{Z}^{\mathbb{Z}}$). *A generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ has the following properties.*

1. ϕ is a surjection.
2. $\phi^{-1}(n)$ is infinite for all $n \in \mathbb{Z}$.
3. Every component of Γ_{ϕ} contains exactly one cycle.
4. For all $k \in \mathbb{N}$, Γ_{ϕ} has infinitely many components with cycles of length k .
5. Γ_{ϕ} has infinitely many infinite components and no finite component.

THEOREM 3.4 (Structure of Almost Every Mapping in $\mathbb{Z}^{\mathbb{Z}}$). *Almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$ has the following properties.*

1. ϕ is not a surjection and $\mathbb{Z} \setminus \phi(\mathbb{Z})$ is infinite.
2. $P_{n,\phi}$ is finite for all $n \in \mathbb{Z}$. Moreover, there exists a finite set $F_{\phi} \subseteq \mathbb{Z}$ such that ϕ is injective on $\mathbb{Z} \setminus F_{\phi}$.
3. A component of Γ_{ϕ} contains a cycle if and only if it is a finite component.
4. Γ_{ϕ} has only finitely many cycles.
5. Γ_{ϕ} has infinitely many infinite components and only finitely many finite components.

The proofs of these theorems require a series of lemmas and corollaries, which comprise much of the remainder of the chapter. Lemmas 3.1-3.4 and Corollary 3.1 will be used to prove Theorem 3.3. Proposition 3.4, below, will be used in several of the proofs that follow; the proof is straightforward and is not included here.

PROPOSITION 3.4. Fix $n, n_1, n_2 \in \mathbb{Z}$ and $k \in \mathbb{N}$.

The following sets are open in $\mathbb{Z}^{\mathbb{Z}}$.

1. $A_n = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : n \in \phi(\mathbb{Z})\}$
2. $B_{n,k} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : |\phi^{-1}(\phi(n))| \geq k\}$
3. $C_n = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : [n]_{\phi} \text{ contains a cycle}\}$
4. $D_{n,k} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \Gamma_{\phi} \text{ contains } n \text{ cycles of length } k\}$

LEMMA 3.1. The set of surjections in $\mathbb{Z}^{\mathbb{Z}}$, given by

$$\mathcal{S} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(\mathbb{Z}) = \mathbb{Z}\},$$

is comeager in $\mathbb{Z}^{\mathbb{Z}}$.

Proof. Fix $n \in \mathbb{N}$ and let A_n be defined as in Proposition 3.4. It is not difficult to see that A_n is dense in $\mathbb{Z}^{\mathbb{Z}}$: For any basis element $[\sigma]$, let τ be any function in $\mathbb{Z}^{\mathbb{Z}}$ that agrees with σ on the domain of σ and satisfies $\tau(m) = n$ for some $m \notin \text{dom}(\sigma)$. Then $\tau \in [\sigma] \cap A_n$. Since A_n has nonempty intersection with every basis element, A_n is dense in $\mathbb{Z}^{\mathbb{Z}}$. Now by Baire's Theorem, $\mathcal{S} = \bigcap_{n \in \mathbb{Z}} A_n$ is a dense G_{δ} in $\mathbb{Z}^{\mathbb{Z}}$, and by Proposition 2.1, \mathcal{S} is comeager in $\mathbb{Z}^{\mathbb{Z}}$. \square

The next lemma, when combined with Lemma 3.1, will be used to show that a generic ϕ has the property that $\phi^{-1}(n)$ is infinite for all $n \in \mathbb{Z}$.

LEMMA 3.2. Let \mathcal{T} be the set of functions ϕ in $\mathbb{Z}^{\mathbb{Z}}$ such that every element of $\phi(\mathbb{Z})$ has infinite preimage under ϕ . The set \mathcal{T} is comeager in $\mathbb{Z}^{\mathbb{Z}}$.

Proof. Fix $n \in \mathbb{Z}$. Let $B_n = \bigcap_{k=1}^{\infty} B_{n,k}$, where $B_{n,k}$ is defined as in Proposition 3.4. Observe that B_n is the set of all functions ϕ such that $\phi(n)$ has infinite preimage. We will show that B_n is comeager in $\mathbb{Z}^{\mathbb{Z}}$. Given any basis element $[\sigma]$,

define a function $\tau : \mathbb{Z} \rightarrow \mathbb{Z}$ as follows. Let $\tau(x) = \sigma(x)$ for all $x \in \text{dom}(\sigma)$. If $n \notin \text{dom}(\sigma)$, choose some value for $\tau(n)$, say $\tau(n) = 1$. After $\tau(n)$ has been defined, let $\tau(m) = \tau(n)$ for all $m \notin \text{dom}(\sigma)$. Now $\tau \in B_n \cap [\sigma]$. Thus B_n is a dense G_δ in $\mathbb{Z}^{\mathbb{Z}}$ and is comeager in $\mathbb{Z}^{\mathbb{Z}}$ by Proposition 2.1. It follows that the countable intersection $\mathcal{T} = \bigcap_{n \in \mathbb{Z}} B_n$ is comeager in $\mathbb{Z}^{\mathbb{Z}}$ as well. \square

COROLLARY 3.1. *The set of functions ϕ in $\mathbb{Z}^{\mathbb{Z}}$ such that every integer has infinite preimage under ϕ is comeager in $\mathbb{Z}^{\mathbb{Z}}$.*

Proof. Let \mathcal{S}, \mathcal{T} be defined as in Lemmas 3.1 and 3.2. Then $\mathcal{S} \cap \mathcal{T}$ is comeager in $\mathbb{Z}^{\mathbb{Z}}$, and any $\phi \in \mathcal{S} \cap \mathcal{T}$ has the property that $\phi^{-1}(n)$ is infinite for all $n \in \mathbb{Z}$. \square

LEMMA 3.3. *The set of functions ϕ in $\mathbb{Z}^{\mathbb{Z}}$ such that every component of Γ_ϕ contains exactly one cycle is comeager in $\mathbb{Z}^{\mathbb{Z}}$.*

Proof. It was shown in Proposition 3.3 that, given any $\phi \in \mathbb{Z}^{\mathbb{Z}}$, each component of Γ_ϕ contains at most one cycle. Let C_n be defined as in Proposition 3.4. It is not difficult to show that C_n is dense in $\mathbb{Z}^{\mathbb{Z}}$. It follows that $\bigcap_{n \in \mathbb{Z}} C_n$ is a dense G_δ in $\mathbb{Z}^{\mathbb{Z}}$. \square

LEMMA 3.4. *The set of functions ϕ in $\mathbb{Z}^{\mathbb{Z}}$ such that Γ_ϕ contains infinitely many components with cycles of length k for all $k \in \mathbb{N}$ is comeager in $\mathbb{Z}^{\mathbb{Z}}$.*

Proof. Let $D_{n,k}$ be defined as in Proposition 3.4 for some fixed k, n . We will show that $D_{n,k}$ is dense in $\mathbb{Z}^{\mathbb{Z}}$. Let $[\sigma]$ be a basic element, with

$$\text{dom}(\sigma) \cup \text{im}(\sigma) \subseteq \{0, \pm 1, \dots, \pm(m-1)\}$$

for some $m \in \mathbb{N}$. (Here $\text{im}(\sigma)$ denotes the set $\sigma(\text{dom}(\sigma))$, the image of $\text{dom}(\sigma)$ under σ .) Let τ be any extension of σ such that Γ_τ contains the n cycles

$$\begin{aligned} &\{m, m+1, \dots, m+k-1\}, \\ &\{m+k, m+k+1, \dots, m+2k-1\}, \end{aligned}$$

⋮

$$\{m + (n - 1)k, m + (n - 1)k + 1, \dots, m + nk - 1\}.$$

Then $\tau \in [\sigma] \cap D_{n,k}$, and so $D_{n,k}$ is dense in $\mathbb{Z}^{\mathbb{Z}}$. Now $D_k = \bigcap_{n=1}^{\infty} D_{n,k}$, the set of functions ϕ such that Γ_{ϕ} contains infinitely many cycles of length k , is a dense G_{δ} in $\mathbb{Z}^{\mathbb{Z}}$ and is thus comeager in $\mathbb{Z}^{\mathbb{Z}}$. Since each component contains exactly one cycle, the set of functions ϕ such that Γ_{ϕ} contains infinitely many components with cycles of length k is comeager in $\mathbb{Z}^{\mathbb{Z}}$. Finally, we intersect the sets D_k over all $k \in \mathbb{N}$ to obtain the lemma. \square

Proof. (Proof of Theorem 3.3) Let \mathcal{G} be the intersection of the sets defined in Lemma 3.1, Corollary 3.1, Lemma 3.3, and Lemma 3.4. Then \mathcal{G} is comeager in $\mathbb{Z}^{\mathbb{Z}}$. Let $\phi \in \mathcal{G}$. Clearly ϕ has properties (1) – (4) of Theorem 3.3; we need only show that ϕ has property (5). Observe that, since every $n \in \mathbb{Z}$ has infinite preimage under ϕ , it must be the case that every component of Γ_{ϕ} is infinite. Also, since Γ_{ϕ} has infinitely many cycles by property (4), and every component contains exactly one cycle by property (3), Γ_{ϕ} has infinitely many components. So ϕ satisfies all of the properties of the theorem. \square

This completes the proof of Theorem 3.3. The next series of lemmas will be used to prove Theorem 3.4. In order to prove that a property holds for almost every ϕ , we must prove that the subset of $\mathbb{Z}^{\mathbb{Z}}$ on which the property does not hold is Haar null. To do so, we must define a Borel probability measure on $\mathbb{Z}^{\mathbb{Z}}$. Throughout the remainder of this section, we will use the test measure μ , defined as follows. For each $n \in \mathbb{Z}$, let μ_n be the uniform probability measure on the set $\{1, 2, \dots, 2^{|n|}\}$. Let μ be the product measure on $\mathbb{Z}^{\mathbb{Z}}$, so that for any basic open set $[\sigma]$, we have

$$\mu([\sigma]) = \prod_{n \in \text{dom}(\sigma)} \mu_n(\sigma(n)).$$

Then μ is a measure on all Borel subsets of $\mathbb{Z}^{\mathbb{Z}}$ [27].

The following notation will be used in the proofs below. We let $\sigma_{i,j}$ denote the function in $\mathbb{Z}^{<\mathbb{Z}}$ which satisfies $\text{dom}(\sigma_{i,j}) = \{i\}$ and $\sigma_{i,j}(i) = j$. Observe that $\mu([\sigma_{i,j}]) = 0$ if $j \notin \{1, 2, \dots, 2^{|i|}\}$ and $\mu([\sigma_{i,j}]) = \frac{1}{2^{|i|}}$ if $j \in \{1, 2, \dots, 2^{|i|}\}$.

LEMMA 3.5. *The set \mathcal{S} of surjections in $\mathbb{Z}^{\mathbb{Z}}$ is Haar null.*

Proof. For each pair $k, n \in \mathbb{Z}$, define $S_{k,n} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(k) = n\}$. Note that

$$\mathcal{S} = \bigcap_{n \in \mathbb{Z}} \bigcup_{k \in \mathbb{Z}} S_{k,n}.$$

Let $\psi \in \mathbb{Z}^{\mathbb{Z}}$ be arbitrarily chosen. The claim is that $\mu(\mathcal{S} + \psi) = 0$. We will show that for all $\epsilon > 0$, there exists $n \in \mathbb{Z}$ such that $\mu(\bigcup_{k \in \mathbb{Z}} S_{k,n} + \psi) < \epsilon$.

Let $\epsilon > 0$. Let $l \in \mathbb{N}$ be such that $2^{1-l} < \epsilon$. Choose $n \in \mathbb{Z}$ satisfying

$$\begin{aligned} n + \psi(0) &> 1, \\ n + \psi(1), n + \psi(-1) &> 2, \\ n + \psi(2), n + \psi(-2) &> 4, \\ &\vdots \\ n + \psi(l), n + \psi(-l) &> 2^l. \end{aligned}$$

Observe that, if $\phi + \psi \in S_{i,n} + \psi$ where $|i| \leq l$, then $(\phi + \psi)(i) = n + \psi(i) > 2^{|i|}$, and so $\mu(S_{i,n} + \psi) = 0$. Now

$$\mu\left(\bigcup_{k \in \mathbb{Z}} S_{k,n} + \psi\right) = \mu\left(\bigcup_{|k| \geq l+1} S_{k,n} + \psi\right) \leq \sum_{|k| \geq l+1} \mu(S_{k,n} + \psi) \leq \sum_{k=l+1}^{\infty} \frac{2}{2^k} = 2^{1-l} < \epsilon.$$

Since $\mu(\bigcup_{k \in \mathbb{Z}} S_{k,n} + \psi)$ is arbitrarily small depending on the choice of n , and $\mathcal{S} + \psi \subseteq \bigcup_{k \in \mathbb{Z}} S_{k,n} + \psi$ for all n , it follows that $\mu(\mathcal{S} + \psi) = 0$ and \mathcal{S} is Haar null. \square

LEMMA 3.6. *The set*

$$\mathcal{A} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : |\phi^{-1}(n)| = \omega \text{ for some } n \in \mathbb{Z}\}$$

is Haar null.

Proof. Fix $n \in \mathbb{Z}$. Let

$$A_n = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : |\phi^{-1}(n)| = \omega\},$$

and for each $k \in \mathbb{N}$, let

$$A_{k,n} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(k) = n \text{ or } \phi(-k) = n\}.$$

Observe that $A_n \subseteq \bigcup_{k=l}^{\infty} A_{k,n}$ for all $l \in \mathbb{N}$. Let $\psi \in \mathbb{Z}^{\mathbb{Z}}$ be arbitrarily chosen. Now for any k , we have

$$A_{k,n} + \psi \subseteq [\sigma_{k,n+\psi(k)}] \cup [\sigma_{-k,n+\psi(-k)}],$$

so $\mu(A_{k,n} + \psi) \leq \mu([\sigma_{k,n+\psi(k)}] \cup [\sigma_{-k,n+\psi(-k)}]) \leq \frac{2}{2^k}$. Then

$$\mu\left(\bigcup_{k=l}^{\infty} A_{k,n} + \psi\right) \leq \sum_{k=l}^{\infty} \mu(A_{k,n} + \psi) \leq \sum_{k=l}^{\infty} \frac{2}{2^k} = \frac{4}{2^l} \rightarrow 0 \text{ as } l \rightarrow \infty.$$

Since $A_n + \psi \subseteq \bigcup_{k=l}^{\infty} A_{k,n} + \psi$ for all l , we have that $\mu(A_n + \psi) = 0$, and so A_n is Haar null. It follows that $\mathcal{A} = \bigcup_{n \in \mathbb{Z}} A_n$ is Haar null as well. □

LEMMA 3.7. *The set*

$$\mathcal{L} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \Gamma_{\phi} \text{ contains infinitely many cycles}\}$$

is Haar null.

Proof. We say that a vertex n in a graph Γ_{ϕ} is a *minimal vertex* of a cycle $\{n_0, \dots, n_{k-1}\}$ if $n \in \{n_0, \dots, n_{k-1}\}$ and $|n| \leq |n_i|$ for all $i = 0, \dots, k-1$. For each $n \in \mathbb{N}$, let

$$L_n = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \Gamma_{\phi} \text{ contains a cycle whose minimal vertex is } n \text{ or } -n\}.$$

Observe that $\mathcal{L} = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} L_j$.

Let $\psi \in \mathbb{Z}^{\mathbb{Z}}$ be arbitrarily chosen. Fix $m \in \mathbb{N}$. Let $\phi + \psi \in L_m + \psi$.

Then $\exists m' \in \mathbb{Z}$ such that $|m'| \geq m$ and either $(\phi + \psi)(m') = m + \psi(m')$, in which case $\phi + \psi \in [\sigma_{m', m + \psi(m')}]$, or $(\phi + \psi)(m') = -m + \psi(m')$, in which case $\phi + \psi \in [\sigma_{m', -m + \psi(m')}]$. It follows that $L_m + \psi \subseteq \bigcup_{|k| \geq m} [\sigma_{k, m + \psi(k)}] \cup [\sigma_{k, -m + \psi(k)}]$. So

$$\mu(L_m + \psi) \leq \mu \left(\bigcup_{|k| \geq m} [\sigma_{k, m + \psi(k)}] \cup [\sigma_{k, -m + \psi(k)}] \right) \leq \sum_{|k| \geq m} \frac{2}{2^{|k|}} = \frac{8}{2^m},$$

and for any $n \in \mathbb{N}$, we have

$$\mu \left(\bigcup_{j=n}^{\infty} (L_j + \psi) \right) \leq \sum_{j=n}^{\infty} \mu(L_j + \psi) \leq \sum_{j=n}^{\infty} \frac{8}{2^j} = \frac{16}{2^n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $\mathcal{L} + \psi \subseteq \bigcup_{j=n}^{\infty} (L_j + \psi)$ for all n , we have that $\mu(\mathcal{L} + \psi) = 0$, and \mathcal{L} is Haar null. \square

LEMMA 3.8. *Fix $l \in \mathbb{N}$. The set*

$$\mathcal{C}_l = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : |\phi(n)| \leq |n| + l \text{ for infinitely many } n \in \mathbb{Z}\}$$

is Haar null.

Proof. Observe that $\mathcal{C}_l \subseteq \bigcap_{n=1}^{\infty} \bigcup_{|i| \geq n} \left(\bigcup_{j=-(|i|+l)}^{|i|+l} [\sigma_{i,j}] \right)$. To see that this is true, note that, for any fixed n , if $\phi \in \mathcal{C}_l$, then there exists $i_0 \in \mathbb{Z}$, $|i_0| \geq n$ such that $|\phi(i_0)| \leq |i_0| + l$; i.e., $\phi(i_0) \in \{-(|i_0| + l), \dots, 0, \dots, |i_0| + l\}$. Then $\phi \in \bigcup_{j=-(|i_0|+l)}^{|i_0|+l} [\sigma_{i_0,j}]$, and it follows that $\phi \in \bigcup_{|i| \geq n} \left(\bigcup_{j=-(|i|+l)}^{|i|+l} [\sigma_{i,j}] \right)$. This is true for any n , so \mathcal{C}_l is in the intersection of all such unions.

Let $\psi \in \mathbb{Z}^{\mathbb{Z}}$. Now $\mu \left(\bigcup_{j=-(|i|+l)}^{|i|+l} [\sigma_{i,j}] + \psi \right) \leq \frac{2(|i|+l)+1}{2^{|i|}}$, so

$$\mu \left(\bigcup_{|i| \geq n} \left(\bigcup_{j=-(|i|+l)}^{|i|+l} [\sigma_{i,j}] + \psi \right) \right) \leq \sum_{|i| \geq n} \frac{2(|i| + l) + 1}{2^{|i|}} = \frac{3 + 2l + 2n}{2^{n-2}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Since $\mathcal{C}_l + \psi$ is contained in this union for every n , we have $\mu(\mathcal{C}_l + \psi) = 0$ and \mathcal{C}_l is Haar null. \square

We saw in Lemma 3.6 that almost every ϕ has the property that the preimage of every point is finite. In the next lemma, we will use Lemma 3.6 to prove that almost every ϕ has the property that $P_{n,\phi}$, the predecessor set of n under ϕ , is finite for all $n \in \mathbb{Z}$. Since $\phi^{-1}(n) \subseteq P_{n,\phi}$, Lemma 3.9 is a stronger result than Lemma 3.6.

LEMMA 3.9. *The set*

$$\mathcal{F} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \exists n \in \mathbb{Z} \ni |P_{n,\phi}| = \omega\}$$

is Haar null.

Proof. Let

$$F_1 = \{\phi \in \mathcal{F} : |\phi^{-1}(n)| < \omega \text{ for all } n\}$$

and

$$F_2 = \{\phi \in \mathcal{F} : |\phi^{-1}(n)| = \omega \text{ for some } n\}.$$

Then $\mathcal{F} = F_1 \cup F_2$. The set F_2 is Haar null, as it is a subset of the Haar null set \mathcal{A} of Lemma 3.6, so we need only show that F_1 is Haar null to prove the lemma.

We will show that $F_1 \subseteq \mathcal{C}_1$, where \mathcal{C}_1 is the set defined in Lemma 3.8 with $l = 1$.

Let $\phi \in F_1$ and let $n \in \mathbb{Z}$ be such that $P_{n,\phi}$ is infinite. By König's Lemma (see [27]) there exists a sequence of distinct integers $(m_k)_{k \in \mathbb{N}}$ such that $\phi^k(m_k) = n$ and $\phi(m_{k+1}) = m_k$. Define an increasing sequence $(N_j) \subseteq \mathbb{N}$ and subsequence $(m_{k_j}) \subseteq (m_k)$ inductively as follows. Choose N_1 so that $N_1 > |n|$. Let $k_1 = \min\{k : |m_k| \geq N_1\}$. For $j \in \{2, 3, \dots\}$, choose $N_j > |m_{k_{j-1}}|$ and $k_j = \min\{k : |m_k| \geq N_j\}$. Now $|\phi(m_{k_j})| < |m_{k_j}|$ for all $j \in \mathbb{N}$, and so $F_1 \subseteq \mathcal{C}_1$. Thus F_1 is Haar null. It follows that \mathcal{F} is Haar null. \square

As a corollary to Lemma 3.9, we obtain the following result.

COROLLARY 3.2. *The set*

$$\mathcal{K} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \Gamma_{\phi} \text{ contains an infinite component which contains a cycle}\}$$

is Haar null.

Proof. Let $\phi \in \mathcal{K}$. Let $\{n_0, \dots, n_{k-1}\}$ be a cycle in Γ_{ϕ} such that the component containing $\{n_0, \dots, n_{k-1}\}$ is infinite. Then it must be the case that for some $n_i \in \{n_0, \dots, n_{k-1}\}$, $P_{n_i, \phi}$ is infinite. Now $\mathcal{K} \subseteq \mathcal{F}$, where \mathcal{F} is defined as in Lemma 3.9, and so \mathcal{K} is Haar null. \square

In the next lemma, we see that almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$ is injective on a co-finite set.

LEMMA 3.10. *Let \mathcal{J} be the set of all $\phi \in \mathbb{Z}^{\mathbb{Z}}$ such that there are infinitely many pairs $k_1 \neq k_2 \in \mathbb{Z}$ satisfying $\phi(k_1) = \phi(k_2)$. Then \mathcal{J} is Haar null.*

Proof. For any $\phi \in \mathcal{J}$, there are two possible cases which may occur.

CASE 1. There exists a fixed $k \in \mathbb{Z}$ such that there are infinitely many $k' \in \mathbb{Z} \setminus \{k\}$ satisfying $\phi(k) = \phi(k')$. Observe that, if this occurs, then $P_{\phi(k), \phi}$, the predecessor set of $\phi(k)$ under ϕ , is an infinite set. Let J be the set all elements of \mathcal{J} which satisfy Case 1. Then J is a subset of \mathcal{F} , where \mathcal{F} is defined as in Lemma 3.9, and so J is Haar null.

CASE 2. Given any pair $k_1 \neq k_2$ with $\phi(k_1) = \phi(k_2)$, there are at most finitely many k' satisfying $\phi(k_1) = \phi(k_2) = \phi(k')$. Let T be the subset of \mathcal{J} consisting of all $\phi \in \mathcal{J}$ which satisfy Case 2. We will show that T is Haar null. For each $m \in \mathbb{N}$, let

$$T_m = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(k_1) = \phi(k_2) \text{ for some } k_1 \neq k_2, |k_2| \geq |k_1| \geq m\}.$$

Then $T \subseteq \bigcap_{m=1}^{\infty} T_m$. Let $\psi \in \mathbb{Z}^{\mathbb{Z}}$ be arbitrarily chosen.

Fix $k_1, k_2 \in \mathbb{Z}$ where, without loss of generality, we may assume $|k_2| \geq |k_1|$.

Consider the μ -measure of the set $\{\phi + \psi \in \mathbb{Z}^{\mathbb{Z}} + \psi : \phi(k_1) = \phi(k_2)\}$. Observe that if

$$\phi + \psi \in \{\phi + \psi : \phi(k_1) = \phi(k_2)\} \cap \text{supp}(\mu)$$

then

$$(\phi + \psi)(k_1) = n \text{ for some } n \in \text{supp}(\mu_{k_1}), \text{ and } (\phi + \psi)(k_2) = n - \psi(k_1) + \psi(k_2).$$

Then $\phi + \psi \in \bigcup_{n \in \text{supp}(\mu_{k_1})} [\sigma_n]$, where $\sigma_n \in \mathbb{Z}^{<\mathbb{Z}}$ satisfies $\text{dom}(\sigma_n) = \{k_1, k_2\}$, $\sigma_n(k_1) = n$, and $\sigma_n(k_2) = n - \psi(k_1) + \psi(k_2)$. Now

$$\mu \left(\bigcup_{n \in \text{supp}(\mu_{k_1})} [\sigma_n] \right) \leq \sum_{n \in \text{supp}(\mu_{k_1})} \mu([\sigma_n]) \leq \sum_{n \in \text{supp}(\mu_{k_1})} \frac{1}{2^{|k_1|}} \frac{1}{2^{|k_2|}} = \frac{1}{2^{|k_2|}}.$$

Thus the μ -measure of the set $\{\phi + \psi : \phi(k_1) = \phi(k_2)\}$ is no more than $\frac{1}{2^{|k_2|}}$. Now

$$T_m + \psi = \bigcup_{|k_1| \geq m} \bigcup_{|k_2| \geq |k_1|} \{\phi + \psi : \phi(k_1) = \phi(k_2)\},$$

so

$$\begin{aligned} \mu(T_m + \psi) &\leq \sum_{|k_1| \geq m} \sum_{|k_2| \geq |k_1|} \mu(\{\phi + \psi : \phi(k_1) = \phi(k_2)\}) \\ &\leq \sum_{|k_1| \geq m} \sum_{|k_2| \geq |k_1|} \frac{1}{2^{|k_2|}} \\ &= \frac{16}{2^m} \rightarrow 0 \text{ as } m \rightarrow \infty. \end{aligned}$$

Observe that $T + \psi \subseteq T_m + \psi$ for all m , so we have $\mu(T + \psi) = 0$ and T is Haar null. Finally, since $\mathcal{J} = J \cup T$, where J and T are Haar null, we have the lemma. \square

When components in a graph Γ_ϕ are viewed as analogous to cycles in a permutation $\sigma \in S_\infty$, the following lemma provides a Dougherty-Mycielski-like result for the space $\mathbb{Z}^{\mathbb{Z}}$.

LEMMA 3.11. *Let \mathcal{M} be the set of all $\phi \in \mathbb{Z}^{\mathbb{Z}}$ such that either Γ_ϕ has infinitely many finite components, or Γ_ϕ has only finitely many components. Then, \mathcal{M} is Haar null.*

Proof. Let \mathcal{L} be defined as in Lemma 3.7. Observe that, for any $\phi \in \mathbb{Z}^{\mathbb{Z}}$, if a component of Γ_ϕ is finite, it must contain a cycle. So the set of functions ϕ such that Γ_ϕ contains infinitely many finite components is contained in \mathcal{L} , a set which is Haar null by Lemma 3.7.

Now fix $m \in \mathbb{N}$ and let $S_m = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \Gamma_\phi \text{ contains exactly } m \text{ components}\}$.

We will show that S_m is Haar null. Write S_m as the union

$$(S_m \cap (\mathcal{C}_{m+1} \cup \mathcal{J})) \cup (S_m \cap (\mathcal{C}_{m+1} \cup \mathcal{J})^c),$$

where \mathcal{C}_{m+1} and \mathcal{J} are defined as in Lemmas 3.8 and 3.10, respectively. Now $S_m \cap (\mathcal{C}_{m+1} \cup \mathcal{J})$ is Haar null, as it is contained in the Haar null set $\mathcal{C}_{m+1} \cup \mathcal{J}$. We claim that $S_m \cap (\mathcal{C}_{m+1} \cup \mathcal{J})^c$ is empty. To obtain a contradiction, suppose not. Let $\phi \in S_m \cap (\mathcal{C}_{m+1} \cup \mathcal{J})^c$. Choose $N \in \mathbb{N}$ large enough so that

- $|\phi(i)| > |i| + m + 1$ for all $|i| \geq N$, and
- $\phi(i_1) \neq \phi(i_2)$ for all $|i_1|, |i_2| \geq N$.

Fix $n \geq N$ and consider the vertices $n, n+1, \dots, n+m$ in Γ_ϕ . Since Γ_ϕ has exactly m components, at least two of these vertices must lie in the same component; say, $v \in [u]_\phi$ where $u, v \in \{n, n+1, \dots, n+m\}$ and $u < v$. By the choice of N , $|\phi(u)| > u + m + 1 > v$, and $(|\phi^k(u)|)_{k \in \mathbb{N}}$ is a strictly increasing sequence. So $\phi^k(u) \neq v$ for any k . Similarly, $\phi^k(v) \neq u$ for any k . Finally, since $|\phi^{k-1}(u)|, |\phi^{l-1}(v)| \geq N$ for all $k, l \geq 1$, we have $\phi^k(u) = \phi(\phi^{k-1}(u)) \neq \phi(\phi^{l-1}(v)) = \phi^l(v)$. So $v \notin [u]_\phi$, a contradiction. Thus $S_m \cap (\mathcal{C}_{m+1} \cup \mathcal{J})^c = \emptyset$, and the set S_m is Haar null. Now $\bigcup_{m=1}^{\infty} S_m$, the set of functions ϕ such that Γ_ϕ has only finitely many components,

is Haar null as well. Then \mathcal{M} , the union of the set of all $\phi \in \mathbb{Z}^{\mathbb{Z}}$ such that Γ_ϕ has infinitely many finite components with the set of all $\phi \in \mathbb{Z}^{\mathbb{Z}}$ such that Γ_ϕ has only finitely many components, is Haar null. \square

We are now ready to prove the second main theorem of the chapter, Theorem 3.4.

Proof. (Proof of Theorem 3.4) Let

$$\mathcal{H} = (\mathcal{S} \cup \mathcal{L} \cup \mathcal{F} \cup \mathcal{K} \cup \mathcal{J} \cup \mathcal{M})^c,$$

where the sets in the union are defined as in Lemmas 3.5, 3.7, 3.9, Corollary 3.2, and Lemmas 3.10 and 3.11. The set \mathcal{H} is co-Haar null. Let $\phi \in \mathcal{H}$. We claim that ϕ satisfies properties (1) – (5) of Theorem 3.4. We can easily see that properties (2) and (3) are satisfied by the definitions of the sets \mathcal{F} , \mathcal{J} , and \mathcal{K} . Since $\phi \notin \mathcal{M}$, Γ_ϕ has infinitely many components and at most finitely many of these components are finite, so property (5) is satisfied. Since a component of Γ_ϕ contains a cycle if and only if it is finite, and there are at most finitely many finite components, we have that property (4) holds. It remains to show that property (1) holds. Clearly ϕ is not surjective, since $\phi \notin \mathcal{S}$. Now let $[n]_\phi$ be any infinite component of Γ_ϕ . Suppose that every vertex in $[n]_\phi$ has an edge entering it. Then it must be the case that $P_{n,\phi}$ is an infinite set, contradicting that $\phi \notin \mathcal{F}$. So every infinite component of Γ_ϕ contains a vertex v such that no edge of Γ_ϕ enters v ; each such vertex v lies in $\mathbb{Z} \setminus \phi(\mathbb{Z})$, and so $\mathbb{Z} \setminus \phi(\mathbb{Z})$ is an infinite set. Thus, property (1) is satisfied. \square

The final propositions of this chapter are included to show that the results of Theorem 3.4 are the strongest possible. We will show that certain subsets of $\mathbb{Z}^{\mathbb{Z}}$ are neither Haar null nor co-Haar null; i.e., these subsets are H-ambivalent. To show that a set is not Haar null, we will use Lemma 2.1. For the remainder of the chapter,

we will use K to denote an arbitrary compact subset of $\mathbb{Z}^{\mathbb{Z}}$, and we will assume that K is contained in a set of the form $\prod_{i \in \mathbb{Z}} K_i$, where $K_i = \{p_1^i, p_2^i, \dots, p_{n_i}^i\}$ and $p_j^i < p_{j+1}^i \forall i \in \mathbb{Z}$.

Consider Property (1) of Theorem 3.4. We have that almost every ϕ has the property that $\phi(\mathbb{Z})$ is infinite, but $\phi(\mathbb{Z}) \neq \mathbb{Z}$ and $\phi(\mathbb{Z})$ “misses” infinitely many points of \mathbb{Z} . Can we say that for almost every ϕ , $\phi(\mathbb{Z})$ is bounded above or below? Or is $\phi(\mathbb{Z})$ unbounded in both directions? In the following proposition, we answer these questions in the negative as we show that the sets of functions with these properties are H-ambivalent.

PROPOSITION 3.5. *Let*

$$S_1 = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(\mathbb{Z}) \text{ is neither bounded above nor bounded below}\},$$

$$S_2 = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(\mathbb{Z}) \text{ is bounded below}\},$$

$$S_3 = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(\mathbb{Z}) \text{ is bounded above}\}.$$

The sets S_i are H-ambivalent.

Proof. First we show that none of the sets is Haar null. Let K be a compact subset of $\mathbb{Z}^{\mathbb{Z}}$. Choose a sequence $\{c_i\}_{i \in \mathbb{Z}}$ as follows. Let $c_0 = 0$, and choose the c_i so that $p_1^i + c_i > p_{n_{i-1}}^{i-1} + c_{i-1}$ for all $i \in \mathbb{Z}$. Let $\psi : \mathbb{Z} \rightarrow \mathbb{Z}$ be the function defined by $\psi(i) = c_i$ for all $i \in \mathbb{Z}$. Then $K + \psi \subseteq S_1$, and so S_1 is not Haar null.

Fix $N \in \mathbb{Z}$ and let $S_{2,N} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(\mathbb{Z}) \subseteq \{N, N+1, \dots\}\}$. Define a function $\psi : \mathbb{Z} \rightarrow \mathbb{Z}$ as follows. For each $i \in \mathbb{Z}$, choose $\psi(i)$ satisfying $p_1^i + \psi(i) \geq N$. Let $\gamma \in K$ and $i \in \mathbb{Z}$. Then $(\gamma + \psi)(i) \geq p_1^i + \psi(i) \geq N$. Thus $S_{2,N}$ is not Haar null, and so $S_2 = \bigcup_{N \in \mathbb{Z}} S_{2,N}$ is not Haar null. By a similar argument we are able to show that S_3 is not Haar null.

Now the complement of S_1 contains the non-Haar null set S_2 , and so S_1 is

not co-Haar null. By the same argument, S_1 is contained in the complements of S_2 and S_3 , so neither S_2 nor S_3 is co-Haar null.

□

By the second property in Theorem 3.4, almost every ϕ is injective on the complement of a finite set. Is almost every ϕ actually injective on \mathbb{Z} ? We see in the next proposition that we cannot say that almost every ϕ is injective, as the set of injections in $\mathbb{Z}^{\mathbb{Z}}$ is H-ambivalent.

PROPOSITION 3.6. *The set $\mathcal{I} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \phi(n_1) \neq \phi(n_2) \text{ for all } n_1 \neq n_2\}$ is H-ambivalent.*

Proof. To see that \mathcal{I} is not co-Haar null, observe that \mathcal{I}^c is not Haar null because it does not have empty interior. (For example, let $\sigma \in \mathbb{Z}^{<\mathbb{Z}}$ with $\text{dom}(\sigma) = \{1, 2\}$ and $\sigma(1) = \sigma(2) = 0$. Then $[\sigma]$ is an open set contained in the complement of \mathcal{I} .)

Now we show that \mathcal{I} is not Haar null. Let K be a compact subset of $\mathbb{Z}^{\mathbb{Z}}$. Let $\{c_i\}_{i \in \mathbb{Z}}$ be the sequence defined in the proof of Proposition 3.5, and let $\psi(i) = c_i$ for all $i \in \mathbb{Z}$. Then $K + \psi \subseteq \mathcal{I}$, and so \mathcal{I} is not Haar null. □

Finally, by Property (4) of Theorem 3.4, almost every ϕ has the property that Γ_ϕ has only finitely many cycles. Does almost every ϕ have the property that Γ_ϕ has at least one cycle? We see in the next proposition that we can draw no conclusion as to whether or not almost every Γ_ϕ has a cycle.

PROPOSITION 3.7. *The set $\mathcal{N} = \{\phi \in \mathbb{Z}^{\mathbb{Z}} : \Gamma_\phi \text{ contains no cycle}\}$ is H-ambivalent.*

Proof. To see that \mathcal{N} is not co-Haar null, we simply show that \mathcal{N}^c contains an open set. For example, if we define $\sigma \in \mathbb{Z}^{<\mathbb{Z}}$ such that $\text{dom}(\sigma) = \{1, 2\}$, $\sigma(1) = 2, \sigma(2) = 1$, then we have $[\sigma] \in \mathcal{N}^c$, and \mathcal{N} is not co-Haar null.

Now let K be a compact subset of $\mathbb{Z}^{\mathbb{Z}}$. Choose a sequence $\{c_i\}_{i \in \mathbb{Z}}$ as follows.

For all $i \in \mathbb{Z}$, choose c_i large enough so that $|i| < p_1^i + c_i$. Let $\psi(i) = c_i$ for all $i \in \mathbb{Z}$. The claim is that $K + \psi \subseteq \mathcal{N}$. Let $\gamma \in K$. Then for any $i \in \mathbb{Z}$, we have $|i| < p_1^i + c_i \leq (\gamma + \psi)(i)$, and so $\Gamma_{\gamma+\psi}$ can have no cycle.

□

CHAPTER 4

COMEAGER SUBSETS OF $\mathcal{C}(\mathbb{R}^n), n \geq 1$

In this chapter we will study properties of a generic $f \in \mathcal{C}(\mathbb{R})$ and a generic $f \in \mathcal{C}(\mathbb{R}^n), n \geq 1$. Many of the results of this chapter concern the behavior of a generic $f \in \mathcal{C}(\mathbb{R})$. These results are given in Theorems 4.1 and 4.2 in Section 4.1, below. In Section 4.2, we will state and prove Theorem 4.3. This theorem concerns the behavior of a generic $f \in \mathcal{C}(\mathbb{R}^n)$. We will see that several of the properties which hold for a generic $f \in \mathcal{C}(\mathbb{R})$ hold in the more general setting of $\mathcal{C}(\mathbb{R}^n), n \geq 1$. In fact, most of Theorem 4.1 is implied by Theorem 4.3. However, the proof techniques used in the proof of Theorem 4.3 are different than those used in the proof of Theorem 4.1, and so we present the theorems separately in this paper.

4.1 Comeager Subsets of $\mathcal{C}(\mathbb{R})$

Before we state and prove the two main theorems of this section, we provide the necessary background information pertaining to the notation that will be used. The definitions and notation are standard; see [7], for example. $\mathcal{C}(\mathbb{R})$ is a nonlocally compact abelian Polish group with the group operation of pointwise addition, endowed with the compact-open topology, as discussed in Chapter 2. Let $f \in \mathcal{C}(\mathbb{R})$. For any $x \in \mathbb{R}$, we define $f^0(x) = x$, and for all $n \in \mathbb{N}$, $f^{n+1}(x) = f(f^n(x))$. We denote by $orb(f, x)$ the set

$$\{x, f(x), f^2(x), \dots\} = \bigcup_{n=0}^{\infty} f^n(x).$$

The ω -limit set of x under f is denoted by $\omega(f, x)$, and is the set of all subsequential limits of $orb(f, x)$, when $orb(f, x)$ is viewed as a sequence. We say that x is a *periodic point* of f if $f^n(x) = x$ for some $n \in \mathbb{N}$; the point x has period n if $f^n(x) = x$ and $f^m(x) \neq x$ for all $m < n$. A point x is said to be *eventually periodic* if there exists $l \geq 0$ such that $f^l(x)$ is a periodic point. In the case that $l = 0$, then x is both periodic and eventually periodic. Observe that if $|orb(f, x)| < \infty$, then x is either periodic or eventually periodic. Let $P_n(f), P(f)$ denote the set of periodic points of period n under f and the set of periodic points of f , respectively; i.e.,

$$P_n(f) = \{x \in \mathbb{R} : f^n(x) = x \text{ and } f^m(x) \neq x \text{ for all } m < n\},$$

$$P(f) = \{x \in \mathbb{R} : f^n(x) = x \text{ for some } n \in \mathbb{N}\}.$$

Observe that $P(f) = \bigcup_{n=1}^{\infty} P_n(f)$. We denote by $f^{-1}(x)$ the preimage of x under f , and this set is defined as $f^{-1}(x) = \{t \in \mathbb{R} : f(t) = x\}$, as in the previous chapter.

In Theorems 4.1 and 4.2, we give the main results of the section. The first of these theorems is of particular interest in light of the results of the previous chapter; we see that there are several similarities between the properties of a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ and a generic $f \in \mathcal{C}(\mathbb{R})$. Both are surjective, for example. We saw in Chapter 3 that a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ has the property that the preimage of every point is infinite (hence, unbounded), and here we see that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the preimage of every point is unbounded and uncountable. Recall that a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$ has the property that every component of Γ_ϕ contains a cycle, and there are infinitely many cycles of length n for all $n \in \mathbb{N}$. Every cycle of a graph Γ_ϕ corresponds to a periodic point of the function ϕ , so for all n , the set of periodic points of period n is infinite, hence unbounded, for a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$. The same is true of a generic $f \in \mathcal{C}(\mathbb{R})$. Finally, note that for a generic $\phi \in \mathbb{Z}^{\mathbb{Z}}$, given any $n \in \mathbb{Z}$, the forward orbit of n under ϕ eventually ends in a cycle, so $orb(\phi, n)$ is finite. Here, we have that for a generic $f \in \mathcal{C}(\mathbb{R})$, given any $x \in \mathbb{R}$, $orb(f, x)$ is

bounded.

In Theorem 4.2, we further investigate the properties of the orbits and ω -limit sets of a generic f . From Theorem 4.1, we have that $orb(f, x)$ is bounded for every x , but is it also finite? If not, is its associated ω -limit set finite? We will show that for a generic $f \in \mathcal{C}(\mathbb{R})$, $orb(f, x)$ is finite for all x in a c -dense meager subset of \mathbb{R} . (A set $D \subseteq \mathbb{R}$ is c -dense in \mathbb{R} if for any open $U \subseteq \mathbb{R}$, the set $U \cap D$ has the cardinality of the continuum.) Although the set of points with finite orbit, known as the eventually periodic points, is c -dense in \mathbb{R} for a generic f , we will show in Corollary 4.4 and Proposition 4.4 that $P(f)$, the set of periodic points, is uncountable but not dense in \mathbb{R} for a generic f . We will also show that $orb(f, x)$ is infinite and $\omega(f, x)$ is finite for all x in a c -dense meager subset of \mathbb{R} , and that $\omega(f, x)$ is a perfect nowhere dense set for all x in a comeager subset of \mathbb{R} . Finally, we will show that $\omega(f, x)$ is a non-perfect infinite set for all x in an unbounded subset of \mathbb{R} .

THEOREM 4.1 (Properties of a Generic Mapping in $\mathcal{C}(\mathbb{R})$). *A generic $f \in \mathcal{C}(\mathbb{R})$ has the following properties.*

1. f is a surjection.
2. $f^{-1}(x)$ is unbounded and uncountable for all $x \in \mathbb{R}$.
3. $P_n(f)$ is unbounded, dense in itself, and not dense in \mathbb{R} for all $n \in \mathbb{N}$.
4. $orb(f, x)$ is bounded for all $x \in \mathbb{R}$.

THEOREM 4.2 (Classification of Orbital Structures and ω -Limit Sets of a Generic $f \in \mathcal{C}(\mathbb{R})$). *A generic $f \in \mathcal{C}(\mathbb{R})$ has the following properties.*

1. The set of $x \in \mathbb{R}$ such that $\omega(f, x)$ is perfect and nowhere dense is comeager in \mathbb{R} .

2. The set of $x \in \mathbb{R}$ such that $\text{orb}(f, x)$ is finite is c -dense and meager in \mathbb{R} .
3. The set of $x \in \mathbb{R}$ such that $\text{orb}(f, x)$ is infinite and $\omega(f, x)$ is finite is c -dense and meager in \mathbb{R} .
4. The set of $x \in \mathbb{R}$ such that $\omega(f, x)$ is infinite and not perfect is unbounded and meager in \mathbb{R} .

For the remainder of this section, we will state and prove a series of propositions, lemmas, and corollaries which will be used to prove Theorems 4.1 and 4.2. We include some helpful observations before we begin. We will use Proposition 2.2 often in the proofs below. To use this proposition, given an arbitrary $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$, we must construct a function g satisfying $\rho(f, g) < \epsilon$. Observe that if $N \in \mathbb{N}$ is chosen so that $\frac{1}{N} < \epsilon$, and g is a function in $\mathcal{C}(\mathbb{R})$ which satisfies $\|f - g\|_{[-N, N]} < \epsilon$, then we have $\rho(f, g) < \epsilon$, regardless of how g is defined outside the interval $[-N, N]$.

We will say that an interval I is a rational open interval if $I = (p, q)$ for some $p, q \in \mathbb{Q}$. The set of all rational open intervals forms a countable basis for \mathbb{R} . A rational closed interval is defined analogously.

The first lemma will be used to prove Property 1 of Theorem 4.1.

LEMMA 4.1. *Let S be the set of all surjections in $\mathcal{C}(\mathbb{R})$. Then, S is comeager in $\mathcal{C}(\mathbb{R})$.*

Proof. Let $I = (p, q)$ be a rational open interval, and let

$$S_I = \{f \in \mathcal{C}(\mathbb{R}) : I \subseteq f(\mathbb{R})\}.$$

We will show that S_I contains a dense open subset of $\mathcal{C}(\mathbb{R})$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. Let $g(x) = f(x)$ for all $x \in [-N, N]$. Let $g(N + 1) = p - 1$ and $g(N + 2) = q + 1$. Extend g continuously to all of

\mathbb{R} . Now we have $g \in B_\epsilon(f)$. Choose $0 < \eta < \frac{1}{N+2}$. Let $h \in B_\eta(g)$. Then $h(N+1) < p$ and $h(N+2) > q$, and since $h([N+1, N+2])$ is an interval, we have $(p, q) \subseteq h([N+1, N+2])$. Thus $B_\eta(g) \subseteq S_I$. It follows from Baire's Theorem and Proposition 2.1 that the set of surjections $S = \bigcap_I S_I$, the intersection being taken over all rational open intervals, is comeager in $\mathcal{C}(\mathbb{R})$. \square

The next set of lemmas will be used to prove Property 2 of Theorem 4.1. We will prove that $f^{-1}(x)$ is unbounded for all $x \in \mathbb{R}$ for a generic f using a straightforward argument in Lemma 4.2. To prove that $f^{-1}(x)$ is uncountable for all x , we will use a result of Bruckner and Garg.

LEMMA 4.2. *There is a comeager subset U of $\mathcal{C}(\mathbb{R})$ with the property that for all $f \in U$, $f^{-1}(x)$ is unbounded for all $x \in \mathbb{R}$.*

Proof. For each $M, K \in \mathbb{N}$, let

$$U_{M,K} = \{f \in \mathcal{C}(\mathbb{R}) : f^{-1}(x) \not\subseteq [-M, M] \forall x \in [-K, K]\}.$$

We will show that $U_{M,K}$ contains a dense open subset of $\mathcal{C}(\mathbb{R})$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$ be arbitrary. Choose $L \in \mathbb{N}$ so that $L - 1 > M$ and $\frac{1}{L-1} < \epsilon$. Define a function $g \in \mathcal{C}(\mathbb{R})$ as follows. Let $g(x) = f(x)$ for all $x \in (-\infty, L - 1]$. For all $x \in [L, L + 1]$, let $g(x) = -2(K + 1)x + (1 + 2L)(K + 1)$; i.e., on $[L, L + 1]$, g is the line segment connecting the points $(L, K + 1)$ and $(L + 1, -K - 1)$. Finally, extend g so that it is a continuous function defined on \mathbb{R} . Note that $g \in B_\epsilon(f)$. Choose $0 < \eta < \frac{1}{L+1}$. We claim that $B_\eta(g) \subseteq U_{M,K}$. Let $h \in B_\eta(g)$. Since $\|h - g\|_{[-L-1, L+1]} < \frac{1}{2}$, we have $h(L) > K + \frac{1}{2}$ and $h(L + 1) < -K - \frac{1}{2}$. Then, because $h([L, L + 1])$ is an interval, we have $[-K, K] \subseteq h([L, L + 1])$. So, for all $x \in [-K, K]$, there exists $p_x \in [L, L + 1]$ such that $h(p_x) = x$ and $p_x \notin [-M, M]$. Then $B_\eta(g) \subseteq U_{M,K}$, and $U_{M,K}$ contains a dense open subset of $\mathcal{C}(\mathbb{R})$. Now the set $U_K = \bigcap_{M \in \mathbb{N}} U_{M,K}$, the set of all functions f such that $f^{-1}(x)$ is unbounded

for all $x \in [-K, K]$, is comeager in $\mathcal{C}(\mathbb{R})$. Finally, by setting $U = \bigcap_{K \in \mathbb{N}} U_K$, we obtain a comeager subset U of $\mathcal{C}(\mathbb{R})$ with the property that if $f \in U$, then $f^{-1}(x)$ is unbounded for all $x \in \mathbb{R}$. \square

For $f \in \mathcal{C}(\mathbb{R})$ and $c \in \mathbb{R}$, we define the *level of f at c* to be the set $\{x \in \mathbb{R} : f(x) = c\}$. (Observe that $f^{-1}(c)$ is the level of f at c .) Bruckner and Garg in [11] proved that a generic $f \in C([0, 1], \mathbb{R})$ has the property that (i) the top and bottom levels of f are singletons, (ii) there are at most countably many levels of f which are the union of a nonempty perfect set and a singleton, and (iii) all other levels of f are perfect. We will say that, given a function $f \in \mathcal{C}(\mathbb{R})$ and a closed interval I , if the level sets of $f|_I$ have properties (i) – (iii), then f has the *Bruckner-Garg property* on I . We will use the result of Bruckner and Garg, together with Lemma 4.1, to prove that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the preimage of every point is uncountable.

LEMMA 4.3. *There is a comeager subset W of $\mathcal{C}(\mathbb{R})$ with the property that for all $f \in W$, $f^{-1}(x)$ is uncountable for all $x \in \mathbb{R}$.*

Proof. For each $N \in \mathbb{N}$, let W_N be the subset of $\mathcal{C}(\mathbb{R})$ consisting of all f which have the Bruckner-Garg property on $[-N, N]$, and let S be the set of surjections in $\mathcal{C}(\mathbb{R})$. Let $W = S \cap (\bigcap_{N \in \mathbb{N}} W_N)$. Observe that each W_N is comeager in $\mathcal{C}(\mathbb{R})$ by the result of Bruckner and Garg, and S is comeager in $\mathcal{C}(\mathbb{R})$ by Lemma 4.1, so the set W is comeager in $\mathcal{C}(\mathbb{R})$. Let $f \in W$ and $x \in \mathbb{R}$. Since $f \in S$, we may choose N large enough so that the level of f at x is neither the top nor bottom level of $f|_{[-N, N]}$, and $f^{-1}(x) \cap [-N, N] \neq \emptyset$. Then since $f \in W_N$, the level of f at x is uncountable. \square

Next we will show that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the set of periodic points of period n is unbounded for all $n \in \mathbb{N}$.

LEMMA 4.4. *There is a comeager subset V of $\mathcal{C}(\mathbb{R})$ with the property that, for all $f \in V$, the set $P_n(f)$ is unbounded for all $n \in \mathbb{N}$.*

Proof. Fix $M, n \in \mathbb{N}$. Let

$$V_{M,n} = \{f \in \mathcal{C}(\mathbb{R}) : \exists x > M \ni x \in P_n(f)\}.$$

Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$ be arbitrarily chosen. Choose $N \in \mathbb{N}$ so that $\frac{1}{N-1} < \epsilon$ and $N - 1 > M$. Let $I_N, I_{N+1}, \dots, I_{N+n-1}$ be disjoint closed intervals of length $\frac{1}{2}$ centered at the points $N, N + 1, \dots, N + n - 1$, respectively. Let $g(x) = f(x)$ for all $x \in [-(N - 1), N - 1]$, $g(x) = N + 1$ for all $x \in I_N$, $g(x) = N + 2$ for all $x \in I_{N+1}, \dots, g(x) = N + n - 1$ for all $x \in I_{N+n-2}$, and $g(x) = N$ for all $x \in I_{N+n-1}$. Then complete the construction of g so that it is continuous and defined on \mathbb{R} . Now $g \in B_\epsilon(f)$. Choose $0 < \eta < \min\{\frac{1}{4}, \frac{1}{N+n}\}$. Let $h \in B_\eta(g)$. Then for any $x \in I_N$, we have $h^n(x) \in I_N$. Since $h^n : I_N \rightarrow I_N$, there exists a point $y \in I_N$ such that $h^n(y) = y$. Since the intervals I_i are disjoint and $\text{orb}(h, y) \cap I_i \neq \emptyset$ for all i , the point y cannot have period less than n . So $y > M$ is a point of period n for the function h , and $B_\eta(g) \subseteq V_{M,n}$. Let V be the intersection of the sets $V_{M,n}$ over all $M, n \in \mathbb{N}$; V is comeager in $\mathcal{C}(\mathbb{R})$ and thus we obtain the lemma. \square

Next we show that a generic f has the property that the orbit of every point is bounded.

LEMMA 4.5. *There is a comeager subset B of $\mathcal{C}(\mathbb{R})$ with the property that for all $f \in B$, $\text{orb}(f, x)$ is bounded for all $x \in \mathbb{R}$.*

Proof. For each $K \in \mathbb{N}$, let

$$B_K = \{f \in \mathcal{C}(\mathbb{R}) : \text{orb}(f, x) \text{ is bounded } \forall x \in [-K, K]\}.$$

We will show that B_K contains a dense open subset of $\mathcal{C}(\mathbb{R})$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. Choose $M \in \mathbb{N}$ such that $M > K$ and $\frac{1}{M} < \epsilon$. Let g be defined as

$$g(x) = \begin{cases} f(M), & x > M, \\ f(x), & x \in [-M, M], \\ f(-M), & x < -M. \end{cases}$$

Observe that $g \in B_\epsilon(f)$. Now choose $L \in \mathbb{N}$ large enough so that $L > M$ and $g(\mathbb{R}) \subseteq [-(L-1), L-1]$. Choose $0 < \eta < \min\{\frac{1}{4}, \frac{1}{L}\}$. Let $h \in B_\eta(g)$ and $x \in [-K, K]$. Since $|h(x) - g(x)| < \eta$ for all $x \in [-L, L]$ and $K < L$, we have that $h(x) \in [-(L - \frac{3}{4}), L - \frac{3}{4}]$. Suppose $\text{orb}(h, x)$ is unbounded, and let n be the smallest number such that $h^n(x) \notin [-L, L]$. Then $h^{n-1}(x) \in [-L, L]$ implies $|h(h^{n-1}(x)) - g(h^{n-1}(x))| < \eta$ implies $h^n(x) \in [-(L - \frac{3}{4}), L - \frac{3}{4}]$, a contradiction. Thus $B_\eta(g) \subseteq B_K$, and B_K contains a dense open subset of $\mathcal{C}(\mathbb{R})$. It follows that $B = \bigcap_{K \in \mathbb{N}} B_K$ is a comeager subset of $\mathcal{C}(\mathbb{R})$. \square

Before we proceed, we give an example of a dense subset D of $\mathcal{C}(\mathbb{R})$ such that for all $f \in D$, there exists x such that $\text{orb}(f, x)$ is unbounded. This example demonstrates that although the set B of Lemma 4.5 is comeager in $\mathcal{C}(\mathbb{R})$, it has empty interior. In a sense, this proves that the result of Lemma 4.5 is the strongest possible result concerning the size of B , topologically speaking.

EXAMPLE 1. *There exists a dense set $D \subseteq \mathcal{C}(\mathbb{R})$ such that $\forall f \in D$, there exists $x \in \mathbb{R}$ such that $\text{orb}(f, x)$ is unbounded.*

Proof. Let $D = \{f_k\}_{k \in \mathbb{N}}$ be a countable dense subset of $\mathcal{C}(\mathbb{R})$ and let $(\epsilon_l)_{l \in \mathbb{N}}$ be a decreasing sequence of positive numbers which converge to 0. For each pair f_k, ϵ_l , let $g_{k,l}$ be any function in $\mathcal{C}(\mathbb{R})$ satisfying $g_{k,l} \in B_{\epsilon_l}(f_k)$ and $g_{k,l}(x) = x + 1$ for all x outside an appropriately large interval. Then $D = \bigcup_{k,l \in \mathbb{N}} g_{k,l}$ is a dense subset of $\mathcal{C}(\mathbb{R})$ with the desired property. \square

The proof of Theorem 4.1 is almost complete; it remains to prove that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $P_n(f)$ is dense in itself and not dense in \mathbb{R}

for all n . We will need to prove some other results first.

The next proposition will be valuable in that it will allow us to simplify the argument in many of the proofs that follow. This proposition says that, for any $f \in \mathcal{C}(\mathbb{R})$, $\epsilon > 0$, and $a \in \mathbb{R}$, there exists $f^* \in B_\epsilon(f)$ such that $\text{orb}(f^*, a)$ is finite.

PROPOSITION 4.1. *For every $a \in \mathbb{R}$, there exists a dense set $F_a \subseteq \mathcal{C}(\mathbb{R})$ such that $|\text{orb}(f, a)| < \infty$ for all $f \in F_a$.*

Proof. Fix $a \in \mathbb{R}$ and let $F = \{f \in \mathcal{C}(\mathbb{R}) : |\text{orb}(f, a)| < \infty\}$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. We will show that $F \cap B_\epsilon(f) \neq \emptyset$. If $f \in F$, we are done. Assume that $f \notin F$.

CASE 1. $\text{orb}(f, a)$ is bounded.

Choose $N \in \mathbb{N}$ so that the sequence $(f^k(a))_{k=0}^\infty \subseteq [-N, N]$. By the Bolzano-Weierstrass Property, there exists a convergent subsequence; call the limit of this subsequence p . Let

$$k_1 = \min\{j : f^j(a) \in B_\epsilon(p)\}, \text{ and}$$

$$k_2 = \min\{j : f^j(a) \in B_\epsilon(p) \text{ and } j > k_1\}.$$

Choose $\delta > 0$ so that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2} \text{ for all } x, y \in [-N, N], \text{ and}$$

$$a, f(a), \dots, f^{k_2-2}(a) \notin B_\delta(f^{k_2-1}(a)).$$

Define a function g as follows. Let $g(x) = f(x)$ for all $x \notin B_\delta(f^{k_2-1}(a))$. Let $g(f^{k_2-1}(a)) = f^{k_1}(a)$. Define g on the remainder of the interval $(f^{k_2-1}(a) - \frac{\delta}{2}, f^{k_2-1}(a) + \frac{\delta}{2})$ so that g is continuous and $\rho(f, g) < \epsilon$. Then $g \in F \cap B_\epsilon(f)$.

CASE 2. $\text{orb}(f, a)$ is unbounded.

By Lemma 4.5, the set of functions in $\mathcal{C}(\mathbb{R})$ which have bounded orbit at every point is dense in $\mathcal{C}(\mathbb{R})$. Thus, we may choose $f^* \in B_{\frac{\epsilon}{3}}(f)$ such that $\text{orb}(f^*, a)$ is

bounded. By Case 1, we may construct $g \in B_{\frac{\epsilon}{3}}(f^*)$ so that $\text{orb}(g, a)$ is finite. Then $g \in F \cap B_{\epsilon}(f)$. \square

The next proposition will be used in several of the proofs below; the proof is not included here.

PROPOSITION 4.2. *Let $f \in \mathcal{C}(\mathbb{R})$ and $n, N \in \mathbb{N}$. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that if $g \in \mathcal{C}(\mathbb{R})$ and $\rho(f, g) < \delta$, then $\|f^k - g^k\|_{[-N, N]} < \epsilon$ for each $k = 1, 2, \dots, n$.*

We now introduce some notation that will be used for the remainder of the chapter. Let $f \in \mathcal{C}(\mathbb{R})$ and $a \in \mathbb{R}$ be such that $\text{orb}(f, a)$ is finite. Then we will write the orbit of a under f as

$$\text{orb}(f, a) = \{a_0, a_1, \dots, a_k, \dots, a_{n-1}\},$$

where $a = a_0$, $f(a_i) = a_{i+1}$ for $0 \leq i \leq n-1$, and $a_n = a_k$.

The techniques used in the proof of the next proposition, Proposition 4.3, are inspired by the techniques used in [2] to prove that a generic $f \in C([0, 1])$ has the property that the set of points with finite orbit under f is dense in I . In the current setting, we prove a stronger statement. We obtain as corollaries that a generic $f \in \mathcal{C}(\mathbb{R})$ has the properties that (i) the set of points with finite orbit is c -dense in \mathbb{R} , (ii) the set $\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\}$ is perfect $\forall n$, (iii) the set $P_n(f)$ is dense in itself $\forall n$, and (iv) $\{x \in \mathbb{R} : f^n(x) = x\}$ is perfect $\forall n$. (It was proven in [2] that the set $\{x \in [0, 1] : f^n(x) = x\}$ is perfect for a generic $f \in C([0, 1])$, and in [40] that $P_n(f)$ is dense in itself for a generic $f \in C([0, 1])$; however, the proof techniques differ from the ones here.) Proposition 4.3 will also be useful as a tool to simplify some of the proofs that follow. Roughly speaking, Proposition 4.3 says that, given any function f with finite orbit at a point a , then arbitrarily close to f we can find an open ball in $\mathcal{C}(\mathbb{R})$ such that for any h in the open ball, there exist

at least two distinct points with orbital structure identical to that of a under f , and these points may be chosen as close to a as we like.

PROPOSITION 4.3. *Let $f \in \mathcal{C}(\mathbb{R})$ be a function with the property that for some $a \in \mathbb{R}$,*

$$\text{orb}(f, a) = \{a_0, a_1, \dots, a_k, \dots, a_{n-1}\}.$$

Let $\epsilon, \delta > 0$. Then, there exist $g \in B_\epsilon(f)$ and $\eta > 0$ such that for all $h \in B_\eta(g)$, there exist distinct points $a^h, b^h \in (a, a + \delta)$ [or, $(a - \delta, a)$] such that

$$\text{orb}(h, a^h) = \{a_0^h, a_1^h, \dots, a_k^h, \dots, a_{n-1}^h\}$$

and

$$\text{orb}(h, b^h) = \{b_0^h, b_1^h, \dots, b_k^h, \dots, b_{n-1}^h\}.$$

Proof. Let $f \in \mathcal{C}(\mathbb{R})$ and $a \in \mathbb{R}$ be such that $\text{orb}(f, a)$ is finite, and let $\epsilon, \delta > 0$. First assume that f is constant on no interval. Let I be a closed interval containing a such that

- $I \subseteq (a - \delta, a + \delta)$,
- $|f^k(I)| < \frac{\epsilon}{2}$,
- $I, f(I), \dots, f^{n-1}(I)$ are pairwise disjoint intervals,
- $|f(x) - f(y)| < \frac{\epsilon}{2}$ for all $x, y \in f^{n-1}(I)$.

Let J, K be disjoint subintervals of $(a, a + \delta)$. Let $g(x) = f(x)$ for all $x \notin f^{n-1}(I)$. On the interval $f^{n-1}(I)$, define g to be a slight perturbation of f with the property that

$$g^k(J) \subseteq \text{Int}(g^n(J)) \text{ and } g^k(K) \subseteq \text{Int}(g^n(K)).$$

By Lemma 4.2, we may choose $\eta > 0$ so that, for all $h \in B_\eta(g)$, the sets $h^i(I)$ where $i = 0, \dots, n - 1$, are pairwise disjoint intervals, $h^k(J) \subseteq \text{Int}(h^n(J))$ and

$h^k(K) \subseteq \text{Int}(h^n(K))$. Choose such an η and let $h \in B_\eta(g)$. Since $h^{n-k} : h^k(J) \rightarrow h^k(J)$, there is a point $x \in h^k(J)$ which is fixed under the mapping h^{n-k} . Since $h^k(J), \dots, h^{n-1}(J)$ are pairwise disjoint, this point x must be periodic under h , with period $n - k$. Moreover, since $x \in h^k(J)$, there exists $a^h \in J$ such that $h^k(a^h) = x$. Now

$$\text{orb}(h, a^h) = \{a_0^h, a_1^h, \dots, a_k^h, \dots, a_{n-1}^h\},$$

where $a_0^h = a^h$, $a_k^h = x$, $h(a_i^h) = a_{i+1}^h$ for $0 \leq i \leq n - 1$, and $a_n^h = a_k^h$. By the same argument, there is a point $\tilde{x} \in h^k(K)$ and $b^h \in K$ such that $h^k(b^h) = \tilde{x}$ and

$$\text{orb}(h, b^h) = \{b_0^h, b_1^h, \dots, b_k^h, \dots, b_{n-1}^h\}.$$

Now $J \cap K = \emptyset$, so $a^h \neq b^h$, and since $J, K \subseteq (a, a + \delta)$, we have $a^h, b^h \in (a, a + \delta)$. Observe that by requiring that J, K be disjoint subintervals of $(a - \delta, a)$ rather than $(a, a + \delta)$, we can produce distinct points $a^h, b^h \in (a - \delta, a)$ with the desired properties.

Now suppose that f is constant on some interval. Then let f^* be a function that is constant on no interval, $\rho(f^*, f) < \frac{\epsilon}{3}$, and $\text{orb}(f, a) = \text{orb}(f^*, a)$. Proceed as above to construct an appropriate g and η so that $g \in B_{\frac{\epsilon}{3}}(f^*)$. Then we have $g \in B_\epsilon(f)$ and $\eta > 0$ satisfying the proposition. \square

COROLLARY 4.1. *Generic $f \in \mathcal{C}(\mathbb{R})$ has the property that*

$$\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\}$$

is perfect for all $n \in \mathbb{N}$.

Proof. Fix $n \in \mathbb{N}$. Fix a closed rational interval I and let $G_{I,n}$ be the set of all $f \in \mathcal{C}(\mathbb{R})$ such that

$$|\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\} \cap I| \neq 1.$$

We will show that $G_{I,n}$ contains a dense open subset of $\mathcal{C}(\mathbb{R})$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. There are three cases to consider.

$$\text{CASE 1. } |\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\} \cap I| = 0$$

Let $g \equiv f$. Choose $N \in \mathbb{N}$ so that $I, g(I), \dots, g^n(I) \subseteq (-N, N)$. Let $\alpha > 0$ be such that $B_\alpha(g^i(I)) \subseteq (-N, N)$ for $0 \leq i \leq n$, and $\alpha < \min_{0 \leq r < s \leq n} \{|g^r(x) - g^s(x)| : x \in I\}$. By Lemma 4.2, we may choose $\eta > 0$ so that $\rho(h, g) < \eta$ implies that $\|h^k - g^k\|_{[-N, N]} < \frac{\alpha}{2}$ for $1 \leq k \leq n$. Choose such an η ; then for any $h \in B_\eta(g)$, the points $x, h(x), \dots, h^n(x)$ are distinct for all $x \in I$. Thus $g \in B_\epsilon(f)$ and $B_\eta(g) \subseteq G_{I,n}$.

$$\text{CASE 2. } |\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\} \cap \text{Int}(I)| \geq 1$$

Let $a \in I$ be such that $|\text{orb}(f, a)| \leq n$. Let $\delta > 0$ be chosen so that $(a - \delta, a + \delta) \subseteq I$. By Proposition 4.3, we may construct $g \in B_\epsilon(f)$ and $\eta > 0$ so that for all $h \in B_\eta(g)$, there exist distinct points $a^h, b^h \in (a - \delta, a + \delta)$ such that $|\text{orb}(h, a^h)| = |\text{orb}(h, b^h)| = |\text{orb}(f, a)| \leq n$. Thus for all $h \in B_\eta(g)$, $|\{x \in \mathbb{R} : |\text{orb}(h, x)| \leq n\} \cap I| \geq 2$, and so $B_\eta(g) \subseteq G_{I,n}$.

$$\text{CASE 3. } |\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\} \cap \partial I| \geq 1$$

By Proposition 4.3, we may choose $f^* \in B_{\frac{\epsilon}{3}}(f)$ so that $|\text{orb}(f^*, a^*)| \leq n$ for some $a^* \in \text{Int}(I)$. By Case 2, construct $g \in B_{\frac{\epsilon}{3}}(f^*)$ and $\eta > 0$ so that $B_\eta(g) \subseteq G_{I,n}$. Then $g \in B_\epsilon(f)$.

We have proven that $G_{I,n}$ contains an open dense subset of $\mathcal{C}(\mathbb{R})$. Let $G_n = \bigcap_I G_{I,n}$, where the intersection is taken over all closed rational intervals. The set G_n is comeager in $\mathcal{C}(\mathbb{R})$, and if $f \in G_n$, then the set $\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\}$ has no isolated points. Moreover, by the continuity of f , the set is closed in \mathbb{R} ; thus it is perfect. Finally, let $G = \bigcap_{n \in \mathbb{N}} G_n$. G is comeager in $\mathcal{C}(\mathbb{R})$ and has the desired properties. \square

From the following corollary, Corollary 4.2, we obtain that Property (2) of

Theorem 4.2 holds for a generic $f \in \mathcal{C}(\mathbb{R})$.

COROLLARY 4.2. *Generic $f \in \mathcal{C}(\mathbb{R})$ has the property that*

$$\{x \in \mathbb{R} : |\text{orb}(f, x)| < \infty\}$$

is c -dense in \mathbb{R} .

Proof. First we will show that there is a comeager subset A of $\mathcal{C}(\mathbb{R})$ with the property that, for all $f \in A$, the set of points with finite orbit under f is dense in \mathbb{R} . Fix a rational open interval I and let

$$A_I = \{f \in \mathcal{C}(\mathbb{R}) : |\text{orb}(f, x)| < \infty \text{ for some } x \in I\}.$$

Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. Fix $a \in I$. By Proposition 4.1, there exists $f^* \in B_{\frac{\epsilon}{3}}(f)$ with the property that $|\text{orb}(f^*, a)| < \infty$. Choose $\delta > 0$ so that $(a - \delta, a + \delta) \subseteq I$. By Proposition 4.3, there exist $g \in B_{\frac{\epsilon}{3}}(f^*)$ and $\eta > 0$ such that, for all $h \in B_\eta(g)$, there exists $a^h \in (a - \delta, a + \delta)$ satisfying $|\text{orb}(h, a^h)| = |\text{orb}(f^*, a)| < \infty$. Thus $g \in B_\epsilon(f)$ and $B_\eta(g) \subseteq A_I$. Now let $A = \bigcap_I A_I$, where the intersection is taken over all rational open intervals. The set A is comeager in $\mathcal{C}(\mathbb{R})$ and has the desired property.

Observe that the set $A \cap G$, where G is defined as in Corollary 4.1, is comeager in $\mathcal{C}(\mathbb{R})$. Let $f \in A \cap G$. Let I be an arbitrary open interval in \mathbb{R} . Then, since $f \in A$, there exists $a \in I$ such that $|\text{orb}(f, a)| = n$ for some n . Since $f \in G$, there are uncountably many $x \in I$ such that $|\text{orb}(f, x)| \leq n$. It follows that the set of points with finite orbit is c -dense in \mathbb{R} for all $f \in A \cap G$. \square

The proof of Corollary 4.3 below is brief, as the techniques used are very similar to those used in the proof of Corollary 4.1.

COROLLARY 4.3. *Generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $P_n(f)$ is dense in itself for all $n \in \mathbb{N}$.*

Proof. Fix $n \in \mathbb{N}$ and a closed rational interval I . Let $G_{I,n}$ be the set of all $f \in \mathcal{C}(\mathbb{R})$ such that $|P_n(f) \cap I| \neq 1$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$ be arbitrarily chosen.

CASE 1. $|P_n(f) \cap I| = 0$

Proceed exactly as in Case 1 in the proof of Corollary 4.1 with the choice of g, N, α , and η , the only modification being that “ $\alpha < \min_{0 \leq r < s \leq n} \{|g^r(x) - g^s(x)| : x \in I\}$ ” should be replaced with “ $\alpha < \min\{|x - g^n(x)| : x \in I\}$.” Then for $h \in B_\eta(g)$ and $x \in I$, if $h^n(x) = x$, we have $|x - g^n(x)| \leq |x - h^n(x)| + |h^n(x) - g^n(x)| < \frac{\alpha}{2}$, contradicting the choice of α . Thus $g \in B_\epsilon(f)$ and $B_\eta(g) \subseteq G_{I,n}$.

CASE 2. $|P_n(f) \cap I| \geq 1$

Let $a \in P_n(f) \cap I$. Without loss of generality, we may assume by Proposition 4.3 that $a \in \text{Int}(I)$ (see Case 3 in the proof of Corollary 4.1). Choose $\delta > 0$ so that $(a - \delta, a + \delta) \subseteq I$, and use Proposition 4.3 to construct $g \in B_\epsilon(f)$ and $\eta > 0$ as in Case 2 in the proof of Corollary 4.1. Then for any $h \in B_\eta(g)$, we have $|P_n(h) \cap I| \geq 2$.

Now as in the proof of Corollary 4.1, the set $G = \bigcap_{n \in \mathbb{N}} \bigcap_I G_{I,n}$ is comeager in $\mathcal{C}(\mathbb{R})$ and has the desired properties.

□

COROLLARY 4.4. *Generic $f \in \mathcal{C}(\mathbb{R})$ has the property that*

$$\{x \in \mathbb{R} : f^n(x) = x\}$$

is perfect for all $n \in \mathbb{N}$.

Proof. Note that if for some f and n , the set $\{x \in \mathbb{R} : f^n(x) = x\}$ has an isolated point, then $P_k(f)$ has an isolated point for some $k \leq n$. Thus it follows from Corollary 4.3 that for a generic f , $\{x \in \mathbb{R} : f^n(x) = x\}$ has no isolated points for any n . Moreover, this set is closed in \mathbb{R} by the continuity of f^n , so it is perfect. □

In order to prove that a generic f has the property that the set of points with finite orbit,

$$\{x \in \mathbb{R} : |\text{orb}(f, x)| < \infty\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\},$$

is c -dense in \mathbb{R} , we used the facts that the set on the left hand side of the equation is dense in \mathbb{R} , and each set in the union on the right hand side of the equation has no isolated points. We cannot obtain a similar result for the set of periodic points, given by

$$P(f) = \bigcup_{n=1}^{\infty} P_n(f),$$

because although each set in the union on the right hand side of the equation has no isolated points, the set on the left hand side is not dense in \mathbb{R} . We prove in the proposition below that a generic function f has the property that $P(f)$ is not dense in \mathbb{R} . (Observe that we actually prove a stronger result: that the set of all f with the property that $P(f)$ is not dense in \mathbb{R} contains an open dense subset of $\mathcal{C}(\mathbb{R})$.)

PROPOSITION 4.4. *A generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $\overline{P(f)} \neq \mathbb{R}$.*

Proof. Let $U = \{f \in \mathcal{C}(\mathbb{R}) : \overline{P(f)} \neq \mathbb{R}\}$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$ be arbitrarily chosen. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$. Choose $M \in \mathbb{N}$ so that $M > N$ and $f([-N, N]) \subseteq (-M, M)$. Define a function g as follows. Let

$$g(x) = \begin{cases} f(-N), & x \in (-\infty, -N), \\ f(x), & x \in [-N, N], \\ f(N), & x \in (N, M), \\ -f(N)x + f(N)(M+1), & x \in [M, M+1], \\ 0, & x \in (M+1, \infty). \end{cases}$$

Clearly $g \in B_\epsilon(f)$. Choose $0 < \eta < \frac{1}{M+2}$ so that

$$B_\eta(f([-N, N])) = B_\eta(g([-M, M])) \subseteq (-M, M).$$

The claim is that $B_\eta(g) \subseteq U$. Observe that if $h \in B_\eta(g)$ and $y \in [-M, M]$, then $h(y) \in B_\eta(g([-M, M])) \subseteq (-M, M)$. Let $I = [M + 1, M + 2]$. Let $h \in B_\eta(g)$ and $x \in I$. Then $h(x) \in (-M, M)$. It follows that $h^k(x) \in (-M, M)$ for all k , and so $x \notin P(h)$. Thus $P(h) \cap I = \emptyset$, and $h \in U$. The result follows. \square

Observe that it follows immediately from Proposition 4.4 that a generic f has the property that $P_n(f)$ is not dense in \mathbb{R} for any n . This is so because $P_n(f) \subseteq P(f)$. We are now ready to prove Theorem 4.1.

Proof. (Proof of Theorem 4.1) Let f be an element of the intersection of the sets comeager subsets of $\mathcal{C}(\mathbb{R})$ defined in Lemmas 4.1, 4.2, 4.3, 4.4, and 4.5, Corollary 4.3, and Proposition 4.4. Then the intersection is comeager in $\mathcal{C}(\mathbb{R})$ as well, and clearly any element f of the intersection of these sets has properties (1) – (4) of the theorem. \square

We now turn our attention to the proof of Theorem 4.2. The proof of Theorem 4.2 is longer and more technical than the proof of Theorem 4.1. The main results which will be needed to prove Theorem 4.2 are Lemmas 4.6, 4.7, 4.8, and 4.9 and Corollary 4.2.

In Lemmas 4.6 and 4.7, we will prove that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $\omega(f, x)$ is nowhere dense and perfect for a generic $x \in \mathbb{R}$. In the space $C([0, 1])$, Agronsky, Bruckner, Ceder, and Pearson proved in [1] that a closed subset C of $[0, 1]$ is an ω -limit set of some function $f \in C([0, 1])$ if and only if C is either nowhere dense, or C is the union of finitely many nondegenerate closed intervals. Agronsky, Bruckner, and Lasczkovich proved in [2] that a generic $f \in C([0, 1])$ has the property that $\omega(f, x)$ is a nowhere dense perfect set for all x in a comeager subset of $[0, 1]$. Lehning used the Tietze Extension Theorem and the Kuratowski-Ulam Theorem (see [26] and [37], respectively) to offer a simpler proof of the latter result in a more general setting [30]. He proved that a generic $f \in C(X)$, where

X is a compact N -dimensional manifold, has the property that $\omega(f, x)$ is nowhere dense and perfect for a generic $x \in X$. We must be cautious in assuming that results such as those of Agronsky, Bruckner, Lasczkovich, and Lehning hold for the space $\mathcal{C}(\mathbb{R})$, as we are not working with self-maps of a compact space in this setting. There are several assumptions that one has for a function $f \in C([0, 1])$ that are certainly not true of $f \in \mathcal{C}(\mathbb{R})$. For example, every ω -limit set for a function $f \in C([0, 1])$ is nonempty; however, we can construct a function $f \in \mathcal{C}(\mathbb{R})$ such that no ω -limit set is nonempty. As another example, every $f \in C([0, 1])$ has a fixed point in $[0, 1]$, but this is not true of every $f \in \mathcal{C}(\mathbb{R})$.

Nevertheless, we have found that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $\omega(f, x)$ is nowhere dense and perfect for a generic $x \in \mathbb{R}$. We state and prove these facts in the lemmas below. We will use techniques similar to those of Lehning, although our proofs are made simpler through the use of Propositions 4.1 and 4.3. The proofs of Lemmas 4.6 and 4.7 require the Kuratowski-Ulam Theorem (see [37]).

THEOREM (Kuratowski-Ulam). *Let X, Y be topological spaces such that Y has a countable basis. If $E \subseteq X \times Y$ is comeager in $X \times Y$, then the set*

$$E_x = \{y \in Y : (x, y) \in E\}$$

is comeager in Y for a generic $x \in X$.

In Lemmas 4.6 and 4.7, let $d : \mathcal{C}(\mathbb{R}) \times \mathbb{R} \rightarrow [0, \infty)$ be defined by

$$d((f, a), (g, b)) = \max\{\rho(f, g), |a - b|\}.$$

Then $\mathcal{C}(\mathbb{R}) \times \mathbb{R}$ is a complete metric space with the metric d .

LEMMA 4.6. *Generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $\omega(f, x)$ is nowhere dense for a generic $x \in \mathbb{R}$.*

Proof. For all $k \in \mathbb{N}$, let E_k be the subset of $C(\mathbb{R}) \times \mathbb{R}$ consisting of all (f, x) such that $\omega(f, x)$ is contained in finitely many disjoint intervals, each of length less than $\frac{1}{k}$. Using Proposition 2.2, we will show that E_k contains a dense open subset of $C(\mathbb{R}) \times \mathbb{R}$. Let $(f, a) \in C(\mathbb{R}) \times \mathbb{R}$ and $\epsilon > 0$. By Proposition 4.1, we may assume without loss of generality that $\text{orb}(f, a)$ is finite. Choose $N \in \mathbb{N}$ so that $\text{orb}(f, a) \subseteq (-N, N)$ and $\frac{1}{N} < \epsilon$. Choose $\delta > 0$ so that for all $x, y \in [-N, N]$, $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Choose pairwise disjoint closed intervals I_0, I_1, \dots, I_{n-1} centered at a_0, a_1, \dots, a_{n-1} , respectively, so that $|I_i| = |I_j| < \min\{\frac{1}{k}, \frac{\delta}{2}\}$ for all i, j , and $\bigcup_{i=0}^{n-1} I_i \subseteq (-N, N)$. Let g be a slight perturbation of f such that, for $i = 0, 1, \dots, n-2$, $g(x) = a_{i+1}$ for all $x \in I_i$, $g(x) = a_k$ for all $x \in I_{n-1}$, and $g \in B_\epsilon(f)$. Note that $(g, a) \in B_\epsilon((f, a))$. Choose $0 < \eta < \min\{\frac{|I_0|}{2}, \frac{1}{N}\}$. Let $(h, c) \in B_\eta((g, a))$. Then it is readily verified that $\text{orb}(h, c) \subseteq \bigcup_{i=0}^{n-1} I_i$, and since the intervals I_i are closed, we have $\omega(h, c) \subseteq \bigcup_{i=0}^{n-1} I_i$.

Let $E = \bigcap_{k \in \mathbb{N}} E_k$, a set which is comeager in $C(\mathbb{R}) \times \mathbb{R}$. For each $(f, x) \in E$, $\omega(f, x)$ is contained in finitely many disjoint intervals of arbitrarily small length, so $\omega(f, x)$ is a closed set with empty interior; i.e., $\omega(f, x)$ is nowhere dense. It follows from the Kuratowski-Ulam Theorem that a generic $f \in C(\mathbb{R})$ has the property that $\omega(f, x)$ is nowhere dense for a generic $x \in \mathbb{R}$. \square

LEMMA 4.7. *Generic $f \in C(\mathbb{R})$ has the property that $\omega(f, x)$ is perfect for a generic $x \in \mathbb{R}$.*

Proof. Fix a closed interval $J = [p, q] \subseteq \mathbb{R}$, and let $P_J \subseteq C(\mathbb{R}) \times \mathbb{R}$ be the set of all (f, x) such that $|\omega(f, x) \cap J| \neq 1$. We will show that P_J contains a dense open subset of $C(\mathbb{R}) \times \mathbb{R}$. Let $(f, a) \in C(\mathbb{R}) \times \mathbb{R}$ and $\epsilon > 0$. Without loss of generality, we may assume by Proposition 4.1 that $\text{orb}(f, a)$ is finite. Observe that, with the notation given above,

$$\text{orb}(f, a) = \{a_0, \dots, a_k, \dots, a_{n-1}\},$$

we have $\omega(f, a) = \{a_k, \dots, a_{n-1}\}$. Choose N and δ as in the proof of Lemma 4.6. There are three cases to consider.

CASE 1. $|\omega(f, a) \cap J| = 0$.

Let I_0, I_1, \dots, I_{n-1} be closed pairwise disjoint intervals centered at a_0, a_1, \dots, a_{n-1} , respectively, so that $\bigcup_{i=0}^{n-1} I_i \subseteq (-N, N)$, $(\bigcup_{i=k}^{n-1} I_i) \cap J = \emptyset$, and $|I_i| = |I_j| < \frac{\delta}{2}$ for all i, j . Construct g so that $g(x) = a_{i+1}$ for all $x \in I_i$, where $i = 0, \dots, n-2$, $g(x) = a_k$ for all $x \in I_{n-1}$, and g satisfies $g \in B_\epsilon(f)$. Now we have $(g, a) \in B_\epsilon((f, a))$. Choose $0 < \eta < \min\left\{\frac{1}{N}, \frac{|I_0|}{2}\right\}$. Let $(h, c) \in B_\eta((g, a))$. Then $\omega(h, c) \subseteq \bigcup_{i=k}^{n-1} I_i$, and so $|\omega(h, c) \cap J| = 0$. It follows that $B_\eta((g, a)) \subseteq P_J$.

CASE 2. $|\omega(f, a) \cap (p, q)| \geq 1$.

Fix l so that $a_l \in (p, q)$. Choose I_0, I_1, \dots, I_{n-1} to be closed pairwise disjoint intervals centered at a_0, a_1, \dots, a_{n-1} , respectively, so that $\bigcup_{i=0}^{n-1} I_i \subseteq (-N, N)$, $I_l \subseteq (p, q)$, and $|I_i| = |I_j| < \min\left\{\frac{\epsilon}{2}, \frac{\delta}{2}\right\}$ for all i, j . We will construct $(g, b) \in B_\epsilon(f)$ and $\eta > 0$ so that, for all $(h, c) \in B_\eta((g, b))$, we have $|\omega(h, c) \cap (p, q)| \geq 2$. To do so, we need to define subintervals of the I_i . For each i , let $I_{i,1}$ be the lower third of the interval I_i , and let $I_{i,2}$ be the upper third of I_i . Let $b_{i,1}$ and $b_{i,2}$ be the midpoints of $I_{i,1}$ and $I_{i,2}$, respectively. Observe that $I_{i,1} \cap I_{i,2} = \emptyset$ and $|I_{i,1}| = |I_{i,2}| = \frac{|I_i|}{3}$ for all i . Construct g as follows. For $i = 0, \dots, k-1$, let $g(x) = b_{i+1,1}$ for all $x \in I_{i,1}$. For $i = k, k+1, \dots, n-2$, let $g(x) = b_{i+1,1}$ for all $x \in I_{i,1}$, and $g(x) = b_{i+1,2}$ for all $x \in I_{i,2}$. Let $g(x) = b_{k,2}$ for all $x \in I_{n-1,1}$, and let $g(x) = b_{k,1}$ for all $x \in I_{n-1,2}$. Complete the construction of g so that g is continuous on \mathbb{R} and $g \in B_\epsilon(f)$. Let $b = b_{0,1}$. Observe that $(g, b) \in B_\epsilon((f, a))$. Choose $0 < \eta < \min\left\{\frac{1}{N}, \frac{|I_{0,1}|}{2}\right\}$. Let $(h, c) \in B_\eta((g, b))$. Then $\text{orb}(h, c)$ intersects both $I_{l,1}$ and $I_{l,2}$ infinitely many times, where $I_{l,1}$ and $I_{l,2}$ are disjoint and contained in (p, q) , so $|\omega(h, c) \cap (p, q)| \geq 2$.

CASE 3. $|\omega(f, a) \cap (p, q)| = 0$ and $|\omega(f, a) \cap \{p, q\}| \geq 1$.

Without loss of generality, assume that $a_l = p$ for some $l \in \{k, \dots, n-1\}$. By

Proposition 4.3, we may choose $f^* \in B_{\frac{\epsilon}{3}}(f)$ such that $\omega(f^*, a^*) \cap (p, q) \neq \emptyset$ for some $a^* \in B_{\frac{\epsilon}{3}}(a)$. Note that $(f^*, a^*) \in B_{\frac{\epsilon}{3}}((f, a))$. By Case 2, we may construct $(g, b) \in B_{\frac{\epsilon}{3}}((f^*, a^*))$ and $\eta > 0$ so that $B_\eta((g, b)) \subseteq P_J$. It follows that $(g, b) \in B_\epsilon((f, a))$ and $B_\eta((g, b)) \subseteq P_J$.

Now P_J contains an open dense subset of $C(\mathbb{R}) \times \mathbb{R}$. Let $P = \bigcap_J P_J$, where the intersection is taken over all closed rational intervals. The set P is comeager in $C(\mathbb{R}) \times \mathbb{R}$, and for any $(f, x) \in P$, $\omega(f, x)$ is perfect. Thus by the Kuratowski-Ulam Theorem, a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $\omega(f, x)$ is perfect for a generic $x \in \mathbb{R}$.

□

Before we proceed, we remark that by Lemmas 4.6 and 4.7, a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $\omega(f, x)$ is nowhere dense and perfect for a generic $x \in \mathbb{R}$. This is true by the following argument. Suppose $f \in \mathcal{C}(\mathbb{R})$ has the property that there exist comeager subsets G_{nwd} and G_p of \mathbb{R} such that $\omega(f, x)$ is nowhere dense for all $x \in G_{nwd}$ and $\omega(f, x)$ is perfect for all $x \in G_p$. Then $G = G_{nwd} \cap G_p$ is comeager in \mathbb{R} , and $\omega(f, x)$ is nowhere dense and perfect for all $x \in G$.

Note that with Lemmas 4.6 and 4.7 and Corollary 4.2, we have shown that Properties (1) and (2) of Theorem 4.2 are true of a generic $f \in \mathcal{C}(\mathbb{R})$. Our next objective is to prove Property (3) of Theorem 4.2; i.e., a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the set of points with infinite orbit and finite ω -limit set is c -dense in \mathbb{R} . We will prove this result in Lemma 4.8. Before we do so, we will need several definitions and propositions.

DEFINITION. Let $f \in \mathcal{C}(\mathbb{R})$. The function f is *nondecreasing* at a point $x \in \mathbb{R}$ if there exists $\delta > 0$ such that $\frac{f(x) - f(t)}{x - t} \geq 0$ for all $t \in (x - \delta, x + \delta) \setminus \{x\}$. The function f is *nonincreasing* at x if $-f$ is nondecreasing at x . We say that f

is *monotone* at x if f is either nonincreasing or nondecreasing at x .

DEFINITION. Let $f \in \mathcal{C}(\mathbb{R})$. A point $x \in \mathbb{R}$ is said to be a point of (relative) *maximum* of f if there exists $\delta > 0$ such that $f(t) \leq f(x)$ for all $t \in (x - \delta, x + \delta)$, and x is a point of *proper maximum* of f if there exists $\delta > 0$ such that $f(t) < f(x)$ for all $t \in (x - \delta, x + \delta) \setminus \{x\}$. If x is a point of maximum or proper maximum of $-f$, then f is said to have a point of *minimum* or *proper minimum* at x , respectively. Moreover, x is said to be a point of *extremum* of f if f has a maximum or minimum at x , and a point of *proper extremum* of f if f has a proper maximum or proper minimum at x .

We used results from Bruckner and Garg [11] in the proof of Lemma 4.3 earlier in the chapter. We return to the results of [11] for additional information concerning the behavior of a generic $f \in \mathcal{C}(\mathbb{R})$. Bruckner and Garg proved that a generic $f \in C([0, 1], \mathbb{R})$ is monotone at no point, and no level set of f contains more than one point of extremum of f . For each $N \in \mathbb{N}$, let A_N be the subset of $\mathcal{C}(\mathbb{R})$ consisting of all $f \in \mathcal{C}(\mathbb{R})$ with the properties that (i) f is monotone at no point of $[-N, N]$, (ii) no level set of $f|_{[-N, N]}$ contains more than one point of extremum of $f|_{[-N, N]}$, and (iii) f has the Bruckner-Garg property on $[-N, N]$. Each A_N is comeager in $\mathcal{C}(\mathbb{R})$, and so $A = \bigcap_N A_N$ is comeager in $\mathcal{C}(\mathbb{R})$ as well. Fix $f \in A$. It is clear that f is monotone at no point of \mathbb{R} . Suppose some level of f contains more than one point of extremum of f , say, p_1 and p_2 . Choose N large enough so that $p_1, p_2 \in (-N, N)$; then some level of $f|_{[-N, N]}$ contains more than one point of extremum of $f|_{[-N, N]}$, contradicting that $f \in A_N$. So each level of f contains at most one point of extremum of f . It follows that every point of extremum of f is a point of proper extremum of f , for if not, some level of f contains more

than one point of extremum of f . Since the set of points of proper extremum of any function is countable, the set of points of extremum of f is countable. Now for each $N \in \mathbb{N}$, let $T_N = \{x \in \mathbb{R} : f|_{[-N, N]}^{-1}(x) \text{ is not perfect}\}$. Each set T_N is countable, so

$$\{x \in \mathbb{R} : f^{-1}(x) \text{ is not perfect}\} = \bigcup_{N \in \mathbb{N}} T_N$$

is countable as well. Thus $f^{-1}(x)$ is not perfect for only countably many $x \in \mathbb{R}$.

We summarize the results of the preceding discussion in Proposition 4.5.

PROPOSITION 4.5. *A generic function $f \in \mathcal{C}(\mathbb{R})$ has the property that*

1. *f is monotone at no point.*
2. *The set of points of extremum of f is countable.*
3. *$f^{-1}(x)$ is not perfect for only countably many $x \in \mathbb{R}$.*

We will now state and prove Propositions 4.6, 4.7, and 4.8, which will be needed in the proof of Lemma 4.8.

PROPOSITION 4.6. *Generic $f \in \mathcal{C}(\mathbb{R})$ has the property that, given any open interval I , there exist uncountably many $x \in I$ such that $|\text{orb}(f, x)| < \infty$ and no point of $\text{orb}(f, x)$ is a point of extremum of f .*

Proof. Fix an open interval I . Let S be the subset of $\mathcal{C}(\mathbb{R})$ consisting of all functions f with the following properties:

1. $\{x \in \mathbb{R} : |\text{orb}(f, x)| \leq n\}$ is perfect for all n .
2. $\{x \in \mathbb{R} : |\text{orb}(f, x)| < \infty\}$ is c -dense in \mathbb{R} .
3. $\{x \in \mathbb{R} : f^n(x) = x\}$ is perfect for all n .
4. f is monotone at no point.

5. The set of points of extremum of f is countable.

6. $f^{-1}(x)$ is not perfect for only countably many $x \in \mathbb{R}$.

Then S is comeager in $\mathcal{C}(\mathbb{R})$ by Corollaries 4.1, 4.2, 4.4, and Proposition 4.5. Let $f \in S$. The claim is that f satisfies the proposition. Let E denote the set of points of extremum of f . There are two cases to consider.

CASE 1. $P(f) \cap I \neq \emptyset$.

Since f has a periodic point in I , by Property (3), there are uncountably many periodic points of f in I . So the set $P(f) \cap I$ is uncountable. Observe that f must be injective on $P(f)$, for if not, then f is not a well-defined function. Remove from $P(f) \cap I$ all points whose orbits have nonempty intersection with E ; since E is countable, we have removed only a countable set. We are left with uncountably many points in $P(f) \cap I$ whose orbits do not intersect E .

CASE 2. $P(f) \cap I = \emptyset$.

By Properties (1) and (2), there exists $n \in \mathbb{N}$ such that $|\text{orb}(f, x)| \leq n$ for uncountably many $x \in I$. We will show in the following paragraph that for some $l \geq 1$, there exists $x \in I$ such that $f^l(x) \in P(f) \cap \text{Int}(f^l(I))$; then using Property (6) and Case 1, we will complete the proof of the proposition.

Claim: For some $l \geq 1$, there exists $x \in I$ such that

$$f^l(x) \in P(f) \cap \text{Int}(f^l(I)).$$

Let $O_n(f)$ denote the set of all $x \in \mathbb{R}$ satisfying $|\text{orb}(f, x)| \leq n$. Observe that, by Property (4), f is constant on no interval, so the sets $f(I), \dots, f^{n-1}(I)$ are intervals. Now if some $x \in O_n(f) \cap I$ has the property that $f(x) \in \partial f(I)$, then either $f(x) \leq f(y)$ for all $y \in I$, or $f(x) \geq f(y)$ for all $y \in I$, and so $x \in E$. So, there must be uncountably many $x \in O_n(f) \cap I$ satisfying $f(x) \in O_{n-1}(f) \cap \text{Int}(f(I))$. If one of these points is periodic, then $l = 1$ and we are done. Suppose not. Then, since the set $O_{n-1}(f) \cap \text{Int}(f(I))$ is nonempty, it is uncountable by Property

(1). As before, there are uncountably many $\tilde{x} \in O_{n-1}(f) \cap \text{Int}(f(I))$ satisfying $f(\tilde{x}) \in O_{n-2}(f) \cap \text{Int}(f^2(I))$. If one such point $f(\tilde{x})$ is periodic, then we have $f(\tilde{x}) = f^2(x)$ for some $x \in O_n(f) \cap I$, and $l = 2$ and we are done. If no such point $f(\tilde{x})$ is periodic, then repeat the argument as many times as is necessary until a periodic (possibly fixed) point is found in $\text{Int}(f^l(I))$ for some $1 \leq l \leq n - 1$. This completes the proof of the claim.

Now choose $x \in O_n(f) \cap I$ so that $f^l(x) \in P(f) \cap \text{Int}(f^l(I))$. By Case 1, there are uncountably many points in $P(f) \cap \text{Int}(f^l(I))$ whose orbits do not intersect E . Since there are only countably many levels of f which are not perfect, we may choose a point $z \in P(f) \cap \text{Int}(f^l(I))$ such that $f^{-1}(z)$, the level of f at z , is perfect, $\text{orb}(f, z) \cap E = \emptyset$, and $f^{-1}(z) \cap \text{Int}(f^{l-1}(I)) \neq \emptyset$. Now we will “pull back” the point z through the intervals $f^{l-1}(I), f^{l-2}(I), \dots, f(I), I$ to complete the proof. Choose $z_{-1} \in f^{l-1}(I) \cap f^{-1}(z)$ so that $z_{-1} \notin E$, $f^{-1}(z_{-1}) \cap \text{Int}(f^{l-2}(I)) \neq \emptyset$, and $f^{-1}(z_{-1})$ is perfect. Continue to choose points $z_{-2} \in f^{-1}(z_{-1}) \cap f^{l-2}(I)$, $z_{-3} \in f^{-1}(z_{-2}) \cap f^{l-3}(I)$, and so on, in a similar manner. Finally we have a point $z_{l-1} \in f(I)$ with the property that $f^{-1}(z_{l-1}) \cap I$ is uncountable. Then for any $x \in (f^{-1}(z_{l-1}) \cap I) \setminus E$, we have $|\text{orb}(f, x)| < \infty$ and $\text{orb}(f, x) \cap E = \emptyset$, and this set is uncountable. \square

PROPOSITION 4.7. *Suppose that $f \in \mathcal{C}(\mathbb{R})$ is monotone at no point, and $x \in \mathbb{R}$ is not a point of extremum of f . Then there exists a unilateral convergent sequence of distinct points $(p_j)_{j \in \mathbb{N}}$ such that $p_j \rightarrow x$ in \mathbb{R} , $f(p_j) = f(x)$ for all j , and for every open interval J containing some $p_j \in (p_j)_{j \in \mathbb{N}}$, we have $f(x) \in \text{Int}(f(J))$.*

Proof. Let $f \in \mathcal{C}(\mathbb{R})$ be a function which is monotone at no point, and let x be a point which is not a point of extremum of f . Since f is not monotone at x , we may assume without loss of generality that for all $\delta > 0$, there exist $t_1, t_2 \in (x - \delta, x)$ such that $t_1 < t_2$ and $f(t_1) < f(x) < f(t_2)$. Choose $t_1^{(1)} < t_2^{(1)} < x$ so that

$f(t_1^{(1)}) < f(x) < f(t_2^{(1)})$. By the Intermediate Value Theorem, we may choose $p_1 \in (t_1^{(1)}, t_2^{(1)})$ such that $f(p_1) = f(x)$; moreover, we may require that this point p_1 not be a point of extremum of f . Continue choosing points p_j as follows: Choose $t_2^{(j-1)} < t_1^{(j)} < t_2^{(j)} < x$, and choose $p_j \in (t_1^{(j)}, t_2^{(j)})$ so that $f(p_j) = f(x)$ and p_j is not a point of extremum of f . Clearly the sequence $(p_j)_{j \in \mathbb{N}}$ is a unilateral sequence of distinct points which converges to x in \mathbb{R} and satisfies $f(p_j) = f(x)$ for all j . Also, since f is not monotone at any p_j and no p_j is a point of extremum of f , any open interval J containing some p_j has the property that $f(J)$ is an interval (since f is constant on no interval), and $f(J)$ contains points both larger and smaller than $f(x)$, so $f(x) \in \text{Int}(f(J))$. \square

PROPOSITION 4.8. *Suppose that $f \in \mathcal{C}(\mathbb{R})$ is monotone at no point. Then given any periodic point x such that no point of $\text{orb}(f, x)$ is a point of extremum of f , and given any open interval I containing x , there exist uncountably many $y \in I$ such that $\text{orb}(f, y)$ is infinite and $\omega(f, y)$ is finite.*

Proof. Let $f \in \mathcal{C}(\mathbb{R})$ be monotone at no point, and let $x \in \mathbb{R}$ be a periodic point of f such that no point of $\text{orb}(f, x)$ is a point of extremum of f . Let $\text{orb}(f, x) = \{x_0, x_1, \dots, x_{n-1}\}$, where $f(x_i) = x_{i+1}$ for $i = 0, \dots, n-1$ and $x_n = x_0 = x$. For each x_i , let $(p_j^i)_{j \in \mathbb{N}}$ be the corresponding sequence with all of the properties listed in Proposition 4.7.

Let I be an open interval containing x_0 . We will construct uncountably many pairwise disjoint subsets of I in the following way.

Step 1. Choose $q_0, q_1 \in I \cap (p_j^0)_{j \in \mathbb{N}}$, and let $J_0, J_1 \subseteq I$ be disjoint intervals centered at q_0, q_1 , respectively, which do not contain x_0 . Note that $f(J_0)$ and $f(J_1)$ are intervals which contain $x_{1(\text{mod } n)}$ as an interior point, so both $f(J_0)$ and $f(J_1)$ contain infinitely many points of the sequence $(p_j^{1(\text{mod } n)})$.

Step 2. For each 2-tuple $(a_1, a_2) \in \{0, 1\}^2$, choose a point $q_{a_1, a_2} \in (p_j^{1(\text{mod } n)})_{j \in \mathbb{N}}$

and an interval J_{a_1, a_2} centered at q_{a_1, a_2} such that

- $J_{a_1, a_2} \subseteq f(J_{a_1})$,
- $x_{1 \pmod n} \notin \bigcup_{(a_1, a_2) \in \{0, 1\}^2} J_{a_1, a_2}$, and
- the intervals J_{a_1, a_2} and J_{a_1} , where $a_1, a_2 \in \{0, 1\}$, are pairwise disjoint.

Observe that each $f(J_{a_1, a_2})$ is an interval containing $x_{1 \pmod n}$ as an interior point.

In general, the k^{th} step of the construction is defined as follows.

Step k . From the previous step, for each $(k-1)$ -tuple $(a_1, \dots, a_{k-1}) \in \{0, 1\}^{k-1}$, we have an interval of the form $f(J_{a_1, \dots, a_{k-1}})$, and for some fixed $i \in \{0, 1, \dots, n-1\}$, each interval $f(J_{a_1, \dots, a_{k-1}})$ contains x_i as an interior point. Choose 2^k distinct points from the sequence $(p_j^i)_{j \in \mathbb{N}}$ as follows. For each k -tuple $(a_1, a_2, \dots, a_k) \in \{0, 1\}^k$, choose a point $q_{a_1, a_2, \dots, a_k} \in (p_j^i)_{j \in \mathbb{N}}$ and an interval J_{a_1, a_2, \dots, a_k} centered at that point such that

- $J_{a_1, a_2, \dots, a_k} \subseteq f(J_{a_1, a_2, \dots, a_{k-1}})$
- $x_i \notin \bigcup_{(a_1, a_2, \dots, a_k) \in \{0, 1\}^k} J_{a_1, a_2, \dots, a_k}$, and
- the intervals J_{a_1, a_2, \dots, a_k} , $J_{a_1, a_2, \dots, a_{k-1}}$, \dots , J_{a_1, a_2} , and J_{a_1} , where $a_1, a_2, \dots, a_{k-1}, a_k \in \{0, 1\}$, are pairwise disjoint.

Continue for all $k \in \mathbb{N}$. Observe that at each step k , the intervals J_{a_1, a_2, \dots, a_k} are all nonempty.

Now for each $(a_j)_{j \in \mathbb{N}} \in \{0, 1\}^\omega$, let

$$A_{(a_j)_{j \in \mathbb{N}}} = \{x \in \mathbb{R} : x \in J_{a_1}, f(x) \in J_{a_1, a_2}, \dots, f^k(x) \in J_{a_1, \dots, a_k}, \dots\}.$$

Since $J_0, J_1 \subseteq I$, each set $A_{(a_j)_{j \in \mathbb{N}}}$ is a subset of I . Since the intervals J were chosen to be pairwise disjoint, the sets $A_{(a_j)_{j \in \mathbb{N}}}$ are pairwise disjoint. We will show that each set $A_{(a_j)_{j \in \mathbb{N}}}$ is nonempty. Suppose that some $A_{(a_j)_{j \in \mathbb{N}}} = \emptyset$ for some

$(a_j)_{j \in \mathbb{N}} \in \{0, 1\}^\omega$. Let $m \in \mathbb{N}$ be the smallest number such that $f^m(t) \notin J_{a_1, \dots, a_{m+1}}$ for any $t \in J_{a_1}$. Since $J_{a_1} \neq \emptyset$, m must be at least 1. The interval J_{a_1, \dots, a_m} is nonempty, so let $y \in J_{a_1, \dots, a_m}$. Since $J_{a_1, \dots, a_m} \subseteq f(J_{a_1, \dots, a_{m-1}})$, there exists $t_{m-1} \in J_{a_1, \dots, a_{m-1}}$ such that $y = f(t_{m-1})$. Continue working backwards to obtain points $t_{m-2} \in J_{a_1, \dots, a_{m-2}}, \dots, t_1 \in J_{a_1}$ with $f(t_i) = t_{i+1}$. Then $t_1 \in J_{a_1}$ has the property that $f^m(t_1) = y \in J_{a_1, \dots, a_{m+1}}$, a contradiction. Now there are continuum many nonempty pairwise disjoint subsets $A_{(a_j)_{j \in \mathbb{N}}}$ of I , and for any y in some $A_{(a_j)_{j \in \mathbb{N}}}$, we have $\text{orb}(f, y)$ is infinite and $\omega(f, y) = \{x_0, \dots, x_{n-1}\}$. \square

We are now ready to prove that Property (3) of Theorem 4.2 holds on a comeager subset of $\mathcal{C}(\mathbb{R})$.

LEMMA 4.8. *Generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the set of $x \in \mathbb{R}$ such that $\text{orb}(f, x)$ is infinite and $\omega(f, x)$ is finite is c -dense in \mathbb{R} .*

Proof. Let $G \subseteq \mathcal{C}(\mathbb{R})$ be the set of all f satisfying:

1. f is monotone at no point,
2. $f^{-1}(x)$ is not perfect for only countably many $x \in \mathbb{R}$, and
3. for any open interval $I \subseteq \mathbb{R}$, there exist uncountably many $x \in I$ such that $\text{orb}(f, x)$ is finite and no point of $\text{orb}(f, x)$ is a point of extremum of f .

The set G is comeager in $\mathcal{C}(\mathbb{R})$ by Propositions 4.5 and 4.6. Let $f \in G$. Fix an open interval I , and let $a \in I$ be such that $|\text{orb}(f, a)| < \infty$ and no point of $\text{orb}(f, a)$ is a point of extremum of f . So, we have $\text{orb}(f, a) = \{a_0, \dots, a_k, \dots, a_{n-1}\}$ for some k, n . Since f is not monotone at any of the points a_0, \dots, a_{k-1} and none of these points is a point of extremum of f , we have that $f^k(I)$ is an interval with $a_k \in \text{Int}(f^k(I))$. Now since $\text{Int}(f^k(I))$ is an open interval containing the periodic point a_k of f , by Lemma 4.8 there are uncountably many points $y \in \text{Int}(f^k(I))$

such that $\text{orb}(f, y)$ is infinite and $\omega(f, y)$ is finite. For uncountably many such y , we can find uncountably many $x \in I$ such that $f^k(x) = y$. Thus, every $f \in G$ has the property that there are uncountably many $x \in I$ such that $|\text{orb}(f, x)| = \infty$ and $|\omega(f, x)| < \infty$. Since I was an arbitrarily chosen interval, we have the lemma. \square

The final few results of this chapter are used to prove that Property (4) of Theorem 4.2 holds on a comeager subset of $\mathcal{C}(\mathbb{R})$. The idea behind Propositions 4.9 and 4.10 and Lemma 4.9 is the following. We will show in Proposition 4.9 that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that for any $M \in \mathbb{N}$, there exists a closed interval I such that $I \cap [-M, M] = \emptyset$ and $f|_I$ exhibits the same behavior as a function in $C([0, 1])$. In Proposition 4.10 we will state and give a proof for a known result concerning a generic $f \in C([0, 1])$. Finally we will use the result of Proposition 4.10, together with Proposition 4.9, to prove Lemma 4.9

PROPOSITION 4.9. *For each $M \in \mathbb{N}$, let U_M be the set of $f \in \mathcal{C}(\mathbb{R})$ such that, for all $f \in U_M$, there exists a closed interval $I \subseteq \mathbb{R}$ such that $f(I) \subseteq I$ and $I \cap [-M, M] = \emptyset$. Each set U_M contains an open dense subset of $\mathcal{C}(\mathbb{R})$.*

Proof. Fix $M \in \mathbb{N}$. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$ and $N > M$. Construct a function g as follows. Let $g(x) = f(x)$ for all $x \in [-N, N]$. Let $I = [N + 1, N + 2]$. Let $g(x) = N + \frac{1}{2}$ for all $x \in [N + 1, N + 2]$. Extend g continuously to \mathbb{R} . Observe that $g \in B_\epsilon(f)$. Choose $0 < \eta < \min\{\frac{1}{N+2}, \frac{1}{4}\}$. Let $h \in B_\eta(g)$. Then $h(I) \subseteq I$, and so $B_\eta(g) \subseteq U_M$. \square

PROPOSITION 4.10. *A generic $f \in C([0, 1])$ has the property that there exists $x \in [0, 1]$ such that $\omega(f, x)$ is infinite and not perfect.*

Proof. [43] $C([0, 1])$ is an abelian Polish group with the metric of supremum norm. Let W be the set of all $f \in C([0, 1])$ such that $\omega(f, x)$ is infinite and not perfect for some $x \in [0, 1]$. Let $f \in C([0, 1])$ and $\epsilon > 0$. Since $f : [0, 1] \rightarrow [0, 1]$, f has

a fixed point. Let g be a slight perturbation of f such that $g \in B_\epsilon(f)$ and g has a point a of period 3 which lies in a small neighborhood of the fixed point of f . Let $\eta > 0$ be chosen so that for all $h \in B_\eta(g)$, h has a point of period 3. Since h has a point of period 3, it follows from Theorems 2.2 and 3.8 of [10] that h has a non-perfect infinite ω limit set. Thus $B_\eta(g) \subseteq W$. \square

LEMMA 4.9. *A generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the set of all $x \in \mathbb{R}$ such that $\omega(f, x)$ is infinite and not perfect is unbounded in \mathbb{R} .*

Proof. For each $M \in \mathbb{N}$, let

$$S_M = \{f \in \mathcal{C}(\mathbb{R}) : \exists x > M \ni \omega(f, x) \text{ is infinite and not perfect}\}.$$

Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. By Proposition 4.9, we may choose $f^* \in B_{\frac{\epsilon}{3}}(f)$ with the property that $f^*(I) \subseteq I$ for some closed interval $I \subseteq \mathbb{R}$ satisfying $I \cap [-M, M] = \emptyset$. By the argument used in the proof of Proposition 4.10, we may choose $g \in B_{\frac{\epsilon}{3}}(f^*)$ and $\eta > 0$ such that, for all $h \in B_\eta(g)$, there exists $x \in I$ such that $\omega(h, x)$ is infinite and non-perfect. Now we have $g \in B_\epsilon(f)$ and $B_\eta(h) \subseteq S_M$. To complete the proof of the lemma, observe that $\bigcap_{M \in \mathbb{N}} S_M$ is comeager in $\mathcal{C}(\mathbb{R})$. \square

We close the section with a proof of Theorem 4.2.

Proof. (Proof of Theorem 4.2) Let \mathcal{G} be the intersection of the sets defined in Lemmas 4.6, 4.7, Corollary 4.2, and Lemmas 4.8 and 4.9. This set is comeager in $\mathcal{C}(\mathbb{R})$. \square

4.2 Preliminary Results in $\mathcal{C}(\mathbb{R}^n)$, $n \geq 1$

In this section, we will present our preliminary results concerning properties which hold for a generic $f \in \mathcal{C}(\mathbb{R}^n)$. In comparing Theorems 4.1 and 4.3, we see that several of the properties which hold for a generic $f \in \mathcal{C}(\mathbb{R})$ also hold in the more

general setting of $\mathcal{C}(\mathbb{R}^n)$, $n \geq 1$, although the proofs require different techniques. In the theorem below, $f^{-1}(\mathbf{x})$ and $orb(f, \mathbf{x})$ are defined as in the previous section.

THEOREM 4.3. *Generic $f \in \mathcal{C}(\mathbb{R}^n)$ has the property that:*

1. f is a surjection.
2. $f^{-1}(\mathbf{x})$ is uncountable and unbounded for all $\mathbf{x} \in \mathbb{R}^n$.
3. $orb(f, \mathbf{x})$ is bounded for all $\mathbf{x} \in \mathbb{R}^n$.
4. The set of periodic points of period k is unbounded for all $k \in \mathbb{N}$.

We will prove the theorem using a series of lemmas. In the first lemma, we will prove that a generic $f \in \mathcal{C}(\mathbb{R}^n)$ is surjective, and has the property that $f^{-1}(\mathbf{x})$ is unbounded for all $\mathbf{x} \in \mathbb{R}^n$. The proof of the lemma uses the well-known Brouwer Fixed Point Theorem (see, for example, page 275 of [32]).

THEOREM (Brouwer Fixed Point Theorem). *Let B be the open unit ball in \mathbb{R}^n . Then every continuous map $f : \bar{B} \rightarrow \bar{B}$ has a fixed point.*

LEMMA 4.10. *A generic $f \in \mathcal{C}(\mathbb{R}^n)$ is surjective and has the property that $f^{-1}(\mathbf{x})$ is unbounded for all $\mathbf{x} \in \mathbb{R}^n$.*

Proof. For each basis element $B = B_r(\mathbf{p})$ of $\mathcal{C}(\mathbb{R}^n)$ and each $M \in \mathbb{N}$, let

$$S_{M,B} = \{f \in \mathcal{C}(\mathbb{R}^n) : \exists \mathbf{q} \in \mathbb{R}^n \ni B \subseteq f(B_r(\mathbf{q})) \text{ and } B_r(\mathbf{q}) \cap [-M, M]^n = \emptyset\}.$$

We will use the Brouwer Fixed Point Theorem to show that each set $S_{M,B}$ contains a dense open subset of $\mathcal{C}(\mathbb{R}^n)$.

Let $B = B_r(\mathbf{p})$ be a fixed open ball in \mathbb{R}^n , and fix $M \in \mathbb{N}$. Let $f \in \mathcal{C}(\mathbb{R}^n)$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$ and $N \geq M$. Choose $\mathbf{q} \in \mathbb{R}^n$ so that $\bar{B}_r(\mathbf{q}) \cap [-N, N]^n = \emptyset$. Let $U = B_r(\mathbf{q})$. Define a function g as follows. Let

$g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in [-N, N]^n$. Let $g(\mathbf{x}) = 2\mathbf{x} - 2\mathbf{q} + \mathbf{p}$ for all $\mathbf{x} \in \overline{B}_r(\mathbf{q})$. Then by the corollary to the Tietze Extension given in [26] (Corollary 1, page 82), we may extend g continuously to \mathbb{R}^n . Observe that $g \in B_\epsilon(f)$, $U \cap [-M, M]^n = \emptyset$, and $g(U) = B_{2r}(\mathbf{p})$.

Choose $0 < \eta < \min\{\frac{r}{2}, \frac{1}{L}\}$, where $L \in \mathbb{N}$ is chosen large enough so that $U \subseteq [-L, L]^n$. Let $h \in B_\eta(g)$. We will show that if $B \not\subseteq h(U)$, then we can construct a map $c : \overline{U} \rightarrow \overline{U}$ which has no fixed point, a contradiction of the Brouwer Fixed Point Theorem. To this end, suppose that there exists some point $\mathbf{w} \in B \setminus h(U)$. Note that $\mathbf{w} \notin h(\overline{U})$. Define mappings c_1, \dots, c_5 as follows.

- Let $c_1 : \mathbb{R}^n \rightarrow \overline{B}$ be the radial projection mapping of \mathbb{R}^n onto $\overline{B} = \overline{B}_r(\mathbf{p})$, defined as follows. If $\mathbf{x} \in \overline{B}$, then $c_1(\mathbf{x}) = \mathbf{x}$. If $\mathbf{x} \notin \overline{B}$, then let $c_1(\mathbf{x})$ be the point in ∂U which intersects the line segment whose endpoints are \mathbf{x} and \mathbf{p} . Observe that $\mathbf{w} \notin c_1 h(\overline{U})$.
- Let $c_2 : \overline{B} \rightarrow \overline{B}$ be a homeomorphism of \overline{B} such that $c_2(\mathbf{w}) = \mathbf{p}$ and c_2 leaves all points in ∂B fixed. Now $\mathbf{p} \notin c_2 c_1 h(U)$.
- Let $c_3 : \overline{B} \rightarrow \overline{U}$ be a translation of \overline{B} onto \overline{U} , given by $c_3(\mathbf{x}) = \mathbf{x} - \mathbf{p} + \mathbf{q}$. So $\mathbf{q} \notin c_3 c_2 c_1 h(U)$.
- Let $c_4 : \overline{U} \setminus \{\mathbf{q}\} \rightarrow \overline{U}$ be the outward radial projection of $\overline{U} \setminus \{\mathbf{q}\}$ onto ∂U .
- Finally, let $c_5 : \overline{U} \rightarrow \overline{U}$ be the map given by $c_5(\mathbf{x}) = -\mathbf{x} + 2\mathbf{q}$.

Let $c = c_5 c_4 c_3 c_2 c_1 h$. It is easily verified that c is continuous and well-defined on \overline{U} , and $c : \overline{U} \rightarrow \overline{U}$. Suppose that $c(\mathbf{x}) = \mathbf{x}$ for some $\mathbf{x} \in \overline{U}$. Since $c(\overline{U}) \subseteq \partial U$, it must be the case that $\mathbf{x} \in \partial U$. Observe that since $d(h(\mathbf{x}), g(\mathbf{x})) < \frac{r}{2}$, we have $d(h(\mathbf{x}), \mathbf{p}) > \frac{3r}{2}$, so $c_1 h(\mathbf{x})$ lies on the boundary of B . Thus $c_2 c_1 h(\mathbf{x}) \in \partial B$, and $c_3 c_2 c_1 h(\mathbf{x}) \in \partial U$. Now $c(\mathbf{x}) = \mathbf{x}$, so $c_4 c_3 c_2 c_1 h(\mathbf{x}) = 2\mathbf{q} - \mathbf{x}$. Since the

boundary points of U are fixed under c_4 , we have $c_3c_2c_1h(\mathbf{x}) = 2\mathbf{q} - \mathbf{x}$. Then $c_2c_1h(\mathbf{x}) = \mathbf{p} + \mathbf{q} - \mathbf{x}$. The map c_2 leaves the boundary points of B fixed, so $c_1h(\mathbf{x}) = \mathbf{p} + \mathbf{q} - \mathbf{x}$. Then $h(\mathbf{x}) = \mathbf{p} + t(\mathbf{q} - \mathbf{x})$ for some $t \geq 1$. Now

$$\begin{aligned} d(h(\mathbf{x}), g(\mathbf{x})) &= d(\mathbf{p} + t(\mathbf{q} - \mathbf{x}), \mathbf{p} + 2(\mathbf{x} - \mathbf{q})) \\ &= (2 + t)d(\mathbf{x}, \mathbf{q}) \\ &= (2 + t)r \\ &\geq 3r, \end{aligned}$$

contradicting that $h \in B_\eta(g)$. Hence, $B \subseteq h(U)$ and $B_\eta(g) \subseteq S_{M,B}$. It follows that $S_{M,B}$ contains a dense open subset of $\mathcal{C}(\mathbb{R}^n)$, as was to be proven.

Now let $S = \bigcap_{M,B} S_{M,B}$, where the intersection is taken over all $M \in \mathbb{N}$ and all basis elements B belonging to a countable basis for \mathbb{R}^n . The set S is comeager in $\mathcal{C}(\mathbb{R}^n)$. Fix $f \in S$. Let $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$. Let $B = B_r(\mathbf{x})$. Then for each $M \in \mathbb{N}$, there exists an open ball $U \subseteq \mathbb{R}^n$ such that $U \cap [-M, M]^n = \emptyset$ and $\mathbf{x} \in B \subseteq f(U)$. Thus $f^{-1}(\mathbf{x})$ is unbounded. Since \mathbf{x} was arbitrarily chosen, we have that for all $\mathbf{x} \in \mathbb{R}^n$, $f^{-1}(\mathbf{x})$ is unbounded, and hence nonempty, so f is a surjection. \square

In the second lemma of this section, we will prove that a generic $f \in \mathcal{C}(\mathbb{R}^n)$ has the property that the preimage of every point in \mathbb{R}^n under f is uncountable. We will use the following theorem of B. Kirchheim in the proof. In the theorem, \mathcal{H}^s denotes the s -dimensional Hausdorff measure on \mathbb{R}^n . (See [19] for a definition of the Hausdorff measure.)

THEOREM ([28]). *Let $n \geq m \geq 1$. Then a generic $f : [0, 1]^n \rightarrow \mathbb{R}^m$ has the property that for any $\mathbf{x} \in \mathbb{R}^m$, the level set $f^{-1}(\mathbf{x})$ is of non- σ -finite \mathcal{H}^{n-m} -measure whenever \mathbf{x} lies in the interior of $f([0, 1]^n)$.*

LEMMA 4.11. *A generic $f \in \mathcal{C}(\mathbb{R}^n)$ has the property that $f^{-1}(\mathbf{x})$ is uncountable for all $\mathbf{x} \in \mathbb{R}^n$.*

Proof. Let $n = m \geq 1$, and observe that \mathcal{H}^0 , the zero-dimensional Hausdorff measure, is the counting measure ([19]). Now if for some \mathbf{x} , $f^{-1}(\mathbf{x})$ is countable, then $f^{-1}(\mathbf{x})$ can be written as the countable union of singleton sets whose \mathcal{H}^0 -measure is one, and so $f^{-1}(\mathbf{x})$ is of σ -finite \mathcal{H}^0 -measure. It follows from the theorem of Kirchheim that a generic $f \in \mathcal{C}(\mathbb{R}^n)$ has the property that $f^{-1}(\mathbf{x})$ is uncountable for all $\mathbf{x} \in \text{Int}(f([0, 1]^n))$.

For each $N \in \mathbb{N}$, let G_N be the set of all $f \in \mathcal{C}(\mathbb{R}^n)$ with the property that $f^{-1}(\mathbf{x})$ is uncountable for all $\mathbf{x} \in \text{Int}(f([-N, N]^n))$. Each set G_N is comeager in $\mathcal{C}(\mathbb{R}^n)$ by Kirchheim's theorem. Let $\mathcal{G} = (\bigcap_{N \in \mathbb{N}} G_N) \cap S$, where S is the intersection of the sets $S_{M,B}$ as defined in the proof of Lemma 4.10. Observe that \mathcal{G} is comeager in $\mathcal{C}(\mathbb{R}^n)$. Let $f \in \mathcal{G}$ and $\mathbf{x} \in \mathbb{R}^n$. Let $r > 0$. Then since $f \in S$, we have $B_r(\mathbf{x}) \subseteq f(U)$ for some open $U \subseteq \mathbb{R}^n$. Choose $L \in \mathbb{N}$ so that $U \subseteq [-L, L]^n$. Then $B_r(\mathbf{x}) \subseteq f([-L, L]^n)$, so $\mathbf{x} \in \text{Int}(f([-L, L]^n))$. Then since $f \in G_L$, $f^{-1}(\mathbf{x})$ is uncountable. \square

Finally, to prove that properties (3) and (4) of Theorem 1 hold for a generic f , we generalize the techniques used in Chapter 4.

LEMMA 4.12. *A generic $f \in \mathcal{C}(\mathbb{R}^n)$ has the property that $\text{orb}(f, \mathbf{x})$ is bounded for all $\mathbf{x} \in \mathbb{R}^n$.*

Proof. Let $K \in \mathbb{N}$ be fixed. Let

$$A_K = \{f \in \mathcal{C}(\mathbb{R}^n) : \text{orb}(f, \mathbf{x}) \text{ is bounded } \forall \mathbf{x} \in [-K, K]^n\}.$$

We will show that A_K contains a dense open subset of $\mathcal{C}(\mathbb{R}^n)$; then by intersecting over all $K \in \mathbb{N}$ we will have a comeager subset of $\mathcal{C}(\mathbb{R}^n)$ with the desired properties. Let $f \in \mathcal{C}(\mathbb{R}^n)$ and $\epsilon > 0$ be arbitrarily chosen. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$ and $N > K$. Let $P : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the map given by $P(\mathbf{x}) = \mathbf{x}_0$, where \mathbf{x}_0 is the nearest point to \mathbf{x} satisfying $\mathbf{x}_0 \in [-N, N]^n$. Observe that, since $[-N, N]^n$ is

convex, P is a well-defined continuous map. Define a function $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $g(\mathbf{x}) = f(P(\mathbf{x}))$. Now g is continuous as it is the composition of the continuous maps f and P . Moreover, since g and f agree on $[-N, N]^n$ and $\frac{1}{N} < \epsilon$, we have that $g \in B_\epsilon(f)$. Now choose $L \in \mathbb{N}$ large enough so that $g([-N, N]^n) \cup [-N, N]^n \subseteq [-L, L]^n$. Choose $0 < \eta < \min\{\frac{1}{L+1}, \frac{1}{2}\}$. Let $h \in B_\eta(g)$ and let $\mathbf{x} \in [-K, K]^n$. So $h(\mathbf{x}) \in [-(L + \frac{1}{2}), L + \frac{1}{2}]^n$, and then $d(h^2(\mathbf{x}), g(h(\mathbf{x}))) < \eta$ implies that $h^2(\mathbf{x}) \in [-(L + \frac{1}{2}), L + \frac{1}{2}]^n$. Proceeding inductively, we see that $h^k(\mathbf{x}) \in [-(L + \frac{1}{2}), L + \frac{1}{2}]^n$ for all $k \in \mathbb{N}$; i.e., $\text{orb}(h, \mathbf{x})$ is bounded. It follows that $h \in A_K$. Now we have $B_\eta(g) \subseteq A_K$, and so A_K contains a dense open subset of $\mathcal{C}(\mathbb{R}^n)$. We obtain the lemma by intersecting the sets A_K over all $K \in \mathbb{N}$. \square

LEMMA 4.13. *A generic $f \in \mathcal{C}(\mathbb{R}^n)$ has the property that the set of periodic points of period k is unbounded for all $k \in \mathbb{N}$.*

Proof. Fix $M \in \mathbb{N}$ and $k \in \mathbb{N}$. Let

$$P = \{f \in \mathcal{C}(\mathbb{R}^n) : \exists \mathbf{x} \notin [-M, M]^n \ni \mathbf{x} \text{ has period } k \text{ under } f\}.$$

Let $f \in \mathcal{C}(\mathbb{R}^n)$ and $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $\frac{1}{N} < \epsilon$ and $N \geq M$. Choose $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{k-1} \in \mathbb{R}^n$ and $r > 0$ so that the closed balls $\overline{B}_r(\mathbf{p}_i)$ are pairwise disjoint and $\overline{B}_r(\mathbf{p}_i) \cap [-M, M]^n = \emptyset$ for each i . Construct a function g as follows. Let $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in [-N, N]^n$, and for each i , let $g(\mathbf{x}) = \mathbf{p}_{i+1(\text{mod } k)}$ for all $\mathbf{x} \in \overline{B}_r(\mathbf{p}_i)$. By the corollary to the Tietze Extension Theorem given in [26] (Corollary 1, page 82), we may extend g continuously to \mathbb{R}^n . Note that $g \in B_\epsilon(f)$. Let $L \in \mathbb{N}$ be such that $\bigcup_i \overline{B}_r(\mathbf{p}_i) \subseteq [-L, L]^n$. Choose $\eta > 0$ so that if $\rho(g, h) < \eta$, then $\|g^j - h^j\|_{[-L, L]^n} < \frac{r}{2}$ for $j = 1, \dots, k$. Let $h \in B_\eta(g)$. Then for any $\mathbf{x} \in \overline{B}_r(\mathbf{p}_0)$, we have $h^k(\mathbf{x}) \in B_{\frac{r}{2}}(\mathbf{p}_0)$. Since $h^k : \overline{B}_r(\mathbf{p}_0) \rightarrow \overline{B}_r(\mathbf{p}_0)$, by the Brouwer's Fixed Point Theorem, there is some $\mathbf{p} \in \overline{B}_r(\mathbf{p}_0)$ such that $h^k(\mathbf{p}) = \mathbf{p}$. Moreover, since $h(\mathbf{p}) \in B_r(\mathbf{p}_1), \dots, h^{k-1}(\mathbf{p}) \in B_r(\mathbf{p}_{k-1})$ and the balls are pairwise disjoint, \mathbf{p}

cannot have period less than k under h . Hence $B_\eta(g) \subseteq P$. Now P contains an open dense subset of $\mathcal{C}(\mathbb{R}^n)$. To complete the proof of the lemma, intersect all such sets P over $M \in \mathbb{N}$ and $k \in \mathbb{N}$. \square

Proof. (Proof of Theorem 4.3). Take the intersection of the sets with the properties given in Lemmas 4.10, 4.11, 4.12, and 4.13. This intersection is comeager in $\mathcal{C}(\mathbb{R}^n)$ as well. \square

In Theorem 4.2, we obtained results concerning properties of the orbits and ω -limit sets of a generic $f \in \mathcal{C}(\mathbb{R})$. We remarked that for any closed subset C of $[0, 1]$, C is an ω -limit set of some function $f \in C([0, 1])$ if and only if C is either nowhere dense, or C is the union of finitely many nondegenerate closed intervals [1]. In higher dimensions, the problem of classifying which types of ω -limit sets may occur for functions in the space $C([0, 1]^n)$ is much more complicated. Some partial results are given in [3], although the authors note that it is unknown whether even some simple sets, such as the union of a line segment and a disk in $[0, 1]^2$, can be an ω -limit set. To simplify the problem in $C([0, 1]^n)$, some have restricted their study of ω -limit sets to special types of mappings such as triangular [21] and antitriangular maps [5]. In the current setting, we merely remark that extending the results of Theorem 4.2 to the space $\mathcal{C}(\mathbb{R}^n)$ is a difficult problem which will require further study.

CHAPTER 5

CO-HAAR NULL AND H-AMBIVALENT SUBSETS OF $\mathcal{C}(\mathbb{R})$

In this chapter we are interested in studying properties of functions in $\mathcal{C}(\mathbb{R})$ which hold on a co-Haar null subset of $\mathcal{C}(\mathbb{R})$. We will see that in many cases, a given property holds on a subset of $\mathcal{C}(\mathbb{R})$ which is neither Haar null nor co-Haar null. Using the terminology of [44], we say that such a property is *H-ambivalent*; in addition, we will say that a set on which an H-ambivalent property holds is H-ambivalent. In particular, we will see that there exist comeager subsets of $\mathcal{C}(\mathbb{R})$ which are H-ambivalent. (Equivalently, there exist meager subsets of $\mathcal{C}(\mathbb{R})$ which are H-ambivalent.) We will also see that there exist subsets of $\mathcal{C}(\mathbb{R})$ which are both comeager and co-Haar null. However, we have found no subset of $\mathcal{C}(\mathbb{R})$ which is both comeager in $\mathcal{C}(\mathbb{R})$ and Haar null. Thus, although we showed in Chapter 3 that $\mathbb{Z}^{\mathbb{Z}}$ may be decomposed into two disjoint sets, one meager in $\mathbb{Z}^{\mathbb{Z}}$ and the other Haar null, we have found no such natural decomposition for $\mathcal{C}(\mathbb{R})$.

Before we state the main results of this chapter, we give the necessary definitions and terminology. These definitions are standard and may be found in [11] or [9]. We say that $f \in \mathcal{C}(\mathbb{R})$ is *nondecreasing at* $x \in \mathbb{R}$ if there exists $\delta > 0$ such that $\frac{f(t)-f(x)}{t-x} \geq 0$ for all $t \in (x - \delta, x + \delta) \setminus \{x\}$, and f is *nonincreasing at* $x \in \mathbb{R}$ if $-f$ is nondecreasing at x . We say that f is *monotone at* x if f is either nondecreasing or nonincreasing at x . (These definitions were given in the previous chapter but are restated here for completeness.) A function f is *monotone on an interval* I if f is either nondecreasing at all points of I , or f is nonincreasing at all points of I . We say that f is *of monotonic type at* x if the function $f(x) + mx$ is

monotone at x for some $m \in \mathbb{R}$. Finally, we say that f is of *monotonic type on an interval* I if the function $f(x) + mx$ is monotone on I for some $m \in \mathbb{R}$. Using the notation of [9], we define the following subsets of $\mathcal{C}(\mathbb{R})$.

$$MNI = \{f \in \mathcal{C}(\mathbb{R}) : f \text{ is monotone on no interval}\}$$

$$MTNI = \{f \in \mathcal{C}(\mathbb{R}) : f \text{ is of monotonic type on no interval}\}$$

$$MNP = \{f \in \mathcal{C}(\mathbb{R}) : f \text{ is monotone at no point}\}$$

$$MTNP = \{f \in \mathcal{C}(\mathbb{R}) : f \text{ is of monotonic type at no point}\}$$

It follows from Theorem 1 of [9] that

$$MTNP \subseteq MNP \subseteq MTNI \subseteq MNI,$$

and each of the inclusions is nonreversible.

For $f \in \mathcal{C}(\mathbb{R})$ and $x \in \mathbb{R}$, we define the upper and lower derivatives of f from the right at x as

$$D^+f(x) = \overline{\lim}_{t \rightarrow x^+} \frac{f(x) - f(t)}{x - t} \text{ and } D_+f(x) = \underline{\lim}_{t \rightarrow x^+} \frac{f(x) - f(t)}{x - t},$$

respectively. The upper and lower derivatives from the left of f at x , denoted by $D^-f(x)$ and $D_-f(x)$, are defined analogously. We define $\underline{D}f(x)$ as the infimum of $D_+f(x)$ and $D_-f(x)$, and $\overline{D}f(x)$ as the supremum of $D^+f(x)$ and $D^-f(x)$. We say that f has a *knot point* at x if $\overline{D}f(x) = +\infty$ and $\underline{D}f(x) = -\infty$. (The definition of knot point may vary depending on the author. For example, Zajíček in [44] defines a knot point to be a point x at which $D^+f(x) = D^-f(x) = +\infty$ and $D_+f(x) = D_-f(x) = -\infty$. Our definition is weaker and follows the example of Bruckner and Garg in [11].) It is well-known that f is not of monotonic type at x if and only if f has a knot point at x . Thus, $MTNP$ is exactly the set of those functions f for which every $x \in \mathbb{R}$ is a knot point of f .

The main results of this chapter are given in the following two theorems.

THEOREM 5.1 (H-ambivalent properties in $\mathcal{C}(\mathbb{R})$). *The following properties of a function $f \in \mathcal{C}(\mathbb{R})$ are H-ambivalent.*

1. f is a surjection.
2. $f^{-1}(x)$ is unbounded for all $x \in \mathbb{R}$.
3. $f^{-1}(x)$ is bounded for all $x \in \mathbb{R}$.
4. $f \in \text{MTNP}$.
5. $f \in \text{MNP}$.
6. For fixed $a \in \mathbb{R}$, f has derivative $+\infty$ at a and f has a knot point at all $x \neq a$.
7. $\text{orb}(f, x)$ is unbounded for all $x \in \mathbb{R}$.
8. For any compact subset C of \mathbb{R} , $\text{orb}(f, x)$ is bounded for all $x \in C$.

THEOREM 5.2 (Properties of Almost Every Mapping in $\mathcal{C}(\mathbb{R})$). *Almost every $f \in \mathcal{C}(\mathbb{R})$ has the following properties.*

1. $f \in \text{MTNI}$.
2. $f \in \text{MNI}$.
3. For any bounded set $F \subseteq \mathbb{R}$, $f|_{\mathbb{R} \setminus F}$ is not injective.
4. $f^{-1}(x)$ is perfect for all x in a comeager subset of \mathbb{R} .
5. $f(\mathbb{R})$ is unbounded.
6. For fixed $a \in \mathbb{R}$, f has neither a fixed point nor a point of period 2 at a .

When Theorem 5.1 above is compared to Theorem 4.1, we see that many comeager subsets of $\mathcal{C}(\mathbb{R})$ are H-ambivalent. Recall that a generic $f \in \mathcal{C}(\mathbb{R})$ is surjective; it cannot be said that almost every f is surjective or that almost every f is not surjective. However, by Property (5) of Theorem 5.2, we can say that almost every f has the property that $f(\mathbb{R})$ is either a line or a ray in \mathbb{R} . We also saw in Chapter 4 that a generic f has the property that $f^{-1}(x)$ is uncountable and unbounded for all x . Here we will see that we can draw no conclusions about the boundedness of the preimage of every x under almost every f . However, by Property (4) of Theorem 5.2, we have that almost every f has the property that $f^{-1}(x)$ is either empty or uncountable for a generic $x \in \mathbb{R}$. We show that the sets $MTNP$ and MNP , which are both comeager in $\mathcal{C}(\mathbb{R})$, are H-ambivalent, and the sets $MTNI$ and MNI , which are also comeager in $\mathcal{C}(\mathbb{R})$, are co-Haar null. Property (6) of Theorem 5.1 is of interest because it provides an explicit example of uncountably many pairwise disjoint universally measurable non-Haar null subsets of $\mathcal{C}(\mathbb{R})$; the existence of such a family of subsets in any nonlocally compact Polish abelian group was proven by S. Solecki in [41]. Properties (7) and (8) of Theorem 5.1 address the properties of the orbit of a point x under a function f . While a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that $orb(f, x)$ is bounded for all x , this property does not hold for almost every f . We see in Property (8) of Theorem 5.1 that if we have some compact set $C \subseteq \mathbb{R}$, then the set of all f which have bounded orbit for all $x \in C$ is H-ambivalent. However, it is not known if almost every f has the property that there exists a compact set C_f such that $orb(f, x)$ is unbounded for all $x \notin C_f$.

5.1 H-ambivalent Properties in $\mathcal{C}(\mathbb{R})$

In this section we will prove that each of the properties in Theorem 5.1 is

H-ambivalent. The following lemma will be used extensively in this section. It was originally stated in Chapter 2; for ease of reading, we restate it below in the present context.

LEMMA 5.1. *Let $S \subseteq \mathcal{C}(\mathbb{R})$. If for any compact subset K of $\mathcal{C}(\mathbb{R})$, there exists a function $h_K \in \mathcal{C}(\mathbb{R})$ such that $K + h_K \subseteq S$, then S is not Haar null.*

Given a set $S \subseteq \mathcal{C}(\mathbb{R})$ and a compact $K \subseteq \mathcal{C}(\mathbb{R})$, in order to define a function h such that $K + h \subseteq S$, we will use the fact that K is bounded above and below by continuous functions. This fact is stated below as Proposition 5.1, and the proof follows easily from the notion of equicontinuity and the well-known Arzela-Ascoli Theorem. (See Theorem 1.23 of [15], for example.)

Let $F \subseteq \mathcal{C}(\mathbb{R})$. We say that F is *equicontinuous at a point* $x \in \mathbb{R}$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(t)| < \epsilon$ whenever $|x - t| < \delta$ and $f \in F$. F is *uniformly equicontinuous on* $C \subseteq \mathbb{R}$ if for all $\epsilon > 0$ there exists $\delta > 0$ such that, for all $f \in F$ and all $x, x' \in C$ satisfying $|x - x'| < \delta$, we have $|f(x) - f(x')| < \epsilon$. The set F is equicontinuous on \mathbb{R} if F is equicontinuous at every $x \in \mathbb{R}$. By the Arzela-Ascoli Theorem, if K is a compact subset of $\mathcal{C}(\mathbb{R})$, then K is equicontinuous on \mathbb{R} . Moreover, K is uniformly equicontinuous on any compact subset of \mathbb{R} .

PROPOSITION 5.1. *Every compact subset K of $\mathcal{C}(\mathbb{R})$ is bounded above and below by continuous functions.*

Proof. Let K be a compact subset of $\mathcal{C}(\mathbb{R})$. We will show that K is bounded above by a continuous function. By a symmetric argument, it will follow that K is bounded below by a continuous function. Define a mapping $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ by $\alpha(x) = \sup_{\gamma \in K} \{\gamma(x)\}$. To show that α is continuous, fix $a \in \mathbb{R}$ and let $\epsilon > 0$. By the Arzela-Ascoli Theorem, K is equicontinuous at a , so we may choose $\delta > 0$ so

that for all $\gamma \in K$, $|\gamma(x) - \gamma(a)| < \frac{\epsilon}{2}$ whenever $|x - a| < \delta$. The claim is that $|\alpha(x) - \alpha(a)| < \epsilon$ whenever $|x - a| < \delta$. For a contradiction, suppose the contrary. Then there exists $z \in B_\delta(a)$ such that $|\alpha(z) - \alpha(a)| \geq \epsilon$. There are two cases to consider.

CASE 1. $\alpha(z) \geq \alpha(a) + \epsilon$.

Choose $g \in K$ so that $|g(z) - \alpha(z)| < \frac{\epsilon}{4}$. Since $g(a) \leq \alpha(a)$ and $|z - a| < \delta$, we have $g(z) < g(a) + \frac{\epsilon}{2} \leq \alpha(a) + \frac{\epsilon}{2}$. But $\alpha(z) \geq \alpha(a) + \epsilon$, so $g(z) > \alpha(a) + \frac{3\epsilon}{4}$, a contradiction.

CASE 2. $\alpha(z) \leq \alpha(a) - \epsilon$.

Choose $h \in K$ so that $|h(a) - \alpha(a)| < \frac{\epsilon}{4}$. Then, since $h(z) \leq \alpha(z)$, we have that $|h(a) - h(z)| > \frac{3\epsilon}{4}$, a contradiction of the choice of δ .

Thus α is a well-defined continuous function with the property that $\gamma(x) \leq \alpha(x)$ for all $x \in \mathbb{R}$ and $\gamma \in K$. To find a continuous function that is a lower bound for K , define $\beta(x) = \inf_{\gamma \in K} \{\gamma(x)\}$ and proceed analogously.

□

Since Proposition 5.1 will be used often, we will define the following notation before we proceed. Given any compact $K \subseteq \mathcal{C}(\mathbb{R})$, we will use α_K and β_K to denote the continuous functions which bound K above and below, respectively. When no confusion will arise, we will omit the subscripts and simply refer to the functions as α and β .

We will use Proposition 5.2, below, combined with Lemma 5.1 to prove that Properties (1)-(3) of Theorem 5.1 are H-ambivalent. Observe that in Proposition 5.2, we have decomposed the set $\{f \in \mathcal{C}(\mathbb{R}) : f^{-1}(x) \text{ is bounded for all } x\}$ into two disjoint subsets, given by S_5 and S_6 in the proposition. The purpose of this decomposition is to demonstrate that, even when additional restrictions are placed on the set $\{f : f^{-1}(x) \text{ is bounded } \forall x\}$, the resulting set remains non-Haar null.

PROPOSITION 5.2. *Let K be a compact subset of $\mathcal{C}(\mathbb{R})$ and fix $N \in \mathbb{Z}$. Define subsets S_1, \dots, S_6 of $\mathcal{C}(\mathbb{R})$ as follows:*

$$S_1 = \{f : f(\mathbb{R}) \subseteq (-\infty, N]\}$$

$$S_2 = \{f : f(\mathbb{R}) \subseteq [N, \infty)\}$$

$$S_3 = \{f : f^{-1}(x) \text{ is unbounded } \forall x\}$$

$$S_4 = \{f : f(\mathbb{R}) = \mathbb{R}\}$$

$$S_5 = \{f : f^{-1}(x) \text{ is bounded } \forall x \text{ and } f(\mathbb{R}) \neq \mathbb{R}\}$$

$$S_6 = \{f : f^{-1}(x) \text{ is bounded } \forall x \text{ and } f(\mathbb{R}) = \mathbb{R}\}$$

Then, for each $i = 1, \dots, 6$, there exists $h_i \in \mathcal{C}(\mathbb{R})$ such that $K + h_i \subseteq S_i$.

Proof. Let $K \subseteq \mathcal{C}(\mathbb{R})$ be compact and fix $N \in \mathbb{Z}$. Define $h_1 : \mathbb{R} \rightarrow \mathbb{R}$ by

$$h_1(x) = \begin{cases} N - \alpha(x), & \text{if } \alpha(x) > N \\ 0, & \text{if } \alpha(x) \leq N. \end{cases}$$

It is clear that h_1 is a well-defined continuous map. Moreover, if $\gamma \in K$, then $(\gamma + h_1)(x) \leq (\alpha + h_1)(x) \leq N$ for all x , so $K + h_1 \subseteq S_1$. We define h_2 analogously; i.e., if

$$h_2(x) = \begin{cases} 0, & \text{if } \beta(x) > N \\ N - \beta(x), & \text{if } \beta(x) \leq N, \end{cases}$$

then we have $K + h_2 \subseteq S_2$.

Define $h_3 : \mathbb{R} \rightarrow \mathbb{R}$ as follows. For all $n \in \mathbb{N}$, if n is odd, let

$$h_3(n) = -\beta(n) + n,$$

and if n is even, let

$$h_3(n) = -\alpha(n) - n.$$

Extend h_3 continuously to \mathbb{R} . Let $\gamma \in K$ and $x \in \mathbb{R}$. The claim is that $(\gamma + h_3)^{-1}(x)$ is unbounded. Choose an odd positive integer m satisfying $-(m+1) < x < m$. Observe that, for all $k \in \mathbb{N} \cup \{0\}$, we have

$$(\gamma + h_3)(m + 2k) \geq (\beta + h_3)(m + 2k) = m + 2k,$$

and

$$(\gamma + h_3)(m + 2k + 1) \leq (\alpha + h_3)(m + 2k + 1) = -(m + 2k + 1).$$

It follows from the Intermediate Value Theorem that $(\gamma + h_3)^{-1}(x)$ is unbounded, and so $K + h_3 \subseteq S_3$. Now observe that $S_3 \subseteq S_4$. By setting $h_4 \equiv h_3$, we have $K + h_4 \subseteq S_4$.

Let $h_5(x) = -\beta(x) + |x|$. Then $h_5 \in \mathcal{C}(\mathbb{R})$, and $K + h_5$ is bounded below by the function $(\beta + h_5)(x) = |x|$, so clearly $K + h_5 \subseteq S_5$. To construct h_6 , we modify h_5 . Define $h_6 : \mathbb{R} \rightarrow \mathbb{R}$ as

$$h_6(x) = \begin{cases} -\beta(x) + x, & \text{if } x \geq 0, \\ -\beta(0) + \alpha(0) - \alpha(x) + x, & \text{if } x < 0. \end{cases}$$

To see that h_6 is continuous, one need only verify that $\lim_{x \rightarrow 0^+} h_6(x) = \lim_{x \rightarrow 0^-} h_6(x)$. Now for $x \geq 0$, $K + h_6$ is bounded below by the function $(\beta + h_6)(x) = x$, and for $x < 0$, $K + h_6$ is bounded above by the function $(\alpha + h_6)(x) = -\beta(0) + \alpha(0) + x$, and thus $K + h_6 \subseteq S_6$.

□

LEMMA 5.2. *Each of the sets S_1, \dots, S_6 as defined in Proposition 5.2 is H -ambivalent.*

Proof. It follows immediately from Lemma 5.1 and Proposition 5.2 that none of the sets is Haar null. The complements of the sets S_1 and S_2 each contain the non-Haar null set S_4 , so neither S_1 nor S_2 is co-Haar null. The complement of S_3

contains S_5 , so S_3 cannot be co-Haar null. Since $S_1 \subseteq S_4^c$, S_4 is not co-Haar null. Finally, $S_5 \subseteq S_4^c$ and $S_6 \subseteq S_4$, and S_4 is H-ambivalent, so neither S_5 nor S_6 is co-Haar null. \square

Now since S_3 and S_4 are neither co-Haar null nor Haar null, Properties (1) and (2) of Theorem 5.1 are H-ambivalent. Clearly the set $\{f \in \mathcal{C}(\mathbb{R}) : f^{-1}(x) \text{ is bounded } \forall x\}$ is not Haar null, and it cannot be co-Haar null because its complement contains S_3 . Thus Property (3) of Theorem 5.1 is H-ambivalent as well.

In the next part of this section, we will show that Properties (4)-(6) of Theorem 5.1 are H-ambivalent. To do so, we will show that MNP is not co-Haar null and $MTNP$ is not Haar null. Since $MTNP \subseteq MNP$, it will follow that $MTNP$ and MNP are H-ambivalent.

The next lemma, which will be used to show that MNP is not co-Haar null, follows immediately from a theorem of Zajíček, who proved in [44] that for any fixed $a \in (0, 1)$, the set of all $f \in C([0, 1], \mathbb{R})$ such that f has derivative $+\infty$ at a is H-ambivalent. Although the lemma follows from Zajíček's result, we provide a proof below because the techniques used in our proof will be used again later in the chapter.

LEMMA 5.3. *Fix $a \in \mathbb{R}$. Then, the set*

$$\{f \in \mathcal{C}(\mathbb{R}) : f \text{ is increasing at } a\}$$

is not Haar null.

Proof. Without loss of generality, let $a = 0$, and let

$$A = \{f \in \mathcal{C}(\mathbb{R}) : f \text{ is increasing at } 0\}.$$

Let K be a compact subset of $\mathcal{C}(\mathbb{R})$. We will construct $M \in \mathcal{C}(\mathbb{R})$ such that $K + M \subseteq A$ and apply Lemma 5.1. Let $H : \mathcal{C}(\mathbb{R}) \rightarrow \mathcal{C}(\mathbb{R})$ be defined by

$$H(f)(x) = \begin{cases} \max\{0, f(x) - f(0)\}, & x \leq 0, \\ \min\{0, f(x) - f(0)\}, & x > 0. \end{cases}$$

It is clear that H is well-defined; we will show that H is continuous. Let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$. Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$, and choose δ satisfying $0 < 2\delta < \min\{\frac{1}{N}, \epsilon\}$. The claim is that $\rho(H(f), H(g)) < \epsilon$ for all $g \in B_\delta(f)$. Let $g \in B_\delta(f)$. Observe that, for any $x \in [-N, N]$, we have

$$|(f(x) - f(0)) - (g(x) - g(0))| \leq |f(x) - g(x)| + |f(0) - g(0)| < 2\delta.$$

So $\|f - f(0), g - g(0)\|_{[-N, N]} < 2\delta$. Suppose that $x \in [-N, 0]$. We will consider three cases.

CASE 1. $f(x) - f(0), g(x) - g(0) < 0$.

Then $H(f)(x) = H(g)(x) = 0$, so clearly $|H(f)(x) - H(g)(x)| < \epsilon$.

CASE 2. $f(x) - f(0), g(x) - g(0) > 0$.

We have $|H(f)(x) - H(g)(x)| = |(f(x) - f(0)) - (g(x) - g(0))| < 2\delta < \epsilon$.

CASE 3. $f(x) - f(0) \geq 0, g(x) - g(0) \leq 0$, or vice versa.

Then, $|H(f)(x) - H(g)(x)| = |f(x) - f(0)| \leq |(f(x) - f(0)) - (g(x) - g(0))| \leq |f(x) - g(x)| + |f(0) - g(0)| < \epsilon$.

In any case, for all $x \in [-N, 0]$, we have $|H(f)(x) - H(g)(x)| < \epsilon$. By a symmetric argument, for all $x \in [0, N]$, $|H(f)(x) - H(g)(x)| < \epsilon$. Thus

$\|H(f) - H(g)\|_{[-N, N]} < \epsilon$, and since $\frac{1}{N} < \epsilon$, we have that $\rho(H(f), H(g)) < \epsilon$.

Now define $M : \mathbb{R} \rightarrow \mathbb{R}$ by

$$M(x) = \begin{cases} -\sup_{\gamma \in K}(H(\gamma)(x)) + x, & x \leq 0, \\ -\inf_{\gamma \in K}(H(\gamma)(x)) + x, & x > 0. \end{cases}$$

Again, M is clearly a well-defined mapping; we wish to show that M is continuous. Since K is a compact subset of $\mathcal{C}(\mathbb{R})$ and H is continuous, the set

$H(K)$ is a compact subset of $\mathcal{C}(\mathbb{R})$. It follows easily from the fact that $H(K)$ is an equicontinuous family of functions that M is continuous at any $x \neq 0$, so we need only verify that M is continuous at 0. Let $\epsilon > 0$. Since $H(K)$ is equicontinuous at 0, we may choose $0 < \delta < \frac{\epsilon}{2}$ so that $|M(\gamma)(y)| < \frac{\epsilon}{2}$ for all $|y| < \delta$ and $\gamma \in K$. If $y > 0$, we have

$$|M(y) - M(0)| = |M(y)| = -\inf_{\gamma \in K} (H(\gamma)(y)) + y < \frac{\epsilon}{2} + \delta < \epsilon,$$

and if $y \leq 0$, we have

$$|M(y)| = |-\sup_{\gamma \in K} (H(\gamma)(y)) + y| \leq |-\sup_{\gamma \in K} (H(\gamma)(y))| + |y| < \epsilon.$$

Thus M is continuous at 0, and $M \in \mathcal{C}(\mathbb{R})$.

It remains to show that $K + M \subseteq A$. Let $f \in K$. Let $x > 0$. Then $f(x) - f(0) \geq \inf_{\gamma \in K} (H(\gamma)(x))$, so $f(x) - f(0) > \inf_{\gamma \in K} (H(\gamma)(x)) - x = -M(x)$. Since $M(0) = 0$, it follows that $(f + M)(x) > (f + M)(0)$. By a symmetric argument, for $x < 0$ we have $(f + M)(x) < (f + M)(0)$. Hence, $K + M \subseteq A$, and by Lemma 5.1, A is not Haar null. \square

Observe that the definition of the function $M : \mathbb{R} \rightarrow \mathbb{R}$ in the previous proof depended on the choice of K and a . For future reference, we will use $M_{K,a}$ to denote the function defined in the proof above. In particular, observe that given any compact $K \subseteq \mathcal{C}(\mathbb{R})$ and $a \in \mathbb{R}$, every $f \in K + M_{K,a}$ has the property that f is increasing at a . This function $M_{K,a}$ will be used in the proof of Lemma 5.5.

COROLLARY 5.1. *MNP is not co-Haar null.*

Proof. For any fixed $a \in \mathbb{R}$,

$$\{f \in \mathcal{C}(\mathbb{R}) : f \text{ is increasing at } a\} \subseteq MNP^c.$$

Since the set on the left hand side is not Haar null by Lemma 5.3, the set MNP^c is not Haar null. The result follows. \square

Since $MTNP$ is comeager in $\mathcal{C}(\mathbb{R})$, a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the set of non-knot points of f is empty. Is it also the case that ae $f \in \mathcal{C}(\mathbb{R})$ has the property that the set of non-knot points is empty? In Corollary 5.2, we will answer this question in the negative. To show that $MTNP$ is not Haar null using Lemma 5.1, we must show that given any compact $K \subseteq \mathcal{C}(\mathbb{R})$ there exists $h \in \mathcal{C}(\mathbb{R})$ such that $K + h \subseteq MTNP$. The existence of such a function h for each compact K is guaranteed by the following, much stronger, result.

LEMMA 5.4. *Given any compact set $K \subseteq \mathcal{C}(\mathbb{R})$, there exists a comeager subset \mathcal{G} of $\mathcal{C}(\mathbb{R})$ such that $K + h \subseteq MTNP$ for all $h \in \mathcal{G}$.*

Proof. Let K be a compact subset of $\mathcal{C}(\mathbb{R})$. Fix an open interval I . Without loss of generality, let $I = (0, 1)$. For each $n \in \mathbb{N}$, we define sets S_n and G_n as follows. Let S_n be the set of all $f \in \mathcal{C}(\mathbb{R})$ such that, for all $x \in [\frac{1}{n}, 1 - \frac{1}{n}]$, there exist $x_1, x_2 \in (x - \frac{1}{n}, x + \frac{1}{n})$ satisfying $\frac{f(x)-f(x_1)}{x-x_1} < -n$ and $\frac{f(x)-f(x_2)}{x-x_2} > n$. Let G_n be the set of all $h \in \mathcal{C}(\mathbb{R})$ such that $K + h \subseteq S_n$. Observe that, if we show that G_n contains a dense open subset of $\mathcal{C}(\mathbb{R})$, then we are done, by the following argument. Suppose that G_n contains a dense open subset of $\mathcal{C}(\mathbb{R})$. Note that

$$\bigcap_{n=1}^{\infty} S_n = \{f \in \mathcal{C}(\mathbb{R}) : \forall x \in I, x \text{ is a knot point of } f\}.$$

Let $\mathcal{G}_I = \bigcap_{n=1}^{\infty} G_n$. Then \mathcal{G}_I is comeager in $\mathcal{C}(\mathbb{R})$, and $K + h \subseteq \bigcap_{n=1}^{\infty} S_n$ for all $h \in \mathcal{G}_I$. Now let $\mathcal{G} = \bigcap_I \mathcal{G}_I$, where the intersection is taken over all rational open intervals. The set \mathcal{G} is also comeager in $\mathcal{C}(\mathbb{R})$, and for any $h \in \mathcal{G}$, we have

$$K + h \subseteq \{f \in \mathcal{C}(\mathbb{R}) : \forall x \in \mathbb{R}, x \text{ is a knot point of } f\} = MTNP.$$

We wish to prove that G_n contains a dense open subset of $\mathcal{C}(\mathbb{R})$. To this end, let $f \in \mathcal{C}(\mathbb{R})$ and $\epsilon > 0$ be arbitrarily chosen. We will construct $g \in B_\epsilon(f)$ and $\eta > 0$ such that $B_\eta(g) \subseteq G_n$; this technique was used often in the previous

chapter. Since K is uniformly equicontinuous on $[0, 1]$, we may choose $\delta > 0$ so that for all $\gamma \in K$ and $x, x' \in [0, 1]$, we have

$$|x - x'| < \delta \Rightarrow |\gamma(x) - \gamma(x')| < \frac{\epsilon}{16}.$$

Choose $m \in \mathbb{N}$ so that

- $\frac{2}{m} < \min \left\{ \delta, \frac{1}{n} \right\}$,
- $|x - x'| < \frac{2}{m} \Rightarrow |f(x) - f(x')| < \frac{\epsilon}{16}$ for all $x, x' \in [0, 1]$,
- $\frac{m\epsilon}{32} > n$.

Define g as follows. Let $g(x) = f(x)$ for all $x \notin \bigcup_{k=1}^{m-1} \left(\frac{k}{m} - \frac{1}{5m}, \frac{k}{m} + \frac{1}{5m} \right)$. For $k = 1, \dots, m-1$, if k is odd, let

$$g\left(\frac{k}{m}\right) = f\left(\frac{k}{m}\right) + \frac{\epsilon}{4},$$

and if k is even, let

$$g\left(\frac{k}{m}\right) = f\left(\frac{k}{m}\right) - \frac{\epsilon}{4}.$$

To complete the construction of g , on each interval $\left(\frac{k}{m} - \frac{1}{5m}, \frac{k}{m} \right)$, let g be the line passing through the points $\left(\frac{k}{m} - \frac{1}{5m}, g\left(\frac{k}{m} - \frac{1}{5m}\right) \right)$ and $\left(\frac{k}{m}, g\left(\frac{k}{m}\right) \right)$, and on each interval $\left(\frac{k}{m}, \frac{k}{m} + \frac{1}{5m} \right)$, let g be the line passing through the points $\left(\frac{k}{m}, g\left(\frac{k}{m}\right) \right)$ and $\left(\frac{k}{m} + \frac{1}{5m}, g\left(\frac{k}{m} + \frac{1}{5m}\right) \right)$. Now $g \in B_\epsilon(f)$.

Choose $0 < \eta < \frac{\epsilon}{32}$ and let $h \in B_\eta(g)$. The claim is that $K + h \subseteq S_n$. Let $\gamma \in K$ and $x \in \left[\frac{1}{n}, 1 - \frac{1}{n} \right]$. Since $\frac{1}{m} + \frac{1}{5m} < \frac{1}{n}$, we have

$$\frac{k}{m} + \frac{1}{5m} \leq x \leq \frac{k+2}{m} - \frac{1}{5m}$$

for some $1 \leq k \leq m-3$. First assume that k is odd. Let $x_1 = \frac{k}{m}$ and $x_2 = \frac{k+2}{m}$.

Observe that, since $\frac{2}{m} < \frac{1}{n}$, we have $x_1, x_2 \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right)$. Note that since

$$(\gamma + g)(x_1) - (\gamma + g)(x), (\gamma + g)(x_2) - (\gamma + g)(x) > \frac{\epsilon}{8}$$

and $h \in B_\eta(g)$, we have

$$(\gamma + h)(x_1) - (\gamma + h)(x), (\gamma + h)(x_2) - (\gamma + h)(x) > \frac{\epsilon}{16}.$$

Since $x_2 - x < \frac{2}{m}$ and $x_1 - x < -\frac{2}{m}$, we have

$$\begin{aligned} \frac{(\gamma + h)(x_1) - (\gamma + h)(x)}{x_1 - x} &< \frac{\frac{\epsilon}{16}}{-\frac{2}{m}} = -\frac{m\epsilon}{32} < -n \text{ and} \\ \frac{(\gamma + h)(x_2) - (\gamma + h)(x)}{x_2 - x} &> \frac{\frac{\epsilon}{16}}{\frac{2}{m}} = \frac{m\epsilon}{32} > n. \end{aligned}$$

If k is even, set $x_1 = \frac{k+2}{m}$ and $x_2 = \frac{k}{m}$ and proceed in the same way. Thus $K + h \subseteq S_n$. It follows that G_n contains a dense open subset of $\mathcal{C}(\mathbb{R})$, as was to be proven. \square

COROLLARY 5.2. *MTNP is not Haar null.*

Proof. The corollary follows immediately from Lemmas 5.1 and 5.4. \square

By the next lemma, we have that Property (6) of Theorem 5.1 holds on an H-ambivalent subset of $\mathcal{C}(\mathbb{R})$.

LEMMA 5.5. *For a fixed $a \in \mathbb{R}$, let S be the set of all f such that f has derivative $+\infty$ at a and f has a knot point at all $x \neq a$. Then S is H-ambivalent.*

Proof. The fact that S is not co-Haar null follows immediately from Zajíček's result, mentioned before Lemma 5.3 above. Now without loss of generality, let $a = 0$, and let K be a compact subset of $\mathcal{C}(\mathbb{R})$. By Lemma 5.4, we may choose $h_1 \in \mathcal{C}(\mathbb{R})$ satisfying $K + h_1 \subseteq MTNP$. Let $M_{K+h_1,0}$ be the continuous function defined above. Let h_2 be a continuous real-valued function on \mathbb{R} such that $h_2(0) = 0$ and h_2 has the following properties:

- $h_2(x) \leq M_{K+h_1,0}(x)$ for all $x < 0$,
- $h_2(x) \geq M_{K+h_1,0}(x)$ for all $x > 0$,

- h_2 is differentiable at all $x \neq 0$, and
- h_2 has derivative $+\infty$ at 0.

Let $\gamma \in K$. Let $x \neq 0$. Then since $\gamma + h_1$ has a knot point at x and h_2 has finite derivative at x , it must be the case that $\gamma + h_1 + h_2$ has a knot point at x . Now observe that by the definition of $M_{K+h_1,0}$ and choice of h_2 , $\gamma + h_1 + h_2$ is increasing at 0, and since h_2 has derivative $+\infty$ at 0, we have that $\gamma + h_1 + h_2$ has derivative $+\infty$ at 0. So, if we define $h = h_1 + h_2 \in \mathcal{C}(\mathbb{R})$, then $K + h \subseteq S$, and by Lemma 5.1, S is not Haar null. □

In [12], Christensen posed the following question: in a Polish abelian group, is any family of mutually disjoint universally measurable non-Haar null sets at most countable? Dougherty answered this question in the negative in [16], where he showed that in many nonlocally compact abelian Polish groups, there exists an uncountable family of mutually disjoint non-Haar null universally measurable sets. Solecki strengthened this result considerably in [41]; he proved that such a family exists in every nonlocally compact abelian Polish group. In the following example, we give an explicit example of such a family of non-Haar null sets in $\mathcal{C}(\mathbb{R})$.

EXAMPLE 2. *For each $a \in \mathbb{R}$, let S_a be the set of all f such that f has derivative $+\infty$ at a , and f has a knot point at all $x \neq a$. Then for any $a_1 \neq a_2$, the sets S_{a_1} and S_{a_2} are necessarily disjoint. Moreover, each set S_a is non-Haar null by Lemma 5.5, and there are continuum many such sets.*

The final two lemmas in this section are used to show that Properties (7) and (8) of Theorem 5.1 are H-ambivalent.

LEMMA 5.6. *The set*

$$T_1 = \{f \in \mathcal{C}(\mathbb{R}) : \text{orb}(f, x) \text{ is unbounded for all } x \in \mathbb{R}\}$$

is not Haar null.

Proof. Let K be a compact subset of $\mathcal{C}(\mathbb{R})$, and let β be a continuous function which is a lower bound for K . Define a continuous real-valued function h on \mathbb{R} in such a way that $(\beta + h)(x) > x + 1$ for all $x \in \mathbb{R}$. Let $\gamma \in K$ and $x \in \mathbb{R}$ be fixed. Then for all $k \in \mathbb{N}$,

$$(\gamma + h)^k(x) = (\gamma + h)((\gamma + h)^{k-1}(x)) \geq (\beta + h)((\gamma + h)^{k-1}(x)) > (\gamma + h)^{k-1}(x) + 1,$$

and so $\text{orb}(\gamma, x)$ is unbounded. Thus $K + h \subseteq T_1$, and by Lemma 5.1, T_1 is not Haar null. \square

LEMMA 5.7. *Let C be a compact subset of \mathbb{R} . The set*

$$T_2 = \{f \in \mathcal{C}(\mathbb{R}) : \text{orb}(f, x) \text{ is bounded for all } x \in C\}$$

is not Haar null.

Proof. Let C be a compact subset of \mathbb{R} , and choose $M \in \mathbb{N}$ so that $C \subseteq [-M, M]$. Let g be the zero function in $\mathcal{C}(\mathbb{R})$. Choose $\epsilon < \frac{1}{M+1}$. Observe that if $f \in B_\epsilon(g)$, then $\|f - g\|_{[-(M+1), M+1]} < \frac{1}{M+1}$. In particular, for any $f \in B_\epsilon(g)$ and $x \in [-M, M]$, we have $|f(x)| < \frac{1}{M+1}$, and so $\text{orb}(f, x)$ is bounded. Since T_2 contains the open set $B_\epsilon(g)$, T_2 is not Haar null. \square

COROLLARY 5.3. *The sets T_1 and T_2 of Lemmas 5.6 and 5.7 are H -ambivalent.*

Proof. By Lemmas 5.6 and 5.7, neither set is Haar null. The complements of T_1 and T_2 contain the sets T_2 and T_1 , respectively, so neither set is co-Haar null. \square

5.2 Co-Haar Null Subsets of $\mathcal{C}(\mathbb{R})$

In this section we will prove Theorem 5.2. In Lemma 5.8 we will prove that Property (1) holds on a co-Haar null subset of $\mathcal{C}(\mathbb{R})$.

LEMMA 5.8. *The set $MTNI$ is a co-Haar null subset of $\mathcal{C}(\mathbb{R})$.*

Proof. Suppose that $g \notin MTNI$. Then g is of monotonic type on some interval J . Now there exists $m \in \mathbb{R}$ such that the function $g_m(x) = g(x) + mx$ is monotone on J . Thus the function g_m is differentiable at almost every $x \in J$; it follows that g is differentiable at almost every $x \in J$. Then g is an element of the set of all $f \in \mathcal{C}(\mathbb{R})$ such that f has finite derivative at at least one point, a set which is Haar null by Hunt's result in [25]. Since the complement of $MTNI$ is contained in a Haar null set, the set $MTNI$ is co-Haar null. □

The following two corollaries are direct results of Lemma 5.8. The first, Corollary 5.4, gives us Property (2) of Theorem 5.2. The second, Corollary 5.5, gives us Property (3) of Theorem 5.2 and is of interest primarily as a contrast to the property of ae $\phi \in \mathbb{Z}^{\mathbb{Z}}$ that ϕ is injective on a co-finite subset of \mathbb{Z} .

COROLLARY 5.4. *The set MNI is a co-Haar null subset of $\mathcal{C}(\mathbb{R})$.*

Proof. Since the set MNI contains the set $MTNI$, it follows immediately from Lemma 5.8 that MNI is co-Haar null. □

COROLLARY 5.5. *Given any bounded set $F \subseteq \mathbb{R}$, ae f has the property that $f|_{\mathbb{R} \setminus F}$ is not injective.*

Proof. Let $F \subseteq [-M, M]$ for some $M \in \mathbb{N}$. If f is injective on $\mathbb{R} \setminus F$, then f is either strictly increasing or strictly decreasing on $(-\infty, -M) \cup (M, \infty)$, in which case $f \in MNI^c$. □

In an unpublished work (2005), U. Darji proved that in the space of absolutely continuous real-valued functions on $[0, 1]$, almost every f has the property that $f^{-1}(x)$ is perfect for all x in a comeager subset of $f([0, 1])$. We have adapted

the techniques used in that proof to prove that an analogous result, which is Property (4) of Theorem 5.2, is true in this setting. Note that we have chosen to state that $f^{-1}(x)$ is perfect for all x in a comeager subset of \mathbb{R} , rather than for all x in a comeager subset of $f(\mathbb{R})$. This is because we are allowing that the set $f^{-1}(x)$ may be an empty set. Since the set of surjections in $\mathcal{C}(\mathbb{R})$ is H-ambivalent, it may or may not be the case that $f(\mathbb{R}) = \mathbb{R}$, and so the lemma may be rephrased to state that almost every f has the property that $f^{-1}(x)$ is nonempty and perfect for all x in a comeager subset of $f(\mathbb{R})$.

LEMMA 5.9. *As f has the property that $f^{-1}(x)$ is perfect for all x in a comeager subset of \mathbb{R} .*

Proof. We will show that if $f \in \mathcal{C}(\mathbb{R})$ has the property that $f^{-1}(x)$ contains an isolated point for all x in a nonmeager subset of \mathbb{R} , then f is monotone on some interval. Let $f \in \mathcal{C}(\mathbb{R})$ be such that the set

$$C = \{x \in \mathbb{R} : f^{-1}(x) \text{ is not perfect}\}$$

is nonmeager in \mathbb{R} . For each rational open interval I , let

$$C_I = \{x \in \mathbb{R} : |f^{-1}(x) \cap I| = 1\}.$$

Observe that if C_I is meager in \mathbb{R} for all I , then the set $C = \bigcup_I C_I$ is meager in \mathbb{R} as well, contrary to assumption. So there exists at least one such interval I such that C_I is nonmeager in \mathbb{R} ; fix such an interval I . Now C_I is nonmeager in \mathbb{R} . Then there exists an open set $U \subseteq \mathbb{R}$ such that C_I is categorically dense in U ; i.e., given any open subset V of U , the set $C_I \cap V$ is nonmeager in \mathbb{R} . Now it follows from the continuity of f that $U \subseteq f(I)$, and so the set $f^{-1}(U) \cap I$ is nonempty and open in \mathbb{R} . Let J be an interval contained in $f^{-1}(U) \cap I$. If f is not monotone on J , then there exist $z_1 < z_2 < z_3 \in J$ so that either $f(z_1) < f(z_2)$

and $f(z_2) > f(z_3)$, or $f(z_1) > f(z_2)$ and $f(z_2) < f(z_3)$. Without loss of generality, assume the former. Choose a, b such that $\max\{f(z_1), f(z_3)\} < a < b < f(z_2)$. Now $(a, b) \subseteq U$, and by the Intermediate Value Theorem, $|f^{-1}(x) \cap I| \geq 2$ for all $x \in (a, b)$, so $C_I \cap (a, b) = \emptyset$, contradicting that C_I is categorically dense in U . So it must be the case that f is monotone on J .

Now we have that if $f \in \mathcal{C}(\mathbb{R})$ has the property that $f^{-1}(x)$ is not perfect for all x in a nonmeager subset of \mathbb{R} , then $f \notin MNI$. Since MNI^c is Haar null by Corollary 5.4, the result follows. □

LEMMA 5.10. *Fix $M \in \mathbb{N}$. Then*

$$A = \{f \in \mathcal{C}(\mathbb{R}) : f(\mathbb{R}) \subseteq [-M, M]\}$$

is Haar null.

Proof. For each $k \in [0, 1]$, let $\gamma_k \in \mathcal{C}(\mathbb{R})$ be defined by $\gamma_k(x) = kx^3$. For all Borel subsets B of $\mathcal{C}(\mathbb{R})$, define

$$\mu(B) = \lambda(\{k : \gamma_k \in B\}),$$

where λ is the Lebesgue measure. Now μ is a Borel probability measure on $\mathcal{C}(\mathbb{R})$ with $\text{supp}(\mu) = \{\gamma_k : k \in [0, 1]\}$. Let $h \in \mathcal{C}(\mathbb{R})$. The claim is that $\mu(A + h) = 0$. Observe that if $|(A + h) \cap \text{supp}(\mu)| \leq 1$, we are done. Suppose that there exist functions $f_1, f_2 \in A$ such that $f_1 + h = \gamma_{k_1}$ and $f_2 + h = \gamma_{k_2}$, where $k_1, k_2 \in [0, 1]$. Then,

$$f_1 - f_2 = \gamma_{k_1} - \gamma_{k_2}.$$

Now, for all $x \in \mathbb{R}$ we have $-2M \leq (f_1 - f_2)(x) \leq 2M$. But $-2M \leq (k_1 - k_2)x^3 \leq 2M$ for all x if and only if $k_1 = k_2 = 0$. So, $|(A + h) \cap \text{supp}(\mu)| = 1$. Thus A is Haar null. □

COROLLARY 5.6. *Almost every $f \in \mathcal{C}(\mathbb{R})$ has the property that $f(\mathbb{R})$ is unbounded.*

Proof. For each $M \in \mathbb{N}$, let $A_M = \{f \in \mathcal{C}(\mathbb{R}) : f(\mathbb{R}) \subseteq [-M, M]\}$. By Lemma 5.10, A_M is Haar null. Thus the union $\bigcup_{M \in \mathbb{N}} A_M$ is Haar null as well, and the result follows. \square

We conclude this chapter by proving that Property (5) of Theorem 5.2 holds on a co-Haar null subset of $\mathcal{C}(\mathbb{R})$.

LEMMA 5.11. *For fixed $a \in \mathbb{R}$, almost every $f \in \mathcal{C}(\mathbb{R})$ has neither a fixed point nor a point of period 2 at a .*

Proof. Fix $a \in \mathbb{R}$. Let P_1 be the set of all $f \in \mathcal{C}(\mathbb{R})$ such that f has a fixed point at a , and let P_2 be the set of all f which have a point of period 2 at a . For each $k \in [0, 1]$, let $\gamma_k, \psi_k : \mathbb{R} \rightarrow \mathbb{R}$ be given by $\gamma_k \equiv k$ and $\psi_k(x) = k(x - a)$. For all Borel subsets B of $\mathcal{C}(\mathbb{R})$, let

$$\mu_1(B) = \lambda(\{k : \gamma_k \in B\}) \text{ and } \mu_2(B) = \lambda(\{k : \psi_k \in B\}),$$

where λ is the Lebesgue measure. Let $h \in \mathcal{C}(\mathbb{R})$. Fix $f \in P_1$ and $k \in [0, 1]$ so that $f_1 + h = \gamma_k$. Suppose there exist $g \in P_1$ and $l \in [0, 1]$ satisfying $g + h = \gamma_l$. Then

$$a = (f - g)(a) = (\gamma_k - \gamma_l)(a) = k - l,$$

so $l = k - a$. It follows that given any $h \in \mathcal{C}(\mathbb{R})$, $|(P_1 + h) \cap \text{supp}(\mu_1)| \leq 2$, and $\mu(P_1 + h) = 0$.

Now let $h \in \mathcal{C}(\mathbb{R})$, and fix $f \in P_2$ and $k \in [0, 1]$ such that $f + h = \psi_k$, and let $g \in P_2$, $l \in [0, 1]$ satisfy $g + h = \psi_l$. Then we have $f = g + \psi_k - \psi_l$, and in particular, we have

$$a = f^2(a)$$

$$\begin{aligned}
&= (g + \psi_k - \psi_l)^2(a) \\
&= (g + \psi_k - \psi_l)(g(a)) \\
&= a + k(g(a) - a) - l(g(a) - a).
\end{aligned}$$

Then $k(g(a) - a) = l(g(a) - a)$, and since a is a point of period 2 under g , $g(a) - a$ is nonzero, so it must be the case that $k = l$. Thus, where h is an arbitrarily chosen function, we have $|(P_2 + h) \cap \text{supp}(\mu_2)| \leq 1$ and $\mu_2(P_2 + h) = 0$. Therefore P_1 and P_2 are Haar null. □

CHAPTER 6

CONCLUSIONS AND OPEN QUESTIONS

In this paper, we have studied properties of generic and almost every mappings in the nonlocally compact Polish abelian groups $\mathbb{Z}^{\mathbb{Z}}$ and $\mathcal{C}(\mathbb{R})$, and properties of generic mappings in $\mathcal{C}(\mathbb{R}^n)$, $n \geq 1$. In the space $\mathbb{Z}^{\mathbb{Z}}$, we proved that a generic ϕ has the property that $\text{orb}(\phi, n)$ is finite for all $n \in \mathbb{Z}$ and ϕ has infinitely many points of period k for all $k \in \mathbb{N}$, and almost every ϕ has the property that $\text{orb}(\phi, n)$ is finite for only finitely many $n \in \mathbb{Z}$. These results are interesting in light of the well-known fact that a generic $\sigma \in S_{\infty}$ has the property that $\text{orb}(\sigma, n)$ is finite for all $n \in \mathbb{N}$ and σ has infinitely many points of period k for all $k \in \mathbb{N}$, and the more recent result [17] that almost every $\sigma \in S_{\infty}$ has the property that $\text{orb}(\sigma, n)$ is finite for only finitely many $n \in \mathbb{N}$. We also studied other properties of generic and almost every mappings in $\mathbb{Z}^{\mathbb{Z}}$. We found that when some prescribed behavior occurred for a generic ϕ , it was often the case that the opposite behavior occurred for almost every ϕ . (See Theorems 3.3 and 3.4.) While we have obtained complete descriptions of generic and almost every mappings in $\mathbb{Z}^{\mathbb{Z}}$, we have found one opportunity for further study in $\mathbb{Z}^{\mathbb{Z}}$. S. Solecki has proven that in every nonlocally compact Polish abelian group G , there exists an uncountable family of non-Haar null universally measurable mutually disjoint subsets of G [41]. Most of the subsets of $\mathbb{Z}^{\mathbb{Z}}$ that we studied are either Haar null or co-Haar null, although in the final three propositions of the chapter we identified three subsets of $\mathbb{Z}^{\mathbb{Z}}$ which are H-ambivalent. It would be interesting to find an explicit example of uncountably many pairwise disjoint non-Haar null subsets of $\mathbb{Z}^{\mathbb{Z}}$.

After obtaining the results for the space $\mathbb{Z}^{\mathbb{Z}}$ given in Chapter 3, we began to study another group of functions, that of the continuous real-valued mappings on \mathbb{R} . We began by answering relatively simple questions about a generic $f \in \mathcal{C}(\mathbb{R})$ (e.g., is a generic f onto?), and we showed that several of the properties of a generic $\phi \in \mathcal{C}(\mathbb{R})$ are also true of a generic $f \in \mathcal{C}(\mathbb{R})$. A generic f is surjective, has infinitely many points of period k for all $k \in \mathbb{N}$, and has the property that $\text{orb}(f, x)$ is bounded for all $x \in \mathbb{R}$ (Theorem 4.1). We studied the properties of the orbits and ω -limit sets of a generic f in greater detail in Theorem 4.2. In particular, we found that $\omega(f, x)$ is perfect for a generic $x \in \mathbb{R}$, $\text{orb}(f, x)$ is finite for all x in a c -dense subset of \mathbb{R} , $\text{orb}(f, x)$ is infinite and $\omega(f, x)$ is finite for all x in a c -dense subset of \mathbb{R} , and $\omega(f, x)$ is infinite and not perfect for all x in an unbounded subset of \mathbb{R} . However, the size of the set $\{x \in \mathbb{R} : \omega(f, x) \text{ is infinite and not perfect}\}$ for a generic f is not yet known. Of course it is an infinite set, but is it uncountable? Is it c -dense in \mathbb{R} ?

We have also studied properties of generic mappings in $\mathcal{C}(\mathbb{R}^n)$, $n \geq 1$ (Theorem 4.3). We have found that several of the properties which are true of a generic $f \in \mathcal{C}(\mathbb{R})$ are also true of a generic f in the more general setting of $\mathcal{C}(\mathbb{R}^n)$, although the proofs are more difficult and require different techniques. While in $\mathcal{C}(\mathbb{R})$, we determined which types of orbits and ω -limit sets might occur for a generic f , we have not obtained such results for the space $\mathcal{C}(\mathbb{R}^n)$. This is a more difficult problem which will require further study.

In Chapter 5, we studied the properties of almost every $f \in \mathcal{C}(\mathbb{R})$. We found that many of the properties that are true of a generic f hold only on an H-ambivalent subset of $\mathcal{C}(\mathbb{R})$. Thus, although in the space $\mathbb{Z}^{\mathbb{Z}}$, there are several properties P of which one can say “a generic ϕ has property P and almost every ϕ does not have property P ,” we have not yet found such a property for mappings

in $\mathcal{C}(\mathbb{R})$. In particular, this frustrated our attempt to find a decomposition of $\mathcal{C}(\mathbb{R})$ into two “small” sets, one meager and the other Haar null. By Theorem 5.1, the set $\{f \in \mathcal{C}(\mathbb{R}) : \text{orb}(f, x) \text{ is unbounded } \forall x \in \mathbb{R}\}$ is H-ambivalent. However, it might be the case that almost every $f \in \mathcal{C}(\mathbb{R})$ has the property that $\text{orb}(f, x)$ is unbounded for all x in a co-compact set. If this were true, then this result would be analogous to the property of almost every $\phi \in \mathbb{Z}^{\mathbb{Z}}$ that $\text{orb}(\phi, n)$ is unbounded for all n in a co-finite set. Moreover, if this set is a co-Haar null subset of $\mathcal{C}(\mathbb{R})$, then we will have produced a decomposition of $\mathcal{C}(\mathbb{R})$ into two small sets, as the set $\{f \in \mathcal{C}(\mathbb{R}) : \text{orb}(f, x) \text{ is unbounded for some } x \in \mathbb{R}\}$ would be both comeager and Haar null in $\mathcal{C}(\mathbb{R})$.

The set *MTNP* was one of the comeager subsets of $\mathcal{C}(\mathbb{R})$ that we found to be H-ambivalent. Recall that *MTNP* is the set of $f \in \mathcal{C}(\mathbb{R})$ such that every point is a knot point. So, in other words, for a generic f , the set of non-knot points of f is empty, but almost every f may or may not have a knot point. How “big” can the set of non-knot points be for almost every f ? Does almost every f have the property that the set of non-knot points is at most countable?

In Theorem 4.1, we proved that a generic $f \in \mathcal{C}(\mathbb{R})$ has the property that the preimage of every point is uncountable and unbounded. The set of functions in $\mathcal{C}(\mathbb{R})$ which have unbounded preimage at every point was shown to be H-ambivalent in Theorem 5.1, and we proved in Theorem 5.2 that almost every f has the property that $f^{-1}(x)$ is a perfect (possibly empty) set for a generic $x \in \mathbb{R}$. It would be interesting to find out what happens for almost every f in the case that x is not in this comeager subset of \mathbb{R} . Does there exist some x such that $f^{-1}(x)$ contains an isolated point?

As this work progresses, we will continue to extend these results to other non-locally compact abelian Polish groups. The ultimate goal of this research is to

find results which hold for any nonlocally compact abelian Polish group. The proof techniques given in this paper are each very specific to the space being studied, so new proof techniques must be found if we hope to obtain results which apply to any non-locally compact abelian Polish group. This leads to many promising opportunities for further research.

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CURRICULUM VITAE

NAME: Susan L. Calcote White

ADDRESS: 1114 Windsong Way
Louisville, KY 40207

DOB: Lake Charles, Louisiana – July 31, 1971

EDUCATION: B.S., Secondary Mathematics Education
Summa Cum Laude
McNeese State University
1988-1992

M.A., Mathematics
University of Louisville
2001-2003

Ph.D., Mathematics
University of Louisville
2003-2007

HONORS & AWARDS: GEMS Fellowship
University of Louisville, 2005-2006

University Fellowship
University of Louisville, 2003-2005

Graduate Dean's Citation,
December 2003

PROFESSIONAL Member of Mathematical Association of America
ACTIVITIES: 2001-present

Presented a talk at the XII Meeting on Real Analysis
and Measure Theory, a biannual measure theory
conference in Ischia, Italy
July 2006

Presented a talk at the 30th Annual Summer Symposium
in Real Analysis at the
University of North Carolina in Asheville
June 2006

Presented a talk at the 25th Annual Mathematics Symposium
at Western Kentucky University
November 2005