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Extending difference of votes rules on three voting models.

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EXTENDING DIFFERENCE OF VOTES RULES ON THREE VOTING MODELS

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DEDICATION

Dedicated to my daughter, Kameron Morgan King, who has taught me more about myself in three years than I could ever have learned without her.

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ABSTRACT

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In a voting situation where there are only two competing alternatives, simple majority rule outputs the alternatives with the most votes or declares a tie if both alternatives receive the same number of votes. For any nonnegative integer k, the difference of votes rule M_k outputs the alternative that beats the competing alternative by more than k votes. Llamazares (2006) gives a characterization of the difference of votes rules in terms of five axioms. In this thesis, we extend Llamazares' result by completely describing the class of voting rules that satisfy only two out of his five axioms. In addition, we state and prove Llamazares' theorem in voting models where either there is an infinite number of votes or each voter is allowed to express an intensity level for one alternative over the other. Finally, we will use a computer simulation to compare different voting methods to simple majority rule, in order to analyze the probability that the voting rules would output different results.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

In 1952 Kenneth May characterized simple majority rule (SMR) in terms of three axioms [12]. This characterization of (SMR) is known as May's Theorem and it is considered a fundamental result in the area of social choice theory. In fact, one goal of social choice theory is to help provide an in-depth understanding of how various voting functions work. While (SMR) is a popular voting method, often times, (SMR) does not allow for the proper threshold of support before winning, since an alternative only has to win by one vote. More classes of majority rule functions have been introduced in the literature by Fishburn [6], Asan and Sanver [2], Llamazares [7] and Houy [10], and most have been characterized axiomatically. One class of majority rules introduced by Bonifacio Llamazares was coined the "difference of votes" rules [11]. These rules require the difference between the number of votes in favor of any alternative and the number of votes in favor of the other alternative to be greater than some fixed integer k in order to choose that alternative. The axiomatic characterization provided by Llamazares included five axioms. The first goal of this dissertation is to extend the difference of votes rules, by reducing the number of axioms in Llamazares' characterization. After that, we will further extend the difference of votes rules, by characterizing the same rules as Llamazares in an infinite model based on a countably infinite set of voters. Moreover, we will extend some of our results from the finite model to the infinite model. We will then examine a third model, known as the fuzzy voting model, or fuzzy aggregation model. Llamazares and Garcia-Lapresta extended Llamazares' results to this model [8] in 2010. Our goal will be to redefine these rules more intuitively and to characterize these rules through the use of two new lemmas. Lastly, we will analyze a computer simulation of voting rules that was developed to compare various voting rules on different population sizes. This simulation will compare some of the voting methods discussed in this paper, as well as popular voting methods in the political world.

When extending Llamazares' difference of votes rules in the finite domain, the first idea was to remove neutrality. The axiom of neutrality captures the idea that a voting rule should not depend on the labeling of the two alternatives. While it is commonly understood that the function that always outputs the same alternative is a notable example of a non-neutral rule that satisfied many of Llamazares' axioms, but not neutrality, we find that the constant rule is actually part of a whole class of functions. This class of functions will be what we refer to as the difference of votes rules. So our version of difference of votes rules is more general than Llamazares' version. The general version of the difference of votes rules will require the number of votes in favor of one alternative and the number of votes in favor of the other alternative to be greater than some fixed integer k in order to be chosen, and require the number of votes in favor of the other alternative and the number in favor of the first alternative to be greater than some fixed integer l in order to be chosen. We will explore the unique relationship between the integers k and l, as well as completely characterize these rules in this model. The next step in this model is to remove the axiom of anonymity. The axiom of anonymity captures the idea that a voting rule should not depend on the labeling of the voters. Since we do still require the function to be canceallive, we will show that anonymity is implied on a large subset of the domain. One of the main results is a complete characterization of the class of aggregation functions with only two axioms: cancellation and monotonicity.

The infinite model has been less explored than the finite model and brings with it new challenges. In 2006, Mark Fey [5] extended May's Theorem to an infinite model where he labeled the alternatives differently from May. In this thesis, we shift the labeling of the alternatives to match May's labeling, and we then extend many of the results of Llamazares as well as our own results given in the finite model. We are able to define both types of difference of votes rules in the infinite model, and extend Llamazares' result, as well as part of our finite model result in the infinite model.

The third voting model is the fuzzy aggregation model and it is based on fuzzy set theory [13]. This domain has been defined and studied by researchers in the area of fuzzy set theory [14], as well as those in the area of social choice theory [7]. The model was first introduced by H. Nurmi in 1981, when examining the idea of preference intensities [13], where the votes can give a partial favor of one alternative over the other without expressing full favor. The most notable results, for our purposes, in the fuzzy model have been done by Garcia-Lapresta and Llamazares [7], [8]. They characterized a class of majority type rules in the fuzzy model as well as a fuzzy version of the difference of votes rules. We introduce and prove two lemmas that help in the characterization of the fuzzy difference of votes rules given by Garcia-Lapresta and Llamazares [8]. Our new proof of the Garcia-Lapresta and Llamazares result will hopefully lead to other results in the fuzzy model.

The final goal of this dissertation is to analyze the outputs of a computer program that was written to help compare and contrast different voting rules. This program generates random votes for N voters and runs the same set of votes through two different voting methods and then gives a comparison of the outputs. Each run of the program generates 100 comparisons and then analyzes the actual probability of agreement of the two voting methods. We first compare simple majority rule to a difference of votes rule and find that the probability of agreement is much less than expected. Next, we use the program to compare simple majority rule to that of the Electoral College, a voting method used in the United States to elect the President. These results were surprising, as well, in that we had three hypotheses and two of them seemed to be very much not the case. The third hypothesis is that as N grew, the disagreement between (SMR) and the Electoral College would grow as well. This hypothesis was not thrown out by our data, but was only confirmed by a small margin. More tests will need to be run to solidify this result.

A precise description of the finite, infinite, and fuzzy voting models is given in the next chapter. Since each model has a different domain, the sets of voters, and could have different possible outputs, making comparisons among these three models is not always straightforward.

CHAPTER 2
VOTING MODELS

A FINITE VOTING MODEL

The focus of this chapter is to introduce three voting models based on a slate of two candidates or alternatives. For example, a group of graduate students want to have a party where they will serve either wine or coke, but not both. Each graduate student votes for either wine, coke, or neither. By voting neither, the student abstains, maybe because they wanted to serve beer, but it was not put as an option, or they simply do not care what is served. The collection of votes is aggregated in some way and a winner is determined. This type of voting situation has been studied extensively in the social choice literature, for example Part I of Fishburn's book [6]. In our first two models, the alternatives are denoted by the integers 1 and -1. In the example above, we could denote a vote for coke with a 1 and a vote for wine with a -1. The integer 0 will be used to represent an abstention vote.

Definition 2.1. *Any function of the form*

$$f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$$

where $n \geq 2$ is an integer is called a finite aggregation function*, or finite aggregation rule. An element in the domain, denoted by*

$$\pi = (x_1, x_2, \ldots, x_n),$$

is called a profile.

A finite aggregation function has three possible outputs, alternative 1, alternative -1 or a tie, denoted by 0. In our example, an output of 1 means that coke is chosen, an output of -1 means that wine is chosen and an output of 0 means we choose neither. What happens in the event of a tie can vary. Often a vote is recast or a person in leadership makes the final decision. From this point forward, we will refer to functions in this model as aggregation functions, dropping the "finite" adjective.

Here is a simple example of an aggregation function to help the reader understand Definition 2.1 more clearly.

Example 2.1. *Define* $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ *as follows. For any profile* $\pi = (x_1, x_2, ..., x_n)$,

$$f(\pi) = x_1 \cdot x_2 \cdot ... \cdot x_n$$

where the output of f *is determined by the multiplication of each entry of* π, *or each voter's choice.*

We will define various sets of the voting profile in order to help with analysis of the finite aggregation functions we will study.

Definition 2.2. *The set of voters in a profile* $\pi = (x_1, x_2, ..., x_n)$, $n \geq 2$, *will be denoted by* $\mathcal{N}$. *So* $\mathcal{N}$ *is the integers in the interval* $[1, n]$. *The set of voters who voted for each alternative will be denoted as follows:*

$$N_+(\pi) = \{i \in \mathcal{N} : x_i = 1\},\ N_-(\pi) = \{i \in \mathcal{N} : x_i = -1\},\ \textit{and}\ N_0(\pi) = \{i \in \mathcal{N} : x_i = 0\}.$$

Finally, $n_+(\pi) = |N_+(\pi)|$, $n_-(\pi) = |N_-(\pi)|$, *and* $n_0(\pi) = |N_0(\pi)|$.

Notice that the aggregation function f given in Example 2.1, can be defined by $f(\pi) = 0$ if $n_0(\pi) > 0$ and $f(\pi) = -1^{n_-(\pi)}$, if $n_0(\pi) = 0$.

Definition 2.3. *The notation E^- represents the profile where all voters choose -1, E^+ represents the profile where all voters choose 1, and $\vec{0}$ represents the profile where all voters abstain.*

The next step is to introduce some properties that a given aggregation function may or may not satisfy.

Definition 2.4. *An aggregation function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ satisfies* Anonymity*, or is anonymous, if for all profiles π and permutations σ of the set of voters $\mathcal{N}$, $f(\pi_\sigma) = f(\pi)$ where $\pi_\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$.*

That is to say, if a function is anonymous, the order of the voters does not effect the outcome.

Definition 2.5. *An aggregation function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ satisfies* Neutrality*, or is neutral, if $f(-\pi) = -f(\pi)$ for all profiles $\pi = (x_1, x_2, ..., x_n)$, where $-\pi = (-x_1, -x_2, ..., -x_n)$.*

If a function in neutral, then the labeling of the alternatives does not effect the outcome of the voting.

In order to look at the next two axioms, we need to define profile comparisons. For any two profiles $\pi = (x_1, x_2, ..., x_n)$ and $\pi' = (x'_1, x'_2, ..., x'_n)$, $\pi' \geq \pi$ if $x'_i \geq x_i$ for all integers $i \in \mathcal{N}$.

Definition 2.6. *An aggregation function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ satisfies* Monotonicity*, or is monotone, if for all profiles π, π', if $\pi' \geq \pi$ then $f(\pi') \geq f(\pi)$.*

If a function is monotone, then increasing favor for any alternative will not negatively effect that alternative's chance of being chosen.

Definition 2.7. *An aggregation function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ satisfies* Strict Monotonicity*, or is strictly monotone, if f is monotone, and furthermore, if*

$\pi' \geq \pi$ *and* $\pi' \neq \pi$, *then*

$$f(\pi) = 0 \Rightarrow f(\pi') = 1 \text{ and } f(\pi') = 0 \Rightarrow f(\pi) = -1.$$

The strict monotonicity condition adds a tie breaking condition to the basic monotonicity axiom. This tie breaking axiom means that if a profile generates an output of 0, a tie, and then one voter changes their choice, then that choice will be the output.

In 1952, Kenneth May completely characterized the class of aggregation functions that satisfy Strict Monotonicity, Neutrality, and Anonymity. It turns out that this class only contains one function called the simple majority rule function (SMR) [12]. The definition of (SMR) is given below:

Definition 2.8. *An aggregation rule* $f_m : \{-1,0,1\}^n \to \{-1,0,1\}$ *is simple majority rule (SMR), if for all* $\pi \in \{-1,0,1\}^n$

$$f_m(\pi) = \begin{cases} -1, & n_-(\pi) > n_+(\pi), \\ 1, & n_+(\pi) > n_-(\pi), \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

In 2006, Bonifacio Llamazares extended May's work by characterizing a wider class of majority type rules [11]. He introduced a new axiom called cancellation, which he defined as follows:

Definition 2.9. *An aggregation function* $f : \{-1,0,1\}^n \to \{-1,0,1\}$ *satisfies* Cancellation, *or is cancellative, if for any profile* $\pi = (x_1, x_2, ..., x_n)$ *such that there exists* $i, j \in \mathcal{N}$ *where* $x_i = 1$ *and* $x_j = -1$, *and there exists another profile* $\pi' = (x'_1, x'_2, ...x'_n)$ *such that* $x'_k = x_k$ *for all* $k \neq i, j$ *and* $x'_i = x'_j = 0$ *then* $f(\pi) = f(\pi')$.

The cancellation axiom indicates that a vote of -1 will "cancel out" a vote of 1.

Llamazares completely characterized the class of aggregation functions which satisfy Anonymity, Neutrality, Cancellation, and Monotonicity [11]. He called the aggregation functions belonging to this class as "difference of votes" rules since the outputs determined by $n_+(\pi) - n_-(\pi)$ or $n_-(\pi) - n_+(\pi)$. They are defined below.

Definition 2.10. *An aggregation rule* $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ *is said to be an* M_k *rule, if there exists an integer* $k \in [0, n]$, *such that for all* $\pi \in \{-1, 0, 1\}^n$,

$$M_k(\pi) = \begin{cases} 1, & n_+(\pi) > n_-(\pi) + k, \\ -1, & n_-(\pi) > n_+(\pi) + k, \\ 0, & \textit{otherwise.} \end{cases} \tag{2.2}$$

In order to make the understanding completely clear, Llamazares' Theorem is stated below.

Theorem 2.1. *If an aggregation function* $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ *satisfies Anonymity, Neutrality, Cancellation, and Monotonicity, then there exists an integer* k *in the interval* $[0, n]$ *such that* $f(\pi) = M_k(\pi)$ *for all* π.

The M_k rules are restricted to values of k in the interval $[0, n]$. If k is negative, then the resulting function is not well-defined. Also, k is defined to be an integer, since for any real number j and integer k such that $k \leq j < k + 1$, $M_j = M_k$. Also for any value $k > n$, $M_k = M_n$. Thus to be well-defined and avoid repetition, we restrict k to the integers in $[0, n]$. Notice that if $k = 0$, then $M_0 = f_m$.

AN INFINITE VOTING MODEL

We are now ready to introduce our second voting model. In this model, we consider a countably infinite set of voters. This type of infinite voting model has been studied by others in the area of voting theory such as Mark Fey [5]. The set of voters $\mathcal{N}$ will be of replaced by $\mathbb{N}$. Here $\mathbb{N}$ is defined to be the set of natural numbers, $\{1,2,3,...\}$. We will define an aggregation function on a countably infinite set of voters. We are now ready to extend Definition 2.1 to the countably infinite set of voters.

Definition 2.11. *An infinite aggregation function is any function of the form* $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$. *The function* f *will take profiles* $\pi = (x_1, x_2, ...) \in \{-1,0,1\}^{\mathbb{N}}$, *where* $1, -1$ *are alternatives and* 0 *represents abstention or indifference in the domain and a tie in the output.*

We now extend the notation given in Definition 2.2 the set of countable voters.

Definition 2.12. *The set of voters in a profile* $\pi = (x_1, x_2, ...)$ *who voted for each alternative will be denoted as follows:*

$$N_+(\pi) = \{i \in \mathbb{N} : x_i = 1\},\ N_-(\pi) = \{i \in \mathbb{N} : x_i = -1\},\ \text{and } N_0(\pi) = \{i \in \mathbb{N} : x_i = 0\}.$$

Finally, $n_+(\pi) = |N_+(\pi)|$, $n_-(\pi) = |N_-(\pi)|$, *and* $n_0(\pi) = |N_0(\pi)|$. *We will write* $n_*(\pi) = \infty$, *if* $n_*(\pi)$ *is not finite, and follow the convention that* $\infty + k = \infty$ *for any finite number* k.

Here is an example of an infinite aggregation function on a countably infinite set of voters.

Example 2.2. *An infinite aggregation rule* $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *will be called an* M_∞ *rule if for all profiles* π, $f(\pi) = M_\infty(\pi)$ *where*

$$M_\infty(\pi) = \begin{cases} 1, & n_+(\pi) = \infty \text{ and } n_-(\pi) < \infty, \\ -1, & n_-(\pi) = \infty \text{ and } n_+(\pi) < \infty, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Next, we need to extend our axioms to the countably infinite set of voters. While some of these axioms extend naturally, others require a more careful approach. Monotonicity, Neutrality, and Cancellation extend quite naturally.

Definition 2.13. *An infinite aggregation function* $f : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ *is said to be* Neutral*, or satisfies neutrality, if* $f(-\pi) = -f(\pi)$ *for all* $\pi = (x_1, x_2, ...) \in \{-1, 0, 1\}^{\mathbb{N}}$ *where* $-\pi = (-x_1, -x_2, ..)$.

In the countably infinite set of voters, for profiles $\pi = (x_1, x_2, ...), \pi' = (x'_1, x'_2, ...)$, we write $\pi \geq \pi'$ if $x_i \geq x'_i$, for all $i \in \mathbb{N}$.

Definition 2.14. *An infinite aggregation function* $f : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ *is said to be* Monotone*, or satisfies monotonicity, if for* $\pi, \rho \in \{-1, 0, 1\}^{\mathbb{N}}$*,* $\pi \geq \rho$ *implies* $f(\pi) \geq f(\rho)$.

Since cancellation only applies to a finite set within the profile, it extends easily as well.

Definition 2.15. *An infinite aggregation function* $f : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ *is said to be* Cancellative*, or satisfies cancellation, if for* $\pi = (x_1, x_2, ...), \pi' = (x'_1, x'_2, ...) \in \{-1, 0, 1\}^{\mathbb{N}}$*, such that* $x_k = x'_k$ *for all* $k \neq i, j$ *and* $x_i = 1$*,* $x_j = -1$ *and* $x'_i = x'_j = 0$ *implies that* $f(\pi) = f(\pi')$.

The above three axioms carried over nicely to the countably infinite set from the finite set of voters; however, there are a number of different types of anonymity to consider in our second voting model. We will only cover two of them, as they are the most intuitive.

Definition 2.16. *An infinite aggregation function* $f : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ *is said to be* Finite Anonymous*, or satisfies finite anonymity, if for any permutation* δ *of* $\mathbb{N}$ *where* $|\{\ i : \delta(i) \neq i\}| < \infty$*, then* $f(\pi_\delta) = f(\pi)$*. We will denote the set of all such permutations by* Σ.

Definition 2.17. *An infinite aggregation function* $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *is said to be* Strongly Anonymous,*or satisfies strong anonymity, if for any permutation* σ *of* $\mathbb{N}$, $f(\pi_\sigma) = f(\pi)$.

When looking at infinite aggregation functions, we will also examine a new axiom that is helpful in understanding these functions. To state this condition, we say that a subset N of $\mathbb{N}$ is co-finite if $N^c = \mathbb{N}\backslash N$ is finite. For example, $N = \{x \in \mathbb{N} : x \geq 100\}$ is co-finite, since $\mathbb{N}\backslash N = \{x \in \mathbb{N} : x < 100\}$ is finite. However, the set $2\mathbb{N} = \{x \in \mathbb{N} : 2 \text{ divides } x\}$ is neither finite nor co-finite since $\mathbb{N}\backslash 2\mathbb{N}$, the set of all odd natural numbers, is also infinite.

Definition 2.18. *An aggregation function* f *is said to be* Zero Co-finite *if there exists a profile* $\pi \in \{-1,0,1\}^{\mathbb{N}}$, *such that* $N_0(\pi)$ *is co-finite and* $f(\pi) \neq 0$.

The zero co-finite axiom is not met by the function M_∞ defined in Example 2.2, where if only a finite set of the countably infinite subset of $\mathbb{N}$ do not abstain, no winner is chosen. There could be many instances where one would want to choose a winner even if only finitely many voters cast a vote.

FUZZY VOTING MODEL

Our third voting model involves fuzzy aggregation functions. These functions are based on two alternatives, x and y, and each voter can choose any number in the range $[0,1]$, based on how strongly they feel towards one alternative or another. A vote of 0.5 is complete indifference or abstention, a vote of 1 is complete favor for y over x, and a vote of 0 is complete favor for x over y. A vote d_i such that $0.5 < d_i < 1$ shows a level of favor towards y over x, with respect to an abstention. A vote of 0.75 shows a 50% favor towards y over x. A graduate student in our original example, with x = *coke*, y = *wine* might vote 0.75 if they like wine, but not coke, but are not necessarily super happy with wine as a choice, because they would rather have beer. The output options are either 0, 1 or 0.5, indicating a choice of x, y, or a tie, respectively. In this model, we will assume the set of voters is $\mathcal{N}_m = \{1, 2, ..., m\}$, where $m \geq 2$ is an integer. It should be observed that 0 in the first two models corresponds to $\frac{1}{2}$ in this model.

Definition 2.19. *A* fuzzy aggregation function*, or fuzzy decision rule, is a mapping* $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ *that assigns* $0, 0.5,$ *or* 1 *to each profile* $\pi = (d_1, ..., d_m) \in [0,1]^m$*, depending on whether* 0 *defeats* 1*,* 0 *and* 1 *tie, or* 1 *defeats* 0*, respectively.*

In order to help better understand these rules, we will look at the fuzzy version of simple majority rule defined in Definition 2.8.

Example 2.3. *An aggregation rule* $\widetilde{f_m} : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ *is fuzzy simple majority rule, if for all* $\pi \in [0,1]^m$

$$\widetilde{f_m}(\pi) = \begin{cases} 1, & \text{if } \sum_{i=1}^{m} d_i > \frac{1}{2}m, \\ \frac{1}{2}, & \text{if } \sum_{i=1}^{m} d_i = \frac{1}{2}m, \\ 0, & \text{if } \sum_{i=1}^{m} d_i < \frac{1}{2}m. \end{cases} \tag{2.4}$$

We can extend our axioms, but they will look slightly different in this domain than in the other two models. Our axioms are the same as those used by Garcia-Lapresta and Llamazares in their 2010 paper [8]. Anonymity and monotonicity look similar to the versions of anonymity and monotonicity given in the previous models.

Definition 2.20. *A fuzzy aggregation function* $F : [0,1]^m \to \{0, 0.5, 1\}$ *is* Anonymous *if for all permutations of* $\mathcal{N}_m$ *and all profiles* $\pi = (d_1, ..., d_m)$*,* $F(d_{\sigma(1)}, ..., d_{\sigma(m)}) = F(d_1, ..., d_m)$*.*

Now, just as in our previous two cases, for any $\pi = (d_1, ..., d_m)$ and $\pi' = (d'_1, ..., d'_m)$, $\pi \geq \pi'$ if $d_i \geq d'_i$ for all $i \in \mathcal{N}_m$.

Definition 2.21. *A fuzzy aggregation function* $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ *is* Monotone *if for all profiles* π *and* π', $\pi' \geq \pi \Rightarrow F(\pi') \geq F(\pi)$.

In order to describe the negative of a profile, we will first define a function on $[0,1]$.

Definition 2.22. *For any* $d \in [0,1]$, *define* $N(d) = 1 - d$. *In other words,* $d + N(d) = 1$ *for all* $d \in [0,1]$.

Now, we can define neutrality in the following way. Since in our previous models, a vote of 1 was "neutralized" by negating it to -1, since 1 and -1 average to 0, an abstention. In this model, a vote of d is "neutralized' by $N(d)$, since they average to $\frac{1}{2}$, an abstention in this context.

Definition 2.23. *A fuzzy aggregation function* $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ *is* Neutral *if for all profiles* $(d_1, ...d_m)$, $F(N(d_1), ..., N(d_m)) = N(F(d_1, ..., d_m))$.

When we define cancellation, it will look the most different, since the alternatives have different labeling. However, the idea is the same.

Definition 2.24. *A fuzzy aggregation function* $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ *is* Cancellative *if for all pairs of profiles* $(d_1, ...d_m), (d'_1, ..., d'_m) \in [0,1]^m$, *such that* $d'_i = d_i + \epsilon$ *and* $d'_j = d_j - \epsilon$, *for some* $i, j \in \{1, ..., m\}$ *and* $\epsilon > 0$, *and* $d'_k = d_k$ *for all* $k \neq i, j$, *it holds that* $F(d_1, ...d_m) = F(d'_1, ..., d'_m)$.

We can see, with little proof, that as with the finite aggregation functions, the fuzzy simple majority rule, $\widetilde{f_m}$, is anonymous, neutral, monotone and cancellative. Garcia-Lapresta and Llamazares showed that for fuzzy decisions rules cancellation completely implies anonymity [8]. We also need to introduce a new axiom for these rules called Pareto.

Definition 2.25. *A fuzzy aggregation function* $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ *is* Pareto *if*

$$F(1, ..., 1) = 1 \text{ and } F(0, ..., 0) = 0.$$

It is not hard to notice that $\widetilde{f_m}$ is Pareto as well. Garcia-Lapresta and Llamazares also defined fuzzy difference of votes rules. However, there were two different variations of these rules [8]. They are listed below.

Definition 2.26. *Given a real number* $k \in [0, m)$*, the* fuzzy $\widetilde{M_k}$ majority *is the fuzzy decision rule defined by:*

$$\widetilde{M_k}(\pi) = \begin{cases} 1, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i > \frac{1}{2} + \frac{k}{2m}, \\ \frac{1}{2}, & \text{if } |\frac{1}{m}\sum_{i=1}^{m} d_i - \frac{1}{2}| \le \frac{k}{2m}, \\ 0, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i < \frac{1}{2} - \frac{k}{2m}. \end{cases} \tag{2.5}$$

Definition 2.27. *Given a real number* $k \in (0, m]$*, the* fuzzy $\widetilde{M'_k}$ majority *is the fuzzy decision rule defined by:*

$$\widetilde{M'_k}(\pi) = \begin{cases} 1, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i \ge \frac{1}{2} + \frac{k}{2m}, \\ \frac{1}{2}, & \text{if } |\frac{1}{m}\sum_{i=1}^{m} d_i - \frac{1}{2}| < \frac{k}{2m}, \\ 0, & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i \le \frac{1}{2} - \frac{k}{2m}. \end{cases} \tag{2.6}$$

Though these fuzzy aggregation rules were defined in this way by Garcia-Lapresta and Llamazares, we will multiply each line through by m, and rewrite them to make them more clear. The rewritten rules are listed below.

$$\widetilde{M_k}(\pi) = \begin{cases} 1, & \text{if } \sum_{i=1}^{m} d_i > \frac{1}{2}(m+k), \\ \frac{1}{2}, & \text{if } |\sum_{i=1}^{m} d_i - \frac{m}{2}| \le \frac{1}{2}k, \\ 0, & \text{if } \sum_{i=1}^{m} d_i < \frac{1}{2}(m-k). \end{cases} \tag{2.7}$$

$$\widetilde{M}'_k(\pi) = \begin{cases} 1, & \text{if } \sum_{i=1}^{m} d_i \geq \frac{1}{2}(m+k), \\ \frac{1}{2}, & \text{if } |\sum_{i=1}^{m} d_i - \frac{m}{2}| < \frac{1}{2}k, \\ 0, & \text{if } \sum_{i=1}^{m} d_i \leq \frac{1}{2}(m-k). \end{cases} \tag{2.8}$$

While $\widetilde{M}'_k$ and $\widetilde{M}_k$ look similar, since the profiles can contain votes that are any real number, and not just integers, the outputs could vary even with the same profile. For $m = 3$ and $k = 1$, where $\pi = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, $\widetilde{M}_k(\pi) = \frac{1}{2}$ and $\widetilde{M}'_k(\pi) = 1$. Notice that $\widetilde{M}_0 = \widetilde{f_m}$, the fuzzy simple majority rule.

Now that all of our voting models have been introduced, the next few chapters will discuss the M_k rules as they behave in each domain. We will highlight some previous findings, as well as introduce new results in each model that have allowed us to extend the definition of difference of votes rules in various ways.

CHAPTER 3
DIFFERENCE OF VOTES RULES ON A FINITE VOTING MODEL

The purpose of this chapter is to look further into the class of difference of votes rules for finite aggregation rules. We will introduce our version of the class of difference of votes rules. This class of functions contains both neutral and non-neutral aggregation functions. We will prove that this class of functions completely characterizes the set of functions that are cancellative, anonymous, and monotone. We will then further extend these results, removing anonymity and looking at aggregations functions that satisfy cancellation and monotonicity and give a complete characterization of such rules.

First, we will introduce the function that we define as the difference of votes rules. We will use the symbol $\mathbb{Z}$ to denote the set of integers. That is $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$.

Definition 3.1. *An aggregation rule $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ is said to be an $M_{k,l}$ rule, if there exists $k, l \in \mathbb{Z} \cap [-n-1, n]$, such that $k+l \geq -1$ for all $\pi \in \{-1, 0, 1\}^n$,*

$$M_{k,l}(\pi) = \begin{cases} 1 & n_+(\pi) > n_-(\pi) + k, \\ -1 & n_-(\pi) > n_+(\pi) + l, \\ 0 & \textit{otherwise}. \end{cases} \tag{3.1}$$

In the definition of $M_{k,l}$ rules, there were restrictions listed on k and l. In order to understand the necessity of those restrictions, we will look at a lemma that explains when this class of functions is well-defined.

Lemma 3.1. *If $k, l \in \mathbb{Z} \cap [-n-1, n]$ then $M_{k,l}$ is well defined if and only if $k+l \geq -1$.*

Proof. First, assume $k + l \leq -2$, and $M_{k,l}$ is well-defined. If $k < 0$ and $l < 0$, then it follows for $\rho = (0, 0, ...0)$ that $n_+(\rho) > n_-(\rho) + k$ and $n_-(\rho) > n_+(\rho) + l$, so $M_{k,l}$ would not be well defined. Therefore, $\max\{k, l\} \geq 0$. Assume without loss of generality $k \geq 0$. Since $l \geq -n-1$ and $k + l \leq -2$, it follows that $k \leq -2 - l$, so $k \leq n-1$ or $k < n$. Let π be a profile such that $n_+(\pi) = k+1$ and $n_-(\pi) = 0$. Observe that $n_+(\pi) > n_-(\pi) + k$ and $n_-(\pi) > n_+(\pi) + l$. Therefore, for that same π both $M_{k,l}(\pi) = 1$ and $M_{k,l}(\pi) = -1$ holds true. This is a contradiction, so $k + l > -2$, hence $k + l \geq -1$.

Now, assume that for some k, l, $M_{k,l}$ is not well-defined. Then there exists π such that both $n_+(\pi) \geq n_-(\pi) + k + 1$ and $n_-(\pi) \geq n_+(\pi) + l + 1$. Thus $k \leq n_+(\pi) - n_-(\pi) - 1$ and $l \leq n_-(\pi) - n_+(\pi) - 1$. Hence $k + l \leq -2$. □

If $k = l$, then equation 3.1 is the same equation described by Llamazares [11], and can be found in Definition 2.1. Since we allow $k \neq l$, this is a broader class of functions. If $k = n$ and $l = -n-1$, then we get the constant function $M_{n,-n-1}(\pi) = -1$ for all π. If $k = -n-1$ and $l = n$, then $M_{-n-1,n}(\pi) = 1$ for all π. Thus, while the difference of votes rules defined by Llamazares are neutral aggregation functions, the class of $M_{k,l}$ contains some non-neutral aggregation rules. In fact, we can show that an $M_{k,l}$ rule is only neutral if $k = l$.

Lemma 3.2. *The function $M_{k,l}$ is neutral if and only if $k = l$.*

Proof. If $k = l$, then since $k + l > -2$, $2k > -1$, and $k \geq 0$. Therefore, by definition, we have an M_k rule which is neutral by Theorem 2.1.

Now, assume that $M_{k,l}$ is neutral. Then anonymity and neutrality allow us to see that $M_{k,l}(0, 0, 0, ...) = 0$, so $k \geq 0$ and $l \geq 0$.

If $k < n$, then consider a profile π such that $n_+(\pi) = k+1$ and $n_-(\pi) = 0$, then $M_{k,l}(\pi) = 1$. By neutrality, $M_{k,l}(-\pi) = -1$, where $n_-(-\pi) = k+1$ and $n_+(-\pi) = 0$.

Therefore, $k + 1 > n_+(-\pi) + l = l$. Thus $k \geq l$. Similarly, if $l < n$ consider ρ such that $n_-(\rho) = l + 1$ and $n_+(\rho) = 0$, then $M_{k,l}(\rho) = -1$. Considering $-\rho$, $n_+(-\rho) = l + 1$ and $n_-(-\rho) = 0$, and by neutrality $M_{k,l}(-\rho) = 1$. Therefore, $l + 1 > n_-(-\rho) + k = k$, so $l \geq k$. Thus, it follows that $k = l$. □

We will look at an example to help understand the motivation and workings of this class of functions.

EXAMPLE 3.1. *Suppose there is a committee in the Senate of ten Senators, and in order to overturn an old decision, they want a super-majority. It is common that Senators are out of town from time to time. To insure that such decisions are not made when only a few Senators are there, they develop an aggregation rule that in order to overturn a previous decision the difference between those in favor and those against must be greater than 3. Formally, if* 1 *represents a decision being overturned and* -1 *represents keeping the old decision, then* 1 *wins if* $n_+(\pi) > n_-(\pi) + 3$. *The senators' aggregation rule is the* $M_{k,l}$ *rule with* $n = 10$ *and defined by:*

$$M_{3,-4}(\pi) = \begin{cases} 1 & n_+(\pi) > n_-(\pi) + 3, \\ -1 & n_-(\pi) > n_+(\pi) - 4, \\ 0 & \textit{otherwise.} \end{cases} \tag{3.2}$$

When 6 *Senators vote, the only way to get an output of* 1 *and overturn the ruling would be to have* 5 *or* 6 *votes in favor. Since* $n_+(\pi) = 5 > n_-(\pi) + 3$ *and* $n_+(\pi) = 6 > n_-(\pi) + 3$. *Otherwise,* 2 *or more "no"'s will result in not changing the rule, as* $n_-(\pi) = 2 > n_+(\pi) - 4$. *With* 4 *or* 5 *Senators voting, it would require all present to vote in favor of overturning the rule in order to do so. If* 3 *or fewer Senators are voting, the rule cannot pass. With all* 10 *voting, the order would require* 7 *or more in favor to pass, but one abstention will actually lower the threshold to* 6. *This voting rule also prevents Senators from abstaining in order to prevent decisions, as it would require half of the committee to abstain to insure there is no change made.*

It is notable that the aggregation function defined in Example 3.1 cannot output 0. This happens to be specific to the relationship between k and l. It turns out that an $M_{k,l}$ rule cannot output 0 when their sum is minimal. This occurs when $k + l = -1$.

Proposition 3.1. *For any $M_{k,l}$, $k + l = -1$ if and only if $M_{k,l}(\pi) \neq 0$ for any $\pi \in \{-1, 0, 1\}^n$.*

Proof. First, if $l + k = -1$, then assume, by way of contradiction, that there exists $\rho \in \{-1, 0, 1\}^n$ such that $M_{k,l}(\rho) = 0$. Then $n_+(\rho) \leq n_-(\rho) + k$ and $n_-(\rho) \leq n_+(\rho) + l$. Thus, $n_+(\rho) \leq n_+(\rho) + l + k$, and so $0 \leq l + k$. This contradicts that $l + k = -1$, therefore, $M_{k,l}(\pi) \neq 0$ for all $\pi \in \{-1, 0, 1\}^n$.

Next, assume that there exists an $M_{k,l}$ rule such that $M_{k,l}(\pi) \neq 0$ for any $\pi \in \{-1, 0, 1\}^n$. First notice that if $k = -n-1$, then $M_{k,l}(\pi) = 1$ for all π. In order for $M_{k,l}$ to be well defined, $l = n$. Similarly, if $l = -n-1$, then $k = n$. Thus, we can assume $k > -n-1$. Now for any π, either $n_+(\pi) > n_-(\pi) + k$ or $n_-(\pi) > n_+(\pi) + l$. Since both cannot hold at the same time, assume π is such that $n_+(\pi) - n_-(\pi) = k + 1$. Then $M_{k,l}(\pi) = 1$. Consider π' such that $n_+(\pi') - n_-(\pi') = k$. Then $n_+(\pi') \not> n_-(\pi') + k$, so $M_{k,l}(\pi') \neq 1$. Hence, $M_{k,l}(\pi') = -1$, since there exists no profile π such that $M_{k,l}(\pi) = 0$. Therefore, $n_-(\pi') > n_+(\pi') + l$, which implies $n_-(\pi') - n_+(\pi') > l$. Thus $-k > l$ or $k + l < 0$. Since we already showed that $k + l \geq -1$, it is clear that $k + l = -1$. □

Even though $M_{k,l}$ rules are not neutral, there is a relationship between $M_{k,l}$ and $M_{l,k}$ that extends neutrality.

Proposition 3.2. *For any $k, l \in \mathbb{Z} \cap [-n-1, n]$ such that $k \neq l$ and $k + l \geq -1$, and any $\pi \in \{-1, 0, 1\}^n$, $M_{k,l}(-\pi) = -M_{l,k}(\pi)$ or $M_{k,l}(\pi) = -M_{l,k}(-\pi)$.*

Proof. Let $M_{k,l}(\pi) = 1$, then $n_+(\pi) > n_-(\pi) + k$. Thus, $n_-(-\pi) > n_+(-\pi) + k$, so $M_{l,k}(-\pi) = -1$. Similarly, if $M_{k,l}(\pi) = -1$, then $n_-(\pi) > n_+(\pi) + l$. Then $n_+(-\pi) >$

$n_-(-\pi)+l$, hence $M_{l,k}(-\pi) = 1$. Lastly, if $M_{k,l} = 0$, then neither $n_+(\pi) > n_-(\pi)+k$ nor $n_-(\pi) > n_+(\pi) + l$. Therefore neither $n_-(-\pi) > n_+(-\pi) + k$ nor $n_+(-\pi) > n_-(-\pi) + l$. Thus $M_{l,k}(-\pi) = 0$. Hence $M_{k,l}(\pi) = -M_{l,k}(-\pi)$. □

Now that we have a basic understanding of the roles of k and l, we can look at which axioms from Chapter 2 the $M_{k,l}$ rules satisfy. The first theorem we have shows that $M_{k,l}$ rules where $k \neq l$ satisfy all of the axioms that M_k rules satisfy, with the exception of neutrality.

Theorem 3.1. *For any $k, l \in \mathbb{Z} \cap [-n-1, n]$ such that $k \neq l$ and $k + l \geq -1$ the function $M_{k,l} : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ is anonymous, cancellative, monotone, but is not neutral.*

Proof. Notice that for any permutation σ of $\mathcal{N}$ and $\pi \in \{-1, 0, 1\}^n$, $n_+(\pi) = n_+(\pi_\sigma)$ and $n_-(\pi) = n_-(\pi_\sigma)$. Therefore, $M_{k,l}(\pi) = M_{k,l}(\pi_\sigma)$. Hence, $M_{k,l}$ is anonymous.

Let $\pi = (x_1, x_2, ..., x_n)$ and $\pi' = (x'_1, x'_2, ..., x'_n)$ be profiles such that $x_k = x'_k$ for all $k \neq i, j$, $x_i = 1$, $x_j = -1$ and $x'_i = x'_j = 0$, then $N_+(\pi') = N_+(\pi)\backslash\{i\}$ and $N_-(\pi') = N_-(\pi)\backslash\{j\}$. If $M_{k,l}(\pi) = 1$, then $n_+(\pi) - n_-(\pi) > k$. In this case $n_+(\pi') - n_-(\pi') = n_+(\pi) - n_-(\pi) - 1 + 1 > k$. Hence $M_{k,l}(\pi') = 0$. Alternatively, if $M_{k,l}(\pi) = -1$, then $n_-(\pi) - n_+(\pi) > l$ and $n_-(\pi') - n_+(\pi') > l$. Hence $M_{k,l}(\pi') = -1$. By completeness, if $M_{k,l}(\pi) = 0$, then $M_{k,l}(\pi') = 0$. Thus $M_{k,l}$ is cancellative.

Next, consider two profiles π and π' such that $\pi \leq \pi'$ and $M_{k,l}(\pi) = 1$, so $n_+(\pi) > n_-(\pi) + k$. Then $N_+(\pi) \subseteq N_+(\pi')$and $N_-(\pi') \subseteq N_-(\pi)$. Thus $n_+(\pi') \geq n_+(\pi)$ and $n_-(\pi') \leq n_-(\pi)$. Hence, $n_+(\pi') > n_-(\pi') + k$, implies $M_{k,l}(\pi') = 1$. Similarly, if ρ and ρ' are such that $\rho \geq \rho'$ and $M_{k,l}(\rho) = -1$, then $n_-(\rho) > n_+(\rho) + l$. It follows that $N_-(\rho) \subseteq N_-(\rho')$and $N_+(\rho') \subseteq N_+(\rho)$. Thus $n_-(\rho') > n_-(\rho)$ and $n_+(\rho') < n_+(\rho)$, so $M_{k,l}(\rho') = -1$ and $M_{k,l}$ is monotone.

By Lemma 3.2, since $k \neq l$ $M_{k,l}$ is not neutral. □

Now that we have proven that $M_{k,l}$ rules are anonymous, cancellative, and monotone, our next step is show that this class of functions completely characterize all functions that satisfy these three axioms. We start by proving the following:

THEOREM 3.2. *An aggregation function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ satisfies anonymity, cancellation, monotonicity and is not neutral if and only if there exists $k, l \in \mathbb{Z} \cap [-n-1, n]$ such that $k \neq l$ and $k + l \geq -1$ such that $f = M_{k,l}$.*

Proof. Assume that f satisfies our four conditions. We will define the integers k and l in the following ways. If $f(\pi) \neq 1$ for all $\pi \in \{-1, 0, 1\}^n$, let $k = n$; otherwise, let

$$k = \min\{n_+(\pi) - n_-(\pi) : f(\pi) = 1\} - 1.$$

Similarly, if $f(\pi) \neq -1$ for all $\pi \in \{-1, 0, 1\}^n$, let $l = n$; otherwise, let

$$l = \min\{n_-(\pi) - n_+(\pi) : f(\pi) = -1\} - 1.$$

Notice that $k, l \in \mathbb{Z} \cap [-n-1, n]$. If $k < n$, then let κ be a profile such that $n_+(\kappa) - n_-(\kappa) = k + 1$ and $f(\kappa) = 1$. If $l < n$, then let λ be a profile such that $n_-(\lambda) - n_+(\lambda) = l + 1$. Our first goal is to verify that $k + l \geq -1$. If $k = n$, then since $l \geq -n - 1$, $k + l \geq -1$. Similarly, if $l = n$, then $k + l \geq -1$. Thus, from now on we will assume that $k < n$ and $l < n$. Since f is cancellative, we may assume that κ satisfies one of the follow equations:

$$n_+(\kappa) = 0 \text{ and } n_-(\kappa) = -k - 1, \tag{3.3}$$

or

$$n_-(\kappa) = 0 \text{ and } n_+(\kappa) = k + 1. \tag{3.4}$$

Similarly, we may assume that λ satisfies one of the follow equations:

$$n_+(\lambda) = 0 \text{ and } n_-(\lambda) = l + 1, \tag{3.5}$$

or

$$n_-(\lambda) = 0 \text{ and } n_+(\lambda) = -l - 1. \tag{3.6}$$

Suppose Equations (3.3) and (3.6) hold. Then $\vec{0} \geq \kappa$ and $\vec{0} \leq \lambda$. By monotonicity, $f(\vec{0}) \geq f(\kappa) = 1$ and $f(\vec{0}) \leq f(\lambda) = -1$, which is impossible. Thus Equations (3.3) and (3.6) cannot hold at the same time.

We cannot have $k = l = -1$, since then $\kappa = \lambda = \vec{0}$ and $f(\kappa) = 1$, but $f(\lambda) = -1$. Therefore, if Equations (3.4) and (3.5) hold then $k + 1 \geq 0$ and $l + 1 \geq 0$. So $k \geq -1$ and $l \geq -1$. Since $k = l = -1$ is impossible, it follows that $k + l \geq -1$.

If Equation (3.3) and (3.5) hold, then $n_+(\kappa) = n_+(\lambda) = 0$. Since $n_-(\kappa) \geq 0$, we know that $-k - 1 \geq 0$. Similarly, $l + 1 \geq 0$. But f is monotone, and $f(\lambda) < f(\kappa)$, so $\lambda < \kappa$. That implies that $l + 1 > -k - 1$ or $l + k > -2$; hence, $l + k \geq -1$.

If Equations (3.4) and (3.6) hold, then $n_-(\kappa) = n_-(\lambda) = 0$ and we can argue like above that

$$k + 1 > -l - 1 \text{ and so } k + l \geq -1.$$

At this point, by Lemma 3.1, the $M_{k,l}$ rule is well defined.

Now, let $f(\pi) = 1$. Then $n_+(\pi) - n_-(\pi) > k$ by the definition of k. Thus $n_+(\pi) > n_-(\pi) + k$, so $M_{k,l}(\pi) = 1$. Similarly, let $f(\pi) = -1$. Then $n_-(\pi) - n_+(\pi) > l$ by the definition of l. So, $n_-(\pi) > n_+(\pi) + l$ and $M_{k,l}(\pi) = -1$.

Next, assume that $M_{k,l}(\pi) = 1$. We will show that $f(\pi) = 1$. Now $M_{k,l}(\pi) = 1$ implies that $n_+(\pi) > n_-(\pi) + k$. So $n_+(\pi) - n_-(\pi) > k$. Since $M_{k,l}$ is cancellative, there exists π' such that $M_{k,l}(\pi') = M_{k,l}(\pi)$ and one of the following holds:

$$n_+(\pi') = 0 \text{ and } -n_-(\pi') > k, \tag{3.7}$$

or

$$n_-(\pi') = 0 \text{ and } n_+(\pi') > k. \tag{3.8}$$

Therefore there are four cases to analyze.

1. Assume Equations (3.3) and (3.7) hold. Then $n_+(\kappa) = n_+(\pi) = 0$, $n_-(\kappa) = -k-1$ and $n_-(\pi') \le -k-1$. Since $n_-(\pi') \le n_-(\kappa)$ and $n_+(\kappa) = n_+(\pi) = 0$, there exists a permutation σ of $\mathcal{N}$ such that $\kappa \le \pi'_\sigma$. Therefore, by the monotonicity and anonymity, $f(\kappa) \le f(\pi') = 1$. Since f is cancellative, $f(\pi') = f(\pi) = 1$.

2. Assume Equations (3.4) and (3.7) hold. Therefore, $n_-(\kappa) = n_+(\pi') = 0$, $n_+(\kappa) = k+1$ and $n_-(\pi') \le -k-1$. We find then,

$$k+1 \ge 0 \text{ and } 0 \le -k-1,$$

$$k \ge -1 \text{ and } k \le -1.$$

   So $k = -1$ and $\kappa = \pi' = \vec{0}$, so $f(\kappa) = f(\pi') = 1$. Since f is cancellative, $f(\pi') = f(\pi) = 1$.

3. Assume Equations (3.3) and (3.8) hold. Then $n_-(\pi') = n_+(\kappa) = 0$, $n_+(\pi') > k$ and $n_-(\kappa) = -k-1$. There exists a permutation σ of $\mathcal{N}$ such that $\pi'_\sigma \ge \kappa$. Therefore, by monotonicity, $f(\pi'_\sigma) \ge f(\kappa) = 1$. Since f is anonymous and cancellative, $f(\pi) = f(\pi') = f(\pi'_\sigma) = 1$.

4. Assume Equations (3.4) and (3.8) hold. Then $n_-(\pi') = n_-(\kappa) = 0$, $n_+(\pi') > k$ and $n_+(\kappa) = k+1$. Using anonymity, if necessary, we can assume that $\pi' \ge \kappa$, so by monotonicity and cancellation, $f(\pi') = f(\pi) = 1$.

Now, assume that $M_{k,l}(\pi) = -1$. We will show that $f(\pi) = -1$. Keeping in mind that $M_{k,l}$ is cancellative, we may simplify the profile π by canceling pairs of the form $(-1,1)$ or $(1,-1)$ to create π' as above such that $M_{k,l}(\pi') = -1$ and one of the following holds:

$$n_+(\pi') = 0 \text{ and } n_-(\pi') > l, \tag{3.9}$$

or

$$n_-(\pi') = 0 \text{ and } -n_+(\pi') > l. \tag{3.10}$$

Therefore, we can compare λ and π' with four cases as above.

1. Assume Equations (3.5) and (3.9) hold. Then $n_+(\lambda) = n_+(\pi') = 0$, $n_-(\lambda) = l+1$ and $n_-(\pi') \geq l+1$. So, by anonymity, if necessary, we may assume that $\pi' \leq \lambda$ and by monotonicity and cancellation $f(\pi) = f(\pi') \leq f(\lambda) = -1$.

2. Assume Equations (3.5) and (3.10) hold. Therefore, $n_+(\lambda) = n_-(\pi') = 0$, $n_-(\lambda) = l+1$ and $n_-(\pi') \leq -l-1$. We find then,

$$l+1 \geq 0 \text{ and } 0 \leq -l-1$$

$$l \geq -1 \text{ and } l \leq -1$$

   So $l = -1$ and $\lambda = \pi' = \vec{0}$, so $f(\lambda) = f(\pi') = -1$. Since f is cancellative, $f(\pi') = f(\pi) = -1$.

3. Assume Equations (3.6) and (3.9) hold. Then $n_+(\pi') = n_-(\lambda) = 0$, $n_-(\pi') \geq l+1$ and $n_+(\lambda) = -l-1$. Now, $-l-1 \geq 0$ implies $l \leq -1$. By using anonymity, if necessary, we may assume that $\pi' \leq \lambda$, so by monotonicity and cancellation, we can find that $f(\pi) = f(\pi') \leq f(\lambda) = -1$.

4. Assume Equations (3.6) and (3.10) hold. Then $n_-(\pi') = n_-(\lambda) = 0$, $n_+(\pi') \leq -l-1$ and $n_+(\lambda) = -l-1$. Using anonymity, if necessary, we can see that $\pi' \leq \lambda$, so by monotonicity and cancellation $f(\pi') = f(\pi) = -1$.

Thus, we have shown that for any $\pi \in \{-1,0,1\}^n$, $f(\pi) = 1$ if and only if $M_{k,l}(\pi) = 1$ and $f(\pi) = -1$ if and only if $M_{k,l}(\pi) = -1$. Therefore, it can be concluded that $f(\pi) = 0$ if and only if $M_{k,l}(\pi) = 0$. So $f = M_{k,l}$. $\square$

From our results above and those outlined in Theorem 2.1, we have the following corollary. This corollary is the complete characterization of aggregation functions that are monotone, anonymous and cancellative.

Corollary 3.1. *An aggregation function* $f : \{-1,0,1\}^n \to \{-1,0,1\}$ *satisfies monotonicity, anonymity, and cancellation if and only if there exists* $k,l \in \mathbb{Z} \cap [-n-1, n]$ *such that* $k + l \geq -1$ *and* $f = M_{k,l}$

In order to show the necessity of each of our axioms, we will look at a few examples. In Llamazares' paper, he gives examples of functions are given that are neutral as well as satisfying various other axioms[11]. Also, note that the example given by Llamazares as monotone, anonymous, and cancellative, but not neutral (labeled SWP #2) is an $M_{k,l}$ rule where $k = n-2$ and $l = n-1$. The next two examples are not neutral by Lemma 3.2. First we will define a function that satisfies anonymity and monotonicity, but is not cancellative.

Example 3.2. *Consider the aggregation function* $H : \{-1,0,1\}^n \to \{-1,0,1\}$*, for* $n \geq 5$*, defined by*

$$H(\pi) = \begin{cases} M_{1,3}(\pi), & n_+(\pi) \geq 4 \text{ or } n_-(\pi) \geq 4, \\ 0, & \text{otherwise.} \end{cases} \tag{3.11}$$

We will prove that the aggregation function H *is anonymous, monotone, but is not cancellative.*

Proof. First, we will show that H is not cancellative. Consider the profiles $\pi = (x_1, x_2, ..., x_n)$ and $\pi' = (x'_1, x'_2, ..., x'_n)$ such that $n_+(\pi) = 4$, $n_-(\pi) = 1$. Then $H(\pi) = 1$, since $n_+(\pi) - n_-(\pi) = 3 > 1$ and $n_+(\pi) \geq 4$. Define π' via $x_k = x'_k$ for all $k \neq i, j$ and $x_i = 1, x_j = -1$ with $x'_i = x'_j = 0$. Then $H(\pi') = 0$, since $n_+(\pi') = 3$ and $n_-(\pi) = 0$. Therefore, since $H(\pi) \neq H(\pi')$, H is not cancellative.

For $n_+(\pi) \geq 4$ or $n_-(\pi) \geq 4$, H is anonymous, since $H(\pi) = M_{1,3}(\pi)$ and $M_{k,l}$ rules are anonymous. If $n_+(\pi) < 4$ and $n_-(\pi) < 4$, then $H(\pi) = 0$ is anonymous.

Now, to show that H is monotone, let ρ and ρ' be profiles such that $\rho' \geq \rho$. If $n_+(\rho) \geq 4$ or $n_-(\rho') \geq 4$, then $H(\rho') = M_{1,3}(\rho') \geq M_{1,3}(\rho) = H(\rho)$. If $n_+(\rho') < 4$ or $n_-(\rho) < 4$, then $H(\rho') = 0 \geq 0 = H(\rho)$. So we need only to compare two profiles where one meets the criteria for $M_{1,3}$ and one meets the criteria for the trivial part of H.

Let ρ and ρ' be profiles such that $n_+(\rho) \leq 3$, $n_-(\rho) \leq 3$ and $\rho' \geq \rho$. Thus $H(\rho) = 0$. Since $\rho' \geq \rho$, $n_+(\rho') \geq n_+(\rho)$ and $n_-(\rho') \leq n_-(\rho) \leq 3$. If $n_+(\rho') \geq 4$, then $H(\rho') = 1$. If $n_+(\rho') \leq 3$, then $H(\rho') = 0$. Hence $H(\rho') \geq H(\rho)$. Now, let π' and π be such that $n_+(\pi) \leq 3$, $n_-(\pi) \leq 3$ and $\pi \geq \pi'$. So $H(\pi) = 0$. Also, $n_-(\pi') \geq 3$ and $n_+(\pi') \leq 3$. Thus, $H(\pi') \leq 0$ and $H(\pi') \leq H(\pi)$. Hence, we have confirmed that H is monotone. □

The aggregation function defined in the next example is cancellative and anonymous, but is not monotone.

EXAMPLE 3.3. *Consider the function $-M_{1,3} : \{-1,0,1\}^n \to \{-1,0,1\}$, for $n \geq 4$, defined by:*

$$-M_{1,3}(\pi) = \begin{cases} -1, & n_+(\pi) > n_-(\pi) + 1, \\ 1, & n_-(\pi) > n_+(\pi) + 3, \\ 0, & \textit{otherwise.} \end{cases} \tag{3.12}$$

We will show that $-M_{1,3}$ is cancellative, anonymous, but not monotone.

Proof. It follows that $-M_{1,3}$ is not neutral as well. Since the output is determined by $n_-(\pi)$ and $n_+(\pi)$, $-M_{1,3}$ is anonymous. Let π be a profile such that $\pi(i) = -1$ and $\pi(j) = 1$. Consider π' such that $\pi'(i) = \pi'(j) = 0$ and $\pi'(k) = \pi(k)$ for all $k \neq i, j$. We know that $M_{1,3}(\pi) = M_{1,3}(\pi')$, so $-M_{1,3}(\pi) = -M_{1,3}(\pi')$. Thus, $-M_{1,3}$ is cancellative.

Lastly, consider $\pi \le \rho$, such that $n_+(\pi) = n_-(\pi) = 0$, $n_+(\rho) = 3$ and $n_-(\rho) = 0$. In this case, $-M_{1,3}(\pi) = 0$ and $-M_{1,3}(\rho) = -1$. Hence $-M_{1,3}(\pi) \nleq -M_{1,3}(\rho)$. So $-M_{1,3}$ is not monotone. □

In order to look at aggregation functions that are not anonymous, we have to introduce some new subsets of the domain. We define the set $\mathbb{U}$ as follows:

$$\mathbb{U}^+ = \{\pi \in \{-1,0,1\}^n : n_+(\pi) = n-1 \text{ and } n_-(\pi) = 0\},$$

$$\mathbb{U}^- = \{\pi \in \{-1,0,1\}^n : n_-(\pi) = n-1 \text{ and } n_+(\pi) = 0\},$$

$$\mathbb{U} = \mathbb{U}^+ \cup \mathbb{U}^-.$$

There is a close interaction between cancellation and anonymity, as Llamazares stated this in his result [11].

Theorem 3.3. *Let f be aggregation function that satisfies cancellation. Then for any profile π such that $\pi \in \{-1,0,1\}^n \backslash \mathbb{U}$ and any permutation σ of $\mathcal{N}$, $f(\pi_\sigma) = f(\pi)$.*

Additionally, we can take this a step further and understand what happens on subsets $\mathbb{U}^+$ and $\mathbb{U}^-$, if a function is cancellative, but not anonymous.

Corollary 3.2. *Let $f : \{-1,0,1\}^n \to \{-1,0,1\}$ be cancellative, monotone, but not anonymous, then f is non-constant on $\mathbb{U}^+$ or $\mathbb{U}^-$ or both.*

Proof. Since f is cancellative, f is anonymous on $\{-1,0,1\}^n \backslash \mathbb{U}$. Since f is not anonymous there exist π, π' such that $n_+(\pi) = n_+(\pi')$ and $n_-(\pi) = n_-(\pi')$ but $f(\pi) \neq f(\pi')$. Therefore, $\pi, \pi' \in \mathbb{U}^+$ or $\pi, \pi' \in \mathbb{U}^-$. Hence f is not constant on $\mathbb{U}^+$ or $\mathbb{U}^-$. □

Therefore, we have three possibilities to consider if f is not anonymous. We will start with a lemma to help understand the interaction between cancellation and monotonicity.

Lemma 3.3. *Let $f : \{-1,0,1\}^n \to \{-1,0,1\}$ be an aggregation function satisfying cancellation and monotonicity. For any profiles π and ρ, if*

$$n_+(\pi) < n_+(\rho) \text{ and } n_-(\rho) \leq n_-(\pi), \tag{3.13}$$

or

$$n_+(\pi) \leq n_+(\rho) \text{ and } n_-(\rho) < n_-(\pi), \tag{3.14}$$

then $f(\pi) \leq f(\rho)$.

Proof. Assume that f satisfies cancellation and monotonicity. Let π, ρ be profiles such that $n_+(\pi) < n_+(\rho)$ and $n_-(\rho) \leq n_-(\pi)$ or $n_+(\pi) \leq n_+(\rho)$ and $n_-(\rho) < n_-(\pi)$. Then, since either $n_+(\pi) \neq n_+(\rho)$ or $n_-(\pi) \neq n_-(\rho)$ or both, we know that π and ρ are not both elements of $\mathbb{U}^-$ or both elements of $\mathbb{U}^+$. Therefore, we have four cases.

Case 1: Neither π or ρ are in $\mathbb{U}$. Then f is anonymous on $\{-1,0,1\}\backslash\mathbb{U}$, so there exists π' such that $n_+(\pi') = n_+(\pi)$, $n_-(\pi') = n_-(\pi)$, and $\pi' \leq \rho$. Thus $f(\pi') \leq f(\rho)$ by monotonicity, and $f(\pi') = f(\pi)$ by anonymity. Thus $f(\pi) \leq f(\rho)$.

Case 2: The profile $\pi \in \mathbb{U}$ and the profile $\rho \notin \mathbb{U}$. We can choose ρ' such that $n_+(\rho') = n_+(\rho)$ and $n_-(\rho') = n_-(\rho)$. Additionally, if Equation (3.13) holds, then pick ρ' such that $N_+(\pi) \subset N_+(\rho')$ and $N_-(\rho') \subseteq N_-(\pi)$, and if Equation (3.14) hold, then $N_+(\pi) \subseteq N_+(\rho')$ and $N_-(\rho') \subset N_-(\pi)$. Since ρ, $\rho' \notin \mathbb{U}$, $f(\rho) = f(\rho')$ by anonymity. By choice of ρ', $\pi \leq \rho'$, so $f(\pi) \leq f(\rho')$. Thus, $f(\pi) \leq f(\rho)$.

Case 3: The profile $\rho \in \mathbb{U}$ and the profile $\pi \notin \mathbb{U}$. We can choose π' such that $N_+(\pi') \subset N_+(\rho)$ and $N_-(\rho) \subseteq N_-(\pi')$ or $N_+(\pi') \subseteq N_+(\rho)$ and $N_-(\rho) \subset N_-(\pi')$. Since π, $\pi' \notin \mathbb{U}$, $f(\pi) = f(\pi')$ by anonymity. By choice of π', $\pi' \leq \rho'$, so $f(\pi') \leq f(\rho)$. Thus, $f(\pi) \leq f(\rho)$.

Case 4: The profile $\pi \in \mathbb{U}^-$ and the profile $\rho \in \mathbb{U}^+$. We can easily see that $\pi < \rho$. Therefore, by monotonicity, $f(\pi) \leq f(\rho)$.

If $\rho \in \mathbb{U}^-$ and $\pi \in \mathbb{U}^+$, then it could not hold that $n_+(\pi) \leq n_+(\rho)$. Hence we have exhausted our case analysis, so $f(\pi) \leq f(\rho)$. $\square$

We can think of the conditions found in Equations (3.13) and (3.14) implying $f(\pi) \leq f(\rho)$ as a weak anonymously monotone condition.

Now we will explore which aggregation functions are cancellative and monotone, but not anonymous, looking first at what happens with f is not constant on both $\mathbb{U}^+$ and $\mathbb{U}^-$.

Theorem 3.4. *Let $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ be cancellative, monotone and non-constant on both $\mathbb{U}^+$ and $\mathbb{U}^-$. Then for all $\pi \notin \mathbb{U}$,*

$$f(\pi) = M_{n-1,\ n-1}(\pi).$$

Proof. Given that f satisfies the hypothesis of our theorem, there exist profiles π, π' that are both in $\mathbb{U}^+$, such that $f(\pi) = 1$ and $f(\pi') = 0$. If $f(\chi) = -1$ for any profile $\chi \in \mathbb{U}^+$, then by monotonicity, $f(\alpha) = -1$ for all $\alpha \in \mathbb{U}^-$. However, f is non-constant on $\mathbb{U}^-$, so no such χ exists. Similarly, there exists profiles ρ, ρ' that are both in $\mathbb{U}^-$, such that $f(\rho) = -1$ and $f(\rho') = 0$. If there exists $\alpha = (x_1, x_2, ..., x_n) \in \mathbb{U}^-$ such that $f(\alpha) = 1$, then by monotonicity $f(\chi) = 1$ for all $\chi \in \mathbb{U}^+$. However, f is not constant on $\mathbb{U}^+$, so no such α exists.

Next, consider E^+. Since $E^+ > \pi$, by monotonicity $f(E^+) \geq f(\pi) = 1$. So $f(E^+) = 1$. Also, consider E^-. Since $E^- < \rho$, by monotonicity $f(E^-) \leq f(\rho) = -1$. So $f(E^-) = -1$.

Notice, if $n_+(\alpha) = n-1$ and $n_-(\alpha) = 1$, then there exists $i, j \in [1, n]$ such that $x_i = 1$ and $x_j = -1$. Thus, we can create $\alpha' = (x_1', x_2', ..., x_n')$ such that $x_i' = x_j' = 0$ and $x_k' = x_k$ for all $k \neq i,\ j$. Then $n_+(\alpha') = n - 2$ and $n_-(\alpha') = 0$. By cancellation $f(\alpha) = f(\alpha')$. Similarly, if $n_-(\alpha) = n - 1$ and $n_+(\alpha) = 1$, we could follow the same process to show that $f(\alpha) = f(\alpha')$ for α' such that $n_-(\alpha') = n - 2$ and $n_+(\alpha') = 0$.

Now, to consider all profiles not in $\mathbb{U}$ and E^+ or E^-, we assume that α is a profile such that $n_+(\alpha) \le n-2$ and $n_-(\alpha) \le n-2$. First, we will compare α to π' from above. Since $\pi' \in \mathbb{U}^+$, $n_+(\pi') = n-1$ and $n_-(\pi') = 0$. Therefore, by Lemma 3.3, $f(\alpha) \le f(\pi') = 0$. Thus $f(\alpha) \le 0$. Next, we will compare α to ρ' from above. Since $\rho' \in \mathbb{U}^-$, $n_-(\rho') = n-1$ and $n_+(\rho') = 0$. Therefore, by Lemma 3.3, $f(\alpha) \ge f(\rho') = 0$. Thus $f(\alpha) \ge 0$. Hence $f(\alpha) = 0$.

Thus for $\pi \notin \mathbb{U}$, $f(\pi) = M_{n-1,\ n-1}(\pi)$. □

Secondly, we will look at the aggregation rules that are constant on $\mathbb{U}^+$, but non-constant on $\mathbb{U}^-$.

THEOREM 3.5. *Let $f : \{-1,0,1\}^n \to \{-1,0,1\}$ be cancellative, monotone, and constant on $\mathbb{U}^+$, but non-constant on $\mathbb{U}^-$. Then for any profile ρ not belonging to $\mathbb{U}^-$,*

$$f(\rho) = M_{k,l}(\rho)$$

$$\textit{where } l = n + f(E^-),$$

and

$$k = min\{n+1, n_+(\pi) - n_-(\pi) : \pi \in \{-1,0,1\}^n \text{ and } f(\pi) = 1\} - 1.$$

Proof. Since f is monotone and not constant on $\mathbb{U}^-$ it follows that $f(E^-) \ne 1$. Therefore, $f(E^-) = 0$ or $f(E^-) = -1$. This in turn implies that $l = n + f(E^-) \in \{n, n-1\}$.

Case 1: Assume that $f(E^-) = 0$. Then $l = n$. By monotonicity, $f(\pi) \ge 0$ for any profile π. Since f is not constant on $\mathbb{U}^-$, there exists $\pi' \in \mathbb{U}^-$ such that $f(\pi') = 1$. Notice that $n_+(\pi') - n_-(\pi') = -n+1$ and so $k = -n$. Let ρ be a profile not belonging to $\mathbb{U}^-$ and assume that $\rho \ne E^-$. Then either $n_-(\rho) \le n-2$ or $n_-(\rho) = n-1$ and $n_+(\rho) = 1$. In either case, observe that for profiles π' and ρ either Equation (3.13) or (3.14) holds. By Lemma 3.3, $f(\pi') \le f(\rho)$ and so $f(\rho) = 1$. Since $n_+(\rho) - n_-(\rho) \ge -n+2$, it follows that $M_{k,l}(\rho) = M_{-n,n}(\rho) = 1$. We have that $f(\rho) = M_{k,l}(\rho)$ where $k = -n$ and $l = n$.

Case 2: Assume that $f(E^-) = -1$. Then $l = n - 1$. Choose a profile π' belonging to $\mathbb{U}^-$ such that $f(\pi') \geq f(\pi'')$ for all $\pi'' \in \mathbb{U}^-$. Since f is non-constant on $\mathbb{U}^-$, $f(\pi') \geq 0$. If $f(\pi') = 1$, then $k = -n$ and the argument in Case 1 shows that $f(\rho) = M_{k,l}(\rho)$ where $k = -n$ and $l = n - 1$, for all profiles ρ not belonging to $\mathbb{U}^-$. Now, we can assume that $f(\pi') = 0$. In other words, $f(\pi'') \leq 0$ for all π'' belonging to $\mathbb{U}^-$.

We define $\tilde{f} : \{-1,0,1\}^n \to \{-1,0,1\}$ by

$$\tilde{f}(\pi) = \begin{cases} 0 & \text{if } \pi \in \mathbb{U}^-, \\ f(\pi) & \text{otherwise.} \end{cases} \tag{3.15}$$

Since both f and the zero function are monotone and cancellative, and $f(E^-) = -1$, $\tilde{f}$ is monotone and cancellative. Since $\tilde{f}$ is constant on $\mathbb{U}^-$ and $\mathbb{U}^+$, $\tilde{f}$ is anonymous. Therefore, from Theorem 3.2 and its proof, $\tilde{f} = M_{k',l'}$ such that

$$k' = min\{n+1, n_+(\pi) - n_-(\pi) : \pi \in \{-1,0,1\}^n \text{ and } \tilde{f}(\pi) = 1\} - 1$$

and

$$l' = min\{n+1, n_-(\pi) - n_+(\pi) : \pi \in \{-1,0,1\}^n \text{ and } \tilde{f}(\pi) = -1\} - 1.$$

Since $\tilde{f}(E^-) = f(E^-) = -1$ and $f(\pi'') \leq \tilde{f}(\pi'') = 0$ for all π'' belonging to $\mathbb{U}^-$, it follows that $\tilde{f}(\pi) = 1$ if and only if $f(\pi) = 1$. Thus, $k' = min\{n+1, n_+(\pi) - n_-(\pi) : \pi \in \{-1,0,1\}^n \text{ and } f(\pi) = 1\} - 1$, so $k' = k$. Since $\tilde{f}(E^-) = -1$, $l' \leq n-1$, since $\tilde{f}(\pi) = 0$ for all $\pi \in \mathbb{U}^-$ and $\tilde{f}$ is monotone, $l' \geq n - 1$. Thus $l' = n - 1$. Since $f(\rho) = \tilde{f}(\rho) = M_{k',l'}$ for all $\rho \notin \mathbb{U}^-$, $l = l'$, and $k' = k$, then $f(\rho) = M_{k,l}(\rho)$ with $k = min\{n+1, n_+(\pi) - n_-(\pi) : \pi \in \{-1,0,1\}^n \text{ and } f(\pi) = 1\} - 1$ and $l = n - 1$ for all ρ not belonging to $\mathbb{U}^-$. Therefore, $f(\pi) = M_{k,n+f(E^-)}(\pi)$ for all π not in $\mathbb{U}^-$. □

Now, we will look at the possibility that f is constant on $\mathbb{U}^-$, but non-constant on $\mathbb{U}^+$.

THEOREM 3.6. *Let $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ be cancellative, monotone, constant on $\mathbb{U}^-$, and non-constant on $\mathbb{U}^+$. Then for any profile ρ not belonging to $\mathbb{U}^+$,*

$$f(\rho) = M_{k,l}(\rho)$$

$$\text{where } k = n - f(E^+) \text{ and}$$
$$l = min\{n + 1, n_-(\pi) - n_+(\pi) : \pi \in \{-1, 0, 1\}^n \text{ and } f(\pi) = -1\} - 1.$$

Proof. Let f be cancellative, monotone, constant on $\mathbb{U}^-$, and non-constant on $\mathbb{U}^+$. Define $\hat{f} : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ as

$$\hat{f}(\pi) = -f(-\pi). \tag{3.16}$$

Since $\pi \in \mathbb{U}^+$ if and only if $-\pi \in \mathbb{U}^-$, then $\hat{f}$ is constant on $\mathbb{U}^+$ and non-constant on $\mathbb{U}^-$. If $f(-\pi) = 1$, then $\hat{f}(\pi) = -1$. If $f(-\pi) = -1$, then $\hat{f}(\pi) = 1$. Suppose $\pi \leq \pi'$, then $-\pi \geq -\pi'$. Since f is monotone, $f(-\pi) \geq f(-\pi')$, so $\hat{f}(\pi) = -f(-\pi) \leq -f(-\pi') = \hat{f}(\pi')$. We now know that $\hat{f}$ is monotone.

If π, π' are profiles such that $N_+(\pi) \subset N_+(\pi')$, $N_-(\pi) \subset N_-(\pi')$, $n_+(\pi) = n_+(\pi') - 1$ and $n_-(\pi) = n_-(\pi') - 1$, then since f is cancellative, $f(-\pi) = f(-\pi')$, so $\hat{f}(\pi) = -f(-\pi) = -f(-\pi') = \hat{f}(\pi')$. Hence $\hat{f}$ is monotone and cancellative, as well. Thus, by Theorem 3.5, $\hat{f}$ is an $M_{k',n+\hat{f}(E^-)}$ rule, where $k' = min\{n+1, n_+(\pi) - n_-(\pi) : \pi \in \{-1, 0, 1\}^n$ and $\hat{f}(\pi) = 1\} - 1$. Therefore, by Proposition 3.2 and the definition of $\hat{f}$,

$$\begin{aligned} f =& M_{n+\hat{f}(E^-),k'} \\ =& M_{n-f(E^+),k'} \end{aligned}$$

such that

$$k' = min\{n+1, n_+(-\pi) - n_-(-\pi) : \pi \in \{-1,0,1\}^n \text{ and } f(-\pi) = -1\} - 1$$
$$= min\{n+1, n_-(\pi) - n_+(\pi) : \pi \in \{-1,0,1\}^n \text{ and } f(\pi) = -1\} - 1.$$

Hence $k' = l$. Furthermore, $f = M_{n-f(E^+),\ l}$ where

$$l = min\{n+1, n_-(\pi) - n_+(\pi) : \pi \in \{-1,0,1\}^n \text{ and } f(\pi) = -1\} - 1.$$

□

The next step to completely characterize anonymous and cancellative aggregation functions is to prove the converses of the above three theorems. Now, we will state and prove the converse of Theorem 3.4.

Theorem 3.7. *Let $\phi^+ : \mathbb{U}^+ \to \{0,1\}$ and $\phi^- : \mathbb{U}^- \to \{-1,0\}$ be any two non-constant mappings. The function $f : \{-1,0,1\}^n \to \{-1,0,1\}$ defined by*

$$f(\pi) = \begin{cases} \phi^+(\pi) & \pi \in \mathbb{U}^+, \\ \phi^-(\pi) & \pi \in \mathbb{U}^-, \\ M_{n-1,n-1}(\pi) & \textit{otherwise}, \end{cases}$$

satisfies cancellation, monotonicity, but not anonymity.

Proof. Since ϕ^+ is non-constant on $\mathbb{U}^+$ and ϕ^- is non-constant on $\mathbb{U}^-$, thus f is not anonymous. Let π be a profile such that $\pi(i) = -1$ and $\pi(j) = 1$. Consider π' such that $\pi'(i) = \pi'(j) = 0$ and $\pi'(k) = \pi(k)$ for all $k \neq i, j$. Thus neither π nor π' are elements of $\mathbb{U}$, so $f(\pi) = M_{n-1,n-1}(\pi) = M_{n-1,n-1}(\pi') = f(\pi')$. Thus f is cancellative.

In order to show monotonicity, consider $\pi \leq \rho$, then both profiles cannot be in $\mathbb{U}^+$ and both profiles cannot be in $\mathbb{U}^-$, so we have five cases.

Case 1: If $\pi \in \mathbb{U}^+$ then $f(\pi) = 0$ or $f(\pi) = 1$. Since $\rho \geq \pi$, either $\rho = \pi$ or $\rho = E^+$. Since $f(E^+) = M_{n-1,n-1}(E^+) = 1$, $f(\pi) \leq f(\rho)$.

Case 2: If $\pi \in \mathbb{U}^-$ then $f(\pi) = 0$ or $f(\pi) = -1$. Since $\rho \geq \pi$, either $\rho = \pi$ or $\rho > \pi$.

For all $\rho > \pi$, $f(\rho) = 0$ or $f(\rho) = 1$. Hence, $f(\pi) \le f(\rho)$.

Case 3: If $\rho \in \mathbb{U}^+$ then $f(\rho) = 0$ or $f(\rho) = 1$. Either $\rho = \pi$ or $\rho > \pi$. For all $\pi < \rho$, either $f(\pi) = 0$ or $f(\pi) = -1$. So $f(\pi) \le f(\rho)$.

Case 4: If $\rho \in \mathbb{U}^-$ then $f(\rho) = 0$ or $f(\rho) = -1$. Either $\rho = \pi$ or $\pi = E^-$. Since $f(E^-) = M_{n-1,n-1}(E^-) = -1$, $f(\pi) \le f(\rho)$.

Case 5: If neither π nor ρ are in $\mathbb{U}$. Then $f(\pi) = M_{n-1,n-1}(\pi) \le M_{n-1,n-1}(\rho) = f(\rho)$.

Therefore, f is monotone. □

Next, to look at the converse of Theorem 3.5, we will state three separate theorems, as there are multiple functions that need to be verified as monotone, cancellative, but not anonymous.

THEOREM 3.8. *Let $\phi_1^- : \mathbb{U}^- \to \{0,1\}$ be any surjective mapping and let l be an integer such that $l \in [n, n-1]$. The function $f_1 : \{-1,0,1\}^n \to \{-1,0,1\}$ defined by*

$$f_1(\pi) = \begin{cases} \phi_1^-(\pi) & \pi \in \mathbb{U}^-, \\ M_{-n,l}(\pi) & \textit{otherwise}, \end{cases}$$

satisfies cancellation, monotonicity, but not anonymity.

THEOREM 3.9. *Let $\phi_2^- : \mathbb{U}^- \to \{-1,0,1\}$ be any surjective mapping. The function $f_2 : \{-1,0,1\}^n \to \{-1,0,1\}$ defined by*

$$f_2(\pi) = \begin{cases} \phi_2^-(\pi) & \pi \in \mathbb{U}^-, \\ M_{-n,n-1}(\pi) & \textit{otherwise}, \end{cases}$$

satisfies cancellation, monotonicity, but not anonymity.

THEOREM 3.10. *Let $\phi_3^- : \mathbb{U}^- \to \{-1,0\}$ be any surjective mapping, and let k be an integer such that $k \in [-n+1, n]$. The function $f_3 : \{-1,0,1\}^n \to \{-1,0,1\}$ defined by*

$$f_3(\pi) = \begin{cases} \phi_3^-(\pi) & \pi \in \mathbb{U}^-, \\ M_{k,n-1}(\pi) & \textit{otherwise}, \end{cases}$$

satisfies cancellation, monotonicity, but not anonymity.

Proof. We will prove all of the above three theorems concurrently. Since ϕ_1^-, ϕ_2^-, and ϕ_3^- are defined to be non-constant on $\mathbb{U}^-$, the functions f_1, f_2, and f_3 are not anonymous. Let π be a profile such that $\pi(i) = -1$ and $\pi(j) = 1$. Consider π' such that $\pi'(i) = \pi'(j) = 0$ and $\pi'(k) = \pi(k)$ for all $k \neq i, j$. Thus neither π nor π' are elements of $\mathbb{U}$, so $f_1(\pi) = M_{-n,l}(\pi) = M_{-n,l}(\pi') = f_1(\pi')$, $f_2(\pi) = M_{-n,n-1}(\pi) = M_{-n,n-1}(\pi') = f_2(\pi')$, and $f_3(\pi) = M_{k,n-1}(\pi) = M_{k,n-1}(\pi') = f_3(\pi')$. Thus f_1, f_2, and f_3 are cancellative.

In order to show monotonicity, consider $\pi \leq \rho$, then both profiles cannot be in $\mathbb{U}^-$ unless $\pi = \rho$. If $\pi = \rho$ then $f_1(\pi) = f_1(\rho)$, $f_2(\pi) = f_2(\rho)$, and $f_3(\pi) = f_3(\rho)$. We will assume $\pi < \rho$ and consider three cases.

Case 1: Assume $\pi \in \mathbb{U}^-$. Now $\pi \in \mathbb{U}^-$ along with $\rho > \pi$ implies that $n_+(\rho) - n_-(\rho) \geq -n + 2$, and so

$$f_1(\rho) = M_{-n,l}(\rho) = 1 \text{ and } f_2(\rho) = M_{-n,n-1}(\rho) = 1$$

Hence, $f_1(\pi) \leq f_1(\rho)$ and $f_2(\pi) \leq f_2(\rho)$.

Now $\pi \in \mathbb{U}^-$ along with $\rho > \pi$ implies that $n_-(\rho) - n_+(\rho) \leq n - 2$. It follows that

$$f_3(\rho) = M_{k,n-1}(\rho) \neq -1$$

Since $f_3(\pi) = \phi_3^-(\pi) \leq 0$, and $f_3(\rho) \neq -1$, it follows that $f_3(\pi) \leq f_3(\rho)$.

Case 2: Assume $\rho \in \mathbb{U}^-$. Now $\rho \in \mathbb{U}^-$ along with $\rho > \pi$ implies that $\pi = E^-$. So $n_-(\pi) - n_+(\pi) = n$. Therefore,

$$f_2(\pi) = M_{-n,n-1}(\pi) = -1 \text{ and } f_3(\pi) = M_{k,n-1}(\pi) = -1$$

Thus, $f_2(\pi) \leq f_2(\rho)$ and $f_3(\pi) \leq f_3(\rho)$.

Since, $n_+(\pi) - n_-(\pi) = -n$, it follows that

$$f_1(\pi) = M_{-n,l}(\pi) \leq 0 \leq \phi_1^-(\rho) = f_1(\rho)$$

Hence $f_1(\pi) \leq f_1(\rho)$.

Case 3: Assume neither π nor ρ are in $\mathbb{U}^-$. Then $f_1(\pi) = M_{-n,l}(\pi) \leq M_{-n,l}(\rho) = f_1(\rho)$, $f_2(\pi) = M_{-n,n-1}(\pi) \leq M_{-n,n-1}(\rho) = f_2(\rho)$, and $f_3(\pi) = M_{k,n-1}(\pi) \leq M_{k,n-1}(\rho) = f_3(\rho)$. Therefore, f is monotone. □

Lastly, we can add three similar theorems, in order to state and prove the converse of Theorem 3.6. These can be proven using Proposition 3.2 and Theorems 3.8, 3.9, and 3.10.

Theorem 3.11. *Let $\phi_1^+ : \mathbb{U}^+ \to \{-1, 0\}$ be any surjective mapping and let k be an integer such that $k \in [n, n-1]$. The function $g_1 : \{-1,0,1\}^n \to \{-1,0,1\}$ defined by*

$$g_1(\pi) = \begin{cases} \phi_1^+(\pi) & \pi \in \mathbb{U}^+, \\ M_{k,-n}(\pi) & \textit{otherwise}, \end{cases}$$

satisfies cancellation, monotonicity, but not anonymity.

Theorem 3.12. *Let $\phi_2^+ : \mathbb{U}^+ \to \{-1, 0, 1\}$ be any surjective mapping. The function $g_2 : \{-1,0,1\}^n \to \{-1,0,1\}$ defined by*

$$g_2(\pi) = \begin{cases} \phi_2^+(\pi) & \pi \in \mathbb{U}^+, \\ M_{n-1,-n}(\pi) & \textit{otherwise}, \end{cases}$$

satisfies cancellation, monotonicity, but not anonymity.

Theorem 3.13. *Let $\phi_3^+ : \mathbb{U}^+ \to \{0, 1\}$ be any surjective mapping, and let l be an integer such that $l \in [-n+1, n]$. The function $g_3 : \{-1,0,1\}^n \to \{-1,0,1\}$ defined by*

$$g_3(\pi) = \begin{cases} \phi_3^+(\pi) & \pi \in \mathbb{U}^+, \\ M_{n-1,l}(\pi) & \textit{otherwise}, \end{cases}$$

satisfies cancellation, monotonicity, but not anonymity.

Proof. Since ϕ_1^+, ϕ_2^+, and ϕ_3^+ are defined to be non-constant on $\mathbb{U}^+$, g_1, g_2, and g_3 are not anonymous. Let π be a profile such that $\pi(i) = -1$ and $\pi(j) = 1$. Consider π' such that $\pi'(i) = \pi'(j) = 0$ and $\pi'(k) = \pi(k)$ for all $k \neq i, j$. Thus neither π nor π' are elements of $\mathbb{U}^+$, so $g_1(\pi) = M_{k,-n}(\pi) = M_{k,-n}(\pi') = g_1(\pi')$, $g_2(\pi) = M_{n-1,-n}(\pi) = M_{n-1,-n}(\pi') = g_2(\pi')$, and $g_3(\pi) = M_{n-1,l}(\pi) = M_{n-1,l}(\pi') = g_3(\pi')$. Thus g_1, g_2, and g_3 are cancellative.

Now, given g_1 and the mapping ϕ_1^+, define

$$f_1 = \begin{cases} -\phi_1^+(-\pi) & \pi \in \mathbb{U}^- \\ -g_1(-\pi) & \text{otherwise.} \end{cases}$$

Notice that $\phi_1^- : \mathbb{U}^- \to \{-1, 0, 1\}$ defined by $\phi_1^-(\pi) = -\phi_1^+(-\pi)$ has range $\{-1, 0\}$, since ϕ_1^+ has range $\{0, 1\}$. If $\pi \notin \mathbb{U}^-$, then $-\pi \notin \mathbb{U}^+$. So by proposition 3.2, $f_1(\pi) = -g_1(-\pi) = -M_{k,-n}(-\pi) = M_{-n,k}(\pi)$. Therefore,

$$f_1(\pi) = \begin{cases} \phi_1^-(\pi) & \pi \in \mathbb{U}^- \\ M_{-n,k}(\pi) & \text{otherwise.} \end{cases}$$

By Theorem 3.8, f_1 is monotone. If $\pi \leq \rho$, then $-\pi \geq -\rho$ and so $f_1(-\rho) \leq f_1(-\pi)$. Therefore, $-f_1(-\rho) \geq -f_1(-\pi)$. It then follows that $g_1(\rho) \geq g_1(\pi)$. Hence g_1 is monotone. Similar arguments can be made to show that g_2 and g_3 are monotone. $\square$

Combining Theorems 3.4, 3.5, 3.6 and their converses, the following corollary arises.

Corollary 3.3. *An aggregation function $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$ satisfies cancellation and monotonicity if and only if there exists integers $k, l \in [-n-1, n]$ such that $k + l \geq -1$ and there exists mappings $\phi^+ : \mathbb{U}^+ \to \{-1, 0, 1\}$ and $\phi^- : \mathbb{U}^- \to \{-1, 0, 1\}$*

such that

$$f(\pi) = \begin{cases} \phi^+(\pi) & \textit{if } \pi \in \mathbb{U}^+ \\ \phi^-(\pi) & \textit{if } \pi \in \mathbb{U}^- \\ M_{k,l}(\pi) & \textit{otherwise} \end{cases} \tag{3.17}$$

where for all $\pi \in \mathbb{U}^-$

$$M_{k,l}(E^-) \leq \phi^-(\pi) \leq M_{k,l}(\pi), \tag{3.18}$$

and for all $\pi' \in \mathbb{U}^+$

$$M_{k,l}(E^+) \geq \phi^+(\pi') \geq M_{k,l}(\pi'). \tag{3.19}$$

Proof. First, we will show that f, as defined above is cancellative. Let π be a profile such that $\pi(i) = -1$ and $\pi(j) = 1$. Consider π' such that $\pi'(i) = \pi'(j) = 0$ and $\pi'(k) = \pi(k)$ for all $k \neq i, j$. Thus neither π nor π' are elements of $\mathbb{U}$, thus $f(\pi) = M_{k,l}(\pi) = M_{k,l}(\pi') = f(\pi')$, since $M_{k,l}$ is cancellative. Next, we will show that f is monotone. Let $\pi \leq \rho$. If $\pi \in \mathbb{U}^-$ and $\rho \in \mathbb{U}^+$, then using equations (3.18) and (3.19), along with the fact that $M_{k,l}$ is monotone,

$$f(\pi) = \phi^-(\pi) \leq M_{k,l}(\pi) \leq M_{k,l}(\rho) \leq \phi^+(\rho) = f(\rho).$$

If $\pi \in \mathbb{U}^-$ and $\rho \notin \mathbb{U}$, then by equation (3.18),

$$f(\pi) = \phi^-(\pi) \leq M_{k,l}(\pi) \leq M_{k,l}(\rho) = f(\rho).$$

If $\pi \notin \mathbb{U}$ and $\rho \in \mathbb{U}^+$, then be equation (3.19),

$$f(\pi) = M_{k,l}(\pi) \leq M_{k,l}(\rho) \leq \phi^+(\rho) = f(\rho).$$

If $\pi, \rho \notin \mathbb{U}$, then $f(\pi) = M_{k,l}(\pi) \leq M_{k,l}(\rho) = f(\rho)$. Therefore, for any $\pi \leq \rho$, $f(\pi) \leq f(\rho)$. Hence f is monotone.

Now, assume that f satisfies cancellation and monotonicity. Define $\phi^- : \mathbb{U}^- \to \{-1,0,1\}$ and $\phi^+ : \mathbb{U}^+ \to \{-1,0,1\}$ by the following:

$$\phi^-(\pi) = f(\pi), \text{ for all } \pi \in \mathbb{U}^-, \text{ and } \phi^+(\pi) = f(\pi), \text{ for all } \pi \in \mathbb{U}^+.$$

Recall, that if f is monotone and cancellative, but not anonymous then by Corollary 3.2, f is either non-constant on $\mathbb{U}^+$, $\mathbb{U}^-$ or both. So, we will examine four cases.

Case 1: Assume f is anonymous. By Corollary 1, $f = M_{k,l}$ for some $k,l \in \mathbb{Z}$ such that $k,l \in [-n-1,n]$ and $k+l \geq -1$. Thus f satisfies Equation (3.17). Since $\phi^- = M_{k,l}$ and $\phi^+ = M_{k,l}$ it follows that f satisfies Equations (3.18) and (3.19).

Case 2: Assume f is non-constant on both $\mathbb{U}^+$ and $\mathbb{U}^-$. By Theorem 3.4, f satisfies Equation (3.17) with $k = l = n-1$. It was shown in the proof of Theorem 3.4 that $f(\pi) \leq 0$ for all $\pi \in \mathbb{U}^-$ and $f(\pi') \geq 0$ for all $\pi' \in \mathbb{U}^+$. Since $M_{n-1,n-1}(\pi) = M_{n-1,n-1}(\pi') = 0$ for all $\pi \in \mathbb{U}^-$ and $\pi' \in \mathbb{U}^+$ it follows that Equations (3.18) and (3.19) hold.

Case 3: Assume f is non-constant on $\mathbb{U}^-$. By Theorem 4, there exist integers k and l belonging to the interval $[-n-1,n]$ such that $k+l \geq -1$ and $f(\rho) = M_{k,l}(\rho)$ for any profile ρ not belonging to $\mathbb{U}^-$. So Equation (3.17) holds. Since $\phi^+(\pi) = f(\pi) = M_{k,l}(\pi)$ for all $\pi \in \mathbb{U}^+$, Equation (3.19) also holds.

Notice that if $k \leq -n$, then $\phi^-(\pi) \leq 1 = M_{k,l}(\pi)$ for all $\pi \in \mathbb{U}^-$ and so Equation (3.18) holds. If $k > -n$, then, by Case 2 in the proof of Theorem 4, $l = n-1$ and $f(\pi'') \leq 0$ for all π'' belonging to $\mathbb{U}^-$. Therefore, $\phi^-(\pi) = f(\pi) \leq 0 \leq M_{k,l}(\pi)$ for all $\pi \in \mathbb{U}^-$ and, again, Equation (3.18) also holds.

Case 4: Assume f is non-constant on $\mathbb{U}^+$. By Theorem 5, there exist integers k and l belonging to the interval $[-n-1, n]$ such that $k + l \geq -1$ and $f(\rho) = M_{k,l}(\rho)$ for any profile ρ not belonging to $\mathbb{U}^+$. So Equation (3.17) holds. Since $\phi^-(\pi) = f(\pi) = M_{k,l}(\pi)$ for all $\pi \in \mathbb{U}^-$, Equation (3.18) also holds.

We now want to show that Equation (3.19) holds. If $f(\pi) = -1$ for some $\pi \in \mathbb{U}^+$, then, by definition of l given in the statement of Theorem 5, $l \leq -n$. Now $l \leq -n$ implies that $M_{k,l}(\pi') = -1$ for all $\pi' \in \mathbb{U}^+$. Thus, $\phi^+(\pi') \geq -1 = M_{k,l}(\pi')$ for all $\pi' \in \mathbb{U}^+$ and so Equation (3.19) holds. We may now assume that $f(\pi') \geq 0$ for all $\pi' \in \mathbb{U}^+$. Since f is monotone, non-constant on $\mathbb{U}^+$, and $\pi' \leq E^+$ for all $\pi' \in \mathbb{U}^+$ it follows that $f(E^+) \neq -1$. By Theorem 5, we know that $k = n - f(E^+)$ and so $k = n - 1$ or $k = n$. Therefore, $M_{k,l}(\pi') \leq 0 \leq \phi^+(\pi') = f(\pi')$ for all $\pi' \in \mathbb{U}^+$ and so in this case Equation (3.19) holds.

In all cases, we have shown that Equations (3.17), (3.18), and (3.19) hold. $\square$

That gives us a complete characterization of the entire class of functions that are cancellative and monotone. These functions may be called the extended difference of votes rules. From this point, we will extend these functions in to our infinite aggregation model. We will look at which axioms these functions satisfy in the countably infinite model, as well as which axioms are needed to characterize these functions completely with a countable infinite set of voters.

CHAPTER 4
DIFFERENCE OF VOTES RULES ON AN INFINITE VOTING MODEL

In this Chapter, we extend the M_k and $M_{k,l}$ rules to the infinite aggregation model. As we look at these functions, we will determine which infinite axioms are needed to completely characterize these classes of rules in the infinite model. While many of the hypotheses from the previous Chapter hold in this model, it is important to note that not all do. For this reason, we must be careful to evaluate and prove each in the infinite model. In fact, the class of M_k rules require an additional axiom in this model, than was needed in the finite model, and the class of $M_{k,l}$ rules have a whole class of exceptions that must be characterized.

The first order of business is to define M_k rules in the domain where the set of voters is the countably infinite set of natural numbers $\mathbb{N}$. We will us the notation $\mathbb{N}^0$ to indicate the set of natural numbers adjoin 0.

Definition 4.1. *An infinite aggregation rule $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ is said to be an M_k rule, if there exists $k \in \mathbb{N}^0$, such that for all $\pi \in \{-1,0,1\}^{\mathbb{N}}$,*

$$M_k(\pi) = \begin{cases} 1 & n_+(\pi) > n_-(\pi) + k \\ -1 & n_-(\pi) > n_+(\pi) + k \\ 0 & \textit{otherwise.} \end{cases} \tag{4.1}$$

Now we need to verify what axioms these M_k rules satisfy. The second section of Chapter 2 can assist the reader in recalling the definitions of each axiom in the infinite model. Before we clarify which axioms are satisfied, we will state and prove a few lemmas regarding axiom interaction, to help with proving axiom satisfaction later in this Chapter.

This first lemma allows us to only prove one direction of monotonicity, rather than having to prove both directions, if the function is neutral.

Lemma 4.1. *Suppose the infinite aggregation rule $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ satisfies neutrality. If for any profiles π, and π'*

$$\pi \le \pi' \text{ and } f(\pi) = 1 \Rightarrow f(\pi') = 1$$

then f satisfies Monotonicity.

Proof. Let f satisfy neutrality and assume that for $\pi \le \pi'$, then $f(\pi) = 1$ implies that $f(\pi') = 1$. Consider profiles ρ and ρ' such that $\rho' \le \rho$ and $f(\rho) = -1$, then $f(-\rho) = 1$ and $-\rho \le -\rho'$, so $f(-\rho') = 1$ by hypothesis, and $f(\rho') = -1$ by neutrality. Thus, if there exist profiles π and ρ such that $\pi \ge \rho$, if $f(\rho) = -1$, we are done, if $f(\rho) = 1$, we have shown, that $f(\pi) = 1$. If $f(\rho) = 0$, then $f(\pi) \ne -1$, since that would violate $\rho' \le \rho$ and $f(\rho) = -1 \Rightarrow f(\rho') = -1$. Thus $f(\pi) = 0$ or $f(\pi) = 1$. Hence $f(\rho) \le f(\pi)$, and so f is Monotone. □

The next lemma allows us to determine the output of an anonymous and neutral function, if the cardinality of the set of 1's and −1's are equal. That is:

Lemma 4.2. *If an infinite aggregation rule f satisfies strong anonymity and neutrality, and $|N_+(\pi)| = |N_-(\pi)|$, then $f(\pi) = 0$.*

Proof. Let $|N_+(\pi)| = |N_-(\pi)|$. Then there exists a bijection $\phi : N_+(\pi) \to N_-(\pi)$. Define a permutation σ on $\mathbb{N}$ via

$$\sigma(i) = \begin{cases} \phi(i) & \text{if } i \in N_+(\pi) \\ \phi^{-1}(i) & \text{if } i \in N_-(\pi) \\ i & \text{otherwise.} \end{cases}$$

Notice that $\pi_\sigma = -\pi$, by construction of σ. Thus, $f(-\pi) = f(\pi_\sigma)$. By Strong Anonymity, $f(\pi) = f(\pi_\sigma) = f(-\pi)$. By Neutrality, $f(-\pi) = -f(\pi)$. Hence $f(\pi) = -f(\pi)$. Therefore, $f(\pi) = 0$. □

Next we are able to show that M_k rules satisfy the same axioms in the infinite model as in the finite model. We can easily notice that finitely anonymous infinite aggregation rules are a subset of strongly anonymous infinite aggregation rules. We prove that M_k is strongly anonymous below, as it will be necessary in characterizing these rules later in this chapter.

THEOREM 4.1. *For all nonnegative integers k, $M_k : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ satisfies strong anonymity, neutrality, monotonicity, and cancellation.*

Proof. First, note that for any permutation σ of $\mathbb{N}$, $n_+(\pi) = n_+(\pi_\sigma)$ and $n_-(\pi) = n_-(\pi_\sigma)$. Therefore, by the definition of M_k, $M_k(\pi) = M_k(\pi_\sigma)$. Hence, M_k is strongly anonymous. Also, notice that $n_+(\pi) = n_-(-\pi)$ and $n_-(\pi) = n_+(-\pi)$. Thus, $M_k(-\pi) = -M_k(\pi)$ by the definition of M_k. Thus, M_k is neutral. Let $\pi \geq \pi'$, then $n_+(\pi) \geq n_+(\pi')$, and $n_-(\pi) \leq n_-(\pi')$. Therefore, $M_k(\pi) \geq M_k(\pi')$, and M_k is monotone by definition. Let $\pi = (x_1, x_2, ...)$ and $\pi' = (x'_1, x'_2, ...)$ agree everywhere except at i, j, and let $x_i = 1$, $x_j = -1$, and $x'_i = x'_j = 0$. Then, $n_+(\pi) = n_+(\pi') + 1$ and $n_-(\pi) = n_-(\pi') + 1$. Thus, if $n_+(\pi) > n_-(\pi) + k$, then $n_+(\pi') > n_-(\pi') + k$. Similarly, if $n_-(\pi) > n_+(\pi) + k$, then $n_-(\pi') > n_+(\pi') + k$. Hence, M_k is cancellative. □

Recall Equation (2.3) that defined $M_\infty : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$. It turns out that this function also satisfies all of the axioms stated in Theorem 4.1.

PROPOSITION 4.1. *The infinite aggregation rule $M_\infty : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ satisfies strong anonymity, neutrality, monotonicity, and cancellation.*

Proof. First, we will show that M_∞ is anonymous. Notice that for any permutation σ of $\mathbb{N}$, $n_+(\pi) = n_+(\pi_\sigma)$ and $n_-(\pi) = n_-(\pi_\sigma)$. Therefore, since M_∞ is completely determined by $n_+(\pi)$ and $n_-(\pi)$, $M_\infty(\pi) = M_\infty(\pi_\sigma)$.

Next, we will show that M_∞ is neutral. Notice that for any π, $N_+(\pi) = N_-(-\pi)$ and $N_-(\pi) = N_+(-\pi)$. So

$$n_+(\pi) = n_-(-\pi) \text{ and } n_-(\pi) = n_+(-\pi). \tag{4.2}$$

Hence $M_\infty(-\pi) = -M_\infty(\pi)$ for any profile π.

Now we will show M_∞ is monotone. Let $\pi' \geq \pi$. Now consider π such that $M_\infty(\pi) = 1$. Since $n_+(\pi) = \infty$, $n_+(\pi') = \infty$. Also, since $n_-(\pi) < \infty$ and $n_-(\pi') \leq n_-(\pi)$, therefore $n_-(\pi') < \infty$. Hence $M_\infty(\pi') = 1$. Therefore, since M_∞ is neutral, M_∞ is monotone by Lemma 4.1.

Finally, we can show that M_∞ is cancellative. Let $\pi = (x_1, x_2, ...)$ such that $x_i = 1$ and $x_j = -1$. Assume $M_\infty(\pi) = 1$ and consider $\pi' = (x_1', x_2', ...)$ where $x_i' = x_j' = 0$ and $x_k' = x_k$ for all $k \neq i, j$. Then $N_+(\pi') = N_+(\pi)\backslash\{i\}$, so $n_+(\pi') = \infty$. Also, $N_-(\pi') = N_-(\pi)\backslash\{j\}$, so $n_-(\pi') = n_-(\pi) - 1 < \infty$. Hence, $M_\infty(\pi') = 1$. By neutrality, the same holds if $M_\infty(\pi) = -1$. Now, if $M_\infty(\pi) = 0$, then either $n_+(\pi) < \infty$ and $n_-(\pi) < \infty$ or $n_+(\pi) = n_-(\pi) = \infty$. First, assume $n_+(\pi) < \infty$ and $n_-(\pi) < \infty$. Consider $\pi'' = (x_1', x_2', ...)$ where $x_i' = x_j' = 0$ and $x_k' = x_k$, then $n_+(\pi') = n_+(\pi) - 1 < \infty$ and $n_-(\pi') = n_-(\pi) - 1 < \infty$. Hence $M_\infty(\pi') = 0$. Next, assume $n_+(\pi) = n_-(\pi) = \infty$. Consider $\pi'' = (x_1', x_2', ...)$ where $x_i' = x_j' = 0$ and $x_k' = x_k$, then $n_+(\pi') = n_+(\pi) - 1 = \infty$ and $n_-(\pi') = n_-(\pi) - 1 = \infty$. Hence $M_\infty(\pi') = 0$. $\square$

A new axiom was defined in Chapter 2 for the infinite aggregation model. This axiom was called the zero co-finite axiom. Now, we can prove that M_k rules satisfy the zero co-finite axiom, but the M_∞ rule does not.

Lemma 4.3. *For all nonnegative integers k, M_k satisfies the zero co-finite axiom.*

Proof. Let π be a profile such that $|N_+(\pi)| = k + 1$, and $|N_-(\pi)| = 0$. Then $N_0(\pi)$ is co-finite and $M_k(\pi) = 1 \neq 0$. Thus M_k satisfies the zero co-finite axiom. $\square$

Lemma 4.4. *The infinite aggregation rule M_∞ does not satisfy the zero co-finite axiom.*

Proof. Let π be a profile where $N_0(\pi)$ is co-finite. Thus $n_+(\pi) < \infty$ and $n_-(\pi) < \infty$. Therefore, $M_\infty(\pi) \neq 1$ and $M_\infty(\pi) \neq -1$, hence $M_\infty(\pi) = 0$. Therefore, M_∞ is not zero co-finite. $\square$

When seeking to characterize the M_k rules in the infinite case, we discovered the function M_∞ also meets all of the criteria for the finite M_k rules, but is not an M_k rule, unless you allow $k = \infty$. For that reason, it was necessary to introduce the zero co-finite axiom. It turns out the M_∞ function part of entire class of functions that are not zero co-finite, but satisfy the other axioms. We will look further into these after we discuss the M_k rules in this model.

Before we characterize the M_k rules in this model, there is an important lemma regarding infinite aggregation functions that satisfy all of our axioms listed in Theorem 4.1.

Lemma 4.5. *If an infinite aggregation function f satisfies strong anonymity, cancellation, neutrality, and monotonicity, then for any $\pi \in \{-1,0,1\}^{\mathbb{N}}$ such that $N_0(\pi)$ is co-finite and $f(\pi) = 1$, then there exists $\pi' \in \{-1,0,1\}^{\mathbb{N}}$ such that $n_-(\pi') = 0$, $f(\pi') = 1$ and $n_+(\pi') = n_+(\pi) - n_-(\pi)$.*

Proof. Let $\pi \in \{-1,0,1\}^{\mathbb{N}}$ such that $N_0(\pi)$ is co-finite and $f(\pi) = 1$. Let $n_+(\pi) = p$ and $n_-(\pi) = q$. There exists a permutation σ of $\mathbb{N}$ such that $\pi_\sigma = (x_1, x_2, ...)$ where $x_i = 1$ for the first p odd $i \in \mathbb{N}$ and $x_i = -1$ for the first q even $i \in \mathbb{N}$ and 0 elsewhere. Since f is anonymous, $f(\pi_\sigma) = f(\pi)$. If $x_i = 0$ for all i, $\pi = \vec{0}$, and since f is neutral, $f(\vec{0}) = 0$. Thus there is at least one non-zero x_i. Now, for each pair x_i, x_{i+1} in π_σ, we can use the cancellation property of f to create π_i where $x_i = x_{i+1} = 0$, and $f(\pi_i) = f(\pi_\sigma) = f(\pi)$ for any i. Furthermore, we can start with π_1 and then repeat this process, creating $\pi_{1,3} = (\pi_1)_3$ that has $0's$ in the first 4 terms. Continuing in this manner $\min\{p,q\}$ times to create π' has $0's$ in the first $2(\min\{p,q\})$ terms. Notice, that since f satisfies (C), $f(\pi') = f(\pi) = 1$. From the construction of π', either

$n_+(\pi') = 0$ or $n_-(\pi') = 0$. Since $f(\vec{0}) = 0$ by neutrality, if $n_+(\pi') = 0$, then $\pi' < \vec{0}$; therefore, by monotonicity, $f(\pi') \leq f(\vec{0})$. But $f(\pi') = 1$, by (A) and (C). Hence, $n_-(\pi') = 0$, and $n_+(\pi') = n_+(\pi) - \min\{p, q\}$. Further, since $n_-(\pi') = 0$, $n_+(\pi) > n_-(\pi)$, and $\min\{p, q\} = q = n_-(\pi)$. Thus, as desired, $n_+(\pi') = n_+(\pi) - n_-(\pi)$. □

Notice that since f is neutral, if $f(\pi) = -1$, this will still hold for a similar π' with $f(\pi') = -1$.

Using the same axioms as listed in Theorem 2.1 in the finite characterization, and the addition of the zero co-finite axiom, we can now fully characterize the M_k rules in the infinite model with domain $\{-1, 0, 1\}^{\mathbb{N}}$.

Theorem 4.2. *An aggregation rule $f : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ satisfies strong anonymity, neutrality, monotonicity, cancellation, and zero co-finite if and only if f is an M_k-rule for some $k \in \mathbb{N}^0$.*

Proof. If f is an M_k rule, we showed above that it has these properties. Now, assume that $f : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ satisfies strong anonymity, neutrality, monotonicity, cancellation, and zero co-finite. Since f satisfies the zero co-finite axiom, there is a profile such ρ such that $N_0(\rho)$ is co-finite and $f(\rho) \neq 0$. Since f is neutral, $f(-\rho) = -f(\rho)$; therefore, $\{f(\rho), -f(\rho)\} = \{1, -1\}$. Thus, without loss of generality, we can assume that there exists a profile ρ such that $f(\rho) = 1$. Then by Lemma 4.5, there exists ρ' such that $f(\rho') = 1$, $n_-(\rho') = 0$, and $n_+(\rho') = n_+(\rho) - n_-(\rho)$. We can then define

$$k = \min\{n_+(\pi) : n_-(\pi) = 0 \text{ and } f(\pi) = 1\} - 1. \tag{4.3}$$

We know that this minimum exists, since $n_+(\rho')$ is in the set $\{n_+(\pi) : n_-(\pi) = 0 \text{ and } f(\pi) = 1\}$. We can insure that $\min\{n_+(\pi) : n_-(\pi) = 0 \text{ and } f(\pi) = 1\} \geq 1$ since if $n_+(\pi) = 0$, then by Lemma 4.2, $f(\pi) = 0$. Let α be a profile that obtains this minimum and has all 1's appearing first, followed by all zeros.

Let M_k be defined as earlier, based on k above, and assume $M_k(\pi) = 1$. Then $n_+(\pi) > n_-(\pi) + k$. To determine $f(\pi)$, since M_k satisfies strong anonymity, neutrality, monotonicity, cancellation, and zero co-finite, if $n_+(\pi) + n_-(\pi) < \infty$, by Lemma 4.5 we can create π' such that $M_k(\pi') = 1$, $n_-(\pi')$, and $n_+(\pi') = n_+(\pi) - n_-(\pi)$. Then, by definition of k, and since both π' and α have all 1's moved to the front of the profile, $\alpha \leq \pi'$. Then by monotonicity, $f(\pi') \geq f(\alpha) = 1$. Since f is also strong anonimity and cancellation, we can then infer that $f(\pi) = 1$ in a similar manner as in the proof of Lemma 4.5. If $N_0(\pi)$ is not co-finite, since $M_k(\pi) = 1$, we cannot have $n_+(\pi) = n_-(\pi) = \infty$, because then $M_k(\pi) = 0$, by Lemma 4.2. Therefore, either $n_+(\pi) < \infty$ or $n_-(\pi) < \infty$. Since one of these sets is finite, we can follow in the same manner of cancellation as in Lemma 4.5 and create a π' such that $n_-(\pi') = 0$ or $n_+(\pi') = 0$, and $M_k(\pi') = M_k(\pi)$. Since M_k is anonymous and neutral, by Lemma 4.2, $f(\vec{0}) = 0$. Since $\pi' \geq \vec{0}$ and M_k is monotone,$M_k(\pi') \geq 0$ so $M_k(\pi') \neq -1$ implying $M_k(\pi') = 1$. Hence, $n_-(\pi') = 0$. Furthermore, since f is also monotone, and $n_-(\pi') = 0$, then $\pi' > \rho'$ implies $f(\pi') \geq f(\rho') = 1$. Therefore $f(\pi') = 1$ and by cancellation and a similar argument as in Lemma 4.5, $f(\pi) = 1$.

By neutrality, we then can say that if $M_k(\pi) = -1$, then $f(\pi) = -1$. If $M_k(\pi) = 0$, then $n_+(\pi) \leq n_-(\pi) + k$ and $n_-(\pi) \leq n_+(\pi) + k$, or both are infinite. If both are not infinite, then, by minimality of $k + 1$, $f(\pi) = 0$. If both are infinite, by Lemma 4.2 $f(\pi) = M_k(\pi) = 0$.

Now, let $f(\pi) = 1$, then if we assume $M_k(\pi) = 0$. This would imply from above that $f(\pi) = 0$, so $M_k(\pi) \neq 0$. Similarly, $M_k(\pi) \neq -1$. Thus, $M_k(\pi) = 1$. The same argument will show that $f(\pi) = 0 \to M_k(\pi) = 0$ and $f(\pi) = -1 \to M_k(\pi) = -1$. □

Since the characterization from Llamazares did not include the zero co-finite axiom, we wished to remove this axiom. In this infinite model, there are many different functions that satisfy neutrality, strong anonymity, monotonicity, and cancellation, but are not zero co-finite. We mentioned the infinite aggregation rule M_∞ earlier,we no introduce two new classes of functions: the f^k class of functions and the $\mathcal{I}$ function.

Definition 4.2. *For any integer $k \geq -1$, define the infinite aggregation function*

$$f^k : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$$

by

$$f^k(\pi) = \begin{cases} 1 & \text{if } 2n_-(\pi) + n_0(\pi) \leq k; \\ -1 & \text{if } 2n_+(\pi) + n_0(\pi) \leq k; \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

Notice that if $k = -1$, then $f^k(\pi) \neq 1$ for any π and $f^k(\pi) \neq -1$ for any π. Therefore, f^{-1} is the constant 0 function.

Proposition 4.2. *For any integer $k \geq -1$, the infinite aggregation function f^k defined above satisfies neutrality, strong anonymity, monotonicity, and cancellation, but is not zero co-finite.*

Proof. First, notice that if $2n_-(\pi) + n_0(\pi) \leq k$, then $n_+(\pi) = \infty$, and if $2n_+(\pi) + n_0(\pi) \leq k$, then $n_-(\pi) = \infty$. It follows then that if $N_0(\pi)$ is co-finite, then both $N_+(\pi)$ and $N_-(\pi)$ are finite, and when both $N_-(\pi)$ and $N_+(\pi)$ are finite $f^k(\pi) = 0$ for all k. Thus the class of f^k rules are not zero co-finite.

Next, assume that $f^k(\pi) = 1$, then $2n_-(\pi) + n_0(\pi) \leq k$. Consider now $-\pi$ and observe that $n_+(-\pi) = n_-(\pi)$ and $n_0(\pi) = n_0(-\pi)$. It follows then that $2n_+(-\pi) + n_0(-\pi) \leq k$, so $f^k(-\pi) = -1$. Similarly, if $f^k(\pi) = -1$, then $f^k(-\pi) = 1$. Hence f^k rules satisfy neutrality.

Let π be any profile in $\{-1,0,1\}^{\mathbb{N}}$ and σ a permutation of $\mathbb{N}$. Then consider the profile π_σ. It can be seen clearly that $n_-(\pi_\sigma) = n_-(\pi)$, $n_+(\pi_\sigma) = n_+(\pi)$, and

$n_0(\pi_\sigma) = n_0(\pi)$. Therefore, if $2n_-(\pi) + n_0(\pi) \le k$, then $2n_-(\pi_\sigma) + n_0(\pi_\sigma) \le k$ and if $2n_+(\pi) + n_0(\pi) \le k$, then $2n_+(\pi_\sigma) + n_0(\pi_\sigma) \le k$. Thus $f^k(\pi_\sigma) = f^k(\pi)$.

Now consider profiles π, π' such that $\pi \le \pi'$. Assume that $f^k(\pi) = 1$, then $2n_-(\pi) + n_0(\pi) \le k$. This means the set of voters who didn't vote for alternative 1 is at most k. Since $\pi' \ge \pi$, $N_+(\pi) \subseteq N_+(\pi')$, $n_-(\pi') \le n_-(\pi)$, and $n_-(\pi') + n_0(\pi') \le n_-(\pi) + n_0(\pi)$. Thus $2n_-(\pi') + n_0(\pi') \le 2n_-(\pi) + n_0(\pi) \le k$. Thus $f^k(\pi') = 1$ and by Lemma 4.1, f^k is Monotone.

Lastly, let the profiles $\pi = (x_1, x_2, \ldots)$ and $\pi' = (x'_1, x'_2, \ldots)$ be such that $x_k = x'_k$ for all $k \ne i, j$, $x_i = 1$, $x_j = -1$, and $x'_i = x'_j = 0$. If $f^k(\pi) = 1$, $2n_-(\pi) + n_0(\pi) \le k$, then $2n_-(\pi') + n_0(\pi') = 2[n_-(\pi) - 1] + n_0(\pi) + 2 = 2n_-(\pi) + n_0(\pi) \le k$. Thus $f^k(\pi') = f^k(\pi) = 1$. If instead, $2n_+(\pi) + n_0(\pi) \le k$, then similarly, $2n_+(\pi') + n_0(\pi') = 2[n_+(\pi) - 1] + n_0(\pi) + 2 = 2n_+(\pi) + n_0(\pi) \le k$. Hence $f^k(\pi') = f^k(\pi) = -1$. Thus it follows that f^k is cancellative. □

Definition 4.3. *Now, define the infinite aggregation function*

$$\mathcal{I} : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$$

by

$$\mathcal{I}(\pi) = \begin{cases} 1 & \textit{if } N_+(\pi) \textit{ is co-finite;} \\ -1 & \textit{if } N_-(\pi) \textit{ if co-finite;} \\ 0 & \textit{otherwise.} \end{cases} \tag{4.5}$$

Proposition 4.3. *The infinite aggregation function $\mathcal{I}$ defined above satisfies neutrality, strong anonymity, monotonicity, and cancellation, but is not zero co-finite.*

Proof. First, notice that if $N_+(\pi)$ is co-finite, then $n_+(\pi) = \infty$, and if $N_-(\pi)$ is co-finite, then $n_-(\pi) = \infty$. If $N_0(\pi)$ is co-finite, then both $N_+(\pi)$ and $N_-(\pi)$ are finite. When both $N_-(\pi)$ and $N_+(\pi)$ are finite, then $\mathcal{I}(\pi) = 0$. It follows then that function $\mathcal{I}$ is not zero co-finite.

Next, assume that $\mathcal{I}(\pi) = 1$, then $N_+(\pi)$ is co-finite. Consider now $-\pi$ and observe that $N_-(-\pi) = N_+(\pi)$. It follows that $N_-(-\pi)$ is co-finite, so $\mathcal{I}(-\pi) = -1$. Similarly, if $\mathcal{I}(\pi) = -1$, then $\mathcal{I}(-\pi) = 1$. Hence $\mathcal{I}$ satisfies neutrality.

Let π be any profile in $\{-1, 0, 1\}^{\mathbb{N}}$ and σ a permutation of $\mathbb{N}$. Then consider the profile π_σ. It can be seen clearly that $n_-(\pi_\sigma) = n_-(\pi)$, $n_+(\pi_\sigma) = n_+(\pi)$, and $n_0(\pi_\sigma) = n_0(\pi)$. Therefore, if $N_+(\pi)$ is co-finite, then $N_+(\pi_\sigma)$ is as well and if $N_-(\pi)$ is co-finite, then $N_-(\pi_\sigma)$ is also co-finite. Thus $\mathcal{I}(\pi_\sigma) = \mathcal{I}(\pi)$.

Now consider profiles π, π' such that $\pi \leq \pi'$. Assume that $\mathcal{I}(\pi) = 1$, then $N_+(\pi)$ is co-finite. This means the set of voters who didn't vote for alternative 1 is finite. Since $\pi' \geq \pi$, $N_+(\pi) \subseteq N_+(\pi')$, so $N_+(\pi')$ is also co-finite. Thus $\mathcal{I}(\pi') = 1$ and by Lemma 4.1, $\mathcal{I}$ is Monotone.

Lastly, let the profiles $\pi = (x_1, x_2, ...)$ and $\pi' = (x_1', x_2', ...)$ be such that $x_k = x_k'$ for all $k \neq i, j$, $x_i = 1$, $x_j = -1$, and $x_i' = x_j' = 0$. If $\mathcal{I}(\pi) = 1$, $N_+(\pi)$ is co-finite, then $N_+(\pi') = N_+(\pi) \backslash i$ and $N_-(\pi') \cup N_0(\pi') = N_-(\pi) \backslash j \cup N_0(\pi) \cup \{i, j\}$. Thus $N_+(\pi')$ infinite, and $N_-(\pi') \cup N_0(\pi')$ is finite, so $N_+(\pi')$ is co-finite. Hence, $\mathcal{I}(\pi') = \mathcal{I}(\pi) = 1$. If instead, $N_-(\pi)$ is co-finite, then similarly, $N_-(\pi') = N_-(\pi') \backslash i$ and $N_+(\pi') \cup N_0(\pi') = N_-(\pi) \backslash j \cup N_0(\pi) \cup \{i, j\}$. Hence $\mathcal{I}(\pi') = \mathcal{I}(\pi) = -1$. Thus it follows that $\mathcal{I}$ is cancellative. □

Theorem 4.3. *An infinite aggregation function f satisfies strong anonymity, monotonicity, neutrality, and cancellation, and is not zero co-finite if and only if $f = f^k$ for some integer $k \geq -1$ or $f = \mathcal{I}$ or $f = M_\infty$.*

Proof. We have shown that f^k, $\mathcal{I}$ and M_∞ satisfy all of our axioms in Propositions 4.2, 4.3, and 4.1, respectively. Therefore, we only need to prove that if the axioms are satisfied, then we have one of these three classes of rules.

Now, assume that f satisfies the axioms listed. Consider the set

$$O = \{n_0(\pi) : \pi \in \{-1,0,1\}^{\mathbb{N}} \text{ and } f(\pi) = 1\}$$

If $O = \varnothing$, then $f = f^{-1}$, and we are done. Now we may assume that O is non-empty and we have two cases.

Case 1. Assume that the set O is bounded above by some integer k, such that $\max O = k$. We will show that $f(\pi) = f^k(\pi)$ for all $\pi \in \{-1,0,1\}^{\mathbb{N}}$.

Let π be any profile such that $n_-(\pi) > 0$ and $2n(\pi) + n_0(\pi)$ is finite. Let A be a subset of $N_+(\pi)$ such that $|A| = n_-(\pi)$. Let π' be the profile satisfying

$$N_-(\pi) = \varnothing \text{ and } N_0(\pi') = N_0(\pi) \cup N_-(\pi) \cup A$$

Notice that $n_-(\pi') = 0$ and $n_0(\pi') = 2n_(\pi) + n_0(\pi)$. Moreover, since both f and f^k satisfy cancellation

$$f(\pi') = f(\pi) \text{ and } f^k(\pi') = f^k(\pi).$$

Assume that $f(\pi) = 1$. If $n_-(\pi) = 0$, then $2n_-(\pi) + n_0(\pi) = n_0(\pi) \leq k$ and so $f^k(\pi) = 1$. If $n_-(\pi) > 0$, then we can work with the profile π'. Now $f(\pi) = 1$ implies that $f(\pi') = 1$. Since $f(\pi') = 1$ it follows that $n_0(\pi') \leq k$. Since $n_0(\pi') = 2n_-(\pi) + n_0(\pi)$ we get that $2n_-(\pi) + n_0(\pi) \leq k$ and so $f^k(\pi') = 1$. Hence $f^k(\pi) = 1$.

Now suppose that $f^k(\alpha) = 1$ for some profile α. Then $2n_-(\alpha) + n_0(\alpha) \leq k$. Let ρ be a profile where the maximum value of the set O is achieved. So $f(\rho) = 1$ and $n_0(\rho) = k$. By the previous paragraph, $f(\rho) = 1$ implies $f^k(\rho) = 1$ and so $2n_-(\rho) + n_0(\rho) \leq k = n_0(\rho)$. It follows that $n_-(\rho) = 0$. By the argument given above, we know that there exists a profile α' such that $n_-(\alpha') = 0$, $n_0(\alpha') = 2n_-(\alpha) + n_0(\alpha)$, $f(\alpha') = f(\alpha)$, and $f^k(\alpha') = f^k(\alpha)$. Since

$$n_-(\alpha') = n_-(\rho) = 0 \text{ and } n_0(\alpha') \leq n_0(\rho),$$

it follows that there exists a permutation σ of $\mathbb{N}$ such that $\rho_\sigma \leq \alpha'$. Applying strong anonymity and monotonicity we get

$$f(\rho) = f(\rho_\sigma) \leq f(\alpha').$$

Now $f(\rho) = 1$ implies that $f(\alpha') = 1$. Hence $f(\alpha) - 1$.

We can say that, for any profile π, $f(\pi) = 1$ if and only if $f^k(\pi) = 1$. Since f and f^k are neutral, $f(\pi) = f^k(\pi)$ for all $\pi \in \{-1, 0, 1\}^{\mathbb{N}}$.

Case 2. Assume that the set O is not bounded above. Then we have two sub-cases: either **A.** for any profile π, $f(\pi) = 1$ implies that $n_0(\pi) < \infty$ or **B.** there exists a profile α such that $f(\alpha) = 1$ and $n_0(\alpha) = \infty$.

If **A.** holds, then we will show that $f(\pi) = \mathcal{I}(\pi)$ for all $\pi \in \{-1, 0, 1\}^{\mathbb{N}}$. First, assume that $f(\pi) = 1$. Then $n_0(\pi)$ is finite. Since f is not zero co-finite, $n_-(\pi) + n_+(\pi) = \infty$. Since f is anonymous neutral, if $n_+(\pi) = n_-(\pi)$, then $f(\pi) = 0$. Since f is monotone, $n_+(\pi) > n_-(\pi)$. Therefore, $n_+(\pi) = \infty$. Thus $n_-(\pi) < \infty$, and it follows that $N_+(\pi)$ is co-finite. Hence $\mathcal{I}(\pi) = 1$. Since f and $\mathcal{I}$ are neutral, if $f(\pi) = -1$, then $\mathcal{I}(\pi) = -1$.

Next, we want to show that if $\mathcal{I}(\pi) = 1$, then $f(\pi) = 1$. But first, we must observe the following result.

Let l be any nonnegative integer and ρ be any profile such that $n_-(\rho) = l$ and $n_0(\rho) = 0$. Since O is unbounded from above there exists a profile ρ' such that $2l \leq n_0(\rho') < \infty$ and $f(\rho') = 1$. Now choose a profile ρ'' such that $N_+(\rho'') = N_+(\rho') \cup A$ and $N_-(\rho'') = N_-(\rho') \cup B$, where A and B are disjoint subsets of $N_0(\rho')$ such that $|A| = |B| = l$. By cancellation, $f(\rho'') = f(\rho') = 1$. Notice $n_-(\rho'') \geq l$. We can find a permutation σ of $\mathbb{N}$ such that $\rho''_\sigma \leq \rho$. By strong anonymity and monotonicity, $f(\rho'') = f(\rho''_\sigma) \leq f(\rho) = 1$.

Now assume $\mathcal{I}(\pi) = 1$. Then $N_+(\pi)$ is co-finite. Then there exists a non-negative integer l such that $l = |N_0(\pi) \cup N_-(\pi)|$. Let ρ be the profile where $N_-(\rho) =$

$N_0(\pi) \cup N_-(\pi)$ and $N_+(\rho) = N_+(\pi)$.It follows from above that $f(\rho) = 1$. Since $\rho \leq \pi$, it follows from monotonicity that $f(\pi) = 1$. Since both $\mathcal{I}$ and f are neutral, we have that $f(\pi) = \mathcal{I}(\pi)$ for all $\pi \in \{-1, 0, 1\}^{\mathbb{N}}$.

If **B.** holds, then we will show that $f(\pi) = M_\infty(\pi)$ for all $\pi \in \{-1, 0, 1\}^{\mathbb{N}}$.

Assume first that $f(\pi) = 1$. Since f is not zero co-finite, either $n_+(\pi) = \infty$ or $n_-(\pi) = \infty$. If $n_+(\pi) = n_-(\pi) = \infty$, then, by Lemma 4.2, $f(\pi) = 0$ contrary to $f(\pi) = 1$. If $n_-(\pi) = \infty$ and $n_+(\pi) < \infty$, then we can remove $n_+(\pi)$ 1's and $n_+(\pi)$ -1's from the profile π and create a profile π' such that $n_+(\pi') = 0$ and $f(\pi') = f(\pi)$. So $f(\pi') = -1$. Observe that $\pi' \leq \vec{0}$, so by monotonicity, $f(\pi') \leq f(\vec{0})$, and $f(\vec{0}) = 0$ by Lemma 4.2. However, this contradicts that $f(\pi') = 1$. Therefore, $n_+(\pi) = \infty$ and $n_-(\pi) < \infty$. Thus $M_\infty(\pi) = 1$.

If $f(\pi) = -1$, then by neutrality $f(-\pi) = 1$. From above, $f(-\pi) = 1$ implies $M_\infty(-\pi) = 1$. Since M_∞ is neutral, $M_\infty(\pi) = -1$.

Assume now that $M_\infty(\pi) = 1$. Consider α, the profile described previously. Notice that if $n_-(\pi) = n_+(\pi)$, then by Lemma 4.2, $f(\pi) = 0$. Since $f(\alpha) = 1$, $n_(\alpha) \neq n_+(\alpha)$. Since f is not zero co-finite, either $n_+(\alpha) = \infty$ or $n_-(\alpha) = \infty$. It follows by monotonicity that $n_+(\alpha) = \infty$ and $n_-(\alpha) < \infty$. Since $n_=(\alpha) = \infty$, and f satisfies cancellation, there exists a profile α' such that $n_-(\alpha') = 0$, $n_+(\alpha') = \infty$, and $f(\alpha') = f(\alpha) = 1$. Thus $N_0(\alpha') \cup N_+(\alpha') = \mathbb{N}$. Since $M_\infty(\pi) = 1$, we know that $n_-(\pi) < \infty$. By cancellation of M_∞, there exists π' such that $n_-(\pi') = 0$, $n_+(\pi') = n_+(\pi) = \infty$, and $M_\infty(\pi') = M_\infty(\pi)$. Now we have that $N_0(\pi') \cup N_+(\pi') = \mathbb{N}$ and either $n_0(\pi') < \infty$ or $n_0(\pi') = \infty$. If $n_0(\pi') = \infty$, then there exists a permutation σ of $\mathbb{N}$ such that $\pi'_\sigma > \alpha'$, then $f(\pi'_\sigma) = f(\pi') = f(\pi)$, by strong anonymity and cancellation. Also, $f(\pi'_\sigma) \geq f(\alpha') = f(\alpha) = 1$. Thus $f(\pi) = f(\pi') = f(\pi'_\sigma) = 1$,

If instead, both $n_0(\pi') = n_+(\pi') = \infty$, then there exists two bijections σ_1 and σ_2 defined as follows:

$$\sigma_1 : N_0(\pi') \to N_0(\alpha')$$

and

$$\sigma_2 : N_+(\pi') \to N_+(\alpha')$$

.

Now, define the permutation σ of $\mathbb{N}$ as follows:

$$\sigma(i) = \begin{cases} \sigma_1(i) & \text{if } i \in N_0(\pi') \\ \sigma_2(i) & \text{if } i \in N_+(\pi'). \end{cases}$$

Observe that $\pi' = \alpha'_\sigma$. By strong anonymity, $f(\pi') = f(\alpha'_\sigma) = f(\alpha') = 1$. We now know that for any profile π, $M_\infty(\pi) = 1$ if and only if $f(\pi) = 1$. Therefore by neutrality, $f = M_\infty$ and we are done. □

From the preceding results, we can now characterize all infinite aggregation functions that satisfy strong anonymity, monotonicity, neutrality, and cancellation. The following corollary comes as a result of Theorems 4.1, 4.2, and 4.3, as well as Propositions 4.1, 4.2, and 4.3,

Corollary 4.1. *If $f : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ satisfies strong anonymity, monotonicity, neutrality, and cancellation than one of the following holds:*

1. *There exists an integer $k \geq 0$ such that $f = M_k$;*
2. *There exists an integer $k \geq -1$ such that $f = f^k$;*
3. *The function $f = \mathcal{I}$;*
4. *The function $f = M_\infty$.*

Now that we have completely extended the characterization of M_k rules to the infinite model, we will look at what happens when we reduce the axioms used in this characterization. In order to do so, let us look at the relationship between cancellation and anonymity in the infinite model. First, by introducing subsets of the domain, as we did in the finite model.

DEFINITION 4.4. *Define the following subsets of* $\{-1,0,1\}^{\mathbb{N}}$*:*

$$\mathcal{U}^{+} = \{\pi \in \{-1,0,1\}^{\mathbb{N}} : n_0(\pi) = 1, n_-(\pi) = 0\}$$

$$\mathcal{U}^{-} = \{\pi \in \{-1,0,1\}^{\mathbb{N}} : n_0(\pi) = 1, n_+(\pi) = 0\}$$

$$\textit{and} \quad \mathcal{U} = \mathcal{U}^{+} \cup \mathcal{U}^{-}$$

While cancellation does not imply strong anonymity of this model, we can show that it implies finite anonymity on a large subset of the domain. We give an example below of a function that is not finite anonymous, to clarify that cancellation does not imply finite anonymity

EXAMPLE 4.1. *There exists integers* $k \geq 0$*,* $l \geq 1$ *and an infinite aggregation function* $F : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *where* F *is defined as follows:*

$$F(\pi) = \begin{cases} M_k(\pi) & \text{if } \pi \in \{-1,0,1\}^{\mathbb{N}} \backslash \mathcal{U} \\ 1 & \text{if } \pi \in \mathcal{U} \text{ and } x_i = 0 \text{ for } i \leq l \\ -1 & \text{if } \pi \in \mathcal{U} \text{ and } x_i = 0 \text{ for } i > l \end{cases} \tag{4.6}$$

Notice, if a profile π *is in* $\mathcal{U}$*, there is only one* 0 *in a profile, then the placement of that* 0 *uniquely determines the output, thus* F *is not finite anonymous. If* $\pi \notin \mathcal{U}$*,* $F = M_k$*, so* F *is cancellative. However, if* $\pi \in \mathcal{U}$ *there exists no such integers* i, j *such that* $x_i = -1$*, and* $x_j = 1$ *or* r, s *such that* $x_r = x_s = 0$*, thus cancellation is trivially satisfied.*

Now, we can show that the cancellation implies finite anonymity, when the domain is restricted to $\{-1,0,1\}^{\mathbb{N}} \backslash \mathcal{U}$. Recall that Σ is the set of all finite permutations of $\mathbb{N}$.

THEOREM 4.4. *If an infinite aggregation rule* $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *is cancellative, then for all profiles* $\pi \in \{-1,0,1\}^{\mathbb{N}} \backslash \mathcal{U}$ *and all permutations* $\sigma \in \Sigma$*,* $f(\pi) = f(\pi_\sigma)$*.*

Proof. Since any finite permutation can be written as a product of transpositions, we may assume that $\sigma = (i\ j)$, for some $i, j \in \mathbb{N}$. We will show that for any profile $\pi \in \{-1, 0, 1\}^{\mathbb{N}} \backslash \mathcal{U}$ that $f(\pi) = f(\pi_\sigma)$. We have three cases.

Case 1 If $x_i = x_j$, then $\pi_\sigma = \pi$, and it follows that $f(\pi) = f(\pi_\sigma)$.

Case 2 If $x_i = 1$ and $x_j = -1$. Then, since f is cancellative, we can create profiles π' and π'', successively, as shown in Table 1, with $k \neq i$ and $k \neq j$.

Table 4.1: Case 2

term / profile	i	j	k
π	1	-1	x_k
π'	0	0	x_k
π''	-1	1	x_k

Then, we can see easily that $\pi'' = \pi_\sigma$, and since, by cancellation $f(\pi) = f(\pi') = f(\pi'')$, then $f(\pi) = f(\pi_\sigma)$.

Case 3 If $x_i = 0$ and $x_j = 1$, we will assume this is the same as if $x_i = 0$ and $x_j = -1$, as the proof is almost identical. Since $\pi \in \{-1, 0, 1\}^{\mathbb{N}} \backslash \mathcal{U}$, either there exists a profile element x_r such that $x_r = 0$ and $r \neq i$ or there exists a profile element x_s such that $x_s = -1$. If there exists an $x_r = 0$, then we will successively create profiles π' and π'', using cancellation, as in Table 2.

Table 4.2: Case 3i

term / profile	i	j	r	k
π	0	1	0	x_k
π'	1	1	-1	x_k
π''	1	0	0	x_k

If there exists $x_s = -1$, then we will successively create profiles π' and π'' as in Table 3. In either case, we can see that $\pi'' = \pi_\sigma$ and by cancellation $f(\pi) = f(\pi') = f(\pi'')$.

Table 4.3: Case 3ii

$\frac{\text{term}}{\text{profile}}$	i	j	s	k
π	0	1	-1	x_k
π'	0	0	0	x_k
π''	1	0	-1	x_k

Thus $f(\pi) = f(\pi_\sigma)$ as desired. □

Next we will prove the necessity of our axioms listed in Theorem 4.2. In order to do this, we will give examples of functions that satisfy some, but not all, of the axioms.

The first example will be proven to be anonymous, neutral, cancellative, and zero co-finite, but not monotone.

EXAMPLE 4.2. *The aggregation function N will be defined as follows.*

$$N(\pi) = \begin{cases} M_\infty(\pi) & N_0 \textit{ is not co-finite.} \\ -M_0(\pi) & N_0 \textit{ is co-finite.} \end{cases} \tag{4.7}$$

THEOREM 4.5. *The function $N : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ is strongly anonymous, neutral, cancellative, and zero co-finite, but is not monotone.*

Proof. First, we will show that N is strongly anonymous. For any permutation σ of $\mathbb{N}$ and $\pi \in \{-1,0,1\}^{\mathbb{N}}$, if $N_0(\pi)$ is co-finite, then $N_0(\pi_\sigma)$ is co-finite. Also $n_+(\pi_\sigma) = n_+(\pi)$ and $n_-(\pi_\sigma) = n_-(\pi)$. Since both $M_\infty(\pi)$ and $M_0(\pi)$ depend completely on $N_0(\pi)$, $n_-(\pi)$ and $n_+(\pi)$, $N(\pi) = N(\pi_\sigma)$.

Next we will show that N is neutral. Let $N(\pi) = 1$, then either (1) $n_-(\pi) = \infty$ and $n_+(\pi) < \infty$ or (2) $N_0(\pi)$ is co-finite and $n_+(\pi) > n_-(\pi)$. We will assume first that that (1) holds. Since $N_-(\pi) = N_+(-\pi)$ and $N_+(\pi) = N_-(-\pi)$ so $n_+(-\pi) = \infty$ and $n_-(-\pi) < \infty$. Therefore $f(-\pi) = -1$. Now assume that (2) holds, then $N_0(\pi)$ is co-finite; thus, $N_0(-\pi)$ is co-finite. Again, $N_-(\pi) = N_+(-\pi)$ and $N_+(\pi) = N_-(-\pi)$, so $n_+(-\pi) < n_-(\pi)$ and $N(-\pi) = -1$. Hence N is neutral.

Now we will show that N is cancellative. Notice first, if $N_0(\pi)$ is co-finite (not co-finite), and $x_i, x_j \in \pi$ such that $x_i = 1$ and $x_j = -1$, we can create π' with $x'_k = x_k$ for all $k \neq i, j$, $x'_i = x'_j = 0$, and $N_0(\pi')$ is also co-finite (not co-finite). Then for any profile π there again are two cases:

Case 1 $N_0(\pi)$ is co-finite. Then consider π such that $f(\pi) = 1$ with $x_i = 1$ and $x_j = -1$ and π' with $x'_k = x_k$ for all $k \neq i, j$, $x'_i = x'_j = 0$. Since $N(\pi) = 1$, $n_+(\pi) = \infty$ and it follows that $n_+(\pi') = \infty$. Also, $n_-(\pi') = n_-(\pi) - 1 < \infty$, thus $N(\pi') = 1$. Since N is neutral, this will hold with $N(\pi) = -1$.

Case 2 $N_0(\pi)$ is not co-finite. Then let $N(\pi) = 1$, thus $n_+(\pi) \leq n_-(\pi)$. Then for any pair $x_i, x_j \in \pi$ such that $x_i = 1$ and $x_j = -1$, if we create π' such that $x'_i = x'_j = 0$ and $x'_k = x_k$ for all $k \neq i, j$. Then $n_+(\pi') = n_+(\pi) - 1$ and $n_-(\pi') = n_-(\pi) - 1$. Thus $n_+(\pi') < n_-(\pi')$, so $N(\pi') = 1$. Again, neutrality holds, so the $N(\pi) = -1$ case follows.

We can show that N satisfies the zero co-finite axiom quickly by noting that if π is such that $n_+(\pi) = 1$ and $n_-(\pi) = 0$, then $N(\pi) = 1$. $N_0(\pi)$ is obviously co-finite, and $N(\pi) \neq 0$.

Lastly, we will show the N is not monotone. Let π be such that N_0 is co-finite and $N(\pi) = 1$. Thus, $n_+(\pi) < n_-(\pi)$. We know $N(-\pi) = -1$ and $n_-(-\pi) < n_+(-\pi)$ by neutrality. Also, $N_+(\pi) = N_-(-\pi)$ and $N_-(\pi) = N_+(-\pi)$; hence, $-\pi \geq \pi$. Under the monotonicity condition, this would imply $N(-\pi) \geq N(\pi)$, but this is not the case. Thus, N is not monotone. $\square$

Next we will define an aggregation function that is finitely anonymous, monotone, cancellative, neutral and zero co-finite, but is not strongly anonymous.

EXAMPLE 4.3. *Let* $E = \{z \in \mathbb{N}^0 : \frac{z}{2} \in \mathbb{N}^0\}$ *be the set of positive even integers, and define* $F : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *via*

$$F(\pi) = \begin{cases} 1 & n_+(\pi) = n_-(\pi) = \infty \text{ and } |E \backslash N_+(\pi)| < \infty \\ -1 & n_+(\pi) = n_-(\pi) = \infty \text{ and } |E \backslash N_-(\pi)| < \infty \\ M_0(\pi) & \text{otherwise.} \end{cases} \tag{4.8}$$

We will prove that the aggregation function F satisfies all of the axioms of Theorem 4.2 except strong anonymity.

THEOREM 4.6. *The function* $F : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *is zero co-finite, neutral, cancellative, monotone, and finite anonymous (but not strongly anonymous) .*

Proof. First, note the M_0 is zero co-finite, so F is as well. Before continuing this proof, it is important to notice that since M_0 is both strongly anonymous and neutral, if $n_+(\pi) = n_-(\pi)$, $M_0(\pi) = 0$. Therefore, if $F(\pi) \neq 0$ we must be in the case where $F \neq M_0$, so $n_+(\pi) = n_-(\pi) = \infty$ and either $|E \backslash N_+(\pi)| < \infty$ or $|E \backslash N_-(\pi)| < \infty$. Alternatively, if $n_+(\pi) = n_-(\pi)$ and $F(\pi) = 0$, then neither $|E \backslash N_+(\pi)| < \infty$ nor $|E \backslash N_-(\pi)| < \infty$ hold, so $F(\pi) = M_0(\pi) = 0$.

Next we will show neutrality. If we are in the case where π is such that $F(\pi) = M_0(\pi)$, we are done, since M_0 is neutral. Observe that $E \backslash N_+(\pi) = E \backslash N_-(-\pi)$ and $E \backslash N_+(-\pi) = E \backslash N_-(\pi)$. Therefore, if $F(\pi) \neq M_0(\pi)$, then $F(\pi) = -F(\pi)$. Thus F is neutral.

Cancellation holds for M_0 as well, so if π is such that $F(\pi) = M_0(\pi)$, cancellation holds. If $F(\pi) \in \{-1,1\}$ and $n_+(\pi) = n_-(\pi) = \infty$, then either $|E \backslash N_+(\pi)| < \infty$ or $|E \backslash N_-(\pi)| < \infty$. Without loss of generality, assume $F(\pi) = 1$ then $|E \backslash N_+(\pi)| < \infty$. Let $\pi = (x_1, x_2, ...)$, such that $x_i = 1$ and $x_j = -1$. Consider $\pi' = (x'_1, x'_2, ...)$ such that $x'_k = x_k$ for all $k \neq i,j$ and $x_i = x_j = 0$. Then $n_+(\pi') = n_-(\pi') = \infty$ and $N_+(\pi) \subseteq N_+(\pi')$. Thus $|E \backslash N_+(\pi')| < |E \backslash N_+(\pi)| < \infty$, so $F(\pi') = F(\pi) = 1$. Since cancellation is a finite process, no amount would make $n_+(\pi) = n_-(\pi) < \infty$, thus F is cancellative.

For profiles π and π' such that $\pi \le \pi'$, if $F(\pi) = M_0(\pi)$ and $F(\pi') = M_0(\pi')$, then $F(\pi) \le F(\pi')$ since M_0 is monotone. Now, let $\pi \le \pi'$ such that $F(\pi) = 1$ and $n_+(\pi) = n_-(\pi) = \infty$, then either $n_+(\pi') = n_-(\pi') = \infty$ or $n_+(\pi') = \infty$ and $n_-(\pi') < \infty$, since monotonicity is point-wise. If we are in the latter case, then $F(\pi') = M_0(\pi')$ and $n_+(\pi') > n_-(\pi')$, so $F(\pi) = 1$. If we are in the former case, then we know that $|E\backslash N_+(\pi)| < \infty$. Since $N_+(\pi) \subseteq N_+(\pi')$, so $|E\backslash N_+(\pi')| \le |E\backslash N_+(\pi)| < \infty$. Hence $F(\pi) = 1$. Thus, by Lemma 4.1, F is monotone.

Since F is cancellative, by Theorem 4.4, for all $\pi \in \{-1, 0, 1\}^{\mathbb{N}}\backslash\mathcal{U}$ and $\tau \in \Sigma$, $F(\pi) = F(\pi_\tau)$. Let $\rho \in \mathcal{U}$, then either (i) $n_+(\rho) = \infty$ and $n_-(\rho) = 0$ or (ii) $n_-(\rho) = \infty$ and $n_+(\rho) = 0$. In either case, for any permutation $\sigma \in \Sigma$, we find $\rho_\sigma \in \mathcal{U}$ as well. So $F(\rho) = M_0(\rho)$ and $F(\rho_\sigma) = M_0(\rho_\sigma)$. Furthermore, since M_0 is strongly anonymous, $F(\rho) = M_0(\rho) = M_0(\rho_\sigma) = F(\rho_\sigma)$ for $\rho \in \mathcal{U}$. Hence F is finite anonymous.

However, consider the following permutation ϕ of $\mathbb{N}$ and profile $\alpha = (a_1, a_2, ...)$

$$\phi(i) = \begin{cases} i+1 & \text{if } i \in \mathbb{N}\backslash E \\ i-1 & \text{if } i \in E \end{cases}$$

$$a_i = \begin{cases} 1 & \text{if } i \in E \\ -1 & \text{if } i \in \mathbb{N}\backslash E \end{cases}$$

Then $n_+(\alpha) = n_-(\alpha) = \infty$ and $|E\backslash N_+(\alpha)| = 0 < \infty$, so $F(\alpha) = 1$. If we apply ϕ to α, then $n_+(\alpha_\phi) = n_-(\alpha_\phi) = \infty$, but $|E\backslash N_-(\alpha)| = 0 < \infty$, so $F(\alpha_\phi) = -1$. Thus F is not strongly anonymous. $\square$

The next example is strongly anonymous, neutral, monotone, and zero co-finite, but is not cancellative.

Example 4.4. *Consider now the aggregation function* $H : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$,

$$H(\pi) = \begin{cases} M_0(\pi) & n_+(\pi) \geq 4 \text{ or } n_-(\pi) \geq 4 \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

Theorem 4.7. *The aggregation function* $H : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *is strongly anonymous, neutral, monotone, and zero co-finite, but is not cancellative.*

Proof. First, we will show that H is not cancellative and zero co-finite. Consider the profiles $\pi = (x_1, x_2, ...)$ and $\pi' = (x_1', x_2', ...)$ such that $n_+(\pi) = 4$, $n_-(\pi) = 1$, $x_k = x_k'$ for all $k \neq i, j$ and $x_i = 1, x_j = -1$ with $x_i' = x_j' = 0$. Then $H(\pi) = 1$, showing H is zero co-finite. Also, since $n_+(\pi) = 4$ and $n_-(\pi) < 4$, but $H(\pi') = 0$, since $n_+(\pi') = 3$ and $n_-(\pi) = 0$. Therefore, since $H(\pi) \neq H(\pi')$, H is not cancellative.

For $n_+(\pi) \geq 4$ or $n_-(\pi) \geq 4$, H is strongly anonymous and neutral, since $H(\pi) = M_0(\pi)$. Otherwise, the function H is the trivial function, and is trivially strongly anonymous, neutral, and monotone. Therefore, H is strongly anonymous, and neutral. Now, we need to show that H is monotone. We need only to consider what happens when comparing two profiles where one meets the criteria for M_0 and one meets the criteria for the trivial part of H.

Let ρ and ρ' be profiles such that $n_+(\rho) \leq 3$, $n_-(\rho) \leq 3$ and $\rho' \geq \rho$. Thus $H(\rho) = 0$. Since $\rho' \geq \rho$, $n_+(\rho') \geq n_+(\rho)$ and $n_-(\rho') \leq n_-(\rho) \leq 3$. If $n_+(\rho') \geq 4$, then $H(\rho') = 1$. If $n_+(\rho') \leq 3$, then $H(\rho') = 0$. Hence $H(\rho') \geq H(\rho)$. Since H is neutral, it follows that H is monotone. □

Our final example is the extension of the $M_{k,l}$ rule defined in the finite model in Equation 3.1. This rule will satisfy all of our axioms, except neutrality.

For this example, we need to introduce some new notation. For our purposes, the notation $-\infty$ represents the opposite of ∞, and so $\infty - \infty = 0$.

Example 4.5. *Consider now the function* $M_{k,l} : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ *for* $k, l \in$

$\mathbb{Z} \cup \{-\infty, \infty\}$ *such that* $k \neq l$ *and* $k + l \geq -1$.

$$M_{k,l}(\pi) = \begin{cases} 1 & n_+(\pi) > n_-(\pi) + k \\ -1 & n_-(\pi) > n_+(\pi) + l \\ 0 & \textit{otherwise.} \end{cases} \tag{4.10}$$

Before proving that the $M_{k,l}$ rules satisfy our axioms, we extend Lemmas 3.1 and 3.2 to this model.

Lemma 4.6. *If* $k, l \in \mathbb{Z} \cup \{-\infty, \infty\}$ *then* $M_{k,l}$ *is well defined if and only if* $k + l \geq -1$.

Proof. First, if both k and l are less than 0, it follows that both $n_+(\rho) > n_-(\rho) + k$ and $n_-(\rho) > n_+(\rho) + l$ would hold for the profile $\rho = (0, 0, 0, ...)$; therefore, both k and l cannot be negative. In order for $M_{k,l}$ to be well defined, we must restrict the values of k and l further. Assume that there exists π such that both $n_+(\pi) > n_-(\pi) + k$ and $n_-(\pi) > n_+(\pi) + l$ hold. Then, $n_+(\pi) \geq n_-(\pi) + k + 1$ and $n_-(\pi) \geq n_+(\pi) + l + 1$. By substitution, $n_+(\pi) \geq n_+(\pi) + k + l + 2$, and $k + l \leq -2$. Hence, from the contrapositive, we have that if $k + l \geq -1$, then $M_{k,l}$ is well-defined. □

Lemma 4.7. *The function* $M_{k,l}$ *is neutral if and only if* $k = l$.

Proof. If $k = l$, then since $k + l > -2$, $2k > -1$, and $k \geq 0$. Therefore, by definition, we have an M_k rule which is neutral by Theorem 4.1. Now, assume that $M_{k,l}$ is neutral, then $M_{k,l}(0, 0, 0, ...) = 0$ by Lemma 4.2, so $k \geq 0$ and $l \geq 0$.

First, let $k < \infty$, the consider π such that $n_+(\pi) = k + 1$ and $n_-(\pi) = 0$, then $M_{k,l}(\pi) = 1$. By neutrality, $M_{k,l}(-\pi) = -1$, when $n_-(-\pi) = k + 1$ and $n_+(-\pi) = 0$. Therefore, $k + 1 > n_+(-\pi) + l = l$. Thus $k \geq l$. Similarly, consider ρ such that $n_-(\rho) = l + 1$ and $n_+(\rho) = 0$, then $M_{k,l}(\rho) = -1$. Considering $-\rho$, $n_+(-\rho) = l + 1$ and $n_-(-\rho) = 0$, and by neutrality $M_{k,l}(-\rho) = 1$. Therefore, $l + 1 > n_-(-\rho) + k = k$, so $l \geq k$. Thus, it follows that $k = l$.

Next, let $k = \infty$, then in order for $M_{k,l}(\pi) = 1$, $n_+(\pi) > n_-(\pi) + \infty$, which cannot occur. Therefore, there exists no π such that $M_{k,l}(\pi) = 1$. Then by neutrality, there is no π such that $M_{k,l}(\pi) = -1$. Hence $M_{k,l}(\pi) = 0$ for all π, thus $k = l = \infty$. □

We can now prove that the class of $M_{k,l}$ rules is strongly anonymous, cancellative, zero co-finite, monotone, but is not neutral.

Theorem 4.8. *For any $k, l \in \mathbb{Z} \cup \{-\infty, \infty\}$ such that $k \neq l$ and $k + l \geq -1$ the function $M_{k,l} : \{-1, 0, 1\}^{\mathbb{N}} \to \{-1, 0, 1\}$ is strongly anonymous, cancellative, zero co-finite, monotone, but is not neutral.*

Proof. Next, notice that $M_{k,l}$ is zero co-finite, for finite k and l since the profile with $n_+(\pi) - n_-(\pi) = k + 1$ yields $M_{k,l}(\pi) = 1$, but $|N_0(\pi)| = \infty$ and $|N_+(\pi) \cup N_-(\pi)| < \infty$. Since $k \neq l$ and $k + l \geq -1$, if $k = \infty$, then $l = -\infty$, and visa versa. In either case, $M_{k,l}(\pi) \neq 0$ for all π and thus $M_{k,l}$ is zero co-finite.

Now, notice that for any permutation σ of $\mathbb{N}$ and $\pi \in \{-1, 0, 1\}^{\mathbb{N}}$, $n_+(\pi) = n_+(\pi_\sigma)$ and $n_-(\pi) = n_-(\pi_\sigma)$. Therefore, for $M_{k,l}(\pi) = M_{k,l}(\pi_\sigma)$. Hence, $M_{k,l}$ is strongly anonymous.

Let $\pi = (x_1, x_2, \ldots)$ and $\pi' = (x'_1, x'_2, \ldots)$ be profiles such that $x_k = x'_k$ for all $k \neq i, j$, $x_i = 1$, $x_j = -1$ and $x'_i = x'_j = 0$, then $N_+(\pi') = N_+(\pi) \backslash \{i\}$ and $N_-(\pi') = N_-(\pi) \backslash \{j\}$. Letting $M_{k,l}(\pi) = 1$, we know $n_+(\pi) - n_-(\pi) > k$. Thus, $n_+(\pi') - n_-(\pi') = n_+(\pi) - n_-(\pi) - 1 + 1 > k$. Alternatively, if $M_{k,l}(\pi) = -1$, then $n_-(\pi) - n_+(\pi) > l$ and $n_-(\pi') - n_+(\pi') > l$. Hence $M_{k,l}(\pi') = -1$. By completeness, if $M_{k,l}(\pi) = 0$, then $M_{k,l}(\pi') = 0$. Thus $M_{k,l}$ is cancellative.

Next, consider two profiles π and π' such that $\pi \leq \pi'$ and $M_{k,l}(\pi) = 1$, so $n_+(\pi) > n_-(\pi) + k$. Then $N_+(\pi) \subseteq N_+(\pi')$and $N_-(\pi') \subseteq N_-(\pi)$. Thus $n_+(\pi') \geq n_+(\pi)$ and $n_-(\pi') \leq n_-(\pi)$. Hence, $n_+(\pi') > n_-(\pi') + k$, so $M_{k,l}(\pi') = 1$. Similarly, for ρ and ρ' such that $\rho \geq \rho'$ and $M_{k,l}(\rho) = -1$, then $n_-(\rho) > n_+(\rho) + l$. It follows that $N_-(\rho) \subseteq N_-(\rho')$and $N_+(\rho') \subseteq N_+(\rho)$. Thus $n_-(\rho') > n_-(\rho)$ and $n_+(\rho') < n_+(\rho)$, so

$M_{k,l}(\rho') = -1$ and $M_{k,l}$ is monotone. □

While some of our results carried over from the finite model, not all of them did. You may notice that if $k = \infty$ and $l = -\infty$, that while $k + l = 0 \neq -1$, $M_{k,l}(\pi) \neq 0$ for any $\pi \in \{-1, 0, 1\}^{\mathbb{N}}$. This shows that the Proposition 3.1 does not hold in the infinite model. However, we can extend Proposition 3.2 quite naturally.

Proposition 4.4. *For any $k, l \in \mathbb{Z} \cup \{-\infty, \infty\}$ such that $k \neq l$ and $k + l \geq -1$, and any $\pi \in \{-1, 0, 1\}^{\mathbb{N}}$, $M_{k,l}(-\pi) = -M_{l,k}(\pi)$ or $M_{k,l}(\pi) = -M_{l,k}(-\pi)$.*

Proof. First, assume both k and l are finite. Let $M_{k,l}(\pi) = 1$, then $n_+(\pi) > n_-(\pi)+k$. Thus, $n_-(-\pi) > n_+(-\pi) + k$, so $M_{l,k}(-\pi) = -1$. Similarly, if $M_{k,l}(\pi) = -1$, then $n_-(\pi) > n_+(\pi) + l$. Then $n_+(-\pi) > n_-(-\pi) + l$, hence $M_{l,k}(-\pi) = 1$. Lastly, if $M_{k,l} = 0$, then neither $n_+(\pi) > n_-(\pi) + k$ nor $n_-(\pi) > n_+(\pi) + l$. Therefore, neither $n_-(-\pi) > n_+(-\pi) + k$ nor $n_+(-\pi) > n_-(-\pi) + l$. Thus $M_{l,k}(-\pi) = 0$. Hence $M_{k,l}(\pi) = -M_{l,k}(-\pi)$.

Next, assume $l = \infty$ and $k = -\infty$. Let $M_{k,l}(\pi) = 1$, then $n_+(\pi) > n_-(\pi) - \infty$. Therefore, $n_-(-\pi) > n_+(-\pi) - \infty$, so $M_{l,k}(-\pi) = -1$. If $M_{k,l}(\pi) \neq -1$ for any π. It follows, if $M_{l,k}(\pi) = -1$, then it follows $M_{k,l}(-\pi) = 1$. Notice $M_{k,l}(\pi) \neq -1$ for any π, and $M_{l,k}(\pi) \neq 1$ for any π. Assume that $M_{k,l}(\pi) = 0$, then neither $n_+(\pi) > n_-(\pi) - \infty$ nor $n_-(\pi) > n_+(\pi) + \infty$. Also, it follows neither $n_-(-\pi) > n_+(-\pi) - \infty$ nor $n_+(-\pi) > n_-(-\pi) + \infty$, so $M_{l,k}(-\pi) = 0$.

Lastly, assume $k = \infty$ and $l < \infty$. Let $M_{k,l}(\pi) = 1$, then $n_+(\pi) > n_-(\pi) + \infty$. Thus, $n_-(-\pi) > n_+(-\pi) + \infty$, so $M_{l,k}(-\pi) = -1$. Now, if $M_{k,l}(\pi) = -1$, then $n_-(\pi) > n_+(\pi) + l$. Then $n_+(-\pi) > n_-(-\pi) + l$, hence $M_{l,k}(-\pi) = 1$. Lastly, if $M_{k,l} = 0$, then neither $n_+(\pi) > n_-(\pi)+\infty$ nor $n_-(\pi) > n_+(\pi)+l$. Hence neither $n_-(-\pi) > n_+(-\pi)+\infty$ nor $n_+(-\pi) > n_-(-\pi) + l$. Thus $M_{l,k}(-\pi) = 0$. □

Since not all of the propositions carried over from the finite model into the infinite model, is also happens that we have not found a full characterization for all of the $M_{k,l}$ rules in the infinite model. However, if we restrict our k, l to finite integers, we can include a new axiom that allows for a full characterization of this subset of $M_{k,l}$ rules.

Definition 4.5. *An infinite aggregation function $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ satisfies the* Strong Zero Co-finite *axiom if there exists $\pi, \pi' \in \{-1,0,1\}^{\mathbb{N}}$ such that $N_0(\pi)$ and $N_0(\pi')$ are co-finite and $\{f(\pi), f(\pi')\} = \{-1,1\}$.*

Before characterizing the finite k, l $M_{k,l}$ rules, we show that they satisfy the strong zero co-finite axiom.

Proposition 4.5. *An infinite aggregation function defined by $M_{k,l}$ such that $k, l \in \mathbb{Z}$ and $k + l \geq -1$ satisfies the strong zero co-finite axiom.*

Proof. Define π as the profile such that $n_+(\pi) = k + 1$ and $n_-(\pi) = 0$. Then $M_{k,l}(\pi) = 1$ and $N_0(\pi)$ is co-finite. Define π' to be the profile such that $n_-(\pi') = l + 1$ and $n_+(\pi) = 0$. Then $M_{k,l}(\pi') = -1$ and $N_0(\pi')$ is co-finite. Therefore $\{M_{k,l}(\pi), M_{k,l}(\pi')\} = \{-1,1\}$, so $M_{k,l}$ satisfies the strong zero co-finite axiom. □

Now we can characterize the class of finite k, l $M_{k,l}$ rules.

Theorem 4.9. *An infinite aggregation function $f : \{-1,0,1\}^{\mathbb{N}} \to \{-1,0,1\}$ satisfies strong anonymity, cancellation, monotonicity, and strong zero co-finite axiom, but is not neutral if and only if there exists $k, l \in \mathbb{Z}$ such that $k \neq l$ and $k + l \geq -1$ such that $f = M_{k,l}$.*

Proof. We have already shown that $M_{k,l}$ satisfies all of the axioms listed. Thus, we can now assume that f satisfies all of our axioms, and we will prove that $f = M_{k,l}$. Now, define k and l as follows.

$$k = \min\{n_+(\pi) - n_-(\pi) : f(\pi) = 1\} - 1.$$

and

$$l = \min\{n_-(\pi) - n_+(\pi) : f(\pi) = -1\} - 1$$

Since f satisfies strong zero co-finite, neither set is empty, so the minimum exists. Also, since f satisfies strong zero co-finite, neither k nor l is infinite, as if $k(l) = \infty$, then $f(\pi) \neq 1(-1)$ for any π. Thus $k, l \in \mathbb{Z}$.

Let κ be a profile such that $n_+(\kappa) - n_-(\kappa) = k + 1$ and $f(\kappa) = 1$. Then let λ be a profile such that $n_-(\lambda) - n_+(\lambda) = l + 1$ and $f(\lambda) = -1$. Our first goal is to verify that $k + l \geq -1$. Since f is cancellative, we may assume that κ satisfies one of the follow equations:

$$n_+(\kappa) = 0 \text{ and } n_-(\kappa) = -k - 1 \tag{4.11}$$

or

$$n_-(\kappa) = 0 \text{ and } n_+(\kappa) = k + 1. \tag{4.12}$$

Similarly, we may assume that λ satisfies one of the follow equations:

$$n_+(\lambda) = 0 \text{ and } n_-(\lambda) = l + 1 \tag{4.13}$$

or

$$n_-(\lambda) = 0 \text{ and } n_+(\lambda) = -l - 1. \tag{4.14}$$

Suppose Equations (4.11) and (4.14) hold. Then $\vec{0} \geq \kappa$ and $\lambda \geq \vec{0}$. By monotonicity, $f(\vec{0}) \geq f(\kappa) = 1$ and $f(\lambda) = -1 \geq f(\vec{0})$, which is impossible. Thus Equations (4.11) and (4.14) cannot hold at the same time.

We cannot have $k = l = -1$, since then $\kappa = \lambda = \vec{0}$ and $f(\kappa) = 1$, but $f(\lambda) = -1$. Therefore, if Equations (4.12) and (4.13) hold then $k + 1 \geq 0$ and $l + 1 \geq 0$. So $k \geq -1$ and $l \geq -1$. Since $k = l = -1$ is impossible, it follows that $k + l \geq -1$.

Now, suppose Equations (4.11) and (4.13) hold. Then $n_+(\kappa) = n_+(\lambda) = 0$, so $n_-(\kappa) = -k - 1$ and $n_-(\lambda) = l + 1$. If $k = l = -1$, the $\kappa = \lambda$, which cannot hold and

$M_{k,l}$ be well defined. Since $n_-(\kappa) \geq 0$, $-k \geq -1$. Also, since $n_-(\lambda) \geq 0$, $l \geq -1$. Then it follows that, since both k and l cannot be -1 at the same time, that

$$k + l \geq -1.$$

If Equations (4.12) and (4.14) hold, then $n_-(\kappa) = n_+(\lambda) = 0$ and we can argue like above that k and l cannot both be negative one, and result to find

$$k + 1 > -l - 1 \text{ and so } k + l \geq -1.$$

At this point, by Lemma 4.6, the $M_{k,l}$ rule is well defined.

Now, let $f(\pi) = 1$. Then $n_+(\pi) - n_-(\pi) > k$ since $k + 1$ is minimal. Thus $n_+(\pi) > n_-(\pi) + k$, so $M_{k,l}(\pi) = 1$. Similarly, let $f(\pi) = -1$. Then $n_-(\pi) - n_+(\pi) > l$ by the minimality of $l + 1$. So, $n_-(\pi) > n_+(\pi) + l$ and $M_{k,l}(\pi) = -1$.

Next, assume that $M_{k,l}(\pi) = 1$. This implies that $n_+(\pi) > n_-(\pi) + k$. So $n_+(\pi) - n_-(\pi) > k$. Since $M_{k,l}$ is cancellative, there exists π' such that $M_{k,l}(\pi') = M_{k,l}(\pi)$ and either $n_+(\pi') = 0$ and $-n_-(\pi') > k$ or $n_-(\pi') = 0$ and $n_+(\pi') > k$. Since $k \neq \pm\infty$, κ, a profile that attains the minimum for $k + 1$ as defined above, exists. Now, f is cancellative as well, so there exists κ' such that $f(\kappa) = f(\kappa')$ and either $n_+(\kappa') = 0$ and $-n_-(\kappa') = k + 1$ or $n_-(\kappa') = 0$ and $n_+(\kappa') = k + 1$. Therefore, we can compare κ' and π' with four cases.

Case 1 Assume $n_+(\pi') = 0$, $-n_-(\pi') > k$ and $n_+(\kappa') = 0$, $-n_-(\kappa') = k + 1$, then $0 \leq n_-(\pi') < -k$ and $0 \leq n_-(\kappa') = -k - 1$, hence $k \leq -1$. Notice that $n_-(\pi') \leq n_-(\kappa)$ and so there exists a permutationσ of $\mathbb{N}$ such that $\pi'_\sigma \geq \kappa'$. Therefore, by monotonicity of f, $f(\pi'_\sigma) \geq f(\kappa') = f(\kappa) = 1$. Since f is strongly anonymous, $f(\pi') = f(\pi'_\sigma) = 1$. Since f is cancellative, $f(\pi') = f(\pi) = 1$.

Case 2 Assume $n_+(\pi') = 0$, $-n_-(\pi') > k$ and $n_-(\kappa') = 0$, $n_+(\kappa') = k + 1$; therefore, $0 \leq n_-(\pi') < -k$, so $k < 0$, and $n_+(\kappa') = k + 1 \geq 0$. Thus, $k \geq -1$, but $k < 0$, so $k = -1$. Hence, $n_-(\pi') < 1$, so $n_-(\pi') = 0$. Similarly, $n_+(\kappa') = k + 1 = 0$, so

$n_+(\pi') = n_-(\pi') = n_-(\kappa') = n_+(\kappa') = 0$. Thus, $\kappa' = \pi'$, so $f(\kappa') = f(\pi') = 1$. Since f is cancellative, $f(\pi') = f(\pi) = 1$.

Case 3 Assume $n_-(\pi') = 0$, $n_+(\pi') > k$ and $n_-(\kappa') = 0$, $-n_+(\kappa') = k+1$. Since both f and $M_{k,l}$ are strongly anonymous, there exists $\sigma \in S_{\mathbb{N}}$ such that $\pi'_\sigma \geq \kappa'$. Therefore, by monotonicity of f, $f(\pi'_\sigma) \geq f(\kappa') = f(\kappa) = 1$. Since f is anonymous, $f(\pi') = f(\pi'_\sigma) = 1$. Since f is cancellative, $f(\pi') = f(\pi) = 1$.

Case 4 Assume $n_-(\pi') = 0$, $n_+(\pi') > k$ and $n_+(\kappa') = 0$, $-n_-(\kappa') = k+1$. Then $0 \leq n_-(\kappa') = -k-1$, so $k \leq -1$, but $n_+(\pi') > k$. Therefore, $\pi' \geq (0,0,0,...)$ and $(0,0,0,...) \geq \kappa'$. Thus by transitivity, $\pi' \geq \kappa'$, so by monotonicity $f(\pi') \geq f(\kappa') = f(\kappa) = 1$. Since f is cancellative, $f(\pi) = f(\pi') = 1$.

Similarly, assume that $M_{k,l}(\pi) = -1$. Therefore, it follows that $n_-(\pi) - n_+(\pi) > l$. Since $M_{k,l}$ is cancellative, there exists π' such that $M_{k,l}(\pi') = M_{k,l}(\pi)$ and either $n_-(\pi') = 0$ and $-n_+(\pi') > l$ or $n_-(\pi') = 0$ and $n_-(\pi') > k$. Since $l \neq \pm\infty$, λ, a profile that attains the minimum for $l+1$ as defined above, exists. Now, f is cancellative as well, so there exists λ' such that $f(\lambda) = f(\lambda')$ and either $n_+(\lambda') = 0$ and $n_-(\lambda') = k+1$ or $n_-(\lambda') = 0$ and $-n_+(\lambda') = k+1$. Therefore, we can compare λ' and π' with four cases.

Case 1 Assume $n_-(\pi') = 0$, $-n_+(\pi') > l$ and $n_-(\lambda') = 0$, $-n_+(\lambda') = l+1$. Therefore, $0 \leq n_+(\pi') < -l$ and $0 \leq n_+(\lambda') = -l-1$. Since both f and $M_{k,l}$ are strongly anonymous, there exists $\sigma \in S_{\mathbb{N}}$ such that $\pi'_\sigma \leq \lambda'$. Therefore, by monotonicity of f, $f(\pi'_\sigma) \leq f(\lambda') = f(\lambda) = -1$. Since f is anonymous, $f(\pi') = f(\pi'_\sigma) = -1$. Since f is cancellative, $f(\pi') = f(\pi) = -1$.

Case 2 Assume $n_+(\pi') = 0$, $n_-(\pi') > l$ and $n_-(\lambda') = 0$, $-n_+(\lambda') = l+1$. So $0 \leq n_+(\lambda') = -l-1$ and $n_-(\pi') > l$. So $\pi' \leq (0,0,0,...)$ and $\lambda' \geq (0,0,0,...)$, so by transitivity

$\pi' \leq \lambda'$. Therefore, since f is monotone, $f(\pi') \leq f(\lambda') = f(\lambda) = -1$. Since f is cancellative, $f(\pi) = f(\pi') = -1$.

Case 3 Assume $n_+(\pi') = 0$, $n_-(\pi') > l$ and $n_+(\lambda') = 0$, $n_-(\lambda') = l+1$. Since both f and $M_{k,l}$ are strongly anonymous, there exists $\sigma \in S_{\mathbb{N}}$ such that $\pi'_\sigma \leq \lambda'$. Therefore, by monotonicity of f, $f(\pi'_\sigma) \leq f(\lambda') = f(\lambda) = -1$. Since f is anonymous, $f(\pi') = f(\pi'_\sigma) = -1$. Since f is cancellative, $f(\pi') = f(\pi) = -1$.

Case 4 Assume $n_-(\pi') = 0$, $-n_+(\pi') > l$ and $n_+(\lambda') = 0$, $n_-(\lambda') = l+1$. Then $0 \leq n_+(\pi') < -l$, so $l < 0$, and $0 \leq n_-(\lambda') = l+1$, so $l+1 \geq 0$. Thus $l \geq -1$ and $l < 0$, so $l = -1$. Therefore, $n_+(\pi') < 1$ and $n_-(\lambda') = 0$. So $n_-(\pi') = n_+(\pi') = n_+(\lambda') = n_-(\lambda') = 0$. Hence $\pi' = \lambda'$, so $f(\pi') = f(\lambda') = f(\lambda) = -1$. Since f is cancellative, $f(\pi) = f(\pi') = -1$.

Thus, we have shown that for all $\pi \in \{-1,0,1\}^{\mathbb{N}}$, $f(\pi) = 1$ if and only if $M_{k,l}(\pi) = 1$ and $f(\pi) = -1$ if and only if $M_{k,l}(\pi) = -1$. Therefore, it can be concluded that $f(\pi) = 0$ if and only if $M_{k,l}(\pi) = 0$, so $f(\pi) = M_{k,l}(\pi)$. □

The Theorems in this Chapter have completely characterized the M_k rules for the infinite model with a countably infinite set of voters. While this extends many of the results from Chapter 3, we have not yet looked deeper into characterizing the infinite aggregation functions that do not satisfy neutrality. Doing so, and extending Corollary 3.3 and its preceding Theorems, has been left for future work.

CHAPTER 5
DIFFERENCE OF VOTES RULES ON THE FUZZY VOTING MODEL

Now that the M_k and $M_{k,l}$ rules have been characterized in our first two models, we will look at the third and final model which we call the fuzzy voting model. Extending the M_k rules to the fuzzy model was first done by Garcia-Lapresta and Llamazares in 2010 [8]. They were able to completely characterize these rules in the fuzzy model. Our goal in this chapter is to give a new proof of the main result given in [8]. We will first state and prove two new lemmas and use them to prove the characterization of the M_k rules in the fuzzy domain, similar to our previous models.

Recall that in the fuzzy decision model, there are two definitions for the difference of votes rules. These two classes of functions are known as the $\widetilde{M_k}$ rules defined in Definition 2.26 and the $\widetilde{M'_k}$ rules defined in Definition 2.27. They are stated again below for convenience. In this section, we will use the notation $sum(\pi)$ to indicate the sum of all the entries in the profile π. That is if $\pi = (d_1, d_2, ..., d_m)$, then $sum(\pi) = \sum_{i=1}^{m} d_i$.

Definition 5.1. *Given a real number $k \in [0, m)$, the* fuzzy $\widetilde{M_k}$ majority *is the fuzzy decision rule defined by:*

$$\widetilde{M_k}(\pi) = \begin{cases} 1 & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i > \frac{1}{2} + \frac{k}{2m}, \\ \frac{1}{2} & \text{if } |\frac{1}{m}\sum_{i=1}^{m} d_i - \frac{1}{2}| \leq \frac{k}{2m}, \\ 0 & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i < \frac{1}{2} - \frac{k}{2m}. \end{cases} \tag{5.1}$$

DEFINITION 5.2. *Given a real number $k \in (0, m]$, the* fuzzy $\widetilde{M}'_k$ majority *is the fuzzy decision rule defined by:*

$$\widetilde{M}'_k(\pi) = \begin{cases} 1 & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i \geq \frac{1}{2} + \frac{k}{2m}, \\ \frac{1}{2} & \text{if } |\frac{1}{m}\sum_{i=1}^{m} d_i - \frac{1}{2}| < \frac{k}{2m}, \\ 0 & \text{if } \frac{1}{m}\sum_{i=1}^{m} d_i \leq \frac{1}{2} - \frac{k}{2m}. \end{cases} \tag{5.2}$$

Before stating our lemmas, we will confirm that both the $\widetilde{M_k}$ and $\widetilde{M}'_k$ classes of fuzzy aggregation functions satisfy the axioms necessary for the characterization.

THEOREM 5.1. *For any $k \in [0, m)$ the fuzzy aggregation function $\widetilde{M_k}$ satisfies cancellation, Pareto, monotonicity and neutrality.*

Proof. For this proof, we will use the definition of $\widetilde{M_k}$ given in Equation (2.7). First, note that the output of $\widetilde{M_k}$ is completely determined by the $sum(\pi)$ and the value of k.

We will now show that $\widetilde{M_k}$ is cancellative. Consider $\pi = (d_1, d_2, ..., d_m)$, such that $sum(\pi) = M$. Suppose $\pi' = (d'_1, d'_2, ...d'_m)$ satisfies $d'_i = d_i + \epsilon$, $d'_j = d_j - \epsilon$, and $d'_k = d_k$ for all $k \neq i, j$. Then it follows that $sum(\pi') = M - \epsilon + \epsilon = M = sum(\pi)$. Since k is fixed, it follows that $\widetilde{M_k}(\pi) = \widetilde{M_k}(\pi')$.

Next we will show that $\widetilde{M_k}$ is Pareto. Consider $\widetilde{M_k}(0, 0, ..., 0)$. It follows that $sum(0, 0, ..., 0) = 0$, and $\frac{1}{2}(m - k)$ is minimal as k approaches m, since $k < m$. Thus, for all k, $\frac{1}{2}(m - k) > 0 = sum(0, 0, ..., 0)$, hence $\widetilde{M_k}(0, 0, ..., 0) = 0$. Now consider $\widetilde{M_k}(1, 1, ..., 1)$. It follows that $sum(1, 1, ..., 1) = m$, and $\frac{1}{2}(m + k)$ is maximal as k approaches m, since $k < m$. Therefore, $\frac{1}{2}(m + k) < m = sum(1, 1, ..., 1)$. Hence $\widetilde{M_k}(1, 1, ..., 1) = 1$ and $\widetilde{M_k}$ is Pareto.

Now we will show that $\widetilde{M_k}$ is neutral. Consider $N(\pi)$. We can easily see that $sum(N(\pi)) = \sum_{i=1}^{m}(1 - d_i) = m - sum(\pi)$. Furthermore, if $\widetilde{M_k}(\pi) = 1$, then $sum(\pi) > \frac{1}{2}(m + k)$, so $sum(N(\pi)) = m - sum(\pi) < m - \frac{1}{2}(m + k) = \frac{1}{2}(m - k)$. Thus

$\widetilde{M_k}(N(\pi)) = -1$. Similarly, if $\widetilde{M_k}(\pi) = -1$, then by the same algebra, $\widetilde{M_k}(N(\pi)) = 1$. Lastly, if $\widetilde{M_k}(\pi) = 0$, then $|sum(\pi) - \frac{m}{2}| \leq \frac{1}{2}k$. It follows that

$$
\begin{aligned}
|sum(N(\pi)) - \frac{m}{2}| &= |m - sum(\pi) - \frac{m}{2}| \\
&= |\frac{m}{2} - sum(\pi)| \\
&= |sum(\pi) - \frac{m}{2}|.
\end{aligned}
$$

Therefore, $|sum(N(\pi)) - \frac{m}{2}| = |sum(\pi) - \frac{m}{2}| \leq \frac{1}{2}k$, so $\widetilde{M_k}(N(\pi)) = 0$. Hence $\widetilde{M_k}$ is neutral.

Finally, we will show that $\widetilde{M_k}$ is monotone. Let $\pi' \geq \pi$, then it follows that $sum(\pi') \geq sum(\pi)$. Therefore, by the definition of $\widetilde{M_k}$ it is clear that $\widetilde{M_k}(\pi') \geq \widetilde{M_k}(\pi)$. □

Next, we need to show that class of fuzzy aggregation functions $\widetilde{M'_k}$ rules satisfy cancellation, Pareto, monotonicity and neutrality, as well. This class of fuzzy aggregation functions is markedly different than the $\widetilde{M_k}$ rules, even though that may not be obvious at first. Before proving the theorem about this class of functions, consider this remark, to understand the differences and necessity of two proofs.

Remark 5.1. *The $\widetilde{M'_k}$ rules allow for values of $k \in (0, m]$, which allows for $k = m$. In this case, notice that $\widetilde{M'_k}(\pi) = 1$ only if $sum(\pi) = m$, and $\widetilde{M'_k}(\pi) = 0$ only if $sum(\pi) = 0$. However, $\widetilde{M_k}$ allows for values of $k \in [0, m)$, so while $k \neq m$, it does allow $k = 0$, which cannot happen in the $\widetilde{M'_k}$ rules. With $k = 0$, the $\widetilde{M_k} = \widetilde{f_m}$, the fuzzy simple majority rule. There are other slight differences in the $\widetilde{M_k}$ and $\widetilde{M'_k}$ rules, based on whether the sum is strictly less(greater) than or whether its allowed to be less(greater) than or equal to. While these differences seem slight at first, the difference is distinct and necessary when considering the range of k for each class of functions. So, while they are inherently similar, these two classes are functions are indeed two different classes of functions.*

Now, we may continue with the proof for $\widetilde{M}'_k$ rules.

THEOREM 5.2. *For any $k \in (0, m]$ the fuzzy aggregation functions $\widetilde{M}'_k$ satisfies cancellation, Pareto, monotonicity and neutrality.*

Proof. For this proof, we will use the definition of $\widetilde{M}'_k$ given in Equation (2.8). First, note that, as with the previous proof, the output of $\widetilde{M}'_k$ is completely determined by the $sum(\pi)$ and the value of k.

We will now show that $\widetilde{M}'_k$ is cancellative. Consider $\pi = (d_1, d_2, ..., d_m)$, such that $sum(\pi) = S$. Then for $\pi' = (d'_1, d'_2, ...d'_m)$ such that $d'_i = d_i + \epsilon$, $d'_j = d_j - \epsilon$, and $d'_k = d_k$ for all $k \neq i, j$. Then it follows that $sum(\pi') = S - \epsilon + \epsilon = S = sum(\pi)$. Since k is fixed, it follows that $\widetilde{M}'_k(\pi) = \widetilde{M}'_k(\pi')$.

Next we will show that $\widetilde{M}'_k$ is Pareto. Consider $\widetilde{M}'_k(0, 0, ..., 0)$. It follows that $sum(0, 0, ..., 0) = 0$. Also $\frac{1}{2}(m-k)$ is minimal when $k = m$, and $\frac{1}{2}(m-m) = 0$. Thus, for all k, $\frac{1}{2}(m-k) \geq 0 = sum(0, 0, ..., 0)$, hence $\widetilde{M}'_k(0, 0, ..., 0) = 0$. Now consider $\widetilde{M}'_k(1, 1, ..., 1)$. It follows that $sum(1, 1, ..., 1) = m$, and $\frac{1}{2}(m+k)$ is maximal when $k = m$, and $\frac{1}{2}(m+m) = m$. Therefore, $\frac{1}{2}(m+k) \leq m = sum(1, 1, ..., 1)$. Hence $\widetilde{M}'_k(1, 1, ..., 1) = 1$ and $\widetilde{M}'_k$ is Pareto.

Now we will show that $\widetilde{M}'_k$ is neutral. Consider $N(\pi)$. We can easily see that $sum(N(\pi)) = \sum_{i=1}^{m}(1 - d_i) = m - sum(\pi)$. Furthermore, if $\widetilde{M}'_k(\pi) = 1$, then $sum(\pi) \geq \frac{1}{2}(m+k)$, so $sum(N(\pi)) = m - sum(\pi) \leq m - \frac{1}{2}(m+k) = \frac{1}{2}(m-k)$. Thus $\widetilde{M}'_k(N(\pi)) = -1$. Similarly, if $\widetilde{M}'_k(\pi) = -1$, then by the same algebra, $\widetilde{M}'_k(N(\pi)) = 1$. Lastly, if $\widetilde{M}'_k(\pi) = 0$, then $|sum(\pi) - \frac{m}{2}| < \frac{1}{2}k$. It follows that

$$\begin{aligned}
|sum(N(\pi)) - \frac{m}{2}| &= |m - sum(\pi) - \frac{m}{2}| \\
&= |\frac{m}{2} - sum(\pi)| \\
&= |sum(\pi) - \frac{m}{2}|.
\end{aligned}$$

Therefore, $|sum(N(\pi)) - \frac{m}{2}| = |sum(\pi) - \frac{m}{2}| < \frac{1}{2}k$, so $\widetilde{M}'_k(N(\pi)) = 0$. Hence $\widetilde{M}'_k$ is neutral.

Finally, we will show that $\widetilde{M'_k}$ is monotone. Let $\pi' \geq \pi$, then it follows that $sum(\pi') \geq sum(\pi)$. Therefore, by the definition of $\widetilde{M'_k}$ it is clear that $\widetilde{M'_k}(\pi') \geq \widetilde{M'_k}(\pi)$. □

The definition of the various axioms were presented in Chapter 2. To begin looking at the fuzzy model, we will prove a few lemmas based on these axioms. The first lemma, requires only that the function be cancellative, as it was proven by Garcia-Lapresta and Llamazares that cancellation completely implies anonymity in this model [8].

Lemma 5.1. *Let the fuzzy decision rule $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ satisfy cancellation. For any two profiles $\pi = (x_1, ..., x_m)$ and $\rho = (y_1, ..., y_m)$ such that $sum(\pi) = sum(\rho)$, then $F(\pi) = F(\rho)$.*

Proof. We will use induction on the length of the profile m to prove this lemma. First, assume that $m = 2$, then $x_1 + x_2 = y_1 + y_2$. Without loss of generality, let $x_1 \geq y_1$, if $x_1 = y_1$, then $x_2 = y_2$, and we are done, so assume the contrary. Then there exists some $\epsilon > 0$ such that $x_1 = y_1 + \epsilon$ and $x_2 = y_2 - \epsilon$. Thus, by cancellation, $F(\pi) = F(\rho)$.

Now, we will assume that the cancellation condition holds for $m - 1$ length profiles, $m \geq 3$.

Case 1: Suppose there exists j such that $x_j = y_j$, define the mapping $G : [0,1]^{m-1} \to \{0, \frac{1}{2}, 1\}$ by $G(z_1, ..., z_{m-1}) = F(\alpha)$, where $\alpha = (a_1, a_2, ..., a_m)$ such that $a_j = x_j$, $a_i = z_i$ for $i < j$ and $\alpha(i) = z_{i-1}$ for $i \geq j + 1$. G is cancellative, since F is cancellative. So, define $\pi' = (x'_1, ..., x'_m)\rho' = (y'_1, ..., y'_m) \in [0,1]^{m-1}$ such that $x'_i = x_i$ for $i < j$, and $x'_i = x_{i+1}$ for $i \geq j$, and ρ' is defined similarly. Then $G(\pi') = F(\pi)$ and $G(\rho') = F(\rho)$. Now, $sum(\pi') = sum(\pi) - x_j$ and $sum(\rho') = sum(\rho) - y_j$. Since $x_j = y_j$ and $sum(\pi) = sum(\rho)$, it follow that $sum(\pi') = sum(\rho')$. Thus, by our inductive hypothesis, $G(\pi') = G(\rho')$. Hence, by our definition of G, $G(\pi') = F(\pi) = F(\rho) = G(\rho')$.

Case 2: Suppose no such j exists such that $x_j = y_j$, that is $x_i \neq y_i$ for all i. Then, consider $E = \{|x_j - y_j| : i = 1, 2, ..., m\}$ and let $\epsilon = \min E$, and call one of the terms where this occurs j. Without loss of generality, let $x_j > y_j$. Then there exists $i \in \mathcal{N}_m$ such that $x_i < y_i$, since $sum(\pi) = sum(\rho)$. Create ρ' such that $y'_j = y_j + \epsilon$, $y'_i = y_i - \epsilon$ and $y_k = y_k$, for all $k \neq i, j$. Then, by cancellation, $F(\rho') = F(\rho)$. Now, consider $F(\pi)$ and $F(\rho')$, by construction, $y'_j = x_j$. Then, we can use Case 1 to assert that $F(\pi) = F(\rho')$. Thus, by transitivity, $F(\pi) = F(\rho)$. □

Next, monotonicity is added to the axioms the fuzzy aggregation function satisfies. Once that is done, we can compare the output of the function on two profiles, π and ρ, based solely on $sum(\pi)$ and $sum(\rho)$.

Lemma 5.2. *Let the fuzzy decision rule $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ satisfy cancellation and monotonicity. For any two profiles $\pi = (x_1, ..., x_m)$ and $\rho = (y_1, ..., y_m)$ such that $sum(\pi) \leq sum(\rho)$, then $F(\pi) \leq F(\rho)$.*

Proof. If $sum(\pi) = sum(\rho)$, we know from Lemma 5.1 that $F(\pi) = F(\rho)$. So we will assume that $sum(\pi) < sum(\rho)$. Then there exists some $\epsilon > 0$, such that $sum(\rho) = sum(\pi) + m\epsilon$. Create $\rho' = (y'_1, ..., y'_m)$ such that $y'_i = \min\{x_i + \epsilon, 1\}$ for all $i \in \mathcal{N}_m$. Then note that $\pi \leq \rho'$ and $sum(\rho) \geq sum(\rho') \geq sum(\pi)$. Then let $\delta = \sum_{i=1}^{m} \epsilon - (y_i - x_i)$. Notice
$\delta = sum(\pi) + m\epsilon - sum(\rho')$. If $\delta = 0$, then let $\widetilde{\rho} = \rho'$.

If $\delta > 0$, create $\widetilde{\rho}$ as follows, so that $sum(\widetilde{\rho}) = sum(\rho') + \delta = sum(\rho)$. For $i = 1, 2, ..., m-1$, let $\delta_i = y'_i + \delta_{i-1} - 1$ with the convention $\delta_0 = \delta$. For example $\delta_1 = \rho'(1) + \delta - 1$. If $\delta_1 \leq 0$, then $\widetilde{\rho}$ is defined by

$$\widetilde{\rho} = y'_1 + \delta, y + 2', y'_3, ..., y'_m).$$

If $\delta_1 > 0$ and $\delta_2 \leq 0$, the $\widetilde{\rho}$ is defined by

$$\widetilde{\rho} = (1, y + 2' + \delta_1, y_3', ..., y_m').$$

This pattern continues until we either find the first index i such that $\delta_i > 0$ and $\delta_{i+1} \leq 0$ or we get $\delta_i > 0$ for all $i = 1, 2, ..., m-1$. Observe that

$$\begin{aligned}
\delta_{m-1} &= y_{m-1}' + \delta_{m-2} - 1 \\
&= (y_{m-1}' - 1) + +\rho'(m-2) + \delta_{m-3} \\
&= y_{m-1}' + y_{m-2}' - 2 + \delta_{m-3} \\
&. \\
&. \\
&. \\
&= \sum_{i=1}^{n} y_i' - (m-1) + \delta
\end{aligned}$$

Since $\delta = sum(\rho) - sum(\rho')$, we can determine

$$\begin{aligned}
\delta_{m-1} &= \sum_{i=1}^{m-1} y_i' - (m-1) + sum(\rho) - sum(\rho') \\
&= - y_m' - m + 1 + sum(\rho) \\
y_m' + \delta_{m-1} &= sum(\rho) - m + 1 \\
&\leq m - m + 1 \\
&= 1.
\end{aligned}$$

If $\delta_{m-1} \geq 0$, then $\widetilde{\rho} = (z_1, ..., z_m) = (1, 1, ..., 1, y_m' + \delta_{m-1})$. If $\delta_{m-1} \leq 0$, then there exists a first index j such that $1 \leq j \leq m-1$, $\delta_j \leq 0$, and $\delta_i > 0$ for all integers $i \in [0, j-1]$. In this case, we let $z_i = 1$ for all $i < j$, $z_j = y_j' + \delta_{j-1}$, and $z_k = y_k'$ for all integers $k \in [j+1, m]$. Therefore, $sum(\rho) = sum(\rho') + \delta = sum(\widetilde{\rho})$, so by Lemma5.1, $F(\rho) = F(\widetilde{\rho})$. Furthermore, each $z_i \geq x_i$, it follows from monotonicity that $F(\pi) \leq F(\widetilde{\rho})$. Hence $F(\pi) \leq F(\rho)$. □

These two lemmas allow us to prove the Llamazares and Garcia-Lapresta characterization result of the $\widetilde{M_k}$ and $\widetilde{M'_k}$ rules[8]. Our proof uses a different approach than Llamazares and Garcia-Lapresta, but has the same conclusion.

THEOREM 5.3. *Let $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ satisfy cancellation, Pareto, monotonicity and neutrality, then F is either an $\widetilde{M_k}$ or $\widetilde{M'_k}$ rule.*

Proof. Consider $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ and let $k = \inf\{2sum(\pi) - m : F(\pi) = 1\}$. Now since F is Pareto, $F(1,1,...,1) = 1$ and since F is neutral, $F(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) = \frac{1}{2}$. We know $sum(\frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}) = \frac{1}{2}m$ and $sum(1,1,...,1) = 1$ so by Lemma 5.2, if $F(\pi) = 1$, $\frac{1}{2}m < sum(\pi) \le m$. So $0 < k \le m$. If there exists a profile ρ such that $F(\rho) = 1$ and $2sum(\rho) - m = k$, then we will show that $F = \widetilde{M'_k}$ rule with this k. Otherwise, we will show that $F = \widetilde{M_k}$ rule.

First, assume that F attains its infimum. Let $\pi = (d_1, d_2, ..., d_m)$ and assume that $F(\pi) = 1$. By Lemma 5.2 and the nature of the infimum, $2sum(\pi) - m \ge k$. Since $sum(\pi) \ge \frac{1}{2}(m+k)$, it follows from Equation (2.8) that $\widetilde{M_k}'(\pi) = 1$.

Now assume that $F(\pi) = 0$. Since F is neutral, $F(N(\pi)) = 1$. Therefore, $2sum(N(\pi)) - m \ge k$, so $m - 2sum(\pi) \ge k$. Since $sum(\pi) \le \frac{1}{2}(m-k)$ it follows from Equation (2.27) that $\widetilde{M_k}'(\pi) = 0$.

Now, let $F(\pi) = \frac{1}{2}$, then neither $2sum(\pi) - m \ge k$ nor $m - 2sum(\pi) \ge k$; therefore, $|sum(\pi) - \frac{m}{2}| < \frac{k}{2}$ and so $\widetilde{M_k}'(\pi) = \frac{1}{2}$.

Finally, assume that there does not exist a profile ρ such that $F(\rho) = 1$ and $2sum(\rho) - m = k$. Let $F(\pi) = 1$, then from Lemma 5.2 and the nature of the infimum, $2sum(\pi) - m > k$. Thus $\frac{1}{m}\sum_{i=1}^{m} d_i > \frac{1}{2} + \frac{k}{2m}$, so $\widetilde{M_k}(\pi) = 1$.

Now, let $F(\pi) = 0$, then since F is neutral, $F(N(\pi)) = 1$. Therefore, $2sum(N(\pi)) - m > k$, so $m - 2sum(\pi) > k$, so $\widetilde{M_k}(\pi) = 0$.

Lastly, let $F(\pi) = \frac{1}{2}$, then neither $2sum(\pi) - m > k$ nor $m - 2sum(\pi) > k$; therefore, $|sum(\pi) - \frac{m}{2}| \le \frac{k}{2}$, so $\widetilde{M_k}(\pi) = \frac{1}{2}$. $\square$

Combining Theorems 5.1 and 5.2 with Theorem 5.3, the following corollary falls out as a consequence and confirms the main result from the Garcia-Lapresta and Llamazares [8].

COROLLARY 5.1. *A fuzzy aggregation function* $F : [0,1]^m \to \{0, \frac{1}{2}, 1\}$ *satisfies cancellation, Pareto, monotonicity and neutrality if and only if* $F = \widetilde{M_k}$ *or* $F = \widetilde{M'_k}$.

The fuzzy aggregation model is the third and final model where we explored the difference of votes rules. It is notable that when characterizing these rules in this model and the infinite model, an additional axiom was needed as opposed to characterizing these rules in the finite model. In the fuzzy model, we had the addition of Pareto, and in our infinite model, we had the addition of the zero co-finite axiom. We were not able to look into the extended difference of votes rules that are not neutral, as we did in the previous two models. Though, we believe a similar characterization can be proven. This characterization will be left for future work. The next Chapter will discuss the application of such rules, by the use of computer simulations and statistics to predict the variability of such rules from the simple majority rule function in the appropriate model.

CHAPTER 6
SIMULATING DIFFERENCE OF VOTES RULES AND OTHER VOTING RULES

In this chapter, we take a turn and look at making comparisons between two voting rules in the finite voting model. Theoretically, we can calculate the probability that Llamazares' difference of votes rule (M_k) would have an output that is different than that of the simple majority rule function (SMR). However, through computer simulations we can look at the actual difference that occurs. It turns out the probability we thought would hold did not hold in simulation and indicates there is more going on than we immediately thought. Furthermore, we can use computer simulations to compare the output of simple majority rule with the output of the Electoral College. The Electoral College is the voting method used by the United States to elect its President. First, we will compare the hypothesized agreement to the simulation agreement of the M_k rule with (SMR). Then we will compare the Electoral College to simple majority rule. While the Electoral College is not in the class of functions we discussed in the previous chapters, we wanted to use it since it is such an important voting method in the United States. Our hope is that the simulation and analysis given in this chapter will provide some insight into the electoral system in the United States. For this chapter, we will be assuming that we are in the finite aggregation model, but will not have a fixed value of the size of the voting population $\mathcal{N}$, in fact we will look at what happens as this number changes.

First, let's take a look at the theoretical set of profiles where an M_k rule, with $k \geq 1$, would differ from (SMR). Recall that (SMR) can be defined using the difference of vote rule M_0. We will let $|\mathcal{N}| = N$. We can see that for each N, there are 3^N profiles. However, for all $\pi = (x_1, x_2, ..., x_N) \in \{-1, 0, 1\}^N$, both aggregation functions can be determined solely by $n_+(\pi) - n_-(\pi) = \sum_{i=1}^{N} x_i$. Thus we assumed that we need to only count the profiles in π that have different sums. So initially, the idea was this. There are $2N + 1$ different possible sums. For each k, there are $2k$ sums where M_k rules differ from (SMR). Therefore, we can say the probability that the outputs differ is

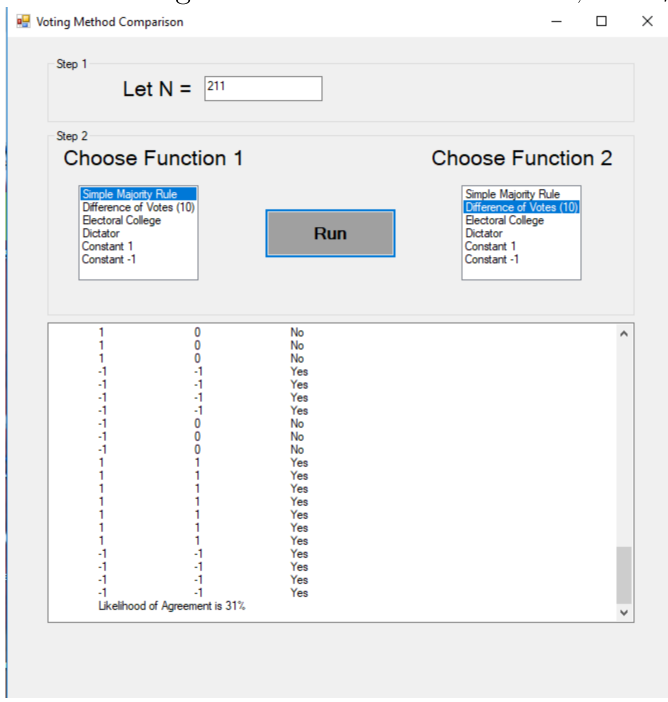
$$\frac{2k}{2N+1} \tag{6.1}$$

If we want the outputs to differ at a level that is statistically significant, then for some $0 < \alpha < 1$, we need

$$\begin{aligned} \frac{2k}{2N+1} &> \alpha \\ \text{or } N &< \frac{k}{\alpha} - \frac{1}{2}. \end{aligned}$$

That is to say, we want N to be smaller than $\frac{k}{\alpha} - \frac{1}{2}$ forM_k and (SMR) to have a statistically significant difference of their outputs at a level of $\alpha = .05$. With $k = 10$ and $\alpha = .05$, that would require your population to be less than 200 for the M_k to vary from (SMR) significantly. However, in our simulation, this looks to be off. This program was run ten times, generating $1,000$ comparisons with $N > 200$ and $k = 10$, and the difference in the outputs was highly significant. Whereas, this hypothesis would indicate the difference would be statistically insignificant. An example of one of the runs of the program can be noticed below in Figure 6.1.

Figure 6.1: For $N = 211$ and $k = 10$, the M_k v. (SMR).



Notice that with $N = 211$, as in Figure 6.1, we get a probability of agreement of 0.31. This is a very low probability of agreement. As mentioned above, this same simulation was run many times, with similar results. When looking further into this, it was determined that we were not including the number of different times when each sum occurs, but how to include this seems unclear. Notice that while there is only one way to get a sum of N, there are N ways to get a sum of $N - 1$, and $\frac{N(N+1}{2}$ ways to get a sum of $N - 2$. Now, most of these cancel out, as they will get the same output in M_k as in (SMR). Determining this statistically proved to be harder than expected. Therefore, it was easier to use simulations to get an estimate of the relationship between k and N. For the results to be meaningful, we must fix k and adjust the values of N. Therefore, we ran the simulation with fixed $k = 10$ and growing numbers of N, we can see that N must be much larger in order for the results to not be significant. The reader can look at the code that generated these results in the Appendix. Some of the results are highlighted below.

Figure 6.2: For $N = 1000$ and $k = 10$, the M_k v. (SMR).

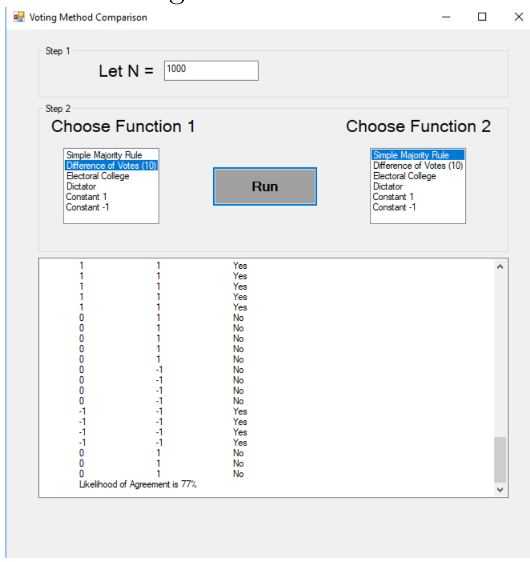


Figure 6.3: For $N = 1750$ and $k = 10$, the M_k v. (SMR).

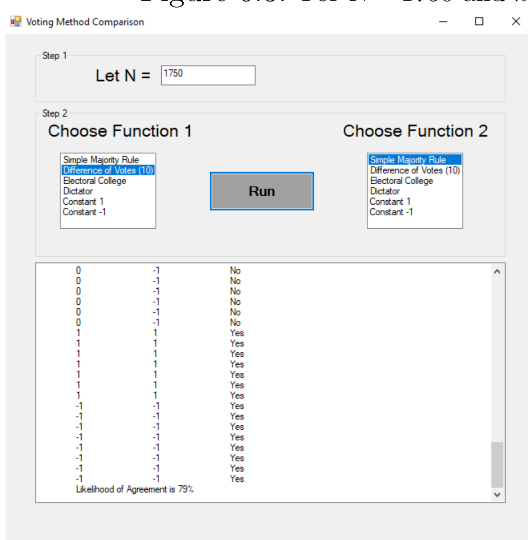


Figure 6.4: For $N = 2000$ and $k = 10$, the M_k v. (SMR).

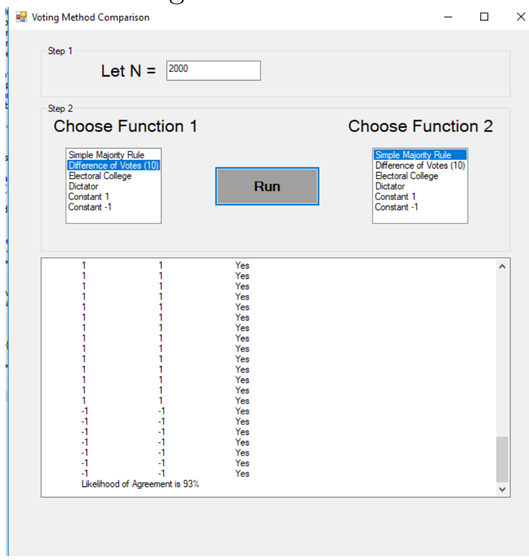


Figure 6.5: For $N = 2000$ and $k = 10$, the M_k v. (SMR).

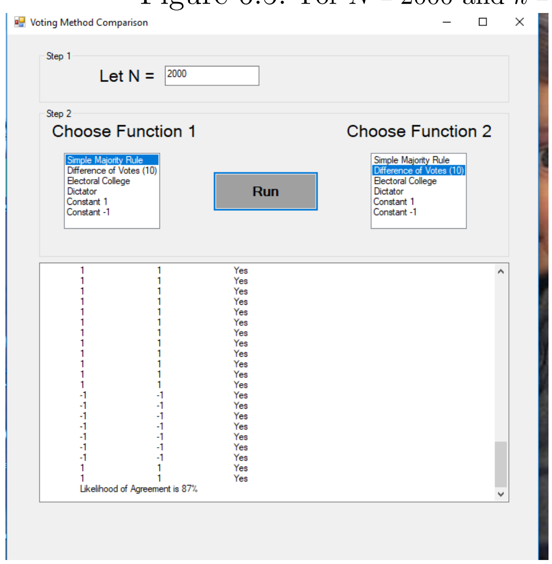


Figure 6.6: For $N = 5000$ and $k = 10$, the M_k v. (SMR).

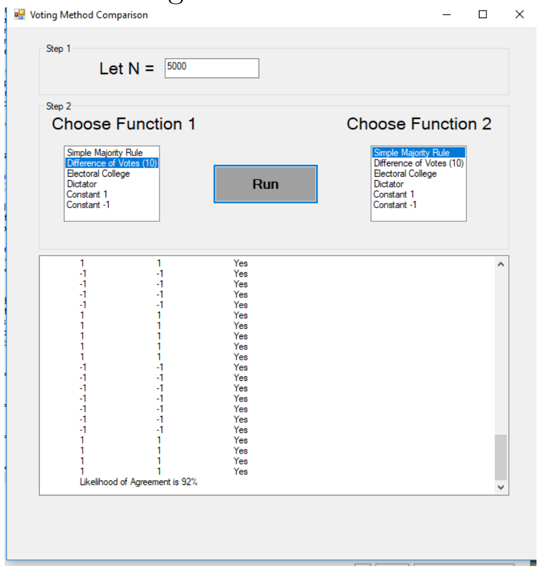


Figure 6.7: For $N = 8500$ and $k = 10$, the M_k v. (SMR).

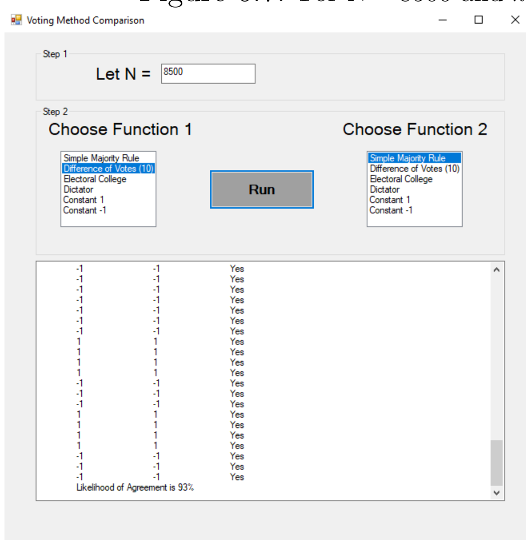


Figure 6.8: For $N = 9500$ and $k = 10$, the M_k v. (SMR).

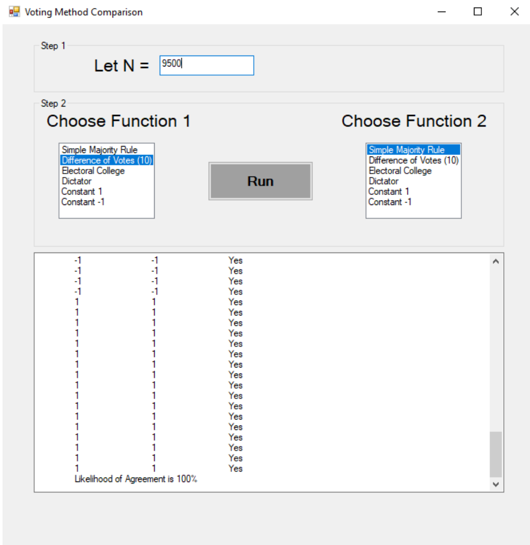
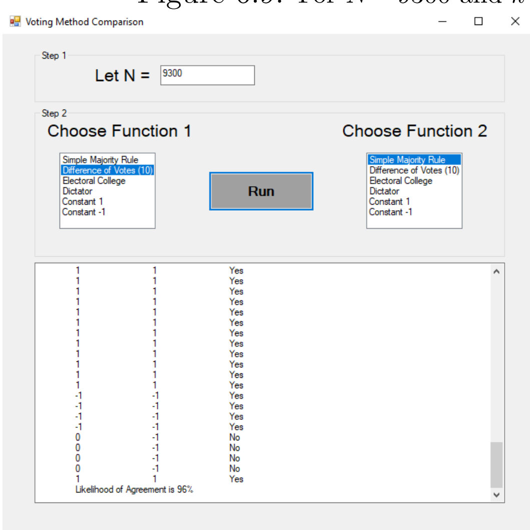


Figure 6.9: For $N = 9300$ and $k = 10$, the M_k v. (SMR).



The above figures show that the outputs of the M_k rule and (SMR) did not become statistically close to identical until $N = 9500$. It was not until $N = 9500$, that in 500 trials we were able to get similarity with 95% probability. In this simulation, we see that while the difference of votes rules looks to be very close to (SMR), the two aggregation functions differ quite significantly for $N < 90k$. While this is not a mathematically proven result, it is an applicable modeling that we can use to estimate the relationship between k and N.

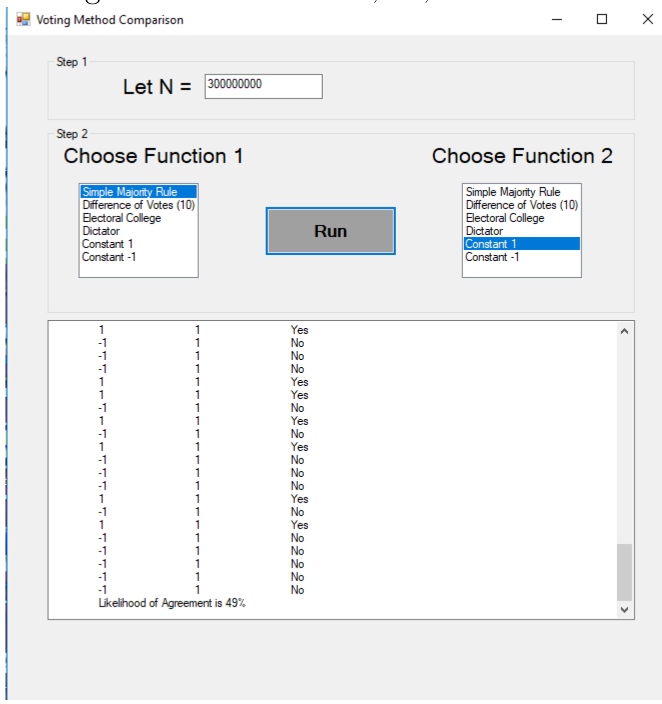
Now, we would like to give an application of this simulator on two aggregation functions that come up quite often when discussing voting theory. One of these two voting methods is commonly referred to as "popular vote," but is the same as (SMR). The other voting method is known as the Electoral College. In the Electoral College, voters vote for electors that then vote for the winner. The Appendix contains all of the C# code used for this simulation, if the reader wants to see the entirety of how the Electoral College is calculated. In this section, we will explain some assumptions made, then give a brief overview of the calculations, as they are quite involved.

When analyzing the results, it is important to note there were a few assumptions made. The first assumption is that all of the states use winner take all elector appointment. While there are a few states that do not, this has not affected the outcome of any election in history. The second assumption made was that the electors use (SMR) to pick the winner of the election. In actuality, the winner has to receive 270 of the 538 votes, which is one more that 50%. Our assumption allows for the output of 0 if both candidates receive 269 votes, but in all of our simulations, the Electoral College function has never output 0. Also, our simulation makes one large profile and then partitions the profile based on the population of each state, alphabetically. Therefore, another assumption is that the population of each state was fixed with the population determined by the 2010 Census. The Census population determines the number of electors that each state receives, but if a number of people moved after the Census, this could change where the partition should occur in the profile. However, the state populations have not changed drastically in recent years. In the case of extreme movement away from or to any state(s), this simulation should be reworked.

For example, we started with Alabama where 1.5% of the population of the United States live, so we multiplied our N by 0.015, and made the Alabama partition go from the first voter to the $0.015 \cdot N$ voter. If the ending number was not an integer, the program rounded down, because if we rounded up, there could be an error in the program not being able to run on the states at the end. Once the Alabama partition was made, (SMR) was run on the votes in that partition. The alternative chosen by the Alabama partition was then inputted into a weighted majority rule function, with all the other states partitions' outcomes. Each state's choice, 1, -1, or 0, was then multiplied by the number of Electoral College votes it has. In example, Alabama's choice would be multiplied by six. All of these weighted alternative choices are then summed together and the sign function determines the output. That is, if the sign is positive, 1 is chosen, and if the sign is negative, -1 is chosen.

With each run of the simulation, our simulator generates 100 profiles at a time of size N and runs each of them through both aggregation functions. The constant function 1 and the constant function -1 were used as control functions to make sure that the profiles we were working with were truly a random sample. Below we have included one of these tests, that shows the constant function 1 agrees with (SMR) about 50% of the time. This comparison was also run with the Electoral College, generating a similar result.

Figure 6.10: For $N = 300,000,000$ the Constant Function 1 v. (SMR).



When starting this simulation, it was hypothesized that as N grew, the probability of agreement between the two voting methods would become more precise, or that the variance in the simulations would shrink. This was thought because we assumed that the true probability of difference would be more distinct for larger N. It was also hypothesized that as N grew the probability of disagreement would increase. This hypothesis was based on the fact that out of the 58 presidential elections in the history of the United States of America, only five have resulted in the Electoral College picking a different winner than the candidate that won the popular vote. Moreover, two of these five elections happened in the past 20 years, when the voter population of the United States is now more than $5,000$ times the size of the voter population in the first election in 1792. There were only $28,579$ voters in the first presidential election and in 2016 there were nearly $139,000,000$ [15]. Since we can see from historical evidence that the popular vote and the Electoral College disagreed on 5 times in 55 elections, we would hypothesize that the overall probability of agreement would be close to 90%, since that is the historical average.

When we started to run the simulations, the averages for each value of N seemed to be fairly close to each other, with outliers for both small and large N. Some notable outliers include an agreement as low as 49% and as high as 92%. It is important to note, we ran the program 10 times generating 1000 comparisons at $N = 2000$ and 10 times generating 1000 comparisons at $N = 300,000,000$. Once all of these were run, the percent of agreement for $N = 2000$ was 82.1% with a standard deviation of 10.47, and for $N = 300,000,000$ was 75.5% with a standard deviation of 9.65. These values were calculated with the help of Microsoft Excel. This Excel sheet also ran a T-Test analysis to determine if there was a statistically significant difference in the means, as well as an F-Test to see if their was a statistically significant difference in the variances. With an α level of 0.10, the T-Test showed that the means were statistically significant. This indicates that the probability of agreement for $N = 300,000,000$ is actually statistically smaller than the probability of agreement for $N = 2000$. While that seems to affirm our hypothesis, this would not hold true for smaller α levels. Therefore, a more powerful CPU would be needed to generate more simulations, faster, in order to get a more affirming result. While the T-Test seems to affirm our hypothesis, the F-Test clearly rejected the hypothesis that the variance would get smaller for larger values of N. The results of these tests are listed below.

Figure 6.11: Statistical Values Used for Calculations and Conclusions

	A	B	C	D	E	F
1	SIMULATION RESULTS	α=0.10				
2		N= 2000	N= 300,000,000			
3	1	84	76			
4	2	81	76			
5	3	70	78			
6	4	88	78			
7	5	86	49			
8	6	58	77			
9	7	87	78			
10	8	84	83			
11	9	91	83			
12	10	92	77			
13	Average	82.1	75.5			
14	Standard Deviation	10.471655	9.652288157			
15						
16						
17	Testing one tailed T-test	with null hypothesis that means are equal				
18	alternative hypothesis that mean is smaller when N=300,000,000					
19	T-test Value	α				
20	0.093314748	< 0.10	significant, null hypothesis rejected			
21						
22	Testing an F-test for variance, null hypothesis that variance is equal					
23	0.812175231	>> α	not significant, null hypothesis is kept			
24						
25						

In each of our simulations, whether $N = 2,000$ or $N = 500,000,000$, the likelihood of agreement was almost always between 75% and 88%, with a few outliers. We expected that maybe this likelihood would get closer to a more constant number as N grew, however, that was not the case. Below are three of these runs of the simulation.

Figure 6.12: For $N = 2000$ the Electoral College v. (SMR).

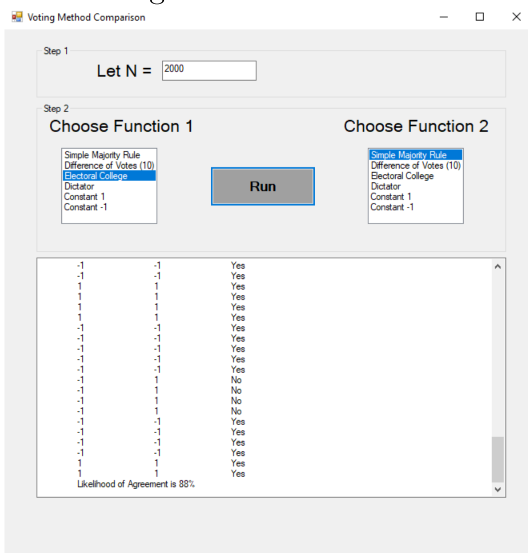


Figure 6.13: For $N = 300,000,000$ the Electoral College v. (SMR).

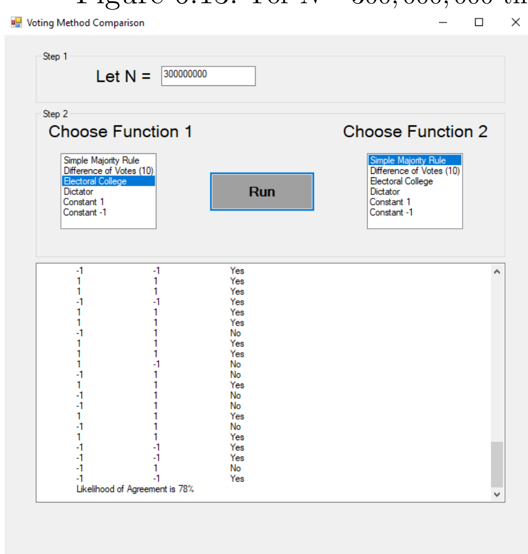
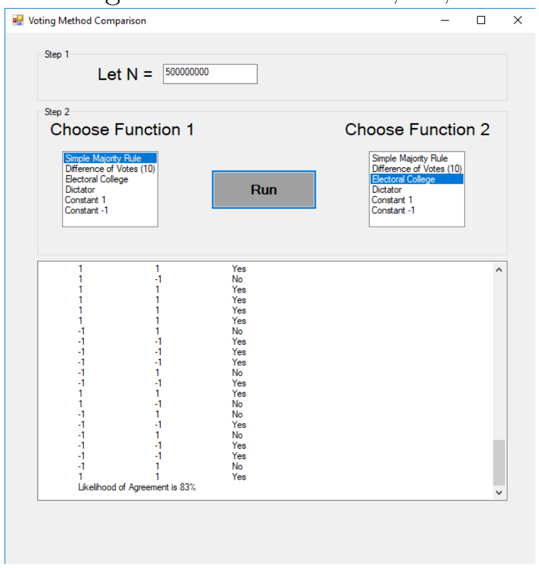


Figure 6.14: For N = 300, 000, 000 the Electoral College v. (SMR).



For these large values of N, the simulator took 20-30 minutes to run. While the simulation was run on 300, 000, 000 multiple times, it was not run as many times on 500, 000, 000, since the result seemed to be similar after two runs, or 200 comparisons. The average, over 1, 000 comparisons, with varying values of N was 79% agreement. This percentage was lower than hypothesized, but not unbelievable. However, this is more than 10% smaller than our historical average.

In order to truly determine if N is significant in the agreement between (SMR) and the Electoral College, we need a computer with a larger memory and more powerful CPU. This will allow for many more comparisons to be run at once. Once we do that, we can better compare the percent of agreement. Hopefully we can get access to such a computer in the future to help verify the results we see from the simulation.

CHAPTER 7
REMARKS AND CONCLUSION

In this dissertation we explored three different voting models, analyzing the classes of difference of votes rules in each. In the finite aggregation model, we characterized a new extension, the $M_{k,l}$ rules, as well as characterizing all finite aggregation rules that satisfied only two axioms: cancellation and monotonicity. From there, these difference of votes rules were extended into the infinite model. There was no previous characterization of the difference of votes rules in the infinite model. So we first defined and characterized the M_k rules and then extended the definition of the $M_{k,l}$ rules. In the third model, the fuzzy aggregation model, the M_k rules were defined and characterized in previous research, but we re-examined them to find a proof of the characterization in line with the previous two models. Lastly, a computer simulation was used to analyze the probability of agreement between the outputs of an M_k rule with $k = 10$ and simple majority rule (SMR), as well as other voting methods that are commonly used. This simulation showed that the difference of votes rules agreed with (SMR) less often than we expected.

In the finite voting model, a notable characterization was that of May in 1952, which used neutrality, anonymity and strict monotonicity to characterize (SMR). All of the characterizations in this paper have utilized the axiom of cancellation, which was not part of May's characterization, but is an axiom satisfied by (SMR). While cancellation can easily seem like an axiom one would want an aggregation function to satisfy, cancellation does not allow for the voting rule to demand a certain percentage of the population to not abstain. This means, for any voter set

$\mathcal{N}$ of size n and integer k, we could determine a winner with only $k+1$ voters not abstaining. This could present a problem when n is significantly larger than k.

There have been a number of characterizations of various classes of voting rules given in the finite model where the axiom of cancellation was not used. Examples of such characterizations can be found in the work by Hoots and Powers [9], Dasgupta and Maskin [4], as well as Perry and Powers [16], and many others [17], [10]. These characterizations look further into anonymity and monotonicity without including cancellation. Other extensions of May's Theorem can be found in the work by Asan and Sanver[1], [2], as well as Cato [3]. With the addition of our characterizations, the only axiom that has yet to be completely eliminated is anonymity. Our research has eliminated anonymity, but by including cancellation, we did not completely eliminate it. The next step is to possibly characterize the class of voting rules that satisfy only strict monotonicity and neutrality.

May's Theorem was extended to the infinite voting model by Fey [5], but his characterization used a different representation for the alternatives than was used in the finite voting model. By defining the alternatives in this model to mimic the finite model, we were able to redefine the axioms to extend naturally from the finite model. After doing this, we were able to extend many of the results from the finite model. In particular, we were able to characterize the M_k and $M_{k,l}$ rules in the finite model. However, we were not able to completely describe the class of infinite aggregation rules that satisfy only monotonicity and cancellation. A solution to this problem will require more analysis and is significantly more complex since one needs to consider cases where a subset of voters is infinite.

The fuzzy aggregation model has been used to deal with preference intensities and was first introduced by H. Nurmi [13]. Garcia-Lapresta and Llamazares defined the M_k rules in the fuzzy model and were able to characterize them. While we were able to redefine these rules to be more in line with our definition of the M_k rules in the finite model, and verify the Garcia-Lapresta and Llamazares' characterization of the fuzzy M_k rules, we were not able to characterize a fuzzy version of the $M_{k,l}$ rules. However, the lemmas that were introduced and proven in this thesis should be of assistance in solving this problem.

The computer simulation modeled a truly random sampling of voting profiles. This data showed that while historically the Electoral College only disagreed with popular vote roughly 9% of the time, its probability of disagreement is actually much higher. While this simulator gives profiles that are truly random, that randomness could be part of the reason that the probability is different from the actual practice of the voting methods. Another step would be to try to make the voting profiles more in-line with what we see in the United States, with no more than 60% choosing any one alternative and no less than 40% choosing any one alternative. This may or may not effect the comparison of the voting methods, but may be able to give some additional insight into the matter.

The focus of this thesis has been on three voting models and yet there are other models one could consider. For example, we could allow the size of the voting population to be the cardinality of $\mathbb{R}$, the real numbers. For more information on this model see the article by K. Surekha and K.P.S Bhaskara Rao [18].

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APPENDIX

Below is the C# code used to calculate the various aggregation functions used in the simulations in Chapter 6.

```
using System;
using System.Collections.Generic;
using System.Linq;
using System.Text;
using System.Threading.Tasks;

namespace kin2
{
    public class CalculationClass
   {
        int N;
        int[] vec;

        public int SMR(int[] rho, int start, int stop, out int fs)
        {
            int total = 0;
            for (var index = start; index <= stop; index++)
            {
                total += rho[index];
            }

            if (total == 0)
            {
                fs = total;
            }
            else if (total > 0)
            {
                fs = 1;
            }
            else if (total < 0)
            {
                fs = -1;
            }
            else
            {
                fs = 100;
            }
            return fs;
        }

        public int DVR(int[] rho, out int dr)
```

```
{
    int pos = 0;
    int neg = 0;
    int count;

    foreach (int i in rho)
    {
        if (i == -1)
        {
            neg++;
        }
        else if (i == 1)
        {
            pos++;
        }
        else
        {

        }
    }

    count = Math.Abs(pos - neg);
    int choose = pos - neg;

    if (count < 10)
    {
        dr = 0;
    }
    else if (choose > 0)
    {
        dr = 1;
    }
    else if (choose < 0)
    {
        dr = -1;
    }
    else
        dr = 101;

    return dr;
}

public int WMR(int[] rho, out int fb)
{
    //state variables, alphabetical
    int total = 0;
    int A = 0;
    int B = 0;
    int C = 0;
    int D = 0;
    int E = 0;
    int f = 0;
    int g = 0;
```

```
int h = 0;
int i = 0;
int j = 0;
int k = 0;
int l = 0;
int m = 0;
int n = 0;
int o = 0;
int p = 0;
int q = 0;
int r = 0;
int s = 0;
int t = 0;
int u = 0;
int v = 0;
int w = 0;
int x = 0;
int y = 0;
int Aa = 0;
int Bb = 0;
int Cc = 0;
int Dd = 0;
int Ee = 0;
int ff = 0;
int gg = 0;
int hh = 0;
int ii = 0;
int jj = 0;
int kk = 0;
int ll = 0;
int mm = 0;
int nn = 0;
int oo = 0;
int pp = 0;
int qq = 0;
int rr = 0;
int ss = 0;
int tt = 0;
int uu = 0;
int vv = 0;
int ww = 0;
int xx = 0;
int yy = 0;
int zz = 0; //District of Columbia

//state population variables by percentage
double Ap = (N * 0.015);
double Bp = (Ap + (N * 0.0023));
double Cp = (Bp + (N * 0.0212));
double Dp = (Cp + (N * 0.0093));
double ep = (Dp + (N * 0.1218));
double fp = (ep + (N * 0.0170));
double gp = (fp + (N * 0.0113));
```

```
double hp = (gp + (N * 0.0029));
double ip = (hp + (N * 0.0631));
double jp = (ip + (N * 0.0318));
double kp = (jp + (N * 0.0045));
double lp = (kp + (N * 0.0051));
double mp = (lp + (N * 0.04));
double np = (mp + (N * 0.0206));
double op = (np + (N * 0.0097));
double ppo = (op + (N * 0.0091));
double qp = (ppo + (N * 0.0138));
double rp = (qp + (N * 0.0145));
double sp = (rp + (N * 0.0041));
double tp = (sp + (N * 0.0187));
double up = (tp + (N * 0.0211));
double vp = (up + (N * 0.0311));
double wp = (vp + (N * 0.0171));
double xp = (wp + (N * 0.0093));
double yp = (xp + (N * 0.0189));
double Aap = (yp + (N * 0.0032));
double Bbp = (Aap + (N * 0.0059));
double Ccp = (Bbp + (N * 0.0090));
double Ddp = (Ccp + (N * 0.0041));
double Eep = (Ddp + (N * 0.0279));
double ffp = (Eep + (N * 0.0065));
double ggp = (ffp + (N * 0.0616));
double hhp = (ggp + (N * 0.0312));
double iip = (hhp + (N * 0.0024));
double jjp = (iip + (N * 0.0361));
double kkp = (jjp + (N * 0.0122));
double llp = (kkp + (N * 0.0125));
double mmp = (llp + (N * 0.0398));
double nnp = (mmp + (N * 0.0033));
double oop = (nnp + (N * 0.0152));
double ppp = (oop + (N * 0.0027));
double qqp = (ppp + (N * 0.0205));
double rrp = (qqp + (N * 0.0855));
double ssp = (rrp + (N * 0.0093));
double ttp = (ssp + (N * 0.0019));
double uup = (ttp + (N * 0.0261));
double vvp = (uup + (N * 0.0223));
double wwp = (vvp + (N * 0.0057));
double xxp = (wwp + (N * 0.018));
double yyp = (xxp + (N * 0.0018));
double zzp = (yyp + (N * 0.0021));

//for (var index = 0; index < Ap; index++)
//{
int Alabama = 0;
int place = Convert.ToInt32(Ap);
SMR(vec, Alabama, place, out A);
//}

//for (var index = Convert.ToInt32(Ap); index < Bp; index++)
```

```
//{
int Alaska = Convert.ToInt32(Ap);
place = Convert.ToInt32(Bp);
SMR(vec, Alaska, place, out B);
// }


//for (var index = Convert.ToInt32(Bp); index < Cp; index++)
//{
int Arizona = Convert.ToInt32(Bp);
place = Convert.ToInt32(Cp);
SMR(vec, Arizona, place, out C);
// }

//for (var index = Convert.ToInt32(Cp); index < Dp; index++)
//{
int Arkansas = Convert.ToInt32(Cp);
place = Convert.ToInt32(Dp);
SMR(vec, Arkansas, place, out D);
//}

//for (var index = Convert.ToInt32(Dp); index < ep; index++)
//{
int California = Convert.ToInt32(Dp);
place = Convert.ToInt32(ep);
SMR(vec, California, place, out E);
//}

//for (var index = Convert.ToInt32(ep); index < fp; index++)
//{
int Colorado = Convert.ToInt32(ep);
place = Convert.ToInt32(fp);
SMR(vec, Colorado, place, out f);
// }

//for (var index = Convert.ToInt32(fp); index < gp; index++)
//{
int Connecticut = Convert.ToInt32(fp);
place = Convert.ToInt32(gp);
SMR(vec, Connecticut, place, out g);
//}
//for (var index = Convert.ToInt32(gp); index < hp; index++)
//{
int Delaware = Convert.ToInt32(gp);
place = Convert.ToInt32(hp);
SMR(vec, Delaware, place, out h);
//}
//for (var index = Convert.ToInt32(hp); index < ip; index++)
//{
int Florida = Convert.ToInt32(hp);
place = Convert.ToInt32(ip);
SMR(vec, Florida, place, out i);
//}
```

```
//for (var index = Convert.ToInt32(ip); index < jp; index++)
//{
int Georgia = Convert.ToInt32(ip);
place = Convert.ToInt32(jp);
SMR(vec, Georgia, place, out j);
//}
//for (var index = Convert.ToInt32(jp); index < kp; index++)
//{
int Hawaii = Convert.ToInt32(jp);
place = Convert.ToInt32(kp);
SMR(vec, Hawaii, place, out k);
//}
//for (var index = Convert.ToInt32(kp); index < lp; index++)
//{
int Idaho = Convert.ToInt32(kp);
place = Convert.ToInt32(lp);
SMR(vec, Idaho, place, out l);
//}
//for (var index = Convert.ToInt32(lp); index < mp; index++)
//{
int Illinois = Convert.ToInt32(lp);
place = Convert.ToInt32(mp);
SMR(vec, Illinois, place, out m);
//}
//for (var index = Convert.ToInt32(mp); index < np; index++)
//{
int Indiana = Convert.ToInt32(mp);
place = Convert.ToInt32(np);
SMR(vec, Indiana, place, out n);
//}
//for (var index = Convert.ToInt32(np); index < op; index++)
//{
int Iowa = Convert.ToInt32(np);
place = Convert.ToInt32(op);
SMR(vec, Iowa, place, out o);
//}
//for (var index = Convert.ToInt32(op); index < ppo; index++)
//{
int Kansas = Convert.ToInt32(op);
place = Convert.ToInt32(ppo);
SMR(vec, Kansas, place, out p);
//}
//for (var index = Convert.ToInt32(ppo); index < qp; index++)
//{
int Kentucky = Convert.ToInt32(ppo);
place = Convert.ToInt32(qp);
SMR(vec, Kentucky, place, out q);
//}
//for (var index = Convert.ToInt32(qp); index < rp; index++)
//{
int Louisiana = Convert.ToInt32(qp);
place = Convert.ToInt32(rp);
SMR(vec, Louisiana, place, out r);
```

```
//}
//for (var index = Convert.ToInt32(rp); index < sp; index++)
//{
int Maine = Convert.ToInt32(rp);
place = Convert.ToInt32(sp);
SMR(vec, Maine, place, out s);
//}
//for (var index = Convert.ToInt32(sp); index < tp; index++)
//{
int Maryland = Convert.ToInt32(sp);
place = Convert.ToInt32(tp);
SMR(vec, Maryland, place, out t);
//}
//for (var index = Convert.ToInt32(tp); index < up; index++)
//{
int Massachusetts = Convert.ToInt32(tp);
place = Convert.ToInt32(up);
SMR(vec, Massachusetts, place, out u);
//}
//for (var index = Convert.ToInt32(up); index < vp; index++)
//{
int Michigan = Convert.ToInt32(up);
place = Convert.ToInt32(vp);
SMR(vec, Michigan, place, out v);
//}
//for (var index = Convert.ToInt32(vp); index < wp; index++)
//{
int Minnesota = Convert.ToInt32(vp);
place = Convert.ToInt32(wp);
SMR(vec, Minnesota, place, out w);
//}
//for (var index = Convert.ToInt32(wp); index < xp; index++)
//{
int Mississippi = Convert.ToInt32(wp);
place = Convert.ToInt32(xp);
SMR(vec, Mississippi, place, out x);
//}
//for (var index = Convert.ToInt32(xp); index < yp; index++)
//{
int Missouri = Convert.ToInt32(xp);
place = Convert.ToInt32(yp);
SMR(vec, Missouri, place, out y);
//}
//for (var index = Convert.ToInt32(yp); index < Aap; index++)
//{
int Montana = Convert.ToInt32(yp);
place = Convert.ToInt32(Aap);
SMR(vec, Montana, place, out Aa);
//}
//for (var index = Convert.ToInt32(Aap); index < Bbp; index++)
//{
int Nebraska = Convert.ToInt32(Aap);
place = Convert.ToInt32(Bbp);
```

```
SMR(vec, Nebraska, place, out Bb);
//}
//for (var index = Convert.ToInt32(Bbp); index < Ccp; index++)
//{
int Nevada = Convert.ToInt32(Bbp);
place = Convert.ToInt32(Ccp);
SMR(vec, Nevada, place, out Cc);
//}
//for (var index = Convert.ToInt32(Ccp); index < Ddp; index++)
//{
int NewHampshire = Convert.ToInt32(Ccp);
place = Convert.ToInt32(Ddp);
SMR(vec, NewHampshire, place, out Dd);
//}
//for (var index = Convert.ToInt32(Ddp); index < Eep; index++)
//{
int NewJersey = Convert.ToInt32(Ddp);
place = Convert.ToInt32(Eep);
SMR(vec, NewJersey, place, out Ee);
//}
//for (var index = Convert.ToInt32(Eep); index < ffp; index++)
//{
int NewMexico = Convert.ToInt32(Eep);
place = Convert.ToInt32(ffp);
SMR(vec, NewMexico, place, out ff);
//}
//for (var index = Convert.ToInt32(ffp); index < ggp; index++)
//{
int NewYork = Convert.ToInt32(ffp);
place = Convert.ToInt32(ggp);
SMR(vec, NewYork, place, out gg);
//}
//for (var index = Convert.ToInt32(ggp); index < hhp; index++)
//{
int NorthCarolina = Convert.ToInt32(ggp);
place = Convert.ToInt32(hhp);
SMR(vec, NorthCarolina, place, out hh);
//}
//for (var index = Convert.ToInt32(hhp); index < iip; index++)
//{
int NorthDakota = Convert.ToInt32(hhp);
place = Convert.ToInt32(iip);
SMR(vec, NorthDakota, place, out ii);
//}
//for (var index = Convert.ToInt32(iip); index < jjp; index++)
//{
int Ohio = Convert.ToInt32(iip);
place = Convert.ToInt32(jjp);
SMR(vec, Ohio, place, out jj);
//}
//for (var index = Convert.ToInt32(jjp); index < kkp; index++)
//{
int Oklahoma = Convert.ToInt32(jjp);
```

```
place = Convert.ToInt32(kkp);
SMR(vec, Oklahoma, place, out kk);
//}
//for (var index = Convert.ToInt32(kkp); index < llp; index++)
//{
int Oregon = Convert.ToInt32(kkp);
place = Convert.ToInt32(llp);
SMR(vec, Oregon, place, out ll);
//}
//for (var index = Convert.ToInt32(llp); index < mmp; index++)
//{
int Pennsylvania = Convert.ToInt32(llp);
place = Convert.ToInt32(mmp);
SMR(vec, Pennsylvania, place, out mm);
//}
//for (var index = Convert.ToInt32(mmp); index < nnp; index++)
//{
int RhodeIsland = Convert.ToInt32(mmp);
place = Convert.ToInt32(nnp);
SMR(vec, RhodeIsland, place, out nn);
//}
//for (var index = Convert.ToInt32(nnp); index < oop; index++)
//{
int SouthCarolina = Convert.ToInt32(nnp);
place = Convert.ToInt32(oop);
SMR(vec, SouthCarolina, place, out oo);
//}
//for (var index = Convert.ToInt32(oop); index < qqp; index++)
//{
int SouthDakota = Convert.ToInt32(oop);
place = Convert.ToInt32(qqp);
SMR(vec, SouthDakota, place, out qq);
//}
//for (var index = Convert.ToInt32(qqp); index < rrp; index++)
//{
int Texas = Convert.ToInt32(qqp);
place = Convert.ToInt32(rrp);
SMR(vec, Texas, place, out rr);
//}
//for (var index = Convert.ToInt32(rrp); index < ssp; index++)
//{
int Utah = Convert.ToInt32(rrp);
place = Convert.ToInt32(ssp);
SMR(vec, Utah, place, out ss);
//}
//for (var index = Convert.ToInt32(ssp); index < ttp; index++)
//{
int Vermont = Convert.ToInt32(ssp);
place = Convert.ToInt32(ttp);
SMR(vec, Vermont, place, out tt);
//}
//for (var index = Convert.ToInt32(ttp); index < uup; index++)
//{
```

```
            int Virginia = Convert.ToInt32(ttp);
            place = Convert.ToInt32(uup);
            SMR(vec, Virginia, place, out uu);
            //}
            //for (var index = Convert.ToInt32(uup); index < vvp; index++)
            //{
            int Washington = Convert.ToInt32(uup);
            place = Convert.ToInt32(vvp);
            SMR(vec, Washington, place, out vv);
            //}
            //for (var index = Convert.ToInt32(vvp); index < wwp; index++)
            //{
            int WestVirginia = Convert.ToInt32(vvp);
            place = Convert.ToInt32(wwp);
            SMR(vec, WestVirginia, place, out ww);
            //}
            //for (var index = Convert.ToInt32(wwp); index < xxp; index++)
            //{
            int Wisconsin = Convert.ToInt32(wwp);
            place = Convert.ToInt32(xxp);
            SMR(vec, Wisconsin, place, out xx);
            //}
            //for (var index = Convert.ToInt32(xxp); index < yyp; index++)
            //{
            int Wyoming = Convert.ToInt32(xxp);
            place = Convert.ToInt32(yyp);
            SMR(vec, Wyoming, place, out yy);
            //}
            //for (var index = Convert.ToInt32(yyp); index < zzp; index++)
            //{
            int DC = Convert.ToInt32(yyp);
            place = N - 1;
            SMR(vec, DC, place, out zz);
            // }
 total = ((9 * A) + (3 * B) + (11 * C) + (6 * D) + (55 * E) + (9 * f) + (7 * g) + (3 * h)
+ (29 * i) + (16 * j) + (4 * k) + (l * 4) + (m * 20) + (11 * n) + (o * 6) + (p * 6)
+ (q * 8) + (r * 8) + (s * 4) + (t * 10) + (u * 11) + (v * 16) + (w * 10) + (x * 6)
+ (y * 10) + (Aa * 3) + (Bb * 5) + (Cc * 6) + (Dd * 4) + (Ee * 14) + (ff * 5) + (gg * 29)
+ (hh * 15) + (ii * 3) + (jj * 18) + (kk * 7) + (ll * 7) + (mm * 20) + (nn * 4) + (oo * 9)
+ (pp * 3) + (qq * 11) + (rr * 38) + (ss * 6) + (tt * 3) + (uu * 13) + (vv * 12) + (ww * 5)
+ (xx * 10) + (yy * 3) + (zz * 3));

            if (total == 0)
            {
                fb = 0;
            }
            else if (total > 0)
            {
                fb = 1;
            }
            else if (total < 0)
            {
                fb = -1;
```

```
    }
    else fb = 102;

    return fb;
}

public int D(int[] profile, out int d)
{
    d = profile[1];
    return d;

}

public void NewSample(string fun1, string fun2, int numberN, out string outputText, out
{
    N = numberN;
    vec = new int[N];
    Random rnd = new Random();
    int num = rnd.Next(-1, 2);
    vec[0] = num;
    int out1;
    int out2;
    count = 0;
    outputText = string.Empty;

    for (int i = 1; i < N; i++)
    {
        num = rnd.Next(-1, 2);
        vec[i] = num;
    }

    if (fun1.Equals("Simple Majority Rule"))
    {
        SMR(vec, 0, (N - 1), out out1);
        outputText = outputText + "\r\n\t" + out1.ToString();


        if (fun2.Equals("Difference of Votes (10)"))
        {
            DVR(vec, out out2);
            outputText = outputText + "\t\t" + out2.ToString();

            if (out1 == out2)

            {
                outputText = outputText + "\t\t" + "Yes";
                count = count + 1;
            }
            else
            {
                outputText = outputText + "\t\t" + "No";
            }
        }
```

```
else if (fun2.Equals("Electoral College"))
{
    WMR(vec, out out2);
    outputText = outputText + "\t\t" + out2.ToString();

    if (out1 == out2)

    {
        outputText = outputText + "\t\t" + "Yes";
        count = count + 1;
    }
    else
    {
        outputText = outputText + "\t\t" + "No";
    }
}
else if (fun2.Equals("Constant 1"))
{
    out2 = 1;
    outputText = outputText + "\t\t" + out2.ToString();

    if (out1 == out2)
    {
        outputText = outputText + "\t\t" + "Yes";
        count = count + 1;
    }
    else
    {
        outputText = outputText + "\t\t" + "No";
    }
}
else if (fun2.Equals("Constant -1"))
{
    out2 = -1;
    outputText = outputText + "\t\t" + out2.ToString();

    if (out1 == out2)
    {
        outputText = outputText + "\t\t" + "Yes";
        count = count + 1;
    }
    else
    {
        outputText = outputText + "\t\t" + "No";
    }
}
else if (fun2.Equals("Dictator"))
{
    D(vec, out out2);
    outputText = outputText + "\t\t" + out2.ToString();
    if (out1 == out2)
    {
```

```
                outputText = outputText + "\t\t" + "Yes";
                count = count + 1;
            }
            else
            {
                outputText = outputText + "\t\t" + "No";
            }
        }
    }
    else if (fun1.Equals("Difference of Votes (10)"))
    {
        DVR(vec, out out1);
        outputText = outputText + "\r\n\t" + out1.ToString();

        if (fun2.Equals("Simple Majority Rule"))
        {
            SMR(vec, 0, (N - 1), out out2);
            outputText = outputText + "\t\t" + out2.ToString();
            if (out1 == out2)

            {
                outputText = outputText + "\t\t" + "Yes";
                count = count + 1;
            }

            else
            {
                outputText = outputText + "\t\t" + "No";
            }
        }
        else if (fun2.Equals("Electoral College"))
        {
            WMR(vec, out out2);
            outputText = outputText + "\t\t" + out2.ToString();
            if (out1 == out2)

            {
                outputText = outputText + "\t\t" + "Yes";
                count = count + 1;
            }

            else
            {
                outputText = outputText + "\t\t" + "No";
            }
        }

        else if (fun2.Equals("Constant 1"))
        {
            out2 = 1;
            outputText = outputText + "\t\t" + out2.ToString();
            if (out1 == out2)
```

```
        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }

        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

    else if (fun2.Equals("Constant -1"))
    {
        out2 = -1;
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }

        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

    else if (fun2.Equals("Dictator"))
    {
        D(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }
}


else if (fun1.Equals("Electoral College"))
{
    WMR(vec, out out1);
    outputText = outputText + "\r\n\t" + out1.ToString();

    if (fun2.Equals("Simple Majority Rule"))
    {
```

```
        SMR(vec, 0, (N - 1), out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }

        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

    else if (fun2.Equals("Difference of Votes (10)"))
    {
        DVR(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();

    }

    else if (fun2.Equals("Constant 1"))
    {
        out2 = 1;
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

    else if (fun2.Equals("Constant -1"))
    {
        out2 = -1;
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }

        else
        {
            outputText = outputText + "\t\t" + "No";
        }
```

```
    }

    else if (fun2.Equals("Dictator"))
    {
        D(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

}
else if (fun1.Equals("Constant 1"))
{
    out1 = 1;
    outputText = outputText + "\r\n\t" + out1.ToString();



    if (fun2.Equals("Simple Majority Rule"))
    {
        SMR(vec, 0, (N - 1), out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

    else if (fun2.Equals("Difference of Votes (10)"))
    {
        DVR(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
```

```
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

    else if (fun2.Equals("Electoral College"))
    {
        WMR(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }

    }
    else if (fun2.Equals("Constant -1"))
    {
        out2 = -1;
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }

    }
    else if (fun2.Equals("Dictator"))
    {
        D(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }
}
```

```
else if (fun1.Equals("Constant -1"))
{
    out1 = -1;
    outputText = outputText + "\r\n\t" + out1.ToString();


    if (fun2.Equals("Simple Majority Rule"))
    {
        SMR(vec, 0, (N - 1), out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }
    else if (fun2.Equals("Difference of Votes (10)"))
    {
        DVR(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }

    else if (fun2.Equals("Electoral College"))
    {
        WMR(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
```

```
    }
    else if (fun2.Equals("Constant 1"))
    {
        out2 = 1;
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }
    else if (fun2.Equals("Dictator"))
    {
        D(vec, out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }
}

else if (fun1.Equals("Dictator"))
{
    D(vec, out out1);
    outputText = outputText + "\r\n\t" + out1.ToString();

    if (fun2.Equals("Simple Majority Rule"))
    {
        SMR(vec, 0, (N - 1), out out2);
        outputText = outputText + "\t\t" + out2.ToString();
        if (out1 == out2)

        {
            outputText = outputText + "\t\t" + "Yes";
            count = count + 1;
        }
        else
        {
            outputText = outputText + "\t\t" + "No";
        }
    }
```

```
else if (fun2.Equals("Difference of Votes (10)"))
{
    DVR(vec, out out2);
    outputText = outputText + "\t\t" + out2.ToString();
    if (out1 == out2)

    {
        outputText = outputText + "\t\t" + "Yes";
        count = count + 1;
    }
    else
    {
        outputText = outputText + "\t\t" + "No";
    }
}

else if (fun2.Equals("Electoral College"))
{
    WMR(vec, out out2);
    outputText = outputText + "\t\t" + out2.ToString();
    if (out1 == out2)

    {
        outputText = outputText + "\t\t" + "Yes";
        count = count + 1;
    }
    else
    {
        outputText = outputText + "\t\t" + "No";
    }
}

else if (fun2.Equals("Constant 1"))
{
    out2 = 1;
    outputText = outputText + "\t\t" + out2.ToString();
    if (out1 == out2)

    {
        outputText = outputText + "\t\t" + "Yes";
        count = count + 1;
    }
    else
    {
        outputText = outputText + "\t\t" + "No";
    }

}
else if (fun2.Equals("Constant -1"))
{
    out2 = -1;
    outputText = outputText + "\t\t" + out2.ToString();
```

```
                    if (out1 == out2)

                    {
                        outputText = outputText + "\t\t" + "Yes";
                        count = count + 1;
                    }
                    else
                    {
                        outputText = outputText + "\t\t" + "No";
                    }
                }
            }
        }
    }
}
```

The C# code below was the code used to create the profiles and run the aggregation functions written above.

```
using System;
using System.Collections.Generic;
using System.ComponentModel;
using System.Data;
using System.Drawing;
using System.Linq;
using System.Text;
using System.Threading.Tasks;
using System.Windows.Forms;

namespace kin2
{
    public partial class VotingMethodComp : Form
    {
        int N;

        public VotingMethodComp()
        {
            InitializeComponent();
            //Setup the progress bar
            progressBar1.Visible = false;
            progressBar1.Minimum = 1;
            progressBar1.Maximum = 100;
            progressBar1.Step = 1;
        }

        private void Run_Click(object sender, EventArgs e)
        {
            bool res = int.TryParse(txtN.Text, out N);

    //Check if a numeric value has been entered for N
   if (!res)
            {
    MessageBox.Show("Please enter a valid numeric value for N", "Error",
    MessageBoxButtons.OK, MessageBoxIcon.Error);
```

```
            }
    //Check if both options are chosen , if not return an error message
    else if (f1.SelectedIndex == -1 || f2.SelectedIndex == -1)
            {
     MessageBox.Show("Please make selection for both function 1
     and function 2",
     "Error", MessageBoxButtons.OK, MessageBoxIcon.Error);
            }
       else
            {
    //Clear out any previous output
    Output.Text = string.Empty;

    string fun1 = f1.SelectedItem.ToString();
    string fun2 = f2.SelectedItem.ToString();
    string sampleOutputs = string.Empty;
    int count = 0;
     decimal prob = 0;
       if (fun1.Equals(fun2))
                {
                    Output.Text = "Functions agree 100%";
                }
                else
                {
     //Display the progress bar
         progressBar1.Visible = true;
         progressBar1.Value = 1;

 Output.Text = "Function 1 Value      Function 2 Value         Agree(Y/N)";

   for(int j = 1; j < 101; j++)
                    {
string newOutputText = string.Empty;
//get the output text for this current sample
 int countSample = 0;
 //gets the count for current sample
     //Invoke the method to create a new sample
        CalculationClass newCalcInstance = new CalculationClass();
        newCalcInstance.NewSample(fun1, fun2, N,
        out newOutputText, out countSample);
        //Aggregate values collected from current sample
          Output.Text = Output.Text + newOutputText;
          count = count + countSample;
                    //update the progressbar
                     progressBar1.PerformStep();
                    }
prob = (count / 100);
Output.Text = Output.Text + "\r\n\tLikelihood of Agreement is "
 + Convert.ToString(count) + "%";

                    //hide the progress bar again
                    progressBar1.Visible = false;
                }
```

```
                }
            }
        }
    }
```

CURRICULUM VITAE

Sarah Schulz King

Academic Record

University of Louisville, Louisville, KY

Ph.D. in Applied and Industrial Mathematics Expected August 2017

Areas of Concentration: Social Choice Theory

Specialty Areas: Operations Research, Optimization, Efficiency, Statistical Reasoning, Social Choice and Voting Theory

Advisor: Dr. Robert Powers

Qualifiers Passed: Real Analysis, Modern Algebra, Mathematical Statistics

Topics Studied: Adv. Combinatorics, Adv. Graph Theory, Foundations of Optimization, Adv. Modern Algebra, Probability and Measure Theory, Mathematical Modeling, Adv. Topics in Social Choice Theory, Complex Analysis, Mathematical Programming, Simulating Discrete Systems, Statistical Inference, Linear Statistical Modeling, Regression Analysis with R full transcripts available upon request.

Eastern Kentucky University, Richmond, KY

M.A. Mathematics July 2013

Master's Thesis: Eigenvalue Comparisons and Positive Solutions to a Fourth-Order Three-Point Boundary Value Problem

Advisor: Dr. Jeffrey Neugebauer

Topics Studied: Advanced Seminar in Differential Equations, Multivariate Statistics, Modern Algebra, Mathematical Statistics, Complex Analysis, Advanced Real Analysis

University of Evansville, Evansville, IN

B.A. Mathematics May 2010

Undergraduate Thesis: Using Differential Equations to Model Human Evacuation and Determine Room Capacities Based on Seconds Allotted to Evacuate.

Advisor: Dr. Talitha Washington

B.S. Political Science May 2010

Undergraduate Research Area: Revamping Neoliberalism in Columbia

Concentration: International Political Economy

Advisor: Dr. Wesley Milner

American University, Washington Semester Program Spring 2008

Economic Policy and Procedure
Research Intern at Hudson Institute, Supervisor: Richard Weitz
Concentration: Central Asian Economics and the Need for Kazakhstani Leadership

Presentations and Papers

EKU Mathematics Symposium / Eastern Kentucky University / April 21, 2017: Beyond Neutrality: Extending Difference of Votes Rules. Advisor: Dr. Robert Powers

EKU Mathematics Colloquium / Eastern Kentucky University / April 19, 2017: Extending Difference of Votes Rules of Three Domains. Advisor: Dr. Robert Powers

S. S. King, R. Powers, Beyond Neutrality: Extended Difference of Votes Rules, in progress.

KyMAA Kentucky State Meeting / Northern Kentucky University / April 2016: Extending Mk Rules to a Countable Population. Advisor: Dr. Robert Powers

EKU Mathematics Symposium / Eastern Kentucky University / April 2015: Mays Theorem; Majority Rule and Extensions through Axiomatic Modifications. Advisor: Dr. Robert Powers

S. S. King, J. T. Neugebauer, Smallest eigenvalues, extremal points, and positive solutions of a fourth order three point boundary value problem, Dynam. Systems Appl., 23 (2014) no. 4

AMS Regional Conference / Louisville, KY / October 2013: Special Section: Extremal points of a fourth order three point boundary value problem. Advisor: Dr. Jeffrey Neugebauer

SEARCDE / University of Tennessee / September 2013 Smallest eigenvalues and positive solutions of a fourth order three point boundary value problem. Advisor: Dr. Jeffrey Neugebauer

Mathematics in Everyday / University of Evansville / May 2010: Using Differential Equations to Model Human Evacuation and Determine Room Capacities Based on Seconds Allotted to Evacuate.

MESCON / University of Evansville / March 2010: Using Differential Equations to Model the Cellular Takeover, Growth, and Regeneration of DNA by the HIV Viron. Advisor: Dr. Talitha Washington

NAM Math Fest 2009: University of the District of Columbia, November

2009: Using Differential Equations to Model the Cellular Takeover, Growth, and Regeneration of DNA by the HIV Viron.

Teaching Experience

University of Louisville, August 2013-Present, Graduate Fellow
Courses Resuscitated: College Algebra, Elements of Calculus, Elementary Statistics
Courses Taught: College Algebra, Contemporary Mathematics

Christian Academy of Indiana, August 2015-June 2016, Instructor
Courses Taught: AP and Dual Credit Calculus, Dual Credit offered through Ivy Tech in Sellersburg, IN

Eastern Kentucky University, January 2012-July 2013, Graduate Assistant
Courses Assisted: Mathematics with Application, College Algebra, Pre-Calculus, Calculus I, Calculus II, Introduction to Statistical Reasoning, Applied Statistics, Business Statistics
Courses Taught: Pre-Algebra, Algebra

University of Evansville, August 2006-May 2010, Official Department Tutor
Aided students with sight and hearing impairments in Calculus II and Calculus III
Tutored Students in College Algebra, Calculus I, World History, U.S. History, Psychology, American Government, and World Politics

Leadership and Volunteer Experience

Hope Southern Indiana: New Albany, IN, July 2015-Present
Organizing, planning and strategizing block parties and other outreach events to the local, impoverished community, to help raise awareness for the services offered at Hope. Teaching and developing new leaders to carry on the tasks of the block party, so myself and other leaders can focus on scaling the events to have the greatest community impact.

Fitness 19 KY 200: Lexington KY, August 2010-Present
Club Manager: Organized and Analyzed Personnel and Membership Data, Facilitated Payroll, and Managed Accounts Payable and Receivable, Co-Lead Monthly Team Meetings, cooperated with other Managers in the area to set up Promotional Events, Marketing Strategies, and Advertising Campaigns

Senior Representative: Model U.N, University of Evansville, 2007-2010
Aided in the Arrangements of Mock Debates and Student events, Competed in multiple State-Wide Events, and winning awards for our school

Kids Club Supervisor: Crossroads Christian Church, 2006-2010
Managed, scheduled and processed payroll for child caregivers during multiple events

offered at the church. Planned and organized children's crafts, stories, and snacks.

Admissions Ambassador: University of Evansville, 2007- 2010
Facilitated tours and festivities for potential and future students visiting campus through events such as "Road Trip" and Open Houses, as well as individual daily visits

Junior Statesmen of America: Stanford University, 2005; Georgetown University, 2004
Participated in congressional style debates, went to numerous political functions and took Collegiate Level American and Comparative Politics at the Host School

Technical Skills and Other Abilities

Highly experienced and trained in Adobe, C and C++ Languages, Intuit, Microsoft Office Suite, LINGO/LINDO, MATLAB, OPNET, Oracle Database, R, and SAS.
Able to learn new skills quickly and with little direction
Self-motivated, task orientated with a focus on the bigger picture, and able to work collaboratively

Honors, Recognitions, and Awards

April 17, 2015, Eastern Kentucky University Symposium - Best Graduate Presentation
2014, University of Louisville, School of Graduate Studies - Graduate Fellowship
2012, Eastern Kentucky University, Dept. of Mathematics - Graduate Assistantship Scholarship
2010, University of Evansville - Mathematics Presentation of the Year
2009, NAM Math Fest - Best Undergraduate Presentation
2009, University of Evansville - College of Arts and Sciences Undergraduate Research Grant
2008, American University Washington Semester Program - Dean's Scholarship
2006-2010, University of Evansville - Academic Achievement Scholarship