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# STRONG QUOTA PAIR SYSTEMS AND MAY'S THEOREM ON MEDIAN SEMILATTICES

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STRONG QUOTA PAIR SYSTEMS AND MAY'S THEOREM ON MEDIAN SEMILATTICES

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DEDICATION

For My Parents

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ABSTRACT

STRONG QUOTA PAIR SYSTEMS AND MAY'S THEOREM ON MEDIAN SEMILATTICES

Lucas Hoots

May 12, 2015

Kenneth May [16], in 1952, characterized simple majority rule in terms of three conditions: anonymity, neutrality, and positive responsiveness. In this thesis, we remove the condition of neutrality and obtain a characterization of the class of voting rules that satisfy anonymity and positive responsiveness. The key concept in this characterization is the notion of a strong quota pair system. The situation with two alternatives studied by May can be thought of as a very simple example of a finite median semilattice. The main result of this thesis is an extension of May's theorem to the domain of all finite median semilattices.

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# CHAPTER 1
# MAY'S THEOREM

In 1952, Kenneth May gave an elegant characterization of simple majority decision on two alternatives [16]. This represents the classic notion of deciding between exactly two candidates by declaring as the winner the candidate who receives the most votes from among a collection of voters. In the following this idea will be presented more precisely, along with the terms and notational conventions required for an easy understanding.

Let $X = \{-1, 0, 1\}$ where $-1$ and $1$ represent two competing alternatives, or candidates; and $0$ represents abstention, or a vote of non-preference. Let $N = \{1, \ldots, n\}$ be the set of $n$ voters with $n \geq 2$. We will call any $P = (x_1, \ldots, x_n) \in X^n$ a *profile* and we will sometimes denote any $x_i$ in $P$ as $P(i)$.

**Definition 1.1.** *A function of the form $f : X^n \to X$ will be called a* **social aggregation function**.

This function receives some profile $P \in X^n$ as its input, representing the votes of $n$ individuals, and outputs a single element of $X$. An output of $0$ indicates a tie between the two candidates, while any other output identifies the "winner" of the election.

Our first example of a social aggregation function is given below, where

$$sign(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0. \end{cases}$$

**Definition 1.2.** *The* **simple majority rule** *function, $f_s$, is defined as:*

$$f_s(P) = sign(\sum_{i=1}^{n} P(i)).$$

It will only be the case that $f_s(P) = 1$ when $P$ contains more ones than negative ones, $f_s(P) = -1$ only when $P$ contains more negative ones than ones, and $f_s(P) = 0$ only when $P$ contains exactly as many ones as negative ones (a tie). This is the version of simple majority decision studied by Kenneth May [16]. May introduced a set of axioms that any reasonably "fair" social aggregation function should satisfy.

**Axiom 1.1.** *A social aggregation function $f : X^n \to X$ satisfies* **anonymity** *if, for any permutation $\sigma$ of $N = \{1, \ldots, n\}$, we have $f(P) = f(\sigma P)$, where $\sigma P = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.*

An anonymous function treats all votes equally, so no voter or group of voters receives undue consideration. Since we will be dealing with anonymous functions, all we are really concerned about is the number of votes each candidate receives. We consider the following notation to help facilitate that idea:

$$K_1(P) = \{i : P(i) = 1\} \text{ and } K_{-1}(P) = \{i : P(i) = -1\}.$$

**Axiom 1.2.** *A social aggregation function $f : X^n \to X$ satisfies* **neutrality** *if, for any $P, P' \in X^n$,*

$$K_1(P) = K_{-1}(P') \text{ and } K_{-1}(P) = K_1(P') \Rightarrow [f(P) = 1 \Leftrightarrow f(P') = -1].$$

A neutral function treats all candidates equally. If candidate 1 wins over candidate $-1$, then switching all the votes should cause a switch in the winning outcome, and vice versa. If the result is a tie, then switching all the votes should cause no change and 0 should still be the outcome. Notice $P'$ in the above can be thought of as $-P$ since it has a $-1$ wherever $P$ has a 1 and a 1 wherever $P$ has a

$-1$. Since our alternatives are 1, $-1$, and 0 we can say that a social aggregation function $f$ satisfies neutrality if and only if, for any profile $P \in X^n$,

$$f(-P) = -f(P).$$

**Definition 1.3.** *For two profiles $P, P' \in X^n$ we say $P \leq P'$ if $P(i) \leq P'(i)$ for all $i \in \{1, \ldots, n\}$.*

In this way we can describe $P'$ as "favoring 1" more than $P$, since $P'$ will have at least as many votes for 1 as $P$ and it can have at most as many votes for $-1$ as $P$. This notion is expressed symbolically below.

$$P \leq P' \text{ iff } K_1(P) \subseteq K_1(P') \text{ and } K_{-1}(P') \subseteq K_{-1}(P). \tag{1.1}$$

It is easy to verify that "$\leq$" forms a partial order on $X^n$.

**Definition 1.4.** *A profile $P \in X^n$ is* **almost equal** *to a profile $P' \in X^n$, denoted $P(ae)P'$, when there exists $i_0 \in N$ such that $P(i) = P'(i)$ for all $i \neq i_0$ and $P(i_0) \neq P'(i_0)$.*

This simply means that the two profiles are identical in every position save exactly one, where a difference must occur.

**Axiom 1.3.** *A social aggregation function $f : X^n \to X$ satisfies* **positive responsiveness** *if $f(P) \in \{0, 1\}$, $P \leq P'$, and $P(ae)P'$ implies $f(P') = 1$.*

This notion is not as complicated as it may appear. If a function outputs a 1 or 0 for a particular profile and we change a single vote in such a way as to favor candidate 1 (by either giving 1 an extra vote or taking a vote away from $-1$), then the function that satisfies positive responsiveness will output a 1 on this new profile.

With these axioms formally in place, we can state May's Theorem.

**Theorem 1.1** (May's Theorem)**.** *A social aggregation function $f$ satisfies anonymity, neutrality, and positive responsiveness if and only if $f = f_s$.*

Our goal is to generalize May's Theorem by proving it for a much broader domain, namely all finite median semilattices. The first step is to refine May's Theorem a bit, so as to make it amenable to generalization. We begin by splitting up positive responsiveness into two simpler conditions. First we present a simple implication of positive responsiveness.

**Lemma 1.1.** *If $f$ satisfies positive responsiveness and $P \leq P'$, $P(ae)P'$ for some $P, P' \in X^n$, then $f(P') = -1$ implies $f(P) = -1$.*

*Proof.* Suppose $f$ satisfies positive responsiveness. Consider $P, P' \in X^n$ such that $P \leq P'$ and $P(ae)P'$. If $f(P') = -1$ then it must be the case that $f(P) = -1$ as well. If not, i.e. $f(P) = 0$ or $f(P) = 1$, then positive responsiveness implies that $f(P') = 1$, contradicting the assumption that $f(P') = -1$. Since $X = \{-1, 0, 1\}$ it must be the case that $f(P) = -1$, as desired. □

**Axiom 1.4.** *A social aggregation function $f : X^n \to X$ satisfies* **monotonicity** *if, for $P, P' \in X^n$, $K_1(P) \subseteq K_1(P')$ and $K_{-1}(P') \subseteq K_{-1}(P)$ implies*

$$f(P) = 1 \Rightarrow f(P') = 1 \text{ and } f(P') = -1 \Rightarrow f(P) = -1.$$

Using equation (1.1) we can express Axiom 1.4 in the following form:

$$P \leq P' \Rightarrow f(P) \leq f(P').$$

Monotonicity handles the situation in positive responsiveness where the social aggregation function outputs a 1. This means that a monotonic function is one that will select the same candidate as the winner for a particular profile whenever it selects that candidate as the winner for a less favorable profile. Here, a "less favorable" profile is one in which that candidate receives fewer votes and/or one in which the alternative candidate receives more; since each voter has the option to abstain (a vote of 0), these scenarios do not necessarily imply one another. The alternative case presented in positive responsiveness, where the social aggregation function outputs a 0, is treated below in its own axiom.

**Axiom 1.5.** *A social aggregation function $f : X^n \to X$ satisfies* **tie-breaking** *if, for $P, P' \in X^n$ such that $P(ae)P'$,*

$$f(P) = 0 \Rightarrow f(P') \in \{-1, 1\}.$$

When a social aggregation function outputs a 0 we consider this a tie between the two candidates. A function that satisfies tie-breaking is simply that, a function that breaks ties. If a function outputs a 0 for a particular profile and we change exactly one vote, then a tie-breaking function would output anything but 0 for the modified profile, thus breaking the tie.

Since the goal was to split positive responsiveness into two seperate conditions, we have the following proposition:

**Proposition 1.1.** *If a social aggregation function $f : X^n \to X$ satisfies positive responsiveness then $f$ satisfies monotonicity and tie-breaking.*

*Proof.* Suppose we have two profiles $P, P' \in X^n$ such that $P \leq P'$, $P \neq P'$, and $f(P) = 1$. If $P(ae)P'$ then positive responsiveness implies $f(P') = 1$. If it is not the case that $P(ae)P'$ then let $i_0$ be the first position where $P(i_0) \neq P'(i_0)$. Now we can consider $P'' \in X^n$ such that $P''(i) = P(i)$ for all $i \neq i_0$ and $P''(i_0) = P'(i_0)$. Thus, $P \leq P''$ and $P(ae)P''$, hence positive responsivness implies $f(P'') = 1$ as well. If $P''(ae)P'$ then we can apply positive responsiveness again to get the desired result, if not, then we can repeat this procedure on $P''$. Since the length of each profile is finite, we can repeatedly apply this procedure to eventually achieve the desired result of $f(P') = 1$.

Alternatively, suppose we have two profiles $P, P' \in X^n$ such that $P \leq P'$, $P \neq P'$, and $f(P') = -1$. If $P(ae)P'$ then Lemma 1.1 implies $f(P) = -1$. If it is not the case that $P(ae)P'$ then let $i_0$ be the first position where $P'(i_0) \neq P(i_0)$. Now we can consider $P'' \in X^n$ such that $P''(i) = P'(i)$ for all $i \neq i_0$ and $P''(i_0) = P(i_0)$. Thus, $P'' \leq P'$ and $P''(ae)P'$, hence Lemma 1.1 implies $f(P'') = -1$ as well. If $P(ae)P''$ then we can apply Lemma 1.1 again to get the desired result, if not, then we can repeat this procedure on $P''$. Since the length of each profile is finite, we can repeatedly apply this procedure to eventually achieve the desired result of $f(P) = -1$.

Suppose instead that $P(ae)P'$ and $f(P) = 0$. Since $P(ae)P'$ we know that either $P \leq P'$ or $P' \leq P$. If $P \leq P'$, then, since $f(P) \in \{0, 1\}$ and $P(ae)P'$, $f(P') = 1$ by positive responsiveness. If $P' \leq P$ and $f(P') = 0$, then positive responsiveness implies $f(P) = 1$, contradicting the assumption that $f(P) = 0$. Therefore, in either case $f(P') \in \{-1, 1\}$. □

It is also the case that the converse of this statement holds.

**Proposition 1.2.** *If a social aggregation function $f : X^n \to X$ satisfies monotonicity and tie-breaking then $f$ satisfies positive responsiveness.*

*Proof.* Suppose we have two profiles $P, P' \in X^n$ such that $P \leq P'$, $P(ae)P'$, and $f(P) = 1$. We can immediately apply monotonicity to get the desired result of $f(P') = 1$. If, on the other hand, $f(P) = 0$ then tie-breaking implies $f(P') \in \{-1, 1\}$ and monotonicity implies $f(P') \in \{0, 1\}$. Thus, it must be the case that $f(P') = 1$ as desired. □

Now that the equivalence of these axioms has been demonstrated, we can restate May's Theorem using the new axioms.

**Theorem 1.2** (May's Theorem)**.** *A social aggregation function $f$ satisfies anonymity, neutrality, monotonicity, and tie-breaking if and only if $f = f_s$.*

There are multiple directions one can take in the attempt to generalize May's Theorem. Before we consider expanding the domain to broader mathematical structures, we consider manipulating the axioms. This kind of thinking is in line with other work that has been done on May's Theorem. Aşan and Sanver replace positive responsiveness with Maskin monotonicity and characterize the absolute q-majority rules in [2]. For more on axiomatic modifications see [1], [4], [12], [22], and [24]. In the next chapter we discuss a specific generalization of May's Theorem, in which the axiom of neutrality is dropped and a characterization of all aggregation functions which satisfy anonymity and positive responsiveness is presented. After that, we will extend simple majority rule to an order theoretic domain. For more on simple majority rule with more than two alternatives see [3], [9], and [23].

# CHAPTER 2
# A SIMPLE EXTENSION OF MAY'S THEOREM

## 2.1 Background

As we saw in the previous chapter, Kenneth May characterized all aggregation functions that satisfy the properties of anonymity, neutrality, and positive responsiveness and reached the result that they are all, in fact, equivalent to simple majority rule. An obvious question to ask is what our social aggregation function would look like if we drop one of these axioms. In 1995 Young, Taylor, and Zwicker characterized all aggregation functions that satisfy anonymity, neutrality, and monotonicity, dropping the tie-breaking portion of positive responsiveness[24]. Their answer was based on a concept called quota systems. Perry and Powers built on this concept and gave a characterization of all aggregation functions that satisfy just anonymity and monotonicity, removing neutrality[22]. They based their work on a concept known as a quota pair system. In the following section we build on this work to characterize all aggregation functions that satisfy anonymity, monotonicity, and tie-breaking i.e. all aggregation functions that satisfy anonymity and positive responsiveness.

**Definition 2.1.** *A* **quota system** *based on $n \geq 2$ voters is a non-increasing sequence of integers*

$$q_0 \geq q_1 \geq \ldots \geq q_n$$

*such that*

**Q1:** $\frac{n-k}{2} < q_k \leq n - k + 1$ *for all* $k \in \{0, 1, \ldots, n\}$*;*

**Q2:** $q_{k+1} \in \{q_k, q_k - 1\}$ *for all* $k \in \{0, 1, \ldots, n-1\}$*.*

Similar to the notation given in the previous section, we define a way to keep track of the abstention votes; for any $P \in X^n$:

$$K_0(P) = \{i : P(i) = 0\}.$$

Thus, $K_0(P) \cup K_1(P) \cup K_{-1}(P) = \{1, \ldots, n\}$.

**Definition 2.2.** *A social aggregation function* $f : X^n \to X$*, where* $X = \{-1, 0, 1\}$*, is determined by a quota system*

$$q_0 \geq q_1 \geq \ldots \geq q_n$$

*if, for any* $P \in X^n$*, we have*

$$f(P) = \begin{cases} 1 & \text{if } |K_1(P)| \geq q_k \text{ and } |K_0(P)| = k \\ -1 & \text{if } |K_{-1}(P)| \geq q_k \text{ and } |K_0(P)| = k \\ 0 & \text{otherwise.} \end{cases}$$

Young, Taylor, and Zwicker[24] proved the following theorem:

**Theorem 2.1.** *A social aggregation function* $f : X^n \to X$ *satisfies anonymity, neutrality, and monotonicity if and only if* $f$ *is determined by a quota system.*

The lower bound given in **Q1** ensures the aggregation function is well-defined, preventing 1 and $-1$ from both receiving enough votes to win in a single profile. The second bound in **Q1** eliminates redundant quota sytems since any quota higher than the number of non-abstaining voters is unobtainable by either candidate; a situation we still want to account for. Monotonicity is ensured by **Q2**. A quota system can be used to create several common voting methods. Setting each $q_k$ to be a specific fraction of the non-abstaining votes creates a supermajority function (such as the two-thirds and three-fifths majority used in our government [10]), where some level of support greater than a half is required to select a winner. When one-half is used (setting each $q_k$ equal to the least integer greater than half the number of non-abstaining votes) the quota system is identical to May's simple majority rule as stated in the previous chapter. An alternative characterization is of these rules was given by Fishburn, for more information see [8] and [14].

The notion of a quota system was expanded upon by Perry and Powers in [22] to deal with the case where neutrality is dropped.

**Definition 2.3.** *A* **quota pair system** *based on $n \geq 2$ voters is a pair of non-increasing sequences of integers*

$$q_0 \geq q_1 \geq \ldots \geq q_n \quad \textit{and} \quad l_0 \geq l_1 \geq \ldots \geq l_n$$

*such that*

**QP1:** $0 \leq q_k, l_k \leq n + 1 - k$ *for all* $k \in \{0, 1, \ldots, n\}$*;*

**QP2:** $q_k + l_k \geq n + 1 - k$ *for all* $k \in \{0, 1, \ldots, n\}$*;*

**QP3:** $q_{k+1} \in \{q_k, q_k - 1\}$ *and* $l_{k+1} \in \{l_k, l_k - 1\}$ *for all* $k \in \{0, 1, \ldots, n - 1\}$.

In a quota pair system the absence of neutrality forces the inclusion of a second non-increasing sequence of integers so that each candidate has their own set of quotas. The lower bound given in **Q1** is replaced by the lower bound given in **QP2** in order to ensure our social aggregation function is well-defined. We will illustrate a quota pair system with the following example:

**Example 2.1.** *Let $n = 2$ and consider the non-increasing sequences*

$$3 \geq 2 \geq 1 \quad \textit{and} \quad 1 \geq 0 \geq 0.$$

Here $q_0 = 3,\ q_1 = 2,\ q_2 = 1,\ l_0 = 1,\ l_1 = 0,$ and $l_2 = 0$. Notice that $0 \leq q_k, l_k \leq 3-k$ for $k \in \{0,1,2\}$, thus **QP1** is satisfied. Observe that $q_k + l_k \geq 3-k$ for $k \in \{0,1,2\}$, thus **QP2** is satisfied. Also $q_{k+1} \in \{q_k, q_k - 1\}$ and $l_{k+1} \in \{l_k, l_k - 1\}$ for $k \in \{0,1\}$, thus **QP3** is satisfied as well.

**Definition 2.4.** *A social aggregation function $f : X^n \to X$ is determined by a quota pair system*

$$q_0 \geq q_1 \geq \ldots \geq q_n \quad \textit{and} \quad l_0 \geq l_1 \geq \ldots \geq l_n$$

*if, for $P \in X^n$, we have*

$$f(P) = \begin{cases} 1 & \textit{if } |K_1(P)| \geq q_k \textit{ and } |K_0(P)| = k \\ -1 & \textit{if } |K_{-1}(P)| \geq l_k \textit{ and } |K_0(P)| = k \\ 0 & \textit{otherwise.} \end{cases}$$

Notice that, due to the properties of a quota pair system, this is a well-defined function.

Perry and Powers[22] proved the following theorem:

**Theorem 2.2.** *A social aggregation function $f : X^n \to X$ satisfies anonymity and monotonicity if and only if $f$ is determined by a quota pair system.*

We build on their notion of a quota pair system to create what we call a strong quota pair system.

## 2.2 Strong Quota Pair Systems

The material in this section has been submitted for publication [11].

**Definition 2.5.** *A* **strong quota pair system** *based on $n \geq 2$ voters, SQP system for short, is a pair of non-increasing sequences of integers*

$$q_0 \geq q_1 \geq \ldots \geq q_n \quad \textit{and} \quad l_0 \geq l_1 \geq \ldots \geq l_n$$

*such that*

**SQP1:** $0 \leq q_k, l_k \leq n+1-k$ *for all* $k \in \{0, 1, \ldots, n\}$*;*

**SQP2:** $n+1-k \leq q_k + l_k \leq n+2-k$ *for all* $k \in \{0, 1, \ldots, n\}$*;*

**SQP3:** $q_{k+1} \in \{q_k, q_k - 1\}$ *and* $l_{k+1} \in \{l_k, l_k - 1\}$ *for all* $k \in \{0, 1, \ldots, n-1\}$*;*

**SQP4:** $q_k + l_k = n+2-k \Rightarrow q_{k-1} = q_k$ *and* $l_{k-1} = l_k$ *for all* $k \in \{1, 2, \ldots, n\}$*.*

The upper bound given in **SQP2** and the new condition **SQP4** were added to the definition of a quota pair system to ensure our aggregation function satisfies tie-breaking. Observe that since the conditions on a quota pair system are all conditions on a strong quota pair system we can conclude that any strong quota pair system is itself a quota pair system. We will refer to a social aggregation function being determined by a strong quota pair system in exactly the same way that a function is determind by a quota pair system in Definition 2.4. We illustrate the definition of a strong quota pair system with the following example:

**Example 2.2.** *Let* $n = 2$ *and consider the non-increasing sequences*

$$2 \geq 2 \geq 1 \quad \textit{and} \quad 1 \geq 1 \geq 0.$$

Here $q_0 = 2$, $q_1 = 2$, $q_2 = 1$, $l_0 = 1$, $l_1 = 1$, and $l_2 = 0$. Notice that $0 \leq q_k, l_k \leq 3-k$ for $k \in \{0, 1, 2\}$, thus **SQP1** is satisfied. Observe that $3-k \leq q_k + l_k \leq 4-k$ for $k \in \{0, 1, 2\}$, thus **SQP2** is satisfied. Also $q_{k+1} \in \{q_k, q_k - 1\}$ and $l_{k+1} \in \{l_k, l_k - 1\}$ for $k \in \{0, 1\}$, thus **SQP3** is satisfied. Finally, $q_k + l_k = n+2-k$ only when $k = 1$, that is $q_1 + l_1 = 2+1 = n+2-k$. Since $q_0 = q_1$ and $l_0 = l_1$ **SQP4** is satisfied. Hence this pair of non-increasing sequences is a strong quota pair system on 2 voters. With this new definition we can state our theorem.

**Theorem 2.3.** *A social aggregation function* $f : X^n \to X$ *satisfies anonymity and positive responsiveness if and only if* $f$ *is determined by a strong quota pair system.*

Before we prove this theorem a lemma will be presented which offers some insight as to when exactly this function will result in a tie.

**Lemma 2.1.** *If a social aggregation function $f : X^n \to X$ is determined by a strong quota pair system*

$$q_0 \geq q_1 \geq \ldots \geq q_n \quad \textit{and} \quad l_0 \geq l_1 \geq \ldots \geq l_n,$$

*then $f(P) = 0$ for some profile $P \in X^n$ if and only if $q_k + l_k = n + 2 - k$ with $|K_0(P)| = k$, $|K_1(P)| = q_k - 1$, and $|K_{-1}(P)| = l_k - 1$.*

*Proof.* Suppose $f(P) = 0$, thus $q_k - 1 \geq |K_1(P)|$ and $l_k - 1 \geq |K_{-1}(P)|$ where $|K_0(P)| = k$, hence $q_k + l_k - 2 \geq |K_1(P)| + |K_{-1}(P)| = n - k$. Thus $q_k + l_k \geq n + 2 - k$; by **SQP2** above we know $q_k + l_k \leq n + 2 - k$ therefore $q_k + l_k = n + 2 - k = |K_1(P)| + |K_{-1}(P)| + 2$. Recall that $q_k - 1 \geq |K_1(P)|$ and $l_k - 1 \geq |K_{-1}(P)|$ hence $q_k \geq |K_1(P)| + 1$ and $l_k \geq |K_{-1}(P)| + 1$, since $q_k + l_k = |K_1(P)| + |K_{-1}(P)| + 2$ we get $q_k = |K_1(P)| + 1$ and $l_k = |K_{-1}(P)| + 1$.

If, for some profile $P \in X^n$, we have $|K_0(P)| = k$, $|K_1(P)| = q_k - 1$, and $|K_{-1}(P)| = l_k - 1$ then clearly $|K_1(P)| < q_k$ and $|K_{-1}(P)| < l_k$. Thus $f(P) = 0$. □

In Example 2.2 when $k = 1$ we have $q_k + l_k = n + 2 - k$ but if we consider $P = (-1, 0)$ then $|K_1(P)| = 0 \neq q_1 - 1$ and $|K_{-1}(P)| = 1 = l_1 \neq l_1 - 1$ thus $f(P) = -1$, not 0. If, on the other hand, we consider $P = (1, 0)$ then $|K_0(P)| = 1$, $|K_1(P)| = q_1 - 1$, and $|K_{-1}(P)| = l_1 - 1$. Thus, $f(P) = 0$.

Armed with Lemma 2.1, we are ready to prove Theorem 2.3.

*Proof of Theorem 2.3.* Let $f$ be a social aggregation function determined by a strong quota pair system as above. Since a strong quota pair system is a quota pair system it follows from Theorem 2.2 that $f$ satisfies both anonymity and monotonicity. By Proposition 1.2 all that remains to be shown is that $f$ satisfies tie-breaking.

Let $P, P' \in X^n$ be two profiles such that $P(ae)P'$ and $f(P) = 0$. Want to show $f(P') \in \{-1, 1\}$. Lemma 2.1 implies $q_k + l_k = n + 2 - k$ with $|K_0(P)| = k$, $|K_1(P)| = q_k - 1$, and $|K_{-1}(P)| = l_k - 1$. Since $P(ae)P'$ we know $|K_1(P)| \neq |K_1(P')|$ and/or $|K_{-1}(P)| \neq |K_{-1}(P')|$, furthermore, we know that $|K_0(P)| - 1 \leq |K_0(P')| \leq |K_0(P)| + 1$. Let $k' = |K_0(P')|$, thus we can write $k - 1 \leq k' \leq k + 1$.

If $f(P') = 0$ then Lemma 2.1 implies $q_{k'} + l_{k'} = n + 2 - k'$ and $|K_1(P')| = q_{k'} - 1$ as well as $|K_{-1}(P')| = l_{k'} - 1$.

If $k' = k$ then we know $|K_1(P')| = q_{k'} - 1 = q_k - 1 = |K_1(P)|$ and $|K_{-1}(P')| = l_{k'} - 1 = l_k - 1 = |K_{-1}(P)|$, contradicting the above observation that this is not possible. Hence, $k' \neq k$.

If $k' = k - 1$ then we know $q_{k'} + l_{k'} = n + 2 - k' = n + 2 - (k - 1) = n + 3 - k$. But **SQP4** says that $q_k + l_k = n + 2 - k \Rightarrow q_{k-1} = q_k$ and $l_{k-1} = l_k$ which implies $q_{k'} + l_{k'} = q_k + l_k = n + 2 - k$, so we can't have $f(P') = 0$ in this case without contradicting **SQP4**. Hence, $k' \neq k - 1$.

If $k' = k + 1$ then **SQP4** implies $q_{k'} + l_{k'} = n + 2 - k' \Rightarrow q_{k'-1} = q_{k'}$ and $l_{k'-1} = l_{k'}$ which implies $q_k + l_k = n + 2 - k' = n + 2 - (k + 1) = n + 1 - k$, contradicting Lemma 2.1 which implies $q_k + l_k = n + 2 - k$, so we know $k' \neq k + 1$ either.

Thus, in any case, the tie is broken. Therefore, $f$ satisfies positive responsivness.

Let $f$ be an aggregation function that satisfies anonymity and positive responsiveness. Proposition 1.1 implies $f$ satisfies monotonicity. Therefore, by Theorem 2.2, $f$ is determined by a quota pair system

$$q_0 \geq q_1 \geq \ldots \geq q_n \quad \text{and} \quad l_0 \geq l_1 \geq \ldots \geq l_n.$$

Our goal is to show that $f$ is determined by a *strong* quota pair system. As noted earlier, the differences between a quota pair system and a strong quota pair system are the upper bound in **SQP2** and the new condition **SQP4**. Since the rest of the conditions of a strong quota pair system are inherited from a quota pair system, these two conditions are all we need to show.

Assume, for the sake of contradiction, that there exists $k \in \{0, 1, \ldots, n\}$ such that $q_k + l_k > n + 2 - k$. Since $q_k, l_k \leq n + 1 - k$ it follows that $q_k \geq 2$ and $l_k \geq 2$. Choose $P \in X^n$ such that

$$|K_1(P)| = q_k - 2 \quad \text{and} \quad |K_0(P)| = k.$$

Then

$$|K_{-1}(P)| = n - (q_k - 2 + k) = n + 2 - k - q_k.$$

Since $q_k \leq n + 1 - k$ it follows that

$$|K_{-1}(P)| \geq 1.$$

Since $q_k + l_k > n + 2 - k$ it follows that

$$|K_{-1}(P)| = n + 2 - k - q_k < (q_k + l_k) - q_k = l_k.$$

Thus, $f(P) = 0$ since $|K_1(P)| < q_k$ and $|K_{-1}(P)| < l_k$. Choose $i_0 \in \{1, \ldots, n\}$ such that $P(i_0) = -1$ and pick $P' \in X^n$ such that $P'(ae)P$ and $P'(i_0) = 1$. Observe that $k' = k$ where $|K_0(P)| = k$ and $|K_0(P')| = k'$. Also notice that $|K_1(P')| = |K_1(P)| + 1 = q_k - 1$ and that $|K_{-1}(P')| = |K_{-1}(P)| - 1$. Thus, $|K_1(P')| = q_k - 1 = q_{k'} - 1 < q_{k'}$ and $|K_{-1}(P')| = |K_{-1}(P)| - 1 < l_k = l_{k'}$. It follows that $f(P') = 0$ but this contradicts tie-breaking. Hence $q_k + l_k \leq n + 2 - k$ for all $k \in \{0, 1, \ldots, n\}$ and so, **SQP2** holds.

Assume that there exists $k \in \{1, \ldots, n\}$ such that $q_k + l_k = n + 2 - k$ and, for the sake of contradiction, that either

$$q_k = q_{k-1} - 1 \text{ and } l_k = l_{k-1} \quad \textit{or} \quad q_k = q_{k-1} \text{ and } l_k = l_{k-1} - 1.$$

It can't be the case that both $q_k = q_{k-1} - 1$ and $l_k = l_{k-1} - 1$, since **SQP3** implies $q_{k-1} + l_{k-1} \leq n+3-k$. Suppose we have the case where $q_k = q_{k-1} - 1$ and $l_k = l_{k-1}$. Choose $P \in X^n$ such that $|K_0(P)| = k$ and $|K_1(P)| = q_k - 1$ (such a $P$ exists since $k + q_k - 1 \leq n$ and $q_k \geq 1$). Pick $i_0 \in \{1, \ldots, n\}$ such that $P(i_0) = 0$ and choose $P' \in X^n$ such that $P'(ae)P$ and $P'(i_0) = 1$. First, notice that

$$|K_{-1}(P)| = n - (k + q_k - 1) = n + 1 - k - q_k < (q_k + l_k) - q_k = l_k.$$

So $f(P) = 0$ since $|K_{-1}(P)| < l_k$ and $|K_1(P)| = q_k - 1$. Now notice that

$$\begin{gathered} |K_0(P')| = k - 1, \\ |K_1(P')| = q_k = q_{k-1} - 1 = q_{k'} - 1, \text{ and} \\ |K_{-1}(P')| = |K_{-1}(P)| < l_k = l_{k-1} = l_{k'}. \end{gathered}$$

Thus, $f(P') = 0$, contradicting tie-breaking. The case where $q_k = q_{k-1}$ and $l_k = l_{k-1} - 1$ can be solved symmetrically (choose $P'$ with $P'(i_0) = -1$ instead). Thus when $q_k + l_k = n + 2 - k$ we know $q_{k-1} = q_k$ and $l_{k-1} = l_k$ which is exactly **SQ4**. Hence, the quota pair system that determines $f$ when $f$ satisfies anonymity and positive responsiveness is indeed a strong quota pair system. □

### 2.3 Counting Strong Quota Pair Systems

It's easy enough to count the number of social aggregation functions that satisfy anonymity, neutrality, and positive responsiveness; there's only one, simple majority rule. In the previous section we altered the axioms and characterized social aggregation functions with quota systems, quota pair systems, and strong quota pair systems. It is a natural question to ask how many of each of these there are. Young, Taylor, and Zwicker[24] counted the number of quota systems.

**Theorem 2.4.** *The number of quota systems for $n \geq 0$ voters is given by*

$$Z_Q(n) = \binom{n+1}{\lfloor \frac{n}{2} \rfloor + 1},$$

*where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.*

The number of quota systems has a connection with the Catalan numbers. Let

$$C(n) = \frac{1}{n+1}\binom{2n}{n},$$

where $C(n)$ denotes the $n$th Catalan number. The Catalan numbers can be used to formulate a recursive definition of $Z_Q(n)$:

**Theorem 2.5.**

$$Z_Q(n+1) = \begin{cases} 2Z_Q(n) & \text{if } n \text{ is even,} \\ 2Z_Q(n) - C(\frac{n+1}{2}) & \text{if } n \text{ is odd.} \end{cases}$$

Perry and Powers[22] counted the number of quota pair systems.

**Theorem 2.6.** *The number of quota pair systems for $n \geq 2$ voters is given by*

$$Z_{QP}(n) = \binom{2n+3}{n+1}.$$

Next we will present a lemma that illustrates how each strong quota pair system contains a smaller strong quota pair system, and will give us a means to systematically construct all quota pair systems; making them amenable to enumeration.

**Lemma 2.2.** *If $q_0 \geq q_1 \geq \ldots \geq q_n$ and $l_0 \geq l_1 \geq \ldots \geq l_n$ is a strong quota pair system based on $n \geq 3$ voters, then $q_1 \geq \ldots \geq q_n$ and $l_1 \geq \ldots \geq l_n$ is a strong quota pair system based on $n-1$ voters.*

*Proof.* Let $q'_k = q_{k+1}$ and $l'_k = l_{k+1}$ for $k \in \{0, 1, \ldots, n-1\}$, and let $n' = n-1$ as well. To verify **SQP1** we need to show that

$$0 \leq q'_k, l'_k \leq n' + 1 - k = n - k \text{ for } k \in \{0, 1, \ldots, n'\}.$$

We know the $q_i s$ and $l_i s$ form a strong quota pair, hence

$$0 \leq q'_k, l'_k = q_{k+1}, l_{k+1} \leq n + 1 - (k+1) = n - k \text{ for } k \in \{0, 1, \ldots, n'\}$$

as desired. To verify **SQP2** we need to show that

$$n' + 1 - k \leq q'_k + l'_k \leq n' + 2 - k \text{ for } k \in \{0, 1, \ldots, n'\}.$$

Since the $q_i s$ and $l_i s$ form a strong quota pair, we know

$$n + 1 - (k+1) \leq q_{k+1} + l_{k+1} \leq n + 2 - (k+1) \text{ for } k \in \{0, 1, \ldots, n\}.$$

If we substitute $q'_k$ for $q_{k+1}$, $l'_k$ for $l_{k+1}$, and $n'$ for $n-1$ we get

$$n' + 1 - k \leq q'_k + l'_k \leq n' + 2 - k \text{ for } k \in \{0, 1, \ldots, n'\}$$

as desired. To verify **SQP3** we need to show that

$$q'_{k+1} \in \{q'_k, q'_k - 1\} \text{ and } l'_{k+1} \in \{l'_k, l'_k - 1\} \text{ for } k \in \{0, 1, \ldots, n' - 1\}.$$

Since $q'_k = q_{k+1}$ and $l'_k = l_{k+1}$ we get $q'_{k+1} = q_{k+2}$ and $l'_{k+1} = l_{k+2}$. Since the $q_i s$ and $l_i s$ form a strong quota pair we know

$$q_{k+2} \in \{q_{k+1}, q_{k+1} - 1\} \text{ and } l_{k+2} \in \{l_{k+1}, l_{k+1} - 1\} \text{ for } k \in \{0, 1, \ldots, n-2\}.$$

Substituting leads us to

$$q'_{k+1} \in \{q'_k, q'_k - 1\} \text{ and } l'_{k+1} \in \{l'_k, l'_k - 1\} \text{ for } k \in \{0, 1, \ldots, n' - 1\}$$

as desired. To verify **SQP4** we need to show that

$$q'_k + l'_k = n' + 2 - k \Rightarrow q'_{k-1} = q'_k \text{ and } l'_{k-1} = l'_k \text{ for } k \in \{1, 2, \ldots, n'\}.$$

As the $q_i s$ and $l_i s$ form a strong quota pair system, we know

$$q_{k+1} + l_{k+1} = n + 2 - (k+1) \Rightarrow q_k = q_{k+1} \text{ and } l_k = l_{k+1} \text{ for } k \in \{0, 1, \ldots, n-1\}.$$

Substituting leads us to

$$q'_k + l'_k = n' + 2 - k \Rightarrow q'_{k-1} = q'_k \text{ and } l'_{k-1} = l'_k \text{ for } k \in \{1, 2, \ldots, n'\}$$

as desired. $\square$

The previous lemma suggests a strategy to modify the strong quota pair systems for $n$ voters to create all strong quota pair systems for any $n+1$ voters. Our method will involve shifting each quota "up" (i.e. increasing all subscripts) and assigning a new $q_0$ and $l_0$. As long as we are careful with our choices for $q_0$ and $l_0$, making sure our new strong quota pair systems meet all the criteria of the definition, then we will only create legitimate strong quota pair systems. This is accomplished in the following lemma.

**Lemma 2.3.** *If the pair of non-increasing sequences of integers* $q_0 \geq q_1 \geq \ldots \geq q_n$ *and* $l_0 \geq l_1 \geq \ldots \geq l_n$ *is a strong quota pair system based on* $n \geq 2$ *voters,* $\epsilon = (n+2) - (q_0 + l_0)$*,* $q_0' \in \{q_0, q_0 + \epsilon\}$ *and* $l_0' \in \{l_0, l_0 + \epsilon\}$*, and* $q_0' + l_0' \geq q_0 + l_0 + \epsilon$*, then*

$$q_0' \geq q_0 \geq q_1 \geq \ldots \geq q_n \quad \textit{and} \quad l_0' \geq l_0 \geq l_1 \geq \ldots \geq l_n$$

*is a strong quota pair system on* $n+1$ *voters.*

*Proof.* Suppose $q_0 \geq q_1 \geq \ldots \geq q_n$ and $l_0 \geq l_1 \geq \ldots \geq l_n$ is a strong quota pair system based on $n$ voters. Consider $q_0' \geq q_1' \geq \ldots \geq q_m'$ and $l_0' \geq l_1' \geq \ldots \geq l_m'$, (hereafter referred to as the *prime* sequences), where $m = n+1$, such that $q_i' = q_{i-1}$ and $l_i' = l_{i-1}$ for $i \in \{1, \ldots, m\}$, $q_0' \in \{q_0, q_0 + \epsilon\}$ and $l_0' \in \{l_0, l_0 + \epsilon\}$. We know from **SQP1** that

$$0 \leq q_k, l_k \leq n + 1 - k \text{ for all } k \in \{0, 1, \ldots, n\}.$$

This can be re-indexed to obtain

$$0 \leq q_{k-1}, l_{k-1} \leq n + 1 - (k-1) \text{ for all } k \in \{1, \ldots, n+1\},$$

which can be rewritten as

$$0 \leq q_{k-1}, l_{k-1} \leq (n+1) + 1 - k \text{ for all } k \in \{1, \ldots, n+1\}.$$

We can make substitutions based on the values defined above to get

$$0 \leq q_k', l_k' \leq m + 1 - k \text{ for all } k \in \{1, \ldots, m\}.$$

Since $\epsilon = (n+2) - (q_0 + l_0)$, **SQP2** tells us that $\epsilon \in \{0, 1\}$. From **SQP1**, we can see that $0 \leq q_0, l_0 \leq n+1$, thus $0 \leq q'_0, l'_0 \leq (n+1)+1 = m+1$, thus

$$0 \leq q'_k, l'_k \leq m+1-k \text{ for all } k \in \{0, \ldots, m\}.$$

Hence, the prime sequences satisfy **SQP1**. We know from **SQP2** that

$$n+1-k \leq q_k + l_k \leq n+2-k \text{ for all } k \in \{0, 1, \ldots, n\}.$$

This can be re-indexed to obtain

$$n+1-(k-1) \leq q_{k-1} + l_{k-1} \leq n+2-(k-1) \text{ for all } k \in \{1, \ldots, n+1\},$$

which can be rewritten as

$$(n+1)+1-k \leq q_{k-1} + l_{k-1} \leq (n+1)+2-k \text{ for all } k \in \{1, \ldots, n+1\}.$$

Substituting as above yields

$$m+1-k \leq q'_k + l'_k \leq m+2-k \text{ for all } k \in \{1, \ldots, m\}.$$

From **SQP2** we can see that $n+1 \leq q_0+l_0 \leq n+2$. If $q_0+l_0 = n+2$ then $\epsilon = 0$, hence $q'_0 = q_0$ and $l'_0 = l_0$. Thus it is clearly the case that $(n+1)+1 \leq q'_0+l'_0 \leq (n+1)+2$, that is, $m+1 \leq q'_0 + l'_0 \leq m+2$. If $q_0 + l_0 = n+1$ then $\epsilon = 1$. We assumed $q'_0 + l'_0 \geq q_0 + l_0 + \epsilon$ thus $q'_0 + l'_0 \geq (n+1)+1$ and the largest that $q'_0 + l'_0$ could be is $q_0 + l_0 + 2 = (n+1)+2$. Thus $(n+1)+1 \leq q'_0 + l'_0 \leq (n+1)+2$, that is, $m+1 \leq q'_0 + l'_0 \leq m+2$, thus

$$m+1-k \leq q'_k + l'_k \leq m+2-k \text{ for all } k \in \{0, 1, \ldots, m\}.$$

Hence, the prime sequences satisfy **SQP2**. We know from **SQP3** that

$$q_{k+1} \in \{q_k, q_k - 1\} \text{ and } l_{k+1} \in \{l_k, l_k - 1\} \text{ for all } k \in \{0, 1, \ldots, n-1\}.$$

This can be re-indexed to obtain

$$q_k \in \{q_{k-1}, q_{k-1} - 1\} \text{ and } l_k \in \{l_{k-1}, l_{k-1} - 1\} \text{ for all } k \in \{1, \ldots, n\}.$$

Substituting as above yields

$$q'_{k+1} \in \{q'_k, q'_k - 1\} \text{ and } l'_{k+1} \in \{l'_k, l'_k - 1\} \text{ for all } k \in \{1, \ldots, m-1\}.$$

Since $q'_0 \in \{q_0, q_0 + \epsilon\}$ and $l'_0 \in \{l_0, l_0 + \epsilon\}$ we can say $q'_1 \in \{q'_0, q'_0 - 1\}$ and $l'_1 \in \{l'_0, l'_0 - 1\}$, thus

$$q'_{k+1} \in \{q'_k, q'_k - 1\} \text{ and } l'_{k+1} \in \{l'_k, l'_k - 1\} \text{ for all } k \in \{0, 1, \ldots, m-1\}.$$

Hence, the prime sequences satisfy **SQP3**. We know from **SQP4** that

$$q_k + l_k = n + 2 - k \Rightarrow q_{k-1} = q_k \text{ and } l_{k-1} = l_k \text{ for all } k \in \{1, 2, \ldots, n\}.$$

This can be re-indexed to obtain

$$q_{k-1} + l_{k-1} = n + 2 - (k-1) \Rightarrow q_{k-2} = q_{k-1} \text{ and } l_{k-2} = l_{k-1} \text{ for all } k \in \{2, \ldots, n+1\},$$

which can be rewritten as

$$q_{k-1} + l_{k-1} = (n+1) + 2 - k \Rightarrow q_{k-2} = q_{k-1} \text{ and } l_{k-2} = l_{k-1} \text{ for all } k \in \{2, \ldots, n+1\}.$$

Substituting as above yields

$$q'_k + l'_k = m + 2 - k \Rightarrow q'_{k-1} = q'_k \text{ and } l'_{k-1} = l'_k \text{ for all } k \in \{2, \ldots, m\}.$$

If $q'_1 + l'_1 = m + 2 - 1$ then $q_0 + l_0 = n + 2$, thus $\epsilon = 0$ and we have $q'_0 = q_0 = q'_1$ and $l'_0 = l_0 = l'_1$. Thus, we can write

$$q'_k + l'_k = m + 2 - k \Rightarrow q'_{k-1} = q'_k \text{ and } l'_{k-1} = l'_k \text{ for all } k \in \{1, 2, \ldots, m\}.$$

Hence the prime sequences satisfy **SQP4**. Thus,

$$q'_0 \geq q_0 \geq q_1 \geq \ldots \geq q_n \quad \text{and} \quad l'_0 \geq l_0 \geq l_1 \geq \ldots \geq l_n$$

is a strong quota pair sytem on $m = n + 1$ voters, as desired. $\square$

To help understand this lemma, consider the following examples.

**Example 2.3.** *Consider the strong quota pair system on* 2 *voters from example 1:*

$$(3, 2, 1) \quad \textit{and} \quad (1, 0, 0).$$

This strong quota pair system attains its upper bound in the $k = 0$ case;

$$q_0 + l_0 = 3 + 1 = 4 = 2 + 2 - 0 = n + 2 - 0.$$

If those values belonged to the $q_1$ and $l_1$ position in a strong quota pair system on 3 voters then **SQP4** implies that they could only be preceded by $q_0$ and $l_0$ creating the strong quota pair system:

$$(3, 3, 2, 1) \quad \text{and} \quad (1, 1, 0, 0).$$

**Example 2.4.** *Consider the strong quota pair system on* 2 *voters:*

$$(2, 2, 1) \quad \textit{and} \quad (1, 1, 0).$$

This strong quota pair system attains it's lower bound in the $k = 0$ case;

$$q_0 + l_0 = 2 + 1 = 3 = 2 + 1 - 0 = n + 1 - 0.$$

If those values belonged to the $q_1$ and $l_1$ position in a strong quota pair system on 3 voters then **SQP2** implies that they could be preceded by any of three combinations of $q_0s$ and $l_0s$ since in this case $q_0 = q_1 + 1$ or $l_0 = l_1 + 1$ or both, creating the following strong quota pair systems:

$$(3, 2, 2, 1) \quad \text{and} \quad (1, 1, 1, 0),$$
$$(2, 2, 2, 1) \quad \text{and} \quad (2, 1, 1, 0),$$
$$(3, 2, 2, 1) \quad \text{and} \quad (2, 1, 1, 0).$$

Let $Z_{SQP}(n)$ be the number of strong quota pair systems on $n \geq 2$ voters. Our goal is to find a formula for $Z_{SQP}(n)$.

**Lemma 2.4.** $Z_{SQP}(2) = 17$.

*Proof.* We present a complete list of all strong quota pair systems on $n = 2$ voters:

$$\begin{array}{llllll}
(0,0,0) \text{ and } (3,2,1), & (1,0,0) \text{ and } (2,2,1), & (1,1,0) \text{ and } (2,1,1), \\
(3,2,1) \text{ and } (0,0,0), & (2,2,1) \text{ and } (1,0,0), & (2,1,1) \text{ and } (1,1,0), \\
(1,1,0) \text{ and } (2,2,1), & (1,1,1) \text{ and } (2,1,0), & (1,1,1) \text{ and } (2,1,1), \\
(2,2,1) \text{ and } (1,1,0), & (2,1,0) \text{ and } (1,1,1), & (2,1,1) \text{ and } (1,1,1), \\
(1,0,0) \text{ and } (3,2,1), & (2,1,0) \text{ and } (2,1,1), & (2,1,1) \text{ and } (2,1,1), \\
(3,2,1) \text{ and } (1,0,0), & (2,1,1) \text{ and } (2,1,0). &
\end{array}$$

Observe that there are 17 SQP systems on $n = 2$ voters presented in the list above. □

We now come to the main theorem of this section.

**Theorem 2.7.** *The number of strong quota pair systems for $n \geq 2$ voters is given by*

$$Z_{SQP}(n) = \left(\frac{3}{2} + \sqrt{2}\right)(1 + \sqrt{2})^n + \left(\frac{3}{2} - \sqrt{2}\right)(1 - \sqrt{2})^n.$$

*Proof.* Let $A(n)$ be the set of all strong quota pair systems on $n \geq 2$ voters,

$$q_0 \geq q_1 \geq \ldots \geq q_n \quad \text{and} \quad l_0 \geq l_1 \geq \ldots \geq l_n$$

such that $q_0 + l_0 = n + 2$. Similarly, let $B(n)$ be the set of all strong quota pair systems such that $q_0 + l_0 = n + 1$. We will let

$$a_n = |A(n)| \quad \text{and} \quad b_n = |B(n)|.$$

Consider a function $f : A(n+1) \to B(n)$ defined by

$$f(q'_0 \geq q'_1 \geq \ldots \geq q'_{n+1},\ l'_0 \geq l'_1 \geq \ldots \geq l'_{n+1}) = q'_1 \geq q'_2 \geq \ldots \geq q'_{n+1},\ l'_1 \geq l'_2 \geq \ldots \geq l'_{n+1}.$$

It follows from Lemma 2.2 that $q_1' \geq q_2' \geq \ldots \geq q_{n+1}',\ l_1' \geq l_2' \geq \ldots \geq l_{n+1}'$ is a SQP system on $n$ voters. We know that $q_0' + l_0' = (n+1) + 2$. By **SQP2** we know $q_1' + l_1' \leq (n+1) + 2 - 1$. Since $q_1' + l_1' < q_0' + l_0'$, it follows from the contrapositive of **SQP4** that $q_1' + l_1' \neq (n+1) + 2 - 1$. Therefore, by **SQP2**, we have $q_1' + l_1' = (n+1) + 1 - 1 = n+1$. Thus, $f$ is well-defined.

Also, $f$ is invertible, with the mapping $f^{-1} : B(n) \to A(n+1)$ defined by

$$f^{-1}(q_0' \geq q_1' \geq \ldots \geq q_n',\ l_0' \geq l_1' \geq \ldots \geq l_n') = q_0'{+}1 \geq q_0' \geq \ldots \geq q_n',\ l_0'{+}1 \geq l_0' \geq \ldots \geq l_n'.$$

It follows from Lemma 2.3 that $q_0' + 1 \geq q_0' \geq \ldots \geq q_n',\ l_0' + 1 \geq l_0' \geq \ldots \geq l_n'$ is a SQP system on $n+1$ voters. We know that $q_0' + l_0' = n+1$, thus

$$(q_0' + 1) + (l_0' + 1) = n + 3 = (n+1) + 2.$$

Hence, $f^{-1}$ is well-defined. Consider the composition

$$f(f^{-1}(q_0' \geq q_1' \geq \ldots \geq q_n',\ l_0' \geq l_1' \geq \ldots \geq l_n')).$$

Clearly this is equal to $q_0' \geq q_1' \geq \ldots \geq q_n',\ l_0' \geq l_1' \geq \ldots \geq l_n'$. Consider the composition

$$f^{-1}(f(q_0' \geq q_1' \geq \ldots \geq q_{n+1}',\ l_0' \geq l_1' \geq \ldots \geq l_{n+1}'))$$

which is equal to $q_1' + 1 \geq q_1' \geq \ldots \geq q_{n+1}',\ l_1' + 1 \geq l_1' \geq \ldots \geq l_{n+1}'$. We saw above that $q_1' + l_1' = (n+1) + 1 - 1 = n+1$ and we assumed $q_0' + l_0' = (n+1) + 2$ thus, according to **SQP3**, we know

$$q_0' = q_1' + 1 \quad \text{and} \quad l_0' = l_1' + 1.$$

We can conclude from this that $f$ is a bijection between $A(n+1)$ and $B(n)$. Thus,

$$a_{n+1} = b_n.$$

Let $\hat{B}(n)$ be the set of all SQP systems on $n \geq 2$ voters such that $q_0 + l_0 = n+1$ where

$$q_0 = q_1 + 1 \quad \text{and} \quad l_0 = l_1.$$

Let $\tilde{B}(n)$ be the set of all SQP systems on $n \geq 2$ voters such that $q_0 + l_0 = n+1$ where

$$q_0 = q_1 \quad \text{and} \quad l_0 = l_1 + 1.$$

Let $\bar{B}(n)$ be the set of all SQP systems on $n \geq 2$ voters such that $q_0 + l_0 = n+1$ where

$$q_0 = q_1 \quad \text{and} \quad l_0 = l_1.$$

These three sets are clearly pairwise disjoint by definition. Moreover, **SQP3** implies that the only other possible SQP system that can exist is one where

$$q_0 = q_1 + 1 \quad \text{and} \quad l_0 = l_1 + 1,$$

but, in this case, it cannot be that $q_0+l_0 = n+1$ since this would imply $q_1+l_1 = n-1$, which directly contradicts **SQP2**. Thus, we can say that

$$\hat{B}(n) \cup \tilde{B}(n) \cup \bar{B}(n) = B(n).$$

Consider a function $g : \hat{B}(n+1) \to B(n)$ defined by

$$g(q'_0 \geq q'_1 \geq \ldots \geq q'_{n+1},\ l'_0 \geq l'_1 \geq \ldots \geq l'_{n+1}) = q'_1 \geq q'_2 \geq \ldots \geq q'_{n+1},\ l'_1 \geq l'_2 \geq \ldots \geq l'_{n+1}.$$

It follows from Lemma 2.2 that $q'_1 \geq q'_2 \geq \ldots \geq q'_{n+1},\ l'_1 \geq l'_2 \geq \ldots \geq l'_{n+1}$ is a SQP system on $n$ voters. We know that $q'_0 + l'_0 = (n+1)+1$. Since $q'_0 = q'_1 + 1$ and $l'_0 = l'_1$ we know $q'_1 + l'_1 = n+1$. Thus, $g$ is well-defined.

Also, $g$ is invertible, with the mapping $g^{-1} : B(n) \to \hat{B}(n+1)$ defined by

$$g^{-1}(q'_0 \geq q'_1 \geq \ldots \geq q'_n,\ l'_0 \geq l'_1 \geq \ldots \geq l'_n) = q'_0{+}1 \geq q'_0 \geq \ldots \geq q'_n,\ l'_0 \geq l'_0 \geq \ldots \geq l'_n.$$

It follows from Lemma 2.3 that $q_0' + 1 \geq q_0' \geq \ldots \geq q_n',\ l_0' \geq l_0' \geq \ldots \geq l_n'$ is a SQP system on $n+1$ voters. We know that $q_0' + l_0' = n+1$, thus

$$(q_0' + 1) + l_0' = n + 2 = (n+1) + 1.$$

Hence, $g^{-1}$ is well-defined. Consider the composition

$$g(g^{-1}(q_0' \geq q_1' \geq \ldots \geq q_n',\ l_0' \geq l_1' \geq \ldots \geq l_n')).$$

Clearly this is equal to $q_0' \geq q_1' \geq \ldots \geq q_n',\ l_0' \geq l_1' \geq \ldots \geq l_n'$. Consider the composition

$$g^{-1}(g(q_0' \geq q_1' \geq \ldots \geq q_{n+1}',\ l_0' \geq l_1' \geq \ldots \geq l_{n+1}')),$$

which is equal to $q_1' + 1 \geq q_1' \geq \ldots \geq q_{n+1}',\ l_1' \geq l_1' \geq \ldots \geq l_{n+1}'$. According to how $\hat{B}(n)$ was defined, we know $q_1' + 1 = q_0'$ and $l_1' = l_0'$. Thus, $q_1' + 1 \geq q_1' \geq \ldots \geq q_{n+1}',\ l_1' \geq l_1' \geq \ldots \geq l_{n+1}' = q_0' \geq q_1' \geq \ldots \geq q_{n+1}',\ l_0' \geq l_1' \geq \ldots \geq l_{n+1}'$. We can conclude that $g$ is a bijection between $\hat{B}(n+1)$ and $B(n)$. Thus,

$$|\hat{B}(n+1)| = b_n.$$

Consider a function $h : \tilde{B}(n+1) \to B(n)$ defined by

$$h(q_0' \geq q_1' \geq \ldots \geq q_{n+1}',\ l_0' \geq l_1' \geq \ldots \geq l_{n+1}') = q_1' \geq q_2' \geq \ldots \geq q_{n+1}',\ l_1' \geq l_2' \geq \ldots \geq l_{n+1}'.$$

A symmetric argument to the one presented above for $g$ can be used to show that $h$ is a bijection between $\tilde{B}(n+1)$ and $B(n)$. Thus,

$$|\tilde{B}(n+1)| = b_n.$$

Consider a function $j : \bar{B}(n+1) \to A(n)$ defined by

$$j(q_0' \geq q_1' \geq \ldots \geq q_{n+1}',\ l_0' \geq l_1' \geq \ldots \geq l_{n+1}') = q_1' \geq q_2' \geq \ldots \geq q_{n+1}',\ l_1' \geq l_2' \geq \ldots \geq l_{n+1}'.$$

It follows from Lemma 2.2 that $q'_1 \geq q'_2 \geq \ldots \geq q'_{n+1},\ l'_1 \geq l'_2 \geq \ldots \geq l'_{n+1}$ is a SQP system on $n$ voters. We know that $q'_0 + l'_0 = (n+1)+1$. Since $q'_0 = q'_1$ and $l'_0 = l'_1$ we know $q'_1 + l'_1 = n+2$. Thus, $j$ is well-defined.

Also, $j$ is invertible, with the mapping $j^{-1} : A(n) \to \bar{B}(n+1)$ defined by

$$j^{-1}(q'_0 \geq q'_1 \geq \ldots \geq q'_n,\ l'_0 \geq l'_1 \geq \ldots \geq l'_n) = q'_0 \geq q'_0 \geq \ldots \geq q'_n,\ l'_0 \geq l'_0 \geq \ldots \geq l'_n.$$

It follows from Lemma 2.3 that $q'_0 \geq q'_0 \geq \ldots \geq q'_n,\ l'_0 \geq l'_0 \geq \ldots \geq l'_n$ is a SQP system on $n+1$ voters. We know that $q'_0 + l'_0 = n+2$, thus

$$q'_0 + l'_0 = n+2 = (n+1)+1.$$

Hence, $j^{-1}$ is well-defined. Consider the composition

$$j(j^{-1}(q'_0 \geq q'_1 \geq \ldots \geq q'_n,\ l'_0 \geq l'_1 \geq \ldots \geq l'_n)).$$

Clearly this is equal to $q'_0 \geq q'_1 \geq \ldots \geq q'_n,\ l'_0 \geq l'_1 \geq \ldots \geq l'_n$. Consider the composition

$$j^{-1}(j(q'_0 \geq q'_1 \geq \ldots \geq q'_{n+1},\ l'_0 \geq l'_1 \geq \ldots \geq l'_{n+1})),$$

which is equal to $q'_1 \geq q'_1 \geq \ldots \geq q'_{n+1},\ l'_1 \geq l'_1 \geq \ldots \geq l'_{n+1}$. According to how $\bar{B}(n)$ was defined, we know $q'_1 = q'_0$ and $l'_1 = l'_0$. Thus, $q'_1 \geq q'_1 \geq \ldots \geq q'_{n+1},\ l'_1 \geq l'_1 \geq \ldots \geq l'_{n+1} = q'_0 \geq q'_1 \geq \ldots \geq q'_{n+1},\ l'_0 \geq l'_1 \geq \ldots \geq l'_{n+1}$. We can conclude that $j$ is a bijection between $\bar{B}(n+1)$ and $A(n)$. Thus,

$$|\bar{B}(n+1)| = a_n.$$

Recall that $\hat{B}(n) \cup \tilde{B}(n) \cup \bar{B}(n) = B(n)$. Thus, using the relations just shown, we can write

$$b_{n+1} = a_n + 2b_n.$$

It was show earlier that $a_{n+1} = b_n$, we can use this fact along with the equality just established and the definition of $Z_{SQP}(n)$ to write the following equalities:

$$Z_{SQP}(n+2) = a_{n+2} + b_{n+2} = b_{n+1} + a_{n+1} + 2b_{n+1} = Z_{SQP}(n+1) + 2b_{n+1}.$$

We can continue to apply our known equalities to the right-hand side to obtain:

$Z_{SQP}(n+2) = Z_{SQP}(n+1) + b_{n+1} + a_n + 2b_n = Z_{SQP}(n+1) + b_{n+1} + a_n + b_n + a_{n+1}$.

If we regroup this we can rewrite $Z_{SQP}(n+2)$ as the following homogeneous linear recurrence relation:

$$Z_{SQP}(n+2) = 2Z_{SQP}(n+1) + Z_{SQP}(n).$$

This recurrence has the following characteristic equation:

$$x^2 - 2x - 1 = 0.$$

The roots of the characteristic equation can be found to be $1 \pm \sqrt{2}$, thus we can write $Z_{SQP}(n)$ as:

$$Z_{SQP}(n) = c_1(1+\sqrt{2})^n + c_2(1-\sqrt{2})^n.$$

where $c_1, c_2 \in \mathbb{R}$ are some constants. We can solve for these constants if we can establish sufficient base cases. Lemma 2.4 tells us that $Z_{SQP}(2) = 17$, and we can further inspect this list to find that $a_2 = 5$ and $b_2 = 12$. These values can be used with the previously determined equalities to find that $a_3 = 12$ and $b_3 = 29$, thus $Z_{SQP}(3) = 41$. These base cases enable us to set up a system of linear equations in two variables:

$$17 = c_1(1+\sqrt{2})^2 + c_2(1-\sqrt{2})^2$$
$$41 = c_1(1+\sqrt{2})^3 + c_2(1-\sqrt{2})^3$$

which can be used to find

$$c_1 = \frac{3}{2} + \sqrt{2} \quad \text{and} \quad c_2 = \frac{3}{2} - \sqrt{2}.$$

Using this information, we can write the general formula for $Z_{SQP}(n)$ as follows:

$$Z_{SQP}(n) = \left(\frac{3}{2} + \sqrt{2}\right)(1+\sqrt{2})^n + \left(\frac{3}{2} - \sqrt{2}\right)(1-\sqrt{2})^n.$$

$\square$

Table 2.1 – Number of functions on $n$ voters

| Axioms | n=2 | n=3 | n=4 | n=5 |
|---|---|---|---|---|
| A,N,Mon,TB | 1 | 1 | 1 | 1 |
| A,N,Mon | 3 | 6 | 10 | 20 |
| A,Mon,TB | 17 | 41 | 99 | 239 |
| A,Mon | 35 | 126 | 462 | 1716 |

The sequence generated by this formula for the number of strong quota pair systems on $n$ voters is a known one. It has several different interpretations that can be found in [21]. Table 2.1 lists the number of distinct functions satisfying the various combinations of axioms discussed in this chapter. This gives a general impression of how restrictive some of the axioms are.

# CHAPTER 3
# MEDIAN SEMILATTICES

Before we can discuss generalizing May's Theorem to an order theoretic domain some background must be provided. This is a brief chapter on language and concepts needed to understand the next chapter. For more information on median semilattices see [5] and [20], the latter having an emphasis on the related idea of median graphs.

**Definition 3.1.** *A* **partial order** *is a relation on a set $X$ that is reflexive, transitive, and antisymmetric.*

We will henceforth refer to the set $X$ as a partially ordered set (poset), taking it as implied that there is a reflexive, transitive, and antisymmetric relation on it. The least upper bound, or supremum, of two elements is referred to as their join. For any two elements $a, b \in X$ their join is denoted as:

$$a \vee b = sup\{a, b\}.$$

The greatest lower bound, or infimum, of two elements is referred to as their meet. For any two elements $a, b \in X$ their meet is denoted as:

$$a \wedge b = inf\{a, b\}.$$

**Definition 3.2.** *If $X$ is a partially ordered set in which any two elements have a join and a meet, then $X$ is a* **lattice***.*

If $X$ is finite then this definition implies $X$ has a maximum and a minimum element, conventionally denoted as 1 and 0 respectively.

**Definition 3.3.** *If $X$ is a partially ordered set in which any two elements have a meet, then $X$ is a* **meet semilattice***.*

It follows from this definition that every lattice is a meet semilattice as well. A distributive lattice is one in which the conventional notion of distributivity applied to joins and meets holds for all elements in the lattice. Meaning, for all elements $a, b, c \in X$ we have

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \text{ and } a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

**Definition 3.4.** *A meet semilattice $X$ is* **distributive** *if, for all $x \in X$, the set $\{y \in X | y \leq x\}$ is a distributive lattice.*

The following definitions round out the majority of the vocabulary needed to understand the next chapter.

**Definition 3.5.** *A meet semilattice $X$ satisfies the* **join-Helly property** *if, for all $x, y, z \in X$, whenever $x \vee y$, $x \vee z$, and $y \vee z$ exist, then $x \vee y \vee z$ exists as well.*

In Figure 3.1 we can see an example of a finite meet semilattice that does not satisfy the join-Helly property, although it is distributive.

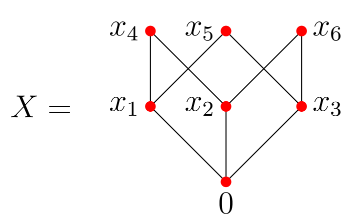



Figure 3.1: A meet semilattice that is not a median semilattice

**Definition 3.6.** *A meet semilattice is a* **median semilattice** *if it is distributive and satisfies the join-Helly property.*

In Figure 3.2 we can see the median semilattice that represents the two-alternative version of May's Theorem discussed in the previous chapters. Another important class of median semilattices are hierarchies, discussed in Chapter 5.

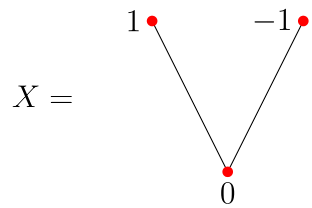



Figure 3.2: May's Case

**Definition 3.7.** *An element $j$ of a meet semilattice $X$ is* **join irreducible** *if $j \neq 0$ and $j = x \vee y \Rightarrow j = x$ or $j = y$ for all $x, y \in X$.*

The definition of median semilattice is the key concept needed for the generalization in the next chapter.

# CHAPTER 4
# MAY'S THEOREM ON MEDIAN SEMILATTICES

## 4.1 Introduction

In the previous chapter some definitions and ideas concerning ordered sets were presented. Prior to that, May's Theorem and and some generalizations of May's Theorem were given in the case of two alternatives. This two-alternative case can be thought of as a particular example of a median semilattice. In this chapter we will state and prove our main result, a generalization of May's theorem to an arbitrary finite median semilattice.

Let $X$ be a finite median semilattice. This $X$ represents a collection of alternatives from which one will be selected. We call any $P = (x_1, \ldots, x_n) \in X^n$ a *profile* for some $n \geq 2$. This profile represents the preferences of $N = \{1, \ldots, n\}$ individuals from amongst the alternatives in $X$. As in Chapter 1, we will denote the preference of a particular individual, $x_i$, as $P(i)$.

**Definition 4.1.** *A function of the form $f : X^n \to X$ will be called a* **consensus function**.

Let $J$ be the set of all join irreducible elements of $X$. Next we establish some important notation. For any $P \in X^n$ and any join irreducible $j \in J$ let

$$K_j(P) = \{i \in N : j \leq P(i)\} \text{ and } \overline{K}_j(P) = \{i \in N : j \vee P(i) \text{ D.N.E.}\}.$$

Here, the abbreviation "D.N.E." means "does not exist." Put simply, $K_j(P)$

represents the individuals who favor $j$, and $\overline{K}_j(P)$ represents the individuals who do not favor $j$. Observe that when $X$ is lattice, $\overline{K}_j(P) = \varnothing$ for all $j \in J$. It is worth noting that since $X$ is a meet semilattice, it has a 0 element. This represents the notion of abstention or non-preference, as distinguished from the elements of $\overline{K}_j(P)$, which are better thought of as preferences for a competitor to $j$. We keep track of these abstentions with a special case of our established notation, where, for $P \in X^n$ we have:

$$K_0(P) = \{i \in N : P(i) = 0\}.$$

Now we define an extremely important example of a consensus function.

**Definition 4.2.** *The* **simple majority rule function**, *$f_s : X^n \to X$, is defined such that for any $P \in X^n$:*

$$f_s(P) = \bigvee\{j \in J : |K_j(P)| > |\overline{K}_j(P)|\}.$$

Presented below is a useful lemma concerning the simple majority rule function. The proof of this lemma uses the concept of a join prime element. An element $j \in X$ is join prime if $j \leq x \vee y \Rightarrow j \leq x$ or $j \leq y$ for all $x, y \in X$.

**Lemma 4.1.** *For any $t \in J$ and $P \in X^n$, $t \leq f_s(P) \Rightarrow |K_t(P)| > |\overline{K}_t(P)|$.*

*Proof.* Suppose $t \leq f_s(P) = \bigvee\{j \in J : |K_j(P)| > |\overline{K}_j(P)|\}$. Since $X$ is distributive and $t$ is a join irreducible, we know $t$ is join prime; thus it follows that $t \leq j$ such that $|K_j(P)| > |\overline{K}_j(P)|$. Since $t \leq j$ we know that $|K_t(P)| \geq |K_j(P)|$ and, moreover, that $|\overline{K}_j(P)| \geq |\overline{K}_t(P)|$. To get this last relation, observe that if $t \leq j$ and $t \vee x$ does not exist for some $x \in X$ then $j \vee x$ does not exist either. Thus,

$$|K_t(P)| \geq |K_j(P)| > |\overline{K}_j(P)| \geq |\overline{K}_t(P)|$$

as desired. □

**Remark 4.1.** *It is worth mentioning that Lemma 4.1 is indeed a true biconditional, as the reverse direction is a direct consequence of the definition of* $f_s$*.*

**Remark 4.2.** *Observe that if* $X$ *is a lattice, then, for any* $P = (x_1, \ldots, x_n)$*,*

$$f_s(P) = x_1 \vee x_2 \vee \ldots \vee x_n.$$

Since $X$ is a lattice, it follows that $\overline{K}_j(P) = \varnothing \; \forall j \in J$ and so

$$f_s(P) = \bigvee \{j \in J : |K_j(P)| > 0\}. \tag{4.1}$$

Since $X$ is distributive, we know that every element of $X$ is the join of all the join irreducibles less than or equal to it, thus if $t \in J$ and $t \leq x_1 \vee x_2 \vee \ldots \vee x_n$ then we know $t \leq x_i$ for some $i$. In this case, $|K_t(P)| > 0$ which implies $t \leq f_s(P)$ by Remark 4.1, so it follows that $x_1 \vee x_2 \vee \ldots \vee x_n \leq f_s(P)$. If $j \in J$ satisfies $j \leq f_s(P)$, then by Lemma 4.1, $|K_j(P)| > 0$ and so $j \leq x_i$ for some $i$. Thus, $j \leq x_1 \vee x_2 \vee \ldots \vee x_n$. It follows that $f_s(P) \leq x_1 \vee x_2 \vee \ldots \vee x_n$. Hence $f_s(P) = x_1 \vee x_2 \vee \ldots \vee x_n$.

Below is another consensus function defined only in the case when the finite median semilattice $X$ is also a lattice.

**Definition 4.3.** *The* **top function***,* $f_T : X^n \to X$*, is defined such that for any* $P \in X^n$*:*

$$f_T(P) = 1 \; \forall P \in X^n.$$

The top function will be important in this chapter.

## 4.2 Axioms

**Axiom 4.1.** *A consensus function* $f : X^n \to X$ *satisfies* ***Anonymity (A)*** *if, for any* $P \in X^n$ *and any* $\sigma$*, a permutation of* $N = \{1, \ldots, n\}$*;* $f(\sigma P) = f(P)$*.*

This means that $f(P)$ is determined only by the elements of $P_i$ that appear, regardless of how they are assigned; i.e. an anonymous function makes a decision based on what the ballots say, not who submitted each one. Also called *equality* or *egalitarian* [16].

**Axiom 4.2.** *A consensus function $f : X^n \to X$ satisfies **Monotonicity (MON)** if, for any $P, P' \in X^n$ and any $j \in J$; if $K_j(P) \subseteq K_j(P')$ and $\overline{K}_j(P') \subseteq \overline{K}_j(P)$ then $j \leq f(P) \Rightarrow j \leq f(P')$.*

This means that $f$ responds to changes of preference in a "positive" way. That is, if $f$ favors $j$ and any individual or group of individuals changes their vote in such a way that they now favor $j$ then $f$ will still favor $j$.

**Axiom 4.3.** *A consensus function $f : X^n \to X$ satisfies **Competitive Decisive Neutrality (CDN)** if, for any $P, P' \in X^n$ and any $j, j' \in J$;*

$$K_j(P) = K_{j'}(P') \text{ and } \overline{K}_j(P) = \overline{K}_{j'}(P') \Rightarrow [j \leq f(P) \Leftrightarrow j' \leq f(P')].$$

This means $f$ does not favor any of the alternatives over another. Thus, if we replaced all the votes consistently, $f$ would favor whichever alternative replaced the previous winner. This axiom is an extention of May's *Neutrality* and first appeared in [17].

**Axiom 4.4.** *A consensus function $f : X^n \to X$ satisfies **Tie Breaking (TB)** if, for every $P, P' \in X^n$ and $j, j' \in J$ such that*

$$\{P\} \cup \{P'\} \subseteq \{j, j', 0\}$$
$$\text{and } P(ae)P'$$

*we have*

$$j \not\leq f(P) \text{ and } j' \not\leq f(P) \Rightarrow j \leq f(P') \text{ or } j' \leq f(P').$$

This is what it sounds like; if the result is a tie and a single vote changes then $f$ will pick a winner, breaking the tie. It is worth remarking that it is possible for $j = j'$, in which case **TB** becomes simplified quite a bit. Meaning, if $j = j'$ then

$$\{P\} \cup \{P'\} \subseteq \{j, 0\}$$
$$\text{and } P(ae)P'$$

implies

$$j \nleq f(P) \Rightarrow j \leq f(P').$$

**Theorem 4.1.** *The simple majority rule function, $f_s$, satisfies Anonymity, Monotonicity, Competitive Decisive Neutrality, and Tie-Breaking.*

*Proof.* Since $f_s(P)$ is determined by the values $|K_j(P)|$ and $|\overline{K}_j(P)|$ it follow that $f_s$ satisfies **A**.

Suppose $P, P' \in X^n$ are two profiles such that $K_j(P) \subseteq K_j(P')$, $\overline{K}_j(P') \subseteq \overline{K}_j(P)$, and $j \leq f_s(P)$ for some $j \in J$. To show $f_s$ satisfies **MON** we need to show that $j \leq f_s(P')$. Observe that,

$$K_j(P) \subseteq K_j(P') \Rightarrow |K_j(P)| \leq |K_j(P')|$$
$$\overline{K}_j(P') \subseteq \overline{K}_j(P) \Rightarrow |\overline{K}_j(P')| \leq |\overline{K}_j(P)|.$$

By Lemma 4.1 we have $j \leq f_s(P) \Rightarrow |K_j(P)| > |\overline{K}_j(P)|$. Thus,

$$|K_j(P')| \geq |K_j(P)| > |\overline{K}_j(P)| \geq |\overline{K}_j(P')|$$

which, by the definition of $f_s$, implies $j \leq f_s(P')$ as desired.

Suppose $P, P' \in X^n$ are two profiles such that $K_j(P) = K_{j'}(P')$, $\overline{K}_j(P) = \overline{K}_{j'}(P')$, and $j \leq f_s(P)$ for some $j, j' \in J$. To show $f_s$ satisfies **CDN** we need to show that $j' \leq f_s(P')$.

By Lemma 4.1 we have $j \leq f_s(P) \Rightarrow |K_j(P)| > |\overline{K}_j(P)|$. Thus,

$$|K_{j'}(P')| = |K_j(P)| > |\overline{K}_j(P)| = |\overline{K}_{j'}(P')|.$$

Hence $j' \leq f_s(P')$ as desired. The reverse direction, $j' \leq f_s(P') \Rightarrow j \leq f_s(P)$ is done in a similar way.

To show $f_s$ satisfies **TB**, consider $P, P' \in X^n$ such that for $j, j' \in J$ we have

$$\begin{gathered} \{P\} \cup \{P'\} \subseteq \{j, j', 0\} \\ P(ae)P', \\ j \nleq f_s(P), \\ \text{and } j' \nleq f_s(P). \end{gathered}$$

We want to show that $j \leq f_s(P')$ or $j' \leq f_s(P')$; this will be handled in two cases.

**Case 1:** $j \vee j'$ exists.

In this case, since $\{P\} \cup \{P'\} \subseteq \{j, j', 0\}$, we know $\overline{K}_j(P) = \varnothing = \overline{K}_{j'}(P)$. Lemma 4.1 tells us that

$$\begin{gathered} j \nleq f_s(P) \Rightarrow |K_j(P)| \leq |\overline{K}_j(P)|, \\ j' \nleq f_s(P) \Rightarrow |K_{j'}(P)| \leq |\overline{K}_{j'}(P)|. \end{gathered}$$

Thus $K_j(P) = \varnothing = K_{j'}(P)$, hence $P = P_0$ where $\{P_0\} = \{0\}$. Since $P(ae)P'$ and $\{P\} \cup \{P'\} \subseteq \{j, j', 0\}$ we know either $P'(i) = j$ or $P'(i) = j'$ for some $i \in \{1, \ldots, n\}$. If $P'(i) = j$ then $|K_j(P')| = 1 > 0 = |\overline{K}_j(P')|$ implying $j \leq f(P')$. If $P'(i) = j'$ instead, then $|K_{j'}(P')| = 1 > 0 = |\overline{K}_{j'}(P')|$ implying $j' \leq f(P')$. Thus, $j \leq f(P')$ or $j' \leq f(P')$ as desired.

**Case 2:** $j \vee j'$ does not exist.

In this case, since $\{P\} \cup \{P'\} \subseteq \{j, j', 0\}$ we know

$$K_j(P) = \overline{K}_{j'}(P),\ K_{j'}(P) = \overline{K}_j(P),$$
$$K_j(P') = \overline{K}_{j'}(P'),\ K_{j'}(P') = \overline{K}_j(P').$$

Using Lemma 4.1 again, we know

$$j \not\leq f_s(P) \Rightarrow |K_j(P)| \leq |\overline{K}_j(P)|,$$
$$j' \not\leq f_s(P) \Rightarrow |K_{j'}(P)| \leq |\overline{K}_{j'}(P)|.$$

We can combine these two sets of facts to conclude that

$$|\overline{K}_{j'}(P)| = |K_j(P)| \leq |\overline{K}_j(P)| = |K_{j'}(P)| \text{ and}$$
$$|\overline{K}_j(P)| = |K_{j'}(P)| \leq |\overline{K}_{j'}(P)| = |K_j(P)|.$$

Hence,

$$|K_j(P)| = |\overline{K}_j(P)| = |K_{j'}(P)| = |\overline{K}_{j'}(P)|.$$

Since $P(ae)P'$, the difference between $P$ and $P'$ can be divided in to three possible subcases:

If $P(i_0) = j'$, then $P'(i_0) = 0$ or $P'(i_0) = j$.

If $P'(i_0) = 0$ then $|K_{j'}(P')| = |K_{j'}(P)| - 1$ and $|K_j(P')| = |K_j(P)|$. Hence

$$|K_j(P')| = |K_j(P)| = |K_{j'}(P)| > |K_{j'}(P)| - 1 = |K_{j'}(P')| = |\overline{K}_j(P')|.$$

Thus $j \leq f_s(P')$.

If $P'(i_0) = j$ then $|K_{j'}(P')| = |K_{j'}(P)| - 1$ and $|K_j(P')| = |K_j(P)| + 1$. Hence

$$|K_j(P')| = |K_j(P)| + 1 > |K_j(P)| - 1 = |K_{j'}(P)| - 1 = |K_{j'}(P')| = |\overline{K}_j(P')|.$$

Thus $j \leq f_s(P')$.

If $P(i_0) = j$ then an argument symmetric to the one just given can be used to conclude $j' \leq f_s(P')$.

If $P(i_0) = 0$, then $P'(i_0) = j$ or $P'(i_0) = j'$. If $P'(i_0) = j$ then $|K_j(P')| = |K_j(P)|+1$ and $|K_{j'}(P')| = |K_{j'}(P)|$, thus

$$|K_j(P')| = |K_j(P)| + 1 > |K_j(P)| = |K_{j'}(P)| = |K_{j'}(P')| = |\overline{K}_j(P')|.$$

Thus $j \leq f_s(P')$.

If $P'(i_0) = j'$, then an equivalent argument will conclude $j' \leq f_s(P')$, as desired. $\square$

**Theorem 4.2.** *When $X$ is a lattice, the top function, $f_T$, satisfies Anonymity, Monotonicity, Competitive Decisive Neutrality, and Tie-Breaking.*

*Proof.* Since $f_T(P) = 1 \ \forall P \in X^n$ we have $f_T(P) = f_T(\sigma P) = 1$ for any $\sigma$, hence, $f_T$ satisfies **A**.

Suppose $P, P' \in X^n$ are two profiles such that $K_j(P) \subseteq K_j(P')$, $\overline{K}_j(P') \subseteq \overline{K}_j(P)$, and $j \leq f_T(P)$ for some $j \in J$. To show $f_T$ satisfies **MON** we need to show that $j \leq f_T(P')$. Since $f_T(P') = 1$ it is clearly the case that $j \leq f_T(P')$.

Suppose $P, P' \in X^n$ are two profiles such that $K_j(P) = K_{j'}(P')$, $\overline{K}_j(P) = \overline{K}_{j'}(P')$, and $j \leq f_T(P)$ for some $j, j' \in J$. To show $f_T$ satisfies **CDN** we need to show that $j' \leq f_T(P')$. Since $f_T(P') = 1$ it is clearly the case that $j \leq f_T(P')$.

Since $f_T(P) = 1 \ \forall P \in X^n$, it will never be the case that $j \not\leq f_T(P)$ for any $j$ or $P$. This means the conditions of **TB** will never be met, thus it is vacuously satisfied. $\square$

Not only do these two functions satisfy our axioms, they are characterized by them. This fact is the main result of this thesis and is proven below.

## 4.3 Main Result

**Theorem 4.3.** *If $X$ is a finite median semilattice and $f : X^n \to X$ satisfies Anonymity, Monotonicity, Competitive Decisive Neutrality, and Tie-Breaking then $f = f_s$ or $f = f_T$.*

To prove this theorem we first consider the profile $P_0$, where $\{P_0\} = \{0\}$, in a lemma. We will build up to less trivial profiles in a series of lemmas, culminating in the proof of the theorem.

**Lemma 4.2.** *$f(P_0) = 0$ or $f(P_0) = 1$.*

*Proof.* Suppose $f(P_0) \neq 0$. Since $X$ is finite there exists $j \in J$ such that $j \leq f(P_0)$. For any $j' \in J$ notice that

$$K_j(P_0) = K_{j'}(P_0) = \varnothing \text{ and } \overline{K}_j(P_0) = \overline{K}_{j'}(P_0) = \varnothing.$$

Thus, by **CDN**, $j \leq f(P_0) \Leftrightarrow j' \leq f(P_0)$. Since $j'$ was an arbitrary element of $J$, this implies $t \leq f(P_0)\ \forall t \in J$, hence $f(P_0) = 1$. □

Observe that the previous lemma follows directly from **CDN** and does not use the other axioms.

**Lemma 4.3.** *If $j \in J$ satisfies $j \leq f(P)$ for some $P \in X^n$, and $f(P_0) = 0$, then $K_j(P) \neq \varnothing$.*

*Proof.* Suppose not, i.e. $j \leq f(P)$ and $K_j(P) = \varnothing$ for some $P \in X^n$. Then

$$K_j(P) = K_j(P_0) = \varnothing \text{ and } \overline{K}_j(P_0) \subseteq \overline{K}_j(P).$$

Thus, **MON** implies $j \leq f(P_0)$, which contradicts the assumption that $f(P_0) = 0$. Therefore $K_j(P) \neq \varnothing$. □

Notice that the previous lemma follows directly from **MON** and does not use the other axioms.

**Lemma 4.4.** *If* $\{P\} \subseteq \{j, 0\}$ *for some* $P \in X^n$ *and* $j \in J$ *such that* $K_j(P) \neq \varnothing$, *and* $f(P_0) = 0$, *then* $j \leq f(P)$.

*Proof.* Consider $P' \in X^n$ such that $P'(i_0) = j$ for some $i_0 \in K_j(P)$ and $P'(i) = 0$ for all $i \neq i_0$. Thus

$$\{P'\} \cup \{P_0\} \subseteq \{j, 0\}$$
$$\text{and } P'(ae)P_0$$

Since $j \nleq f(P_0)$ **TB** implies $j \leq f(P')$. Since $K_j(P') \subseteq K_j(P)$ and $\overline{K}_j(P) = \overline{K}_j(P') = \varnothing$, **MON** implies $j \leq f(P)$. $\square$

The previous lemma follows from **MON** and **TB**.

**Lemma 4.5.** *If* $\{P\} \subseteq \{j_1, j_2, 0\}$ *and* $|K_{j_1}(P)| = |K_{j_2}(P)| > 0$ *for* $P \in X^n$ *and* $j_1, j_2 \in J$ *such that* $j_1 \vee j_2$ *does not exist, then*

$$f(P) = \bigvee\{t \in J : t \leq j_1 \text{ and } t \vee j_2 \text{ exists}$$
$$\text{or } t \leq j_2 \text{ and } t \vee j_1 \text{ exists}\}.$$

*Proof.* Consider $t \in J$ such that $t \leq j_1$ and $t \vee j_2$ exists. Also consider $P' \in X^n$ such that $\{P'\} \subseteq \{j_1, 0\}$ and $K_{j_1}(P') = K_{j_1}(P)$. By Lemma 4.4, $j_1 \leq f(P')$.

Consider $P'' \in X^n$ such that $\{P''\} \subseteq \{t, 0\}$ and $K_t(P'') = K_{j_1}(P')$. Which implies $t \leq f(P'')$, again by Lemma 4.4.

Observe $K_t(P'') \subseteq K_t(P)$ and $\overline{K}_t(P'') = \overline{K}_t(P) = \varnothing$, thus, **MON** implies $t \leq f(P)$.

Suppose $u \in J$ such that $u \leq f(P)$. Since we assumed the existence of $j_1, j_2 \in J$

such that $j_1 \vee j_2$ does not exist, we know $X$ can't be a lattice. Since $X$ is not a lattice it cannot be the case that $f(P_0) = 1$, thus Lemma 4.2 implies $f(P_0) = 0$, hence, Lemma 4.3 implies $K_u(P) \neq \varnothing$. Thus $u \leq j_1$ or $u \leq j_2$; without loss of generality, let $u \leq j_1$. We need to show $u \vee j_2$ exists.

If $u \vee j_2$ does not exist then

$$K_u(P) = K_{j_1}(P) \text{ and } \overline{K}_u(P) = \overline{K}_{j_1}(P),$$

hence **CDN** implies $j_1 \leq f(P)$. Consider $P''' \in X^n$ such that $\{P'''\} \subseteq \{j, j', 0\}$ where

$$\begin{aligned} K_{j_1}(P''') &= K_{j_2}(P), \\ K_{j_2}(P''') &= K_{j_1}(P), \\ \text{and } K_0(P''') &= K_0(P). \end{aligned}$$

It was assumed that $|K_{j_1}(P)| = |K_{j_2}(P)|$, hence, we have that $P'''$ is simply a permutation of the elements of $P$, thus, **A** implies $j_1 \leq f(P''')$. Observe that

$$K_{j_1}(P) = K_{j_2}(P''') \text{ and } \overline{K}_{j_1}(P) = \overline{K}_{j_2}(P''').$$

Since $j_1 \leq f(P)$ it follows from **CDN** that $j_2 \leq f(P''')$. But now $f(P''')$ is an upper bound for $\{j_1, j_2\}$, contrary to the fact that $j_1 \vee j_2$ does not exist. Thus $u \vee j_2$ exists, as desired. $\square$

The last lemma utilized all the axioms to arrive at a proof. With these lemmas established, we can complete the proof of Theorem 4.3.

*Proof of Theorem 4.3.* It follows from Lemma 4.2 that $f(P_0) = 0$ or $f(P_0) = 1$.

If $f(P_0) = 1$ then consider $P \in X^n$ such that $P \neq P_0$. Observe that for any $j \in J$ we have $K_j(P_0) = \varnothing$, thus $K_j(P_0) \subseteq K_j(P)$. If $f(P_0) = 1$ then

$\overline{K}_j(P_0) = \overline{K}_j(P) = \varnothing$. Furthermore, if $f(P_0) = 1$, then $j \leq f(P_0)$. Thus, by **MON**, $j \leq f(P)$. Since $j$ is an arbitrary element of $J$, this implies $t \leq f(P)\ \forall t \in J$, hence $f(P) = 1$. Thus, in this case, $f = f_T$.

If $f \neq f_T$ then $f(P_0) = 0$, which is equal to $f_s(P_0)$, since the join of the empty set is 0. Suppose $j \leq f(P)$ for some $P \in X^n$. Want to show $j \leq f_s(P)$. Lemma 4.3 implies $|K_j(P)| > 0$. If $|\overline{K}_j(P)| = 0$ then we're done.

If $|\overline{K}_j(P)| > 0$ consider $P' \in X^n$ and $j' \in J$ such that $j \vee j'$ does not exist, where

$$P'(i) = j \text{ for } i \in K_j(P),$$
$$P'(i) = j' \text{ for } i \in \overline{K}_j(P), \text{ and}$$
$$P'(i) = 0 \text{ otherwise.}$$

Thus, $K_j(P') = K_j(P)$ and $\overline{K}_j(P') = K_{j'}(P') = \overline{K}_j(P)$, hence, **CDN** implies $j \leq f(P')$. This implies

$$|K_j(P')| > |\overline{K}_j(P')| = |K_{j'}(P')|.$$

If not (that is, if $|K_j(P')| \leq |K_{j'}(P')|$) then consider $\{P''\} \subseteq \{j, j', 0\} \in X^n$ such that

$$K_j(P') \subseteq K_j(P''),$$
$$\overline{K}_j(P'') \subseteq \overline{K}_j(P'), \text{ and}$$
$$|K_{j'}(P'')| = |K_j(P'')|.$$

Thus, **MON** implies $j \leq f(P'')$. Lemma 4.5 implies

$$f(P'') = \bigvee\{t \in J : t \leq j \text{ and } t \vee j' \text{ exists}$$
$$\text{or } t \leq j' \text{ and } t \vee j \text{ exists}\}.$$

Since $j \leq f(P'')$ and $j$ is join prime, we know that $j \leq t$ such that either

$$t \leq j \text{ and } t \vee j' \text{ exists} \Rightarrow j \vee j' \text{ exists, or}$$

$$t \leq j' \text{ and } t \vee j \text{ exists} \Rightarrow j \vee j' \text{ exists.}$$

Both cases contradict the assumption that $j \vee j'$ does not exist. Hence, $|K_j(P')| > |K_{j'}(P')|$. Thus, by the definition of $f_s$, $|K_j(P)| > |\overline{K}_j(P)| \Rightarrow j \leq f_s(P)$.

Now we will assue $j \leq f_s(P)$ and show $j \leq f(P)$. Lemma 4.1 implies $|K_j(P)| > |\overline{K}_j(P)|$. If $|\overline{K}_j(P)| = 0$ then consider $P' \subseteq \{j, 0\}$ such that $K_j(P') = K_j(P)$. Lemma 4.4 implies $j \leq f(P')$ hence, by **CDN**, $j \leq f(P)$. If $|\overline{K}_j(P)| > 0$ then there exists $j' \in J$ such that $j \vee j'$ does not exist. Consider $P' \subseteq \{j, j', 0\}$ with

$$K_{j'}(P') = \overline{K}_j(P), \ K_j(P') \subseteq K_j(P), \text{ and } |K_j(P')| = |K_{j'}(P')|.$$

Thus, by Lemma 4.5,

$$f(P') = \bigvee \{t \in J : t \leq j \text{ and } t \vee j' \text{ exists}$$
$$\text{or } t \leq j' \text{ and } t \vee j \text{ exists}\}.$$

If $j \leq f(P')$ then, since $j$ is a join irreducible, we know that $j \leq t$ such that either

$$t \leq j \text{ and } t \vee j' \text{ exists} \Rightarrow j \vee j' \text{ exists, or}$$

$$t \leq j' \text{ and } t \vee j \text{ exists} \Rightarrow j \vee j' \text{ exists.}$$

Both cases contradict the fact that $j \vee j'$ does not exist, hence $j \nleq f(P')$. Similarly, $j' \nleq f(P')$. Consider $P'' \subseteq \{j, j', 0\}$ where

$$K_{j'}(P'') = K_{j'}(P'), \ K_j(P') \subseteq K_j(P''), \text{ and } |K_j(P'')| = |K_j(P')| + 1.$$

Observe that $P'(ae)P''$. Since $j \nleq f(P')$ and $j' \nleq f(P')$, **TB** tells us that either $j \leq f(P'')$ or $j' \leq f(P'')$. Since $K_{j'}(P'') \subseteq K_{j'}(P')$ and $\overline{K}_{j'}(P') \subseteq \overline{K}_{j'}(P'')$, if

$j' \leq f(P'')$ then **MON** would imply $j' \leq f(P')$; a contradiction. Hence, $j \leq f(P'')$ and we can use **A** and **MON** to get $j \leq f(P)$.

Since $j \leq f(P) \Rightarrow j \leq f_s(P)$ and $j \leq f_s(P) \Rightarrow j \leq f(P)$, we have $f = f_s$ as desired. □

It is obviously the case that $f_T$ only makes sense as a function when $X$ is a lattice, thus when $X$ is not a lattice **A**, **CDN**, **MON**, and **TB** *uniquely* characterize the simple majority rule function $f_s$. Since we would like a unique characterization on any finite median semilattice $X$, the case when $X$ is a lattice will require some further study; which is done in the following sections. It will be shown in the next section that, when $X$ is a lattice, **A** is implied by the other axioms. To see the importance of Anonymity when $X$ is not a lattice we will look at a particular consensus function that uses the following notation.

$$|K_j(P)|^1 = \begin{cases} |K_j(P)| + 1 & \text{if } j \leq P(1); \\ |K_j(P)| & \text{if } j \not\leq P(1); \end{cases}$$

$$|\overline{K}_j(P)|^1 = \begin{cases} |\overline{K}_j(P)| + 1 & \text{if } j \vee P(1) \ D.N.E.; \\ |\overline{K}_j(P)| & \text{otherwise.} \end{cases}$$

**Example 4.1.** *Let the function $\widehat{f_s} : X^n \to X$ be defined by:*

$$\widehat{f_s}(P) = \bigvee \{j \in J : |K_j(P)|^1 > |\overline{K}_j(P)|^1\}.$$

This function satisfies **MON**, **CDN**, and **TB**, but not **A**. This function can be thought of as simple majority rule only the first voters vote is counted twice. The argument for $\widehat{f_s}$ satisfying **MON**, **CDN**, and **TB** is very similar to the argument for $f_s$ satisfying those same axioms. To see that $\widehat{f_s}$ does not satisfy **A**, consider $P = (x_1, x_2, 0, \ldots, 0)$ and $P' = (x_2, x_1, 0, \ldots, 0)$, where $x_1 \vee x_2$ does not exist; such an $x_1$ and $x_2$ can be found because $X$ is not a lattice.

It turns out (as we would hope) that when $X$ is not a lattice, all of our axioms are independent of one another. The function presented above demonstrates the independence of Anonymity; below, functions will be presented which demonstrate the independence of the remaining three axioms.

**Example 4.2.** *The function $f_{j^*} : X^n \to X$ where $X$ is any finite median semilattice besides the two element chain, and $j^* \in J$ a fixed atom, is defined by:*

$$f_{j^*}(P) = \begin{cases} f_s(P) \vee j^* & \text{if } \overline{K}_{j^*}(P) = \varnothing \\ f_s(P) & \text{otherwise.} \end{cases}$$

Since $f_s$ is a well defined function all we have to check to make sure $f_{j^*}$ is well defined is that $f_s(P) \vee j^*$ exists when $\overline{K}_{j^*}(P) = \varnothing$. If $f_s(P) = 0$, then it is clearly the case that $f_s(P) \vee j^*$ exists. If $f_s(P) \neq 0$ then observe that since $X$ is finite, we know $f_s(P) = \bigvee\{j \in J : j \leq f_s(P)\}$. Recall that $f_s$ satisfies all the axioms and that $f_s(P_0) = 0$, thus we can apply Lemma 4.3 and conclude that for any $j \in J$, if $j \leq f_s(P)$ then $K_j(P) \neq \varnothing$, thus there exists an $i \in N$ such that $j \leq P(i)$. Since $\overline{K}_{j^*}(P) = \varnothing$ we have that $P(i) \vee j^*$ exists, and is thus an upper bound for both $j^*$ and any $j \in J$ such that $j \leq f_s(P)$, hence, $j \vee j^*$ exists. If $|\{j \in J : j \leq f_s(P)\}| = 1$ then $\{j \in J : j \leq f_s(P)\} = \{f_s(P)\}$ and we have that $f_s(P) \vee j^*$ exists. If $|\{j \in J : j \leq f_s(P)\}| \geq 2$ then the join-Helly property implies $f_s(P) \vee j^*$ exists. Thus the function $f_{j^*}$ is well defined, it is also the case that $f_{j^*}$ satisfies **A**, **MON**, and **TB**, but not **CDN**.

It follows from the fact that $f_s$ satisfies **A** that $f_{j^*}$ does as well. The fact that $f_s$ satisfies **TB** and the observation that $f_s(P) \leq f_{j^*}(P)$ for all profiles $P \in X^n$, that $f_{j^*}$ satisfies **TB** as well. To see that $f_{j^*}$ satisfies **MON**, consider $P, P' \in X^n$ such that $K_j(P) \subseteq K_j(P')$, $\overline{K}_j(P') \subseteq \overline{K}_j(P)$, and $j \leq f_{j^*}(P)$ for some $j \in J$. We want to show $j \leq f_{j^*}(P')$. Since $j^*$ is an atom, it follows that $j \leq f_{j^*}(P)$ implies $j \leq f_s(P)$ or $j = j^*$ (with $\overline{K}_{j^*}(P) = \varnothing$). If $j \leq f_s(P)$ then $j \leq f_s(P')$

since $f_s$ satisfies **MON**, and thus $j \leq f_{j^*}(P')$. If $j = j^*$ and $\overline{K}_{j^*}(P) = \varnothing$ then $\overline{K}_j(P') \subseteq \overline{K}_j(P)$ implies $\overline{K}_{j^*}(P') = \varnothing$ and we have $f_{j^*}(P) = f_s(P) \vee j^*$, thus $j \leq f_{j^*}(P')$ as desired. To see that $f_{j^*}$ does not satisfy **CDN** consider any $j \in J$ such that $j \neq j^*$; $j^* \leq f_{j^*}(P_0)$ but $j \nleq f_{j^*}(P_0)$ even though the conditions of **CDN** are met at $P_0$.

**Example 4.3.** *Let the function $f_0 : X^n \to X$ be defined by:*

$$f_0(P) = 0$$

This function satisfies **A**, **CDN**, and **MON** but not **TB**. This function clearly satisfies **A**. $f_0$ vacuously satisfies **CDN** and **MON**. Since this function always outputs a 0 the tie can never be broken.

We will see an example of a function that satisfes **A**, **CDN**, **TB**, but not **MON** in Chapter 5.

## 4.4 Lattice Case

It was mentioned in the preceding sections that things simplify quite a bit when $X$ is a lattice. In this section $X$ will be restricted to being a distributive lattice in order to study these simplifications. Just as we previously defined the profile $\{P_0\} = \{0\}$, it will be convenient to define $\{P_1\} = \{1\}$, which is a potential profile now that $X$ is a lattice. We begin with the axioms of Monotonicity and Competitive Decisive Neutrality, which reduce to axioms studied by Monjardet in [19]; these axioms, Decisive Monotonicity and Decisive Neutrality respectively, are presented below.

**Axiom 4.5.** *A consensus function $f : X^n \to X$ satisfies **Decisive Monotonicity (DM)** if, for any $P, P' \in X^n$ and any $j \in J$; if $K_j(P) \subseteq K_j(P')$ then $j \leq f(P) \Rightarrow j \leq f(P')$.*

When $X$ is a lattice and $P \in X^n$ is a profile, then $\overline{K}_j(P) = \varnothing$ for all $j \in J$. One way of looking at this is that **MON** isn't actually changed, but that the condition requiring $\overline{K}_j(P') \subseteq \overline{K}_j(P)$ is vacuously satisfied by all profiles $P$ and $P' \in X^n$.

**Axiom 4.6.** *A consensus function $f : X^n \to X$ satisfies **Decisive Neutrality (DN)** if, for any $P, P' \in X^n$ and any $j, j' \in J$;*

$$K_j(P) = K_{j'}(P') \ \Rightarrow [j \leq f(P) \Leftrightarrow j' \leq f(P')].$$

As with **MON**, the requirement that $\overline{K}_j(P) = \overline{K}_{j'}(P')$ is vacuously satisfied by all profiles when $X$ is a lattice.

Restricting $X$ to being a lattice does not change the presentation of Anonymity or Tie Breaking, however, when $X$ is a lattice **DM**, **DN**, and **TB** imply Anonymity.

**Theorem 4.4.** *If $X$ is a distributive lattice and $f : X^n \to X$ satisfies **DM**, **DN**, and **TB** then $f$ satisfies **A**.*

*Proof.* Since $X$ is a lattice, **DM** and **DN** are equivalent to **MON** and **CDN**. Thus, it follows from Lemma 4.2 that $f(P_0) = 0$ or $f(P_0) = 1$. It follows from the same argument given in the proof of Theorem 4.3 that if $f(P_0) = 1$ then $f(P) = 1$ for all $P \in X^n$, thus, since $f$ is constant, it clearly satisfies Anonymity. If $f(P_0) = 0$ consider $P, P' \in X^n$ such that $P' = \sigma P$ for some permutation $\sigma$ and $j \leq f(P)$ for some $j \in J$. Lemma 4.3 implies $K_j(P) \neq \varnothing$ thus $K_j(P') \neq \varnothing$. Consider $P'' \in X^n$ such that $\{P''\} \subseteq \{j, 0\}$ and $K_j(P'') = K_j(P')$. Lemma 4.4 implies $j \leq f(P'')$ and **DM** implies $j \leq f(P')$. Since $f(P) = \bigvee\{j : j \leq f(P) \textit{ and } j \in J\}$, it follows that $f(P) \leq f(P')$. An argument symmetric to the one just presented can be used to show $f(P') \leq f(P)$, thus $f(P) = f(P')$ as desired. □

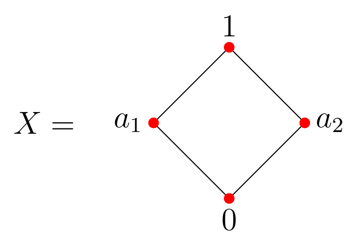
This interesting result allows us to remove the axiom of Anonymity from Theorem 4.3 when $X$ is a lattice. This result is presented below as a corollary.

**Corollary 1.** *If $X$ is a distributive lattice and $f : X^n \to X$ is a consensus function that satisfies **DM**, **DN**, and **TB** then $f = f_s$ or $f = f_T$.*

To see that **A** is the only axiom implied by the others we can consider the following examples of consensus functions that satisfy all but one of the remaining axioms.

In the following example let $X = B_2$, the Boolean lattice with two atoms, pictured in Figure 4.1.

Figure 4.1: The lattice $B_2$




Define the function $f_1 : X^n \to X$ by:

$$f_1(P) = x_1 \wedge x_2 \wedge \ldots \wedge x_n \quad \forall P = (x_1, \ldots, x_n) \in X^n.$$

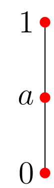
We now show that the function $f_1$ satisfies **DM** and **DN**, but not **TB**. Since $j \leq f_1(P)$ if and only if $K_j(P) = \{1, \ldots, n\}$, for any $j \in J$ and $P \in X^n$, **DM** is vacuously satisfied. To see that **DN** is satisfied, consider $P, P' \in X^n$ such that $K_j(P) = K_{j'}(P')$ for some $j, j' \in J$ and $j \leq f_1(P)$. This implies $K_j(P) = N$ thus $\{P'\} = K_{j'}(P')$ and it follows that $j' \leq f_1(P')$. To see that **TB** is *not* satisfied, consider $P_0 = (0, 0, \ldots, 0)$ and $P = (a_1, 0, \ldots, 0)$. It follows from the definition of

$f_1$ that $f_1(P_0) = f_1(P) = 0$, hence the tie is not broken.

The next example function is defined on the chain with two join irreducibles, $a$ and 1, pictured in Figure 4.2.

Figure 4.2: The 3 element chain

$X =$




The following function satisfies Decisive Monotonicity and Tie-Breaking, but not Decisive Neutrality. Define $f_2 : X^n \to X$ by:

$$f_2(P) = \begin{cases} x_1 \vee x_2 \vee \ldots \vee x_n & \text{if } P \neq P_0 \\ a & \text{if } P = P_0. \end{cases}$$

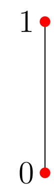
To see that this function satisfies **DM** observe that, when $j = a$, **DM** is satisfied automatically; if $j = 1$ then $1 \leq f_2(P)$ implies $P(i) = 1$ for some $i \in K$, thus $K_1(P) \subseteq K_1(P')$ implies $P'(i) = 1$ for some $i \in K$, thus $1 \leq f_2(P')$. A similar argument illustrates that $f_2$ satisfies **TB**, since $a \leq f_2(P)$ for all profiles $P$ the only case we need to consider is when $j = j' = 1$. If $1 \nleq f_2(P)$ for some $P$ such that $\{P\} \subseteq \{0, 1\}$ then we know $P = P_0$. Thus, $P(ae)P'$ implies $P'(i) = 1$ for some $i \in K$, hence, $1 \leq f_2(P')$ and $f_2$ satisfies **TB**. Lastly, consider $P = P_0$ and $P' = (a, 0, \ldots, 0)$, $K_a(P) = K_1(P') = \varnothing$ and $a \leq f_2(P)$ but $1 \nleq f_2(P')$, thus $f_2$ does not satisfy **DN**.

For the last example, consider the two element chain seen in Figure 4.3.

Define $f_3 : X^n \to X$ as:

Figure 4.3: The 2 element chain

$X =$




$$f_3(P) = \begin{cases} x_1 \vee x_2 \vee \ldots \vee x_n & \text{if } P \neq P_1 \\ 0 & \text{if } P = P_1. \end{cases}$$

The function $f_3$ satisfies Decisive Neutrality and Tie-Breaking, but not Decisive Monotonicity. To see that it satisfies **DN** consider the fact that 1 is the only join irreducible and $X$ consists of only two elements, so $K_1(P) = K_1(P')$ implies $P = P'$ for all profiles $P, P' \in X^n$, thus $1 \leq f_3(P) \Leftrightarrow 1 \leq f_3(P')$, so **DN** is satisfied. To see that $f_3$ satisfies **TB** observe that $1 \not\leq f_3(P)$ implies $P = P_0$ or $P = P_1$, in either case if $P(ae)P'$ then $1 \leq f_3(P')$. The function $f_3$ does not satisfy **DM**, which can be seen by considering the profiles $P = (0, 1, \ldots, 1)$ and $P' = P_1$, thus $K_1(P) \subseteq K_1(P')$, however, $1 \leq f_3(P)$ but $1 \not\leq f_3(P')$.

Thus **DN**, **DM**, and **TB** are independent of one another. Since the lattices in the preceeding examples are also all finite median semilattices, these functions demonstrate the independence of **CDN**, **MON**, and **TB** when $X$ is any finite median semilattice. Combined with the example at the end of the previous section, we can see that all the axioms in Theorem 4.3 are independent of one another.

## 4.5 Unanimity

To get the unique characterization that we want we will need to further restrict what functions we are describing. Following is the definition of another common axiom from voting theory, unanimity.

**Axiom 4.7.** *A consensus function $f : X^n \to X$ satisfies **Unanimity (U)** if, for any $P \in X^n$ and $x \in X$ such that $P = (x, \ldots, x)$, then $f(P) = x$.*

Thus, if a profile indicates that all the individuals favor the same alternative, a unanimous function will select that alternative as the output.

**Lemma 4.6.** *The simple majority rule function $f_s$ satisfies Unanimity.*

*Proof.* Let $X$ be a finite median semilattice. Let $P \in X^n$ be a profile such that $\{P\} = \{x\}$ for some $x \in X$. Recall that $f_s$ is defined as:

$$f_s(P) = \bigvee \{j \in J : |K_j(P)| > |\overline{K}_j(P)|\}.$$

Since $K_j(P) = \{i \in N : j \leq P(i)\}$ and $\overline{K}_j(P) = \{i \in N : j \vee P(i) \text{ D.N.E.}\}$, we know that if $j \in J$ and $j \leq x$ then $K_j(P) = N$ and $\overline{K}_j(P) = \varnothing$, thus $|K_j(P)| > |\overline{K}_j(P)|$ and we have $j \in \{j \in J : |K_j(P)| > |\overline{K}_j(P)|\}$. If $j \in J$ and $j \not\leq x$ then $K_j(P) = \varnothing$, thus $|K_j(P)| = 0$ and we have $j \notin \{j \in J : |K_j(P)| > |\overline{K}_j(P)|\}$. Hence $\{j \in J : |K_j(P)| > |\overline{K}_j(P)|\} = \{j \in J : j \leq x\}$. It follows from this that

$$\bigvee \{j \in J : |K_j(P)| > |\overline{K}_j(P)|\} = \bigvee \{j \in J : j \leq x\}$$

but we know $X$ is finite, so $\bigvee \{j \in J : j \leq x\} = x$; thus $f_s(P) = x$. □

It is worth remarking at this point that our other important example of a consensus function, the top function $f_T$, does not satisfy unanimity. To see this, consider $f_T(P_0)$. The profile $P_0 = (0, \ldots, 0)$, thus $\{P_0\} = \{0\}$ but $f_T(P_0) = 1$.

Just as it turned out that restricting $X$ to being a lattice caused Anonymity to be implied by the other axioms, adding Unanimity makes Decisive Neutrality redundant when $X$ is a lattice.

**Theorem 4.5.** *If $X$ is a distributive lattice and $f : X^n \to X$ satisfies **U**, **DM**, and **TB** then $f$ satisfies **DN**.*

*Proof.* Consider $P, P' \in X^n$ such that $K_j(P) = K_{j'}(P')$ for some $j, j' \in J$ where $j \leq f(P)$. We want to show $j' \leq f(P')$. Observe that **U** implies $f(P_0) = 0$, satisfying the conditions of Lemmas 4.3 and 4.4; hence, by Lemma 4.3, $K_j(P) \neq \varnothing$. Since $K_j(P) = K_{j'}(P')$ there is some $i \in N$ such that $i \in K_{j'}(P')$. Consider $P'' \in X^n$ such that $P''(i) = j'$ and $P''(k) = 0$ for $k \neq i$. It follows from Lemma 4.4 that $j' \leq f(P'')$. Since $K_{j'}(P'') \subseteq K_{j'}(P')$, **DM** implies $j' \leq f(P')$ as desired. $\square$

**Theorem 4.6.** *Let $X$ be a finite median semilattice and let $f : X^n \to X$.*

*(a) If $X$ is not a lattice, then $f$ satisfies **A**, **CDN**, **MON**, and **TB*** iff *$f = f_s$.*

*(b) If $X$ is a lattice, then $f$ satisfies **U**, **DM**, and **TB*** iff *$f = f_s$.*

*Proof.* If $X$ is a finite median semilattice that is not a lattice and $f : X^n \to X$ is a consensus function satisfying **A**, **CDN**, **MON**, and **TB**, then Theorem 4.3 implies $f = f_s$ or $f = f_T$. Since $X$ is not a lattice, then it cannot be the case that $f = f_T$ since this function is only defined on a lattice, thus $f = f_s$. If $f = f_s$ then it follows from Theorem 4.1 that $f$ satisfies **A**, **CDN**, **MON**, and **TB**.

If $X$ is a finite median semilattice that is a lattice and $f : X^n \to X$ is a consensus function satisfying **U**, **DM**, and **TB**, then Theorem 4.5 implies $f$ satisfies **DN** as well. Thus Corollary 1 implies $f = f_s$ or $f = f_T$, but since $f$ also satisfies **U** then it cannot be the case that $f = f_T$ since we saw above that $f_T$ does not satisfy Unanimity, thus $f = f_s$. If $f = f_s$ then it follows from Theorem 4.1 and Lemma 4.6 that $f$ satisfies **U**, **DN**, **DM**, and **TB**. $\square$

To summarize, if $X$ is a finite median semilattice and we know whether or not it is a lattice, then we can uniquely characterize the simple majority rule function with four axioms or less. Monotonicity, and Tie-Breaking are required for either case; Unanimity if $X$ is a lattice, Anonymity and Competitive Decisive Neutrality if $X$ is not a lattice. If $X$ is a general finite median semilattice then all five are required for a unique characterization of $f_s$.

# CHAPTER 5
# MAY'S THEOREM ON HIERARCHIES

The true power of Theorem 4.3 can be seen by looking at a particular example of a median semilattice. In this chapter the median semilattice of hierarchies is considered. We will see that the characterization of simple majority rule given in Chapter 4 can be applied to the median semilattice of hierarchies to obtain a version of May's Theorem on hierarchies.

**Definition 5.1.** *A **hierarchy** $H$, on a finite set $S$ of size $n \geq 3$, is a collection of subsets of $S$ such that*

- $\varnothing \notin H$ *and* $S \in H$*;*
- $\{x\} \in H$ *for all* $x \in S$*;*
- $A \cap B \in \{A, B, \varnothing\}$ *for all* $A, B \in H$*.*

The application of the hierarchy concept in the field of clustering is illustrated in [13]. An element of a hierarchy is called a **cluster**; in particular, a cluster $A \in H$ is a **nontrivial cluster** if $1 < |A| < n$, whereas $A$ is a **trivial cluster** if it is a singleton subset of $S$ or the entire set $S$. Let

$$H_{\varnothing} = \{\{x\}, S : x \in S\} \quad \text{and} \quad H_A = H_{\varnothing} \cup \{A\}.$$

In Figure 5.1, the hierarchy $H_{\{a,b,c\}}$ with $S = \{a, b, c, d\}$ is shown. Notice that $H_{\varnothing}$ is a hierarchy on $S$ containing only the trivial clusters and $H_A$ is a hierarchy on $S$ containing a single nontrivial cluster, $A$. For any two hierarchies $H$ and $J$ on $S$, we

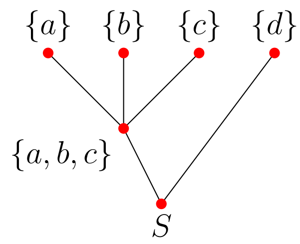



Figure 5.1: The hierarchy $H_{\{a,b,c\}}$

say $H \leq J$ if $H \subseteq J$. It is easy to check that $\leq$ is a partial order on the set of all hierarchies. The first two items in the definition of a hierarchy $H$ are equivalent to $H_\varnothing \leq H$.

We denote the set of all hierarchies on a set $S$ by $\mathcal{H}(S)$, or just $\mathcal{H}$ when the set $S$ is understood from context.

**Proposition 5.1.** *The pair $(\mathcal{H}(S), \leq)$ is a median semilattice.*

*Proof.* By the definition of hierarchy,

$$H_\varnothing \leq H \quad \forall H \in \mathcal{H}(S)$$

Next, for any $H, H' \in \mathcal{H}(S)$, we have $H_\varnothing \leq H \cap H'$. For any nontrivial clusters $A, B \in H \cap H'$ we know $A, B \in H$, since $H \cap H'$ is a subset of $H$. Since $H$ is a hierarchy, we know $A \cap B \in \{A, B, \varnothing\}$, hence,

$$H \cap H' \in \mathcal{H}(S).$$

So $(\mathcal{H}(S), \leq)$ is a meet semilattice with set intersection as the meet operation.

Let $H \in \mathcal{H}(S)$ and consider the set

$$\{J \in \mathcal{H}(S) : J \leq H\}.$$

Because we are considering only hierarchies that are subsets of a fixed hierarchy, this set is a lattice with intersection as meet and union as join. Since intersection distributes over union and vice versa, it follows that this lattice is distributive.

Let $H, H', H'' \in \mathcal{H}(S)$ such that

$$H \cup H', H \cup H'', H' \cup H'' \in \mathcal{H}(S).$$

Now consider $H \cup H' \cup H''$. As before, $H_\varnothing \leq H \cup H' \cup H''$. For any nontrivial clusters $A, B \in H \cup H' \cup H''$ it follows that $A$ is in one of $H$, $H'$, or $H''$; likewise with $B$. Since each of $H$, $H'$, and $H''$ is a hierarchy, as well as each of their pairwise unions, we have $A \cap B \in \{A, B, \varnothing\}$, so $H \cup H' \cup H'' \in \mathcal{H}(S)$ and $\mathcal{H}(S)$ satisfies the join-Helly property. Thus, $\mathcal{H}(S)$ is a median semilattice. □

As one might expect, the set of all hierarches on a gives set grows large very quickly as the size of the base set increases. For a diagram of the median semilattice derived from the set of all hierarchies on a set of size 4, see [5]. It will also be useful to be able to talk about the join irreducible elements of the median semilattice $\mathcal{H}(S)$, this prompts the following proposition:

**Proposition 5.2.** *For any median semilattice $\mathcal{H}(S)$,*

*(a) If $A$ is a nontrivial subset of $S$, then $H_A$ is a join irreducible element of $\mathcal{H}(S)$.*

*(b) If $J$ is a join irreducible element of $\mathcal{H}(S)$, then $J = H_A$ for some nontrivial subset $A$.*

*Proof.* $(a)$ Suppose $H_A = J \cup J'$, where $J, J' \in \mathcal{H}(S)$. Without loss of generality we can assume $A \in J$. We also know $H_\varnothing \in J$ thus $H_A \subseteq J$. it is clearly the case that $J \subseteq J \cup J' = H_A$ hence $J = H_A$.

$(b)$ Suppose $J$ is a join irreducible element of $\mathcal{H}(S)$. This implies $J \neq H_\varnothing$, thus there exists some non trivial subset $A$ such that $A \in J$. Now suppose there exists some non trivial subset $B \in J$ such that $A \neq B$, then we can say $J = H_A \cup (J \setminus A)$, contradicting the assumption that $J$ is a join irreducible. Thus $J = H_A$. □

If we let $K = \{1, \ldots, k\}$ be a set of individuals, then a function of the form

$$f : \mathcal{H}^k \to \mathcal{H}$$

is called a **consensus function**, and a $k$-tuple $P = (H_1, \ldots, H_k) \in \mathcal{H}^k$ in the domain of $f$ is called a **profile**. Here $k \geq 2$ is the length of a profile. For any $P = (H_1, \ldots, H_k) \in \mathcal{H}^k$, let $\{P\} = \{H \in \mathcal{H} : H = H_i \ for\ some\ i \in K\}$. For any profile $P$ and any cluster $A$ let

$$K_A(P) = \{i \in K : A \in H_i\} \text{ and } \overline{K}_A(P) = \{i \in K : H_i \cup \{A\} \notin \mathcal{H}\}.$$

The **majority rule** consensus function $Maj : \mathcal{H}^k \to \mathcal{H}$ is defined by

$$Maj(P) = \left\{A : |K_A(P)| > \frac{k}{2}\right\}.$$

A simple pigeonhole argument shows that $Maj$ is well defined [15]. The **simple majority rule** consensus function $Maj^+ : \mathcal{H}^k \to \mathcal{H}$ is defined by

$$Maj^+(P) = \{A : |K_A(P)| > |\overline{K}_A(P)|\}.$$

This function is called Majority Rule + in [6] where the domain is $\bigcup_{k \geq 1} \mathcal{H}^k$, this is why we use the notation $Maj^+$.

Presented below are the axioms used in Theorem 4.3 but explicitly expressed for hierarchies.

**Axiom 5.1.** *A function $f : \mathcal{H}^k \to \mathcal{H}$ satisfies **anonymity (A)** if, for any permutation $\sigma$ of $K = \{1, \ldots, k\}$, we have $f(P) = f(\sigma P)$, where $\sigma P = (H_{\sigma(1)}, \ldots, H_{\sigma(n)})$.*

**Axiom 5.2.** *A function $f : \mathcal{H}^k \to \mathcal{H}$ satisfies **competitive decisive neutrality (CDN)** if, for any $P, P' \in \mathcal{H}^k$ and any nontrivial clusters $A$ and $B$;*

$$K_A(P) = K_B(P') \text{ and } \overline{K}_A(P) = \overline{K}_B(P') \Rightarrow [A \in f(P) \Leftrightarrow B \in f(P')].$$

**Axiom 5.3.** *A function $f : \mathcal{H}^k \to \mathcal{H}$ satisfies **monotonicity (MON)** if, for any $P, P' \in \mathcal{H}^k$ and any nontrivial cluster $A$; if $K_A(P) \subseteq K_A(P')$ and $\overline{K}_A(P') \subseteq \overline{K}_A(P)$ then $A \in f(P) \Rightarrow A \in f(P')$.*

The two functions mentioned above, $Maj$ and $Maj^+$, satisfy all of these axioms. They differ in that $Maj^+$ satisfies the following axiom of tie breaking, whereas $Maj$ does not.

**Definition 5.2.** *Two profiles $P = (H_1, \ldots, H_k)$ and $P' = (H'_1, \ldots, H'_k)$ are **almost equal**, denoted $P(ae)P'$, if there exists $i_0 \in K$ such that $H_i = H'_i$ for all $i \neq i_0$ and $H_{i_0} \neq H'_{i_0}$.*

**Axiom 5.4.** *A function $f : \mathcal{H}^k \to \mathcal{H}$ satisfies **tie breaking (TB)** if, for every $P, P' \in \mathcal{H}^k$ and nontrivial clusters $A$ and $B$ such that*

$$\{P\} \cup \{P'\} \subseteq \{H_A, H_B, H_\varnothing\}$$
$$\text{and } P(ae)P'$$

*we have*

$$A \notin f(P) \text{ and } B \notin f(P) \Rightarrow A \in f(P') \text{ or } B \in f(P').$$

**Theorem 5.1.** *A consensus function $F : \mathcal{H}^k \to \mathcal{H}$ satisfies Anonymity, Competitive Decisive Neutrality, Monotonicity, and Tie Breaking if and only if $F = Maj^+$.*

*Proof.* We saw in Proposition 5.1 that $(\mathcal{H}(S), \leq)$ is a median semilattice, moreover, because $|S| \geq 3$ it is easy to see that $(\mathcal{H}(S), \leq)$ is not a lattice. Since the axioms above are equivalent to those in Theorem 4.3, we get that Theorem 5.1 follows from Theorem 4.3, Proposition 5.1, and the observation that $\mathcal{H}$ is not a lattice. □

The $Maj^+$ function has been characterized before [7], but this characterization highlights the parallels between $Maj^+$ and $f_s$ by using extensions of the axioms originally used by May to characterize simple majority rule. We can see that all the axioms are independent of one another by considering the following examples of consensus functions that satisfy all but one of the axioms. The first example uses a unique notation defined as follows:

$$|K_A(P)|^1 = \begin{cases} |K_A(P)| + 1 & \text{if } A \in H_1; \\ |K_A(P)| & \text{if } A \notin H_1; \end{cases}$$

$$|\overline{K}_A(P)|^1 = \begin{cases} |\overline{K}_A(P)| + 1 & \text{if } \{A\} \cup H_1 \notin \mathcal{H}; \\ |\overline{K}_A(P)| & \text{if } \{A\} \cup H_1 \in \mathcal{H}. \end{cases}$$

**Example 5.1.** *The function $f_1 : \mathcal{H}^k \to \mathcal{H}$ defined by*

$$f_1(P) = \{A : |K_A(P)|^1 > |\overline{K}_A(P)|^1\}$$

*satisfies* **CDN**, **MON**, **TB**, *but not* **A**. *Since we can think of $f_1$ as $Maj^+$ on profiles of length $k+1$, $f_1$ is well-defined.*

**Example 5.2.** *The function $L : \mathcal{H}^k \to \mathcal{H}$ defined by*

$$L(P) = Maj(P) \cup \{X : |K_X(P)| > 0 \text{ and } |\overline{K}_X(P) = 0\}$$

*satisfies* **A**, **CDN**, **MON**, *but not* **TB**. *$L(P)$ is the "loose consensus function" [18]. Recall that $Maj$ also satisfies* **A**, **CDN**, **MON**, *but not* **TB**.

**Example 5.3.** *The function $NU : \mathcal{H}^k \to \mathcal{H}$, with $k \geq 3$, defined by*

$$NU(P) = H_\varnothing \cup \{X : k > |K_X(P)| > |\overline{K}_X(P)|\}$$

*satisfies* **A**, **CDN**, **TB**, *but not* **MON**.

**Example 5.4.** *The function $Maj^{+A} : \mathcal{H}^k \to \mathcal{H}$, with a fixed nontrivial cluster $A$, defined by*

$$Maj^{+A}(P) = \begin{cases} Maj^+(P) \cup H_A & \text{if } \overline{K}_A(P) = \varnothing \\ Maj^+(P) & \text{otherwise.} \end{cases}$$

*satisfies* **A**, **MON**, **TB**, *but not* **CDN**.

Using the fact that the collection of all hierarchies on a given set forms a median semilattice, we were able to offer a new characterization of $Maj^+$ that is essentially a corollary of Theorem 4.3. In this way, we can see that $Maj^+$ truly is May's simple majority rule on hierarchies.

# CHAPTER 6
# APPLICATIONS AND FUTURE WORK

## 6.1 Applications

A direct application of the main result to the field of hierarchical clustering was presented in Chapter 5. Now we will discuss an idea on how to apply the main result to the area of social science. Most governments have some notion of what is called a "bill." This bill is what the representatives in the legislature create, discuss, debate, revise, harangue, and ultimately vote upon; the outcome of said vote determining whether or not the bill becomes a law.

The process of creating a law is lengthy, partly because each bill is a lengthy document composed of many component parts. No one single law is getting voted on in a typical bill, instead each bill addresses numerous issues and is full of numerous provisions, provisos, addendums, and amendments that are added, removed, and modified during the revision process in the attempt to create a law that will get a majority approval.

If we consider each of the component parts that any particular law maker might like to see in a bill as separate entities, then we can look at every possible combination of those parts as a distinct variation of the bill. Since all we are really considering here is some subset of the set of all possible combinations of these components, we can think of each possible bill as an element in a partially ordered set with union and intersection as the join and meet operations, respectively. It would inherit it's distibutive property from the power set lattice on which it is based. It is possible that this structure is a distributive lattice or, more generally, a median semilattice. So we could apply our main result to this situation as a way of designing a bill; each representative would vote for the version of the bill they found most agreeable and the output would be a new version of the bill that has the most overall approval. This relatively simplistic idea can be readily applied to almost any situation where people are trying to chose a subset from among a larger set of alternatives.

## 6.2 Future Work

In the case with two alternatives, the simple majority rule function is uniquely determined by anonymity, neutrality and positive responsiveness. It has been demonstrated that anonymity and positive responsiveness alone characterize a larger class of functions based on the idea of strong quota pair systems. A more ambitious yet worthwhile endeavor would be to fully characterize the class of functions satisfying any combination of these axioms, comparing these results would undoubtedly shed insight into the nature and restrictiveness of each condition.

It has also been demonstrated that the simple majority rule function can be characterized on finite median semilattices, yielding a true generalization of May's Theorem to an order theoretic domain. What is not yet known is how the notion of strong quota pair systems could be generalized to the domain of finite median semilattices or what this class of functions would look like.

Finally, the generalization of May's theorem on median semilattices could be pushed further. The concept of medain semilattices is very closely related to that of median graphs, an undirected graph where each set of three vertices has a unique vertex belonging to the shortest paths between each pair of the three vertices. Translating the axioms and theorems presented in this dissertation to the realm of graph theory is a more than feasible task and the result would be a further generalization of May's theorem to the realm of graph theory.

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CURRICULUM VITAE

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## EDUCATION

*Doctor of Philosophy*

Mathematics, University of Louisville (Expected August 2015)

Dissertation: Strong Quota Pair Systems and May's Theorem on Median Semilattices

Advisor: Dr. Robert C. Powers

*Master of Arts*

Mathematics, University of Louisville (2011)

*Bachelor of Science*

Mathematics and Philosophy, Centre College (2009)

## TEACHING EXPERIENCE

### Instructor

| | |
|---|---|
| Calculus I | Summer '15, Fall '14 |
| Mathematics for Elementary Education II | Summer '14 |
| Mathematics for Elementary Education I | Fall '13, Summer '13 |
| Contemporary Mathematics | Summer '12 |
| Finite Mathematics | Summer '11 |

**Teaching Assistant**

| | |
|---|---|
| College Algebra | Spring ‘15, Spring ‘14, Fall ‘13, Spring ‘11, Spring ‘10 |
| Finite Mathematics | Spring ‘13, Fall ‘09 |
| Contemporary Mathematics | Spring ‘12, Fall ‘11 |
| Elements of Calculus | Fall ‘10 |

## PRESENTATIONS

*Anonymous and Positively Responsive Aggregation Rules* March 2015

KYMAA Section Meeting, Morehead State University

*May's Theorem on Hierarchis* March 2014

KYMAA Section Meeting, Murray State University

*May's Theorem on Median Semilattices* April 2013

KYMAA Section Meeting, Transylvania University

*May's Theorem on Median Semilattices* April 2013

Algebra and Combinatorics Seminar, University of Louisville

*A Simple Extension of May's Theorem* October 2012

WKY Mathematics Symposium, Western Kentucky University

## PUBLICATIONS

Hoots, L., Powers, R.C. *Anonymous and positively responsive aggregation rules*

Submitted to Mathematical Social Sciences

## RESEARCH INTERESTS

Applied discrete mathematics, combinatorics, and voting theory; in particular, the axiomatic characterization of consensus functions on various algebraic structures.