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Functional equations with involution related to sine and cosine functions.

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FUNCTIONAL EQUATIONS WITH INVOLUTION RELATED TO SINE AND
COSINE FUNCTIONS

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DEDICATION

For my parents, Doug and Susan, and George for all their love and support.

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ABSTRACT

FUNCTIONAL EQUATIONS WITH INVOLUTION RELATED TO SINE AND COSINE FUNCTIONS

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Let G be an abelian group, $\mathbb{C}$ be the field of complex numbers, $\alpha \in G$ be any fixed, nonzero element and $\sigma : G \to G$ be an involution. In Chapter 2, we determine the general solution $f, g : G \to \mathbb{C}$ of the functional equation $f(x + \sigma y + \alpha) + g(x + y + \alpha) = 2f(x)f(y)$ for all $x, y \in G$.

Let G be an arbitrary group, z_0 be any fixed, nonzero element in the center $Z(G)$ of the group G, and $\sigma : G \to G$ be an involution. The main goals of Chapter 3 are to study the functional equations $f(x\sigma y z_0) - f(xy z_0) = 2f(x)f(y)$ and $f(x\sigma y z_0) + f(xy z_0) = 2f(x)f(y)$ for all $x, y \in G$ and some fixed element z_0 in the center $Z(G)$ of the group G.

In Chapter 4, we consider some properties of the general solution to $f(xy)f(x\sigma y) = f(x)^2 - f(y)^2$. We also find the solution to this equation when G is a 2-divisible, perfect group. We end the chapter by discussing the periodicity of the solutions to both the sine functional equation and the sine inequality.

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CHAPTER 1
PRELIMINARIES AND DEFINITIONS

1.1 Introduction

The cosine function satisfies the following well-known identity

$$\cos(x - y) + \cos(x + y) = 2\cos(x)\cos(y)$$

for all $x, y \in \mathbb{R}$. If we define $f(x) = \cos(x)$, then we have the functional equation

$$f(x - y) + f(x + y) = 2f(x)f(y) \tag{1.1}$$

for all $x, y \in \mathbb{R}$ (the set of real numbers). The functional equation (1.1) is known as the d'Alembert functional equation as d'Alembert himself studied it in 1769 (see [7]). The research continued into the next two centuries as Poisson investigated the equation in 1804 (see [19]) and Picard worked with it in the 1920s (see [17] and [18]). The importance of (1.1) is seen in determining the sum of two vectors in Euclidean and non-Euclidean geometries. The continuous solutions of the d'Alembert equation were determined by Cauchy in 1821 (see [2]). Furthermore, in [10] Kannappan gave the general solutions of (1.1) on arbitrary groups assuming that the unknown function f is an abelian function.

In 1910, Van Vleck [25] (see also [26] and [21]) studied the following equation with restricted argument α and proved the following result: The continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$f(x - y + \alpha) - f(x + y + \alpha) = 2f(x)f(y) \tag{1.2}$$

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for all $x, y \in \mathbb{R}$ and fixed nonzero $\alpha \in \mathbb{R}$ if and only if f is given by either $f \equiv 0$ or

$$f(x) = \cos\left(\frac{\pi}{2\alpha}(x - \alpha)\right), \quad \text{for all } x \in \mathbb{R}. \tag{1.3}$$

Notice that a nonzero continuous function satisfying (1.2) also satisfies $f(2\alpha) = 0$ and is periodic with period 4α. Using these properties, (1.3) can be written as

$$f(x) = (-1)^n \sin\left(\frac{(2n + 1)\pi x}{2\alpha}\right)$$

for $x \in \mathbb{R}$ and $n \in \mathbb{Z}$ (the set of integers). Therefore, functional equation (1.2) characterizes the sine function.

In [9], Kannappan considered the functional equation

$$f(x - y + \alpha) + f(x + y + \alpha) = 2f(x)f(y) \tag{1.4}$$

which is similar to (1.2) and proved the following result: The general solution $f : \mathbb{R} \to \mathbb{C}$ (the set of complex numbers) of functional equation (1.4) is either $f \equiv 0$ or $f(x) = g(x - \alpha)$, where g is an arbitrary solution of the cosine functional equation $g(x + y) + g(x - y) = 2g(x)g(y)$ for all $x, y \in \mathbb{R}$ with period 2α. The only continuous real-valued solutions of (1.4) (see [12], Corollary 3.14a, p. 118) are $f \equiv 0$, $f \equiv 1$, $f(x) = \cos\left(\frac{4n\pi x}{\alpha}\right)$, $f(x) = \cos\left(\frac{2(2n+1)\pi x}{\alpha}\right)$ and $f(x) = -\cos\left(\frac{(2n+1)\pi x}{\alpha}\right)$.

In [20], Sahoo studied the following generalization

$$f(x - y + \alpha) + g(x + y + \alpha) = 2f(x)f(y) \text{ for all } x, y \in G \tag{1.5}$$

of the functional equations (1.2) and (1.4). He determined the general solutions of this equation on an abelian group G and raised the following problem: Given an involution $\sigma : G \to G$, find all functions $f, g : G \to \mathbb{C}$ satisfying

$$f(x + \sigma y + \alpha) + g(x + y + \alpha) = 2f(x)f(y) \tag{1.6}$$

for all $x, y \in G$. The goal of Chapter 2 is to provide an answer to the above question posed in [20] by determining the general solutions $f, g : G \to \mathbb{C}$ of functional

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equation (1.6) for all $x, y \in G$. The functional equation (1.2) is a special case of the functional equation (1.6) where $g = -f$ and $G = \mathbb{R}$ with $\sigma(x) = -x$. If $G = \mathbb{R}$ and $g = f$ with $\sigma(x) = -x$, then the functional equation (1.6) reduces to the functional equation (1.4) studied by Kannappan in [9]. Hence the solution of (1.2) and (1.4) can be obtained from our results in Chapter 2.

Other similar functional equations solved in literature are

$$f(x + y + \alpha) f(x - y + \alpha) = f(x)^2 - f(y)^2 \tag{1.7}$$

and

$$f(x + y + \alpha) f(x - y + \alpha) = f(x)^2 + f(y)^2 - 1, \tag{1.8}$$

for all $x, y \in \mathbb{R}$. The functional equation (1.7) was considered by Kannappan in [11] (see also [12]) while (1.8) was considered by Etigson in [8]. The functional equations (1.2), (1.4), (1.5), (1.6), (1.7) and (1.8) are examples of functional equations with restricted arguments where at least one of the variables is restricted to a certain discrete subset of the domain of the other variable(s). In particular, the subset may consist of a single element.

In Chapter 3, we determine the general solutions of Van Vleck's equation with involution $f(x\sigma y z_0) - f(xyz_0) = 2f(x)f(y)$ and Kannappan's equation with involution $f(x\sigma y z_0) + f(xyz_0) = 2f(x)f(y)$ for all x and y in some arbitrary group G and some fixed element z_0 in the center $Z(G)$ of G. All results obtained in Chapter 3 have been accepted as a paper [16] for publication in the journal *Aequationes Mathematicae*.

The sine functional equation is motivated by the following well-known trigonometric identity

$$\sin(x + y) \sin(x - y) = \sin^2(x) - \sin^2(y)$$

for all $x, y \in \mathbb{R}$. By defining $f(x) = \sin(x)$, one obtains the functional equation

$$f(x + y) f(x - y) = f(x)^2 - f(y)^2 \tag{1.9}$$

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for all $x, y \in \mathbb{R}$. The functional equation (1.9) is known as the sine functional equation and on arbitrary groups takes the form

$$f(xy) \, f(xy^{-1}) = f(x)^2 - f(y)^2. \tag{1.10}$$

In 1920, Wilson [27] (see also [1]) found the following result: If $f : \mathbb{C} \to G$ satisfies (1.10) where G is a 2-divisible abelian group and $\mathbb{C}$ is the field of complex numbers, then f is given by

$$f(x) = \frac{\psi(x) - \psi(x^{-1})}{2\alpha} \quad \text{or} \quad f(x) = \phi(x),$$

where $\psi \colon G \to \mathbb{C}^*$ is a multiplicative homomorphism, $\phi \colon G \to \mathbb{C}$ is an additive homomorphism and α is an arbitrary nonzero element of $\mathbb{C}$.

Wilson's result holds true if $\mathbb{C}$ is replaced by a quadratically closed field $\mathbb{K}$ of characteristic different from two (see [1]). Kannappan [9] found the general solution $f : G \to \mathbb{C}$ of (1.10) when G is a cyclic group. Only a few results are known for the sine functional equation on arbitrary groups. Corovei [5] proved the following: Let G be a group whose elements are of odd order, $\mathbb{K}$ be a field of characteristic different from 2 and $f : G \to \mathbb{K}$ be a nonzero solution of the sine functional equation (1.10). Then f has the form

$$f(x) = \frac{\psi(x) - \psi(x^{-1})}{2\alpha} \quad \text{or} \quad f(x) = \phi(x),$$

where $\psi \colon G \to \mathbb{K}^*$ is a multiplicative homomorphism, $\phi \colon G \to \mathbb{K}$ is an additive homomorphism and α is an arbitrary nonzero element of $\mathbb{K}$.

The following result of Corovei [6] generalized the theorem of Aczél and Dhombres [1]. Let $\mathbb{K}$ be a quadratically closed field with char $\mathbb{K} \neq 2$ and G be a 2-divisible group. The function $f \colon G \to \mathbb{K}$ satisfies the sine functional equation (1.10) if and only if

$$f(x) = \frac{\psi(x) - \psi(x^{-1})}{2\alpha} \quad \text{or} \quad f(x) = \phi(x),$$

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where $\psi \colon G \to \mathbb{K}^*$ is a multiplicative homomorphism, $\phi \colon G \to \mathbb{K}$ is an additive homomorphism and α is an arbitrary nonzero element of $\mathbb{K}$.

Stetkaer (see [24]) proved that if G is a group such that G and its commutator subgroup $[G, G]$ are generated by squares, then the function $f \colon G \to \mathbb{C}$ satisfying (1.10) takes one of the following forms:

$$f(x) = c(\chi(x) - \check{\chi}(x))$$

where χ is a character on G and c is a nonzero complex constant, or $f \colon G \to (\mathbb{C}, +)$ is an additive function.

In Chapter 4, we examine the sine functional equation with involution

$$f(xy)f(x\sigma y) = f(x)^2 - f(y)^2. \tag{1.11}$$

We present some properties for any function f satisfying (1.11) and show that the only solution of (1.11) on perfect groups is trivial. We also discuss the periodicity of the solution of the sine functional equation.

1.2   Notation and Terminology

If G is an arbitrary group, the group operation will be denoted by $\cdot$ and we write $x \cdot y$ simply as xy. For an arbitrary group G, e will denote neutral (or identity) element of G. The *center* of a group G is the set of elements $c \in G$ that commute with every other element in G; i.e. c is in the center of G if and only if $xc = cx$ for all $x \in G$. This set will be denoted by $Z(G)$. A group G is said to be *2-divisible* if for every $g \in G$ there exists an $h \in G$ such that $h^2 = g$. Let $Z_0(G) = \{e\}$, $Z_1(G) = Z(G)$ and Z_{i+1} be a subgroup of G containing $Z_i(G)$ such that $Z_{i+1}(G)/Z_i(G) = Z\big(G/Z_i(G)\big)$. A group G is called *nilpotent* if $Z_c(G) = G$ for some $c \in \mathbb{Z}$. The nilpotent groups are groups that are almost abelian. For a group G, the *commutator subgroup*, denoted $[G, G]$, is the subgroup generated by

$ghg^{-1}h^{-1}$ for all $g, h \in G$. It is well known that $[G, G]$ is a normal subgroup of G. Futhermore, a group is said to be *perfect* if $G = [G, G]$. An example of a perfect group is the alternating group, A_n, for $n \geq 5$.

Let $(G, \cdot)$ and $(H, \star)$ be arbitrary groups throughout, unless otherwise stated. A function $f : G \to H$ is said to be a *homomorphism* if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in G$. The set of all homomorphisms from G into H will be denoted by $Hom(G, H)$. In the case where H is an abelian group, the group operation will be denoted by addition. When the group H is abelian, the homomorphism f will be called an additive homomorphism. If H is an arbitrary group with multiplication as the binary operation, then the homomorphism f will be called a multiplicative homomorphism. A function $f : G \to H$ is said to be an *anti-homomorphism* if $f(x \cdot y) = f(y) \star f(x)$ for all $x, y \in G$. A mapping $\sigma : G \to G$ is said to be an *involution* if it is an anti-homomorphism and satisfies $\sigma(\sigma(x)) = x$ for all $x \in G$. For convenience we denote $\sigma(x)$ as simply σx. Furthermore, a subgroup H of a group G is called σ-*involutive* if $\sigma(H) \subset H$.

A function $f : G \to H$ is said to be an *abelian function* if and only if $f(xyz) = f(xzy)$ for all $x, y, z \in G$ [24]. Let $GL(n, \mathbb{R})$ be the set of all $n \times n$ invertible matrices with real entries. Let $f : GL(n, \mathbb{R}) \to \mathbb{R}$ be defined by $f(A) = det(A)$. Then f is an abelian function. However, $f(A) = trace(A)$ is not an abelian function. A function $f : G \to H$ is said to be *central* if and only if $f(xy) = f(yx)$ for all $x, y \in G$. The function $f : GL(n, \mathbb{R}) \to \mathbb{R}$ defined by $f(A) = trace(A)$ is an example of a central function. Note that every abelian function is central but the converse is not true. A function $f : G \to \mathbb{C}$ is said to be σ-*odd* with respect to an involution $\sigma : G \to G$ if and only if $f(\sigma x) = -f(x)$ for all $x \in G$. If f is a σ-odd function, then $f(e) = 0$. Similarly, $f : G \to \mathbb{C}$ is said to be σ-*even* with respect to an involution $\sigma : G \to G$ if and only if $f(\sigma x) = f(x)$ for all $x \in G$. A function f on an arbitrary group G is *periodic* with period $\alpha \neq e$ if $f(x\alpha) = f(x)$ for all $x \in G$.

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Let $\mathbb{K}$ be a field. A field $\mathbb{K}$ is said to be *quadratically closed* if every element of the field has a square root in $\mathbb{K}$. The additive group of $\mathbb{K}$ will be denoted by $\mathbb{K}$, while the multiplicative group will be denoted by $\mathbb{K}^\star$. Hence if $\mathbb{K}$ is a field, $Hom(G, \mathbb{K})$ will denote the group of homomorphisms from the group G to the additive group of the field $\mathbb{K}$, while $Hom(G, \mathbb{K}^\star)$ will denote the group of homomorphisms from the group G to the multiplicative group of the field $\mathbb{K}$. An element $f \in Hom(G, \mathbb{C}^\star)$ is called a (group) *character*. Thus a character is a nonzero multiplicative homomorphism from group G into the multiplicative group of nonzero complex numbers. If χ is a group character of G, then by $\check{\chi}(x)$ we denote $\chi(x^{-1})$. It is easy to see that $\check{\chi}$ is also a character of G.

A Lie group is a set G with two structures: G is a group and G is a (smooth, real) manifold. These structures agree in the following sense: multiplication and inversion are smooth maps. The circles and spheres are examples of smooth manifolds. Lie groups were studied by the Norwegian mathematician Sophus Lie at the end of the 19$^{\text{th}}$ century. The orthogonal $n \times n$ matrices

$$O(n, \mathbb{R}) = \{\Phi \in \mathbb{R}^{n \times n} \mid \Phi^T \Phi = \mathbb{I}\}$$

form a Lie group. This group has two components distinguished by the determinant $\det \Phi = \pm 1$ and the component of the identity is denoted by

$$SO(n, \mathbb{R}) = \{\Phi \in O(n, \mathbb{R}) \mid \det \Phi = 1\}.$$

The group $SO(n, \mathbb{R})$, called the special orthogonal group, is compact and connected. $SL(n, \mathbb{C})$, $SL(n, \mathbb{R})$ and $SO(n, \mathbb{R})$ are other examples of perfect groups. It is well known that every compact connected Lie group is 2-divisible. Hence $SO(n, \mathbb{R})$ is an example of 2-divisible perfect group. However the group $SL(n, \mathbb{R})$ is not 2-divisible but it is generated by its squares.

CHAPTER 2

GENERALIZED VAN VLECK'S EQUATION ON ABELIAN GROUPS

2.1 Introduction

The main goal of this chapter is to find the general solutions $f, g : G \to \mathbb{C}$ of the generalized Van Vleck's functional equation with involution, namely

$$f(x + \sigma y + \alpha) + g(x + y + \alpha) = 2f(x)f(y) \tag{2.1}$$

for all $x, y \in G$, where G is an abelian group, α is a fixed nonzero element in G and $\sigma : G \to G$ is a homomorphism satisfying $\sigma(\sigma x) = x$ for all $x \in G$.

In order to find the solutions of the above functional equation we also study Van Vleck's equation with involution, that is

$$\ell(x + y + 2\alpha) + \ell(x + \sigma y + 2\alpha) = 2\ell(x)\ell(y)$$

for all $x, y \in G$.

Note that the generalized Van Vleck's functional equation (2.1) contains Van Vleck's functional equation (1.2) and Kannappan's functional equation (1.4) as special cases.

2.2 Some Preliminary Results

It is easy to see that if ϕ is a zero function and ψ is an arbitrary function,

then they are the solutions of the functional equation

$$\phi(x + y) + \phi(x + \sigma y) = 2\,\phi(x)\,\psi(y) \tag{2.2}$$

for all $x, y \in G$. The following result from [3] gives the solution of (2.2) when ϕ is not identically zero.

LEMMA 2.1. *Let $\phi, \psi : G \to \mathbb{C}$ satisfy the functional equation (2.2) for all $x, y \in G$ where $\sigma : G \to G$ is an involution. Then there exists a multiplicative homomorphism $h : G \to \mathbb{C}^\star$ such that*

$$\psi(x) = \frac{h(x) + h(\sigma x)}{2}$$

for all $x \in G$. If $h \neq h \circ \sigma$, then ϕ has the form

$$\phi(x) = a\,h(x) + b\,h(\sigma x)$$

for all $x \in G$ and for some $a, b \in \mathbb{C}$. If $h = h \circ \sigma$, then ϕ has the form

$$\phi(x) = h(x)\,[A(x - \sigma x) + \gamma],$$

where $A : G \to \mathbb{C}$ is an additive homomorphism and $\gamma \in \mathbb{C}$.

The following lemma generalizes the result of Kannappan in [9] and will be used to prove our main result.

LEMMA 2.2. *Let $\ell : G \to \mathbb{C}$ satisfy the functional equation*

$$\ell(x + y + 2\alpha) + \ell(x + \sigma y + 2\alpha) = 2\ell(x)\ell(y), \tag{2.3}$$

where $\alpha \in \mathbb{C}^\star$ is fixed and $\sigma : G \to G$ is an involution. Then ℓ is of the form

$$\begin{cases} \ell(x) = & 0 \\ \ell(x) = & \frac{1}{2}\,h(2\alpha)\,\big[h(x) + h(\sigma x)\big], \end{cases} \tag{2.4}$$

where $h : G \to \mathbb{C}^\star$ is a multiplicative homomorphism satisfying $h(2\alpha) = h(2\sigma\alpha)$.

Proof. Clearly, $\ell(x) \equiv 0$ is a solution. We assume from here that ℓ is not trivial. By interchanging x with y in (2.3) we get

$$\ell(x + y + 2\alpha) + \ell(y + \sigma x + 2\alpha) = 2\ell(x)\ell(y). \tag{2.5}$$

Comparing the last equation (2.5) to our original equation (2.3) we see that

$$\ell(x + \sigma y + 2\alpha) = \ell(y + \sigma x + 2\alpha). \tag{2.6}$$

Substituting $y = 0$ in (2.6) gives us

$$\ell(x + 2\alpha) = \ell(\sigma x + 2\alpha) \tag{2.7}$$

for all $x \in G$. If we let $x = 0$ in (2.3) we have

$$\ell(y + 2\alpha) + \ell(\sigma y + 2\alpha) = 2\ell(0)\ell(y). \tag{2.8}$$

Further, if we use (2.7) in (2.8), we have

$$\ell(y + 2\alpha) = \ell(0)\ell(y) \tag{2.9}$$

for all $y \in G$. Now, let $y = -2\alpha$ in (2.9) to yield

$$\ell(0)\left[\ell(-2\alpha) - 1\right] = 0.$$

Hence, we have two cases, either $\ell(0) = 0$ or $\ell(-2\alpha) = 1$.

CASE 1: Suppose that $\ell(0) = 0$. By substituting $y = 0$ in (2.3) we see that

$$\ell(x + 2\alpha) = 0$$

for all $x \in G$. So, by replacing x by $x - 2\alpha$ in the previous equation, we conclude that $\ell(x) = 0$ for all $x \in G$. Thus, this case leads us to the trivial solution.

CASE 2: Let $\ell(-2\alpha) = 1$. Substitution of $y = -2\alpha$ in (2.3) yields

$$\ell(x + 2\alpha - 2\sigma\alpha) = \ell(x) \tag{2.10}$$

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for all $x \in G$. Now, if we let $x = x - 2\alpha$ in (2.10) then we have

$$\ell(x - 2\sigma\alpha) = \ell(x - 2\alpha) \qquad (2.11)$$

for all $x \in G$. Also, by letting $x = x + 2\alpha$ in (2.10) we have

$$\ell(x + 4\alpha - 2\sigma\alpha) = \ell(x + 2\alpha) \qquad (2.12)$$

for all $x \in G$. Next, we substitute $x = x + 2\alpha$ and $y = y - 2\sigma\alpha$ in (2.3) and rewrite the resulting equation using (2.11) and (2.12) to obtain

$$\ell(x + y + 2\alpha) + \ell(x + \sigma y + 2\alpha) = 2\ell(x + 2\alpha)\,\ell(y - 2\alpha).$$

Define $\phi(x) = \ell(x + 2\alpha)$ and $\psi(x) = \ell(x - 2\alpha)$. Then the previous equation becomes

$$\phi(x + y) + \phi(x + \sigma y) = 2\phi(x)\,\psi(y)$$

for all $x, y \in G$. Using Lemma 2.1, we obtain

$$\phi(x) = \begin{cases} h(x)\,[A(x - \sigma x) + \gamma] & \text{if } h = h \circ \sigma, \\[2mm] a\,h(x) + b\,h(\sigma x) & \text{if } h \neq h \circ \sigma \end{cases} \qquad (2.13)$$

and

$$\psi(x) = \frac{h(x) + h(\sigma x)}{2} \qquad (2.14)$$

for all $x \in G$. Here $h : G \to \mathbb{C}^\star$ is a multiplicative homomorphism, $A : G \to \mathbb{C}$ is an additive homomorphism, and $\gamma, a, b \in \mathbb{C}$ are constants. By the definitions of $\psi(x)$ and $\phi(x)$ we have

$$\psi(x) = \ell(x - 2\alpha) = \ell(x - 2\alpha - 2\alpha + 2\alpha) = \ell(x - 4\alpha + 2\alpha) = \phi(x - 4\alpha).$$

Hence,

$$\psi(x) = \phi(x - 4\alpha). \qquad (2.15)$$

CASE 1: Suppose $h = h \circ \sigma$, then from (2.13) and (2.14) with (2.15), we obtain

$$\frac{h(x) + h(\sigma x)}{2} = h(x - 4\alpha)\,[A(x - 4\alpha - \sigma x + 4\sigma\alpha) + \gamma]$$

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which in turn implies

$$h(x) \left[h(\alpha)^{-4} \left[A(x - \sigma x) - 4A(\alpha - \sigma \alpha) + \gamma \right] - 1 \right] = 0$$

If $h(x) = 0$, then we have a trivial solution $\ell(x) = 0$ for all $x \in G$. If $h(x)$ is not identically zero, then we have

$$h(\alpha)^4 = A(x - \sigma x) - 4A(\alpha - \sigma \alpha) + \gamma \qquad (2.16)$$

for all $x \in G$. Replacing x by $x + y$ in (2.16) and using the fact that $A : G \to \mathbb{C}$ is an additive homomorphism, we obtain

$$h(\alpha)^4 = A(x - \sigma x) + A(y - \sigma y) - 4A(\alpha - \sigma \alpha) + \gamma \qquad (2.17)$$

for all $x, y \in G$. Comparing (2.16) with (2.17), we have

$$0 = A(y - \sigma y)$$

for all $y \in G$ and (2.16) becomes

$$\gamma = h(\alpha)^4.$$

Thus, the solution of the functional equation (2.3) for the case when $h = h \circ \sigma$ is

$$\begin{cases} \phi(x) = h(x)\,h(\alpha)^4 \\ \psi(x) = h(x) \end{cases} \qquad (2.18)$$

for some multiplicative homomorphism $h \colon G \to \mathbb{C}^\star$.

CASE 2: Suppose that $h \neq h \circ \sigma$. Then from (2.13) and (2.14) with (2.15), we obtain

$$h(x) + h(\sigma x) = 2\,a\,h(x)\,h(\alpha)^{-4} + 2\,b\,h(\sigma x)\,h(\sigma \alpha)^{-4}$$

which in turn simplifies to

$$\left(2ah(\alpha)^{-4} - 1 \right) h(x) + \left(2bh(\sigma \alpha)^{-4} - 1 \right) h(\sigma x) = 0. \qquad (2.19)$$

for all $x \in G$. Next we determine the coefficients of $h(x)$ and $h(\sigma x)$ in the last equation. Letting $x = 0$ in (2.19) and using the fact that $h(0) = 1$, we obtain

$$\left(2ah(\alpha)^{-4} - 1\right) + \left(2bh(\sigma\alpha)^{-4} - 1\right) = 0. \tag{2.20}$$

Using (2.20) in (2.19), we see that $a = \frac{h(\alpha)^4}{2}$ and $b = \frac{h(\sigma\alpha)^4}{2}$. So the solution of the functional equation (2.3) when $h \neq h \circ \sigma$ is

$$\begin{cases} \phi(x) = \frac{1}{2}[h(x)h(\alpha)^4 + h(\sigma x)h(\sigma\alpha)^4] \\[2mm] \psi(x) = \dfrac{h(x) + h(\sigma x)}{2}, \end{cases} \tag{2.21}$$

where $h \colon G \to \mathbb{C}^\star$ is a multiplicative homomorphism. Notice that when $h = h \circ \sigma$ then (2.21) is the same solution as (2.18). Hence, (2.21) is the only solution.

Using (2.21) and the definition of $\phi(x)$ we have the form of $\ell(x)$ as

$$\ell(x) = \phi(x - 2\alpha) = \frac{1}{2}[h(x)h(\alpha)^2 + h(\sigma x)h(\sigma\alpha)^2]$$

for all $x \in G$. Using this form of $\ell(x)$ in the functional equation (2.3) we see that $\ell(x)$ is a solution if $h(2\alpha) = h(2\sigma\alpha)$. Hence we have the asserted solution (2.4). Since there are no cases left, the proof of the lemma is now complete. $\qquad\square$

2.3 Solution of Generalized Van Vleck's Equation

In this section we find the solution of the generalized Van Vleck equation through variable manipulation. Setting one of our variables to 0 easily reduces our equation from two functions to one. By interchanging our variables, substituting new variables and comparing the resulting equations we are able to discover properties held by our solutions f and g. We use these properties and variable substitution to find relationships between certain function values. This helps to reduce our final solution. Knowing facts (σ-oddness, periodicity, etc.) about f and g we are able to

reduce our unknown equation down to an equation that has previously been solved (such as the equations in Lemmas 2.1 and 2.2). We can then work backwards to find the solutions of f and g for the generalized Van Vleck equation.

THEOREM 2.1. *Let G be an abelian group, $\alpha \in G$ be a fixed element and $\sigma : G \to G$ be an involution. Suppose the functions $f, g : G \to \mathbb{C}$ satisfy the functional equation*

$$f(x + \sigma y + \alpha) + g(x + y + \alpha) = 2f(x)f(y)$$

for all $x, y \in G$. Then there exist multiplicative homomorphisms $h_1, h_2 : G \to \mathbb{C}^\star$ such that the solutions f and g are given by

$$f(x) = \gamma, \quad g(x) = \gamma\,(2\gamma - 1) \tag{2.22}$$

$$f(x) = -\frac{h_1(\alpha)}{2}\big[h_1(x) - h_1(\sigma x)\big], \quad g(x) = -f(x), \tag{2.23}$$

$$\begin{cases} f(x) = \begin{cases} f(0)\,h_2(x) & \text{if } h_2 = h_2 \circ \sigma \\[2mm] \frac{1}{2}\big[h_2(x)h_2(\sigma\alpha) + h_2(\sigma x)h_2(\alpha)\big], & \text{if } h_2 \neq h_2 \circ \sigma \end{cases} \\[8mm] g(x) = \begin{cases} f(0)\big[2f(0)h_2(\alpha)^{-1} - 1\big]h_2(x) & \text{if } h_2 = h_2 \circ \sigma \\[2mm] \frac{1}{2}\big[\frac{a}{b}h_2(x)h_2(\sigma\alpha) + \frac{b}{a}h_2(\sigma x)h_2(\alpha)\big], & \text{if } h_2 \neq h_2 \circ \sigma, \end{cases} \end{cases} \tag{2.24}$$

where $h_1(\alpha) = -h_1(\sigma\alpha)$, and $\gamma, a, b \in \mathbb{C}$ are arbitrary constants satisfying $a + b = 1$ together with $a\,h_2(\alpha) = b\,h_2(\sigma\alpha) = 2abf(0)$.

Moreover, if $f(0) = 0$ and $\sigma\alpha = -\alpha$, then f and g are periodic functions of period 4α.

Proof. First, suppose $f(x) = \gamma$ for all $x \in G$. Then (2.1) becomes

$$\gamma + g(x + y + \alpha) = 2\gamma^2.$$

This implies that

$$g(x + y + \alpha) = \gamma(2\gamma - 1)$$

and by letting $y = -\alpha$ we have $g(x) = \gamma(2\gamma - 1)$ for all $x \in G$. This is (2.22).

From now on, we assume that f is nonconstant. Let $y = 0$ in (2.1), then we have

$$f(x + \alpha) + g(x + \alpha) = 2f(x)f(0).$$

This implies

$$g(x) = 2f(x - \alpha)f(0) - f(x) \tag{2.25}$$

for all $x \in G$. Using (2.25) in (2.1), we see that

$$f(x + \sigma y + \alpha) - f(x + y + \alpha) = 2f(x)f(y) - 2f(0)f(x + y) \tag{2.26}$$

for all $x, y \in G$.

CASE 1: Suppose $f(0) = 0$. Then (2.25) becomes

$$g(x) = -f(x) \tag{2.27}$$

and (2.26) becomes

$$f(x + \sigma y + \alpha) - f(x + y + \alpha) = 2f(x)f(y) \tag{2.28}$$

for all $x, y \in G$. Interchanging y with σy in (2.28), we obtain

$$f(x + y + \alpha) - f(x + \sigma y + \alpha) = 2f(x)f(\sigma y) \tag{2.29}$$

for all $x, y \in G$. Adding (2.28) and (2.29) yields

$$f(x)[f(y) + f(\sigma y)] = 0$$

for all $x, y \in G$. Since f is nonconstant this means

$$f(\sigma y) = -f(y) \quad \text{for all } y \in G. \tag{2.30}$$

Hence, f is a σ-odd function. By interchanging x with y in (2.28) we see that

$$f(y + \sigma x + \alpha) - f(y + x + \alpha) = 2f(y)f(x) \tag{2.31}$$

for all $x, y \in G$. Comparing (2.28) and (2.31) shows

$$f(x + \sigma y + \alpha) = f(y + \sigma x + \alpha). \tag{2.32}$$

Using (2.30) with (2.32) we see that

$$f(x + \sigma y + \alpha) = f(y + \sigma x + \alpha) = f(\sigma(\sigma y + x + \sigma \alpha)) = -f(x + \sigma y + \sigma \alpha).$$

and by letting $y = 0$ in the above equation we have

$$f(x + \alpha) = -f(x + \sigma \alpha) \tag{2.33}$$

for all $x, y \in G$.

SUBCASE 1.1: Suppose $\sigma \alpha \neq -\alpha$. Then replace x by $x + \alpha$ and y by $y + \alpha$ in (2.28) to obtain

$$f(x + \sigma y + \sigma \alpha + 2\alpha) - f(x + y + 3\alpha) = 2f(x + \alpha)f(y + \alpha)$$

for all $x, y \in G$. With the help of (2.33) the previous equation becomes

$$-f(x + \sigma y + 3\alpha) - f(x + y + 3\alpha) = 2f(x + \alpha)f(y + \alpha)$$

for all $x, y \in G$. Define $\ell(x) = -f(x + \alpha)$ then, from the above equation, we have

$$\ell(x + \sigma y + 2\alpha) + \ell(x + y + 2\alpha) = 2\,\ell(x)\,\ell(y)$$

and we can determine the solutions using Lemma 2.2. If $\ell = 0$, then $f = 0$ and since f is nonconstant this case does not mature. If

$$\ell(x) = \frac{1}{2}h_1(\alpha)^2\big[h_1(x) + h_1(\sigma x)\big],$$

then, since $h_1(\alpha)^2 = h_1(\sigma\alpha)^2$ by Lemma 2.2,

$$\ell(x) = \frac{1}{2}\big[h_1(x)h_1(\alpha)^2 + h_1(\sigma x)h_1(\sigma\alpha)^2\big],$$

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where $h_1 : G \to \mathbb{C}^\star$. Therefore

$$f(x) = -\ell(x - \alpha) = -\frac{1}{2}\left[h_1(x)h_1(\alpha) + h_1(\sigma x)h_1(\sigma \alpha)\right] \qquad (2.34)$$

and by (2.27)

$$g(x) = -f(x) = \frac{1}{2}\left[h_1(x)h_1(\alpha) + h_1(\sigma x)h_1(\sigma \alpha)\right].$$

From the form of f in (2.34), we obtain

$$f(\sigma x) = -\frac{1}{2}[h_1(\sigma x)h_1(\alpha) + h_1(x)h_1(\sigma \alpha)] \qquad (2.35)$$

for all $x \in G$. Using (2.34), (2.35) and the fact that f is σ-odd, we have

$$\left[h_1(\alpha) + h_1(\sigma \alpha)\right]\left[h_1(x) + h_1(\sigma x)\right] = 0$$

so either $h_1(\alpha) + h_1(\sigma \alpha) = 0$ or $h_1(x) + h_1(\sigma x) = 0$. If $h_1(x) + h_1(\sigma x) = 0$, then it is true for all x, and in particular it is true that $h_1(\alpha) + h_1(\sigma \alpha) = 0$. Hence, in both cases, $h_1(\alpha) + h_1(\sigma \alpha) = 0$. This means that $h_1(\alpha) = -h_1(\sigma \alpha)$ and our solution for f becomes

$$f(x) = -\frac{1}{2}h_1(\alpha)\left[h_1(x) - h_1(\sigma x)\right]. \qquad (2.36)$$

Now, g becomes

$$g(x) = \frac{1}{2}h_1(\alpha)\left[h_1(x) - h_1(\sigma x)\right]. \qquad (2.37)$$

Next, we will verify that the forms of f and g given in (2.36) and (2.37), respectively, are solutions of (2.1). Since

$$f(x + \sigma y + \alpha) + g(x + y + \alpha)$$
$$= -\frac{h_1(\alpha)}{2}[h_1(x)h_1(\sigma y)h_1(\alpha) - h_1(\sigma x)h_1(y)h_1(\sigma \alpha)]$$
$$+ \frac{h_1(\alpha)}{2}[h_1(x)h_1(y)h_1(\alpha) - h_1(\sigma x)h_1(\sigma y)h_1(\sigma \alpha)]$$

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$$= -\frac{h_1(\alpha)}{2}[h_1(x)h_1(\sigma y)h_1(\alpha) + h_1(\sigma x)h_1(y)h_1(\alpha)]$$

$$+ \frac{h_1(\alpha)}{2}[h_1(x)h_1(y)h_1(\alpha) + h_1(\sigma x)h_1(\sigma y)h_1(\alpha)]$$

$$= \frac{h_1(\alpha)^2}{2}[h_1(x)h_1(y) - h_1(\sigma x)h_1(y) - h_1(x)h_1(\sigma y) + h_1(\sigma x)h_1(\sigma y)]$$

$$= \frac{h_1(\alpha)^2}{2}[h_1(x) - h_1(\sigma x)][h_1(y) - h_1(\sigma y)]$$

and

$$2f(x)f(y) = \frac{h_1(\alpha)^2}{2}[h_1(x) - h_1(\sigma x)][h_1(y) - h_1(\sigma y)]$$

therefore f in (2.36) and g in (2.37) are the solutions of (2.1) given in (2.23).

SUBCASE 1.2: We now suppose that $\sigma\alpha = -\alpha$. Hence (2.33) becomes

$$f(x + \alpha) = -f(x - \alpha) \tag{2.38}$$

The substitution of $x + \alpha$ for x in (2.38) yields

$$f(x + 2\alpha) = -f(x) \tag{2.39}$$

and the same substitution in (2.39) leaves us

$$f(x + 3\alpha) = -f(x + \alpha). \tag{2.40}$$

Using (2.38) in (2.40), and replacing x by $x + \alpha$ in the resulting expression, we obtain

$$f(x + 4\alpha) = f(x) \tag{2.41}$$

for all $x \in G$. Substituting (2.41) in (2.27) gives us

$$g(x) = -f(x) = -f(x + 4\alpha) = g(x + 4\alpha)$$

for all $x \in G$. Thus, like $g(x)$ and $f(x)$ are both periodic with period 4α. Next, we replace x by $x + \alpha$ and y by $y + \alpha$ in (2.28) we have

$$f(x + \sigma y + 2\alpha + \sigma\alpha) - f(x + y + 3\alpha) = 2f(x + \alpha)f(y + \alpha). \tag{2.42}$$

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Since $\sigma\alpha = -\alpha$, the previous equation (2.42) becomes

$$f(x + \sigma y + \alpha) - f(x + y + 3\alpha) = 2f(x + \alpha)f(y + \alpha).$$

Using (2.40) with the above we have

$$f(x + \sigma y + \alpha) + f(x + y + \alpha) = 2f(x + \alpha)f(y + \alpha). \qquad (2.43)$$

Define $\ell : G \to \mathbb{C}$ by

$$\ell(x) = f(x + \alpha). \qquad (2.44)$$

Using this new function definition from (2.44), the equation (2.43) becomes

$$\ell(x + y) + \ell(x + \sigma y) = 2\,\ell(x)\,\ell(y) \qquad (2.45)$$

for all $x, y \in G$. The solution of (2.45) can be obtained either from Lemma 2.1 or from [23] as

$$\ell(x) = \frac{1}{2}\big[h_1(x) + h_1(\sigma x)\big], \qquad (2.46)$$

where $h_1 : G \to \mathbb{C}^\star$ is a multiplicative homomorphism. The definition (2.44) and the solution (2.46) imply

$$f(x) = \frac{1}{2}[h_1(x - \alpha) + h_1(\sigma x - \sigma\alpha)] = \frac{1}{2}\big[h_1(x)h_1(\alpha)^{-1} + h_1(\sigma x)h_1(\alpha)\big], \quad (2.47)$$

and by (2.27)

$$g(x) = -f(x) = -\frac{1}{2}\big[h_1(x)h_1(\alpha)^{-1} + h_1(\sigma x)h_1(\alpha)\big]. \qquad (2.48)$$

From the form of f in (2.47), we obtain

$$f(\sigma x) = \frac{1}{2}\big[h_1(\sigma x)h_1(\alpha)^{-1} + h_1(x)h_1(\alpha)\big] \qquad (2.49)$$

for all $x \in G$. Using (2.47), (2.49) and the fact that f is σ-odd we have

$$\big[h_1(x) + h_1(\sigma x)\big]\big[h_1(\alpha) + h_1(\alpha)^{-1}\big] = 0.$$

If $h_1(x) + h_1(\sigma x) = 0$ for all x, then it is true that $h_1(\alpha) + h_1(\sigma\alpha) = 0$. Hence, we have $h_1(\alpha) = -h_1(\alpha)^{-1}$ which yields

$$h_1(\alpha)^2 = -1. \tag{2.50}$$

Using (2.50) in (2.47), we obtain

$$f(x) = \frac{1}{2h_1(\alpha)}\left[h_1(x) - h_1(\sigma x)\right]. \tag{2.51}$$

Similarly, using (2.50) in (2.48) shows

$$g(x) = -\frac{1}{2h_1(\alpha)}\left[h_1(x) - h_1(\sigma x)\right]. \tag{2.52}$$

We now verify that (2.51) and (2.52) are solutions of (2.1). Since $h_1(\alpha)^2 = -1$, we have

$$f(x + \sigma y + \alpha) + g(x + y + \alpha) - 2f(x)f(y)$$

$$= \frac{1}{2h_1(\alpha)}[h_1(x)h_1(\sigma y)h_1(\alpha) - h_1(\sigma x)h_1(y)h_1(\alpha)^{-1}]$$

$$\quad - \frac{1}{2h_1(\alpha)}[h_1(x)h_1(y)h_1(\alpha) - h_1(\sigma x)h_1(\sigma y)h_1(\alpha)^{-1}]$$

$$\quad - 2\left(\frac{1}{2h_1(\alpha)}\right)\left(\frac{1}{2h_1(\alpha)}\right)[h_1(x) - h_1(\sigma x)][h_1(y) - h_1(\sigma y)]$$

$$= \frac{1}{2}\left[h_1(x)h_1(\sigma y) - \frac{h_1(\sigma x)h_1(y)}{h_1(\alpha)^2}\right] - \frac{1}{2}\left[h_1(x)h_1(y) - \frac{h_1(\sigma x)h_1(\sigma y)}{h_1(\alpha)^2}\right]$$

$$\quad - \frac{1}{2h_1(\alpha)^2}[h_1(x) - h_1(\sigma x)][h_1(y) - h_1(\sigma y)]$$

$$= \frac{1}{2}[h_1(x)h_1(\sigma y) + h_1(\sigma x)h_1(y)] - \frac{1}{2}[h_1(x)h_1(y) + h_1(\sigma x)h_1(\sigma y)]$$

$$\quad + \frac{1}{2}[h_1(x) - h_1(\sigma x)][h_1(y) - h_1(\sigma y)]$$

$$= -\frac{1}{2}[h_1(x) - h_1(\sigma x)][h_1(y) - h_1(\sigma y)] + \frac{1}{2}[h_1(x) - h_1(\sigma x)][h_1(y) - h_1(\sigma y)]$$

$$= 0.$$

Thus our solution set for (2.1) when $f(0) = 0$ and $\sigma\alpha = -\alpha$ is

$$\begin{cases} f(x) & = \frac{1}{2h_1(\alpha)}\left[h_1(x) - h_1(\sigma x)\right], \\[2mm] g(x) & = -\frac{1}{2h_1(\alpha)}\left[h_1(x) - h_1(\sigma x)\right], \end{cases}$$

where $h_1 : G \to \mathbb{C}^*$ is a multiplicative homomorphism with $h_1(\alpha) = -h_1(\alpha)^{-1}$. Notice that since $h(\alpha)^2 = -1$ this is the same solution we found in Subcase 1.1. Hence our only solution for the case when $f(0) = 0$ is the one asserted in (2.23).

CASE 2: Next, suppose $f(0) \neq 0$. Replace y with σy in (2.26) to obtain

$$f(x + y + \alpha) - f(x + \sigma y + \alpha) = 2f(x)f(\sigma y) - 2f(0)f(x + \sigma y) \qquad (2.53)$$

for all $x, y \in G$. By adding (2.53) to (2.26) we see that

$$f(0)[f(x + y) + f(x + \sigma y)] = f(x)[f(y) + f(\sigma y)] \qquad (2.54)$$

Let $\phi(x) = \frac{f(x)}{f(0)}$, then (2.54) can be rewritten as

$$\phi(x + y) + \phi(x + \sigma y) = \phi(x)\big[\phi(y) + \phi(\sigma y)\big]$$

which becomes

$$\phi(x + y) + \phi(x + \sigma y) = 2\phi(x)H(y), \qquad (2.55)$$

where

$$H(y) = \frac{\phi(y) + \phi(\sigma y)}{2}. \qquad (2.56)$$

From Lemma 2.1, the solution of (2.55) is given by

$$\phi(x) = \begin{cases} h_2(x)\left[A(x - \sigma x) + \gamma\right] & \text{if } h_2 = h_2 \circ \sigma \\[2mm] a\,h_2(x) + b\,h_2(\sigma x) & \text{if } h_2 \neq h_2 \circ \sigma \end{cases} \qquad (2.57)$$

and

$$H(x) = \frac{h_2(x) + h_2(\sigma x)}{2} \qquad (2.58)$$

where $h_2 : G \to \mathbb{C}^*$ is a multiplicative homomorphism, $A : G \to \mathbb{C}$ is an additive homomorphism and $a, b \in \mathbb{C}$ are arbitrary constants. Hence,

$$f(x) = \begin{cases} f(0)\,h_2(x)\left[A(x - \sigma x) + \gamma\right] & \text{if } h_2 = h_2 \circ \sigma \\[2mm] f(0)\left[a\,h_2(x) + b\,h_2(\sigma x)\right] & \text{if } h_2 \neq h_2 \circ \sigma. \end{cases} \qquad (2.59)$$

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Interchanging x and y in (2.26) we have

$$f(y + \sigma x + \alpha) - f(x + y + \alpha) = 2f(x)f(y) - 2f(0)f(x + y). \tag{2.60}$$

By comparing (2.26) and (2.60) we see

$$f(y + \sigma x + \alpha) = f(x + \sigma y + \alpha).$$

The substitution of $y = 0$ in the previous equation yields

$$f(x + \alpha) = f(\sigma x + \alpha) \tag{2.61}$$

for all $x \in G$.

SUBCASE 2.1: Suppose $h_2 = h_2 \circ \sigma$. Using (2.56) and (2.58) we see that

$$h_2(x) + h_2(\sigma x) = \phi(x) + \phi(\sigma x)$$

Since $h_2 = h_2 \circ \sigma$, the use of (2.57) in the last equation yields

$$2h_2(x) = h_2(x)[A(x - \sigma x) + \gamma - A(x - \sigma x) + \gamma]$$

which simplifies to

$$2\, h_2(x)\, (1 - \gamma) = 0.$$

Hence $\gamma = 1$. This means our solution for f is of the form

$$f(x) = f(0)\, h_2(x)\, \big[A(x - \sigma x) + 1\big], \tag{2.62}$$

where $h : G \to \mathbb{C}^*$ is a multiplicative homomorphism and $A : G \to C$ is an additive homomorphism. Using (2.62) we obtain

$$f(x + \alpha) = f(0)h_2(x)h_2(\alpha)[A(x + \alpha - \sigma x - \sigma \alpha) + 1] \tag{2.63}$$

and

$$f(\sigma x + \alpha) = f(0)h_2(x)h_2(\alpha)[A(\sigma x + \alpha - x - \sigma \alpha) + 1]. \tag{2.64}$$

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Thus, (2.61), (2.63) and (2.64) yield

$$A(x - \sigma x) = 0 \quad forall\, x \in G$$

and (2.62) reduces to

$$f(x) = f(0)\, h_2(x) \tag{2.65}$$

when $h_2 = h_2 \circ \sigma$. Using (2.65) in (2.25) we see

$$g(x) = 2f(0)f(x - \alpha) - f(x) = f(0)\, h_2(x)\left[2f(0)h_2(\alpha)^{-1} - 1\right]. \tag{2.66}$$

Now we show that f in (2.65) and g in (2.66) are solutions of (2.1) for this subcase. Consider

$$f(x + \sigma y + \alpha) + g(x + y + \alpha) - 2f(x)f(y)$$

$$= f(0)h_2(x)h_2(\sigma y)h_2(\alpha) + f(0)h_2(x)h_2(y)h_2(\alpha)[2f(0)h_2(\alpha)^{-1} - 1]$$

$$- 2f(0)^2 h_2(x)h_2(y)$$

$$= f(0)h_2(x)h_2(y)h_2(\alpha) + 2f(0)^2 h_2(x)h_2(y) - f(0)h_2(x)h_2(y)h_2(\alpha)$$

$$- 2f(0)^2 h_2(x)h_2(y)$$

$$= 0.$$

Hence, $f(x + \sigma y + \alpha) + g(x + y + \alpha) = 2f(x)f(y)$ for all $x, y \in G$. Thus for the subcase when $h_2 = h_2 \circ \sigma$

$$\begin{cases} f(x) = f(0)\, h_2(x), \\[2mm] g(x) = f(0)\left[2\, f(0)\, h_2(\alpha)^{-1} - 1\right] h_2(x) \end{cases}$$

is the solution of (2.1), where $h_2 : G \to \mathbb{C}^\star$ is a multiplicative homomorphism. This solution is included in the asserted solution (2.24).

SUBCASE 2.2: Suppose $h_2 \neq h_2 \circ \sigma$. Using (2.56) and (2.58), we get

$$h_2(x) + h_2(\sigma x) = \phi(x) + \phi(\sigma x). \tag{2.67}$$

The use of (2.57) in (2.67) yields

$$h_2(x) + h_2(\sigma x) = (a + b)h_2(x) + (a + b)h_2(\sigma x).$$

Hence, we have $a + b = 1$. From (2.59)

$$f(x) = f(0)\left[a\, h_2(x) + b\, h_2(\sigma x)\right], \tag{2.68}$$

where $a, b \in \mathbb{C}$ are arbitrary constants with $a + b = 1$. This f is the form of the solution when $h_2 \neq h_2 \circ \sigma$. Using (2.68) in (2.61), we obtain

$$\left[a\, h_2(\alpha) - b\, h_2(\sigma\alpha)\right]\left[h_2(x) - h_2(\sigma x)\right] = 0.$$

Since $h_2 \neq h_2 \circ \sigma$, this means that

$$ah_2(\alpha) = bh_2(\sigma\alpha). \tag{2.69}$$

Use of (2.68) and (2.69) on the left side of equation (2.26) yields

$$f(x + \sigma y + \alpha) - f(x + y + \alpha)$$
$$= f(0)[ah_2(x)h_2(\sigma y)h_2(\alpha) + bh_2(\sigma x)h_2(y)h_2(\sigma\alpha)]$$
$$\qquad - f(0)[ah_2(x)h_2(y)h_2(\alpha) + bh_2(\sigma x)h_2(\sigma y)h_2(\sigma\alpha)]$$
$$= -f(0)[h_2(x) - h_2(\sigma x)][ah_2(y)h_2(\alpha) - ah_2(\sigma y)h_2(\alpha)]$$
$$= -f(0)\, a\, h_2(\alpha)\left[h_2(x) - h_2(\sigma x)\right]\left[h_2(y) - h_2(\sigma y)\right].$$

Similarly, using (2.68) and (2.69) on the right side of the equation (2.26), and then simplifying we obtain

$$2f(x)f(y) - 2f(0)f(x + y)$$
$$= 2f(0)^2[ah_2(x) + bh_2(\sigma x)][ah_2(y) + bh_2(\sigma y)]$$
$$\qquad - 2f(0)^2[ah_2(x)h_2(y) + bh_2(\sigma x)h_2(\sigma y)]$$
$$= 2f(0)^2[a^2h_2(x)h_2(y) + abh_2(x)h_2(\sigma y) + abh_2(\sigma x)h_2(y) + b^2h_2(\sigma x)h_2(\sigma y)$$
$$\qquad - ah_2(x)h_2(y) - bh_2(\sigma x)h_2(\sigma y)]$$

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$$= 2f(0)^2[(a^2 - a)h_2(x)h_2(y) + (b^2 - b)h_2(\sigma x)h_2(\sigma y)$$

$$+ abh_2(x)h_2(\sigma y) + abh_2(\sigma x)h_2(y)]$$

$$= -2f(0)^2[abh_2(x)h_2(y) + abh_2(\sigma x)h_2(\sigma y) - abh_2(x)h_2(\sigma y) - abh_2(\sigma x)h_2(y)]$$

$$= -2\,a\,b\,f(0)^2\left[h_2(x) - h_2(\sigma x)\right]\left[h_2(y) - h_2(\sigma y)\right].$$

From the last equalities we see that

$$h_2(\alpha) = 2\,b\,f(0), \tag{2.70}$$

and from (2.69),

$$h_2(\sigma\alpha) = 2\,a\,f(0). \tag{2.71}$$

Using (2.70) and (2.71) we obtain

$$f(0)^2 = \frac{h_2(\alpha)\,h_2(\sigma\alpha)}{4\,a\,b}. \tag{2.72}$$

We use (2.68), (2.70), (2.71) and (2.72) in (2.25) to obtain

$$g(x) \tag{2.73}$$

$$= 2f(0)f(x - \alpha) - f(x)$$

$$= 2f(0)^2[ah_2(x)h_2(\alpha)^{-1} + bh_2(\sigma x)h_2(\sigma\alpha)^{-1}] - f(0)[ah_2(x) + bh_2(\sigma x)]$$

$$= \frac{1}{2ab}h_2(\alpha)h_2(\sigma\alpha)[ah_2(x)h_2(\alpha)^{-1} + bh_2(\sigma x)h_2(\sigma\alpha)^{-1}] - f(0)[ah_2(x) + bh_2(\sigma x)]$$

$$= \frac{1}{2ab}[ah_2(x)h_2(\sigma\alpha) + bh_2(\sigma x)h_2(\alpha)] - \frac{ah_2(\alpha)}{2ab}[ah_2(x) + bh_2(\sigma x)]$$

$$= \frac{1}{2ab}[ah_2(x)h_2(\sigma\alpha) + bh_2(\sigma x)h_2(\alpha) - a^2h_2(x)h_2(\alpha) - abh_2(\sigma x)h_2(\alpha)]$$

$$= \frac{1}{2ab}[a^2h_2(x)h_2(\sigma\alpha) + b^2h_2(\sigma x)h_2(\alpha)]$$

$$= \frac{1}{2}\left[\frac{a}{b}h_2(x)h_2(\sigma\alpha) + \frac{b}{a}h_2(\sigma x)h_2(\alpha)\right].$$

Now, using (2.69) and (2.70), we rewrite (2.68) as

$$f(x) = \frac{h_2(\alpha)}{2b}[ah_2(x) + bh_2(\sigma x)] = \frac{1}{2}[h_2(x)h_2(\sigma\alpha) + h_2(\sigma x)h_2(\alpha)]. \tag{2.74}$$

We now check that $f(x)$ in (2.74) and $g(x)$ in (2.73) satisfy equation (2.1). The left side of the equation (2.1) yields

$$
\begin{aligned}
&f(x + \sigma y + \alpha) + g(x + y + \alpha) \\
&= \frac{1}{2}[h_2(x)h_2(\sigma y)h_2(\alpha)h_2(\sigma\alpha) + h_2(\sigma x)h_2(y)h_2(\sigma\alpha)h_2(\alpha)] \\
&\quad + \frac{1}{2}\left[\frac{a}{b}h_2(x)h_2(y)h_2(\alpha)h_2(\sigma\alpha) + \frac{b}{a}h_2(\sigma x)h_2(\sigma y)h_2(\sigma\alpha)h_2(\alpha)\right] \\
&= \frac{h_2(\alpha)h_2(\sigma\alpha)}{2}\left[h_2(x)h_2(\sigma y) + h_2(\sigma x)h_2(y) + \frac{a}{b}h_2(x)h_2(y) + \frac{b}{a}h_2(\sigma x)h_2(\sigma y)\right] \\
&= \frac{h_2(\alpha)h_2(\sigma\alpha)}{2ab}[ah_2(x) + bh_2(\sigma x)][ah_2(y) + bh_2(\sigma y)].
\end{aligned}
$$

Similarly, the right side of the equation (2.1) yields

$$
\begin{aligned}
2f(x)f(y) &= \frac{1}{2}[h_2(x)h_2(\sigma\alpha) + h_2(\sigma x)h_2(\alpha)][h_2(y)h_2(\sigma\alpha) + h_2(\sigma y)h_2(\alpha)] \\
&= \frac{1}{2}\Big[\frac{a}{b}h_2(x)h_2(y)h_2(\sigma\alpha)h_2(\alpha) + h_2(x)h_2(\sigma y)h_2(\alpha)h_2(\sigma\alpha) \\
&\quad + h_2(\sigma x)h_2(y)h_2(\alpha)h_2(\sigma\alpha) + \frac{b}{a}h_2(\sigma x)h_2(\sigma y)h_2(\alpha)h_2(\sigma\alpha)\Big] \\
&= \frac{h_2(\alpha)h_2(\sigma\alpha)}{2}[ah_2(x) + bh_2(\sigma x)]\left[\frac{1}{b}h_2(y) + \frac{1}{a}h_2(\sigma y)\right] \\
&= \frac{h_2(\alpha)h_2(\sigma\alpha)}{2ab}[ah_2(x) + bh_2(\sigma x)][ah_2(y) + bh_2(\sigma y)].
\end{aligned}
$$

Hence, for this subcase, (2.74) and (2.73) are solutions of the functional equation (2.1) for some multiplicative homomorphism $h_2 : G \to \mathbb{C}^\star$ satisfying $h_2 \neq h_2 \circ \sigma$ and some constants $a, b \in \mathbb{C}$ satisfying $ah_2(\alpha) = bh_2(\sigma\alpha) = 2abf(0)$. This completes Case 2. Since there are no cases left, the proof of the theorem is now complete. $\square$

As a consequence of Theorem 2.1 one can obtain the following corollaries.

COROLLARY 2.1. *The continuous solutions* $f : \mathbb{R} \to \mathbb{R}$ *of* (1.2) *is given by either* $f \equiv 0$ *or* $f(x) = (-1)^n \sin\left(\dfrac{(2n+1)\pi x}{2\alpha}\right)$ *for all* $x \in \mathbb{R}$ *and* $n \in \mathbb{Z}$.

Proof. Follows from Theorem 2.1 and [12], pages 170-171. $\square$

COROLLARY 2.2. *The continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of Kannappan's equation (1.4) is given by either $f \equiv 0$, $f \equiv 1$, $f(x) = \cos\left(\dfrac{4n\pi x}{\alpha}\right)$, $f(x) = \cos\left(\dfrac{2(2n+1)\pi x}{\alpha}\right)$ or $f(x) = -\cos\left(\dfrac{(2n+1)\pi x}{\alpha}\right)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.*

Proof. Follows from Theorem 2.1 and Corollary 3.14a in [12]. □

The material presented in this chapter is taken from the author's paper [15].

CHAPTER 3
VAN VLECK'S AND KANNAPPAN'S EQUATIONS ON GROUPS

3.1 Introduction

This chapter is devoted to the study of Van Vleck's functional equation with involution as well as Kannappan's functional equation with involution on groups (not necessarily abelian).

In this chapter, G will always denote an arbitrary group unless otherwise stated. Similarly, z_0 will be a fixed element in the center $Z(G)$ of the group G, and $\sigma : G \to G$ is an involution.

3.2 Solution of Van Vleck's Equation

In this section, we determine the solutions to Van Vleck's equation with involution on arbitrary groups. Once again, we manipulate the variables to find properties of our solution f. Using these properties we can reduce our unknown equation so that our solution f is in terms of a solution to a known equation.

THEOREM 3.1. *Let G be a group, $\mathbb{C}$ be the field of complex numbers and $\sigma : G \to G$ be an involution. If $f : G \to \mathbb{C}$ satisfies the functional equation*

$$f(x\sigma y z_0) - f(xy z_0) = 2f(x)f(y) \tag{3.1}$$

for all $x,y \in G$ and fixed $z_0 \in Z(G)$, then either $f \equiv 0$ or f is given by

$$f(x) = g(x z_0^{-1})f(z_0), \tag{3.2}$$

where $g : G \to \mathbb{C}$ *is a nonzero solution of the cosine functional equation with involution*

$$g(x\sigma y) + g(xy) = 2g(x)g(y) \qquad \text{for all } x, y \in G. \tag{3.3}$$

Proof. Interchanging y with σy in (3.1), we obtain

$$f(xyz_0) - f(x\sigma yz_0) = 2f(x)f(\sigma y)$$

for all $x, y \in G$. By adding (3.1) with the above equation we see that

$$2f(x)\big[f(y) + f(\sigma y)\big] = 0 \qquad \text{for all } x, y \in G.$$

Hence, either $f \equiv 0$ or $f(y) = -f(\sigma y)$ for $y \in G$. Assume from now that f is not identically zero, so f is σ-odd and thus $f(e) = 0$. Letting $x = \sigma z_0$ in (3.1), we see that

$$f(\sigma z_0 \sigma y z_0) - f(\sigma z_0 y z_0) = 2f(\sigma z_0)f(y)$$

for all $y \in G$. From the last equation and the fact that f is σ-odd, we obtain

$$f(\sigma z_0 y z_0) = f(z_0)f(y) \quad \text{for all } y \in G. \tag{3.4}$$

Setting $y = z_0$ in (3.1) gives us

$$f(x\sigma z_0 z_0) - f(xz_0^2) = 2f(x)f(z_0).$$

Now, we apply (3.4) to the above equation to obtain

$$f(z_0)f(x) - f(xz_0^2) = 2f(x)f(z_0)$$

which implies

$$-f(xz_0^2) = f(x)f(z_0) \tag{3.5}$$

for all $x \in G$. Notice from above that if $f(z_0) = 0$ then replacing x by xz_0^{-2} in (3.5) yields $f \equiv 0$. Since f is not identically zero, this means that $f(z_0) \neq 0$. Now, let $x = e$ in (3.1) and we have

$$f(\sigma y z_0) = f(y z_0) \tag{3.6}$$

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for all $y \in G$. The substitution of $x = xz_0$ and $y = yz_0$ in (3.1) yields

$$f(x\sigma z_0\sigma yz_0^2) - f(xyz_0^3) = 2f(xz_0)f(yz_0)$$

for all $x, y \in G$. By (3.5) the above equation becomes

$$-f(x\sigma z_0\sigma y)f(z_0) + f(xyz_0)f(z_0) = 2f(xz_0)f(yz_0)$$

which, by using the fact that f is σ-odd, can be rewritten as

$$f(y\sigma xz_0)f(z_0) + f(xyz_0)f(z_0) = 2f(xz_0)f(yz_0).$$

Finally, using (3.6) we have

$$f(z_0)\big[f(x\sigma yz_0) + f(xyz_0)\big] = 2f(xz_0)f(yz_0) \tag{3.7}$$

for all $x, y \in G$. Define a function $g : G \to \mathbb{C}$ by

$$g(x) = \frac{f(xz_0)}{f(z_0)} \tag{3.8}$$

for all $x \in G$. Then (3.7) reduces to

$$g(x\sigma y) + g(xy) = 2g(x)g(y)$$

for all $x, y \in G$. Thus from (3.8) we get the asserted solution (3.2). $\quad\square$

We collect the following facts from the above theorem for later use.

REMARK 3.1. *If f is a nonzero solution of the functional equation* (3.1), *then (i)* $f(e) = 0$, *(ii)* $f(z_0) \neq 0$, *and (iii)* $f(xz_0^2) = -f(z_0)f(x)$ *for all* $x \in G$.

The following result can be found in [24] (see Lemma 9.2, p. 137).

PROPOSITION 3.1. *Let* $g : G \to \mathbb{C}$ *be an abelian function. Then the nonzero solution of*

$$g(xy) + g(x\sigma y) = 2g(x)g(y)$$

is given by

$$g(x) = \frac{1}{2}\big[\chi(x) + \chi(\sigma x)\big]$$

where χ is a character of G.

It is clear that $f \equiv 0$ is a solution to (3.1) so in the following corollaries we are only concerned with the nonzero solutions. Using the above result we have the following corollary.

COROLLARY 3.1. *Let $f : G \to \mathbb{C}$ be an abelian function. Then the nonzero solution of (3.1) is given by*

$$f(x) = -\chi(z_0)\frac{\big[\chi(x) - \chi(\sigma x)\big]}{2}$$

where χ is a character of G.

Proof. From item (ii) of Remark 3.1 we see that $f(z_0) \neq 0$. From Theorem 3.1, the nonzero solution of (3.1) is $f(x) = g(xz_0^{-1})f(z_0)$, where g is a nonzero solution of (3.3). Since f is abelian, g given by $g(x) = \frac{f(xz_0)}{f(z_0)}$ is also abelian which follows from

$$g(xyz) = \frac{f(xyzz_0)}{f(z_0)} = \frac{f(z_0xyz)}{f(z_0)} = \frac{f(z_0xzy)}{f(z_0)} = \frac{f(xzyz_0)}{f(z_0)} = g(xzy)$$

for all $x, y, z \in G$. From Proposition 3.1 we have the solution for $g(x)$ which gives us $f(x)$ as

$$f(x) = \frac{f(z_0)}{2}\left[\frac{\chi(x)}{\chi(z_0)} + \frac{\chi(\sigma x)}{\chi(\sigma z_0)}\right]. \tag{3.9}$$

From item (i) of Remark 3.1, we have $f(e) = 0$. Using this with (3.9) we see that

$$f(e) = \frac{f(z_0)}{2}\left[\frac{1}{\chi(z_0)} + \frac{1}{\chi(\sigma z_0)}\right] = 0$$

and since $f(z_0) \neq 0$ we conclude that

$$\chi(z_0) = -\chi(\sigma z_0). \tag{3.10}$$

From item (iii) of Remark 3.1 we also have $f(xz_0^2) = -f(z_0)f(x)$. Using (3.9) with this remark we have

$$\frac{f(z_0)}{2}\left[\chi(x)\chi(z_0) + \chi(\sigma x)\chi(\sigma z_0)\right] = -\frac{f(z_0)^2}{2}\left[\frac{\chi(x)}{\chi(z_0)} + \frac{\chi(\sigma x)}{\chi(\sigma z_0)}\right].$$

Hence by using (3.10) and simplifying the resulting equation, we get

$$\left[\chi(z_0)^2 + f(z_0)\right]\left[\chi(x) - \chi(\sigma x)\right] = 0 \tag{3.11}$$

for all $x \in G$. Substituting $x = z_0$ in (3.11) and using (3.10), we conclude that $\left[\chi(z_0)^2 + f(z_0)\right]2\chi(z_0) = 0$. Since $\chi(z_0) \neq 0$, we must have $f(z_0) = -\chi(z_0)^2$. Using this with (3.9) and (3.10) we obtain

$$f(x) = -\frac{\chi(z_0)}{2}\left[\chi(x) - \chi(\sigma x)\right] \tag{3.12}$$

for all $x \in G$. Now we verify that $f(x)$ given by (3.12) is the solution of (3.1). Since

$$\begin{aligned}
f(x\sigma y z_0) &- f(xy z_0) \\
&= \frac{\chi(z_0)^2}{2}\left[-\chi(x)\chi(\sigma y) - \chi(\sigma x)\chi(y) + \chi(x)\chi(y) + \chi(\sigma x)\chi(\sigma y)\right] \\
&= \frac{\chi(z_0)^2}{2}\left[\chi(x) - \chi(\sigma x)\right]\left[\chi(y) - \chi(\sigma y)\right] \\
&= 2f(x)f(y),
\end{aligned}$$

$f(x)$ given by (3.12) is indeed the solution of (3.1). This completes the proof of the corollary. $\qquad\square$

The following proposition was proved in [24] (see Proposition 9.23, p. 150).

PROPOSITION 3.2. *Let $g : G \to \mathbb{C}$ be a nonzero solution of*

$$g(xy) + g(xy^{-1}) = 2g(x)g(y)$$

for all $x, y \in G$. If $g(z_0)^2 \neq 1$ for some $z_0 \in Z(G)$, then there exists a character $\chi : G \to \mathbb{C}^*$ with $\chi(x) \neq \check{\chi}(x)$, such that $g(x) = \dfrac{\chi(x) + \check{\chi}(x)}{2}$.

The following corollary can be found in [24] as an exercise (see Exercise 9.18, p. 156) with ample hints. Here we present its proof for the sake of completeness.

COROLLARY 3.2. *Let $\sigma : G \to G$ be an involution with $\sigma x = x^{-1}$ for all $x \in G$. Then the nonzero solution $f : G \to \mathbb{C}$ for (3.1) is*

$$f(x) = \frac{\chi(x) - \check{\chi}(x)}{2i},$$

where χ is a character of G and i is the imaginary unit.

Proof. From (3.8), we obtain

$$g(z_0) = \frac{f(z_0^2)}{f(z_0)}.$$

Substituting $y = e$ in (3.5) we see that

$$f(z_0^2) = -f(e)f(z_0) = 0.$$

Hence $g(z_0)^2 = 0$ and, by Proposition 3.2, we have $g(x) = \frac{\chi(x) + \check{\chi}(x)}{2}$. Using these two facts together yields the following:

$$g(z_0) = \frac{\chi(z_0) + \chi(z_0^{-1})}{2} = 0$$

which is

$$\chi(z_0)^2 + 1 = 0.$$

Thus,

$$\chi(z_0) = i \qquad or \qquad \chi(z_0) = -i. \tag{3.13}$$

Since $\sigma x = x^{-1}$ for all $x \in G$ and $z_0 \in Z(G)$, (3.4) reduces to $f(y) = f(y)f(z_0)$. We are only considering nonzero solutions, so $f(z_0) = 1$. Now using (3.13) and the definition of $g(x)$ in (3.8) we have

$$f(x) = g(xz_0^{-1})f(z_0) = g(xz_0^{-1}) = \frac{\chi(xz_0^{-1}) + \chi(z_0 x^{-1})}{2} = \frac{\chi(x) - \chi(x^{-1})}{2\chi(z_0)}.$$

Now we assume that $\chi(z_0) = i$. Notice this means $\check{\chi}(z_o) = \chi(z_0^{-1}) = -i$. Thus, we have

$$f(x) = \frac{\chi(x) - \check{\chi}(x)}{2i}. \tag{3.14}$$

Now we verify that (3.14) is the solution of (3.1) when $\sigma x = x^{-1}$. Since

$$
\begin{aligned}
f(xy^{-1}z_0) - f(xyz_0) &= \frac{\chi(xy^{-1}z_0) - \chi(z_0^{-1}yx^{-1})}{2i} - \frac{\chi(xyz_0) - \chi(z_0^{-1}y^{-1}x^{-1})}{2i} \\
&= \frac{i\chi(x)\chi(y^{-1}) + i\chi(y)\chi(x^{-1})}{2i} - \frac{i\chi(x)\chi(y) + i\chi(x^{-1})\chi(y^{-1})}{2i} \\
&= \frac{1}{2}[\chi(y^{-1}) - \chi(y)][\chi(x) - \chi(x^{-1})] \\
&= 2\frac{[\chi(x) - \chi(x^{-1})]}{2i}\frac{[\chi(y) - \chi(y^{-1})]}{2i} \\
&= 2f(x)f(y),
\end{aligned}
$$

$f(x)$ given by (3.14) is indeed the solution for the case when $\chi(z_0) = i$.

In the case, $\chi(z_0) = -i$, then

$$f(x) = \frac{\check{\chi}(x) - \chi(x)}{2i} \tag{3.15}$$

and replacing the character $\check{\chi}$ by χ in (3.15), we have the asserted solution. Our proof is now complete. $\square$

3.3   Solution of Kannappan's Equation

In this section, we find the solutions to Kannappan's equation with involution on arbitrary groups. As one might imagine, the proofs in this section are almost identical to those in the previous section. Since the equations differ by a sign, most of our properties will as well. For example, we found before that our solution was σ-odd and now we will see that f is σ-even. Despite the sign differences, we can still reduce our unknown equation to the same solved equation.

THEOREM 3.2. *Let G be an arbitrary group, $\mathbb{C}$ be the field of complex numbers and $\sigma : G \rightarrow G$ be an involution. If $f : G \rightarrow \mathbb{C}$ satisfies the functional equation*

$$f(x\sigma y z_0) + f(xy z_0) = 2f(x)f(y) \tag{3.16}$$

for all $x, y \in G$ and fixed $z_0 \in Z(G)$, then either $f \equiv 0$ or $f(x) = g(xz_0^{-1})f(z_0)$ where $g : G \rightarrow \mathbb{C}$ is a nonzero solution of the cosine functional equation with involution

$$g(x\sigma y) + g(xy) = 2g(x)g(y) \quad \text{for all } x, y \in G.$$

Proof. It is easy to check that $f \equiv 0$ is a solution of (3.16). Hence from now we assume that f is not identically zero. Interchanging y with σy in (3.16), we have

$$f(xy z_0) + f(x\sigma y z_0) = 2f(x)f(\sigma y).$$

Subtraction of the last equation from (3.16) yields

$$2f(x)[f(y) - f(\sigma y)] = 0$$

for all $x, y \in G$. Since f is not identically zero, we must have f is σ-even. Letting $x = \sigma z_0$ in (3.16), we obtain

$$f(\sigma z_0 \sigma y z_0) + f(\sigma z_0 y z_0) = 2f(\sigma z_0)f(y)$$

for all $x, y \in G$. Since f is σ-even, the previous equation yields

$$f(\sigma z_0 y z_0) = f(z_0)f(y) \tag{3.17}$$

for all $y \in G$. Letting $y = z_0$ in (3.16) yields

$$f(x\sigma z_0 z_0) + f(xz_0^2) = 2f(x)f(z_0).$$

Applying (3.17) to the previous equation we have

$$f(x)f(z_0) + f(xz_0^2) = 2f(x)f(z_0)$$

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and this implies

$$f(xz_0^2) = f(x)f(z_0) \tag{3.18}$$

for all $x \in G$. Now, let $y = e$ in (3.16) and we see that

$$f(xz_0) = f(x)f(e) \tag{3.19}$$

for all $x \in G$. By substituting $x = xz_0$ and $y = yz_0$ in (3.16) we obtain

$$f(x\sigma z_0 \sigma y z_0^2) + f(xyz_0^3) = 2f(xz_0)f(yz_0)$$

which, by (3.18), becomes

$$f(x\sigma z_0 \sigma y)f(z_0) + f(xyz_0)f(z_0) = 2f(xz_0)f(yz_0).$$

And since f is σ-even this becomes

$$f(y\sigma x z_0)f(z_0) + f(xyz_0)f(z_0) = 2f(xz_0)f(yz_0).$$

Then by using (3.19) we see that this is

$$f(y\sigma x)f(e)f(z_0) + f(xyz_0)f(z_0) = 2f(xz_0)f(yz_0).$$

Once again σ-evenness yields

$$f(x\sigma y)f(e)f(z_0) + f(xyz_0)f(z_0) = 2f(xz_0)f(yz_0)$$

which, by applying (3.19) once more, becomes

$$f(x\sigma yz_0)f(z_0) + f(xyz_0)f(z_0) = 2f(xz_0)f(yz_0).$$

Define $g : G \to \mathbb{C}$ by

$$g(x) = \frac{f(xz_0)}{f(z_0)} \tag{3.20}$$

then the above gives us

$$g(x\sigma y) + g(xy) = 2g(x)g(y)$$

for all $x, y \in G$. This completes the proof. $\qquad\square$

REMARK 3.2. *If f is a nonzero solution of* (3.16), *then (i) f is a σ-even function, and (ii) $f(z_0^2) = f(e)f(z_0)$.*

REMARK 3.3. *In the case when f is not identically zero, it is easy to see from* (3.18) *and* (3.19) *that $f(e) \neq 0$ and $f(z_0) \neq 0$. From* (3.19) *we get $f(z_0^2) = f(z_0)f(e)$ and by the definition of g given in the proof of Theorem 3.2, we have $g(z_0) = \frac{f(z_0^2)}{f(z_0)} = f(e) \neq 0$.*

It is obvious that $f \equiv 0$ is a solution to equation (3.16), so we will only consider the nonzero solutions in the following corollaries. From Proposition 3.1 we can derive our next corollary.

COROLLARY 3.3. *Let $f : G \to \mathbb{C}$ be an abelian function. Then the nonzero solution of* (3.16) *is given by*

$$f(x) = \chi(z_0) \frac{\chi(x) + \chi(\sigma x)}{2},$$

where χ is a character of G.

Proof. We know from (3.20) of Theorem 3.2 that

$$g(x) = \frac{f(xz_0)}{f(z_0)}. \tag{3.21}$$

Since f is abelian,

$$g(xyz) = \frac{f(xyzz_0)}{f(z_0)} = \frac{f(z_0xyz)}{f(z_0)} = \frac{f(z_0xzy)}{f(z_0)} = \frac{f(xzyz_0)}{f(z_0)} = g(xzy)$$

and g is also an abelian function. The nonzero solution $g(x)$ of the functional equation $g(xy) + g(x\sigma y) = 2g(x)g(y)$ can be obtained from Proposition 3.1. From (3.21) we have $f(x) = f(z_0)g(xz_0^{-1})$, and hence

$$f(x) = \frac{f(z_0)}{2}[\chi(x)\chi(z_0)^{-1} + \chi(\sigma x)\chi(\sigma z_0)^{-1}]. \tag{3.22}$$

From (3.22) we obtain the following:

$$f(z_0^2) = \frac{f(z_0)}{2}[\chi(z_0) + \chi(\sigma z_0)] \tag{3.23}$$

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and

$$f(e) = \frac{f(z_0)}{2}[\chi(z_0)^{-1} + \chi(\sigma z_0)^{-1}].$$ (3.24)

From (3.19) of Theorem 3.2 (or item (ii) of Remark 3.2) we have $f(z_0^2) = f(e)f(z_0)$.
Using (3.23) and (3.24) we see that

$$\frac{f(z_0)}{2}\left[\chi(z_0) + \chi(\sigma z_0)\right] = \frac{f(z_0)^2}{2}\left[\chi(z_0)^{-1} + \chi(\sigma z_0)^{-1}\right].$$

Simplifying the last equality, we obtain

$$f(z_0)\left[\chi(z_0) + \chi(\sigma z_0)\right]\left[\frac{f(z_0)}{\chi(z_0)\chi(\sigma z_0)} - 1\right] = 0.$$

From Remark 3.3 we know $f(z_0) \neq 0$ and $\chi(x) + \chi(\sigma x) = 0$ leads to $f(e) = 0$
which is impossible since (3.19) yields $f(e)^2 = f(z_0) \neq 0$. Hence, we must have
$f(z_0) = \chi(z_0)\chi(\sigma z_0)$. Using this in (3.22) yields

$$f(x) = \frac{1}{2}[\chi(x)\chi(\sigma z_0) + \chi(\sigma x)\chi(z_0)]$$ (3.25)

and from the above we also have

$$f(\sigma x) = \frac{1}{2}[\chi(\sigma x)\chi(\sigma z_0) + \chi(x)\chi(z_0)].$$ (3.26)

From item (i) of Remark 3.2 we know that f is σ-even. Thus, by (3.25) and (3.26),
we have

$$\frac{1}{2}[\chi(\sigma x)\chi(\sigma z_0) + \chi(x)\chi(z_0)] = \frac{1}{2}[\chi(x)\chi(\sigma z_0) + \chi(\sigma x)\chi(z_0)]$$

which simplifies to

$$[\chi(x) - \chi(\sigma x)][\chi(z_0) - \chi(\sigma z_0)] = 0.$$

Notice that if $\chi(x) = \chi(\sigma x)$ for all x in G, then it is true that $\chi(z_0) = \chi(\sigma z_0)$ so we
may assume that only the latter is true. Now (3.25) becomes

$$f(x) = \frac{\chi(z_0)}{2}[\chi(x) + \chi(\sigma x)].$$ (3.27)

We now verify that f given by (3.27) is the solution of (3.16). Since

$$f(x\sigma y z_0) + f(xyz_0)$$
$$= \frac{\chi(z_0)}{2} \Big[\chi(x)\chi(\sigma y)\chi(z_0) + \chi(\sigma x)\chi(y)\chi(\sigma z_0)$$
$$+ \chi(x)\chi(y)\chi(z_0) + \chi(\sigma x)\chi(\sigma y)\chi(\sigma z_0) \Big]$$
$$= \frac{\chi(z_0)^2}{2} \big[\chi(x) + \chi(\sigma x) \big] \big[\chi(y) + \chi(\sigma y) \big]$$
$$= 2f(x)f(y)$$

f given by (3.27) is the solution. This completes the proof of this corollary. $\quad\square$

REMARK 3.4. *When* $\sigma x = x^{-1}$ *for all* $x \in G$ *then* (3.17) *reduces to* $f(y) = f(z_0)f(y)$. *Hence, for nonzero* f, *we have* $f(z_0) = 1$. *Also, from* (3.19) *we obtain* $f(z_0) = f(e)^2 = 1$.

When $\sigma x = x^{-1}$ for all $x \in G$, from Remarks 3.3 and 3.4 we see that $g(z_0)^2 = 1$. Thus one can not use Proposition 3.2. Hence we use the following result due to Corovei (see Theorem 2 in [4]).

PROPOSITION 3.3. *Let* G *be a nilpotent group whose elements are of odd order and* $\mathbb{C}$ *be the field of complex numbers. If* $g : G \to \mathbb{C}$ *is a nonzero solution of*

$$g(xy) + g(xy^{-1}) = 2g(x)g(y) \quad \text{for all } x, y \in G$$

then g *has the form*

$$g(x) = \frac{\chi(x) + \check{\chi}(x)}{2} \quad \text{for all } x \in G,$$

where χ *is a character of* G.

COROLLARY 3.4. *Let* G *be a nilpotent group whose elements are of odd order and* $\mathbb{C}$ *be the field of complex numbers, and let* $\sigma : G \to G$ *be defined such that* $\sigma x = x^{-1}$ *for all* x. *If* $f : G \to \mathbb{C}$ *is a nonzero solution of* (3.16) *then* f *has the form*

$$f(x) = \frac{\chi(x) + \check{\chi}(x)}{2} \tag{3.28}$$

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where χ a character of G.

Proof. From Remark 3.4 we know $f(z_0) = 1$. Using this fact, (3.21) and Proposition 3.3 we see

$$f(x) = \frac{1}{2}[\chi(x)\chi(z_0^{-1}) + \chi(z_0)\chi(x^{-1})]. \tag{3.29}$$

Using $f(z_0) = 1$ in (3.18) we obtain $f(xz_0^2) = f(x)$. Applying this to the form of f in (3.29) and simplifying resulting expression, we have

$$\chi(x)\chi(z_0) + \chi(x^{-1})\chi(z_0^{-1}) = \chi(x)\chi(z_0^{-1}) + \chi(z_0)\chi(x^{-1})$$

The equality further simplifies to

$$\left[\chi(x)^2 - 1\right]\left[\chi(z_0)^2 - 1\right] = 0.$$

Notice that if $\chi(x)^2 = 1$ for all x then it is true that $\chi(z_0)^2 = 1$ so we can assume just the latter. This means $\chi(z_0) = 1$ or $\chi(z_0) = -1$. Suppose $\chi(z_0) = -1$. Since $z_0 \in G$, there exists a $k \in \mathbb{N}$ such that $z_0^{2k+1} = e$. This implies

$$\chi(z_0^{2k+1}) = \chi(e) = 1.$$

However,

$$\chi(z_0^{2k+1}) = \chi(z_0)^{2k+1} = (-1)^{2k+1} = -1.$$

Therefore we have a contradiction and $\chi(z_0) \neq -1$ so we must have $\chi(z_0) = 1$. Thus, $f(x)$ becomes

$$f(x) = \frac{1}{2}[\chi(x) + \chi(x^{-1})].$$

Since

$$
\begin{aligned}
f(xy^{-1}z_0) &+ f(xyz_0) \\
&= \frac{1}{2}[\chi(x)\chi(y^{-1}) + \chi(x^{-1})\chi(y) + \chi(x)\chi(y) + \chi(x^{-1})\chi(y^{-1})] \\
&= \frac{1}{2}[\chi(x) + \chi(x^{-1})][\chi(y) + \chi(y^{-1})] \\
&= 2f(x)f(y),
\end{aligned}
$$

$f(x)$ given by (3.28) is the solution and the proof of the corollary is complete. $\square$

CHAPTER 4
SINE FUNCTIONAL EQUATION AND PERIODICITY ON GROUPS

4.1 Introduction

In this chapter, we consider the sine functional equation with involution, namely

$$f(xy) f(x\sigma y) = f(x)^2 - f(y)^2 \tag{4.1}$$

for all $x, y \in G$. Notice that $f(x) \equiv 0$ is a trivial solution of (4.1). We will assume from now that all solutions of (4.1) are nonzero. We will discuss some properties of any solution f, solve this equation on perfect groups and discuss the periodicity of the solutions.

4.2 Some Properties of the Solution of Sine Equation

LEMMA 4.1. *Let G be a group and $\mathbb{C}$ be the field of complex numbers. Let $\sigma : G \to G$ be an involution. If $f : G \to \mathbb{C}$ is a nonzero solution of (4.1), then the following hold:*

$$f(e) = 0, \tag{4.2}$$

$$f(\sigma x) = -f(x), \tag{4.3}$$

$$f(x)^2 = f(x^{-1})^2, \tag{4.4}$$

$$f(x\sigma x) = 0, \tag{4.5}$$

$$f(xy\sigma y)^2 = f(x)^2 \tag{4.6}$$

for all $x, y \in G$.

Proof. Letting $x = y = e$ in (4.1) it is easy to see that $f(e) = 0$. Next, letting $x = e$ in (4.1) and using (4.2), we have

$$f(y) \left[f(\sigma y) + f(y) \right] = 0$$

for all $y \in G$. Since $f \not\equiv 0$ we have $f(\sigma y) = -f(y)$ hence, (4.3) is proven.

In order to establish (4.4), let $y = x^{-1}$ in (4.1). By using (4.2), we obtain $f(x)^2 = f(x^{-1})^2$.

To prove (4.5), let $x = e$ in (4.1) to see

$$f(y) f(\sigma y) = -f(y)^2.$$

Replacing y by $y\sigma y$ in the last equality, we have

$$f(y\sigma y) f(y\sigma y) = -f(y\sigma y)^2.$$

Hence $2f(y\sigma y)^2 = 0$ and (4.5) follows.

Next, replacing y by $y\sigma y$ in (4.1), we get

$$f(xy\sigma y) f(xy\sigma y) = f(x)^2 - f(y\sigma y)^2 \tag{4.7}$$

for all $x, y \in G$. Using (4.5) in (4.7), we have the relation (4.6) and now the proof of the lemma is complete. $\square$

4.3   Solution of Sine Equation on Perfect Groups

Let

$$A_f(G) = \left\{ u \in G \mid f(u) = 0 \right\}$$

be the set of zeros of the solution f of the sine functional equation with involution (4.1). The importance of this set is already observed in Paranami and Vasudev [14]

and Corovei [6]. From Lemma 4.1 we have $f(\sigma x) = -f(x)$ so it is easy to see that if $f(x) = 0$, then $f(\sigma x) = 0$ as well. Hence, $A_f(G)$ is a σ-involutive subgroup of G.

LEMMA 4.2. *Let G be a group and $\mathbb{C}$ be the field of complex numbers. Let $\sigma : G \to G$ be an involution. If $f : G \to \mathbb{C}$ is a nonzero solution of* (4.1), *then*
(i) $A_f(G)$ is a subgroup of G;
(ii) If $xy \in A_f(G)$, then $yx \in A_f(G)$;
(iii) $A_f(G)$ is a normal subgroup of G.

Proof. First we show that $A_f(G)$ is a subgroup of G. $A_f(g)$ is nonempty since $f(e) = 0$. Suppose $x \in A_f(G)$. Then $f(x) = 0$ and from (4.1) we see that

$$f(xy)f(x\sigma y) = -f(y)^2.$$

Letting $y = x^{-1}$ in the above equation yields

$$f(e)f(x\sigma x^{-1}) = -f(x^{-1})^2.$$

Hence by (4.2), we have $f(x^{-1}) = 0$ which implies $x^{-1} \in A_f(G)$ and $A_f(G)$ is closed under inverses.

Suppose $x, y \in A_f(G)$. Replacing x by xy and y by y^{-1} in (4.1) and using closure under inverses, we obtain

$$f(x)f(xy\sigma y^{-1}) = f(xy)^2$$

for $x, y \in G$. But $x \in A_f(G)$, so we have

$$f(xy)^2 = 0$$

and hence $f(xy) = 0$. Thus $xy \in A_f(G)$ when $x, y \in A_f(G)$ and $A_f(G)$ is closed under multiplication. Therefore $A_f(G)$ is a subgroup of G.

To prove (ii), suppose $xy \in A_f(G)$. Then from (4.1), we have

$$f(x)^2 = f(y)^2. \tag{4.8}$$

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Replacing x by σx and y by σy in (4.1), we obtain

$$f(\sigma x \sigma y) f(\sigma xy) = f(\sigma x)^2 - f(\sigma y)^2$$

which by (4.3) reduces to

$$f(\sigma x \sigma y) f(\sigma xy) = f(x)^2 - f(y)^2.$$

Using (4.8) with the last equality yields

$$f(\sigma x \sigma y) f(\sigma xy) = 0 \qquad (4.9)$$

for all $x, y \in G$. Using properties of σ, (4.9) reduces to

$$f(\sigma(yx)) f(\sigma xy) = 0$$

which by (4.3) yields

$$f(yx) f(\sigma xy) = 0$$

for all $x, y \in G$. Notice by interchanging x with y in (4.1) and using (4.8), we also have

$$f(yx) f(y\sigma x) = f(y)^2 - f(x)^2 = 0.$$

Hence, either $f(yx) = 0$ or $f(\sigma xy) = f(y\sigma x) = 0$. Now we consider two cases.

CASE 1: Suppose $f(yx) = 0$. Then $yx \in A_f(G)$. Therefore (ii) holds.

CASE 2: Suppose $f(\sigma xy) = f(y\sigma x) = 0$. Then $\sigma xy, \ y\sigma x \in A_f(G)$. Since $A_f(G)$ is a subgroup of G, we also have

$$y^{-1}\sigma x^{-1}, \ \ \sigma x^{-1}y^{-1} \in A_f(G).$$

Recall that $xy \in A_f(G)$, and since A_f is σ-involutive subgroup we also have $\sigma(xy) \in A_f(G)$ and hence

$$\sigma y \, \sigma x \in A_f(G).$$

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Using the subgroup properties of $A_f(G)$ we have

$$\sigma y \sigma x \sigma x^{-1} y^{-1} = \sigma y y^{-1} \in A_f(G).$$

Since $A_f(G)$ is a σ-involutive subgroup and $\sigma xy \in A_f(G)$, we see

$$\sigma(\sigma xy) = \sigma yx \in A_f(G).$$

$A_f(G)$ is closed under inverses hence $x^{-1}\sigma y^{-1} \in A_f(G)$. Using closure under multiplication, we see that

$$x^{-1}\sigma y^{-1}\sigma yy^{-1} = x^{-1}y^{-1} \in A_f(G).$$

And since $A_f(G)$ is closed under inverses, we have

$$yx \in A_f(G).$$

This completes the proof of item (ii).

The proof of (iii) follows from (ii). To this see, let u be any arbitrary element of the subgroup $A_f(G)$. Using (ii) with any $g \in G$ we have

$$u = ug^{-1}g = gug^{-1} \in A_f(G).$$

Hence $A_f(G)$ is a normal subgroup of G. The proof of the lemma is now complete.

$\square$

THEOREM 4.1. *Let G be a 2-divisible group. If $f\colon G \to \mathbb{C}$ is a solution of* (4.1) *then $f(u) = 0$, for all $u \in [G, G]$.*

Proof. Interchanging x and y in (4.1) yields

$$f(yx)f(y\sigma x) = f(y)^2 - f(x)^2$$

which by (4.3) is

$$f(yx)f(x\sigma y) = f(x)^2 - f(y)^2.$$

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If we subtract the above equation from (4.1) we obtain

$$[f(xy) - f(yx)]f(x\sigma y) = 0 \quad \text{for all } x, y \in G.$$

Hence, either $f(xy) = f(yx)$ or $f(x\sigma y) = 0$ for all $x, y \in G$.

CASE 1: If $f(xy) = f(yx)$ then by letting $x = xy$ and $y = yx$ in (4.1) we have

$$f(xy^2 x)f(xy\sigma(yx)) = f(xy)^2 - f(yx)^2 = 0.$$

Now we have two subcases:

SUBCASE 1.1: Let $xy\sigma(yx) = xy\sigma x\sigma y \in A_f$ for all $x, y \in G$. By letting $y = e$ in our assumption we see that $x\sigma x \in A_f$ and similarly, when $x = e$, $y\sigma y \in A_f$. Therefore, $\sigma y^{-1}y^{-1} \in A_f$ and $\sigma x^{-1}x^{-1} \in A_f$ since A_f is a subgroup. Hence,

$$(xy\sigma x\sigma y)(\sigma y^{-1}y^{-1}) = xy\sigma xy^{-1} \in A_f$$

And since A_f is a normal subgroup this means that $y^{-1}xy\sigma x \in A_f$. Also, we have

$$(y^{-1}xy\sigma x)(\sigma x^{-1}x^{-1}) = y^{-1}xyx^{-1} \in A_f.$$

Again, since A_f is a normal subgroup this means $xyx^{-1}y^{-1} \in A_f$ for all $x, y \in G$.

SUBCASE 1.2: Let $xy^2 x \in A_f$ for all $x, y \in G$. Since G is 2-divisible there exist $u, v \in G$ such that $u^2 = x$ and $v^2 = y$. And since $xy^2 x \in A_f$ holds for all elements in G we have

$$uv^2 u \in A_f \implies u^2 v^2 \in A_f \implies xy \in A_f.$$

This implies that both $yx \in A_f$ and $x^{-1}y^{-1} \in A_f$. Hence, $xyx^{-1}y^{-1} \in A_f$

CASE 2: If $x\sigma y \in A_f$ then by letting $x = xyx^{-1}$ and $y = \sigma y^{-1}$ in our assumption we see that $xyx^{-1}y^{-1} \in A_f$.

Therefore, we have $[G, G] \subset A_f$ when G is 2-divisible. $\qquad \square$

Notice that the previous lemma gives us that on a 2-divisible perfect group G, that is when $G = [G, G]$, the only solution to (4.1) is $f \equiv 0$.

EXAMPLE 4.1. *The special orthogonal group of $n \times n$ matrices $SO(n, \mathbb{R})$ is an example of a 2-divisible perfect group. Hence the sine functional equation with involution has only the trivial solution on $SO(n, \mathbb{R})$.*

4.4   Periodicity of the Solution of Sine Equation

Since we were unable to determine the general solution of the sine functional equation with involution on arbitrary groups, it seems appropriate to study the periodicity of the solution of the sine functional equation.

The following lemma is from Kulosman in [13].

LEMMA 4.3. *Let $\mathbb{K}$ be a field and $\sigma : G \to G$ be an involution. Let $f : G \to \mathbb{K}$ satisfy (4.1). Let H be a subgroup of G defined by $H = \{\alpha \sigma \alpha^{-1} | \alpha \in A_f(G)\}$. Then*

$$f(xh) = f(x)$$

for all $x \in G$ and $h \in H$. Hence, f is h-periodic.

Proof. Let $\alpha \in A_f(G)$. In (4.1) let $x = x\alpha$ and $y = \sigma x^{-1}$. Then we have

$$f(x\alpha \sigma x^{-1})f(x\alpha x^{-1}) = f(x\alpha)^2 - f(\sigma x^{-1})^2.$$

Recall from property (iii) of Lemma 4.2 that $A_f(G)$ is a normal subgroup. Hence, $x\alpha x^{-1} \in A_f(G)$ and the previous equation becomes $f(x\alpha)^2 - f(\sigma x^{-1})^2 = 0$. Now, $f(x^{-1}) = f(x)^2$ for every $x \in G$ follows from Lemma 4.1. Using this property as well as the fact that f is σ-odd we obtain

$$[f(x\alpha) + f(x)][f(x\alpha) - f(x)] = 0 \tag{4.10}$$

for all $x \in G$ and $\alpha \in A_f(G)$.

Now let $y = \alpha$ in (4.1) to see

$$f(x\alpha)f(x\sigma\alpha) = f(x)^2 \tag{4.11}$$

for all $x \in G$ and $\alpha \in A_f(G)$. Comparing (4.10) and (4.11) we see there are two cases. For all $x \in G \setminus A_f(G)$ and $\alpha \in A_f(G)$, we have either

$$f(x\alpha) = -f(x) = f(x\sigma\alpha)$$

or

$$f(x\alpha) = f(x) = f(x\sigma\alpha).$$

Hence, in both cases we have

$$f(x\alpha\sigma\alpha^{-1}) = f(x)$$

for all $x \in G \setminus A_f(G)$ and $\alpha \in A_f(G)$. And clearly this also holds when $x \in A_f(G)$ thus, the lemma is proven. $\qquad\square$

EXAMPLE 4.2. *The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, is a solution of the equation*

$$f(x + y)f(x - y) = f(x)^2 - f(y)^2.$$

Here $\sigma x = -x$. We have $A_f(\mathbb{R}) = \{k\pi \mid k \in \mathbb{Z}\}$ and

$$H = \{\alpha - \sigma\alpha \mid \alpha \in A_f(\mathbb{R})\} = \{2\alpha \mid \alpha \in A_f(\mathbb{R})\} = \{2k\pi \mid k \in \mathbb{Z}\}.$$

By Lemma 4.3 we have

$$\sin(x + 2k\pi) = \sin x$$

for all $x \in \mathbb{R}$, $k \in \mathbb{Z}$. This is precisely the periodicity of the function $f(x) = \sin x$.

THEOREM 4.2. *Let $\mathbb{K}$ be a field. Let $f, g : G \to \mathbb{K}$ be functions satisfying*

$$g(xyz_0)g(x\sigma yz_0) = f(x)^2 - f(y)^2 \tag{4.12}$$

where $z_0 \in G$ is a nonzero constant. Then $g(x) = \phi(xz_0^{-1})$ where ϕ satisfies (4.1).

Furthermore, g is periodic with period $z_0^{-1}hz_0$ and f^2 is periodic with period h, where $h \in H = \{\alpha\sigma\alpha^{-1} \mid \alpha \in A_\phi(G)\}$ is a period of ϕ.

Proof. Letting $y = e$ in (4.12) yields

$$g(xz_0)^2 = f(x)^2 - f(e)^2. \tag{4.13}$$

We can use (4.13) to rewrite (4.12) as

$$g(xyz_0)g(x\sigma yz_0) = g(xz_0)^2 - g(yz_0)^2.$$

If we define $\phi(x) = g(xz_0)$ then the above equation becomes

$$\phi(xy)\phi(x\sigma y) = \phi(x)^2 - \phi(y)^2$$

and from Lemma 4.3 we have $\phi(xh) = \phi(x)$ for $h \in H = \{\alpha\sigma\alpha^{-1}|\alpha \in A_\phi(G)\}$. Using the definition of ϕ we see

$$\phi(xh) = \phi(x) \implies g(xhz_0) = g(xz_0) \implies g(xz_0^{-1}hz_0) = g(x).$$

Thus, g has period $z_0^{-1}hz_0$, where h is a period of (4.1).

Replacing x by $xz_0^{-1}h$ in (4.13) we have

$$f(xz_0^{-1}h)^2 = g(xz_0^{-1}hz_0)^2 + f(e)^2.$$

Because of the periodicity of g the previous equation is equal to

$$f(xz_0^{-1}h)^2 = g(x)^2 + f(e)^2.$$

And finally, replacement of x by xz_0 shows

$$f(xh)^2 = g(xz_0)^2 + f(e)^2 = f(x)^2.$$

Hence, the period of f^2 is the same as that of the sine functional equation with involution. $\square$

EXAMPLE 4.3. *The function $\phi : \mathbb{R} \to \mathbb{R}$, $\phi(x) = \sin(x)$ is a solution of the equation*

$$\phi(x + y)\phi(x - y) = \phi(x)^2 - \phi(y)^2$$

and ϕ has period $2k\pi$ where $k \in \mathbb{Z}$. Here $\sigma x = -x$.

Theorem 4.2 tells us that for

$$g(x + y + z_0)g(x - y + z_0) = f(x)^2 - f(y)^2$$

a solution is $g : \mathbb{R} \to \mathbb{R}$ and $g(x) = \phi(x - z_0) = \sin(x - z_0)$ for some constant z_0. And the period of g is $z_0 + 2k\pi - z_0 = 2k\pi$. Which is precisely the period of $\sin(x - z_0)$.

We also know that $f(x)^2 = g(x + z_0)^2 + f(e)^2 = \sin(x)^2 + f(e)^2$. Using this we have

$$f(x + 2k\pi)^2 = \sin(x + 2k\pi)^2 + f(e)^2 = \sin(x)^2 + f(e)^2 = f(x)^2$$

and $2k\pi$ is a period of f^2.

4.5 Periodicity of the Solution of Sine Inequality

A function $f : G \to \mathbb{R}$ is central if $f(xy) = f(yx)$ for all $x, y \in G$. In the following theorem, we generalize a result of S. L. Segal [22]. The essential idea in method of proof is due to Segal [22].

THEOREM 4.3. *Let G be a group, $\sigma : G \to G$ be an involution and $f : G \to \mathbb{R}$ be a central function satisfying the functional inequality*

$$f(xy) f(x\sigma y) \leq f(x)^2 - f(y)^2 \tag{4.14}$$

for all $x, y \in G$. Then f satisfies the sine functional equation with involution (4.1).

Proof. Letting $x = e = y$ in (4.14), we obtain $f(e)^2 \leq 0$ and hence we have

$$f(e) = 0. \tag{4.15}$$

Letting $y = x^{-1}$ in (4.14) we have

$$f(e) f(x\sigma x^{-1}) \leq f(x)^2 - f(x^{-1})^2$$

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which, by (4.15), implies

$$f(x^{-1})^2 \le f(x)^2 \tag{4.16}$$

for all $x \in G$. Replacing x by x^{-1} in (4.16), we obtain

$$f(x)^2 \le f(x^{-1})^2 \tag{4.17}$$

for all $x \in G$. From (4.16) and (4.17) we have

$$f(x^{-1})^2 \le f(x)^2 \le f(x^{-1})^2.$$

Hence

$$f(x)^2 = f(x^{-1})^2 \tag{4.18}$$

for all $x \in G$.

Next, letting σy^{-1} in place of x in (4.14), we obtain

$$f(\sigma y^{-1})f(e) \le f(\sigma y^{-1})^2 - f(y)^2.$$

From (4.15), we have

$$f(y)^2 \le f(\sigma y^{-1})^2.$$

Hence

$$f(y^{-1})^2 \le f(\sigma y)^2 \tag{4.19}$$

for all $y \in G$. Replacing y by $\sigma(y)$, we have

$$f(\sigma y^{-1})^2 \le f(y)^2$$

for all $y \in G$ and therefore

$$f(\sigma y)^2 \le f(y^{-1})^2 \tag{4.20}$$

for all $y \in G$. From (4.19) and (4.20) we get

$$f(y^{-1})^2 \le f(\sigma y)^2 \le f(y^{-1})^2$$

and therefore

$$f(\sigma y)^2 = f(y^{-1})^2 \tag{4.21}$$

for all $y \in G$. Using (4.18) in (4.21), we obtain

$$f(\sigma y)^2 = f(y)^2 \tag{4.22}$$

for all $y \in G$. The relation (4.22) implies that either

$$f(\sigma y) = -f(y)$$

or

$$f(\sigma y) = f(y)$$

for all $y \in G$. Suppose for a particular $y_0 \in G$

$$f(\sigma y_0) = f(y_0). \tag{4.23}$$

Letting $x = e$ and $y = y_0$ in (4.14), we get

$$f(y_0)f(\sigma y_0) \le -f(y_0)^2. \tag{4.24}$$

Using (4.23) in (4.24), we obtain

$$f(y_0)^2 \le -f(y_0)^2.$$

Hence $2f(y_0)^2 \le 0$. Therefore

$$f(y_0) = 0$$

and by (4.23)

$$f(\sigma y_0) = 0.$$

Thus

$$f(\sigma y) = -f(y) \tag{4.25}$$

for all $y \in G$. Using (4.14), (4.25) and the fact that f is central, we get

$$f(y)^2 \leq f(x)^2 - f(xy)f(x\sigma y)$$
$$= f(x)^2 + f(xy)f(\sigma(x\sigma y))$$
$$= f(x)^2 + f(yx)f(y\sigma x)$$
$$\leq f(x)^2 + f(y)^2 - f(x)^2$$
$$= f(y)^2$$

for all $x, y \in G$. Hence

$$f(y)^2 \leq f(x)^2 - f(xy)f(x\sigma(y)) \leq f(y)^2$$

for all $x, y \in G$. Therefore

$$f(xy)f(x\sigma y) = f(x)^2 - f(y)^2$$

for all $x, y \in G$ and the proof of the theorem is now complete. □

The following theorem follows from the above theorem and Lemma 4.3.

THEOREM 4.4. *Let $\mathbb{R}$ be a field of real numbers. Let $\sigma : G \to G$ be an involution. Let the central function $f : G \to \mathbb{R}$ satisfy (4.14) for all $x, y \in G$. Let H be a subgroup of G defined by $H = \{\alpha\sigma\alpha^{-1} | \alpha \in A_f(G)\}$. Then*

$$f(xh) = f(x)$$

for all $x \in G$ and $h \in H$.

Proof. From Theorem 4.3 we know that f satisfies (4.1). Lemma 4.3 tells us that $f(xh) = f(x)$ for all $x \in G$ and $h \in H = \{\alpha\sigma\alpha^{-1} | \alpha \in A_f(G)\}$. □

CHAPTER 5
SUMMARY AND FUTURE PLAN

5.1   Summary

In each chapter we have studied functional equations with involutions. Recall that for a group G, an involution $\sigma : G \to G$ is a function such that $\sigma(xy) = \sigma y \sigma x$ and $\sigma(\sigma x) = x$ for every $x \in G$. Notice that $\sigma x = x^{-1}$ is an example of an involution. By replacing all x^{-1}, or in abelian groups, $-x$, with σx we have generalized equations studied in each chapter in a new way.

On the set of real numbers, the generalized Van Vleck's equation has the form

$$f(x - y + \alpha) + g(x + y + \alpha) = 2f(x)f(y)$$

where $f, g : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ is a constant. In Chapter 2 we generalize this equation by examining it with an involution σ namely,

$$f(x + \sigma y + \alpha) + g(x + y + \alpha) = 2f(x)f(y)$$

for $f, g : G \to \mathbb{C}$. So we have expanded from the real line to an abelian group G. Our theorem results in four different solution sets for f and g and the solutions of equation (1.2) found by Van Vleck, the solutions of equation (1.4) found by Kannappan, and the solutions of equation (1.5) found by Sahoo can all be obtained from Theorem 2.1.

In Chapter 3 we find the solutions $f : G \to \mathbb{C}$ to Van Vleck's equation with

involution

$$f(x\sigma y z_0) - f(xyz_0) = 2f(x)f(y) \qquad (5.1)$$

on an arbitrary group G. This expands on the work from Chapter 2 a bit by no longer requiring G to be abelian. However, we do put the restriction that the constant z_0 be in the center $Z(G)$ of G. Using our results from Theorem 3.1 we are able to prove a couple corollaries. Corollary 3.1 gives us the solution of (5.1) when f is an abelian function. And Corollary 3.2 shows us the solution of (5.1) when $\sigma x = x^{-1}$ for all $x \in G$.

We also consider Kannappan's equation with involution

$$f(x\sigma y z_0) + f(xyz_0) = 2f(x)f(y) \qquad (5.2)$$

for $f : G \to \mathbb{C}$. Kannappan previously found the solutions when G is the additive group of reals and z_0 is a fixed constant in G. We have moved to an arbitrary group G, but again we have the restriction that $z_0 \in Z(G)$. Theorem 3.2 leads to two corollaries. Corollary 3.3 gives the solutions of (5.2) when f is an abelian function. And Corollary 3.4 finds the solutions to (5.2) when G is a nilpotent group with elements of odd order. In this case, $\sigma x = x^{-1}$ for all $x \in G$.

We move from functional equations related to the cosine function to functional equations related to the sine function for our work in Chapter 4. Specifically, we study the sine functional equation with involution

$$f(xy)f(x\sigma y) = f(x)^2 - f(y)^2 \qquad (5.3)$$

for $f : G \to \mathbb{C}$ and all $x, y \in G$. Again, we are generalizing previous results by replacing y^{-1} with an arbitrary involution σ. We are working towards solving this equation on an arbitrary group G.

For now, we have proved in Lemma 4.1 a few properties of any solution to (5.3). From the work of Paranami and Vasudev [14] and Corovei [6] we know the

importance of the set

$$A_f(G) = \{u \in G \mid f(u) = 0\}.$$

We are able to show that $A_f(G)$ is a normal subgroup of G in Lemma 4.2 and further that $[G, G] \subset A_f(G)$ in Theorem 4.1. At this juncture, due to time constraint we are unable to determine the solutions of (5.3) on arbitrary groups so we began looking into the periodicity of such solutions. For Theorem 4.2 we consider a generalized form of (5.3)

$$g(xyz_0)g(x\sigma yz_0) = f(x)^2 - f(y)^2$$

for $f, g : G \to \mathbb{K}$, where $\mathbb{K}$ is a field and $z_0 \in G$ is a nonzero constant. We found that the period of g is $z_0^{-1}hz_0$ and the period of f^2 is h, where h is a period of the solution of (5.3). Finally, we examine the sine inequality

$$f(xy)f(x\sigma y) \le f(x)^2 - f(x)^2,$$

where $f : G \to \mathbb{R}$. We determine that any central function f, that is if $f(xy) = f(yx)$ for all $x, y \in G$, that satisfies the above sine inequality has the same period as the solution of the sine functional equation with involution (5.3).

5.2   Future Plan

Ideally, we would like to solve the sine functional equation with involution

$$f(xy)f(x\sigma y) = f(x)^2 - f(y)^2$$

on arbitrary groups. We have been working on this for a while but without any success. So far our experience tells us that this is a very difficult problem to solve on arbitrary groups. Thus we would like to determine the general solutions of sine functional equation with involution on "near-abelian" groups such as nilpotent groups, p-groups or any other nonabelian group with a large center.

Another goal is to study the generalized Van Vleck's equation with involution

$$f(x\sigma y\alpha) + g(x\sigma y\alpha) = 2f(x)f(y),$$

on arbitrary groups, where $\alpha \in G$ is a fixed element instead of an element of the center $Z(G)$. There are also a couple of new, but related, equations that we would like to study. The first equation has been studied by Kannappan in [11]

$$f(x + y + \alpha)f(x - y + \alpha) = f(x)^2 - f(y)^2, \qquad \text{for all } x, y \in \mathbb{R}$$

and similarly,

$$f(x + y + \alpha)f(x - y + \alpha) = f(x)^2 + f(y)^2 - 1, \qquad \text{for all } x, y \in \mathbb{R}$$

which was studied by Etigson in [8]. These two equations can be generalized by introducing an involution σ and also by finding the solutions on an arbitrary group.

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