An–Najah National University
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On Best Approximation Problems In Normed Spaces With S-property

By
Ghadeer Ghanem Fayez Qwadreh

Supervisor
Dr. Abdallah Hakawati

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This thesis was defended successfully on 7–August–2008 and approved *by*

Committee Members	**Signature**
1. Dr. Abdallah Hakawati / Supervisor	**…………………….**
2. Dr. Saed Mallak / External Examiner	**…………………**
3. Dr. Fawaz Abu-Diyak / Internal Examiner	**…………………**

Dedication

I dedicate this work to my father and my mother, also to my sisters and my brothers.

Acknowledgement

First of all, I thank God for assistance in writing this thesis.

Next, I thank Dr. Justin Heavilin for his major contribution which was recorded in Remark 3.1.2 and for all his valuable suggestions and the time he took in reading the thesis over and over again, especially in the preparation of the final version.

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الإقــرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

On Best Approximation Problems In Normed Spaces With S-property

أفضل تقريب في فضاءات قياسية تحقق خاصية (S)

اقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء مـــا تمــت الإشارة إليه حيثما ورد، وان هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أية درجة علمية أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's name:	**اسم الطالب:**
Signature:	**التوقيع:**
Date:	**التاريخ:**

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Abstract

The problem of best approximation is the problem of finding, for a given point $x \in X$ and a given set G in a normed linear space ($X, \|\cdot\|$), a point $g_0 \in G$ which should be nearest to x among all points of the set G.

This thesis contains properties of best approximations in spaces with the S−property. We provide original results about Orlicz subspaces, and about $L^p(\mu, X)$, $1 \leq p \leq \infty$ subspaces with the S−property.

As a major result we prove that: if G is a closed subspace of X and has the S−property. Then the following are equivalent:

1. G is a Chebyshev subspace of X.

2. $L^{\phi}(\mu,G)$ is a Chebyshev subspace of $L^{\phi}(\mu,X)$.

3. $L^{p}(\mu,G)$ is a Chebyshev subspace of $L^{p}(\mu,X)$, $1 \leq p < \infty$.

Preface

The problem of best approximation is the problem of finding, for a given point $x \in X$ and a given set G in a normed linear space ($X, \|\cdot\|$) a point $g_0 \in G$ which should be nearest to x among all points of the set G.

We shall denote by $P(x,G)$, the set of all elements of best approximants (approximations) of x in G,

i.e. $P(x,G) = \{ g_0 \in G : \|x - g_0\| = \inf\{\|x - g\| : g \in G\} \}$.

The problem of best approximation began, in 1853, with P.L. Chebyshev who was led to state, the problem of finding for a real continuous function $x(t)$ on a segment $[a,b]$, an algebraic polynomial $g_0(t) = \sum_{i=1}^{n} \alpha_i^{(0)} t^{i-1}$ of degree $\leq n-1$, such that the deviation of the polynomials from the function $x(t)$ on the segment $[a,b]$ be the least possible among the deviations of all algebraic polynomials $g(t) = \sum_{i=1}^{n} \alpha_i t^{i-1}$ of degree $\leq n-1$. In other words; the problem of best approximation of the function $x(t)$ by algebraic polynomials $g(t)$ of degree $\leq n-1$ [9].

Many remarkable results appeared in Al–Dwaik's Masters Thesis [7]. Al–Dwaik gave the following definition : given a Banach space X, and a closed subspace G, then the subspace G is said to have the S–property in X if $z_1 \in P(x_1,G)$ and $z_2 \in P(x_2,G)$ imply that $z_1 + z_2 \in P(x_1 + x_2,G)$ $(x_1, x_2 \in X)$.

Chapter four of Al–Dwaik's thesis contains the following results:

1- Let X be any Banach space and G be a closed subspace of X with the S–property, then $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$ if and only if $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$.

2- Let X be any Banach space and G be a closed subspace of X which has the S–property. The following are equivalent :

(i)- G is proximinal in X.

(ii)- $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

3- If G has the S–property in X, then $L^\phi(\mu,G)$ has the S–property in $L^\phi(\mu,X)$.

Many other results can be found in there.

In this thesis we adopt the same definition as in Al–Dwaik [7], but X is a metric linear space, instead of a Banach space. My thesis consists of three chapters; each chapter is divided into sections. A triple like 1.3.2 indicates item (definitions, theorems, corollary, lemma …etc) number two in section three of chapter one. At the end of the thesis we present a collection of references, an appendix and abstract in Arabic.

In chapter one, we introduce the basic results and definitions which shall be needed in the following chapters. The topics include metric linear spaces, Hilbert spaces, Banach spaces, projections, orthogonality, measurable spaces and integrable functions.

Chapter two will be devoted to an introduction to fundamental ideas behind best approximations in normed linear spaces, Orlicz spaces, and the spaces $L^{p}(\mu, X)$, $1 \leq p \leq \infty$, which we need in chapter three. Section (2.1) contains some properties of $P(x,G)$ and theorems on best approximation. In Section (2.2) we define the 1–complemented subspace and $L^{p}-$summand subspace, $1 \leq p < \infty$. We also have theorems on best approximations in these subspaces and prove that if G is an $L^{p}-$summand subspace, then G is a Chebyshev subspace. In Section (2.3) we define the modulus function (ϕ), Orlicz space, and will have theorems on best approximations in subspaces of Orlicz space and $L^{p}(\mu,X)$, $1 \leq p \leq \infty$, which we need in section (3.2).

Chapter three is the main part of the thesis and contains two sections. Section (3.1) contains some theorems and consequences from Al–Dwaik [7], and the following new results:

1. In Example (3.1.3) we will see that if G is proximinal in X, it does not necessarily follow that G has the S–property.

2. In Remark (3.1.2) we will see that if G has the S–property, it does not necessarily follow that G is proximinal.

3. If G is an L^{p}–summand, $1 \leq p < \infty$, then G has the S–property.

4. In Example (3.1.19) we will see that if $P_{G}^{-1}(0)$ is proximinal in X which has the S–property and G has the S–property, and then G is proximinal.

5. Let X be a normed linear space, then any closed subspace G of X which has the S–property is a semi–Chebyshev subspace of X.

6. Let X be a normed linear space. If $P_G^{-1}(0)$ is a closed subspace of X, then G has the S−property in X.

7. Let G be a closed subspace of a normed linear space X which has the S−property. If G is proximinal, then G is a Chebyshev subspace of X.

There are more results which can be found in section (3.1).

In section (3.2) we have many results about the Orlicz subspaces and $L^p(\mu,X)$, $1\leq p \leq\infty$, subspaces with the S−property. The following are the main theorems in this section:

1. Theorem (3.1.4) and Theorem (3.1.5) imply that $L^p(\mu,G)$ has the S−property in $L^p(\mu,X)$ $1\leq p <\infty \Leftrightarrow$ G has the S−property in X.

2. $L^\infty(\mu,G)$ has the S−property in $L^\infty(\mu,X)$ $\Rightarrow$ G has the S−property in X.

3. Let $L^\infty(\mu,G)$ be a Chebyshev subspace of $L^\infty(\mu,X)$. If G has the S−property, then $L^\infty(\mu,G)$ has the S−property in $L^\infty(\mu,X)$.

4. $L^\phi(\mu,G)$ has the S−property in $L^\phi(\mu,X)$ $\Rightarrow G$ has the S−property in X.

5. If $\ell_\infty(S,G)$ has the S−property in $\ell_\infty(S,X)$, then G has the S−property in X.

The most important consequence of the above theorems is that: if G is a closed subspace of X and has the S−property. Then the following are equivalent:

a) G is a Chebyshev subspace of X.

b) $L^{\phi}(\mu,G)$ is a Chebyshev subspace of $L^{\phi}(\mu,X)$.

c) $L^{p}(\mu,G)$ is a Chebyshev subspace of $L^{p}(\mu,X)$, $1\leq p<\infty$.

Finally; I ask God to be our assistant always we do remain.

Chapter One
Preliminaries

1. Introduction

In this chapter we present some definitions and theorems on metric topics which will be needed in the next chapters. These definitions and theorems can be found in the texts and are foundational to the study of best approximations and the S−property.

1.1. Metric and Normed Linear Spaces

The following are the definitions and theorems regarding metric and normed linear spaces and they are essential to prove properties of best approximations. These can be found in textbooks of functional analysis by Kantorovich et al. and Lebedev et al. and Singer on best approximations in normed linear spaces [11, 12, and 9].

Definition 1.1.1: (Akilov [11]). A set X is called a metric space if to each pair of elements x, $y \in X$ there is associated a real number $d(x,y)$, the distance between x and y, subject to the following conditions:

M1:- $d(x,y) \geq 0$, and $d(x,y) = 0$ iff $x = y$.

M2:- $d(x,y) = d(y,x)$.

M3:- $d(x,y) \leq d(x,z) + d(z,y)$ for any $z \in X$ (This is the triangle inequality).

Such function d: $X \times X \rightarrow \boldsymbol{R}$ is called a metric on X. ■

Theorem 1.1.2: (Akilov [11]). $d(x,y)$ is a continuous function on its arguments, that is, if $x_n \to x$ and $y_n \to y$, then $d(x_n, y_n) \to d(x,y)$.

Definition 1.1.3: (Akilov [11]). A set $G \subseteq X$ is said to be closed if every convergent sequence $\{x_n\} \subset G$ converges to a point in G. ■

Definition 1.1.4: (Akilov [11]). The distance of a point x_0 from a set $G \subset X$ such that (X,d) is a metric space, is given by

$$d(x_0, G) = \inf\{d(x_0, g) : g \in G\}. \quad ■$$

Definition 1.1.5: (Lebedev [12]). Let $\boldsymbol{K}$ be the field of real or complex numbers (the field of scalars). A set X is called a vector (or linear) space over $\boldsymbol{K}$ if for every two of its elements x and y there is defined a sum $x + y$ an element of X, and if for every element $x \in X$ and every scalar $\lambda \in \boldsymbol{K}$ there is defined a product λx also an element of X, such that the following axioms are satisfied for all elements $x, y, z \in X$ and all scalars $\lambda, \mu \in \boldsymbol{K}$:

1. $x + y = y + x$;

2. $x + (y + z) = (x + y) + z$;

3. there is a zero element, $0 \in X$, such that $x + 0 = x$;

4. $\lambda(\mu x) = (\lambda\mu)x$;

5. $\lambda(x + y) = \lambda x + \lambda y$;

6. $(\lambda + \mu)x = \lambda x + \mu x$. ■

Remark 1.1.6: (Akilov [11]). Let X be a vector space. Then:

1. $x = y$ is equivalent to $x - y = 0$.

2. For all $x \in X$, there is a unique $x' \in X$ such that $x + x' = 0$. In fact

$x' = (-1)x$ and x' is usually written as $-x$ and is called the negative of x.

Definition 1.1.7: (Lebedev [12]). $\|\cdot\|$ is called a norm of x in a linear space X if it is a real valued function defined for every $x \in X$ which satisfies the following norm axioms:

N1: $\|x\| \geq 0$, and $\|x\| = 0$ iff $x = 0$.

N2: $\|\lambda x\| = |\lambda| \, \|x\|$.

N3: $\|x + y\| \leq \|x\| + \|y\|$ (This is the triangle inequality).

A vector space X having a fixed norm on it is called a normed linear space. ■

Remark 1.1.8: (Akilov [11]). Let X be a normed linear space, then:

1. If we set $d(x,y) = \|x - y\|, \forall x, y \in X$, then d is a metric on X.

2. $\big| \|x\| - \|y\| \big| \leq \|x - y\|$.

3. $\|x\|$ is a continuous function of x, that is, if $x_n \to x$, then $\|x_n\| \to \|x\|$.

Definition 1.1.9: (Lebedev [12]). Let X be a normed linear space and suppose $G \subset X$, G is called a subspace of X if it is a linear space; i.e. one which satisfies conditions (1–6) listed in definition 1.1.5, and has the norm on G obtained by restricting the norm on X to the subset G. The norm on G is said to be induced by the norm on X. ■

Theorem 1.1.10 : (Singer [9]). Let X be a normed linear space and G a linear subspace of X. Then:-

1. $d(x + g,G) = d(x,G)$ $\quad (x \in X,\ g \in G)$.
2. $d(x + y,G) \leq d(x,G) + d(y,G)$ $\quad (x, y \in X)$.
3. $d(\alpha x,G) = |\alpha|\ d(x,G)$ $\quad (x \in X,\ \alpha$ scalar).
4. $|d(x,G) - d(y,G)| \leq \|x - y\|$ $\quad (x, y \in X)$.
5. $d(x,G) \leq \|x\|$ $\quad (x \in X)$.

Proof: - For (1). Let $x \in X$, $g \in G$ and $\varepsilon > 0$ be arbitrary, by the definition of $d(x,G) = \inf \{ \|x - g\| : g \in G \}$ there exists $g_0 \in G$ such that

$$\|x - g_0\| \leq d(x,G) + \varepsilon \tag{1.1}$$

Consequently, we have

$$d(x + g,G) \leq \|x + g - (g_0 + g)\| = \|x - g_0\| \leq d(x,G) + \varepsilon$$

But $x \in X$, $g \in G$ and $\varepsilon > 0$ were arbitrary, hence

$$d(x + g,G) \leq d(x,G) \qquad (x \in X,\ g \in G). \tag{1.2}$$

Applying these relations for $x+g \in X$ instead of x and for $-g \in G$ instead of $g \in G$, we obtain

$$d(x,G) \le d(x+g,G) \qquad (x \in X,\ g \in G). \tag{1.3}$$

From (1.2) and (1.3) we get $d(x+g,G) = d(x,G)$, $x \in X$, $g \in G$.

For (2) of the theorem: Let $x, y \in X$ and $\varepsilon > 0$ be arbitrary. By the definition of $d(x,G)$ and $d(y,G)$ there exist $g_1, g_2 \in G$ such that

$$\|x - g_1\| \le d(x,G) + \varepsilon/2 \qquad\qquad \|y - g_2\| \le d(y,G) + \varepsilon/2$$

Consequently, we have

$$d(x+y,G) \le \|x + y - (g_1 + g_2)\| \le \|x - g_1\| + \|y - g_2\| \le d(x,G) + d(y,G) + \varepsilon$$

But $x, y \in X$, and $\varepsilon > 0$ were arbitrary, hence

$$d(x+y,G) \le d(x,G) + d(y,G) \qquad (x, y \in X).$$

For (3) of the theorem: Let $x \in X$, $\alpha \neq 0$ a scalar, and $\varepsilon > 0$ be arbitrary and take $g_0 \in G$ for which

$$\|x - g_0\| \le d(x,G) + (\varepsilon/|\alpha|). \tag{1.4}$$

We have

$$d(\alpha x, G) \le \|\alpha x - \alpha g_0\| = |\alpha|\,\|x - g_0\| \le |\alpha|\, d(x,G) + \varepsilon.$$

It follows that

$$d(\alpha x, G) \le |\alpha|\, d(x,G). \tag{1.5}$$

Applying this relation for αx instead of x and for $1/\alpha$ instead of α we obtain:

$$d(x,G) = d(1/\alpha\ (\alpha x), G) \le (1/|\alpha|)\ d(\alpha x, G).$$

Hence $|\alpha|\ d(x,G) \le d(\alpha x, G).$ (1.6)

From (1.5) and (1.6) and since $d(0,G) = 0$, we get

$$|\alpha|\ d(x,G) = d(\alpha x, G).$$

For (4): Let $x, y \in X$ and $\varepsilon > 0$ be arbitrary and take $g_0 \in G$ with

$$\|y - g_0\| \le d(y,G) + \varepsilon. \tag{1.7}$$

We have

$$d(x,G) \le \|x - g_0\| \le \|x - y\| + \|y - g_0\| \le \|x - y\| + d(y,G) + \varepsilon.$$

But x, y and $\varepsilon > 0$ were arbitrary, there follows

$$d(x,G) - d(y,G) \le \|x - y\|.$$

In these relations, interchange x and y to yield:

$$d(y,G) - d(x,G) \le \|x - y\|.$$

Hence $|d(x,G) - d(y,G)| \le \|x - y\|.$ (1.8)

For (5) of the theorem: Let $x \in X$ and $y = 0$, then by relation (1.8)

$$|d(x,G) - d(0,G)| \le \|x - 0\|$$

But $d(0,G) = 0$, then $d(x,G) \le \|x\|$, $\quad x \in X$. ■

Theorem 1.1.11: (Singer [9]). Let X be a normed linear space and G a linear subspace of X. Then we have

$$d(x,G) = \inf \ \{\|x-g\| \ : g \in G, \ \|g\| \leq 2\ \|x\| \ \}.$$

Proof: - If $x \in X$, $g_0 \in G$ and $\|g_0\| > 2\ \|x\|$, then taking into account

$d(x,G) \leq \|x\|$, $x \in X$, (part (5) of Theorem 1.1.10). One has

$$\|x-g_0\| \geq \|g_0\| - \|x\| > 2\|x\| - \|x\| = \|x\| \geq d(x,G).$$

Since for all $g_0 \in G$ such that $\|g_0\| > 2\ \|x\|$, we have $d(x,G) < \|x-g_0\|$, then $d(x,G) = \inf \ \{\|x-g\| \ : g \in G, \ \|g\| \leq 2\ \|x\| \ \}$. ■

Definition 1.1.12: (Lebedev [12]). A space S is said to be a linear subspace of a linear space X if S is linear space and S is a subset of X. ■

Definition 1.1.13: (Lebedev [12]). A metric space X is said to be complete if any Cauchy sequence in X has a limit in X; otherwise it is said to be incomplete. ■

1.2. Linear Operators

Essential definitions and fundamental theorems on linear operators that will be required in many places of the thesis can be found in texts by Siddiqi and by Kreyszig, as well as by singer, and by Lebedev et al. [14, 10, 9, 12].

Definition 1.2.1: (Lebedev [12]). The operator A is a linear operator from X into Y and X, Y are linear spaces, if its domain $D(A)$ is a linear subspace of X and for every $x_1, x_2 \in D(A)$, and every α, β (scalars) we have :

$$A(\alpha x_1 + \beta x_2) = \alpha A(x_1) + \beta A(x_2)$$

For a linear operator A, the image $A(x)$ is usually written Ax. ■

The null space $N(A)$ consisting of all $x \in X$ such that $Ax = 0$ is a subspace of X.

Definition 1.2.2: (Kreyszig [10]). Let X and Y be normed linear spaces and T: $D(T) \rightarrow Y$ a linear operator where $D(T) \subset X$. The operator T is said to be bounded if there is a real number k such that for all $x \in D(T)$, $\|Tx\| \leq k\|x\|$. ■

Theorem 1.2.3: (Siddiqi [14]). Let A: $D(A) \rightarrow Y$ be a linear operator where $D(A) \subset X$ and X, Y are normed linear spaces, Then:

1. A is continuous iff A is bounded.

2. If A is bounded, then $N(A)$ is closed subspace.

Definition 1.2.4: (Kreyszig [10]). A linear functional f is a linear operator with domain in a vector space X and range in the scalar field $\boldsymbol{K}$ of X. ■

Definition 1.2.5: (Kreyszig [10]).

1. The set of all linear functionals defined on a vector space X can itself be made into a vector space and is denoted by X^* and is called the dual

space. The sup norm on the unit disc of X will turn X^* into a normed linear space.

2. $B[X]$ is the set of all bounded operators from X into X. ■

Definition 1.2.6: (Singer [9]). A subset H of a vector space X is called a hyperplane if there exists a bounded linear functional, $f \neq 0$, defined on X and a scalar α such that $H = \{x \in X: f(x) = \alpha\}$. ■

Theorem 1.2.7: (Singer [9]). Let X be a normed linear space and

$H=\{x \in X: f(x) = \alpha\}$ be a hyperplane of X then the distance of the point x to the hyperplane H is

$$d(x,H) = \frac{|f(x) - \alpha|}{\|f\|}.$$

Remark 1.2.8: (Siddiqi [14]). An arbitrary $f \in c^*$(The space c consists of all convergent sequences of scalars with the sup norm) can be expressed as $f(x) = y_0 \lim_{n\to\infty} x_n + \sum_{n=1}^{\infty} y_n x_n$ where $x = (x_1, x_2, x_3, \ldots) \in c$ and

$y = (y_0, y_1, y_2, \ldots)$ such that $\sum_{i=1}^{\infty} |y_i| < \infty$ and $\|f\| = |y_0| + \sum_{i=1}^{\infty} |y_i|$.

1.3. Hilbert and Banach Spaces:-

We need to define Hilbert and Banach spaces for the next chapters, and we can find the definitions in texts by Lebedev et al. and by Kreyszig [10, 12].

Definition 1.3.1: (Lebedev [12]). A complete normed linear space is called a Banach space. ■

Definition 1.3.2: (Kreyszig [10]). Let X be a vector space over the field $\boldsymbol{K}$. An inner product on X is a function $<,>$: $X \times X \to \boldsymbol{K}$ such that for all $x, y, z \in X$ and a scalar α, we have:-

P1: $<x + y, z> = <x, z> + <y, z>$

P2: $<\alpha x, y> = \alpha <x, y>$

P3: $<x, y> = \overline{<y, x>}$

P4: $<x, x> \geq 0 \ \forall x \in X$, and $<x, x> = 0$ iff $x = 0$.

An inner product on X defines a norm on X given by $\|x\| = \sqrt{<x, x>}$ and a metric on X given by $d(x, y) = \|x - y\| = \sqrt{<x - y, x - y>}$.

An inner product space is a linear space with an inner product on it. ■

Definition 1.3.3: (Lebedev [12]). A complete inner product space is called a Hilbert space. ■

Remark 1.3.4: (Kreyszig [10]). If H is a Hilbert space, then for elements $x, y \in H$ we have the equation (This is the parallelogram law)

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

1.4. Orthogonal Sets

Kreyszig's book on functional analysis provides necessary definitions and theorems regarding orthogonal sets and direct sums of Hilbert spaces, but the properties of orthogonal elements in normed linear spaces can be found in Singer [10, 9].

Definition 1.4.1: (Singer [9]). An element *x* of a normed linear space *X* is said to be orthogonal to an element $y \in X$, and we write $x \perp y$, if we have $\|x + \alpha y\| \geq \|x\|$ for every scalar α. ■

Remark 1.4.2: (Singer [9]). Two vectors *x* and *y* in an inner product space *X* are called orthogonal, written as ($x \perp y$), if and only if $< x, y > = 0$.

Proof: Let $x, y \in X$ and $< x, y > \neq 0$. Then for $\alpha = -\dfrac{< x, y >}{< y, y >}$ we have

$$
\begin{aligned}
\|x + \alpha y\|^2 &= < x - \frac{< x, y >}{< y, y >} y, x - \frac{< x, y >}{< y, y >} y > \\
&= < x, x > - 2\frac{|< x, y >|^2}{< y, y >} + \frac{|< x, y >|^2}{< y, y >^2} < y, y > \\
&= < x, x > - \frac{|< x, y >|^2}{< y, y >} \\
&< \ < x, x > \\
&= \|x\|^2 .
\end{aligned}
$$

This contradicts our assumption. Therefore $\|x + \alpha y\| \geq \|x\|$, hence $x \perp y$.

For the converse, let $< x, y > = 0$. Then for every scalar α we have

$$\|x+\alpha y\|^2 =< x+\alpha y, x+\alpha y >= \|x\|^2 + |\alpha|^2 \|y\|^2 \geq \|x\|^2.$$

Hence $x \perp y$. ■

Definition 1.4.3: (Kreyszig [10]). A vector space X is said to be the direct sum of two subspaces Y and Z of X, written $X = Y \oplus Z$, if each $x \in X$ has a unique representation $x = y + z$, $y \in Y$, $z \in Z$.

Then Z is called an algebraic complement of Y in X and vice versa, and Y, Z are called a complementary pair of subspaces in X. ■

In the general Hilbert space X, we obtain the interesting representation of X as a direct sum of any closed subspace M of X and its orthogonal complement $M^{\perp} = \{x \in X : x \perp M\}$ which is the set of all vectors orthogonal to each member of M [10].

Theorem 1.4.4: (Kreyszig [10]). If M is a closed subspace of a Hilbert space X, then

$$X = M \oplus M^{\perp}.$$

Theorem 1.4.5: (Kreyszig [10]). If $x \perp y$ in an inner product space X, then

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

1.5. Projections

Requisite theorems on projections on Banach spaces can be found in Limaye [15].

Definition 1.5.1: (Limaye [15]). If X is a normed linear space and $P \in B[X]$ satisfies $P^2 = P$, then P is called a projection. ■

Theorem 1.5.2: (Limaye [15]). If P is a projection on a Banach space X and if M and N are its range and null space, respectively, then M and N are closed subspaces and $X = M \oplus N$.

Theorem 1.5.3: (Limaye [15]). Let X be a Banach space, and M and N be closed subspaces of X such that $X = M \oplus N$. The mapping defined on each $z = x + y$, $x \in M$, $y \in N$, by $P(z) = x$ is a projection on X whose range is M and whose null space is N.

1.6. Measurable Spaces and Integrable Functions

Rudin's Real and Complex analysis contains definitions for measure spaces and integrable functions and we will use a definition from Deeb and Khalil for Bochner p–integrable functions [13, 1].

Definition 1.6.1: (Rudin [13]).

(a) A collection Σ of subsets of a set X is said to be a $\sigma-$algebra in X if it has the following properties:

1. $X \in \Sigma$

2. If $A \in \Sigma$, then $A^c \in \Sigma$ where A^c is the complement of A relative to X

3. If $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \Sigma$ for n = 1,2,3,…., then $A \in \Sigma$

(b) If Σ is a $\sigma-$algebra in *X*, then *X* is called a measurable space, and the members of Σ are called the measurable sets in *X*.

(c) If *X* is a measurable space, *Y* is a topological space, and *f* is a mapping of *X* into *Y*, then *f* is said to be measurable provided that $f^{-1}(V)$ is a measurable set in *X* for each open set *V* in *Y*.

For the subset *E* of *X*, let χ_E denote the characteristic function of *E*. χ_E is measurable iff *E* is measurable. ■

Definition 1.6.2: (Rudin [13]).

(a) A positive measure is a function μ, defined on a $\sigma-$algebra Σ, whose range is in $[0,\infty]$ and which is countably additive. This means that if $\{A_n\}$ is a disjoint countable collection of members of Σ, then

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

(b) A measure space is a measurable space which has a measure defined on the $\sigma-$algebra of its measurable sets.

A property which is true except for a set of measure zero is said to hold almost everywhere (a.e.). ■

Definition 1.6.3: (Rudin [13]). A function *f*: $\Omega \to X$ is said to be simple if its range contains only finitely many points $x_1, x_2, \ldots, x_n$ and if $f^{-1}(x_i)$ is measurable for *i* = 1,2,3,…,*n*. Such a function then can be written as $f = \sum_{i=1}^{n} x_i \chi_{E_i}$ where for each *i*, $E_i = f^{-1}(x_i)$. We define $\int_E f\, d\mu = \sum_{i=1}^{n} x_i \mu(E_i \cap E)$.

If f is a non-negative measurable function on E, then we define

$\int_E f\, d\mu$ =sup{ $\int_E s d\mu$: $0 \leq s \leq f$, and s is a simple and measurable function on E}.

Remark 1.6.4: (Rudin [13]). The following propositions are immediate consequences of the definition. Functions and sets are assumed to be measurable on a measure space E:

1. If $A \subset B$ and $f \geq 0$, then $\int_A f\, d\mu \leq \int_B f\, d\mu$.

2. If $f \geq 0$ and $\int_E f\, d\mu = 0$, then $f = 0$ a.e. on E.

3. If c is constant, then $\int_E c\, d\mu = c\, \mu(E)$.

4. If $0 \leq f \leq g$, then $\int_E f\, d\mu \leq \int_E g\, d\mu$.

5. If $E = E_1 \bigcup E_2$, where E_1 and E_2 are disjoint, then

$$\int_E f\, d\mu = \int_{E_1} f\, d\mu + \int_{E_2} f\, d\mu.$$

Definition 1.6.5: (Deeb [1]). Let X be a real Banach space, and (Ω, μ) be a finite measure space. The space of Bochner p–integrable functions defined on (Ω, μ) with values in the Banach space X is denoted $L^p(\mu, X)$. For $f \in L^p(\mu, X)$, we define the norm

$$\|f\|_p = \begin{cases} \left(\int_\Omega \|f(t)\|^p \quad d\mu(t) \right)^{1/p} & 1 \leq p < \infty \\ \int_\Omega \|f(t)\|^p\, d\mu(t) & 0 < p < 1 \\ ess.\sup_{t \in \Omega} \|f(t)\| & p = \infty \end{cases}$$

Where $\|f\|_\infty = ess.\sup_{t\in\Omega}\|f(t)\| = \inf\{M : \mu\{t : \|f(t)\| > M\} = 0\}$. ■

It is clear that if $f \in L^\infty(\mu, X)$, then $\|f(t)\| \le \|f\|_\infty$ a.e. t. by the definition of essential supremum in Rudin [13].

Although the preceding is not an exhaustive list of theorems and proofs concerning the study of the S−property, they are central in the following discussion. We will use these tools to build our understanding of properties of the set of best approximations.

Chapter Two

Best Approximation in Normed Linear Space

2. Introduction.

Let X be a normed linear space and G be a subspace of X and $x \in X$; then the problem of best approximation consists of finding an element $g_0 \in G$ such that $\|x - g_0\| = d(x,G) = \inf \{\|x - g\| \ : \ g \in G\}$.

Every $g_0 \in G$ with this property is called an element of best approximation of x, or g_0 is a best approximant of x in G. We see that for all $x \in X$ a best approximation of x in G is an element of minimal distance from the given x. Such a $g_0 \in G$ may or may not exist. We shall denote the set of all elements of best approximation of x by elements of the set G by $P(x,G)$, i.e. $P(x,G) = \{ g_0 \in G : \|x - g_0\| = d(x,G)\}$.

2.1 . The Proximinal Sets and The Set of Best Approximations

Singer's book and Al–Dwaik's thesis provide the basic theorems on proximinal sets and the set of best approximations which we will need in our study [9, 7].

First, we will begin with some properties of $P(x,G)$.

Theorem 2.1.1: (Singer [9]). Let G be a subspace of a normed linear space X:

1. If $x \in G$, then $P(x,G) = \{x\}$.

2. If G is not closed and $x \in \overline{G}/G$, then $P(x,G) = \varnothing$.

Theorem 2.1.2: (Al–Dwaik [7]). Let G be a subspace of a normed space X, then, for $x \in X$:

1. $P(x,G)$ is a bounded set.

2. If G is a closed subspace of X, then $P(x,G)$ is a closed set.

Proof: - For (1), let $g_0 \in P(x,G)$, then $\|g_0\| \leq 2\|x\|$ by Theorem 1.1.11.

Thus $P(x,G)$ is a bounded set.

For (2), we show that if $\{g_n\}$ is a sequence in $P(x,G)$, such that $g_n \to g$, then $g \in P(x,G)$. Now $g_n \in P(x,G)$ $\forall\, n \in \boldsymbol{N}$, so $\|x - g_n\| = d(x,G) = \delta$**,** $\forall\, n \in \boldsymbol{N}$. Also $g_n \in G$. Since G is a closed subspace, then $g \in G$.

But the function $F_x : G \to \boldsymbol{R}$ defined by $F_x(g) = \|x - g\|$ $\forall\, g \in G$ is continuous by part (3) of Remark 1.1.8 .So $F_x(g_n) \to F_x(g)$ implies that

$$\|x - g_n\| \to \|x - g\|. \tag{2.1}$$

But $\|x - g_n\| = \delta$, $\forall\, n \in \boldsymbol{N}$, so $\|x - g\| = \delta$. Therefore $g \in P(x,G)$. ■

The following theorem is proved by Al–Dwaik, but we will provide another proof.

Theorem 2.1.3: (Al–Dwaik [7]). Let G be a subspace of a normed linear space X:

1. If $z \in P(x,G)$, then $\alpha z \in P(\alpha x,G)$ for all scalars α.

2. If $z \in P(x,G)$, then $z + g \in P(x + g,G)$ for all $g \in G$.

Proof: For part (1): Let $x \in X$ and $z \in P(x,G)$; we want to show that $\alpha z \in P(\alpha x,G)$ for any scalar α.

$$\begin{aligned}
\|\alpha x - \alpha z\| &= \|\alpha(x - z)\| \\
&= |\alpha| \ \|x - z\| \\
&= |\alpha| \ d(x,G) && \text{because } z \in P(x,G). \\
&= d(\alpha x,G) && \text{by part (3) of Theorem 1.1.10.}
\end{aligned}$$

Consequently, for $x \in X$ and scalar α we have

$$\|\alpha x - \alpha z\| = d(\alpha x,G). \tag{2.2}$$

Therefore $\alpha z \in P(\alpha x,G)$, for $x \in X$ and scalar α.

For part (2): Let $x \in X$, $g \in G$ and $z \in P(x,G)$; we want to show that $z + g \in P(x + g,G)$.

$$\begin{aligned}
\|x + g - (z + g)\| &= \|x - z\| \\
&= d(x,G) && \text{because } z \in P(x,G). \\
&= d(x + g,G) && \text{by part (1) of Theorem 1.1.10.}
\end{aligned}$$

Consequently, we have

$$\|x + g - (z + g)\| = d(x + g,G), \ \forall g \in G. \tag{2.3}$$

Therefore $z + g \in P(x + g, G)$, for any $g \in G$. ■

Any set $G \subset X$ which has the property that $P(x,G) \neq \emptyset, \forall x \in X$, is called a proximinal set in X. We call G a semi–Chebyshev set if for every $x \in X$, the set $P(x,G)$ contains at most one element. G is called Chebyshev if it is simultaneously proximinal and semi–Chebyshev, i.e. if for every $x \in X$ the set $P(x,G)$ contains exactly one element [9].

Lemma 2.1.4: (Singer [9]). Let X be a normed linear space, G a linear subspace of X, $x \in X \setminus \overline{G}$ and $g_0 \in G$. We have $g_0 \in P(x,G)$ if and only if $x - g_0 \perp G$.

Proof: By the definition of orthogonality, we have

$$\|x - g_0 + \alpha g\| \geq \|x - g_0\| \qquad (g \in G,\ \alpha \text{ being scalar}) \tag{2.4}$$

and this is obviously equivalent to $g_0 \in P(x,G)$. ■

Theorem 2.1.5: (Singer [9]). Let X be a normed linear space and H a hyperplane in X, passing through the origin. H is proximinal if and only if there exists an element $z \in X \setminus \{0\}$ such that $0 \in P(z,H)$ (i.e. such that $z \perp H$).

Theorem 2.1.6: (Al–Dwaik [7]). For a subspace G of a normed linear space X, the following are equivalent:

1. G is proximinal in X.

2. $X = G + P_G^{-1}(0)$ where $P_G^{-1}(0) = \{ x \in X : 0 \in P(x,G)\}$.

Proof: $(1)\to(2)$. If G is proximinal and $x\in X$, then

$$x = g_0 + (x-g_0) \in G + P_G^{-1}(0), \text{ where } g_0 \in P(x,G).$$

$(2) \to (1)$. Let $x\in X$ and $x = g_0 + y\in G + P_G^{-1}(0)$ where $g_0 \in G$ and $y\in P_G^{-1}(0)$ then $0\in P(y,G) = P(x-g_0,G)$. This implies that

$$d(x-g_0,G) = \|x-g_0\| \Rightarrow d(x,G) = \|x-g_0\|.$$

Hence $g_0 \in P(x,G)$, so G is proximinal. ■

2.2. 1–complemented and L^p–summand Subspaces

Deeb and Khalil defined the 1–complemented subspace and L^p–summand subspace, $1\le p<\infty$, and gave theorems on best approximations in these spaces [6, 1].

Definition 2.2.1: (Deeb [1]). A subspace G of a Banach space X is called 1–complemented in X if there is a closed subspace W in X such that $X = G\oplus W$ and the projection P: $X\to W$ is a contractive projection, (i.e. $\|Px\|\le\|x\|, \forall x\in X$). ■

Lemma 2.2.2: (Deeb [1]). If G is 1–complemented in X, then G is proximinal in X.

Proof: Let $X= G\oplus W$ and $x\in X$. Then $x = g + w$, where $g\in G$, $w\in W$ and $\|w\|\le\|x\|$, we show that $\|x-g\|\le\|x-y\|\ \ \forall y\in G$. Assume that there exists $g_1\neq g\in G$ such that $\|x-g_1\|<\|x-g\|$. Set $w_1 = x - g_1$. By the uniqueness of the representation of x we have $w_1\notin W$.

Hence $w_1 = g_2 + w_2$, where $g_2 \in G$, $w_2 \in W$ and $\|w_2\| \leq \|w_1\|$. Therefore

$$x = w_1 + g_1 = (g_2 + w_2) + g_1 = (g_1 + g_2) + w_2$$

and consequently $g = g_1 + g_2$ and $w = w_2$. Thus

$$\|w\| = \|w_2\| \leq \|w_1\| \tag{2.5}$$

But by assumption, $\|w_1\| = \|x - g_1\| < \|x - g\| = \|w\|$

This contradicts the assumption. Consequently $\|x - g\| \leq \|x - y\| \quad \forall y \in G$. Hence G is proximinal in X. ■

Now we need the following definition of L^p –summand subspaces.

Definition 2.2.3: (Khalil [6]). A closed subspace G of a Banach space X is called an L^p –summand, $1 \leq p < \infty$, if there is a bounded projection $P: X \to G$ which is onto, and $\|x\|^p = \|P(x)\|^p + \|x - P(x)\|^p$. ■

Theorem 2.2.4: (Khalil [6]). If G is an L^p –summand, then G is proximinal in X.

Proof: Let $x \in X$, for every $g \in G$ we have

$$\|x - g\|^p = \|P(x - g)\|^p + \|x - g - P(x - g)\|^p$$

$$= \|P(x) - g\|^p + \|x - P(x)\|^p$$

$$\geq \|x - P(x)\|^p$$

Hence $\|x - g\| \geq \|x - P(x)\|$, i.e. $P(x) \in P(x,G)$. Thus G is proximinal in X. ■

Now, we prove this new result on L^p –summand, $1 \leq p < \infty$.

Theorem 2.2.5: Let *X* be a Banach space and *G* be a closed subspace of *X*. If *G* is an L^p –summand, $1 \leq p < \infty$, then *G* is a Chebyshev subspace.

Proof: Let $x \in X$ and *G* be an L^p –summand, then there exists a bounded projection $T: X \to G$ which is onto, and so by the proof of Theorem 2.2.4 we have $T(x) \in P(x,G)$.

Now, assume that $g_0 \in P(x,G)$, then

$$\|x - g_0\|^p = \|x - T(x)\|^p \tag{2.6}$$

So, for $1 \leq \mathrm{p} < \infty$, we have

$$\|x - g_0\|^p = \|T(x - g_0)\|^p + \|x - g_0 - T(x - g_0)\|^p \tag{2.7}$$

$$= \|T(x) - g_0\|^p + \|x - T(x)\|^p$$

Consequently, we have

$$\|T(x) - g_0\| = 0 \Rightarrow T(x) = g_0 \text{ (by the definition of the norm).}$$

Therefore *G* is Chebyshev. ■

2.3 Approximation in Orlicz Spaces and in $L^p(\mu, X)$, $1 \leq p \leq \infty$

This section contains the concept of a modulus function and Orlicz spaces, in which we have some theorems on best approximation. Moreover, many of theorems and definitions in this section can be found in the articles by Deeb and by Khalil, also by Al–Dwaik and by Cheney et al. [1, 2, 3, 4, 5, 7, 8].

Definition 2.3.1:

a) (Deeb [5]**)**. A function $\phi : [0,\infty) \to [0,\infty)$ is called a modulus function if:

(i) ϕ is continuous at 0 and is increasing.

(ii) $\phi(x) = 0 \Leftrightarrow x = 0$.

(iii) $\phi(x + y) \leq \phi(x) + \phi(y)$, ($\phi$ is a subadditive function).

Examples of such functions are $\phi(x) = x^p$, $0 < p \leq 1$, and $\phi(x) = \ln(1+x)$. In fact if ϕ is a modulus function, then $\psi(x) = \dfrac{\phi(x)}{1+\phi(x)}$ is also a modulus function.

b) (Deeb [5]). Let X be a real Banach space, and (Ω, μ) be a finite measure space. For a given modulus function ϕ, we define the Orlicz space

$L^{\phi}(\mu, X)$ = {measurable function f: $\Omega \to X$: $\int_{\Omega} \phi(\|f(t)\|) d\mu(t) < \infty$ }.

The function d: $L^{\phi}(\mu, X) \times L^{\phi}(\mu, X) \to [0,\infty)$, given by

$$d(f,g) = \int_{\Omega} \phi(\|f(t) - g(t)\|) d\mu(t) \tag{2.8}$$

defines a metric on $L^{\phi}(\mu, X)$, under which it becomes a complete metric linear space. For $f \in L^{\phi}(\mu, X)$, we write $\|f\|_{\phi} = \int_{\Omega} \phi(\|f(t)\|) d\mu(t)$.

Then $\left(L^{\phi}(\mu, X), \|\cdot\|_{\phi}\right)$ is a complete metric linear space. If $\phi(x) = x^{p}$, $0<p\leq 1$, then $L^{\phi}(\mu, X)$ is the space $L^{p}(\mu, X)$, $0 < p \leq 1$ (p-Bochner space) and if ϕ is bounded, then $L^{\phi}(\mu, X)$ becomes the space of all measurable functions.

c) (Deeb [4]). For a Banach space X, we define

$$\ell^{\phi}(X) = \{ (f(n)) : \sum_{n=1}^{\infty} \phi(\|f(n)\|) < \infty ,\ f(n) \in X,\ \forall n \in N \}$$

For $f \in \ell^{\phi}(X)$, set $\|f\|_{\phi} = \sum_{n=1}^{\infty} \phi\|f(n)\|$. Then $\left(\ell^{\phi}(X), \|\cdot\|_{\phi}\right)$ is a complete metric linear space. ■

Clearly; for every nonnegative integer m we have: $\phi(mx) \leq m\phi(x)$ $\forall x \geq 0$ and we will use this result in proof the following theorem; then turn to a list of useful facts which we will need.

Theorem 2.3.2: (Deeb [5]). If ϕ is a modulus function and X is a Banach space, then $L^{1}(\mu, X) \subset L^{\phi}(\mu, X)$.

Proof: For each real number $y \geq 0$, we have $[y] \leq y < [y]+1$, where [] denotes the greatest integer function. But ϕ is increasing and subadditive, then:

$$\phi(y) \leq \phi([y]+1) \leq \phi([y]) + \phi(1) \leq [y]\phi(1) + \phi(1) \leq y\phi(1) + \phi(1) \leq (y+1)\phi(1)$$

If $y > 1$, then $\phi(y) < 2y\phi(1)$; and if $y \leq 1$, then $\phi(y) \leq \phi(1)$

Now, let $f \in L^{1}(\mu, X)$ and $A = \{t \in \Omega : \|f(t)\| \leq 1\}$ and $B = \{ t \in \Omega : \|f(t)\| > 1\}$

Then we have

$$\|f\|_{\phi} = \int_{\Omega} \phi(\|f(t)\|) d\mu(t)$$

$$= \int_{A} \phi(\|f(t)\|)\ d\mu(t) + \int_{B} \phi(\|f(t)\|)\ d\mu(t)$$

$$\leq \int_{\Omega} \phi(1)\ d\mu(t) + \int_{\Omega} 2\|f(t)\|\ \phi(1)\ d\mu(t)$$

$$\leq \phi(1)\ \ \mu(\Omega) + 2\phi(1)\ \|f\|_{1} < \infty$$. Hence $f \in L^{\phi}(\mu, X)$. ■

Theorem 2.3.3: (Deeb [4]). Let G be a closed subspace of a Banach space X. If g is a best approximant of f in $L^{\phi}(\mu, G)$, then $g(t)$ is a best approximant of $f(t)$ in G for almost all $t \in \Omega$.

Corollary 2.3.4: (Cheney [8]). Let G be a closed subspace of a Banach space X. If g is a best approximant of f in $L^{1}(\mu, G)$, then $g(t)$ is a best approximant of $f(t)$ in G for almost all $t \in \Omega$.

Theorem 2.3.5: (Deeb [4]). Let G be a closed subspace of a Banach space X and ϕ be a strictly increasing modulus function. If G is a proximinal subspace of X, then $\ell^{\phi}(G)$ is a proximinal subspace of $\ell^{\phi}(X)$.

Proof: - Let a sequence $f = \{f(n)\} \in \ell^{\phi}(X)$, since G is a proximinal in X, for each n, there exists $g(n) \in G$ such that $d(f(n), G) = \|f(n) - g(n)\|$.

Furthermore

$$\|g(n)\| \leq \|g(n) - f(n)\| + \|f(n)\| \leq \|0 - f(n)\| + \|f(n)\| = 2\|f(n)\|. \qquad (2.9)$$

Consequently, $g = \{g(n)\} \in \ell^{\phi}(G)$.

Now, we claim that g is a best approximation for f in $\ell^{\phi}(G)$. To see that, let h be any element of $\ell^{\phi}(G)$, then

$$\|f-h\|_{\phi} = \sum_{n=1}^{\infty}\phi\left(\|f(n)-h(n)\|\right) \geq \sum_{n=1}^{\infty}\phi\left(\|f(n)-g(n)\|\right) = \|f-g\|_{\phi}.$$

Hence $d(f,\ell^{\phi}(G)) = \|f-g\|_{\phi}$, and $g \in P(f,\ell^{\phi}(G))$. ■

Theorem 2.3.6: (Deeb [4]). Let G be a proximinal subspace of X. Then for every simple function $f \in L^{\phi}(\mu,X)$, $P(f, L^{\phi}(\mu,G))$ is not empty.

Proof: Let $f = \sum_{i=1}^{n} \chi_{E_i} x_i$, where E_i are disjoint measurable sets in Ω. Set $g = \sum_{i=1}^{n} \chi_{E_i} y_i$, where $y_i \in P(x_i, G)$. If h is an arbitrary element in $L^{\phi}(\mu,G)$ then we have

$$\begin{aligned}
\|f-h\|_{\phi} &= \int_{\Omega}\phi(\|f(t)-h(t)\|)\ d\mu(t) \\
&= \sum_{i=1}^{n}\int_{E_i}\phi(\|f(t)-h(t)\|)\ d\mu(t) \\
&= \sum_{i=1}^{n}\int_{E_i}\phi(\|x_i-h(t)\|)\ d\mu(t) \\
&\geq \sum_{i=1}^{n}\int_{E_i}\phi(\|x_i-y_i\|)\ d\mu(t) \\
&= \sum_{i=1}^{n}\int_{E_i}\phi(\|f(t)-g(t)\|)\ d\mu(t) \\
&= \int_{\Omega}\phi(\|f(t)-g(t)\|)\ d\mu(t) = \|f-g\|_{\phi}.
\end{aligned}$$

Hence $\|f-g\|_{\phi} = \inf\{\|f-h\|_{\phi} : h \in L^{\phi}(\mu,G)\} = d(f, L^{\phi}(\mu,G))$.

Therefore $g \in P(f, L^{\phi}(\mu,G))$. ■

Theorem 2.3.7: (Deeb [4]). Let G be a closed subspace of X. Then the following are equivalent:

(i) $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$.

(ii) $L^{1}(\mu,G)$ is proximinal in $L^{1}(\mu,X)$.

Proof: (i)$\rightarrow$(ii). Let $f\in L^{1}(\mu,X)$. Since $L^{1}(\mu,X)\subset L^{\phi}(\mu,X)$, then $f\in L^{\phi}(\mu,X)$, but $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$ so there exists $g\in L^{\phi}(\mu,G)$ such that

$$\|f-g\|_{\phi}\leq\|f-h\|_{\phi}\quad\forall\, h\in L^{\phi}(\mu,G).$$

Theorem 2.3.3 implies that

$$\|f(t)-g(t)\|\leq\|f(t)-y\|\quad\forall\, y\in G,\text{ a.e. } t\in\Omega. \tag{2.10}$$

Theorem 1.1.10 part (5) implies that $\|f(t)-g(t)\|\leq\|f(t)\|$, a.e. $t\in\Omega$.

Hence $\|g(t)\|\leq\|g(t)-f(t)\|+\|f(t)\|\leq 2\|f(t)\|$ a.e. $t\in\Omega$, thus $g\in L^{1}(\mu,G)$.

From (2.10) we get

$$\|f(t)-g(t)\|\leq\|f(t)-k(t)\|\quad\forall\, k\in L^{1}(\mu,G)\ \text{ a.e }\ t. \tag{2.11}$$

Integrating both sides we get:

$$\|f-g\|_{1}\leq\|f-k\|_{1}\quad\forall\, k\in L^{1}(\mu,G). \tag{2.12}$$

Therefore $L^{1}(\mu,G)$ is proximinal in $L^{1}(\mu,X)$.

Conversely, (ii) $\rightarrow$ (i). Define the map J: $L^{\phi}(\mu, X) \rightarrow L^{1}(\mu, X)$ by $J(f) = \hat{f}$ where

$$\hat{f}\,(t) = \begin{cases} f(t)\,\phi(\|f(t)\|)/\|f(t)\| & f(t) \neq 0 \\ 0 & f(t) = 0 \end{cases} \tag{2.13}$$

At first we show that $\hat{f} \in L^{1}(\mu, X)$.

$$\|\hat{f}\|_1 = \int_{\Omega} \|\hat{f}(t)\|\, d\mu(t)$$

$$= \int_{\Omega} \frac{\phi(\|f(t)\|)}{\|f(t)\|} \|f(t)\|\, d\mu(t)$$

$$= \int_{\Omega} \phi(\|f(t)\|)\, d\mu(t) = \|f\|_{\phi} < \infty .$$

Second, we claim that J is onto.

Let $g \in L^{1}(\mu, X)$ and let $f(t) = \begin{cases} g(t)\,\phi^{-1}(\|g(t)\|)/\|g(t)\| & g(t) \neq 0 \\ 0 & g(t) = 0 \end{cases}$

Then $\|f\|_{\phi} = \int_{\Omega} \phi(\|f(t)\|)\, d\mu(t)$

$$= \int_{\Omega} \phi\left(\frac{\phi^{-1}(\|g(t)\|)}{\|g(t)\|} \|g(t)\| \right) d\mu(t)$$

$$= \int_{\Omega} \|g(t)\|\, d\mu(t)$$

$$= \|g\|_1 < \infty.$$

Hence $f \in L^{\phi}(\mu, X)$ and $J(f) = g$.

Finally since ϕ is one to one it follows that J is one-to-one. It is now clear that $J(L^{\phi}(\mu, G)) = L^{1}(\mu, G)$.

Now, let $f \in L^{\phi}(\mu, X)$.Then $J(f) = \hat{f} \in L^{1}(\mu, X)$and there exists $\hat{g} \in L^{1}(\mu, G)$ such that $\|\hat{f} - \hat{g}\|_{1} \leq \|\hat{f} - \hat{h}\|_{1}$ for all $\hat{h} \in L^{1}(\mu, G)$. By Corollary 2.3.4; we have

$$\|\hat{f}(t) - \hat{g}(t)\| \leq \|\hat{f}(t) - y\| \quad \forall \ y \in G \ \text{ a.e. } t. \tag{2.14}$$

Since $\hat{g} \in L^{1}(\mu, G)$ and J is onto, there exists $g \in L^{\phi}(\mu, G)$such that $J(g) = \hat{g}$.

Hence $\left\| f(t) - \dfrac{\phi(\|g(t)\|)\|f(t)\| g(t)}{\phi(\|f(t)\|)\|g(t)\|} \right\| \leq \left\| f(t) - \dfrac{\|f(t)\|}{\phi(\|f(t)\|)} y \right\|$ a.e. t. and $\forall y \in G$.

Now take $h \in L^{\phi}(\mu, G)$. Then $\dfrac{\phi(\|f(t)\|)}{\|f(t)\|} h(t) \in G$ a.e. t.

Hence $\|f(t) - w(t)\| \leq \|f(t) - h(t)\|$ a.e. t. and $\forall h \in L^{\phi}(\mu, G)$, where

$w(t) = \dfrac{\phi(\|g(t)\|)\|f(t)\|}{\phi(\|f(t)\|)\|g(t)\|} g(t)$.

Using the fact that $\|g(t)\| \leq 2 \cdot \|f(t)\|$ we see that $w \in L^{\phi}(\mu, G)$ as follows

$$\|w(t)\| = \frac{\phi(\|g(t)\|) \ \ \|f(t)\|}{\phi(\|f(t)\|) \ \ \|g(t)\|} \|g(t)\|. \tag{2.15}$$

$$\leq \frac{2\phi(\|f(t)\|) \ \ \|f(t)\|}{\phi(\|f(t)\|)}$$

$$= 2\|f(t)\|.$$

Hence $w \in L^{\phi}(\mu, G)$. Thus $L^{\phi}(\mu, G)$ is proximinal in $L^{\phi}(\mu, X)$. ■

In a similar way we can prove the following theorem.

Theorem 2.3.8: (Deeb [2]). Let G be a closed subspace of a Banach space X. If $1<p<\infty$, the following are equivalent:

(i) $L^p(\mu,G)$ is proximinal in $L^p(\mu,X)$.

(ii) $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$.

Theorem 2.3.9: (Deeb [1]). Let G be a closed subspace of a Banach space X. If $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$, then $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$.

Proof: Let $f\in L^\infty(\mu,X)$. Since $L^\infty(\mu,X)\subset L^1(\mu,X)$, $f\in L^1(\mu,X)$ and there exists $g\in L^1(\mu,G)$ such that $\|f-g\|_1=d(f,\ L^1(\mu,G))$.

By Corollary 2.3.4, it follows that

$$\|f(t)-g(t)\|=d(f(t),G) \qquad \text{a.e. } t. \tag{2.16}$$

Hence

$$\|f(t)-g(t)\|\le\|f(t)-y\| \qquad \text{a.e. } t,\ \forall y\in G. \tag{2.17}$$

In particular

$$\|f(t)-g(t)\|\le\|f(t)-h(t)\| \qquad \text{a.e. } t,\ \forall h\in L^1(\mu,G). \tag{2.18}$$

But $L^\infty(\mu,G)\subset L^1(\mu,G)$, and hence, for every k in $L^\infty(\mu,G)$, we have

$$\|f(t)-g(t)\|\le\|f(t)-k(t)\| \qquad \text{a.e. } t. \tag{2.19}$$

Now since $\|g(t)\|\le 2\|f(t)\|$ a.e. t. Hence $g\in L^\infty(\mu,G)$.

And so it follows from relation (2.19) that

$$\|f-g\|_{\infty} \leq \|f-k\|_{\infty} \qquad \forall\, k \in L^{\infty}(\mu,G)\,.$$

Consequently, $L^{\infty}(\mu,G)$ is proximinal in $L^{\infty}(\mu,X)$. ■

Theorem 2.3.10: (Al–Dwaik [7]). Let G be a closed subspace of a Banach space X. If $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$, then G is proximinal in X.

Proof: Let $x \in X$, we define $f(t) = x, \forall t \in \Omega$, then $f \in L^{\phi}(\mu,X)$(because $\int\limits_{\Omega}\phi(\|f(t)\|)d\mu(t) = \int\limits_{\Omega}\phi(\|x\|)d\mu(t) = \phi(\|x\|)\mu(\Omega) < \infty$ (since $\|x\| < \infty$ and (Ω,μ) is a finite measure space). Since $L^{\phi}(\mu,G)$ is proximinal in $L^{\phi}(\mu,X)$, there exists $g \in L^{\phi}(\mu,G)$ such that $\|f-g\|_{\phi} = d(f, L^{\phi}(\mu,G))$. Theorem 2.3.3 implies

$$\|f(t)-g(t)\| \leq \|f(t)-y\| \qquad \text{a.e. } t, \text{ and } \forall\, y \in G. \tag{2.20}$$

Hence $\|x-g(t)\| \leq \|x-y\|$ a.e. t, and $\forall\, y \in G$.

Consequently, G is proximinal in X. ■

The following theorem is proved by Al–Dwaik [7], but we will provide another proof.

Theorem 2.3.11: Let G be a closed subspace of a Banach space X. If $L^{\infty}(\mu,G)$ is proximinal in $L^{\infty}(\mu,X)$, then G is proximinal in X.

Proof: Let $x \in X$ and $\|\cdot\|$ denote the norm of X. Consider the function $f(t) = x$, $\forall t \in \Omega$. Then $f \in L^{\infty}(\mu,X)$. Since $L^{\infty}(\mu,G)$ is proximinal in $L^{\infty}(\mu,X)$, then $\exists g \in L^{\infty}(\mu,G)$ such that

$$ess.\sup_{t\in\Omega}\|(f-g)(t)\| = \|f-g\|_{\infty} = d\,(f, L^{\infty}(\mu,G)).$$

Hence for all $h \in L^{\infty}(\mu,G)$ we have $\|f-g\|_{\infty} \leq \|f-h\|_{\infty}$. And so we have

$$\|f(t)-g(t)\| = \|x-g(t)\| \leq \|f-g\|_{\infty} \leq \|f-h\|_{\infty} \quad \text{a.e. } t. \tag{2.21}$$

In particular, let $h_y(t) = y$, $\forall t\in\Omega$ and $y\in G$, then $h_y \in L^{\infty}(\mu,G)$ and so by (2.21) we have

$$\|x-g(t)\| \leq \|f-h_y\|_{\infty} \quad \text{a.e. } t. \tag{2.22}$$

But $\|f-h_y\|_{\infty}$ = inf $\{M: \mu\{t : \|f(t)-h_y(t)\| > M\} = 0\}$

= inf $\{M: \mu\{t : \|x-y\| > M\} = 0\}$

$= \|x-y\|$.

By relation (2.22) we have

$$\|x-g(t)\| \leq \|x-y\| \quad \text{a.e. } t, \text{ and } y\in G. \tag{2.23}$$

Since $y\in G$ is arbitrary, then relation (2.23) is true for all $y\in G$ and so

$$g(t) \in P(x,G) \quad \text{a.e. } t, \text{ and } \forall x\in X.$$

We have proved that $P(x,G)$ contains $g(t)$ almost every t, and all what we need is just one such $g(t)$. Hence G is proximinal in X. ■

Theorem 2.3.12: (Deeb [3]). Let G be a closed subspace of a Banach space X. If $L^{p}(\mu,G)$ is proximinal in $L^{p}(\mu,X)$ for $1< p <\infty$, then G is proximinal in X.

Proof: If $L^p(\mu,G)$ is proximinal in $L^p(\mu,X)$ for 1 < p <∞, then theorem 2.3.8 implies that $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$. Theorem 2.3.9 implies $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$. Now, theorem 2.3.11 implies that G is proximinal in X. ■

Theorem 2.3.13: (Al–Dwaik [7]). If G is 1–complemented in X, then $L^1(\mu,G)$ is 1–complemented in $L^1(\mu,X)$.

Proof: Let $X = G \oplus W$ and let P: $X \to W$ be a contractive projection. Hence $x = (I-P)(x) + P(x)$ and $\|P(x)\| \le \|x\|$. For $f \in L^1(\mu,X)$, set

$f_1 = (I-P)\circ f$, $f_2 = P\circ f$, then

$$\|f_2\|_1 = \int_\Omega \|f_2(t)\|\, d\mu(t) = \int_\Omega \|P(f(t))\|\, d\mu(t) \le \int_\Omega \|f(t)\|\, d\mu(t) = \|f\|_1 < \infty .$$

Hence $f_2 \in L^1(\mu,W)$. Also

$$\|f_1\|_1 = \int_\Omega \|f_1(t)\|\, d\mu(t) = \int_\Omega \|(I-P)(f(t))\|\, d\mu(t) = \int_\Omega \|f(t) - P(f(t))\|\, d\mu(t)$$

$$\le \int_\Omega \|f(t)\|\, d\mu(t) + \int_\Omega \|P(f(t))\|\, d\mu(t) \le \int_\Omega \|f(t)\|\, d\mu(t) + \int_\Omega \|f(t)\|\, d\mu(t)$$

$$= 2\,\|f\|_1 < \infty .$$

Hence $f_1 \in L^1(\mu,G)$. Clearly $f = f_1 + f_2$.

Since W is a closed subspace of X, then $L^1(\mu,W)$ is a closed subspace of $L^1(\mu,X)$. Also if $f \in L^1(\mu,W) \cap L^1(\mu,G)$, then $f \in L^1(\mu,W)$ and $f \in L^1(\mu,G)$.

Thus $f(t) \in W$ and $f(t) \in G, \forall t \in \Omega$, but $G \cap W = \{0\}$,

so $f(t) = 0, \forall t \in \Omega \Rightarrow f \equiv 0$.

Hence $L^1(\mu, X) = L^1(\mu, G) \oplus L^1(\mu, W)$. Define $\hat{P} : L^1(\mu, X) \to L^1(\mu, W)$ by $\hat{P}(f) = P \circ f = f_2 \quad \forall f \in L^1(\mu, X)$, $\hat{P}$ is a contractive projection.

So $L^1(\mu, G)$ is 1–complemented in $L^1(\mu, X)$. ■

Corollary 2.3.14: (Al–Dwaik [7]). If G is 1–complemented in X, then $L^1(\mu, G)$ is proximinal in $L^1(\mu, X)$.

Proof: The Corollary follows from the above Theorem and Lemma 2.2.2.

Definition 2.3.15: (Deeb [4]). A closed subspace G of a Banach space X is called a ϕ–summand of X if there is a bounded projection P: $X \to G$ such that

$$\phi(\|x\|) = \phi(\|P(x)\|) + \phi(\|x - P(x)\|) \qquad \forall\ x \in X; \tag{2.24}$$

where ϕ is a modulus function. ■

Theorem 2.3.16: (Deeb [4]). If G is a ϕ–summand of a Banach space X, then G is proximinal in X.

Proof: - Let $x \in X$, for every $g \in G$ we have

$$\begin{aligned}\phi(\|x-g\|) &= \phi(\|P(x-g)\|) + \phi(\|(x-g) - P(x-g)\|)\\ &= \phi(\|P(x-g)\|) + \phi(\|x - P(x)\|)\\ &\geq \phi(\|x - P(x)\|).\end{aligned}$$

Hence $\|x-g\| \geq \|x - P(x)\|$ (i.e. $P(x) \in P(x,G)$). Thus G is proximinal in X. ■

Remark 2.3.17: (Al–Dwaik [7]). If G is a ϕ–summand of a Banach space X, then G is a Chebyshev subspace.

Proof: Assume that G is a ϕ–summand of X. The above theorem implies that $P(x) \in P(x,G)$.

Now suppose $g_0 \in P(x,G)$ i.e. $\|x - g_0\| = \|x - P(x)\|$. But $x - g_0 \in X$, then

$$\phi(\|x - g_0\|) = \phi(\|P(x - g_0)\|) + \phi(\|x - g_0 - P(x - g_0)\|)$$

$$= \phi(\|P(x) - g_0\|) + \phi(\|x - P(x)\|).$$

Hence $\phi(\|P(x) - g_0\|) = 0 \Rightarrow P(x) = g_0$.

Therefore $P(x)$ is the unique best approximant of x in G. Thus G is Chebyshev. ■

In the remaining part of the thesis, we will assume that the modulus function ϕ is positive homogeneous (i.e. for $\lambda \geq 0$ $\phi(\lambda x) = \lambda\phi(x)$ [11]) to make Orlicz spaces normed linear.

Theorem 2.3.18: Let X be a Banach space and G be a closed subspace of X. If G is a ϕ–summand of X, then $\ell^{\phi}(G)$ is a 1–summand of $\ell^{\phi}(X)$.

Proof: Let $P: X \to G$ be a bounded linear projection with $\phi(\|x\|) = \phi(\|P(x)\|) + \phi(\|x - P(x)\|)$, $\forall\, x \in X$.

Let Q: $\ell^{\phi}(X) \to \ell^{\phi}(G)$ be defined as $Q(f) = Q(\{f(n)\}) = \{P(f(n))\}$. We claim that $\{P(f(n))\} \in \ell^{\phi}(G)$. Clearly; $P(f(n)) \in G$ $\forall\, n$.

Since P is bounded, then there exists a real number k such that $\forall\, x \in X$ we have, $\|Px\| \leq k\|x\|$. And so

$$\|Q(f)\|_\phi = \|\{P(f(n))\}\|_\phi = \sum \phi\|P(f(n))\| \leq \sum \phi(k\|f(n)\|) = k\sum \phi\|f(n)\| = k\|f\|_\phi < \infty$$

Therefore $Q(f) = \{P(f(n))\} \in \ell^\phi(G)$, and Q is a bounded linear projection.

Since $f(n) \in X,\ \forall n$, and since G is a ϕ–summand of X;

$$\phi\|f(n)\| = \phi\|P(f(n))\| + \phi\|(I-P)(f(n))\|,\ \forall n. \qquad (2.25)$$

$$\Rightarrow \sum_{n=1}^{\infty} \phi\|f(n)\| = \sum_{n=1}^{\infty} \phi\|P(f(n))\| + \sum_{n=1}^{\infty} \phi\|(I-P)(f(n))\|. \qquad (2.26)$$

$\Rightarrow \quad \|f\|_\phi = \|Q(f)\|_\phi + \|(I-Q)(f)\|_\phi$; and consequently, Q is the required projection.

Therefore $\ell^\phi(G)$ is a 1–summand of $\ell^\phi(X)$. ■

Corollary 2.3.19: If G is a ϕ–summand of X**,** then $\ell^\phi(G)$ is a Chebyshev subspace in $\ell^\phi(X)$.

Proof: - Let G be a ϕ–summand of X. Theorem 2.3.18 implies $\ell^\phi(G)$ is a 1–summand of $\ell^\phi(X)$. Theorem 2.3.17 implies $\ell^\phi(G)$ is a Chebyshev subspace in $\ell^\phi(X)$. ■

For the general case we have the following theorem:

Theorem 2.3.20: (Deeb [4]). Let G be a ϕ–summand of X, then $L^\phi(\mu, G)$ is 1–summand of $L^\phi(\mu, X)$.

Proof: Let $P: X \rightarrow G$ be the associated projection for G. Let

$\widetilde{P} : L^\phi(\mu, X) \rightarrow L^\phi(\mu, G)$, be defined by $\widetilde{P}(f)(t) = P(f(t))$.

Clearly $\widetilde{P}(f) \in L^\phi(\mu, G)$. Furthermore

$$\phi\|f(t)\| = \phi\|P(f(t))\| + \phi\|(I-P)(f(t))\|. \qquad (2.27)$$

Hence

$$\int_{\Omega}\phi\|f(t)\|d\mu(t) = \int_{\Omega}\phi\|P(f(t))\|d\mu(t) + \int_{\Omega}\phi\|(I-P)(f(t))\|d\mu(t)$$

So, $\|f\|_{\phi} = \left\|\tilde{P}(f)\right\|_{\phi} + \left\|(I-\tilde{P})(f)\right\|_{\phi}$.

Consequently, $\tilde{P}$ is the required projection and so $L^{\phi}(\mu,G)$ is a 1–summand of $L^{\phi}(\mu,X)$. ■

Corollary 2.3.21: If G is a ϕ–summand of X, then $L^{\phi}(\mu,G)$ is a Chebyshev subspace of $L^{\phi}(\mu,X)$.

Proof: Let G be a ϕ–summand of X. Theorem 2.3.20 implies $L^{\phi}(\mu,G)$ is a 1–summand of $L^{\phi}(\mu,X)$. Theorem 2.3.17 implies $L^{\phi}(\mu,G)$ is a Chebyshev subspace $L^{\phi}(\mu,X)$. ■

Definition 2.3.22: (Cheney [8]). If S is a compact Hausdorff space and X is a Banach space; $C(S,X)$ denotes the Banach space of all continuous maps f from S into X with norm defined by $\|f\|_{\infty} = \sup_{\|s\|\le 1}\|f(s)\|$, and we define $\ell_{\infty}(S,X)$ by the set of all bounded maps from S into X with norm defined by $\|f\|_{\infty} = \sup_{\|s\|\le 1}\|f(s)\|$. ■

Theorem 2.3.23: (Cheney [8]). Let G be a closed subspace of a Banach space X. Let S be a compact Hausdorff space. For each $f \in C(S,X)$ we have

$$d(f,C(S,G)) = d(f,\ell_{\infty}(S,G)) = \sup_{s} d(f(s),G). \qquad (2.28)$$

Chapter Three
The S-property

3. Introduction

Let X be a linear metric space and G a closed subspace of X. The space G is said to have the S–property in X if $z_1 \in P(x_1, G)$ and $z_2 \in P(x_2, G)$ imply that $z_1 + z_2 \in P(x_1 + x_2, G)$ $\forall x_1, x_2 \in X$. This chapter has many new results which will be proved. We also give examples that answer some open questions. More results are found in Al–Dwaik's thesis [7].

3.1. The S–property and Best Approximation

Many interesting theorems on the S–property can be found in [7], and here we present new results which describe the relation between the S–property and best approximations. Also between the S–property and $P_G^{-1}(0)$ [7].

In the following example we see a subspace $G \subseteq X$ which has the S–property.

Example 3.1.1 (Al–Dwaik [7]) Let $X = R^3$ and G be the xy–plane, then for a given point $h = (x_0, y_0, z_0)$, the unique best approximation of h in G is $g_0=(x_0, y_0, 0)$ and the distance from h to G is $|z_0|$ (i.e. $d(h,G) = |z_0|$) and since h is arbitrary, then $P(h,G) \neq \varnothing$, $\forall h \in X$

Now assume $g_1 \in P(h_1,G)$ and $g_2 \in P(h_2,G)$ where $h_1 = (x_1, y_1, z_1)$ and $h_2 = (x_2, y_2, z_2)$, then $g_1 = (x_1, y_1, 0)$ and $g_2 = (x_2, y_2, 0)$.

Since $h_1 + h_2 = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ and $g_1 + g_2 = (x_1 + x_2, y_1 + y_2, 0)$. Therefore $g_1 + g_2 \in P(h_1 + h_2, G)$, thus G has the S–property. ■

Remark 3.1.2: In a Banach space *X*, if *G* has the S–property, it does not necessarily follow that *G* is proximinal in *X*. For example, take $X = c_0$(the space of all sequences of scalars converging to zero) and here we use the real sequences with $\|\boldsymbol{x}\| = \sup\limits_{\boldsymbol{n}} |x_n|$ and $G = \{\boldsymbol{x} = \{x_n\} \in c_0 : \sum\limits_{n=1}^{\infty} 2^{-n} x_n = 0\}$. First, we claim that $G \neq \{\boldsymbol{0}\}$, ($\boldsymbol{0}$ is the zero sequence)."This construction is due to Dr. Justin Heavilin who was visiting at An-Najah University in the year 2007/2008". To begin with; choose a real sequence $\boldsymbol{x} \in c_0 \backslash G$ such that $\sum\limits_{n=1}^{\infty} 2^{-n} x_n < \infty$, and assume

$$\alpha = \sum_{n=1}^{\infty} 2^{-n} x_n \neq 0. \tag{3.1}$$

Now consider the sequence $\boldsymbol{y} = \{y_n\}$, with

$$y_n = \begin{cases} -\alpha & , \quad n = 1 \\ x_{n-1} & , \quad n \geq 2. \end{cases}$$

It is clear that $\boldsymbol{y} \neq \boldsymbol{0}$ and we want to show that $\boldsymbol{y} \in G$.

$$\sum_{n=1}^{\infty} 2^{-n} y_n = -\frac{\alpha}{2} + \sum_{n=2}^{\infty} 2^{-n} y_n$$

$$= -\frac{\alpha}{2} + \sum_{m=1}^{\infty} 2^{-m-1} x_m$$

$$= -\frac{\alpha}{2} + \frac{1}{2} \sum_{m=1}^{\infty} 2^{-m} x_m$$

$$= -\frac{\alpha}{2} + \frac{\alpha}{2} = 0 \text{ (from (3.1) we have } \frac{\alpha}{2} = \frac{1}{2} \sum_{m=1}^{\infty} 2^{-m} x_m \text{).}$$

Therefore $\boldsymbol{y} \in G$. Thus $G \neq \{\mathbf{0}\}$ (i.e. G is not trivial).

Now, consider the linear functional $f(\boldsymbol{x}) = \sum_{n=1}^{\infty} 2^{-n} x_n$, such that $f \in c_0{}^*$ and by Remark 1.2.8 we have $\|f\|=1$. It is clear that $N(f) = G = \{\boldsymbol{x} : f(\boldsymbol{x}) = 0\}$, and so by Theorem 1.2.3, we have G is a closed hyperplane in c_0.

Letting $e^{(1)} = (1, 0, 0,\ldots) \in c_0 \backslash G$. Theorem 1.2.7 implies that

$d(\boldsymbol{e}^{(1)}, G) = \frac{1}{2}$. Assume that $\exists \boldsymbol{g} = (g_n) \in G$ satisfies $\|\boldsymbol{e}^{(1)} - \boldsymbol{g}\| = \frac{1}{2}$. Then

$$\|\boldsymbol{e}^{(1)} - \boldsymbol{g}\| = \sup\left\{ \left|e_k^{(1)} - g_k\right| : k = 1,2,3,\ldots\right\}$$

$$= \sup\left\{ |1 - g_1|,\ |g_2|,\ |g_3|,\ldots\right\} = \frac{1}{2}.$$

Therefore $|1 - g_1| \leq \frac{1}{2} \Rightarrow 1 - |g_1| \leq \frac{1}{2} \Rightarrow |g_1| \geq \frac{1}{2}$ and $|g_k| \leq \frac{1}{2}$, $\forall k \geq 2$.

Since $\sum_{n=1}^{\infty} 2^{-n} g_n = 0$, we get that

$\frac{1}{2} g_1 + \sum_{n=2}^{\infty} 2^{-n} g_n = 0 \Rightarrow \sum_{n=2}^{\infty} 2^{-n} g_n = -\frac{1}{2} g_1 \Rightarrow \left|\sum_{n=2}^{\infty} 2^{-n} g_n\right| = \frac{1}{2}|g_1|$, then we have

$$\frac{1}{4} \leq \frac{1}{2}|g_1| = \left|\sum_{n=2}^{\infty} 2^{-n} g_n\right| \leq \sum_{n=2}^{\infty} 2^{-n} |g_n| \leq \sum_{n=2}^{\infty} 2^{-n} \left(\frac{1}{2}\right) = \frac{1}{4}\ . \tag{3.2}$$

$$\Rightarrow |g_1| = \frac{1}{2} \text{ and } \sum_{n=2}^{\infty} 2^{-n} |g_n| = \frac{1}{4}.$$

Since $\sum_{n=2}^{\infty} 2^{-n} |g_n| = \frac{1}{4}$, then we have

$$\frac{1}{4}|g_2| + \sum_{n=3}^{\infty} 2^{-n} |g_n| = \frac{1}{4}$$

$$\sum_{n=3}^{\infty} 2^{-n} |g_n| = \frac{1}{4} - \frac{1}{4}|g_2|$$

$$\Rightarrow \frac{1}{4}(1-|g_2|) \geq \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}.$$

$$\Rightarrow \frac{1}{8} \leq \frac{1}{4}(1-|g_2|) = \sum_{n=3}^{\infty} 2^{-n}|g_n| \leq \frac{1}{2}\sum_{n=3}^{\infty} 2^{-n} = \frac{1}{8}.$$

$$\Rightarrow \frac{1}{4}(1-|g_2|) = \frac{1}{8}.$$

$$\Rightarrow |g_2| = \frac{1}{2}.$$

So we must have equality in (3.2), and that can happen only if $|g_n| = \frac{1}{2}$ for all n. But this contradicts our assumption that $\boldsymbol{g} \in c_0$. Thus G is not proximinal in c_0.

Finally; let $x \in c_0 \backslash G$ and suppose that $P(x,G) \neq \emptyset$, i.e. $\exists z \in P(x,G)$; so $0 \in P(x-z,G)$ by part (2) of Theorem 2.1.3.

This means that there exists $x-z \in c_0 \backslash \{0\}$ such that $0 \in P(x-z,G)$. Theorem 2.1.5 implies that G is proximinal in c_0 which is a contradiction to the above discussion for $\boldsymbol{e}^{(1)}$. Therefore $P(x,G) = \emptyset \;\; \forall x \in c_0 - G$. Hence G has the S–property "vacuously" in c_0. ■

In the following example we see that if X is a Banach space and G is a proximinal subspace, it is not necessarily implied that G has the S–property.

Example 3.1.3: Let $X=\boldsymbol{R}^2$ with $\|x\| = |x_1| + |x_2|$ and let $G = \{(\alpha,\alpha): \alpha \in \boldsymbol{R}\}$. We claim that G is proximinal in X. Let $x = (x_1,x_2) \in X$ and for any $g = (\alpha,\alpha) \in G$, we have

$$\|x-g\| = |x_1-\alpha| + |x_2-\alpha|$$

$$= |x_1-\alpha| + |\alpha - x_2|$$

$$\geq |x_1 - x_2|$$

$\Rightarrow d(x,G) \geq |x_1 - x_2|$ (3.3)

Since $(x_2, x_2) \in G$, then

$$d(x,G) \leq \|x-(x_2,x_2)\|$$

$$= |x_1 - x_2| + |x_2 - x_2| = |x_1 - x_2|.$$

Hence $d(x,G) \leq |x_1 - x_2|$. (3.4)

By (3.3) and (3.4) we have

$$d(x,G) = |x_1 - x_2|, \ \forall x = (x_1, x_2) \in X.$$

Therefore $P(x,G) \neq \emptyset$, $\forall x \in X$. Thus G is proximinal in X.

Now let $x = (1,-1)$ and $y = -x = (-1, 1)$ in $X \setminus G$.

It is clear $(1,1), (-1,-1) \in P(x,G)$. Part (1) of Theorem 2.1.3 implies

$(1,1), (-1,-1) \in P(y,G)$.

Now take $(1,1) \in P(x,G)$ and $(1,1) \in P(y,G)$, so that

$(1,1) + (1,1) = (2,2)$ and $x + y = x + (-x) = 0$. But $(2,2) \notin P(x+y,G)$ i.e. $(2,2) \notin P(0,G) = \{0\}$. Therefore G does not have the S– property . ■

Theorem 3.1.4: Let X be a normed linear space, then any closed subspace G with the S–property is a semi–Chebyshev.

Proof: Let $x \in X \setminus G$ and $z_1, z_2 \in P(x,G)$, then $-z_1, -z_2 \in P(-x,G)$ by part (1) of Theorem 2.1.3.

Since G has the S–property and $z_1 \in P(x,G)$, $-z_2 \in P(-x,G)$, then

$$z_1 + (-z_2) \in P(x + (-x), G) \Rightarrow z_1 - z_2 \in P(0,G)$$

But $P(0,G) = \{0\}$, since $0 \in G \Rightarrow z_1 - z_2 = 0 \Rightarrow z_1 = z_2$.

Therefore G is a semi–Chebyshev subspace of X. ■

We know that a Chebyshev subspace is a special case of a semi–Chebyshev subspace and so we have the following corollary.

Corollary 3.1.5: Let X be a normed linear space and G be a closed subspace of X and G has the S–property. If G is proximinal, then G is a Chebyshev subspace.

Proof: Let G be a closed subspace which is proximinal and has the S–property in X. Theorem 3.1.4 implies G is proximinal and semi–Chebyshev, then G is a Chebyshev subspace. ■

We need the following theorem from Al–Dwaik's thesis to show that G is a closed subspace of a Banach space X which has the S–property if and only if $P_G^{-1}(0)$ is a subspace of X.

Theorem 3.1.6: (Al–Dwaik [7]). Let X be a Banach space, and G a closed subspace of X which has the S–property, then $P_G^{-1}(0)$ is a closed subspace of X and $P_G^{-1}(0) \bigcap G = \{0\}$.

Proof: Let $x_1, x_2 \in P_G^{-1}(0)$, so; $0 \in P(x_1, G)$ and $0 \in P(x_2, G)$. Since G has the S–property we get $0 \in P(x_1 + x_2, G)$. Hence

$$x_1 + x_2 \in P_G^{-1}(0). \tag{3.5}$$

Let $x \in P_G^{-1}(0)$ and α be any scalar. Then

$$d(\alpha x, G) = |\alpha|\, d(x,G) = |\alpha|\, \|x\| = \|\alpha x\| \Rightarrow 0 \in P(\alpha x, G).$$

$$\Rightarrow \alpha x \in P_G^{-1}(0). \tag{3.6}$$

By (3.5) and (3.6) $P_G^{-1}(0)$ is a subspace of X.

Now let (x_n) be a sequence in $P_G^{-1}(0)$ and $x \in X$ such that $\lim_{n\to\infty} x_n = x$. By part (5) of Theorem 1.1.10 we have $d(x,G) \le \|x\|$.

Given $\varepsilon > 0$. There exists a natural number $N(\varepsilon)$ such that $\|x_n - x\| < \varepsilon$ for all $n > N(\varepsilon)$. Fixing $n > N(\varepsilon)$ we have :

$$\begin{aligned}
\|x\| = \|x - x_n + x_n\| \le \|x - x_n\| + \|x_n\| \\
&< \varepsilon + \|x_n\| \\
&\le \varepsilon + \|x_n - g\| \\
&\le \varepsilon + \|x_n - x\| + \|x - g\| \\
&\le 2\varepsilon + \|x - g\|, \ \forall g \in G.
\end{aligned}$$

Since ε > 0 was arbitrary, then $\|x\| \le \|x-g\|\ \forall g \in G$; and so by the definition of best approximation we have $0 \in P(x,G)$, hence $x \in P_G^{-1}(0)$.

Thus $P_G^{-1}(0)$ is a closed subspace in *X*.

Let $g \in G \bigcap P_G^{-1}(0) \Rightarrow g \in P_G^{-1}(0)$ and $g \in G$

$\Rightarrow g \in G$ and $\|g-0\| = d(g,G) = 0 \Rightarrow g = 0$

Therefore $G \bigcap P_G^{-1}(0) = \{0\}$. ■

Now, we will prove the converse of Theorem 3.1.6 for normed linear spaces.

Theorem 3.1.7: Let *X* be a normed linear space. If $P_G^{-1}(0)$ is a closed subspace of *X*, then *G* has the S–property in *X*.

Proof: Let $x, y \in X$ and $g_1 \in P(x,G)$, $g_2 \in P(y,G)$, then $x - g_1 \in P_G^{-1}(0)$ and $y - g_2 \in P_G^{-1}(0)$. Since $P_G^{-1}(0)$ is a closed subspace of *X* and $x - g_1 \in P_G^{-1}(0)$ $y - g_2 \in P_G^{-1}(0)$, then we have

$$x + y - g_1 - g_2 \in P_G^{-1}(0) \Rightarrow 0 \in P(x + y - g_1 - g_2, G) \Rightarrow g_1 + g_2 \in P(x + y, G).$$

Therefore *G* has the S–property. ■

The following theorem can be proved immediately by the S–property, but here we will provide another proof.

Theorem 3.1.8: Let *X* be a normed linear space. If $P_G^{-1}(0)$ is a linear subspace of *X*, then *G* is semi–Chebyshev.

Proof: Let $x\in X$ and $g_1, g_2\in P(x,G)$, then $x-g_1\in P_G^{-1}(0)$ and $x-g_2\in P_G^{-1}(0)$.

Since $P_G^{-1}(0)$ is a closed subspace of *X*, then

$x-g_1-(x-g_2)\in P_G^{-1}(0) \Rightarrow g_1-g_2\in P_G^{-1}(0)$ and since $g_1,g_2\in G$, then

$g_1-g_2\in P_G^{-1}(0)\cap G \quad \Rightarrow g_1-g_2=0$

$$\Rightarrow \; g_1=g_2, \quad \text{by Remark 1.1.6.}$$

Hence *G* is a semi–Chebyshev subspace of *X*. ■

In chapter two we see that if *G* is 1–complemented, then *G* is proximinal subspace, Al–Dwaik proved that the converse is true if G has the S–property.

Theorem 3.1.9: (Al–Dwaik [7]). Let *X* be any Banach space, and *G* a closed subspace of *X* which has the S–property. *G* is proximinal in *X* if and only if *G* is 1–complemented in *X*.

Proof: If *G* is 1–complemented in *X*, then by Lemma 2.2.2 it is proximinal in *X*. Now suppose that *G* is proximinal in *X*. Theorem 2.1.6 implies $X=G+P_G^{-1}(0)$. Theorem 3.1.6 implies that $P_G^{-1}(0)$ is a closed subspace of *X* and $P_G^{-1}(0)\cap G=\{0\}$. Hence $X=G\oplus P_G^{-1}(0)$.

Now define $P\colon X\to P_G^{-1}(0)$ by

$P(x)=P(g+z)=z$ where $x=g+z,\ g\in G,\ z\in P_G^{-1}(0)$.

$$\|x\| \geq d(x,G) = d(g + z, G)$$

$$= d(z, G)$$

$$= \|z\|.$$

Therefore $\|x\| \geq \|z\|$.

Hence P is a contractive projection. Thus G is 1–complemented in X. ■

Corollary 3.1.10: Let X be a Banach space, and G be a closed subspace of X which has the S–property. G is a Chebyshev subspace in X if and only if it is 1–complemented in X.

Proof: Suppose G has the S–property**,** then G is Chebyshev if and only if G is proximinal in X by Corollary 3.1.5. Theorem 3.1.9 implies G is Chebyshev if and only if it is 1–complemented in X. ■

Theorem 3.1.11: Let X be a Banach space and G be a closed subspace of X. If G is an L^p–summand subspace of X, $1 \leq p < \infty$, then G has the S–property**.**

Proof: Let $x_1, x_2 \in X$ and G be an L^p–summand subspace, then there exists a bounded projection E: $X \rightarrow G$ which is onto G. By Theorem 2.2.5 we have $P(x_1,G) = \{E(x_1)\}$ and $P(x_2,G) = \{E(x_2)\}$. $(x_1, x_2 \in X)$.

Since $P(x_1 + x_2, G) = \{E(x_1 + x_2)\} = \{E(x_1) + E(x_2)\}$. Therefore G has the S–property**.** ■

Corollary 3.1.12: Let G be a closed subspace of a Hilbert space X, then G has the S–property.

Proof: Let X be a Hilbert space and G be a closed subspace. Theorem 1.4.4 implies $X = G \oplus G^{\perp}$ such that $G^{\perp} = \{x \in X: x \perp G\}$. Theorem 1.5.3 implies there exists a bounded projection $E: X \to G$. Theorem 1.4.5 implies that for all $x \in X$, we have

$$\|x\|^2 = \|E(x)\|^2 + \|x - E(x)\|^2 \quad \forall x \in X.$$

By the definition of L^p–summand subspace, G is an L^2–summand subspace of X, and consequently Theorem 3.1.11 implies G has the S–property. ■

Theorem 3.1.13: (Al–Dwaik [7]). Let X be a Banach space and G a closed subspace of X. If G is a ϕ–summand of X, then G has the S–property.

Proof: Let $z_1 \in P(x_1, G)$, $z_2 \in P(x_2, G)$. Since G is a ϕ–summand of X then there exists a projection $E: X \to G$ such that $E(x)$ is a unique best approximant of x in G $\forall x \in X$ by Theorem 2.3.16 and Remark 2.3.17. Hence

$z_1 = E(x_1)$ and $z_2 = E(x_2)$. But $z_1 + z_2 = E(x_1) + E(x_2) = E(x_1 + x_2)$ since E is linear.

This implies that $z_1 + z_2 \in P(x_1 + x_2, G)$. Thus G has the S–property. ■

Theorem 3.1.14: (Al–Dwaik [7]). Let X be a Banach space and G a closed subspace of X. If G is 1–complemented and Chebyshev in X, then G has the S–property.

Proof: Let $z_1 \in P(x_1,G)$, $z_2 \in P(x_2,G)$. Since G is 1–complemented in X then there exists a closed subspace $W \subset X$ such that $X = G \oplus W$. This implies that x_1 and x_2 can be written uniquely in the form

$$x_1 = g_1 + w_1 \,,\; x_2 = g_2 + w_2$$

where $g_1, g_2 \in G$, and $w_1, w_2 \in W$. From the proof of Lemma 2.2.2 and the assumption that G is Chebyshev we get that $z_1 = g_1$, $z_2 = g_2$.

Now $x_1 + x_2 = (g_1 + g_2) + (w_1 + w_2)$. Since G is a subspace, $g_1 + g_2 \in G$. Also W is a subspace, $w_1 + w_2 \in W$. It now follows that

$$z_1 + z_2 = g_1 + g_2 \in P(x_1 + x_2, G).$$

Thus G has the S–property. ■

Theorem 3.1.15: (Al–Dwaik [7]). Let X be a Banach space and G a closed subspace of X which has the S–property. If G is proximinal in X then $P_G^{-1}(0)$ is proximinal in X and has the S–property.

Proof: Let $x \in X$. The proof of Theorem 3.1.9 implies that x can be written uniquely, in the form

$$x = g + z \quad g \in G,\, z \in P_G^{-1}(0) \tag{3.7}$$

Now $g \in G \Rightarrow g \perp w \;\; \forall w \in P_G^{-1}(0) \Rightarrow g \perp P_G^{-1}(0) \Rightarrow 0 \in P(g, P_G^{-1}(0))$

$$\Rightarrow d(g, P_G^{-1}(0)) = \|g\|. \tag{3.8}$$

From (3.7) and (3.8) we get

$d(x-z, P_G^{-1}(0)) = \|x-z\| \Rightarrow d(x, P_G^{-1}(0)) = \|x-z\|$.

Therefore $z \in P(x, P_G^{-1}(0))$ i.e. $z = x - g$ where $g \in P(x,G)$ (3.9)

Thus, $P_G^{-1}(0)$ is proximinal in X. To show, $P_G^{-1}(0)$ has the S−property, let $z_1 \in P(x_1, P_G^{-1}(0))$ and $z_2 \in P(x_2, P_G^{-1}(0))$.From (3.9) we get

$x_1 - z_1 \in P(x_1, G)$ and $x_2 - z_2 \in P(x_2, G)$.

Since G has the S−property, then

$$x_1 + x_2 - (z_1 + z_2) \in P(x_1 + x_2, G) \Rightarrow z_1 + z_2 \in P(x_1 + x_2, P_G^{-1}(0)).$$

Thus $P_G^{-1}(0)$ has the S−property. ■

Now we have the following three corollaries.

Corollary 3.1.16: Let X be a Banach space and G a closed subspace of X which has the S−property. If G is a Chebyshev subspace, then $P_G^{-1}(0)$ is Chebyshev and has the S−property.

Proof: Let G have the S−property. Theorem 3.1.15 implies $P_G^{-1}(0)$ is proximinal and has the S−property. Corollary 3.1.5 implies $P_G^{-1}(0)$ is a Chebyshev subspace of X and has the S−property. ■

Corollary 3.1.17: Let X be a Banach space and G a closed subspace of X and Chebyshev. If $P_G^{-1}(0)$ is a closed subspace of X, then $P_G^{-1}(0)$ is a Chebyshev subspace of X.

Proof: Let $P_G^{-1}(0)$ be a closed subspace of X and G is Chebyshev. Theorem 3.1.7 implies G has the S−property and Chebyshev. Corollary 3.1.16 implies $P_G^{-1}(0)$ is a Chebyshev subspace . ■

Corollary 3.1.18: Let X be a Banach space and G be a $\phi-$summand of X, then $P_G^{-1}(0)$ has the S−property and is a Chebyshev subspace.

Proof: Let G be a $\phi-$summand of X. Theorem 3.1.13 implies that G has the S−property and Chebyshev. Corollary 3.1.16 implies $P_G^{-1}(0)$ has the S−property and is a Chebyshev subspace. ■

In the following example we show that the converse of Theorem 3.1.15 is not true.

Example 3.1.19: Let $X = c_0$ and $G = \{(x_n) : \sum_{n=1}^{\infty} 2^{-n} x_n = 0\}$, with $\|x\| = \sup_n |x_n|$. Remark 3.1.2 shows that G is not proximinal in X and has the S−property and that $P(x,G) =\emptyset$, $\forall x \in X \setminus G$; so $0 \notin P(x,G), \forall x \in X \setminus G$ and hence $P_G^{-1}(0)$={0}. Clearly $P_G^{-1}(0)$ is a closed linear subspace of X and Chebyshev (being proximinal with the S−property). Moreover,

$$d(x,P_G^{-1}(0))=\|x-0\| = \|x\| \ \ \forall x \in X \Rightarrow P(x, P_G^{-1}(0))=\{0\}.$$

Now let $x, y \in X$, then $P(x, P_G^{-1}(0))=\{0\}$, $P(y, P_G^{-1}(0))=\{0\}$ and for $x + y \in X$ $P(x + y, P_G^{-1}(0)) = \{0\} \Rightarrow P_G^{-1}(0)$ has the S−property.

Therefore G has the S−property and $P_G^{-1}(0)$ has the S−property and is Chebyshev, but G is not proximinal. ■

Theorem 3.1.20: (Al–Dwaik [7]). Let X be a Banach space and G be proximinal in X. If G has the S–property, then $P^{-1}_{P_G^{-1}(0)}(0) = G$.

Proof : Let $g \in G \Rightarrow z \perp g \ \ \forall\ z \in P_G^{-1}(0) \Rightarrow g \perp P_G^{-1}(0) \Rightarrow 0 \in P(g, P_G^{-1}(0))$ $\Rightarrow g \in P^{-1}_{P_G^{-1}(0)}(0)$.

Therefore

$$G \subset P^{-1}_{P_G^{-1}(0)}(0) \tag{3.10}$$

Now let $x \in P^{-1}_{P_G^{-1}(0)}(0)$. Then by the proof of Theorem 3.1.9 we have

$$x = x_1 + x_2 \qquad \text{where } x_1 \in G \text{ and } x_2 \in P_G^{-1}(0). \tag{3.11}$$

Since $G \subset P^{-1}_{P_G^{-1}(0)}(0)$, $x_1 \in P^{-1}_{P_G^{-1}(0)}(0)$, then $x_2 = x - x_1 \in P^{-1}_{P_G^{-1}(0)}(0)$.

But $x_2 \in P_G^{-1}(0)$. Theorem 3.1.9 implies $x_2 = x - x_1 = 0$.

$$\Rightarrow x = x_1 \Rightarrow x \in G$$

Therefore

$$P^{-1}_{P_G^{-1}(0)}(0) \subset G \tag{3.12}$$

From (3.10) and (3.12) we have $P^{-1}_{P_G^{-1}(0)}(0) = G$. ■

Theorem 3.1.21: (Al–Dwaik [7]). If G is a semi–Chebyshev hyperplane in a Banach space X passing through the origin, then G has the S–property.

Proof: case (1): G is proximinal in X. Let $f \in X^*$ so that

$G = \{y \in X: f(y) = 0\}$. Fix an arbitrary $z \in X \backslash G$ so $f(z) \neq 0$, and let

$y_0 = x - z\,\dfrac{f(x)}{f(z)}$ where $x \in X$, so $f(y_0) = 0$, hence $y_0 \in G$. Consequently

$$X = G \oplus [z] \text{ where } [z] = \{\alpha z : \alpha \text{ scalar}\}. \tag{3.13}$$

Now let $z_1 \in P(x_1, G)$, $z_2 \in P(x_2, G)$. It will be shown that

$$z_1 + z_2 \in P(x_1 + x_2, G). \tag{3.14}$$

By (3.13) every $x_1, x_2 \in X$ can be written uniquely in the form

$$x_1 = g_1 + \alpha_1 z, \quad x_2 = g_2 + \alpha_2 z \tag{3.15}$$

where $g_1, g_2 \in G$ and α_1, α_2 are scalars.

Now assume that $g' \in P(x_1 + x_2, G)$, then by (3.15)

$g' \in P(g_1 + g_2 + (\alpha_1 + \alpha_2)z, G)$. Theorem 2.1.3 implies

$g' = g_1 + g_2 + (\alpha_1 + \alpha_2)w = g_1 + \alpha_1 w + g_2 + \alpha_2 w$ where $w \in P(z, G)$.

Since $w \in P(z, G)$, Theorem 2.1.3 implies that

$g_1 + \alpha_1 w \in P(g_1 + \alpha_1 z, G) = P(x_1, G)$ and $g_2 + \alpha_2 w \in P(g_2 + \alpha_2 z, G) = P(x_2, G)$.

Hence $g_1 + \alpha_1 w = z_1$, $g_2 + \alpha_2 w = z_2$ and $g' = z_1 + z_2$.

Therefore $z_1 + z_2 \in P(x_1 + x_2, G)$. Thus G has the S−property.

Case (2): If G is not proximinal in X, then theorem 2.1.5 implies $P(x, G) = \emptyset$ $\forall x \in X - G$. Thus G has the S−property. ■

Let X be a normed linear space, and G be proximinal in X, then any map which associates with each element of X one of its best approximations in G is called a proximity map. This mapping is, in general, nonlinear.

Theorem 3.1.22: (Al–Dwaik [7]). Let X be a Banach space, and G be a Chebyshev subspace of X. There exists a linear proximity map if and only if G has the S–property.

Proof: Let T be a linear proximity map. We claim that G has the S–property.

Let $z_1 \in P(x_1,G)$ and $z_2 \in P(x_2,G)$.

Now $z_1 + z_2 = T(x_1) + T(x_2) = T(x_1+x_2) \in P(x_1+x_2,G)$. Therefore G has the S–property.

Conversely, assume that G has the S–property.

Define T: $X \to G$ such that $T(x) \in P(x,G)$. Now we claim that T is linear.

Let $x_1, x_2 \in X$, we show that $T(x_1+x_2) = T(x_1) + T(x_2)$.

Now $T(x_1) \in P(x_1,G)$, $T(x_2) \in P(x_2,G)$. Since G has the S–property, then $T(x_1) + T(x_2) \in P(x_1+x_2,G)$.

Also $T(x_1+x_2) \in P(x_1+x_2,G)$ and Since G is Chebyshev, then

$$T(x_1+x_2) = T(x_1) + T(x_2) \tag{3.16}$$

Let $x \in X$, α scalar then $T(x) \in P(x,G)$. Theorem 2.1.3 implies that

$$\alpha T(x) \in P(\alpha x,G), \text{ also } T(\alpha x) \in P(\alpha x,G).$$

Since G is a Chebyshev subspace of X, then

$$\alpha T(x) = T(\alpha x) \tag{3.17}$$

By (3.16) and (3.17) we have T is a linear map. ■

3.2 The S−property of Subspaces of Orlicz Spaces and $L^p(\mu, X)$

In this section we have many new consequences about the Orlicz subspaces and $L^p(\mu, X)$ subspaces which have the S−property. First, we need the following theorem from Al−Dwaik [7].

Theorem 3.2.1: (Al−Dwaik [7]). Let X be a Banach space and G be a closed subspace of X. If G has the S−property in X, then $L^{\phi}(\mu,G)$ has the S−property in $L^{\phi}(\mu,X)$.

Proof: Let $g_1 \in P(f_1, L^{\phi}(\mu,G))$ and $g_2 \in P(f_2, L^{\phi}(\mu,G))$, we will show that

$g_1 + g_2 \in P(f_1 + f_2, L^{\phi}(\mu,G))$. Now $g_1 \in P(f_1, L^{\phi}(\mu,G))$, Theorem 2.3.3 implies $g_1(t) \in P(f_1(t),G)$ a.e $t \in \Omega$. Also $g_2 \in P(f_2, L^{\phi}(\mu,G))$, Theorem 2.3.3 implies $g_2(t) \in P(f_2(t),G)$ a.e. $t \in \Omega$.

Hence

$$d((f_1 + f_2)(t),G) = \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \text{ a.e. } t. \tag{3.18}$$

And we have

$$\|(f_1+f_2)(t)-(g_1+g_2)(t)\| \le \|(f_1+f_2)(t)-y\| \text{ a.e. } t \text{ and } \forall\, y \in G. \qquad (3.19)$$

In particular

$$\|(f_1+f_2)(t)-(g_1+g_2)(t)\| \le \|(f_1+f_2)(t)-h(t)\| \text{ a.e. } t \text{ and } \forall\, h \in L^{\phi}(\mu,G).$$

Since ϕ is increasing, then

$$\phi(\|(f_1+f_2)(t)-(g_1+g_2)(t)\|) \le \phi(\|(f_1+f_2)(t)-h(t)\|) \text{ a.e. } t, \forall\, h \in L^{\phi}(\mu,G).$$

Integrating both sides we get

$$\|(f_1+f_2)-(g_1+g_2)\|_{\phi} \le \|(f_1+f_2)-h\|_{\phi} \quad \forall\, h \in L^{\phi}(\mu,G). \qquad (3.20)$$

Hence $d(f_1+f_2,G) = \|(f_1+f_2)-(g_1+g_2)\|_{\phi}$.

Therefore $g_1+g_2 \in P(f_1+f_2, L^{\phi}(\mu,G))$. Thus $L^{\phi}(\mu,G)$ has the S–property in $L^{\phi}(\mu,X)$. ■

Now, we will present our results on Orlicz subspaces and $L^{p}(\mu,G)$ with the S–property, and will start with the converse of the last theorem.

Theorem 3.2.2: Let X be a Banach space and G be a closed subspace of X. If $L^{\phi}(\mu,G)$ has the S–property in $L^{\phi}(\mu,X)$, then G has the S–property in X.

Proof: Suppose $L^{\phi}(\mu,G)$ has the S–property in $L^{\phi}(\mu,X)$, and let $z_i \in P(x_i,G)$ for i=1, 2; we want to show $z_1+z_2 \in P(x_1+x_2,G)$.

Now let $f_i(t) = x_i$ and $g_i(t) = z_i, \forall t \in \Omega$ and for i=1,2; and since $\|x\| < \infty$, $\forall x \in X$ (by definition of the norm), then $f_1, f_2, g_1, g_2 \in L^1(\mu, X)$, Theorem 2.3.2 implies $f_1, f_2, g_1, g_2 \in L^\phi(\mu, X)$ such that $g_i \in L^\phi(\mu, G)$ i=1, 2.

First, we show that $g_i \in P(f_i, L^\phi(\mu, G))$ (i = 1, 2).

Now for i =1, 2, we have

$$z_i \in P(x_i, G) \Rightarrow \|x_i - z_i\| \le \|x_i - y\| \qquad \forall y \in G.$$

$$\Rightarrow \|f_i(t) - g_i(t)\| \le \|f_i(t) - y\| \qquad \forall y \in G \text{ and } \forall\ t \in \Omega.$$

$$\Rightarrow \|f_i(t) - g_i(t)\| \le \|f_i(t) - h(t)\| \quad \forall\ t \in \Omega \text{ and } \forall h \in L^\phi(\mu, G).$$

Since ϕ is strictly increasing, then we have

$$\phi(\|f_i(t) - g_i(t)\|) \le \phi(\|f_i(t) - h(t)\|) \ \forall\ t \in \Omega \text{ and } \forall h \in L^\phi(\mu, G). \tag{3.21}$$

$$\Rightarrow \|f_i - g_i\|_\phi \le \|f_i - h\|_\phi \ \forall h \in L^\phi(\mu, G) \Rightarrow g_i \in P(f_i, L^\phi(\mu, G))\ i = 1, 2.$$

Since $L^\phi(\mu, G)$ has the S–property in $L^\phi(\mu, X)$,

$$g_1 + g_2 \in P(f_1 + f_2, L^\phi(\mu, G)) \tag{3.22}$$

Theorem 2.3.3 implies

$$(g_1 + g_2)(t) \in P((f_1 + f_2)(t), L^\phi(\mu, G))) \qquad \text{a.e. } t. \tag{3.23}$$

Then $z_1 + z_2 \in P(x_1 + x_2, G)$. Therefore G has the S–property in X. ■

This section has many equivalent relations, and the following corollary is the first.

Corollary 3.2.3: Let X be a Banach space and G a closed subspace of X then the following are equivalent:

1. G has the S–property in X.

2. $L^{\phi}(\mu, G)$ has the S–property in $L^{\phi}(\mu, X)$.

Proof: This corollary follows From Theorem 3.2.1 and Theorem 3.2.2 immediately. ■

Moreover, some important results on $L^{p}(\mu, X)$ subspaces with the S–property will now follow.

First, if $p = 1$.

Theorem 3.2.4: Let X be a Banach space and G a closed subspace of X, then the following are equivalent:

1. G has the S–property in X.

2. $L^{1}(\mu, G)$ has the S–property in $L^{1}(\mu, X)$.

Proof: $(1) \rightarrow (2)$. Suppose (1) and let $f_i \in L^{1}(\mu, X)$ and $g_i \in P(f_i, L^{1}(\mu, G))$ such that i=1, 2.We want to show that $g_1 + g_2 \in P(f_1 + f_2, L^{1}(\mu, G))$.

Now if $g_1 \in P(f_1, L^{1}(\mu, G))$. Corollary 2.3.4 implies

$$g_1(t) \in P(f_1(t), G) \quad \text{a.e.} \quad t \in \Omega \tag{3.24}$$

Also $g_2 \in P(f_2, L^{1}(\mu, G))$. Corollary 2.3.4 implies

$$g_2(t) \in P(f_2(t), G) \quad \text{a.e.} \quad t \in \Omega \tag{3.25}$$

Since G has the S–property, from (3.24) and (3.25) we have

$$(g_1 + g_2)(t) \in P((f_1 + f_2)(t), G) \quad \text{a.e.} \quad t \in \Omega. \tag{3.26}$$

Hence

$$d((f_1 + f_2)(t), G) = \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \quad \text{a.e.} \quad t \in \Omega. \tag{3.27}$$

Then we have

$$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \le \|(f_1 + f_2)(t) - y\| \quad \forall y \in G, \text{ a.e.} \quad t \in \Omega.$$

Consequently, we have

$$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \le \|(f_1 + f_2)(t) - h(t)\|, \text{ a.e. } t \in \Omega, \forall h \in L^1(\mu, G).$$

$$\Rightarrow \|(f_1 + f_2) - (g_1 + g_2)\|_1 \le \|(f_1 + f_2) - h\|_1 \quad \forall h \in L^1(\mu, G). \tag{3.28}$$

Therefore $g_1 + g_2 \in P(f_1 + f_2, L^1(\mu, G))$. Thus $L^1(\mu, G)$ has the S–property in $L^1(\mu, X)$.

(2)$\to$(1). Suppose (2) and let $x_1, x_2 \in X$ and $z_1 \in P(x_1, G)$, $z_2 \in P(x_2, G)$. We want to show that $z_1 + z_2 \in P(x_1 + x_2, G)$.

Consider the constant functions f_1, f_2, g_1, g_2 defined as follows $f_1(t) = x_1$, $f_2(t) = x_2$, $g_1(t) = z_1$, $g_2(t) = z_2$ $\forall t \in \Omega$. Clearly

$f_1, f_2 \in L^1(\mu, X)$ and $g_1, g_2 \in L^1(\mu, G)$.

First we show that $g_i \in P(f_i, L^1(\mu, G))$, $i = 1, 2$.

Now for $i = 1, 2$ we have

$$\| f_i(t) - g_i(t)\| = \|x_i - z_i\| \qquad \forall\, t \in \Omega$$

$$\leq \|x_i - y\| \qquad \forall y \in G$$

$$= \| f_i(t) - y\| \qquad \forall y \in G \text{ and } \forall\, t \in \Omega .$$

Then $\forall\ h \in L^1(\mu,G)$ and i =1, 2, we have

$$\| f_i(t) - g_i(t)\| \leq \| f_i(t) - h(t)\|,\ \forall\, t \in \Omega\ . \tag{3.29}$$

$$\Rightarrow \| f_i - g_i\|_1 \leq \| f_i - h\|_1,\ \forall\ h \in L^1(\mu,G) \text{ and } i = 1, 2. \tag{3.30}$$

Thus $g_i \in P(f_i, L^1(\mu,G)),\ i$=1, 2.

Since $L^1(\mu,G)$ has the S−property in $L^1(\mu, X),$ then

$$g_1 + g_2 \in P(f_1 + f_2, L^1(\mu,G)). \tag{3.31}$$

By Corollary 2.3.4, we have $(g_1 + g_2)\ (t) \in P((f_1 + f_2)(t),G) \quad \forall\, t \in \Omega .$

Then $z_1 + z_2 \in P(x_1 + x_2 ,G)$. Therefore G has the S−property in X. ■

Second, if $1 < p < \infty$

Theorem 3.2.5: Let X be a Banach space and G be a closed subspace of X then the following are equivalent:

(i) $L^p(\mu ,G)$ has the S−property in $L^p(\mu ,X)$, $1 < p < \infty$.

(ii) G has the S−property in X.

Proof: (i)$\rightarrow$(ii). Let $x_i \in X$ and $z_i \in P(x_i ,G)$ for i =1, 2. We want to show that $z_1 + z_2 \in P(x_1 + x_2 ,G)$.

Consider the constant functions $f_i(t) = x_i$ and $g_i(t) = z_i$, for $i = 1, 2$ and $\forall t \in \Omega$. Clearly $f_i \in L^p(\mu, X)$, $1 < p < \infty$, and $g_i \in L^p(\mu, G)$ for i=1, 2.

We claim that $g_i \in P(f_i, L^p(\mu, G))$ for i =1, 2.

$$\|f_i - g_i\|_p^p = \int_\Omega \|f_i(t) - g_i(t)\|^p d\mu(t)$$

$$= \int_\Omega \|x_i - z_i\|^p d\mu(t)$$

$$\le \int_\Omega \|x_i - y\|^p d\mu(t),\ \forall y \in G \text{ because } z_i \in P(x_i, G).$$

And so for all $h \in L^p(\mu, G)$ and i=1, 2, we get

$$\|f_i - g_i\|_p^p \le \int_\Omega \|x_i - h(t)\|^p d\mu(t)$$

$$= \int_\Omega \|f_i(t) - h(t)\|^p d\mu(t)$$

$$= \|f_i - h\|_p^p.$$

Then, for all $h \in L^p(\mu, G)$, we have $\|f_i - g_i\|_p \le \|f_i - h\|_p$, i =1, 2.

Hence $g_i \in P(f_i, L^p(\mu, G))$, i =1, 2.

Since $L^p(\mu, G)$ has the S–property in $L^p(\mu, X)$ $1 < p < \infty$, then

$$g_1 + g_2 \in P(f_1 + f_2, L^p(\mu, G))$$

Thus for all $h \in L^p(\mu, G)$, we have

$$\|f_1 + f_2 - (g_1 + g_2)\|_p \le \|f_1 + f_2 - h\|_p$$

And $\|f_1 + f_2 - (g_1 + g_2)\|_p^p \le \|f_1 + f_2 - h\|_p^p \quad (1 < p < \infty)$.

Now we have

$$\|f_1 + f_2 - (g_1 + g_2)\|_p^p = \int_\Omega \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|^p d\mu(t)$$

$$= \int_\Omega \|(x_1 + x_2) - (z_1 + z_2)\|^p d\mu(t)$$

$$= \|(x_1 + x_2) - (z_1 + z_2)\|^p \ \mu(\Omega) \tag{3.32}$$

$$\|f_1 + f_2 - h\|_p^p = \int_\Omega \|(f_1 + f_2)(t) - h(t)\|^p d\mu(t)$$

$$= \int_\Omega \|(x_1 + x_2) - h(t)\|^p d\mu(t). \tag{3.33}$$

From (3.32) and (3.33), we have

$$\|(x_1 + x_2) - (z_1 + z_2)\|^p \ \mu(\Omega) \ \le \int_\Omega \|(x_1 + x_2) - h(t)\|^p d\mu(t), \ \forall h \in L^p(\mu, G).$$

In particular, for $y \in G$, let $h_y(t) = y, \ \forall \ t \in \Omega$ be a constant function, and clearly $h_y \in L^p(\mu, G)$, and so we have

$$\|(x_1 + x_2) - (z_1 + z_2)\|^p \ \mu(\Omega) \ \le \int_\Omega \|(x_1 + x_2)(t) - y\|^p d\mu(t)$$

$$= \|(x_1 + x_2) - y\|^p \ \mu(\Omega).$$

Since (μ, Ω) is a finite measure space (i.e. $\mu(\Omega) < \infty$) and assume $\mu(\Omega) > 0$, then

$$\|(x_1 + x_2) - (z_1 + z_2)\|^p \le \|(x_1 + x_2) - y\|^p. \tag{3.34}$$

Since $y \in G$ was arbitrary,

$$\|(x_1 + x_2) - (z_1 + z_2)\| \le \|(x_1 + x_2) - y\|, \ \forall \ y \in G. \tag{3.35}$$

Hence $z_1 + z_2 \in P(x_1 + x_2, G)$. Therefore G has the S−property in X.

Conversely. Let $f_i \in L^p(\mu, X)$ and $g_i \in P(f_i, L^p(\mu, G))$ for $i = 1, 2$ and $1 < p < \infty$. Then for any $h \in L^p(\mu, G)$ we have $\|f_i - g_i\|_p \leq \|f_i - h\|_p$.

Using the same arguments as in Lemma 2.10 of Light and Cheney [8] we get

$$\|f_i(t) - g_i(t)\| \leq \|f_i(t) - y\| \quad \text{a.e.} \quad t, \forall y \in G \text{ and } i = 1, 2. \tag{3.36}$$

Then we have $g_i(t) \in P(f_i(t), G)$ a.e. t, and for $i = 1, 2$.

Since G has the S−property in X, then

$$(g_1 + g_2)(t) \in P((f_1 + f_2)(t), G) \quad \text{a.e.} \quad t. \tag{3.37}$$

Hence, for all $y \in G$, we have

$$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - y\| \quad \text{a.e.} \quad t. \tag{3.38}$$

Hence $\forall h \in L^p(\mu, G)$ we have

$$\|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \leq \|(f_1 + f_2)(t) - h(t)\| \quad \text{a.e.} \quad t.$$

$$\Rightarrow \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\|^p \leq \|(f_1 + f_2)(t) - h(t)\|^p \quad \text{a.e.} \quad t, 1 < p < \infty.$$

$$\Rightarrow \|f_1 + f_2 - (g_1 + g_2)\|_p^p \leq \|f_1 + f_2 - h\|_p^p$$

$$\Rightarrow \|f_1 + f_2 - (g_1 + g_2)\|_p \leq \|f_1 + f_2 - h\|_p \quad \forall h \in L^p(\mu, G).$$

Hence $g_1 + g_2 \in P(f_1 + f_2, L^p(\mu, G))$, $1 < p < \infty$. Therefore $L^p(\mu, G)$ has the S−property in $L^p(\mu, X)$, $1 < p < \infty$. ■

From the previous theorems we have the following interesting result.

Theorem 3.2.6: Let G be a closed subspace of a Banach space X which has the S–property in X, then the following are equivalent:

1. G is a Chebyshev subspace of X

2. $L^{\phi}(\mu,G)$ is a Chebyshev subspace of $L^{\phi}(\mu,X)$.

3. $L^{p}(\mu,G)$ is a Chebyshev subspace of $L^{p}(\mu,X)$ $1 \le p < \infty$.

Proof: $(1)\rightarrow(2)$. Let G have the S–property in X. Theorem 3.2.4 implies $L^1(\mu,G)$ has the S–property in $L^1(\mu,X)$. Since G is a Chebyshev (proximinal)$\Rightarrow$G is 1–complemented in X by Corollary 3.1.10. Theorem 2.3.13 implies that $L^1(\mu,G)$ is 1–complemented in $L^1(\mu,X)$, then by Lemma 2.2.2 we have $L^1(\mu,G)$ is proximinal in $L^1(\mu,X)$. Since $L^1(\mu,G)$ is proximinal and has the S–property in $L^1(\mu,X)$, then $L^1(\mu,G)$ is a Chebyshev subspace of $L^1(\mu,X)$, (by Corollary 3.1.5). And so $L^{\phi}(\mu,G)$ is a proximinal subspace of $L^{\phi}(\mu,X)$, (by Theorem 2.3.7); and since G has the S–property in X, then $L^{\phi}(\mu,G)$ has the S–property in $L^{\phi}(\mu,X)$ (by Theorem 3.2.1). Hence $L^{\phi}(\mu,G)$ is a Chebyshev subspace of $L^{\phi}(\mu,X)$ (by Corollary 3.1.5).

$(2)\rightarrow(3)$. Let $L^{\phi}(\mu,G)$ be a Chebyshev subspace of $L^{\phi}(\mu,X)$ which has the S–property. Theorem 2.3.7 implies $L^1(\mu,G)$ is proximinal (Chebyshev) subspace of $L^1(\mu,X)$ and Theorem 2.3.8 implies $L^{p}(\mu,G)$ is a proximinal subspace of $L^{p}(\mu,X)$. Theorem 3.2.5 implies $L^{p}(\mu,G)$ has the S–property in $L^p(\mu,X)$, then by Corollary 3.1.5 we have $L^p(\mu,G)$ is a Chebyshev subspace of $L^p(\mu,X)$.

(3)$\rightarrow$(1). Let $L^p(\mu,G)$ be a Chebyshev subspace of $L^p(\mu,X)$, then by Theorem 2.3.12, G is proximinal in X but also G has the S–property in X, so G is a Chebyshev subspace of X (by Corollary 3.1.5). ■

Finally, if $p=\infty$; we have:

Theorem 3.2.7: Let X be a Banach space and G be a closed subspace of X. If $L^\infty(\mu,G)$ has the S–property in $L^\infty(\mu,X)$, then G has the S–property.

Proof: Suppose $L^\infty(\mu,G)$ has the S–property in $L^\infty(\mu,X)$ and let $x_i \in X$ and $z_i \in P(x_i,G)$ for i = 1, 2. We want to show that $z_1 + z_2 \in P(x_1 + x_2, G)$.

Now consider the constant functions $f_i(t) = x_i$ and $g_i(t) = z_i$ for i =1, 2 and $\forall t \in \Omega$. It is clear $f_i \in L^\infty(\mu,X)$ and $g_i \in L^\infty(\mu,G)$ for i = 1, 2.

We claim that $g_i \in P(f_i, L^\infty(\mu,G))$ for i =1, 2.

Now for i =1, 2 and $\forall t \in \Omega$ we have

$$\|f_i(t) - g_i(t)\| = \|x_i - z_i\|$$

$$\leq \|x_i - y\| \qquad \forall y \in G, \text{ because } z_i \in P(x_i,G) \text{ for } i=1,2.$$

$$= \|f_i(t) - y\| \quad \forall y \in G,\ i=1,2.$$

Then for all $h \in L^\infty(\mu,G)$ we have

$$\|f_i(t) - g_i(t)\| \leq \|f_i(t) - h(t)\| \ \ \forall t \in \Omega \text{ and } i=1,2.$$

Then we have $\|f_i - g_i\|_\infty \le \|f_i - h\|_\infty$ i =1, 2, $\forall h \in L^\infty(\mu, G)$.

Hence $g_i \in P(f_i, L^\infty(\mu, G))$ for i =1, 2.

Because $L^\infty(\mu, G)$ has the S–property in $L^\infty(\mu, X)$, then

$g_1 + g_2 \in P(f_1 + f_2, L^\infty(\mu, G))$

$\Rightarrow \|f_1 + f_2 - (g_1 + g_2)\|_\infty \le \|f_1 + f_2 - h\|_\infty \ \forall h \in L^\infty(\mu, G)$.

But we have

$$\|f_1 + f_2 - (g_1 + g_2)\|_\infty = \inf\{M : \mu\{t : \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| > M\} = 0\}$$

$$= \inf\{M : \mu\{t : \|(x_1 + x_2) - (z_1 + z_2)\| > M\} = 0\}$$

$$= \|(x_1 + x_2) - (z_1 + z_2)\|$$

$$= \|(f_1 + f_2)(t) - (g_1 + g_2)(t)\| \quad \forall t \in \Omega .$$

In particular, let $h_y(t) = y$, $\forall t \in \Omega$ such that $y \in G$ be a constant function, then $\|f_1 + f_2 - h_y\|_\infty = \inf\{M : \mu\{t : \|(f_1 + f_2)(t) - h_y(t)\| > M\} = 0\}$

$$= \inf\{M : \mu\{t : \|(x_1 + x_2) - y\| > M\} = 0\}$$

$$= \|(x_1 + x_2) - y\|$$

$$= \|(f_1 + f_2)(t) - h_y(t)\| \quad \forall t \in \Omega .$$

Hence

$$\|(x_1+x_2)-(z_1+z_2)\| = \|f_1+f_2-(g_1+g_2)\|_\infty$$
$$\leq \|f_1+f_2-h_y\|_\infty$$
$$=\|(x_1+x_2)-y\|,\ y\in G.$$

Since $y\in G$ was arbitrary, then

$$\|(x_1+x_2)-(z_1+z_2)\| \leq \|(x_1+x_2)-y\|,\quad \forall\, y\in G. \tag{3.39}$$

Therefore $z_1+z_2 \in P(x_1+x_2, G)$. Thus G has the S–property in X. ■

We saw in the previous theorem that if $L^\infty(\mu,G)$ has the S–property in $L^\infty(\mu,X)$, then G has the S–property. Now what about the converse?; to answer this question we assume that $L^\infty(\mu,G)$ is a Chebyshev subspace of $L^\infty(\mu,X)$.

Theorem 3.2.8: Let G be a closed subspace of a Banach space X and suppose $L^\infty(\mu,G)$ is a Chebyshev subspace of $L^\infty(\mu,X)$. If G has the S–property in X, then $L^\infty(\mu,G)$ has the S–property in $L^\infty(\mu,X)$.

Proof: Let $f_i \in L^\infty(\mu,X)$, i=1,2. Since $L^\infty(\mu,G)$ is Chebyshev $\Rightarrow L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X) \Rightarrow G$ is proximinal in X $\Rightarrow G$ is Chebyshev (because G has the S–property in X). Theorem 3.2.4 and Theorem 3.2.6 imply that $L^1(\mu,G)$ is Chebyshev and has the S–property in $L^1(\mu,X)$. Then for i = 1, 2, $\exists!\ h_i \in L^1(\mu,G)$ such that $h_i \in P(f_i, L^1(\mu,G))$; and since $\|h_i(t)\| \leq 2.\|f_i(t)\|$ a.e. t, then $h_i \in L^\infty(\mu,G)$. Using the same arguments as in Theorem 2.3.9 we have $h_i \in P(f_i, L^\infty(\mu,G))$.

Now since $h_i \in P(f_i, L^1(\mu,G))$, i =1,2, and $L^1(\mu,G)$ has the S–property in $L^1(\mu,X)$, then $h_1+h_2 \in P(f_1+f_2, L^1(\mu,G))$.

Since $f_1 + f_2 \in L^{\infty}(\mu, G)$ and $\|h_1(t) + h_2(t)\| \leq 2.\|f_1(t) + f_2(t)\|$ a.e. t, then once again; using the same arguments as in Theorem 2.3.9 we have

$h_1 + h_2 \in P(f_1 + f_2, L^{\infty}(\mu, G))$; and since $L^{\infty}(\mu, G)$ is Chebyshev, then $L^{\infty}(\mu, G)$ has the S–property in $L^{\infty}(\mu, X)$. ■

Now, we have:

Corollary 3.2.9: Let G be a closed subspace of a Banach space X which has the S–property. If $L^{\infty}(\mu, G)$ is a Chebyshev subspace of $L^{\infty}(\mu, X)$, then G is a Chebyshev subspace of X.

Proof: By Theorem 2.3.11 and Theorem 3.2.8. ■

Theorem 3.2.10: Let G be a closed subspace of the Banach space X and S be a compact Hausdorff space. If $\ell_{\infty}(S,G)$ has the S–property in $\ell_{\infty}(S,X)$ then G has the S–property in X.

Proof: Let $x_i \in X$ and $z_i \in P(x_i, G)$, i =1, 2. Consider the constant functions $f_i(s) = x_i$, $g_i(s) = z_i$ for i = 1, 2 and $\forall s \in S$. Then

$f_i, g_i \in C(S,X) \subset \ell_{\infty}(S,X)$ $\forall i$ =1, 2; and for each i, we show that

$$g_i \in P(f_i, \ell_{\infty}(S,G)). \text{ For } i = 1, 2.$$

For i = 1, 2, we have

$$\|f_i - g_i\|_{\infty} = \sup_s \|f_i(s) - g_i(s)\| = \sup_s \|x_i - z_i\| = \|x_i - z_i\| \leq \|x_i - y\| \quad \forall y \in G.$$

Because $z_i \in P(x_i, G)$, i =1, 2. Then, for all $h \in \ell_{\infty}(S,G)$ and i = 1, 2, we have

$$\|f_i - g_i\|_\infty \leq \|x_i - h(s)\| \leq \sup_s \|x_i - h(s)\| \leq \|f_i - h\|_\infty$$

Therefore $g_i \in P(f_i, \ell_\infty(S,G))$.

Since $\ell_\infty(S,G)$ has the S–property in $\ell_\infty(S,X)$, then

$$g_1 + g_2 \in P(f_1 + f_2, \ell_\infty(S,G)).$$

Now, Theorem 2.3.23 implies that

$$\begin{aligned}\|f_1 + f_2 - (g_1 + g_2)\|_\infty &= \sup_s d((f_1 + f_2)(s), G) \quad , \; s \in S \\ &= \sup_s d(x_1 + x_2, G) \quad , \; s \in S \\ &= d(x_1 + x_2, G). \qquad (3.40)\end{aligned}$$

By definition of the norm of $\ell_\infty(S,X)$, we have

$$\begin{aligned}\|f_1 + f_2 - (g_1 + g_2)\|_\infty &= \sup_s \|(f_1 + f_2)(s) - (g_1 + g_2)(s)\| \\ &= \sup_s \|x_1 + x_2 - (z_1 + z_2)\|, \; s \in S. \\ &= \|x_1 + x_2 - (z_1 + z_2)\| \qquad (3.41)\end{aligned}$$

By relations (3.40) and (3.41) we have $d(x_1 + x_2, G) = \|x_1 + x_2 - (z_1 + z_2)\|$.

Therefore $z_1 + z_2 \in P(x_1 + x_2, G)$. Hence G has the S–property in X. ■

Conclusion

This thesis contains a few properties of best approximations and the S−property. We conclude from Remark 3.1.2 and Example 3.1.3 that if a subspace, G, has the S−property, then G is not necessarily proximinal in X; and moreover, if G is proximinal in X, G does not necessarily possess the S−property. However, we see that every closed subspace G with the S−property is a semi–Chebyshev subspace. Furthermore, from Theorem 3.1.6 and Theorem 3.1.7 we conclude that G has the S−property if and only if $P_G^{-1}(0)$ is a closed subspace of X where X is a Banach space.

In section 3.2 we have the most important results about Orlicz and $L^p(\mu,X)$ subspaces. If G is a closed subspace of a Banach space X, then we have the following:

1. G has the S−property $\Leftrightarrow$ $L^\phi(\mu,G)$ has the S−property.
2. G has the S−property $\Leftrightarrow$ $L^1(\mu,G)$ has the S−property.
3. G has the S−property $\Leftrightarrow$ $L^p(\mu,G)$ has the S−property.
4. $L^\infty(\mu,G)$ has the S−property $\Rightarrow$ G has the S−property.
5. $\ell_\infty(S,G)$ has the S−property $\Rightarrow$ G has the S−property.

Other results can also be found in the thesis.

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Appendix

It was pointed out that Al-Dwaik's thesis is not published, so this appendix is intended to include the theorems that are not mentioned in this thesis, or which we give different proofs.

Theorem 1: The infinite dimensional subspace c_0 of c is proximinal in c.

Proof: On c define the linear functional f by $f(\boldsymbol{x}) = \lim x_n = y$.

Then $c_0 = \{\boldsymbol{g} = \{g_n\} \in c : f(\boldsymbol{g}) = 0\}$ is the hyperplane of c and if $\boldsymbol{x} \in c$, then $d(\boldsymbol{x}, c_0) = |y|$ by Theorem 1.2.6 and Remark 1.2.7.

Let $\boldsymbol{g} = \{g_n\}$ be defined as, $g_n = x_n - y$.

Now $\boldsymbol{g} \in c_0$ and $\|\boldsymbol{x} - \boldsymbol{g}\| = \sup\{|x_n - g_n| : n \in N\}$

$$= \sup\{|x_n - (x_n - y)| : n \in N\} = |y|.$$

Hence $d(\boldsymbol{x}, c_0) = \|\boldsymbol{x} - \boldsymbol{g}\|$ and so; $\boldsymbol{g} \in P(\boldsymbol{x}, c_0)$.

Since $\boldsymbol{x}$ was arbitrary; c_0 is proximinal in c.

Theorem 2: Every modulus function is continuous on $[0,\infty)$.

Proof: Let $x_0 \in [0,\infty)$. We show that ϕ is continuous at x_0, i.e. $\lim_{x \to x_0} \phi(x) = \phi(x_0)$. At first we show that

$$|\phi(x) - \phi(y)| \le \phi(|x - y|) \ \forall x, y \in [0,\infty).$$

Now $|x| = |x - y + y| \le |x - y| + |y|$, since ϕ is increasing and subadditive we get $\phi(|x|) \le \phi(|x - y|) + \phi(|y|)$.

So $\phi(|x|)-\phi(|y|)\le\phi(|x-y|)$ (A1)

If we interchange *x* and *y*, then we have

$$\phi(|y|)-\phi(|x|)\le\phi(|x-y|) \qquad \text{(A2)}$$

By (A1) and (A2) we have

$$\left|\phi(|x|)-\phi(|y|)\right|\le\phi(|x-y|) \quad \forall x,y\in[0,\infty).$$

Now given ε > 0, there exists δ(ε) > 0 such that if 0<*x*<δ(ε), then $|\phi(x)|<\varepsilon$, because ϕ is continuous at 0. But $|\phi(x)-\phi(x_0)|\le\phi(|x-y|)<\varepsilon$ if $|x-x_0|<\delta(\varepsilon)$. Hence ϕ is continuous at x_0 and since x_0 is arbitrary, then ϕ is continuous on [0,∞).

Theorem 3: Let *G* be a closed subspace of a Banach space *X*. If $L^\infty(\mu,G)$ is proximinal in $L^\infty(\mu,X)$, then *G* is proximinal in *X*.

Proof: Let $x\in X$. Consider the function $f(t)=x$ $\forall t\in\Omega$, then $f\in L^\infty(\mu,X)$. Hence there exists $g\in L^\infty(\mu,G)$ such that $\|f-g\|_\infty=d(f,L^\infty(\mu,G))$.

By theorem [11, p.36] $\|f-g\|_\infty=\sup\limits_t d(f(t),G)$.

Hence $\|f-g\|_\infty=\sup\limits_t d(x,G)$, since $f(t)=x$ $\forall t\in\Omega$.

$$\|f-g\|_\infty=d(x,G).\ \text{But } d(x,G)=\sup\{\|x-g(t)\|:t\in\Omega\}.$$

$$\Rightarrow \|x-g(t)\|\le d(x.G)$$

Therefore *G* is proximinal in *X*. ■

Theorem 4: Let G be a closed subspace of a Hilbert space X, then G has the S−property.

Proof: Let $x_i \in X$, $z_i \in P(x_i, G)$ for i=1, 2. We show $z_1 + z_2 \in P(x_1 + x_2, G)$. Theorem 2.1.4 implies $x_i - z_i \perp G$ for i=1, 2. Hence

$< x_i - z_i, g > = 0, \ \forall g \in G.$

Now, $< x_1 + x_2 - (z_1 + z_2), g > \ = \ < x_1 - z_1, g > + < x_2 - z_2, g > \ = \ 0, \ \forall \ g \in G.$ Hence $x_1 + x_2 - (z_1 + z_2) \perp G$.

Theorem 2.1.4 implies that $z_1 + z_2 \in P(x_1 + x_2, G)$. Thus G has the S−property. ■

Theorem 5: Let G be a closed subspace of a Banach space X. If G has the S−property, then $L^{\phi}(\mu, P_G^{-1}(0)) \subset P_{L^{\phi}(\mu,G)}^{-1}(0)$.

Proof: Let $f \in L^{\phi}(\mu, P_G^{-1}(0))$ i.e. $f(t) \in P_G^{-1}(0) \ \forall\, t \in \Omega$, and so $\|f\|_{\infty} < \infty$. Now we have

$0 \in P(f(t), G) \Rightarrow d(f(t), G) = \|f(t)\|, \forall t \in \Omega,$ i.e. $\|f(t)\| \le \|f(t) - g\| \ \ \forall g \in G$

and $\forall\, t \in \Omega$.

In particular:

$$\|f(t)\| \le \|f(t) - h(t)\|, \quad \forall\, h \in L^{\phi}(\mu, G).$$

$$\Rightarrow \phi(\|f(t)\|) \le \phi(\|f(t) - h(t)\|), \quad \forall h \in L^{\phi}(\mu, G).$$

$$\Rightarrow \|f\|_{\phi} \le \|f - h\|_{\phi}, \quad \forall h \in L^{\phi}(\mu, G).$$

Hence $d(f,L^{\phi}(\mu,G))=\|f\|_{\phi}$, therefore $0\in P(f,L^{\phi}(\mu,G))\Rightarrow f\in P^{-1}_{L^{\phi}(\mu,G)}(0)$.

Thus $L^{\phi}(\mu,P_G^{-1}(0))\subset P^{-1}_{L^{\phi}(\mu,G)}(0)$. ■

Remark 6: If G is 1–complemented in X, then G may not be a Chebyshev subspace.

For Example let X = $\boldsymbol{R}^2$ and G = $\{(g,g)$: $g\in G\}$ with $\|(x,y)\|=|x|+|y|$, then G is proximinal and not Chebyshev.

Now, let $W = \{(0,w) : w\in\boldsymbol{R}\}$, then $(x,y)=(x,x)+(0,y-x)$.

Clearly $\boldsymbol{R}^2 = G\oplus W$.

We define $P : X\to W$ as $P(x,y) = P((x,x)+(0,y-x)) = (0,y-x)$

Now $\|w\|=\|(0,y-x)\|=|y-x|\leq|y|+|x|=\|(x,y)\|$.

Hence P is a contractive projection.

Therefore G is 1–complemented in X. ■

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كلية الدراسات العليا

أفضل تقريب في فضاءات قياسية تحقق خاصية (S)

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قدمت هذه الأطروحة استكمالا لمتطلبات نيل درجة الماجستير في الرياضيات بكلية الدراســات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

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الملخص

إذا كان x عنصراً في الفضاء الخطي القياسي X و كان G فضاءً جزئياً من X. فإن أفضل تقريب للعنصر x بالنسبة إلى الفضاء الجزئي هو(إن وجد) عنصرٌ في G يكون الأقرب إلى x.

تحتوي الأطروحة على العديد من النتائج الخاصة بإيجاد أفضل تقريب وبشكل خاص في فضاءات أورلكز و بشكل أخص حين تحقق الفضاءات الجزئية خاصية "S".