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Mathematical Analysis of Heat Radiation Problems

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Dedication

I dedicate this thesis to my beloved Palestine, my parents, my brothers, my sister Ghada, and my friends, who stood by me and believed in me.

الإقرار

انا الموقعة أدناه مقدمة الرسالة التي تحمل العنوان:

Mathematical Analysis of Heat Radiation Problems

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Abstract

In this thesis we focus on the analytical aspects of the Fredholm integral equation a rising as a result of the heat energy exchange inside a convex and non-convex enclosure geometries. The phenomenon of heat radiation plays a very dominant role in a wide range of applications in science and technology. The general characteristicsof heat transfer modes ; namely: heat conduction, heat convection and heat radiation will be addressed.

A systematic derivation of the heat radiation equation using some physical properties of heat transfer modes will be presented. Some important properties of the integral operator for the Fredholm integral equation will be investigated. The Banach fixed-point theorem will be introduced and used to show the existence and the uniqueness of the solution of the integral equation. A problem of coupling radiation with conduction will be solved and analyzed.

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Introduction

All bodies at temperature above absolute zero emit energy in the form of electromagnetic waves. A portion of this energy flux when impinging other bodies is absorbed. As a result, net energy flux occurs from a body of higher temperature to a body having lower temperature. This mode of energy transfer is known as heat radiation. The wavelength range encompassed by thermal radiation is approximately $0.1 - 100\ \mu m$. Heat radiation is, as each wave propagation phenomenon, of dual nature. It possesses the continuity properties of electromagnetic waves and the corpuscular properties characteristic for photons. Heat radiation plays a dominant role in engineering and modern technology. These applications includes the heat transfer in furnaces and combustion chambers and in the energy emission from a nuclear explosion. Also heat radiation must often be considered when calculating thermal effects in devices such as a rocket nozzle, a nuclear power plant, or nuclear rockets. One of the factors that accounts for the importance of the thermal radiation in some applications is the manner in which radiant emission depends on temperature. Another characteristic feature of radiation is that it can be transferred directly from one location to another only when the point can be seen when looking from another, i.e., it does not lay in the shadow zone. The presence of shadow zones should be taken into account in heat radiation calculation [3, 13, 23]. This leads to a rather complex algorithm and long computing times.

It is evident that almost all phenomena that modelers deal with are governed by differential equations, however, heat radiation is one of the few phenomena governed by an integral equation. Due to this nice feature, the idea of solving this Fredholm integral equation by the boundary element method naturally arises. Boundary element method together with other numerical methods for different types of heat radiation problems have been addressed in [4, 5, 13, 17, 18, 22, 23].

Moreover, the problem of coupling radiation with other heat transfer modes such as conduction and convection was also studied by many authors [4, 5, 11, 15, 19, 25]. Concerning the simplest nontrivial case of conductive body with nonconvex opaque radiating surface, we are aware of the work [31] and other previous work [4, 5]. They all studied some properties of the operators related to the radiative transfer and showed the existence of a weak solution under some restrictions. The basic case has been extended to cover several conductive bodies and time dependent problems [14]. In the case of semitransparent material the analysis has been carried out in one-dimensional case with nonreflecting surfaces [16] and in two and three-dimensional with diffusively reflecting surfaces [12].

This thesis is organized as follows:

Chapter one contains general characteristics of heat transfer modes. These include heat conduction, heat convection and heat radiation.

The aim of chapter two is to present a systematic derivation of the equation of heat radiation. This equation is given in the form of a Fredholm integral equation of the second kind.

In chapter three, we investigate some important analytical results concerning the integral operator of the Fredholm integral equation of the second kind.

In chapter four, we present the Banach fixed point theorem and show how this theorem can be used to show the existence and the uniqueness of the solution of the radiosity integral equation.

In chapter five, we consider a conductive-radiative heat transfer model and investigate the question of existence and uniqueness of a weak solution for this problem.

Chapter One

Introduction to Heat Transfer Modes

Chapter One

Introduction to Heat Transfer Modes

In this chapter, we summarize the general characteristics of heat transfer modes and some of their physical properties.

Heat Transfer is thermal energy in transit due to a spatial temperature difference.When a temperature gradient exists in a stationary medium, which may be a solid or fluid, we use the term conduction to refer to the heat transfer that will occur across the medium. In contrast, the term convection refers to heat transfer that will occur between a surface and a moving fluid when they are at different temperatures. The third mode of heat transfer is termed thermal radiation. All surfaces of finite temperature emit energy in the form of electromagnetic waves. Hence, in the absence of an intervening medium, there is net heat transfer by radiation between two surfaces at different temperature.

1.1 Heat Conduction

The most efficient method of heat transfer is conduction. This mode of heat transfer occurs when there is a temperature gradient across a body. In this case, the energy is transferred from a high temperature region to a low temperature region due to random molecular motion (diffusion). Conduction occurs similarly in liquids and gases. Regions with greater molecular kinetic energy will pass their thermal energy to regions with less molecular energy through direct molecular collisions. Different materials have varying abilities to conduct heat. Materials that conduct heat poorly (wood) are often called insulator. Unlike radiation and convection, conduction requires the presence of intervening medium [29, 30].

Fourier Law

The basic equation for the analysis of heat conduction is Fourier law :

$$q_n = -k_n \frac{\partial T}{\partial n} \tag{1.1}$$

where the heat flux q_n is the heat transfer rate in the n direction per unit area perpendicular to the direction of the heat flow (W), k_n is the thermal conductivity in the direction n $(W/m.K)$, $\partial T\ /\ \partial n$ is the temperature gradient in the direction n (K/m).The thermal conductivity is a thermo physical property of the material, which is in general, a function of both temperature and location, that is $k = k(T,n)$.

For isotropic materials, k is the same in all directions. For anisotropic materials such as wood and laminated materials.

The temperature difference resulting from the steady-state diffusion of heat is thus related to the thermal conductivity of the material, the cross sectional area and the path length L, according to:

$$T_1 - T_2 = q\ \frac{L}{k_n} \tag{1.2}$$

The form of equation (1.2), where k_n is constant that is equivalent to Ohm's Law governing electrical current flow through a resistance. It is possible to define a conduction thermal resistance as:

$$R = \frac{T_1 - T_2}{q} = \frac{L}{k_n} \tag{1.3}$$

1.2 Heat Convection

Heat energy transferred between a surface and a moving fluid at different temperature is known as convection. Convection heat transfer may be categorized into two forms according to the nature of the flow, natural convection and forced convection.

In natural "free" convection, the fluid motion is driven by density differences associated with temperature changes generated by heating or cooling. Thus the heat transfer itself generates the flow which conveys energy away from the point at which the transfer occurs.

In forced convection, the fluid motion is driven by some external influences. For example: the flow of air induced by a fan, by the wind, or by the motion of a vehicle, and the flow of water within heating, cooling, supply and drainage systems.

If T_1 is the temperature of the surface receiving or giving heat, and T_∞ is the average temperature of the stream of fluid adjacent to the surface, then the convective heat transfer q is governed by Newton's Law:

$$q = h_c A \, (T_1 - T_\infty) \tag{1.4}$$

where h_c is the convective heat transfer coefficient,$(W/m^2.K)$, A is the heat transfer surface area (m^2).

Heat transfer by convection is more difficult to analyze than heat transfer by conduction because no single property of the heat transfer medium, such as thermal conductivity, can be defined to describe the mechanism. Heat transfer by convection varies from situation to situation (upon the fluid flow conditions), and it is frequently coupled with the mode of fluid flow [4].

1.3HeatRadiation

All bodies at temperatures above absolute zero emit energy in a form of electromagnetic waves. A portion of this energy flux when impinging other bodies is absorbed. As a result, net energy flow occurs from a body of higher temperature to a body having low temperature. This mode of energy transfer is termed heat radiation. Radiation plays a dominant rule in energy transfer at elevated temperatures and in the presence of rarefied gases. One of the factors that accounts for the importance of the thermal radiation in some applications is the manner in which radiant emission depends on temperature. For conduction and convection, the transfer of energy between two locations depends on temperature difference of the locations. The transfer of energy by thermal radiation, however, depends on the individual absolute temperatures of the bodies, each raised to a power in the range of about 4 or 5. It is clear that the importance of radiation becomes intensified at high absolute temperature levels. Consequently, radiation contributes substantially to the heat transfer in furnaces, composition chambers and in the energy emission from a nuclear explosion. Also heat radiation must often be considered when calculating thermal effects and devices such as a rocket nozzle, nuclear power plants and nuclear rockets.

Blackbody radiation

Each body emits radiative energy. Radiative energy flux is defined as the amount of radiative energy passing a unit surface in a unit time:

$$e^r = \frac{d^2E^r}{dsdt} \tag{1.5}$$

where e^r is the radiative energy flux $(W.m^{-2})$, E^r is the radiative energy(J),ds is the infinitesimal surface(m^2) and $dt$is the infinitesimal time increment(s).

Radiation is transferred via electromagnetic waves whose lengths (spectrum) range from 0 to ∞. The amount of energy transported at a given wavelength is a function of this length. Functions associated with transfer at a certain wavelength are referred to as spectral or monochromatic quantities. These quantities will be denoted by appending a subscript λ to appropriate symbol.Functions associated with the entire spectrum are referred to as total or panchromatic quantities. Let A_λ denote arbitrary spectral quantity and Astand for its panchromatic (total) counterpart. These two functions are related by the relationships:

$$A_\lambda = \frac{dA}{d\lambda} \tag{1.6}$$

$$A = \int_0^\infty A_\lambda \, d\,\lambda \tag{1.7}$$

Transport of radiation takes place a long straight lines referred to as lines of sight.

To describe the transfer of heat radiation a long a line of sight, a notion of intensity of radiation is introduced. Intensity of radiation is defined as a radiative flux passing through an infinitesimal surface orthogonal to the line of sight and subtended within an infinitesimal solid angle centered around the line of sight.

$$I = \frac{d^3 E^r}{dS_\perp dt d\Omega} \quad (1.8)$$

where I is the intensity of radiation (W/m^2), $d\Omega$ is the differential solid angle, and $dS_\perp$ is the differential surface normal to the line of sight;

$dS_\perp = ds \cos\emptyset$, $\emptyset$ is the angle with which the line of sight is inclined with respect to the surface normal .

The intensity is connected with the energy flux and hence, it can be interpreted as a vector having a direction of the line of sight. As an infinite number of lines of sight cross at a given point, an infinite number of intensities can be assigned to a chosen point.

A perfect emitter of radiant energy is called a blackbody. From all bodies of the same temperature a blackbody emits maximum energy. Kirchhoff's Law states that a blackbody is also a perfect absorber of radiant energy. Thus it absorbs the entire incident radiative energy. Quantities associated with this reference body will be marked by appending a subscript b to appropriate symbols.

A blackbody, being by definition an ideal emitter of radiation, emits energy uniformly in all directions. Hence, the intensity of blackbody radiation I_b is independent of direction. This property of a blackbody is referred to as the isotropy of blackbody emission.

The flux of radiative energy emitted by an elemental blackbody surface is termed blackbody emissive powere_b. This flux can be computed upon integrating the normal component of the intensity vectors over the entire hemisphere centered at that surface:

$$e_b = \int_{2\pi} I_b \cos\emptyset \; d\Omega \tag{1.9}$$

where I_b is the blackbody intensity of radiation, e_b is the blackbody emissive power.

A differential solid angle can be related to the polar angle $\emptyset$ and the angle θ by:

$$d\Omega = sin\emptyset d\emptyset d\theta \tag{1.10}$$

Taking into account equation (1.10) and performing an appropriate integration, equation (1.9) yields a relationship linking the intensity and the emissive power of the blackbody:

$$e_b = I_b \int_{\theta=0}^{2\pi} \int_{\emptyset=0}^{\pi/2} \cos\emptyset \sin\emptyset d\emptyset d\theta = \pi I_b \tag{1.11}$$

Quantum mechanics yields an equation expressing the blackbody spectral emissive power as a function of temperature and wavelength. This relationship is known as the Planck's function and has the form:

$$e_{b\lambda} = \frac{2\pi C_1}{\lambda^5\left[\exp\left(\frac{C2}{\lambda T}\right)-1\right]} \tag{1.12}$$

where λ is the wavelength(m), T is the temperature(K),

$C_1 = 0.59544 \times 10^{-16} W.m^2$, $C_2 = 1.4388 \times 10^{-2} m.K.$

Planck's function also depends on the refraction index of the medium. This index is defined as the ratio of the speed of light in vacuum and in the medium. The refraction index of gases is very close to 1. For simplicity, radiation transfer within media having a refraction index equal to 1 will be considered.

Wien's displacement law is obtained by differentiating equation (1.12) to find the wavelength at which the emission $e_{b\lambda}$is a maximum :

$$\lambda_{max} T = C_3 \quad (1.13)$$

where $C_3 = 2.8978.10^{-3} m.K$.

The energy emitted by a unit blackbody surface in a unit time within the entire spectrum can be computed from a relationship known as the Stefan-Boltzmann law :

$$e_b = \int_0^{\infty} e_{b\lambda} d\lambda = \sigma T^4 \quad (1.14)$$

where σis the Stefan-Boltzmann constant ($\sigma = 5.669 \times 10^{-8} W/m^2.K^4$)

For two blackbodies at temperature T_1 and T_2 , the maximum radiative interchange between them is given by:

$$e_{b1} = \sigma T_1^4 \ , \quad e_{b2} = \sigma T_2^4 \, , \quad e_{b,1-2} = \sigma(T_1^4 - T_2^4) \quad (1.15)$$

Radiative Properties

Radiation can be absorbed by a medium it transverses. Certain materials including vacuum do not interact with the radiation. Such media are termed transparent. Remaining media are said to be participating. In this case radiation is absorbed in a very thin layer in the vicinity of the surface of the medium and the material is called an opaque medium. Most solids are treated as opaque substances. Real substances differ from the ideal blackbody model because at the same temperature they emit less radiative energy. The fraction of blackbody emission radiated by a real surface of an opaque body is termed emissivity and is denoted by ε. Similarly, a real surface absorbs only a fraction of the incident radiant energy. This fraction is called absorptivity and is denoted as α. From Kirchhoff's law it follows that :

$$\varepsilon = \alpha \tag{1.16}$$

Incident radiation that is not absorbed on an opaque surface must be reflected. The fraction of incident energy reflected from a surface is termed reflectivity and is denoted as ρ. The energy balance yields the following relationship :

$$\rho = 1 - \alpha = 1 - \varepsilon \tag{1.17}$$

Emissivity, absorptivity and reflectivity are material properties of a surface. Surfaces whose radiative properties are independent of the wavelength are termed grey. Equations (1.16) and (1.17) concern such surfaces.

Chapter Two

Formulation and The Derivation of The Radiosity Fredholm Integral Equation

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Formulation and The Derivation of The Radiosity Fredholm Integral Equation

One of the most important features about the heat radiation between two points on a surface is its formulation as a Fredholm integral equation of the second kind. Another important feature of radiation is that it can be transferred directly from one point to another only when the point can be seen when looking from another, that means it does not lay in the visibility zone. The presence of visibility zones should be taken into account in heat radiation analysis.

We consider an enclosure geometries of domain $\Omega\subset\mathbb{R}^3$, with boundary $\partial\Omega$. Assume that the boundary$\partial\Omega$ of the enclosure is composed of n elements.

The heat balance for an element j of area dA_j is given as:

$$R_j = q_j\, dA_j = \left(q_{o,j} - q_{i,j}\right) dA_j \qquad (2.1)$$

where $q_{i,j}$ is the rate of incoming energy per unit area on element j, $q_{o,j}$ is the rate of heat per unit area emitted by an element j, dA_j is the area of the element j, and q_j is the heat energy brought to the element j by conduction.

For diffuse and grey surfaces the outgoing flux has the form:

$$q_{o,j} = \varepsilon_j \sigma T_j^4 + \rho_j\, q_{i,j} \qquad (2.2)$$

where the first part corresponds to the Stefan-Boltzmann radiative law.

Here ε_j is the emissivity coefficient $\left(0 < \varepsilon_j < 1\right), \sigma$ is the Stefan-Boltzmann constant which has the value $5.669 \times 10^{-8} W/m^2.K^4$, ρ_j is the reflection coefficient with the relation $\rho_j = 1 - \varepsilon_j$ for diffuse grey surfaces.

The incoming energy $q_{i,j}$ is composed of the portions of the energy leaving the viewable surfaces of the enclosure and arriving at the jth surface. If the jthsurface is nonconvex, then the incident energy is given as:

$$dA_j\, q_{i,j} = dA_1 q_{o,1} V_{1j}\ \alpha(1,j) + dA_2 q_{o,2} V_{2j}\ \alpha(2,j) + \cdots + dA_r\, q_{o,r} V_{rj}\ \alpha(r,j)$$

$$+ \cdots + dA_j\, q_{o,j} V_{jj}\ \alpha(j,j) + \cdots + dA_n\, q_{o,n} V_{nj}\ \alpha(n,j) \qquad (2.3)$$

Upon using the reciprocity relation for the view factor [29, 30] we get:

$$dA_1 V_{1j}\, \alpha(1,j) = dA_j V_{j1} \alpha(j,1),$$

$$dA_2 V_{2j}\, \alpha(2,j) = dA_j V_{j2} \alpha(j,2),$$

$$\vdots$$

$$dA_n V_{nj}\, \alpha(n,j) = dA_j V_{jn}\, \alpha(j,n)\,. \qquad (2.4)$$

In virtue of (2.4) we can rewrite (2.3) as:

$$dA_j\, q_{i,j} = dA_j V_{j1} \alpha(j,1) q_{o,1} + dA_j V_{j2} \alpha(j,2) q_{o,2}$$

$$+ \cdots + dA_j V_{jr}\, \alpha(j,r) q_{o,r} + \cdots + dA_j V_{jj}\, \alpha(j,j) q_{o,j}$$

$$+ \cdots + dA_j V_{jn} \alpha(j,n) q_{o,n} \qquad (2.5)$$

Then the incoming energy can be written as:

$$q_{i,j} = \sum_{r=1}^{n} V_{jr}\ \alpha(j,r) q_{o,r} \tag{2.6}$$

The shadow factor $\alpha(j,r)$is defined as [4]:

$$\alpha(j,r) = \begin{cases} 1 & \text{when surface element j and surface element r} \\ & \qquad \text{exchange heat directly.} \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

Inserting (2.6) into (2.2) and using the relation $\rho_j = 1 - \varepsilon_j$, we then obtain:

$$q_{o,j} = \varepsilon_j\ \sigma T_j^4 + \left(1 - \varepsilon_j\right) \sum_{r=1}^{n} V_{jr}\ \alpha(j,r) q_{o,r} \tag{2.8}$$

2.1 The computation of the view factor V_{jr}.

The total energy per unit time leaving the surface element dA_j and incident on the element dA_r is given as:

$$R_{jr} = L_j\, dA_j \cos\theta_j\ dw_j \tag{2.9}$$

where dw_j is the solid angle inclined by dA_r when viewed from dA_j and L_j is the total intensity of a blackbody for the surface element dA_j.

The solid angle dw_j is related to the projected area of dA_r and the distance S_{jr} between the element dA_j and dA_r and can be calculated as:

$$dw_j = \frac{dA_r \cos\ \theta_r}{S_{jr}^2} \tag{2.10}$$

where θ_{r} denotes the angle between the normal vector n_{r} and the distance vector S_{jr}. Inserting (2.10) into (2.9) gives the following equation for the total energy per unit time leaving dA_j and arriving at dA_r:

$$R_{jr} = \frac{L_j \, dA_j \cos\theta_j \, dA_r \cos\theta_r}{S_{jr}^2} \tag{2.11}$$

In [29], we have the relation between the total intensity L_j and the total emissivity E_j of a black body, that is

$$L_j = \frac{E_j}{\pi} = \frac{\sigma \, T_j^4}{\pi} \tag{2.12}$$

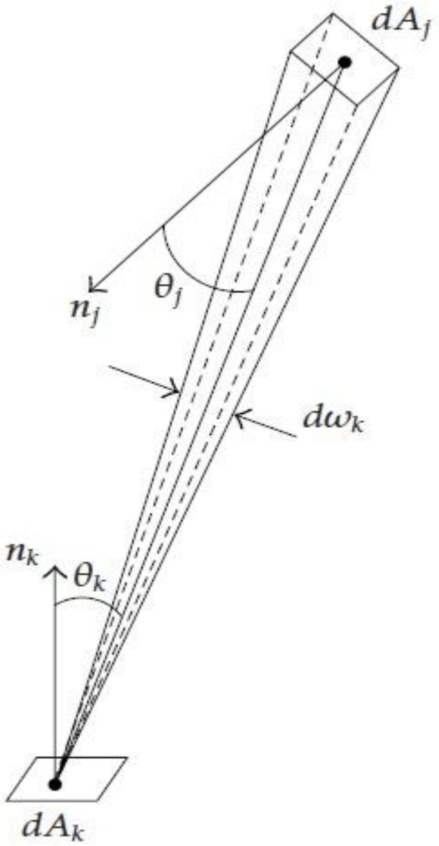

Figure 2.1 Calculation of the view factor.

and consequently (2.11) becomes:

$$R_{jr} = \frac{\sigma T_j^4 \cos\theta_j \cos\theta_r \, dA_j \, dA_r}{\pi S_{jr}^2} \tag{2.13}$$

From the definition of the view factor V_{jr} [4], together with (2.13) we get:

$$V_{jr} = \frac{R_{jr}}{\sigma T_j^4 dA_j} = \frac{\cos\theta_j \cos\theta_r dA_r}{\pi S_{jr}^2} \quad (2.14)$$

2.2 The boundary Fredholm integral equation.

Now we are able to derive the boundary Fredholm integral equation describing the heat balance in a grey and diffuse surfaces. The substitution of (2.14) into (2.8) leads to:

$$q_{0,j} = \varepsilon_j \sigma T_j^4 + \left(1 - \varepsilon_j\right) \sum_{r=1}^{n} \frac{\cos\theta_j \cos\theta_r dA_r}{\pi S_{jr}^2} \alpha(j,r)\, q_{0,r} \quad (2.15)$$

If the number of the area elements $n \to \infty$, then for all $x \in dA_j$, we obtain the boundary integral equation:

$$q_0(x) = \varepsilon(x)\sigma T^4(x) + \left(1 - \varepsilon(x)\right) \int_{\partial\Omega} M(x,y)\, q_0(y) d\,\partial\Omega_y \quad for\ x \in \partial\Omega \quad (2.16)$$

where the kernel $M(x,y)$denotes the view factor between the points x and y of $\partial\Omega$.

From the above considerations and for general enclosure geometries, $M(x,y)$ is given through:

$$M(x,y) = M^*(x,y)\alpha(x,y) = \frac{\left[n_y\ .\ (y-x)\right]\ .\ [n_x\ .\ (x-y)]}{\pi|x-y|^4}\ \alpha(x,y)$$

where :

$$M^*(x,y) = \frac{\left[n_y\ .\ (y-x)\right]\ .\ [n_x\ .\ (x-y)]}{\pi|x-y|^4} \quad (2.17)$$

For convex enclosure geometries, $\alpha(x,y) \equiv 1$. If the enclosure is not convex, then we have to take into account the shadow function $\alpha(x,y)$ defined as ,

$$\alpha(x,y) = \begin{cases} 1 & if\ x\ and\ y\ can\ see\ each\ other \\ 0 & otherwise \end{cases} \tag{2.18}$$

Chapter Three

Investigation of the Fredholm Integral Operator

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Investigation of the Fredholm Integral Operator

Consider the Fredholm integral equation of the second kind(2.16) derived in chapter 2. Introducing the Fredholm integral operator K defined as:

$$\widetilde{K} q_0(x) = \int_{\partial\Omega} L^*(x, y)\, q_0(y)\, d\Omega_y \qquad \text{for } x \in \partial\Omega \tag{3.1}$$

Before investigating the properties of this integral operator we are going to state the following definitions:

Definition 3.1:

A linear operator K that maps a normal vector space X into a normal vector space Y is said to be bounded if there exists a real constant $M > 0$ such that $\|Kv\|_Y \leq M\|v\|_X$, for all $v \in X$.

Definition 3.2 :

A sequence $\{S_n\}$ is said to be convergent or converges to S if for every $\epsilon > 0$, there exist a positive integer $N = N(\epsilon)$ such that: $|S_n - S| < \epsilon$ for all $n > N(\epsilon)$. In this case we write $\lim_{n\to\infty} S_n = S$ and S is called the limit of S_n.

Definition 3.3:

A compact operator K is a linear operator that maps a Banach space X into a Banach space Y with the property that the image under K of any bounded subset of X is a relatively compact subset of Y.

Definition 3.4:

TheL^p spaces are function spaces, called Lebesgue spaces defined as a space of all measurable functions for which the $p-th$ power of the absolute value is Lebesgue integrable, with

$$\|f\|_{L^p[a,b]} = \left(\int_a^b |f|^p \, dx\right)^{\frac{1}{p}}$$

$$\|f\|_{L^\infty} = inf\{c \geq 0, |f(x)| \leq c \;\; for\ almost\ every\ x\}$$

Definition 3.5:

The duality between L^p_μ and L^q_μ for a Borel measure μ is defined as:

$$\langle f_1, f_2 \rangle_{\boldsymbol{\mu}} = \int f_1 f_2 \, d\mu \;\; , \;\; f_1 \in \; L^p_\mu \; and \; f_2 \; \in L^q_\mu$$

For $1 \leq p \leq \infty$ with $p \; and \; q$ are the conjugate exponents, that is $\frac{1}{p}+\frac{1}{q}=1$.

Definition 3.6:

An operator K is said to be positive if $g \geq 0$ implies $K \geq 0$. We denote the positive and negative parts of a function by either sub – or superscript

$$g^+ = g_+ = \max(g, 0) \; and \; g^- = g_- = \min(-g, 0).$$

Lemma 3.1 :

Suppose that$\partial\Omega$ is a piecewise surface of class $C^{1,m}$ with $m \in [0,1)$, then for any $x \in \partial\Omega$ we can show that:

$$\int_{\partial\Omega} L^*(x, y) d \, \Omega_y = 1 \tag{3.2}$$

Proof: Our goal is to show that $L^*(x,y)$ is a weakly singular kernel and hence integrable. To do that we select a local coordinate in $x \in \partial\Omega$ such that both $x=(0,0,0)$ and the plane (r_1,r_2) are tangent to $\partial\Omega$ in x. Moreover, we let $y=(r_1,r_2,g(r_1,r_2))$ to be in the neighborhood of $r_1=r_2=0$. Since $\partial\Omega \in C^{1,m}$ then with the use of Taylor expansion of y in the local coordinate we obtain [17, 23]

$$\left|\frac{n_x.(y-x)}{|y-x|^2}\right| \leq M_1\left|r_\beta\right|^{m-1}, \left|\frac{n_y.(x-y)}{|x-y|^2}\right| \leq M_1\left|r_\beta\right|^{m-1} \tag{3.3}$$

with $\beta \in [1,2]$.

Equation (3.3) implies that

$$|L^*(x,y)| \leq M\left|r_\beta\right|^{-2(1-m)} \tag{3.4}$$

This proves that the kernel $L^*(x,y)$ is weakly singular and hence integrable.

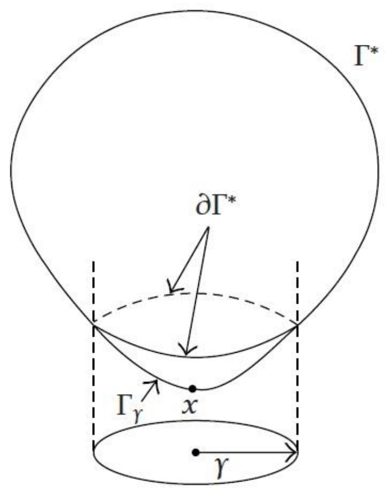

Figure 3.1 Convex case.

Next, to calculate $\int_{\partial\Omega} L^*(x,y)d\Omega_y$, we use the divergence theorem [29]. Let $\partial\Omega$ bea closed surface and choose $y=(y_1,y_2,y_3) \in \partial\Omega$. Moreover, let

$Q_1(y), Q_2(y)$ and $Q_3(y)$be twice differentiable functions of y_1, y_2 and y_3 and n is the normal. The divergence theorem implies that:

$$\int_{\partial\Omega} (Q_1 dy_1 + Q_2 dy_2 + Q_3 dy_3)$$

$$= \int_{\Omega} \left[\left(\frac{\partial Q_3}{\partial y_2} - \frac{\partial Q_2}{\partial y_3}\right) n_1(y) + \left(\frac{\partial Q_1}{\partial y_3} - \frac{\partial Q_3}{\partial y_1}\right) n_2(y) + \left(\frac{\partial Q_2}{\partial y_1} - \frac{\partial Q_1}{\partial y_2}\right) n_3(y)\right] d\Omega. \quad (3.5)$$

Consider the surface $\partial\Omega$ as shown in Fig. 3.1, let $\partial\Omega_y = Z(x,y) \cap \partial\Omega$ be a small neighbourhood of the point x, and define $\partial\Omega^* = \partial\Omega \setminus \partial\Omega_y$ where $Z(x,y)$ is a cylinder given by $x_1^2 + x_2^2 \leq y^2$. Hence, the integral $\int_{\partial\Omega} L^*(x,y) d\Omega_y$can bewritten as:

$$E_y(x) = \int_{\partial\Omega} L^*(x,y) d\Omega_y = \int_{\partial\Omega_y} L^*(x,y) d\Omega_y + \int_{\partial\Omega^*} L^*(x,y) d\Omega_y \quad (3.6)$$

since $L^*(x,y)$ is a weakly singular kernel then the first integral vanishes as $y \to 0$. Consequently (3.6) becomes

$$E_y(x) = \lim_{y\to 0} \int_{\partial\Omega^*} L^*(x,y) d\Omega_y \quad (3.7)$$

Using the divergence theorem we have

$$E_y(x) = \lim_{y\to 0} \int_{\partial\Omega^*} \nabla \times \vec{Q}(y). n_y dy$$

$$= \lim_{y\to 0} \oint_{\partial\Omega^*} (Q_1 dy_1 + Q_2 dy_2 + Q_3 dy_3),$$

(3.8)

where

$$Q_1(y) = \frac{-n_2(x_3 - y_3) + n_3(x_2 - y_2)}{2\pi |x-y|^2},$$

$$Q_2(y) = \frac{n_1(x_3 - y_3) - n_3(x_1 - y_1)}{2\pi |x-y|^2},$$

$$Q_3(y) = \frac{-n_1(x_2 - y_2) + n_2(x_1 - y_1)}{2\pi|x - y|^2}, \tag{3.9}$$

Since the normal to the area element is perpendicular to both the x_1 and x_2-axes and parallel to the x_3- axis, then (3.8) yields

$$E_y(x) = \frac{1}{2\pi} \lim_{y \to 0} \oint_{\partial\Omega^*} \frac{(x_2 - y_2)dy_1 - (x_1 - y_1)dy_2}{|x - y|^2}$$

$$= \frac{1}{2\pi} \lim_{y \to 0} \oint_{\partial\Omega^*} \frac{-y_2 dy_1 + y_1 dy_2}{y_1^2 + y_2^2 + y_3^2}, \tag{3.10}$$

Upon using $y_1^2 + y_2^2 = \gamma^2$. We can write

$$E_y(x) = I_1 + I_2$$

where

$$I_1 = \frac{1}{2\pi} \lim_{y \to 0} \oint_{\partial\Omega^*} \frac{1}{\gamma^2} (-y_2 dy_1 + y_1 dy_2)$$

and

$$I_2 = \frac{1}{2\pi} \lim_{y \to 0} \oint_{\partial\Omega^*} \frac{-y_3^2 (-y_2 dy_1 + y_1 dy_2)}{(y^2 + y_3^2)\gamma^2} \tag{3.11}$$

I_1 can be evaluated by using the polar coordinates $y_1 = y\cos\theta$ and $y_2 = y\sin\theta$, then obtain

$$I_1 = \frac{1}{2\pi} . \frac{1}{\gamma^2} \int_0^{2\pi} \gamma^2 d\theta = 1. \tag{3.12}$$

Moreover, one can show that $I_2 = 0$ [13,14,18]. Hence, $\int_{\partial\Omega} L^*(x, y) d\, \Omega_y = 1$.

Lemma 3.2 :

Suppose that $\partial\Omega$ is a closed surface of class C^2. Then $L^*(x,y)$ is bounded, i.e:

$$|L^*(x,y)| \leq M \tag{3.13}$$

Proof: Let θ be the angle between the normals at the points x *and* y *on* $\partial\Omega$ and d be the distance between these two points, then we have [14,22]

$$|\theta| < c_1 d, \qquad \theta \in (0, 2\pi) \tag{3.14}$$

where c_1 is a positive number.

Introducing the orthonormal system (m_1, m_2, m_3) where m_1 is the axis normal at the surface point x_0, with m_2 and m_3 are tangential plane containing the point x_0. Let the corresponding unit vectors be e_1, e_2 *and* e_3. If we denote the part of the surface that lies inside the sphere by the form $m_1 = \emptyset(m_1, m_2)$ and r is the radius of a sphere around a point x_0 then if r is sufficiently small then

$$c_1 r \leq 1 \tag{3.15}$$

If we denote the distance $|x_0 - y_0|$ with d_0 *and* θ_0 is the angle between the normal at x_0 and the normal at any point of the surface, then [15, 17]

$$\cos\theta_0 \geq 1 - \frac{1}{2}\theta_0^2 \geq 1 - \frac{1}{2}{c_1}^2 d_0^2 > \frac{1}{2} \tag{3.16}$$

Also we have

$$\frac{1}{\cos\,\theta_0} = \sqrt{1 + \emptyset_{m_1}^2 + \emptyset_{m_2}^2} \leq 1 + {c_1}^2 d_0^2 \leq 2 \tag{3.17}$$

Hence we obtain ,

$$\emptyset_{m_1}^2 + \emptyset_{m_2}^2 \leq 2\, {c_1}^2 d_0^2 + {c_1}^4 d_0^2 \qquad (3.18)$$

Using the polar coordinates $m_1 = \zeta \cos\theta\,, m_2 = \zeta \sin\theta$yields

$$\emptyset_\zeta^2 = (\emptyset_{m_1}\cos\theta + \emptyset_{m_2}\sin\theta)^2 \leq \emptyset_{m_1}^2 + \emptyset_{m_2}^2 \qquad (3.19)$$

In virtue of (3.18) and $|\emptyset| \leq \sqrt{3}\zeta$ and $d_0 \leq 2\zeta$, we obtain

$$\left|\emptyset_\zeta\right| \leq 2\sqrt{3}\, c_1 \zeta. \qquad (3.20)$$

Also (3.16) yields

$$1 - \cos\theta_0 \leq 2{c_1}^2\zeta^2 \qquad (3.21)$$

Consequently

$$|\cos(n, e_1)| \leq \left|\emptyset_{m_1}\right| \leq \sqrt{3} c_1 d_0 \qquad (3.22)$$

$$|\cos(n, e_2)| \leq \sqrt{3} c_1 d_0\,, \quad |\cos(n, e_3)| = \cos\theta_0 \qquad (3.23)$$

Summarizing the above estimates, we have

$$|\emptyset| \leq c_2\zeta^2 |\cos(n, e_1)| \leq c_2\zeta$$

$$|\cos(n, e_2)| \leq c_2\zeta |\cos(n, e_3)| \geq \frac{1}{2} \qquad (3.24)$$

From (3.22), we get

$$|\cos((x-y), n(x))| = \left|\frac{n_x.(x-y)}{d}\right| \leq \emptyset_{m_1} \leq M_1 d \qquad (3.25)$$

and

$$|\cos((y-x), n(y))| = \left|\frac{n_y.(y-x)}{d}\right| \leq M_1 d \qquad (3.26)$$

with $M_1 = \sqrt{3}c_1$. Finally

$$|L^*(x,y)| = \left|\frac{\cos((x-y),n_x).\cos((y-x),n_y)}{d^2}\right| \leq M \qquad (3.27)$$

where $M = \frac{3C_1{}^2}{\pi}$.

Lemma 3.3 :

The mapping $K: L^P(\partial\Omega) \longrightarrow L^P(\partial\Omega)$ is compact for $1 \leq p \leq \infty$. Furthermore, we obtain:

(a) $||\widetilde{K}||$ *= 1 in* L^P *for* $1 \leq p \leq \infty$.

(b) The spectral radius $\rho(\widetilde{K}) = 1$.

Proof: Since $L^*(x,y)$ is integrable and $\widetilde{K}$ is a weakly singular operator (see Lemma 3.1) then the mapping

$\widetilde{K} : L^P(\partial\Omega) \rightarrow L^P(\partial\Omega)$ is compact. For 1 < P < ∞ and $q_0 \in L^P(\partial\Omega)$ and using $1/p + 1/q = 1$, we have

$$\left|\widetilde{K}q_0(x)\right| = \left|\int_{\partial\Omega_y} L^*(x,y)^{\frac{1}{p}+\frac{1}{q}} q_0(y)d\Omega_y\right|$$

$$\leq \left(\int_{\partial\Omega_y} L^*(x,y)d\Omega_y\right)^{1/q} \left(\int_{\partial\Omega_y} L^*(x,y)|q_0(y)|^p d\Omega_y\right)^{1/p} \quad (3.28)$$

Since $\int_{\partial\Omega_y} L^*(x,y)d\Omega_y = 1$ then

$$\left|\widetilde{K}q_0(x)\right| \leq \left(\int_{\partial\Omega_y} L^*(x,y)|q_0(y)|^p d\Omega_y\right)^{1/p} \quad (3.29)$$

Moreover, we obtain

$$\left\|\widetilde{K}q_0(x)\right\|_{L^P}^P = \int_{\partial\Omega_x} \left|\widetilde{K}q_0(x)\right|^P d\Omega_x$$

$$\leq \int_{\partial\Omega_y} |q_0(y)|^{\mathrm{p}} \int_{\partial\Omega_x} L^*(x,y)d\Omega_x d\Omega_y = \|q_0(x)\|_{L^P}^P \quad (3.30)$$

Hence we get $\|\widetilde{K}\| \leq 1$ in L^P, $1 \leq \mathrm{p} \leq \infty$. Equality can be achieved by taking $q_0 = 1$ which is obviously the eigenvector of $\widetilde{K}$ with the eigenvalue 1.

Finally, the Hilbert theorem together with $\widetilde{K}1 = 1$ implies that the Fredholm integral operator $\widetilde{K}$ has an eigenvalue λ $with$ $|\lambda| = \left\|\widetilde{K}\right\| = 1$.

Chapter Four

Fixed Point Theorem and Applications to the Radiosity Integral Equation

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Fixed Point Theorem and Applications to the Radiosity Integral Equation

Definition 4.1: The number z is called a fixed point for a function f if $f(z) = z$.

Examples 4.1:

Assume we want to determine the fixed points of the function $f(x) = x^2 - 2$. A fixed point z for the function f has the properties that

$z = f(z) = z^2 - 2$. This implies that:

$$0 = z^2 - z - 2 = (z + 1)(z - 2)$$

A fixed point for f occurs exactly when the graph of $y = f(x)$ intersects the graph of $y = x$, then f has the two fixed points, namely;

$$z_1 = -1 \ \ and \ \ z_2 = 2.$$

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point:

Theorem 4.1:[6]If $f \in C[a, b]$ and $f(x) \in [a, b]$ for every $x \in [a, b]$, then f has at least one fixed point in $[a, b]$. Moreover, if, in addition, $\acute{f}(x)$exists on (a, b) and a positive constant k, $0 < k < 1$ exists such that

$$\left|\acute{f}(x)\right| \leq k \qquad\qquad for\ every\ x \in (a, b)$$

Then, there is precisely one fixed point z in $[a, b]$.

Proof: If $f(a) = a \, and \, f(b) = b$ then f has a fixed point at the end points of $[a, b]$. Otherwise, $f(a) > a \, and \, f(b) < b$. The function $g(x) = f(x) - x$ is continuous on $[a, b]$ with $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$. Consequently, the intermediate value theorem implies that there exists $z \in (a, b)$ such that $g(z) = 0$. z is a fixed point for f because $0 = g(z) = f(z) - z$ which implies $f(z) = z$.

Moreover, we assume that $\left|\acute{f}(x)\right| \leq k < 1$ and z and m are both fixed points in $[a, b]$. If $z \neq m$ then the Mean Value theorem states that a number L exists between $z \, and \, m$ and hence $[a, b]$ such that

$$\frac{f(z) - f(m)}{z - m} = \acute{g}(L)$$

Thus

$$|z - m| = |f(z) - f(m)| = \acute{g}(L)|z - m|$$

$$\leq k|z - m| < |z - m|$$

which is a contradiction. This proves that $z = m$ and the fixed point in $[a, b]$ is unique.

The Banach Fixed Point Theorem

Definition 4.2: Let (Z,d) be a metric space. A contraction mapping on Z is a function $g: Z \to Z$ that satisfies

$$d\big(g(\acute{z}), g(z)\big) \leqq c_1 d(\acute{z}, z) \qquad\qquad \forall z, \acute{z} \in Z$$

for some real number $0 < c_1 < 1$.

Example 4.1: Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable real function. If a real number $c_1 < 1$ exists such that the derivative g' satisfies $|g'(z)| \leqq c_1$ for all $z \in \mathbb{R}$, then g is a contraction with respect to the metric on $\mathbb{R}$ and c_1 is a modulus of contraction for g. This is a consequence of the Mean Value Theorem: let $z, z' \in \mathbb{R}$ and suppose $z<z'$; the MVT states that there is a number $\zeta \in (z,z')$ such that $g(z') - g(z) = g'(\zeta)(z' - z)$, hence

$$|g(z') - g(z)| = |g'(\zeta)||(z' - z)| \leqq c_1 |(z' - z)|.$$

Theorem 4.2: Every contraction mapping is continuous.

Proof: Suppose that $W: Z \to Z$ is a contraction mapping on a metric space (Z,d), with modulus c_1, and let $\bar{z} \in Z$. Moreover, let $\epsilon>0$, and $\delta = \epsilon$, then

$$d(z,\bar{z}) < \delta \Longrightarrow d(Wz, W\bar{z}) \leqq c_1\delta < \epsilon.$$

Hence, W is continuous at $\bar{z}$. Since $\bar{z}$ is arbitrary, then W is continuous on Z. The above proof actually assures that a contraction mapping is uniformly continuous.

Definition 4.3: Suppose that (A,d_A) and (B,d_B) are metric spaces. A function $h : A \to B$ is uniformly continuous if for every $\epsilon > 0$ there is $\delta > 0$ such that

$$d_A(\acute{a}, a) < \delta \Longrightarrow d_B\big(h(a), h(\acute{a})\big) < \epsilon. \qquad \forall a, \acute{a} \in A$$

Theorem 4.3: [12] Every contraction mapping is uniformly continuous.

Theorem4.4. (Banach Fixed Point Theorem):

Every contraction mapping on a complete metric space has a unique fixed point. (This is also called the Contraction Mapping Theorem).

Proof: Suppose that$W$: $Z \to Z$ is contraction mapping on the complete metric space (Z,d), and let c_1 be a contraction modulus of W. First we want to show that W can have at most one fixed point. We construct a sequence which converges and show that its limit is a fixed point of W.

(a) Assume thatz and z' are fixed points of W. Then

$d(z, z') = d(Wz,Wz') \leqq c_1\, d(z,z')$; since $c_1 < 1$, this implies that

$d(z, z') = 0$. *i.e.*, $z = z'$.

(b) Let $z_0 \in Z$, and define a sequence $\{z_n\}$ as follows:

$$z_1 = Wz_0\,, \qquad z_2 = Wz_1 = W^2 z_0\,, \quad \ldots \qquad , z_n = Wz_{n-1} = W^n z_0, \ldots$$

We show that adjacent terms of $\{z_n\}$ grow arbitrarily close to one another – in particular, $d(z_n, z_{n+1}) \leqq c_1{}^n d(z_0, z_1)$:

$$d(z_1, z_2) \leqq c_1 d(z_0, z_1)$$

$$d(z_2, z_3) \leqq c_1 d(z_1, z_2) \leqq c_1{}^2 d(z_0, z_1)$$

$$\ldots$$

$$d(z_n, z_{n+1}) \leqq c_1 d(z_{n-1}, z_n) \leqq c_1{}^n d(z_0, z_1)$$

Next we show that if$n < m$ then $d(z_n\,, z_m) < c_1{}^n \frac{1}{1-c_1}\, d(z_0\,, z_1)$:

$d(z\,,z_{n+1}) \leqq {c_1}^n d(z_0\,,z_1)$

$d(z_n\,,z_{n+2}) \leqq d(z_n\,,z_{n+1}) + \; d(z_{n+1}\,,z_{n+2})$

$$\leqq {c_1}^n d(z_0\,,z_1) + {c_1}^{n+1} d(z_0\,,z_1) = d({c_1}^n + {c_1}^{n+1}) d(z_0\,,z_1)$$

…..

$d(z_n\,,z_m) \leqq ({c_1}^n + {c_1}^{n+1} + \cdots + {c_1}^{m-1}) d(z_0\,,z_1)$

$$= {c_1}^n (1 + c_1 + {c_1}^2 + \cdots + {c_1}^{m-1-n}) d(z_0\,,z_1)$$

$< {c_1}^n (1 + c_1 + {c_1}^2 + \cdots) d(z_0\,,z_1)$

$$= {c_1}^n \frac{1}{1 - c_1}\; d(z_0\,,z_1).$$

Therefore $\{z_n\}$ is Cauchy sequence, that is, : for $\epsilon > 0$, let N be large enough such that ${c_1}^N \frac{1}{1-c_1} d(z_0\,,z_1) < \epsilon$, which ensures that $n,m > N$ $\Longrightarrow d(z_n\,,z_m) < \epsilon$.Since the metric space (Z,d) is complete, the Cauchy sequence$\{z_n\}$ converges to a point $z^* \in Z$. We show that z^* is a fixed point of W: since $z_n \rightarrow z^*$and W is continuous, we have $Wz_n \rightarrow Wz^*$ — *i.e.*, $z_{n+1} \rightarrow Wz^*$.Since $z_{n+1} \rightarrow z^*$and $z_{n+1} \rightarrow Wz^*$, we have $Wz^* = z^*$.

First Cournot Equilibrium Example:

Suppose that two factories are producing goods at output levels $p_1$and p_2. Each factory responds to the other factory's production level when choosing its own level of output. In particular (with c_1, c_2, d_1, d_2 all positive), we have

$$p_1 = r_1(p_2) = c_1 - d_1 p_2$$

$$p_2 = r_2(p_1) = c_2 - d_2 p_1$$

but $p_i = 0$if the above expression for p_i is negative. The function $r_i : \mathbb{R}_+ \rightarrow \mathbb{R}_+$is factory$i$'s reaction function. Define $\bar{r} : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2$ by $\bar{r} : (p_1, p_2) = \big(r_1(p_2), r_2(p_1)\big)$. The function$\bar{r}$ is a contraction with respect to the city-block metric if $d_1, d_2 < 1$:

$$d(\bar{r}(p), \bar{r}(\acute{p})) = |\bar{r}_1(p) - \bar{r}_1(\acute{p})| + |\bar{r}_2(p) - \bar{r}_2(\acute{p})|$$

$$= |(c_1 - d_1 p_2) - (c_1 - d_1 \acute{p}_2)| + |(c_2 - d_2 p_1) - (c_2 - d_2 \acute{p}_1)|$$

$$= d_1 |\acute{p}_2 - p_2| + d_2 |\acute{p}_1 - p_1|$$

$$\leqq \max\{d_1, d_2\}(|p_1 - \acute{p}_1|) + |p_2 - \acute{p}_2|)$$

$$= \max\{d_1, d_2\} \mathrm{d}(p, \acute{p}).$$

Therefore we have an "existence and uniqueness result" for Cournot equilibrium in this example: $\bar{r}$ has a unique fixed point p^*— a unique Cournt equilibrium — if each d_i<1.

4.1. Successive Approximation Method(Picard Iteration)

For any initial point$z_0 \in Z$, the sequence $z_n = W(z_{n-1})$ converges to the fixed point z^*, than the previous one. Consequently, starting from any arbitrary point in Z, we can iteratively apply the function W to the current approximationof $z^*$to obtain a better approximation and each approximation converges to z^*. This provides a straightforward computation of z^*.

Theorem 4.5: [20] *Let* (X, d) be a complete metric and $T: X \rightarrow X$ a strict contraction, that is, there exists a, $0 \leq a < 1$ such that

$$d(Tx, Ty) \leq a.\, d(x, y) \text{ for all } x, y \in X$$

Then the Picard iteration (the sequence of successive approximation) (x_n), given by

$$x_n = T(x_{n-1}) = T^n(x_0), n = 1, 2, \ldots.$$

converges to the unique fixed point x^* of T,

$$d(x_n, x^*) \leq \frac{a^n}{1-a}.\, d(x_0, x_1), n \geq 1;$$

$$d(x_n, x^*) \leq \frac{a}{1-a}.\, d(x_n, x_{n-1}), n \geq 1;$$

Banach fixed point theorem for the radiosity Fredholm integral equation:

A straightforward method for the existence of the solution of the integral equation (2.16) is the application of Banach's fixed point theorem. The successive approximation method (Picard's iteration) can be used and the convergence of the Neumann series can be proved. We want to show first that the integral operator

$$K = (1-\varepsilon)\widetilde{K}: L^P(\partial\Omega) \to L^P(\partial\Omega) \text{ for } 1 < P < \infty. \tag{4.1}$$

defines a contraction mapping, that is, there exists a constant $0 \leq c_1 < 1$ such that

$$\|Kq_0 - K\widetilde{q_0}\|_{L^P(\partial\Omega)} \leq c_1 \|q_0 - \widetilde{q_0}\|_{L^P(\partial\Omega)} \tag{4.2}$$

is satisfies. From the definition of

$$Kq_0 - K\widetilde{q_0} = (1-\varepsilon)\int_{\partial\Omega} M(x,y).\big(q_0(y) - \widetilde{q_0}(y)\big)d\Omega_y \tag{4.3}$$

and the application of Holder's inequality follows

$$|Kq_0 - K\widetilde{q_0}| \leq |(1-\varepsilon)| \left(\int_{\partial\Omega} M(x,y) d\Omega_y\right)^{1/q} . \left(\int_{\partial\Omega} M(x,y) |q_0 - \widetilde{q_0}|^P d\Omega_y\right)^{1/P} \tag{4.4}$$

with the conjugate exponent 1/p+1/q = 1. Since $\int_{\partial\Omega} M(x,y) d\Omega_y = 1$ (see Lemma (3.1), we get

$$|Kq_0 - K\widetilde{q_0}| \leq |(1-\varepsilon)| \left(\int_{\partial\Omega} M(x,y) |q_0(y) - \widetilde{q_0}(y)|^P d\Omega_y\right)^{1/P} \tag{4.5}$$

Then we obtain

$$\|Kq_0 - K\widetilde{q_0}\|^P_{L^P(\partial\Omega)} \leq |(1-\varepsilon)|^P . \int_{\partial\Omega_y} |q_0(y) - \widetilde{q_0}(y)|^P \int_{\partial\Omega_x} M(x,y) d\, \partial\Omega_x d\Omega_y \tag{4.6}$$

Consequently we have

$$\|Kq_0 - K\widetilde{q_0}\|^P_{L^P(\partial\Omega)} \leq |1-\varepsilon|^P . \left\|q_0(y) - \widetilde{q}_0(y)\right\|_{L^P(\partial\Omega)}. \tag{4.7}$$

Since 0 <ε<1, then for the constant c_1, we obtain

$c_1 := |1-\varepsilon|^P < 1$Hence the integral operator K is contractive on $L^p(\Gamma)$ and the iteration scheme $q_{0,n+1} = Kq_{0,n}$ for n= 1, 2, … is convergent. The sequence $\{q_{0,n}\}$ converges to some q_0 in the space $L^P(\partial\Omega)$, which solves the equation $Kq_0 = q_0$ in $L^P(\partial\Omega)$. The uniqueness of $q_0 \in L^P(\partial\Omega)$ follows directly from the contraction of K due to

$$0 < \|q_0 - \widetilde{q_0}\|_{L^P(\partial\Omega)} = \|Kq_0 - K\widetilde{q_0}\|_{L^P(\partial\Omega)} \leq c_1 \|q_0 - \widetilde{q_0}\|_{L^P(\partial\Omega)}, \; c_1 < 1 \tag{4.8}$$

Consequently, we have

$$(1-c_1). \left||q_0 - \widetilde{q_0}|\right|_{L^P(\partial\Omega)} \leq 0 \tag{4.9}$$

Sinceq_0 and $\widetilde{q_0}$ are two fixed point of K with (1−c_1)> 0 and

$\|q_0 - \widetilde{q_0}\| > 0$, then implies $q_0 = \widetilde{q_0}$ and we obtain the result.

Chapter Five

The Conductive – Radiative Heat Transfer Model

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The Conductive – Radiative Heat Transfer Model

Consider a heat radiative exchange in a non-convex enclosure $\Omega \subset \mathbb{R}^3$, which consist of two conducting enclosures that are bounded by diffuse-grey surfaces. In this case $\Omega = \Omega_1 \cup \Omega_2$. The boundary $\partial\Omega$ of Ω can be represented as $\partial\Omega = S \cup \Gamma$ where $\Gamma$is a part of the boundary where the heat radiation is taking place. Let T be the absolute temperature and q_0 is the reflected energy leaving the surface, then the boundary value problem (BVP):

$$-\nabla.(k \nabla T) = g_0 \qquad \text{in} \qquad \Omega, \tag{5.1}$$

where k is the heat conductivity coefficient and g_0 is the internal heat source.

Also, we have the boundary condition

$$T = T_0 \tag{5.2}$$

with T is the absolute temperature and T_0 is the effective external radiation temperature.

On the boundary Γ we have

$$-k\frac{\partial T}{\partial n} = q_r = q_0 - q_i \quad \text{on} \quad \Gamma \tag{5.3}$$

The outgoing radiation q_0 and the incoming radiation q_i are related by

$$q_i = \widetilde{K}q_0, \; q_0 = \varepsilon\sigma T^4 + \rho q_i = \varepsilon\sigma T^4 + (1-\varepsilon)\,\widetilde{K}q_0 \tag{5.4}$$

Lemma 5.1:

Let $1 \leq p \leq \infty$ and $0 < \varepsilon \leq 1$. Then the operator $I - (1-\varepsilon)\widetilde{K}$ from $L^p(\Gamma)$into itself is invertible and this inverse is nonnegative.

Proof: We will show that the spectral radius of the operator $(1-\varepsilon)$ $\widetilde{K}$isstrictly less than one. Then the inverse exists and can be written as Neumann series

$$(I-(1-\varepsilon)\widetilde{K})^{-1} = \sum_{i=0}^{\infty}((1-\varepsilon)\widetilde{K})^{i}$$

which shows that the inverse is positive.

Since $(1-\varepsilon)\widetilde{K}$is compact, it is sufficient to prove that 1 is not an eigenvalue of $(1-\varepsilon)\widetilde{K}$. Thus, suppose that there is $q_0 \in L^p(\Gamma)$such that $(1-\varepsilon)\widetilde{K}q_0 = q_0$. Then since $(1-\varepsilon)\widetilde{K}$ is a positive and compact operator, we have $q_0 \geq 0$ and hence according to Lemma 3.3 we obtain

$$\int_{\Gamma} q_0 d\Gamma = \int_{\Gamma} (1-\varepsilon)\widetilde{K} q_0 d\Gamma = \int_{\Gamma} q_0 d\Gamma - \int_{\Gamma} \varepsilon \widetilde{K} q_0 d\Gamma \qquad (5.5)$$

Therefore, $\varepsilon\widetilde{K}q_0 = 0$ on Γ and hence $\widetilde{K}q_0 = q_0$. Moreover from Lemma 3.3 and sine the eigenvalue λ_0 of $\widetilde{K}$ is simple it follows that q_0 is a constant, but since $(1-\varepsilon)\widetilde{K}q_0 = q_0$, this constant has to be zero.

5.1 The weak form and some existence results

The weak formulation

Before we consider the weak formulation of the problem (5.1) – (5.4) we can solve for the intensity q_0 in equation (5.4) to obtain

$$q_r = (I - \widetilde{K})q_0 = (I - \widetilde{K})(I - (1-\varepsilon)\widetilde{K})^{-1}\varepsilon\sigma T^4 = G\sigma T^4 \ , \qquad (5.6)$$

An alternative formulation can be obtained by rearranging the terms in equation (5.4)

$$q_r = \varepsilon\sigma T^4 - \varepsilon q_i = \varepsilon\sigma T^4 - \varepsilon\widetilde{K}(I - (1-\varepsilon)\widetilde{K})^{-1}\varepsilon\sigma T^4, \qquad (5.7)$$

which physically means that q_r is composed of the difference between the emitted and the absorbed radiation. The non-local operator G in equation (5.6) is called the Gebhart factor [31,32]. Consequently, the boundary condition (5.3) can now with the use of (5.6) be rewritten to yield the non-local condition

$$k\frac{\partial T}{\partial n} + G(\sigma T^4) = 0 \quad \text{on}\Gamma \qquad (5.8)$$

The weak formulation of the system (5.1) – (5.4) together with (5.8) reads:

$$a(T, w_0) + b(T, w_0) = \langle g, w_0 \rangle, \qquad \forall w_0 \in X \qquad (5.9)$$

with

$$a(T, w_0) = \int_\Gamma k \, \nabla T \, \nabla w_0 \qquad (5.10)$$

$$b(T, w_0) = \int_\Gamma G(\sigma|T|^3 T) w_0, \qquad (5.11)$$

and $\langle g, w_0 \rangle$ is the duality pairing between X and $\acute{X}$.

If was set $X = L^2(\Gamma)$ then the weak form (5.9) is well defined (in the three-dimensional case) and T is defined in X [31, 32].

Lemma 5.2:[16] The operator G from $L^p(\Gamma)$ into itself is positive semi-definite.

Lemma 5.3:[31]The problem

$$a(T,w_0) + \int_\Gamma \ \varepsilon\,\sigma|T|^3 T w_0 = \langle g, w_0\rangle, \qquad \forall w_0, T \in X, \tag{5.12}$$

has a unique solution $T \in X$ for all $f \in \acute{X}$.

Theorem 5.1. Let Ω be a three dimensional enclosure with Γ is $C^{1,m}$ a Lyapunov surface where $m \in [0,1]$, and suppose that $T_0 \in L^2(\Gamma), g \in \acute{X}$ and there exist two functions $\emptyset_1 \leq \emptyset_2, \emptyset_1, \emptyset_2 \in L^2(\Gamma)$, such that

$$a(\emptyset_1, w_0) + b(\emptyset_1, w_0) \leq \langle g, w_0\rangle \qquad \forall w_0 \in X^+ \tag{5.13}$$

$$a(\emptyset_2, w_0) + b(\emptyset_2, w_0) \geq \langle g, w_0\rangle \qquad \forall w_0 \in X^+ \tag{5.14}$$

Then (5.9) has a unique solution *T*. Furthermore $\emptyset_1 \leq T \leq \emptyset_2$ in Ω and $\emptyset_1 \leq T_0 \leq \emptyset_2$ on Γ. We denote by X^+ the cone of the non-negative elements of *X*: $X^+ = \{w_0 \in X, w_0 \geq 0\}$.

The proof of theorem (5.1) is a consequence of the following Lemmas:

Lemma 5.4: [16] *Let* $[u] = \max((u - \emptyset_2)^+, \min(0, -(\emptyset_1 - u)^+))$ and $c(u, w_0) = \int_\Gamma \ [u]^4 w_0$ then the modified problem reads as:

$$A(T,w_0) = a(T,w_0) + b(T,w_0) + c(T,w_0) = \langle g, w_0\rangle, \quad T, w_0 \in X \tag{5.15}$$

and has a unique solution.

Proof. We shall prove that A is monotone. *Let* $T_1, T_2 \in X$ be arbitrary temperatures. Then the following estimate holds:

$$A(T_1, T_1 - T_2) - A(T_2, T_1 - T_2) = a(T_1 - T_2, T_1 - T_2) \tag{5.16}$$

$$+\int_{\Gamma} G(\sigma(T_1^4 - T_2^4))(T_1 - T_2)$$

$$+\int_{\Gamma} ([T_1]^4 - [T_2]^4)(T_1 - T_2)$$

$$\geq c\|T_1 - T_2\|^2_{L^2(\Gamma),}$$

where c is the coercivity constant of a. In (5.16) the estimation of the first and third term is quite obvious [16], however, for the non-local term we use the monotony law simply by writing $T_1^4 - T_2^4 = 4\left|\tilde{T}\right|^3(T_1 - T_2)$. Then one obtains

$$\int_{\Gamma} G(\sigma(T_1^4 - T_2^4))(T_1 - T_2) = \int_{\Gamma} G(\sigma f(T_1 - T_2))(T_1 - T_2) \geq 0$$

Since $f = 4\left|\tilde{T}\right|^3 \geq 0$ on Γ and Lemma (5.2) holds for G. As A is hemicontinuous and the additional term is also coercive in X, it implies the existence of at least one solution [16]. Moreover, because A is strictly coercive in L^2, the solutions is also unique.

Lemma 5.5:

The solution of the modified problem (5.15) is also a solution of the original problem (5.9).

Proof. We need to show that the solution T of the modified problem (5.15) satisfies $\emptyset_1 \leq T \leq \emptyset_2$ in $\Omega \cup \Gamma$. Thus it is sufficient to prove that $T \leq \emptyset_2$ then the other inequality follows directly using similar procedure. Subtracting the modified problem (5.15) from the condition imposed on$\emptyset_2$ yields

$$a(\emptyset_2 - T, w_0) + b(\emptyset_2, w_0) - b(T, w_0) \geq c(T, w_0), \quad \forall w_0 \in X^+. \quad (5.17)$$

Now,

$$b(\emptyset_2, w_0) - b(T, w_0) = \int_\Gamma G\left(\sigma\left({\emptyset_2}^4 - T^4\right)\right) w_0.$$

Since$\emptyset_2$ and T are defined in $L^2(\Gamma)$, there exists a function $g \geq 0$ such that ${\emptyset_2}^4 - T^4 = f(\emptyset_2 - T)$ on Γ. Let $\tilde{b}(u, w_0) = \int_\Gamma G(\sigma f u) w_0$ then (5.17) takes the form

$$a(\emptyset_2 - T, w_0) + \tilde{b}(\emptyset_2 - T, w_0) \geq c(T, w_0) \qquad \forall w_0 \in X^+ \qquad (5.18)$$

If we choose $w_0 = (\emptyset_2 - T)^-$then (5.18) gives

$$\begin{aligned} -a((\emptyset_2 - T)^-, (\emptyset_2 - T)^-) - \tilde{b}((\emptyset_2 - T)^-, (\emptyset_2 - T)^-) \\ + \tilde{b}((\emptyset_2 - T)^+, (\emptyset_2 - T)^-) \\ \geq \int_\Gamma ((\emptyset_2 - T)^-)^5 \\ \geq 0. \end{aligned} \qquad (5.19)$$

Again,

$$\begin{aligned} \tilde{b}((\emptyset_2 - T)^+, (\emptyset_2 - T)^-) = \int_\Gamma f\sigma\varepsilon(\emptyset_2 - T)^+(\emptyset_2 - T)^- \\ - \int_\Gamma H(f\sigma\varepsilon(\emptyset_2 - T)^+)(\emptyset_2 - T)^- \\ \leq 0. \end{aligned}$$

as H is nonnegative operator [31]. Hence it follows from the coercivity of a and the semi-coercivity of $\tilde{b}$ that $(\emptyset_2 - T)^- = 0$.

It must be indicated that when Ω is a two-dimensional enclosure then the non-linear boundary term is well defined for all $L^2(\Omega)$ temperature fields and according to [14, 16] there exists a unique solution T for (5.9).

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التحليل الرياضي لمشاكل الإشعاع الحراري

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الملخص

تم في هذه الأطروحة التركيز على الجوانب التحليلية لمعادلة فريدهولم التكاملية من النوع الثاني، والتي هي نتيجة لتغيرات الإشعاع الحراري الساقط على الأجسام المحدبة وغير المحدبة.

ظاهرة الإشعاع الحراري تلعب دوراً مهماً وعلى نطاق واسع في الكثير من التطبيقات في مجال العلوم والتكنولوجيا.

لقد تم دراسة موسعة لهذه المعادلة التكاملية وخاصة التحليلية منها.

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