An-Najah National University

Faculty of Graduate Studies

A Comparative Study in Cone Metric Spaces and Cone Normed Spaces

By

Dua'a Abdullah Mohammad Al-Afghani

Supervisor

Dr. Abdallah A.Hakawati

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This thesis was defended successfully on 11 / 8 /2016 and approved by:

Defense committee Members	**Signature**
– **Dr. Abdallah A.Hakawati /Supervisor**	…………………..
– **Dr. Ibrahim Almasri /External Examiner**	…………………
– **Dr. Muath Karaki /Internal Examiner**	…………………

Dedication

To my dearest people, who believed in me and led me to the road of success, to my husband, my son, my mother, my father, my sisters and brothers, also I'll never forget my best friends.

All dears, to the wonder of your hearts I send this dedication.

الاقرار

انا الموقعة ادناه مقدمة الرسالة التي تحمل العنوان:

A Comparative Study in Cone Metric Spaces and Cone Normed Spaces

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Signature: دعاء الأفغاني **التوقيع**

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Abstract

Cone metric spaces are, not yet proven to be generalization of metric spaces. In many occasions the answer was proved, not to be affirmative.

In this thesis we made a comparison between(Cone Metric Spaces and Cone Normed Spaces) *and* (Ordinary Metric Spaces and Normed Spaces) as a way to find an answer for our main contribution.

We choose the most important branches of mathematics to make a comparison as in: convergence, topology and best approximation theory. We also tried to transplant the idea of cone metric spaces in Orlicz's spaces.

We obtained new results while we investigate some properties which were proven to be incorrect in cone metric spaces but hold in ordinary case like as Sandwich Theorem, which gives us a sense of generality here.

Introduction

Cone metric spaces were defined in [1] by substituting an ordered normed space for the real numbers, by the means of partial ordering " $\leq$ " on Banach space (E, ‖ . ‖) via a cone P, The authors of this article and in [6], introduced the notion of cone normed spaces, where bounded linear operators between cone normed spaces were studied. It has been proven in [3] that every cone metric defined on a Banach space is really equivalent to a metric.

Recently, in [8], the authors proved that cone metric spaces are also topological spaces. Moreover, compactness, boundedness, first countability were discussed there.

This thesis is organized as follows:

Chapter one contains definitions and some examples which shall be needed in the following chapters. The topics include cones, cone metric spaces and cone normed spaces. This chapter is absolutely fundamental. A reader who is familiar with these topics may skip this chapter and refer to it only when necessary.

Chapter two has two purposes. First, we present some examples in cone metric and cone normed spaces and investigated them deeply to find out more properties of these spaces. Second, we introduce some properties of cone metric and cone normed spaces.

In chapter three we made a comparison between cone metric spaces and cone normed spaces *and* metric spaces and normed spaces in three branches; namely: convergence, topology, and best approximation.

In chapter four we introduce finite dimensional cone normed spaces, and make a comparison between these spaces and finite dimensional normed spaces, where genuine match were found. Also we try to give a definition of Orlicz's cone normed spaces by trying to insert the idea of cone metric in Orlicz spaces.

The main results of this thesis are:

1) We conclude that Sandwich Theorem holds in cone metric spaces if and only if the cone P is normal, and we provide a proof for this result.
2) We introduce a proposition that the comparison test holds in cone normed spaces if and only if the cone P is normal, and prove it.
3) We make a comparison between finite dimensional normed spaces and finite dimensional cone normed spaces, and find a noticable match in results under the condition that P is a normal cone.

Chapter One

Preliminaries and definitions

Chapter One

preliminaries

This chapter contains some definitions and basic results about cones, cone metric spaces, normed spaces and cone normed spaces which will be used in the subsequent chapters.

1.Cones:

Definition 1.1:[1]

Let E be a real Banach space with norm ‖ . ‖ and let P be a subset of E. then P is called a cone if:

1) P is closed, nonempty and $P \neq \{0\}$.
2) If a, b ≥ 0 , and x, y $\in$ P then ax + by $\in$ P.
3) If x $\in$ P and –x $\in$ P then x = 0.

Definition 1.2: [1]

Let P be a cone in E. we define a partial ordering $\leq$ with respect to P on E as:

1) $x \leq y$ if and only if $y - x \in P$.
2) $x < y$ if $x \leq y$ but $x \neq y$.
3) $x \ll y$ if $y - x \in P^{\circ}$. (P° is the interior of P).

Example1.1:[1]

Let R^2= E, and P= $\{ (x, y): x\geq 0, y\geq 0 \}$. P is indeed a cone.

Definition 1.3:

There are many types of cones, here we mention some of the frequently used ones:

1) normal cones:

P is called normal if $\exists$ $k \geq 0$, such that:

If $0 \leq x \leq y$, then $\| x \| \leq k \| y \|$. The least such k is called the normal constant of P.

2) regular cones :

P is called regular if every increasing sequence in E, which is bounded above, is convergent. Equivalently, the cone P is regular if and only if every decreasing sequence in E which is bounded from below is convergent in E.

3) minihedral cones:

P is called minihedral if sup {x, y} exists for every $x, y \in E$.

4) strongly minihedral cones :

P is called strongly minihedral if every set which is bounded above has a supremum.

5) positive cone of E:

Let $(E, \leq)$ be an ordered vector space, then $E^+ = \{ x \in E: x \geq 0 \}$ is called positive cone of E, members of E^+ are called positive elements of E, the non-zero elements of E^+ are called the strictly positive elements of E.

Definition 1.4:

The norm $\| . \|$ is called monotonic if $\forall$ x, y $\in$ E, $0 \leq x \leq y \Longrightarrow \| x \| \leq \| y \|$.

and called semi-monotonic if $\forall$ x, y $\in$ E, $\exists$ $k \geq 0$, such that

$0 \leq x \leq y \Longrightarrow \| x \| \leq k \| y \|$.

Example 1.2:[11]

Let E = $C_R^1[0, 1]$, be the space of real valued functions on [0, 1] which have continuous derivatives, with the supremum norm

$\| . \|_\infty = \max_{[0,1]}\{f \in E\}$, and P = {f $\in$ E: $f \geq 0$}.

Then P is a cone with normal constant of K = 1.

Example 1.3:[9]

The cone $[0, \infty)$ in (R, $| . |$), and the cone P={(x, y): x, $y \geq 0$} in R^2 are normal cones with normal constant K = 1.

Example1.4:[9]

Let E be the real Banach space, R^2, with the cone

P = { (x, 0) : $x \geq 0$ }. So P is a positive cone of E , P has an empty interior.

Example1.5[12]:

Let E = $C_R^2[0, 1]$, be the space of real valued functions on [0, 1] which have continuous second derivatives, with the norm

$\| f \| = \| f \|_\infty + \| f` \|_\infty$ and the cone P = {f $\in$ E: f (t) > 0}.

this cone is not normal cone, and not minihedral.

Example1.6:[12]

Let E = R^2 and P ={(x, y): x, $y \geq 0$}. The cone P is strongly minihedral in which each subset of P has an infimum.

Example1.7:[12]

let $E = R^2$ and $P = \{(x, 0): x \geq 0\}$. The cone P is strongly minihedral but not minihedral.

Lemma1.1:[11]

Every regular cone is normal.

Proof:

On the contrary, let P be a regular cone which is not normal.

For each $n \geq 1$, choose t_n and $s_n \in P$ such that $t_n - s_n \in P$,

but $n^2 \|t_n\| < \|s_n\|$.

For each $n \geq 1$, put $x_n = \frac{s_n}{\|t_n\|}$ and $y_n = \frac{t_n}{\|t_n\|}$, so x_n, y_n and $y_n - x_n \in P$.

$\|y_n\| = 1$ and $\|x_n\| > n^2$, for all $n \geq 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2} \| y_n \| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is uniformly convergent, by Wierstrass-M test, there is $y \in P$ s.t $\sum_{n=1}^{\infty} \frac{1}{n^2} y_n = y$.

We now see that :

$0 \leq x_1 \leq x_1 + \frac{1}{2^2} x_2 \leq x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_3 \leq \ldots\ldots \leq y$.

Because P is regular, the series $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$ is convergent. Hence, $\lim_{n\to\infty} \frac{\|x_n\|}{n^2} = 0$,

which is a contradiction.

Now according to example1.2 consider the following sequence of elements of E which decreasing and bounded from below but is not convergent in E.

$y \geq y^2 \geq y^3 \geq y^4 \geq \ldots\ldots \geq 0$.

therefore, the converse of lemma1.1 is not true.

Lemma 1.2:[11]

There is no normal cone with normal constant $K<1$.

Proof:

Suppose on the contrary, that E is a Banach space, and P is a normal cone with normal constant $K< 1$.

Take $x \in P$, $x \neq 0$ and $0 < \varepsilon < 1$, where $K< (1-\varepsilon)$.

Then, $(1-\varepsilon)\, x \leq x$

but $(1-\varepsilon) \| x \| > K \| x \|$, which is a contradiction.

2. Cone metric spaces:

Definition 2.1[1]:

A cone metric space is a pair (X, d) where X is any set and $d: X \times X \to E$ is a map , with the following satisfied :

1) $d(x, y) \geq 0 \ \forall x, y \in X$ and $d(x, y) = 0$ iff $x = y$.
2) $d(x, y) = d(y, x) \ \forall x, y \in X$.
3) $d(x, y) \leq d(x, z) + d(z, y) \quad \forall x, y, z \in X$.

Example 2.1:[12]

let $E=R^2$ and $P=\{(x, y): x, y \geq 0\}$, $X=R$, and $d: X \times X \to E$ such that

$d(x, y)= (|x-y|, a|x-y|)$ where $a > 0$ is a constant, then (X, d) is a cone metric space.

Example 2.2[12]

Let $E = R^n$ with

$P = \{(x_1, x_2, x_3, \ldots, x_n) : x_i \geq 0, \forall i =1,2,\ldots,n\}$

$X = R$ and $d: X \times X \to E$ such that d (x, y)= (|x-y|, a_1| x -y|,….. ,a_{n-1} | x- y|). where $a_i \geq 0$ for all l ≤ i ≤ n-1, Then (X, d) is a cone metric space.

Example2.3:[12]

Let E = $C_R^1[0,1]$ with the supremum norm and

P = {f ∈E: f (t) ≥ 0}.

Then P is a normal cone with normal constant K = l. Define

d: X×X→E by d(x, y)=|x−y| ϕ, where X=R, and

ϕ: [0,1]→R^+ such that ϕ(t) = e^t. Then d is a cone metric on X.

Definition 2.2[1]:

Let (X ,d) be a cone metric space and $\{x_n\}$be a sequence in X , then:

1) { x_n} is said to be convergent to x if ∀ e >> 0,

   ∃ n_0 ∈ N such that n ≥ n_0 ⟹ d(x_n, x) << e.

   in this case we write x_n→ x.

2) $\{x_n\}$ is called a Cauchy sequence in X whenever for every e >> 0 there is n_0 s.t for all m, n ≥ n_0 ,d(x_n,x_m) << e .

3) (X , d) is called a complete cone metric space if every Cauchy sequence is convergent.

Definition 3.2[8]:

A subset A of a cone metric space (X, d) is called sequentially closed if for every sequence $\{x_n\}$ ⊆ A, with x_n→ x we have x ∈ A.

Definition 4.2[8]:

A subset A of a cone metric space (X, d) is called sequentially compact , if for any sequence{ x_n} in A there is a subsequence{ x_{n_k} } of {x_n }such that{ x_{n_k}} is convergent in A.

proposition 2.2[8]:

let (X, d) be a cone metric space. Then every sequentially compact subset A ⊆ X is compact.

For the proof, we refer the reader to [8].

Definition 2.3:[6]

Let (X , d) be a cone metric space and A ⊆ X, Then:

1) A is said to be bounded above if ∃e ∈ E , e >> 0 s.t

d (x , y) << e ∀ x ,y ∈ A.

2) A is called bounded if δ (A) = sup { d(x , y) :x , y ∈ A} exists in E, thus if P is strongly minihedral, then being bounded is the same as being bounded above.

Proposition 2.2:[6]

Every Cauchy sequence in a cone metric space over a strongly minihedral cone is bounded.

3. Cone Normed Spaces:

Definition 3.1:[10]

Let X be a linear space over a field K, a norm on X is a function

$\|.\| : X \to R$ such that $\forall$ x, y $\in$X and a $\in$K , we have :

1) $\| x \| \geq 0$ and $\| x \| = 0$ iff $x = 0$.
2) $\| x+ y \| \leq \|x\| + \|y\|$.
3) $\| ax \| = |a| \|x\|$.

A normed linear space (X, $\|.\|$) is a linear space X with a norm on it .

Proposition3.1:[10]

1) Every normed space is a metric space with respect to the metric $d(x , y) = \|x - y\|$, and is called the metric induced by the norm.
2) For any two elements x and y of a normed space we have, $\big| \|x\| - \|y\| \big| \leq \|x - y\|$
3) A norm is a real valued continuous function.

Definition 3.2[1] :

Let X be a real vector space and E be a real Banach space ordered by the strongly minihedral cone P, then a cone normed space is an ordered pair (X , $\|.\|_c$) where $\|.\|_c : X \to E$. such that :

1) $\| x \|_c \geq 0$ and $\| x \|_c = 0$ iff x=0 .
2) $\| ax \|_c = |a| \| x \|_c$ $\forall$ a$\in$ R and all x $\in$ X.
3) $\| x + y \|_c \leq \| x \|_c + \| y \|_c$.

Definition 3.4:[1]

In a cone normed space (X , $\|.\|_c$) over (E , P, $\|.\|$) the sequence $\{x_n\}$ is said to be:

(1) convergent if $\exists x \in X$ s.t $\forall e \in E$ with $e >> 0$, $\exists\, n_0 \in N$ such that

$\forall$ n > n_0, $\| x_n - x \|_c \ll e$.

(2) Cauchy if for each $e \gg 0$ $\exists$ $n_0 \in N$ s.t for m, n $\geq n_0$ we have

$\| x_n - x_m \|_c \ll e$.

Definition3.5[9]:

A cone normed space (X , $\|.\|_c$) is called a cone Banach space if every Cauchy sequence in X is convergent in X.

Definition3.6[16] : (equivalent cone-norms)

Let X be a real vector space, P is a normal cone with normal constant k, $\| . \|_{c_1}: X \to E$ and $\| . \|_{c_2}: X \to E$ be two cone norms on X. $\| . \|_{c_1}$ is said to be equivalent to $\| . \|_{c_2}$ if there exist α, β > 0 such that:

$\alpha \| x \|_{c_1} \leq \| x \|_{c_2} \leq \beta \| x \|_{c_1}$.   For each $x \in X$

Definition 3.7[9]:

Let (X, $\| . \|_c$) be a cone-normed space, a subset A of X is said to be bounded if sup { $\| x - y \|_c$:x, y ∈ A } exists in E.

Example 3.1[13]:

Let X = R^2, P = {(x, y): x ≥ 0, y ≥ 0} ⊂ R^2 and

$\| (x, y) \|_c = (\alpha|x|, \beta|y|)$, α > 0, β > 0. Then, (X , $\| \cdot \|_c$) is a cone normed space over R^2.

Chapter Two

Examples and some properties

Chapter Two

Examples and some properties

This chapter presents some examples and some theorems on cone metric spaces and cone normed spaces.

1.Examples:

One of our main contributions is to characterize a comparison satisfaction. Specifically, we show that the Sandwich theorem holds if and only if P is normal.

Example 1.1 [12]:

Let E = $C_R^1[0, 1]$, be the space of real valued functions on [0, 1] which have continuous derivatives, P = { x ∈ E : x (t) ≥ 0 }. Let

$\| x \| = \| x \|_\infty + \| x` \|_\infty$, where $\| x \|_\infty = \max \{x(t) : t \in [0,1] \}$

and $\| x` \|_\infty = \max\{x`(t) : t \in [0, 1]\}$.

Let x (t) = t, y (t) = t^{2k} where k ≥ 1.

Here $0 \leq y \leq x \quad \forall\, t \in [0, 1]$

$\| x \| = \max \{t: t \in [0, 1] \} + \max \{ 1: t \in [0,1] \}$

$= 1 + 1 = 2.$

$\| y \| = \max \{t^{2k} : t \in [0,1] \} + \max \{ 2k\, t^{2k-1} : t \in [0,1] \}$

$= 1 + 2k$. So,

$0 \leq y \leq x$ but, $\| y \| > k \| x \|$.

Since k was arbitrary, P is not normal.

Example 1.2[3]:

Let E = C_R^1 [0, 1], P = {f ∈ E: f (t) ≥ 0} and ‖ f ‖ = ‖ f ‖$_\infty$.

Then P is a normal cone with normal constant 1.

Proof :

If f and g are bounded on a compact set, both are continuous.

Now if 0 ≤ f ≤ g then : ‖ f ‖$_\infty$ ≤ ‖ g ‖$_\infty$.

We conclude that the normality of the cone depends on the norm of E.

Example 1.3 [12]:

Let E = C_R^1[0,1], P = { x∈ E : x (t) > 0 } , and ‖ x ‖ = ‖ x ‖$_\infty$+‖ x` ‖$_\infty$.

Let $x_n(t) = \frac{x^{2n}}{n}$, $y_n(t) = \frac{1}{n}$ where $0 \le x_n \le y_n$.

$\lim_{n\to\infty} y_n = 0$, but

‖ x_n ‖ = max {$\frac{t^{2n}}{n}$: t ∈ [0,1] } + max { 2 t^{2n-1} :t∈ [0,1] }

$= \frac{1}{n} + 2.$

Hence x_n doesn't converge to zero.

It has been noticed in [12] that the sandwich theorem doesn't hold in cone metric spaces, and this is another example.

Example 1.4 [12]:

Let E = C_R^1[0,1], P = { x∈ E : x(t) ≥ 0 }, and ‖ x ‖ = ‖ x ‖$_\infty$+‖ x`‖$_\infty$.

Let $x_n(t) = \frac{1-\sinh(nt)}{n+2}$, and $y_n(t) = \frac{1+\sinh(nt)}{n+2}$

Clearly $0 \le x_n \le x_n + y_n$.

And $\| x_n \| = \| y_n \| = 1$, ∀ n.

now, $x_n + y_n = \frac{2}{2+n} \to 0$

but x_n doesn't converge to zero.

we see from these examples that the Sandwich theorem doesn't hold in non-normal cone metric spaces. This leads to the question:

If cones are normal, would we still have the same result?

The answer is negative. The proof will be given in proposition (1.1).

Example 1. 5:

Let E = $C_R^1[0,1]$, P = { x∈ E : x (t) > 0 } ,

and ‖ x ‖ = ‖ x $\|_\infty$. (our cone is normal)

Let $x_n(t) = \frac{x^{2n}}{n}$, $y_n(t) = \frac{1}{n}$ where $0 \leq x_n \leq y_n$.

$\lim_{n\to\infty} y_n = 0$, but

$\| x_n \| = \max \{\frac{t^{2n}}{n} : t \in [0,1] \}$

$= \frac{1}{n}$.

$\| x_n \| \to 0$ as $n \to \infty$.

Hence x_n converge to zero.

Example 1.6:

Let E = $C_R^1[0,1]$, P = { x∈ E : x(t) ≥ 0 },

and ‖ x ‖ = ‖ x ‖∞. (our cone is normal)

Let $x_n(t) = \frac{1-\sinh(nt)}{n+2}$, and $y_n = \frac{1+\sinh(nt)}{n+2}$

Clearly $0 \leq x_n \leq x_n + y_n$.

now, $x_n + y_n = \frac{2}{2+n} \to 0$ as $n \to \infty$.

$\| x \| = \max\{ x_n(t) : t \in [0, 1]\}$.

$= \max \{ \frac{1-sinsh\,(nt)}{n+2} : t \in [0, 1] \}$

$= \frac{2}{n+2} \to 0$ as $n \to \infty$.

thus x_n converge to zero.

That is to say the Sandwich theorem holds here.

Example 1.7:

Let E = $C_R^1[0,1]$, P = { x∈ E : x(t) > 0 }, and ‖ x ‖ = ‖ x ‖∞+‖ x\` ‖∞.

∀ n >1, let $x_n(t) = \frac{sin^2 nt}{n}$, , $y_n(t) = \frac{1}{n}$, where $0 \le x_n \le y_n$.

$\lim_{n\to\infty} y_n = 0$, but

$\| x_n \| = \max \{\frac{sin^2 nt}{n}: t \in [0,1] \} + \max \{ 2 \sin(nt) \cos(nt) \ :t\in [0,1] \}$

$= \frac{1}{n} + c.$ (c = max { 2 sin(nt) cos(nt) :t∈ [0,1] })

Hence x_n doesn't converge to zero.

So in this case Sandwich theorem doesn't hold.

But, if ‖ x ‖ = ‖ x ‖∞, then $\| x \| = \max \{ \frac{sin^2 nt}{n} : t \in [0, 1] \}$.

$\| x \| = \frac{1}{n} \to 0$ as $n \to \infty$.

So Sandwich theorem holds here.

This gave us the motivation to introduce and prove the following theorem.

Theorem1.2:

In Cone-Normed Spaces, the Sandwich theorem holds if and only if the cone P is normal.

Proof:

⟹ suppose the Sandwich theorem holds and P is not normal.

∀ n ≥ 1, choose t_n and s_n ∈ P such that

$S_n \le t_n$ but $n^2 \| t_n \| < \| s_n \|$.

For each n ≥ 1, put $x_n = \frac{s_n}{\|s_n\|}$, $y_n = \frac{t_n}{\|s_n\|}$, here $x_n \le y_n$.

And $\| x_n \| = 1$, $\| y_n \| < \frac{1}{n^2}$ for all n ≥ 1.

So, $\| y_n \| \to 0$ as $n \to \infty$.

But $\| x_n \| = 1$ for all n.

Which is a contradiction.

$\Leftarrow$ Let P be normal.

Thus for $0 \leq x_n \leq y_n$ there is $k \geq 1$ s.t $\| x_n \| \leq k \| y_n \|$.

Let y_n converge to zero.

i.e $\| y_n \| \to 0$ as $n \to \infty$.

since, $0 \leq \| x_n \| \leq 0$ as $n \to \infty$

thus $\| x_n \| \to 0$ as $n \to \infty$, that is to say x_n converge to zero.

Thus Sandwich theorem holds.

Example 1.8[12]:

Let $E = C_R^1[0,1]$, $P = \{ y \in E : y(t) \geq 0 \}$, and $\| y \| = \| y \|_\infty + \| y` \|_\infty$.

this is not a normal cone.

For all $n \geq 1$ and $t \in [0, 1]$ put

$x_n(t) = \frac{t^{(n-1)^2}}{(n-1)^2+1} - \frac{t^{n^2}}{n^2+1}$ and $y_n(t) = \frac{2}{n^2}$.

So, $0 \leq x_n \leq y_n$ and $s_n(t) = \sum_{k=1}^{n} x_k(t) = 1 - \frac{t^{n^2}}{n^2+1}$

Therefore, $\| s_n - s_m \| = \| \frac{t^{m^2}}{m^2+1} - \frac{t^{n^2}}{n^2+1} \|_\infty + \| \frac{m^2 t^{m^2-1}}{m^2+1} - \frac{n^2 t^{n^2-1}}{n^2+1} \|_\infty$

$$= \frac{1}{m^2+1} + \frac{m^2}{m^2+1} = 1.$$

For all m, n so $\{s_n\}$ is not Cauchy sequence, namely

$\sum_{k=1}^{\infty} x_k(t)$ is divergent

But, $\sum_{k=1}^{\infty} y_k(t) = \sum_{k=1}^{\infty} \frac{2}{k^2}$ is convergent.

It has been noticed in [12] that comparison test doesn't hold for series in some cone metric spaces.

The question here is, what would the result be if we impose normality?

For the same example let ‖ x ‖ = ‖ x ‖∞, which makes P normal.

Thus, $\| s_n - s_m \| = \| \frac{t^{m^2}}{m^2+1} - \frac{t^{n^2}}{n^2+1} \|_\infty$.

$= \frac{1}{m^2+1} \quad \rightarrow 0 \quad$ as m, n → ∞.

So the series $\sum_{n=1}^{\infty} x_n$ is convergent, so comparison test holds for this normal cone metric space.

Example1.9:

Let E = $C_R^1[0,1]$, with the norm ‖ x ‖ = ‖ x ‖∞+‖ x\` ‖∞.

and P = { x∈ E : x(t) ≥ 0 }.

For all n ≥ 1 and t ∈ [0, 1] put,

$a_n(t) = \frac{sin^2 nt}{n^2}$, $b_n(t) = \frac{1}{n^2}$.

so $0 \leq a_n \leq b_n$.

$\| a_n \| = \| \frac{sin^2 nt}{n^2} \|_\infty + \| \frac{2}{n} \sin(nt)\cos(nt) \|_\infty$.

$= \frac{1}{n^2} + \frac{2c}{n}$ (c = max { sin (nt) cos(nt) : t ∈ [0, 1]})

Thus $\sum_{n=1}^{\infty} a_n(t)$ is divergent,

since $\| a_n \|$ doesn't converge to zero as n → ∞,

and $\sum_{n=1}^{\infty} b_n(t)$ is convergent. Thus, the comparison test fails here .

For the same example let ‖ x ‖ = ‖ x ‖∞ which makes P normal.

Then, $\| a_n \| = \| \frac{sin^2 nt}{n^2} \|_\infty$.

$= \frac{1}{n^2} \rightarrow 0$ as n → ∞ .

Thus $\sum_{n=1}^{\infty} a_n(t)$ is convergent.

Hence comparison test holds for this example when the cone is normal.

Thus, again, we introduce the following proposition of ours.

Proposition1.3:

In cone normed spaces comparison test holds if and only if the cone P is normal.

Proof:

Suppose comparison test holds but P is not normal.

For each n $\geq$ 1 take a_n , $b_n \in$ P such that $a_n \leq b_n$

But $n^2 \| b_n \| < \| a_n \|$.

For each n $\geq$ 1, put $x_n = \frac{a_n}{\|b_n\|}$, $y_n = \frac{b_n}{\|b_n\|}$, here $x_n \leq y_n$.

here $\| y_n \| = 1$, $\| x_n \| > n^2$ for all n $\geq$ 1. ___________ (1)

The series $\sum_{n=1}^{\infty} \frac{1}{n^2} \| y_n \| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is uniformly convergent. Since P is closed,

There is y $\in$ P s.t $\sum_{n=1}^{\infty} \frac{1}{n^2} y_n = y$.

We now see that :

$0 \leq x_1 \leq x_1 + \frac{1}{2^2} x_2 \leq x_1 + \frac{1}{2^2} x_2 + \frac{1}{3^2} x_3 \leq \ldots.. \leq y$.

Because the comparison test holds, the series $\sum_{n=1}^{\infty} \frac{1}{n^2} x_n$ is convergent.

Hence, $\lim_{n\to\infty} \frac{\|x_n\|}{n^2} = 0$, a contradiction to (1).

Conversely, suppose comparison test doesn’t hold.

For each n $\geq$ 1 choose a_n , $b_n \in$ P such that $0 \leq a_n \leq b_n$.

Where $\sum_{n=1}^{\infty} b_n$ is divergent but $\sum_{n=1}^{\infty} a_n$ is convergent,

That is to say $\lim_{n\to\infty} \|b_n\| = 0$ and $\lim_{n\to\infty} \|a_n\| \neq 0$.

Thus Sandwich theorem doesn’t hold here,

Therefore the cone P is not normal.

Example 1.10 [12,11]:

let E be the real vector space:

E = {ax+ b: a, b ∈ R; x∈ [.5,1] }, with the supremum norm

$\| . \|_\infty$ = max{ | ax+ b | : x∈[0.5, 1]} and P = {ax+ b: a ≤ 0, b ≥ 0}.

So P is a normal cone in E with normal constant k ≥1,

Now, define:

f (x) = -4x + 20 and g (x) = -12x + 22

Then f ≤ g, since g (x) – f (x) = -8x + 2 ∈ P.

But, ‖ f ‖= f (.5) = 18 and ‖ g ‖ = g (0.5) = 16

therefore f ≤ g but ‖ f ‖ > ‖ g ‖.

It has been noticed in [12] that we can find two elements of normal cone where f ≤ g but ‖ f ‖ > ‖ g ‖.

We see here that this doesn't agree with the normality with k =1, thus we shall investigate this example analytically.

Is P normal?

The answer has been found in [11] as follows:

Let $\{a_n x + b_n\}_{n\geq 1}$ be an increasing sequence which is bounded above in E.

then, there is an element cx + d ∈ E such that:

$a_1 x + b_1 \leq a_2 x + b_2 \leq a_3 x + b_3 \leq \ldots \leq a_n x + b_n \leq cx + d$. $\forall$ x ∈ [0.5, 1].

Then :

$\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are two sequences in R such that:

$b_1 \leq b_2 \leq b_3 \leq \ldots \leq d$ and $a_1 \geq a_2 \geq a_3 \geq \ldots \geq c$.

thus $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are convergent by the monotone convergent theorem .

let $a_n \to$ a and $b_n \to$ b , but

ax+ b ∈ P, thus P is regular, hence normal.

But we are wondering, does ax + b necessarily belong to P?

Here we introduce this example, which agrees with the proof , but the limit doesn't belong to P.

Let $y_n = \frac{n+1}{n}$ x + $\frac{n}{n+1}$.

Where $\{\frac{n+1}{n}\}_{n\geq 1}$ is a decreasing sequence of numbers which is bounded below, and

$\{\frac{n}{n+1}\}$ is an increasing sequence of numbers which is bounded above.

Here, $y_2 - y_1 = -\frac{1}{2}x + \frac{1}{6} \in P$, thus $y_2 \geq y_1$.

But, $\lim_{n\to\infty} y_n = x + 1 \notin P$.

Although $\lim_{n\to\infty} y_n \notin$ P, but by the definition of regularity the cone P is regular, thus is normal.

We see here that in a normal cone with normal constant k ≠ 1, we find two elements in P where f ≤ g but ‖ f ‖ > ‖ g ‖.

Example 1.9[15] : (Cones may be non-minihedral)

Let E = $C_R^1[0,1]$, P = { g∈ E : g (t) ≥ 0 } , and ‖ g ‖ = ‖ g $\|_\infty$+‖ g` $\|_\infty$.

Let f (x) = sin x and g(x) = cos x . Both f and g in E.

but h = sup {f , g } doesn't belong to E

Since h is not differentiable at $\frac{\pi}{4}$.

Example 1.10 [9] :

Let E = $C_R^2[0, 1]$ with the norm ‖ f ‖ = ‖ f $\|_\infty$+‖ f` $\|_\infty$ and

P= { f ∈ E : f(t) ≥ 0},

take x_n (t) = $\frac{1-\sinh(nt)}{n+2}$ so $x_n \in$ E $\forall$n.

And let d: E×E → P (i.e X= E) be defined as :

$$d(f,g) = \begin{cases} f+g, & f \neq g \\ 0, & f = g \end{cases}$$

Clearly d is a cone metric on X, and ‖ x_n ‖ = 1, so x_n doesn't converge to zero.

Now, we will show that d (x_n , 0) → 0.

Let c >> 0 be arbitrary.

(so, c is an element in P° , i.e c (t) > 0 $\forall$ t ∈ [0,1])

The range of c is bounded below,

Let δ_o = inf {c(t): t ∈ [0, 1] }.

Choose $n_o \in$ N such that $\frac{1}{2+n} < \delta_o$.

Now $\forall$ t∈ [0, 1] and n $\geq$ n_o , we have

$$\begin{aligned} c(t) - x_n(t) &= c(t) - \frac{1-\sinh(nt)}{2+n} \\ &\geq \delta_o - \frac{1}{2+n} - \frac{\sinh(nt)}{2+n} \\ &\geq \delta_o - \frac{1}{2+n} > \delta_o - \frac{1}{2+n_0} > 0 \end{aligned}$$

Since t was arbitrary, we have: c >> x_n $\forall$ n ≥ n_o .

So, d (x_n, 0) → 0 thus x_n → 0.

We conclude from this example that if E = X and d is a cone metric on X then convergence in the norm is different from convergence in d.

2. Some properties of Cone Metric and Cone Normed Spaces:

Proposition2.1 [1]:

Every Cauchy sequence in a cone metric space over a strongly minihedral cone is bounded.

Proof:

Suppose $\{x_n\}$ is Cauchy.

Fix e >> 0.

Choose $n_0 \in N$ such that m, n $\geq n_0 \Longrightarrow$ d $(x_m\ x_n)$ << e.

Let e` = sup {e, d (x_m, x_n): m, n < n_0}.

e` exists since P is strongly minihedral .

$\Longrightarrow$ d $(x_m\ x_n)$ << e` for all m, n .

So, $\{x_n\}$ is bounded.

Theorem 2.1: (Translation invariance)

A cone metric space (X, d) induced by a cone norm on a cone normed space (X, $\|\ .\ \|_c$) satisfies:

1) d (x+ a, y+ a) = d (x, y)
2) d (ax, ay) = $|a|$ d(x, y) $\forall$ x, y $\in$ X and $\forall$ scalar a .

Proof:

We have d(x+ a, y+ a) = $\|(x+a) - (y+a)\|_c = \|x - y\|_c$ = d (x, y)

d (ax, ay) = $\|ax - ay\|_c = \|a(x-y)\|_c = |a|\,\|x-y\|_c = |a|$ d(x, y).

we see in previous theorems that the translation invariance in cone normed spaces is just the same as in normed spaces.

Chapter Three

Comparative remarks

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Comparative remarks

1. Convergence in cone metric spaces compared to convergence in metric spaces :

In this section we compare some properties of convergence of sequences in cone metric spaces with metric spaces, where we have the same results.

Theorem1.1 [1]:

Let (X, d) be a cone metric space with a strongly minihedral normal cone P, then:

1) a convergent sequence in X is bounded and its limit is unique.

2) if $x_n \overset{d}{\rightarrow} x$ and $y_n \overset{d}{\rightarrow} y$ in X , then $d(x_n , y_n) \rightarrow d(x , y)$.

Proof:

1) Fix $e \gg 0$.

choose $n_0 \in N$ such that $\forall\, n \geq n_0 \Longrightarrow d(x_n , x) \ll e$.

Let $e` = \sup \{e, d(x_n , x): n > n_0\}$

e` exists since P is strongly minihedral cone.

Then $d(x_n , x) \ll e`$ for all n .

Thus x_n is bounded.

Assume that $x_n \overset{d}{\rightarrow} w$ and $x_n \overset{d}{\rightarrow} z$ then we get

$0 \leq d(w, z) \leq d(w, x_n) + d(x_n, z) \rightarrow 0$ as $n \rightarrow \infty$

$\Longrightarrow d(w, z) = 0$.(by normality of P since the Sandwich theorem holds)

And the uniqueness w = z of the limit follows.

2) let $e \gg 0$ be given , let $\in > 0$ be arbitrary

Choose n_1 and $n_2 \in$ N such that :

n $\geq n_1 \rightarrow$ d(x_n, x) << $\frac{\epsilon}{2K\|e\|}$ .e and

n $\geq n_2 \rightarrow$ d(y_n, y) << $\frac{\epsilon}{2K\|e\|}$. e

where k is the normal constant of the cone P.

take n_0 = max$\{n_1, n_2\}$.

for n $\geq n_0$, we have :

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y_n, y)$$

$$<< \frac{\epsilon}{2K\|e\|} . e + \frac{\epsilon}{2K\|e\|} .e + d(x, y)$$

$$\Longrightarrow d(x_n, y_n) - d(x, y) << \frac{\epsilon}{K\|e\|} .e \quad \forall n \geq n_0 .$$

By the normality of P we have:

$$\| d(x_n, y_n) - d(x, y)\| \leq \frac{\epsilon}{K\|e\|} .\| e \| . K = \epsilon \quad \forall n \geq n_0.$$

$$\Longrightarrow \| d(x_n, y_n) - d(x, y) \| \rightarrow 0.$$

$$\Longrightarrow d(x_n\ y_n) \rightarrow d(x, y).$$

This theorem has a classical copy in metric spaces.

Here, we introduce this example which agrees with part two of the previous theorem.

Example1.1 :

Let E be the real Banach space R^2, with the cone

P = {(x, y): x, y $\geq$ 0}

And of course, here,we have, order $(x_1, y_1) \leq (x_2, y_2)$ if and only if

$x_1 \leq x_2$ and $y_1 \leq y_2$.

Let X = R^2 and define d: X×X → E, as:

$d((x_1, y_1), (x_2, y_2)) = (\alpha |x_1\text{-}x_2| , \beta |y_1\text{-}y_2|)$ where $\alpha , \beta \geq 0$.

This indeed makes (X, d) a cone metric space.

Suppose $z_n = (x_n, y_n)$ is a sequence in X and z = (x, y) ∈ E

Then $z_n \to$ z if and only if $x_n \overset{R}{\to}$ x and $y_n \overset{R}{\to}$y.

Proof :

Suppose $z_n \to$ z, and let ∈ > 0 be given.

Choose $n_0 \in N$ such that $n \geq n_0 \Longrightarrow d(z_n, z) \ll (\alpha\epsilon, \beta\epsilon)$

Now for $n \geq n_0$ $(\alpha|x_n\text{-x}|, \beta|y_n\text{-y}|) \ll (\alpha\epsilon, \beta\epsilon)$. Now,

$\alpha\epsilon - \alpha|x_n - x| > 0$ and

$\beta\epsilon - \beta|y_n - y| > 0 \quad \forall\, n \geq n_0$.

i.e $|x_n - x| < \epsilon$ and $|y_n - y| < \epsilon \quad \forall\, n \geq n_0$.

thus $x_n \overset{R}{\to}$ x and $y_n \overset{R}{\to}$ y.

Conversely, suppose $x_n \overset{R}{\to}$ x and $y_n \overset{R}{\to}$ y.

let $e = (e_1, e_2) \gg 0$ in R^2

for $\frac{e_1}{\alpha}$ and $\frac{e_2}{\beta}$, there is n_1 and $n_2 \in N$ such that

$n \geq n_1 \Longrightarrow |x_n - x| < \frac{e_1}{\alpha}$ and $|y_m - y| < \frac{e_2}{\beta} \quad \forall\, m \geq n_2$

let $n_0 = \max\{n_1, n_2\}$

for $n \geq n_0$, we have

$|z_n - z| = (\alpha|x_n - x|, \beta|y_n - y|) \ll (e_1, e_2)$

Since $e_1 - \alpha|x_n - x| > 0$ and $e_2 - \beta|y_n - y| > 0$.

$d(z_n, z) \ll e \quad \forall\, n \geq n_0$,

thus, $z_n \overset{d}{\to}$ z.

of course, this pointwise convergence is similar to that in R^n.

Theorem 1.2 [1] :

Every convergent sequence $\{x_n\}$ in a cone metric space is a Cauchy sequence.

Proof:

For any e ∈ E with e >> 0 , there is N such that ∀ n, m > N
d (x_n, x) << $\frac{e}{2}$, and d (x_m, x) << $\frac{e}{2}$.

hence, d (x_n, x_m) ≤ d(x_n, x) + d(x_m, x) << e .

therefore $\{x_n\}$ is a Cauchy sequence.

Proposition 1.1[9] :

Let $\{y_n\}$ be a Cauchy sequence in a cone metric space (X, d),

suppose { y_n} has a convergent subsequence y_{n_k}→ y

then y_n → y .

Proof:

Let e >> 0, e ∈ E.

There is n_0 ∈ N such that ∀ m, n ≥ n_0 ⟹ d(y_m, y_n) << $\frac{1}{2}$ e

We may choose n_0 such that for n ≥ n_0 we have d (y_{n_k}, y) << $\frac{1}{2}$ e.

Now, for n ≥ n_0 we have:

d (y_n, y) ≤ d (y_n, y_{n_k}) + d(y_{n_k}, y) = e

So, y_n → y as n → ∞ .

We close this section by this lemma which gave us a match with the classical case, but under the condition of normality of the cone P.

Lemma1.1 [1]:

let (X, d) be a cone metric space, P is a normal cone with normal constant k, and $\{x_n\}$ be a sequence in X, then :

1) $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

2) $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

Proof:

1) suppose $\{x_n\}$ converges to x . ∀ ∈ > 0 choose e ∈ E where

0 << e and K ‖e‖ < ∈ .

There is N, such that for all n > N d(x_n, x) << e .

So that when n > N, ‖ d(x_n, x) ‖ ≤ k ‖e‖ < ∈ .

this means d (x_n, x) → 0 when (n → ∞).

Conversely, suppose d (x_n, x) → 0 when (n → ∞).

For e ∈ E with e >> 0, there is δ > 0 s.t

‖x‖ < δ ⟹ e-x ∈ P°.

For this δ there is N, such that ∀ n > N ‖d (x_n, x) ‖ < δ ,

So e - d (x_n, x) ∈ P° , so d(x_n, x) << e ,

therefore $\{x_n\}$ converges to x .

2) suppose $\{x_n\}$ is a Cauchy sequence . ∀ ∈ > 0 choose e ∈ E where:

e >> 0 and K ‖ e ‖ < ∈ .

There is N∈ N, s.t for all n, m > N d (x_n, x_m) << e.

So that when n, m > N, ‖ d(x_n, x_m) ‖ ≤ k ‖ e ‖ < ∈ .

this means d(x_n, x_m) → 0 when (n, m → ∞).

Conversely, suppose d (x_n, x_m) → 0 when (n, m → ∞).

For e ∈ E with e >> 0, there is δ > 0 such that:

$\|x\| < \delta \Longrightarrow e - x \in P^\circ$.

For this δ there is N, such that ∀ n, m > N $\|d(x_n, x_m)\| < \delta$,

So $e - d(x_n, x_m) \in P^\circ$, so $d(x_n, x_m) << e$, therefore $\{x_n\}$ is a Cauchy sequence .

We conclude from all this that convergence in cone-metric spaces agrees with the definition of convergence in metric spaces.

2.Topologies on Cone Metric Spaces

Just like any metric the cone metric d induces a topology. To do so we need to introduce the following two lemmas.

Lemma 1.2 [8]:

Let (X, d) be a cone metric space with cone P and a real Banach space E. then for each $e \in E$ with $e >> 0$ there is a real number $\in > 0$ such that for any $x \in E$ with $\| x \| < \in$, we have $x << e$.

Lemma 2.2 [8]:

Let (X, d) be a cone metric space. Then for each $e_1 >> 0$ and $e_2 >> 0$, there is $e >> 0$ such that $e << e_1$ and $e << e_2$.

Theorem 1.2 [8] :

Every cone metric space (X, d) is a topological space.

Proof:

Let (X, d) be a cone metric space.

For $x \in X$ and $e >> 0$, let $B(x, e) = \{y \in X : d(x, y) << e \}$

Let $\mathbb{B}$= {B(x, e): x∈ X and e >> 0}, where B(x, e) is the usual ball of center x and radius e. also, $\mathbb{B}$ is the usual base for our " cone metric" topology.

Call G open if ∀ x∈ G there is B∈$\mathbb{B}$ such that x ∈ B ⊆ G .

This defines a topology on X,

1) ϕ is vacuously open , also X is open , since for any c_0∈ P° choose

x ∈ X , (so $c_0 \geq 0$) ; B(x, c_0) ⊆ X .

2) let G_1 and G_2 be open, and x ∈ $G_1 \cap G_2$ be arbitrary so there is e_1 >> 0 and e_2 >> 0 such that x ∈ B(x, e_i) ⊆ G_i , i = 1, 2 .

By lemma (2.2) choose e >> 0 such that c_1 >> e and c_2 >> e

Now, x∈ B (x, e) ⊆ B (x, c_1) ∩ B (x, c_2) ⊆ $G_1 \cap G_2$.

Thus $G_1 \cap G_2$ is an open set.

3) let $\mathcal{G}$ = { G_α: α∈A } be a family of open sets ; and let x ∈ $\cup\mathcal{G}$ be arbitrary.

So there is α_0∈ A such that x ∈ B(x, c) ⊆ G_{α_0}

Pick e >> 0 such that x ∈ B(x, e) ⊆ G_{α_0} ⊆ $\cup\mathcal{G}$

Thus $\bigcup_{\alpha\in A} G_\alpha$ is open.

Therefore

the collection T_d = {G ⊆ X : G is open }Is indeed a topology on X.

Furthermore,

Any cone metric space (X, d) is Hausdorff, and first countable.

3. Metrizability of cone metric spaces:

Here we will show that every cone metric defined on a real Banach space is really equivalent to a metric.

Theorem 3.1[2]:

For every cone metric $M : X \times X \rightarrow E$ there exists metric

$m : X \times X \rightarrow R+$ which is sequentially equivalent to M on X.

Proof:

Define $m(x, y) = \inf\{\| u \| : M(x, y) \leq u\}$. We shall prove that m

is an equivalent metric to M. If $m(x, y) = 0$ then there exists $\{u_n\}$ such that

$\| u_n \| \rightarrow 0$ and $M(x, y) \leq u_n$ $\forall n$. So $u_n \rightarrow 0$ and consequently for all $c >> 0$

there exists $N \in N$ such that $u_n << c$ for all $n \geq N$.

Thus for all $c >> 0$, $0 \leq M(x, y) << c$. that is to say $x = y$.

If $x = y$ then $M(x, y) = 0$ which implies that $m(x, y) \leq \| u \|$ for all $0 \leq u$.

Put $u = 0$ it implies $m(x, y) \leq \| 0 \| = 0$,

on the other hand $0 \leq m(x, y)$, therefore $m(x, y) = 0$.

It is clear that $m(x, y) = m(y, x)$.

To prove the triangle inequality,

for $x, y, z \in X$ we have,

$\forall \delta > 0$ there is u_1 s.t $\|u_1\| < m(x, z) + \delta$, $M(x, z) \leq u_1$, and

$\forall \delta > 0$ there is u_2 s.t $\|u_2\| < m(z, y) + \delta$, $M(z, y) \leq u_2$.

But $M(x, y) \leq M(x, z) + M(z, y) \leq u_1 + u_2$, therefore

$$m(x, y) \leq \| u_1 + u_2\| \leq \| u_1\| + \| u_2\| \leq m(x, z) + m(z, y) + 2\delta.$$

Since $\delta > 0$ was arbitrary so $m(x, y) \leq m(x, z) + m(z, y)$.

Now we shall prove that, for all $\{x_n\} \subseteq X$ and $x \in X$,

$x_n \rightarrow x$ in (X, m) if and only if $x_n \rightarrow x$ in (X,M).

We have:

$\forall n, m \in N \; \exists \; u_{nm}$ such that $\| u_{nm} \| < m(x_n, x) + \frac{1}{m}$, $M(x_n, x) \leq u_{nm}$.

Put $v_n = u_{nn}$ then $\| v_n \| < m(x_n, x) + \frac{1}{n}$

and $M(x_n, x) \leq v_n$. Now if $x_n \rightarrow x$ in (X, m) then

$m(x_n, x) \rightarrow 0$ and so $v_n \rightarrow 0$, therefore,

for all $c >> 0$ there exists $N \in \mathbb{N}$ such that $v_n << c$ for all $n \geq N$.

This implies that $M(x_n, x) << c$

for all $n \geq N$. Namely $x_n \rightarrow x$ in (X,M).

Conversely, for every real $\delta > 0$, choose $c \in E$ with $c >> 0$ and $\| c \| < \delta$.

Then there exists $N \in \mathbb{N}$ such that $M(x_n, x) << c$ for all $n \geq N$. This mean that for all $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$d(x_n, x) \leq \| c \| < \delta$.

Since, by [2], mutual generations of metrics and cone metrics produce sequentially equivalent topologies, the fact that both topologies are first countable implies that they are the same topology. See [5].

4. Best Approximation in Cone Normed Spaces:

Introduction:

Let $X = (X, \| . \|_c)$ be a cone normed space, G a subset of X, and $x \in X$,

An element $g_0 \in G$ is called a best approximant of x in G if

$\| x - g_0 \|_c = d_c(x, G) = \inf \{ \| x - g \|_c : g \in G \}$.

We see that for $x \in X$ a best approximant $g_0 \in G$ is an element of minimal cone-distance from the given x. such a g_0 may or may not exist.

We shall denote the set of all elements of best approximants of x in G by $p_c(x, G)$ i.e $p_c(x, G) = \{ g \in G : \| x - g \|_c = d_c(x, G) \}$.

Example1.4:

Take X= R^2 with the cone {(x, 0) : x ≥ 0}, with the usual metric on R

d(x, y) = $\sqrt[2]{(x_{1-}\ y_1) + (x_2 -\ y_2)}$.

Take G = {(x, y) : x =y, 0 ≤ x < 1}, take x = (2,2),

Then G has no best approximant for x.

Take y = ($\frac{1}{2\sqrt[2]{2}}$, 0) , then p_0= ($\frac{1}{2}$, $\frac{1}{2}$) is a best approximant of x.

Definition1.4 [5]:

Let (X, ‖ . $‖_c$) be a cone-normed space , and let G be a non-empty set in X, and x∈ X, we say that g_0 ∈ G is a cone-best approximation of x if

‖ x – g_0 $‖_c$ ≤ ‖ x – g $‖_c$ ∀ g ∈ G. we denote the set of best approximant of x in G by P_c(x, G).

Definition 2.4 [5]:

Let (X, ‖ . $‖_c$) be a cone-normed space , and let G be a non-empty set in X , for x ∈ X, we define the cone distance

d_c (x, G)= inf { ‖ x – g $‖_c$: g ∈ G }. The definition makes sense because every subset of P has an infimum.

{ - ‖ x – g $‖_c$: g ∈ G } has a supremum and this is the required infimum.

The following theorem transforms word for word it’s dual from classical approximation theory.

Theorem1. 4 [5]:

Let (X, ‖ . $‖_c$) be a cone normed space with a minihedral cone P and G a subspace in X, then:

1) $d_c(y + g, G) = d_c(x, G)$ $\forall$ $y \in X$, $g \in G$.

2) $d_c(x + y, G) \leq d_c(x, G) + d_c(y, G)$ $\forall$ $x, y \in X$.

3) $d_c(\alpha y, G) = |\alpha| d_c(y, G)$ $\forall$ $\alpha \in R$, $y \in X$.

4) $\| d_c(x, G) - d_c(y, G) \|_c \leq \| x - y \|_c$.

Proof:

1) let $y \in X$, $g \in G$ and $a >> 0$.

Then by the definition of the infimum, there is $g_0 \in G$ such that

$\|y - g_0\|_c \leq d_c(y, G) + a$. so we have:

$d_c(y + g, G) \leq \|y + g - (g + g_0)\|_c = \|y - g_0\|_c \leq d_c(y, G) + a$.

since x, g were arbitrary and by the minihedrality of the cone P, we get that:

$d_c(y + g, G) \leq d_c(y, G)$ $\forall(y \in X, g \in G)$ …………(1)

now, replacing y by y+ g and g by – g , we get:

$d_c(y, G) \leq d_c(y + g, G)$ $\forall(y \in X, g \in G)$…………(2)

combining (1) and (2) we get the equality.

2) let $x, y \in X$, and $e >> 0$, so $\frac{1}{2} e >> 0$.

There is $g_1, g_2 \in G$ such that:

$\|x - g_1\|_c < d_c(x, G) + \frac{e}{2}$, and $\|y - g_2\|_c < d_c(y, G) + \frac{e}{2}$.

So, $d_c(x + y, G) \leq \| (x + y) - (g_1 + g_2)\|_c$

$$\leq \|x - g_1\|_c + \|y - g_2\|_c$$

$$\leq d_c(x, G) + \frac{e}{2} + d_c(y, G) + \frac{e}{2}$$

$$= d_c(x, G) + d_c(y, G) + e.$$

Since e were arbitrary, we get that :

$d_c(x + y, G) \leq d_c(x, G) + d_c(y, G)$.

3) let $y \in X$ and $B \neq 0$ be any scalar, and let $e >> 0$.

Pick $g_0 \in G$ for which $\|y - g_0\|_c \leq d_c(y, G) + \frac{e}{|B|}$.

So, $d_c(By, G) \leq \|By - Bg_0\|_c$

$= |B| \; \|y - g_0\|c$

$\leq |B| \; d_c(x, G) + e$.

Since e was arbitrary, $d_c(by, G) \leq |B| \; d_c(y, G)$ …….(1)

Now, applying this relation to bx in place of x, and $\frac{1}{B}$ in place of B , we get that:

$d_c(y, G) = d_c(\frac{1}{e} . Bx, G) \leq \frac{1}{|B|} d_c(Bx, G)$, and hence:

$|B| \; d_c(x, G) \leq d_c(cx, G)$ …………(2)

Combining (1) and (2) gives $|B| \; d_c(x, G) = d_c(Bx, G)$.

4) let $x, y \in X$ and let $e >> 0$.

Take $g_0 \in G$ so that $\|y - g_0\|_c \leq d_c(y, G) + e$

So, $d_c(x, G) \leq \| x - g_0\|_c \leq \|x - y\|_c + \|y - g_0\|_c$

$\leq \|x - y\|_c + d_c(y, G) + e$

Since e was arbitrary, $d_c(x, G) - d_c(y, G) \leq \|x - y\|_c$.

Similarly, we get $d_c(y, G) - d_c(x, G) \leq \|x - y\|_c$. thus,

$\| d_c(y, G) - d_c(x, G)\|_c \leq \|x - y\|_c$.

Once more, the next result stands firm, and as a mimic of what occurs in the classical setting.

Theorem 2. 4 [5]:

Let $(X, \| . \|_c)$ be a cone normed space with strongly minihedral cone P, and G is a subspace in X. then:

1) if $z \in G$ then $p_c(z, G) = \{z\}$.

2) if G is not closed then $p_c(x, G)$ is empty.

3) $p_c(x, G)$ is a convex set

Proof:

1) let $z \in G$, then the cone-distance between x and G must be zero.

Thus, if $g \in p_c(z, G)$ then $d_c(z, g) = 0 \Longrightarrow x=g$.

2) Suppose that G is not closed.

Pick $y \in \bar{G} \setminus G$. thus, for each e >> 0, there is $y_e \in G$ s.t

$\| y - y_e \|_c \leq e$.

Since P is strongly minihedral, then

$\| y - y_e \|_c = 0$, so $y = y_e$,

which implies that $y \in G$, a contradiction.

3) let $\mu = d_c(x, G)$.

The statement holds if $P_c(x, G)$ is empty or a singleton.

Suppose that $y, z \in P_c(x, G)$ and $y \neq z$.

For $0 \leq \alpha \leq 1$, let $w = \alpha y + (1 - \alpha) z$, then:

$$
\begin{aligned}
\| x - w \|_c &= \| x - (\alpha y + (1 - \alpha) z) \|_c \\
&= \| x - \alpha y - (1 - \alpha) z + \alpha x - \alpha x \|_c \\
&= \| \alpha(x - y) + (1 - \alpha)(x - z) \|_c \\
&\leq \alpha \| x - y \| + (1 - \alpha) \| x - z \|_c \\
&= \alpha\mu + (1 - \alpha) \mu \\
&= \mu.
\end{aligned}
$$

Since G is a subspace of X, $w \in G$, which implies that

$\mu \leq \| x - w \|_c$.

Therefore, $\| x - w \|_c = \mu$ and so $P_c(x, G)$ is convex.

Theorem 3.4:

Let G be a subspace of a cone normed space $(X, \| . \|_c)$, for $x \in X$:

1) If $z \in P_c(x, G)$ then $az \in P_c(ax, G)$ for all scalar a.
2) If $z \in P_c(x, G)$ then $z + g \in P_c(x + g , G)$ for all $g \in G$.

Proof:

for (1) if $g \in G$ and a is a scalar $\neq 0$, we have:

$$\| ax - g \|_c = |a| \, \| x - \frac{1}{a} g \|_c \geq |a| \, \| x - z \|_c$$

$$= \| ax - az \|_c.$$

Thus $az \in P_c(ax, G)$.

For (2) if $h \in G$ we have:

$\| x + g - h \|_c \geq \| x - z \|_c = \| x + g - (z+g) \|_c$

Hence $z + g \in P_c(x+ g, G)$.

We close this chapter with the following true copy of the classical theory in normed spaces.

Theorem 4.4 [5]:

Let $(X, \| . \|_c)$ be a cone normed space, and let G be a subspace of X, then for any $x \in X$,

1) $p_c(x, G)$ is a bounded set.

2) if G is closed then $p_c(x, G)$ is a closed set.

Proof:

1) Let $a \in P_c(x, G)$.

$\| a \|_c = \| a - x + x \|_c$

$\leq \| a - x \|_c + \| x \|_c$

$\leq \| 0 - x \|_c + \| x \|_c$ (since $0 \in G$)

$= 2 \| x \|_c \in E$.

So, $p_c(x, G)$ is bounded.

2) suppose that $\mu = d_c(x, G)$, and (a_n) be a sequence in $P_c(x, G)$ which converges in $(X, \| . \|_c)$ to a .

Since G is closed then, $a \in G$.

Now for each $n \in N$, $\| x - a_n \|_c = \mu$.

But since the cone norm is continuous, then $\| x - g \|_c = \mu$.

Thus, $p_c(x, G)$ is closed.

We see here that in the previous theorems on cone normed spaces we have the same results as in normed spaces, in the sense of best approximation.

Chapter Four

Finite dimensional cone normed spaces and compactness in cone normed spaces

Chapter Four

1.Finite dimensional cone normed spaces:

In this section we will consider the finite dimensional cone normed spaces, and again we see that results in cone metric space match results in metric spaces.

Definition:

Let (X, $\| \, . \, \|_c$) be a cone normed space where the real vector space X is of finite dimension. Then we say that (X, $\| \, . \, \|_c$) is a finite dimensional cone normed space.

Lemma1.1 :(linear combinations)

Let $\{x_1, x_2, x_3, \ldots\ldots., x_n\}$ be a linearly independent set of vectors in a cone-normed space X (of any dimension n) with a normal cone P. Then there is e ∈ E with e >> 0 such that for every choice of scalars $\alpha_1, \ldots\ldots, \alpha_n$ we have

$\|\alpha_1 x_1 + \ldots\ldots + \alpha_n x_n\|_c \geq e\,(|\alpha_1| + \ldots + |\alpha_n|)$ e >> 0.

Proof:

Define S= $(|\alpha_1| + \ldots + |\alpha_n|)$

If S = 0 , then $a_i = 0 \ \forall$ i = 1, 2, ……., n.

If S > 0, then $\|\alpha_1 x_1 + \ldots\ldots + \alpha_n x_n\|_c \geq c\,.S$ is equivalent to

$\|\beta_1 x_1 + \ldots\ldots + \beta_n x_n\|_c \geq c$, where $\beta_j = \frac{a_j}{S}$ with $\sum_{j=1}^{n} |\beta_j| = 1$.

Now, by the normality of the cone we have:

k $\| \|\beta_1 x_1 + \ldots\ldots + \beta_n x_n\|_c \| \geq \| c \|$, where k is the normal constant of P.

$\| \|\beta_1 x_1 + \ldots\ldots + \beta_n x_n\|_c \| \geq \frac{\| c \|}{k}$, where $\frac{\| c \|}{k} > 0$.

Suppose that the statement is false.

Then there is $y_m \in$ X such that,

$y_m = \beta_1^m x_1 + \ldots\ldots + \beta_n^m x_n$. Where $\sum_{j=1}^{n} \left|\beta_j^m\right| = 1$.

$\|\|y_m\|_c\| \to 0$ as m → ∞.

Since $\sum_{j=1}^{n} \left|\beta_j^m\right| = 1$, then, $\left|\beta_j^m\right| \leq 1$.

Fix j, $\beta_j^m = (\beta_j^1, \beta_j^2, \ldots\ldots, \beta_j^m)$ is bounded in R,

then by Bolzano-Weiestrass theorem, $\beta_j^{mr} \to \beta_j$ as r → ∞.

After n steps we obtain a sequence $y_{n,m} = (y_{n,1}, y_{n,2}, \ldots)$ of y_m whose terms of the form $y_{n,m} = \sum_{j=1}^{n} \gamma_j^m x_j$ where $\sum_{j=1}^{n} \left|\gamma_j^m\right| = 1$.

With scalars γ_j^m satisfying $\gamma_j^m \to \beta_j$ as m → ∞.

Hence as m → ∞,

$y_{n,m} \to$ y = $\sum_{j=1}^{n} \beta_j x_j$

$\Longrightarrow \|\|y_{n,m}\|_c\| \to \|\|y\|_c\|$ by continuity of the norm.

Since $\|\|y_m\|_c\| \to 0$ by assumption and $y_{n,m}$ is a subsequence of y_m we must have $\|\|y_{n,m}\|_c\| \to 0$.

Hence $\|\|y\|_c\| = 0$ this contradicts y ≠ 0, and the lemma is proved.

Under the assumption that the previous lemma and its proof are correct we introduce the following theorem.

Theorem1.1:

Every finite dimensional cone-normed space with a normal cone P, is complete.

Proof:

Let $\{x_n\}$ be arbitrary Cauchy sequence in X, with dim X= n

And $\{e_1, e_2, \ldots\ldots, e_n\}$ be a basis for X.

Then $\forall\ x_m \in X,\ x_m = \alpha_1^m e_1 + \ldots + \alpha_n^m e_n$.

Since $\{x_n\}$ is a Cauchy, then for each $\delta \in E$ with $\delta \gg 0 \exists\ n_0 \in N$ such that $\forall$ r, m $\geq n_0$, we have $\| x_r - x_m \|_c \ll \delta$.

From previous lemma we have:

$\delta \gg \| x_r - x_m \|_c = \| \sum_{i=1}^{n} (\alpha_i^r - \alpha_i^m) e_i \|_c \geq c \sum_{i=1}^{n} \left| \alpha_i^r - \alpha_i^m \right|$, $c \gg 0$

(m, r >N) .

$\delta \geq c \sum_{i=1}^{n} \left| \alpha_i^r - \alpha_i^m \right|$

By the normality of the cone P we have

$\| \delta \| k \geq \| c \| \sum_{i=1}^{n} \left| \alpha_i^r - \alpha_i^m \right|$.

Division by $\| c \| > 0$ gives

$\left| \alpha_i^m - \alpha_i^r \right| \leq \sum_{i=1}^{n} \left| \alpha_i^m - \alpha_i^r \right| \leq \frac{\| \delta \| k}{\| c \|}$. (m, r > N)

Thus, each of the n sequences $(\alpha_i^m) = (\alpha_i^1, \alpha_i^2, \ldots\ldots)$ i = 1, …… , n.

is Cauchy in R, Hence is converget.

Let $x = a_1 e_1 + a_2 e_2 + \ldots\ldots + a_n e_n$. where each a_i denotes the limit of α_i^m.

Clearly x ∈ X . furthermore,

$\| x_m - x \| = \| \sum_{i=1}^{n} (\alpha_i^m - \alpha_i)\, e_i \| \leq \sum_{i=1}^{n} \left| \alpha_i^m - \alpha_i \right| \| e_i \|$.

On the right $\alpha_i^m \to \alpha_i$.

hence $\| x_m - x \| \to 0$, that is $x_m \to x$.

This shows that $\{x_m\}$ is convergent in X, therefore X is complete.

Thus we conclude that every finite dimensional cone-normed space with a normal cone P, is a cone-Banach space.

Theorem 1.2:

Every finite dimensional subspace Y of a cone normed space X ordered with a normal cone P, is closed in X.

Proof:

Let Y be a finite dimensional cone normed space ordered with a normal cone P, then Y is complete. That is every Cauchy sequence in Y is convergent in Y ⟹ Y is closed.

Theorem1.3: (its classical version occurs in [14,p75])

On a finite dimensional cone normed space $(X, \| . \|_c)$, with a normal cone P, any cone norm $\| . \|_{c_1}$ is sequentially equivalent to any other cone norm $\| . \|_{c_2}$.

Proof:

Let X be a finite dimensional real vector space,

dim X = n, and basis for X = $\{ e_1, \ldots, e_n\}$

then ∀ x ∈ X, x = $a_1 e_1 + \ldots\ldots + a_n e_n$.

so, there is a b ∈ E, b >> 0 such that

$\| x\|_{c_1} \geq b (|\alpha_1| + \ldots + |\alpha_n|)$. by the normality of the cone P.

$k \| \| x\|_{c_1} \| \geq \| b \| (|\alpha_1| + \ldots + |\alpha_n|)$, where k >0 is the normal constant.

On the other hand, the triangular inequality gives:

$\| x\|_{c_2} \leq \sum_{j=1}^{n} |\alpha_j| \; \| e_j \|_{c_2} \leq M \sum_{j=1}^{n} | \alpha_j |$, $M = \max_j \| e_j\|_{c_2}$.

Again, by normality of the cone.

$\| \| x\|_{c_2} \| \leq \| M \| \, k \sum_{j=1}^{n} | \alpha_j |$

Together, $\| \| x \|_{c_2} \| \leq B \| \| x\|_{c_1} \|$, where $B = \frac{k^2 \| M \|}{\| b \|} > 0$.

By an interchange of the roles of $\| \cdot \|_{c_1}$ and $\| \cdot \|_{c_2}$ we get the inequality

$\alpha \| \| x \|_{c_1} \| \leq \| \| x \|_{c_2} \| \leq \beta \| \| x \|_{c_{c_1}} \|$.

This shows that the convergence of a sequence in finite dimensional cone normed space (with a normal cone P) doesn't depend on the particular choice of the norm of the space.

2. Compactness in cone normed spaces:

Proposition 2.2:

A subset M in a cone-normed space is bounded if and only if there is an $h \gg 0$ such that $\| x \|_c \leq h \ \forall x \in M$.

proof:

Let M be bounded, and suppose that

$\delta (M) = sup_{x,y \in M} \| x - y \|_c$ exists in E .

let $\delta (M) = b$.

fix $x_0 \in M$ and set $h = b + \| x_0 \|_c$.

$\| x \|_c = \| x - x_0 + x_0 \|_c \leq \| x - x_0 \|_c + \| x_0 \|_c$

$\| x \|_c \leq \ = b + \| x_0 \|_c = h$.

Conversely, suppose that for some $h \gg 0$.

$\| x \|_c \leq h \ \forall x \in M$. then,

$\| x - y \|_c \leq \| x \|_c + \| y \|_c = 2h$.

$\| x - y \|_c \leq 2h$,

and $\delta (M) < 2h$ where $h \in E$.

Thus M is bounded.

Lemma 2.2:

A compact subset M of a cone metric space (X, d) is closed and bounded.

Proof:

For every x ∈ $\overline{M}$. There is a sequence x_n in M such that:

$x_n \xrightarrow{d} x$.

since M is compact, x∈ M.

hence M is closed.

If M is unbounded, it would contain an unbounded sequence (y_n).

Let m be any fixed element in M,

We may assume that d (y_n, m) > n.

y_n cannot have a convergent subsequence, since a convergent sequence must be bounded.

So M is bounded.

Theorem 2.1:

In a finite dimensional cone normed space X with a normal cone P, any subset M⊆X is compact if and only if M is closed and bounded.

Proof:

Compactness implies closedness and boundedness.

To prove the converse.

Let M be a closed and bounded set in X.

dim X = n, and {e_1, e_2, ….., e_n}is a basis for X.

consider x_m in M, where,

$x_m = a_1^m e_1 + a_2^m e_2 + \ldots.. + a_n^m e_n$.

Since M is bounded so $\{x_m\}$.

Let h >> 0 , h ≥ ‖x_m ‖c = ‖$\sum_{j=1}^{n} a_j^m e_j$‖c ≥ e $\sum_{j=1}^{n} |a_j^m|$

where e >> 0.

h ≥ e $\sum_{j=1}^{n} |a_j^m|$, by the normality of the cone P

‖ h ‖ k ≥ ‖ e ‖ $\sum_{j=1}^{n} |a_j^m|$, where k is the normal constant of P.

Hence the sequence of numbers a_j^m is bounded,

And by Bolzano-Weiestrass theorem, has a limit a_j where 1≤ j ≤ n.

We conclude that $x_m$has a subsequence $z_m$which converges to z

Where z = $\sum a_j e_j$.

Since M is closed then z ∈ M.

The arbitrary sequence x_m in M has a convergent subsequence in M.

Hence M is compact.

This shows that in any finite dimensional cone-normed space, with a normal cone the compact subsets are precisely those which are closed and bounded.

3.Orlicz cone normed space

Orlicz spaces are Banach spaces , and in order to study them,it is necessary to introduce the definition of modulus function.

Definition 3.1[7]:

A function ϕ: [0 ,∞)→[0 , ∞) is called a modulus function if the following are satisfied :

1) ϕ is continuous at 0 from the right and strictly increasing .

2) ϕ(0) = 0 .

3) ϕ is a sub additive that is to say $\phi(x + y) \leq \phi(x) + \phi(y)$,

$\forall$ x, y $\in [0, \infty)$.

Examples of such functions are $\phi(x) = x^p$, $0 < p \leq 1$.

and $\phi(x) = \ln(1 + x)$.

Theorem 3.1[7]:

Every modulus function is continuous on $[0,\infty)$.

Definition 3.2[7]:

Let X be a real Banach space, and (T, μ) be a finite measure space. For a given modulus function ϕ, we define the Orlicz space as:

$L^{\phi}(\mu, X) = \{ f{:}T \to X : \int \phi(\|f(t)\|) d_{\mu}(t) < \infty \}$.

The function d: $L^{\phi}(\mu, X) \times L^{\phi}(\mu, X) \to [0, \infty)$ given by:

$d(f, g) = \int \phi(\|f(t) - g(t)\|) d_{\mu}(t)$.

defines a metric on $L^{\phi}(\mu, X)$.

For $f \in L^{\phi}(\mu, X)$ we write $\| f \|_{\phi} = \int \phi(\|f(t)\|) d_{\mu}(t)$.

Definition 3.3:

Let (Ω, Ƒ, μ) be a measure space, where Ω any set, Ƒ the measurable sets in Ω, and μ is a measure. And let E be a real Banach space, and P is a cone in E. let 1 be a non-zero fixed element of P.

Let the indicator function $I_A(w) = \begin{cases} 1 & , w \in A \\ 0 & w \notin A \end{cases}$.

So $I_A(w)$ is a function: $\Omega \to E$.

A simple functions on w is one which takes $\Omega \to E$, and takes the form

$s(w) = \sum_{k=1}^{n} \alpha_k I_{A_k}(w)$, where $\forall$ k = 1, 2, ……, n. $\alpha_k \in R$, $A_k \in$ Ƒ.

For a non-negative simple function s ($\alpha_k \geq 0$, ∀ k)

$\int s \, d\mu = \sum_{k=1}^{n} \alpha_k \, \mu(A_k)$.

Suppose f ≥ 0 is a measurable function: Ω → E, let S_f = {s: s is a simple measurable non-negative function Ω → E with s (w) ≤ f (w) ∀ w ∈ Ω }.

$\int f \, d\mu = \sup \{ \int S \, d\mu : s \in S_p \}$.

To ensure the well definition of this integral, we assume P is strongly minihedral, f is bounded in Ω (i.e there is z ∈ P s.t f(w) ≤ z ∀ w ∈ Ω) and that μ is a finite measure.

Now, for arbitrary function f, let f^+(w) = sup { f(w), 0}

f^-(w) = sup { -f(w), 0}.

So that f (w) = f^+(w) – f^-(w), here we can define

$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$.

Definition 3.4 :

A function ϕ_c: P →P is called a cone-modulus function if the following are satisfied :

1) ϕ_c is continuous at 0 from the right and strictly increasing .

2) $\phi_c(0) = 0$.

3) ϕ_c is a sub additive that is to say $\phi_c(x + y) \leq \phi_c(x) + \phi_c(y)$,

∀ x, y ∈ P.

Definition 3.5:

let E be a real Banach space ordered by a strongly minihedral positive cone P, and (T,μ) be a finite measure space . for a given cone-modulus function ϕ_c, we define the Orlicz cone-normed space as:

$\boldsymbol{L_c^{\phi}}$ (μ,X) = { f : T→E : $\int \phi_{\boldsymbol{c}}(\| \boldsymbol{f(t)} \|_{\boldsymbol{c}})\, \boldsymbol{d}\mu < \infty$ } .

Where $\|f_\phi\|_c$ = sup { $\int \phi_c(\| f(t) \|_c)\, d\mu$ }.

= sup { $\int s\, d\mu$: s is a simple measurable non-negative function: T → E, with $s(t) \leq \phi_c(\| f(t) \|_c)$ }.

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جامعة النجاح الوطنية

كلية الدراسات العليا

دراسة مقارنة في فضاءات القياس المخروطية وفضاءات المعايير المخروطية

إعداد

دعاء عبدالله محمد الأفغاني

إشراف

د. عبدالله حكواتي

قدمت هذه الاطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية ، نابلس- فلسطين.

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بإشراف

د. عبدالله حكواتي

الملخص

لم يتم حتى الان تقديم اثبات قاطع بان فضاءات القياس المخروطية هي تعميم لفضاءات القياس العادية .

في محاولة منا لايجاد اجابة لهذا الموضوع الجدلي فقد قمنا باجراء دراسة مقارنة بين فضاءات القياس و فضاءات االمعايير المخروطية و فضاءات القياس و المعايير العادية .

قمنا باختيار عدة افرع مهمة لاجراء المقارنة فيها و هي : التقارب في هذه الفضاءات و البناء التبولوجي لها و نظرية التقريب الامثل. كما حاولنا زرع فكرة فضاءات القياس المخروطي في فضاءات اورليكس .

توصلنا الى بعض النتائج الجديدة اثناء تقصي بعد الخصائص التي لا تكون صحيحة في فضاءات القياس المخروطي و لكنها تتحقق في فضاءات القياس العادية مثل نظرية الشطيرة، وهذا يعطينا انطباعا بوجود التعميم في هذا المجال .