An-Najah National University

Faculty of Graduate Studies

Topological Characters of Complete Metric Spaces

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This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus, Palestine.

2015

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Dedication

TO MY SISTER GHADEER GANIM

Acknowledgment

First and foremost I would like to thank Allah for giving me the strength and determination to carry on this thesis.

I would also like to express my special thanks of gratitude to my supervisor Dr. Fawas Abu diak for the true effort in supervising and directing me to come with this thesis.

Thanks are also due to all faculty members in the Department of Mathematics at An-Najah National University, and for the examiner Dr. Bassam Manasrah, Hebron University . And thanks also to the presence of Dr. Mohammad ALamlah Attorney academic, As an internal examiner

Finally, I would like to express the appreciation for my family who always give me the support and the concern.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل عنوان:

Topological Characters of Complete Metric Spaces

أقر أن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص باستثناء ما تمت الإشارة إليه حيثما ورد، وإن هذه الرسالة ككل أو جزء منها لم يقدم من قبل لنيل أية درجة أو بحث علمي أو بحثي لدى أي مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the Researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

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Abstract

In this thesis the topological aspects of complete metric spaces are studied.

Complete metric spaces, Characterizations of complete metric spaces and examples of complete metric spaces are presented. A completion of a metric space is discussed.

Function spaces and their topologies are defined and studied. Completeness of function spaces is considered. As an application, a construction of the well-known Peano space –filling curve is discussed.

Finally, Theorems of topological characters concerning complete spaces such as Heine-Borel theorem, Ascoli's theorem and Baire's theorem with their proofs are introduced, in addition to other matters concerning the subject. The existence of continuous nowhere-differentiable real-valued functions is proved.

Introduction

The concept of metric spaces was first introduced by M. Fréchet in 1906 in his paper [10]. He formulated the abstract notion of compactness. After that, many mathematicians studied the concept of completeness of metric spaces that is basic of all aspects of analysis. Although completeness is a metric property rather than a topological one, there are a number of theorems involving complete metric spaces that are topological in character. In this thesis, we study complete metric spaces with the most important examples and then explain deeply theorems of topological characters concerning complete metric spaces. We found that the study of topological aspects of complete metric spaces has a huge place in topology.

In chapter one we concentrate on the concept of complete metric spaces and provide characterizations of complete metric spaces. Also, we present a characterization of complete subspaces of complete metric spaces. Then we shed light on examples that play a pivotal role in analysis. Finally, we show that a non complete metric space has a completion (that can be made into a complete metric space) and any two completions are isometric to each other.

In chapter two we make a study of topologies defined on a given set of functions: the product topology, the set-set topology, and the uniform metric topology. Then we discuss the idea of completeness of a function space. The completeness of the spaces $C(X,Y)$, $B(X,Y)$ and $BC(X,Y)$ is studied where $C(X,Y)$ denotes the set of all continuous mappings of the set

X into a space Y, $B(X,Y)$ denotes the set of all bounded mappings of the set X into a space Y and $BC(X,Y)$ denotes the set of all bounded continuous mappings of the set X into a space Y . Finally, we construct the well-known Peano space –filling curve.

In chapter three we introduce the most important theorems of topological characters concerning complete metric spaces. We prove a theorem that characterizes compactness of a metric space and use it to prove Heine-Borel theorem and a classical version of Ascoli's theorem. Then we state and prove one form of Baire's theorem. Finally, we use Baire's theorem to prove the existence of continuous nowhere-differentiable real-valued functions.

Chapter One

Complete Spaces

Introduction

In 1906 M. Fréchet introduced the concept of metric space in [10]. The importance of completeness is to prove that a sequence converges without a prior knowledge of its limit. Therefore, completeness can be used to prove existence of the limits which is important in proving some theorems of topological characters.

1.1.Complete Metric Space:

In this section some definitions and theorems concerning metric spaces are provided. Characterizations of complete metric spaces are also presented.

Definition (1.1.1) [22]**:** Let X be a nonempty set. A function, $\rho: X \times X \to [0, \infty)$ that has the following properties:

(a) (positive definiteness) $\rho(x, y) = 0$ iff $x = y$ for $x, y \in X$;

(b) (symmetry) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(c)(triangle inequality) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$ is called a metric , or distance function, on X .

Definition (1.1.2) [52]**:** Let X be a nonempty set, and ρ be a metric on X. Then the pair (X, ρ) is called a metric space.

Definition (1.1.3) [52]**:** Let (x_n) be a sequence in a metric space (X,ρ), and let $c \in X$. Then the sequence (x_n) converges to c iff $\forall\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ (set of natural numbers) such that $\rho(x_n\ ,c) < \epsilon$ for all $n \geq n_0$.

Definition (1.1.4) [20]**:** A sequence (x_n) in a metric space (X,ρ) is called Cauchy if the following is true: For any $\epsilon > 0$, $\exists n_0 \in \mathbb{N}$ such that $\rho(x_n\ ,x_m) < \epsilon$ for all $n,m \geq n_0$.

Definition(1.1.5)[7]**:** Consider a sequence (a_n). Let (n_k) be a sequence of natural numbers that is strictly increasing; that is, $n_1 < n_2 < n_3 < \cdots$. Then the sequence (b_k) defined by $b_k = a_{n_k}$ for every index k is called a subsequence of the sequence (a_n), it is denoted by (a_{n_k}) .

Definition(1.1.6)[7]**:** A subsequence (a_{n_k}) of a sequence (a_n) in a metric space (X,ρ) converges to $a \in X$ if for any $\epsilon > 0$, $\exists m \in \mathbb{N}$ such that $\rho(a_{n_i}, a) < \epsilon$, for all $i \geq m$.

Theorem (1.1.7)[34]**:** If a Cauchy sequence (x_n) in a metric space (X,ρ) has a subsequence that converges to $x \in X$, then the whole sequence converges to x.

Proof: Let a subsequence$(x_{k_n} : k_1 < k_2 < \cdots)$ of a Cauchy sequence (x_k) in a metric space (X,ρ) be convergent to a point $x \in X$. Then for any $\epsilon > 0$, $\exists m_1 \in \mathbb{N}$ such that $\rho(x_{k_i}, x) < \frac{\epsilon}{2}$, for all $i \geq m_1$ ... (1) but (x_k) is Cauchy, so $\exists m_2 \in \mathbb{N}$ with $\rho(x_i\ ,x_j) < \frac{\epsilon}{2}$, for all $i\ ,j \geq m_2$ ... (2). Let $m = max\{m_1, m_2\}$, then for all $i \geq m$ we have :

$\rho\left(x_{k_i}, x\right) < \frac{\epsilon}{2}$ (by (1) , as $k_i \geq i$) and

$\rho(x_i, x_{k_i}) < \frac{\epsilon}{2}$ (by (2) , as $k_i \geq i$) .The result is that , for all $i \geq m$, we have:$\rho(x_i , x) \leq \rho(x_i , x_{k_i}) + \rho(x_{k_i} , x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Thus the sequence (x_k) converges to x. ■

Definition (1.1.8) [52]**:** The metric space (X , ρ) is called a complete space if every Cauchy sequence in X converges to a point in X .

Definition (1.1.9) [6]**:**If the metric space (X , ρ) is complete, then ρ is called complete metric on X.

The following theorems (**1.1.10**, **1.1.19** and **1.1.21**)characterize the completeness of a metric space:

Theorem (1.1.10)[35]**:** A metric space (X, ρ) is complete if every Cauchy sequence in (X, ρ) has a convergent subsequence in X.

Proof: Let $(\boldsymbol{x_n})$ be a Cauchy sequence in (X, ρ) that has a convergent subsequence. Then the whole sequence converges (by theorem **1.1.7**). Then (X, ρ) is complete.■

Definition(1.1.11)[35]**:** Let (X, ρ) be a metric space and $\epsilon > 0$**,** then the set $B(x, \epsilon) = \{y \in X : \rho(x, y) < \epsilon\}$ is called ϵ-ball centered at x.

Definition(1.1.12)[35]**:** If ρ is a metric on the set X, then the collection of all ϵ-balls $B(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, called the metric topology induced by ρ.

Definition (1.1.13)[13]**:** For a set E in the metric space (X, ρ) , the closure of E, denoted by $\bar{E}$, is $Int(E) \cup Bdy(E)$ where $Int(E)$ is the interior of E and $Bdy(E)$ is the boundary of E.

Theorem (1.1.14)[13]**:** Let (X, d) be a metric space and $E \subseteq X$. Then :

(i) $x \in \bar{E}$ iff $B(x, \epsilon) \cap E \neq \emptyset, \forall \epsilon > 0$.

(ii)$\bar{E} = \{x \in X : d(x, E) = 0\}$ where $d(x, E) = inf\{d(x, e) : e \in E\}$.

Proof: (i) Since $\bar{E} = Int(E) \cup Bdy(E)$, then it is clear that $x \in \bar{E}$ iff every open ball with center x intersects E.

(ii) By(i), if $x \in \bar{E}$, then for every $\epsilon > 0$ there exists a $y \in B(x, \epsilon) \cap E$ and, therefore, $d(x, E) = 0$. If $d(x, E) = 0$ then for every $\epsilon > 0$ there is a $y \in E$ such that $d(x, y) < \epsilon$; that is , $B(x, \epsilon) \cap E \neq \emptyset$, and thus, $x \in \bar{E}$.■

Theorem (1.1.15)[13]**:** A set E in a metric space (X, d) is closed iff every sequence of points in E which is convergent in X converges to a point in E.

Proof: Suppose E is closed in X and let (x_n) be a sequence of points in E such that (x_n) converges to $x \in X$. Then, by definition (**1.1.3**), for any $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$. So $x_n \in B(x, \epsilon)$ for all $n \geq n_0$ and any $\epsilon > 0$, but $x_n \in E$,$\forall n \in \mathbb{N}$. Thus, $B(x, \epsilon) \cap E \neq \emptyset, \forall \epsilon > 0$,and so $x \in \bar{E} = E$. For the converse: Let $x \in \bar{E}$. Then, $\forall n \in \mathbb{N}$, $B\left(x, \frac{1}{n}\right) \cap E \neq \emptyset$. Take $x_n \in B\left(x, \frac{1}{n}\right) \cap E$, then (x_n) is a sequence of points in E converges to $x \in X$.So by the condition it converges to $x \in E$. Hence E is closed.■

Corollary (1.1.16): Let (X, d) be a metric space , $E \subseteq X$ .If $x \in \bar{E}$ then there exists a sequence $(x_n)$of points in E which converges to x .

Proof: Clear by the proof of theorem(**1.1.15**).■

Definition (1.1.17) [45]**:**Let (X, ρ) be a metric space ,for $A \subseteq X$, $diam(A) = \sup\{\rho(x, y): x, y \in A\}$.

Theorem (1.1.18)[45]**:** Let (X, d) be a metric space ,for $A \subseteq X$, $diam(A) = diam(\bar{A})$.

Proof: Since $A \subseteq \bar{A}$, then the inequality $diam(A) \leq diam(\bar{A})$ is immediate. For the other inequality, let $x, y \in \bar{A}$, then there exist sequences (x_n) and (y_n) in A such that $d(x, x_n) < \frac{\epsilon}{2}$ and $d(y, y_n) < \frac{\epsilon}{2}$ for $n \geq n_0$, say, where $\epsilon > 0$ is arbitrary. Now for $n \geq n_0$, we have:

$$
\begin{aligned}
d(x, y) &\leq d(x, x_n) + d(x_n, y_n) + d(y_n, y) \\
&\leq \frac{\epsilon}{2} + d(x_n, y_n) + \frac{\epsilon}{2} \\
&\leq diam(A) + \epsilon
\end{aligned}
$$

so, $diam(\bar{A}) \leq diam(A)$, since $\epsilon > 0$ is arbitrary.

Thus , $diam(A) = diam(\bar{A})$.■

Theorem (Cantor's Intersection Theorem) (1.1.19)[34]**:** A metric space (X, d) is complete iff for any descending sequence (F_n) of nonempty closed sets such that $diam(F_n) \to 0 \; as \; n \to \infty$, the intersection $F = \bigcap_{n=1}^{\infty} F_n$ consists of exactly one point.

Proof: Let (X, d) be a complete metric space and (F_n) be a descending sequence of nonempty closed sets in X such that $diam(F_n) \to 0$ as $n \to \infty$. Since each F_n is nonempty, choose a point $x_n \in F_n$ for each $n \in \mathbb{N}$. Since $diam(F_n) \to 0$, then $\forall \epsilon > 0, \exists m_0 \in \mathbb{N}$ such that $diam(F_{m_0}) < \epsilon$. For $n, m \geq m_0$, $F_n , F_m \subseteq F_{m_0}$ since (F_n) is descending sequence. Now $d(x_n , x_m) \leq diam(F_{m_0}) < \epsilon$. So the sequence (x_n) is Cauchy in X . By completeness of the space (X, d), the sequence (x_n) converges to a point (say) $x_0 \in X$ .To show that $x_0 \in \bigcap_{n=1}^{\infty} F_n$, let m be any positive integer. Then for $n \geq m \Rightarrow x_n \in F_m$. The sequence (x_n) converges to x_0 , then $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $d(x_n, x_0) < \epsilon$ for all $n \geq n_0$ which means that $x_n \in B(x_0 , \epsilon)$.

Take $s = max\{m, n_0\}$. Then $x_n \in B(x_0 , \epsilon)$ for all $n \geq s$. So $B(x_0 , \epsilon) \cap F_m \neq \emptyset$ .Hence $x_0 \in \overline{F_m}$, and then $x_0 \in F_m$ since F_m is closed. Since m was arbitrarily chosen, $x_0 \in \bigcap_{n=1}^{\infty} F_n$.

Now, suppose there is another point $y \in \bigcap_{n=1}^{\infty} F_n$.

Then $d(x_0 , y) \leq diam(F_n)$, for every n .Since $diam(F_n) \to 0$, therefore $d(x_0 , y) = 0$, hence $x_0 = y$.

Conversely, let the given condition hold and (x_n) be a Cauchy sequence in X . For each $n \in \mathbb{N}$, let $A_n = \{x_n, x_{n+1}, x_{n+2}, \ldots\}$. Obviously $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ and hence $\overline{A_1} \supseteq \overline{A_2} \supseteq \overline{A_3}$ .... Since (x_n) is Cauchy , $diam(A_n) \to 0$,therefore $diam(\overline{A_n}) \to 0$ .By the hypothesis, $\bigcap_{n=1}^{\infty} \overline{A_n}$ consists of a single point say x_0 .Thus, $d(x_0, x_n) \leq diam(\overline{A_n})$ since $x_0, x_n \in \overline{A_n}$, but

$diam(\overline{A_n}) \to 0$, then $d(x_0, x_n) \to 0$ as $n \to \infty$. Hence, (x_n) converges to x_0 in (X, d) . ■

Cantor's Intersection theorem is extended to characterize completeness of a 2-metric space which is defined as follows:

Definition(1.1.20)[28]**:** Let X be a non-empty set and let σ be a mapping from $X \times X \times X$ to $[0, \infty)$ i.e. $\sigma: X^3 \to [0, \infty)$ satisfying the following conditions:

(i) For every pair of distinct points a, b in X there exists a point $c \in X$ such that $\sigma(a, b, c) \neq 0$.

(ii) $\sigma(a, b, c) = 0$ only if at least two of the three points are same.

(iii)$\sigma(a, b, c) = \sigma(a, c, b) = \sigma(b, c, a)$ for all $a, b, c \in X$.

(iv)$\sigma(a, b, c) \leq \sigma(a, b, d) + \sigma(a, d, c) + \sigma(d, b, c)$ for all a, b, c and $d \in X$.

Then σ is called a 2-metric on X and (X, σ) is called a 2-metric space.

Theorem(1.1.21)[49]**:** Let (X, ρ) be a metric space, then (X, ρ) is complete if for every continuous function $F: X \to \mathbb{R} \cup \{+\infty\}, F \not\equiv +\infty$, bounded from below, and for every $\epsilon > 0$, there is a point $v \in X$ satisfying:

(i)$F(v) \leq inf_X\, F + \epsilon$ and

(ii)For all $w \neq v$, $F(w) + \epsilon\rho(v, w) > F(v)$.

Proof: Let (y_n) be a Cauchy sequence in X, and let $F: X \to \mathbb{R}$ given by $F(x) = \lim_{n\to\infty} \rho(y_n, x)$. This function is continuous, and $inf_X F = 0$, since (y_n) is Cauchy. We need to show that (y_n) converges in X. Choose any $0 < \epsilon < 1$, then by (i) we have $F(v) \le \epsilon$ for a point $v \in X$. Also for all $w \ne v$ we have $F(w) + \epsilon\rho(v, w) > F(v)$. Now by the definition of F and the fact that (y_n) is Cauchy, we can take $w = y_p$ for p large enough such that $F(w)$ is arbitrary small and thus: $\rho(w, v) \le \epsilon + \eta$ for any $\eta > 0$ Now using (ii), the result is $F(v) \le \epsilon^2$. Repeating the argument : $F(v) \le \epsilon^2$ and $\rho(w, v) \le \epsilon^2 + \eta$ for any $\eta > 0$. So we get $F(v) \le \epsilon^3$ Repeating this till concluding that : $F(v) \le \epsilon^n$, for all $n \ge 1$ where $n \in \mathbb{N}$. Since $0 < \epsilon < 1$, we have $F(v) = 0$, i.e $\lim_{n\to\infty} \rho(y_n, v)$, this means that (y_n) converges to v by definition **(1.1.3)** .So the metric space (X, ρ) is complete.■

Other characterizations of the metric completeness can be found in [16,38,55] .

Theorem (Cauchy Convergence Criterion) (1.1.22) [47]**:** In the metric space $(\mathbb{R}, d)$ where d is the usual metric($d(x, y) = |x - y|, \forall\, x, y \in \mathbb{R}$), a real sequence is convergent iff it is a Cauchy sequence .

Proof: Let (x_n) be a convergent sequence in $\mathbb{R}$,then $\lim_{n\to\infty} x_n = L$ for some real number L. That is, $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $d(x_n, L) < \frac{\epsilon}{2}$. So if $n, m \ge N$ we have

$$d(x_n, x_m) \le d(x_n, L) + d(L, x_m)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Which means that the sequence (x_n) is Cauchy.

Conversely, Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that for all $n, m \geq N, d(x_n, x_m) < 1$. So for all $n \geq N$ we have $d(x_n, x_N) < 1$ which implies that $|x_n| < |x_N| + 1$.

Let $M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |x_N| + 1\}$, then $|x_n| \leq M$ for all $N \in \mathbb{N}$, that is the sequence is bounded. But every real sequence contains a monotonic subsequence. Now, this subsequence is both monotonic and and bounded, hence convergent. By theorem **(1.1.7)**, the whole sequence converges. ■

Example (1.1.23)[52]**:** the metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|, \forall\, x, y \in \mathbb{R}$ is a complete metric space since every Cauchy sequence converges in $\mathbb{R}$ by theorem (**1.1.22**).

1.2. Subspaces of Complete Metric Spaces:

Subspaces of a complete metric space are characterized in this section. Definitions and theorems concerning subspaces of a metric space are presented before discussing this topic.

Definition (1.2.1)[6]**:** (Y, d_Y) is a metric subspace of the metric space (X, d) when $Y \subseteq X$ and $d_Y(a, b) = d(a, b)$ for all $a, b \in Y$.

Theorem (1.2.2)[34]**:** let (y_n) be a sequence in a metric subspace (Y, d_Y) of a metric space (X, d). Let $y \in Y$, the sequence (y_n) converges to y in (Y, d_Y) iff it converges to y in (X, d).

Proof: For $y \in Y, A \subseteq Y$ is an open set in the metric subspace (Y, d_Y) containing y iff $A = B \cap Y$ for some open set B in (X, d) such that $y \in B$. Thus, if (y_n) is a sequence in Y , then the sequence is eventually in every open set containing y in (Y, d_y) iff it is also in every open set containing y in (X, d). ■

Theorem (1.2.3)[34]: let (y_n) be a sequence in a metric subspace (Y, d_Y) of a metric space (X, d). Then (y_n) is Cauchy in Y iff (y_n) is Cauchy in X.

Proof: For any $y, y' \in Y$, we have $d_Y(y, y') = d(y, y')$. Hence, if (y_n) is a sequence in Y then $d_Y(y_i, y_j) = d(y_i, y_j)$, $\forall i, j \in \mathbb{N}$.■

Theorem (1.2.4)[43]: A convergent sequence of points in a metric space (X, d) is a Cauchy sequence.

Proof: let (x_n) be a sequence of points in (X, d) that converges to $x \in X$. Then, for any $\epsilon > 0, \exists\ N \epsilon\, \mathbb{N}$ such that $d(x, x_n) < \frac{\epsilon}{2}$, for all $n \geq N$. Hence if $n, m \geq N$, then $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.■

Definition (1.2.5)[34]: The metric space (X, d) is called incomplete if it is not complete.

Example(1.2.6)[34]: Consider the metric subspace $X = (0,1]$ of the metric space $\mathbb{R}$ with its usual metric d . Then (X, d_X) is incomplete .

Proof: Consider the sequence $\left(\frac{1}{n}\right), n \in \mathbb{N}$ .It is a sequence of points in X The sequence $\left(\frac{1}{n}\right)$ converges to $0 \in \mathbb{R}$, then it is Cauchy in $\mathbb{R}$ by theorem

(**1.2.4**). It is also Cauchy in X by theorem (**1.2.3**).But it doesn't converges to any point in X as $0 \notin X$. ■

Example(1.2.7)[34]**:** Consider the metric subspace $X = \mathbb{Q}$ of the metric space $\mathbb{R}$ with its usual metric d . Then, (X, d_X) is incomplete.

Proof: Consider the sequence (x_n) in X, where $x_n = (1 + \frac{1}{n})^n$.This sequence converges to e in $\mathbb{R}$, hence it is Cauchy in $\mathbb{R}$ by theorem(**1.2.4**), so it is also Cauchy sequence in X by theorem (**1.2.3**) .But it doesn't converges to any point in X since e is irrational number.■

Theorem(1.2.8)[34]**:** A convergent sequence has a unique limit in a metric space (X, d) .

Proof: Let (x_n) be a convergent sequence in the metric space (X, d) to two distinct points x , y.Then $\exists m_1 \in \mathbb{N}$ such that $d(x_n, x) < \frac{\epsilon}{2}, \forall \epsilon > 0$. Also, $\exists m_2 \in \mathbb{N}$ such that $d(x_n, y) < \frac{\epsilon}{2}, \forall \epsilon > 0$ .Let $m = max\{m_1, m_2\}$. Then we have :

$d(x, y) \leq d(x, x_n) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So, $0 \leq d(x, y) < \epsilon, \forall \epsilon > 0$. Then $d(x, y) = 0 \Rightarrow x = y$ which is a contradiction.■

The following theorem characterizes complete metric subspaces of a complete metric space.

Theorem (1.2.9)[52]**:** Let (X, d) be a complete metric space. Let (Y, d_Y) be a metric subspace of X . Then, Y is closed in X iff (Y, d_Y) is complete .

Proof : Let (X, d) be a complete metric space , Y be closed subset of X .Let (y_n) be a Cauchy sequence in Y, then it is also a Cauchy sequence in X. But $(X$,d) is complete, so (y_n) converges to a point $x \in X$ then(by theorem(**1.1.15**)) the sequence (y_n) converges to a point $x \in Y$ .Thus (Y, d_Y) is complete. For the converse: Let (Y, d_Y) be complete. If $c \in \bar{Y}$ then there exists a sequence (y_n) of points in Y that converges to $c \in X$ (by corollary **1.1.16**) . Then, (y_n) is a Cauchy sequence in X, hence Cauchy in Y which is complete. So (y_n) converges to $c \in Y$. Thus Y is closed in X.■

Remark(1.2.10): It is now easy to prove that the subspaces $(0,1]$ and $\mathbb{Q}$ discussed in the previous examples (**1.2.6**) and (**1.2.7**) respectively are incomplete, since $(0,1]$ and $\mathbb{Q}$ are not closed in $(\mathbb{R}, d)$ where d is the usual metric on $\mathbb{R}$.

1.3. Metrically Topologically Complete Space:

In spite of the fact that completeness is a metric property rather than a topological one, completeness can be considered also in topological spaces. This section discusses and elaborates the former idea supported with examples.

Definition (1.3.1)[56]**:** A topological space (X, τ) is called metrizable if there exist a metric d on the set X that induces the topology τ of X.

Definition (1.3.2)[6]**:** Two metrics d , ρ on a set X are called equivalent if they induce the same topology.

Remark(1.3.3)[56,6]:

(i)A topological space (X, τ) may not be metrizable, that is no metric on X induces the topology of X. For example let $X = \{a, b\}$ and let $\tau = \{\emptyset, \{a\}, X\}$. Then, τ is a topology for X, and it is not metrizable. For suppose ρ is a metric on X which produces τ. $a \neq b$ so, $\rho(a, b) = r > 0$. Now, $B\left(b, \frac{r}{2}\right) = \{b\}$, so $\{b\}$ is an open set, contrary to the definition of τ. Hence, no ρ can produce this topology on X. With this topology, X is sometimes called the Sierpinski space.

(ii)A topological space that is induced by a metric d, can also be induced by other equivalent metrics. For example, the metrics

$\rho_\mu(x, y) = min\{\mu, d(x, y)\}$ for $\mu > 0$ and $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ are equivalent to the metric d on a set X.

Definition (1.3.4)[19]: A topological space is said to be metrically topologically complete if there exists a complete metric inducing the given topology on it. These spaces are also called topologically complete.

In other words , if a topological space X is metrizable and induced by a metric d , then X is topologically complete if d is complete or if an equivalent complete metric for X exists.

Example (1.3.5)[6]: Consider the topological space $(\mathbb{R}, \tau)$ where τ is the usual topology on $\mathbb{R}$. This topological space is metrizable and is induced by the following equivalent metrics:

$d_e(x,y) = |x-y|$ and $d_\theta(x,y) = \left|\frac{x}{1+|x|} - \frac{y}{1+|y|}\right|$ .It is proven in example (**1.1.23**) that the metric d_e is complete on $\mathbb{R}$ while the metric d_θ is incomplete metric on $\mathbb{R}$, for the sequence (n) is Cauchy with respect to d_θ, but it doesn't converge to any point in $\mathbb{R}$ .The topological space $(\mathbb{R},\tau)$ in this example is metrically topologically complete .The existence of one complete metric for the topological space is enough to call it metrically topologically complete.

Some subspaces of a metrically topologically complete space are metrically topologically complete while others are not.

Example (1.3.6)[19]**:** Consider the subspace $X = (0,1)$ of the space $(\mathbb{R},\tau)$ where τ is the usual topology on $\mathbb{R}$. The space $(\mathbb{R},\tau)$ is metrically topologically complete induced by the usual metric d_e which is complete. The metric d_{e_X} is not complete on X because X is not closed set in the metric space $(\mathbb{R}, d_e)$. But this subspace X is topologically complete (this is proven in the following pages) ,that is a complete metric for X must exist.

Definition (1.3.7)[56]**:** A G_δ-set in a space is a set which can be expressed as the intersection of a countable family of open sets.

The following theorem characterizes the topologically complete subspaces of a topologically complete space.

Theorem (1.3.8)[6]**:** If Y is a topologically complete space then, a subset A of Y is topologically complete iff A is a G_δ-set in Y .

Proof: [6 page 307].

Corollary (1.3.9): If A is a closed subset of a topologically complete space X, then A is a topologically complete subspace.

Proof[56]**:** Any closed set in a metric space is a G_δ-set .Then it is topologically complete by theorem (**1.3.8**). More over, if d is a complete metric for X, then d_A is a complete metric for A .■

Corollary (1.3.10): Any open subset of a topologically complete space is topologically complete.

Proof: An open set in a metric space is a G_δ-set .Then it is topologically complete by theorem (**1.3.8**).■

Remark (1.3.11): The subspace $(0,1)$ discussed in example (**1.3.6**) is topologically complete by corollary (**1.3.10**) since$(0,1)$ is open set in $(\mathbb{R},\tau)$where τ is the usual topology.

Example (1.3.12)[19]**:** The set X of irrationals is a topologically complete subspace of the topologically complete space $(\mathbb{R},\tau)$where τ is the usual topology since X is a G_δ-set in $\mathbb{R}$. While the set $\mathbb{Q}$ is not topologically complete since $\mathbb{Q}$ is not a G_δ-set in $\mathbb{R}$.

1.4.Examples:

The aim of this section is to shed light on three examples of complete metric spaces. These examples play a pivotal role in analysis.

Lemma(1.4.1)[5]**:** If (x_n) is a sequence in X , where (X, d) is a discrete metric space, then the sequence (x_n) converges in X iff there exist $N \in \mathbb{N}$ such that $x_n = x$, $\forall n \geq N$,for some $x \in X$.

Proof: Let (x_n) be a sequence of points in X that converges to $x \in X$, then, $\exists N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon, \forall \epsilon > 0$ and $\forall\, n \geq N$. Let $\epsilon = \frac{1}{2}$, then $d(x_n, x) < \frac{1}{2} \Rightarrow x_n = x, \forall\, n \geq N$.

Conversely, If $x_n = x$, $\forall n \geq N$, for some $N \in \mathbb{N}$, then $d(x_n, x) = 0 < \epsilon, \forall \epsilon > 0$. So (x_n) converges to x. ■

Lemma (1.4.2)[5]**:** Any discrete metric space (X, d) is complete.

Proof: If (x_n) is Cauchy sequence in (X, d), then $\forall \epsilon > o, \exists n_0 \in \mathbb{N}$ such that $n, m \geq n_0 \Rightarrow d(x_n, x_m) < \epsilon$. For $0 < \epsilon < 1 \Rightarrow x_n = x_m = x$ for a point $x \in X$. Thus, $\forall n \geq n_0 \Rightarrow x_n = x$, and so (x_n) converges to x by lemma **(1.4.1)**,hence the discrete metric space is complete. ■

Corollary(1.4.3)[56]**:** The discrete topology on a set X is metrizable, being the topology produced by the discrete metric on X which is complete by lemma (**1.4.2**).Thus, the discrete topology is metrically topologically complete.

Theorem(1.4.4)[45]**:** Let $x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n)$ be two points of $\mathbb{R}^n$, then $d_2(x, y) = [\sum_{i=1}^{n} (x_i - y_i)^2]^{\frac{1}{2}}$ is a metric on $\mathbb{R}^n$, called the Euclidean metric .

Theorem(1.4.5)[33]: Let $(x^k), k \in \mathbb{N}$, be a sequence in $(\mathbb{R}^n, d_2)$, $x = (x_1, x_2, \dots, x_n)$ where $x_i \in \mathbb{R}$,$\forall i = 1,2, \dots, n$. Then, the sequence (x^k) converges to x iff the sequence (x_i^k) converges to x_i in $(\mathbb{R}, d_e)$ for each $i = 1,2, \dots, n$.

Proof: Suppose (x^k) converges to x, then $\forall \epsilon > 0, \exists N \in \mathbb{N}$,such that $d_2(x^k, x) < \epsilon$ for $k \geq N$. $|x_i^k - x_i| = \sqrt{(x_i^k - x_i)^2} \leq d_2(x^k, x) < \epsilon$, then (x_i^k) converges to x_i for each $i = 1,2, \dots, n$.

Conversely, if the sequence (x_i^k) converges to x_i in $(\mathbb{R}, d_e)$ for each $i = 1,2, \dots, n$, then for every $\epsilon > 0$, there are positive integers N_i such that $|x_i^k - x_i| < \frac{\epsilon}{\sqrt{n}}$ for all $k \geq N_i$.

Letting $N = max\{N_i : i = 1,2, \dots, n\}$, we have $|x_i^k - x_i| < \frac{\epsilon}{\sqrt{n}}$, for all $k \geq N$.Now, $d_2(x^k, x) = \left[\sum_{i=1}^n (x_i^k - x_i)^2\right]^{\frac{1}{2}} < \left[\sum_{i=1}^n \left(\frac{\epsilon}{\sqrt{n}}\right)^2\right]^{\frac{1}{2}} = \epsilon$.

Therefore, $d_2(x^k, x) < \epsilon$, for all $k \geq N$ and (x^k) converges to x.■

Theorem (Completeness of the Euclidean Space$(\mathbb{R}^n, d_2)$)(1.4.6)[18]: $(\mathbb{R}^n, d_2)$ is a complete metric space.

Proof: Let (a^k) be a Cauchy sequence in $\mathbb{R}^n$. Let $\epsilon > 0$. Then there exist a positive integer N such that, for $k, m \geq N$, $d_2(a^k, a^m) = [\sum_{i=1}^n (x_i - y_i)^2]^{\frac{1}{2}} < \epsilon$. If j is appositive integer with $1 \leq j \leq n$, we have $|a_j^k - a_j^m| \leq d(a^k, a^m) < \epsilon$, for $k, m \geq N$. Thus for $1 \leq j \leq n$, the real sequence $(a_j^k)_{k=1}^\infty$ is Cauchy sequence and hence converges , for all

$j = \{1,2,\dots,n\}$. Then the sequence (a^k) converges by theorem (1.4.5) .Therefore $\mathbb{R}^n$ is complete.■

The Euclidean space is also complete in the square metric $d_s(x,y) = \max\{|x_i - y_i|, i = 1,2,\dots n\}, x, y \in \mathbb{R}^n$ for the details see [35].

Remark (1.4.7)[35]**:** The metric d_2 induces the product topology on $\mathbb{R}^n$, so the product topology on $\mathbb{R}^n$ is metrically topologically complete.

It is already proven that $\mathbb{R}^n$ is a complete space. Analogously, $\mathbb{R}^\omega$,the set of all real sequences, is also complete when considered with

the metric:

$$d(x,y) = \sum_{n=1}^{\infty}\left(\frac{1}{n!}\left[\frac{|x_n - y_n|}{1 + |x_n - y_n|}\right]\right)$$

for $x = (x_n)_{n\geq 1}, y = (y_n)_{n\geq 1}$. The following example explains the details.

Example (Completeness of Frechet's Sequence Space) (1.4.8)[34]**:** Let (X,d) denote the frechet's sequence space, where X consist of all real sequences, and the metric d is given by

$$d(x,y) = \sum_{n=1}^{\infty}\left(\frac{1}{n!}\left[\frac{|x_n - y_n|}{1 + |x_n - y_n|}\right]\right), \text{for } x = (x_n)_{n\geq 1}, y = (y_n)_{n\geq 1}$$

Then:

(i) d is a metric on X.

(ii) (X,d) is a complete metric space .

Proof (i) (Same proof can be found in [45]): The series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n!}\left[\frac{|x_n - y_n|}{1+|x_n - y_n|}\right]\right)$$

converges . In fact

$$\frac{1}{n!}\left[\frac{|x_n - y_n|}{1+|x_n - y_n|}\right] \leq \frac{1}{n!}$$

The series $\sum_{n=1}^{\infty}\left(\frac{1}{n!}\right)$ converges, and so, by the Weierstrass M-test,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n!}\left[\frac{|x_n - y_n|}{1+|x_n - y_n|}\right]\right)$$

converges. It is immediate that $d(x, y) \geq 0$ and that $d(x, y) = 0$ iff $x = y$. Also, $d(x, y) = d(y, x)$.

Let

$$d_n(x_n, y_n) = \frac{|x_n - y_n|}{1+|x_n - y_n|}$$

Since d_n is a metric on $\mathbb{R}$, then for all x_n, y_n and z_n in $\mathbb{R}$,

$$d_n(x_n, z_n) \leq d_n(x_n, y_n) + d_n(y_n, z_n)$$

So for $x = (x_n)_{n\geq 1}$, $y = (y_n)_{n\geq 1}$, $z = (z_n)_{n\geq 1}$,we have:

$$d(x, z) \leq d(x, y) + d(y, z)$$

Thus d is a metric on $\mathbb{R}^{\omega}$.

(ii) (This proof can be found in [34]):Let $\{x_n : n \in \mathbb{N}\}$ be a Cauchy sequence in $\mathbb{R}^\omega$, where $x_n = (x_1^n, x_2^n, \dots)$ for $n = 1,2,\dots$ Let $\epsilon > 0$ be given and let $r \in \mathbb{N}$ be arbitrary. Choose ϵ_1 such that $0 < \epsilon_1 < \frac{\epsilon}{r!(1+\epsilon)}$. Then there exists $n_1 \in \mathbb{N}$ such that for $m, n > n_1$, $d(x_m, x_n) < \epsilon_1$, i.e

$$\sum_{k=1}^{\infty}\left(\frac{1}{k!}\left[\frac{|x_k^m - x_k^n|}{1 + |x_k^m - x_k^n|}\right]\right) < \epsilon_1$$

Then for each $r \in \mathbb{N}$,

$$\frac{1}{r!}\left[\frac{|x_r^m - x_r^n|}{1 + |x_r^m - x_r^n|}\right] < \epsilon_1 \Rightarrow |x_r^m - x_r^n| < \frac{\epsilon_1 r!}{1 - r!\,\epsilon_1} < \epsilon$$

for all $m, n > n_1$ (since $\epsilon_1 < \frac{\epsilon}{r!(1+\epsilon)}$).Hence for each $r \in \mathbb{N}$, the sequence $\{x_r^n : n \in \mathbb{N}\}$ is Cauchy in $(\mathbb{R}, d_e)$. By completeness of $\mathbb{R}$, the sequence converges to a limit t_r (say) in $\mathbb{R}$. Let $x = \{t_r : r \in \mathbb{N}\} \in \mathbb{R}^\omega$.

Now, let $\epsilon > 0$ be given. Since the series $\sum_{r=1}^{\infty}\frac{1}{r!}$ converges, choose $m \in \mathbb{N}$ sufficiently large such that

$$\sum_{r=m+1}^{\infty}\frac{1}{r!} < \frac{\epsilon}{2}$$

Then

$$\sum_{r=m+1}^{\infty}\left(\frac{1}{r!}\left[\frac{|t_r - x_r^n|}{1 + |t_r - x_r^n|}\right]\right) < \frac{\epsilon}{2}$$

for $n = 1,2,\dots$ (1)

Since $\lim_{n\to\infty} x_r^n = t_r$ (for $r = 1,2,\dots$), $\exists k \in \mathbb{N}$ such that

$|t_r - x_r^n| < \frac{\epsilon}{2m}$ $\forall n \geq k$ and for all $r = 1,2,\dots,m$. Therefore,

$$\sum_{r=1}^{m}\left(\frac{1}{r!}\left[\frac{|t_r - x_r^n|}{1+|t_r - x_r^n|}\right]\right) < m\frac{\epsilon}{2m} = \frac{\epsilon}{2} \dots (2)$$

hold for all $n > k$.

Adding (1) and (2), then $d(x, x_n) < \epsilon$, $\forall n \geq k$, proving that $\lim_{n\to\infty} x_n = x$. Hence, the space $(\mathbb{R}^\omega, d)$ is complete.■

1.5. Completion:

Cauchy sequences are all convergent; they converge either to points in the metric space under consideration or to points that do not belong to the space under consideration, but they are members of another larger metric space. So, imbedding an incomplete metric space in to a larger complete metric space that preserving the distance function, and the metrical properties of the incomplete space is possible.

Definition (1.5.1) [6]**:** A map $f: X \to X'$ between metric spaces (X, d) and (X', d') is called an isometry if $d'(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. The mapping f is also called an isometric embedding of X into X'.

Definition (1.5.2) [48]**:** The spaces (X, d) and (X', d') are said to be isometric spaces if there exists a surjective isometry $f: X \to X'$.

Remark (1.5.3)[45]**:** An isometry $f: X \to X'$ between metric spaces (X, d) and (X', d') is one-to-one.

Proof: Let $x, y \in X$, $f(x) = f(y) \Rightarrow d'\big(f(x), f(y)\big) = 0$, then $d(x, y) = 0$ and so, $x = y$.■

Definition (1.5.4)[6]**:** A binary relation R in a set A is a subset $R \subseteq A \times A$. $(a, b) \in R$ is written $a \sim b$.

Definition (1.5.5)[6]**:** A binary relation R in A is called an equivalence relation if:

(i). $\forall a \in A : a \sim a$ (reflexive).

(ii). $(a \sim b) \Rightarrow (b \sim a)$ (symmetric).

(iii). $(a \sim b) \wedge (b \sim c) \Rightarrow (a \sim c)$ (transitive).

If $a \sim b$, we say that a and b are equivalent.

Definition (1.5.6)[6]**:** Let R be an equivalence relation in A. For each $a \in A$, the subset $[a] = \{b \in A |\ b \sim a\}$ is called the equivalence class of a.

Lemma (1.5.7)[6]**:** Let R be an equivalent relation in A, and let $a, b \in A$. Then:

(i). $\cup \{[a] |\ a \in A\} = A$.

(ii). If a and b are equivalent, then $[a] = [b]$.

(iii). If a and b are not equivalent, then $[a] \cap [b] = \emptyset$.

Lemma (1.5.8)[57]**:**Let (X, d) be a metric space, then for any quadruple of points a, b, u and v of X, the following inequality holds:

$$|d(a,b) - d(u,v)| \leq d(a,u) + d(b,v)$$

Proof: By the triangle inequality and the symmetry properties:

$$d(a,b) \leq d(a,u) + d(u,v) + d(b,v).$$

Therefore, $\quad d(a,b) - d(u,v) \leq d(a,u) + d(b,v)$

Also:

$$d(u,v) \leq d(b,v) + d(a,b) + d(a,u).$$

From which, $\quad d(u,v) - d(a,b) \leq d(a,u) + d(b,v).$

Thus: $\quad |d(a,b) - d(u,v)| \leq d(a,u) + d(b,v).$ ■

The importance of complete metric spaces is much more than incomplete ones. By adding points that are the limits of the noncom vergent Cauchy sequences, an incomplete metric space can be imbedded in to a complete metric space.

Definition (1.5.9) [34]**:** A subset A of a metric space (X, d) is called dense in X if $\bar{A} = X$.

Definition(1.5.10)[15]**:** A metric space $(\tilde{X}, \rho)$ is called a completion of the metric space (X, d) if the following conditions are satisfied:

(a) there is an isometric embedding $f: X \to \tilde{X}$

(b) the image space $f(X)$ is dense in $\tilde{X}$

(c) the space $(\tilde{X}, \rho)$ is complete.

The main theorem about the completion of a metric space is the following:

Theorem (1.5.11) [45]**:** Every metric space has a completion and any two completions are isometric to each other.

Proof: The proof will be provided in steps. Let (X, d) be a metric space. **Step(1):** Let $\hat{X}$ denote the set of all Cauchy sequences in X, and let $\sim$ be a relation in $\hat{X}$ defined as follows :

For $(x_n), (y_n) \in \hat{X}$, $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

This relation is :

(i) reflexive: For$(x_n) \in \hat{X}$, $(x_n) \sim (x_n)$, since $d(x_n, x_n) = 0$ for every $n \in \mathbb{N}$ and so, $\lim_{n\to\infty} d(x_n, x_n) = 0$.

(ii)symmetry: For $(x_n), (y_n) \in \hat{X}$, if $(x_n) \sim (y_n)$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$, therefore, $\lim_{n\to\infty} d(y_n, x_n) = 0$, so that $(y_n) \sim (x_n)$.

(iii) transitivity: For $(x_n), (y_n), (z_n) \in \hat{X}$, if $(x_n) \sim (y_n)$ and $(y_n) \sim (z_n)$, then $\lim_{n\to\infty} d(y_n, x_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$, but:

$$0 \leq d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n)$$

For all $n \in \mathbb{N}$, it follows that

$$0 \leq \lim_{n\to\infty} d(x_n, z_n) \leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) = 0$$

so $\lim_{n\to\infty} d(x_n, z_n) = 0$. Thus $(x_n) \sim (z_n)$.

By **(i)**,**(ii)** and **(iii)**; $\sim$ is an equivalence relation and $\hat{X}$ splits into equivalence classes.

Step(2): Let $\tilde{X}$ denote the set of all equivalence classes; the elements of $\tilde{X}$ will be denoted by $\tilde{x}, \tilde{y}$, etc. If a Cauchy sequence (x_n) has a limit $x \in X$, and if (y_n) is equivalent to (x_n), then $\lim_{n\to\infty} y_n = x$,since $d(y_n, x) \leq d(y_n, x_n) + d(x_n, x)$.For the nonequivalent sequences (x_n) and (y_n), then $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} y_n$. For if

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x$$

then , $0 \leq d(x_n, y_n) \leq d(x_n, x) + d(x, y_n)$

and so, $\lim_{n\to\infty} d(x_n, y_n) = 0$, contradicting the fact that (x_n) and (y_n) are two nonequivalent sequences.

Step(3): An element $(x_n) \in \tilde{x}$ of an equivalence class $\tilde{x} \in \tilde{X}$ is called a representative of $\tilde{x}$.

Define: $\rho: \tilde{X} \times \tilde{X} \to \mathbb{R}$ by

$$\rho(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n),$$

where (x_n) and (y_n) are two representatives of $\tilde{x}$ and $\tilde{y}$, respectively. By lemma (**1.5.8**), $|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_m, y_n)$, and so, the sequence $(d(x_n, y_n))$ is a Cauchy sequence of real numbers in the complete metric space $(\mathbb{R}, d_e)$ where d_e is the usual metric on $\mathbb{R}$. Hence, $\lim_{n\to\infty} d(x_n, y_n)$ exists.

Now suppose that $(x_n), (x_n^{'})$ represent $\tilde{x}$ and $(y_n), (y_n^{'})$ represent $\tilde{y}$.Then:

$$d(x_n^{'}, y_n^{'}) \leq d(x_n^{'}, x_n) + d(x_n, y_n) + d(y_n, y_n^{'})$$

and,

$$d(x_n, y_n) \leq d(x_n, x_n^{'}) + d(x_n^{'}, y_n^{'}) + d(y_n^{'}, y_n),$$

Taking the limits as $n \to \infty$ of these inequalities, and using the assumption that $(x_n) \sim (x_n^{'})$ and $(y_n) \sim (y_n^{'})$, it follows that

$$\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x_n^{'}, y_n^{'})\ .$$

Thus, ρ is well-defined.

ρ satisfies the following properties on $\tilde{X}$:

(i) $\rho(\tilde{x}, \tilde{y}) \geq 0$, since $d(x_n, y_n) \geq 0$ for all n, it follows that $\lim_{n\to\infty} d(x_n, y_n) \geq 0$. If $\tilde{x} = \tilde{y}$, then $\rho(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$,

Where $(x_n) \in \tilde{x}, (y_n) \in \tilde{y}$ and $(x_n) \sim (y_n)$. So, $\lim_{n\to\infty} d(x_n, y_n) = 0$. Therefore, $\rho(\tilde{x}, \tilde{y}) = 0$. Conversely, if $\rho(\tilde{x}, \tilde{y}) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$ and hence $(x_n) \sim (y_n)$, so that $\tilde{x} = \tilde{y}$.

(ii) $\rho(\tilde{x}, \tilde{y}) = \rho(\tilde{y}, \tilde{x})$ as $d(x_n, y_n) = d\big(y_{n,}x_n\big)$.

(iii) For $(x_n) \in \tilde{x}, (y_n) \in \tilde{y}$ and $(z_n) \in \tilde{z}$,where $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$,

$$\rho(\tilde{x}, \tilde{z}) = \lim_{n\to\infty} d(x_n, z_n)$$

$$\leq \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n)$$

$$= \rho(\tilde{x}, \tilde{y}) + \rho(\tilde{y}, \tilde{z}).$$

Thus ρ is a metric on $\tilde{X}$.

Step(4): Define a mapping $f: X \to \tilde{X}$ as follows: $f(x) = \tilde{x}$, where $\tilde{x}$ denotes the equivalence class each of whose members converges to x. Thus the constant sequence $x_n = x, \forall n \in \mathbb{N}$ is a representative of $\tilde{x}$. This map is one-to-one. The metric ρ has the property that

$\rho(\tilde{x}, \tilde{y}) = \rho\big(f(x), f(y)\big) = d(x, y)$ for all $x, y \in X$, i.e, f is an isometric embedding of X into $\tilde{X}$.

Step(5): To show the density of $f(X)$ in $\tilde{X}$, let (x_n) be a representative of an arbitrary point $\tilde{x} \in \tilde{X}$. For any $k \in \mathbb{N}$, there exists a positive integer n_k such that $d(x_n, x_{n_k}) < \frac{1}{k}$ for $n \geq n_k$. Let $\tilde{y}_k$ be the equivalence class containing all Cauchy sequences converging to x_{n_k}, i.e., $\tilde{y}_k = f\big(x_{n_k}\big)$. Then $\rho\left(\tilde{x}, f\big(x_{n_k}\big)\right) = \rho(\tilde{x}, \tilde{y}_k) = \lim_{n\to\infty} d(x_n, x_{n_k}) \leq \frac{1}{k}$

Thus, $\tilde{x} = \lim_{k\to\infty} f(x_{n_k})$.

Step(6): To prove that $\big(\tilde{X}, \rho\big)$ is complete, Let $(\tilde{x}_n)$ be a Cauchy sequence in $\tilde{X}$. Since each $\tilde{x}_n$ is the limit of a sequence in $f(X), \exists \tilde{y}_n \in f(X)$ such that $\rho(\tilde{x}_n , \tilde{y}_n) < \frac{1}{n}$. Then the sequence $(\tilde{y}_n)$ can be shown to be Cauchy in $\tilde{X}$ by arguing as follows:

$$\rho(\tilde{y}_n, \tilde{y}_m) \leq \rho(\tilde{y}_n, \tilde{x}_n) + \rho(\tilde{x}_n, \tilde{x}_m) + \rho(\tilde{x}_m, \tilde{y}_m)$$
$$\leq \frac{1}{n} + \rho(\tilde{x}_n, \tilde{x}_m) + \frac{1}{m}.$$

The right hand side can be made as small as desired by choosing m and n large enough, for $(\tilde{x}_n)$ is Cauchy. Since $\tilde{y}_n \in f(X), \exists y_n \in X$ such that $f(y_n) = \tilde{y}_n$. The sequence (y_n) in X must be Cauchy because $(\tilde{y}_n)$ is a Cauchy sequence in $\tilde{X}$ and f is an isometry. Therefore, (y_n) belongs to some equivalence class $\tilde{x} \in \tilde{X}$. Now, for any $\epsilon > 0$:

$$\rho(\tilde{x}_n, \tilde{x}) \leq \rho(\tilde{x}_n , \tilde{y}_n) + \rho(\tilde{y}_n, \tilde{x}) < \frac{1}{n} + \rho(\tilde{y}_n, \tilde{x})$$

and

$$\rho(\tilde{y}_n, \tilde{x}) = \rho(f(y_n), \tilde{x})) = \lim_{n\to\infty} d(y_n, y_m) \leq \epsilon$$

for sufficiently large n, since (y_n) is a Cauchy sequence in X. This implies that $\lim_{n\to\infty} \rho(\tilde{x}_n, \tilde{x}) = 0$, thus $(\tilde{X}, \rho)$ is complete.

Step(7): Finally, let (X^*, d^*) and (X^{**}, d^{**}) be any two completions of (X, d). To show that (X^*, d^*) and (X^{**}, d^{**}) are isometric:

Let $x^* \in X^*$ be arbitrary. By the definition of completion, $\exists (x_n) \in X$ such that $\lim_{n\to\infty} x_n = x^*$. The sequence (x_n) may be assumed to belong to X^{**}.Since X^{**} is complete, (x_n) converges in X^{**} to x^{**}, say, i.e., $\lim_{n\to\infty} x_n = x^{**}$. Define $\varphi: X^* \to X^{**}$ by setting $\varphi(x^*) = x^{**}$.It is clear that the mapping φ is one-to-one and does not depend on the choice of the sequence (x_n) converging to x^*. Moreover, by construction, $\varphi(x) = x$ for $x \in X$ and $d^{**}(\varphi(x^*), \varphi(y^*) = d^*(x^*, y^*)$ for all $x^*, y^* \in X^*$. Clearly, φ is onto.

This implies that any two completions of a metric space are isometric.■

Example(1.5.12)[41]**:** With respect to the usual metric on $\mathbb{R}$:

(i) The completion of $\mathbb{R}$ is $\mathbb{R}$ itself.

(ii) The completion of $\mathbb{Q}$ is $\mathbb{R}$.

(iii)The completion of $(-\infty, b)$ is $(-\infty, b]$.

(iv) The completion of (a, b) is $[a, b]$.

Chapter Two

Complete Function Spaces

Introduction

A function space is a topological space whose points are functions. There are different kinds of function spaces, and several topologies that can be defined on a given set of functions. Completeness of the function space is a basic property which is the focus of this chapter. As an application, a construction of the well-known Peano space –filling curve is discussed. The set of all functions from a set X to a set Y is denoted by $\mathcal{F}(X,Y)$.

2.1.The Space $\boldsymbol{\mathcal{F}(X,Y)}$:

The overall aim of this section is the study of topologies defined on a given set of functions: the product topology, the set-set topology, and the uniform (metric) topology.

The Product Topology:

To study the topological products of arbitrary families of topological spaces it is necessary to discuss briefly some related definitions and propositions.

Definition(2.1.1)[19]**:** Let $\{X_\alpha : \alpha \in A\}$ be an indexed family of sets. Then, its Cartesian product, denoted by $\prod_{\alpha\in A} X_\alpha$ is defined as the set of all functions x from the indexing set A in to $\bigcup_{\alpha\in A} X_\alpha$ such that $x(\alpha) \in X_\alpha$ for all $\alpha \in A$. That is, $\prod_{\alpha\in A} X_\alpha = \{x: A \to \bigcup_{\alpha\in A} X_\alpha \mid x(\alpha) \in X_\alpha, \forall\alpha \in A\}$.

Definition(2.1.2)[56]: The map $\pi_\beta : \prod_{\alpha\in A} X_\alpha \to X_\beta$, defined by $\pi_\beta(x) = x(\beta)$, is called the projection map of $\prod_{\alpha\in A} X_\alpha$ on X_β, or simply, the βth projection map.

Definition(2.1.3)[19]: A box in $\prod_{\alpha\in A} X_\alpha$ is a subset $\prod_{\alpha\in A} B_\alpha$ of $\prod_{\alpha\in A} X_\alpha$ where $B_\alpha \subseteq X_\alpha$, $\alpha \in A$. For $j \in A$, B_j is called the jth side of the box $\prod_{\alpha\in A} B_\alpha$. A box $\prod_{\alpha\in A} B_\alpha$ is said to be large if all except finitely many of its sides are equal to the respective sets X_α's, that is to say, if there exist $j_1, j_2, \dots, j_n \in A$ such that $B_\alpha = X_\alpha$ for all $\alpha \in A - \{j_1, j_2, \dots, j_n\}$. Thus, a large box is a box which has finitely many 'short' sides.

Definition(2.1.4)[19]: A wall in $\prod_{\alpha\in A} X_\alpha$ is a set of the form $\pi_j^{-1}(B_j)$ for some $j \in A$ and some $B_j \subset X_j$. We also say this set is a wall on B_j.

Proposition(2.1.5)[19]: A subset of $\prod_{\alpha\in A} X_\alpha$ is a box iff it is the intersection of a family of walls. A subset of $\prod_{\alpha\in A} X_\alpha$ is a large box iff it is the intersection of finitely many walls.

Proof: Suppose $B = \prod_{\alpha\in A} B_\alpha$ where $B_\alpha \subset X_\alpha$ for all $\alpha \in A$ is a box in $X = \prod_{\alpha\in A} X_\alpha$. For $\alpha \in A$, let $W_\alpha = \pi_\alpha^{-1}(B_\alpha)$. Then each W_α is a wall in X. Claim: $B = \bigcap_{\alpha\in A} W_\alpha$. For $x \in B$ iff the αth coordinate of x belongs to B_α for all $\alpha \in A$, or, $x \in B$ iff $\pi_\alpha(x) \in B_\alpha$ for all $\alpha \in A$. Hence $x \in B$ iff $x \in W_\alpha$ for all $\alpha \in A$. Thus B can be written as an intersection of walls. Conversely, suppose $\{W_i : i \in I\}$ is a family of walls in X. Then for each $i \in I$, there exists $j(i) \in A$ such that $W_i = \pi_{j(i)}^{-1}(B_{j(i)})$ for some subset $B_{j(i)}$ of $X_{j(i)}$.For each $i \in I$ fix such $j(i) \in A$ and $B_{j(i)} \subseteq X_{j(i)}$.

Now for $j \in A$, let $C_j = \cap \{B_{j(i)}: i \in I, j(i) = j\}$.In case there are no i's in I for which $j(i) = j$, C_j is to be the set X_j. Let B be the box $\prod_{\alpha \in A} C_\alpha$. Claim: $B = \bigcap_{i \in I} W_i$. Suppose $x \in B$ and $i \in I$. Let $j = j(i)$. Then $\pi_{j(i)}(x) = \pi_j(x) \in C_j \subseteq B_{j(i)}$ and so $x \in \pi_{j(i)}^{-1}(B_{j(i)})$,or, $x \in W_i$.Hence $B \subseteq \bigcap_{i \in I} W_i$. Conversely suppose $x \in W_i$ for all $i \in I$. Let $j \in A$. Then $\pi_{j(i)}(x) \in B_{j(i)}$for all $i \in I$ for which $j(i) = j$. Hence $\pi_j(x) \in C_j$for all $j \in A$. Hence $x(j) \in C_j$ for all $j \in A$. So $x \in B = \prod_{\alpha \in A} C_\alpha$. Hence $\bigcap_{i \in I} W_i \subseteq B$. Combining the two together, then the intersection of walls is a box.

The proof of the second assertion is similar to the above except that we take into account only those indices $j \in A$ for which the jth side is possibly not equal to the entire set X_j.■

Proposition(2.1.6)[19]**:**

(i)The intersection of any family of boxes is a box .

(ii)The intersection of a finite number of large boxes is a large box.

Proof: (i)Every box is an intersection of walls. Therefore an intersection of boxes is an intersection of intersections of walls and hence an intersection of walls. But an intersection of walls is a box. So the intersection of a family of boxes is a box.

(ii) A large box is the intersection of finitely many walls. Hence the intersection of finitely many large boxes will again be the intersection of finitely many walls and hence a large box.■

Let each set X_α be a topological space with the topology τ_α. Then a topology can be defined on the product set $\prod_{\alpha\in A} X_\alpha$.

Definition(2.1.7)[56]: Let$\{(X_\alpha, \tau_\alpha): \alpha \in A\}$ be an indexed collection of topological spaces. The topology on $\prod_{\alpha\in A} X_\alpha$ that is obtained by taking sets of the form $\prod_{\alpha\in A} U_\alpha$, where:

(i)U_α is open in X_α, for each $\alpha \in A$.

(ii)For all but finitely many coordinates, $U_\alpha = X_\alpha$.

as a base for the open sets is called the product topology.

Remark(2.1.8)[56]: The set $\prod_{\alpha\in A} U_\alpha$, where $U_\alpha = X_\alpha$ except for $\alpha = \alpha_1, \alpha_2, \dots, \alpha_n, n \in \mathbb{N}$, can be written as:

$\prod_{\alpha\in A} U_\alpha = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \dots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$. Thus, the product topology is that topology which has for a subbase the collection $\{\pi_\alpha^{-1}(U_\alpha): \alpha \in A, U_\alpha \text{ open in } X_\alpha\}$

This topology is defined in terms of large boxes as follows:

Definition(2.1.9)[19]: Let $\{(X_\alpha, \tau_\alpha): \alpha \in A\}$ be an indexed collection of topological spaces. The family of all large boxes all of whose sides are open in the respective spaces is a base for a topology on $\prod_{\alpha\in A} X_\alpha$ called the product topology.

Example(2.1.10)[56]: Consider the case, $\mathrm{X}_\alpha = \mathrm{X}$ for each $\alpha \in \mathrm{A}$, then $\prod_{\alpha\in\mathrm{A}} \mathrm{X}_\alpha$ is just the set $\mathcal{F}(\mathrm{A}, \mathrm{X})$ of all functions from A toX.The product topology on $\mathcal{F}(\mathrm{A},\mathrm{X})$ is obtained by taking the collection: $\{\pi_\alpha^{-1}(\mathrm{U}): \alpha \in \mathrm{A}, \mathrm{U} \text{ open in } \mathrm{X}\}$ as a subbase .

Note(2.1.11)[35]**:** The subset $\pi_\alpha^{-1}(\mathrm{U})$ of $\mathcal{F}(\mathrm{A},\mathrm{X})$ can be written in the following form:

$$\{f: f \in \mathcal{F}(\mathrm{A},\mathrm{X}), f(\alpha) \in \mathrm{U}\}$$

This set is denoted by (α, U).

The set-set topology:

The set-set topology is defined on the set $\mathcal{F}(\mathrm{X},\mathrm{Y})$,where X and Y are topological spaces. The most commonly set-set topologies are discussed.

Definition(2.1.12)[42]**:** Let (X,τ) and (Y,τ^*) be topological spaces. Let U and V be collection of subsets of X and Y, respectively. Let $\mathcal{F}(\mathrm{X},\mathrm{Y})$ be the collection of all functions from X into Y. Define, for $u \in U$ and $v \in V$, $(u,v) = \{f \in \mathcal{F}(\mathrm{X},\mathrm{Y}): f(u) \subseteq v\}$. Let $S(U,V) = \{(u,v): u \in U, v \in V\}$.If $S(U,V)$ is a subbase for a topology on $\mathcal{F}(\mathrm{X},\mathrm{Y})$, then it is called a set-set topology.

Some of the most commonly discussed set-set topologies are the point-open topology, and the compact open topology.

Definition(2.1.13)[42]**:** With the notations of definition **(2.1.12).** If U is the collection of all singletons in X and $V = \tau^*$ then the set-set topology on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ is called the point-open topology.

The point-open topology is also defined on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ even if X is just a set not a topological space.

Notation(2.1.14)[31]**:** Let X be a set and Y be a topological space. For $\alpha \in X$ and $v \subseteq Y$ the notation (α, v) is used to describe the subset $\{f \in \mathcal{F}(\mathrm{X},\mathrm{Y}) : f(\alpha) \in v\}$ of $\mathcal{F}(\mathrm{X},\mathrm{Y})$ determined by the point α and the set v .

Definition(2.1.15)[31]**:** Let X be a set and Y be a topological space. The topology on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ having the subbase:

$\{(\alpha, v) : \alpha$ is a point in X and v is open set in $Y\}$ is called the point-open topology.

Definition(2.1.16)[31]: Let X be a set and Y be a topological space. Let (f_n) be a sequence of functions in $\mathcal{F}(X,Y)$. Then (f_n) converges pointwise to $f_0 \in \mathcal{F}(X,Y)$ iff for each fixed $x_o \in X$ the sequence $\big(f_n(x_0)\big)$ converges in the topological sense to the point $f_0(x_0)$ in Y.

The point-open topology is called the topology of pointwise convergence. The latter's name goes back to the following theorem:

Theorem(2.1.17) [31]**:** Let X be a set and Y be a topological space. Let $\mathcal{F}(X,Y)$ have the point-open topology. Then the sequence (f_n) in $\mathcal{F}(X,Y)$ converges in the topological sense to $f_0 \in \mathcal{F}(X,Y)$ iff the sequence (f_n) converges pointwise to the function $f_0 \in \mathcal{F}(X,Y)$.

Proof: Let(f_n) be a sequence of functions in $\mathcal{F}(X,Y)$ which converges in the topological sense to the function $f_0 \in \mathcal{F}(X,Y)$. Let $x_0 \in X$ and let v be any open set in Y containing $f_0(x_0)$. Then the set (x_0, v) is a subbasic open set in $\mathcal{F}(X,Y)$ containing f_0, which implies there exists an $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $f_n \in (x_0, v)$,or, for all $n > n_0$, $f_n(x_0) \in v$, which proves that the sequence (f_n) converges pointwise to the function $f_0 \in \mathcal{F}(X,Y)$.

Conversely, suppose that the sequence (f_n) converges pointwise to the function $f_0 \in \mathcal{F}(X,Y)$ and let (x_0, v) be any subbasic open set in the space $\mathcal{F}(X,Y)$ which contains f_0.

Then $f_0(x_0) \in v$ by definition of the point-open topology and, due to the fact (f_n) converges pointwise to the function f_0, there exists an $n_0 \in \mathbb{N}$ for which if $n \geq n_0$, then $f_n(x_0) \in v$. So, the sequence (f_n) converges in the topological sense to f_0.

Remark(2.1.18)[35]**:** Let X be a set and Y be a topological space. The topology of pointwise convergence on $\mathcal{F}(X,Y)$ is just the product topology which is discussed in example **(2.1.10)**.

Proof: The subset (α, v); α is a point in X and v is open set in Y (which is the subbasis element for the topology of pointwise convergence), is just the subset $\pi_\alpha^{-1}(\mathrm{U})$ of $\mathcal{F}(\mathrm{A},\mathrm{X})$ (which is the subbasis element for the product topology on $\mathcal{F}(\mathrm{A},\mathrm{X})$ as illustrated in note **(2.1.11)**. ■

A deep study of the point-open topology on the set of all continuous real-valued functions is found in [21].

The other set-set topology defined on the set $\mathcal{F}(\mathrm{X},\mathrm{Y})$ is the compact-open topology which made its appearance in 1945 by R. H. Fox in [9].

Definition(2.1.19)[42]**:** With the notations of definition **(2.1.12).** If U is the collection of all compact subsets of X and $V = \tau^*$, then the set-set topology on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ is called the compact-open topology.

The compact-open topology is also defined on the set of continuous functions from the space X to the space Y, denoted by $C(X,Y)$.

Definition(2.1.20)[37]**:** The sets of the form:

$$(U,V) = \{ f \in C(X,Y) : f(U) \subseteq V \},$$

where the set U is a compact subset of X and V is an open subset of Y, form a subbase for a topology on $C(X,Y)$ called the compact-open topology.

The compact-open topology on $C(X,Y)$is very important; it turns out to have useful properties in algebraic topology. This topology is defined in the following way:

Definition(2.1.21)[44]**:** A family η of subsets of a topological space X is called a network on X if for each point $x \in X$ and each neighbourhood $U$of x there exists $\mathrm{P} \in \eta$ such that $x \in \mathrm{P} \subseteq U$.

Definition(2.1.21)[44]: A network η on a space X is said to be compact if all of its elements are compact subspaces of X.

Definition(2.1.22)[44]: Let X and Y be topological spaces, and η a compact network on X. Let the set $[\mathrm{P}, V] = \{f \in C(X, Y): f(\mathrm{P}) \subseteq \mathrm{V}\}$ where $\mathrm{P} \in \eta$ and V is open set in Y.Then the family $\{[\mathrm{P}, V]\}$ is a subbase for a topology on $C(X, Y)$ called the compact-open topology.

Proposition (2.1.23)[46]: (i)The compact-open topology is always finer than the point-open topology. (ii) If X is discrete space, then the compact-open topology and the point-open topology are identical for all Y.

Proof: (i) This is immediate from the fact that the defining subbase for the compact-open topology contains a subbasis for the point-open topology, since each one-point subset of X is compact.(ii) If X is discrete space, then the only compact sets in X are the finite sets (if A is an infinite subset of X then the collection $C = \{\{x\}: x \in A\}$ is an open cover of A which has no finite subcover, since if we remove any single element of C then it will not cover A, that means A is not compact).

In [26] a study of the compact-open topology on the set of all real-valued functions defined on X, which are continuous on compact subsets of X is presented.

The Uniform Metric Topology:

The uniform metric topology (or the uniform topology) is one of the most important topologies defined on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ where X is a set and Y is a metrizable space.

Definition(2.1.24)[46]**:** Let X be a set and let Y be a metrizable space induced by the standard bounded metric d. Let ρ be a metric on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ defined as $\rho(f,g) = \sup\{d\big(f(x),g(x)\big): x \in X, f,g \in \mathcal{F}(\mathrm{X},\mathrm{Y})\}$. This metric ρ is called the uniform metric corresponding to the metric d, or the sup metric.

Definition(2.1.25)[46]**:**The sup metric in definition **(2.1.24)** induces a topology for $\mathcal{F}(\mathrm{X},\mathrm{Y})$ called the topology of uniform convergence or the uniform topology.

2.2. Completeness of The Space $\mathcal{F}(\mathbf{X},\mathbf{Y})$:

In this section the idea of completeness of a function space is discussed. Note that for a function space to be complete it must be metrizable first.

The space $\mathcal{F}(\mathrm{X},\mathrm{Y})$ equipped with the product topology when X is an uncountable set is not metrizable, hence is not complete. Similarly, The space $\mathcal{F}(\mathrm{X},\mathrm{Y})$ with the point-open topology when X is an uncountable set is not complete.

Example(2.2.1)[39]: Let X be an uncountable set, and let τ be the product topology on $\mathcal{F}(\mathrm{X},\mathbb{R})$. Then the space $\mathcal{F}(\mathrm{X},\mathbb{R})$ is not metrizable.

Proof: Let $A=\{x\in\mathcal{F}(\mathrm{X},\mathbb{R})$: there is a finite subset Γ of X such that $x_\alpha=0$ for all $\alpha\in\Gamma$ and $x_\alpha=1$ for all $\alpha\in X-\Gamma\}$, and let a be the member of $\mathcal{F}(\mathrm{X},\mathbb{R})$ with the property that $a_\alpha=0$ for all $\alpha\in X$. <u>Claim(1):</u> $a\in\bar{A}$. Let U be a basic open set containing a. Then there exists a finite subset $\{\alpha_1,\alpha_2,\dots,\alpha_n\}$ of X and a collection $U_{\alpha_1},U_{\alpha_2},\dots,U_{\alpha_n}$ of open subsets of $\mathbb{R}$ such that $U=\bigcap_{i=1}^{n}\pi_{\alpha_i}^{-1}(U_{\alpha_i})$. Thus, the point x defined by $x_{\alpha_i}=0$ for $i=1,2,\dots,n$ and $x_\alpha=1$ for all $\alpha\neq\alpha_i$ for any $i=1,2,\dots n$ belongs to $U\cap A$. That means, $U\cap A\neq\emptyset$ for any arbitrary basic open set U contains a. Therefore, $a\in\bar{A}$.

<u>Claim(2):</u> no sequence of points in A converges to a. Let (a_n) be a sequence of points in A. Let $X_n=\{\alpha\in X:(a_n)_\alpha=0\}$ for each $n\in\mathrm{N}$. Then, $\bigcup_{n\in\mathbb{N}}X_n$ is the countable union of finite sets and therefore it is countable. Thus there exists $\beta\in X-\bigcup_{n\in\mathbb{N}}X_n$. So for each $n\in\mathrm{N}$, $(a_n)_\beta=1$. Hence, the open set $\pi_\beta^{-1}(-1,1)$ in $\mathcal{F}(\mathrm{X},\mathbb{R})$ is a neighborhood of a that does not contain any member of the sequence (a_n). Thus, the sequence (a_n) does not converges to a. That is, there is a point a in $\bar{A}$ with the property that no sequence in A converges to a which contradicts the fact that in a first countable space, a point belongs to the closure of a set iff there is a sequence of points in the set converges to that point. With the knowledge that every metric space is first countable.■

The topology in which the space $\mathcal{F}(\mathrm{X},\mathrm{Y})$ is complete is the uniform topology.

Theorem(2.2.2)[39]**:** If (Y,d) is a complete metric space where d is the standard bounded metric on Y, X is a nonempty set, and ρ is the uniform metric on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ corresponding to d, then the metric space $(\mathcal{F}(\mathrm{X},\mathrm{Y}),\rho)$ is complete.

Proof: Let (Y,d) be a complete metric space where d is the standard bounded metric on Y. Let (f_n) be a Cauchy sequence in $(\mathcal{F}(\mathrm{X},\mathrm{Y}),\rho)$ and $\alpha \in X$. For each $n,m \in \mathbb{N}$, $d(f_m(\alpha),f_n(\alpha)) \leq \rho(f_m,f_n)$, so the sequence $\big(f_n(\alpha)\big)$ is Cauchy sequence in (Y,d). Hence, this sequence converges to a point say $y_\alpha \in Y$ because the space (Y,d) is complete.

Claim: the sequence (f_n) converges to the function $f \in \mathcal{F}(\mathrm{X},\mathrm{Y})$ defined by $f(\alpha) = y_\alpha$ in the space $(\mathcal{F}(\mathrm{X},\mathrm{Y}),\rho)$.

Let $\epsilon > 0$ be given. Then there exists $n_0 \in \mathrm{N}$ such that $\rho(f_m,f_n) < \frac{\epsilon}{2}$

Which implies that for each $\alpha \in X, d(f_m(\alpha),f_n(\alpha)) < \frac{\epsilon}{2}$ whenever $n,m \geq n_0$. But the sequence $\big(f_n(\alpha)\big)$ converges to $y_\alpha = f(\alpha) \in Y$. Therefore there exists $n_1 \in \mathbb{N}$ such that $d(f_n(\alpha),f(\alpha)) < \frac{\epsilon}{2}$ for all $\alpha \in X$ and all $n \geq n_1$. But $d\big(f_m(\alpha),f(\alpha)\big) \leq d(f_m(\alpha),f_n(\alpha)) + d(f_n(\alpha),f(\alpha))$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $\alpha \in X$ and all $n \geq \max\{n_{0,}n_1\}$.

Hence $\rho(f_m,f) < \epsilon$ whenever $m \geq \max\{n_{0,}n_1\}$. ■

The latter theorem proves completeness of the space $\mathcal{F}(\mathrm{X},\mathrm{Y})$ for an arbitrary set X and a complete metric space (Y,d) where d is the standard bounded metric defined on Y. So this theorem is also valid when X is a topological space not just a set.

2.3. Complete Subspaces of $\mathcal{F}(\mathrm{X},\mathrm{Y})$:

In this section the completeness of the spaces $C(X,Y)$, $B(X,Y)$ and $BC(X,Y)$ is studied where $C(X,Y)$ denotes the set of all continuous mappings of the set X into a space Y, $B(X,Y)$ denotes the set of all bounded mappings of the set X into a space Y, and $BC(X,Y)$ denotes the set of all bounded continuous mappings of the set X into a space Y .

The Space $C(X,Y)$ of Continuous Functions:

The space of continuous functions is basic in several aspects in analysis. Completeness of this space has many applications.

Definition(2.3.1)[3]**:** A sequence $f_1, f_2, \ldots$ of functions from a topological space X to a metric space (Y,d) is said to converge uniformly to a function $f \in \mathcal{F}(\mathrm{X},\mathrm{Y})$ if, for each $\epsilon > 0$, there is a number $n_0 \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \epsilon$ whenever $n \geq n_0$ for all $x \in X$.

Theorem (Uniform limit theorem) (2.3.2)[3]**:** If a sequence $f_1, f_2, \ldots$ of continuous functions from a topological space X to a metric space (Y,d) converges uniformly to a function $f \in \mathcal{F}(\mathrm{X},\mathrm{Y})$, then f is continuous.

Proof: Given $\epsilon > 0$, (by the uniformity of the convergence) choose $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then

$$d\big(f(x), f_n(x)\big) < \frac{\epsilon}{3}$$

For all $x \in X$.

Given a point $x_0 \in X$, the continuity of f_{n_0} implies that there is a neighborhood U of x_0 in X such that if $x \in U$ then

$$d\left(f_{n_0}(x), f_{n_0}(x_0)\right) < \frac{\epsilon}{3}$$

Thus, for any $x \in U$ we have:

$$\begin{aligned} &d\big(f(x), f(x_0)\big) \\ &\leq d\left(f(x), f_{n_0}(x)\right) + d\left(f_{n_0}(x), f_{n_0}(x_0)\right) + d\left(f_{n_0}(x_0), f(x_0)\right) \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

That is, the function f is continuous.■

Now, a theorem giving conditions for the subset $C(X, Y)$ to be closed in the space $\mathcal{F}(\mathrm{X}, \mathrm{Y})$ is proven.

Theorem(2.3.3)[39]**:** Let (X, τ) be a topological space, let (Y, d) be a complete metric space where d is the standard bounded metric on Y, and let ρ be the uniform metric defined on $\mathcal{F}(\mathrm{X}, \mathrm{Y})$ corresponding to d. Then $C(X, Y)$ is closed subset of the metric space $(\mathcal{F}(\mathrm{X}, \mathrm{Y}), \rho)$.

Proof: Let (f_n) be a sequence of members of $\mathcal{F}(\mathrm{X},\mathrm{Y})$ that converges to $f \in \mathcal{F}(\mathrm{X},\mathrm{Y})$ relative to the metric ρ.

Claim: the sequence (f_n) converges uniformly to f relative to the standard bounded metric d.

Suppose that (f_n) converges to $f \in \mathcal{F}(\mathrm{X},\mathrm{Y})$ relative to the metric ρ**,** and let $\epsilon > 0$. Then, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$,then $\rho(f_n, f) < \epsilon$. Therefore, for all $n \geq n_0$ and all $x \in X$,

$$d(f_n(x), f(x)) \leq \rho(f_n, f) < \epsilon$$

Hence, the sequence (f_n) converges uniformly to f relative to the standard bounded metric d.

Let $f \in \mathcal{F}(\mathrm{X},\mathrm{Y})$ and $f \in \overline{C(X,Y)}$. Then, there exists a sequence (f_n) of members of $C(X,Y)$ that converges to f relative to ρ. Then by the claim, the sequence (f_n) converges uniformly to f relative to the standard bounded metric d .Then(By theorem **(2.3.2)**) f is continuous function. Hence, $f \in C(X,Y)$.■

Corollary(2.3.4)[39]**:** Let (X,τ) be a topological space, let (Y,d) be a complete metric space where d is the standard bounded metric on Y, and let ρ be the uniform metric defined on $\mathcal{F}(\mathrm{X},\mathrm{Y})$ corresponding to d. Then $(C(X,Y),\rho)$ is complete.

Proof: By theorem **(2.2.1)**, the metric space $(\mathcal{F}(\mathrm{X},\mathrm{Y}),\rho)$ is complete. By theorem**(2.3.3)**, $C(X,Y)$ is closed subset of the metric space $(\mathcal{F}(\mathrm{X},\mathrm{Y}),\rho)$.

By theorem**(1.2.9)**,a closed subset of a complete metric space is complete. That is, $(C(X,Y),\rho)$ is complete.■

Example(2.3.5): The space $(C(X,\mathbb{R}),\rho)$ (where X is a topological space $\mathbb{R}$ has the usual metric and ρ is the uniform metric corresponding to the standard bounded metric of the usual metric) is complete. This is because the space $\mathbb{R}$ is complete in the usual metric by example**(1.1.23)**.

The Space $B(X,Y)$ of Bounded Functions:

A metric σ is defined on $B(X,Y)$ in which X is a set and (Y,d) is a metric space. Then, the completeness of the space $(B(X,Y),\sigma)$ is proven.

Theorem(2.3.6)[36]**:** The space $B(X,Y)$ of bounded functions from a set X to a metric space (Y,d) is itself a metric space, with distance defined by: For $f,g \in B(X,Y)$ and $x \in X$,

$$\sigma(f,g) = \sup\{d\big(f(x),g(x)\big)\,, x \in X\}$$

Proof: The distance is well-defined because if imf and img are bounded then so is there union, and:

$$d(f(x),g(x)) \leq diam(imf \cup img)$$

Now, σ satisfies the distance axioms follows from the same properties for the metric d;

$$\sigma(f,g)=0 \iff \forall\, x\in X, d\big(f(x),g(x)\big)=0$$
$$\iff \forall\, x\in X, f(x)=g(x)$$
$$\iff f=g$$

For $h\in B(X,Y)$

$$\sigma(f,g)=\sup\{d\big(f(x),g(x)\big)\,, x\in X\}$$
$$\leq \sup\{d\big(f(x),h(x)\big)+d(h(x),g(x))\,, x\in X\}$$
$$\leq \sup\{d\big(f(x),h(x)\big)\,, x\in X\}+\sup\{d\big(h(x),g(x)\big)\,, x\in X\}$$
$$=\sigma(f,h)+\sigma(h,g)$$

The axiom of symmetry is clear since

$$\sigma(f,g)=\sup\{d\big(f(x),g(x)\big)\,,\in X\}$$
$$=\sup\{d\big(g(x),f(x)\big)\,, x\in X\}$$
$$=\sigma(f,g)$$

So σ is a metric on $B(X,Y)$.

The importance of this metric is that the space $B(X,Y)$ is complete in it. This is proven in the following theorem.

Theorem(2.3.7) [36]**:** Let X be a set and (Y,d) be a complete metric space. Then the metric space $(B(X,Y),\sigma)$ where σ is the metric defined in theorem **(2.3.6)** is complete.

Proof: Let (f_n) be a Cauchy in $B(X,Y)$, then for every $x\in X$,

$$d(f_n(x),f_m(x))\leq \sigma(f_n,f_m)\to 0 \text{ as } n,m\to\infty$$

So $\left(f_n(x)\right)$ is a Cauchy sequence in Y. But Y is complete, implies that the sequence $\left(f_n(x)\right)$ converges to, say, $f(x)$ in Y.

Normally, this convergence would expected to depend on x, being slower for some points than others. In this case however, the convergence is uniform, as it is $\sigma(f_n, f_m) = \sup\{d(f_n(x), f_m(x)), x \in X\}$ which converges to 0. So given any $\epsilon > 0$, there is an $n_0 \in \mathbb{N}$, such that: for any $n, m \geq n_0$ and any $x \in X$

$$d(f_n(x), f_m(x)) < \frac{\epsilon}{2}$$

For each $x \in X$, choose $m \geq n_0$, dependent on x and large enough so that:

$$d(f_m(x), f(x)) < \frac{\epsilon}{2}$$

And this implies $\forall x \in X$, for any $n \geq n_0$

$$d\left(f_n(x), f(x)\right) \leq d\left(f_n(x), f_m(x)\right) + d\left(f_m(x), f(x)\right)$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

That is, $f_n \to f$ uniformly in Y , hence $\sup\{d(f_n(x), f(x)), x \in X\} \leq \epsilon$

Since n_0 is independent of x, it follows that $\sigma(f_n, f) \to 0$.

<u>Claim:</u> The function f is bounded: Since $f_n \to f$ uniformly in Y, it is possible to choose $n_0 \in \mathbb{N}$, such that

$$d\left(f_n(x), f(x)\right) < 1$$

for all $x \in X$.

Since f_{n_0} is bounded, there exists a positive number K such that

$$d\left(f_{n_0}(x), f_{n_0}(y)\right) \leq K, \quad \forall x, y \in X$$

$$d(f(x), f(y)) \leq d\left(f(x), f_{n_0}(x)\right) + d\left(f_{n_0}(x), f_{n_0}(y)\right) + d(f_{n_0}(y), f(y))$$

$$< 1 + \mathrm{K} + 1 = 2 + K, \quad \text{for all } x, y \in X$$

With n_0 independent of x and y. That is $f \in B(X, Y)$, which complete the proof. ■

Example(2.3.8): The space $(B(X, \mathbb{R}), \sigma)$ is complete metric space where X is an arbitrary set, $\mathbb{R}$ has the usual metric, and

$$\sigma(f, g) = \sup\{|f(x) - g(x)| : x \in X\}.$$

The Space $BC(X, Y)$ of Bounded Continuous Functions:

Let $BC(X, Y)$ be the set of all bounded continuous functions from the metric space (X, D) to the metric space (Y, d). Then

$\sigma(f, g) = \sup\{d(f(x), g(x)), x \in X\}$ For $f, g \in BC(X, Y)$ and $x \in X$ is a metric on $BC(X, Y)$.

Theorem(2.3.9)[20]**:** Let X be a metric space, (Y, d) be a complete metric space. Then the metric space $(BC(X, Y), \sigma)$ is complete.

Proof: Let (f_i) be a Cauchy in $BC(X, Y)$, then for every $x \in X$,

$$d(f_i(x), f_j(x)) \leq \sigma(f_i, f_j) \to 0 \text{ as } i, j \to \infty$$

So $\left(f_i(x)\right)$ is a Cauchy sequence in Y. But Y is complete, implies that the sequence $\left(f_i(x)\right)$ converges to, say, $f(x)$ in Y.

Claim: $f_i \to f$ uniformly.

Given $\epsilon > 0$ take $N \in \mathbb{N}$ so that $\sigma\left(f_i, f_j\right) < \frac{\epsilon}{2}$ for $i, j \geq N$. Fix x and i for the moment. Since $d\left(f_i(x), f_j(x)\right) < \frac{\epsilon}{2}$ for all j, we can pass to the limit, and we get $d\left(f_i(x), f(x)\right) \leq \frac{\epsilon}{2} < \epsilon$. Hence, the convergence is uniform. By theorem**(2.3.2)** we deduce that f is continuous, i.e., $f \in C(X, Y)$.

Finally, $f \in B(X, Y)$ (By the claim of theorem **(2.3.7)** and since $f_i \in B(X, Y)$ for each i). So, $f \in C(X, Y) \cap B(X, Y)$.That is, $f \in BC(X, Y)$.■

Example(2.3.10): The space $(BC(X, \mathbb{R}), \sigma)$ is complete metric space where X is an any metric space, $\mathbb{R}$ has the usual metric, and

$$\sigma(f, g) = \sup\{|f(x) - g(x)| : x \in X\}.$$

2.4. An Application: Space Filling Curve

The first space-filling curve was discovered and published in 1890 in [40] and called peano's curve. As a simple application of the results in last sections, the existence of a curve that fills the space I^2 where $\mathrm{I} = [0,1]$ is proved.

Definition(2.4.1)[6]**:** A curve in a space I^2 is the image $f(\mathrm{I})$ of a continuous map $f : \mathrm{I} \to \mathrm{I}^2$.

Definition(2.4.2)[6]: A space filling curve in I^2 is a curve going through each point of I^2.

Recall that the space $(\mathbb{R}^n, d_s)$ is complete where d_s is the square metric defined in section four of chapter one. If $n = 2$, the space $(\mathbb{R}^2, d_s)$is complete. Since I^2 is closed in $\mathbb{R}^2$, (I^2, d_s) is complete where $d_s(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}, x, y \in \mathrm{I}^2$. In corollary**(2.3.4)**, let $X = \mathrm{I}$ with the usual topology and Y be the space (I^2, d_s) , then the space $C(\mathrm{I}, \mathrm{I}^2)$ is complete in the uniform metric ρ defined on $C(\mathrm{I}, \mathrm{I}^2)$.

Theorem(2.4.3)[35]: There exists a continuous map $f: \mathrm{I} \to \mathrm{I}^2$ whose image fills up the entire square I^2.

Proof:

Step 1. Construction of triangular paths :

begin with the closed interval $[0,1]$ in the real line and the square I^2 in the plane.

The triangular path $\mathrm{g}: \mathrm{I} \to \mathrm{I}^2$ pictured in Figure 1 is a continuous map. Replace the path g by the path ǵ pictured in Figure 2 . It is made up of four triangular paths, each half the size of g and having the same initial final points as g. This operation can also be applied to any triangular path connecting two adjacent corners of the square. For instance, when applied to the path h pictured in Figure 3, it gives the path $\acute{h}$.

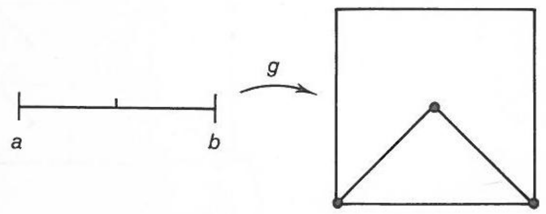

Figure 1

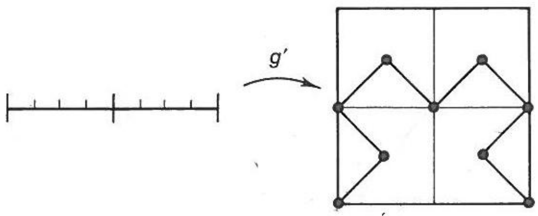

Figure 2

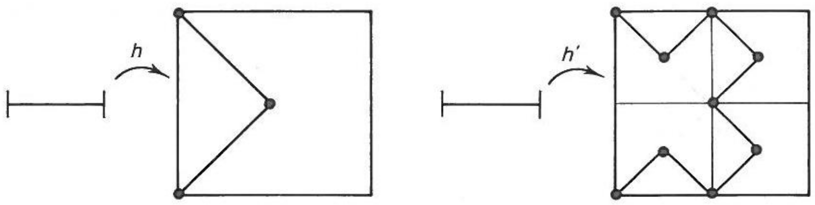

Figure 3

Step 2. Construction of a sequence of continuous functions:

Define a sequence of functions $f_n : \mathrm{I} \to \mathrm{I}^2$. The triangular path pictured in Figure 1, is the first function f_0. The next function f_1 is the function obtained by applying the operation described in Step 1 to the function f_0; it is pictured in Figure 2. The next function f_2 is the function obtained by this same operation to each of four triangular paths that make up f_1. It is pictured in Figure 4. The next function f_3 is obtained by applying the operation to each of the 16 triangular paths that make up f_2; it is pictured in Figure 5. And so on. At the general step, f_n is a path made up of 4^n

triangular paths of the type considered in Step 1, each lying in a square of edge length $\frac{1}{2^n}$. The function f_{n+1} is obtained by applying the operation of Step 1 to these triangular paths, replacing each one by four smaller triangular paths.

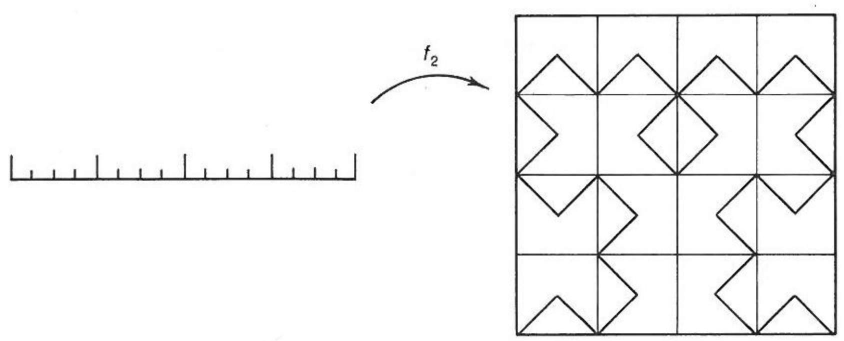

Figure 4

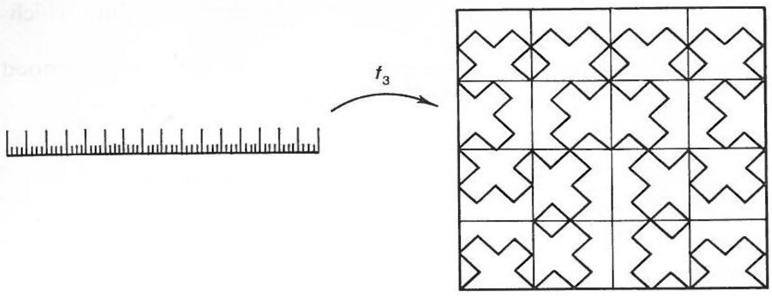

Figure 5

Step 3. Proving that (f_n) is a Cauchy sequence:

To prove that the sequence of functions (f_n) defined in step 2 is a Cauchy sequence in the space $(C(\mathrm{I}, \mathrm{I}^2), \rho)$ where ρ is the uniform metric, take the functions f_n and f_{n+1}. Each small triangular path in f_n lies in a square of edge length $\frac{1}{2^n}$. The operation by which the function f_{n+1} is obtained replaces this triangular path by four triangular paths that lie in the same square. Therefore, in the square metric on I^2, the distance between $f_n(t)$ and $f_{n+1}(t)$ is at most $\frac{1}{2^n}$. That is:

$$d_s(f_n(t), f_{n+1}(t)) \leq \frac{1}{2^n} \ , \forall t \in \mathrm{I}$$

As a result,

$$\rho(f_n, f_{n+1}) \leq \frac{1}{2^n}$$

But, $\forall\, n, m$

$$\rho(f_n, f_{n+m})$$

$$\leq \rho(f_n, f_{n+1}) + \rho(f_{n+1}, f_{n+2}) + \cdots + \rho(f_{n+m-1}, f_{n+m})$$

$$\leq \quad \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+m-1}} < \frac{2}{2^n}$$

So the sequence (f_n) is Cauchy. Hence it converges in $(C(\mathrm{I}, \mathrm{I}^2), \rho)$ where ρ is the uniform metric to a continuous function f because $(C(\mathrm{I}, \mathrm{I}^2), \rho)$ is complete space.

Step 4. Proving that f is surjective:

Let $x \in \mathrm{I}^2$ .Show $x \in f(\mathrm{I})$

Claim: $x \in \overline{f(\mathrm{I})}$

Given $\epsilon > 0$, Let N large enough that $\rho(f_N, f) < \frac{\epsilon}{2}$ and $\frac{1}{2^N} < \frac{\epsilon}{2}$.

Then for all $t \in \mathrm{I}$, $d_s(f_N(t), f(t)) < \frac{\epsilon}{2}$.

Given n, the path f_n comes within a distance of $\frac{1}{2^n}$ of the point x. for the path f_n touches each of the little squares of edge length

$\frac{1}{2^n}$ into which I^2 is divided. So there is a point $t_0 \in \mathrm{I}$ such that

$d_s(x, f_N(t_0)) \leq \frac{1}{2^N} < \frac{\epsilon}{2}$.

But $d_s(f_N(t_0), f(t_0)) < \frac{\epsilon}{2}$. Hence, $d_s(x, f(t_0) \leq d_s\big(x, f_N(t_0)\big) + d_s(f_N(t_0), f(t_0)) < \epsilon$. So, the ϵ-

neighborhood of x intersects $f(\mathrm{I})$. It follows that $x \in \overline{f(\mathrm{I})}$

The set I is compact, so $f(\mathrm{I})$ is compact (continuous image of A compact set is compact) and therefore is closed.

Hence, x belongs to $f(\mathrm{I})$. ■

Chapter Three

Theorems of Topological Characters Concerning Complete Spaces

Introduction

The study of topological aspects of complete metrics has a place in topology. In this chapter, theorems of topological characters concerning complete metric spaces are presented and proved. One of these theorems is to characterize compactness of a metric space that is used to prove Heine-Borel theorem and the classical version of Ascoli's theorem. Another one is to prove that complete metric spaces belong to the class of topological spaces called the Baire spaces. As an application, the existence of a continuous nowhere-differentiable real-valued function is proved.

3.1. Heine-Borel Theorem And Ascoli's Theorem.

In this section, a condition on a metric space to be complete is provided. A theorem characterizes compactness of a metric space is proved and it is then used to prove Heine-Borel theorem and the classical version of Ascoli's theorem.

Definition(3.1.1)[29]**:** A set S in a metric space (X, d) is said to be sequentially compact if every sequence in S contains a subsequence which converges to a point in S .

The terms compact and sequentially compact are equivalent in any metric space.

Theorem(3.1.2)[29]**:** A set S in a metric space (X, d) is compact iff it is sequentially compact.

Proof: Suppose that S is compact, but not sequentially compact. Thus there is an infinite sequence $(x_n) \subset S$ with no subsequence converging to a point in S. Which implies that the points of the sequence (x_n) do not cluster about any point of S. Thus there exists $\epsilon > 0$ in which each point $s \in S$ can be covered by the open ball $B(s, \epsilon)$ which contains at most one point of (x_n).Hence $\{B(s, \epsilon): s \in S\}$ is an open cover for S, which has a finite sub-cover (by compactness of S) $B_1, B_2, \dots, B_N$. Since (x_n) can have at most one point in each such ball, (x_n) is finite, which is impossible.

Conversely, suppose that S is sequentially compact, but not compact, so there is an infinite cover of the set which does not contain a finite sub-cover. Choose $\epsilon > 0$, and a point $s_1 \in S$. Since S cannot be covered by a finite collection of open sets, it cannot be covered by the ball of radius ϵ about s_1. Therefore, we can choose $s_2 \in S$ such that $d(s_1, s_2) \geq \epsilon$. For the same reason we can choose s_3 outside balls of radius ϵ around s_1 and s_2, i.e., so that $d(s_i, s_3) \geq \epsilon$, $i = 1,2$. Continue in this way to define s_n such that $d(s_i, s_n) \geq \epsilon$, $i = 1,2, \dots, n-1$. Since S is sequentially compact, the sequence (s_n) must possess a Cauchy subsequence (s_{n_k}), so that $d(s_{n_j}, s_{n_k}) < \epsilon$ for sufficiently large n_j, n_k which is impossible. This is a contradiction.■

Although completeness of a metric space is not a topological property, there are some topological conditions which implies that a metric space is complete.

Theorem(3.1.3)[20]: Any compact metric space (X, d) is complete.

Proof: Let (x_n) be a Cauchy sequence in X. By theorem **(3.1. 2)** X is compact then it is sequentially compact which implies that the sequence (x_n) has a convergent subsequence, say, to x. By theorem **(1.1.7)** the whole sequence converges to x. So (X, d) is complete.■

Definition(3.1.4)[47]: A subset G of a metric space (X, d) is said to be totally bounded if, given any $\epsilon > 0$, there exists a finite subset $\{x_1, x_2, \dots, x_n\} \subset X$ such that $G \subset \bigcup_{k=1}^{n} B(x_k, \epsilon)$; i.e., for each $\epsilon > 0$, G can be covered by a finite number of open balls of radius ϵ and centers at $x_1, x_2, \dots, x_n$.

Definition(3.1.5)[27]: A finite ϵ-net for a subset G of a metric space (X, d) is a finite collection of points $y_\epsilon = \{y_1, y_2, \dots, y_n\}$ in which for each $x \in G$ there is a point $y_k \in Y_\epsilon$ such that $d(x, y_k) < \epsilon$. That is, if Y_ϵ is a finite ϵ-net for the set $G \subseteq X$, then the set G is covered by the open balls $B(y_i, \epsilon); i = 1,2, \dots, n$ having radius ϵ and center $y_i \in y_\epsilon$ that is, $G \subseteq \bigcup_{i=1}^{n} B(y_i, \epsilon)$.

The ϵ-net can be used to define total boundedness of a subset of a metric space.

Definition(3.1.6)[27]: A subset G of the metric space (X, d) is said to be totally bounded if for every $\epsilon > 0$ there is a finite ϵ-net for G.

Consequences(3.1.7)[36]: In a metric space (X, d) we have the following:

1. Any subset of a totally bounded set is totally bounded.

2. A finite union of totally bounded sets is totally bounded.

Proof: 1. Let G be a totally bounded set of X.Let A be any subset of G. Given $\epsilon > 0$, then the set G is covered by the open balls $B(x_i, \epsilon); i = 1,2, \dots, n$ having radius ϵ and center $x_i \in \{x_1, x_2, \dots, x_n\}$. That is there is a finite ϵ-net for G. But $A \subseteq G$ implies that A is covered by the same finite ϵ-net of the set G.

2. Let $U_i, i = 1,2, \dots, m$ be totally bounded sets of X. Let $A = \bigcup_{i=1}^{m} U_i$. Since U_i is totally bounded for each $i = 1,2, \dots, m$, given $\epsilon > 0$,there exists $\{x_{i1}, x_{i2}, \dots, x_{in_i}\} \subset X$ such that $U_i \subset \bigcup_{k=1}^{n_i} B(x_{ik}, \epsilon)$. Then $\bigcup_{i=1}^{m} U_i \subset \bigcup_{i=1}^{m} \bigcup_{k=1}^{n} B(x_{ik}, \epsilon)$ implies that there exists $\bigcup_{i=1}^{m}\{x_{i1}, x_{i2}, \dots, x_{in_i}\} \subset X$ such that $A \subset \bigcup_{i=1}^{m} \bigcup_{k=1}^{n_i} B(x_{ik}, \epsilon)$.Since finite union of finite sets remains finite, A is totally bounded.

Theorem(3.1.8)[27]: A totally bounded set G in a metric space (X, d) is bounded.

Proof: Pick some $\epsilon > 0$. Since A is totally bounded let $Y_\epsilon = \{y_1, y_2, \dots, y_n\}$ be an ϵ-net for G. Define

$$C = \max\{d(y_i , y_j): i, j = 1,2, \dots, n\}$$

Let x, y be any two points in G. By the definition of an ϵ-net there are two balls $B(y_m, \epsilon)$ and $B(y_n, \epsilon)$ such that

$$x \in B(y_m, \epsilon),$$

and

$$y \in B(y_n, \epsilon),$$

where $y_m, y_n \in Y_\epsilon$. By the triangle inequality,

$$d(x,y) \leq d(x,y_m) + d(y_m,y_n) + d(y_n,y)$$

$$\leq \epsilon + C + \epsilon$$

$$= C + 2\epsilon$$

Since $x, y \in G$ are arbitrary, we have that $diam(G) \leq C + 2\epsilon$.Hence the set G is bounded.■

The converse of this theorem is not always true.

Example(3.1. 9)[23]**:** Consider the metric space (l_2, d), where l_2 is the set of all sequences of real numbers (x_n) such that $\sum_{i=0}^{\infty} x_i{}^2 < \infty$,and $d(x,y) = \sqrt{\left(\sum_{i=0}^{\infty}(x_i - y_i)^2\right)}$ for $x = (x_1, x_2, \dots, x_n, \dots)$ and $y = (y_1, y_2, \dots, y_n, \dots) \in l_2$

The unit sphere with the equation $\sum_{i=1}^{\infty} x_i{}^2 = 1$ is a bounded subset in l_2 but not totally bounded, because the points $e_1 = (1,0,0,\dots), e_2 = (0,1,0,\dots)$ where the i-th coordinate of e_i is one and the other coordinates are all zero all lie on this unit sphere, and the distance between any two of them is $\sqrt{2}$. Hence this unit sphere cannot have a finite ϵ-net when $\epsilon = \frac{\sqrt{2}}{2}$.

In Euclidean space $\mathbb{R}^n$ total boundedness is equivalent to boundedness.

Theorem(3.1.10)[8]: Bounded subsets of $(\mathbb{R}^n, d_2)$ are totally bounded.

Proof: Every bounded set is contained in some cube $Q = [-R, R]^n = \{x \in \mathbb{R}^n : max\{|x_1|, |x_2|, \dots, |x_n|\} \leq R\}$. Since any subset of a totally bounded set is totally bounded, it is enough to show that Q is totally bounded. Given $\epsilon > 0$, choose an integer $k > R\sqrt{n}/\epsilon$, and let Q be the union of k^n identical subcubes by dividing the interval $[-R, R]$ into k equal pieces. The side length of these subcubes is $2R/k$ and hence their diameter is $\sqrt{n}(2R/k) < 2\epsilon$, so they are contained in the balls of radius ϵ about their centers. ■

Theorem(3.1.11)[25]: Every closed and bounded interval of the real line is totally bounded.

Proof: Consider the real line $\mathbb{R}$ with the usual metric. Let V_ρ be any nondegenerate closed and bounded interval, say $V_\rho = [\alpha, \alpha + \rho]$ for some real number α and some $\rho > 0$. Take an arbitrary $\epsilon > 0$ and let n_ϵ be a positive integer large enough so that $\rho < (n_\epsilon + 1)\frac{\epsilon}{2}$. For each integer $k = 0,1, \dots, n_\epsilon$ consider the interval

$A_k = [\alpha + k\frac{\epsilon}{2}, \alpha + (k+1)\frac{\epsilon}{2})$ of diameter $\frac{\epsilon}{2}$. Since $A_j \cap A_i = \emptyset$ whenever $j \neq i$, and $V_\rho \subset [\alpha, \alpha + (n_\epsilon + 1)\frac{\epsilon}{2}) = \bigcup_{k=0}^{n_\epsilon} A_k$, so $\{A_k \cap V_\rho\}_{k=0}^{n_\epsilon}$ is a finite partition of V_ρ into sets of diameter less than ϵ. Thus every closed and bounded interval of the real line is totally bounded. ■

Total boundedness is also characterized in terms of Cauchy sequences.

Theorem(3.1.12)[36]**:**In a metric space (X, d), a set K is totally bounded iff every sequence in K has a Cauchy subsequence.

Proof: Let the totally bounded set K be covered by a finite number of balls of radius 1, and let $\{x_1, x_2, \dots\}$ be an infinite subset of K. (If K is finite, a selected sequence must take some value x_i infinitely often and so has a constant subsequence) A finite number of balls cannot cover an infinite set of points, unless at least one of the balls, $B(a_1, 1)$, has an infinite number of points, say $\{x_{1,1}, x_{2,1}, \dots\}$.

Now cover K with a finite number of balls each of radius $\frac{1}{2}$. For the same reason as above, at least one of these, $B(a_2, \frac{1}{2})$ covers an infinite number of points of $\{x_{n,1}\}$, say the new subset $\{x_{1,2}, x_{2,2}, \dots\}$. Continue this process forming covers of balls each of radius $\frac{1}{m}$ and infinite subsets $\{x_{n,m}\}$ of $B(a_m, \frac{1}{m})$. The sequence $(x_{n,n})$ is Cauchy, since for $m \le n$, both $x_{m,m}$ and $x_{n,n}$ are elements of the set $\{x_{1,m}, x_{2,m}, \dots\}$, and so $d(x_{n,n}, x_{m,m}) < \frac{2}{m} \to 0$ as $n, m \to \infty$.

Conversely, given $\epsilon > 0$, let $a_1 \in K$. If $B(a_1, \epsilon)$ covers K then we are done. If not, pick a_2 in K but not in $B(a_1, \epsilon)$. Continue like this to get a sequence $(a_n)$of distinct points in K with $a_n \notin \bigcup_{i=1}^{n-1} B(a_i, \epsilon)$, all of which are at least ϵ distant from each other. This process cannot continue indefinitely otherwise we get a sequence (a_n) in which $d(a_m, a_n) \ge \epsilon$ for all n, m in $\mathbb{N}$, and so has no Cauchy subsequence. So after some N steps we must have $K \subseteq \bigcup_{i=1}^{N} B(a_i, \epsilon)$. ■

Compactness in metric spaces is characterized by proving the following theorem that deals with two sets of properties one of which is topological while the other is not. If (X, d) is a metric space then X is compact means that X with the induced metric topology is compact.

Theorem(3.1. 13)[51]**:** A metric space (X, d) is compact iff it is complete and totally bounded.

Proof: Suppose X is compact. By theorem **(3.1.3)** the metric space is complete. For total boundedness, let (x_n) be a sequence of points in X. Since X is compact it is sequentially compact by theorem **(3.1. 2)** which means that the sequence (x_n) has a convergent subsequence. So this subsequence is Cauchy. Hence, the sequence has a Cauchy subsequence. By theorem **(3.1.12)** X is totally bounded.

Conversely, suppose that X is complete and totally bounded. If (x_n) is an arbitrary sequence in X, then (x_n) has a Cauchy subsequence, by theorem **(3.1.12)**. Since X is complete this subsequence converges. Thus X is sequentially compact. Hence by theorem **(3.1.2)** X is compact. ■

Theorem **(3.1.13)** is used to prove a theorem characterizes the compact subsets of Euclidean space $\mathbb{R}^n$.

Theorem(Heine-Borel)(3.1.14)[25]**:** Every closed bounded subset of $\mathbb{R}^n$ is compact.

Proof: Let B be an arbitrary closed bounded subset of $\mathbb{R}^n$. By theorem **(3.1.8)**, the set B is totally bounded. It is proven(in theorem **(1.4.6)**) that the space $\mathbb{R}^n$ is complete when equipped with the metric d_2. Hence the set B is complete because closed subset of complete metric space is complete subspace by theorem **(1.2.9)**. So, the set B is complete and totally bounded. Therefore (by theorem**(3.1. 13)**), B is compact. ■

Some definitions and theorems are needed to prove the classical version of Ascoli's theorem.

Definition(3.1.15)[39]**:** Let (X, τ) be a topological space, let (Y, d) be a metric space, let $\mathcal{F} \subseteq C(X, Y)$, and let $x_0 \in X$. Then $\mathcal{F}$ is equicontinuous at x_0 if for each $\epsilon > 0$ there exists a neighborhood U of x_0 such that if $f \in \mathcal{F}$ and $x \in U$, then $d(f(x), f(x_0)) < \epsilon$. If $\mathcal{F}$ is equicontinuous at each point of X, then it is said to be equicontinuous.

Note(3.1.16)[39]**:** The difference between a collection of continuous functions and a collection of equicontinuous functions is that if (X, τ) is a topological space, (Y, d) is a metric space, and $\mathcal{F} \subseteq C(X, Y)$ then $\mathcal{F}$ is a collection of continuous functions if for each $x_0 \in X$, each $\epsilon > 0$, and each $f \in \mathcal{F}$, there exists a neighborhood U_f of x_0 such that $x \in U_f$, then $d\big(f(x), f(x_0)\big) < \epsilon$. But if $\mathcal{F}$ is a collection of equicontinuous functions, then for each $x_0 \in X$, each $\epsilon > 0$, there exists a neighborhood U of x_0 such that if $x \in U$ and f is any member of $\mathcal{F}$, then $d\big(f(x), f(x_0)\big) < \epsilon$. In other words, U works for every member of $\mathcal{F}$.

Theorem(3.1.17)[39]**:** Let (X, τ) be a compact space, let (Y, d) be a compact metric space, and let $\mathcal{F} \subseteq C(X, Y)$. Then $\mathcal{F}$ is equicontinuous iff $\mathcal{F}$ is totally bounded with respect to the sup metric

$\rho(f,g) = \max\{d(f(x),g(x)), f,g \in \mathcal{F}, x \in X\}$

Proof: Suppose $\mathcal{F}$ is totally bounded with respect to ρ. Let $x_0 \in X$ and let $\epsilon > 0$. Let $\epsilon_1 = \frac{\epsilon}{3}$, and let $\{f_1, f_2, \dots, f_n\}$ be an ϵ_1-net for $\mathcal{F}$. For each $i = 1,2,\dots,n$, f_i is continuous. Therefore, for each $i = 1,2,\dots,n$, let U_i be a neighborhood of x_0 such that if $x \in U_i$ then $d(f_i(x), f_i(x_0)) < \epsilon_1$. Let $U = \bigcap_{i=1}^{n} U_i$.

<u>Claim</u>: If $f \in \mathcal{F}$ and $x \in U$, then $d(f(x), f(x_0)) < \epsilon$. Let $f \in \mathcal{F}$ and let $x \in U$. Because f belongs to at least one of the ϵ_1-balls, there exists i $(i = 1,2,\dots,n)$ such that $\rho(f, f_i) < \epsilon_1$. Hence $d(f(x), f_i(x)) < \epsilon_1$ and $d(f(x_0), f_i(x_0)) < \epsilon_1$. Thus

$$d(f(x), f(x_0)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(x_0)) + d(f_i(x_0), f(x_0))$$

$$< \epsilon_1 + \epsilon_1 + \epsilon_1 = \epsilon$$

Therefore, $\mathcal{F}$ is equicontinuous.

Conversely, suppose that $\mathcal{F}$ is equicontinuous, and let $\epsilon > 0$. Let $\epsilon_1 = \frac{\epsilon}{3}$. Since $\mathcal{F}$ is equicontinuous, for each $x \in X$ there is a neighborhood U_x of x such that if $z \in U_x$, then $d(f(z), f(x)) < \epsilon_1$ for all $f \in \mathcal{F}$. Then $\{U_x : x \in X\}$ is a open cover of X. Since X is compact, there exists $x_1, x_2, \dots, x_m$ in X such that $\{U_{x_1}, U_{x_2}, \dots, U_{x_m}\}$ covers X. Now $\{B(y, \epsilon_1) : y \in Y\}$ is an open cover of Y. Since Y is compact, there exist $y_1, y_2, \dots, y_n$ in Y such that $B(y_1, \epsilon_1), B(y_2, \epsilon_1), \dots, B(y_n, \epsilon_1)$ covers Y. Let Λ be the collection of all functions that map $\{1,2,\dots,m\}$ into $\{1,2,\dots,n\}$, let $\alpha \in \Lambda$. If there exists $f \in \mathcal{F}$ such that for each $i = 1,2,\dots,m$, $f(x_i) \in B(y_{\alpha(i)}, \epsilon_1)$ choose one such function and label it f_α. Let $\Gamma = \{\alpha \in \Lambda : f_\alpha$ exists$\}$. Since Λ is finite and $\Gamma \subseteq \Lambda$, Γ is finite. <u>Claim:</u> $\{B(f_\alpha, \epsilon) : \alpha \in \Gamma\}$ covers $\mathcal{F}$. Let $f \in \mathcal{F}$, and for each $i = 1,2,\dots,m$, let $\alpha(i)$ such that $f(x_i) \in B(y_{\alpha(i)}, \epsilon_1)$ then $\alpha \in \Gamma$. To show $f \in B(f_\alpha, \epsilon)$, let $x \in X$ and let $i \in \{1,2,\dots,m\}$ such that $x \in U_{x_i}$. Then $d\big(f(x), f_\alpha(x)\big) \le d\big(f(x), f(x_i)\big) + d\big(f(x_i), f_\alpha(x_i)\big) + d\big(f_\alpha(x_i), f_\alpha(x)\big)$

$$< \epsilon_1 + \epsilon_1 + \epsilon_1 = \epsilon$$

Since this inequality holds for every $x \in X$,
$\rho(f, f_\alpha) = \max\big\{d\big(f(x), f_\alpha(x)\big) : x \in X\big\} < \epsilon$. Hence $f \in B(f_\alpha, \epsilon)$. ■

Theorem(3.1.18)[39]**:** Let (X, τ) be a compact space, and let $\mathcal{F}$ be a bounded subset of $(C(X, \mathbb{R}^n), \rho)$. Then there exists a compact subset Y of $\mathbb{R}^n$ such that if $f \in \mathcal{F}$ and $x \in X$, then $f(x) \in Y$.

Proof: Let $f_0 \in \mathcal{F}$. Since $\mathcal{F}$ is bounded, there exists a positive number M such that $\rho(f_0, f) < M$ for all $f \in \mathcal{F}$. Since X is compact and f_0 is continuous, $f_0(X)$ is compact. Hence, $f_0(X)$ is a bounded subset of $\mathbb{R}^n$, so there is a positive number N such that $f_0(X) \subseteq B((0,0,..,0), N)$. Therefore, if $f \in \mathcal{F}$, then $f(X) \subseteq B((0,0,..,0), N + M)$. Let Y be the closure of the ball $B((0,0,..,0), N + M)$. So, Y is closed and bounded subset of $\mathbb{R}^n$ and hence is compact by theorem **(3.1.14).**■

Theorem(3.1.19)[4]**:** Let A be a subset of a metric space (X, d). If A is compact then A is closed in (X, d).

Proof: Suppose that A is compact, and let (x_n) be a sequence in A that converges to a point $x \in X$. Then, from theorem **(3.1.2)**, (x_n) has a subsequence that converges in A, and hence x must be in A. Thus, A is closed. ■

A characterization of compact subsets of $\mathbb{R}^n$ is that they are closed and bounded. But for the space $C(X, \mathbb{R}^n)$ the standard criterion for compactness is given by the classical version of Ascoli's theorem which is proven using theorem **(3.1.13)**.

Theorem(Ascoli's Theorem: Classical Version)(3.1.20)[39]**:** Let (X, τ) be a compact space. Then, a subset of $(C(X, \mathbb{R}^n), \rho)$ is compact iff it is closed, bounded, and equicontinuous.

Proof: Suppose $\mathcal{F}$ is a compact subset of $(C(X, \mathbb{R}^n), \rho)$.

<u>Closed</u>: By theorem**(3.1.19)** $\mathcal{F}$ is closed.

<u>Bounded</u>: $\mathcal{F}$ is compact then $\mathcal{F}$ is totally bounded by theorem **(3.1.13)**. But a totally bounded set is bounded by theorem **(3.1.8)**.

<u>Equicontinuous</u>: By theorem **(3.1.18)** there exists a compact subset Y of $\mathbb{R}^n$ such that if $f \in \mathcal{F}$, then $f(X) \subseteq Y$. It follows that $\mathcal{F} \subseteq C(X, Y)$ which is totally bounded. Therefore, by theorem **(3.1.17)** $\mathcal{F}$ is equicontinuous.

Conversely, suppose $\mathcal{F}$ is closed, bounded, and equicontinuous subset of $(C(X, \mathbb{R}^n), \rho)$. By theorem **(1.4.6)** the space $(\mathbb{R}^n, d_2)$ is complete. Therefore by corollary **(2.3.4)** $(C(X, \mathbb{R}^n), \rho)$ is complete. Since

$\mathcal{F}$ is closed subset of $(C(X, \mathbb{R}^n), \rho)$, by theorem **(1.2.9)** $\mathcal{F}$ is complete. Since $\mathcal{F}$ is bounded, by theorem **(3.1.18)**, there exists a compact subset Y of $\mathbb{R}^n$ such that if $f \in \mathcal{F}$ then $f(X) \subseteq Y$. It follows that $\mathcal{F} \subseteq C(X, Y)$ which is equicontinuous. Therefore, by theorem **(3.1.17)** $\mathcal{F}$ is totally bounded. Since $\mathcal{F}$ is complete and totally bounded, by theorem **(3.1.13)**, it is compact. ■

3.2. Baire Spaces.

In applying topology to analysis one of the most useful applications of completeness is the Baire's theorem. In this section one form of Baire's theorem is stated and proved.

Definition(3.2.1)[32]**:** A topological space X is called a Baire space if for each sequence (O_n) of dense open subsets of X, $\bigcap_{n \in N} O_n$ is dense in X.

Theorem(Baire's theorem)(3.2.2)[11]**:** In a complete metric space (X, d) the intersection of a countable number of open, dense sets is itself dense. That is, a complete metric space is a Baire space.

Proof: Let $\{G_n\}$ be a countable family of dense open sets in X. Let $x \in X$ and $\epsilon > 0$ be arbitrary. Since G_1 is dense open set in X there exists an $x_1 \in G_1 \cap B(x, \epsilon)$ such that

$\bar{B}(x_1, \epsilon_1) \subseteq G_1 \cap B(x, \epsilon)$ where $0 < \epsilon_1 < \epsilon/2$.

Since G_2 is dense open set in X there exists an $x_2 \in G_2 \cap B(x_1, \epsilon_1)$ such that

$\bar{B}(x_2, \epsilon_2) \subseteq G_1 \cap G_2 \cap B(x, \epsilon) \cap B(x_1, \epsilon_1)$ where $0 < \epsilon_2 < \epsilon/4$

Continuing inductively, there exists an

$x_n \in G_n \cap B(x_{n-1}, \epsilon_{n-1})$ such that

$\bar{B}(x_n, \epsilon_n) \subseteq G_1 \cap G_2 \cap \ldots \cap G_n \cap B(x, \epsilon) \cap B(x_{n-1}, \epsilon_{n-1})$
where $0 < \epsilon_n < \epsilon/2^n$

The sequence $\bar{B}(x_n, \epsilon_n)$ is a sequence of nonempty closed, descending sets such that $diam\bar{B}(x_n, \epsilon_n) \to 0$ as $n \to \infty$. Since the space (X, d) is complete, by Cantor's intersection theorem there exists $y \in \bar{B}(x_n, \epsilon_n)$. But $\bigcap_{n=1}^{\infty} \bar{B}(x_n, \epsilon_n) \subseteq (\bigcap_{n=1}^{\infty} G_n) \cap B(x, \epsilon)$ implies that: $(\bigcap_{n=1}^{\infty} G_n) \cap B(x, \epsilon) \neq \emptyset$. Hence, $x \in \overline{\bigcap_{n=1}^{\infty} G_n}$. ■

The converse of this theorem is not always true. There are incomplete metric spaces that are Baire spaces.

Example(An Incomplete Baire Space)(3.2.3)[34]**:** Let X be the open interval (a, b) with the usual metric. Then (a, b) is dense in $[a, b]$. Let $U_1, U_2, \ldots, U_n, \ldots$ be a sequence of dense open sets in X. Then $U_i = H_i \cap (a, b)$, where H_i is dense and open in $[a, b]$. Now, $(a, b), H_1, H_2, \ldots, H_n, \ldots$ is a sequence of dense open sets in $[a, b]$ and hence (since $[a, b]$ is a Baire space) $(a, b) \cap (\bigcap_{i=1}^{\infty} H_i) = \bigcap_{i=1}^{\infty}\big((a, b) \cap H_i\big) = \bigcap_{i=1}^{\infty} U_i$ is dense in $[a, b]$ and therefore in (a, b) as well. Hence (a, b) with the subspace metric is a Baire space. But (a, b) is incomplete.

In [12] it is discussed that the intersection of countably many dense G_δ-subsets of a Baire space X must be dense in X. A weaker condition is the intersection of any two dense G_δ-sets of X must be dense in X, and that is the definition of a Volterra space. Any Baire space is Volterra. The converse is studied in [12].

Various forms of the Baire's theorem is discussed in [14]. A generalization of Baire's theorem is proved in [30]. Cartesian products of metric Baire spaces is discussed in [24].

3.3. Continuous Nowhere Differentiable Function.

In 1806 Ampere in [1] tried to prove that any continuous function must be differentiable on a set of points. In 1872 Weierstrass presented a continuous nowhere differentiable function which is published in 1875 in [2]. In [17 , 54 , 53 , 50] examples of continuous nowhere differentiable functions are provided.

In this section, the existence of continuous nowhere-differentiable real-valued functions is proved using Baire's theorem.

Theorem(3.3.1)[35]: Let h be a continuous real-valued function defined on the unit interval $I = [0,1]$ and let $\epsilon > 0$ be given. Then, there is a continuous real-valued function g defined on I that is nowhere differentiable with the property $|h(x) - g(x)| < \epsilon$ for all $x \in I$.

Proof: The existence is proven without constructing an example. Let $C(I, \mathbb{R})$ be the set of all continuous real-valued functions defined on I.

By corollary **(2.3.4)** the space $(C(I, \mathbb{R}), \rho)$ where $\rho(f, g) = max\{|f(x) - g(x)| : x \in I\}$ is complete. By theorem**(3.2.2)** the space $(C(I, \mathbb{R}), \rho)$ is a Baire space. This is the main property which is used in the proof.

Step 1:

Defining (U_n) to be a sequence of open dense subsets of $C(I, \mathbb{R})$:

Let $0 < h < \frac{1}{2}$, let $x \in I$ be given and $\alpha > 0$.

Given $f \in C(I, \mathbb{R})$, define its difference quotients as:

$$\Delta f(x, h) = \max\left\{\left|\frac{f(x+h) - f(x)}{h}\right|, \left|\frac{f(x-h) - f(x)}{-h}\right|\right\}$$

If both

$$\left|\frac{f(x+h) - f(x)}{h}\right|$$

and

$$\left|\frac{f(x-h) - f(x)}{-h}\right|$$

are exist. And $\Delta f(x, h)$ is the one that is defined if one of the two is not defined.

At least one of the two is defined since at least one of the numbers $x + h$ and $x - h$ is in the unit interval I.

Let the set

$$U(\alpha,h)=\{f\in C(I,\mathbb{R}):\Delta f(x,h)\geq\alpha\ \forall\ x\in I\}$$

Now, for each $n\in\mathbb{N}$, define the set U_n as :

$$U_n=\bigcup_{\alpha>n,h<\frac{1}{n}}U(\alpha,h)$$

<u>Claim(1):</u> for each $n\in\mathbb{N}$, the set U_n is open in $C(I,\mathbb{R})$.

Let $f\in U_n$ then $f\in U(\alpha,h)$ for some $\alpha>n$ and some $h<\frac{1}{n}$. take

$$r=\frac{h}{4}(\alpha-n)$$

Without loss of generality, let

$$\Delta f(x,h)=\left|\frac{f(x+h)-f(x)}{h}\right|$$

And let $g\in B(f,r)$ such that

$$\Delta g(x,h)=\left|\frac{g(x+h)-g(x)}{h}\right|$$

Now,

$$\left|\frac{f(x+h)-f(x)}{h}-\frac{g(x+h)-g(x)}{h}\right|$$

$$=\left|\frac{\big(f(x+h)-g(x+h)\big)-\big(f(x)-g(x)\big)}{h}\right|$$

$$\leq\frac{2r}{h}=\frac{\alpha-n}{2}$$

Since

$$\left|\frac{f(x+h)-f(x)}{h}\right|$$

is at least α, then

$$\left|\frac{g(x+h)-g(x)}{h}\right|$$

is at least

$$\alpha - \frac{\alpha-n}{2} = \frac{\alpha+n}{2} = \grave{\alpha}$$

For some $\grave{\alpha}$.

Then $\Delta g(x,h) \geq \grave{\alpha}$ and this implies that $g \in U(\acute{\alpha},h)$. But $\alpha' > n$, so $g \in U_n$, i.e., $B(f,r) \subseteq U_n$ which means that the set U_n is open in $C(I,\mathbb{R})$.

<u>Claim(2):</u> for each $n \in \mathbb{N}$, the set U_n is dense in $C(I,\mathbb{R})$.

Let f be arbitrary function in $C(I,\mathbb{R})$. Given $\epsilon > 0, \alpha > n$, we construct a function g that belongs to both sets U_n and $B(f,\epsilon)$.

Let $0 = x_0 < x_1 < x_2 < \cdots < x_k = 1$ be a partition of of the unit interval I . Let $I_i = [x_{i-1}, x_i], i = 1,2, \ldots, k$. Consider g to be the function such that $g|I_i$ is a linear function in which the slope of each line segment is at least α. Let $h < \frac{1}{n}$ and $h \leq \frac{1}{2}\min\{|x_i - x_{i-1}|; i = 1,2, \ldots, k\}$. g is a member of U_n, for if $x \in I$, then $x \in I_i$ for some $i = 1.2, \ldots, k$. If $x \in \left[x_{i-1}, \frac{x_{i-1}+x_i}{2}\right]$, then $x + h \in I_i$ and $\frac{g(x+h)-g(x)}{h}$ equals the slope of the line segment representing the linear function $g|I_i$. And if $x \in \left[\frac{x_{i-1}+x_i}{2}, x_i\right]$, then $x - h \in I_i$ and $\frac{g(x-h)-g(x)}{-h}$ equals the slope of the line segment representing the linear function $g|I_i$. This implies that $\Delta g(x,h) \geq \alpha$, $g \in U(\alpha, h) \subseteq U_n$.

A construction of g to be an element of $B(f, \epsilon)$ is the following:

By uniform continuity of f, we can choose a partition of the unit interval I $0 = t_0 < t_1 < \cdots < t_m = 1$ such that $|f(x) - f(y)| < \frac{\epsilon}{4}\ \forall\, x, y \in [t_{i-1}, t_i], i = 1,2, \dots, m.$

Let $a_i \in (t_{i-1}, t_i), i = 1,2, \dots, m.$ Define

$$g_1(x) = \begin{cases} f(t_{i-1}) & \forall x \in [t_{i-1}, a_i] \\ f(t_{i-1}) + \dfrac{f(t_i) - f(t_{i-1})}{t_i - a_i}(x - a_i) & \forall x \in [a_i, t_i] \end{cases}$$

The graphs of g_1 and f are pictured in Figure 6. If $f(t_i) \neq f(t_{i-1})$ then a_i must be such that

$$t_i - a_i \leq \left|\frac{f(t_i) - f(t_{i-1})}{\alpha}\right|$$

Then g_1 is a piecewise-linear function for which each line segment have slope at least α in absolute value or have slope zero.

Now, for each subinterval I_i the two functions g_1 and f vary from $f(t_{i-1})$ by at most $\epsilon/4$. That is $|g_1(x) - f(x)| < \epsilon/2$ for all $x \in I$ .So $\rho(g_1, f) = \max\{|g_1(x) - f(x)|\} < \epsilon/2.$

The function g that is wanted is just the function g_1 but with replacing each line segment that has slope zero by a "sawtooth" graph for which the absolute value of the slope of each edge is at least α and lies within $\epsilon/2$ of the function g_1. The graphs of f , $g_1$and g are pictured in Figure 7.

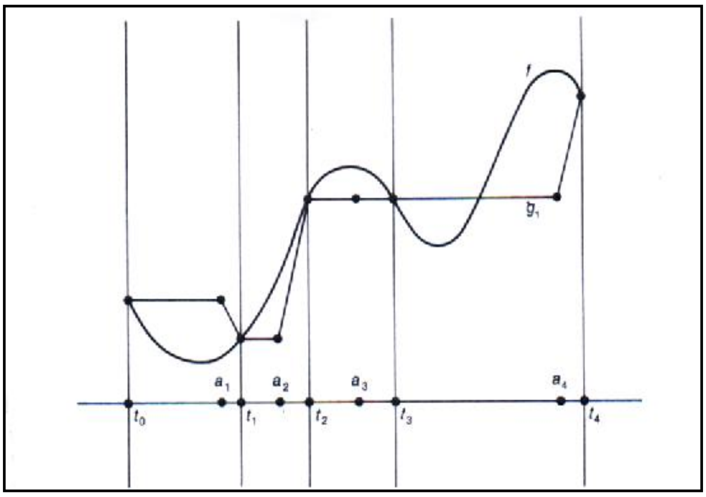

Figure 6

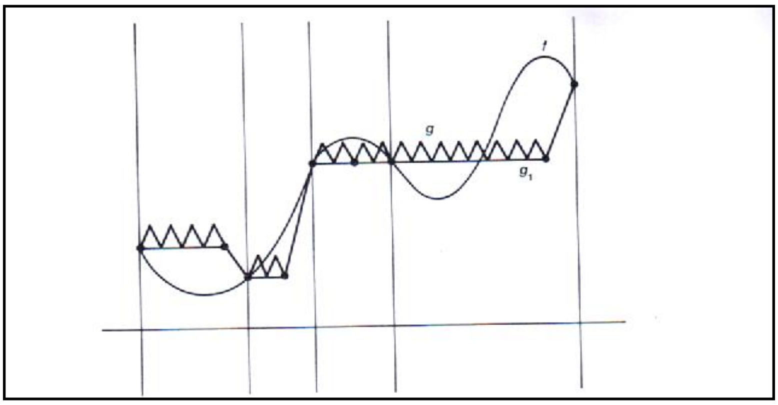

Figure 7

Step 2:

The functions that are in the intersection of the sets $U_n, n \in \mathbb{N}$ are continuous nowhere differentiable:

the space $(C(I,\mathbb{R}),\rho)$ is a Baire space So$\bigcap_{n\in\mathbb{N}} U_n$ is dense in $C(I,\mathbb{R})$ which implies that for each $\epsilon > 0$ there is a function $g \in \bigcap_{n\in\mathbb{N}} U_n$ such that $\rho(h,g) < \epsilon$.

For the end of this proof, let $x \in I$ be given. Let $f \in \bigcap_{n\in\mathbb{N}} U_n$, then $f \in U_n$ for each $n \in \mathbb{N}$. So there is a number $0 < h_n < 1/n$ where $\Delta f(x, h_n) > n$. Clearly the sequence (h_n) converges to zero while the sequence $\big(\Delta f(x, h_n)\big)$ diverge. In other words, $\lim_{h\to 0} \Delta f(x, h)$ does not exist. Hence f is not differentiable at x. Since, x was arbitrary, the function f is nowhere differentiable.■

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جامعة النجاح الوطنية

كلية الدراسات العليا

صفات تبولوجية للفراغات المترية التامة

إعداد

حنين أكرم مصطفى غانم

إشراف

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قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات ، بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس – فلسطين.

2015

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الملخص

في هذه الرسالة نتناول الخصائص التبولوجية للفراغات المترية التامة.

تم عرض الفراغات المترية التامة وصفاتها وأمثله عليها.ثم تم مناقشه تحويل فراغ متري غير تام الى فراغ تام.

عرفنا فراغات الاقترانات والتبولوجيات المعرفة عليها. تم إنشاء فراغ بيانو كتطبيق على ذلك.

أخيرا، نظريات ذات خصائص تبولوجية بالفراغات المترية التامة مثل نظريه هايين بوريل، أزكولي ونظريه بيير تم عرضها مع براهينها بالإضافة إلى أمور أخرى متعلقة بالموضوع . تم برهان وجود اقتران متصل ولكن غي قابل للاشتقاق عند أي نقطه كتطبيق على نظريه بيير.