An-Najah National University

Faculty of Graduated Studies

On The Theory of Convergence Spaces

By

Raed Juma Hassan Shqair

Supervisor

Dr. Mohammad Abu Eideh

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus, Palestine.

2015

On The Theory of Convergence Spaces

By

Raed Juma Hassan Shqair

This thesis was defended successfully on 4/2/2015 and approved by:

Defense Committee Members		**Signature**
– Dr. Mohammad Abu Eideh	(Supervisor)	……………
– Dr. Bassam Manasrah	(External Examiner)	……………
– Dr. Abdallah Hakawati	(Internal Examiner)	……………

Dedication

To my parents and wife.

***To my friends Ahmad Noor* , Abdallah Nassar**

To the souls of the martyrs of Palestine.

ACKNOWLEDGMENTS

First of all , thank God The Almighty for all the blessing He bestowed on me , and continues to bestow .

I would also like to express my special thanks of gratitude to my supervisor Dr. Mohammad Abu Eideh for the greatest effort in supervising and directing me to come with this thesis .

Thanks and appreciations to the defense committee members Dr. Bassam Manasrah and Dr. Abdallah Hakawati for their time and patience .

Thanks are also due to all faculty members in The Department of Mathematics at An-Najah National University .

Thanks and appreciations to Dr. Ali Abdel-Mohsen

I would like to express the appreciation for my parents ,my helpful wife and to all relatives for their fruitful assistance and support to achieve this thesis .

I would like to thank and appreciate all my friends especially Ahmed Noor, Abdallah Nassar , Isaam Sabri , Hisham Salahat , Ahamad Thabaineh, Bassem Mustafa , for their continuous encouragement .

Finally , I would like to thank Horizon bookstore for printing this thesis .

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان :

On The Theory of Convergence Spaces

اقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص ، باستثناء ما تمت الإشارة إليه حيثما ورد ، وأن هذه الرسالة ككل أو جزء منها لم يقدم لنيل أية درجة أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى .

Declaration

The work provided in this thesis , unless otherwise reference , is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification .

Student's name : — **اسم الطالب :** رائد جمعة حسن شقير

Signature : — **التوقيع :** رائد جمعه

Date : — **التاريخ :** ٤ / ٢ / ٢٠١٠

Table of Contents

On The Theory of Convergence Spaces

By

Raed Juma Hassan Shqair

Supervised by

Dr. Mohammad Abu Eideh

Abstract

In this thesis we investigate some information about convergence space concepts such as closure and interior of sets , open sets, closed set, cluster point of a filter , closed adherences of convergence spaces , separation axioms, continuity, homeomorphism, compactness, connectedness spaces and obtain some results about the aforesaid concepts and provide basic ideas of convergence theory, which would enable One to tackle convergence -theoretic without much effort . In this thesis some results on the cluster set of functions in convergence spaces are obtained.

Historical Remarks and Introduction

The study of topological spaces as a formal subject goes back to Hausdorff (1914)[26] and Kuratowski (1922) [27]. There were, of course, several motivations for the introduction and study of general topological spaces and one of the main reasons for doing so was to provide a setting for the investigation of convergence.

However, the concept of convergence in topological spaces is not general enough to cover all interesting cases in analysis, probability theory, etc. In particular, the following is an example of 'non-topological' convergence:

The measure theoretic concept of convergence almost everywhere is well known to be non-topological. Since topological spaces are inadequate for the investigation of certain interesting limit operations, the idea of using the concept of convergence itself as a primitive term arises naturally. As a matter of fact, even before Hausdorff's 1914 work , in 1906 [28] Frechet took the notion of the limit of a sequence as a primitive term and he explored the consequences of a certain set of axioms involving limits. Later, in 1926, Urysohn [29] considered more appropriate axioms for limits of sequences.

But for the study of convergence to reach maturity, the concept of filter was needed, which Cartan [30] provided in 1937.

In 1948, Choquet [4] presented his theory of 'structures pseudotopologiques' and 'structures pretopologiques' in which the concept of convergence of a filter is axiomatized. In 1954, Kowalsky [11] introduced his 'Limesraume' which involve also an axiomatization of the concept of convergence of a filter, but Kowalsky's axioms are both simpler and less restrictive than those of Choquet. Kowalsky, as an example, showed how convergence almost everywhere is precisely the convergence in a certain Limesraüm.

In 1959, Fischer [1] took up the study of Limesraume, but apparently without knowing about Kowalsky's paper. In his work, Fischer used category-theoretical methods and he took a special interest in applications to analysis. In 1965, Cook and Fischer [31] pushed Limesraume further into analysis by proving an Ascoli theorem for convergence spaces and they showed how Hahn's continuous convergence is always given by a convergence structure (i.e. structure of a Limesraum) although it is in general not given by a topology. In 1964,

Kent [32] considered an even more general class of convergence spaces by having axioms weaker than those of his predecessors.

The basic convergence theory was developed by H.R.Fisher[1] (Zurich) in 1959 introduced a convergence concept for filters, in which he associated with each element of a set X, a definite set of filters in X which has to satisfy two conditions of purely algebraic nature .

A convergence space is a generalization of a topological space based on the concept of convergence of filters as fundamental .

However there are convergence spaces which are not topologies as we mentioned earlier. Many topological concepts were easily generated in to convergence spaces.

In this thesis we investigate information about convergence space concepts such as closure and interior of sets , open sets, closed set, cluster point of a filter , closed adherences of filter, the cluster set of functions in convergence spaces , separation axioms, continuity, homeomorphism, compactness, connectedness spaces and obtain some results about the aforesaid concepts and provide basic ideas of convergence theory, which would enable One to tackle convergence -theoretic without much effort.

The dissertation starts with a review in chapter 0 , of the basic concepts of the theory of filters and filter basis , which are needed in the later chapters.

In chapter 1 , convergence spaces and the topological modification of the convergence structure are introduced. Also the concepts of interior , closure operators , the adherence of a filter , and the closed adherences of a convergence spaces are studied.

Continuity of functions on convergence spaces and subspaces are introduced in chapter2.

In chapter 3 ; separation axioms in convergence spaces such as T_1 ,T_2 , Hausdorff , minimal Hausdorff , regular , strongly regular , weakly regular, Π-regular , t-regular convergence spaces are discussed .

Compact, relatively compact and locally compact convergence spaces are introduced in chapter 4.

In chapter 5; connected convergence spaces and their properties are introduced .

Finally in chapter 6 some results on the cluster set of functions in convergence spaces are obtained .

Chapter Zero
Filter and Filter basis

Chapter Zero

Filter and Filter Basis

This preparatory chapter is devoted for preliminaries and terminological conventions which are used in the subsequent chapters .

In order to make the dissertation self contained , we give brief exposition of the parts of the theory of filters .

Definition 0.1 :

Let X be a nonempty set , and $\mathcal{P}$(X) be the power set of X . A nonempty family $\mathcal{F}$of subsets of X is called a filter , if and only if

a) $\emptyset \notin \mathcal{F}$, where $\emptyset$ is the empty set .

b) If $A, B \in \mathcal{F}$ $then$ $A \cap B \in \mathcal{F}$.

c) If $A \in \mathcal{F}$ and $A \subseteq B$ for $some$ $B \in \mathcal{P}$(X)$, thenB \in \mathcal{F}$.

Let F(X) be the set of all filters in X .It is a partially ordered set with respect to the order relation "≤" defined as follows $\mathcal{G} \leq \mathcal{F}$ means that $\mathcal{G} \subseteq \mathcal{F}$.

By this definition , $\mathcal{G} \leq \mathcal{F}$, if and only if for each G ∈ $\mathcal{G}$, there exists some F ∈ $\mathcal{F}$ such that F ⊆ G. If $\mathcal{G} \leq \mathcal{F}$ we say that $\mathcal{F}$ is finer than $\mathcal{G}$ or $\mathcal{G}$ is coarser than $\mathcal{F}$.

One can arrive at filters by another method.

Definition 0.2:

A filter base β on X is a non empty family of subsets of X satisfying the following conditions ;

a) $\emptyset \notin \beta$, and

b) If A,B ∈ β, then ∃ C ∈ β $such$ $that$ $C \subseteq A \cap B$

The class of all supersets of sets in a filter base is a filter in X.

Each nonempty subset A of X defines a filter base {A} in X . Let [A] denote the filter generated by this base i.e [A] = {F $\subseteq$X :A$\subseteq$ F}.In particular if A= {x}, then [{x}] is an ultrafilter in X , denote this filter by $[x]$.

Let { $\mathcal{F}_i$: i $\in$I} be a family of filters in X. Then $\inf_{i\in I} \mathcal{F}_i$ or $\bigwedge_{i\in I} \mathcal{F}_i$ always exists and it is generated by { $\bigcup_{i\in I} F_i$:$F_i \in \mathcal{F}_i$}.

The $\sup_{i\in I} \mathcal{F}_i$ or $\bigvee_{i\in I} \mathcal{F}_i$ exists , if and only if whenever each finite family from { $\mathcal{F}_i$: i $\in$I} possesses an upper bound . Then $\bigvee_{i\in I} \mathcal{F}_i$ is generated by { $\bigcap_{i=1}^{n} F_i$: $F_i \in \mathcal{F}_i$ }.

Now let A $\subseteq$ X , $\mathcal{F} \in$ F(X) . then we say that $\mathcal{F}$ has a trace on A if for each F $\in$ $\mathcal{F}$, F $\cap$ A $\neq \emptyset$. Denote $\mathcal{F}_A$ ={ F $\cap$ A : F $\in \mathcal{F}$ }. It is clear that $\mathcal{F}_A$ is a filter in A .

Let $\mathcal{G}$ be a filter in A . Then $\mathcal{G}$ generates a filter $[\mathcal{G}]_X$ in X i.e $[\mathcal{G}]_X$ = {F $\subseteq$ X : G $\subseteq$ F for some G $\in \mathcal{G}$ }.

Images of Filters Under Mappings

Let X and Y be two nonempty sets and f a mapping from X to Y . Let $\mathcal{F} \in$ F(X). Then { f(F) : F $\in \mathcal{F}$ } is a filter base in Y , which generates a filter $f(\mathcal{F})$ called the image of $\mathcal{F}$ under f . Let $\mathcal{G}$ $\in$ F(Y) , then { f^{-1}(G) : G $\in\mathcal{G}$ } is a filter base in X if and only if f^{-1}(G) $\neq \emptyset$, $\forall$ G $\in\mathcal{G}$.

We also have $f([x])$ = $[f(x)]$. Furthermore , for $\mathcal{G} \leq \mathcal{F}$, $f(\mathcal{G}) \leq f(\mathcal{F})$, where $\mathcal{F}, \mathcal{G} \in$ F(X).

Ultrafilters

Definition 0.3:

Let X $\neq\emptyset$. The maximal elements of F(X) are called ultrafilters in X . That is, a filter $\mathcal{F}\in$F(X) is an ultrafilter if and only if there is no filter $\mathcal{G}\in$ F(X) such that $\mathcal{G} > \mathcal{F}$. The filter $[x]$ is an ultrafilter .

Theorem0.1 :

A filter $\mathcal{F}$ on X is an ultrafilter if and only if for each $E \subset X$, either $E\epsilon\ \mathcal{F}$ or X\E $\epsilon\ \mathcal{F}$.

Theorem 0.2 :

Every filter $\mathcal{F}\in$ F(X) is contained in some ultrafilter in X.

Theorem 0.3:

if $\mathcal{F}$ is an ultrafilter on X and $A \cup B\epsilon\ \mathcal{F}$, then either $A\ \epsilon\ \mathcal{F}$ or $B\ \epsilon\ \mathcal{F}$.

Theorem 0.4 :

If f maps X into Y and $\mathcal{F}$ is an ultrafilter in X , then $f\ (\mathcal{F})$ is an ultrafilter inY.

Definition 0.4:

Two filters are said to be disjoint if they contain disjoint sets.

Theorem 0.5 :-

let U($\mathcal{F}$) be the set of ultrafilters on a set X that are finer than the filter $\mathcal{F}$, then $\mathcal{F} = \wedge_{\mathcal{G}\in U(\mathcal{F})} \mathcal{G}$.

Definition 0.5:

let $\mathcal{F}\epsilon F(X)\ and\ \mathcal{G}\epsilon F(Y)$, then the product filter $\mathcal{F} \times \mathcal{G}$ is the filter on $X \times X$ based on $\{F \times G : F\epsilon\mathcal{F}, G\epsilon\mathcal{G}\}$. Moreover , if $\mathcal{F}_i\in$F(X_i) then $\prod_{i\epsilon I}\mathcal{F}_i$ denotes the Tychonoff product of the filters $\mathcal{F}_i$, i.e , the filter based on $\{\prod F_i :\ F_i\epsilon\mathcal{F}_i$ for all $i\epsilon I$, $F_i \neq X_i$for only finitely many $i\epsilon I$ }.

Chapter One

Convergence Spaces

Chapter One

1.1 Convergence Structure:

Since the topological structure on a topological space is determined by the data of the convergence of filters on the space , the convergence structure has been introduced to generalize the topological structure ([1]) .

For a set X , let $\mathcal{P}$(X) and F(X) denote the power set of X and the set of all filters on X , respectively .

Definition 1.1.1:

For a set X , a map P : X → $\mathcal{P}$ (F(X)) is called a convergence structure on X if it satisfies the following conditions :

1. For any $x \epsilon$X , $[x] \epsilon$ P(x) , where $[x]$ denotes the principal filter generated by $\{x\}$.
2. If $\mathcal{F} \epsilon$ P(x) , then for any $\mathcal{G} \in$ F(X) , if $\mathcal{G} \geq \mathcal{F}$ then $\mathcal{G} \epsilon P(x)$.
3. If $\mathcal{F}$,$\mathcal{G} \epsilon P(x)$, then $\mathcal{F} \cap \mathcal{G} \epsilon P(x)$.

   If P is a convergence structure on X , then (X,P) is called a convergence space .If $\mathcal{F} \epsilon P(x)$, then we say that $\mathcal{F}$ converges to x in X. ([2])

Notice that axiom (3) , along with the principal of mathematical induction , actually tells us that any finite intersection of elements from P(x) is a gain an element of P(x) .

Theorem : 1.1.1:

Every topological structure t on X yields a convergence structure P_t on X . [1].

Proof:-

Define P_t as : $\forall x \in X$, $\mathcal{F} \in P_t(x)$ if and only if $\mathcal{F} \geq N(x)$, where $N(x)$ is the neighbourhood filter of x in X.

1) $[x] \in P_t(x)$ as $[x] \geq N(x)$ $\forall x \in X$.
2) If $\mathcal{F} \in P_t(x)$ and $\mathcal{G} \geq \mathcal{F}$, then $\mathcal{G} \geq \mathcal{F} \geq N(x)$.Hence $\mathcal{G} \geq N(x)$ .Thus $\mathcal{G} \in P_t(x)$.
3) If $\mathcal{F}, \mathcal{G} \in P_t(x)$, then $\mathcal{F}$, $\mathcal{G} \geq N(x)$ .Thus $\mathcal{F} \cap \mathcal{G} \geq N(x)$. $\mathcal{F} \cap \mathcal{G} \in P_t(x)$.

Hence P_t is a convergence structure .

The convergence structure P_t in the above theorem is called the natural convergence structure of the given topology t .

Definition 1.1.2 :

A convergence structure is called topological if the convergence structure is the natural convergence structure of a topology .i.e it is produced from a topology and this means if the convergent filters are precisely those of a topology ([3]) .

For any non empty set X a convergence structure P_α may be defined on X as $P_\alpha(x) = \{[x]\}$ for each $x \in X$. This convergence structure is called the discrete topology on X .

Another way of defining a convergence structure on X is to let P be characterized by each filter on X converges to each $x \in X$ and this convergence is called the indiscrete topology for X .

Note that the above two examples are natural convergence structures produced from the discrete and indiscrete topologies .

Now for each convergence space (X , P) we can construct a topological convergence space (X , P_t) on X to do that we need the following definition.

Definition 1.1.3 :

let (X , P) be a convergence space . A subset A of X is called P-open if $x \in$ A implies that A $\in \mathcal{F}$ for each $\mathcal{F} \in$ P(x). ([1]) .

Theorem 1.1.2 :

let (X , P) be a convergence space and let M_p be the set of all P-open sets in X , then Mp satisfies the axioms of open sets in topological spaces. ([1]) .

Proof :-

1) $\emptyset \in M_p$ trivially by logic .
2) X $\in M_p$ since X $\in \mathcal{F}$, $\forall \mathcal{F} \in$ P(x) , for any $x \in$ X .
3) Let A_1 , $A_2 \in M_p$ and assume that A = $A_1 \cap A_2 \neq \emptyset$, so let $x \in$ A then $x \in A_1$ and $x \in A_2$ this implies A_1 , $A_2 \in \mathcal{F}$, $\forall \mathcal{F} \in$ P(x) .

   Hence , A = $A_1 \cap A_2 \in \mathcal{F}$, $\forall \mathcal{F} \in$ P(x).
4) Let $\mathcal{A}$ be any subfamily of M_p , and let A_o be the union of all A $\in \mathcal{A}$. Let $x \in A_o$, then there exists some A $\in \mathcal{A}$ such that $x \in$ A .

   But A $\in \mathcal{F}$, $\forall \mathcal{F} \in$ P(x) and A $\subseteq A_o$. Hence $A_o \in \mathcal{F}$, $\forall \mathcal{F} \in$ P(x).

   This means that $A_o \in M_P$.

Definition 1.1.4 :

The topology M_P in theorem 1.1.2 is called the topology associated with the given convergence structure P.

Theorem 1.1.3 :

If P is the natural convergence structure of a topology t on a set X then $M_p = t$.

Proof :-

let $A \neq \emptyset$ and $A \in M_P$, then $A \in N(x)$ $\forall x \in A$ this means that $A \in t$, i.e $M_p \subseteq t$.

Let $A \in t$ and let $x \in A$, then $A \in N(x)$. This means that $A \in \mathcal{F}$, $\forall \mathcal{F} \in P(x)$ as $\mathcal{F} \geq N(x)$ .Then , A is P-open . Hence $t \subseteq M_P$.

From theorem 1.1.3 we get if (X , P) is a convergence space then M_P is a topology on X and P_t is the natural convergence structure of M_p .

Definition 1.1.5 :

let (X , P) be a convergence space and let P_t be the natural convergence structure of M_P , then P_t is called the topological modification of P .

Definition1.1.6 :

let (X , P) be a convergence space . For all $x \in X$ the filter $\mathcal{U}(x) = \cap \{ \mathcal{F} : \mathcal{F} \in P(x) \}$ is called the neighbourhood filter of x and its elements are the neighbourhoods of x . [3]

Definition 1.1.7 :

A convergence space X is called Pretopological if $\mathcal{U}(x)$ converges to x in X for every x in X , i.e , if the neighbourhood filter of each point converges to this point . [3]

One can associate to each convergence space (X , P) a Pretopological convergence space (X , Π(P)) in a natural way :

Define $\mathcal{F}$ converges to x in Π(P) if and only if $\mathcal{F} \supseteq \mathcal{U}(x)$.

Π(P) is called the Pretopological modification of X .

It is clear that , the neighbourhood filter of x is the same in (X , P) and (X , Π(P)) .

Note that a set A ⊆ X is open if and only if it is a neighbourhood of each of its points . Hence , A is P-open if and only if A is Π(P)-open .

It is clear that every natural convergence space is pretopological .

The following is an example of a convergence space which is not a pretopological space and hence not a topological space .

Example 1.1.1 :

let X be an infinite set and let A be an infinite proper subset of X . Define P on X as follows : for x in A , $\mathcal{F}$ P-converges to x if and only if $\mathcal{F}$ =[B] , where B is a finite subset of A and for x in X\A , $\mathcal{F}$ p-converges to x if and only if $\mathcal{F}$ = $[x]$. [5]

Note that for each $x \epsilon$ A , $\mathcal{U}(x)$ = [A] which does not converges to $x \epsilon$A.

The following is an example of a pretopological space which is not a topological space .

Example 1.1.2 :

let X={ x_n : n ϵ Z} , and P be the pretopology with neighbourhood filters defined as follows : for each n ϵ $\mathbb{Z}$, $\mathcal{U}(x_n)$ is the filter generated by { x_{n-1} , x_n , x_{n+1}}.[5]

Note that the topological modification of P is the indiscrete topology .

Definition 1.1.8 :

Let P and q be any two convergence structures on X , we say that P is finer than q or that q is coarser than P . "In symbols $P \geq q$" if $P(x) \subseteq q(x)$, $\forall x \in X$.[1]

The order relation induced by $\leq$ on the set of all natural convergence structures on X agrees with usual order of topologies associated . [1]

Theorem 1.1.4 :

Let (X , P) and (X , q) be two convergence spaces such that $q \leq P$ then $M_q \subseteq M_P$.

Proof :

let $A \in M_q$ then $A \in \mathcal{F}$, $\forall \mathcal{F} \in q(x)$, $\forall x \in A$. Since $P(x) \subseteq q(x)$ then $A \in \mathcal{F}$, $\forall \mathcal{F} \in P(x)$, $\forall x \in A$. Hence , $A \in M_p$.

Theorem1.1.5 :

Let (X , P) and (X , q) be two natural convergence spaces. $q \leq P$ if and only if $M_q \subseteq M_P$.

Proof :

First direction holds by theorem 1.1.4.Conversely assume that $M_q \subseteq M_P$.

Let $x \in X$ then $N_q(x) \subseteq N_P(x)$, So $\forall \mathcal{F} \in P(x)$ we get $\mathcal{F} \geq N_P(x)$ as (X,P) is a topological convergence . But $N_P(x) \geq N_q(x)$ then $\mathcal{F} \geq N_q(x)$. Hence , $\mathcal{F} \in q(x)$, i.e $\forall x \in X$, $P(x) \subseteq q(x)$ and this means that $q \leq P$.

Definition1.1.9 :

Let P be a convergence structure on X and let $\mathcal{F}$ be a filter on X , we define $\lim_P \mathcal{F} = \{ x \in X : \mathcal{F} \in P(x) \}$.[7]

Theorem1.1.6 :

Let (X , P) and (X , q) be two convergence spaces . Then , the following are equivalent :-

a) $P \geq q$.

b) $\lim_P \mathcal{F} \subseteq \lim_q \mathcal{F}$, for every $\mathcal{F} \epsilon F(X)$.

Proof :-

$a \Longrightarrow b$

Assume that $P \geq q$ then $p(x) \subseteq q(x)$, $\forall x \epsilon X$. Let $x \epsilon \lim_P \mathcal{F}$ then $\mathcal{F} \epsilon P(x)$. This implies that $\mathcal{F} \epsilon\, q(x)$. Hence $x \epsilon \lim_q \mathcal{F}$

$b \Longrightarrow a$

Let $\mathcal{F} \epsilon P(x)$ then $x \epsilon \lim_P \mathcal{F}$. Clearly $x \epsilon \lim_q \mathcal{F}$ .Then $\mathcal{F} \epsilon\, q(x)$. So $\forall x \epsilon X$, we have $P(x) \subseteq q(x)$. Hence $q \leq P$.

Theorem1.1.7 :

Let (X , P) be a convergence space . Then , $P \geq P_t$.

Proof :

Let $x \epsilon X$ and $\mathcal{F} \epsilon P(x)$. Since $N_{Pt}(x)$ is a filter generated by the set of all P-open sets which contains x , we get $N_{Pt}(x) \subseteq \mathcal{F}$. This means that $\mathcal{F} \epsilon P_t(x)$, i.e $P(x) \subseteq P_t(x)$, $\forall x \epsilon X$. Hence $P \geq P_t$.

Theorem1.1.8 :

Let P and q be convergence structures on X such that $P \geq q$, then $P_t \geq q_t$.[8] .

Proof :

Let $P \geq q$ then $M_p \supseteq M_q$ by theorem 1.1.4 but by theorem 1.1.5 and $M_p = M_{pt}$, $M_q = M_{qt}$ we get $P_t \geq q_t$.

Theorem 1.1.9 :

Let (X , P) be a convergence space . The topological modification P_t of P is the finest topology that is coarser than P .

Proof :-

Assume that $P \geq C$ where C is a topology on X then $P_t \geq C_t$ but $C_t = C$ by theorem 1.1.3 .

Theorem 1. 1.7 tells us that $P \geq P_t$. Hence P_t is the finest topology that is coarser than P .

Since a subset A of X is P-open if and only if A is Π(P)-open we have $M_P = M_{\Pi(P)}$ and $P_t \leq \Pi(P)$. But $\Pi(P) \leq P$ so we get $P_t \leq \Pi(P) \leq P$.

Result :

We can have different convergence spaces which have the same open sets on the contrary of topologies .

Theorem 1.1.10 :

Let (X , P) and (X , q) be convergence spaces . If $q \leq P$ then $\Pi(q) \leq \Pi(P)$.

Proof :

Let $x \epsilon$ X and $q \leq P$. Then $P(x) \subseteq q(x)$. This implies that $\mathcal{U}_P(x) \supseteq \mathcal{U}_q(x)$. If $\mathcal{F} \geq \mathcal{U}_P(x)$, then $\mathcal{F} \geq \mathcal{U}_q(x)$.

Hence $\Pi(P)(x) \subseteq \Pi(q)(x)$

Therefore , $\Pi(q) \leq \Pi(P)$.

In fact , if we replace the condition 3 in definition 1.1.1 by if $\mathcal{F} \in P(x)$ then $\mathcal{F} \cap [x] \in P(x)$ then we call P point deep convergence structure and hence the set of all point deep convergence structures on the set X and the relation $\leq$ is a complete lattice whose inf and sup are respectively defined by :-

1. $\mathcal{F} \epsilon \wedge_{i \in I} \mathrm{P_i}(x)$ if and only if $\exists i \epsilon I$ such that $\mathcal{F} \epsilon \mathrm{P_i}(x)$.
2. $\mathcal{F} \epsilon \vee_{i \in I} \mathrm{Pi(x)}$ if and only if $\forall i \epsilon I, \mathcal{F} \epsilon \mathrm{P_i}(x)$.[7]

The smallest and the largest elements of the set of pont deep convergence structures on a set X are indiscrete topology and the discrete topology respectively .

Theorem 1.1.11 :

The set of topologies on X is closed under supremum [9] .

Proof :-

Let $\{t_i : i \epsilon I\}$ be a family of topologies on X . The supremum of this family in the set of convergence spaces is defined by $\mathcal{F} \epsilon \vee_{i \in I} t_i(\mathrm{x})$ if and only if $\mathcal{F} \epsilon t_i (x)$, $\forall i \epsilon I$. Hence $\mathcal{F} \geq \mathrm{N}_{ti}(x)$ for every $i \in I$.

Define the family $\mathrm{B}(x)$=$\{\bigcap_{i=1}^{m} S : S \subset \bigcup N_{ti}(x) , |S| < w\}$ which is a filter base on X and let $\mathrm{N}(x)$ denote the filter generated by $\mathrm{B}(x)$. It is clear that $\mathcal{F} \epsilon \vee_{i \in I} \mathrm{ti}(x)$ if and only if $\mathcal{F} \geq N(x)$.

It remains to show that for every $u \epsilon N(x)$ there exists $\mathrm{O} \epsilon \mathrm{N}(x)$ such that $u \epsilon N(y)$ for every $\mathrm{y} \epsilon O$.

If $u \epsilon N(x)$, then there exist i_1 , i_2 , …. i_n and $\vee_1 , \ldots,$ $\vee_n$ such that $\vee_J \epsilon \mathrm{N}_{\mathrm{ti}_J}(\mathrm{x})$ and neighbourhood in ti_J and $\bigcap_{i=1}^{n} \vee i \subseteq u$. Let $\mathrm{O} = \bigcap_{i=1}^{n} \vee i$. If $y \epsilon O$ then $y \epsilon \vee_J \epsilon \mathrm{N}_{\mathrm{ti}_J}(y)$ for every $j \epsilon \{1, \ldots n\}$.

Hence $\mathrm{O} = \bigcap_{i=1}^{n} \vee i \epsilon N(y)$. Since $\mathrm{O} \subseteq u$ we get $u \epsilon N(y)$.

Therefore $\vee_{i \in I} ti(x)$ is the natural convergence structure of a topology whose neighbourhoods are $\mathrm{N}(x)$, $\forall x \epsilon \mathrm{X}$.

Theorem 1.1.12 :

Every convergence structure is the infinimum of a set of topologies .[9]

Since we have a convergence space which is not topological and by theorem 1.1.12 we get in general that the set of topologies on X is not closed under infinimum .

Theorem1.1.13:

Let (X , P) be a pretopological space . $\mathcal{F} \epsilon P(x)$ if and only if each ultrafilter finer than $\mathcal{F}$ converges to x . [1] .

Proof :

The conditions is obviously necessary . It is also sufficient because each filter $\mathcal{F}$ is the intersection of all ultrafilters finer than $\mathcal{F}$. So if each such ultrafilter converges to x , then so does $\mathcal{F}$, as P is a pretopological structure .

1.2 Interior and Closure Operators in Convergence Spaces.

If P is a convergence structure on a set X, then we can define the closure and interior operators in the following manner .

Definition 1.2.1 :

Let (X , P) be a convergence space and A ⊆ X , then the closure of A , $CL_p(A) = \{ x\epsilon X : \exists \mathcal{F}\epsilon P(x) and A \epsilon \mathcal{F}\}$. [10]

Definition 1.2.2 :

Let (X , P) be a convergence space , $A \subseteq X$,then the interior of A in $t_p(A) = \{ x\epsilon A : A \epsilon \mathcal{F} for all \mathcal{F}\epsilon P(x)\}$. [10] . It is clear that A is P-open if and only if int(A) = A . [10]

Note that we will write CL(A) for CL_P(A) and so on for interior if there is no ambiguity .

Theorem 1.2.1 :

Let (X , P) be a convergence space . $x \in CL(A)$ if and only if $\exists \mathcal{F} \in P(x)$ and $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$. [1]

Proof :

If $x \in CL(A)$ then $\exists \mathcal{F} \in P(x)$ and $A \in \mathcal{F}$. Hence $A \cap F \neq \emptyset$ for all $F \in \mathcal{F}$.

Conversely assume that $\exists \mathcal{F} \in P(x)$ such that $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}$ then $\{ F \cap A : F \in \mathcal{F}\}$ is a filterbase generating a filter $\mathcal{G}$. It is clear that $A \in \mathcal{G}$ and $\mathcal{F} \leq \mathcal{G}$. Hence $\mathcal{G} \in P(x)$ therefore $x \in CL(A)$.

Theorem 1.2.2 :

Let (X , P) be a convergence space , the map A → CL(A) from $\mathcal{P}$(X) into itself has the following properties :

1) $CL(\emptyset) = \emptyset$.
2) $A \subseteq CL(A)$.
3) $A \subseteq B \textit{ implies } CL(A) \subseteq CL(B)$.
4) $CL(A \cap B) \subseteq CL(A) \cap CL(B)$.
5) $CL(A \cup B) = CL(A) \cup CL(B)$.

Proof :

1) Trivial .
2) Let $x \in A$ then $[x] \in P(x)$and $A \in [x]$ and this means that $x \in CL(A)$. Hence $A \subseteq CL(A)$.
3) Suppose $A \subseteq B$ and $x \in CL(A)$, then $\exists \mathcal{F} \in P(x)$ such that $A \in \mathcal{F}$. Since $A \subseteq B$, $B \in \mathcal{F}$. Hence $x \in CL(B)$.

4) $A \cap B \subseteq A$ and $A \cap B \subseteq B$ by (3) we get $CL(A \cap B) \subseteq CL(A)$ and $CL(A \cap B) \subseteq CL(B)$. Hence $CL(A \cap B) \subseteq CL(A) \cap CL(B)$.

5) $A , B \subseteq A \cup B$, then $CL(A), CL(B) \subseteq CL(A \cup B)$ by (3) . Hence $CL(A) \cup CL(B) \subseteq CL(A \cup B)$.

To prove the other inclusion , let $x \epsilon CL(A \cup B)$, then $\exists \mathcal{F} \epsilon P(x)$ such that $A \cup B \epsilon \mathcal{F}$. Since for each filter there exists an ultrafilter containing it , let $\mathcal{H}$ be an ultrafilter such that $\mathcal{H} \geq \mathcal{F}$ and since P is convergence structure $\mathcal{H} \in$ P(x) . Since $\mathcal{H}$ is an ultrafilter , then either $A \epsilon \mathcal{H}$ or B $\epsilon \mathcal{H}$. Hence $x \epsilon CL(A)$ or $x \in CL(B)$. $CL(A \cup B) \subseteq CL(A) \cup CL(B)$

Theorem 1.2.3 :

Let (X , P) be a convergence space .Then

1. X\CL(A) = int (X\A) .
2. If $A \subseteq B$ then int (A) ⊆ int (B) .
3. int (A) ∩ int (B) = int $(A \cap B)$.[10]

Proof :

1) Let $x \epsilon X \backslash CL(A)$ then $x \notin CL(A)$ so $x \epsilon X \backslash A$ as $A \subseteq CL(A)$ and $\forall \mathcal{F} \epsilon P(x)$, we have $A \notin \mathcal{F}$. Thus $X \setminus A \epsilon \mathcal{F}$ $\forall \mathcal{F} \in P(x)$ as [P is a convergence structure and each filter is the intersection of all ultrafilters finer than it and either A or $X \backslash A \epsilon$ ultrafilters] .

   But $A \notin$ ultrafilters. Hence , $x \epsilon$ int (X\A).

   Thus , $X \backslash CL(A) \subseteq int(X \backslash A)$.

   Conversely let $x \epsilon int (X \backslash A)$ then $x \epsilon X \backslash A$ and $X \backslash A \epsilon \mathcal{F}$, $\forall \mathcal{F} \epsilon P(x)$. This means that there is no $\mathcal{F} \epsilon P(x)$ such that $A \epsilon \mathcal{F}$.

   Hence , $x \notin CL(A)$.Thus , $x \in X \backslash CL(A)$.

Therefore , $int(X\backslash A) \subseteq X\backslash CL(A)$.

2) If $A \subseteq B$, then $X\backslash B \subseteq X\backslash A$. Then $CL(X\backslash B) \subseteq CL(X\backslash A)$ by theorem 1.2.2 .

Hence $X\backslash CL(X\backslash A) \subseteq X\backslash CL(X\backslash B)$. From which it follows that $int\left(X\backslash(X\backslash A)\right) \subseteq int\left(X\backslash(X\backslash B)\right)$ by part(1) of this theorem . Thus , $int\ (A) \subseteq int\ (B)$.

3) $int\ (A \cap B) = X\backslash CL(X\backslash(A \cap B))$ by part(1) of this theorem.

$$= X\backslash CL\big((X\backslash A) \cup (X\backslash B)\big)$$

$$= X\backslash\big(CL(X\backslash A) \cup CL(X\backslash B)\big)\ by\ theorem\ 1.2.2$$

$$= X\backslash CL(X\backslash A) \cap X\backslash CL(X\backslash B)$$

$$= int\big(X\backslash(X\backslash A)\big) \cap int\ (X\backslash(X\backslash B))$$

$$= int\ (A) \cap int\ (B).$$

Definition 1.2.3 :

Let (X , P) be a convergence space , we say that A is P-closed or simply closed if A = CL_p(A) .[1]

Theorem 1.2.4 :

Let (X , P) be a convergence space , A is closed if and only if X\A is open .[1]

Proof:

A is closed if and only if A = CL(A) if and only if $X \setminus A = X\backslash CL(A) = int(X\backslash A)$ if and only if X\A is open by theorem 1.2.3

Theorem1.2.5 :

Let (X , P) be a convergence space and $A \subseteq X$, then $x \epsilon\ CL(A)$ if and only if $\vee \cap A \neq \emptyset$ for each $\vee \epsilon\ \mathcal{U}(x)$.

Proof :

Let $x \in CL(A)$ then $\exists \mathcal{F} \in P(x)$ and $F \cap A \neq \emptyset, \forall F \in \mathcal{F}$ by theorem1.2.1 . But $\mathcal{F} \geq \mathcal{U}(x)$, therefore $\vee \cap A \neq \emptyset$ for each $\vee \in \mathcal{U}(x)$.

Conversely assume that $\vee \cap A \neq \emptyset$ for each $\vee \in \mathcal{U}(x)$ and $x \notin CL(A)$ then $\forall \mathcal{F} \in P(x)$ we have $A \notin \mathcal{F}$. Hence $\forall \mathcal{F} \in P(x)$ there exists a $\vee_{\mathcal{F}}$ such that $\vee_{\mathcal{F}} \cap A = \emptyset$. Let $\vee$ be the union of $\vee_{\mathcal{F}}, \mathcal{F} \in P(x)$.

Now $\vee \in \mathcal{F}$, $\forall \mathcal{F} \in P(x)$. Thus $\vee \in \mathcal{U}(x)$ but $\vee \cap A = \emptyset$ which is a contradiction .

Corollary 1.2.1 :

The convergence spaces (X , P) and (X , $\Pi(P)$) have the same closure operators .[3]

Proof :

Let $A \neq \emptyset \subseteq X$ then , $x \in CL_P(A)$ if and only if $\vee \cap A \neq \emptyset$ for each $\vee \in \mathcal{U}(x)$ if and only if $x \in CL_{\Pi(P)}(A)$ by theorems 1.2.5 and 1.2.1.

The above corollary shows that two different convergence structures may have the same closure operators . While two topologies are identical when they have the same closed sets .

Theorem1.2.6 :

Let (X , P) and (X , q) be convergence spaces such that $q \leq P$, then for each $A \subseteq X, CL_P(A) \subseteq CL_q(A)$. In particular , each q-closed subset is P-closed .

Proof :

Let $x\epsilon\, CL_p(A)$, then by theorem 1.2.1 $\exists\mathcal{F}\epsilon\, P(x)$ such that $F\cap A\neq\ \emptyset$ for all $F\epsilon\mathcal{F}$. But $\mathcal{F}\ \epsilon\ q(x)$ as $P(x)\subseteq q(x)$. Hence , $x\epsilon\ CL_q(A)$ by theorem 1.2.1 . If A is q-closed then $A=CL_q(A)\supseteq CL_P(A)$ but $A\subseteq CL_p(A)$. Hence , $CL_p(A)=A$. Thus , A is P-closed .

Theorem 1.2.7 :

Let (X , P) be a convergence space and $\mathcal{F}\epsilon\ P(x)$ for some $x\epsilon\ X$, then $x\epsilon\ CL(F)$, $\forall F\epsilon\mathcal{F}$.

Proof :

Follows by definition 1.2.1 .

Theorem 1.2.8 :

A convergence space (X , P) is topological if and only if (X , P) is a pretopological space and the closure operator is idempotent.[3] .

Proof :

Let $\mathcal{U}(x)$ be a neighbourhood filter of x in (X , P) and let N(x) be a neighbourhood filter of x in (X , P_t) .

The first direction is trivial .

Conversely assume that (X , P) is a pretopological space where its closure operator is idempotent .

Since $P_t\leq$ P then $P(x)\subseteq P_t(x)$. Hence $N(x)\subseteq\mathcal{U}(x)$.

Let $u\epsilon\mathcal{U}(x)$, then CL(CL(X\\$u$)) = CL(X\\$u$) . This means that CL(X\\$u$) is closed . Therefore , $X\backslash CL(X\backslash u)$ = int (u) is P-open .

Since $x \in \text{int}(u) \subseteq u$ we get that $u \in N(x)$. Hence $\mathcal{U}(x) \subseteq \text{N}(x)$, $\forall x \epsilon X$. Thus , $\text{N}(x) = \mathcal{U}(x)$, $\forall x \epsilon X$. Since both (X,P) and (X, P_t) are pretopological spaces and $\text{N(x)} = \mathcal{U}(\text{x})$, $\forall x \epsilon X$, they must coincide .

conclusions :

Theorem 1.2.8 shows that if we have a pretopological space which is not topological then the closure operator is not idempotent .

1. In general the closure of a set in convergence space is not closed so that , CL(CL(A)) is usually larger than CL(A) .
2. From Corollary 1.2.1 and theorem 1.2.8 the closure operator of a given convergence space P is idempotent if and only if the topological modification and the pretopology modifications are the same i.e $P_t = \Pi(P)$.

1.3 Adherence of a Filter in Convergence Spaces:

Definition 1.3.1 :

Let (X , P) be a convergence space . An element $x \epsilon X$ is said to be an adherent to the filter $\mathcal{F}$ if a filter $\mathcal{G}$ exists such that $\mathcal{G} \geq \mathcal{F}$and $\mathcal{G} \in P(x)$.[1]

Definition 1.3.2 :

Let (X , P) be a convergence space . The set of all points of X which are adherent to a filter $\mathcal{F}$ is called the adherence of $\mathcal{F}$ and denoted by $a_P(\mathcal{F})$, or simply $a(\mathcal{F})$ if there is no ambiguity .[1]

It follows from definition 1.3.1 that $a(\mathcal{F}) = \cup_{\mathcal{G} \geq \mathcal{F}} \, lim\, \mathcal{G}$. And if $\mathcal{G} \leq \mathcal{F}$ then $a(\mathcal{F}) \subseteq a(\mathcal{G})$, and if $\mathcal{U}$ is an ultrafilter , then $a(\mathcal{U}) = \lim \mathcal{U}$.

Theorem 1.3.2 :

Let (X , P) be a convergence space , $\mathcal{F}$and $\mathcal{G}$ be two filters on X then :

1) $a(\mathcal{F}) \cup a(\mathcal{G}) \subseteq a(\mathcal{F} \cap \mathcal{G})$.

2) If $\mathcal{F} \vee \mathcal{G}$ exists then $a(\mathcal{F}) \cap a(\mathcal{G}) \supseteq a(\mathcal{F} \vee \mathcal{G})$.[1]

Proof :

1) Let $x \epsilon a(\mathcal{F}) \cup a(\mathcal{G})$, then either $x\epsilon a(\mathcal{F})$ or $x\epsilon a(\mathcal{G})$. Assume without loss of generality that $x \in a(\mathcal{F})$, then $\exists \mathcal{H}\epsilon F(X)$ such that $\mathcal{F} \leq \mathcal{H}$ and $\mathcal{H}\epsilon P(x)$. But $\mathcal{F} \cap \mathcal{G} \subseteq \mathcal{F} \subseteq \mathcal{H}$. Hence $x \in a(\mathcal{F} \cap \mathcal{G})$.

2) $\mathcal{F}, \mathcal{G} \subseteq \mathcal{F} \vee \mathcal{G}$. Hence $a(\mathcal{F} \vee \mathcal{G}) \subseteq a(\mathcal{F})$ and $a(\mathcal{F} \vee \mathcal{G}) \subseteq a(\mathcal{G})$. Therefore $a(\mathcal{F}) \cap a(\mathcal{G}) \supseteq a(\mathcal{F} \vee \mathcal{G})$.

Theorem 1.3.3 :

Let (X , P) and (X , q) be two convergence spaces such that $q \leq P$, then $\forall \mathcal{F} \epsilon F(X)$ we have $a_P(\mathcal{F}) \subseteq a_q(\mathcal{F})$.

Proof : Let $x \epsilon a_P(\mathcal{F})$, then $\exists \mathcal{G}\epsilon F(X)$ such that $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \epsilon P(x)$. Since $P(x) \subseteq q(x)$, then we get that $\mathcal{G} \in q(x)$. Hence $x \in a_q(\mathcal{F})$.

Corollary 1.3.1 :

Let (X , P) and (X , q) be convergence spaces and $C = P \vee q$, then for $\mathcal{F}\epsilon F(X)$, we have $a_C(\mathcal{F}) \subseteq a_P(\mathcal{F}) \cap a_q(\mathcal{F})$.[1]

Proof :

Since C is the supremum convergence structures of P and q we get by theorem 1.3.3 that $a_C(\mathcal{F}) \subseteq a_P(\mathcal{F})$ and $a_C(\mathcal{F}) \subseteq a_q(\mathcal{F})$.

Hence $a_C(\mathcal{F}) \subseteq a_P(\mathcal{F}) \cap a_q(\mathcal{F})$, $\forall \mathcal{F}\epsilon F(X)$.

Definition 1.3.3 :

A convergence space (X , P) is said to have closed adherences if for every filter $\mathcal{F}$ on X the adherence $a_P(\mathcal{F})$ is a closed subset of (X , P) .[12]

Theorem 1.3.4 :

If (X , P) is a convergence space with closed adherences then (X , P) has a closure operator which is idempotent .

Proof :

Let $A \subseteq X$. It is clear that $CL(A) = a([A])$, where [A] is the filter generated by A . But since (X , P) with closed adherences we get that $CL(A)$ is closed . Hence , $CL\big(CL(A)\big) = CL(A)$.

The converse of theorem 1.3.4 is not true .Evalowen gave in [12] an example of convergence space with an idempotent closure operater but not with closed adherences .

Definition1.3.4 :

A convergence space (X , P) is said to be diagonal if for every $x \epsilon X$, $\mathcal{F} \epsilon P(x)$, for every mapping n from X into F(X) such that $n(y)$ converges to y for every $y \epsilon X$ and for every $F\epsilon \mathcal{F}$ we have that the filter $k_n\mathcal{F} = \sup_{F\epsilon\mathcal{F}} \mathcal{G}(n , F)$ converges to x where
$\mathcal{G}(n , F) = \cap_{y\epsilon F}\, n(y)$.[11]

Kowalski showed in [11] that each diagonal space has a closure operator which is idempotent .

EvaLowen weakened the diagonal condition of Kowalsky and introduced weakly diagonal convergence spaces and showed that these are exactly the convergence spaces with closed adherences .

Definition 1.3.5 :

A convergence space (X , P) is said to be weakly diagonal , if $\forall x \epsilon X$, $\forall$filter $\mathcal{F}$ converges to x , $\forall$ mapping $n$from X into F(X) such that $n(y)$ converges to y $\forall y \epsilon X$ and $\forall F \epsilon \mathcal{F}$ we have that x is an adherence point of the filter $\mathcal{G}(n , F)$ where $\mathcal{G}(n , F) = \cap_{y \epsilon F} ny$. [12]

Note that every diagonal convergence space is weakly diagonal .

Theorem 1.3.5 :

A convergence space (X , P) has closed adherences if and only if it is weakly diagonal .[12]

Proof :

Suppose (X , P) has closed adherences . Let $x \in$ X , $\mathcal{F} \epsilon F(X)$, $\mathcal{F} \epsilon\ P(x)$ and n : X → F(X) a map such that $ny\ \epsilon\ P(y)$ for every $y \epsilon X$. For $F \epsilon \mathcal{F}$ we have $F \subset a\left(\mathcal{G}(n , F)\right)$ since $y\ \epsilon\ F$ implies $ny \supset \mathcal{G}(n , F)$ and thus $y \in a\left(\mathcal{G}(n , F)\right)$. It follows that $a\left(\mathcal{G}(n , F)\right) \epsilon\ \mathcal{F}$. Since $\mathcal{F}$converges to x we have $x \epsilon\ CL\left(a\left(\mathcal{G}(n , F)\right)\right) = a\left(\mathcal{G}(n , F)\right)$.

Conversely suppose that (X , P) is weakly diagonal . Let $\mathcal{F}$ be a filter on X . If $x\ \epsilon\ CL\left(a(\mathcal{F})\right)$, then take a filter $\mathcal{H} \epsilon P(x)$ and containing $a(\mathcal{F})$.

For $y\ \epsilon\ a(\mathcal{F})$ let $\mathcal{H}_y\ \epsilon\ P(y)$ and $\mathcal{H}_y \supseteq \mathcal{F}$. consider $n{:}\,X \to F(X)$, $ny = [y]$ if $y \notin a(\mathcal{F})$ and $ny = \mathcal{H}_y$ if $y \epsilon a(\mathcal{F})$. Then we have $\mathcal{F} \subseteq \mathcal{G}\left(n , a(\mathcal{F})\right)$ and therefore $x\ \epsilon\ a(\mathcal{F})$.

In general , adherences of filters in a convergence space are not closed. This is one of the essential differences between topological spaces and general convergence spaces .

Chapter Two

Continuous Functions on Convergence Spaces

Chapter Two

2.1 Continuous Functions on Convergence Spaces

Definition 2.1.1:

Let (X , P) and (Y , q) be convergence spaces . A mapping f:(X , P) →(Y , q) is called continuous at a point $x \epsilon X$ if $\forall \mathcal{F} \epsilon P(x)$ the filter $f(\mathcal{F}) \epsilon$ q($f(x)$) . The mapping f is called continuous on X if it is continuous at each point of X . f is called a homeomorphism if it is bijective and both f and f^{-1} are continuous .[3]

Theorem 2.1.1:

Let X be a set equipped with two convergence structures P and q . Then , P ≥ q if and only if the identity mapping i : (X , P) →(X , q) is continuous .

Proof :

Suppose that P ≥ q . Let $x \epsilon X$ and $\mathcal{F} \epsilon P(x)$.Then i($\mathcal{F}$) = $\mathcal{F} \epsilon q(x)$ because P(x) ⊆ q(x). Thus , i is continuous .

Conversely if i is continuous then $\forall \mathcal{F} \epsilon P(x)$ we have i($\mathcal{F}$) = $\mathcal{F} \epsilon q(x)$. Thus , P($x$) ⊆ q($x$) $\forall x \epsilon X$. Hence , P ≥ q .

Remark 2.1.1:

It is clear that if (X , t) and (Y , S) are topological spaces , then f: (X , t) →(Y , S) is continuous at a point $x \epsilon X$ if and only if f: (X , P_t)→ (Y, P_s) is continuous at x .

Theorem 2.1.2:

Let f:(X , P) →(Y , q) be continuous and let (X , P̀) and (Y , q̀) be other convergence spaces such that P ≤ P̀ and q̀ ≤ q , then f: (X , P̀) → (Y , q̀) is also continuous .[1]

Proof :

Let $x \epsilon X$ and $\mathcal{F} \epsilon$ P̀(x) . Then $\mathcal{F} \epsilon P(x)$ and $f(\mathcal{F}) \epsilon$ q($f(x)$) as f is continuous from (X , P) to (Y , q). Since q̀ ≤ q , $f(\mathcal{F}) \epsilon$ q̀ (f(x)) . Hence f: (X , P̀) → (Y , q̀) is continuous .

Corollary 2.1.1:

Let (X , P) be a convergence space . Then the following hold :

1) The identity mapping i : (X , P) →(X , P_t) is continuous . It is a homeomorphism if and only if (X , P) is a topological space .
2) The identity mapping i : (X , P) →(X , $\Pi(P)$) is continuous . It is a homeomorphism if and only if (X ,P) is a pretopological space .
3) The identity mapping i :(X , $\Pi(P)$) →(X , P_t) is continuous .

Proof :

Since $P_t \leq \Pi(P) \leq$ P , then the proof follows by Theorem 2.1.1.

Theorem 2.1.3:

Let (X , P) and (Y , q) be convergence spaces such that the mapping f: (X , P) → (Y , q) is continuous , then f^{-1}(A) is a P-open subset of X if A is a q-open subset of Y . [1]

Proof:-

Assume that A is q-open . If $f^{-1}(A) = \emptyset$, then the theorem is true . Let $f^{-1}(A) \neq \emptyset$, $x \epsilon f^{-1}(A)$ and let $\mathcal{F} \epsilon P(x)$.

Since f is continuous at x , $f(\mathcal{F}) \epsilon$ q($f(x)$) . But $f(x) \epsilon$ A and A is q-open this impiles that $A \epsilon f(\mathcal{F})$.

Thus , $\exists F \epsilon \mathcal{F}$ such that f(F) ⊆ A . But F ⊆ $f^{-1}\big(f(F)\big) \subseteq f^{-1}(A)$.

Thus , $f^{-1}(A) \epsilon \mathcal{F}$. Hence , $f^{-1}(A)$ is a P- open subset of X .

Remark 2.1.2

Let (X , P) be any convergence space which is not natural convergence space and let i :(X , P_t) →(X , P) be the identity map . Then i is not continuous since $P_t < P$ and by theorem 2.1.1 . But i^{-1}(A) is a P_t-open subset of X if A is a P-open subset of X .

Remark 2.1.3:

Remark 2.1.2 shows that the converse of theorem 2.1.3 is not true in general . But the converse of theorem 2.1.3 is true if f is taken between any two topological spaces . This is One of the essential differences between topological spaces and general convergence spaces.

Corollary 2.1.2:

Let (X , P) and (Y , q) be convergence spaces . Then if the mapping f : (X , P) → (Y , q) is continuous , then f : (X , P_t) → (Y , q_t) is continuous .[3]

Proof:

Follows by theorem 2.1.3 and Remark 2.1.1 .

Corollary 2.1.3:

Let (X , P) and (Y , q) be convergence spaces .

If f : (X , P) → (Y , q) is a homeomorphism mapping , then a subset A of X is P-open if and only if f(A) is q-open . In particular , f is a topological homeomorphism mapping from (X , P_t) *onto* (Y , q_t) .

Proof:

Follows by corollary 2.1.2 and theorem 2.1.3 .

Theorem 2.1.4:

Let (X , P) and (Y , q) be convergence spaces such that the mapping f: (X , P) → (Y , q) is continuous , then $\mathcal{U}_q(f(x)) \subseteq f\left(\mathcal{U}_p(x)\right)$ $\forall x \in X$.

Proof :

Let $\vee \in \mathcal{U}_q(f(x))$ and $\mathcal{F} \in P(x)$. Since f is continuous , then $f(\mathcal{F}) \in q(f(x))$. Thus , $\vee \in f(\mathcal{F})$. This implies that $f^{-1}(\vee) \in \mathcal{F}$, because $f(u) \subseteq \vee$ for some $u \in \mathcal{F}$. Thus $u \subseteq f^{-1}f(u) \subseteq f^{-1}(\vee) \in \mathcal{F}, \forall \mathcal{F} \in P(x)$. Therefore , $f^{-1}(\vee) \in \mathcal{U}_p(x)$. Since $f(f^{-1}(\vee)) \subseteq \vee$ implies that $\vee \in f(\mathcal{U}_p(x))$. Hence $\mathcal{U}_q(f(x)) \subseteq f\left(\mathcal{U}_p(x)\right)$.

Corollary 2.1.4:

Let (X , P) and (Y , q) be convergence spaces . If the mapping f : (X , P) → (Y , q) is continuous , then f : (X , $\Pi(P)$) →(Y , $\Pi(q)$) is continuous .[3]

Proof:

Let $\mathcal{F} \in \Pi(P)(x)$, then $\mathcal{F} \geq \mathcal{U}_p(x)$ so $f(\mathcal{F}) \geq f\left(\mathcal{U}_p(x)\right)$.

By theorem 2.1.4 , we get $\mathcal{U}_q(f(x)) \leq f\left(\mathcal{U}_p(x)\right) \leq f(\mathcal{F})$.

Thus $f(\mathcal{F}) \epsilon\, \Pi(q) \left(f(x)\right)$. Hence f : (X , $\Pi(P)$) → (Y , $(\Pi)q$) is continuous.

The following remark shows that the converse of theorem 2.1.4 is not true in general .

Remark 2.1.4:

Let (X , P) be any convergence space which is not a pretopological space , then the identity mapping i : (X , $\Pi(P)$) → (X , P) is not continuous by theorem 2.1.1 . But $\mathcal{U}_P\left(f(x)\right) = f\left(\mathcal{U}_{\Pi(P)}(x)\right)$.

Theorem 2.1.5:

Let (X , P) and (Y , q) are pretopological spaces , then the converse of theorem 2.1.4 is true .[10]

Proof :

Let $\mathcal{F} \epsilon P(x)$, then $\mathcal{F} \geq \mathcal{U}_P(x)$.

Since $\mathcal{F} \geq \mathcal{U}_P(x)$, $f(\mathcal{F}) \geq f\left(\mathcal{U}_P(x)\right)$. This implies that

$\mathcal{U}_q\left(f(x)\right) \leq f(\mathcal{F})$. Thus $f(\mathcal{F}) \,\epsilon\, q\left(f(x)\right)$.

Hence f is continuous .

Theorem 2.1.6 :

Let (X , P) and (Y , q) be convergence spaces such that f : (X , P) → (Y , q) is continuous . Then , $f\left(CL(A)\right) \subseteq CL\left(f(A)\right)$ for all $A \subseteq X$. [3]

Proof :

Let $y \,\epsilon\, f\left(CL(A)\right)$,then $\exists x \,\epsilon\, CL(A)$ such that y = f (x) . Now $x \,\epsilon\, CL(A)$ implies that $\exists \mathcal{F} \,\epsilon\, P(x)$ such that A ∈ $\mathcal{F}$. So $f(A) \epsilon\, f(\mathcal{F})$. Since f is continuous $f(\mathcal{F}) \epsilon\, q\left(f(x)\right)$. Thus , y= $f(x) \,\epsilon\, CL\left(f(A)\right)$.

Theorem 2.1.7 :

Let (X , P) and (Y , q) be convergence spaces such that f: (X , P) → (Y , q) is continuous. Then , $f(x) \epsilon a_q\big(f(\mathcal{F})\big)$ if $x \epsilon a_P(\mathcal{F})$.[1]

Proof :

Let $x \epsilon a_P(\mathcal{F})$, then $\exists \mathcal{G} \epsilon F(X)$ such that $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \epsilon P(x)$.

Since f is continuous and $f(\mathcal{F}) \leq f(\mathcal{G})$ we have $f(\mathcal{G}) \epsilon q\big(f(x)\big)$ and $f(x) \epsilon a_q\big(f(\mathcal{F})\big)$.

The following theorem shows that the composition of two continuous functions is continuous .

Theorem2.1.8 :

Let (X , P) , (Y , q) and (Z , C) be convergence spaces such that f : (X , P) → (Y , q) is continuous at $x \epsilon X$ and g: (Y , q) → (Z , C) is continuous at $f(x) \epsilon$ Y, then g of : (X , P)→ (Z , C) is continuous at x.[1]

Proof :-

Let $x \in$X and $\mathcal{F} \epsilon P(x)$ then $f(\mathcal{F}) \ \epsilon q\big(f(x)\big)$ and $g\big(f(\mathcal{F})\big) \epsilon C\Big(g\big(f(x)\big)\Big)$ as f and g are continuous at x and $f(x)$, respectively .

so , $(g\text{o}f)(\mathcal{F})\epsilon C\Big(g\big(f(x)\big)\Big)$. Hence , (g of) is continuous at x .

2.2 Subspaces and Product Convergence Structure

To construct subspaces and product convergence structures we need to introduce the concept of the initial convergence structure .

Let X be a set ,$(X_i , P_i)_{i\epsilon I}$be a collection of convergence spaces and for each $i\epsilon I$, f_i: X → X_i is a mapping . Define P : X →$\mathcal{P}$ (F(X)) by $\mathcal{F}$ ∈ P(x) if and only if $f_i(\mathcal{F})\epsilon\, P_i\big(f_i(x)\big)$∀i ∈ I .

Claim : P is a convergence structure on X .

Since $f_i([x]) = [f_i(x)]\ \epsilon\ P_i\big(f_i(x)\big)\ \forall i\epsilon I$, then $[x]\ \epsilon\ P(x)$.

If $\mathcal{F}\ \epsilon\ P(x)$ and $\mathcal{G}\ \geq \mathcal{F}$, then $f_i(\mathcal{F}) \subseteq f_i(\mathcal{G})\ \forall i\epsilon I$.

Since $f_i(\mathcal{F})\ \epsilon\ P_i\big(f_i(x)\big)$, then $f_i(\mathcal{G})\ \epsilon\ P_i\big(f_i(x)\big)\ \forall i\epsilon I$. Thus $\mathcal{G}\epsilon P(x)$.

If $\mathcal{F},\mathcal{G}\ \epsilon\ P(x)$ then $f_i(\mathcal{F})$, $f_i(\mathcal{G})\ \epsilon\ P_i\big(f_i(x)\big)\ \forall i\epsilon I$.

Since $f_i(\mathcal{F}) \cap f_i(\mathcal{G}) \subseteq\ f_i(\mathcal{F} \cap \mathcal{G})$, then $f_i(\mathcal{F} \cap \mathcal{G})\ \epsilon\ P_i\big(f_i(x)\big), \forall i\epsilon I$. Thus $\mathcal{F} \cap \mathcal{G}\ \epsilon\ P(x)$. Therefore , P is a convergence structure on X . This convergence structure is called the Initial Convergence Structure**.**

It is clear that f_i: (X , P) → (X_i , P_i) is continuous $\forall i\epsilon I$.

Let q be a convergence structure on X making all of the f_i continuous . If $\mathcal{F}\epsilon\ q(x)$, then $f_i(\mathcal{F})\ \epsilon\ P_i\big(f_i(x)\big)$, $\forall i\epsilon I$ and this implies that $\mathcal{F}\epsilon P(x)$. That is , $q(x) \subseteq P(x)\ \ \forall x\ \epsilon X$. Thus $P \leq$ q . Hence the Initial convergence structure P is the coarsest convergence structure on X making all of the f_i continuous .

Note that if we find the Initial convergence structure on X with respect to f_i ,$i\ \epsilon\ I$ separately then the initial convergence structure on X with respect to $(f_i)_{i\epsilon I}$ is equal to the sup of the above convergence structures .

Theorem 2.2.1:

Let (Y , q) be a topological space and let $f: X \to (Y , q)$, then the initial convergence structure P on X with respect to f and q is a topology.[7]

Proof :

$f : (X , P) \to (Y , q)$ is continuous by definition of Initial convergence structure . $f : (X , P_t) \to (Y , q_t)$ is continuous by corollary 2.1.2 .

Since (Y , q) is a topological space we get $(Y , q) = (Y , q_t)$.

Since P is the initial convergence structure on X with respect to f and q we get $P \leq P_t$. But $P_t \leq P$ by theorem 1.1.6 .

Hence $P = P_t$. That is , P is a topology .

It is interesting to note that in the proof of theorem 2.2.1 we did not use any internal description of topologies .

Corollary 2.2.1:

Let $f_i : X \to (Y_i , q_i)$, $i \epsilon I$ and each (Y_i , q_i) is a topological space , then the initial convergence structure P with respect to f_i and q_i , $i \epsilon I$ is a topological convergence structure .

Proof:

Follows by theorem 2.2.1 and theorem 1.1.11 .

Definition 2.2.1 :

Let (X , P) be a convergence space and $A \subseteq X$. The subspace convergence structure P_A on A is the initial convergence structure with respect to the inclusion mapping e : A→ X .[3]

Let $\mathcal{F} \epsilon F(A)$ and $x \in A$. We say that $\mathcal{F} \in P_A(x)$ if and only if $[\mathcal{F}]_X \in P(x)$.

Theorem 2.2.2 :

Let (X , P) and (Y , q) be convergence spaces such that f : (X , P) → (Y , q) is a continuous mapping . Let A ⊆ X . Then the restriction map $f_{/A}$: (A ,P_A) → (f(A) , $q_{f(A)}$) is continuous . [1]

Proof :

Let $x \epsilon A$ and $\mathcal{F} \epsilon P_A(x)$. Then , $[\mathcal{F}]_X \epsilon P(x)$ and therefore $[f_{/A}(\mathcal{F})]_Y = f([\mathcal{F}]_X) \epsilon q\big(f(x)\big)$ and so $f_{/A}(\mathcal{F}) \epsilon q_{f(A)}\big(f(x)\big)$. Hence , $f_{/A}$ is continuous .

Theorem 2.2.3:

Let (X , q) be a convergence space and let $A \subseteq X$, then $CL_{qA}(B) = CL_q(B) \cap A$ for each B ⊆ A .[1]

Proof :

Let $x \epsilon CL_{qA}(B)$ then $\exists \mathcal{F} \epsilon F(A)$ such that $B \in \mathcal{F}$ and $\mathcal{F} \in q_A(x)$.

Clearly $[\mathcal{F}]_X \epsilon q(x)$ and $B \epsilon [\mathcal{F}]_X$.

Then $x \epsilon CL_q(B)$. This implies that $x \epsilon CL_q(B) \cap A$.

Conversely let $x \in CL_q(B) \cap A$, then $x \in A$ and $\exists \mathcal{F} \in F(X)$ such that $B \in \mathcal{F}$ and $\mathcal{F} \in q(x)$. $A \in \mathcal{F}$ so let $\mathcal{F}_A$={F∩A :F $\in \mathcal{F}$} .
$\mathcal{F}_A \in q_A(x)$ by definition 2.2.1 and $B \in \mathcal{F}_A$. Thus $x \in CL_{qA}(B)$.

Theorem 2.2.4 :

Let (X , P) and (Y , q) be convergence spaces and $(Y_i , q_i)_{i \in I}$ be a family of convergence spaces such that q is the Initial Convergence Structure with respect to (f_i :Y→ (Y_i , q_i)) then f : (X , P) → (Y , q) is continuous if and only if for each $i \in I$, f_i of: (X , P) → (Y_i , q_i) is continuous .[3]

Proof :

If f is continuous , then (f_i of) is continuous $\forall i \in I$, as f_i is continuous $\forall i \in I$ and by theorem 2.1.8 the composition of two continuous functions is continuous .

Conversely , assume that (f_iof) is continuous $\forall i \in I$. Let $\mathcal{F} \in P(x)$. Since (f_iof) is continuous we get (f_iof)($\mathcal{F}$) = $f_i\big(f(\mathcal{F})\big) \in q_i\Big(f_i\big(f(x)\big)\Big) \forall i \in I$. $Thus\ f(\mathcal{F}) \in q\ \big(f(x)\big)$ by definition of initial convergence space . Hence f is continuous .

Theorem2.2.5 :

Let (X , P) be a convergence space which carries the initial convergence structure with respect to the convergence spaces $(X_i , P_i)_{i \in I}$ and the mappings $(f_i$: X → $X_i)_{i \in I}$. If all the P_i are pretopological structures then P is a pretopological structure .[3]

Proof :

Let id :(X , $\Pi(P)$) →(X , P) be the identity mapping .

$f_i o$ id = f_i: (X , $\Pi(P)$) → (X_i , P_i) which is continuous for all $i \epsilon I$ by corollary 2.1.4 and since all of the P_i are pretopological structures.

id : $(X, \Pi(P))$ → (X , P) is continuous by theorem 2.2.4 .

$P \leq \Pi(P)$ by theorem 2.1.1 . But $\Pi(P) \leq P$. Hence , $\Pi(P) = P$. Therefore P is a pretopological structure on X .

From theorem 2.2.5 we get that any subspace of a pretopological space is a pretopology , and the sup of the pretopological structures on X is a pretopological structure on X .

If (X , q) is a pretopology and (A , q_A) be a subspace of (X , q) , then the neighbourhood of $x \epsilon A$ in (A , q_A) is $\mathcal{U}_q(x)|_A = \{A \cap u : u \epsilon \mathcal{U}_q(x)\}$.

Theorem 2.2.6:

Let (X , q) be a convergence space and $A \subseteq X$. Then $\Pi(q_A) = [\Pi(q)]_A$ i.e (A , $\Pi(q_A)$) is a subspace of (X , $\Pi(q)$) . [3]

Proof:

Let $\mathcal{U}_A(x)$ denotes the neighbourhood filter of x in $(A, \Pi(q_A))$ and let $\mathcal{U}(x)|A$ denotes the neighbourhood filter of x in (A , $[\Pi(q)]_A$) .

It is sufficient to show that $\mathcal{U}_A(x) = \mathcal{U}(x)|_A \; \forall x \in A$.

Since the inclusion mapping e : (A , $\Pi(q_A)$) → (X , $\Pi(q)$) is continuous we get by theorem 2.2.2 that e : (A , $\Pi(q_A)$) → (A , $[\Pi(q)]_A$) is continuous .

$[\Pi(q)]_A \leq \Pi(q_A)$ by theorem 2.1.1 . Hence $\mathcal{U}_A(x) \supseteq \mathcal{U}(x)|_A$.

Now if $u \in \mathcal{U}_A(x)$, then let the set $\vee = u \cup (X \setminus A)$. We show that $\vee \epsilon \mathcal{U}(x)$.

Take any filter $\mathcal{F} \epsilon\, q(x)$. If $\mathcal{F}$ does not have a trace on A , then $X \setminus$ A $\epsilon\, \mathcal{F}$ and so $\vee \epsilon\, \mathcal{F}$. If $\mathcal{F}$ has a trace on A , the filter $\mathcal{F}_A = \{F \cap A : F \epsilon\, \mathcal{F}\} \epsilon\, q_A(x)$ and so $u \,\epsilon\, \mathcal{F}_A$, i.e there is a set $F \epsilon\, \mathcal{F}$ such that $F \cap A \subseteq u$.

Then $F \subseteq \vee$ and so $\vee \epsilon\, \mathcal{F}$. Hence $\vee \epsilon\, \mathcal{U}(x)$, so $\vee \cap A \,\epsilon\, \mathcal{U}(x)|_A$.

But $\vee \cap A = u$. $\mathcal{U}_A(x) \subseteq \mathcal{U}(x)|_A$.

Let (X , q) be a convergence space and A$\subseteq X$. We denote the topological modification of q_A by t(q_A) .

Theorem 2.2.7 :

Let (X , q) be a convergence space . Then for each A $\subseteq X$, t(q_A) $\geq [\mathrm{q_t}]_\mathrm{A}$.[5]

Proof:

Since the inclusion mapping e : (A , t(q_A)) $\rightarrow$ (X , q_t) is continuous we get by theorem 2.2.2 that e : (A , t(q_A)) $\rightarrow$ (A , $[q_t]_A$) is continuous. Thus $[q_t]_A \leq$ t(q_A) by theorem 2.1.1 .

Theorem2.2.8 :

Let (X , q) be a convergence space . If A is q-closed then t(q_A) $\leq [q_t]_A$.[5]

Proof:

Let u be t(q_A)-closed , then u is q_A-closed .

$CL_{q_A}(u) = u = CL_q(u) \cap A$ by theorem 2.2.3 .

$u = CL_q(u) \cap A = CL_q(u) \cap CL_q(A) \supseteq CL_q(u \cap A) = CL_q(u)$,

as $u \subseteq A$. $u = CL_q(u)$ and this means that u is q-closed .

$u = u \cap A$ is $[q_t]_A$ -closed and from theorem 1.1.5 we get $[q_t]_A \geq t(q_A)$.

From theorems 2.2.7 and 2.2.8 it follows that , if A is q-closed subset of X then $[q_t]_A = t(q_A)$.

Example2.2.1 :

Let X={x_n : $n \epsilon Z$} and q be the pretopology with neighbourhood filters defined as follows : for each $n \epsilon Z$, $\mathcal{U}_q(x_n)$ is the filter generated by {x_{n-1} , x_n , x_{n+1}} . The topology q_t is indiscrete .

Let A = {x_n: n is an even integer}. Then q_A is the discrete topology on A. Hence $t(q_A) = q_A$. But $[q_t]_A$ is indiscrete topology on A .

Theorem2.2.9 :

Let *f*: (X , P) → (Y , q) be a map , where P is the initial convergence structure on X with respect to *f* . If (Y , q) is weakly diagonal convergence space , then (X , P) is weakly diagonal .

Proof:

Let $x \epsilon X$ and $\mathcal{F} \epsilon P(x)$. Let $n : X \to F(X)$ be a map such that $n(z) \epsilon P(z)$ for all $z \epsilon X$. Let $F \epsilon \mathcal{F}$. Want to show that $x \in a\left(\cap_{z \epsilon F} n(z)\right)$.

Since f is continuous , then $f(\mathcal{F}) \epsilon q(f(x))$.

Define $\grave{n} : Y \to F(Y)$ as

$$\grave{n}(y)\begin{Bmatrix} f\big(n(z)\big)\ if\ \exists z \epsilon X\ such\ that\ y = f(z) \\ [y] \quad if \quad y \notin f(X) \end{Bmatrix}$$

Since (Y , q) is weakly diagonal , then there exists a filter $\mathcal{K} \epsilon q\big(f(x)\big)$

such that $\mathcal{K} \supseteq \mathcal{G}\big(\grave{n}, f(F)\big) = \cap_{f(z) \epsilon f(F)} f\big(n(z)\big) \supseteq f\left(\cap_{z \epsilon F} n(z)\right)$.

Let $\mathcal{H}$ be the filter generated by the filter base $\{f^{-1}(k) : k\epsilon\mathcal{K}\}$. Let M= $\mathcal{H} \vee \left(\cap_{z\epsilon F}\, n(z)\right)$, it is clear that M is well defined .

Now $f(\mathrm{M}) \supseteq \mathcal{K}$ and hence $f(\mathrm{M})\epsilon\, q\big(f(x)\big)$.

Since P is the initial convergence structure , then M $\epsilon\, P(x)$.

But M $\supseteq \cap_{z\epsilon F}\, n(z)$. $x\,\epsilon\, a\big(\cap_{z\epsilon F}\, n(z)\big)$. Hence (X , P) is weakly diagonal convergence space .

Corollary 2.2.2:

Let f : (X , P) $\rightarrow$ (Y , q) be a map where P is the initial convergence structure with respect to f . If (Y , q) has closed adherences then (X , P) has closed adherences too . In particular any Subspace of a space with closed adherences has closed adherences .

Proof:

(Y , q) is weakly diagonal by theorem 1.3.5 .

(X , P) is weakly diagonal by theorem 2.2.9 .

(X , P) has closed adherences by theorem 1.3.5 .

The particular case holds since any subspace is the initial convergence space with respect to the inclusion map .

Definition2.2.2 :

Let $(X_i)_{i\epsilon I}$ be a collection of convergence spaces and let $\prod_{i\epsilon I} X_i$ be the product set of the X_i . The product convergence structure on $\prod_{i\epsilon I} X_i$ is the initial convergence structure with respect to the projection mappings $(P_i\colon (\prod_{j\epsilon J} X_j \rightarrow X_i)_{i\epsilon I})$ and the resulting convergence space is called the (Tychonoff) product of the $(X_i)_{i\epsilon I}$. [3]

A filter $\mathcal{F}$ converges to $x = (x_i)_{i\in I}$ in $\prod_{i\epsilon I} X_i$ if and only if for each $i\epsilon I$, $P_i(\mathcal{F})$ converges to $P_i(x)$ in X_i .

Theorem 2.2.10:

Let$(X_i)_{i\epsilon I}$ be a family of convergence spaces . A filter $\mathcal{F}$ on $\prod_{i\epsilon I} X_i$ converges to $(x_i)_{i\in I} = x \epsilon \prod_{i\epsilon I} X_i$ if and only if , for all $i\epsilon I$, there are filters $\mathcal{F}_i$ converging to x_i in X_i, such that $\mathcal{F} \geq \prod_{i\epsilon I} \mathcal{F}_i$.

Here $\prod_{i\epsilon I} \mathcal{F}_i$ denotes the Tychonoff product of the filters $\mathcal{F}_i$, i.e , the filter based on $\{\prod F_i : F_i \epsilon \mathcal{F}_i$ for all $i\epsilon I$, $F_i \neq X_i$ for only finitely many $i\epsilon I \}$. [3]

Proof:

Clearly $P_j(\prod_{i\epsilon I} \mathcal{F}_i) = \mathcal{F}_j$ for all $j\epsilon I$ and so the product filter converges if all components filters do . On the other hand , if $\mathcal{F}$ converges to $x = (x_i)_{i\in I}$ in the product convergence space , then $\mathcal{F} \supseteq \prod_{i\epsilon I} P_i(\mathcal{F})$ gives the reverse implication .

Chapter Three

Separation Axioms in Convergence Spaces

Chapter Three

3.1 Separation Axioms in Convergence Spaces

Definition 3.1.1 :

let (X,P) be a convergence space and x , $y \in$ X . Then , (X , P) is called

a) T_1 if $x \neq y$, then $[y] \notin P(x)$, i.e lim $[y]$= $\{y\}$ ∀y∈X.

b) Hausdorff (separated) if $x \neq y$, then $P(x) \cap P(y) = \emptyset$.

c) T_2 if $\mathcal{U}(x)$ and $\mathcal{U}(y)$ are disjoint filters whenever $x \neq y$.[5]

d) compact if every ultrafilter on X converges in X .

Theorem 3.1.1 :

Let (X,P) be a convergence space . Then ,

a) P is T_1 if and only if $a([x])= \{x\}$,$\forall x \epsilon X$.

b) If P is Hausdorff and $\mathcal{F} \epsilon P(x)$, then $a(\mathcal{F}) = \{x\}$.[1]

c) If P is Hausdorff , then P is T_1

d) If P is T_2 , then P is Hausdorff .

Proof :

a) Assume that P is T_1 then $[x]$ converges only to x and since $[x]$ is an ultrafilter then $a([x])$= lim $[x]$ = $\{x\}$.

Conversely , if $a([x])$ =$\{x\}$, then $[x] \notin P(y)$, if $x \neq y$. Hence P is T_1.

b) Let $\mathcal{F} \epsilon P(x)$ then $x \epsilon a(\mathcal{F})$. Now assume that $y \neq x$ and $y \epsilon a(\mathcal{F})$, then $\exists \mathcal{G} \epsilon P(y)$ such that $\mathcal{G} \geq \mathcal{F}$. Hence , $\mathcal{G} \epsilon P(x) \cap P(y)$ which contradicts our hypothesis . Hence $a(\mathcal{F}) = \{x\}$.

c) If P is Hausdorff , then $a([x]) = \{x\}$ by part(b) . By part (a) P is T_1.

d) suppose (X , P) is T_2 . Let $x,y \in X$ such that $x \neq y$. If $\mathcal{F} \in P(x) \cap P(y)$ then $\mathcal{F} \geq \mathcal{U}(x)$ and $\mathcal{F} \geq \mathcal{U}(y)$. Hence $\mathcal{U}(x)$ and $\mathcal{U}(y)$ are not disjoint filters which is a contradiction . Therefore , (X,P) is Hausdorff.

The following theorem shows that the definition of Hausdorff and T_2 are equivalent in a pretopological convergence space .

Theorem 3.1.2 :

Let (X , P) be a pretopological space , then (X , P) is Hausdorff if and only if it is T_2 space

Proof:-

Let (X , P) be Hausdorff and $x,y \in X$ where $x \neq y$. Suppose that $\mathcal{U}(x)$, $\mathcal{U}(y)$ are not disjoint filters , then $\mathcal{F} = \mathcal{U}(x) \vee \mathcal{U}(y)$ exists and $\mathcal{F} \in P(x) \cap P(y)$ since p is a pretopological structure . But this contradicts the assumption that P is Hausdorff . Hence , P is T_2 .

The converse follows by theorem 3.1.1 (d).

Note , in general the definitions of Hausdorff and T_2 are not equivalent .

In topological spaces the axioms T_1 and T_2 agree with the separation axioms of Frechet and Hausdorff respectively.

Theorem 3.1.3 :

Let (X , P) be a convergence space . Then , P is T_1 if and only if $\{x\}$ is P-closed subset of X , $\forall x \in X$. [3]

Proof :-

Assume that P is T_1 . Let $y \in$ CL({x}) and $y \neq x$, then $\exists \mathcal{F} \in P(y)$ such that $\{x\} \in \mathcal{F}$, and this implies that $[x] = \mathcal{F}$ which contradicts the hypothesis , so CL($\{x\}$) =$\{x\}$. Hence $\{x\}$ is P-closed .

Conversely , if $\{x\}$ is a P-closed subset of X ,$\forall x \in X$. Then CL({x}) $= a([x]) = \{x\}$ so by theorem 3.1.1 (a) , P is T_1.

Theorem 3.1.4 :

Let (X , P) and (X , q) be convergence spaces such that P ≤ q , then :

a) If P is T_1 ,then q is T_1.

b) If P is Hausdorff , then q is Hausdorff.

c) If P is T_2, then q is T_2 .

Proof :

a) By theorems 1.3.3 and theorem 3.1.1(a) we have $\{x\} \subseteq a_q([x]) \subseteq a_p([x]) = \{x\}$,$\forall x \in X$ .So $a_q([x]) = x$ hence q is T_1 .

b) Since $q(x) \subseteq P(x)$ $\forall x \in X$ we have $q(x) \cap q(y) \subseteq P(x) \cap P(y) = \emptyset$ if $x \neq y$ since P is Hausdorff . Hence , $q(x) \cap q(y) = \emptyset$ if $x \neq y$ which means that q is Hausdorff.

c) Let $x \neq y$, since $q(x) \subseteq P(x)$ and $q(y) \subseteq P(y)$, we have $\mathcal{U}_p(x) \subseteq \mathcal{U}_q(x)$ and $\mathcal{U}_p(y) \subseteq \mathcal{U}_q(y)$. If $\mathcal{U}_P(x)$ *and* $\mathcal{U}_P(y)$ are disjoint filters then $\exists\, u$, v such that $u \in \mathcal{U}_p(x)$ and $\vee \in \mathcal{U}_p(y)$ *and* $u \cap$ v $= \emptyset$.

Since $u \in \mathcal{U}_q(x)$ and v $\in \mathcal{U}_q(y)$, Then $\mathcal{U}_q(x)$ and $\mathcal{U}_q(y)$ are disjoint filters . Hence , q is T_2

Theorem 3.1.5 :

Let f : (X , P) → (Y , q) be a continuous and injective mapping from a convergence space (X , P) into a Hausdorff convergence space (Y, q) . Then , (X ,P) is a Hausdorff space . [14]

proof :-

Let x and y ∈ X with $x \neq y$ and $P(x) \cap P(y) \neq \emptyset$, then $\exists \mathcal{F} \epsilon F(X)$ such that $\mathcal{F} \epsilon P(x)$ and $\mathcal{F} \epsilon P(y)$ .Since f is continuous , we get $f(\mathcal{F}) \epsilon q\big(f(x)\big)$ and $f(F) \epsilon q\big(f(y)\big)$. Since f is injective we get $f(x) \neq f(y)$. So q($f(x)$) ∩ q($f(y)$) ≠ ∅ .

But this contradicts the hypothesis as (Y , q) is Hausdorff . Hence $P(x) \cap P(y) = \emptyset$ and thus , (X , P) is a Hausdorff space .

Theorem 3.1.6 :

Let f : (X , P) →(Y , q) be a continuous and injection mapping from a convergence space (X , P) into a T_1- convergence space (Y , q) . Then , (X , P) is a T_1 – convergence space .

Proof :-

Let $x \epsilon X \; with \; [x] \epsilon P(y) \, where \, x \neq y \, , then \; f([x]) = [f(x)] \, \epsilon \mathrm{q}\big(f(y)\big)$ because f is continuous . $f(x) \neq f(y)$ as f is injective . Hence , q is not T_1 which is a contradiction . Therefore , (X , P) is T_1 – space .

Corollary 3.1.1 :

Any subspace of a T_1-space , Hausdorff space is T_1 - space , Hausdorff space , respectively .

Proof :

Follows by Definition 2.2.1 and Theorems 3.1.5 , 3.1.6.

Theorem 3.1.7 :

Let h and f : (X , P) →(Y , q) be two continuous mappings from a convergence space (X, P) into a Hausdorff convergence space (Y, q) .Then,

a) The set A $=\{x \epsilon X: h(x) = f(x)\}$ is a P - closed subset of X.

b) If D is a dense subset of X and $h(x) = f(x)$, $\forall x \epsilon D$ .Then , $f(x) = h(x)\ \forall x \epsilon X$.

Proof:

a) Let $x \epsilon$ CL(A) , then $\exists\, \mathcal{F}_A \epsilon F$ (A) such that $[\mathcal{F}_A]_X \epsilon P(x)$. $h([\mathcal{F}_A]_x) \epsilon q\big(h(x)\big)$ and $f([\mathcal{F}_A]_x) \epsilon q\big(f(x)\big)$ since f , h are continuous. Since $h([\mathcal{F}_A]_x)$ and $f([\mathcal{F}_A]_x)$ are generated by the filter bases $h(\mathcal{F}_A)$ and $f(\mathcal{F}_A)$ respectively and $h(y) = f(y)\ \forall y\ \epsilon$A , we have $h([\mathcal{F}_A]_X) = f([\mathcal{F}_A]_X)$.

Since (Y, q) is a Hausdorff space we have $h(x) = f(x)$. Hence , $x \epsilon$ A . Thus , A is a p – closed subset of X .

b) D $\subseteq A$ then X = CL (D) ⊆ CL (A) = A . Hence X=A .Thus , $h(x) = f(x)\ \forall x \epsilon$X.

Definition 3.1.2:

A convergence space (X, P) is said to be minimal Hausdorff if (X, P) is Hausdorff space and every strictly coarser convergence space (X, q) is not Hausdorff . [15]

Definition 3.1.3:

A convergence space (X , P) is called a pseudotopological convergence space if $\mathcal{F} \in P(x)$ whenever every ultrafilter $\mathcal{G}$ finer than $\mathcal{F}$ converges to x in X .[3],[4]

Theorem 3.1.8 :

The following statements about a convergence space (X , P) are equivalent.

a) (X , P) is minimal Hausdorff .
b) $\{x\} = a(\mathcal{F})$ if and only if $\mathcal{F} \in P(x)$.
c) (X , P) is a compact pseudotopological Hausdorff space . [15]

Proof:

***a* implies b**: If $\mathcal{F} \in P(x)$, then $\{x\} = a(\mathcal{F})$ by theorem 3.1.1(b). Conversely , assume that $a(\mathcal{F}) = \{x\}$ and $\mathcal{F} \notin P(x)$. Define q on X as follows.

$\mathcal{H} \in q(x)$ if and only if $\mathcal{H} \geq \mathcal{G} \cap \mathcal{F}$ where $\mathcal{G} \in P(x)$ and $\mathcal{H} \in q(y)$ if and only if $\mathcal{H} \in P(y)$ where $x \neq y$.

It is clear that q is a convergence structure .

We Claim that q is strictly coarser than P and it is Hausdorff . Let $x,y \in$ X with $x \neq y$, then we have $q(y) = P(y)$. Let $\mathcal{G} \in P(x)$ then $\mathcal{G} \geq \mathcal{F} \cap \mathcal{G}$ and $\mathcal{F} \geq \mathcal{F} \cap \mathcal{G}$. Hence , $\mathcal{G}$ and $\mathcal{F} \in q(x)$, $\forall \mathcal{G} \in P(x)$, and since $\mathcal{F} \notin P(x)$, we get $P(x) \subset q(x)$. Therefore $q < P$.

To show that q is Hausdorff let $x \neq y \neq$ z , then $q(y) \cap q(z) = P(y) \cap P(z) = \emptyset$ as P is Hausdroff.

Assume that $\mathcal{H} \in q(x) \cap q(y)$ where $x \neq y$. Then , $\exists \mathcal{G} \in F(X)$ such that $\mathcal{H} \geq \mathcal{F} \cap \mathcal{G}$ where $\mathcal{G} \in P(x)$ and $\mathcal{H} \in P(y)$. Without loss of generality we take $\mathcal{H}$ to be an ultrafilter , so $\exists F_o \in \mathcal{F}$ such that $X \backslash F_o \in \mathcal{H}$ and $\mathcal{H}$ can not be finer than $\mathcal{G}$ as P is Hausdorff , so $\exists g_0 \in \mathcal{G}$ such that $X \backslash g_o \in \mathcal{H}$. Since $\mathcal{H} \geq \mathcal{F} \cap \mathcal{G}$ and $\{F \cup g : F \in \mathcal{F}$ and $g \in \mathcal{G}\}$ is a filter base generating $\mathcal{F} \cap \mathcal{G}$ we have $(F_o \cup \mathcal{G}_o) \in \mathcal{H}$. But $(X \backslash F_o \cap X \backslash g_o) \in \mathcal{H}$. Hence , $(F_o \cup g_o) \cap (X \backslash F_o \cap X \backslash g_o) = \emptyset \in \mathcal{H}$ which is a contradiction . Hence, (X, q) is a Hausdorff space and since q < P we get a contradiction as (X , P) is a minimal Hausdorff space . Hence , $\mathcal{F} \in P(x)$.

b implies c: (X, P) is Hausdorff, as if $\mathcal{F} \in P(x) \cap P(y)$, then $a(\mathcal{F}) = \{x\} = \{y\}$. Hence $x = y$.

To show compactness of (X, P) , let $\mathcal{F}$ be an ultrafilter . If $a(\mathcal{F}) = \emptyset$, then $a(\mathcal{F} \cap [x]) = \{x\}$. We will show that: now, $x \in a(\mathcal{F} \cap [x])$ as $\mathcal{F} \cap [x] \subseteq [x] \in P(x)$. Assume that $y \in a(\mathcal{F} \cap [x])$ and $y \neq x$, then $\exists \mathcal{G} \in P(y)$ where $\mathcal{F} \cap [x] \subseteq \mathcal{G}$.

Now $\mathcal{G} \nsubseteq \mathcal{F}$ so $\exists g_1 \in \mathcal{G}$ and $X \backslash g_1 \in \mathcal{F}$ as $\mathcal{F}$ is an ultrafilter , and $\mathcal{G} \nsubseteq [x]$ as $a(\mathcal{G}) = \{y\}$ by hypothesis so $\exists g_2 \in \mathcal{G}$ and $X \backslash g_2 \in [x]$ as $[x]$ is an ultrafilter.

But $((X \backslash g_1) \cup (X \backslash g_2)) \in \mathcal{F} \cap [x] \subseteq \mathcal{G}$ and $(g_1 \cap g_2) \in \mathcal{G}$ .So $((X \backslash g_1) \cup (X \backslash g_2)) \cap (g_1 \cap g_2) = \emptyset \in \mathcal{G}$ which is a contradiction as $\mathcal{G}$ is a filter. Hence , $a(\mathcal{F} \cap [x]) = \{x\}$.

Now if $a(\mathcal{F}) = \emptyset$, then $a(\mathcal{F} \cap [x]) = \{x\}$, then $\mathcal{F} \cap [x] \in P(x)$ by hypothesis , so $\mathcal{F} \in P(x)$ as $\mathcal{F} \geq \mathcal{F} \cap [x]$ which is a contradiction as $a(\mathcal{F}) = \emptyset$. The final contradiction shows that $a(\mathcal{F}) \neq \emptyset$. Hence , $\mathcal{F}$ converges in X as $a(\mathcal{F}) = \lim \mathcal{F}$ in X as $\mathcal{F}$ is an ultrafilter . Hence , (X, P) is a compact space .

Finally , we show that (X, P) is a pseudotopological space .

If each ultrafilter $\mathcal{H}$ finer than $\mathcal{G}$ converges to x then lim$\mathcal{H}$= $\{x\}$ for all ultrafilter $\mathcal{H} \geq \mathcal{G}$ then $a(\mathcal{G}) = \{x\}$. Hence , $\mathcal{G} \in P(x)$ by hypothesis . Thus, (X , P) is a pseudotopological space .

c implies a : Assume that $q \leq$ P where q is a Hausdorff convergence structure . Let $\mathcal{F} \in q(x)$, then $a_q(\mathcal{F}) = \{x\}$ as (X, q) is a Hausdorff space and by theorem 3.1.1 (b) . $a_p(\mathcal{F}) \neq \emptyset$ as (X, P) is compact space .

Now $a_p(\mathcal{F}) \subseteq a_q(\mathcal{F}) = \{x\}$ by theorem 1.3.3 . Hence , $a_p(\mathcal{F}) = \{x\}$.

Now , let $\mathcal{H} \geq \mathcal{F}$ where $\mathcal{H}$ is an ultrafilter , then $a_p(\mathcal{H}) \subseteq a_p(\mathcal{F}) = \{x\}$. Since $a_p(\mathcal{H}) \neq \emptyset$ as (X, P) is compact , then $a_p(\mathcal{H}) = \{x\}$. Hence , $\mathcal{H} \in P(x)$. Now $\mathcal{F} \in P(x)$ as (X , P) is a pseudotopological convergence space . Thus , $q(x) \subseteq P(x)$ so P $\leq$ q . Hence , P = q. Therefore , (X , P) is a minimal Hausdorff .

Corollary 3.1.2 :

A Hausdorff topological space is a minimal Hausdorff if and only if it is compact.

Proof :

Follows by theorem 3.1.8.

Theorem 3.1.9:

Let f : (X , P) → (Y , q) be a continuous bijective mapping from a minimal Hausdorff space (X , P) into a Hausdorff space (Y, q) , then f is a homeomorphism map .[15]

Proof :-

Let $\mathcal{F} \epsilon\, q(y)$, since f is bijective , then $y = f(x)$ and x is unique $a_q(\mathcal{F}) = \{f(x)\}$ because (Y, q) is a Hausdorff space and by theorem 3.1.2(a) .

Claim : $a_p(f^{-1}(\mathcal{F})) = \{x\}$.

$a_p(f^{-1}(\mathcal{F})) \neq \emptyset$ as (X, P) is minimal Hausdorff and hence it is compact by theorem 3.1.8 .Let z $\epsilon\, a_p(f^{-1}(\mathcal{F}))$, then $f(z) \epsilon\, a_q f((f^{-1}(\mathcal{F}))$ as f is continuous .Since f is bijective , we have $f((f^{-1}(\mathcal{F})) = \mathcal{F}$. Hence $f(z) \epsilon a_q(\mathcal{F}) = \{f(x)\}$. Thus $f(z)= f(x)$. Then x = z as f is 1.1 .

Therefore , $\{x\} = a_p(f^{-1}(\mathcal{F}))$. Thus $f^{-1}(\mathcal{F}) \epsilon P(x)$ as (X, P) is minimal Hausdorff and by theorem 3.1.8 . Hence , f^{-1} is continuous.Therefore , f is a homeomorphism map .

Corollary 3.1.3 :[Fischer]

Let f : (X , P) → (Y , q) be a continuous bijection mapping from a compact pertopological space (X , P) into a Hausdorff space (Y, q) .Then , f is a homeomorphism .[15] .

Proof :-

(X, P) is Hausdorff by theorem 3.1.5 , (X, P) is a pseudotopological space since every pretopological space is a pseudotopological space.

(X , P) is a compact pseudotopological Hausdorff space .

(X , P) is minimal Hausdorrf by theorem 3.1.8.

f is a homeomorphism map by theorem 3.1.9 .

3.2 Regularity in convergence spaces

Extending the topological property of regularity to convergence spaces has created certain interest among mathematicians studying convergence spaces .see ([19] . [20])

Definition 3.2.1 :

A convergence space (X, P) is regular if $CL_P(\mathcal{F}) \in P(x)$ whenever $\mathcal{F} \in P(x)$.(The filter $CL_P(\mathcal{F})$ is generated by the filter base { $CL_P(F) : F \in \mathcal{F}$ } .[19]

Regular convergence space definition , arises from the paper by Cook and Fischer on regular convergence spaces , [20]. In this paper , the authors define regularity in terms of an iterated limit axiom for filters .

Biesterfelt [19] has shown that the definition given by Cook and Fisher is equivalent to definition 3.2.1

Lemma 3.2.1: Let (X , P) be a convergence space and $\mathcal{F} \in P(x)$, then $x \in CL(F)$, $\forall F \in \mathcal{F}$.

Proof:-

Since $\mathcal{F} \in P(x)$ and $F \in \mathcal{F}$ we have $x \in CL(F)$.

Theorem 3.2.1:

If (X, P) is a regular T_1- convergence space , then it is Hausdorff . [20]

Proof :-

Assume that P is not Hausdorff , then $\exists\, \mathcal{F} \in F(X)$ such that $\mathcal{F} \in P(x) \cap P(y)$ for some $x , y \in X$ and $x \neq y$.

$CL(\mathcal{F}) \in P(x)$ and $P(y)$ by regularity of P .

x ϵ CL(F) , $\forall F \epsilon \mathcal{F}$ by lemma 3.2.1 .This means CL($\mathcal{F}$) $\subseteq [x]$. Hence, $[x] \epsilon P(y)$ which is a contradiction as (X, P) is a T_1 – space .

Therefore , (X , P) is a Hausdorff space .

Note that the definition of regular convergence space gives the usual concept for topologies .

Theorem 3.2.2 :

Let (X , P_t) be the natural convergence space related to the topological space (X , t) . Then , (X , P_t) is regular if and only if (X , t) is regular .

proof :-

Let (X , t) be regular , so it is sufficient to show that for each $x \in$ X we have CL($\mathcal{U}(x)$) converges to x in (X , P_t).

Let A$\epsilon\, \mathcal{U}(x)$, then $\exists$ an open set u such that $x \epsilon\, u \subseteq$ A . So by regularity of (X , t) $\exists$ an open set v such that x ϵ v $\subseteq$ CL(v) $\subseteq u \subseteq$ A. Hence , A ϵ CL($\mathcal{U}(x)$) . Thus , $\mathcal{U}(x) \subseteq$ CL($\mathcal{U}(x)$) . This means that CL($\mathcal{U}(x)$) converges to x in (X , P_t) .

Conversely let (X , P_t) be regular and let u be an open set containing x . Then , u ϵ CL($\mathcal{U}(x)$) . So $\exists$ an open set v containing x such that $x \epsilon$ v $\subseteq$ CL(v) $\subseteq u$ which means that (X , t) is regular .

Theorem 3.2.3 :

Let X and Y be convergence spaces and f : (X , P) →(Y , q) be an initial map , then :

a) If Y is regular , then X is regular .

b) If X is regular and f is a surjection , then Y is regular .[21]

Proof :-

a) Let $\mathcal{F}$ converges to x in X . $f(\mathcal{F})$ converges to $f(x)$ as f is an initial map and hence it is continuous . CL($f(\mathcal{F})$) converges to $f(x)$in Y Since f(CL(F)) ⊆ CL($f(F)$) for all F $\epsilon\,\mathcal{F}$ by theorem 2.1.6 we have CL($f(\mathcal{F})$) ⊆ f (CL($\mathcal{F}$)) .Therefore , f (CL($\mathcal{F}$)) converges to $f(x)$ in Y. Hence , CL $(\mathcal{F})$ converges to x because f is an initial map . Therefore , X is regular

b) Let $\mathcal{F}$ converges to y in Y and since f is surjective y = $f(x)$ for some x ϵ X and $f(f^{-1}(\mathcal{F}))$ = $\mathcal{F}$ .Thus $f^{-1}(\mathcal{F})$ converges to x in X as f is an initial map.

Since X is regular , CL($f^{-1}(\mathcal{F})$) converges to x in X . f(CL($f^{-1}(\mathcal{F})$)) converges to $f(x)$ in Y as f is continuous .

We claim that f(CL($f^{-1}(\mathcal{F})$)) ⊆ CL($\mathcal{F}$)

At first we show that f^{-1}(CL(B)) ⊆ CL(f^{-1}(B)) . Let b ϵ f^{-1}(CL(B)) , then f(b) ϵ CL(B) , so B ϵ $\mathcal{G}$ for some filter $\mathcal{G}$ converges to f(b) in Y . $f(f^{-1}(\mathcal{G}))$ = $\mathcal{G}$ as f is surjective .Since f is an initial map we have $f^{-1}(\mathcal{G})$ converges to b . Since f^{-1}(B) ϵ $f^{-1}(\mathcal{G})$, we get b ϵ CL(f^{-1}(B)).

Note that CL(F) = $f(f^{-1}$(CL(F))) ⊆ f(CL(f^{-1}(F))) . Hence f(CL($f^{-1}(\mathcal{F})$)) ⊆ CL($\mathcal{F}$) . Therefore , CL($\mathcal{F}$) converges to $f(x)$ = y in Y. Hence Y is regular .

Corollary 3.2.1 :

A subspace of a regular convergence space is regular .

Proof :

Follows by theorem 3.2.3

Theorem 3.2.4 :

The closure operator for a compact regular Hausdorff convergence space is idempotent .[22]

Corollary 3.2.2 :

Let (X, P) be a compact regular Hausdorff space , then (X , Π(p)) is Hausdorff and topological .[22]

Proof :

The closure operators for (X , P) and (X , $\Pi(P)$) are the same corollary 1.2.1 and by theorem 1.2.8 (X , $\Pi(P)$) is a topological space as the closure operator of (X , $\Pi(P)$) is idempotent by theorem 3.2.4 .

To see that (X , $\Pi(P)$) is Hausdorff , let $\mathcal{F}$ be an ultrafilter which converges both to x and y with respect to $\Pi(P)$.

By compactness , $\mathcal{F}$ converges to some point z with respet to P , and by regularity of P , $CI_p(\mathcal{F})$ also converges to z with respect to P. But each neighbourhood of x is in $\mathcal{F}$, so x is in each member of $CL_p(\mathcal{F})$. Since P is Hausdorff , x = z . similarly , y = z .Therefore x=y .

Corollary 3.2.3 :

If A is a subspace of a compact regular Hausdorff convergence space (X, P) , then $\Pi(P_A)$ is Hausdorff and topological .[22]

Proof :

Since (A, P_A) is a subspace of (X , P) , then $\Pi(P_A)$ is the subspace of (X , $\Pi(p)$) by theorem 2.2.6 and since $\Pi(P)$ is Hausdorff and topological and by corollary 3.2.2 we get $\Pi(P_A)$ is Hausdorff and topological as a subspace of Hausdorff and topological space is Hausdorff and topological subspace .

Theorem 3.2.5 :

Let X be a compact regular Hausdorff convergence space . Let K be a closed subset of X and z is a point with z $\notin$ K .Let $\mathcal{F}$ be a filter converging to z ,then there is an open set u containing K such that $u \notin \mathcal{F}$.[21]

Proof :

X\K is open and z ϵ X\K .Since X is regular , CL($\mathcal{F}$) converges to z and so X\K ϵ CL($\mathcal{F}$) . Hence CL(F) $\subseteq$ X\K for some F$\epsilon\,\mathcal{F}$. By theorem 3.2.4 CL(F) is closed . Let u= X\CL (F) .Then u is an open set containing K with $u \notin \mathcal{F}$.

Definition 3.2.2 :

Let (X , q) be a convergence space . For A $\subseteq$ X we define $\mathcal{U}_q(A)$ = $\cap$ {$\mathcal{U}_q(x)$: $x\epsilon$A} and $\mathcal{U}_q(A)$ written simply as $\mathcal{U}(A)$ is called the q-neighbourhood filter of A .[15]

Definition 3.2.3 :

Let (X, q) be a convergence space , then (X, q) is weakly regular if $\mathcal{U}(x)$ and $\mathcal{U}(A)$ are disjoint filters whenever A is a q – closed set and $x\,\epsilon$ X\A .[5],[23]

Definition 3.2.4 :

Let (X, q) be a convergence space , then (X , q) is strongly regular if $\mathcal{U}(x)$ has a base of q – closed sets for each $x\,\epsilon$ X .[5]

Definition 3.2.5 :

Let (X, q) be a convergence space , then (X , q) is $\prod$- regular if $\mathcal{U}(x)$ and $\mathcal{U}(A)$ are disjoint for each A ⊆ X and x ϵ X\CL(A) .[5]

Definition 3.2.6 :

Let (X , q) be a convergence space .Then (X , q) is t – regular , if (X , q_t) is a regular topology. [5]

The following theorems give some relations between these concepts .

Theorem 3.2.6 :

Let (X, q) be a pretopological space .Then the following statements are equivalent to each other .

a) (X , q) is regular
b) (X , q) is $\prod$ - regular
c) $\mathcal{U}(x)$ = CL ($\mathcal{U}(x)$) for each x in X . [5]

Proof :-

For equivalence of (b) and (c) , see [24]

To prove that (a) implies (c) , let q be a regular pretopology then $\mathcal{U}(x) \epsilon\, q(x)$.The regularity of q implies that CL($\mathcal{U}(x)$) $\epsilon\, q(x)$.

Thus $\mathcal{U}(x) \leq$ CL($\mathcal{U}(x)$). But CL($\mathcal{U}(x)$) is always coarser than $\mathcal{U}(x)$. Hence CL($\mathcal{U}(x)$) = $\mathcal{U}(x)$for each $x\epsilon$X.

To show that (c) implies (a) assume that $\mathcal{U}(x)$ = CL ($\mathcal{U}(x)$) for each $x\epsilon$X .If $\mathcal{F}\ \epsilon\ q(x)$ then $\mathcal{F} \geq\ \mathcal{U}(x)$. Thus CI ($\mathcal{F}$) $\geq$ CL ($\mathcal{U}(x)$) = $\mathcal{U}(x)$. Hence CL($\mathcal{F}$) $\epsilon\ q(x)$. Therefore (X, q) is a regular space .

Theorem 3.2.7 :

Let (X , q) be a convergence space . Then q is strongly regular (weakly regular , $\prod$ - regular) if and only if $\Pi(q)$ is strongly regular (weakly regular , $\prod$ - regular).[5]

Proof :

Since the closure operators and $\mathcal{U}_q(x) = \mathcal{U}_{\Pi(q)}(x)\ \forall x \epsilon X$ are the same in $\Pi(q)$ *and* q , the theorem holds .

Corollary 3.2.4 :

Let (X, q) be a convergence space . Then q is $\prod$ - regular if and only if $\Pi(q)$ is regular.

Proof :

Follows by theorems 3.2.7 and 3.2.6.

Theorem 3.2.8 :

Let (X, q) be a convergence space . If q is t - regular , then q is weakly regular .[5]

Proof :

Let q be t – regular and A a q – closed subset of X and x ϵ X\A Because q_t is regular , there are disjoint open sets u and v such that x ϵ u and A $\subseteq$ v . But $u \epsilon \mathcal{U}(x)$ and v ϵ $\mathcal{U}(A)$. Thus $\mathcal{U}(x)$ and $\mathcal{U}(A)$ are disjoint filters .

Theorem 3.2.9 :

Let (X, q) be a convergence space . If q is strongly regular , then q is $\prod$ - regular.[5]

Proof :

Let q be strongly regular . Then , $\mathcal{U}(x)$ has a base of q – closed sets for each $x\epsilon$X . Hence $\mathcal{U}(x)$ = CL($\mathcal{U}(x)$) for each x ϵ X and therefore $\Pi(q)$ is $\prod$ - regular by theorem 3.2.6 .Thus , q is $\prod$ - regular by theorem 3.2.7.

Note that if q is $\prod$ - regular then it is weakly regular by definitions 3.2.3 and 3.2.5 . Hence , by theorem 3.2.9 every strongly regular space is weakly regular .

Theorem 3.2.10 :

Let (X, q) be a pretopological convergence space .Then

a) If q is strongly regular , then q is regular
b) If $CL^{(n+1)}$ = $CL^{(n)}$ for some n ϵ N and q is regular , then q is strongly regular (where $CL^{(n+1)}(A)$ = CL($CL^{(n)}(A)$) , $CL^{(1)}(A)$ =CL(A)) .[5]

Proof :

a) If q is a strongly regular , then q is $\prod$ - regular by theorem 3.2.9. Since q is a pretopology , then q is regular by theorem 3.2.6.
b) Assume that q is a regular pretopology and there is n ϵ N such that $CL^{(n+1)}(A)$ = $CL^{(n)}(A)$ for each A $\subseteq X$ .Then $CL^{(n)}(A)$ is a q – closed for each A $\subseteq X$. Since q is a regular pretopologgy , $\mathcal{U}(x)$ = CL($\mathcal{U}(x)$) for each $x\epsilon$X . Let u $\epsilon\,\mathcal{U}(x)$.To show that q is strongly regular it suffice to find a v $\epsilon\,\mathcal{U}(x)$ such that v is q–closed and v $\subseteq u$ .The set u ϵ CL($\mathcal{U}(x)$) , so there is a $v_1 \epsilon\, \mathcal{U}(x)$ such that CL(v_1) $\subset u$. The set v_1 ϵ CL($\mathcal{U}(x)$) so there is a v_2 $\epsilon\,\mathcal{U}(x)$, such that CL(v_2) $\subset$ v_1 . Thus , CL^2(v_2) $\subseteq u$ repeating this argument n times

shows that there is a $v_n \in \mathcal{U}(x)$ such that $CL^n(v_n) \subseteq u$. But $CL^n(v_n)$ is q - closed and the proof is complete.

Theorem 3.2.11 :

Let (X, q) be a convergence space and X is a finite set .Then

a) (X, q) is a pretopological space

b) Strong regularity , regularity and $\prod$ - regularity are equivalent

c) Weak regularity and t – regularity are equivalent.[5]

Proof :

a) Since the set of all filters on a finite set is finite . Then we have $\mathcal{U}(x) = \cap\{\mathcal{F} : \mathcal{F} \in q(x)\} \in q(x)$, $\forall x \in X$. Hence (X , q) is a pretopological space .

b) Regularity and $\prod$ - regularity are equivalent by theorem 3.2.6.

Since X is finite , there is $n \in N$ such that $CL^{(n)} = CL^{(n+1)}$. Hence regularity implies strong regularity by theorem 3.2.10(b) . But strong regularity implies regularity by theorem 3.2.10(a) .

c) t-regularity implies weak regularity by theorem 3.2.8 .

Conversely , assume that q is weakly regular . Let A be a q – closed set and x an element of X\A .To show that x and A can be separated by disjoint q – open sets it is sufficient to show that A is q – open .

For each $z \in X \backslash A$ there exists $A_z \in \mathcal{U}(A)$ and a $B_z \in \mathcal{U}(z)$ such that $B_z \cap A_z = \emptyset$.

Claim : $\cap \{A_z : z \in X \backslash A\} = A$.

Since $A_z \in \mathcal{U}(A)$, then $A_z \in \mathcal{U}(x)$ for all $x \epsilon A$, and hence $x \in A_z$ for all $x \in A$, so $A \subseteq A_z$. Hence $A \subseteq \cap \{A_z : z \in X \backslash A\}$.

If $\exists x \epsilon X$ such that $x \in \cap \{A_z : z \in X \backslash A\}$ and $x \in X \backslash A$, then $x \in A_x$ and $x \in B_x \in \mathcal{U}(x)$ which is a contradiction as $B_x \cap A_x = \emptyset$. Hence $\cap \{A_z : z \epsilon X \backslash A\} \subseteq A$.

The set $X \backslash A$ is finite and so $\cap \{A_z : z \in X \backslash A\} \in \mathcal{U}(A)$. Hence $[A] \subseteq \mathcal{U}(A)$. But $\mathcal{U}(A) \subseteq [A]$, so $\mathcal{U}(A) = [A]$ and A is q – open as $A \in \mathcal{U}(x)$for all $x \epsilon A$.

The following is an example of a pretopological space which is t – regular and weakly regular but not strongly regular . Hence not $\prod$ - regular and not regular .

Example 3.2.1 :

Let X = {a, b, c} and let q be a pretopology with neighborhood filters defined as follows .

$\mathcal{U}(a)$ = { {a, b} , X} , $\mathcal{U}(b)$= { {b, c} , X}, and $\mathcal{U}(c)$= { {a, c} , X}.[5]

The only q – closed sets are X and $\emptyset$. Hence , q_t is the indiscrete topology , q is t – regular . Thus q is also weakly regular by theorem 3.2.11.

Since no q – neighbourhood filter has a base of q – closed sets , q is not strongly regular .Thus , q is also not $\prod$ - regular and not regular by theorem 3.2.11.

The following example is an example of a convergence space which is strongly regular and t – regular but not regular .

Example 3.2.2 [5] :

let q be the convergence structure as in example 1.1.1 , for each $x \in$ A , $\mathcal{U}(x)$= [A] , and for each $x \in$ X\A , $\mathcal{U}(x)$ = $[x]$.

If D is any non empty subset of A , then CL(D) = A , and if D $\subseteq X \setminus A$ then CL(D) = D . Since A and any subset of X\A is q – closed , it follows that q is strongly regular , and thus weakly regular and $\prod$ - regular .

The set A and any subset of X\A are q – open , as well as , q – closed . Thus , $q_t = \Pi(q)$ and q is t – regular .

If $x \in$ A , then $[x] \in q(x)$. But CL($[x]$) = [A] $\notin q(x)$. therefore , q is not regular .

The following is an example of a pretopological space which is not strongly regular , not $\prod$ - regular and not regular . But it is weakly regular and not t – regular .

Example 3.2.3 :

Let X = {a} ∪ {x_n :$n \in$Z} and let q be the pretopology with neighbourhood filters defined as follows :

$\mathcal{U}(a)$ = [B$\subseteq X$: a $\in$ B and X\B is finite]

$\mathcal{U}(x_n)$ = [{ x_{n-1} ,x_n ,x_{n+1}}] ,[5]

{ x_{n-1} , x_n , x_{n+1}} is not q – closed because $x_{n+2} \in$ CL({ x_{n-1} , x_n , x_{n+1}}).

Hence , q is not strongly regular , not regular and not $\prod$ - regular .

CL({a}) = {a} so{a} is q – closed .

$a \in$ CL ({x_n : n $\in$Z}) and if B ⊂X such that ∃ $x_n \notin$ B , x_{n-1} or $x_{n+1} \in$ B , then $x_n \in$ CL(B).

Thus , the only q – closed sets in X are {a} , X and $\emptyset$.

q is weakly regular since $\mathcal{U}(a)$ and $\mathcal{U}(x_n)$, n ϵ Z are disjoint as $\{a\} \cup (\{ x_n : n \epsilon Z \} \backslash \{x_{n-1}, x_n, x_{n+1}\}) \epsilon\, \mathcal{U}(a)$ and $\{x_{n-1}, x_n, x_{n+1}\} \epsilon\, \mathcal{U}(x_n)$

q is not t – regular as the only q – closed sets are {a} , X and $\emptyset$.

Theorem 3.2.12 :

Let (X, q) be a convergence space and $A \subset X$. Then

a) If q is strongly regular , then q_A is strongly regular.

b) If q is $\prod - regular$, then q_A is $\prod - regular$

c) Let $t(q_A) = [q_t]_A$. If q is t – regular , then q_A is t – regular

d) Let $t(q_A) = [q_t]_A$. If q is weakly regular , then q_A is weakly regular.[5]

Proof :

a) Let $x \,\epsilon$ A and v ϵ $\mathcal{U}_{qA}(x)$. Then , v = $u \cap$ A for some u ϵ $\mathcal{U}_q(x)$. Since q is strongly regular , there is q – closed set F ϵ $\mathcal{U}_q(x)$ such that $F \subseteq u$.

The set $F \cap A$ is q_A – closed because $F \cap A$ is $[q_t]_A$ closed and $t(q_A) \geq [q_t]_A$. The set $F \cap A \in \mathcal{U}_{qA}(x)$ and $F \cap A \subseteq v$. Therefore , $\mathcal{U}_{qA}(x)$ has a base of q_A – closed sets.

b) if q is $\prod - regular$, then $\Pi(q)$ is regular by corollary 3.2.4. Therefore , $[\Pi(q)]_A$ is regular by corollary 3.2.1 . But $[\Pi(q)]_A = (\Pi(q_A))$ .Therefore q_A is $\prod - regular$ by corollary 3.2.4.

c) Follows from the heredity of regularity for topologies when $t(q_A) = [q_t]_A$.

d) Let B be a q_A – closed subset of A and let $x \in A \backslash B$ .Then $B = A \cap F$ for some q – closed set F , because $t(q_A) = [q_t]_A$. $x \notin F$ since $x \notin B$, therefore $\mathcal{U}_q(x)$ and $\mathcal{U}_q(F)$ are disjoint filters .

But $\mathcal{U}_{qA}(x) = \{u \cap A : u \in \mathcal{U}_q(x)\}$ and $\mathcal{U}_{qA}(B) = \{u \cap A : u \in \mathcal{U}_q(B)\}$

Therefore , $\mathcal{U}_{qA}(x)$ and $\mathcal{U}_{qA}(B)$ are disjoint .

Remark 3.2.1 :

weak regularity and t – regularity are not hereditary properties . If we let (X , q) be the pretopological space of example 3.2.1 and A = {a , b}. The topology q_t is indiscrete , so q is t – regular and weakly regular . The neighborhood filters of q_A are given by $\mathcal{U}_{qA}(a)=[A]$ and $\mathcal{U}_{qA}(b)= [b]_A$. Therefore , $\emptyset$, {a} and A are the only q_A – closed sets.

Hence the topology $t(q_A)$ is not regular . Thus , q_A is not t – regular and by theorem 3.2.11 q_A is not weakly regular .

Chapter Four

Compactness in Convergence Spaces

Chapter Four

Compactness in Convergence Spaces

Compactness is one of the most important topological property . It is an important notion in convergence space as well .

Definition 4.1:

A convergence space (X , P) is compact if every ultrafilter on X converges in X .[16]

Definition 4.2:

A system ℓ of nonempty subsets of a convergence space (X , P) is called a covering system if each convergent filter on X contains some elements of ℓ .[16]

Theorem 4.1:

Let (X , P) be a convergence space . Then , the following are equivalent :

a) (X , P) is compact .

b) Every filter on X has a point of adherence .

c) In every covering system there are finitely many members of which the union is X .[16]

Proof :

a implies b: We use the fact that for every filter $\mathcal{F}$ on X there is an ultrafilter $\mathcal{G}$ on X with $\mathcal{G} \geq \mathcal{F}$. So , any limit of $\mathcal{G}$ is an adherent to $\mathcal{F}$.

b implies c: Let ℓ be a covering system allowing no finite subcover . Hence $\{X \setminus S : S \epsilon \ell\}$ generates a filter , say $\mathcal{F}$ on X . So , by our hypothesis $\mathcal{F}$ has an adherence point say x .

Then $\exists \mathcal{G} \epsilon F(X)$ such that $\mathcal{G} \geq \mathcal{F}$ and $\mathcal{G} \epsilon P(x)$ for some $x \in$X .

By definition 4.2 , $\exists S \epsilon \ell$ such that $S \epsilon \mathcal{G}$.

$S \cap X \setminus S = \emptyset \in \mathcal{G}$ which is a contradiction . So , in every covering system there are finitely many members of which their union is X .

c implies a: Assume that some ultrafilters on X , say $\mathcal{G}$ doesn't converge in X , then $\mathcal{G}$ cannot be finer than any convergent filter $\mathcal{F}$. Since for any $M \subseteq X$ either M or $X \setminus M$ belongs to $\mathcal{G}$. So we find in any convergent filter $\mathcal{F}$ a member $M_{\mathcal{F}} \in \mathcal{F}$ for which X\ $M_{\mathcal{F}}$ belongs to $\mathcal{G}$. The system $\{M_{\mathcal{F}} | \mathcal{F}$ is convergent in X$\}$ is a covering system of X. So , there exists finitely many members of this system that covers X , then $\mathcal{G}$ would have to contain the empty set . So , every ultrafilter on X is a convergent filter . Hence , (X , P) is compact .

Theorem 4.2:

Let X be a set equipped with two convergence structures P and q such that $P \leq q$.Then , if (X , q) is compact , then(X , P) is compact .[1]

Proof:,

Since $a_q(\mathcal{F}) \subseteq a_P(\mathcal{F})$ for all $\mathcal{F} \epsilon F(X)$ and (X , q) is compact we get $a_q(\mathcal{F}) \neq \emptyset$. Hence , $a_p(\mathcal{F}) \neq \emptyset$ so by theorem 4.1 we get that P is compact .

Theorem 4.3:

Let (X , P) be a compact and Hausdroff pretopological convergence space .Then , a filter $\mathcal{F} \in$ F(X) converges in X if and only $a_P(\mathcal{F})$ is a singleton set .

Proof:

Since every pretopological space is a pseudotopological space, $\{x\} = a(\mathcal{F})$ if and only if $\mathcal{F} \epsilon\, P(x)$ by theorem 3.1.8 .

Theorem 4.4:

Let (X , P) be a compact pretopological space and (X , q) be a Hausdorff convergence space . If q ≤ P , then P= q. [1]

Proof:

(X , P) is Hausdorff , by theorem 3.1.4 .

$a_p(\mathcal{F}) \subseteq a_q(\mathcal{F})$, $\forall \mathcal{F} \,\epsilon\, F(X)$, by theorem 1.3.3 .

Let $\mathcal{F} \,\epsilon\, q(x)$ then $a_q(\mathcal{F}) = \{x\}$ by theorem 3.1.1 .

Since P is compact and $a_p(\mathcal{F}) \subseteq a_q(\mathcal{F}) = \{x\}$ we have $a_p(\mathcal{F}) = \{x\}$. Hence $\mathcal{F} \epsilon P(x)$. Thus for each $x \epsilon X$ we have $q(x) \subseteq P(x)$ which means P ≤ q . Therefore , q = P .

Corollary 4.1 :

Let (X , P) be a compact topological space and q be a Hausdorff convergence structure on X such that q ≤ P , then P = q . In particular , q is a topology .

Proof :

Follows by theorem 4.4 .

Note that theorems 4.3 , 4.4 hold if we replace pretopology by pseudotopology .

Definition 4.3 :

A subset of a convergence space is compact if it is compact with respect to the subspace convergence structure .[3]

Theorem 4.5:

Let (X , P) be a convergence space and A ⊆ X be a subspace . Then , the following hold .

1) If X is compact and A is P-closed , then A is compact .
2) If X is Hausdorff space and A is compact , then A is closed .[3]

Proof:

1) We know that a filter $\mathcal{F}$ converges to $a \epsilon A$ in A if and only if $[\mathcal{F}]_X$ (the filter generated by the filter base $\mathcal{F}$ in X) converges to a in X .

   Let $\mathcal{F}$ be an ultrafilter in A then $[\mathcal{F}]_X$ is an ultrafilter in X which converges in X as X is compact .

   Assume that $a([\mathcal{F}]_X) \cap A = \emptyset$ so $a([\mathcal{F}]_X) \subseteq X \setminus A$. Thus, $X \setminus A \epsilon [\mathcal{F}]_X$ as $X \setminus A$ is P-open and $lim[\mathcal{F}]_X = a([\mathcal{F}]_X) \neq \emptyset$.

   But $A \epsilon [\mathcal{F}]_X$ so $(X \setminus A) \cap A = \emptyset \epsilon [\mathcal{F}]_X$ which is a contradiction . This implies that $\exists a \epsilon A$ such that $[\mathcal{F}]_X$ converges to a . Hence , $\mathcal{F}$ converges to a . Thus A is compact .

2) Let $x \epsilon CL(A)$ then $\exists\ \mathcal{F} \epsilon P(x)$ such that A ∈ $\mathcal{F}$.

   $\mathcal{F}_A = \{A \cap F: F \epsilon \mathcal{F}\}$ is a filter in A .

   Let $\mathcal{G}$ be the ultrafilter in A containing $\mathcal{F}_A$. So $\mathcal{G}$ converges to some $y \epsilon A$. But $[\mathcal{G}]_X$ is an ultrafilter converges to y and $[\mathcal{G}]_X \geq \mathcal{F}$ this leads $[\mathcal{G}]_X$ converges to x too .

   So x = y as X is a Hausdorff space . Thus , $x \epsilon A$. Therefore , $CL(A) = A$. So , A is P-closed .

Corollary 4.2 :

A subspace of a compact Hausdorff convergence space is compact if and only if it is closed .

Proof :

Follows by theorem 4.5 .

Theorem 4.6:

Let $f: X \to Y$ be a continuous surjective mapping from a compact convergence space X onto a convergence space Y . Then , Y is compact .

Proof:

Let $\mathcal{G}$ be an ultrafilter on Y, then $\{f^{-1}(G): G \epsilon \mathcal{G}\}$ is a basis of a filter $\mathcal{F}$ on X . Choose a finer ultrafilter $\mathcal{H} \geq \mathcal{F}$.

Then $\mathcal{H}$ converges and $f(\mathcal{H}) \geq \mathcal{G}$. Since $\mathcal{G}$ is an ultrafilter we get $\mathcal{G} = f(\mathcal{H})$. Since f is continuous , then $\mathcal{G}$ converges in Y . Therefore , Y is compact .

Corollary 4.3 :

Let X and Y be any convergence spaces . If $f: X \to Y$ is a continuous mapping , then the image of a compact subset of X is compact in Y .

Proof :

Let $A \subseteq X$ be a compact set .The restriction mapping $f_A: A \to f(A)$ is continuous . Hence by theorem 4.6 $f(A)$ is compact .

Theorem 4.7:

Let X and Y be convergence spaces and $f: X \to Y$ be a continuous where X is compact and Y is Hausdorff . Then , if A is a closed set in X , then $f(A)$ is a closed set in Y .

Proof :

A closed subset A of X is compact by theorem 4.5(1) , $f(A)$ is compact by corollary 4.3 so $f(A)$ is closed by theorem 4.5(2) .

Theorem 4.8:

Let $f: X \to Y$ be a continuous map from a compact convergence space X onto a Hausdorff convergence space Y .

If $B \subseteq Y$ is compact , then $f^{-1}(B)$ is compact . [4]

Proof :

Let $B \subseteq Y$ be compact then B is closed set in Y by theorem 4.5 (2). $f^{-1}(B)$ is closed since f is continuous . $f^{-1}(B)$ is compact by theorem 4.5(1).

Theorem 4.9:

If P and q are convergence structures on the set X , with q is Hausdorff , P is compact and q ≤ P .Then

1) $q(x) \cap UF(X) = P(x) \cap UF(X)$ .where U(F(X) is the set of ultrafilters on X.
2) The Pretopologies associated to q and P are identical .
3) The topologies associated to q and P are identical . [18]

Proof:

1) $P(x) \subseteq q(x)$ as q ≤ P . Hence , $P(x) \cap UF(X) \subseteq q(x) \cap UF(X)$. Let $\mathcal{F} \epsilon\, q(x) \cap UF(X)$ this means that $\mathcal{F}$ is an ultrafilter converges to x .

Since q is Hausdorff then $lim_q\mathcal{F} = a_q(\mathcal{F}) = \{x\}$ by theorem 3.1.1(b) . P is compact so $a_P(\mathcal{F}) \neq \emptyset$ and since $a_P(\mathcal{F}) \subseteq a_q(\mathcal{F}) = \{x\}$ by theorem1.3.3 we get

$lim_P\mathcal{F} = a_P(\mathcal{F}) = \{x\}$. Hence , $\mathcal{F} \in P(x)$.

$Thus$, $q(x) \cap UF(X) \subseteq P(x) \cap UF(X)$.

2) Since each filter $\mathcal{F}$ is the intersection of all ultrafilters finer than $\mathcal{F}$ and by part1 of this theorem we get $\mathcal{U}_P(x) = \mathcal{U}_q(x)$, $\forall x \epsilon X$. Hence , $\Pi(q) = \Pi(P)$.

3) Since the closure operators of q and $\Pi(q)$ are the same , by Corollary 1.2.1 and $\Pi(q) = \Pi(P)$ we get $CL_q(A) = CL_P(A)$ for all $A \subseteq X$. Hence the set of all closed sets in (X , q) and (X , P) are the same . Therefore , $q_t = P_t$.

Definition 4.3 :

A subset A of a convergence space X is called relatively compact if its closure $CL(A)$ is compact . [3]

Theorem 4.10:

Let X and Y be convergence spaces . Let Y be Hausdorff and let $A \subseteq X$ be a relatively compact set . If $f: X \to Y$ is a continuous mapping then $f(A)$ is relatively compact .

Proof:

Assume that A is relatively compact . Then , $CL(A)$ is compact. $f\big(CL(A)\big)$ is compact and closed by theorems 4.6 and 4.5 .

Now $A \subseteq CL(A)$ then $f(A) \subseteq f\big(CL(A)\big)$.

$$So\ CL(f(A)) \subseteq CL\left(f(CL(A))\right).\ CL(f(A)) \subseteq f(CL(A))$$

$as\ f(CL(A))$ is closed .

$f(CL(A)) \subseteq CL(f(A))\ by$ theorem 2.1.6. Hence $CL(f(A)) = f(CL(A))$.

$CL(f(A))\ is$ compact . So $f(A)$ is relatively compact .

Definition 4.4 :

A convergence space (X , P) is said to be locally compact if each convergent filter contains a compact set .[25]

Lemma 4.1 :

Let $\mathcal{G}$ be a filter on the set X , and let $\{\mathcal{F}_\propto : \propto \in \Delta\}$ denote the family of all ultrafilters finer than $\mathcal{G}$. For each $\propto \in \Delta$, choose $F_\propto \epsilon \mathcal{F}_\propto$. Then there is a finite subset $\{\propto_1, \dots \dots, \propto_2\}$ of Δ such that $\cup \{F_{\propto i} : i = 1, \dots ., n\} \epsilon\ \mathcal{G}$.[25]

Proof:

If the assertion were false then the collection of all sets of the from $H \setminus (\cup \{F_\propto : i = 1, \dots ., n\})$, for $H \epsilon \mathcal{G}$, would constitute a filterbase $\mathfrak{P}$ with the property that no ultrafilter containing $\mathfrak{P}$ could be finer than $\mathcal{G}$, which is a contradiction .

Theorem 4.11:

A convergence space (X , P) is locally compact if and only if each convergent ultrafilter contains a compact set .[25]

Proof :

Suppose that each convergent ultrafilter contains a compact subset , and let $\mathcal{G}$ be any filter converging to x in X , Let $\{\mathcal{F}_\propto : \propto \in \Delta\}$ be the family of all ultrafilters finer than $\mathcal{G}$. For each ultrafilter $\mathcal{F}_\propto$, choose a compact subset $F_{\propto}$. By lemma 4.1 , $\mathcal{G}$ contains a compact subset . Thus, X is locally compact .

Theorem 4.12:

(Tychonoff) The product $\prod(X_i\ ,\ q_i)$ is compact if and only if each $(X_i\ ,\ q_i)$ is compact . [1] .

Proof :

Let (X , q) $= \prod(X_i\ ,\ q_i)$. $X_i = P_{ri}(X)$ and P_{ri} is continuous for each i . Hence , from compactness of (X , q) , follows the compactness of each $(X_i\ ,\ q_i)_{i\epsilon I}$. Let $\mathcal{U}$ be an ultrafilter on X .$P_{ri}(\mathcal{U})$ is an ultrafilter for each i . Hence , by hypothesis it converges , so $\mathcal{U}$ converges in X . Hence , (X , q) is compact .

Chapter Five

Connectedness in Convergence Spaces

Chapter Five

Connectedness in Convergence Spaces

Connectedness is a topological property for which the definition may be extended to convergence spaces . It is known that a topological space is connected if and only if each continuous function from the space onto a discrete space is constant .

Definition5.1 :

A convergence space (X , q) is connected if the only continuous functions from (X , q) onto a discrete topological space are constant functions.[5]

For subsets of a convergence space , connectedness is defined in a manner analogous to the topological definition . In this section we denote the discrete topological space by (T , d) .

Definition 5.2:

If (X , q) is a convergence space and $A \subseteq X$. Then A is a q-connected subset of X if (A , q_A) is a connected convergence space.[5]

Theorem5.1 :

Let (X , q) and (X , P) be convergence spaces with $P \geq q$. If (X , P) is connected , then (X , q) is connected . Furthermore , if A is a P-connected subset of X , then A is a q-connected subset of X . [5]

Proof :-

To prove the first assertion , assume that $f : (X , q) \rightarrow (T , d)$ is a continuous function , then $f : (X , P) \rightarrow (T , d)$ is continuous by theorem 2.1.2 and $P \geq q$. Hence f is constant as (X , P) is connected .

This means that (X , q) is a connected convergence space .

The second assertion follows , because $P_A \geq q_A$ whenever $P \geq q$.

Theorem5.2:

Let (X , q) be a convergence space . Then , the following statements are equivalent :

a) (X , q) is connected .

b) (X , $\Pi(q)$)) is connected .

c) (X , q_t) is connected .[5]

Proof -

(a) is equivalent (c) Since a function f : (X , q) →(T , d) is continuous if and only if f : (X , q_t) →(T , d) is continuous , we get that (X , q) is connected if and only if (X , q_t) is connected . Since $q_t \leq \ \Pi(q) \leq q$ we get by theorem 5.1 that (b) is equivalent to (c) . Hence , (a) , (b) and (c) are equivalent statements .

The next theorem shows that the continuous image of a connected set is connected .

Theorem5.3:

Let f : (X , P) →(Y , q) be a continuous mapping from the connected convergence space (X , P) onto the convergence space (Y , q) . Then , (Y , q) is connected .

Proof :

Let h : (Y , q) → (T , d) be a continuous map, then (h of) : (X , P) → (T , d) is continuous , and since f is onto we get that h is a constant function . Hence , (Y , q) is connected .

Definition 5.3:

Two nonempty and proper subsets A and B of X are separated in (X , q) if $CL(A) \cap B = A \cap CL(B) = \emptyset$. [10]

Theorem 5.4 :

Let (X , q) be a convergence space . Then , the following are equivalent :

a) (X , q) is connected .
b) There is no proper subset of X that is both q-open and q-closed .
c) X cannot be represented as the union of two disjoint q-open sets .
d) X cannot be represented as the union of two disjoint q-closed sets .
e) X cannot be represented as the union of two separated sets .[10]

Proof :-

Follows by theorem 5.2 and theorems from topology .

Theorem5.5 :

Let (X , q) be a convergence space and A ⊆ X , then A is q-connected if and only if A can not be written as the union of two separated sets in X , each of which has a nonempty intersection with A.[14]

Proof :-

Assume that A is q-connected and A = D ∪ B where $A \cap D \neq \emptyset$ and $A \cap B \neq \emptyset$ and $CL_q(D) \cap B = D \cap CL_q(B) = \emptyset$.

$CL_{q_A}(D) \cap B = D \cap CL_{q_A}(B) = \emptyset$ by theorem 2.2.3 .

But B and D are complementary sets relative to A . Hence , $CL_{qA}(D) \subseteq D$ and $CL_{qA}(B) \subseteq B$. Hence D and B are q_A-closed and q_A-open which is a contradiction as A is q-connected if and only if (A , q_A)

is connected if and only if there is no proper subsets of A that is both q_A-open and q_A-closed by theorem 5.4 .

Conversely , assume that A is not q-connected . Hence there is a nonempty proper subset B of A which is q_A-open and q_A-closed by theorem 5.4 and definition 5.2 . Hence , $CL_{qA}(B) = B$ and $CL_{qA}(A \setminus B) = A \setminus B$. Thus $CL_q(B) \cap (A \setminus B) = CL_q(A \setminus B) \cap B = \emptyset$. Hence , B and A\B are separated sets in X each of which has a nonempty intersection with A which is a contradiction of the assumption .Therefore , A is a q-connected .

Theorem5.6:

Let A be a q-connected subset of X and $B \subseteq X$ such that $A \subset B \subset CL_q(A)$. Then , B is q-connected .[14]

Proof:

Assume that $A \subset B \subset Cl_q(A)$ and that A is q-connected .

If B is not q-connected then there is a discrete space (T , d) and a function f from B onto T which is continuous with respect to q_B and d and which is not constant .

Let f_A be the restriction of f to A , then $f_A: (A , q_A) \to (f(A) , d_{f(A)})$ is continuous . The function f_A is constant because A is q-connected and $(f(A) , d_{f(A)})$ is a discrete space . Let $f(A) = \{s\}$.

The function f is not constant , so there is a $b \in B$ such that $f(b) \neq s$.

The point $b \in Cl_q(A)$; therefore , there is $\mathcal{F} \epsilon q(b)$ such that $A \epsilon \mathcal{F}$.

Let $\mathcal{B} = \{B \cap F : F \epsilon \mathcal{F}\}$. The collection $\mathcal{B}$ is a filterbase because $A \epsilon \mathcal{F}$ andA $\subseteq B$. The filter generated by $\mathcal{B}$ on X $= \mathcal{F}$. Hence , the filter generated by $\mathcal{B}$ on B , q_B-converges to b .

Let $\mathcal{H}$ denote the filter generated by $\mathcal{B}$ on B .

$f(\mathcal{H}) = [s]$ as the set $A \epsilon \mathcal{H}$. But $\mathcal{H} q_B$ -converges to b and $f(b) \neq s$ and the topology d is discrete , therefore $f(\mathcal{H})$ does not converge to $f(b)$.

This contradicts the continuity of f . Therefore , B is q-connected .

Theorem5.7 :

Let (X , q) be a convergence space and $A \subseteq X$. Then ,

a) If A is q-connected then A is q_t-connected .

b) If t(q_A) = $[q_t]_A$ and A is q_t –connected then A is q-connected .[5]

Proof:

a) Since $q \geq q_t$we get by theorem 5.1 that if A is q-connected , then A is q_t-connected .

b) If $[q_t]_A$ = t(q_A) and A is q_t-connected then (A , $[q_t]_A$) is connected , Hence (A , t(q_A)) is connected . Therefore , (A , q_A) is connected by theorem 5.2. Consequently , A is q-connected by definition 5.2

Theorem 5.8:

Let (X , q) be a convergence space and $A \subseteq X$. Then if A is q-connected , then $CL_{q_t}(A)$ is q-connected **.**

Proof:

If A is q-connected then A is q_t-connected by theorem 5.1 which implies that $CL_{q_t}(A)$ is q_t-connected by theorem 5.6 and since $CL_{q_t}(A)$ is closed we have $CL_{q_t}(A)$ is q-connected by theorem 5.7 (b) and theorems 2.2.7 , 2.2.8 .

The following example and theorem 5.1 show that in general the set of connected subsets of a convergence space may be strictly subset of the set of connected subsets of its topological modification space .

we can have $A \subset B \subset CL_{q_t}(A)$ and A is q-connected but B is not q-connected .

Example5.1:

Let X=$\{x_n : n \in Z\}$ and q be the pretopology with neighbourhood filters defined as follows : for each $n \in Z$, $\mathcal{U}_q(x_n)$ is the filter generated by $\{x_{n-1} , x_n , x_{n+1}\}$.[5]

1. The topology q_t is indiscrete .

Proof:

Let A be a nonempty q-open subset of X such that $A \neq X$. Then there exist $i \epsilon Z$ such that $x_i \epsilon A$ and x_{i-1} or $x_{i+1} \epsilon X \setminus A$. But A is q-open then $A \epsilon \mathcal{F}$, $\forall \mathcal{F} \geq \mathcal{U}(x_i)$. This implies that $\{x_{i-1} , x_i , x_{i+1}\} \subseteq$ A which is a contradiction as x_{i-1} or $x_{i+1} \in X \setminus A$. Hence , A cannot be q-open . Therefore , q_t is indiscrete topology .

2. Let A = $\{ x_n : n$ is an even integer$\}$. Then , q_A is the discrete topology on A

Proof:

Let $x_i \epsilon A$ then $\mathcal{F} \epsilon q_A(x_i)$ if and only if $[\mathcal{F}]_X \epsilon q(x_i)$.

Since A , $\{x_{i-1}, x_i , x_{i+1}\} \epsilon [\mathcal{F}]_x$ we have ,
$A \cap \{x_{i-1} , x_i , x_{i+1}\}=\{x_i\} \epsilon [\mathcal{F}]_x$ and this means $[\mathcal{F}]_X = [x_i]$ on X . Hence $\mathcal{F}= [x_i]$ on A . Hence (A , q_A) is the discrete topology .

3. $t(q_A) = q_A$ since q_A is the discrete topology on A . The topology $[q_t]_A$ is the indiscrete topology on A . So $[q_t]_A < t(q_A)$ and (A , $[q_t]_A$) is connected but (A , $t(q_A)$) is not connected . Hence , A is q_t-connected subset of X but not a q-connected subset of X.
4. If $B \subseteq A$ where B is the set containing One element . Then B is q-connected also $CL_{q_t}(B) = X$ is q-connected by theorem 5.8 and since q_t is an indiscrete topology . But $B \subset A \subset X = Cl_{q_t}(B)$. But A is not q-connected .

Theorem5.9 :

If H and K are separated in the convergence space (X , q) and E is a q-connected subsets of $H \cup K$. then , E $\subseteq H$ or E $\subseteq K$.

Proof:

Assume that E $\subseteq H \cup K$ and E $\nsubseteq H$ and E $\nsubseteq K$ then E $\cap H$ and E $\cap K$ are separated in (X , q) .

Since $CL_q(E \cap H) \cap (E \cap K) \subseteq CL_q(E) \cap CL_q(H) \cap E \cap K = \emptyset$.
And $(E \cap H) \cap CL_q(E \cap K) \subseteq (E \cap H) \cap CL_q(E) \cap CL_q(K) = \emptyset$ as H , K are separated .

But $(E \cap H) \cup (E \cap K) = E$ so E is not q-connected by theorem 5.5 which is a contradiction as E is q-connected . Hence , $E \subseteq H$ or $E \subseteq K$.

Theorem5.10 :

Let $\{C_{\propto} : \propto \epsilon \Delta\}$ be a family of connected subsets of the convergence space (X , q) . If for every $\propto, B \epsilon \, \Delta$, $C_{\propto} \cap C_B \neq \emptyset$, then $\cup_{\propto \epsilon \Delta} C_{\propto}$ is q-connected .

Proof:

Follows by theorem5.5 and 5.9 .

Definition 5.4:

A set K is a q-component of the convergence space (X , q) if K is a maximal q-connected subset of X .[5]

Since the union of two non-disjoint q-connected sets must be q-connected it follows that each element of X is in one and only one of q-component .

Hence , the q-components of X form a partition of X . This is also true of the q_t-components of X .

The following theorem shows that these two partitions are identical .

Theorem5.11 :

Let (X , q) be a convergence space , $x \, \epsilon X$ and let C denote the q_t-component containing x and let K denote the q-component containing x, then C =K .[5]

Proof:

The set K is q-connected and hence by theorem 5.1 , K is q_t-connected . Thus , $K \subseteq C$.

The set C is q_t-closed and hence it is q-closed and C is q_t-connected. Therefore , by theorem 5.7 (b) C is q-connected . Thus C $\subseteq K$.

Hence K = C .

Note that from theorem 5.11 we can conclude that q-components are q-closed .

Theorem5.12 :

If X is finite and (X , q) is weakly regular and if there exists a proper subset A of X which is q-closed , then (X , q) is disconnected .

Proof:

By the proof of part C of theorem 3.2.11 we get that A is q-open . Hence by theorem 5.4 (b) , (X , q) is disconnected .

Chapter Six

The Cluster Set of Functions in Convergence Spaces

Chapter Six

The Cluster Set of Functions in Convergence Spaces

Let X and Y be topological spaces and f be a map from the space X into the space Y , then the cluster set of f at $x \epsilon X$, denoted $\ell(f;x)$, is defined in [13] as $y \epsilon \ell(f;x)$ if there exists a filter $\mathcal{F}$ on X such that $\mathcal{F}$ converges to x and $f(\mathcal{F})$ converges to y .

In this chapter we generate the above definition into convergence spaces .

Definition 6.1:

Let f: (X , P) $\rightarrow$ (Y , q) be a map from a convergence space (X , P) into a convergence space (Y , q) . A point $y \epsilon Y$ is an element of the cluster set $\ell(f;x)$ of f at x if there exists a filter $\mathcal{F} \epsilon F(X)$ such that $\mathcal{F} \epsilon P(x)$ and $f(\mathcal{F}) \, \epsilon \, q(y)$.

It is clear that $\ell(f;x) \neq \emptyset$ $([x] \epsilon P(x) and\ f([x]) = [f(x)] \epsilon\ q\big(f(x)\big)$, so $f(x) \,\epsilon\, \ell(f;x))$.

Let $\ell^*(f;x)$ be the cluster set of f at x when P and q are replaced by the associated topologies P_t , q_t respectively . Then , it is clear that $\ell(f;x) \subseteq \ell^*(f;x)$. It is proved in [13] that $\ell^*(f;x)$ is a closed subset of (Y , q_t) . Also it's known that q and q_t have the same closed sets , then $\ell^*(f;x)$ is a q-closed set in Y .

Theorem 6.1:

Let f: (X , P) $\rightarrow$ (Y , q) be a map from a convergence space (X , P) into a convergence space (Y , q) . Then the following are equivalent :

a) $y \epsilon \ell(f;x)$.

b) $\exists$ a filter $\mathcal{G} \epsilon F(Y)$ such that $\mathcal{G} \epsilon q(y)$ and $x \epsilon a\big(f^{-1}(\mathcal{G})\big)$ provided $f^{-1}(\mathcal{G})$ exists .

Proof :

a implies b: Let $y \epsilon \ell(f;x)$ then $\exists \mathcal{F} \epsilon F(X)$ such that $\mathcal{F} \epsilon P(x)$ and $f(\mathcal{F}) \epsilon q(y)$. Let $\mathcal{G} = f(\mathcal{F})$, clearly $f^{-1}(\mathcal{G}) \leq \mathcal{F}$ and exists .

Since $\mathcal{F} \epsilon P(x)$, we get $x \epsilon a\big(f^{-1}(\mathcal{G})\big)$.

b implies a: $Let\ x \epsilon a\big(f^{-1}(\mathcal{G})\big)$ then $\exists \mathcal{F} \epsilon P(x)$ such that $\mathcal{F} \geq f^{-1}(\mathcal{G})$.

It is clear that $f(\mathcal{F}) \geq \mathcal{G}$ and $f(\mathcal{F}) \epsilon q(y)$. Hence , $y \epsilon \ell(f;x)$.

Theorem 6.2:

Let f: (X , P) $\rightarrow$ (Y , q) be a map from a convergence space X into the convergence space Y . If $y \epsilon \ell(f;x)$, then $\exists \mathcal{F} \epsilon F(X)$ such that $\mathcal{F} \epsilon P(x)$ and $y \epsilon \cap_{F \epsilon \mathcal{F}} CL_q\big(f(F)\big)$.

Proof:

Let $y \epsilon \ell(f;x)$ then $\exists \mathcal{F} \epsilon F(X)$ such that $\mathcal{F} \epsilon P(x)$ and $f(\mathcal{F}) \epsilon q(y)$. Clearly $y \epsilon CL_q\big(f(F)\big)$, $\forall F \epsilon \mathcal{F}$. Thus , $y \epsilon \cap_{F \epsilon \mathcal{F}} CL_q\big(f(F)\big)$

Theorem 6.3:

Let f: (X , P) $\rightarrow$ (Y , q) be a map from a convergence space (X , P) into a pretopological space (Y , q) .Then the following are equivalent :

a) $y \epsilon \ell(f;x)$.

b) $\exists$ a filter $\mathcal{G} \epsilon F(Y)$ such that $\mathcal{G} \epsilon q(y)$ and $x \epsilon a\big(f^{-1}(\mathcal{G})\big)$ provided $f^{-1}(\mathcal{G})$ exists .

c) $\exists$ a filter $\mathcal{F} \epsilon F(X)$ such that $\mathcal{F} \epsilon P(x)$ and $y \epsilon \cap_{F \epsilon \mathcal{F}} CL_q\big(f(F)\big)$.

Proof:

a implies b follows by theorem 6.1.

b implies c follows by theorem 6.1 and 6.2 .

c implies a given $y \epsilon CL_q\big(f(F)\big)$, $\forall F\epsilon\mathcal{F}$, for some filter $\mathcal{F} \epsilon P(x)$. This implies that $\forall F\epsilon\mathcal{F}$, $\exists$ a filter $\mathcal{G}_F \epsilon F(Y)$ such that $\mathcal{G}_F \epsilon q(y)$ and $f(F) \epsilon \mathcal{G}_F$.

Let $\mathcal{H} = \wedge_{F\epsilon\mathcal{F}} \mathcal{G}_F$.

Let $\mathcal{K} = \mathcal{H} \vee f(\mathcal{F})$ then it is clear that $\mathcal{K}$ is well defined and $f^{-1}(\mathcal{K})$ too .

Let $\mathrm{M} = \mathcal{F} \vee f^{-1}(\mathcal{K})$ then it is clear that M is well defined and $f(\mathrm{M}) \geq K$.

Now since (X , q) is a pretopological then $\mathcal{H}\epsilon\, q(y)$. Thus $\mathcal{K}\epsilon\, q(y)$ and $f(\mathrm{M})\, \epsilon\, q(y)$ as $f(\mathrm{M}) \geq \mathcal{K}$.

$\mathrm{M} \epsilon P(x)$ as $\mathrm{M} \geq \mathcal{F}$ and $\mathcal{F} \epsilon P(x)$. Hence , $y\epsilon\ell(f;x)$.

Theorem 6.4:

Let $f\colon (\mathrm{X}, \mathrm{P}) \to (\mathrm{Y}, \mathrm{q})$ be a continuous map from a convergence space (X , P) into a Hausdorff space (Y , q) , then $\ell(f;x)$ is a singleton .

Proof :

Let $y \epsilon \ell(f;x)$ such that $f(x) \neq y$. Then $\exists\mathcal{F}\epsilon P(x)$ such that $f(\mathcal{F})\, \epsilon\, q(y)$. Since f is continuous , $f(\mathcal{F})\, \epsilon\, q\big(f(x)\big)$.

$q(y) \cap q\big(f(x)\big) \neq \emptyset$ which is a contradiction as q is Hausdorff. Therefore , $f(x) = y$.

Theorem 6.5:

Let $f\colon (\mathrm{X}, \mathrm{P}) \to (\mathrm{Y}, \mathrm{q})$ be a 1-1 map from a convergence space (X , P) into a compact pretopological convergence space (Y , q) . If $\ell(f;x)$ is singleton , then f is continuous at x .

Proof:

Suppose $\ell(f;x) = \{f(x)\}$. $Let\, \mathcal{F} \epsilon P(x)$. We have to prove that $f(\mathcal{F})\, \epsilon\, q(x)$. Since q is a pretopological structure , it is sufficient show that $f(\mathcal{F}) \geq \mathcal{U}_q\big(f(x)\big)$.

Let $u\, \epsilon\, \mathcal{U}_q\big(f(x)\big)$, we have to show that $\exists F\, \epsilon\, \mathcal{F}$ such that $f(F) \cap (Y|u) = \emptyset$. Assume that $f(F) \cap (Y|u) \neq \emptyset, \forall F \epsilon\, \mathcal{F}$.

Let $\mathcal{G}$ be a filter on Y generated by $\{f(F) \cap (Y|u) \colon F \epsilon\, \mathcal{F}\}$.

Since (Y , q) is compact , $a(\mathcal{G}) \neq \emptyset$. Let $y\, \epsilon\, a_q(\mathcal{G})$, then $\exists$ a filter $\mathcal{H}\, \epsilon\, q(y)$ such that $\mathcal{H} \geq \mathcal{G}$.

But $\mathcal{H} \geq \mathcal{U}_q(y)$. Thus G ∩V≠ $\emptyset$, $\forall v\, \epsilon\, \mathcal{U}_q(y)$ and $\forall G\, \epsilon\, \mathcal{G}$. Claim $y \neq f(x)$.

Suppose $y\ = f(x)$, this means that $(f(F) \cap Y|u) \cap u = \emptyset\, \epsilon\, \mathcal{H}$ which is impossible . Thus , $y \neq f(x)$.

Since $\mathcal{H} \geq \mathcal{G} \geq f(\mathcal{F})$, then $f^{-1}(\mathcal{H}) \geq \mathcal{F}$ as f is 1-1 .

This implies that $f^{-1}(\mathcal{H})\, \epsilon\, P(x)$. Since $f(f^{-1}(\mathcal{H}))\ \geq \mathcal{H}$ we have $f(f^{-1}(\mathcal{H}))\, \epsilon\, q(y)$. Hence , $y \epsilon \ell(f;x)$ which is a contradiction .

Thus $f(\mathcal{F})\, \epsilon\, q\big(f(x)\big)$ .Therefore , f is continuous .

Corollary 6.1:

Let $f\colon$(X , P) → (Y , q) be a 1-1 map from a convergence space (X , P) into a compact Hausdorff pretopological convergence space (Y , q) . Then , f is continuous if and only if $\ell(f;x)$ is singleton .

Proof :

Follows by theorem 6.4 and 6.5 .

References

1) H.R.Fischer , **Limesräume** , Math.Ann.137(1959),269-303.
2) Chang-Koolee , **Extentions of Convergence Spaces** , J.Korean Math.Soc.1982Vol . 19 , No . 1 , 47-54 .
3) R.Beattie and H.P. Butzmann , **"Convergence Structures and Applications to Functional Analysis"** . Published by Kluwer Academic Publishers in 2002.
4) G. Choquet, **Convergences**, Ann . Univ. Grenoble Sect . Sci . Math. Phys. (NS) 23 (1948) 57-112.
5) B.V.Hearsey , **On Extending Regularity and Other Topological Properties to Convergence Spaces**, Ph . D. Dissertation,Washington State University , Pullman , Washington , 1968.
6) E.T.Ordman . **Convergence Almost Everywhere is not Topological**.Amer Math. Monthly 73(1966,182-183).
7) Frédéric Mynard , **An Introduction to Convergence Spaces** , Georgia Southern University, 2005 .
8) SZYMON DOLECKI , **AN Initiation into Convergence Theory** , Mathematical Institute of Burgundy, Burgundy University, B.P. 47 870, 21078 DIJON, FRANCE.
9) B.V.Hearsey and D.C.Kent **, Convergence Structures ,** Portugaliae Mathematica,Vol. 31-Fasc. 2-1972**.**
10) BÄRBEL M . R STADLER , PETER F . STADLER , **Basic Properties of Filter Convergence Spaces** , Institute für Theoretische Chemie , Universität Wien , Währingerstraβe 17,A -1090Wien Austria.Eva.

11) H.kowalsky , **Limesräume and Komplettierung ,** Math , Nachr.12(1954),301,340.

12) Eva Lowen-Colebunders , **An Internal and An External Characterization of Convergence Spaces in which Adherences of Filters are Closed ,** Proc. Amer. Math. Soc. 72 (1978), 205-210.

13) T. R. Hamlett**, Cluster Sets in General Topology,** J. London Math. Soc. (2) 12 (1975/76), no. 2, 192–198 .

14) Paul E . Long , **"An Introduction to General Topology"**,Char E.Merrill Publishing Company , 1986.

15) D. C. Kent and G. D. Richardson **, Minimal Convergence Spaces ,** Trans. Amer. Math. Soc., 160:487–499.

16) E.Binz **,Continuous Convergence on C(X) , Lecture Notes in Mathematics,**469,Springer-Verlag,Berlin-Heidelberg-NewYork 1975.

17) J.F.Ramaley and O.Wyler **:Cauchy-Spaces II : Regular Completions and Compactifications** . Math. Ann.187,P.187-199

18) C.H.Cook . **Compact Pseudo-Convergences** . Math.Ann 202 , 193-202(1973).Springer–Verlag1973.

19) Biesterfeldt,H.J.Jr.**Regular Convergence Spaces** .Nederl.Akad. Wetensch.Ingat.Math.28(1966),605-607.

20) Cook ,C.H and H.R.Fischer.**Regular Convergence Spaces.** Math. Ann 174(1967,107).

21) Jin Won Park **. Regular Convergence Spaces.** Korean J. Math. 21 (2013), No. 4, pp. 375–382.4

22) G.D.Richardson and D.C.Kent , **Regular Compactifications of Convergence Spaces ,** Proc.Amer.Math.Soc.31(1972),571-573.

23) ----------**On Extending Topological Concepts to Convergence Spaces** . The University of New Mexico Technical Report No 39.August 1963.

24) Ĉech,E**, Topological Spaces,** Publishing House of the Czechoslovak Academy of Sciences , Inter science,JohnWiley 1966.

25) D.C.Kent and G.D.Richardson**.Locally Compact Convergence Spaces,** Michigan Mathematical Journal,Vol.22(1975),pp.353-360.

26) F. Hausdorff**, Grundzüge der Mengenlehre,** (Leipzig, 1914)

27) K. Kuratowski**, Sur l'opération Ā de l'analysis situs,** Fund. Math. 3 (1922) 182-199.

28) M. Frechet**, Sur quelques points du calcul fonctionnel,** Rend. Circ. Mat. Palermo 22 (1906) l-74.

29) P. Urysohn **, Sur les classes (L) de M.** Frechet, Enseign. Math. 25 (1926) 77-83.

30) H. Cartan**, Theorie des filtres,** C.R. Acad. Sci. Paris 205 (1937) 595-598.

31) C.H. Cook and H.R. Fischer**, On equicontinuity and continuous convergence ,** Math. Ann. 159(1965) 94-104.

32) D.C. Kent**, Convergence functions and their related topologies,** Fund. Math. 54 (1964) 125-133.

جامعة النجاح الوطنية
كلية الدراسات العليا

في نظرية الفضاءات المةقاربة

إعداد
رائد جمعة حسذ شقير

إشراف
د. محمد أبو عيدة

قدمت هذه الأطروحة استكمالا لمتط .ب.. الحصول على درجة الماجستير في الرياضيات، بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس- فلسطين.

2015

ب

في نظرية الفضاءات المتقاربة

إعداد

رائد جمعة حسن شقير

إشراف

مد .دمد أبو عيدة

الملخص

في هذه الرسالة قمنا بدراسة بعض خصائص الفضاء المتقارب من حيث المفهوم وعلاقته بالتبولوجيا . حيث تناولنا خصائص الانغلاق والانفتاح للمجموعات ونقاط التجمع للفلاتر .

وقد درسنا ايضا خاصية التجمع المغلق للفلاتر وكذلك مجموعة التجمع للاقترانات المعرفة على الفضاءات المتقاربة .

ثم تناولنا بعض خصائص مسلمات الانفصال في الفضاءات المتقاربة وتم ربط ذلك باتصال الاقترانات المعرفة بين الفضاءات المتقاربة . حيث تم دراسة بعض خصائص الترابط والتراص للفضاءات المتقاربة. و قد حصلنا على بعض النتائج المتعلقة بالمجموعات المتجمعة للاقترانات المعرفة على الفضاءات المتقاربة . وذلك بهدف تقديم افكار رئيسية للدارسين والباحثين في نظرية التقارب بطريقة توفر الجهد عليهم .