An-Najah National University

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An Analytic and Dynamic Programming Treatment for Solow and Ramsey Models

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Dedication

To my father Dr.Yaser Thabaineh.

To my mother T.Huda Fataftah.

To my brothers Ammar, Suhail, Amer, Najem and Saif.

To my sisters Nagham and Duha.

Acknowledgments

When I decided to join the master program of mathematics at An-Najah National University, it was with the hope that I would be able to work faithfully with several people. I am glad to assert that my hopes were done and I am very lucky to have practiced the partnership and friendship that I found in the department.

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Finally, I thank all who had provided me with advice, aid and guidance.

الإقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

An Analytic and Dynamic Programming Treatment for Solow and Ramsey Models

اقر بأن ما اشتملت عليه هذه الرسالة إنما هو نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وان هذه الرسالة ككل أو جزء منها لم يقدم من قبل لنيل أية درجة أو بحث علمي أو بحثي لدى أية مؤسسة تعليمية أو بحثية أخرى .

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's name:	**اسم الطالب:**
Signature:	**التوقيع:**
Date:	**التاريخ:**

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An Analytic and Dynamic Programming Treatment for Solow and Ramsey Models

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Abstract

In this thesis, we studied two of the most important exogenous economic growth models; Solow and Ramsey models and their effects in microeconomics by using dynamic programming techniques. Dynamic programming (DP) is a general approach to solve economic growth problems.

The main differences between Solow and Ramsey models are discussed in details. Bellman value function for the growth models is applied to the two models and an analytic formula are derived.

Concerning the models under study, we then discussed the steady states for the model and derived a closed formula for the capital. This formula was checked by computer using Python codes where a new concave assumed value function is given; $w = 2(k)^{0.25} - 35$, to be compared with a value function given by other $w = 5\log(k) - 25$. These two initial functions have the same properties of being monotone and concave up.

The comparison shows the excellence and advantages of our assumption. We reached the true value function faster.

Chapter One

Introduction

1.1 Overview

There is a wide literature on macroeconomics which studies the whole economy. Therefore, it is interested in a lot of questions in economics which need answers. Why are some countries richer than others? What are the reasons behind the high and/or low incomes of countries? Why people save? How much should he/she invest/save to keep a steady income? Macroeconomics may find some answers for these questions. Romer (1996) represents a text book of advanced macroeconomics [6]. He had discussed many economic models. An increase in the capacity of an economy to produce goods and services, compared from one period of time to another is known as an **economic dynamic growth**. Many researchers developed many economic growth models such as Ramsey (1928), Harrod (1939), Domar(1946), and Solow (1956). Harrod and Domar worked separately and developed an economic growth model named by Harrod-Domar model. It is used in economics development in order to clarify the rate of economic growth in terms of saving and productivity levels of capital. It argues that there no evidence for having a balanced growth of an economy. A drawback of this model is that it considers the development similar to the economic growth.

Economists interested in the study of economic growth that has practiced markedly in the history of economics. John von Neumann's growth model and Roy Harrods's trial to generalize Keynesian's growth model. Interest in the theory of economic growth calmed in 1970s and early 1980s, just few result were produced Following Solow (1956) and Kador (1961) papers from 1950s until the early of 1970s, growth theory became one of the central topics in economics. While in the middle of 1980s it seems to provide a new beginning for the economics of growth. Once again economic growth becomes a central topic in the theory of economics. The growth economics exploded after Solow's paper. Through 1960s the basic neoclassical growth model (Ramsey model) was extended in several directions by Hirofumi Uzawa, Kenneth Arrow, James Tobin, Peter Diamond and others. There are two types of economic growth models, exogenous growth models, like Ramsey and Solow models and endogenous growth models like Romer model.

The first idea of the exogenous growth model was introduced by Ramsey (1928) when he asked his famous questions (how much of its income should a native save?)[10]. As early as a new complex model of saving was determined by Ramsey. His contribution was theoretically and mathematically and did not have a response from economists until after thirty years. In the neoclassical growth theory, this model became important and the version of this model was finished by Cass (1965)[5] and Koopmans (1965)[24]. Therefore, this model is so called Ramsey-Cass-Koopmans model. It is one of the basic cornerstone models in macroeconomics. It consists of a finite number of completely alike agents with an infinite time horizon, i.e. it is a representative agent model[1].

[1] - Agents: Households and firms

Robert Solow expand the idea of Ramsey and published two articles "a contribution to the theory of economic growth"(1956) and "technical change and the aggregate production function"(1957), so the growth economics and economic theory attracting the attention of a significant part of the economics profession.

In 1956, Swan worked separately without knowing about Solow's work in the same field. So the model is named as Solow-Swan model. Solow and Swan turns to neoclassical production function with varying share of labor and capital input. This approach provides the first neoclassical model of long run economic growth and become the starting point for most studies on economic growth.

The new growth theory worked on the steady-state rate as an endogenous rate, i.e. the steady-state rate is determined within the model. This work was referred to David Warsh and Romer.

The main difference between exogenous and endogenous concepts in economy is that, exogenous model refers to some external factors that affect the production function such that A; the effectiveness of labor, also, these factors are given as constants. But endogenous models refer to internal factors that affected the production function such that capital and labor and these factors determined within the model and changed with time [4].

In our thesis, Ramsey and Solow are exogenous growth models, but we may sometimes use the concept (endogenous) if we are dealing with the internal factors that affected the model.

Dynamic programming is a general approach to solve economic growth problems. This method acts recursively like the routine of computer that calls itself, adding information to stack each time and stopped when it met specific conditions. Once stopped, it finds a solution by deleting information from the stack in the appropriate sequence. One of the important characteristics of dynamic programming is that the problem can be divided into stages. In order to find the next closest node to the origin, each stage contains a new problem to be solved. In some applications the stages are relevant to time and to get efficient solution of the problem we can solve the stages backwards in time, i.e. we go back from point in the future to point in the present, or we can solve it forwards.

The basic attributes that characterize dynamic programming problems are summarized as follows:

The problem divided into stages and there is a policy decision required at each stage. Moreover, each stage has a number of states associated with the beginning of that stage. These states are the possible conditions in which the system might be at that stage of the problem. In addition, the number of states may be either finite or infinite. The effect of the policy decision at each stage is to transform the current state to a state associated with the beginning of the next stage. The final step in the solution of the problem is to find an optimal policy for the whole problem [18].

There are two fields in dynamic programming, stochastic and deterministic dynamic programming. In deterministic dynamic programming, the state of the next stage is completely determined by the state and policy decision of the current stage.

The following figure summarizes the deterministic dynamic programming where:

N: number of stages.

n: label of current stage.

s_n : Current state of stage n.

x_n : Decision variable of stage n.

x_n^* : Optimal value of x_n .

$f_n(s_n, x_n)$:Contribution of stages n, n+1,….N to objective function if the system starts in state s_n at stage n.

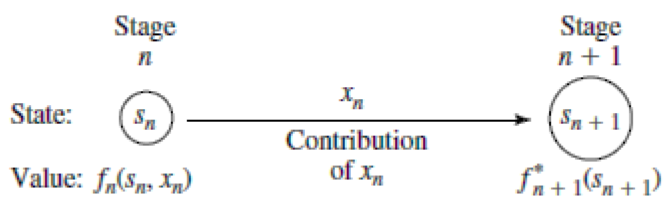

Figure (1.1): The structure of deterministic dynamic programming

$f_n^*(s_n) = f_n(s_n, x_n^*)$

The figure shows us the following:

At stage n, the process will be in some state s_n . If we make a policy decision x_n then the process will move to some state s_{n+1} at stage n+1. Therefore, the contribution to the objective function will be calculated to be $f_{n+1}^*(s_{n+1})$. Also, the policy decision x_n makes some contribution to the objective function. Lastly, the objective function may be used to minimize or maximize the sum of all individual states [29].

The father of dynamic programming is Richard Bellman [27]. In the sense of naming dynamic programming, he said "try thinking some combination that will possibly give it a pejorative meaning. It's impossible. Thus, I thought dynamic programming was a good name. It was something not even a congressman could object to. So I used it as an umbrella for my activities".

A Bellman equation which is also known as a dynamic programming equation is associated with the mathematical optimization methods. It writes the value of a decision problem at a certain point in time in terms of the reward from some initial choices and the value of the remaining decision problem that result from those initial choices. The mathematical style of the Bellman equation in our thesis is:

$$V(k_t) = \max_{c_t, k_{t+1}} \left\{ u(c_t) + \beta V(k_{t+1}) \right\}$$

Where $V(k_t)$, is called the value function.

In this thesis we aim to review a detailed Solow and Ramsey growth models. Furthermore, we aim to use dynamic programming in order to find an appropriate solution for Solow and Ramsey optimization problems. Moreover, it contains some Python codes to solve infinite dynamic programming horizon numerically. Python software is a rapidly maturing into one of the major programming languages and it is favored for many high technology companies.

1.2. Preliminaries

In this thesis we deal with many symbols, these symbols are used frequently.

1. The discount factor β : it is a number that takes value between zero and one. The fact that $\beta < 1$ means that the household cares a little more about current consumption than it cares about future consumption.
   $\beta = \frac{1}{1+r}$ where r is the discount rate.

   There is a calculated table that gives us the appropriate value of β that we shall use. See appendix II.
2. The ratio α: it is a statistical measurement for calculating returns. It is a measure of an investment's performance compared to a benchmark and it is a mathematical estimate of the return, based on the growth of earnings per share. It is also take a value between zero and one.
3. The multifactor productivity (A): it measures the change in output per unit of combined input. A is called technology and it has no unit.
4. The capital amount (K): it is the one of the cornerstone of the production function besides the labor force. It corresponds to the quantity of machines (equipment and structures).
5. The labor force (L): it is the total of employment. It can be measured in different ways; it corresponds to hours of employment or number of employees.
6. The Output (Y): it is the total amount of production of final good.
7. The saving rate (s): it is constant exogenous fraction which household save from their income.

8. Consumption (C): is the amount of goods that consumed, destroyed or used up by individuals, firms in a period of time.
9. Investment (I): it is the purchase of goods or units that are not consumed in the present, but are used in the future to create wealth. Or the amount of goods that used to generate an income in the future.

   In good market,

$$Y_t = C_t + I_t$$

Chapter Two

Models of Economic Growth

Economic growth and development are dynamic processes, focusing on how and why output, capital, consumption and population change over time. Therefore, the study of economic growth needs dynamical models. The most dynamical models which we are studying here are Solow and Ramsey models.

2.1 The Solow Growth Model

The Solow-Swan model named after Robert Solow and Trevor Swan, or simply the Solow model for the more famous of the two economists. Solow and Swan published two articles in the same year,1956, and they introduce the Solow model. After that, Solow developed many implications and applications of this model and awarded the Nobel prize in economics for his work (1987). This model shaped the way to approach the economic growth. So, The Solow growth model is the basic reference point for almost all analysis of growth. Before Solow model, the most common approach to economic growth was built on the model that developed by Harrod (1939) and Domar (1946). The Harrod-Domar model emphasized potential dysfunctional aspects of economic growth, as an **example** on this; how economic growth could go side by side with the increasing in unemployment. But this model in the opinion of Solow is not a good start in the economic growth. The basic difference between Solow model and Harrod-Domar model is that; the Solow model have the neoclassical production function. this function connectes Solow model with macroeconomics. Besides, Solow model is a simple and abstract representation of the economy. The basic assumptions of this model in the economy, there is a single good produced and there is no international trade. That is, the economy is closed to foreign goods and factor flows. Moreover, there is no government and all factor of production (i.e. capital and labor) assumed to be fully employed in the production process. It takes technological progress and investigates the effects of the division of output between consumption and investment on capital accumulation and growth.

The principal conclusion of the Solow model is that the accumulation of physical capital cannot account for either the vast growth over time in output per person or the vast geographic differences in output per person.

The Solow model has no optimization in it; it simply takes the saving rate as exogenous and constant.

Relaxing the Solow model's assumption of a constant saving rate has three advantages. First, and most important for studying growth, it demonstrates that the Solow model's conclusion about the central questions of growth theory do not hinge on its assumption of a fixed saving rate. Second, it allows us to consider welfare issues. Third, infinite and finite horizon models are used to study many issues in economics other than economic growth, thus they are valuable tools [6].

The Solow model can be formulated either in discrete or continuous time. In discrete time the work is simpler than in continuous time and it is more common in macroeconomic applications. In order to distinguish between the two versions of time, we use the notation (x_t) if we are dealing with discrete time and ($x(t)$) in continuous time.

2.1.1 Assumption in the Solow model:

a) Time:

The Solow model is a dynamic model, so the economic variables are evolves through time. The time is partitioned into periods, first period, next period…… and it denoted by a subscript (t). take some variable Y_t , this means the value of Y at the time t, similarly Y_{t+1} means the value of Y at the period (t+1) and it's the same for any variable which has t as a subscript.

b) Variables and parameters

There are five key variables in Solow model which are endogenous and dynamic, these variables are (defined related to time):

Y_t : output, income

K_t : capital

L_t : labor

I_t : investment

C_t : consumption

While the parameters in Solow model are exogenous and constant, these parameters are:

s: Saving rate (always between 0 & 1)

δ : Depreciation rate (always between 0 & 1)

In a good market we have the fact that:

$$Y_t = C_t + I_t$$

This means that the amount of income which produced in the economy and composed to people either consumed or invested.

We may write the investment and consumption in other way like,

I_t = sY_t and C_t = $(1\text{-}s)Y_t$. Consumption is something clear, households or firms use it and disappear, but what happened when we save something? Saving is invested into the capital stock, it is added to the capital K_t.

Example: let's take the capital to be the corn. Corn can be used for consumption and as input, as seeds, for the production of more corn tomorrow. So, capital corresponds to the amount of corn used as seeds for future production [4].

Let's ask a question, how the capital stock changes through time? The capital stock is different from one period to the next, when we want to know the capital stock for the next period we take the capital stock for the current period as exogenous and use it beside investment to find the capital for the next period using the equation $K_{t+1} = K_t - \delta K_t + I_t$, this equation is known as the capital accumulation equation.

Using the fact that I_t = sY_t , so we rewrite the equation as:

$$K_{t+1} = K_t - \delta K_t + sY_t \text{ ,}$$

$$K_{t+1} - K_t = sY_t - \delta K_t \tag{1}$$

This equation shows how much capital changes from one period to the next, if $\Delta K = K_{t+1} - K_t$ then $\Delta K = sY_t - \delta K_t$

c) Input and output in Solow model

The Solow model focuses on four variables: output (Y), capital (K), labor (L) and the "effectiveness of labor" (A). At any time, the economy has some amounts of capital, labor and knowledge; these are combining to produce output. The production function takes the form:

$$Y_t = F(K_t, A_t L_t) \tag{2}$$

Where, t, denote the time.

The effectiveness of labor, A; is a multifactor productivity which measures the output per unit of labor input. Also, it's the amount of goods and services that a worker produces in a given amount of time.

There are a lot of factors that affected the value of (A), like

- Physical-organic, location and technological factors.
- Levels of flexibility in internal labor market.
- Individual rewards and payment system.
- Economic and political-legal environment.

Two features of the production function should be noted. First, time does not enter the production function directly, but only through K, L, and A. that is, output change over time if the inputs into production change. Second, A and L enter multiplicatively.

The central assumption of the Solow model concerns the properties of the production function and the evolution of the three inputs into production (capital, labor and knowledge) over time.

d) The factor A is free; it is publicly available as a non-excludable (a good is non-excludable if it's impossible to preclude the individual from using or consuming it) and a non-rival good (a good in non-rival if it's consumption by others does not prevent my consumption). Also, A, is freely available to all potential firms in the economy and firms do not have to pay for using this factor.

2.1.2 Assumptions concerning the production function

The model's critical assumption concerning the production function is that it has constant returns to scale into its arguments, capital and effective labor. That is, doubling the quantities of the capital and effective labor doubles the amount produced. More generally, multiplying both arguments by any nonnegative constant c causes output to change by the same factor:

$$F(cK, cAL) = cF(K, AL) \quad \text{for all } c \geq 0 \tag{3}$$

The assumption of constant returns can be thought of as combining two assumptions. The first is that the economy is big enough that the gains from specialization have been exhausted. The second assumption is that inputs other than capital, labor, and knowledge are relatively unimportant. In particular, the model neglects land and other natural resources.

The assumption of constant returns allows us to work with the production function in intensive form. Setting c = 1/AL in equation (3) yields:

$$F\left(\frac{K}{AL}, 1\right) = \frac{1}{AL} F(K, AL) \tag{4}$$

K/AL is the amount of capital per unit of effective labor

F(K, AL)/AL is Y/AL , output per unit of effective labor

Define k = K/AL, y = Y/AL and f(k) = F(k, 1). Then we can rewrite (2) as

$$Y = f(k) \tag{5}$$

Let us consider the production function in terms of discrete time;

$$Y_t = F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha} \tag{6}$$

Let $k_t = \dfrac{K_t}{L_t}$, $y_t = \dfrac{Y_t}{L_t}$, $i_t = \dfrac{I_t}{L_t}$, $c_t = \dfrac{C_t}{L_t}$, yields the following

$$y_t = \frac{Y_t}{L_t} = \frac{F(K_t, L_t)}{L_t} = F\left(\frac{K_t}{L_t}, \frac{L_t}{L_t}\right) = F(k_t, 1) = f(k_t) \tag{7}$$

$$\text{But } y = f(k) = \frac{Y}{L} = \frac{AK_t^{\alpha} L_t^{1-\alpha}}{L} = AK_t^{\alpha} L_t^{-\alpha} = \frac{AK_t^{\alpha}}{L_t^{\alpha}} = A\left(\frac{K}{L}\right)^{\alpha} = Ak^{\alpha} \tag{8}$$

That is, we can write output per unit of effective labor as a function of capital per unit of effective labor [6].

Inputs are essential:

$$F(0,0) = F(k,0) = F(0,L) = 0 \tag{9}$$

The production function in Solow model tells us how economy works; we have capital, labor…, if we combine them together then we will get an economic growth. If we increase capital or labor, then the production function will increase (output increase) in a decreasing rate, so the shape of the production function is concave. If we have no capital or no labor, i.e if they destroyed from one period to another then the production is zero.

Marginal productivities are positive and decreasing.

$$\frac{\partial F}{\partial k}, \frac{\partial F}{\partial L} > 0 \tag{10}$$

$$\frac{\partial^2 F}{\partial k^2}, \frac{\partial^2 F}{\partial L^2} < 0 \tag{11}$$

The production function F satisfies the Inada conditions

1) $\lim_{k \to 0} F_k(K, AL) = \infty$ and $\lim_{k \to \infty} F_k(K, AL) = 0$ for all $L \succ 0$. This means that adding any other capital does not change the output.

2) $\lim_{L \to 0} F_L(K, AL) = \infty$ and $\lim_{L \to \infty} F_L(K, AL) = 0$ for all $K \succ 0$. This means that adding any other workers do not change the output.

Given the following **example** for a production function:

The Cobb-Douglas:

$$F(K, AL) = K^{\alpha}(AL)^{1-\alpha} = AK^{\alpha}L^{1-\alpha} \ , 0 < \alpha < 1 \tag{12}$$

To show that the Cobb-Douglas function has constant returns; multiplying both inputs by a constant c gives us

$$F(cK, cAL) = (cK)^{\alpha}(cAL)^{1-\alpha}$$

$$= c^{\alpha}c^{1-\alpha}K^{\alpha}(AL)^{1-\alpha}$$

$$= cF(K, AL) \tag{13}$$

F is linear homogenous since it is exhibits constant returns to scale in K and L, so F is concave.

To see that marginal productivities are positive:

$$\frac{\partial F}{\partial k} = \alpha k^{\alpha-1}(AL)^{1-\alpha} \ > 0 \tag{14}$$

$$\frac{\partial F}{\partial L} = (1-\alpha)k^{\alpha}A^{1-\alpha}L^{-\alpha} \ > 0 \tag{15}$$

It is natural that the level of capital and labor should be positive. So, multiplying positive items gives a positive result [28].

2.1.3 The evolution of the inputs into production

The remaining assumptions of the model concern how the stocks of labor, knowledge, and capital change over time. The model is set in continuous time; that is, the variables of the model are defined at every point in time.

The initial levels of capital, labor, and knowledge are taken as given. Labor and knowledge grow at constant rates:

$$\begin{aligned} \dot{L}(t) &= nL(t) \\ \dot{A}(t) &= gA(t) \end{aligned} \tag{16}$$

Where n and g are exogenous parameters and where a dot over a variable denotes a derivative with respect to time.

Output is divided between consumption and investment. The fraction of output devoted to investment, s, is exogenous and constant. One unit of output devoted to investment yields one unit of new capital. In addition, existing capital depreciates at rate δ. Thus:

$$\dot{k}(t) = sY(t) - \delta k(t) \tag{17}$$

Although no restrictions are placed on n, g, and δ individually, their sum; $(n + g + \delta)$ is assumed to be positive [6].

2.1.4 The low of motion for capital and labor:

Let's begin the work by given the following symbols,

C : consumption, I : Investment , Y : output , L : labor

First of all, we must always be sure that (at any time t), $C_t + I_t \leq Y_t$ while in a good market Y_t must equal C_t plus I_t .

If the population growth is n $\geq$ 0 per period, then the size of the labor force evolves over time as follows:

$L_t = (1+n)L_{t-1} = (1+n)L_o$, take $L_o = 1$

Suppose that existing capital depreciates over time at a fixed rate $\delta \in [0,1]$. The capital stock in the beginning of next period is given by the non-depreciated part of current-period capital plus contemporaneous investment

i.e $K_{t+1} = (1-\delta)K_t + I_t$ (18)

2.1.5 The steady state level for capital

A steady state for the economy is a value of capital per unit of labor,K^*, such that, if the economy has $K_0 = K^*$, then $K_t = K^*$ $\forall$ t > 1. This means that in the steady state $\Delta K = 0$

Now :

$\Delta K = sY_t - \delta K_t$, but $\Delta K = 0$

$0 = sY_t - \delta K_t$

But we show that $Y_t = AK^{\alpha}$

$0 = sAK^{\alpha} - \delta K_t$

Since we are in the steady state, $K = K^*$

$0 = sA(K^*)^{\alpha} - \delta K^*$, solving for K^*, we get that:

$$K^* = \left(\frac{sA}{\delta}\right)^{\frac{1}{1-\alpha}} \tag{19}$$

So, in the law of motion for capital , if $K_t = K^*$ then $\Delta K = 0$.[6]

2.2 The Ramsey-Cass-Koopmans Model

A new complex model of saving was determined by Ramsey (1928). His contribution did not have a response from economists until thirty years. The Ramsey growth model is a neoclassical model of economic growth based on work of the economist and mathematician Frank P. He discussed the problem of optimum saving assuming that we have a closed economy. Ramsey (1928) set out the model as a central planner's problem of maximizing level of consumption over successive generation. Later Ramsey model is adopted by researchers as a description of a decentralized dynamic economy. Cass (1965) and Koopmans elaborate the problem addressed by Ramsey with significant extension which called by Ramsey-Cass-Koopmans growth model [30].

Ramsey-Cass-Koopmans growth model is resembled to Solow model. However, the dynamics of aggregate economics can be determined by decisions at the macroeconomic level, in particular they make the decision how much of their income to consume in the current period and how much to save later. This model continued to treat the growth rates of labor and knowledge as exogenous. But from the interactions of maximizing households and firms form in competitive markets, this model can derive the evolution of the capital stock. Competitive firms ret capital and hire labor in order to produce and sell output, and a fixed number of infinitely households supply labor, hold capital, consume and save.

2.2.1 Assumption of Ramsey model

Firms

i. There are a large number of identical firms. Each has access to the production function Y= F (k, AL), which satisfies the same assumptions as the Solow Model.

ii. The firms hire workers and rent capital in competitive factor markets, and sell their output in a competitive output market.

iii. Firms take A as given; as in Solow model. A grows exogenously as rate g.

iv. The firms maximize profits. They are owned by the households, as any profits they earn accrue to the households.

Households

i. There are also a large number of identical households.

ii. The size of each household grows at rate n.

iii. Each member of the household supplies 1 unit of labor at every point in time.

iv. In addition, the household rents whatever capital it owns to firms. It has initial capital holdings of K(0)/H, where K(0)is the initial amount of capital in the economy and H is the number of households.

v. We assume that there is no depreciation.

vi. The household divides its income (from the labor and capital it supplies and, potentially, from the profits it receives from firms) at each point in time between consumption and saving so as to maximize its lifetime utility.

2.2.2 Utility function

The household's utility function takes the form

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt \,. \tag{20}$$

$C(t)$ is the consumption of each member of the household at time t. u(•) is the instantaneous utility function, which gives each member's utility at a given date. $L(t)$ is the total population of the economy; $\frac{L(t)}{H}$ is therefore the number of members of the household. Thus $u(C(t))\frac{L(t)}{H}$ is the household's total instantaneous utility at t. Finally, ρ is the discount rate.

The instantaneous utility function takes the form

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta}, \; \theta > 0, \;\; \rho - n - (1-\theta)g > 0. \tag{21}$$

This utility function is known as constant-relative-risk-aversion (or CRRA) utility. The reason for the name is that the coefficient of relative risk aversion (which is defined as $-Cu''(C)/u'(C)$) for this utility function is θ, and thus is independent of C.

In other words, $\theta = -Cu''(C)/u'(C)$ (22)

To see this,

$$u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta} \tag{23}$$

$$u'(C) = \frac{1}{C(t)^{\theta}} \tag{24}$$

$$u''(C) = \frac{-\theta C(t)^{\theta-1}}{C(t)^{2\theta}} \tag{25}$$

Now, $\frac{u^{''}(C)}{u'(C)} = \frac{-\theta C(t)^{\theta-1}}{C(t)^{\theta}} = -\theta C(t)^{-1} = \frac{-\theta}{C(t)}$ (26)

Multiply it by $-C(t)$ We get that:

$$\theta = -Cu^{''}(C)/u'(C). \quad (27)$$

Three features of the instantaneous utility function are worth mentioning. First, $C^{1-\theta}$ is increasing in C if θ > 1 but decreasing if θ < 1; dividing $C^{1-\theta}$ by $1-\theta$ thus ensure that the marginal utility of consumption is positive regardless of the value of θ. Second, in the special case of $\theta \to 1$, the instantaneous utility function simplifies to $\ln C$; this is often a useful case to consider. Third, the assumption that $\rho - n - (1-\theta)g > 0$ ensures that lifetime utility does not diverge; if this condition does not hold, the household can attain infinite lifetime utility, and its maximization problem does not have a well-defined solution.

2.2.3 The Behavior of Households and Firms

Firms

The behavior of firms is simple. At each point in time they employ the stocks of labor and capital, pay them their marginal product, and sell the resulting output. Firms earn zero profit; since the production function has constant returns and the economy is competitive.

The marginal product of capital, $\frac{dF(K,AL)}{dK} = f^{'}(k)$ as in Solow model, where $f(\bullet)$ is the intensive form of the production function. But markets are competitive, capital earns its marginal product and there is no depreciation, so the real rate of return on capital equals its earnings per unit of time. Thus the real interest rate at time t is:

$r(t) = f^{'}(k_t)$

The marginal product of labor is, $\frac{dF(K, AL)}{dL} = \frac{AdF(K, AL)}{dAL}$. If we want to deal with it in terms of $f(\bullet)$, it is $A[f(k) - kf^{'}(k)]$. The real wage at time t is:

$$W(t) = A(t)\left[f(k(t)) - k(t)f^{'}(k(t))\right] \quad (28)$$

The wage per unit of effective labor $\left(\frac{dW(t)}{dA(t)} = w(t)\right)$ is:

$$w(t) = f(k(t)) - k(t)f^{'}(k(t)) \quad (29)$$

Household's Budget Constraint (HBC)

The HBC is:

$$\int_{t=0}^{\infty} e^{-R(t)}C(t)\frac{L(t)}{H}dt \le \frac{K(0)}{H} + \int_{t=0}^{\infty} e^{-R(t)}W(t)\frac{L(t)}{H}dt \quad (30)$$

Where R(t) is defined as $\int_{\tau=0}^{t} r(\tau)d\tau$. One unit of the output good invested at time 0 yields $e^{R(t)}$ units of the good at t.

But this integral if difficult to be found, we express the budget constraint as:

$$\int_{t=0}^{\infty} e^{-R(t)}c(t)e^{(n+g)t}dt \le k(0) + \int_{t=0}^{\infty} e^{-R(t)}w(t)e^{(n+g)t}dt \quad (31)$$

Household's maximization problem

The representative household wants to maximize its lifetime utility subject to its budget constraint. As in Solow model, it is easier to work with variables normalized by the quantity of the effective labor. To do this, we need to express both the objective function and the budget constraint in terms of consumption and labor income per unit of effective labor.

Starting by the objective function, define c(t) to be the consumption per unit of effective labor. Thus C(t), consumption per worker, equals A(t)c(t). The household's instantaneous utility is therefore:

$$\frac{C(t)^{1-\theta}}{1-\theta} = \frac{[A(t)c(t)]^{1-\theta}}{1-\theta}$$

$$= \frac{[A(o)e^{gt}]^{1-\theta} c(t)^{1-\theta}}{1-\theta}$$

$$= A(o)^{1-\theta} e^{(1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta} \quad (32)$$

Substitute this result and the fact that $L(t) = L(0)e^{nt}$ into the household's objective function yields to:

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{C(t)^{1-\theta}}{1-\theta} \frac{L(t)}{H} dt$$

$$= \int_{t=0}^{\infty} e^{-\rho t} \left[A(o)^{1-\theta} e^{(1-\theta)gt} \frac{c(t)^{1-\theta}}{1-\theta} \right] \frac{L(o)e^{nt}}{H} dt$$

$$= A(o)^{1-\theta} \frac{L(o)}{H} \int_{t=0}^{\infty} e^{-\rho t} e^{(1-\theta)gt} e^{nt} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

$$= B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt \quad (33)$$

Where $B = A(o)^{1-\theta} \frac{L(o)}{H}$ and $\beta = \rho - n - (1-\theta)g \geq 0$ (34)

Household behavior

We must know that household chooses the path c(t) to maximize lifetime utility by $B\int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt$, in addition with the objective function and the budget constraint $\int_{t=0}^{\infty} e^{-R(t)} c(t) e^{(n+g)t} dt \leq k(0) + \int_{t=0}^{\infty} e^{-R(t)} w(t) e^{(n+g)t} dt$, we can use them to set up the Lagrangian:

$$L = B \int_{t=0}^{\infty} e^{-\beta t} \frac{c(t)^{1-\theta}}{1-\theta} dt + \lambda \left[k(0) + \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} w(t) dt - \int_{t=0}^{\infty} e^{-R(t)} e^{(n+g)t} c(t) dt \right] \quad (35)$$

The first-order condition for an individual c(t) is:

$$B e^{-\beta t} c(t)^{-\theta} = \lambda e^{-R(t)} e^{(n+g)t} \quad (36)$$

$$\ln B - \beta t - \theta \ln c(t) = \ln \lambda - R(t) + (n+g)t$$

$$= \ln \lambda - \int_{\tau=0}^{t} r(\tau) d\tau + (n+g)t$$

Where we use $R(t) = \int_{\tau=0}^{t} r(\tau) d\tau$. (37)

$$-\beta - \theta \frac{\dot{c}(t)}{c(t)} = -r(t) + (n+g)$$

$$\frac{\dot{c}(t)}{c(t)} = \frac{r(t) - n - g - \beta}{\theta}$$

$$= \frac{r(t) - \rho - \theta g}{\theta} \quad (38)$$

Where we use $\beta = \rho - n - (1-\theta)g$. (39)

2.2.4 The key equation of Ramsey model

There are two key equations of Ramsey model,

1. The law of motion for capital accumulation

$$\dot{k} = f(k) - \delta k - C \quad (40)$$

Where

k : capital per worker

$\dot{k}$: change in capital per worker over time

C: consumption per worker

δ : depreciation rate of capital

$f(k)$: output per worker

The equation (40) states the investment or increase in capital per worker, also it seems that, from equation (40), investment is the same as saving.

2. The second equation concerns the saving behavior of household.

   If households are maximizing their consumption, at each point in time they equate the marginal benefit of consumption today with that of consumption in the future.

2.2.5 Derivation of the equation of motion of consumption

We assume that our capital constraint is given by:

$$\frac{d}{dt}(k(t)) = f(k(t)) - c - \delta k(t). \tag{41}$$

This says that the rate of change of capital is given by the output $f(k(t))$ minus the consumption c minus capital depreciation.

Assume that we get utility from consumption u(c). This means that our current value Hamiltonian is given by:

$$H = u(c) + \lambda(f(k) - c - \delta k \tag{42}$$

Also, we now note that:

For the current value Hamiltonian, our first-order conditions are given by the expression below:

i. $$\frac{\partial H}{\partial k} = p.\lambda(t) - \frac{d}{dt}\lambda(t) \tag{43}$$

ii. $\frac{\partial H}{\partial c} = 0$ (44)

iii. $\frac{\partial H}{\partial \lambda} = \frac{d}{dt} k(t)$ (45)

Our first-order conditions are therefore given by

$\frac{\partial H}{\partial k} = p.\lambda(t) - \frac{d}{dt}\lambda(t)$ by (i)

$$\lambda\left(\frac{d}{dk} f(k) - \delta\right) = p.\lambda(t) - \frac{d}{dt}\lambda(t)$$

Solve for $\lambda(t)$ to get that

$$\frac{d}{dt}\lambda(t) = -\lambda\left(\frac{d}{dk} f(k) - \delta\right) + p.\lambda(t)$$

Divide both sides by $\lambda(t)$

$$\frac{\frac{d}{dt}\lambda(t)}{\lambda(t)} = -\left(\frac{d}{dk} f(k) - \delta\right) + p = -\frac{d}{dk} f(k) + \delta + p$$

But by (ii) we have that $\frac{\partial H}{\partial c} = o$, so we get the following

$\frac{d}{dc} u(c) - \lambda = 0$, so $\lambda = \frac{d}{dc} u(c)$

We see before that $u(c) = \frac{c^{1-\theta}}{1-\theta}$, for example if $\theta = 3$, the graph of $u(c)$ will be as

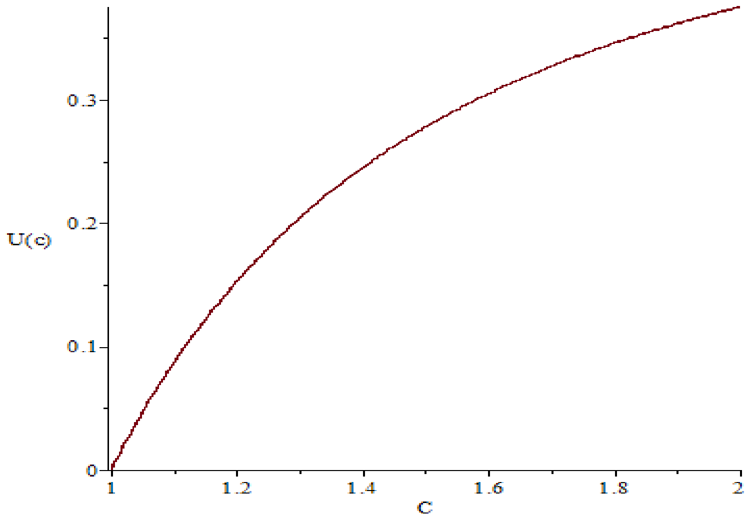

Figure (2.1): example of utility function.

The partial derivative of $u(c)$ with respect to c is therefore given by

$\frac{d}{dc}u(c)=c^{-\theta}$, but $\lambda=\frac{d}{dc}u(c)$ so, $\lambda=c^{-\theta}$. Since λ, c depends on time, $\lambda(t)=c(t)^{-\theta}$

Take the logarithm for both sides

$$\ln\lambda(t)=\ln c(t)^{-\theta}$$

$$\ln\lambda(t)=-\theta\ln c(t)$$

Differentiate both sides with respect to t

$$\frac{d}{dt}\ln\lambda(t)=\frac{d}{dt}(-\theta\ln c(t))$$

$$\frac{1}{\lambda(t)}\frac{d}{dt}\lambda(t)=-\frac{\theta\frac{d}{dt}c(t)}{c(t)}$$

By (iii) we have that $\frac{\partial H}{\partial\lambda}=\frac{d}{dt}k(t)$, so $f(k)-c-\delta k=\frac{d}{dt}k(t)$ which is equal our initial capital constraint.

We see from the previous work the following

$$\frac{\frac{d}{dt}\lambda(t)}{\lambda(t)} = -\frac{d}{dk}f(k) + \delta + p \text{ And } \frac{\frac{d}{dt}\lambda(t)}{\lambda(t)} = -\frac{\theta\frac{d}{dt}c(t)}{c(t)}$$

So $-\frac{\theta\frac{d}{dt}c(t)}{c(t)} = -\frac{d}{dk}f(k) + \delta + p$

Multiply both sides by -1

$$\frac{\theta\frac{d}{dt}c(t)}{c(t)} = \frac{d}{dk}f(k) - \delta - p$$

$$\frac{\frac{d}{dt}c(t)}{c(t)} = \frac{1}{\theta}\left[\frac{d}{dk}f(k) - \delta - p\right]$$

2.3 Differences between Solow model and Ramsey model:

1. The Solow model has no optimization in it, it simply takes the saving rate as exogenous and constant while in Ramsey model the saving rate is endogenous and potentially time-varying.
2. In Solow model, saving and consumption decisions are made by infinitely-lived household, while in Ramsey model, saving and consumption are made by household with finite horizon.
3. In Ramsey model, capital stock is determined by optimization decision of household and firms.
4. The Solow model introduce a plausible consumption function with some empirical support, while Ramsey strategy is to imagine the economy to be populated by a single immortal representative household that optimizes its consumption plans over infinite time in the sort of institutional environment that will translate its wishes into actual resources allocation at any point of time.

Chapter Three

Dynamic programming

Overview

The good understanding of the mathematical concepts leads to effective algorithms for solving real world problems. Dynamic Programming is a powerful technique that can be used to solve many problems in time. Dynamic programming is a method for solving dynamic optimization problems. It becomes an important tool in macroeconomic literature, and has some appealing features in its solution strategies. It is especially suitable for solving problems under uncertainty, and because of its recursive nature computer simulation is easily done when open form solution is hardly to be obtained.

The basic idea of dynamic programming is to collapse a multi-periods problem into a sequence of two period problems at any t using the recursive nature of the problem:

$$V(k_t) = \max_{c_t, k_{t+1}} \sum_{i=0}^{+\infty} \beta^i u(k_{t+i}) \tag{1}$$

$$= \max_{c_t, k_{t+1}} \left\{ u(c_t) + \beta \sum_{i=0}^{+\infty} \beta^i u(c_{t+i+1}) \right\}$$

$$= \max_{c_t, k_{t+1}} \left\{ u(c_t) + \beta V(k_{t+1}) \right\}$$

$$\text{s.t.} \quad k_{t+1} = f(c_t, k_t) \tag{2}$$

Equation $V(k_t) = \max_{c_t, k_{t+1}} \left\{ u(c_t) + \beta V(k_{t+1}) \right\}$ is known as Bellman equation [31]. The value function $V(\cdot)$ is only a function of state variable k_t because the optimal value of c_t is just a function of k_t. Then the original problem can be solved by the methods for two-period problems plus some tricks.

The following is an example to see how we can use dynamic programming in our work. Let us take the deterministic optimal growth model of Cass-Koopmans, which extended the famous Solow model to permit elastic saving rate. In this model, output is produced using capital only; the production technology is given by $f(k_t)$. The representative household or planner chooses sequences of consumption $\{c_t\}_{t=0}^{T}$ and capital $\{k_{t+1}\}_{t=0}^{T}$ to maximize lifetime utility

$$\sum_{t=0}^{T} \beta^t u(c_t) \tag{3}$$

Subject to the budget constraints

$$c_t + k_{t+1} \leq f(k_t) \tag{4}$$

In the next steps, we will use two important ratios in economics; alpha (α) and beta (β). They are risk ratios used as statistical measurements for calculating returns; both are designed to help investors determine the risk-reward profile _profits or losses_. There are differences between them, alpha is a measure of an investment's performance compared to a benchmark and it's a mathematical estimate of the return, based on the growth of earnings per share. The value of alpha is between zero and one.

In the other hand, beta is based on the volatility _extreme ups and downs in prices or trading_ of the stock. β is called here, the discount factor, and its value is between zero and one. Discount factor β is made of two components, namely discount rate and time

$\beta = \frac{1}{(1+r)^n}$, where r: discount factor, n: time

For simplicity, tables are calculated values of beta,(see appendix II)

3.1 The value function V for finite planning horizon

We will show that this approach has value by solving a series of problems with progressively longer horizon, showing that the solutions display simple patterns. Using these patterns we will rewrite our problem recursively; that is, we will write it in a way that only depends on the current state and only has a choice for the current control [32].

Now, we will proceed forward by solving this problem for T=0. Setting lifetime utility to zero after death, this problem becomes

$$v_0(k_0) = \max_{k_1} \{u(f(k_0) - k_1)\} \tag{5}$$

We will impose the condition that $k_1 \geq 0$; that is, capital cannot be negative in the final period. This restriction is necessary for there to exist a solution if u is increasing. Moreover, we will specialize to the following functional forms because it makes the algebraic solution possible:

$$\begin{aligned} f(k) &= Ak^{\alpha} \\ u(c) &= \log(c) \end{aligned} \tag{6}$$

The solution to the above problem is obviously

$$\begin{aligned} k_1 &= 0 \\ c_0 &= Ak_0{}^{\alpha} \end{aligned} \tag{7}$$

The solution is trivial; the planner tells the household to eat everything and then go off to die. The value function is therefore

$$v_0(k_0) = \log(Ak_0{}^{\alpha}) = \log(A) + \alpha \log(k_0) \tag{8}$$

Lifetime utility depends on the existing stock of capital; endowing an economy with more capital will generate more utility for consumers in this static world.

First, we want to examine the problem above for T=1. The problem is

$$v_1(k_0) = \max_{k_1, k_2} \left\{ \log(Ak_0{}^{\alpha} - k_1) + \beta \log(Ak^{\alpha}{}_1 - k_2) \right\} \tag{9}$$

Here we assume that $k_2 \geq 0$(but not necessarily on k_1. If we put $k_2 = 0$ then the first-order condition for $k_1$is:

$$\frac{\partial v_1}{\partial k_1} = 0$$

$$\frac{-1}{Ak_0{}^{\alpha} - k_1} + \frac{\beta\alpha Ak_1{}^{\alpha-1}}{Ak_1{}^{\alpha}} = 0$$

$$\frac{1}{Ak_0{}^{\alpha} - k_1} = \frac{\beta\alpha}{k_1}$$

$$k_1 = \beta\alpha Ak_0{}^{\alpha} - \beta\alpha k_1$$

$$(\alpha\beta + 1)k_1 = \alpha\beta Ak_0{}^{\alpha}$$

$$k_1 = \frac{\alpha\beta}{1 + \alpha\beta} Ak_0{}^{\alpha} \tag{10}$$

Substitute the value of $k_1$in (9)

$$v_1(k_0) = \log(Ak_0{}^{\alpha} - k_1) + \beta \log(Ak_1{}^{\alpha})$$

$$=\log\left(Ak_0^{\alpha}-\frac{\alpha\beta}{1+\alpha\beta}Ak_0^{\alpha}\right)+\beta\log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}Ak_0^{\alpha}\right)^{\alpha}\right)$$

$$=\log\left(k_0^{\alpha}\left(A-\frac{\alpha\beta}{1+\alpha\beta}A\right)\right)+\beta\log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}A\right)^{\alpha}k_0^{\alpha^2}\right)$$

$$=\log k_0^{\alpha}+\log\left(\frac{A}{1+\alpha\beta}\right)+\beta\log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}A\right)^{\alpha}\right)+\beta\log k_0^{\alpha^2}$$

$$=\log\left(\frac{A}{1+\alpha\beta}\right)+\alpha\log k_0+\beta\log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}A\right)^{\alpha}\right)+\beta\alpha^2\log k_0 \qquad (11)$$

Notice that $v_1(k_0)$ is increasing on the initial capital stock k_0; if we give people more capital they will be better off. Also,

$$v_1(k_0)=\log(A)+\alpha\log(k_0)+\log\left(\frac{1}{1+\alpha\beta}\right)+\beta\log\left(A\left(\frac{\alpha\beta}{1+\alpha\beta}A\right)^{\alpha}\right)+\beta\alpha^2\log k_0 \qquad (12)$$

Using (8) to get that

$v_1(k_0)\ =v_0(k_0)+\log\left(\frac{1}{1+\alpha\beta}\right)+\alpha\beta\log\left(\frac{\alpha\beta}{1+\alpha\beta}\right)+(1+\alpha)\beta\log(A)+\beta\alpha^2\log k_0$; that is, the value function for the two-period case is the value function for the static case plus some extra terms. That is,

$$v_1(k_0)=\max_{k_1}\left\{\log(Ak_0^{\alpha}-k_1)+\beta v_0(k_1)\right\} \qquad (13)$$

Now, if we want to examine the case that T=2; the problem is given by

$$v_2(k_0)=\max_{k_1,k_2,k_3}\left\{\log(Ak_0^{\alpha}-k_1)+\beta\log(Ak_1^{\alpha}-k_2)+\beta^2\log(Ak_2^{\alpha}-k_3)\right\} \qquad (14)$$

Subject to $k_3\geq 0$. Let $k_3=0$, the first-order conditions for k_1 and k_2 are given by

$\frac{\partial v_1}{\partial k_1}=0$ And $\frac{\partial v_2}{\partial k_2}=0$

First, $\frac{\partial v_2}{\partial k_2} = 0$

$$\frac{-\beta}{Ak_1{}^{\alpha} - k_2} + \frac{\beta^2 A\alpha k_2{}^{\alpha-1}}{Ak_2{}^{\alpha}} = 0$$

$$\frac{\beta}{Ak_1{}^{\alpha} - k_2} = \frac{\beta^2 A\alpha k_2{}^{\alpha-1}}{Ak_2{}^{\alpha}}$$

$$\frac{\beta}{Ak_1{}^{\alpha} - k_2} = \frac{\beta^2 \alpha}{k_2}$$

$$A\beta^2\alpha k_1{}^{\alpha} - \beta^2\alpha k_2 = \beta k_2$$

$$k_2\left(\beta + \beta^2\alpha\right) = A\beta^2\alpha k_1{}^{\alpha}$$

$$k_2 = \frac{A\beta^2\alpha k_1{}^{\alpha}}{\left(\beta + \beta^2\alpha\right)} = \frac{A\beta^2\alpha k_1{}^{\alpha}}{\beta(1+\beta\alpha)}$$

So, $k_2 = \frac{A\beta\alpha k_1{}^{\alpha}}{(1+\beta\alpha)}$ (15)

Second, $\frac{\partial v_1}{\partial k_1} = 0$,this yields the following

$$\frac{-1}{Ak_0{}^{\alpha} - k_1} + \frac{\beta A\alpha k_1{}^{\alpha-1}}{Ak_1{}^{\alpha} - k_2} = 0$$

$\frac{1}{Ak_0{}^{\alpha} - k_1} = \frac{\beta A\alpha k_1{}^{\alpha-1}}{Ak_1{}^{\alpha} - k_2}$ But $k_2 = \frac{\alpha\beta}{1+\alpha\beta}Ak_1{}^{\alpha}$

So, $\frac{1}{Ak_0{}^{\alpha} - k_1} = \frac{\beta A\alpha k_1{}^{\alpha-1}}{Ak_1{}^{\alpha} - \left(\frac{\alpha\beta}{1+\alpha\beta}Ak_1{}^{\alpha}\right)} = \frac{\beta A\alpha k_1{}^{\alpha-1}}{Ak_1{}^{\alpha}\left(1 - \frac{\alpha\beta}{1+\alpha\beta}\right)} = \frac{\alpha\beta}{\left(\frac{1}{1+\alpha\beta}\right)k_1}$

$\frac{1}{Ak_0{}^{\alpha} - k_1} = \frac{(1+\alpha\beta)\alpha\beta}{k_1} = \frac{\alpha\beta + (\alpha\beta)^2}{k_1}$ putting all of these in terms of k_1

$$k_1 = A\alpha\beta k_0{}^{\alpha} + (\alpha\beta)^2 Ak_0{}^{\alpha} - k_1\alpha\beta - k_1(\alpha\beta)^2$$

$$k_1\left(1+\alpha\beta+(\alpha\beta)^2\right)=\left(\alpha\beta+(\alpha\beta)^2\right)Ak_0{}^{\alpha}$$

$$k_1=\frac{\left(\alpha\beta+(\alpha\beta)^2\right)}{\left(1+\alpha\beta+(\alpha\beta)^2\right)}Ak_0{}^{\alpha} \tag{16}$$

Finally,

$$v_2(k_0)=\log\left(Ak_0{}^{\alpha}-\frac{\left(\alpha\beta+(\alpha\beta)^2\right)}{\left(1+\alpha\beta+(\alpha\beta)^2\right)}Ak_0{}^{\alpha}\right)+$$

$$\beta\log\left(A\left[\frac{\left(\alpha\beta+(\alpha\beta)^2\right)}{\left(1+\alpha\beta+(\alpha\beta)^2\right)}Ak_0{}^{\alpha}\right]^{\alpha}-\frac{\alpha\beta}{1+\alpha\beta}A\left[\frac{\left(\alpha\beta+(\alpha\beta)^2\right)}{\left(1+\alpha\beta+(\alpha\beta)^2\right)}Ak_0{}^{\alpha}\right]^{\alpha}\right)+$$

$$\beta^2\log\left(A\left[\frac{\alpha\beta}{1+\alpha\beta}A\left[\frac{\left(\alpha\beta+(\alpha\beta)^2\right)}{\left(1+\alpha\beta+(\alpha\beta)^2\right)}Ak_0{}^{\alpha}\right]^{\alpha}\right]^{\alpha}\right) \tag{17}$$

$$v_2(k_0)=v_1(k_0)+\log\left(\frac{1}{1+\alpha\beta+(\alpha\beta)^2}\right)+\alpha\beta\log\left(\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\right)+$$

$$\alpha^2\beta^2\log\left(\frac{\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2}\right)+(1+\alpha+\alpha^2)\beta^2\log(A)+\alpha^2\beta^2\log(k_0) \tag{18}$$

And it satisfies

$$v_2(k_0)=\max_{k_1}\left\{\log(Ak_0{}^{\alpha}-k_1)+\beta v_1(k_1)\right\} \tag{19}$$

The general solution to a problem with horizon T is

$$k_1=\frac{\alpha\beta+(\alpha\beta)^2+.....+(\alpha\beta)^T}{1+\alpha\beta+(\alpha\beta)^2+.....(\alpha\beta)^T}Ak_0{}^{\alpha} \tag{20}$$

Here we have two finite geometric series, so we can use the sum of them from the formula $\sum_{t=o}^{T-1}ar^t=a\frac{1-r^T}{1-r}$, where $|r|\prec 1$, to get that

$$k_1 = \alpha\beta\left[\frac{1-(\alpha\beta)^T}{1-(\alpha\beta)^{T+1}}\right]Ak_0^{\alpha} \tag{21}$$

For infinite planning horizon: if $T \to \infty$, we get that $k_1 = \alpha\beta A k_0^{\alpha}$ (22)

Notes on the previous work:

1) When we examine the decision rules from the problems with horizons of 2 and 3 periods, we see that the only thing that matters is the current capital stock k_t.
2) The decision rules depend on the number of periods before the end T – t; that is, a household makes the same decisions n periods from death no matter how long they have been alive, conditional on current capital.

In the infinite horizon case, we may drop the time subscript as the customer will always be infinitely far from death:

$k_{t+1} = \alpha\beta A k_t^{\alpha}$ or $k^* = \alpha\beta A k^{\alpha}$ where k^* denote the capital for next period.

3.2 Determination of the parameters of $v(k)$

The value function in the infinite horizon case may converges to a function of the form

$$v(k) = a + b\log(k) \tag{23}$$

Solving the problem this way is not very fast when we know the form of the value function [32]. If we rewrite (23) as

$$v_1(k) = \max_{k_1}\left\{\log(c_0) + \beta v_0(k_1)\right\} \tag{24}$$

Then, we can solve our problem by turning our utility into the sum of two parts, what we get today and what we get in the future, assuming we make the proper choices tomorrow; we then only need to worry about making the proper choice today. With an infinite horizon we have

$$v(k_t) = \max_{k_{t+1}} \left\{ \log(Ak_t^{\alpha} - k_{t+1}) + \beta v(k_{t+1}) \right\} \tag{25}$$

$v(k_t)$ is the lifetime utility from having k_t units of capital. This equation which is called Bellman equation gives us a convenient method for solving the problem. If we could somehow know the form of the value function we could simply insert it into the above problem and maximize it. If this sounds too good to be true, well it almost is; knowing the form of the value function is generally impossible. For the above case, we could insert a guess of the form $v(k) = a + b\log(k)$ into the Bellman equation and take derivatives, we get

$\dfrac{1}{Ak_t^{\alpha} - k_{t+1}} = \dfrac{\beta b}{k_{t+1}}$ Solving this, we get that

$$k_{t+1} = \frac{\beta b}{1+\beta b} Ak_t^{\alpha} \tag{26}$$

The problem here is that we don't know b. But if we insert our solution into the Bellman equation we get

$$a + b\log(k_t) = \log(Ak_t^{\alpha} - k_{t+1}) + \beta v(k_{t+1})$$

$$= \log\left(Ak_t^{\alpha} - \frac{\beta b}{1+\beta b} Ak_t^{\alpha} \right) + \beta\left(a + b\log\left[\frac{\beta b}{1+\beta b} Ak_t^{\alpha} \right] \right)$$

$$= \log\left(\frac{Ak_t^{\alpha}(1+\beta b) - \beta b Ak_t^{\alpha}}{1+\beta b} \right) + \beta a + \beta b\log\left(\frac{\beta b}{1+\beta b} Ak_t^{\alpha} \right)$$

$$= \log\left(\frac{Ak_t^{\alpha}}{1+\beta b} \right) + \beta a + \beta b\log\left(\frac{\beta b}{1+\beta b} Ak_t^{\alpha} \right)$$

$$= \log\left(\frac{A}{1+\beta b}\right) + \alpha\log(k_t) + \beta a + \beta b\log\left(\frac{\beta b}{1+\beta b}A\right) + \beta b\alpha\log(k_t)$$

$$= \log\left(\frac{A}{1+\beta b}\right) + \beta a + \beta b\log\left(\frac{\beta b}{1+\beta b}A\right) + (\alpha+\beta b\alpha)\log(k_t) \quad (27)$$

From the above equality, we get that

$$b = \alpha + \beta b\alpha \text{ So } b = \frac{\alpha}{1-\beta\alpha} \quad (28)$$

$$a = \log\left(\frac{A}{1+\beta b}\right) + \beta a + \beta b\log\left(\frac{\beta b}{1+\beta b}A\right)$$

$$a - \beta a = \log\left(\frac{A}{1+\beta b}\right) + \beta b\log\left(\frac{\beta b}{1+\beta b}A\right)$$

$$a = \frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right) + \beta b\log\left(\frac{\beta b}{1+\beta b}A\right)\right] \quad (29)$$

If we rewrite a without b, it becomes (see appendix I)

$$a = \frac{1}{(1-\beta)(1-\alpha\beta)}\log\left[A(1-\alpha\beta)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta}\right] \quad (30)$$

The value of b is absolutely greater than zero since the value of both alpha and beta is between zero and one, there multiplication is so. While the value of a depends on the argument of the logarithmic function in (30) $A(1-\alpha\beta)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta}$

To ensure that $\log\left[A(1-\alpha\beta)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta}\right] \succ 0$

$A(1-\alpha\beta)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta}$ Must be greater than one……………….. (*)

In dynamic programming: if we given a known and positive quantity b which we wish to divided it into 2 parts in a way that the product of the 2 parts is to be maximum, we solve the following;

Define $g_2(b)$ which is a maximum value of subdividing b into 2 parts when one part is y and the remaining quantity is b - y.

$$g_2(b) = \max_{0 \le y \le b} \{y - g_1(b-y)\}$$

$$= \max_{0 \le y \le b} \{y(b-y)\}, \; g_1(b-y) = b-y \text{ since we have one part.}$$

To maximize $f = y(b-y)$, by simple calculus:

$$\frac{df}{dy} = b - 2y = 0, \text{ so } y = \frac{b}{2}$$

$$\frac{d^2f}{dy^2} = -2 \prec 0$$

$\therefore y = \dfrac{b}{2}$ is a maximum value of $f = y(b-y)$

$$\Rightarrow b - y = b - \frac{b}{2} = \frac{b}{2}$$

The optimal policy is to subdivide b into two equal parts

$$g_2(b) = \frac{b}{2}(b - \frac{b}{2}) = \frac{b}{2}(\frac{b}{2}) = \frac{b^2}{4}$$

Using the previous work with b=1 and $y = \alpha$ we get that

$$0 \prec \alpha\beta \le \frac{1}{4}$$

Now, $\left(\dfrac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta} = \left(\dfrac{1}{\dfrac{1}{\alpha\beta} - 1}\right)^{\alpha\beta} \prec \left(\dfrac{1}{4-1}\right)^{\frac{1}{4}} = \left(\dfrac{1}{3}\right)^{\frac{1}{4}}$

$$(1-\alpha\beta) \prec \left(1-\frac{1}{4}\right) = \frac{3}{4}$$

$$\Rightarrow A(1-\alpha\beta)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta} \prec A\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{4}}$$

To ensure that $A(1-\alpha\beta)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta} \succ 1,\ A\left(\frac{3}{4}\right)\left(\frac{1}{3}\right)^{\frac{1}{4}} \succ 1$

$$\Rightarrow A \succ \left(\frac{4}{3}\right)(3)^{\frac{1}{4}} = \frac{4}{(3)^{\frac{3}{4}}} = 1.76$$

Let $y = (1-x)\left(\frac{x}{1-x}\right)^{x}$ where $x = \alpha\beta$

The maximum value of x is the minimum value of y; to see this:

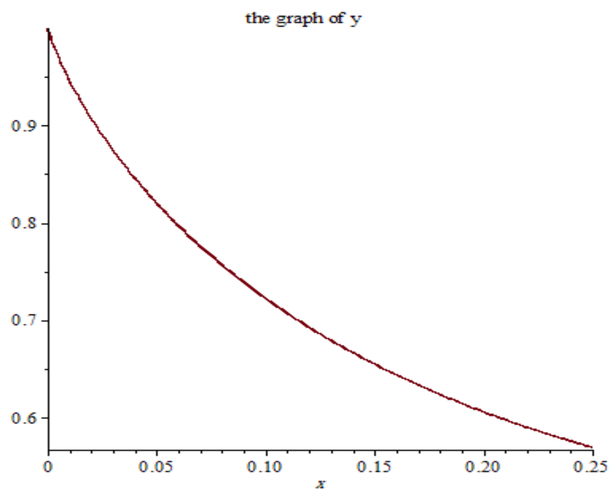

Figure (3.1): the graph of Y.

We have completely solved the consumer's problem; with the given solution for b optimal capital accumulation is given by

$$k_{t+1} = \alpha\beta A k_t^{\alpha} \tag{31}$$

To see what happened in (30), substitute the value of (b) in (26)

$$k_{t+1} = \frac{\beta b}{1+\beta b} A k_t^{\alpha} = \frac{\beta\left(\frac{\alpha}{1-\alpha\beta}\right)}{1+\beta\left(\frac{\alpha}{1-\alpha\beta}\right)} A k_t^{\alpha} = \frac{\left(\frac{\beta\alpha}{1-\alpha\beta}\right)}{1+\left(\frac{\beta\alpha}{1-\alpha\beta}\right)} A k_t^{\alpha} = \frac{\left(\frac{\beta\alpha}{1-\alpha\beta}\right)}{\left(\frac{1-\alpha\beta+\alpha\beta}{1-\alpha\beta}\right)} = \alpha\beta A k_t^{\alpha}$$

The only problem with this method is that there are a very small number of economic problems where we know the form of the value function; of these, some are impossible to solve for the coefficients analytically and others are simply not very interesting from an economic standpoint as they involve odd choices for parameters.

In (30), the value of capital stock in the period t+1depends on the previous period t. suppose that we wish to rewrite it in terms of the period 0,

$$k_1 = \alpha\beta A k_0^{\alpha}$$

$$k_2 = \alpha\beta A k_1^{\alpha} = \alpha\beta A(\alpha\beta A k_0^{\alpha})^{\alpha} = (\alpha\beta A)^{\alpha+1} k_0^{\alpha^2}$$

$$k_3 = \alpha\beta A k_2^{\alpha} = \alpha\beta A((\alpha\beta A)^{\alpha+1} k_0^{\alpha^2})^{\alpha} = (\alpha\beta A)^{\alpha^2+\alpha+1} k_0^{\alpha^3}$$

$$k_4 = \alpha\beta A k_3^{\alpha} = \alpha\beta A((\alpha\beta A)^{\alpha^2+\alpha+1} k_0^{\alpha^3})^{\alpha} = (\alpha\beta A)^{\alpha^3+\alpha^2+\alpha+1} k_0^{\alpha^4}$$

Going this way to get that

$$k_n = \alpha\beta A k_{n-1}^{\alpha} = (\alpha\beta A)^{\sum_{i=0}^{n-1}\alpha^i} k_0^{\alpha^n}$$

Lemma: the capital stock in the steady state depends on the economic ratios $\alpha \,\&\, \beta$, the effectiveness of labor A and the capital stock in the first period and given by

$$k_n = \alpha\beta A k_{n-1}^{\alpha} = (\alpha\beta A)^{\sum_{i=0}^{n-1}\alpha^i} k_0^{\alpha^n}$$

3.3 What happened when the value function is unknown?

If we don't know the value function or it simply does not exist in closed form, we will go back to the general Bellman equation

$$v_t(k_t) = \max_{k_{t+1}} \{u(f(k_t) - k_{t+1}) + \beta v_{t+1}(k_{t+1})\}$$

In our work above, the value function is constant over time but we don't know its form. We try to use the recursive nature of the value function. Suppose we guessed the value in period T+1 was zero, then the Bellman equation would imply that

$$v_T(k_T) = u(f(k_T))$$

But we know that by the Bellman recursion

$$v_{T-1}(k_{T-1}) = \max_{k_T} \{u(f(k_{T-1}) - k_T) + \beta v_T(k_T)\}$$

That is, we update our guess v_T by replacing it with v_{T-1} after solving for k_T as a function of k_{T-1}. That is

$$v_{T-1}(k_{T-1}) = u(f(k_{T-1}) - k_T(k_{T-1})) + \beta v_T(k_T(k_{T-1}))$$

Then, according to the Bellman equation we must have that

$$v_{T-2}(k_{T-2}) = \max_{k_{T-1}} \{u(f(k_{T-2}) - k_{T-1}) + \beta v_{T-1}(k_{T-1})\}$$

And so on. If we let $T \to \infty$, then we get the value function for infinite planning horizon as

$$v(k) = \max_{k'} \{u(f(k) - k') + \beta v(k')\}$$

3.4 The prove which confirms that the value function exist

In this section we will prove that the previous algorithm will converges to the true value function, that is; the value function exists.

The Bellman equation is given by

$$v(x) = \max_{a}\left\{r(x,a) + \beta v(x') : a \in \Gamma(x), x' = t(x,a)\right\}$$

The function v(x) is unknown of this equation. We know that inserting a function into the right-hand side for $v(\cdot)$ and performing the maximization gives us a new function for the left-hand side; moreover, these functions are not necessarily the same. Let our guess is given by w so as not to confuse it with the true value function v, which may not exist and which we certainly do not know. We can view the Bellman equation as mapping functions into functions, a functional operator. Calling this thing L, we have the operator L takes a function $w : X \rightarrow R$ and turns it into a function $Lw : X \rightarrow R$ via the process

$$(Lw)(x) = \max_{a}\left\{r(x,a) + \beta w(x') : a \in \Gamma(x), x' = t(x,a)\right\}$$

Let D, for example, be the differentiation operator. Then $Dx^2 = 2x$, a new function which is related to the old one via the operator D. the Bellman operator works exactly the same way. If the true value function exists, it satisfies equation (1). That is, if we feed the Bellman operator v we get v back; Lv(x) = v(x). in other words, the value function is a fixed point of the L operator in the space of functions.

For example, one fixed point for the differentiation operator is the zero function:

$D0 = 0$

Another one is e^x :

$D\, e^x = e^x$

If we somehow prove that the Bellman operator had a fixed point, we would definitely know that the value function existed. Also, if we could prove that it only had one fixed point, then we know that the value function was that fixed point.

Finally, assuming that we have enough structure that solutions to the Bellman equation exist, there will be (at least) one action for each value of the state that is optimal.

Definition 1: A mapping T from a metric space (Ξ, ρ) into itself is a **strict contraction map** if $\exists \theta \in (0,1)$ such that $\rho(Tf, Tg) \le \theta\rho(f, g), \forall f, g \in \Xi$.

Theorem 2: If $T : (\Xi, \rho) \to (\Xi, \rho)$ is a strict contraction map then T is uniformly continuous.

Proof: if T is a contraction, then for some $\theta \in (0,1)$ we have

$$\frac{|Tx - Ty|}{|x - y|} \le \theta \prec 1$$

$\forall x, y \in S$. Let $\delta = \dfrac{\varepsilon}{\theta}$; then for any $\varepsilon \succ 0$, if $|x - y| \prec \delta$ we have

$$|Tx - Ty| \le \theta |x - y| \prec \theta\delta = \varepsilon$$

Thus, T is uniformly continuous.

We now prove the key theorem in this section.

Definition: (Banach space) is a normed linear space that is complete metric space with respect to the metric derived from its norm.

Theorem 3: (**Contraction mapping theorem)** A strict contraction map on a Banach space has a unique fixed point. Furthermore, the space $\{f, Tf, T^2 f, ...\}$ converges to that unique fixed point [32].

Proof. Let (Ξ,ρ) be a complete metric space and $T:(\Xi,\rho)\to(\Xi,\rho)$ be a strict contraction map. For any $f\in\Xi$ define $f^n = T^n f$. Since T is a strict contraction map, there is $\theta\prec 1$ such that if $n\ge m$ we have

$$\rho(f^n, f^m) \le \theta^m \rho(f^{n-m}, f)$$

This result is obtained by using the contraction property m times. Using the triangle inequality for metrics we must have

$$\rho(f^{n-m}, f) \le \rho(f^{n-m}, f^{n-m-1}) + ... + \rho(f^1, f)$$

We also know that

$$\rho(f^{n-m}, f^{n-m-1}) \le \theta^{n-m-1} \rho(f^1, f)$$

And

$$\rho(f^{n-m-1}, f^{n-m-2}) \le \theta^{n-m-2} \rho(f^1, f)$$

And so on. Putting these together we get that

$$\rho(f^{n-m}, f) \le \rho(f^{n-m}, f^{n-m-1}) + \rho(f^{n-m-1}, f^{n-m-2}) + ... + \rho(f^1, f)$$

$$\le \rho(f^1, f) \times (\theta^{n-m-1} + \theta^{n-m-2} + ... + \theta + 1)$$

We know that

$$\theta^{n-m-1}+\theta^{n-m-2}+...+\theta+1=\frac{1-\theta^{n-m}}{1-\theta}\leq\frac{1}{1-\theta}$$

Combining all our inequalities yields to

$$\rho(f^{n},f^{m})\leq\frac{\theta^{m}}{1-\theta}\rho(f^{1},f\ )$$

The case $m\geq n$ is similar. Therefore, we must have

$$\rho(f^{n},f^{m})\rightarrow 0$$

or $\{f^{n}\}$ is Cauchy. Since Ξ is complete, this sequence has a limit point $f^{*}\in\Xi$.

We simply need to show that f^{*} is a fixed point of T. With strict contraction map being uniformly continuous, it follows that

$$f^{*}=\lim_{n\rightarrow\infty}T^{n}f\ =T\left(\lim_{n\rightarrow\infty}T^{n-1}f\ \right)=Tf^{*}$$

Thus we have a fixed point. This fixed point must be unique, since if g^{*} were another fixed point we must have

$$\rho(f^{*},g^{*})=\rho(Tf^{*},Tg^{*})$$

$$\leq\theta\rho(f^{*},g^{*})$$

So, with $\theta\prec 1$ we must have $f^{*}=g^{*}$.

The contraction mapping theorem proves that a sequence generated by a contraction map converges to a limit point independent of the initial condition for that sequence.

3.5 Properties of the value function

We show that the value function v(k) exists and continuous under the assumptions above. Let us assume that u(c) and f(k) are both increasing, concave, and continuously differentiable. We want to show that the value function will inherit these properties.

If $w : X \to R$ is weakly increasing, then so is

$$(Lw)(k) = \max_{k'} \{u(f(k) - k') + \beta w(k')\}$$

Because if k' is feasible from k it is feasible from any $\bar{k} \succ k$ and the residual consumption cannot decrease the RHS because u and f are increasing. Thus, the Bellman operator maps weakly increasing functions into weakly increasing functions. Since we can initialize our iterations with a weakly increasing functions, the value function itself must be weakly increasing provided this property is preserved in the limit. Since weakly increasing functions are defined by a weak inequality,

$$g(x) \geq g(y)$$

If $x \geq y$, then the set of these functions is closed and therefore preserved in the limit. Thus, the value function is weakly increasing. This approach cannot guarantee that v is strictly increasing, since that set in not closed.

Similarly, note that if $w : X \to R$ is concave, then so is

$$(Lw)(k) = \max_{k'} \{u(f(k) - k') + \beta w(k')\}$$

To see this, take any $\bar{k}, k \geq 0$ and note that the set of feasible $\bar{k}$ is convex, given k. Then

$$(Lw)(\alpha k + (1-\alpha)\bar{k}) \geq u(f(\alpha k + (1-\alpha)\bar{k}) - \alpha k' - (1-\alpha)\bar{k}') + \beta w(\alpha k' + (1-\alpha)\bar{k}')$$

Using the fact that both u and w are concave, we have

$$u(f(\alpha k+(1-\alpha)\overline{k})-\alpha k'-(1-\alpha)\overline{k}')+\beta w(\alpha k'+(1-\alpha)\overline{k}')\geq \alpha(Lw)(k)+(1-\alpha)(Lw)(\overline{k})$$

That is, we know that $(Lw)(k)$ is concave; evidently, the Bellman operator takes concave functions into concave functions. Since the limit of any sequence of functions is the value function, all we need to prove is that the limit of a sequence of concave functions must be concave because we can start our iterations with a concave function. but note that concave functions are defined by a weak inequality

$$g(\alpha x+(1-\alpha)y)\geq \alpha g(x)+(1-\alpha)g(y)$$

If $\alpha \in [0,1]$. Thus, the set defining it must be closed, so the value function is concave. This proof will not establish that the value function is strictly concave since that is not a closed set.

Lemma 4 (**Benveniste-Scheinkman Lemma**) let v be a real-valued, concave function defined on a convex set $D \subset R^n$. If $w \in C^1$is a concave function on a neighborhood N of $x_0 \in D$ such that $w(x_0)=v(x_0)$ and $w(x)\leq v(x)\ \ \forall x \in N$ then $v \in C^1$at x_0.

This lemma states that if we can find a function that is everywhere below v, agrees with v at k^*, and is continuously-differentiable at k^*, then v will be continuously-differentiable at k^*as well. $w(k)$ is that function, so the value function is differentiable.

Chapter Four

Infinite horizon dynamic programming

In this section we will use python for solving simple infinite horizon dynamic programming. Also, we will focus on solving for consumption in an optimal model.

4.1.The growth model

The growth model or the neoclassical growth model is a macro model in which the long-run growth rate of output per workers is determined an exogenous rate of technological progress, like those following from Ramsey (1928), Solow (1956) and Swan (1956). R. Solow identifies its assumption of labor and capital as the cause of an equilibrium growth. In 1956, Solow and Swan turn to neoclassical production function with varying share of labor and capital input. This approach provides the first neoclassical model of long run economic growth and become the starting point for most studies on economic growth.

Consider that at time t an agent owns capital stock $k_t \in R^+$ and produces output $f(k_t) \in R^+$. This output can be either consumed or saved as capital for the next period and denoted by k_{t+1}. So

$$k_{t+1} = f(k_t) - c_t \qquad (1)$$

If we take k_0 as given, our assumption is that the agent wishes to maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$ where u is a given utility function [31] and β is the discount factor. But we should be aware that the agent selects a path $c_0, c_1, c_2, \ldots.$ for consumption that is

i. Nonnegative
ii. Feasible in the sense that the capital path $\{k_t\}$ determined by $\{c_t\}$ is always nonnegative
iii. Optimal and maximize $\sum_{t=0}^{\infty} \beta^t u(c_t)$

The standard theory of dynamic programming states that; any optimal consumption sequence $\{c_t\}$ must be markov, which means that there exist a function σ such that

$$c_t = \sigma(k_t) \text{ For all t.} \tag{2}$$

So,
$$k_{t+1} = f(k_t) - \sigma(k_t) \tag{3}$$

Markov chain specification

Given a set of states, $S = \{s_1, s_2, \ldots., s_r\}$. The process starts in one of these states and moves successively from one state to another. Each move is called a step. If the chain is currently in state s_i, then it moves to state s_j at the next step with a probability denoted by p_{ij}, and this probability does not depend upon which state the chain was in before the current state.

The probabilities p_{ij} are called transition probabilities. The process can remain in the state it is in, and this occurs with probability p_{ii}. An initial probability distribution, defined on S, specifies the starting state. Usually this is done by specifying a particular state as the starting state.

The policy function $\sigma : R^+ \to R^+$ is a feasible consumption policy if

$$0 \le \sigma(k) \le f(k) \quad \forall k \in R^{+} \tag{4}$$

Moreover, if we denote the previous such policies by Σ, then the agent's decision problem may be rewritten as [23]

$$\max_{\sigma \in \Sigma} \left\{ \sum_{t=0}^{\infty} \beta^{t} u(\sigma(k_t)) \right\} \tag{5}$$

We assume that the utility function u is a strictly increasing and concave function.

4.2.Dynamic programming

In this part, we try to use dynamic programming in order to find the optimal policy.

First, the value function associated with this optimization problem defined as

$$v^{*}(k_0) = \sup_{\sigma \in \Sigma} \left\{ \sum_{t=0}^{\infty} \beta^{t} u(\sigma(k_t)) \right\} \tag{6}$$

Where $\{k_t\}$ is given by (3).

The value function gives the supremum amount of the utility which we obtained from the state k_0.

Now, we try to build a recursive reformulation using the bellman equation which takes the form

$$v^{*}(k) = \max_{0 \le c \le k} \left\{ u(c) + \beta v^{*}(f(k) - c) \right\} \text{ for all } k \in R^{+} \tag{7}$$

We will optimize $v^{*}(k)$ by choosing an appropriate c to trade off the current utility function for future utility which depends in the first place on our saving.

Definition: (greedy policy)

A greedy algorithm makes a locally optimal choice at each step as strategy for approximating a global optimum. Correspondingly, given a continuous function w on R^+, we say a policy $\sigma \in \Sigma$ is a w-greedy if $\sigma(k)$ is a solution to

$$\max_{0 \le c \le k} \{u(c) + \beta w(f(k) - c)\} \text{ for all } k \in R^+ \tag{8}$$

To ensure that there is a solution to $v^*(k)$, we will put some assumptions; we assume that f and u are continuous and u is bounded. Depending on these assumptions, we also get that v^* is finite, bounded, continuous and satisfies the Bellman equation.

Proposition: if the map $U : R^+ \to R^+$ is bounded and continuous, the function f is measurable and maps $R^+ \times Z$ into R^+, then the value function v^* is continuous and is the unique function in $b\vartheta(S)$ that satisfies

$$v^*(k) = \max_{0 \le c \le k} \left\{ U(k - c) + \rho \int v^*[f(c,z)]\phi(dz) \right\}$$

This proposition guide us to the following result, a policy is optimal iff it is v^* -greedy.

To find the optimal policy, we will follow the following steps:

i. Compute v^*.

ii. Solve for v^*-greedy policy.

For any k, as soon as we get the second step, we are going to solve a one-dimensional optimization problem on the right-hand side of the bellman equation. We will focus on the first step, which is how we obtain the value function since the second step become trivial once we get v^*.

4.3.Value function iteration

Here, we will start our work by a guess, some initial value function w and try to improve it in order to compute v^* by using an iterative technique.

The bellman operator is the best choice for improving v^*.

The bellman operator maps a function w to a new function Tw, as follows,

$$Tw(k) = \max_{0 \le c \le k} \{u(c) + \beta w(f(k) - c)\} \tag{9}$$

This operator, as we see, is like the bellman equation but in fact it is quite different. If we apply T from some starting function w, then we produce a sequence of functions $w, Tw, T(Tw), \ldots$ which are continuous, bounded and converges uniformly to v^*. To see this,

Lemma: the value function v^* is a unique fixed point of T in $b\vartheta(S)$, moreover $v^* \in b\vartheta(S)$, where $b\vartheta(S)$ is the set of all bounded measurable functions on S.

Definition: (fixed point)

Let $T: S \to S$,where S is any set. x is called a fixed point of T on S if it's a solution of the equation $Tx = x$.

Fixed point and optimization problems are closely related, when we study dynamic programming, an optimization problem will be converted into a fixed point problem, i.e : if $T: S \to S$ has a unique fixed point in a metric space (S, ρ), then finding this point is the same as finding the minimize of $f(x) = \rho(Tx, x)$

Also, let $T: S \to S$. An $x^* \in S$ is called a fixed point of T on S if $Tx^* = x^*$. If S is a subset of R, then fixed points of T are those points in S where T meets the 45 degree line as in the figure:

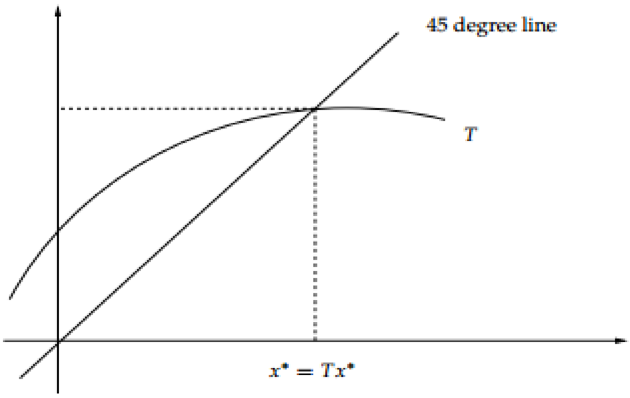

Figure (4.1): Fixed point in xy-plane

All our work made on bounded utility function unless the economists often work with unbounded utility functions, the reason behind our assumption is that; for unbounded utility functions the situation is more complicated.

Fitted value function:

We would like to compute the value function by an iteration procedure using the bellman operator as following:

i. Give an initial guess w.

ii. Solving (9) and obtain the function Tw .

iii. Unless some stopping condition is satisfied, set $w = Tw$ and go to step ii.

However, there is a problem we must notice before we start this algorithm; Tw could not be calculated exactly but could be calculated approximately using the suggested algorithm, also, these values could not be stored in a computer completely because of the huge output of Tw on k .

Unless w is known function, and Tw can be got by some iterates, the only way to store this function is to record the value of $Tw(k)$ for every $k \in R^+$, which is impossible.

So, instead of that, we try to use what we call it the fitted value function iteration [23].

The procedure is to record the value of the function Tw at only finitely many points $\{k_1,......,k_I\} \subset R^+$ which we call them grid points, and reconstruct it from these informations.

For more specific details, the algorithm is

i. Begin with an array of values $\{w_1,......,w_I\}$,typically representing the values of some initial function w on the grid points $\{k_1,......,k_I\}$

ii. Build a function w^* on the space R^+ by interpolating the pointes $\{w_1,......,w_I\}$

iii. By repeatedly solving (9), obtain and record the value $Tw^*(k_i)$ on each grid point k_i

iv. Unless some stopping condition is satisfied, set $\{w_1,......,w_I\} = \{Tw^*(k_1),......,Tw^*(k_I)\}$ and go to step ii.

The most important step for the last algorithm is step 2 which can be accomplished by many ways.

We need a function approximation that produces a good approximation to Tw and also combine well with the algorithm's iteration. Our choice will be a continuous piecewise linear interpolation. In other words, we need continuous piecewise linear interpolation to produce an approximation to Tw .

As an example, the next algorithm and figure illustrate piecewise linear interpolation on the function $f(x) = \sin(2\pi x)$ on the grid points $0, 0.2, 0.4, ..., 1$ using a code in python.

The algorithm:

```
def f(x): return np.sin(math.pi * 2 * x)
grid = np.linspace(0, 1, 6)
def f_interp(x): return np.interp(x, grid, f(grid))
domain = np.linspace(0, 1, 201)
fig, ax = plt.subplots(1,1)
ax.plot(domain, f(domain), 'b-', lw=3, label='f')
ax.plot(domain, f_interp(domain), 'r-', lw=2, alpha=0.9, label='f interpolated')
ax.axhline(lw=0.5, color='0.5')
ax.vlines(grid, np.zeros_like(grid), f(grid), linestyle='dashed')
ax.set_xlim(0, 1)
ax.legend(loc='upper right')
fig.show()
```

The figure:

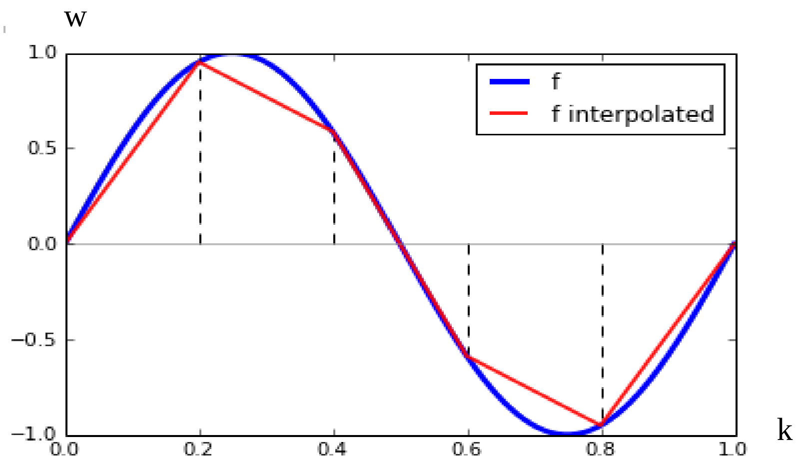

Figure (4.2): example for piecewise linear interpolation

We specially choose the piecewise linear interpolation since it is preserves useful shape properties such as monotonicity and concavity.

The following example gives an exact analytic solution as a special case of the considered problem:

Let $f(k) = k^{\alpha}$ with $\alpha = 0.65$

$u(c) = \ln c$ and $\beta = 0.95$

The exact solution will be

$$v^*(k) = c_1 + c_2 \ln k$$

Where

$$c_1 = \frac{1}{(1-\beta)}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta\ln(\alpha\beta)}{(1-\alpha\beta)}\right] \text{ and } c_2 = \frac{\alpha}{1-\alpha\beta}$$, note that ($\ln = \log_e$)

We will replicate this solution numerically using the fitted value function iteration.

```
def exact_solution(alpha,beta):
    """Return function, the exact solution."""
    ab = alpha * beta
    c1 = (math.log(1 - ab) + math.log(ab) * ab / (1 - ab)) / (1 - beta)
    c2 = alpha / (1 - ab)
    return lambda k: c1 + c2 * np.log(k)

#parameters and grid
prms01 = dict(alpha = 0.65,beta=0.95) #match params in SS
kgrid01 = np.linspace(1e-6, 2, 150) #match grid in SS

#plot the exact solution
v_star = exact_solution(**prms01)
fig, ax = plt.subplots(1,1)
ax.plot(kgrid01, v_star(kgrid01), 'k-', lw=2)
ax.set_title('Value Function')

fig.show()
```

Running the code produces the following figure

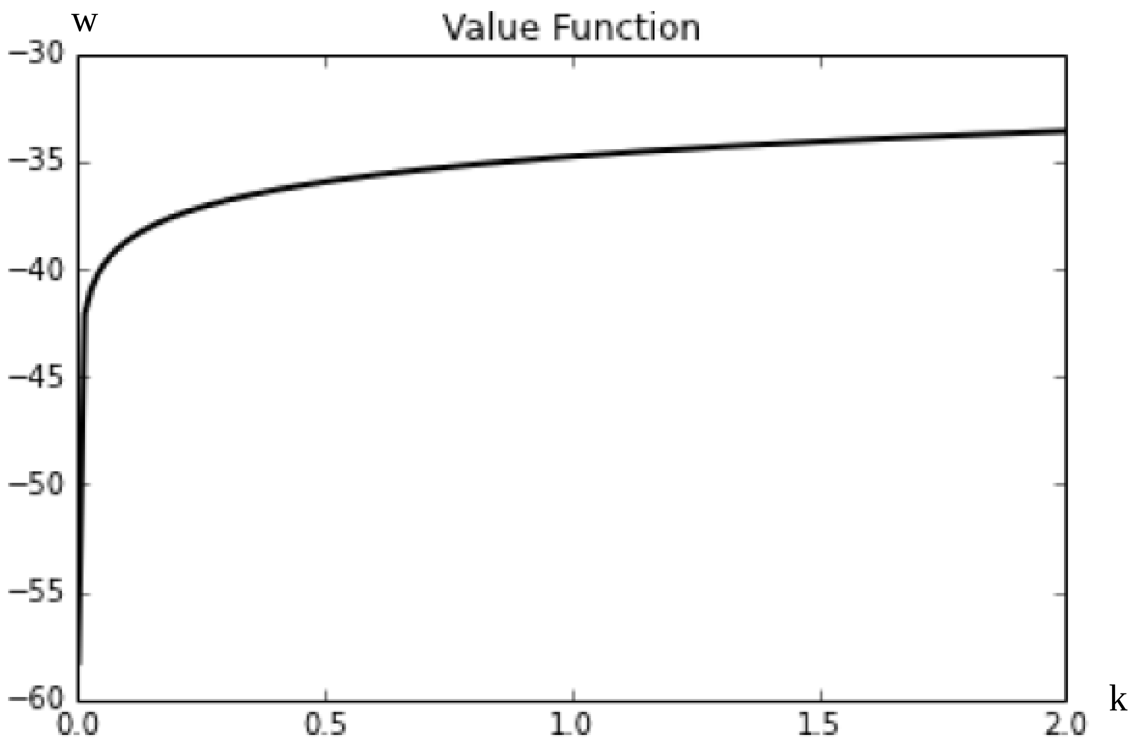

Figure (4.3): the value function

The next code is made by John Stachurski and Thomas J. Sargent in 11\8\2013. In the code they take the initial condition as $w = 5\log(k) - 25$ where k represents the grid points.

They are trying to solve the optimal growth problem via the value function iteration. First, they put out the primitives such as $\alpha, \beta,$ exact solution and the grid points. Second, they use the bellman operator to approximate Tw on the grid points. Note that the vector w in the code represents the value of the input function on the grid points. Third, they apply linear interpolation to w, the initial condition. Finally, they plot the successive functions which produced by the fitted value function iteration. In the figure, the hotter colors represents higher iterates. The true value function v^* is the thick, black line. The sequence of iterates converges toward v^*. Increasing the number of iterations produces further improvement.

The code :

```
from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
from numpy import log
from scipy.optimize import fminbound
from scipy import interp

alpha = 0.65
beta=0.95
grid_max=2
grid_size=150
grid = np.linspace(1e-6, grid_max, grid_size)

ab = alpha * beta
c1 = (log(1 - ab) + log(ab) * ab / (1 - ab)) / (1 - beta)
c2 = alpha / (1 - ab)
def v_star(k):
    return c1 + c2 * log(k)

def bellman_operator(w):

    Aw = lambda x: interp(x, grid, w)

    # === set Tw[i] equal to max_c { log(c) + beta w(f(k_i) - c)} === #
    Tw = np.empty(grid_size)
    for i, k in enumerate(grid):
ax.legend(loc='upper left')

plt.show()
```

If we run the code, the following figure will illustrate the work:

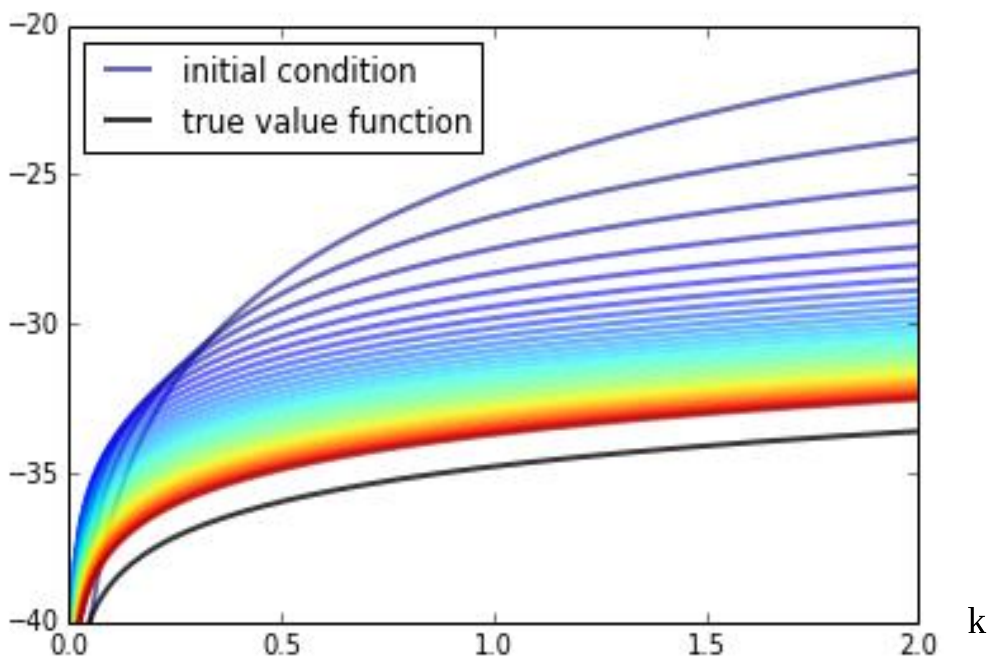

Figure (4.4): the graph of w (the initial guess) against the true value function when n= 35

In the last code, n is the number of functions that generated by the value function iteration algorithm. If we increase n to 75 and run the code again we will see that

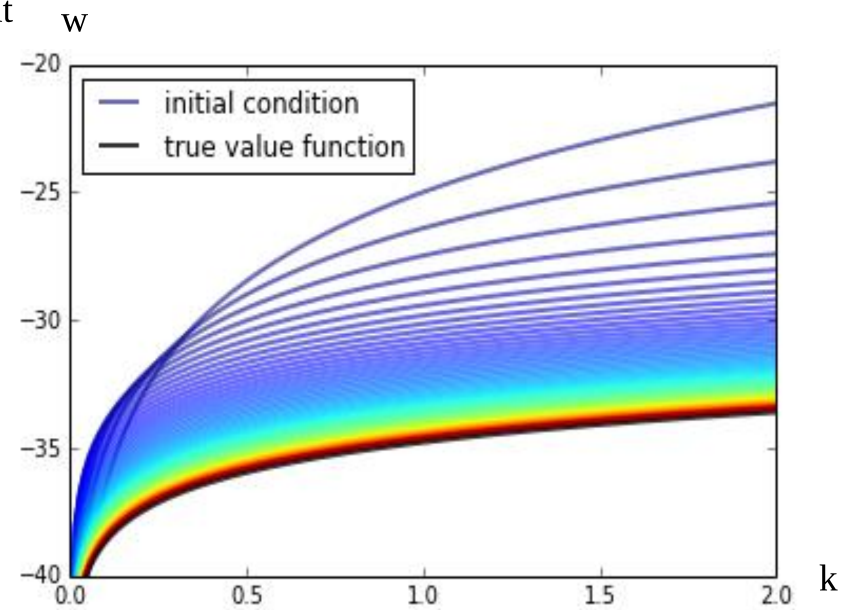

Figure (4.5): the graph of w (the initial guess) against the true value function when n= 75

Another example is to let $w = 2(k)^{0.25} - 35$, this example preserve the concave shape of the value function in the proceeding example

The figure is (n=35)

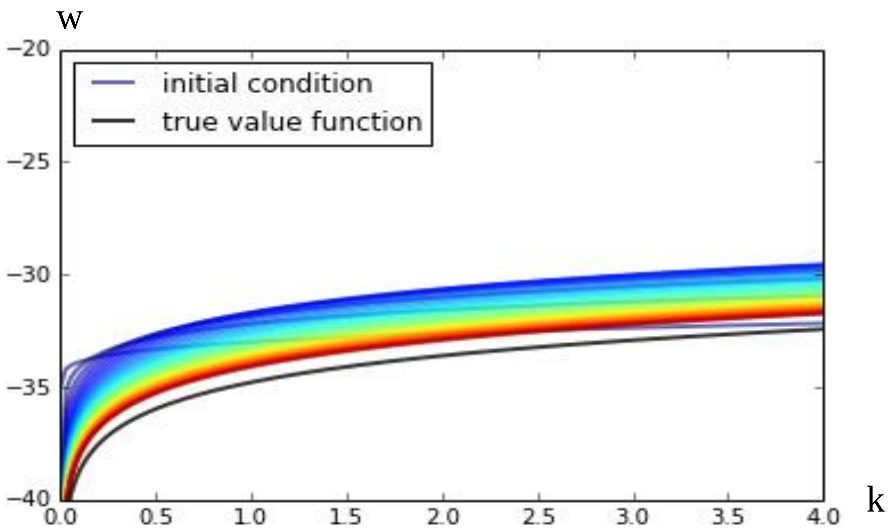

Figure (4.6): another example for w against the true value function with n= 35

If we increase n to 60, the figure will be

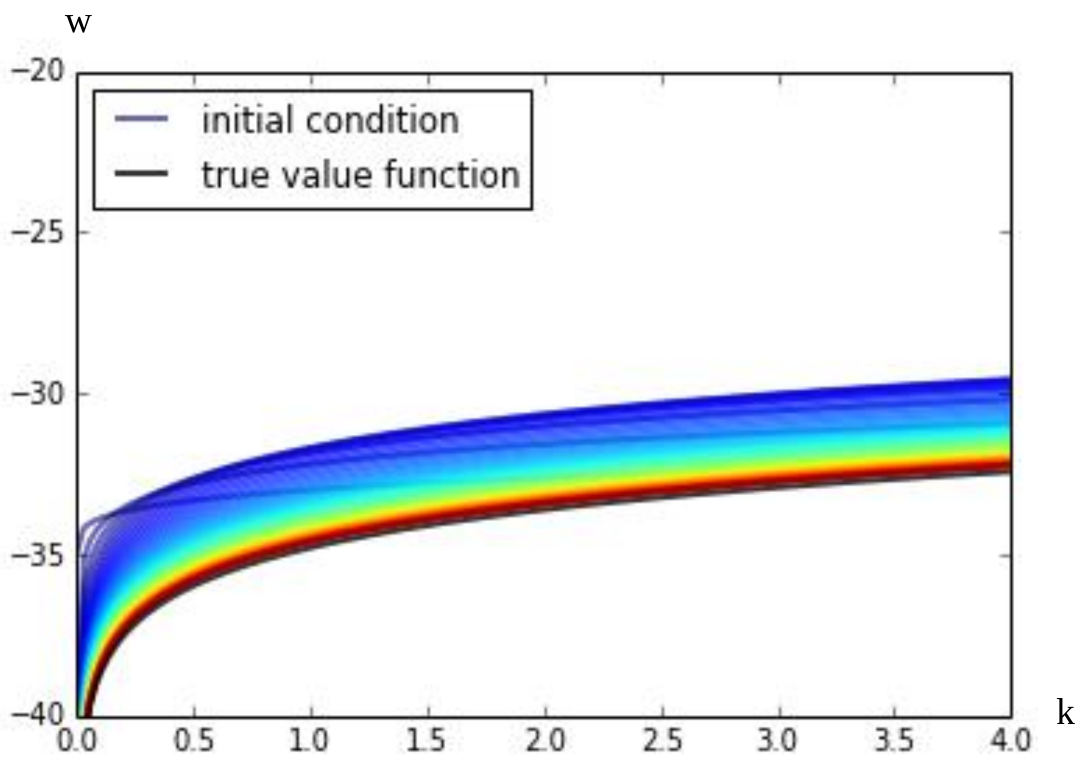

Figure (4.7): another example for w against the true value function with n= 35

Comments on the figures:

1. The two initial functions $w = 5\log(k) - 25$ and $w = 2(k)^{0.25} - 35$ have the same properties of being monotone and concave up.
2. In the first figure we reached the exact value function after 75 iterations while in the next figure we reached it in 60 iterations.
3. The main reasons that guides us to use python are
   i. Python is a general purpose programming language conceived in 1989 by Guildo Van Rossum. It is now one of the most popular programming languages.
   ii. It's free and open source. All libraries of interest are completely free. The most advantages of open source libraries is that you can read them and also you can easily change them.
   iii. Graphs and figures in python are most popular nowadays.
   iv. Python and Matlab are both high quality tools and similar in many respects. But python has some important strengths that are driving its rapid uptake in scientific computing.

Conclusion:

The performance of dynamic programming dealing with Solow and Ramsey models was distinguished among other procedures, but it still open to deduce a value function which makes the computations better and faster to converge. So, we recommended a simulation study to get such value function.

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Appendix I

$$a=\frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right)+\beta b\,\log\left(\frac{\beta b}{1+\beta b}A\right)\right] \text{ and } b=\frac{\alpha}{1-\alpha\beta}$$

If we want to rewrite a without b:

$$a=\frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right)+\beta b\,\log\left(\frac{\beta b}{1+\beta b}A\right)\right]$$

$$a=\frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right)+\log\left(\frac{\beta b}{1+\beta b}A\right)^{\beta b}\right]$$

$$=\frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right)\left(\frac{\beta b}{1+\beta b}A\right)^{\beta b}\right]$$

$$=\frac{1}{1-\beta}\left[\log\left(\frac{A^{(\beta b+1)}}{(1+\beta b)^{(\beta b+1)}}\right)(\beta b)^{(\beta b)}\right]$$

$$=\frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta b}\right)^{(\beta b+1)}(\beta)^{(\beta b)}(b)^{(\beta b)}\right] \text{ but } b=\frac{\alpha}{1-\alpha\beta}$$

$$=\frac{1}{1-\beta}\left[\log\left(\frac{A}{1+\beta\left(\frac{\alpha}{1-\alpha\beta}\right)}\right)^{\left(\beta\frac{\alpha}{1-\alpha\beta}+1\right)}(\beta)^{\left(\beta\frac{\alpha}{1-\alpha\beta}\right)}\left(\frac{\alpha}{1-\alpha\beta}\right)^{\left(\beta\frac{\alpha}{1-\alpha\beta}\right)}\right]$$

$$=\frac{1}{1-\beta}\left[\log\left(\frac{A}{\frac{1}{1-\alpha\beta}}\right)^{\left(\frac{1}{1-\alpha\beta}\right)}\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\left(\frac{\alpha\beta}{1-\alpha\beta}\right)}\right]$$

$$=\frac{1}{1-\beta}\left[\log\left(A(1-\alpha\beta)\right)^{\left(\frac{1}{1-\alpha\beta}\right)}\left[\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta}\right]^{\left(\frac{1}{1-\alpha\beta}\right)}\right]$$

$$=\frac{1}{1-\beta}.\frac{1}{1-\alpha\beta}\left[\log A(1-\alpha\beta)\left(\frac{\alpha\beta}{1-\alpha\beta}\right)^{\alpha\beta}\right]$$

Appendix II

Discount Factor Table

DISCOUNT FACTOR (p.a.) FOR A RANGE OF DISCOUNT RATES

Present Value of $1 in the Future at Discount Rate r%

Year	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%	13%	14%	15%
0	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0.9709	0.9615	0.9524	0.9434	0.9346	0.9259	0.9174	0.9091	0.9009	0.8929	0.8850	0.8772	0.8696
2	0.9426	0.9246	0.9070	0.8900	0.8734	0.8573	0.8417	0.8264	0.8116	0.7972	0.7831	0.7695	0.7561
3	0.9151	0.8890	0.8638	0.8396	0.8163	0.7938	0.7722	0.7513	0.7312	0.7118	0.6931	0.6750	0.6575
4	0.8885	0.8548	0.8227	0.7921	0.7629	0.7350	0.7084	0.6830	0.6587	0.6355	0.6133	0.5921	0.5718
5	0.8626	0.8219	0.7835	0.7473	0.7130	0.6806	0.6499	0.6209	0.5935	0.5674	0.5428	0.5194	0.4972
6	0.8375	0.7903	0.7462	0.7050	0.6663	0.6302	0.5963	0.5645	0.5346	0.5066	0.4803	0.4556	0.4323
7	0.8131	0.7599	0.7107	0.6651	0.6227	0.5835	0.5470	0.5132	0.4817	0.4523	0.4251	0.3996	0.3759
8	0.7894	0.7307	0.6768	0.6274	0.5820	0.5403	0.5019	0.4665	0.4339	0.4039	0.3762	0.3506	0.3269
9	0.7664	0.7026	0.6446	0.5919	0.5439	0.5002	0.4604	0.4241	0.3909	0.3606	0.3329	0.3075	0.2843
10	0.7441	0.6756	0.6139	0.5584	0.5083	0.4632	0.4224	0.3855	0.3522	0.3220	0.2946	0.2697	0.2472
11	0.7224	0.6496	0.5847	0.5268	0.4751	0.4289	0.3875	0.3505	0.3173	0.2875	0.2607	0.2366	0.2149
12	0.7014	0.6246	0.5568	0.4970	0.4440	0.3971	0.3555	0.3186	0.2858	0.2567	0.2307	0.2076	0.1869
13	0.6810	0.6006	0.5303	0.4688	0.4150	0.3677	0.3262	0.2897	0.2575	0.2292	0.2042	0.1821	0.1625
14	0.6611	0.5775	0.5051	0.4423	0.3878	0.3405	0.2992	0.2633	0.2320	0.2046	0.1807	0.1597	0.1413
15	0.6419	0.5553	0.4810	0.4173	0.3624	0.3152	0.2745	0.2394	0.2090	0.1827	0.1599	0.1401	0.1229
16	0.6232	0.5339	0.4581	0.3936	0.3387	0.2919	0.2519	0.2176	0.1883	0.1631	0.1415	0.1229	0.1069
17	0.6050	0.5134	0.4363	0.3714	0.3166	0.2703	0.2311	0.1978	0.1696	0.1456	0.1252	0.1078	0.0929
18	0.5874	0.4936	0.4155	0.3503	0.2959	0.2502	0.2120	0.1799	0.1528	0.1300	0.1108	0.0946	0.0808
19	0.5703	0.4746	0.3957	0.3305	0.2765	0.2317	0.1945	0.1635	0.1377	0.1161	0.0981	0.0829	0.0703
20	0.5537	0.4564	0.3769	0.3118	0.2584	0.2145	0.1784	0.1486	0.1240	0.1037	0.0868	0.0728	0.0611
21	0.5375	0.4388	0.3589	0.2942	0.2415	0.1987	0.1637	0.1351	0.1117	0.0926	0.0768	0.0638	0.0531
22	0.5219	0.4220	0.3418	0.2775	0.2257	0.1839	0.1502	0.1228	0.1007	0.0826	0.0680	0.0560	0.0462
23	0.5067	0.4057	0.3256	0.2618	0.2109	0.1703	0.1378	0.1117	0.0907	0.0738	0.0601	0.0491	0.0402
24	0.4919	0.3901	0.3101	0.2470	0.1971	0.1577	0.1264	0.1015	0.0817	0.0659	0.0532	0.0431	0.0349
25	0.4776	0.3751	0.2953	0.2330	0.1842	0.1460	0.1160	0.0923	0.0736	0.0588	0.0471	0.0378	0.0304
26	0.4637	0.3607	0.2812	0.2198	0.1722	0.1352	0.1064	0.0839	0.0663	0.0525	0.0417	0.0331	0.0264
27	0.4502	0.3468	0.2678	0.2074	0.1609	0.1252	0.0976	0.0763	0.0597	0.0469	0.0369	0.0291	0.0230
28	0.4371	0.3335	0.2551	0.1956	0.1504	0.1159	0.0895	0.0693	0.0538	0.0419	0.0326	0.0255	0.0200
29	0.4243	0.3207	0.2429	0.1846	0.1406	0.1073	0.0822	0.0630	0.0485	0.0374	0.0289	0.0224	0.0174
30	0.4120	0.3083	0.2314	0.1741	0.1314	0.0994	0.0754	0.0573	0.0437	0.0334	0.0256	0.0196	0.0151
31	0.4000	0.2965	0.2204	0.1643	0.1228	0.0920	0.0691	0.0521	0.0394	0.0298	0.0226	0.0172	0.0131
32	0.3883	0.2851	0.2099	0.1550	0.1147	0.0852	0.0634	0.0474	0.0355	0.0266	0.0200	0.0151	0.0114
33	0.3770	0.2741	0.1999	0.1462	0.1072	0.0789	0.0582	0.0431	0.0319	0.0238	0.0177	0.0132	0.0099
34	0.3660	0.2636	0.1904	0.1379	0.1002	0.0730	0.0534	0.0391	0.0288	0.0212	0.0157	0.0116	0.0086
35	0.3554	0.2534	0.1813	0.1301	0.0937	0.0676	0.0490	0.0356	0.0259	0.0189	0.0139	0.0102	0.0075
36	0.3450	0.2437	0.1727	0.1227	0.0875	0.0626	0.0449	0.0323	0.0234	0.0169	0.0123	0.0089	0.0065
37	0.3350	0.2343	0.1644	0.1158	0.0818	0.0580	0.0412	0.0294	0.0210	0.0151	0.0109	0.0078	0.0057
38	0.3252	0.2253	0.1566	0.1092	0.0765	0.0537	0.0378	0.0267	0.0190	0.0135	0.0096	0.0069	0.0049
39	0.3158	0.2166	0.1491	0.1031	0.0715	0.0497	0.0347	0.0243	0.0171	0.0120	0.0085	0.0060	0.0043
40	0.3066	0.2083	0.1420	0.0972	0.0668	0.0460	0.0318	0.0221	0.0154	0.0107	0.0075	0.0053	0.0037

جامعة النجاح الوطنية
كلية الدراسات العليا

معالجة تحليلية وديناميكية لنموذجي سولو و رامسي

إعداد
أحمد ياسر عامر ذباينة

إشراف
د. محمد نجيب أسعد

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية في نابلس، فلسطين.

2014

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الملخص

لقد قام الباحث في هذه الدراسة بدراسة نوعين من اشهر نماذج نظريات النمو الخارجي وهما سولو ورامسي، ومدى تأثيرهم على الاقتصاد الجزئي والكلي باستخدام تقنيات البرمجة الديناميكية التي تعتبر من أهم الطرق المتبعة لحل مشكلات النمو الاقتصادي.

كما وقام الباحث بمناقشة الفروقات بين النموذجين بالتفصيل. بالاضافة الى ذلك، استخدم الباحث اقتران القيمة للعالم بيلمان فيما يتعلق بالنمو وقام بتطبيقه على كلا النموذجين. وأيضا تم اشتقاق معادلتين دلاليتين فيما يخص النماذج بالدراسة خصوصا نموذج سولو.

فيما يخص النموذجين في هذه الدراسة، فقد ناقش الباحث حالة الثبات في رأس المال، وقام باشتقاق صيغة لها ، ومن ثم قام الباحث بفحص صحة هذه الصيغة على الحاسوب باستخدام برنامج البايثون. اذ عمل الباحث على اختيار اقتران قيمة جديد وهو:

$$w = 2(k)^{0.25} - 35$$

وقارن النتائج التي حصل عليها مع نتائج باحثين سابقين قاموا باختيار اقتران القيمة على صورة:

$$w = 5\log(k) - 25$$

وخلصت النتائج الى أن افتراض الباحث للاقتران كان أفضل وأنسب من سابقيه.

التوصيات:

إن أداء البرمجة الديناميكية في التعامل مع النموذجين تحت الدراسة كان مميزا أثناء التحليل، ولكن الفرصة لا تزال متاحة امام الباحثين لاختيار اقتران قيمة جديد اسرع في الوصول الى اقتران القيمة الحقيقي. لذا يوصي الباحث باجراء دراسات أخرى للحصول على اقتران أفضل.