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دراسة مقارنة بين صيغتي "الهاملتونيان واللاجرانج" للأنظمة المقيدة

A Comparison Study Between Hamiltonian And Lagrangian Formulations For The Constrained Systems

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The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any other degree or qualification

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A COMPARISON STUDY BETWEEN HAMILTONIAN AND LAGRANGIAN FORMULATIONS FOR THE CONSTRAINED SYSTEMS

BY

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A comparison Study between Hamiltonian and Lagrangian Formulations for the Constrained Systems

وبعد المناقشة التي تمت اليوم الأحد 22 جمادى اولى 1435هـ، الموافق 2014/03/23م الساعة الواحدة ظهراً، اجتمعت لجنة الحكم على الأطروحة والمكونة من:

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واللجنة إذ تمنحها هذه الدرجة فإنها توصيها بتقوى الله ولزوم طاعته وأن تسخر علمها في خدمة دينها ووطنها.

والله ولي التوفيق،،،

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أ.د. فؤاد علي العاجز

Abstract

A comparison between Hamiltonian and Lagrangian formulations, for constrained system is done. It is shown that the two approaches are equivalent. The Hamiltonian formulation are treated using Dirac's and Guler's methods. A second order Lagrangian dynamics in phase space is studied. Besides, The equivalence between the Hamiltonian and the Lagrangian formulations for the parametrization-invirant theories is done.

ملـــخص

دراسة مقارنة بين صيغتي الهاملتونيان واللاجرانج للأنظمة المقيدة

إعــــداد

الباحث/ وسيم محمد أبوطواحينة

إشـراف

الأستاذ الدكتور/ ناصر فرحـــــات

تم إجراء مقارنة بين صيغة لاجرنجيان وصيغة هاملتونيان للأنظمة المقيدة . ويتضح التكافؤ بين هاتين الصيغتين. صياغة هاملتون تمت باستخدام كل من طريقتي ديراك وجولر. تم دراسة ديناميكا اللاجرانج من الدرجة الثانية في فراغ الطور، اضافة إلى دراسة التكافؤ بين صياغة هاملتون وصياغة لاجرانج للنظريات البارامترية الغير متغايرة .

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Contents

Chapter 1

Introduction

1.1 Historical Preview

The theory of constrained systems is developed by Dirac [1,2] and it is becoming the fundamental tool for the study of classical systems of particles and fields [3,4]. In particular, the equivalence between the Lagrangian and Hamiltonian formulations for constrained systems has been established by Gotay and Nester when considering the geometric version of the dynamical equation [5]. Hamilton-Jacobi approach was developed to study singular first order systems [6 − 9].

The presence of constraints in the singular Lagrangian theories makes one to be careful when applying Dirac's method, especially when first class-constraints arise. Dirac showed that the algebra of Poisson's brackets devide the constraints into two classes: the first-class constraints and the second class ones. The first-class constraints which have zero Poisson's brackets with all other constraints in the subspace of phase space in which constraints hold. Constraints, which are not first-class, are by definition second-class. In the case of second class constraints Dirac introduced a new Poisson brackets, the Dirac brackets, to attain self-consistency However, whenever we adopt the Dirac method, we frequently encounter the problem of the operating ordering ambiguity.

In Dirac's approach, an accurate description of the constraint functions plays a crucial role. Dirac's main aim was to apply this procedure to field theory, indeed, many field theories are singular. Since the first class constraints are generators of gauge transformations which lead to the gauge freedom. In other words, the equations of motion are still degenerate and depend on the functional arbitrariness [10].

1.2 Singular Systems

This section serves as an initiation to the concept of singularities in the Lagrange formalism. We will introduce some basic notions such as constraints arising due to the singularities and the definition of the canonical momenta.

We will start our discussion of constrained systems with the principle of least action. Any physical system can be described by a function L depending on the positions and velocities[11].

$$L = L\big(q_i(t), \dot{q}_i(t)\big), \qquad i = 1, \dots, n \tag{1.1}$$

We assume for the sake of simplicity that this *Lagrange function* exhibits no explicit time dependence. The abbreviations $q(t)$and $\dot{q}(t)$stand for the set of *all* positions $q(t) = \{q_i(t)\}$ andvelocities $\dot{q}(t) = \{\dot{q}_i(t)\}$, respectively, with $i = 1, \dots n$. The system's motion proceeds in a way that the action integral

$$\mathcal{A} = \int_{t_1}^{t_2} dt\, L\big(q_i(t), \dot{q}_i(t)\big), \tag{1.2}$$

becomes stationary under infinitesimal variations $\delta q_i(t)$. Assuming that the end points are fixed during the variation, i.e. $\delta q_i(t_1) = \delta q_i(t_2) = 0$, yields the equations of motion for the classical path, which is called Euler-Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{1.3}$$

Executing the total time derivative gives

$$\ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} = \frac{\partial L}{\partial q_i} - \dot{q}_j \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}. \tag{1.4}$$

In this form we recognize that the accelerations $\ddot{q}_i$ can be uniquely expressed by the position$q_i$and the velocities $\dot{q}_i$ if and only if the Hessian matrix

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, \dots, n, \tag{1.5}$$

is invertible. In other words its determinant must not vanish.

$$\det W_{ij} \neq 0. \tag{1.6}$$

Since we are interested in the Hamiltonian formulation, we have to perform a Legendre transformation from the velocities to the momenta. The latter are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{1.7}$$

In the case that the determinant vanishes, the Lagrangian (1.1) is singular and some of the accelerations are not determined by the velocities and positions. This means that some of the variables are not independent from each other. The singularity of the Hessian is equivalent to the noninvertibility of (1.5). As a consequence, in a singular system we are not able to display the velocities as functions of the momenta and the positions. This gives rise to the existence of r relations between the positions and momenta

$$\phi_m(p_i, q_i) = 0, \tag{1.8}$$

if the rank of the Hessian matrix (1.5) is $(n - r)$. The conditions in Eq.(1.8), which obviously cannot be equations of motion, are called primary constraints [1]. They follow directly from the structure of the Lagrangian and the definition of the momenta (1.7). The interesting point is that these functions are real restrictions on the phase space.

1.3 Dirac Method

The standard quantization methods can't be applied directly to the singular Lagrangian theories. However, the basic idea of the classical treatment and the quantization of such systems were presented along time by Dirac [1,2]. And is now widely used in investigating the theoretical models in a contemporary elementary particle physics and applied in high energy physics, especially in the gauge theories [4].

This is because the first-class constraints are generators of gauge transformation which lead to the gauge freedom [12].

Let us consider a system which is described by the Lagrangian (1.1) such that the rank of the Hessian matrix is $(n - r), r < n$.

The singular system characterized by the fact that all velocities $\dot{q}_i$ are not uniquely determined in terms of the coordinates and momenta only. In other words, not all momenta are independent, and there must exist a certain set of relations among them of the form (1.8).

The generalized momenta corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, \dots, n - r, \tag{1.9}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} \equiv -H_\mu, \qquad \mu = n - r + 1, \dots, n. \tag{1.10}$$

Here $\dot{q}_i$ stands for the total derivative with respect to t.

The equations (1.10) enable us to write the primary constraint as [1,2]

$$H'_\mu = P_\mu + H_\mu = 0. \tag{1.11}$$

In this formulation the total Hamiltonian is defined as

$$H_T = H_0 + \lambda_\mu H'_\mu, \tag{1.12}$$

where the canonical Hamiltonian H_0 is defined as

$$H_0 = p_i \dot{q}_i - L, \qquad i = 1, \dots, n, \tag{1.13}$$

and λ_μ are arbitrary functions. (Throughout this thesis, we use Einstein's summation rule which means that the repeating of indices indicate to summation).

The equations of motion are obtained in term of Poisson brackets as

$$\dot{q}_i = \{q_i, H_T\} = \{q_i, H_0\} + \lambda_\mu \{q_i, H'_\mu\}, \tag{1.14}$$

$$\dot{p}_i = \{p_i, H_T\} = \{p_i, H_0\} + \lambda_\mu \{p_i, H'_\mu\}. \tag{1.15}$$

The consistency conditions, which means that the total time derivative of the primary constrains should be identically zero are given as

$$\dot{H}'_\mu = \{H'_\mu, H_T\} = \{H'_\mu, H_0\} + \lambda_\mu \{H'_\mu, H'_\nu\} \approx 0. \tag{1.16}$$

Where $\mu, \nu = n - r + 1, \dots, n$.

Equations (1.16) may be identically satisfied for the singular system with primary constraints, or lead to the new secondary constraints, repeating this procedure until one arrives at a final set of constraints or specifies some of λ_μ. Primary and secondary constraints are divided into two types: first class constraints which have vanishing poisson brackets with all other constraints, and second class constraints which have non-vanishing poisson brackets.

1.4 Hamilton– Jacobi Approach (Güler's method)

The aim is to obtain a valid and consistent Hamilton-Jacobi theory of singular system. The mathematical method used is the Caratheodory's equivalent Lagrangians method. The main point of the method is to define the equivalent Lagrangian (

variational principle) and then pass to the phase space. This formulation leads us to a set of Hamilton-Jacobi partial differential equation $[6-10]$.

1.4.1 Construction of phase space

The starting point of the Hamilton - Jacobi method is to consider the Lagrangian $L = L(q_i, \dot{q}_i, t),\ i = 1, \dots, n,$ with the Hessian matrix (1.5) of rank $(n-r), r < n$.

Then we can solve (1.9) for $\dot{q}_a$ in term of q_i, $\dot{x}_\mu$, p_a, and t as

$$\dot{q}_a = \dot{q}_a\big(q_i, \dot{x}_\mu, p_a; t\big). \qquad (1.17)$$

Substituting (1.17) into (1.10), we get

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu} = -H_\mu\big(q_i, \dot{x}_\mu, p_a; t\big). \qquad (1.18)$$

Relations (1.18) indicate the fact that the generalized momenta p_μ are not independent of p_a which is a natural result of the singular nature of the Lagrangian.

Although, it seems that $H_\mu$are functions of $\dot{x}_\mu$, it is a task to show that they do not depend on it explicitly.

The fundamental equations of the equivalent Lagrangian method read as

$$p_0 = \frac{\partial S}{\partial t} \equiv -H_0\big(q_i, \dot{x}_\mu, p_a; t\big); \qquad p_a = \frac{\partial S}{\partial q_a}, \qquad p_\mu = \frac{\partial S}{\partial \dot{q}_\mu} = -H_\mu, \qquad (1.19)$$

where the function $S \equiv S(q_i; t)$ is the action. The Hamiltonian H_0 reads as

$$H_0 = p_a \dot{q}_a + p_\mu \dot{x}_\mu|_{p_\nu = -H_\nu} - L(t, q_i, \dot{x}_\nu, \dot{q}_a), \qquad \mu, \nu = n-r+1, \dots, n. \quad (1.20)$$

Like the functions H_μ, the Hamiltonian H_0 is also not an explicit function of $\dot{x}_\mu$.

Therefore, the function $S \equiv S(q_i; t)$ should satisfy the following set of Hamilton–Jacobi partial differential equation (HJPDE′s) which is expressed as

$$H'_0\left(t, x_\mu, q_a, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0, \tag{1.21}$$

$$H'_\mu\left(t, x_\mu, q_a, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0, \tag{1.22}$$

where

$$H'_0 = P_0 + H_0, \qquad\qquad H'_\mu = p_\mu + H_\mu. \tag{1.23}$$

Equations (1.21) and (1.22) may be expressed in a compact form as

$$H'_\alpha\left(t_\beta\ , q_a\ , p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0,$$
$$\alpha, \beta = 0, n - r + 1, \dots, n, \qquad a = 1, \dots, n - r, \tag{1.24}$$

where

$$H'_\alpha = p_\alpha + H_\alpha. \tag{1.25}$$

The equations of motion are written as total differential equations in many variables t_β as follows [9]:

$$dq_i = \frac{\partial H'_\alpha}{\partial p_i} dt_\alpha, \qquad\qquad i = 0,1, \dots, n, \tag{1.26}$$

$$dp_a = \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \tag{1.27}$$

$$dp_\mu = \frac{\partial H'_\alpha}{\partial p_\mu} dt_\alpha, \tag{1.28}$$

where $a = 1, \dots, n-r$ and $\alpha = 0, n-r+1, \dots, n$.

We define

$$Z = S(t_\alpha, q_a) \tag{1.29}$$

and making use of Eq.(1.26) and definitions of generalized momenta (1.19),we obtain:

$$dZ = \frac{\partial S}{\partial t_\alpha} dt_\alpha + \frac{\partial S}{\partial q_a} dq_a = (-H_\alpha dt_\alpha + p_a dq_a) = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial P_a}\right) dt_\alpha. \tag{1.30}$$

Equations $(1.26 - 1.28)$ and (1.30) are called the total differential equations for the characteristics. If these equations form a completely integrable set, the simultaneous solutions of them determine the function $S(t_\alpha, q_a)$ uniquely by the prescribed initial conditions. The set of Equations $(1.26 - 1.28)$ is integrable if the variations of H'_α vanish identically [10, 13], that is:

$$dH'_0 = 0, \tag{1.31}$$

$$dH'_\mu = 0, \qquad \mu = n-r+1, \dots, n. \tag{1.32}$$

If condition (1.31) and (1.32) are not satisfied identically, one considers them as new constraints and again testes the consistency conditions. Hence , the canonical formulation leads to obtain the set of canonical phase space coordinates $q_\alpha$and $p_a$as functions oft_a , besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians H_α are considered as the infinitesimal generators of canonical transformations given by parameters $t_\alpha$respectively.

A general approach for solving the set of HJPDE's for the constrained system (1.21)(1.22) has been studied [7,14]. The general solution is given in the form :

$$S(q_a, q_\mu, t) = f(t) + W_a(E_a, q_a) + f_\mu(q_\mu) + A, \tag{1.33}$$

where E_a are the $(n-r)$ constants of integration and A is some other constant. The equation of motion can be obtained using the canonical transformations as follows:

$$\lambda_a = \frac{\partial S}{\partial E_a}; \qquad p_i = \frac{\partial S}{\partial q_i}, \tag{1.34}$$

where λ_a are constants and can be determined from the initial conditions. The number of λ_a is equal to the rank of the Hessian matrix.

Equation (1.33) can be solved to furnish q_a and p_i as follows :

$$q_a = q_a(\lambda_a, E_a, q_\mu, t), \tag{1.35}$$

$$p_i = p_i(\lambda_a, E_a, q_\mu, t), \tag{1.36}$$

1.5 Mixed of Lagrangian and Hamiltonian Formulation of Constrained System

1.5.1 Singular Lagrangian as Field System

Singular Lagrangian as field system has been studied in Ref [7]. As a natural extension of the Hamiltonian formulation we would like to study the Lagrangian approach of a constrained system. The usual way to pass from the Hamiltonian to the Lagrangian approach is to use Eqs. $(1.26-1.28)$. Since there are additional constraints, Eq (1.25), given in the phase space, they should also appear as constraints in the configuration space. As we have stated before, Eqs. $(1.26-1.28)$ and Eq.(1.25) allow us to treat the system as a continuous or field system. Thus, we propose that the Euler-Lagrange equations of a constrained system are in the form

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial L'}{\partial(\partial_\mu q_a)}\right] - \frac{\partial L'}{\partial q_a} = 0, \tag{1.37}$$

with constraints

$$dG_{\mu} = -\frac{\partial L'}{\partial x_{\mu}} dt, \tag{1.38}$$

where

$$L'(x_{\mu}, \partial_{\mu} q_a, \dot{x}_{\nu}, q_a) \equiv L(q_a, x_{\mu}, \dot{q}_a = (\partial_{\mu} q_a) \dot{x}_{\mu}), \qquad \dot{x}_{\nu} = \frac{dx_{\nu}}{dt}, \tag{1.39}$$

and

$$G_{\mu} = H_{\mu} \left[q_a, x_{\nu}, p_a = \frac{\partial L}{\partial \dot{q}_a} \right]. \tag{1.40}$$

One should notic that variations of constraints should be considered in order to have a consistent theory.

Chapter 2

On Singular Lagrangians and Dirac′s Method

In this chapter we study some singular Lagrangians from the classical mechanics of particles and apply Dirac's method for building the equation of motion. We find then the reason for the singularity, and therefore, we get the Hamilton equations with the familiar procedure, that is without the need of Dirac's procedure. Known cases of singular Lagrangians in special relativity are also presented [15].

2.1 Preliminaries

As it has been shown in chapter one, the transition from the Lagrangian to the Hamiltonian formalism is carried out by expressing the generalized velocities $\dot{q}_i(i = 1, \dots, n)$ in terms of the momenta $p_j = p_j(q_i, \dot{q}_i, t) = \partial L/\partial \dot{q}_j$, and eliminating them in the function $H_0 = \sum P\dot{q} - L$. This is possible if the mathematical condition.

$$\left\|\frac{\partial p_i}{\partial \dot{q}_j}\right\| = \left\|\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right\| \neq 0, \qquad (2.1)$$

is satisfied. This signifies that they build a set of independent variables. But if the determinant vanishes then there exists one or more relations between the p's :

$$\phi_k(p, q, t) = 0, \qquad k = 1, \dots, \alpha, \qquad (2.2)$$

where $n-\alpha$ is the rank of the matrix $\left(\partial^2 L/\partial \dot{q}_i \partial \dot{q}_j\right)$,thus not all p's are independent. In such situation one says that the Lagrangian is degenerat or singular, and the Hamilton equations of motion cannot be obtained by the familiar procedure. In an attempt to generalize the Hamiltonian dynamics, Dirac [1,2,16] developed a method for building the canonical equations starting from the complete Hamiltonian

$$H_T = H_0 + \sum \upsilon_k \phi_k, \tag{2.3}$$

where $H_0 = \sum \dot{q}_l p_l - L$depends on the coordinates and the independent p's, and ν_k are new independent variables. This comes from taking a virtual variation of H_0([17]):

$$\delta H_0 = \sum \left(\dot{q}_i \delta p_i - \frac{\partial L}{\partial q_i}\, \delta q_i \right) = \sum (\dot{q}_i \delta p_i - \dot{p}_i \delta q_i), \tag{2.4}$$

Using Eq. (1.30) and (1.31) we get

$$\sum \left(q_i - \frac{\partial H_0}{\partial p_i} \right) \delta p_i - \left(p_i + \frac{\partial H_0}{\partial q_i} \right) \delta q_i = 0, \tag{2.5}$$

for all $\delta p_i, \delta q_i$, consistent with the restrictions:

$$\sum_{i=1}^{n} \left(\frac{\partial \phi_k}{\partial q_i} \delta q_i + \frac{\partial \phi_k}{\partial p_i} \delta p_i \right) = 0, \qquad k = 1, \ldots, \alpha, \tag{2.6}$$

that is, α δ's of all $\delta p_i, \delta q_i$ depend on the remaining ones. Eliminating them from Eq. (2.5) by the well-known multiplier's procedure, one has

$$\dot{q}_i = \frac{\partial H_0}{\partial p_i} + \sum \upsilon_k \frac{\partial \phi_k}{\partial p_i}, \tag{2.7}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} - \sum \upsilon_k \frac{\partial \phi_k}{\partial q_i}, \qquad i = 1, \dots, n, \tag{2.8}$$

Dirac then imposes $\dot{\phi}_k = 0$ to the primary restrictions ϕ_k the consistency conditions , from which one can obtain additional restrictions. Some of these can be identities $(0 = 0)$, others of the form $f_m(q, p) = 0$ $\left(\text{like of Eq. (2.2)}\right)$, and others of type $g_l(q, p) + \upsilon_l h_l(p, q) = 0$, that can be used to fix some of the unknown variables υ_k. The second possibility is treated in a similar way as conditions $\phi_k = 0$.

2.2 Cases of Singular Lagrangians

Here, we write down a set of particular Lagrangians of a special type:

$$L = \frac{1}{2} m(\dot{q}_1^2 + \dot{q}_2^2 + l^2 \dot{q}_3^2 + 2l\dot{q}_1\dot{q}_3 cos q_3 + 2l\dot{q}_2\dot{q}_3 sin q_3) + V(q_1, q_2, q_3), \tag{2.9}$$

where l and m are constants. This is called the Mittelstaedt's Lagrangian [18],

$$L = \frac{1}{2m}(\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2\mu}\dot{q}_3^2 + V(q_1, q_2, q_3), \tag{2.10}$$

that of Cawley ([19]),

$$L = \dot{q}_1\dot{q}_2 + V(q_1, q_2, q_3), \qquad \left(V = \frac{1}{2} q_2 q_3^2\right), \tag{2.11}$$

the Lagrangian of Deriglazov[20]

$$L = q_2^2 \dot{q}_1^2 + q_1^2 \dot{q}_2^2 + 2q_1 q_2 \dot{q}_1 \dot{q}_2 + V(q_1, q_2), \tag{2.12}$$

where

$$V = q_1^2 + q_2^2. \tag{2.13}$$

They all have in common that the potential energy *V* depends only on the coordinates of the system, and that they are singular. Certainly, it is not difficult to see that there exists one relation between the *p's* in each instance:

$$\phi_1 = p_3 - l\, p_1 cos\, q_3 - l\, p_2 sin q_3 = 0, \tag{2.14}$$

$$\phi_2 = p_2 - p_1 = 0, \tag{2.15}$$

$$\phi_3 = p_3 = 0, \tag{2.16}$$

$$\phi_4 = q_1 p_1 - q_2 p_2 = 0. \tag{2.17}$$

Thus in these cases one cannot arrive at the canonical equations of motion using the well-known procedure, and we are forced to use Dirac's method.

2.3 Dirac′s Method for Mittelstaedt′s and Deriglazov Lagrangians

Actually, we will only give the details for Lagrangian (2.10) because the results for the other are found in the reference [20].

We start by obtaining the momenta of the system, using Eq. (2.10):

$$p_1 = \frac{1}{m}(\dot{q}_1 + \dot{q}_2), \quad p_2 = \frac{1}{m}(\dot{q}_1 + \dot{q}_2), \quad p_3 = \frac{1}{\mu}\dot{q}_3, \tag{2.18}$$

so, P_2 depends on p_1 and only p_1(or p_2) and p_3 are independent. The primary restriction is then Eq. (2.15) $p_2 - p_1$=0. For getting H_0(Eq. (2.3)), we eliminate the velocities from $\sum \dot{q}_i p_i - L$ in favor of the independent p's, resulting

$$H_0 = \frac{m}{2}p_1^2 + \frac{\mu}{2}p_3^2 - V. \tag{2.19}$$

The complete Hamiltonian is then

$$H = \frac{m}{2}p_1^2 + \frac{\mu}{2}p_3^2 - V + \upsilon(P_2 - P_1). \tag{2.20}$$

The consistency condition $\dot{\phi}_1 = \dot{p}_2 - \dot{p}_1 = 0$ leads to the secondary restriction

$$\dot{\phi} \quad = \frac{\partial V}{\partial q_1} - \frac{\partial V}{\partial q_2} = 0. \tag{2.21}$$

This is a relation between $q_1, q_2$and$q_3$which we briefly write as

$$\phi_2 = q_2 - F(q_1, q_3) = 0. \tag{2.22}$$

We then build the consistency condition $\dot{\phi}_2 = 0$, or

$$\dot{\phi}_2 = [\phi_2, H] = 0, \tag{2.23}$$

from which we find

$$\upsilon\left(1 + F_{,1}\right) - m\, p_1 F_{,1} - \mu\, p_3\, F_{,3} = 0, \qquad \left(F_{,i} \equiv \frac{\partial F}{\partial q_i}\right). \tag{2.24}$$

$[\phi_2, H]$ is the Poisson bracket of ϕ_2 and H. Eq. (2.24) allows the fixing of variable υ:

$$\upsilon = \frac{m\, p_1 F_{,1} + \mu\, p_3 F_{,3}}{1 + F_{,1}}. \tag{2.25}$$

With the additional relations Eq. (2.22) and Eq. (2.25), we can now write the canonical equations of motion:

$$\dot{q}_1 = mp_1 - \upsilon\ , \qquad \dot{q}_2 = \upsilon\ , \qquad \dot{q}_3 = \mu p_3\ , \tag{2.26}$$

$$\dot{P}_1 = -V_{,1} \qquad , \qquad \dot{P}_2 = V_{,2} \qquad , \quad \dot{P}_3 = V_{,3} \quad , \tag{2.27}$$

with

$$q_2 = F(q_1, q_3), \qquad\qquad \upsilon = \frac{m\, p_1 F_{,1} + \mu\, p_3 F_{,3}}{1 + F_{,1}}. \tag{2.28}$$

Thus, the independent equations of motion are

$$\dot{q}_1 = \frac{m\, p_1 - \mu\, p_3 F_{,3}}{1 + F_{,1}}, \qquad\qquad \dot{q}_3 = \mu p_3, \tag{2.29}$$

$$\dot{p}_1 = \left(V_{,1}\right)_{q_2=F}, \qquad\qquad \dot{p}_3 = \left(V_{,3}\right)_{q_2=F}. \tag{2.30}$$

Eq.(2.30) can easily be written in newtonian form

$$(1 + F_{,1})\ddot{q}_1 + F_{,3}\ddot{q}_3 + F_{,11}\dot{q}_1^2 + 2F_{,13}\dot{q}_1\dot{q}_3 + F_{,33}\dot{q}_3^2 = m\left(V_{,1}\right)_{q_2=F}, \tag{2.31}$$

$$\ddot{q}_3 = \mu\left(V_{,3}\right)_{q_2=F}. \tag{2.32}$$

On the other hand, for Deriglazov's Lagrangian Eq.(2.24) it is found that

$$H = \frac{p_1^2}{4q_2^2} - V(q_1, q_2) + \upsilon(q_1 p_1 - q_2 p_2). \tag{2.33}$$

and

$$\phi_2 = q_1 V_{,1} - q_2 V_{,2} = 0, \quad or \quad \phi_2 = q_2 - F(q_1) = 0. \tag{2.34}$$

$$\upsilon = -\frac{p_1}{2F^2\left(F + q_1 F_{,1}\right)} F_{,1}. \tag{2.35}$$

Therefore, the independent canonical equations are

$$\dot{q}_1 = -\frac{p_1}{2F^2 + 2q_1 F F_{,1}}, \tag{2.36}$$

$$\dot{P}_1 = -\frac{P_1^2}{2F^2\left(F + q_1 F_{,1}\right)} F_{,1} + \left(V_{,1}\right)_{q_2 = F(q_1)}, \tag{2.37}$$

and from here one also gets the Newton's equations of motion (Deriglazov uses $V(x,y) = x^2 + y^2 \ and \ F(x) = \pm x$)

$$2F\left(F + q_1 F_{,1}\right) q_1 + 2F\left(2F_{,1} + q_1 F_{,11}\right) q_1^2 - \left(V_{,1}\right)_{q_2 = F(q_1)} = 0, \tag{2.38}$$

2.4 An Alternative Procedure to Arrive to The Equations of Motion

The particular cases, here considered, here imply the relations $\phi_1 = 0, (2.14)$ to (2.17) between the momenta. Without regarding the Hamiltonian formalism, we can deduce the consequence of such relations .

For Lagrangian of Equ. (2.9) the p's are given by

$$p_1 = m\,\dot{q}_1 + m\,l\,\dot{q}_3 \cos q_3\,, \tag{2.39}$$

$$p_2 = m\,\dot{q}_2 + m\,l\,\dot{q}_3 \sin q_3\,, \tag{2.40}$$

$$p_3 = m\,l^2 \dot{q}_3 + m\,l(\dot{q}_1 \cos q_3 + \dot{q}_2 \sin q_3), \tag{2.41}$$

After substituting forp_1, p_2,and p_3 from Eqs. $(2.39 - 2.41)$ and telring into account Lagrange's equations, the time derivativs of (2.24) gives. If we now take the time derivative of Eq. (2.24), substitute there p_1 and p_2 from Eqs. $(2.39 - 2.41)$ and take into account Lagrange's equations, we write

$$V_{,1}\, l\, cos\, q_3 + V_{,2}\, l \sin q_3 - V_{,3} = 0. \tag{2.42}$$

In a similar way, the implication of $\phi_1 = 0$ for the remaining cases is

$$V_{,1} = V_{,2} \text{or} q_2 = F(q_1, q_3), \tag{2.43}$$

$$V_{,2} = 0, \tag{2.44}$$

$$q_1\, V_{,1} = q_2\, V_{,2} \text{ or} q_2 = F(q_1), \tag{2.45}$$

Equations $(2.42 - 2.45)$ are relations between the coordinates of each system, thus one coordinate cannot be independent. In these cases, the reason for the Lagrangian to be singular is that the coordinates are not independent, and so the canonical equations cannot be obtained by the familiar procedure, in which it is necessary that the coordinates be generalized (independent). Therefore, eliminating one of the coordinates from the corresponding Lagrangian, it would be possible to build straightforwardly the Hamilton's equations.

Let us do it for Lagrangians (2.10) and (2.12). substituting Eq. (2.43) into (2.10) we get

$$L = \frac{1}{2m}\left(1 + F_{,1}\right)^2 \dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{m}F_{,3} + \frac{1}{\mu}\right)\dot{q}_3^2 + \frac{1}{m}\left(1 + F_{,1}\right)F_{,3}\dot{q}_1\dot{q}_3 + V', \tag{2.46}$$

with

$$V'(q_1, q_3) = V(q_1, F(q_1, q_3), q_3). \tag{2.47}$$

Likewise, the substitution of Eq. (2.45) into (2.12) leads to

$$L = \left(q_1 F_{,1} + F\right)^2 \dot{q}_1^2 + V'(q_1), \qquad V'(q_1) = V\big(q_1, F(q_1)\big), \tag{2.48}$$

$$H_0 = \frac{p_1^2}{4\left(F + q_1 F_{,1}\right)^2} - V'(q_1). \tag{2.49}$$

Let us write the equation of motion for Deriglazov's Lagrangian Eq (2.49), $dp_1/dt = \partial L/\partial q_1$:

$$2\left(F+q_1F_{,1}\right)^2\ddot{q}_1+2\left(F+q_1F_{,1}\right)\left(2\,F_{,1}+q_1F_{,11}\right)\dot{q}_1^2-V'_{,1}=0. \tag{2.50}$$

This equation is equivalent to Eq. (2.38). This can be seen from Eq. (2.45) that we write at $q_2=F(q_1)$:

$$q_1\left(V_{,1}\right)_{q_2=F(q_1)}=\left(q_2V_{,2}\right)_{q_2=F(q_1)}, \tag{2.51}$$

so that

$$\left(V_{,1}\right)_{q_2=F(q_1)}=\frac{1}{q_1}F\left(V_{,2}\right)_{q_2=F(q_1)}, \tag{2.52}$$

$$V'_{,1}=\frac{1}{q_1}\left(F+q_1F_{,1}\right)\left(V_{,2}\right)_{q_2=F(q_1)}, \tag{2.53}$$

and thus factor F cancels out from Eq. (2.38), and $F+q_1F_{,1}$from Eq.(2.50).

Regarding (2.10) we get, after substituting (2.43) into (2.10),

$$L=\frac{1}{2m}A^2\dot{q}_1^2+\left(\frac{B^2}{2m}+\frac{1}{2\mu}\right)\dot{q}_3^2+\frac{A\,B}{m}\dot{q}_1\dot{q}_3+V'\,, \tag{2.54}$$

where we have done the abbreviations

$$A=1+F_{,1},\qquad B=F_{,3}, \tag{2.55}$$

$$V'(q_1,q_3)=V(q_1,q_2=F(q_1,q_3),q_3). \tag{2.56}$$

The two momenta and the generalized velocities are then given by

$$p_1=\frac{\partial L}{\partial\dot{q}_1}=\frac{A^2}{m}\dot{q}_1+\frac{AB}{m}\dot{q}_3, \tag{2.57}$$

$$p_3=\frac{AB}{m}\dot{q}_1+\left(\frac{B^2}{m}+\frac{1}{\mu}\right)\dot{q}_3, \tag{2.58}$$

$$\dot{q}_1 = \frac{m + \mu B^2}{A^2} p_1 - \frac{\mu\, B}{A} P_3, \qquad \dot{q}_3 = -\frac{\mu\, B}{A} p_1 + \mu\, p_3. \tag{2.59}$$

Thus the Hamiltonian is

$$H_1 = \frac{m}{2A^2} p_1 + \frac{\mu}{2A^2} (Bp_1 - Ap_3)^2 - V'(q_1, q_3). \tag{2.60}$$

We can now easily write the canonical equations of motion:

$$\dot{q}_1 = \frac{m + \mu B^2}{A^2} p_1 - \frac{\mu\, B}{A} P_3, \qquad \dot{q}_3 = -\frac{\mu\, B}{A} p_1 + \mu\, p_3. \tag{2.61}$$

$$\dot{p}_1 = V'_{,1} + \frac{m}{A^3} A_{,1} p_1 - \frac{\mu}{A^2} (B\, p_1 - A\, p_3) \left(B_{,1} p_1 - \frac{B}{A} A_{,1} p_3 \right), \tag{2.62}$$

$$\dot{p}_3 = V'_{,3} + \frac{m}{A^3} A_{,3} p_1 - \frac{\mu}{A^2} (B\, p_1 - A\, p_3) \left(B_{,3} p_1 - \frac{B}{A} A_{,3} p_3 \right). \tag{2.63}$$

From here we come to the equations of motion for q_1 and q_3 by eliminating p_1 and p_3 in the two last equations $(2.61 - 2.63)$. For this purpose, we derive Eqs. $(2.57 - 2.58)$ with respect to t and substitute the result in Eqs.$(2.61 - 2.63)$. We get, after solving for $\ddot{q}_1$ and $\ddot{q}_3$ and taking into account that

$$A_{,3} = B_{,1} = F_{,13} \quad , \tag{2.64}$$

the equations

$$A\, \ddot{q}_1 + A_{,1} \dot{q}_1^2 + B_{,3} \dot{q}_3^2 + 2A_{,3} \dot{q}_1 \dot{q}_3 + \mu\, B\, V'_{,3} - \frac{m + \mu\, B^2}{A} V'_{,1} = 0, \tag{2.65}$$

$$q_3 - \mu\, V'_{,3} + \mu \frac{B}{A} V'_{,1} = 0. \tag{2.66}$$

By Eq. (2.43), regarding that

$$\left(V_{,1}\right)_{q_2=F} = \left(V_{,2}\right)_{q_2=F}, \tag{2.67}$$

we can see that Eqs. (2.65) and (2.66) are fully equivalent to Eqs. (2.31) and (2.32).

For the cases presented here we then see that even though there exists a relation between the momenta, it is not necessary to apply Dirac's method for building the canonical equations. We can continue using the conventional procedure, without the need of invoking any generalization of the dynamics.

These Lagrangians are of the type

$$L = L_0\left(Q_m, \dot{Q}_m\right) + V(Q, Q_m) \tag{2.68}$$

which is known to be singular. Of course, this does not change the fact that they are so because one uses more coordinates than the number of degrees of freedom. Lowering the number of coordinates accordingly, the problems reduce to ordinary ones. Moreover, restrictions Eq.(2.43) and Eq.(2.45) are not set 'on the fly', rather they are a consequence of the way we build the Lagrangian.

We can summarize the results in other terms. If we interpret the velocity dependent part of Eq.(2.9) to Eq.(2.12) as the kinetic energy of the system,

$$T = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 = \frac{1}{2}\, m\, g_{ij} \dot{q}_i \dot{q}_j, \tag{2.69}$$

where g_{ij} are the components of the metric tensor, and sum over repeated indices is understood, then the volume element in the space of the system can be written as

$$d\tau = \sqrt{\|g_{mn}\|}\, dq_1\, dq_2\, dq_{3,} \tag{2.70}$$

where $\|g_{mn}\|$ is the determinant of the metric tensor. But if, as it is here the cases, Eq. (2.1) is violated, then the volume element vanishes, and thus the system is restricted to a space of lower dimension (e.g. a surface).

2.5 Relativistic Lagrangians

There are several possibilities to build the free particle relativistic Lagrangian that reduce to the classical expression in the limit $c \longrightarrow \infty$. For instance [15]

$$L = -m\,c\,\sqrt{c^2 - \dot{q}^2}\,, \tag{2.71}$$

for the one dimensional motion is

$$L \approx -m\,c^2 + \frac{1}{2} m\,\dot{q}^2 \tag{2.72}$$

when the velocity of the particle is much smaller than c. The corresponding Hamiltonian is, therefore

$$H = p\,\dot{q} - L = c\,\frac{p^2 + m^2c^2}{\sqrt{p^2 + m^2c^2}} = c\sqrt{p^2 + m^2c^2}, \tag{2.73}$$

where

$$p = \frac{m\,c\,\dot{q}}{\sqrt{c^2 - \dot{q}^2}}, \qquad q = \frac{c\,p}{\sqrt{p^2 + m^2c^2}}. \tag{2.74}$$

One tries to come in another way to the Hamiltonian by using the proper time τ of the particle:

$$c^2\,d\tau^2 = c^2\,dt^2 - dq^2, \tag{2.75}$$

Instead of the coordinates time t. q and t are then functions of the parameterτ: $q(\tau), t(\tau)$, so that the Lagrangian now is

$$L = -m\,c\,\sqrt{c^2 t'^2 - q'^2}, \tag{2.76}$$

where

$$t' = \frac{dt}{d\tau}, \qquad q' = \frac{dq}{d\tau}. \tag{2.77}$$

For the Lagrangian of Eq.(2.76) we can construct two momenta p_0 and p, given by

$$p_0 = \frac{\partial L}{\partial t'} = -\frac{m\,c^3 t'}{\sqrt{c^2 t'^2 - q'^2}}, \tag{2.78}$$

$$p \;\; = \frac{\partial L}{\partial q'} = \frac{m\,c\,q'}{\sqrt{c^2 t'^2 - q'^2}}. \tag{2.79}$$

It is not difficult to see that there exists a relation between them:

$$p_0^2 = c^2 p^2 + m^2 c^4, \tag{2.80}$$

so that Eq.(2.76) is singular. This Lagrangian is peculiar in a certain sense. For all Lagrangians of the form

$$L(t', q') = F(c^2 t'^2 - \ q'^2), \tag{2.81}$$

where F is an arbitrary function of the invariant $c^2 t'^2 - q'^2$, the only function F that violates Eq. (2.1) is just the square root. Indeed, the determinant Eq.(2.1) for the function (2.81) is

$$\left\| \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right\| = -4c^2 F'(F' + 2(c^2 t'^2 - q'^2)F''), \tag{2.82}$$

where F' is the derivative of F with respect of its argument, so that the determinat is zero for a function F satisfying the equation

$$F' + 2(c^2 t'^2 - q'^2)F'' = 0, \tag{2.83}$$

that

$$F(x) = a\sqrt{x} + b, \tag{2.84}$$

where a and b are constants.

Eq.(2.76) is, in this sense, the "worst" choice one can take, much in the equivalent manner as the construction of Lagrangian. If one should have started with the relativistic covariant Newton's second law

$$m\frac{d^2q^i}{d\tau^2} = m\frac{dq'^i}{d\tau} = 0, \tag{2.85}$$

where $q^0 = c\,t,\ q^1 = q$, and the line element is given by $ds = (dq^0, dq)$, and the metric tensor g_{ij} has components

$$g_{00} = 1, \qquad g_{11} = -1, \qquad g_{10} = g_{01} = 0, \tag{2.86}$$

one would have arrived at

$$L = \frac{1}{2}m\,(c^2t'^2 - q'^2), \tag{2.87}$$

which is certainly not singular. With Lagrangian (2.88) one can directly get the Hamiltonian by the familiar procedure:

$$H = \frac{p_0^2}{2m\,c^2} - \frac{p^2}{2m}. \tag{2.88}$$

The equations of motion are according to (2.76)

$$\frac{c^3mq'(q't'' - t'q'')}{(c^2t'^2 - q'^2)^{3/2}} = 0, \tag{2.89}$$

$$\frac{c^3mt'(q't'' - t'q'')}{(c^2t'^2 - q'^2)^{3/2}} = 0, \tag{2.90}$$

and they clearly reduce to only one equation, from which it follows

$$q = c_1t + c_2, \tag{2.91}$$

a relation between q and t. On the contrary, from (2.87) one get the equations

$$t'' = 0, \qquad\qquad q'' = 0, \tag{2.92}$$

or

$$t = a_1\tau + b_1, \qquad\qquad q = a_2\tau + b_2. \tag{2.93}$$

Lagrangian Eq.(2.71) describes a relativistic particle if we demand it to be real, so that $v < c$. In the case represented by Eq. (2.90), one can add the condition $v < c$ for completeness, or demand that the proper time τ (appering in Eq. (2.95), for example) must be real.

We are not diminishing the interesting properties of Lagrangian in equ. (2.76), like invariance, parametrization independence, rather we are only showing here the consequences for the existence of a relation between momenta, and how can one overcome it without the necessity of generalizing the classical dynamics.

There is another example of a (relativistic) singular Lagrangian, namely [20],

$$L = \frac{1}{2q}(\dot{q}_0^2 - \dot{q}_1^2) + \frac{1}{2}m^2 q, \tag{2.94}$$

where $q_0 = q_0(\tau)$, $q_1 = q_1(\tau)$, $q = q(\tau)$ are the unknowns and m is a constant. L is singular because

$$\frac{\partial L}{\partial \dot{q}} = 0, \qquad \text{or} \quad p = 0, \tag{2.95}$$

and this is a relation between p's.

On the other hand, the equations of motion are

$$\frac{d}{d\tau}\left(\frac{\dot{q}_0}{q}\right) = 0, \qquad \frac{d}{d\tau}\left(\frac{\dot{q}_1}{q}\right) = 0, \tag{2.96}$$

$$\frac{1}{q^2}(\dot{q}_0^2 - \dot{q}_1^2) - m^2 = 0, \tag{2.97}$$

from which the third, that is a consequence of Eq. (2.95), can be solved for $q(\tau)$:

$$q = \pm\frac{1}{m}\sqrt{\dot{q}_0^2 - \dot{q}_1^2}. \tag{2.98}$$

The first two Eqs. (2.97) can thus be expressed in the form

$$\frac{d}{d\tau}\left(\frac{\dot{q}_0}{\sqrt{\dot{q}_0^2-\dot{q}_1^2}}\right)=0, \qquad \frac{d}{d\tau}\left(\frac{\dot{q}_1}{\sqrt{\dot{q}_0^2-\dot{q}_1^2}}\right)=0, \tag{2.99}$$

and they are equivalent to the equations of motion resulting from (2.76).

According to Deriglazov, Dirac's method applied to (2.97), leads to the Hamiltonian

$$H=\frac{q}{2}(p_0^2-p_1^2-m^2)+\upsilon\, p, \tag{2.100}$$

and hence the canonical equations are

$$\dot{q}_i=qp_i, \qquad \dot{p}_i=0, \qquad \dot{q}=\upsilon, \qquad \dot{p}=0, \qquad i=0,1, \tag{2.101}$$

with the conditions (primary and secondary)

$$p=0, \qquad p_0^2-p_1^2-m^2=0. \tag{2.102}$$

The secondary condition is similar to the primary one (2.80) for Lagrangian (2.76).

The canonical equations of motion (2.101) contain an undetermined variable υ, that equals $\dot{q}$. One can intend to fix it employing the Eqs.(2.101). From Eqs. (2.101) one sees that p_0 and p_1 are constant, so that

$$\dot{q}_1=\frac{p_1}{p_0}\dot{q}_0=A\,\dot{q}_0, \qquad A=\text{constant}, \tag{2.103}$$

or

$$q_1=A\,q_0+B, \tag{2.104}$$

where B is an arbitrary constant. On the other side, variable q can be written as

$$q^2=\frac{1-A^2}{m^2}\dot{q}_0^2, \tag{2.105}$$

from which it follows

$$\dot{q}=\pm\sqrt{\frac{1-A^2}{m^2}}\,\ddot{q}_0, \tag{2.106}$$

and hence

$$\upsilon = \pm\sqrt{\frac{1-A^2}{m^2}}\,\ddot{q}_0, \tag{2.107}$$

Of course, from the canonical equations of motions q_0 and q_1 cannot be determined as functions of τ, so that υ, like q, remains undetermined.

For Lagrangians (2.76) and (2.94), one cannot avoid the use of Dirac's method for constructing the Hamiltonian, not even by employing the restrictions as was done in sec. 2.4. In the case of Eq.(2.76) the altemative is to take a different Lagrangian, for instance that given by Eq. (2.87).

Chapter 3

Second − Order Lagrangian Dynamics in The Phase − Space

3.1 Preliminaries

By means of class of nondegenerate models with a finite number of degree of freedom, it will be proved that in a Hamiltonian formulation of dynamics, there exists an equivalent second-order Lagrangian formulation whose configuration space coincides with the Hamiltonian phase-space. The above result is extended to scalar field theories in a Lorentz-covariant manner [21].

It is well-known that the Euler-Lagrange and Hamilton equations play a central role in theoretical physics. For system described by nondegenerate Lagrangians [22, 23].

$$det\left(\frac{\partial^2 L_0}{\partial \dot{q}^i \partial \dot{q}^k}\right) \neq 0 \ (locally), \tag{3.1}$$

the Euler-Lagrange equations are equivalent to the Hamilton ones. Indeed, if one defines, in the standard manner, the canonical momenta and the Hamiltonian $H_0(q^i, p_i)$, then, from the Euler-Lagrange equations, one infers the Hamilton equations. Conversely, if one eliminates algebraically the canonical momenta from the Hamiltonian equations, then one deduces the Euler-Lagrange equations. In the case of constrained (degenerate) systems [2,16], the equivalence between the two sets of equations is no longer manifest and must be implemented via the introduction of Lagrange multipliers.

For a nondegenerate system, locally described by the bosonic canonical pairs (q^i, p_i) and the Hamiltonian $H_0(q^i, p_i)$, the Hamilton equations

$$\dot{q}^i = \frac{\partial H_0}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_0}{\partial q^i}, \tag{3.2}$$

can be derived from the first-order varitional principle based on the action

$$S_0[q^i, p_i] = \int_{t_1}^{t_2} dt \left(\dot{q}^i p_i - H_0(q^i, p_i) \right). \tag{3.3}$$

It is straightforward to verify that the Euler-Lagrange equations for the first-order Lagrangian

$$\mathcal{L}(q^i, p_i, \dot{q}^i, \dot{p}_i) = \dot{q}^i p_i - H_0(q^i, p_i) \tag{3.4}$$

coincide with the Hamiltonian equations (3.2).

It is easy to see that the Lagrangian (3.4) is degenerate in the sense of the Dirac approach [1,2,16], but the canonical analysis of this Lagrangian emphasizes only second-class constraints. Then, by passing to the Dirac bracket we find that the dynamics in terms of independent variables is precisely described by Eqs. (3.2) (up to a possible change in variable notation). Thus, given a Hamiltonian formulation of dynamics for a nondegenerate system one can always construct an equivalent degenerate first-order Lagrangian formulation whose configuration space coincides with the Hamiltonian phase-space. On the other hand, the Hamiltonian action of a nondegenerate system is just an example of first-order theory. Indeed, there are many familiar dynamical systems that are described at the Lagrangian level by degenerate first-order actions. For instance, the Lagrangian actions for the Schrodinger and Dirac equations are first-order in the time derivatives and, in consequence, the equations of motion are olso first-order in time. On the other hand, the Klein-Gordon, Maxwell, and Einstein, equations are second-order (in spacetime) [21].

The previous discussion leads, to the following problem: given a first-order formulation of dynamics in terms of some variables, does there exist an equivalent, nondegenerate, second-order Lagrangian formulation in terms of the same variables? It is possible that the answer to this question is not affirmative for any first-order system. Besides the challenging aspect (intellectual or the like), the investigation of the previous problem is important from the point of view of equal footing between first- and second-order formalisms involving exactly the same variables and, implicitly, of establishing a novel equivalence between first-and second-order equations in terms of the same variables. The above-mentioned equivalence between first- and second-order equations may also be of interest to mathematicians[21].

In this chapter we start the investigation of the above problem in the framework of two classes of nondegenerate models (one class with a finite number of degrees of freedom and the other with scalar fields). In this respect we show, for each of the two models that given a Hamiltonian formulation of dynamics, one can find an equivalent second-order Lagarngian formulation whose configuration space coincides with the Hamiltonian phase-space.

In view of this, we start with a class of Hamiltonians of the form

$$H_0(q^i, p_i) = \frac{1}{2}\mu^{ij}p_i p_j + U(q^i), \tag{3.5}$$

where μ^{ij} is a constant, symmetric, and invertible matrix, μ_{ij} is the inverse of μ^{ij}, and $U(q^i)$ is an arbitrary potential. The corresponding Hamilton equations read as

$$\dot{q}^i = \mu^{ij} p_j \,, \tag{3.6}$$

$$\dot{p}_i = -\frac{\partial U}{\partial q^i} \,. \tag{3.7}$$

Now, we take the second-order Lagrangian

$$\bar{L}_0(q^i, p_i, \dot{q}^i, \dot{p}_i) = \dot{q}^i \dot{p}_i - \mu^{ij} p_i \frac{\partial U}{\partial q^j} \,, \tag{3.8}$$

from which we derive the Euler-Lagrange equations

$$\frac{\delta \bar{L}_0}{\delta q^i} \equiv -\ddot{p}_i - \mu^{kj} p_k \frac{\partial^2 U}{\partial q^i \partial q^j} = 0 \,, \tag{3.9}$$

$$\frac{\delta \bar{L}_0}{\delta p^i} \equiv -\ddot{q}^i - \mu^{ij} \frac{\partial U}{\partial q^j} = 0 \,. \tag{3.10}$$

The next theorem represents the first of our main results.

$\mathbf{3.2}$ Theorem 1

The Hamilton equations $(3.6 - 3.7)$ and the (second-order) Euler-Lagrange equations $(3.9 - 3.10)$ describe the same dynamics, i.e.

$$\begin{cases} \dot{q}^i = \mu^{ij} p_j \,, \\ \dot{p}_i = -\dfrac{\partial U}{\partial q^i} \,, \end{cases} \Leftrightarrow \begin{cases} -\ddot{P}_i - \mu^{kj} p_k \dfrac{\partial^2 U}{\partial q^i \partial q^j} = 0 \,, \\ -\ddot{q}^i - \mu^{ij} \dfrac{\partial U}{\partial q^j} = 0, \end{cases} \tag{3.11}$$

Proof. The proof of above theorem is done in three steps.

Step 1. From Eqs. $(3.6-3.7)$ and $(3.9-3.10)$ we deduce the relations

$$\frac{\delta \bar{L}_0}{\delta q^i} \equiv -\frac{d}{dt}\left(\dot{p}_i + \frac{\partial U}{\partial q^i}\right) + \frac{\partial^2 U}{\partial q^i \partial q^j}\left(\dot{q}^j - \mu^{jl} p_l\right), \tag{3.12}$$

$$\frac{\delta \bar{L}_0}{\delta p^i} \equiv -\frac{d}{dt}\left(\dot{q}_i - \mu^{ij} p_j\right) - \mu^{ij}\left(\dot{p}_j + \frac{\partial U}{\partial q^j}\right), \tag{3.13}$$

which prove that: if $\left(q^i(t), p_i(t)\right)$ are solutions of Eqs.$(3.6-3.7)$, then they are also solutions of Eqs.$(3.9-3.10)$.

Step 2. Next, we prove the following statement: if $q^i(t)$ are solutions of Eq.(3.10), then $p_i(t) = \mu_{ij}\dot{q}^j(t)$ are solutions of Eq. (3.9).

Assume that $q^i(t)$ are solutions of Eq. (3.10), i.e.

$$\frac{\delta \bar{L}_0}{\delta p^i} = 0. \tag{3.14}$$

Substituting the relations$p_i(t) = \mu_{ij}\dot{q}^j(t)$in (3.9), we obtain the formulas

$$\frac{\delta \bar{L}_0}{\delta q^i} \equiv \mu_{ij}\frac{d}{dt}\frac{\delta \bar{L}_0}{\delta p_j}, \tag{3.15}$$

which combined with (3.14)

Step 3. Finally,we establish the third conclusion, namely: if $\left(q^i(t), p_i(t)\right)$ are solutions of Equ. $(3.9-3.10)$, then they are also solutions of Eqs. $(3.6-3.7)$. We conclude that the solutions of Eqs. $(3.9-3.10)$ are given by$\left(q^i(t), p_i(t) =\right.$

$\mu_{ij}\dot{q}^j(t)\Big)$, where $q^i(t)$ are the solutions of Eq. (3.10). In consequence, the solutions to Eqs. $(3.9-3.10)$ verify Eq.(3.6). Inserting Eq.(3.6) into Eq.(3.10), we find that the solutions to Eqs. $(3.9-3.10)$ satisfy also Eq.(3.7). Conclusions of $\mathrm{step1}$ and $\mathrm{step3}$ prove the theorem.

In the context of Hamiltonians of the type (3.5) the above theorem emphasizes a new type of relationship between the Lagrangian and Hamiltonian formalisms for nondegenerate systems.

In the sequel, we extend the result of Theorem 1 to field theories in a Lorentz-covariant manner. Since we consider only nondegenerate systems, we take the simplest case of scalar field theories. In order to be specific, we take a class of Hamiltonians of the form

$$H_0 = \int d^{D-1}\mathrm{X}\left(\frac{1}{2}\mu^{ab}\pi_a\pi_b - \frac{1}{2}\mu_{ab}(\partial_i\varphi^a)(\partial^i\varphi^b) + \mathrm{V}(\varphi^a)\right), \qquad (3.16)$$

where μ^{ab} is a constant, symmetric, and invertible matrix, μ_{ab} is the inverse of μ^{ab}, and $V(\varphi^a)$ is an arbitrary function depending only on the undifferentiated scalar fields. The Hamilton equations that follow from (3.16) are given by

$$\dot{\varphi}^a = \mu^{ab}\varphi_b, \qquad (3.17)$$

$$\dot{\pi}_a = -\mu_{ab}\partial_i\partial^i\varphi^b - \frac{\partial V}{\partial\varphi^a}\,. \qquad (3.18)$$

In $(3.16-3.18)$ and in what follows, we use the standard notations $\dot{f} = \partial_0 f = \partial f/\partial t\,, \partial_i g = \partial g/\partial x^i$ and the flat Minkowski metric of "mostly minus" signature, $\sigma_{\mu v} = \sigma^{\mu v} = (+ - \cdots -)$. Comparing Eq.$(3.5)$ with Eq. (3.16), we find that the former role of $U\big(q^i\big)$ is played here by

$$\widehat{U}[\varphi^a] = \int d^{D-1}\mathrm{X}\left(-\frac{1}{2}\mu_{ab}(\partial_i\varphi^a)(\partial^i\varphi^b) + V(\varphi^a)\right) \equiv \int d^{D-1}\mathrm{X}\,\widehat{\mathcal{U}} \quad (3.19)$$

Let us try now a generalization of Eq.(3.8) of the type

$$\bar{L}_0[\varphi^a, \pi_a, \dot{\varphi}^a, \dot{\pi}_a] = \int d^{D-1}\ \mathrm{X}\left(\partial_\mu\varphi^a\partial^\mu\pi_a - \mu^{ab}\pi_a\frac{\delta\widehat{U}}{\delta\varphi^b}\right)$$

$$\equiv \int d^{D-1}\mathrm{X}\,\bar{\mathcal{L}}_0, \quad (3.20)$$

where

$$\frac{\delta\widehat{U}}{\delta\varphi^b} = \frac{\partial\widehat{\mathcal{U}}}{\partial\varphi^b} - \partial_i\frac{\partial\widehat{U}}{\partial(\partial_i\varphi^b)}\,. \quad (3.21)$$

Taking into account formulas (3.19) and (3.21) we find that the Lagrangian density of Eq.(3.20) takes the Lorentz-covariant form

$$\bar{\mathcal{L}}_0 = \left(\partial_\mu\varphi^a\right)(\partial^\mu\pi_a) - \mu^{ab}\pi_a\frac{\partial V}{\partial\varphi^b}\ , \quad (3.22)$$

which further leads to the Euler-Lagrange equations

$$\frac{\delta\bar{\mathcal{L}}_0}{\delta\varphi^a} \equiv -\partial_\mu\partial^\mu\pi_a - \mu^{bc}\pi_b\frac{\partial^2 V}{\partial\varphi^a\partial\varphi^c} = 0\,, \quad (3.23)$$

$$\frac{\delta\bar{\mathcal{L}}_0}{\delta\pi_a} \equiv -\partial_\mu\partial^\mu\varphi^a - \mu^{ab}\frac{\partial V}{\partial\varphi^b} = 0\,. \quad (3.24)$$

Under these considerations, the next theorem represents the second main result of this section.

3.3 Theorem 2

The Hamilton equations $(3.17-3.18)$ and the (second-order) Euler-Lagrange equations $(3.23-3.24)$ are equivalent, i.e.

$$\begin{cases} \dot{\varphi}^a = \mu^{ab}\pi_b \ , \\ \dot{\pi}_a = -\mu_{ab}\partial_i\partial^i\varphi^b - \dfrac{\partial V}{\partial q\varphi^a} \ , \end{cases} \Leftrightarrow \begin{cases} -\partial_\mu\partial^\mu\pi_a - \mu^{bc}\pi_b \dfrac{\partial^2 V}{\partial\varphi^a\partial\varphi^c} = 0 \ , \\ -\partial_\mu\partial^\mu\varphi^a - \mu^{ab}\dfrac{\partial V}{\partial\varphi^b} = 0, \end{cases} \tag{3.25}$$

Proof.

The proof goes along the same line with the proof of Theorem 1.(i)Bymeans of Eq.$(3.17-3.18)$and Eq.$(3.23-3.24)$ we derive the formulas

$$\frac{\delta\bar{\mathcal{L}}_0}{\delta\varphi^a} = -\partial_0\left(\dot{\pi}_a + \mu_{ab}\partial_i\partial^i\varphi^b + \frac{\partial V}{\partial\varphi^a}\right) + \left(\mu_{ab}\partial_i\partial^i\varphi^b + \frac{\partial^2 V}{\partial\varphi^a\partial\varphi^b}\right)(\dot{\varphi}^b - \mu^{bc}\pi_v) \tag{3.26}$$

$$\frac{\delta\bar{\mathcal{L}}_0}{\delta\pi_a} = -\partial_0(\dot{\varphi}^a - \mu^{ab}\pi_b) - \mu^{ab}\left(\dot{\pi}_b + \mu_{bc}\partial_i\partial^i\varphi^c + \frac{\partial V}{\partial\varphi^a}\right). \tag{3.27}$$

From $(3.26-3.27)$ we arrive at the following result: if $\left(\varphi^a(x), \pi_a(x)\right)$ are also solutions of Eqs. $(3.17-3.18)$, then they are also solutions of Eqs. $(3.23-3.24)$.

(ii) In the next step we prove that: if $\varphi^a(x)$ are solutions of Eq. (3.24), then$\pi_a(x) = \mu_{ab}\dot{\varphi}^b(x)$are solutions of Eq. (3.23)

Let $\varphi^a(x)$ be solutions to Eq. (3.24), which means that

$$\frac{\delta\bar{\mathcal{L}}_0}{\delta\pi_a}(x) = 0. \tag{3.28}$$

Replacing the formulas $\pi_a(x) = \mu_{ab}\dot{\varphi}^b(x)$ in Eq. (3.23), we find the relations

$$\frac{\delta \bar{\mathcal{L}}_0}{\delta \varphi^a} = \mu_{ab}\partial_0 \left(\frac{\delta \bar{\mathcal{L}}_0}{\delta \pi_a}(x) \right). \qquad (3.29)$$

(iii) In this step we show that the following result holds: if $\left(\varphi^a(x), \pi_a(x)\right)$ are solutions of Eqs. $(3.23 - 3.24)$,then they are also solutions of Eqs. $(3.17 - 3.18)$.

As we have proved at step (ii), the solutions to Eqs. $(3.23 - 3.24)$ are expressed by $\left(\varphi^a(x), \pi_a(x) = \mu_{ab}\dot{\varphi}^b(x)\right)$, with $\varphi^a(x)$ solutions of Eq. (3.24).Then, the solutions to also satisfied by the solutions to Eqs. $(3.23 - 3.24)$.

Theorem 2 extends to above emphasized new type of relationship between Lagrangian and Hamiltonian formalisms to scalar field theories.

Chapter 4

The Equivalence between The Hamiltonian and Lagrangian Formulation For The Parametrization-Invariant Theories

4.1 Preliminaries

The link between the treatment of singular Lagrangians as field systems and the canonical Hamiltonian approach is studied. It is shown that the singular Lagrangians as field systems are always in exact agreement with the canonical approach for the parametrization invariant theories[24].
This formulation leads us to obtain the set of Hamilton-Jacobi partial differential equations (HJPDE) as in chapter 1.

In this chapter, we study the link between the treatment of singular Lagrangians as field systems and the canonical formalism for the parametrization invariant theories [24].

4.2 Parametrization $-$ Invariant Theories as Singular Systems

Consider a system with the action integral as [25]

$$S(q_i) = \int dt\, \mathcal{L}(q_i, \dot{q}_i, t)\,, \quad i = 1, \dots, n, \tag{4.1}$$

where $\mathcal{L}$ is a regular Lagrangian with Hessian *n*. Parametrize the time $t \to \tau(t)$, with $\dot{\tau} = d\tau/dt > 0$. The velocities $\dot{q}_i$ may be expressed as

$$\dot{q}_i = q_i' \dot{\tau}, \tag{4.2}$$

where q_i' are define as

$$q_i' = \frac{dq_i}{d\tau}. \tag{4.3}$$

Denote $t \equiv q_0$ and $q_\mu \equiv (q_0, q_i), \mu = 0,1, \dots, n,$ then the action integral of Eq.(4.1) may be written as

$$S(q_\mu) = \int d\tau \; \dot{t} \, \mathcal{L}\left(q_\mu, \frac{q_i'}{\dot{t}}\right), \tag{4.4}$$

which is parametrization invariant since $\mathcal{L}$ is homogeneous of first degree in the velocities q_μ' with $\mathcal{L}$ given as

$$\mathcal{L}(q_\mu, \dot{q}_\mu) = \dot{t}\mathcal{L}\left(q_\mu, \frac{q_i'}{\dot{t}}\right). \tag{4.5}$$

The Lagrangian $\mathcal{L}$ is now singular since its Hessian is *n*.

The canonical method in $[8, 9, 12, 13]$ leads us to obtain the set of Hamilton-Jacobi Partial differential equations as follows:

$$H_0' = P_\tau - L(q_0, q_i, \dot{q}_0, (\dot{q}_i = w_i)) + p_i^\tau q_i' + p_t q_0' |_{p_t = -H_t} = 0, \qquad P_\tau = \frac{\partial S}{\partial \tau}, \tag{4.6}$$

$$H_t' = p_t + H_t = 0, \qquad p_t = \frac{\partial S}{\partial t}, \tag{4.7}$$

where H_t is defined as

$$H_t = -\mathcal{L}(q_i, w_i) + p_i^\tau w_i. \tag{4.8}$$

Here, p_i^τ and p_t are the generalized momenta conjugated to the generalized coordinates q_i and *t*, respectively.

The equations of motion are obtained as total differential equations in many variables as follows:

$$dq^i = \frac{\partial H_0'}{\partial p_i} d\tau + \frac{\partial H_t'}{\partial p_i} dq^0 = \frac{\partial H_t'}{\partial p_i} dq^0, \tag{4.9}$$

$$dp^i = -\frac{\partial H_0'}{\partial q_i} d\tau - \frac{\partial H_t'}{\partial q_i} dq^0 = -\frac{\partial H_t'}{\partial q_i} dq^0, \tag{4.10}$$

$$dp_t = -\frac{\partial H_0'}{\partial q_0} d\tau - \frac{\partial H_t'}{\partial q_0} dq^0 = 0. \tag{4.11}$$

Since

$$dH_t' = dp_t + dH_t, \tag{4.12}$$

vanishes identically, this system is integrable and the canonical phase space coordinates q_i and p_i are obtained in terms of the time $(q_0 = t)$.

Now, we look at the Lagrangian (4.5) as a field system. Since the rank of the Hessian matrix is *n*, this Lagrangian can be treated as a field system in the form

$$q_i = q_i(\tau, t), \tag{4.13}$$

thus, the expression

$$q_i' = \frac{\partial q_i}{\partial \tau} + \frac{\partial q_i}{\partial t} \dot{t}, \tag{4.14}$$

can be substituted in (4.5) to obtain the modified Lagrangian L':

$$L' = \dot{t}\mathcal{L}\left(q_\mu, \frac{1}{\dot{t}}\left(\frac{\partial q_i}{\partial \tau} + \frac{\partial q_i}{\partial t}\dot{t}\right)\right). \tag{4.15}$$

Making use of Eq.(1.43), we have

$$\frac{\partial L'}{\partial q_i} - \frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial q_i/\partial t)}\right) - \frac{\partial}{\partial \tau}\left(\frac{\partial L'}{\partial(\partial q_i/\partial \tau)}\right) = 0. \tag{4.16}$$

Calculations show that Eq.(1.43) leads to a well-known Lagrangian equation as

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial(\partial q_i/\partial t)}\right) = 0. \tag{4.17}$$

Using Eq.(4.8), we have

$$H_t = -\mathcal{L} + \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i. \tag{4.18}$$

In order to have a consistent theory, we should consider the total variation of H_t.

In fact,

$$dH_t = -\frac{\partial \mathcal{L}}{\partial t}dt. \tag{4.19}$$

Making use of Eq.(1.44), we find that

$$dH_t = -\frac{\partial L'}{\partial t}d\tau. \tag{4.20}$$

Besides, the quantity H_0 is identically satisfied and does not lead to constraints. The equations of motion (4.9) and (4.10) are equivalent to the equations of motion (4.13) and (4.14). Besides, the variations of constraints (4.19) and (4.20) are identically satisfied and no further constraints arise.

4. 3 Classical Fields as Constrained Systems

In the following sections, we study the Hamiltonian and Lagrangian formulations for classical field systems and demonstrate the equivalence between these two formulations for the reparametrization-invariant fields.

A classical relativistic field $\phi_i = \phi_i(\vec{x}, t)$in four space-time dimensions may be described as the action functional

$$S(\phi_i) = \int dt \int d^3x\{\mathcal{L}(\phi_i, \partial_\mu \phi_i)\}, \quad \mu = 0,1,2,3; \;\; i = 1,2, \dots, n, \qquad (4.21)$$

which leads to the Euler-Lagrange equations of motion as

$$\frac{\partial \mathcal{L}}{\partial \emptyset_i} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right] = 0. \qquad (4.22)$$

We can go over from the Lagrangian description to the Hamiltonian description by using the definition

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \qquad (4.23)$$

then the canonical Hamiltonian is defined as

$$H_0 = \int d^3x\,(\pi_i \dot{\phi}_i - \mathcal{L}). \qquad (4.24)$$

The equations of motion are then obtained

$$\dot{\pi}_i = -\frac{\partial H_0}{\partial \phi_i}, \qquad \dot{\phi}_i = \frac{\partial H_0}{\partial \pi_i}. \qquad (4.25)$$

4.4 Reparametrization − Invariant Fields.

In analogy with the finite dimensional systems,we introduce the reparametrization-invariant action for the field system:

$$S = \int d\tau \int \mathcal{L}_R d^3x, \qquad (4.26)$$

where

$$\mathcal{L}_R = \dot{t}\mathcal{L}(\phi_i, \partial_\mu \phi_i). \tag{4.27}$$

Following the canonical method, we obtain the set of [HJPDE],

$$H'_0 = \pi_\tau + \pi_i^{(\tau)} \frac{d\phi_i}{d\tau} + \pi_t \frac{dt}{d\tau} - \mathcal{L}_R = 0, \quad \pi_\tau = \frac{\partial S}{\partial \tau}, \tag{4.28}$$

$$H'_t = \pi_t + H_t = 0, \quad \pi_t = \frac{\partial S}{\partial t}, \tag{4.29}$$

where H_t is defined as

$$H_t = -\mathcal{L}(\phi_i, \partial_\mu \phi_i) + \pi_i^{(\tau)} \frac{d\phi_i}{dt}, \tag{4.30}$$

and $\pi_i^{(\tau)}, \pi_t$ are the generalized momenta conjugated to the generalized coordinates ϕ_i and t, respectively.

The equations of motion are obtained as follows:

$$d\phi_i = \frac{\partial H'_0}{\partial \pi_i} d\tau + \frac{\partial H'_t}{\partial \pi_i} dt = \frac{\partial H'_t}{\partial \pi_i} dt, \tag{4.31}$$

$$d\pi_i = -\frac{\partial H'_0}{\partial \phi_i} d\tau - \frac{\partial H'_t}{\partial \phi_i} dt = -\frac{\partial H'_t}{\partial \phi_i} dt, \tag{4.32}$$

$$d\pi_t = -\frac{\partial H'_0}{\partial t} d\tau - \frac{\partial H'_t}{\partial t} dt = 0. \tag{4.33}$$

Now the Euler-Lagrangian equations for the field system read as

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial \phi_i / \partial x_\mu)}\right) = 0. \tag{4.34}$$

Chapter 5

Conclusion

The treatment of Lagrangians as field systems is always in exact agreement with the Hamilton-Jacobi treatment for reparametrization-invariant theories. In analogy with the finite-dimensional systems, it is observed that the Lagrangian and the Hamilton-Jacobi treatments for the reparametrization-invariant fields are in exact agreement[24].

In the classical mechanics of particles, there is no case reported of a singular Lagrangian for a real system; all instances that we know are of artificially built system. Thus, it seem that the Lagrangians of classical mechanics are basically non degenerate [15].

Singularities appears first when we generalize to cases where there is not a previously given rule for building L, like in the special relativity. There one has the freedom to choose the Lagrangians among several possibilities, some of which are regular and others singular.

Perhaps it would be more natural to set the condition on new Lagrangians to be regular. One can argue against this by saying that the additional variables υ that appear in the theory can reveal symmetries of the system, like gauges. However, if two Lagrangians, one regular and the other singular, lead to the same set of equations (for example, field equations), they must share comparable symmetries.

To conclude with, in chapter 3 we have proved that given a Hamiltonian formulation of dynamics we can find an equivalent second-order Lagrangian formulation whose configuration space coincides with the Hamiltonian phase-space. This has been done initially in the context of a class of nondegenerate models with a finite number of degrees of freedom. The above result has been extended to scalar field theories in Lorentz-covariant manner. In a future work we hope to generalize the previous results to a generic first-order system whose one-form potential leads to a nondegenerate symplectic two-forms [21].

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