إقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Constrained Hamiltonian Systems with Higher-Order Lagrangians

أنظمة هاملتون المقيدة للدوال اللاغرانجية ذوات الرتب العالية

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CONSTRAINED HAMILTONIAN SYSTEMS with HIGHER–ORDER LAGRANGIANS

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“A thesis submitted in partial fulfillment of the requirements for the degree of master of science in department of physics faculty of science Islamic University of Gaza.”

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﴿ قَالُواْ سُبْحَانَكَ لاَ عِلْمَ لَنَا إِلاَّ مَا عَلَّمْتَنَا إِنَّكَ أَنتَ

الْعَلِيمُ الْحَكِيمُ ﴾

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إلى والدَيَّ.....

إجلالاً لقدر الأبوة.

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ABSTRACT

CONSTRAINED HAMILTONIAN SYSTEMS with HIGHER−ORDER LAGRANGIANS

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The higher−order regular Lagrangian is reduced to first−order singular Lagrangian. Dirac's method of discrete regular systems with higher−order Lagrangian, are studied as singular systems with first−order Lagrangian, and the equations of motion are obtained. It is shown that the Hamilton−Jacobi approach leads to the same equations of motion as obtained by Dirac's method. The second−order and third−order Lagrangian are studied as an examples. The Hamilton−Jacobi formulation for first−order constrained systems has been discussed. In such formalism the equations of motion are written as total differential equations in many variables. We generalize the Hamilton−Jacobi formulation for singular systems with second−order Lagrangians and apply this new formulation to Podolsky electrodynamics, comparing the results with the results obtained through Dirac's method. The equations of motion for the associated Lagrangian to a nonholonomic Lagrangian of second−order are computed in both methods Dirac and Hamilton−Jacobi. Besides, the canonical path integral quantization was obtained to quantize singular systems. All the results obtained using Hamilton−Jacobi method, are in exact agreement with those results obtained using Dirac's method.

ملخص

أنظمة هاملتون المقيدة للدوال اللاغرانجية ذوات الرتب العالية

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الأستاذ الدكتور ناصر إسماعيل فرحات

١٤٣٦-٢٠١٥

٦٠ صفحة

تتناول هذه الرسالة اختزال الدوال اللاغرانجية ذات الرتب العالية للأنظمة العادية إلى دوال لاغرانجية من الدرجة الأولى للأنظمة المقيدة، تم تطبيق طريقة ديراك للأنظمة العادية المنفصلة ذات الرتب العالية وتحويلها لأنظمة مقيدة من الدرجة الأولى ، وتم الحصول على معادلات الحركة. وقد طبقت طريقة أخرى وهي طريقة هاملتون جاكوبي والتي أدت إلى نفس معادلات الحركة التي تم الحصول عليها من خلال طريقة ديراك. وأيضا قمنا بدراسة الدوال اللاغرانجية للأنظمة المقيدة من الدرجة الثانية والدرجة الثالثة باستخدام طريقة هاملتون .

كما نوقشت صياغة هاملتون جاكوبي للأنظمة المقيدة ذات الرتبة الأولى حيث تم كتابة معادلات الحركة على شكل معادلات تفاضلية كلية في العديد من المتغيرات، استخدمنا صياغة هاملتون جاكوبي لدراسة الدالة اللاغرانجية للديناميكا الكهربائية لبودولسكي، وتم مقارنة النتائج مع طريقة ديراك.

وتم الحصول على معادلات الحركة للدوال اللاغرانجية للأنظمة الغير تامة التقييد ذات الدرجة الثانية باستخدام كلتا الطريقتين ديراك وهاملتون جاكوبي. إلى جانب ذلك، تم الحصول على المسار التكاملي لتكميم الأنظمة المقيدة.

كل النتائج التي تم الحصول عليها باستخدام طريقة هاملتون جاكوبي كانت متطابقة مع تلك النتائج التي تم الحصول عليها باستخدام طريقة ديراك.

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Chapter 1

Introduction

1.1 Historical Background

The Hamiltonian formulation of singular systems is usually made through a formalism developed by Dirac[1, 2, 3] who assumed that there is an action integral which depend on the motion, such that, when one varies the motion, and puts down the conditions for the action integral to be stationary, one gets the equation of motion [3] then he showed that, in the presence of constraints, the number of degrees of freedom of the dynamical system was reduced. His approach are subsequently extended to continuous systems [1], this formalism has found a wide range of applications in field theory [4, 5] and it is still the basic tool for the analysis for singular systems. Following Dirac, the path integral quantization of singular theories with first class constraints in canonical gauge was given by Faddeev and Popov [6]. The generalization of the method to theories with second class constraints is given by Senjanovic[7]. In this formalism the constraints caused by the Hessian matrix singularity are added to the canonical Hamiltonian [8, 9], A most powerful approach for treating constrained systems which called the Hamilton−Jacobi approach(Güler method), which has been developed to investigate the constrained systems. Several constrained systems were investigated by using the Hamilton-Jacobi approach [10, 11, 12, 13, 14, 15, 16]. The equivalent Lagrangian method is used to obtain the set of Hamilton-Jacobi Partial Differential Equation (HJPDE). In this approach, the distinction between the first- and second-class constraints is not necessary. The equations of motion are written as total differential equations in many variables, which require the investigation of the integrability conditions. In other words, the integrability conditions may lead to new constraints. Moreover, it is shown that gauge fixing, which is an essential procedure to study singular system by Diracs method, which is not necessary if the Hamilton-Jacobi approach is used.

Following Hamilton-Jacobi approach, there is another approach for quantizing constrained systems of classical singular theories by path integral quantization which is Hamilton-Jacobi quantization [17, 18].

The study of new formalisms for singular systems may provide new tools to investigate these systems. In classical dynamics, different formalisms (Lagrangian, Hamiltonian, Hamilton-Jacobi) provide different approaches to the problems, each formalism having advantages and disadvantages in the study of some features of the systems and being equivalent among themselves. In the same way, different formalisms provide different views of the features of singular systems, which justify the interest in their study.

The Hamilton-Jacobi formalism that was developed [8, 9] to include singular higher−order Lagrangians by Ostrogradsky[19]. The higher−order singular Lagrangian have been studied in many different problems of physics like general relativity, string theories, Diracs model of the radiating electron,Yang−Mills(massive vectors) fields and take a wide range in Refs.[20, 21, 22, 23].

1.2 Constrained Systems

Singular Lagrangian systems represent a special case of a more general dynamics called constrained systems[3]. A general feature of constrained system is characterized by the existence of constraint for its classical configurations. The constraints also place restrictions on the possible choice of boundary conditions for the canonical coordinates.

The dynamics of the physical system is encoded in the Lagrangian, a function of positions and velocities of all degrees of freedoms, which comprise the system[24]. The Lagrangian formulation of classical physics requires the configuration space formed by n generalized coordinates q_i, n generalized velocities $\dot{q}_i$ and parameter τ, defined as

$$L \equiv L(q_i, \dot{q}_i; t), \qquad i = 1, \ldots, n. \tag{1.2.1}$$

where τ is a parameter which will be the time on which the coordinates q_i depend. For a system characterized by this Lagrangian, the action which is a function of path in configuration space reads as

$$S = \int L(q_i, \dot{q}_i; t)\, dt. \tag{1.2.2}$$

The action principle states that the path which satisfies the classical equation is the one which brings the action to extremes

$$\begin{aligned} \delta S &= \delta \int L(q_i, \dot{q}_i; t)\, dt. \\ &= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\, \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\, \delta \dot{q}_i + \frac{\partial L}{\partial t}\, \delta t \right) dt. \end{aligned} \tag{1.2.3}$$

In deriving (1.2.3), it was assumed that $\dot{q}_i$ is dependent of q_i, so that $\delta \dot{q}_i = \frac{d}{dt}\, \delta q_i$. Imposing $\delta S = 0$, we obtain the Euler-Lagrange equations of motion

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0. \tag{1.2.4}$$

So, the Lagrangian equations are of second order.

To go over the Hamiltonian formalism, defining a generalized momentum p_i conjugate to q_i as[24]

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \tag{1.2.5}$$

then the momentum is function of q_j and $\dot{q}_j$ such that,

$$p_i = p_i(q_j, \dot{q}_j) \qquad j = 1, \ldots, n. \tag{1.2.6}$$

The canonical Hamiltonian H_0 is defined by

$$H_0 = \sum_{i=1}^{n} \dot{q}_i \, p_i - L. \tag{1.2.7}$$

Consider the differential of the Lagrange function (1.2.1) and using eqs. (1.2.4), (1.2.5) and (1.2.7), then we read off the Hamilton's equations of motion as

$$\dot{q}_i = \frac{\partial H_0}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H_0}{\partial q_i}. \tag{1.2.8}$$

It is standard national practice to define the Poisson bracket of two functions f and g on phase space by [3]

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \tag{1.2.9}$$

thus the Hamilton's equation may be written as

$$\dot{q}_i = \{q_i, H_0\}, \qquad \dot{p}_i = \{p_i, H_0\}. \tag{1.2.10}$$

So, the time evolution of any function of positions and momenta is given by

$$\frac{dF}{dt} = \{F, H_0\} + \frac{\partial F}{\partial t}. \tag{1.2.11}$$

In order to characterize the constrained systems; one evaluates the time derivative of the momentum as

$$\frac{dp_i}{dt} = \frac{\partial p_i}{\partial q_j} \dot{q}_j + \frac{\partial p_i}{\partial \dot{q}_j} \ddot{q}_j. \tag{1.2.12}$$

We can write the Lagrangian equation of motion (1.2.4) as

$$\frac{\partial L}{\partial q_i} - \frac{dp_i}{dt} = 0, \tag{1.2.13}$$

then by using the definition (1.2.5) and the Lagrangian equation of motion (1.2.4), we get

$$\frac{\partial L}{\partial q_i} = \frac{\partial p_i}{\partial q_j}\,\dot{q}_j + \frac{\partial p_i}{\partial \dot{q}_j}\,\ddot{q}_j. \tag{1.2.14}$$

$$\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_j\,\partial \dot{q}_i}\,\dot{q}_j - \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\,\ddot{q}_j = 0. \tag{1.2.15}$$

Defining Hessian matrix elements A_{ij} of second derivatives of the Lagrangian with respect to velocities as

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \tag{1.2.16}$$

so we can solve (1.2.15) for $\ddot{q}_j$ as

$$\ddot{q}_j = A_{ij}^{-1}\left[\frac{\partial L}{\partial q_i} - \frac{\partial^2 L}{\partial q_i\,\partial \dot{q}_j}\,\dot{q}_j\right]. \tag{1.2.17}$$

A valid phase space is formed if the rank of the Hessian matrix is n. Systems, which posses this property, are called regular and their treatments are found in a standard mechanics books. Systems, which have the rank less than n are called singular systems. Thus, by definition we have[4]

$$Hessian = det\left(\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\right) = \begin{cases} \neq 0 & regular\ system, \\ = 0 & singular\ system. \end{cases} \tag{1.2.18}$$

To clarify the situation of singular systems, it can be investigated by two different approach of quantization.

1.2.1 Dirac's Method of Singular Systems with First−Order Lagrangian

The Hamiltonian formulation for constrained systems is usually made through a formalism developed by Dirac [2, 3]. In this formalism Dirac showed that the algebra of Poisson brackets determines a division of constraints into two classes: so−called first−class and second− class constraints. The first−class constraints are those that have zero Poisson brackets with all other constraints in the subspace of phase space in which constraints hold; constraints which are not first−class are by definition second−class [2, 3].

Now, we will give a brief review in Dirac's approach. For the singular Lagrangian function $L(q_i, \dot{q}_i, t)$, which defined in (1.2.18) with the rank is $n-r$, $r<n$.

The generalized momenta which corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, \ldots, n-r, \tag{1.2.19}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = n-r+1, \ldots, n. \tag{1.2.20}$$

Here, $\dot{q}_i$ stands for the total derivative with respect to time t. The equation (1.2.20) enables us to write the primary constraints as

$$H'_\mu = p_\mu + H_\mu \approx 0. \tag{1.2.21}$$

One can define the total Hamiltonian as

$$H_T = H_o + \lambda_\mu H'_\mu, \tag{1.2.22}$$

where λ_μ are arbitrary functions, H'_μ are the primary constraints and H_o is the canonical (usual) Hamiltonian, which is defined as

$$H_o = p_i \dot{q}_i - L(q_i, \dot{q}_i, t), \qquad i = 1, \ldots, n. \tag{1.2.23}$$

The poisson brackets of the two function are defined as equation (1.2.9), the time variation of any function g is defined in the phase space as

$$\dot{g} = \{g, H_T\} = \{g, H_o\} + \lambda_\mu \{g, H'_\mu\}. \tag{1.2.24}$$

Thus, the equations of motion can be written as

$$\dot{q}_a = \{q_a, H_T\} = \{q_a, H_o\} + \lambda_\mu \{q_a, H'_\mu\}, \tag{1.2.25}$$

$$\dot{p}_a = \{p_a, H_T\} = \{p_a, H_o\} + \lambda_\mu \{p_a, H'_\mu\}. \quad (1.2.26)$$

The consistency conditions, which means that the total derivative of primary constraints should be vanish, are given as

$$\dot{H}'_\mu = \{H'_\mu, H_T\} = \{H'_\mu, H_o\} + \lambda_\mu \{H'_\mu, H'_\mu\} \approx 0. \quad (1.2.27)$$

These equations may be identically satisfied with the help of primary constraints, or lead to new relations which are called secondary constraints, repeating this procedure until one arrives at a final test of constraints or specifies some of λ_μ.

1.2.2 Hamilton−Jacobi Approach

Hamilton−Jacobi approach of singular systems was developed by Güler [8, 9]. One starts from singular Lagrangian $L(q_i, \dot{q}_i, t)$. Since the rank of the Hess. matrix is $n - r, r < n$, one may solves (1.2.19) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a(q_i, \dot{q}_\mu, p_a; t) \equiv \omega_a, \quad (1.2.28)$$

substituting from (1.2.28) in (1.2.20), we get

$$p_\mu = \frac{\partial L}{\partial \dot{q}_\mu}|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\mu, p_a; t). \quad (1.2.29)$$

The canonical Hamiltonian H_o is defined as

$$H_o = -L(q_i, \dot{q}_\nu, \dot{q}_a; t) + p_a \omega_a + p_\mu \dot{q}_\mu|_{p_\nu = -H_\nu}, \quad (1.2.30)$$

$$\mu, \nu = n - r + 1, \ldots, n.$$

The function H_o is not an explicite function of the velocities $\dot{q}_\mu$.Therefor, the Hamilton−Jacobi function $S(t, q_i)$ should satisfy the following set of Hamilton−Jacobi partial differential equations (**HJPDE**) simultaneously for an extremum of the function

$$H'_\alpha(t, q_\alpha, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}) = 0, \quad (1.2.31)$$

$$\alpha = 0, n - r + 1, \ldots, n.$$

where

$$H'_o = p_o + H_o, \tag{1.2.32}$$

$$H'_\mu = p_\mu + H_\mu. \tag{1.2.33}$$

The equations of motion are obtained as total differential equations in many variables as follows

$$dq_a = \frac{\partial H'_\alpha}{\partial p_a}\, dt_\alpha; \tag{1.2.34}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a}\, dt_\alpha; \tag{1.2.35}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu}\, dt_\alpha; \tag{1.2.36}$$

$$dZ = \left(-H_\alpha + P_a \frac{\partial H'_\alpha}{\partial p_a} \right) dt_\alpha; \tag{1.2.37}$$

where $Z = S(t_\alpha, q_a)$ is the action.

The equations of motion (1.2.34 - 1.2.37) are integrable if and only if [8, 9, 11]

$$dH'_\alpha = 0, \qquad \alpha = 0, n - r + 1, \ldots, n. \tag{1.2.38}$$

If dH'_α is not identically zero, we have a new constraint, repeating this procedure until a complete system is obtained. The equation (1.2.38) is the necessary and sufficient condition that the system (1.2.34 - 1.2.37) of total differential equations to be completely integrable. The set of equations of motion (1.2.34 - 1.2.36) may be only integrable, then we call this system as partially integrable[25].

1.2.3 The Canonical Path Integral Quantization

The Hamilton−Jacobi path integral quantization of singular systems can be found in refs.[11, 12, 13]. If the set of equations (1.2.34 - 1.2.37) is integrable, then one can solve them to obtain the trajectories of the motion in the canonical phase space coordinates as

$$q_a \equiv q_a(t, t_\mu), \qquad p_a \equiv p_a(t, t_\mu), \qquad \mu = 1, \ldots, r, \quad a = 1, \ldots, n-r. \tag{1.2.39}$$

Moreover, the canonical action integral is integrable and can be obtained in terms of canonical variables. In this case, the path integral representation may be written as [13, 18]

$$\psi(q'_a, t'_\alpha; q_a, t_\alpha) = \int_{q_a}^{q'_a} \prod_{a=1}^{n-r} dq^a\, dp^a \times \exp\left\{ i \int_{t_\alpha}^{t'_\alpha} \left[-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a} \right] dt_\alpha \right\}; \tag{1.2.40}$$

$\alpha = 0, n-r+1, \ldots, n.$

One should notice that the integral (1.2.40) is an integration over the canonical phase space coordinates (q_a, p_a).

1.3 Theories with Higher Derivatives

Now we shall be concerned with Lagrangian theories with higher derivatives and their canonical quantization[21, 26, 27], the Lagrangians of such theories in general case contain derivatives of higher order than one (higher derivatives) of the coordinates q_i. The Lagrangian formulation of these theories requires the configuration space formed by n generalized coordinates q_i ,$\dot{q}_i$ and $\ddot{q}_i$, etc..., can be written as

$$L = L(q_i, \dot{q}_i, \ddot{q}_i, \ldots) \tag{1.3.1}$$

which it a function of q_i , $i = 1, \ldots, n$ where n number of coordinates , and their time derivatives up to some order k.

In this case , The Euler−Lagrange equation of motion that follow from extremality of the action are of the form

$$\frac{\delta S}{\delta q_i} = \sum_{l=0}^{k} (-1)^l \frac{d^l}{dt^l} \left(\frac{\partial L}{\partial \overset{(l)}{q}_i} \right) = 0 \ . \tag{1.3.2}$$

The generalized momenta $(p_{(s-1)i}, p_{(k-1)i})$ conjugated to the generalized coordinates $(\overset{(s-1)}{q}_i, \overset{(k-1)}{q}_i)$ respectively as

$$p_{(s-1)i} = \frac{\partial L}{\partial \overset{(k)}{q}_i} - \dot{p}(s)_i \tag{1.3.3}$$

$$p_{(k-1)i} = \frac{\partial L}{\partial \overset{(k)}{q}_i}. \tag{1.3.4}$$

However, a valid phase space is formed if the rank of the Hess matrix

$$\frac{\partial^2 L}{\partial \overset{(k)}{q}_i \partial \overset{(k)}{q}_j}, \qquad i, j = 1, \ldots, n, \tag{1.3.5}$$

is n. Systems which have this property are called regular and their treatments are found in a standard mechanics books. Systems which have the rank less than n are called singular systems.

In the following two sections, one investigates singular systems using both Diracs method and the canonical method.

1.3.1 Dirac's Method of Singular Systems with Higher−Order Lagrangian

The equivalence canonical Hamiltonian H_o in (1.2.23) is defined as

$$H_0 = \sum_{u=0}^{k-2} p_{(u)i} \overset{(u+1)}{q_i} + p_{(k-1)i} f_{(k)i} + \sum_{u=0}^{k-1} \overset{(u+1)}{q_\mu} p_{(u)\mu}|_{p_{(s)\nu}=-H_{(s)\nu}} - L(\overset{(s)}{q_i}, \overset{(k)}{q_\mu}, \overset{(k)}{q_i} = f(k)_i), \tag{1.3.6}$$

where $\mu, \nu = 0, n-r+1, \ldots, n \qquad i = 1, \ldots, n-r$. Thus, the extended Hamiltonian is determined as

$$H_T = H_o + \sum_{s=0}^{(k-1)} \lambda_s H'_{(s)\mu}, \tag{1.3.7}$$

where λ_s are unknown coefficients.

The equation of motion can be written as the total derivatives in terms of Poisson brackets as

$$\overset{(u+1)}{q_i} = \{\overset{(u)}{q_i}, H_T\} = \{\overset{(u)}{q_i}, H_0\} + \sum_{s=0}^{(k-1)} \lambda_{(s)\mu}\{\overset{(u)}{q_i}, H'_{(s)\mu}\}, \tag{1.3.8}$$

$$\dot{p}(u)_i = \{p(u)_i, H_T\} = \{p(u)_i, H_0\} + \sum_{s=0}^{(k-1)} \lambda_{(s)\mu}\{p(u)_i, H'_{(s)\mu}\}, \tag{1.3.9}$$

where $u, s = n - r + 1, \ldots, n \qquad \mu, = 0, n - r + 1, \ldots, n \qquad i = 1, \ldots, n - r.$

Due to singular nature of the Hessian, we have α functionally independent relations of the form

$$H'_\mu(q_i, \overset{(k)}{p}_i, p_{(s-1)i}, p_{(k-1)i}) \approx 0. \tag{1.3.10}$$

The consistency conditions

$$\dot{H}'(u)_\mu = \{H'(u)_\mu, H_o\} + \sum_{(s=0)}^{(k-1)} \lambda_s\{H'(u)_\mu, H'(s)_\nu\} \approx 0, \tag{1.3.11}$$

lead to the secondary constraints. Sometimes, there are some difficulties to determine the multiplies λ_s, to remove this arbitrariness, one has to impose the external gauge fixing conditions for each first class constraints.

Fixing gauge is not always an easy task, which make one be careful when applying Dirac's method.

1.3.2 The Canonical Method Quantization for Higher−Order Lagrangian

The canonical method [26, 20] has been developed to investigate singular systems where the equations of motion are obtained as total differential equations in many variables, now we will give a brief review of the canonical method for higher−order Lagrangian . Let us consider a Lagrangian $L(q_i, \dot{q}_i, \ddot{q}_i, \ldots, \overset{(k)}{q}_i, t)$, If the rank of the Hess matrix $\frac{\partial^2 L}{\partial \overset{(k)}{q}_i \partial \overset{(k)}{q}_j}$ is $n - R, R < n$, then the generalized momenta conjugated to the generalized coordinates $\overset{(k)}{q}_i$ are defined as(1.3.3) and(1.3.4)
One can solve the derivatives $\overset{(k)}{q}_a$ in terms of coordinates $\overset{(s)}{q}_i$,and the momenta $p_{(k-1)b}$ and unsolved derivatives $\overset{(k)}{q}_\mu$ as follows:

$$q_a^{(k)} \equiv f_{(k)a}(q_i^{(s)}, q_\mu^{(k)}, p_{(k-1)b}, t) \tag{1.3.12}$$

where $a, b = 1, \ldots, n - r$,and $\mu = 0, n - r + 1, \ldots, n$.
The Hamiltonian is defined as(1.3.6)
Here the Hamiltonian H_0 does not depend explicitly upon the derivatives $\overset{(k)}{q}_\mu$, that is

$$\frac{\partial H_0}{\partial \overset{(k)}{q}_\mu} = 0 \tag{1.3.13}$$

Now let us consider the following notation the time parameter will be called $t_{(s)0} \equiv \overset{(s)}{q}_0$, (for any value of s) the coordinate $\overset{(s)}{q}_\mu$ will be called $t_{(s)\mu}$, the momenta $p_{(s)\mu}$ will be called $P_{(s)\mu}$, and the momentum $p_{(s)0} = P_{(s)}$ will be defined as

$$P_{(s)} = \frac{\partial S}{\partial t} \tag{1.3.14}$$

where S is the action , and $H_{(s)0} \equiv H_0$. Then to obtain an extremum of the action integral, we must find a function $S(t_{(c)\mu}, \overset{(c)}{q}_a, t), (c = 0, \ldots, k - 1)$ that satisfies the following set of Hamilton−Jacobi partial differential equation (HJPDE)

$$H'_{(s)\alpha} = P_{(s)\alpha} + H_{(s)\alpha} = 0. \tag{1.3.15}$$

where $s, u = 0, \ldots, k-1$ and $\alpha, \beta = 0, n-r+1, \ldots, n$.

The equations of motion can be written as the total differential equation as follows

$$d\overset{(u)}{q_i} = \sum_{s=0}^{k-1} \frac{\partial H'_{(s)\alpha}}{\partial p_{(u)i}} dt_{(s)\alpha}, \tag{1.3.16}$$

.

$$dp_{(u)c} = -\sum_{s=0}^{k-1} \frac{\partial H'_{(s)\alpha}}{\partial \overset{(u)}{q_c}} dt_{(s)\alpha}, \tag{1.3.17}$$

where $c = 0, 1, \ldots, n$ $\quad i = 1, \ldots, n,$

Making

$$Z \equiv S(t_{(s)\alpha}, q_a^{(s)}) \tag{1.3.18}$$

and using the momenta definitions together with equation(1.3.17), we have

$$dZ = \sum_{d=0}^{k-1} \left(-H_{(d)\alpha} + \sum_{d=0}^{k-1} P_{(s)a} \frac{\partial H_{(d)\alpha}}{\partial P_{(s)a}} \right) dt_{(d)\alpha}; \tag{1.3.19}$$

the set of equations (1.3.16 , 1.3.17 and 1.3.19) is integrable if and onlly if $dH'_{(\alpha)} = 0$, and if they form a completely integral set, their solutions determine $Z \equiv S(t_{(s)\alpha}, q_a^{(s)})$ uniquely from the initial conditions.

Our aim in this thesis is to deal with higher−order regular Lagrangian as first−order singular one, as will be introduced in the following chapters. In chapter two, the discrete systems of higher−order Lagrangian is discussed. Using Hamilton−Jacobi method, a treatment of singular Lagrangian system as field system is studied, then the canonical path integral quantization is constructed. The results that were obtained, using the Hamilton−Jacobi method are in exact agreement with those obtained using Dirac's method in Ref. [28],[14]. An example with continuous systems with second order Lagrangian is studied, in chapter three.

In chapter four, the discrete systems with higher order regular Lagrangian is treated as first order singular Lagrangian. We used the Hamilton−Jacobi method to construct the equations of motion as total differential equations. These equations are integrable under specified conditions on new coordinates ,quantization of regular Lagrangian of nonholonomic spinning particle are studied as an example.

Chapter 2

Applications on Second And Third Order Lagrangians

Systems with higher−order Lagrangian have been studied with increasing interest because they appear in many relevant physical problems. Many authors studied higher−order singular Lagrangian systems using both Dirac and Hamilton−Jacobi approaches [14, 15, 16, 23, 28]. A treatment of singular Lagrangian system as field system was studied in Ref[28] In this chapter, we will discuss two models of discrete systems with higher−order Lagrangian. In section (2.1), we will make a brief discussion of the canonical method to investigate singular system of the second−order singular Lagrangian and an example is solved. Hamilton−Jacobi method for third−order will be addressed in section (2.2) with an example will be discussed. The canonical path integral quantization is displayed in section (2.3).

2.1 Canonical Formulation of The Second−Order Lagrangian

The second−order Lagrangian is described by the function $L(q_i, \overset{(2)}{q_i}, \overset{(1)}{q_i}{}_i, t)$, where $q_i^{(s)} = \frac{d^s q_i}{dt^s}$, $s = 0, 1, 2$ and $i = 1, ..., N$. The system is regular if the rank of the

Hessian matrix[20, 26]

$$A_{ij} = \frac{\partial^2 L}{\partial q_i^{(2)} \partial q_j^{(2)}} \tag{2.1.1}$$

is N , and singular if the rank is $N - R$, $R < N$. The generalized momenta can be written as

$$p_{(0)i} = \frac{\partial L}{\partial q_i^{(1)}} - \frac{d}{dt}(\frac{\partial L}{\partial q_i^{(2)}}). \tag{2.1.2}$$

$$p_{(1)i} = \frac{\partial L}{\partial q_i^{(2)}} \tag{2.1.3}$$

where $p_{(0)i}$, $p_{(1)i}$ are the momenta conjugated to the coordinates $q_i, q_i^{(1)}$ respectively. Since the rank of Hessian matrix is N-R, one may solve equation (2.1.3) for $q_a^{(2)}$ as a function of $p_{(1)a}, q_\mu^{(2)}$ and t as,

$$q_a^{(2)} \equiv f_{(2)a}(q_i, q_i^{(1)}, q_i^{(2)}, p_{(1)a}, t), \tag{2.1.4}$$

where $a = 1, ..., N - R$ and $\mu = N - R + 1, ..., N$. Since the momenta are not independent; $p_{(s)\mu}$ can be written as

$$p_{(s)\mu} = -H_{(s)\mu}(q_j^{(u)}, p_{(u)a}, t), \tag{2.1.5}$$

where $u, s = 0, 1$, $u \geq s$, $j = 1, 2$.

The canonical method leads us to obtain the set of Hamilton−Jacobi partial differential equations as

$$H_0' = p_0 + H_0(q_i^{(u)}, p_{(u)a}, t) = 0, \tag{2.1.6}$$

$$H_{(s)\mu}' = p_{(s)\mu} + H_{(s)\mu}(q_i^{(u)}, p_{(u)a}, t) = 0. \tag{2.1.7}$$

The usual Hamiltonian H_0 is defined as

$$H_0 = p_{(0)a}q_a^{(1)} + p_{(1)a}f_{(2)a} + q_\mu^{(1)}p_{(0)\mu}|_{p_{(0)\mu}=-H_{(0)\mu}} + q_\mu^{(2)}p_{(1)\mu}|_{p_{(1)\mu}=-H_{(1)\mu}}$$
$$-L(q_i, q_i^{(1)}, q_i^{(2)} = f_{(2)a}). \tag{2.1.8}$$

The Hamiltonian H_0 does not depend on $q_\mu^{(2)}$, so that

$$\frac{\partial H_0}{\partial q_\mu^{(2)}} = 0. \tag{2.1.9}$$

The equations of motion in the canonical method are given as total differential equations in many variables of the forms

$$dq_a = \frac{\partial H_0'}{\partial p_{(0)a}} dt + \frac{\partial H_{(0)\mu}'}{\partial p_{(0)a}} dq_\mu, \tag{2.1.10}$$

$$dq_a^{(1)} = \frac{\partial H_0'}{\partial p_{(1)a}} dt + \frac{\partial H_{(0)\mu}'}{\partial p_{(1)a}} dq_\mu + \frac{\partial H_{(1)\mu}'}{\partial p_{(1)a}} dq_\mu^{(1)}, \tag{2.1.11}$$

$$dp_{(0)_i} = -\frac{\partial H_0'}{\partial q_i} dt - \frac{\partial H_{(0)\mu}'}{\partial q_i} dq_\mu - \frac{\partial H_{(1)\mu}'}{\partial q_i} dq_\mu^{(1)} - \frac{\partial H_{(2)\mu}'}{\partial q_i} dq_\mu^{(2)}, \tag{2.1.12}$$

$$dp_{(1)i} = -\frac{\partial H_0'}{\partial q_i^{(1)}} dt - \frac{\partial H_{(0)\mu}'}{\partial q_i^{(1)}} dq_\mu - \frac{\partial H_{(1)\mu}'}{\partial q_i^{(1)}} dq_\mu^{(1)} - \frac{\partial H_{(2)\mu}'}{\partial q_i^{(1)}} dq_\mu^{(2)}, \tag{2.1.13}$$

The set of total differential equations (2.1.10-2.1.13) is integrable if the variation of the constraints H_0' and $H_{(s)\mu}'$ are identically zero. If the variations of H_0' and $H_{(s)\mu}'$ are not identically zero, then a new constraint will arise and again we test the integrability conditions until we obtain a complete system [3,9].

2.1.1 An Example

In the next point we will use the canonical method and the treatment of singular system as field system to solve the second−order singular Lagrangian[14],

$$L = \frac{1}{2}\beta\left((q_1^{(2)})^2 - 2(q_1^{(2)}q_2^{(2)}) + (q_2^{(2)})^2\right) + \frac{1}{2}K(q_1^{(1)} - q_2^{(1)})^2 + \frac{1}{2}\gamma(q_1 - q_2) \tag{2.1.14}$$

The system is singular since the rank of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial q_i^{(2)} \partial q_j^{(2)}} \tag{2.1.15}$$

is ONE. where $i, j = 1, 2$.

The canonical momenta (2.1.2-2.1.3) are

$$p_{(1)1} = \frac{\partial L}{\partial {q_1}^{(2)}} = \beta({q_1}^{(2)} - {q_2}^{(2)}), \tag{2.1.16}$$

$$p_{(1)2} = \frac{\partial L}{\partial {q_2}^{(2)}} = \beta({q_2}^{(2)} - {q_1}^{(2)}) = -P_{(1)1} = -H_{(1)2}, \tag{2.1.17}$$

$$p_{(0)1} = \frac{\partial L}{\partial {q_1}^{(1)}} - \dot{p}_{(1)1} = K({q_1}^{(1)} - {q_2}^{(1)}) - \beta \frac{d}{dt}({q_1}^{(2)} - {q_2}^{(2)}), \tag{2.1.18}$$

$$p_{(0)2} = \frac{\partial L}{\partial {q_2}^{(1)}} - \dot{p}_{(1)2} = K({q_2}^{(1)} - {q_1}^{(1)}) - \beta \frac{d}{dt}({q_2}^{(2)} - {q_1}^{(2)}) = -p_{(0)1} = -H_{(0)2}, \tag{2.1.19}$$

Equations (2.1.16) can be solved for $q_1^{(2)}$ as

$${q_1}^{(2)} = \frac{p_{(1)1}}{\beta} + {q_2}^{(2)} = f_{(2)1}, \tag{2.1.20}$$

The Hamiltonian (2.1.8), takes the form

$$H_0 = \frac{1}{2\beta}(p_{(1)1})^2 + p_{(0)1}(q_1^{(1)} - q_2^{(1)}) - \frac{1}{2}K(q_1^{(1)} - q_2^{(1)})^2 - \frac{1}{2}\gamma(q_1 - q_2)^2 \tag{2.1.21}$$

The set of Hamilton−Jacobi equations (2.1.6) and (2.1.7) can be read as

$$H'_{(0)2} = p_{(0)2} + H_{(0)2} = p_{(0)2} + p_{(0)1} = 0, \tag{2.1.22}$$

$$H'_{(1)2} = p_{(1)2} + H_{(1)2} = p_{(1)2} + p_{(1)1} = 0, \tag{2.1.23}$$

$$H'_0 = p_0 + H_0 = 0. \tag{2.1.24}$$

The equations of motion (2.1.10-2.1.13) can be written as

$$dq_1 = \frac{\partial H'_0}{\partial p_{(0)1}}dt + \frac{\partial H'_{(0)2}}{\partial p_{(0)1}}dq_2 = ({q_1}^{(1)} + {q_2}^{(1)})dt + dq_2, \tag{2.1.25}$$

$$dq_1^{(1)} = \frac{\partial H'_0}{\partial p_{(1)1}}dt + \frac{\partial H'_{(0)2}}{\partial p_{(1)1}}dq_2 + \frac{\partial H'_{(1)2}}{\partial p_{(1)1}}dq_2^{(1)} = \frac{1}{\beta}p_{(1)1}dt + dq_2^{(1)}, \tag{2.1.26}$$

$$dp_{(0)1} = -\frac{\partial H'_0}{\partial q_1}dt - \frac{\partial H'_{(0)2}}{\partial q_1}dq_2 - \frac{\partial H'_{(1)2}}{\partial q_1}dq_2^{(1)} = \gamma(q_1 - q_2)dt. \tag{2.1.27}$$

$$dp_{(0)2} = -\frac{\partial H'_0}{\partial q_2}dt - \frac{\partial H'_{(0)2}}{\partial q_2}dq_2 - \frac{\partial H'_{(1)2}}{\partial q_2}dq_2^{(1)} = -\gamma(q_1 - q_2)dt. \tag{2.1.28}$$

$$dp_{(1)1} = -\frac{\partial H'_0}{\partial q_1^{(1)}}dt - \frac{\partial H'_{(0)2}}{\partial q_1^{(1)}}dq_2 - \frac{\partial H'_{(1)2}}{\partial q_1^{(1)}}dq_2^{(1)} = \left(-P_{(0)1} + K(q_1^{(1)} - q_2^{(1)})\right) dt. \tag{2.1.29}$$

$$dp_{(1)2} = -\frac{\partial H'_0}{\partial q_2^{(1)}}dt - \frac{\partial H'_{(0)2}}{\partial q_2^{(1)}}dq_2 - \frac{\partial H'_{(1)2}}{\partial q_2^{(1)}}dq_2^{(1)} = \left(P_{(0)1} - K(q_1^{(1)} - q_2^{(1)})\right) dt. \tag{2.1.30}$$

The system of total differential equations (2.1.25-2.1.30) is integrable if the variation of $H'_{(s)2}$ is identically zero.

The variation of $H'_{(0)2}$ is

$$dH'_{(0)2} = dp_{(0)2} + dp_{(0)1} = 0 \tag{2.1.31}$$

The variation of $H'_{(1)2}$ is

$$dH'_{(1)2} = dp_{(1)2} + dp_{(1)1} = 0. \tag{2.1.32}$$

From (2.1,31) and (2.1.32), we conclude that the system is integrable.

2.1.2 Treatment of Second−Order Lagrangian as Field System

The second−order Lagrangian system can be treated as field system where the field q_a are expressed in term of the independent coordinates as

$$q_a \equiv q_a(t, x_\mu), \tag{2.1.33}$$

where $\mu = N - R + 1, ..., N$. The Euler-Lagrange equation of the second−order Lagrangian takes the form

$$\frac{\partial L'}{\partial q_a} - \frac{\partial}{\partial x_\mu}(\frac{\partial L'}{\partial(\partial_\mu q_a)}) + \frac{\partial^2}{\partial x_{\mu_1}\partial x_{\mu_2}}(\frac{\partial L'}{\partial(\partial_{\mu_1}\partial_{\mu_2} q_a)}) = 0, \tag{2.1.34}$$

where

$$\partial_\mu q_a \equiv \frac{\partial q_a}{\partial x_\mu}, \tag{2.1.35}$$

$$\partial_{\mu_1}\partial_{\mu_2} q_a \equiv \frac{\partial^2 q_a}{\partial x_{\mu_1}\partial x_{\mu_2}}, \tag{2.1.36}$$

and the modified Lagrangian L' is defined as [10,14]

$$L'(x_\mu, q_a, \partial_\mu q_a, \partial_{\mu_1}\partial_{\mu_2} q_a, x_\mu^{(1)}, x_\mu^{(2)}) \equiv L(x_\mu, q_a, q_a^{(1)} = (\partial_\mu q_a)x_\mu^{(1)},$$

$$q_a^{(2)} = \partial_{\mu_2}(\partial_{\mu_1} q_a x_{\mu_1}^{(1)}) x_{\mu_2}^{(1)}) \tag{2.1.37}$$

with $x_\mu^{(1)} = \frac{dx_\mu}{dt}$ and $x_0^{(1)} = 1$

The constraint equation can be written as

$$dG_0 = -\frac{\partial L'}{\partial t} dt, \tag{2.1.38}$$

$$dG_\mu = -\frac{\partial L'}{\partial q_\mu} dt. \tag{2.1.39}$$

where

$$G_0 = H_0(q_i^{(u)}, p_{(u)a}, t), \tag{2.1.40}$$

$$G_\mu = H_\mu(q_i^{(u)}, p_{(u)a}, t). \tag{2.1.41}$$

Solving of Euler-Lagrange equation (2.1.34) together with the constraint equations (2.1.38) and (2.1.39) we get the solution of the system. Since the rank of Hessian matrix of the Lagrangian (2.1.15) is one then the system is singular, and can be written as field system in the form

$$q_1 = q_1(q_2, t), \tag{2.1.42}$$

The first and second derivatives with respect to t of (2.1.42) is:

$$q_1^{(1)} = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2} q_2^{(1)}, \tag{2.1.43}$$

$$q_1^{(2)} = \frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t \partial q_2} q_2^{(1)} + \frac{\partial^2 q_1}{\partial q_2^2}(q_2^{(1)})^2 + \frac{\partial q_1}{\partial q_2} q_2^{(2)}, \tag{2.1.44}$$

Substituting (2.1.43) into (2.1.37) we get the modified Lagrangian function

$$L' = \frac{1}{2}\beta\Bigg[\left(\frac{\partial^2 q_1}{\partial t^2}\right)^2 + 4\frac{\partial^2 q_1}{\partial t^2}\frac{\partial^2 q_1}{\partial t \partial q_2} q_2^{(1)} + 2q_2^{(2)}\frac{\partial^2 q_1}{\partial t^2}\frac{\partial q_1}{\partial q_2}$$

$$+2\frac{\partial^2 q_1}{\partial t^2}\frac{\partial^2 q_1}{\partial q_2^2}(q_2^{(1)})^2 + 4\left(\frac{\partial^2 q_1}{\partial t \partial q_2}\right)^2 (q_2^{(1)})^2 + 4\frac{\partial^2 q_1}{\partial t \partial q_2}\frac{\partial q_1}{\partial q_2} q_2^{(1)} q_2^{(2)}$$

$$+2\frac{\partial q_1}{\partial q_2}\frac{\partial^2 q_1}{\partial q_2^2}q_2^{(2)}(q_2^{(1)})^2+4\frac{\partial^2 q_1}{\partial t\partial q_2}\frac{\partial^2 q_1}{\partial q_2^2}(q_2^{(1)})^3+(q_2^{(2)})^2\left(\frac{\partial q_1}{\partial q_2}\right)^2+(q_2^{(1)})^4\left(\frac{\partial^2 q_1}{\partial q_2^2}\right)^2$$

$$-2\frac{\partial^2 q_1}{\partial t^2}q_2^{(2)}-4\frac{\partial^2 q_1}{\partial t\partial q_2}(q_2^{(1)})(q_2^{(2)})-2\frac{\partial q_1}{\partial q_2}(q_2^{(2)})^2-2\frac{\partial^2 q_1}{\partial q_2^2}(q_2^{(1)})^2q_2^{(2)}+(q_2^{(2)})^2\Bigg]$$

$$+\frac{1}{2}K\Bigg[\left(\frac{\partial q_1}{\partial t}\right)^2+2\frac{\partial q_1}{\partial t}\frac{\partial q_1}{\partial q_2}q_2^{(1)}-2\frac{\partial q_1}{\partial t}q_2^{(1)}+\left(\frac{\partial q_1}{\partial q_2}\right)^2(q_2^{(1)})^2$$

$$-2\frac{\partial q_1}{\partial q_2}(q_2^{(1)})^2+(q_2^{(1)})^2\Bigg]+\frac{1}{2}\gamma(q_1-q_2)^2 \tag{2.1.45}$$

The Euler-Lagrange equations (2.1.34) read as

$$\frac{\partial L'}{\partial q_1}-\frac{\partial}{\partial t}\Big(\frac{\partial L'}{\partial(\frac{\partial q_1}{\partial t})}\Big)-\frac{\partial}{\partial q_2}\Big(\frac{\partial L'}{\partial(\frac{\partial q_1}{\partial q_2})}\Big)+\frac{\partial^2}{\partial t^2}\Big(\frac{\partial L'}{\partial(\frac{\partial^2 q_1}{\partial t^2})}\Big)+\frac{\partial^2}{\partial t\partial q_2}\Big(\frac{\partial L'}{\partial(\frac{\partial^2 q_1}{\partial t\partial q_2})}\Big)+\frac{\partial^2}{\partial q_2^2}\Big(\frac{\partial L'}{\partial(\frac{\partial^2 q_1}{\partial q_2^2})}\Big)$$

$$\frac{\partial^2}{\partial q_2\partial t}\Big(\frac{\partial L'}{\partial(\frac{\partial^2 q_1}{\partial q_2\partial t})}\Big)=0 \tag{2.1.46}$$

Making use of (2.1.45), equation (2.1.46) becomes

$$\gamma(q_1-q_2)-\frac{\partial}{\partial t}\Bigg[k\Big(q_1^{(1)}-q_2^{(1)}\Big)\Bigg]-\frac{\partial}{\partial q_2}\Bigg[\beta\Big(q_1^{(2)}-q_2^{(2)}\Big)q_2^{(2)}+k\Big(q_1^{(1)}-q_2^{(2)}\Big)q_2^{(1)}\Bigg]$$

$$+\frac{\partial^2}{\partial t^2}\Bigg[\beta\Big(q_1^{(2)}-q_2^{(2)}\Big)\Bigg]+2\frac{\partial^2}{\partial t\partial q_2}\Bigg[\beta\Big(q_1^{(2)}-q_2^{(2)}\Big)q_2^{(1)}\Bigg]$$

$$+\frac{\partial^2}{\partial q_2^2}\Bigg[\beta\Big(q_1^{(2)}-q_2^{(2)}\Big)(q_2^{(1)})^2\Bigg]=0 \tag{2.1.47}$$

The constraint equation (2.1.39) read as

$$\frac{\partial L'}{\partial q_2}-\frac{dP_{(0)2}}{dt}=0. \tag{2.1.48}$$

Using (2.1.28) and (2.1.45), we get:

$$-\gamma(q_1-q_2)+\frac{\partial}{\partial t}\left\{k\left(q_1^{(1)}-q_2^{(1)}\right)\right\}+\frac{\partial}{\partial q_2}\left\{k\left(q_1^{(1)}-q_2^{(2)}\right)q_2^{(1)}\right\}-\frac{\partial^2}{\partial t^2}\left\{\beta\left(q_1^{(2)}-q_2^{(2)}\right)\right\}$$

$$-2\frac{\partial^2}{\partial t\partial q_2}\left\{\beta\left(q_1^{(2)}-q_2^{(2)}\right)q_2^{(1)}\right\}-\frac{\partial}{\partial q_2}\left\{\beta\left(q_1^{(2)}-q_2^{(2)}\right)q_2^{(2)}\right\}$$

$$-\frac{\partial^2}{\partial q_2^2}\left\{\beta\left(q_1^{(2)}-q_2^{(2)}\right)\left(q_2^{(1)}\right)^2\right\}=0 \tag{2.1.49}$$

Adding (2.1.47)and(2.1.49) , we get

$$\frac{\partial}{\partial q_2}\left\{\beta\left(q_1^{(2)}-q_2^{(2)}\right)\right\}\equiv\frac{\partial}{\partial q_2}(P_{(1)1})=0. \tag{2.1.50}$$

Using (2.1.50)and(2.1.47) become,

$$\gamma(q_1-q_2)-\frac{d}{dt}\left\{k\left(q_1^{(1)}-q_2^{(1)}\right)\right\}+\frac{d^2}{dt^2}\left\{\beta\left(q_1^{(1)}-q_2^{(1)}\right)\right\}=0. \tag{2.1.51}$$

Notice that equation (2.1.50) is equivalent to equation (2.1.26) and equation (2.1.51) is equivalent to the sum of equations (2.1.27) and (2.1.29).

2.2 Canonical Formulation of The Third−Order Lagrangian

The third−order Lagrangian is described by the function $L(q_i, q_i^{(1)}, q_i^{(2)}, q_i^{(3)}, t)$, where $q_i^{(s)} = \frac{d^s q_i}{dt^s}$, $s = 0, 1, 2, 3$ and $i = 1, ..., N$. The system is regular if the rank of the Hessian matrix [20,26]

$$A_{ij}=\frac{\partial^2 L}{\partial q_i^{(3)}\partial q_j^{(3)}} \tag{2.2.1}$$

is N , and singular if the rank is $N-R$, $R<N$. The generalized momenta can be written as

$$p_{(0)i}=\frac{\partial L}{\partial q_i^{(1)}}-\frac{d}{dt}(\frac{\partial L}{\partial q_i^{(2)}})+\frac{d^2}{dt^2}(\frac{\partial L}{\partial q_i^{(3)}})\equiv\frac{\partial L}{\partial q_i^{(1)}}-\dot{p}_{(1)i}, \tag{2.2.2}$$

$$p_{(1)i}=\frac{\partial L}{\partial q_i^{(2)}}-\frac{d}{dt}(\frac{\partial L}{\partial q_i^{(3)}})\equiv\frac{\partial L}{\partial q_i^{(2)}}-\dot{p}_{(2)i}, \tag{2.2.3}$$

$$p_{(2)i}=\frac{\partial L}{\partial q_i^{(3)}}, \tag{2.2.4}$$

where $p_{(0)i}$, $p_{(1)i}$ and $p_{(2)i}$ are the momenta conjugated to the coordinates $q_i, q_i^{(1)}$ and $q_i^{(2)}$ respectively.

Since the rank of Hessian matrix is N-R, one may solve equation (2.1.3) for $q_a^{(3)}$ as a function of $p_{(1)a}, p_{(2)a}, q_\mu^{(3)}$ and t as,

$$q_a^{(3)} \equiv f_{(3)a}(q_i, q_i^{(1)}, q_i^{(2)}, p_{(1)a}, p_{(2)a}, q_\mu^{(3)}, t), \tag{2.2.5}$$

where $a = 1, ..., N - R$ and $\mu = N - R + 1, ..., N$. Since the momenta are not independent; $p_{(s)\mu}$ can be written as

$$p_{(s)\mu} = -H_{(s)\mu}(q_j^{(u)}, p_{(u)a}, t), \tag{2.2.6}$$

where $u, s = 0, 1, 2$, $u \geq s$, $j = 1, 2, 3$.
The canonical method leads us to obtain the set of (HJPDE) as

$$H_0' = p_0 + H_0(q_i^{(u)}, p_{(u)a}, t) = 0, \tag{2.2.7}$$

$$H_{(s)\mu}' = p_{(s)\mu} + H_{(s)\mu}(q_i^{(u)}, p_{(u)a}, t) = 0. \tag{2.2.8}$$

The usual Hamiltonian H_0 is defined as

$$H_0 = p_{(0)a}q_a^{(1)} + p_{(1)a}q_a^{(2)} + p_{(2)a}f_{(3)a} + q_\mu^{(1)}p_{(0)\mu}|_{p_{(0)\mu}=-H_{(0)\mu}} + q_\mu^{(2)}p_{(1)\mu}|_{p_{(1)\mu}=-H_{(1)\mu}}$$

$$+q_\mu^{(3)}p_{(2)\mu}\,|_{p_{(2)\mu}=-H_{(2)\mu}} - L(q_i, q_i^{(1)}, q_i^{(2)}, q_\mu^{(3)}, q_a^{(3)} = f_{(3)a}). \tag{2.2.9}$$

The Hamiltonian H_0 does not depend on $q_\mu^{(3)}$, so that

$$\frac{\partial H_0}{\partial q_\mu^{(3)}} = 0. \tag{2.2.10}$$

The equations of motion in the canonical method are given as total differential equations in many variables of the forms

$$dq_a = \frac{\partial H_0'}{\partial p_{(0)a}}dt + \frac{\partial H_{(0)\mu}'}{\partial p_{(0)a}}dq_\mu, \tag{2.2.11}$$

$$dq_a^{(1)} = \frac{\partial H_0'}{\partial p_{(1)a}}dt + \frac{\partial H_{(0)\mu}'}{\partial p_{(1)a}}dq_\mu + \frac{\partial H_{(1)\mu}'}{\partial p_{(1)a}}dq_\mu^{(1)}, \tag{2.2.12}$$

$$dq_a^{(2)} = \frac{\partial H_0'}{\partial p_{(2)a}}dt + \frac{\partial H_{(0)\mu}'}{\partial p_{(2)a}}dq_\mu + \frac{\partial H_{(1)\mu}'}{dp_{(2)a}}dq_\mu^{(1)} + \frac{\partial H_{(2)\mu}'}{\partial p_{(2)a}}dq_\mu^{(2)}, \tag{2.2.13}$$

$$dp_{(0)_i} = -\frac{\partial H_0'}{\partial q_i}dt - \frac{\partial H_{(0)\mu}'}{\partial q_i}dq_\mu - \frac{\partial H_{(1)\mu}'}{\partial q_i}dq_\mu^{(1)} - \frac{\partial H_{(2)\mu}'}{\partial q_i}dq_\mu^{(2)}, \tag{2.2.14}$$

$$dp_{(1)i} = -\frac{\partial H_0'}{\partial q_i^{(1)}}dt - \frac{\partial H_{(0)\mu}'}{\partial q_i^{(1)}}dq_\mu - \frac{\partial H_{(1)\mu}'}{\partial q_i^{(1)}}dq_\mu^{(1)} - \frac{\partial H_{(2)\mu}'}{\partial q_i^{(1)}}dq_\mu^{(2)}, \tag{2.2.15}$$

$$dp_{(2)i} = -\frac{\partial H_0'}{\partial q_i^{(2)}}dt - \frac{\partial H_{(0)\mu}'}{\partial q_i^{(2)}}dq_\mu - \frac{\partial H_{(1)\mu}'}{\partial q_i^{(2)}}dq_\mu^{(1)} - \frac{\partial H_{(2)\mu}'}{\partial q_i^{(2)}}dq_\mu^{(2)}. \tag{2.2.16}$$

The set of total differential equations (2.1.10−2.1.13) is integrable if the variation of the constraints H_0' and $H_{(s)\mu}'$ are identically zero. If the variations of H_0' and $H_{(s)\mu}'$ are not identically zero, then a new constraint will arise and again we test the integrability conditions until we obtain a complete system [9].

2.2.1 An Example

Now we will use the canonical method and the treatment of singular system as field system to solve the third-order singular Lagrangian[29],

$$L = q_1^{(3)}q_2^{(3)} + q_1^{(2)}(q_2^{(2)} - q_3^{(2)}) + q_1^{(1)}(q_2^{(1)} - q_3^{(1)}) - q_1q_3. \tag{2.2.17}$$

The system is singular since the rank of the Hessian matrix

$$A_{ij} = \frac{\partial^2 L}{\partial q_i^{(3)}\partial q_j^{(3)}} \tag{2.2.18}$$

is two. where $i, j = 1, 2, 3$.

The canonical momenta (2.2.2−2.2.4) are

$$p_{(2)1} = \frac{\partial L}{\partial q_1^{(3)}} = q_2^{(3)}, \tag{2.2.19}$$

$$p_{(2)2} = \frac{\partial L}{\partial q_2^{(3)}} = q_1^{(3)}, \tag{2.2.20}$$

$$p_{(2)3} = \frac{\partial L}{\partial q_3^{(3)}} = 0, \tag{2.2.21}$$

$$p_{(1)1} = \frac{\partial L}{\partial q_1^{(2)}} - \dot{p}_{(2)1} = q_2^{(2)} - q_3^{(2)} - q_2^{(4)}, \tag{2.2.22}$$

$$p_{(1)2} = \frac{\partial L}{\partial q_2^{(2)}} - \dot{p}_{(2)2} = q_1^{(2)} - q_1^{(4)}, \tag{2.2.23}$$

$$p_{(1)3} = \frac{\partial L}{\partial q_3^{(2)}} - \dot{p}_{(2)3} = -q_1^{(2)}, \tag{2.2.24}$$

$$p_{(0)1} = \frac{\partial L}{\partial q_1^{(1)}} - \dot{p}_{(1)1} = q_2^{(1)} - q_3^{(1)} - q_2^{(3)} + q_3^{(3)} + q_2^{(5)}, \tag{2.2.25}$$

$$p_{(0)2} = \frac{\partial L}{\partial q_2^{(1)}} - \dot{p}_{(1)2} = q_1^{(1)} - q_1^{(3)} + q_1^{(5)}, \tag{2.2.26}$$

$$p_{(0)3} = \frac{\partial L}{\partial q_3^{(1)}} - \dot{p}_{(1)3} = -q_1^{(1)} + q_1^{(3)}. \tag{2.2.27}$$

Equations (2.2.19) and (2.2.20) can be solved for $q_1^{(3)}$ and $q_2^{(3)}$ as

$$q_1^{(3)} = p_{(2)2} = f_{(3)1}, \tag{2.2.28}$$

$$q_2^{(3)} = p_{(2)1} = f_{(3)2}. \tag{2.2.29}$$

Since the momenta are not independent, $p_{(u)\mu}$ can be written as

$$p_{(0)3} = -q_1^{(1)} + p_{(2)2} = -H_{(0)3}, \tag{2.2.30}$$

$$p_{(1)3} = -q_1^{(2)} = -H_{(1)3}, \tag{2.2.31}$$

$$p_{(2)3} = 0 = -H_{(2)3}. \tag{2.2.32}$$

The Hamiltonian (2.2.9), takes the form

$$H_0 = p_{(0)1}q_1^{(1)} + p_{(0)2}q_2^{(1)} + p_{(1)1}q_1^{(2)} + p_{(1)2}q_2^{(2)} + p_{(2)1}f_{(3)1} + p_{(2)2}f_{(3)2} + q_3^{(1)}p_{(0)3}$$

$$+q_3^{(2)}p_{(1)3} - L(q_i, q_i^{(1)}, q_i^{(2)}, q_3^{(3)}, q_1^{(3)} = f_{(3)1}, q_2^{(3)} = f_{(3)2}), \tag{2.2.33}$$

or

$$H_0 = p_{(0)1}q_1^{(1)} + p_{(0)2}q_2^{(1)} + p_{(1)1}q_1^{(2)} + p_{(1)2}q_2^{(2)} + p_{(2)1}p_{(2)2} + q_3^{(1)}p_{(2)2}$$
$$-q_1^{(2)}q_2^{(2)} - q_1^{(1)}q_2^{(1)} + q_1q_3 \tag{2.2.34}$$

The set of Hamilton−Jacobi equations (2.2.7) and (2.2.8) can be read as

$$H'_{(0)3} = p_{(0)3} + H_{(0)3} = p_{(0)3} + q_1^{(1)} - p_{(2)2} = 0, \tag{2.2.35}$$

$$H'_{(1)3} = p_{(1)3} + H_{(1)3} = p_{(1)3} + q_1^{(2)} = 0, \tag{2.2.36}$$

$$H'_{(2)3} = p_{(2)3} + H_{(2)3} = p_{(2)3} = 0, \tag{2.2.37}$$

$$H'_0 = p_0 + H_0 = 0. \tag{2.2.38}$$

The equations of motion (2.2.11−2.2.16) can be written as

$$dq_1 = \frac{\partial H'_0}{\partial p_{(0)1}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(0)1}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(0)1}}dq_3^{(1)} + \frac{\partial H'_{(2)3}}{\partial p_{(0)1}}dq_3^{(2)} = q_1^{(1)}dt, \tag{2.2.39}$$

$$dq_2 = \frac{\partial H'_0}{\partial p_{(0)2}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(0)2}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(0)2}}dq_3^{(1)} + \frac{\partial H'_{(2)3}}{\partial p_{(0)2}}dq_3^{(2)} = q_2^{(1)}dt, \tag{2.2.40}$$

$$dq_1^{(1)} = \frac{\partial H'_0}{\partial p_{(1)1}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(1)1}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(1)1}}dq_3^{(1)} + \frac{\partial H'_{(2)3}}{\partial p_{(1)1}}dq_3^{(2)} = q_1^{(2)}dt, \tag{2.2.41}$$

$$dq_2^{(1)} = \frac{\partial H'_0}{\partial p_{(1)2}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(1)2}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(1)2}}dq_3^{(1)} + \frac{\partial H'_{(2)3}}{\partial p_{(1)2}}dq_3^{(2)} = q_2^{(2)}dt, \tag{2.2.42}$$

$$dq_1^{(2)} = \frac{\partial H'_0}{\partial p_{(2)1}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(2)1}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(2)1}}dq_3^{(1)} + \frac{\partial H'_{(2)3}}{\partial p_{(2)1}}dq_3^{(2)} = p_{(2)2}dt, \tag{2.2.43}$$

$$dq_2^{(2)} = \frac{\partial H'_0}{\partial p_{(2)2}}dt + \frac{\partial H'_{(0)3}}{\partial p_{(2)2}}dq_3 + \frac{\partial H'_{(1)3}}{\partial p_{(2)2}}dq_3^{(1)} + \frac{\partial H'_{(2)3}}{\partial p_{(2)2}}dq_3^{(2)} = (p_{(2)1} + q_2^{(1)})dt - dq_3, \tag{2.2.44}$$

$$dp_{(0)1} = -\frac{\partial H'_0}{\partial q_1}dt - \frac{\partial H'_{(0)3}}{\partial q_1}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_1}dq_3^{(1)} - \frac{\partial H'_{(2)3}}{\partial q_1}dq_3^{(2)} = -q_3dt, \tag{2.2.45}$$

$$dp_{(0)2} = -\frac{\partial H'_0}{\partial q_2}dt - \frac{\partial H'_{(0)3}}{\partial q_2}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_2}dq_3^{(1)} - \frac{\partial H'_{(2)3}}{\partial q_2}dq_3^{(2)} = 0, \tag{2.2.46}$$

$$dp_{(0)3} = -\frac{\partial H'_0}{\partial q_3}dt - \frac{\partial H'_{(0)3}}{\partial q_3}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_3}dq_3^{(1)} - \frac{\partial H'_{(2)3}}{\partial q_3}dq_3^{(2)} = -q_1dt, \tag{2.2.47}$$

$$dp_{(1)1} = -\frac{\partial H'_0}{\partial q_1^{(1)}}dt - \frac{\partial H'_{(0)3}}{\partial q_1^{(1)}}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_1^{(1)}}dq_3^{(1)} - \frac{\partial H'_{(2)3}}{\partial q_1^{(1)}}dq_3^{(2)} = -(p_{(0)1} - q_2^{(1)})dt - dq_3, \tag{2.2.48}$$

$$dp_{(1)2} = -\frac{\partial H'_0}{\partial q_2^{(1)}}dt - \frac{\partial H'_{(0)3}}{\partial q_2^{(1)}}dq_3 - \frac{\partial H'_{(1)3}}{\partial q_2^{(1)}}dq_3^{(1)} - \frac{\partial H'_{(2)3}}{\partial q_2^{(1)}}dq_3^{(2)} = -(p_{(0)2} - q_1^{(1)})dt, \tag{2.2.49}$$

$$dp_{(1)3} = -\frac{\partial H_0'}{\partial q_3^{(1)}}dt - \frac{\partial H_{(0)3}'}{\partial q_3^{(1)}}dq_3 - \frac{\partial H_{(1)3}'}{\partial q_3^{(1)}}dq_3^{(1)} - \frac{\partial H_{(2)3}'}{\partial q_3^{(1)}}dq_3^{(2)} = -p_{(2)2}dt, \quad (2.2.50)$$

$$dp_{(2)1} = -\frac{\partial H_0'}{\partial q_1^{(2)}}dt - \frac{\partial H_{(0)3}'}{\partial q_1^{(2)}}dq_3 - \frac{\partial H_{(1)3}'}{\partial q_1^{(2)}}dq_3^{(1)} - \frac{\partial H_{(2)3}'}{\partial q_1^{(2)}}dq_3^{(2)} = -(p_{(1)1} - q_2^{(2)})dt - dq_3^{(1)}, \quad (2.2.51)$$

$$dp_{(2)2} = -\frac{\partial H_0'}{\partial q_2^{(2)}}dt - \frac{\partial H_{(0)3}'}{\partial q_2^{(2)}}dq_3 - \frac{\partial H_{(1)3}'}{\partial q_2^{(2)}}dq_3^{(1)} - \frac{\partial H_{(2)3}'}{\partial q_2^{(2)}}dq_3^{(2)} = -(p_{(1)2} - q_1^{(2)})dt, \quad (2.2.52)$$

$$dp_{(2)3} = -\frac{\partial H_0'}{\partial q_3^{(2)}}dt - \frac{\partial H_{(0)3}'}{\partial q_3^{(2)}}dq_3 - \frac{\partial H_{(1)3}'}{\partial q_3^{(2)}}dq_3^{(1)} - \frac{\partial H_{(2)3}'}{\partial q_3^{(2)}}dq_3^{(2)} = 0. \quad (2.2.53)$$

The system of total differential equations (2.2.39−2.2.53) is integrable if the variation of $H_{(s)3}'$ is identically zero.

The variation of $H_{(0)3}'$ is

$$dH_{(0)3}' = dp_{(0)3} + dq_1^{(1)} - dp_{(2)2} = (-q_1 + p_{(1)2})dt \quad (2.2.54)$$

since $dH_{(0)3}'$ does not identically zero we have a new constraint

$$H_{(0)3}'' = -q_1 + p_{(1)2}. \quad (2.2.55)$$

The variation of $H_{(0)3}''$ is

$$dH_{(0)3}'' = -dq_1 + dp_{(1)2} = -p_{(0)2}dt. \quad (2.2.56)$$

Again $dH_{(0)3}''$is not identically zero we have

$$H_{(0)3}''' = -p_{(0)2}. \quad (2.2.57)$$

Using (2.2.46) the variation of $H_{(0)3}'''$ is

$$dH_{(0)3}''' = -dp_{(0)2} = 0. \quad (2.2.58)$$

We can solve equation (2.2.57)as

$$p_{(0)2} = C. \quad (2.2.59)$$

Using equation (2.2.26) we get

$$q_1^{(1)} - q_1^{(3)} + q_1^{(5)} = C \tag{2.2.60}$$

or by taking the derivatives of (2.2.60)

$$q_1^{(2)} - q_1^{(4)} + q_1^{(6)} = 0. \tag{2.2.61}$$

The variation of $H'_{(1)3}$ is

$$dH'_{(1)3} = dp_{(1)3} + dq_1^{(2)} \equiv 0. \tag{2.2.62}$$

the variation of $H'_{(2)3}$ is

$$dH'_{(2)3} = dp_{(2)3} \equiv 0. \tag{2.2.63}$$

From (2.2.58),(2.2.62) and (2.2.63), we conclude that the system is integrable.

The equivalent partial differential equation of (2.2.44) is

$$\frac{\partial q_2^{(2)}}{\partial q_3} = -1 \tag{2.2.64}$$

which can be solved as

$$q_2^{(2)} = -q_3 + F(t) \tag{2.2.65}$$

or

$$F(t) = q_2^{(2)} + q_3. \tag{2.2.66}$$

The second and forth derivative of (2.2.66)can be written as

$$F^{(2)} = q_2^{(4)} + q_3^{(2)}, \tag{2.2.67}$$

$$F^{(4)} = q_2^{(6)} + q_3^{(4)}. \tag{2.2.68}$$

Equation(2.2.25) and (2.2.45) together take the form,

$$q_2^{(6)} - q_2^{(4)} + q_3^{(4)} + q_2^{(2)} - q_3^{(2)} + q_3 = 0 \tag{2.2.69}$$

Using equations (2.2.66),(2.2.67) and (2.2.68),equation (2.2.69) becomes

$$F^{(4)} - F^{(2)} + F = 0 \tag{2.2.70}$$

2.2.2 Treatment of The Third−Order Lagrangian as Field System

The third−order Lagrangian system can be treated as field system where the field q_a are expressed in term of the independent coordinates as

$$q_a \equiv q_a(t, x_\mu), \tag{2.2.71}$$

where $\mu = N - R + 1, ..., N$. The Euler-Lagrange equation of the third-order Lagrangian takes the form[10]

$$\frac{\partial L'}{\partial q_a} - \frac{\partial}{\partial x_\mu}(\frac{\partial L'}{\partial(\partial_\mu q_a)}) + \frac{\partial^2}{\partial x_{\mu_1}\partial x_{\mu_2}}(\frac{\partial L'}{\partial(\partial_{\mu_1}\partial_{\mu_2} q_a)}) - \frac{\partial^3}{\partial x_{\mu_1}\partial x_{\mu_2}\partial x_{\mu_3}}(\frac{\partial L'}{\partial(\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3} q_a)}) = 0, \tag{2.2.72}$$

where

$$\partial_\mu q_a \equiv \frac{\partial q_a}{\partial x_\mu}, \tag{2.2.73}$$

$$\partial_{\mu_1}\partial_{\mu_2} q_a \equiv \frac{\partial^2 q_a}{\partial x_{\mu_1}\partial x_{\mu_2}}, \tag{2.2.74}$$

$$\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3} q_a = \frac{\partial^3 q_a}{\partial x_{\mu_1}\partial x_{\mu_2}\partial x_{\mu_3}}, \tag{2.2.75}$$

and the modified Lagrangian L' is defined as[10,22]

$$L'(x_\mu, q_a, \partial_\mu q_a, \partial_{\mu_1}\partial_{\mu_2} q_a, \partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3} q_a, x_\mu^{(1)}, x_\mu^{(2)}, x_\mu^{(3)}) \equiv L(x_\mu, q_a, q_a^{(1)} = (\partial_\mu q_a)x_\mu^{(1)},$$

$$, q_a^{(2)} = \partial_{\mu_2}(\partial_{\mu_1} q_a x_{\mu_1}^{(1)})x_{\mu_2}^{(1)}, q_a^{(3)} = \partial_{\mu_3}(\partial_{\mu_2}(\partial_{\mu_1} q_a x_{\mu_1}^{(1)})x_{\mu_2}^{(1)})x_{\mu_3}^{(1)}), \tag{2.2.76}$$

with $x_\mu^{(1)} = \frac{dx_\mu}{dt}$ and $x_0^{(1)} = 1$

The constraint equation can be written as

$$dG_0 = -\frac{\partial L'}{\partial t}dt, \tag{2.2.77}$$

$$dG_\mu = -\frac{\partial L'}{\partial q_\mu}dt. \tag{2.2.78}$$

where

$$G_0 = H_0(q_i^{(u)}, p_{(u)a}, t), \tag{2.2.79}$$

$$G_\mu = H_\mu(q_i^{(u)}, p_{(u)a}, t). \tag{2.2.80}$$

Solving of Euler-Lagrange equation (2.2.72) together with the constraint equations (2.2.77) and (2.2.78) we get the solution of the system. Since the rank of Hessian matrix of the Lagrangian (2.2.17) is two then the system is singular,and can be written as field system in the form

$$q_1 = q_1(q_3, t), \tag{2.2.81}$$

$$q_2 = q_2(q_3, t). \tag{2.2.82}$$

The first, second, and third derivatives with respect to t of (2.2.81) and (2.2.82) respectively are:

$$q_1^{(1)} = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3} q_3^{(1)}, \tag{2.2.83}$$

$$q_2^{(1)} = \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3} q_3^{(1)}, \tag{2.2.84}$$

$$q_1^{(2)} = \frac{\partial^2 q_1}{\partial t^2} + 2\frac{\partial^2 q_1}{\partial t \partial q_3} q_3^{(1)} + \frac{\partial^2 q_1}{\partial q_3^2}(q_3^{(1)})^2 + \frac{\partial q_1}{\partial q_3} q_3^{(2)}, \tag{2.2.85}$$

$$q_2^{(2)} = \frac{\partial^2 q_2}{\partial^2 t} + 2\frac{\partial^2 q_2}{\partial t \partial q_3} q_3^{(1)} + \frac{\partial^2 q_2}{\partial q_3^2}(q_3^{(1)})^2 + \frac{\partial q_2}{\partial q_3} q_3^{(2)}, \tag{2.2.86}$$

and

$$\begin{aligned} q_1^{(3)} = {} & \frac{\partial^3 q_1}{\partial t^3} + 3\frac{\partial^3 q_1}{\partial t^2 \partial q_3} q_3^{(1)} + 3\frac{\partial^3 q_1}{\partial t \partial q_3^2}(q_3^{(1)})^2 + 3\frac{\partial^2 q_1}{\partial t \partial q_3} q_3^{(2)} + \frac{\partial^3 q_1}{\partial q_3^3}(q_3^{(1)})^3 \\ & +3\frac{\partial^2 q_1}{\partial q_3^2} q_3^{(1)} q_3^{(2)} + \frac{\partial q_1}{\partial q_3} q_3^{(3)}. \end{aligned} \tag{2.2.87}$$

$$\begin{aligned} q_2^{(3)} = {} & \frac{\partial^3 q_2}{\partial t^3} + 3\frac{\partial^3 q_2}{\partial t^2 \partial q_3} q_3^{(1)} + 3\frac{\partial^3 q_2}{\partial t \partial q_3^{(2)}}(q_3^{(1)})^2 + 3\frac{\partial^2 q_2}{\partial t \partial q_3} q_3^{(2)} + \frac{\partial^3 q_2}{\partial q_3^3}(q_3^{(1)})^3 \\ & +3\frac{\partial^2 q_2}{\partial q_3^2} q_3^{(1)} q_3^{(2)} + \frac{\partial q_2}{\partial q_3} q_3^{(3)}. \end{aligned} \tag{2.2.88}$$

Substituting (2.2.83–2.2.88) into (2.2.17) we get the modified Lagrangian function

$$L' = (\frac{\partial^3 q_1}{\partial t^3} + 3\frac{\partial^3 q_1}{\partial t^2 \partial q_3} q_3^{(1)} + 3\frac{\partial^3 q_1}{\partial t \partial q_3^2}(q_3^{(1)})^2 + 3\frac{\partial^2 q_1}{\partial t \partial q_3} q_3^{(2)} + \frac{\partial^3 q_1}{\partial q_3^3}(q_3^{(1)})^3 + 3\frac{\partial^2 q_1}{\partial q_3^2} q_3^{(1)} q_3^{(2)}$$

$$+\frac{\partial q_1}{\partial q_3}q_3^{(3)})(\frac{\partial^3 q_2}{\partial t^3}+3\frac{\partial^3 q_2}{\partial t^2\partial q_3}q_3^{(1)}+3\frac{\partial^3 q_2}{\partial t\partial q_3^{(2)}}(q_3^{(1)})^2+3\frac{\partial^2 q_2}{\partial t\partial q_3}q_3^{(2)}+\frac{\partial^3 q_2}{\partial q_3^3}(q_3^{(1)})^3+$$

$$3\frac{\partial^2 q_2}{\partial q_3^2}q_3^{(1)}q_3^{(2)}+\frac{\partial q_2}{\partial q_3}q_3^{(3)})+(\frac{\partial^2 q_1}{\partial t^2}+2\frac{\partial^2 q_1}{\partial t\partial q_3}q_3^{(1)}+\frac{\partial^2 q_1}{\partial q_3^2}(q_3^{(1)})^2+\frac{\partial q_1}{\partial q_3}q_3^{(2)})[(\frac{\partial^2 q_2}{\partial t^2}$$

$$+2\frac{\partial^2 q_2}{\partial t\partial q_3}q_3^{(1)}+\frac{\partial^2 q_2}{\partial q_3^2}(q_3^{(1)})^2+\frac{\partial q_2}{\partial q_3}q_3^{(2)})-q_3^{(2)}]+(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}q_3^{(1)})$$

$$[(q_2^{(1)}=\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_3}q_3^{(1)})-q_3^{(1)}]-q_1q_3. \tag{2.2.89}$$

The Euler-Lagrange equations (2.2.72) read as

$$\frac{\partial L'}{\partial q_1}-\frac{\partial}{\partial t}(\frac{\partial L'}{\partial(\frac{\partial q_1}{\partial t})})-\frac{\partial}{\partial q_3}(\frac{\partial L'}{\partial(\frac{\partial q_1}{\partial q_3})})+\frac{\partial^2}{\partial t^2}(\frac{\partial L'}{\partial(\frac{\partial^2 q_1}{\partial t^2})})+2\frac{\partial^2}{\partial t\partial q_3}(\frac{\partial L'}{\partial(\frac{\partial^2 q_1}{\partial t\partial q_3})})+\frac{\partial^2}{\partial q_3^2}(\frac{\partial L'}{\partial(\frac{\partial^2 q_1}{\partial q_3^2})})-$$

$$\frac{\partial^3}{\partial t^3}(\frac{\partial L'}{\partial(\frac{\partial^3 q_1}{\partial t^3})})-3\frac{\partial^3}{\partial t\partial q_3^2}(\frac{\partial L'}{\partial(\frac{\partial^3 q_1}{\partial t\partial q_3^2})})-3\frac{\partial^3}{\partial t^2\partial q_3}(\frac{\partial L'}{\partial(\frac{\partial^3 q_1}{\partial t^2\partial q_3})})-\frac{\partial^3}{\partial q_3^3}(\frac{\partial L'}{\partial(\frac{\partial^3 q_1}{\partial q_3^3})})=0, \tag{2.2.90}$$

and

$$\frac{\partial L'}{\partial q_2}-\frac{\partial}{\partial t}(\frac{\partial L'}{\partial(\frac{\partial q_2}{\partial t})})-\frac{\partial}{\partial q_3}(\frac{\partial L'}{\partial(\frac{\partial q_2}{\partial q_3})})+\frac{\partial^2}{\partial t^2}(\frac{\partial L'}{\partial(\frac{\partial^2 q_2}{\partial t^2})})+2\frac{\partial^2}{\partial t\partial q_3}(\frac{\partial L'}{\partial(\frac{\partial^2 q_2}{\partial t\partial q_3})})+\frac{\partial^2}{\partial q_3^2}(\frac{\partial L'}{\partial(\frac{\partial^2 q_2}{\partial q_3^2})})$$

$$-\frac{\partial^3}{\partial t^3}(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial t^3})})-3\frac{\partial^3}{\partial t\partial q_3^2}(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial t\partial q_3^2})})-3\frac{\partial^3}{\partial t^2\partial q_3}(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial t^2\partial q_3})})-\frac{\partial^3}{\partial q_3^3}(\frac{\partial L'}{\partial(\frac{\partial^3 q_2}{\partial q_3^3})})=0. \tag{2.2.91}$$

Using equation(2.2.89), equation(2.2.90)and(2.2.91)becomes

$$-q_3-\frac{\partial q_2^{(1)}}{\partial t}+q_3^{(2)}-\frac{\partial q_2^{(3)}}{\partial q_3}q_3^{(3)}-\frac{\partial q_2^{(2)}}{\partial q_3}q_3^{(2)}-\frac{\partial q_2^{(1)}}{\partial q_3}q_3^{(1)}+\frac{\partial^2 q_2^{(2)}}{\partial t^2}-q_3^{(4)}+6\frac{\partial^2 q_2^{(3)}}{\partial t\partial q_3}q_3^{(2)}+4\frac{\partial^2 q_2^{(2)}}{\partial t\partial q_3}q_3^{(1)}$$

$$+3\frac{\partial^2 q_2^{(3)}}{\partial^2 q_3}q_3^{(1)}q_3^{(2)}+\frac{\partial^2 q_2^{(2)}}{\partial q_3^2}(q_3^{(1)})^2-\frac{\partial^3 q_2^{(3)}}{\partial t^3}-9\frac{\partial^3 q_2^{(3)}}{\partial t\partial q_3^2}(q_3^{(1)})^2-9\frac{\partial^3 q_2^{(3)}}{\partial t^2\partial q_3}q_3^{(1)}-\frac{\partial^3 q_2^{(3)}}{\partial q_3^3}(q_3^{(1)})^2=0, \tag{2.2.92}$$

$$-\frac{\partial q_1^{(1)}}{\partial t}-\frac{\partial q_1^{(1)}}{\partial q_3}q_3^{(1)}-\frac{\partial q_1^{(2)}}{\partial q_3}q_3^{(2)}-\frac{\partial q_1^{(3)}}{\partial q_3}q_3^{(3)}+\frac{\partial^2 q_1^{(2)}}{\partial t^2}+6\frac{\partial^2 q_1^{(3)}}{\partial t\partial q_3}q_3^{(2)}+4\frac{\partial^2 q_1^{(2)}}{\partial t\partial q_3}q_3^{(1)}+$$

$$3\frac{\partial^2 q_1^{(3)}}{\partial q_3^2}q_3^{(1)}q_3^{(2)}+\frac{\partial^2 q_1^{(2)}}{\partial q_3^2}(q_3^{(1)})^2-\frac{\partial^3 q_1^{(3)}}{\partial t^3}-9\frac{\partial^3 q_1^{(3)}}{\partial t\partial q_3^2}(q_3^{(1)})^2-9\frac{\partial^3 q_1^{(3)}}{\partial t^2\partial q_3}q_3^{(1)}-\frac{\partial^3 q_1^{(3)}}{\partial q_3^3}(q_3^{(1)})^2=0. \tag{2.2.93}$$

By using the equations (2.2.30),(2.2.83),(2.2.20)and (2.2.87),the constraint equation (2.2.80) reads as

$$G_3 = (\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3} q_3^{(1)}) - (\frac{\partial q_1^{(2)}}{\partial t} + \frac{\partial q_1^{(2)}}{\partial q_3} q_3^{(1)}), \tag{2.2.94}$$

$$dG_3 = [(\frac{\partial q_1^{(1)}}{\partial t} + \frac{\partial q_1^{(1)}}{\partial q_3} q_3^{(1)}) - (\frac{\partial^2 q_1^{(2)}}{\partial t^2} + 2\frac{\partial^2 q_1^{(2)}}{\partial t \partial q_3} q_3^{(1)} + \frac{\partial^2 q_1^{(2)}}{\partial^2 q_3} (q_3^{(1)})^2 + \frac{\partial q_1^{(2)}}{\partial q_3} q_3^{(2)})]dt, \tag{2.2.95}$$

and

$$dG_3 = -\frac{\partial L'}{\partial q_3} dt = q_1. \tag{2.2.96}$$

Using (2.2.95)and(2.2.96)we get,

$$(\frac{\partial^2 q_1^{(2)}}{\partial t^2} + 2\frac{\partial^2 q_1^{(2)}}{\partial t \partial q_3} q_3^{(1)} + \frac{\partial^2 q_1^{(2)}}{\partial^2 q_3} (q_3^{(1)})^2 + \frac{\partial q_1^{(2)}}{\partial q_3} q_3^{(2)}) - (\frac{\partial q_1^{(1)}}{\partial t} + \frac{\partial q_1^{(1)}}{\partial q_3} q_3^{(1)}) + q_1 = 0. \tag{2.2.97}$$

Now equation (2.2.97) can be written as

$$(\frac{\partial^3 q_1^{(3)}}{\partial t^3} + 3\frac{\partial^3 q_1^{(3)}}{\partial t^2 \partial q_3} q_3^{(1)} + 3\frac{\partial^3 q_1^{(3)}}{\partial t \partial q_3^2} (q_3^{(1)})^2 + 3\frac{\partial^2 q_1^{(3)}}{\partial t \partial q_3} q_3^{(2)} + \frac{\partial^3 q_1^{(3)}}{\partial q_3^3} (q_3^{(1)})^3 + 3\frac{\partial^2 q_1^{(3)}}{\partial q_3^2} q_3^{(1)} q_3^{(2)}$$

$$+ \frac{\partial q_1^{(3)}}{\partial q_3} q_3^{(3)}) - (\frac{\partial^2 q_1^{(2)}}{\partial t^2} + 2\frac{\partial^2 q_1^{(2)}}{\partial t \partial q_3} q_3^{(1)} + \frac{\partial^2 q_1^{(2)}}{\partial^2 q_3} (q_3^{(1)})^2 + \frac{\partial q_1^{(2)}}{\partial q_3} q_3^{(2)}) + (\frac{\partial q_1^{(1)}}{\partial t} + \frac{\partial q_1^{(1)}}{\partial q_3} q_3^{(1)}) = 0. \tag{2.2.98}$$

Subtracting (2.2.98) from (2.2.93)we get

$$2\frac{\partial^2 q_1^{(2)}}{\partial t \partial q_3} q_3^{(1)} - 2\frac{\partial q_1^{(2)}}{\partial q_3} q_3^{(2)} - 6\frac{\partial^3 q_1^{(3)}}{\partial t \partial q_3^2} (q_3^{(1)})^2 - 6\frac{\partial^3 q_1^{(3)}}{\partial t^2 \partial q_3} q_3^{(1)} + 9\frac{\partial^2 q_1^{(3)}}{\partial t \partial q_3} q_3^{(2)} + 6\frac{\partial^2 q_1^{(3)}}{\partial q_3^2} q_3^{(1)} q_3^{(2)} = 0. \tag{2.2.99}$$

Using equation (2.2.99) equation (2.2.93) becomes,

$$-\frac{dq_1^{(1)}}{dt} + \frac{d^2 q_1^{(2)}}{dt^2} + \frac{d^3 q_1^{(3)}}{dt^3} = 0. \tag{2.2.100}$$

The constraint equation (2.2.79) take the form

$$G_0 = H_0(q_a, x_\mu, p_a = \frac{\partial L}{\partial q_a}), \tag{2.2.101}$$

$$dG_0 = (q_3^{(4)} - q_3^{(2)} + q_3)(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3} q_3^{(1)}) + [(\frac{\partial^2 q_1^{(2)}}{\partial t^2} + 2\frac{\partial^2 q_1^{(2)}}{\partial t \partial q_3} q_3^{(1)} + \frac{\partial^2 q_1^{(2)}}{\partial q_3^2} (q_3^{(1)})^2$$

$$+\frac{\partial q_1^{(2)}}{\partial q_3}q_3^{(2)}) - (\frac{\partial q_1^{(1)}}{\partial t} + \frac{\partial q_1^{(1)}}{\partial q_3}q_3^{(1)}) + q_1]q_3^{(1)} + [(\frac{\partial^3 q_2^{(3)}}{\partial t^3} + 3\frac{\partial^3 q_2^{(3)}}{\partial t^2 \partial q_3}q_3^{(1)} + 3\frac{\partial^3 q_2^{(3)}}{\partial t \partial q_3^2}(q_3^{(1)})^2$$

$$+3\frac{\partial^2 q_2^{(3)}}{\partial t \partial q_3}q_3^{(2)} + \frac{\partial^3 q_2^{(3)}}{\partial q_3^3}(q_3^{(1)})^3 + 3\frac{\partial^2 q_2^{(3)}}{\partial q_3^2}q_3^{(1)}q_3^{(2)} + \frac{\partial q_2^{(3)}}{\partial q_3}q_3^{(3)}) - (\frac{\partial^2 q_2^{(2)}}{\partial t^2} + 2\frac{\partial^2 q_2^{(2)}}{\partial t \partial q_3}q_3^{(1)}$$

$$+\frac{\partial^2 q_2^{(2)}}{\partial q_3^2}(q_3^{(1)})^2 + \frac{\partial q_2^{(2)}}{\partial q_3}q_3^{(2)}) + (\frac{\partial q_2^{(1)}}{\partial t} + \frac{\partial q_2^{(1)}}{\partial q_3}q_3^{(1)})](\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3}q_3^{(1)}) + [(\frac{\partial^3 q_1^{(3)}}{\partial t^3} +$$

$$3\frac{\partial^3 q_1^{(3)}}{\partial t^2 \partial q_3}q_3^{(1)} + 3\frac{\partial^3 q_1^{(3)}}{\partial t \partial q_3^2}(q_3^{(1)})^2 + 3\frac{\partial^2 q_1^{(3)}}{\partial t \partial q_3}q_3^{(2)} + \frac{\partial^3 q_1^{(3)}}{\partial q_3^3}(q_3^{(1)})^3 + 3\frac{\partial^2 q_1^{(3)}}{\partial q_3^2}q_3^{(1)}q_3^{(2)} + \frac{\partial q_1^{(3)}}{\partial q_3}q_3^{(3)})$$

$$-(\frac{\partial^2 q_1^{(2)}}{\partial t^2} + 2\frac{\partial^2 q_1^{(2)}}{\partial t \partial q_3}q_3^{(1)} + \frac{\partial^2 q_1^{(2)}}{\partial q_3^2}(q_3^{(1)})^2 + \frac{\partial q_1^{(2)}}{\partial q_3}q_3^{(2)}) + (\frac{\partial q_1^{(1)}}{\partial t} + \frac{\partial q_1^{(1)}}{\partial q_3}q_3^{(1)})](\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3}q_3^{(1)}) = 0. \tag{2.2.102}$$

Using equation (2.2.97)and(2.2.98)equation (2.2.102)lead us

$$(\frac{\partial^3 q_2^3}{\partial t^2} + 3\frac{\partial^3 q_2^3}{\partial t^2 \partial q_3}q_3^{(1)} + 3\frac{\partial^3 q_2^3}{\partial t \partial q_3^{(2)}}(q_3^{(1)})^2 + 3\frac{\partial^2 q_2^3}{\partial t \partial q_3}q_3^{(2)} + \frac{\partial^3 q_2^3}{\partial q_3^3}(q_3^{(1)})^3 + 3\frac{\partial^2 q_2^3}{\partial q_3^2}q_3^{(1)}q_3^{(2)} + \frac{\partial q_2^3}{\partial q_3}q_3^{(3)})$$

$$-(\frac{\partial^2 q_2^{(2)}}{\partial^2 t} + 2\frac{\partial^2 q_2^{(2)}}{\partial t \partial q_3}q_3^{(1)} + \frac{\partial^2 q_2^{(2)}}{\partial q_3^2}(q_3^{(1)})^2 + \frac{\partial q_2^{(2)}}{\partial q_3}q_3^{(2)}) + (\frac{\partial q_2^{(1)}}{\partial t} + \frac{\partial q_2^{(1)}}{\partial q_3}q_3^{(1)}) + q_3^{(4)} - q_3^{(2)} + q_3 = 0. \tag{2.2.103}$$

Adding (2.2.92)and(2.2.103)we get

$$-6\frac{\partial^3 q_2^{(3)}}{\partial t^2 \partial q_3}q_3^{(1)} - 6\frac{\partial^3 q_2^{(3)}}{\partial t \partial^2 q_3}(q_3^{(1)})^2 + 9\frac{\partial^2 q^{(3)}}{\partial t \partial q_3}q_3^{(2)} + 6\frac{\partial^2 q_2^{(3)}}{\partial^2 q_3}q_3^{(1)}q_3^{(2)} + 2\frac{\partial^2 q_2^{(2)}}{\partial t \partial q_3}q_3^{(1)} - 2\frac{\partial q_2^{(2)}}{\partial q_3}q_3^{(2)} = 0. \tag{2.2.104}$$

Using equation (2.2.104) equation (2.2.92) becomes

$$q_2^{(6)} - q_2^{(4)} + q_3^{(4)} + q_2^{(2)} - q_3^{(2)} + q_3 = 0, \tag{2.2.105}$$

or

$$F^{(4)} - F^{(2)} + F = 0. \tag{2.2.106}$$

Which is equivalent to (2.2.70). The solution of system is exactly solution by using canonical method and treatment of system as field system.

2.3 The Canonical Path Integral of Higher−Order Lagrangian

Hamilton−Jacobi path integral quantization of singular systems was developed in ref. [11, 12, 13]. If the set (HJPDE's)of equations is integrable, then one can solve them to obtain the trajectories of the motion in the canonical phase space coordinates.Now making

$$Z \equiv S\left(t_{(s)\alpha}, q_a^{(s)}\right) \tag{2.3.1}$$

and using the momenta definitions together with equation (1.3.17), we have

$$dZ = \sum_{d=0}^{k-1}\left(-H_{(d)\alpha} + \sum_{d=0}^{k-1} P_{(s)a}\frac{\partial H_{(d)\alpha}}{\partial P_{(s)a}}\right) dt_{(d)\alpha}; \tag{2.3.2}$$

In this case , the path integral representation may be written as

$$\langle OUT\,|S|\,IN\rangle = \int \prod_{a=1}^{n-r} Dq_a Dp_a \exp\left[i\int_{t_a}^{t'_a}\sum_{d=0}^{k-1}\left(-H_{(d)\alpha} + \sum_{d=0}^{k-1} P_{(s)a}\frac{\partial H_{(d)\alpha}}{\partial P_{(s)a}}\right) dt_{(d)\alpha}\right] \tag{2.3.3}$$

Now for the second−order Lagrangian (2.1.14) ,for $\alpha = 0, 2 \quad K = 2 \quad a = 1, 2$ we have

$$dZ = \left(-H_{(0)} + \sum_{d=0}^{1} P_{(s)a}\frac{\partial H'_{(0)}}{\partial P_{(s)a}}\right) dt_{(0)}$$

$$+\left(-H_{(0)2} + \sum_{d=0}^{1} P_{(s)a}\frac{\partial H'_{(0)2}}{\partial P_{(s)a}}\right) dt_{(0)2}$$

$$+\left(-H_{(1)2} + \sum_{d=0}^{1} P_{(s)a}\frac{\partial H'_{(1)2}}{\partial P_{(s)a}}\right) dt_{(1)2} \tag{2.3.4}$$

or

$$dZ = \left\{\frac{1}{2\beta}\left(p_{(1)1}\right)^2 + \frac{1}{2}K(q_1^{(1)} - q_2^{(1)})^2 + \frac{1}{2}\gamma(q_1 - q_2)^2\right\} dt_{(0)} \tag{2.3.5}$$

So, the path integral for this system will be

$$\langle OUT\,|S|\,IN\rangle = \int \prod_{a=1}^{2} dq_1 dq_2 dq_1^{(1)} dq_2^{(1)} dp_{(0)1} dp_{(0)2} dp_{(1)1} dp_{(1)2} \exp\left[i\int_{t_a}^{t'_a}\left\{\frac{1}{2\beta}\left(p_{(1)1}\right)^2\right.\right.$$

$$+\frac{1}{2}K(q_1^{(1)}-q_2^{(1)})^2+\frac{1}{2}\gamma(q_1-q_2)^2\Big\}dt_{(0)}\Big] \tag{2.3.6}$$

the set of equations(2.1.25 $-$ 2.1.30) are integrable if and onlly if $dH'_{(s)2}=0$, and they form a completely integral set , their solutions determine$Z\equiv S(t_{(s)\alpha},q_a^{(s)})$ uniquely from the initial conditions.

Let us investigate the integrability conditions in terms of the action [25] .

$$dH'_\beta=\{H'_\beta,H'_\alpha\}dt_{(\alpha)} \tag{2.3.7}$$

where $\alpha,\beta=0,2$ then

$$\frac{\partial H'_0}{\partial t_2}=\frac{\partial H'_0}{\partial q_2}=\gamma(q_1-q_2)\neq 0 \tag{2.3.8}$$

so the integrability conditions in terms of the action are not satisfied , hence the action function is not integrable and it has no unique solution .

Now for the third$-$order Lagrangian (2.2.17) ,for $\alpha=0,3\quad K=3\qquad a=1,2,3$ we have :

$$dZ=\left(-H_{(0)}+\sum_{d=0}^{2}P_{(s)a}\frac{\partial H'_{(0)}}{\partial P_{(s)a}}\right)dt_{(0)}$$

$$+\left(-H_{(0)3}+\sum_{d=0}^{2}P_{(s)a}\frac{\partial H'_{(0)3}}{\partial P_{(s)a}}\right)dt_{(0)3}$$

$$+\left(-H_{(1)3}+\sum_{d=0}^{2}P_{(s)a}\frac{\partial H'_{(1)3}}{\partial P_{(s)a}}\right)dt_{(1)3}$$

$$+\left(-H_{(2)3}+\sum_{d=0}^{2}P_{(s)a}\frac{\partial H'_{(2)3}}{\partial P_{(s)a}}\right)dt_{(2)3} \tag{2.3.9}$$

or

$$dZ=\Big[-p_{(2)1}p_{(2)2}-q_3^{(1)}p(2)2+q_1^{(2)}q_2^{(2)}+q_1^{(1)}q_2^{(1)}-q_1q_3\Big]dt_0+p_{(0)3}dq_3+p_{(1)3}dq_3^{(1)} \tag{2.3.10}$$

SO , the path integral for this system will be :

$$\langle OUT\,|S|\,IN\rangle = \int \prod_{a=1}^{3} dq_a dq_a^{(1)} dp_{(0)a} dp_{(1)a} dp_{(2)a} \exp\left[i \int_{t_a}^{t'_a} \left[-p_{(2)1}p_{(2)2} \right.\right.$$

$$\left.\left. -q_3^{(1)}p_{(2)2} + q_1^{(2)}q_2^{(2)} + q_1^{(1)}q_2^{(1)} - q_1q_3 \right] dt_0 + p_{(0)3}dq_3 + p_{(1)3}dq_3^{(1)} \right. \qquad (2.3.11)$$

the set of equations(2.2.39 − 2.2.53) are integrable if and onlly if $dH'_{(s)3} = 0$, and they form a completely integral set , their solutions determine$Z \equiv S(t_{(s)\alpha}, q_a^{(s)})$ uniquely from the initial conditions.

Let us investigate the integrability conditions in terms of the action .

$$dH'_\beta = \{H'_\beta, H'_\alpha\}dt_{(\alpha)} \qquad (2.3.12)$$

where $\alpha, \beta = 0, 3$ then

$$\frac{\partial H'_0}{\partial t_3} = \frac{\partial H'_0}{\partial q_3} = q_1 \neq 0 \qquad (2.3.13)$$

so the integrability conditions in terms of the action are not satisfied , hence the action function is not integrable and it has no unique solution .

Chapter 3

The Lagrangian of Podolosky Electrodynamics

3.1 Introduction

In this chapter we will consider a continuous system with Lagrangian density dependent on the dynamical field variables and its derivatives upon second order $\ell = \ell(\psi, \partial\psi, \partial^2\psi)$, so we consider the case of the generalized electromagnetic theory of Podolosky which is developed in the 1948 [31] as a generalization of Maxwells electromagnetism. Besides the fact that the quantum electrodynamics, based on Maxwells theory, is the most successful theory of the modern physics, it suffers from problems of divergences as the infinite self energy of the punctual electron and a divergent vacuum polarization current, difculties that come from the fact that the classical electrodynamics presents a r^{-1} singularity in the electrostatic potential. Podoloskys theory adds a higher order derivative term in Maxwells Lagrangian, which maintains the most important features of the classical electromagnetism, and also gives linear field equations. We adopt the metric $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$, as stated previously, the generalization of the formalism presented in chapter 1 is straightforward.That the Euler−Lagrange equation of motion are now given by

$$\frac{\partial \ell}{\partial \psi^a} - \partial_\mu \left(\frac{\partial \ell}{\partial(\partial_\mu \psi^a)} \right) + \partial_\mu \partial_\upsilon \left(\frac{\partial \ell}{\partial(\partial_\mu \partial_\upsilon \psi^a)} \right) = 0 \tag{3.1.1}$$

and that the momenta conjugated respectively to $\dot{\psi}^a$ and $\ddot{\psi}^a$, are

$$p_a = \frac{\partial \ell}{\partial \dot{\psi}^a} - 2\partial_k \left(\frac{\partial \ell}{\partial(\partial_k \dot{\psi}^a)} \right) - \partial_0 \left(\frac{\partial \ell}{\partial(\ddot{\psi}^a)} \right) \tag{3.1.2}$$

$$\pi_a = \frac{\partial \ell}{\partial(\ddot{\psi}^a)} \tag{3.1.3}$$

The Hessian matrix is now

$$A_{ij} = \frac{\partial^2 \ell}{\partial \ddot{\psi}^a \partial \ddot{\psi}^b} \tag{3.1.4}$$

3.2 Dirac's Method

The Podolsky electrodynamics Lagrangian is given by [26]

$$\ell = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + a^2 \partial_\lambda F^{\alpha\lambda} \partial_p F_\alpha{}^p \tag{3.2.1}$$

where the factor$-\frac{1}{4}$ is conventional, and a is a dimensional parameter can be written as $a = \frac{1}{m}$, where m is mass parameter, a generalized field tensor$F_{\mu\nu}$ is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.2.2}$$

An analysis of the Hamiltonian formalism for this theory was carried out in ref.[20,30],the Euler−Lagrange equations are

$$(1 + 2a^2 \partial_m \partial^m) \partial_p F^p_\alpha = 0 \tag{3.2.3}$$

where $\partial_m \partial^m = \eta^{\mu\nu} \partial_\mu \partial_\nu = \frac{\partial^2}{\partial t^2} - \nabla^2$ is the d'Alembertian operator.
with our dynamical variables chosen as A^μ and $\dot{A}^\mu$. The conjugated momenta given by definitions (3.1.2) and (3.1.3) are

$$p_\mu = -F_{0\mu} - 2a^2 (\partial_k \partial_\lambda F^{0\lambda} \delta^k_\mu - \partial_0 \partial_\lambda F^\lambda_\mu) \tag{3.2.4}$$

$$\pi_\mu = 2a^2(\partial_\lambda F^{0\lambda}\delta^0_\mu - \partial_\lambda F^\lambda_\mu) \tag{3.2.5}$$

Let $\mu = 0$,the primary constraints are

$$\phi_1 = \pi_0 \approx 0 \tag{3.2.6}$$

$$\phi_2 = p_0 - \partial^k \pi_k \approx 0 \tag{3.2.7}$$

Using the definition ofπ we can write the accelerations $\ddot{A}^i$ as

$$\ddot{A}^i = \frac{1}{2a^2}\pi^i + \partial_k F^i_k + \partial^i \dot{A}_o \tag{3.2.8}$$

The canonical Hamiltonian is given by

$$H_c = \int d^3x \left[p_\mu \dot{A}^\mu + \pi_\mu \ddot{A}^i - \ell\right]. \tag{3.2.9}$$

Using equation (3.2.8) we get

$$H_c = \int d^3x \Big[\dot{A}^o \partial^i \pi_i + p_i \dot{A}^i + \frac{1}{2a^2}\pi_i \pi^i + \pi_i \partial_k F_k{}^i + \pi_i \partial^i \dot{A}_o + \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$+\frac{1}{2}(\dot{A}_i - \partial_i A_0)(\dot{A}_i - \partial^i A_0) - a^2(\partial_k \dot{A}^k - \partial_k \partial^k A_0)(\partial_i \dot{A}^i - \partial_i \partial^i A_0)\Big] \tag{3.2.10}$$

According to Diracs formalism the total Hamiltonian is

$$H_T = H_C + \int d^3x (C_1(X)_{\phi 1} + C_2(X)_{\phi 2}) \tag{3.2.11}$$

where $C_1(X)$ and $C_2(X)$ are multipliers.

The consistency conditions result in

$$\dot{\phi}_1 = \{\phi_1, H_T\} \approx 0 \tag{3.2.12}$$

$$\dot{\phi}_2 = \{\phi_2, H_T\} \tag{3.2.13}$$

So we have a secondary constraint given by

$$\phi_3 = \partial^k p_k \approx 0 \tag{3.2.14}$$

and the consistency condition results in $\dot{\phi}_3 = \{\phi_3, H_T\} \approx 0$. All constraints are first class so the extended Hamiltonian is

$$H_E = H_C + \int d^3x(C_1(x)_{\phi 1} + C_2(x)_{\phi 2} + C_3(x)_{\phi 3}) \qquad (3.2.15)$$

The equations of motion for the dynamical variables, given by $\dot{A}^\alpha = \{A^\alpha, H_E\}$are

$$\dot{A}^0 = \dot{A}^0 + C_2, \qquad \dot{A}^i = \dot{A}^i - \partial^i C_3, \qquad (3.2.16)$$

Besides, $\ddot{A}^\alpha = \left\{\dot{A}^\alpha, H_E\right\}$ gives

$$\ddot{A}^0 = C_1, \qquad \ddot{A}^i = \frac{1}{2a_2}\pi^i + \partial_k F^{ik} + \partial^i \dot{A}_0 \qquad (3.2.17)$$

which mean that both $\bar{A}^0$and A^0 are arbitrary while we obtained again equation (3.2.8).

For the momenta variables $\dot{\pi}_i = \{\pi_i, H_E\}$ and$\dot{p}_\alpha = \{p_\alpha, H_E\}$ gives

$$\dot{\pi}_i = -F_{0i} - 2a^2 \partial_i \partial_k F_0^k - p_i \qquad (3.2.18)$$

$$\dot{p}_0 = -\partial_i F^{0i} - 2a^2 \partial^i \partial_i \partial_k F_0^k \qquad (3.2.19)$$

$$\dot{p}_i = -\partial_i \partial^k \pi_k + \partial_k \partial^k \pi_i - \partial_k F_i^k \qquad (3.2.20)$$

Equation (3.2.18) is the definition of p_i given by equation (3.2.4) and together with (3.2.19) it gives constraint ϕ_3.

3.3 Hamilton−Jacobi Formalism

According to the primary constraints (3.2.6) and (3.2.7), the set of Hamilton−Jacobi partial differential equation can be obtained as

$$H_0' = H_C + P_0, P_0 = \frac{\partial S}{\partial t} \qquad (3.3.1)$$

$$H_1' = \pi_0, H_2' = P_0 - \partial^k \pi_k \qquad (3.3.2)$$

and the total differential equation for A^i is

$$dA^i = \frac{\partial H'_0}{\partial p_i}dt + \frac{\partial H'_1}{\partial p_i}d\dot{A}_0 + \frac{\partial H'_2}{\partial p_i}dA_0 \tag{3.3.3}$$

$$dA^i = \frac{\partial H'_0}{\partial p_i}dt = \frac{\partial H_C}{\partial p_i}dt \Rightarrow dA^i = \dot{A}^i dt \tag{3.3.4}$$

which is completely equivalent to equation (3.2.16) since C_3 is arbitrary. For$\dot{A}^i$ we have

$$d\dot{A}^i = \frac{\partial H'_0}{\partial \pi_i}dt + \frac{\partial H'_1}{\partial \pi_i}d\dot{A}_0 + \frac{\partial H'_2}{\partial \pi_i}dA_0 = \frac{\partial H'_0}{\partial \pi_i}dt = \frac{\partial H_C}{\partial \pi_i}dt \tag{3.3.5}$$

$$d\dot{A}^i = (\frac{1}{2a^2}\pi^i + \partial_k F^{ik} + \partial^i \dot{A}_0)dt \tag{3.3.6}$$

Again we have a result in agreement with Diracs method result given in (3.2.17). For the momenta p_i and p_0 we have

$$dp^i = -\frac{\partial H'_0}{\partial A_i}dt - \frac{\partial H'_1}{\partial A_i}d\dot{A}_0 - \frac{\partial H'_2}{\partial A_i}dA_0 = \frac{\partial H'_0}{\partial A_i}dt = -\frac{\partial H_C}{\partial A_i}dt \tag{3.3.7}$$

$$dp^i = -\int d^3x \left[\pi_j \partial_k (\frac{\partial F^{jk}}{\partial A_i}) - \frac{1}{2}F^{jn}\frac{\partial F_{jn}}{\partial A_i}\right] dt \tag{3.3.8}$$

$$dp^i = -[\partial^i \partial^k \pi_k + \partial_k \partial^k \pi^i - \partial_k F^{ik}]dt \tag{3.3.9}$$

$$dp^0 = -\frac{\partial H'_0}{\partial A_0}dt - \frac{\partial H'_1}{\partial A_0}d\dot{A}_0 - \frac{\partial H'_2}{\partial A_0}dA_0 = -\frac{\partial H_C}{\partial A_0}dt \tag{3.3.10}$$

$$dp^0 = -\int d^3x \left[(\bar{A}_i - \partial^i A_0)\frac{\partial(\bar{A}_i - \partial_i A_0)}{\partial A_0}\right.$$

$$\left. -2a^2(\partial_i \dot{A}^i - \partial_i \partial^i A_0)\frac{\partial(\partial_k \dot{A}^k - \partial_k \partial^k A_0)}{\partial A_0}dt \right. \tag{3.3.11}$$

$$dp^0 = \left[-\partial_i(\dot{A}^i - \partial^i A_0) - 2a^2 \partial^k \partial^k (\partial_i \dot{A}^i - \partial_i \partial^i A_0)\right] dt \tag{3.3.12}$$

Finally for π^i we have:

$$d\pi^i = -\frac{\partial H'_0}{\partial \dot{A}_i}dt - \frac{\partial H'_1}{\partial \dot{A}_i}d\dot{A}_0 - \frac{\partial H'_2}{\partial \dot{A}_i}dA_0 = \frac{\partial H'_0}{\partial \dot{A}_i}dt = -\frac{H_C}{\partial \dot{A}_i}dt \tag{3.3.13}$$

$$d\pi^i = -\int d^3x \left[p^j \frac{\partial \bar{A}_j}{\partial \bar{A}_i} + (\bar{A}^j - \partial^j A_0)\frac{\partial(\bar{A}_j - \partial_j A_0)}{\partial A_0}\right.$$

$$-2a^2(\partial_i \bar{A}^j - \partial_j \partial^j A_0)\frac{\partial(\partial_k \bar{A}^k - \partial_k \partial^k A_0)}{\partial A_0}dt \tag{3.3.14}$$

$$d\pi^i = \left[-p^i - F^{0i} - 2a^2\partial^i\partial_k F^{0k}\right] dt \tag{3.3.15}$$

Equations (3.3.9), (3.3.12) and (3.3.15) are completely equivalent to (3.2.18), (3.2.19) and (3.2.20) respectively; consequently equations (3.3.12) and (3.3.15) give us the secondary constraint that isn't present in the total differential equations.

Chapter 4

Quantization of Regular Lagrangian of Nonholonomic Spinning Particle

The Hamiltonian formulation for systems with higher order regular Lagrangians was first developed by Ostrogradski[19]. This led to Euler's and Hamiltons equations of motion. However, in Ostrogradski's construction the structure of phase space and in particular of its local simplistic geometry is not immediately transparent which leads to confusion when considering canonical path integral quantization. In Ostrogradski's construction, this problem can be resolved within the well established context of constrained systems [32] described by Lagrangians depending on coordinates and velocities only. Therefore, higher−order systems can be set in the form of ordinary constrained systems . These new systems will be functions only of first order time derivative of the degrees of freedom and coordinates which can be treated using the theory of constrained systems[2]. The purpose of the present chapter is to study the canonical path integral quantization for singular systems with arbitrary higher order Lagrangian. where the path integral for certain kinds of higher order Lagrangian systems has been obtained, we consider an application of Nonholonomic regular Lagrangian of second order which will be quantized, using both the Hamilton Jacobi method and Dirac's method. Section (4.1), a brief review of the reduction of higher order regular Lagrangians to the first order singular Lagrangians. Both methods,the Hamilton Jacobi and dirac will be establish in section(4.2),besides this we calculate the canonical path integral quantization.

4.1 Review of The Reduction of Higher−Order Regular Lagrangian To First Order Singular One

Given a system of degrees of freedom $q_{n(t)}$,$(n = 1, \ldots, N)$ with higher−order regular Lagrangian $L_0(q_n, \dot{q}_n, \ldots, q_n^m)$ where $q_n^{(s)} = \frac{d^s q_n}{dt^s}$, $(s = 0, ..., m)$ Now let us introduce new independent variables $(q_{n,m-1}, q_{n,i}; (i = 0, \ldots, m - 2))$ such that the following relations would hold [19].

$$\dot{q}_{n,i} = q_{n,i+1} \tag{4.1.1}$$

Clearly, the variables $(q_{n,m-1}, q_{n,i},(i = 0, \ldots, m - 2))$ would then correspond to the time -derivatives $(q_n^{(m-1)}, q_n^{(i)})$ respectively, that is,

$$q_n^{(0)} = q_{n,0}, \qquad \dot{q}_n = q_{n,1}, \qquad q_n^{(m-1)} = q_{n,m-1}, \qquad q_n^{(m)} = \dot{q}_{n,m-1}. \tag{4.1.2}$$

Equation(4.1.1) represents relations between the new variables. In order to enforce these relations for independent variables $(q_{n,m-1}, q_{n,i})$, additional Lagrange multipliers $\lambda_{n,i}$ are introduced [19]. The variables $q_{n,m-1}, q_{n,i}, \lambda_{n,i}$, thus, determine the set of independent degrees of freedom of the extended Lagrangian system. The extended Lagrangian of this auxiliary description of the system is given by

$$L_T(q_{n,i}, q_{n,m-1}, \dot{q}_{n,i}, \dot{q}_{n,m-1}, \lambda_{n,i}) = L_0(q_{n,i}, q_{n,m-1}, \dot{q}_{n,i}) + \sum_{i=0}^{m-2} \lambda_{n,i}(\dot{q}_{n,i} - q_{n,i+1}) \tag{4.1.3}$$

The new Lagrangian in (4.1.3) is singular, and one can use the standard methods of singular systems like Diracs method or the canonical approach to investigate this Lagrangian.

Upon introducing the canonical momenta

$$p_{n,m-1} = \frac{\partial L_T}{\partial \dot{q}_{n,m-1}}, \tag{4.1.4}$$

$$p_{n,i} = \frac{\partial L_T}{\partial \dot{q}_{n,i}} = \lambda_{n,i} = -H_{n,i}, \tag{4.1.5}$$

$$\pi_{n,i} = \frac{\partial L_T}{\partial \dot{\lambda}_{n,i}} = 0 = -\phi_{n,i}. \tag{4.1.6}$$

The canonical Hamiltonian for the new first order singular Lagrangian can be written as

$$H_0(q_{n,i}, q_{n,m-1}, p_{n,m-1}, \lambda_{n,i}) =$$

$$p_{n,m-1}\dot{q}_{n,m-1} + \sum_{i=0}^{m-2} p_{n,i}\dot{q}_{n,i} + \sum_{i=0}^{m-2} \pi_{n,i}\dot{\lambda}_{n,i} - L_T(q_{n,i}, q_{n,m-1}, \dot{q}_{n,i}, \dot{q}_{n,m-1}, \lambda_{n,i}). \tag{4.1.7}$$

Equations(4.1.5) and (4.1.6) represent primary constraints [2,3]. Their Hamilton−Jacobi partial differential equations can be obtained as

$$H_0' = p_0 + H_0(q_{n,i}, q_{n,m-1}, p_{n,m-1}, \lambda_{n,i}) = 0, \tag{4.1.8}$$

$$\phi_{n,i}' = \pi_{n,i} = 0, \tag{4.1.9}$$

$$H_{n,i}' = p_{n,i} - \lambda_{n,i}. \tag{4.1.10}$$

Thus, The equations of motion can be written as total differential equations in many variables as follows:

$$dq_{n,j} = \frac{\partial H_0'}{\partial p_{n,j}}dt + \frac{\partial \phi_{n,i}'}{\partial p_{n,j}}d\lambda_{n,i} + \frac{\partial H_{n,i}'}{\partial p_{n,j}}dq_{n,i}, \tag{4.1.11}$$

$$dq_{n,m-1} = \frac{\partial H_0'}{\partial p_{n,m-1}}dt + \frac{\partial \phi_{n,i}'}{\partial p_{n,m-1}}d\lambda_{n,i} + \frac{\partial H_{n,i}'}{\partial p_{n,m-1}}dq_{n,i}, \tag{4.1.12}$$

$$d\lambda_{n,j} = \frac{\partial H_0'}{\partial \pi_{n,j}} dt + \frac{\partial \phi_{n,i}'}{\partial \pi_{n,j}} d\lambda_{n,i} + \frac{\partial H_{n,i}'}{\partial \pi_{n,j}} dq_{n,i}, \tag{4.1.13}$$

$$dp_{n,j} = -\frac{\partial H_0'}{\partial q_{n,j}} dt - \frac{\partial \phi_{n,i}'}{\partial q_{n,j}} d\lambda_{n,i} - \frac{\partial H_{n,i}'}{\partial q_{n,j}} dq_{n,i}, \tag{4.1.14}$$

$$dp_{n,m-1} = -\frac{\partial H_0'}{\partial q_{n,m-1}} dt - \frac{\partial \phi_{n,i}'}{\partial q_{n,m-1}} d\lambda_{n,i} - \frac{\partial H_{n,i}'}{\partial q_{n,m-1}} dq_{n,i}, \tag{4.1.15}$$

$$d\pi_{n,j} = -\frac{\partial H_0'}{\partial \lambda_{n,j}} dt - \frac{\partial \phi_{n,i}'}{\partial \lambda_{n,j}} d\lambda_{n,i} - \frac{\partial H_{n,i}'}{\partial \lambda_{n,j}} dq_{n,i}, \tag{4.1.16}$$

where $j = 0, 1, \ldots, m-2$.

The total differential equations are integrable if and only if

$$dH_0' = dp_0 - dH_0 = 0, \tag{4.1.17}$$

$$dH_{n,j}' = dp_{n,j} - d\lambda_{n,j} = 0, \tag{4.1.18}$$

$$d\phi_{n,j}' = d\pi_{n,j} = 0, \tag{4.1.19}$$

4.1.1 The Path Integral Quantization of the First Order Singular Lagrangian

If the coordinates t, $q_{n,i}$,$\lambda_{n,i}$ are denoted by t_α, that is,

$$t_\alpha = t, q_{n,i}, \lambda_{n,i}, \tag{4.1.20}$$

then the set of primary constraints (4.1.8), (4.1.9), and (4.1.10) can be written in a compact form as

$$H_\alpha' = 0, \tag{4.1.21}$$

where

$$H_\alpha' = H_0', H_{n,i}', \phi_{n,i}'. \tag{4.1.22}$$

The canonical path integral for the extended Lagrangians can be obtained as

$$K(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t)$$

$$= \int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} (Dq_{n,m-1} Dp_{n,m-1}) \exp\left[\frac{i}{\hbar}\int_{t_\alpha}^{t'_\alpha}\left(-H'_\alpha + p_{n,m-1}\frac{\partial H'_\alpha}{\partial p_{n,m-1}}\right) dt_\alpha\right], \tag{4.1.23}$$

where,$n = 1, \ldots, N$, $i = 0, \ldots, m-2$.

Note that (4.1.12) gives

$$\frac{\partial H'_\alpha}{\partial p_{n,m-1}} dt_\alpha = \frac{\partial H'{}_0}{\partial p_{n,m-1}} dt + \frac{\partial \phi'_{n,i}}{\partial p_{n,m-1}} d\lambda_{n,i} \frac{\partial H'_{n,i}}{\partial p_{n,m-1}} dq_{n,i} = dq_{n,m-1}. \tag{4.1.24}$$

Therefore, (4.1.23) can be written as

$$K(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t)$$

$$= \int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} (Dq_{n,m-1} Dp_{n,m-1}) \exp\left[\frac{i}{h}\int_{t_\alpha}^{t'_\alpha} (-H'_\alpha dt_\alpha + p_{n,m-1} dq_{n,m-1})\right], \tag{4.1.25}$$

However, according to (4.1.5) and (4.1.6), we get

$$H_{n,i} = -\lambda_{n,i} \tag{4.1.26}$$

$$\phi_{n,i} = 0, \tag{4.1.27}$$

so, it can be found that

$$H'_\alpha dt_\alpha = H'_0 dt + H_{n,i} dq_{n,i} + \phi'_{n,i} d\lambda_{n,i} = H'_0 dt - \lambda_{n,i} dq_{n,i}. \tag{4.1.28}$$

Then the transition amplitude can be written in the final form as

$$K(q'_{n,m-1}, q'_{n,i}, \lambda'_{n,i}, t'; q_{n,m-1}, q_{n,i}, \lambda_{n,i}, t)$$

$$= \int_{q_{n,m-1}}^{q'_{n,m-1}} \prod_{n=1}^{N} (Dq_{n,m-1} Dp_{n,m-1}) \exp\left[\frac{i}{h}\int_{t_\alpha}^{t'_\alpha} (-H'_0 dt + \lambda_{n,i} dq_{n,i} + p_{n,m-1} dq_{n,m-1})\right]. \tag{4.1.29}$$

Equation (4.1.29) represents the canonical path integral quantization of higher−order regular Lagrangians as first-order singular Lagrangians.

4.2 Nonholonomic Lagrangian of Second−Order

The constraint is a relation between the coordinate (some times and time), there are two types of constraint, holonomic constraint which is a relation in the form $f(\vec{r_1}, \vec{r_2}, \ldots, \vec{r_n}, t)$, other wise it is called nonholonomic constraint. Now the Lagrangian of nonholonomic spinning particle is

$$L(q,\dot{q},\ddot{q}) = \frac{1}{2}\sum_{i=0}^{3}(\dot{q}_i)^2 - \frac{1}{2}\sum_{i=0}^{3}(\ddot{q}_i)^2 \tag{4.2.1}$$

Consider the Nonholonomic constraint,

$$\dot{q}_3 = q_2\dot{q}_1 \tag{4.2.2}$$

which gives

$$\ddot{q}_3 = \dot{q}_2\dot{q}_1 + q_2\ddot{q}_1 \tag{4.2.3}$$

so the constrained Lagrangian is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + (q_2\dot{q}_1)^2) - \frac{1}{2}(\ddot{q}_1^2 + \ddot{q}_2^2 + (\dot{q}_2\dot{q}_1 + q_2\ddot{q}_1)^2) \tag{4.2.4}$$

where $L$is regular Lagrangian: To reduce this Lagrangian into first order singular Lagrangian, we use constrained auxiliary description , using the equations and (4.1.2) where $n = 1, 2$, $m = 2$ the higher derivatives, and $i = 0$, we get

$$q_1^{(0)} = q_{1,0}, \qquad q_2^{(0)} = q_{2,0} \tag{4.2.5}$$

$$q_1^{(1)} = q_{1,1}, \qquad q_2^{(1)} = q_{2,1} \tag{4.2.6}$$

$$q_1^{(2)} = \dot{q}_{1,1}, \qquad q_2^{(2)} = \dot{q}_{2,1} \tag{4.2.7}$$

Hence, the Lagrangian (4.2.4) becomes:

$$L = \frac{1}{2}\left(q_{1,1}^2 + q_{2,1}^2 + (q_{2,0}q_{1,1})^2\right) - \frac{1}{2}\left(\dot{q}_{1,1}^2 + \dot{q}_{2,1}^2 + (q_{2,0}\dot{q}_{1,1} + q_{2,1}q_{1,1})^2\right) \tag{4.2.8}$$

Upon using (4.1.1), the recursion relations is $\dot{q}_{1,0} = q_{1,1}$ and $\dot{q}_{2,0} = q_{2,1}$ and (4.1.3) the extended Lagrangian is simply

$$L_T = \frac{1}{2}\left(q_{1,1}^2+q_{2,1}^2+(q_{2,0}q_{1,1})^2\right)-\frac{1}{2}\left(\dot{q}_{1,1}^2+\dot{q}_{2,1}^2+(q_{2,0}\dot{q}_{1,1}+q_{2,1}q_{1,1})^2\right)+\lambda_{1,0}(\dot{q}_{1,0}-q_{1,1})+\lambda_{2,0}(\dot{q}_{2,0}-q_{2,1}) \tag{4.2.9}$$

The conjugate momenta can be obtained as

$$p_{1,1} = \frac{\partial L_T}{\partial \dot{q}_{1,1}} = -(1+q_{2,0}^2)\dot{q}_{1,1} - q_{2,0}q_{2,1}q_{1,1} \tag{4.2.10}$$

$$p_{2,1} = \frac{\partial L_T}{\partial \dot{q}_{2,1}} = -\dot{q}_{2,1} \tag{4.2.11}$$

$$p_{1,0} = \frac{\partial L_T}{\partial \dot{q}_{1,0}} = \lambda_{1,0} = -H_{1,0} \tag{4.2.12}$$

$$p_{2,0} = \frac{\partial L_T}{\partial \dot{q}_{2,0}} = \lambda_{2,0} = -H_{2,0} \tag{4.2.13}$$

Also

$$\pi_{1,0} = \frac{\partial L_T}{\partial \dot{\lambda}_{1,0}} = 0 = -\phi_{1,0} \tag{4.2.14}$$

$$\pi_{2,0} = \frac{\partial L_T}{\partial \dot{\lambda}_{2,0}} = 0 = -\phi_{2,0} \tag{4.2.15}$$

It is obvious that the equations (4.2.12−4.2.15) are constraints.Therefore, the coordinates $q_{1,0}, q_{2,0}$, $\lambda_{1,0}$ and $\lambda_{2,0}$ represent the restricted coordinates.

4.2.1 Hamilton−Jacobi Formalism

Using (4.1.7) and (4.2.10) the canonical Hamiltonian takes the form

$$H_0 = -\frac{p_{1,1}p_{1,1}}{(1+q_{2,0}^2)} - \frac{q_{2,0}q_{2,1}q_{1,1}p_{1,1}}{(1+q_{2,0}^2)} - \frac{1}{2}p_{2,1}^2$$

$$-\frac{1}{2}\left(q_{1,1}^2+q_{2,1}^2+(q_{2,0}q_{1,1})^2\right) + \frac{1}{2}\frac{(p_{1,1}+q_{2,0}q_{2,1}q_{1,1})}{(1+q_{2,0}^2)^2}$$

$$+\frac{1}{2}\left(-\frac{q_{2,0}p_{1,1}}{(1+q_{2,0}^2)} - \frac{q_{2,0}q_{2,1}q_{1,1}p_{1,1}}{(1+q_{2,0}^2)} - \frac{q_{2,0}^2q_{2,1}q_{1,1}}{(1+q_{2,0}^2)} + q_{2,1}q_{1,1}\right)^2 \tag{4.2.16}$$

Accordingly, the set of HJPDEs can be written as

$$H'_0 = P_0 + H_0, \tag{4.2.17}$$

$$H'_{1,0} = p_{1,0} - \lambda_{1,0}, \tag{4.2.18}$$

$$H'_{2,0} = p_{2,0} - \lambda_{2,0}, \tag{4.2.19}$$

$$\phi'_{1,0} = \pi_{1,0}, \tag{4.2.20}$$

$$\phi'_{2,0} = \pi_{2,0}, \tag{4.2.21}$$

and the equations of motion are

$$dq_{1,0} = \dot{q}_{1,0} dt, \tag{4.2.22}$$

$$dq_{2,0} = \dot{q}_{2,0} dt, \tag{4.2.23}$$

$$dq_{1,1} = -\frac{(p_{1,1} + q_{2,0}q_{2,1}q_{1,1})}{(1+q_{2,0}^2)^2} dt, \tag{4.2.24}$$

$$dq_{2,1} = -p_{2,1} dt, \tag{4.2.25}$$

$$dp_{1,0} = 0, \tag{4.2.26}$$

$$\begin{aligned} dp_{2,0} = \Bigg[&(q_{2,0}q_{1,1})q_{1,1} + \frac{p_{1,1}q_{2,1}q_{1,1}}{(1+q_{2,0}^2)} - q_{2,0}\frac{(p_{1,1}^2 - q_{2,1}^2 q_{1,1}^2)}{(1+q_{2,0}^2)^2} \\ &- \frac{(3p_{1,1}q_{2,1}q_{1,1}q_{2,0}^2)}{(1+q_{2,0}^2)^3} + \frac{(q_{2,1}^2 q_{1,1}^2 q_{2,0}^2)}{(1+q_{2,0}^2)^3} \Bigg] dt, \end{aligned} \tag{4.2.27}$$

$$dp_{1,1} = (q_{1,1} - \lambda_{1,0} + q_{1,1}q_{2,0}^2 - q_{2,1}^2 q_{1,1} - q_{2,0}q_{2,1}\frac{(p_{1,1} + q_{2,0}q_{2,1}q_{1,1})}{(1+q_{2,0}^2)^2}) dt, \tag{4.2.28}$$

$$dp_{2,1} = (q_{2,1} - \lambda_{2,0} - q_{2,1}q_{1,1}^2 - q_{2,0}q_{1,1}\frac{(p_{1,1} + q_{2,0}q_{2,1}q_{1,1})}{(1+q_{2,0}^2)^2}) dt, \tag{4.2.29}$$

$$d\pi_{1,0} = 0, \tag{4.2.30}$$

$$d\pi_{2,0} = 0, \tag{4.2.31}$$

$$d\lambda_{1,0} = \dot{\lambda}_{1,0} dt, \tag{4.2.32}$$

$$d\lambda_{2,0} = \dot{\lambda}_{2,0} dt, \tag{4.2.33}$$

The equations of motion are integrable if the following are identically satisfied

$$dH'_0 = dP_0 + dH_0 = 0 \tag{4.2.34}$$

$$dH'_{1,0} = dP_{1,0} - d\lambda_{1,0} = 0 \tag{4.2.35}$$

$$dH'_{2,0} = dP_{2,0} - d\lambda_{2,0} = 0 \tag{4.2.36}$$

$$d\phi'_{1,0} = d\pi_{1,0} = 0 \tag{4.2.37}$$

$$d\phi'_{2,0} = d\pi_{2,0} = 0 \tag{4.2.38}$$

4.2.2 The Path Integral Quantization of The System

From (4.1.29) the canonical path integral quantization for this system is

$$K = \int \prod_{n=1}^{2} (Dq_{n,1} Dp_{n,1}) \exp\left[\frac{i}{\hbar}\int (-H_0 dt + \lambda_{n,0} dq_{n,0} + p_{n,1} dq_{n,1})\right]. \tag{4.2.39}$$

For $n = 1, 2$, then

$$K = \int \prod_{n=1}^{2} (Dq_{1,1} Dq_{2,1} Dp_{1,1} Dp_{2,1}) \exp\left[\frac{i}{\hbar}\int \left(\frac{p_{1,1}p_{1,1}}{(1+q_{2,0}^2)} + \frac{q_{2,0}q_{2,1}q_{1,1}p_{1,1}}{(1+q_{2,0}^2)} + \frac{1}{2}p_{2,1}^2\right.\right.$$

$$+\frac{1}{2}\left(q_{1,1}^2 + q_{2,1}^2 + (q_{2,0}q_{1,1})^2\right) - \frac{1}{2}\frac{(p_{1,1} + q_{2,0}q_{2,1}q_{1,1})}{(1+q_{2,0}^2)^2}$$

$$-\frac{1}{2}\left(-\frac{q_{2,0}p_{1,1}}{(1+q_{2,0}^2)} - \frac{q_{2,0}q_{2,1}q_{1,1}p_{1,1}}{(1+q_{2,0}^2)} + q_{2,1}q_{1,1}\right)^2 - \lambda_{1,0}q_{1,1} - \lambda_{2,0}q_{2,1}\Bigg) dt$$

$$+\lambda_{1,0}dq_{1,0} + \lambda_{2,0}dq_{2,0} + p_{1,1}dq_{1,1} + p_{2,1}dq_{2,1}. \tag{4.2.40}$$

Equation (4.2.40)can be written in a compact form as

$$K = \int \prod_{n=1}^{2} (Dq_{1,1} Dq_{2,1} Dp_{1,1} Dp_{2,1}) \exp\left[\frac{i}{\hbar}\int \left(\frac{1}{2}\left(q_{1,1}^2 + q_{2,1}^2 + (q_{2,0}q_{1,1})^2\right)\right.\right.$$

$$\left.\left.-\frac{1}{2}\left(\dot{q}_{1,1}^2 + \dot{q}_{2,1}^2 + (q_{2,0}\dot{q}_{1,1} + q_{2,1}q_{1,1})^2\right) + \lambda_{1,0}(\dot{q}_{1,0} - q_{1,1}) + \lambda_{2,0}(\dot{q}_{2,0} - q_{2,1})\right) dt\right] \tag{4.2.41}$$

Finally we have:

$$K = \int \prod_{n=1}^{2} (Dq_{1,1} Dq_{2,1} Dp_{1,1} Dp_{2,1}) \exp\left[\frac{i}{\hbar}\int L_T dt\right] \tag{4.2.42}$$

4.2.3 Dirac's Method

Now we will use Dirac method to study the same system discussed in previous subsection with higher order singular Lagrangians, the total Hamiltonian can be written as

$$H_T = H_0 + \lambda_1\phi_1' + \lambda_2\phi_2' + \lambda_1' H_{1,0}' + \lambda_2' H_{2,0}' \tag{4.2.43}$$

were $\lambda_1, \lambda_2, \lambda_1', \lambda_2'$ are arbitrary functions. The equation of motion can be written as:

$$\dot{q}_{n,i} = \{q_{n,i}, H_T\} \tag{4.2.44}$$

$$\dot{q}_{n,m-1} = \{q_{n,m-1}, H_T\} \tag{4.2.45}$$

$$\dot{p}_{n,i} = \{p_{n,i}, H_T\} \tag{4.2.46}$$

$$\dot{p}_{n,m-1} = \{p_{n,m-1}, H_T\} \tag{4.2.47}$$

were$n = 1, 2$, $m = 2$ and $i = 0$ to determined the arbitrary functions, we use the consistency conditions

$$\dot{\phi'}_1 = \{\phi_1', H_T\} = \{\phi_1', H_0\} + \lambda_1\{\phi_1', \phi_1'\} + \lambda_2\{\phi_1', \phi_2'\} + \lambda_1'\{\phi_1', H_{1,0}'\} + \lambda_2'\{\phi_1', H_{2,0}'\} \tag{4.2.48}$$

$$\lambda_1' = q_{1,1} \tag{4.2.49}$$

$$\dot{\phi'}_2 = \{\phi_2', H_T\} = \{\phi_2', H_0\} + \lambda_1\{\phi_2', \phi_1'\} + \lambda_2\{\phi_2', \phi_2'\} + \lambda_1'\{\phi_2', H_{1,0}'\} + \lambda_2'\{\phi_2', H_{2,0}'\} \tag{4.2.50}$$

$$\lambda_2' = q_{2,1} \tag{4.2.51}$$

$$\dot{H'}_{1,0} = \{H_{1,0}', H_T\} = \{H_{1,0}', H_0\}+\lambda_1\{H_{1,0}', \phi_1'\}+\lambda_2\{H_{1,0}', \phi_2'\}+\lambda_1'\{H_{1,0}', H_{1,0}'\}+\lambda_2'\{H_{1,0}', H_{2,0}'\} \tag{4.2.52}$$

$$\lambda_1 = 0 \tag{4.2.53}$$

$$\dot{H'}_{2,0} = \{H_{2,0}', H_T\} = \{H_{2,0}', H_0\}+\lambda_1\{H_{2,0}', \phi_1'\}+\lambda_2\{H_{2,0}', \phi_2'\}+\lambda_1'\{H_{2,0}', H_{1,0}'\}+\lambda_2'\{H_{2,0}', H_{2,0}'\} \tag{4.2.54}$$

$$\lambda_2 = (q_{2,0}q_{1,1})q_{1,1} - (q_{2,0}\dot{q}_{1,1} + q_{2,1}q_{1,1})\dot{q} \tag{4.2.55}$$

The equation of motion (4.2.44) to (4.2.47) becomes

$$\dot{q}_{1,0} = \{q_{1,0}, H_T\} = \{q_{1,0}, H_0\} + \lambda_2\{q_{1,0}, \phi'_2\} + \lambda'_1\{q_{1,0}, H'_{1,0}\} + \lambda'_2\{q_{1,0}, H'_{2,0}\} = q_{1,1} \tag{4.2.56}$$

$$\dot{q}_{2,0} = \{q_{2,0}, H_T\} = \{q_{2,0}, H_0\} + \lambda_2\{q_{2,0}, \phi'_2\} + \lambda'_1\{q_{2,0}, H'_{1,0}\} + \lambda'_2\{q_{2,0}, H'_{2,0}\} = q_{2,1} \tag{4.2.57}$$

$$\dot{p}_{1,0} = \{p_{1,0}, H_T\} = \{p_{1,0}, H_0\} + \lambda_2\{p_{1,0}, \phi'_2\} + \lambda'_1\{p_{1,0}, H'_{1,0}\} + \lambda'_2\{p_{1,0}, H'_{2,0}\} = 0 \tag{4.2.58}$$

$$\dot{p}_{2,0} = \{p_{2,0}, H_T\} = \{p_{2,0}, H_0\} + \lambda_2\{p_{2,0}, \phi'_2\} + \lambda'_1\{p_{2,0}, H'_{1,0}\} + \lambda'_2\{p_{2,0}, H'_{2,0}\} =$$

$$\Bigg[(q_{2,0}q_{1,1})q_{1,1} + \frac{p_{1,1}q_{2,1}q_{1,1}}{(1+q_{2,0}^2)} - q_{2,0}\frac{(p_{1,1}^2 - q_{2,1}^2q_{1,1}^2)}{(1+q_{2,0}^2)^2}$$
$$-\frac{(3p_{1,1}q_{2,1}q_{1,1}q_{2,0}^2)}{(1+q_{2,0}^2)^3} + \frac{(q_{2,1}^2q_{1,1}^2q_{2,0}^2)}{(1+q_{2,0}^2)^3}\Bigg] \tag{4.2.59}$$

$$\dot{p}_{1,1} = \{p_{1,1}, H_T\} = \{p_{1,1}, H_0\} + \lambda_2\{p_{1,1}, \phi'_2\} + \lambda'_1\{p_{1,1}, H'_{1,0}\} + \lambda'_2\{p_{1,1}, H'_{2,0}\} =$$

$$(q_{1,1} - \lambda_{1,0} + q_{1,1}q_{2,0}^2 - q_{2,1}^2q_{1,1} - q_{2,0}q_{2,1}\dot{q}_{1,1}) \tag{4.2.60}$$

$$\dot{p}_{2,1} = \{p_{2,1}, H_T\} = \{p_{2,1}, H_0\} + \lambda_2\{p_{2,1}, \phi'_2\} + \lambda'_1\{p_{2,1}, H'_{1,0}\} + \lambda'_2\{p_{2,1}, H'_{2,0}\} =$$

$$(q_{2,1} - \lambda_{2,0} - q_{2,1}q_{1,1}^2 - q_{2,0}q_{1,1}\dot{q}_{1,1}) \tag{4.2.61}$$

$$\dot{\pi}_{1,0} = \{\pi_{1,0}, H_T\} = \{\pi_{1,0}, H_0\}+\lambda_2\{\pi_{1,0}, \phi'_2\}+\lambda'_1\{\pi_{1,0}, H'_{1,0}\}+\lambda'_2\{\pi_{1,0}, H'_{2,0}\} = -q_{1,1}+q_{1,1} = 0 \tag{4.2.62}$$

$$\dot{\pi}_{2,0} = \{\pi_{2,0}, H_T\} = \{\pi_{2,0}, H_0\}+\lambda_2\{\pi_{2,0}, \phi'_2\}+\lambda'_1\{\pi_{2,0}, H'_{1,0}\}+\lambda'_2\{\pi_{2,0}, H'_{2,0}\} = -q_{2,1}+q_{2,1} = 0 \tag{4.2.63}$$

Notice that the equations (4.2.56) to (4.2.63) are equivalent to equations (4.2.22) to (4.2.31)

Chapter 5

CONCLUSION

The purpose of this thesis is to study the higher-order singular Lagrangian systems by using the Hamiltonian formulation (Dirac's method and the canonical method), and the Lagrangian treatment of singular systems as field (continuous) systems. Besides, the canonical path integral has been investigated. The second and the third order singular Lagrangian were studied using previews methods, all methods give us the same equation of motion .

The second order Lagrangian of Podolosky electrodynamics was studied by Dirac's method and Hamilton Jacobi method , the equations of motion that obtained are the same.

Dirac's method and the Hamilton Jacobi formulation represent the Hamilton treatment of the constraints systems , where Dirac's method introduces a primary constraints, then we construct the total Hamiltonian which is the primary constraints multiplied by Lagrange multiplier to the usual Hamiltonian. The consistency condition is checked on the primary constraints, were they are classified into two types : first and second classes constraints. The first−class constraints which have vanishing Poisson brackets, but the second−class constraints have non−vanishing Poisson brackets. The equations of motion are obtained as total derivative in terms of Poisson brackets. In the Hamilton Jacobi formulation, the equations of motion are written as total differential equations in many variables. If the integrability conditions are not identically satisfied, then these will be considered as new constraints. This process will be continued until we obtain a complete system. the singular systems have two types of integrable systems, completely integrable systems were the integrability conditions are identically satisfied, and partially integrable systems were integrability conditions are not satisfied. In the canonical method it is not necessary to distinguish between the primary and secondary constraints, also it is no need to introduce the Lagrange multiplierλ_{μ}. Another approach is the Lagrangian treatment of singular systems as field system, in this treatment we solved the Euler$-$Lagrangian equation with some constraints. This treatment is applied in many applications for the singular Lagrangian systems and the results are in exact agreement with the results obtained using Dirac's method and the Hamilton Jacobi formulation. This treatment unified the Hamiltonian and Lagrangian formulations of constrained systems.

As in Hamilton$-$Jacobi method, the Lagrangian treatment of singular systems as field system, there is no need to introduce arbitrary Lagrange multipliers, which should be determined in the Dirac's method. The existence of this kind of arbitrary Lagrange multipliers is inevitable and they should be determined by imposing a new gauge condition, which is not easy task in Dirac's method. The Hamilton Jacobi formulation and the Lagrangian treatment of singular systems as field system need more investigations for some another physical models.

The last method, the reducing of higher$-$order regular Lagrangian systems to first order singular Lagrangian. Both Dirac's and Hamilton$-$Jacobi formulations were represented to study the reduction form of the first order singular Lagrangian. The higher order (regular) lagrangian with order m is transformed to first order singular by introducing an auxiliary fields,and the equations of them were re−written as a set of m-constraints.

The main advantages of using Hamilton−Jacobi method is that we have no difference between first and second class constraints and we don't need gauge fixing term, because the gauge variables are separated in the process of constructing an integrable system of total differential equations. Besides, the integrable action function provided by the formalism can be used in the process of the path integral quantization method of the constrained systems.

Bibliography

[1] K. Sundermeyer, *Constrained dynamics, Lect. notes in phys.*, 169 (Springer,Verlag) 1982.

[2] P. A. M. Dirac ,*Generalized Hamiltonian Dynamics* ,(Can.J.Math.), 2(1950)129.

[3] P. A. M. Dirac , *Lectures on quantum mechanics* (Belfer Graduate school of science, Yashiv University, New York), 1964.

[4] D. M. Gitman and I. V. Tyuin, *Quantization of Fields with Constraints*, (Spring-Verlag, Berlin Heidelberg), (1990).

[5] Jan Govaerts, *Hamiltonian Quantisation and Constrsined Dynamics*, Volume 4, Leuven University Press, (1991).

[6] L. D. Faddeev, *Theor. Math. Phys.*, 1 (1970)1.

[7] P. Senjavonic, *Ann. Phys. N. Y.*, 100(1976) 227.

[8] Y. Güler, *Nouvo Cimento* **B**, 107 (1992) 1389.

[9] Y. Güler, *Nouvo Cimento* **B**, 107 (1992) 1143.

[10] Farahat N. I. and Güler Y.,*Phys.Rev.* **A**,(1995) .

[11] S. Muslih, Y. Güler, *Nuovo Cimento* **B**, 113 (1998) 277.

[12] S. Muslih, *Nuovo Cimento* **B**, 115 (2000)1.

[13] S. Muslih, *Nuovo Cimento* **B**, 115 (2000)7.

[14] N. Farahat and Kh. Ramadan, *Hadronic J.*, 25 (2002)291.

[15] N. Farahat and Z. Nassar, J. of Islamic University, 13 (2005) 149.

[16] O.A. Jarabah,E.H.Hasan,*European Scientific Journal.*,10 (2014) 9..

[17] Muslih S. I.,*Nuovo Cimento* **B**, 117 (2002)4..

[18] Muslih S. I., *Path Intehral Formulation of Constrained Systems ,Hadonic* ,23 (2000) 203.18.

[19] J.M.PONS, *Ostrogradskis theorem for higher−order sigular lagrangians,Lett.*17 (1989) 181−189. .

[20] B. M. Pimental and R. G. Teixeira, *Nuovo Cimento* **B**, 113 (1998) 805.

[21] S. Muslih, *math−ph/0010020* **v1**, *Quantization of singular systems with second order Lagrangian*, (2000)1.

[22] S. Muslih, N. Farahat and Heles, *Nuovo Cimento* **B**, 119(2004) 531.

[23] O. Shihada, S. Muslih and N. Farahat, *Quantization of constrained systems with higher−order Lagrangian.*Thesis, The Islamic University of Gaza, (2006)

[24] Goldshtein H., *Classical Mechanics*, 2nd edition, Addision-Wesley, (1980).24.

[25] S. Muslih, *Nuovo Cimento* **B**, 118 (2003) 505. .

[26] B. M. Pimental and R. G. Teixeira, *Nuovo Cimento* **B**,111,841(1996)..

[27] M. Rashid and S. Khalil, IC, 420 (1993).

[28] M. R. Helles, *Singular Systems with Higher Order Lagrangian*, M. Sc. Thesis, The Islamic University of Gaza, (2004)

[29] Y.Saito,R.Sugano,T.Ohta,and T.Kimura,J.Math.Phys.30(1989)1122

[30] J. Govaerts and M. Rashid, hep−th/9403009 **v**2, (1994)1.

[31] B. Podolsky, P. Schwed, Rev. Mod. Phys.20,40 (1948).

[32] K.I.Nawafleh,A.A.ALsoub,**v**1, (2008)73-78.