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Hamilton-Jacobi Formulation of Constrained Particles and Strings

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بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

يَرْفَعِ اللَّهُ الَّذِينَ آمَنُوا مِنكُمْ وَالَّذِينَ أُوتُوا الْعِلْمَ دَرَجَاتٍ

وَاللَّهُ بِمَا تَعْمَلُونَ خَبِيرٌ

To my parents ,,, to those who taught me

Hana'a

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Abstract

In this dissertation, we explore various aspects of the singular physical systems, which are treated by using the Hamiltonian formulation of constrained systems.

The Hamilton-Jacobi formalism, are presented for supersymmetry models of supersymmetric particles and string. The formalism is applied to study the dynamics of massive Brink-Schwarz superparticle and massless Seigel superparticle with simple supersymmetry $N = 1$, possesses in the most popular superspace in physics $\mathbb{R}^{4|4}$. The integrability conditions are satisfied, so the systems are integrable. We have applied the method to obtain the classical dynamics in phase-space of massless spinning superparticles. By proceeding in the same way, we analyze the dynamics of superparticle with extended supersymmetric $N = 2$ coupled with an external superpotential, and various applications of superstring. Firstly, we generalize this approach for a string that is propagating in $4D$ superspace. Secondly, the example of Green-Schwarz superstring, which provides the most realistic string models, has been formulated. Besides, a mechanical model of $D = 11$ superstring is constructed, and in terms of the Hamilton-Jacobi formalism the equations of motion are derived and discussed, as total differential equations in many variables.

ملـــــــــــــخص

صياغة هاملتون – جاكوبي للجسيمات والأوتار المقيدة

إعـــداد

الباحثة/ هناء عبد الكريم رزق العجلة

إشـراف

الدكتور/ ناصر فرحـــــات

تعتمد هذه الأطروحة على دراسة الأنظمة المقيدة باستخدام صياغة هاملتون – جاكوبي، حيـــث تمـــت دراسة الأنظمة المتناظرة للأجسام المثالية، و الأوتار الفائقة التنـــاظر. حيـــثُ نُوقِشـــتْ نمـــاذِج مختلفـــة للجسيمات المثالية الهائلة و المهملة الكتلة، بالإضافة إلى الجسيمات المثالية ذات الحركة المِغزَليـــة، و إيجـــاد معادلات الحركة واختبار شروط التكامل لكلٍ منها.

كما تحتوي نموذجاً لجســـيم مثـــالي يتحـــرك في مجـــال كهرومغناطيســـي ذو درجـــات عاليـــة مـــن التناظر $(N = 2)$ كنظام مقيد وتم الحصول على معادلات الحركة لوصف حركة ذلك الجسيم عن طريق المعادلات التفاضلية الجزئية لهاملتون – جاكوبي.

كما تم التعرف على نظرية الأوتار و الأوتار الفائقة واختبار مدى قابليـــة تطبيـــق صـــياغة هـــاملتون – جاكوبي للحصول على معادلات الحركة واختبار شروط التكامل لمجموعة من الأمثلة المختلفـــة للأوتـــار الفائقة $(N = 1)$.

Chapter 1

Introduction

1.1 Historical Background

The Hamiltonian formulation of singular systems is usually made through the formalism developed by Dirac [1, 2]. In this formalism, the constraints caused by the singularity of Hess matrix are added to the canonical Hamiltonian, and then the consistency conditions are worked out, being possible to eliminate some degrees of freedom of the system. Dirac also showed that the gauge freedom is caused by the presence of first class constraints. This formalism has a wide range of applications in field theory and it is still the main tool for the analysis of singular systems [3]-[5]. Despite the success of Dirac's method, it is always interesting to apply different formalisms to the analysis of singular systems.

The study of new formalisms for singular systems may provide new tools to investigate these systems. In classical dynamics, different formalisms (Lagrangian, Hamiltonian, Hamilton-Jacobi) provide different approaches to the problems, each formalism has advantages and disadvantages in the study of some features of the systems and being equivalent among themselves. In the same way, different formalisms provide different views of the features of singular systems, which justify the interest in their study.

In this thesis, we generalize the Hamilton-Jacobi formalism that was developed by Güler [6, 7]. This approach based on Carathéodory's equivalent Lagrangian method [8] to write down the Hamilton-Jacobi equations for the system and make use of its singularity to write the equations of motion as total differential equations in many variables . The Hamilton-Jacobi formalism that we study in this work was applied only to a few number of physical examples as the electromagnetic field, relativistic particle in an external electromagnetic field, the Young-Mills field and the Einstein gravitational field [9]-[18]. The advantage of the Hamilton-Jacobi formalism is that we have no difference between first and second class constraints and we do not need gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations.

1.2 Hamilton-Jacobi Formalism

In this section, we study the singular systems using the Hamilton-Jacobi formulation or 'Canonical Method'.

The system that is described by singular Lagrangian $L(q_i, \dot{q}_i, t)$ with $i = 1, ..., N$, has a rank of Hess matrix

$$A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, \ldots, n, \tag{1.1}$$

equal to $(N - p)$, $p < N$. In this case we have p momenta which are dependent on each other. The generalized momenta P_i corresponding to the generalized coordinates q_i are defined as,

$$P_a = \frac{\partial L}{\partial \dot{q}_a}, \qquad a = 1, \ldots, N - p, \tag{1.2}$$

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}, \qquad \mu = N - p + 1, \ldots, N. \tag{1.3}$$

Since, the rank of the Hess matrix is $(N - p)$, one may solve (1.2) for $\dot{q}_a$ as

$$\dot{q}_a = \dot{q}_a \left(q_i, \dot{q}_\mu, P_b\right) \equiv \omega_a. \tag{1.4}$$

Substituting (1.4) into (1.3), we obtain relations in q_i, P_a, $\dot{q}_\nu$ and t in the form

$$P_\mu = \frac{\partial L}{\partial \dot{q}_\mu}\bigg|_{\dot{q}_a = \omega_a} \equiv -H_\mu(q_i, \dot{q}_\nu, \dot{q}_a = \omega_a, P_a, t), \qquad \nu = N - p + 1, \ldots, N. \tag{1.5}$$

By mean of (1.4) and (1.5) the canonical Hamiltonian H_0 is defined as

$$H_0 = -L\big(q_i, \dot{q}_\mu, \dot{q}_a = \omega_a, t\big) + P_a \omega_a + \dot{q}_\mu P_\mu\big|_{P_\nu = -H_\nu}. \tag{1.6}$$

The set of Hamilton-Jacobi partial differential equations (HJPDE) is expressed as

$$H'_\alpha \left(q_\beta; q_a; P_a = \frac{\partial s}{\partial q_a}; P_\mu = \frac{\partial s}{\partial q_\mu} \right) = 0, \qquad \alpha, \beta = 0, 1, \ldots, p. \tag{1.7}$$

where

$$H_0' = P_0 + H_0; \tag{1.8}$$

and

$$H_\mu' = P_\mu + H_\mu. \tag{1.9}$$

with $q_0 \equiv t$ and S being the action. The equations of motion are obtained as total differential equations in many variables such as,

$$dq_a = \frac{\partial H_\alpha'}{\partial P_a} dt_\alpha, \tag{1.10}$$

$$dP_r = -(-1)^{n_r n_\alpha} \frac{\partial H_\alpha'}{\partial q_r} dt_\alpha, \qquad r = 0, 1, \ldots, N, \tag{1.11}$$

$$dZ = \left(-H_\alpha + P_a \frac{\partial H_\alpha'}{\partial P_a} \right) dt_\alpha, \tag{1.12}$$

where $n_i = 0\,, 1$, $(i = r\,, \alpha)$ define the Grassmann parity of the corresponding quantity, and $Z = S\,(t_\alpha, q_a)$. These equations are integrable if and only if [19, 20]

$$dH_0' = 0, \tag{1.13}$$

and

$$dH_\mu' = 0, \qquad \mu = N - p + 1, \ldots, N. \tag{1.14}$$

If the conditions (1.13) and (1.14) are not satisfied identically, we consider them as new constraints and we examine their variations. Thus repeating this procedure, one may obtain a set of constraints such that all the variations of them vanish. We have two types of integrable systems: The first is completely integrable systems, where the set of equations of motion and the action function are integrable. The second is partially integrable systems, where the set of equations of motion is only integrable. The importance of classification systems as they are completely integrable or partially is evident in quantization methods, to obtain the path integral as an integration over the canonical phase space coordinates.

1.3 Superparticle

Supersymmetries play a very important role in all field theories being relevant in physics, for example: Yang-Mills theory, string theory and gravity. Supersymmetry is a symmetry connecting the properties of bosons and those of fermions. This symmetry can be realized on ordinary fields (functions of space-time) by transformations that mix bosons and fermions. Supersymmetry proposes that for every ordinary particle there exists a "superpartner" having similar properties except the spin which differs by $\frac{1}{2}$.

The relativistic particle of mass m is described by the world line action

$$S = \frac{1}{2}\int d\tau \Big(e^{-1}\dot{x}^2 - em^2\Big), \tag{1.15}$$

where$e(\tau)$ is the "einbein"; which is an auxiliary field takes the role of the metric tensor on the world-line such as $g(\tau) = e^2(\tau)$ [4]. Here, we achieve space-time supersymmetry by generalizing Minkowski space, with bosonic coordinates x^μ, to a superspace with fermonic coordinates. If there are to be N supersymmetries, we introduce N anticommuting spinor coordinates

θ_i^α $i = 1, 2, ..., N$, and the index α is that of a space-time spinor appropriate to D dimensions. Now we generalize the bosonic point particle, which propagates in Minkowski space, to supersymmetric point particle propagating in superspace. The simplest and most straightforword generalization of (1.15) is written as

$$S = \frac{1}{2}\int d\tau \left\{ e^{-1}\left(\dot{x}^\mu - i\bar{\theta}\gamma^\mu\dot{\theta}\right)^2 - em^2 \right\}. \tag{1.16}$$

The classical dynamics of superparticles are considered by applying Hamilton-Jacobi formalism in chapter*2*.

1.4 String Theory

String theory is a model of fundamental physics whose building blocks are one-dimensional extended objects (strings) rather than the zero-dimensional points (particles) that are the basis of the standard model of particle physics. For this reason, string theories are able to avoid problems associated with the presence of point-like particles in a physical theory. Interest in string theory is driven largely by the hope that it will serve to be a theory of everything. It is a possible solution of the quantum gravity problem, and in addition to gravity it can naturally describe interactions similar to electromagnetism and the other forces of nature [22, 23]. There are two different types of strings. Open strings (figure 1.1) are simply one-dimensional structures that have two endpoints. Thus, an open string can be thought of as a line that has the capability of moving flexibly. Closed strings (figure 1.2) are one-dimensional

Figure 1.1: open string

structures that lack endpoints; therefore, equating them with flexible circles.

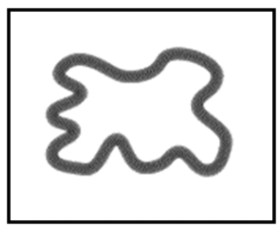

Figure 1.2: closed string

The original version of string theory is now known as ”bosonic string theory” and involved twenty-six spacetime dimensions. The naming of this theory is due to its use of only bosonic particles; bosons are those particles which have an integer spin. This meant that the theory was lacking fermions; fermion is any particle that has an odd half-integer spin, and could not be a grand unification theory. Superstring theory or supersymmetric string theory added to the previous version of string theory by incorporating supersymmetry and realizing that bosonic patterns and fermionic patterns came in pairs. That is, there was ”symmetry” between the bosonic and fermionic patterns.

1.5 Outline

The main argument of this thesis is devoted to make a formal generalization of Hamilton-Jacobi formalism for singular systems to be applicable to new areas in physics, *e.g.* relativistic extended objects, such as strings and membranes. We have tried to develop a Hamilton-Jacobi formulation to study the dynamics of supersymmetric singular Lagrangian systems.

The organization is as follow. Chapter *2* contains a discussion of different models of superparticle which treated as a singular system to be investigated by Hamilton-Jacobi approach. The third chapter opens with the Hamilton-Jacobi formulation of superparticle with extended supersymmetric coupled with an external superpotential. Chapter *4* is devoted to analyze the different examples of superstring by using similar formulation. Finally, Chapter *5*, contains the discussion of the obtained results and a general summary of this thesis.

Chapter 2

Hamilton-Jacobi Formulation of Superparticles

Supersymmetric particles "superparticles" was stimulated by a dynamical developing research in the supersymmetry in the present decade with such a new models presented by: Brink-Schwarz [24] and Siegel [25]. This aspect of the superparticle models makes that they are instructive toy models used to understand the superstrings and the variety of their quantization procedures. In this chapter we try to apply the treatment of Hamilton-Jacobi for different models of superparticle.

2.1 Brink-Schwarz Superparticle

One may write an action for a particle moving in a superspace; which is an extension of ordinary spacetime to include extra anticommuting coordinates in the form of N two-component Weyl spinors $\theta, \bar{\theta}$, where $\bar{\theta}$ is the conjugate of θ. Such action, firstly is written by Brink-Schwarz with simple supersymmetry $N = 1$ [24],

$$S_{BS} = \frac{1}{2}\int d\tau \left\{ e^{-1}\left(\dot{x}^{\mu} - i\bar{\theta}\gamma^{\mu}\dot{\theta}\right)^{2} + em^{2} \right\}. \tag{2.1}$$

where m is the mass of the superparticle and γ^{μ} is Dirac gamma matrices. Then a manifestly Lagrangian is

$$L = \frac{1}{2}\left\{ e^{-1}\left(\dot{x}^{\mu} - i\bar{\theta}\gamma^{\mu}\dot{\theta}\right)^{2} + em^{2} \right\}. \tag{2.2}$$

The singularity of the Lagrangian follows from the fact that the rank of the Hessian A_{ij} is one.

To find the Hamiltonian H_0, one typically begins by finding the conjugate momenta. Our conjugate momenta are defined in (1.2) and (1.3) read as

$$P_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}} = e^{-1}\left(\dot{x}_{\mu} - i\bar{\theta}\gamma_{\mu}\dot{\theta}\right), \tag{2.3}$$

$$\pi_{\theta} = \frac{\partial_r L}{\partial \dot{\theta}} = -i\bar{\theta}P_{\mu}\gamma^{\mu} = -H_{\theta}, \tag{2.4}$$

$$\bar{\pi}_{\bar{\theta}} = \frac{\partial_r L}{\partial \dot{\bar{\theta}}} = 0 = -H_{\bar{\theta}}, \tag{2.5}$$

$$P_{e} = \frac{\partial L}{\partial \dot{e}} = 0 = -H_{e}. \tag{2.6}$$

where ∂_r means right derivatives (A.13).

Since the rank of the Hess matrix is one, we can solve (2.3) for $\dot{x_\mu}$ in terms of P_μ and other coordinates, in the form

$$\dot{x}_\mu = eP_\mu + i\bar{\theta}\gamma_\mu\dot{\theta}. \tag{2.7}$$

The canonical Hamiltonian H_0 is

$$\begin{aligned} H_0 &= -L + P_\mu\dot{x^\mu} + \pi_\theta\dot{\theta} + \bar{\pi}_{\bar{\theta}}\dot{\bar{\theta}} + P_e\dot{e} \\ &= \frac{1}{2}e\Big(P^2 - m^2\Big). \end{aligned} \tag{2.8}$$

The corresponding set of (HJPDE)'s according to (1.7) is

$$H'_0 = P_0 + \frac{1}{2}e\Big(P^2 - m^2\Big), \tag{2.9}$$

$$H'_\theta = P_\theta + i\bar{\theta}P_\mu\gamma^\mu, \tag{2.10}$$

$$H'_{\bar{\theta}} = P_{\bar{\theta}}, \tag{2.11}$$

$$H'_e = P_e. \tag{2.12}$$

Equations (2.9)-(2.12) are the constraints restricting the system. The total differential equations for the characteristics (1.10) and (1.11) read as

$$\begin{aligned} dx_\mu &= \frac{\partial H'_0}{\partial P^\mu}\,d\tau + \frac{\partial H'_\theta}{\partial P^\mu}\,d\theta + \frac{\partial H'_{\bar{\theta}}}{\partial P^\mu}\,d\bar{\theta} + \frac{\partial H'_e}{\partial P^\mu}\,de \\ &= eP_\mu d\tau + i\bar{\theta}\gamma_\mu d\theta, \end{aligned} \tag{2.13}$$

$$\begin{aligned} dP_0 &= -\frac{\partial H'_0}{\partial \tau}\,d\tau - \frac{\partial H'_\theta}{\partial \tau}\,d\theta - \frac{\partial H'_{\bar{\theta}}}{\partial \tau}\,d\bar{\theta} - \frac{\partial H'_e}{\partial \tau}\,de \\ &= 0, \end{aligned} \tag{2.14}$$

$$dP_\mu = -\frac{\partial H'_0}{\partial x^\mu}\, d\tau - \frac{\partial H'_\theta}{\partial x^\mu}\, d\theta - \frac{\partial H'_{\bar\theta}}{\partial x^\mu}\, d\bar\theta - \frac{\partial H'_e}{\partial x^\mu}\, de$$
$$= 0, \tag{2.15}$$

$$d\pi_\theta = -\frac{\partial H'_0}{\partial \theta}\, d\tau - \frac{\partial H'_\theta}{\partial \theta}\, d\theta - \frac{\partial H'_{\bar\theta}}{\partial \theta}\, d\bar\theta - \frac{\partial H'_e}{\partial \theta}\, de$$
$$= 0, \tag{2.16}$$

$$d\bar\pi_{\bar\theta} = -\frac{\partial H'_0}{\partial \bar\theta}\, d\tau - \frac{\partial H'_\theta}{\partial \bar\theta}\, d\theta - \frac{\partial H'_{\bar\theta}}{\partial \bar\theta}\, d\bar\theta - \frac{\partial H'_e}{\partial \bar\theta}\, de$$
$$= (-iP_\mu\gamma^\mu)d\theta, \tag{2.17}$$

$$dP_e = -\frac{\partial H'_0}{\partial e}\, d\tau - \frac{\partial H'_\theta}{\partial e}\, d\theta - \frac{\partial H'_{\bar\theta}}{\partial e}\, d\bar\theta - \frac{\partial H'_e}{\partial e}\, de$$
$$= -\frac{1}{2}\Big(P^2 - m^2\Big)d\tau. \tag{2.18}$$

To check whether the set of equations (2.13) to (2.18) are integrable or not, let us consider the total variations of the set of (HJPDE)'s. The variation of constraints (2.9) to (2.12) respectively are:

$$dH'_0 = dP_0 + \frac{1}{2}\Big(P^2 - m^2\Big)de + eP_\mu dP^\mu, \tag{2.19}$$

$$dH'_\theta = dP_\theta + id\bar\theta P_\mu\gamma^\mu, \tag{2.20}$$

$$dH'_{\bar\theta} = dP_{\bar\theta}, \tag{2.21}$$

and

$$dH'_e = dP_e. \tag{2.22}$$

Making use of (2.13) to (2.18) we obtain that

$$dH'_0 = 0, \tag{2.23}$$

$$dH'_\theta = 0, \tag{2.24}$$

and

$$dH'_{\bar{\theta}} = 0, \tag{2.25}$$

are identically zero, whereas

$$dH'_e = -\frac{1}{2}\left(P^2 - m^2\right)d\tau \equiv H''_e d\tau, \tag{2.26}$$

are not, where

$$H''_e = \frac{1}{2}\left(P^2 - m^2\right) = 0. \tag{2.27}$$

is a new constraint. We notice that the total differential of H''_e vanish identically, *i.e.*

$$dH''_e = P_\mu dP^\mu = 0. \tag{2.28}$$

Thus the equations of motion (2.13)-(2.18) and the new constraint (2.27) represent an integrable system. According to (1.12) the action can be written as

$$\begin{aligned} dZ &= -H_0 d\tau - H_\theta d\theta - H_{\bar{\theta}} d\bar{\theta} - H_e de + P_\mu dx^\mu \\ &= \left\{-\frac{1}{2}\, e\left(P^2 - m^2\right) + P_\mu\left(\dot{x} - i\bar{\theta}\gamma^\mu\dot{\theta}\right)\right\}d\tau, \end{aligned} \tag{2.29}$$

and the canonical action integral becomes

$$S = \int\left\{-\frac{1}{2}\, e\left(P^2 - m^2\right) + P_\mu\left(\dot{x} - i\bar{\theta}\gamma^\mu\dot{\theta}\right)\right\}d\tau. \tag{2.30}$$

As a result, the phase space action (2.30) involves momenta conjugated to the particle position, and there are constraints on the phase space variables. By using (2.3) we obtain the original action (2.1).

2.2 Siegel Superparticle

The most popular superspace in physics is $\mathbb{R}^{4|4}$; (standard superspace $\mathbb{R}^{4|4N} = (x^\mu, \theta_i^\alpha, \bar{\theta}^{\dot{\alpha}i}), \quad i = 1, 2, ..., N)$, which is the direct sum of four real bosonic dimensions and four real Grassmann dimensions (see the appendix A). The action of massless Siegel superparticle moving in $\mathbb{R}^{4|4}$ flat superspace is [25, 26]

$$S = \int \left\{ \frac{1}{2e} \left(\dot{x}^\mu + i\theta\gamma^\mu \dot{\bar{\theta}} - i\dot{\theta}\gamma^\mu\bar{\theta} + i\psi\gamma^\mu\bar{\rho} - i\rho\gamma^\mu\bar{\psi} \right)^2 - \rho^\alpha \dot{\theta}_\alpha - \bar{\rho}_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}} \right\} d\tau. \tag{2.31}$$

The variables $(\psi,\ \bar{\psi})$ are the gauge fields and the pair $(\rho^\alpha,\ \bar{\rho}_{\dot{\alpha}})$ provides the terms corresponding to (mixed) covariant propagator for fermions. Then the Lagrangian can be written in the following way:

$$L = \frac{1}{2e} \left(\dot{x}^\mu + i\theta\gamma^\mu \dot{\bar{\theta}} - i\dot{\theta}\gamma^\mu\bar{\theta} + i\psi\gamma^\mu\bar{\rho} - i\rho\gamma^\mu\bar{\psi} \right)^2 - \rho^\alpha \dot{\theta}_\alpha - \bar{\rho}_{\dot{\alpha}} \dot{\bar{\theta}}^{\dot{\alpha}}. \tag{2.32}$$

The momenta, canonically conjugated to the coordinates of the superparticle, take the forms

$$P^\mu = \frac{\partial L}{\partial \dot{x}_\mu} = \frac{1}{e} \left(\dot{x}^\mu + i\theta\gamma^\mu \dot{\bar{\theta}} - i\dot{\theta}\gamma^\mu\bar{\theta} + i\psi\gamma^\mu\bar{\rho} - i\rho\gamma^\mu\bar{\psi} \right), \tag{2.33}$$

$$\pi_\theta = \frac{\partial_r L}{\partial \dot{\theta}} = -iP_\mu\gamma^\mu\bar{\theta} - \rho^\alpha = -H_\theta, \tag{2.34}$$

$$\bar{\pi}_{\bar{\theta}} = \frac{\partial_r L}{\partial \dot{\bar{\theta}}} = i\theta\gamma^\mu P_\mu - \bar{\rho}_{\dot{\alpha}} = -H_{\bar{\theta}}, \tag{2.35}$$

$$\pi_\psi = \frac{\partial_r L}{\partial \dot{\psi}} = 0 = -H_\psi, \tag{2.36}$$

$$\bar{\pi}_{\bar{\psi}} = \frac{\partial_r L}{\partial \dot{\bar{\psi}}} = 0 = -H_{\bar{\psi}}, \tag{2.37}$$

$$\pi_\rho = \frac{\partial_r L}{\partial \dot{\rho}} = 0 = -H_\rho, \tag{2.38}$$

$$\bar{\pi}_{\bar{\rho}} = \frac{\partial_r L}{\partial \dot{\bar{\rho}}} = 0 = -H_{\bar{\rho}}, \tag{2.39}$$

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e. \tag{2.40}$$

Since the rank of the Hess matrix is one, we can solve (2.33) for $\dot{x}^\mu$ in terms of P^μ and other coordinates

$$\dot{x}^\mu = eP^\mu - i\theta\gamma^\mu\dot{\bar{\theta}} + i\dot{\theta}\gamma^\mu\bar{\theta} - i\psi\gamma^\mu\bar{\rho} + i\rho\gamma^\mu\bar{\psi}. \tag{2.41}$$

The canonical Hamiltonian H_0 of this system is given by

$$\begin{aligned} H_0 &= -L + P_\mu\dot{x^\mu} + \pi_\theta\dot{\theta} + \bar{\pi}_{\bar{\theta}}\dot{\bar{\theta}} + \pi_\psi\dot{\psi} + \bar{\pi}_{\bar{\psi}}\dot{\bar{\psi}} + \pi_\rho\dot{\rho} + \bar{\pi}_{\bar{\rho}}\dot{\bar{\rho}} + P_e\dot{e} \\ &= \frac{1}{2}eP^2 - i\psi\gamma^\mu\bar{\rho}P_\mu + i\rho\gamma^\mu\bar{\psi}P_\mu. \end{aligned} \tag{2.42}$$

As a consequence, the set of (HJPDE)'s is obtained as,

$$H_0' = P_0 + \frac{1}{2}eP^2 - i\psi\gamma^\mu\bar{\rho}P_\mu + i\rho\gamma^\mu\bar{\psi}P_\mu, \tag{2.43}$$

$$H_\theta' = \pi_\theta + P_\mu\gamma^\mu\bar{\theta} + \rho^\alpha, \tag{2.44}$$

$$H_{\bar{\theta}}' = \bar{\pi}_{\bar{\theta}} - i\theta\gamma^\mu P_\mu + \bar{\rho}_{\dot{\alpha}}, \tag{2.45}$$

$$H_\psi' = \pi_\psi, \tag{2.46}$$

$$H_{\bar{\psi}}' = \bar{\pi}_{\bar{\psi}}, \tag{2.47}$$

$$H'_\rho = \pi_\rho, \tag{2.48}$$

$$H'_{\bar{\rho}} = \bar{\pi}_{\bar{\rho}}, \tag{2.49}$$

and

$$H'_e = P_e. \tag{2.50}$$

The equations of motion (1.10) and (1.11) can be written as

$$\begin{aligned} dx^\mu &= \frac{\partial H'_0}{\partial P_\mu}\, d\tau + \frac{\partial H'_\theta}{\partial P_\mu}\, d\theta + \frac{\partial H'_{\bar{\theta}}}{\partial P_\mu}\, d\bar{\theta} + \frac{\partial H'_\psi}{\partial P_\mu}\, d\psi \\ &\quad + \frac{\partial H'_{\bar{\psi}}}{\partial P_\mu}\, d\bar{\psi} + \frac{\partial H'_\rho}{\partial P_\mu}\, d\rho + \frac{\partial H'_{\bar{\rho}}}{\partial P_\mu}\, d\bar{\rho} + \frac{\partial H'_e}{\partial P_\mu}\, de \\ &= \left(eP^\mu - i\psi\gamma^\mu\bar{\rho} + i\rho\gamma^\mu\bar{\psi} \right) d\tau + i\gamma^\mu\bar{\theta} d\theta \\ &\quad - i\theta\gamma^\mu d\bar{\theta}, \end{aligned} \tag{2.51}$$

$$\begin{aligned} dP_0 &= -\frac{\partial H'_0}{\partial \tau}\, d\tau - \frac{\partial H'_\theta}{\partial \tau}\, d\theta - \frac{\partial H'_{\bar{\theta}}}{\partial \tau}\, d\bar{\theta} - \frac{\partial H'_\psi}{\partial \tau}\, d\psi \\ &\quad - \frac{\partial H'_{\bar{\psi}}}{\partial \tau}\, d\bar{\psi} - \frac{\partial H'_\rho}{\partial \tau}\, d\rho - \frac{\partial H'_{\bar{\rho}}}{\partial \tau}\, d\bar{\rho} - \frac{\partial H'_e}{\partial \tau}\, de \\ &= 0, \end{aligned} \tag{2.52}$$

$$\begin{aligned} dP_\mu &= -\frac{\partial H'_0}{\partial x^\mu}\, d\tau - \frac{\partial H'_\theta}{\partial x^\mu}\, d\theta - \frac{\partial H'_{\bar{\theta}}}{\partial x^\mu}\, d\bar{\theta} - \frac{\partial H'_\psi}{\partial x^\mu}\, d\psi \\ &\quad - \frac{\partial H'_{\bar{\psi}}}{\partial x^\mu}\, d\bar{\psi} - \frac{\partial H'_\rho}{\partial x^\mu}\, d\rho - \frac{\partial H'_{\bar{\rho}}}{\partial x^\mu}\, d\bar{\rho} - \frac{\partial H'_e}{\partial x^\mu}\, de \\ &= 0, \end{aligned} \tag{2.53}$$

$$\begin{aligned} d\pi_\theta &= -\frac{\partial H'_0}{\partial \theta}\, d\tau - \frac{\partial H'_\theta}{\partial \theta}\, d\theta - \frac{\partial H'_{\bar{\theta}}}{\partial \theta}\, d\bar{\theta} - \frac{\partial H'_\psi}{\partial \theta}\, d\psi \\ &\quad - \frac{\partial H'_{\bar{\psi}}}{\partial \theta}\, d\bar{\psi} - \frac{\partial H'_\rho}{\partial \theta}\, d\rho - \frac{\partial H'_{\bar{\rho}}}{\partial \theta}\, d\bar{\rho} - \frac{\partial H'_e}{\partial \theta}\, de \\ &= \left(i\gamma^\mu P_\mu \right) d\bar{\theta}, \end{aligned} \tag{2.54}$$

$$
\begin{aligned}
d\bar{\pi}_{\bar{\theta}} &= -\frac{\partial H_0'}{\partial \bar{\theta}}\, d\tau - \frac{\partial H_\theta'}{\partial \bar{\theta}}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial \bar{\theta}}\, d\bar{\theta} - \frac{\partial H_\psi'}{\partial \bar{\theta}}\, d\psi \\
&\quad - \frac{\partial H_{\bar{\psi}}'}{\partial \bar{\theta}}\, d\bar{\psi} - \frac{\partial H_\rho'}{\partial \bar{\theta}}\, d\rho - \frac{\partial H_{\bar{\rho}}'}{\partial \bar{\theta}}\, d\bar{\rho} - \frac{\partial H_e'}{\partial \bar{\theta}}\, de \\
&= \Big(-iP_\mu \gamma^\mu \Big) d\theta, \qquad (2.55)
\end{aligned}
$$

$$
\begin{aligned}
d\pi_\psi &= -\frac{\partial H_0'}{\partial \psi}\, d\tau - \frac{\partial H_\theta'}{\partial \psi}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial \psi}\, d\bar{\theta} - \frac{\partial H_\psi'}{\partial \psi}\, d\psi \\
&\quad - \frac{\partial H_{\bar{\psi}}'}{\partial \psi}\, d\bar{\psi} - \frac{\partial H_\rho'}{\partial \psi}\, d\rho - \frac{\partial H_{\bar{\rho}}'}{\partial \psi}\, d\bar{\rho} - \frac{\partial H_e'}{\partial \psi}\, de \\
&= \Big(iP_\mu \gamma^\mu \bar{\rho} \Big) d\tau, \qquad (2.56)
\end{aligned}
$$

$$
\begin{aligned}
d\bar{\pi}_{\bar{\psi}} &= -\frac{\partial H_0'}{\partial \bar{\psi}}\, d\tau - \frac{\partial H_\theta'}{\partial \bar{\psi}}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial \bar{\psi}}\, d\bar{\theta} - \frac{\partial H_\psi'}{\partial \bar{\psi}}\, d\psi \\
&\quad - \frac{\partial H_{\bar{\psi}}'}{\partial \bar{\psi}}\, d\bar{\psi} - \frac{\partial H_\rho'}{\partial \bar{\psi}}\, d\rho - \frac{\partial H_{\bar{\rho}}'}{\partial \bar{\psi}}\, d\bar{\rho} - \frac{\partial H_e'}{\partial \bar{\psi}}\, de \\
&= \Big(-i\rho \gamma^\mu P_\mu \Big) d\tau, \qquad (2.57)
\end{aligned}
$$

$$
\begin{aligned}
d\pi_\rho &= -\frac{\partial H_0'}{\partial \rho}\, d\tau - \frac{\partial H_\theta'}{\partial \rho}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial \rho}\, d\bar{\theta} - \frac{\partial H_\psi'}{\partial \rho}\, d\psi \\
&\quad - \frac{\partial H_{\bar{\psi}}'}{\partial \rho}\, d\bar{\psi} - \frac{\partial H_\rho'}{\partial \rho}\, d\rho - \frac{\partial H_{\bar{\rho}}'}{\partial \rho}\, d\bar{\rho} - \frac{\partial H_e'}{\partial \rho}\, de \\
&= \Big(-i\gamma^\mu \bar{\psi} P_\mu \Big) d\tau - d\theta, \qquad (2.58)
\end{aligned}
$$

$$
\begin{aligned}
d\bar{\pi}_{\bar{\rho}} &= -\frac{\partial H_0'}{\partial \bar{\rho}}\, d\tau - \frac{\partial H_\theta'}{\partial \bar{\rho}}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial \bar{\rho}}\, d\bar{\theta} - \frac{\partial H_\psi'}{\partial \bar{\rho}}\, d\psi \\
&\quad - \frac{\partial H_{\bar{\psi}}'}{\partial \bar{\rho}}\, d\bar{\psi} - \frac{\partial H_\rho'}{\partial \bar{\rho}}\, d\rho - \frac{\partial H_{\bar{\rho}}'}{\partial \bar{\rho}}\, d\bar{\rho} - \frac{\partial H_e'}{\partial \bar{\rho}}\, de \\
&= \Big(i\psi \gamma^\mu P_\mu \Big) d\tau - d\bar{\theta}, \qquad (2.59)
\end{aligned}
$$

$$\begin{aligned} dP_e = & -\frac{\partial H_0'}{\partial e}\,d\tau - \frac{\partial H_\theta'}{\partial e}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial e}\,d\bar\theta - \frac{\partial H_\psi'}{\partial e}\,d\psi \\ & - \frac{\partial H_{\bar\psi}'}{\partial e}\,d\bar\psi - \frac{\partial H_\rho'}{\partial e}\,d\rho - \frac{\partial H_{\bar\rho}'}{\partial e}\,d\bar\rho - \frac{\partial H_e'}{\partial e}\,de \\ = & \left(\frac{1}{2}P^2\right)d\tau. \end{aligned} \tag{2.60}$$

The integrability conditions (1.13) and (1.14) imply that the variations of the constraints (2.43)-(2.50) should be identically zero. One notices that

$$dH_0' = 0, \tag{2.61}$$

$$dH_\theta' = 0, \tag{2.62}$$

and

$$dH_{\bar\theta}' = 0. \tag{2.63}$$

are identically zero, whereas the variations of

$$dH_\psi' = \left(iP_\mu\gamma^\mu\bar\rho\right)d\tau \equiv H_\psi'' d\tau, \tag{2.64}$$

$$dH_{\bar\psi}' = \left(-i\rho\gamma^\mu P_\mu\right)d\tau \equiv H_{\bar\psi}'' d\tau, \tag{2.65}$$

$$dH_\rho' = \left(-i\gamma^\mu\bar\psi P_\mu\right)d\tau - d\theta \equiv H_\rho'' d\tau, \tag{2.66}$$

$$dH_{\bar\rho}' = \left(i\psi\gamma^\mu P_\mu\right)d\tau - d\bar\theta \equiv H_{\bar\rho}'' d\tau, \tag{2.67}$$

and

$$dH_e' = \left(\frac{1}{2}P^2\right)d\tau \equiv H_e'' d\tau. \tag{2.68}$$

are not. Therefore we obtain the following set of additional constraints;

$$H''_{\psi} = iP_{\mu}\gamma^{\mu}\bar{\rho}, \tag{2.69}$$

$$H''_{\bar{\psi}} = -i\rho\gamma^{\mu}P_{\mu}, \tag{2.70}$$

$$H''_{\rho} = -i\gamma^{\mu}\bar{\psi}P_{\mu}, \tag{2.71}$$

$$H''_{\bar{\rho}} = i\psi\gamma^{\mu}P_{\mu}, \tag{2.72}$$

and

$$H''_{e} = \frac{1}{2}P^{2}. \tag{2.73}$$

Calculations show that the total differential of H''_{ψ}, $H''_{\bar{\psi}}$, H''_{ρ}, $H''_{\bar{\rho}}$ and H''_{e} vanish identically, *i.e.*

$$dH''_{\psi} = 0, \tag{2.74}$$

$$dH''_{\bar{\psi}} = 0, \tag{2.75}$$

$$dH''_{\rho} = 0, \tag{2.76}$$

$$dH''_{\bar{\rho}} = 0, \tag{2.77}$$

and

$$dH''_{e} = 0. \tag{2.78}$$

Thus the equations of motion (2.51) - (2.60) and the new constraints (2.69) - (2.73) represent an integrable system. Since the equations of motion are integrable, the action can be written as

$$
\begin{aligned}
dZ &= -H_0 d\tau - H_\theta d\theta - H_{\bar{\theta}} d\bar{\theta} - H_\psi d\psi - H_{\bar{\psi}} d\bar{\psi} \\
&\quad - H_\rho d\rho - H_{\bar{\rho}} d\bar{\rho} - H_e de + P_\mu dx^\mu \\
&= \Bigg\{ P_\mu \Bigg(\dot{x}^\mu + i\theta\gamma^\mu \dot{\bar{\theta}} - i\dot{\theta}\gamma^\mu\bar{\theta} + i\psi\gamma^\mu\bar{\rho} - i\rho\gamma^\mu\bar{\psi} \Bigg) \\
&\quad - \rho^\alpha \dot{\theta}_\alpha - \bar{\rho}_{\dot{\alpha}} \dot{\bar{\theta}} - \frac{1}{2} e P^2 \Bigg\} d\tau. \qquad (2.79)
\end{aligned}
$$

We now present a phase-space action for the superparticle,

$$
\begin{aligned}
S = \int \Bigg\{ & P_\mu \Bigg(\dot{x}^\mu + i\theta\gamma^\mu \dot{\bar{\theta}} - i\dot{\theta}\gamma^\mu\bar{\theta} + i\psi\gamma^\mu\bar{\rho} - i\rho\gamma^\mu\bar{\psi} \Bigg) \\
& - \rho^\alpha \dot{\theta}_\alpha - \bar{\rho}_{\dot{\alpha}} \dot{\bar{\theta}} - \frac{1}{2} e P^2 \Bigg\} d\tau. \qquad (2.80)
\end{aligned}
$$

In the Hamiltonian framework, there are first class constraints, which are combined with second class constraints, which are simply formulated by applying Hamilton- Jacobi rather than Dirac's method. Since the system is integrable, we can obtain the canonical reduced phase-space coordinates without using any gauge fixing condition, which would be suitable for quantization.

2.3 Spinning Superparticle

This section is concerned with theories describing spinning particles that are formulated in terms of actions possessing local world-line supersymmetry. These classical actions are obtained by adding a finite number of spinor or vector coordinates to the usual space-time coordinates. Generalizing to superspace leads to corresponding types of "spinning superparticle". The spinning superparticle model is described by the action [27]

$$S = \int \Big\{ \frac{1}{2e}\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big)^2 + \frac{i}{2}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\frac{d}{d\tau}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big) + \frac{i}{e}\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big)\Big\} d\tau. \tag{2.81}$$

In such a space-time one can define, in addition to the usual coordinates, a spinor of real fermionic supermultiplets (θ, h), which define an anti-commuting and a commuting Majorana spinor; is a real spinor which is equal to its charge conjugate $\bar{\theta} = \theta$, in the target space-time, respectively, and $(\psi^\mu,\ \chi)$ are Grassman (fermionic or odd) variables which describe spinning degrees of freedom.

From the action (2.81), we have the singular Lagrangian

$$L = \frac{1}{2e}\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big)^2 + \frac{i}{2}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\frac{d}{d\tau}\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big) + \frac{i}{e}\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big). \tag{2.82}$$

The momenta, canonically conjugated to the coordinates of the spinning superparticle, are

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{e}\Big(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h\Big) + \frac{i}{e}\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big) \tag{2.83}$$

$$\pi_\theta = \frac{\partial_r L}{\partial \dot{\theta}} = -i\Big(P_\mu \gamma^\mu \bar{\theta} + \frac{1}{2}\psi_\mu \gamma^\mu \bar{h}\Big) \equiv -H_\theta, \tag{2.84}$$

$$\bar{\pi}_{\bar{\theta}} = \frac{\partial_r L}{\partial \dot{\bar{\theta}}} = 0 \equiv -H_{\bar{\theta}}, \tag{2.85}$$

$$\pi_h = \frac{\partial_r L}{\partial \dot{h}} = 0 \equiv -H_h, \tag{2.86}$$

$$\bar{\pi}_{\bar{h}} = \frac{\partial_r L}{\partial \dot{\bar{h}}} = -\frac{i}{2}\psi_\mu \gamma^\mu \theta \equiv -H_{\bar{h}}, \tag{2.87}$$

$$P_\psi^\mu = \frac{\partial L}{\partial \dot{\psi}_\mu} = \frac{i}{2}\Big(\psi^\mu - \bar{h}\gamma^\mu \theta\Big) \equiv -H_\psi, \tag{2.88}$$

$$P_\chi = \frac{\partial L}{\partial \dot{\chi}} = 0 \equiv -H_\chi, \tag{2.89}$$

and

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 \equiv -H_e. \tag{2.90}$$

Since the rank of the Hess matrix is one, we can solve (2.83) for $\dot{x}_\mu$ in terms of P_μ and other coordinates as

$$\dot{x}_\mu = eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu \theta\Big) + i\bar{\theta}\gamma_\mu \dot{\theta} + e\bar{h}\gamma_\mu h. \tag{2.91}$$

The canonical Hamiltonian H_0 is obtained as

$$\begin{aligned} H_0 &= -L + P_\mu \dot{x^\mu} + \pi_\theta \dot{\theta} + \bar{\pi}_{\bar{\theta}}\dot{\bar{\theta}} + \pi_h \dot{h} + \bar{\pi}_{\bar{h}}\dot{\bar{h}} + P_\psi \dot{\psi} + P_\chi \dot{\chi} + P_e \dot{e} \\ &= \frac{1}{2e}\Big\{eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu \theta\Big)\Big\}^2 + eP_\mu\Big(\bar{h}\gamma^\mu h\Big). \end{aligned} \tag{2.92}$$

By means of relations (2.84) - (2.90) the set of (HJPDE)'s is,

$$H_0' = P_0 + \frac{1}{2e}\Big\{eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu \theta\Big)\Big\}^2 + eP_\mu\Big(\bar{h}\gamma^\mu h\Big) = 0, \tag{2.93}$$

$$H'_\theta = \pi_\theta + i\Big(P_\mu \gamma^\mu \bar{\theta} + \frac{1}{2}\psi_\mu \gamma^\mu \bar{h}\Big) = 0, \tag{2.94}$$

$$H'_{\bar{\theta}} = \bar{\pi}_{\bar{\theta}} = 0, \tag{2.95}$$

$$H'_h = \pi_h = 0, \tag{2.96}$$

$$H'_{\bar{h}} = \bar{\pi}_{\bar{h}} + \frac{i}{2}\psi_\mu \gamma_\mu \theta = 0, \tag{2.97}$$

$$H'_\psi = P^\mu_\psi - \frac{i}{2}\Big(\psi^\mu - \bar{h}\gamma^\mu \theta\Big) = 0, \tag{2.98}$$

$$H'_\chi = P_\chi = 0, \tag{2.99}$$

$$H'_e = P_e = 0. \tag{2.100}$$

The equations of motion (1.10) and (1.11) can be written as

$$\begin{aligned} dx_\mu = & \frac{\partial H'_0}{\partial P^\mu}\, d\tau + \frac{\partial H'_\theta}{\partial P^\mu}\, d\theta + \frac{\partial H'_{\bar{\theta}}}{\partial P^\mu}\, d\bar{\theta} + \frac{\partial H'_h}{\partial P^\mu}\, dh + \frac{\partial H'_{\bar{h}}}{\partial P^\mu}\, d\bar{h} \\ & + \frac{\partial H'_\psi}{\partial P^\mu}\, d\psi_\mu + \frac{\partial H'_\chi}{\partial P^\mu}\, d\chi + \frac{\partial H'_e}{\partial P^\mu}\, de \\ = & \Big\{eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big) + e\Big(\bar{h}\gamma_\mu h\Big)\Big\}d\tau \\ & + i\bar{\theta}\gamma_\mu d\theta, \end{aligned} \tag{2.101}$$

$$\begin{aligned} dP_0 = & -\frac{\partial H'_0}{\partial \tau}\, d\tau - \frac{\partial H'_\theta}{\partial \tau}\, d\theta - \frac{\partial H'_{\bar{\theta}}}{\partial \tau}\, d\bar{\theta} - \frac{\partial H'_h}{\partial \tau}\, dh - \frac{\partial H'_{\bar{h}}}{\partial \tau}\, d\bar{h} \\ & - \frac{\partial H'_\psi}{\partial \tau}\, d\psi_\mu - \frac{\partial H'_\chi}{\partial \tau}\, d\chi - \frac{\partial H'_e}{\partial \tau}\, de \\ = & \, 0, \end{aligned} \tag{2.102}$$

$$\begin{aligned} dP_\mu = & -\frac{\partial H'_0}{\partial x_\mu}\, d\tau - \frac{\partial H'_\theta}{\partial x_\mu}\, d\theta - \frac{\partial H'_{\bar{\theta}}}{\partial x_\mu}\, d\bar{\theta} - \frac{\partial H'_h}{\partial x_\mu}\, dh - \frac{\partial H'_{\bar{h}}}{\partial x_\mu}\, d\bar{h} \\ & - \frac{\partial H'_\psi}{\partial x_\mu}\, d\psi_\mu - \frac{\partial H'_\rho}{\partial x_\mu}\, d\chi - \frac{\partial H'_e}{\partial x_\mu}\, de \\ = & \, 0, \end{aligned} \tag{2.103}$$

$$
\begin{aligned}
d\pi_\theta &= -\frac{\partial H_0'}{\partial\theta}\,d\tau - \frac{\partial H_\theta'}{\partial\theta}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial\theta}\,d\bar\theta - \frac{\partial H_h'}{\partial\theta}\,dh - \frac{\partial H_{\bar h}'}{\partial\theta}\,d\bar h \\
&\quad - \frac{\partial H_\psi'}{\partial\theta}\,d\psi_\mu - \frac{\partial H_\chi'}{\partial\theta}\,d\chi - \frac{\partial H_e'}{\partial\theta}\,de \\
&= -\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\bar h\gamma^\mu\chi d\tau - \frac{i}{2}\psi_\mu\gamma^\mu d\bar h - \frac{i}{2}\bar h\gamma^\mu d\psi_\mu, \qquad (2.104)\\
d\bar\pi_{\bar\theta} &= -\frac{\partial H_0'}{\partial\bar\theta}\,d\tau - \frac{\partial H_\theta'}{\partial\bar\theta}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial\bar\theta}\,d\bar\theta - \frac{\partial H_h'}{\partial\bar\theta}\,dh - \frac{\partial H_{\bar h}'}{\partial\bar\theta}\,d\bar h \\
&\quad - \frac{\partial H_\psi'}{\partial\bar\theta}\,d\psi_\mu - \frac{\partial H_\chi'}{\partial\bar\theta}\,d\chi - \frac{\partial H_e'}{\partial\bar\theta}\,de \\
&= \Big(-iP_\mu\gamma^\mu\Big)d\theta, \qquad (2.105)\\
d\pi_h &= -\frac{\partial H_0'}{\partial h}\,d\tau - \frac{\partial H_\theta'}{\partial h}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial h}\,d\bar\theta - \frac{\partial H_h'}{\partial h}\,dh - \frac{\partial H_{\bar h}'}{\partial h}\,d\bar h \\
&\quad - \frac{\partial H_\psi'}{\partial h}\,d\psi_\mu - \frac{\partial H_\chi'}{\partial h}\,d\chi - \frac{\partial H_e'}{\partial h}\,de \\
&= \Big(-eP_\mu\bar h\gamma^\mu\Big)d\tau, \qquad (2.106)\\
d\bar\pi_{\bar h} &= -\frac{\partial H_0'}{\partial\bar h}\,d\tau - \frac{\partial H_\theta'}{\partial\bar h}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial\bar h}\,d\bar\theta - \frac{\partial H_h'}{\partial\bar h}\,dh - \frac{\partial H_{\bar h}'}{\partial\bar h}\,d\bar h \\
&\quad - \frac{\partial H_\psi'}{\partial\bar h}\,d\psi_\mu - \frac{\partial H_\chi'}{\partial\bar h}\,d\chi - \frac{\partial H_e'}{\partial\bar h}\,de \\
&= -\bigg\{\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\chi\gamma^\mu\theta + eP_\mu\gamma^\mu h\bigg\}d\tau - \frac{i}{2}\psi_\mu\gamma^\mu d\theta \\
&\quad - \frac{i}{2}\gamma^\mu\theta d\psi_\mu, \qquad (2.107)\\
dP_\psi^\mu &= -\frac{\partial H_0'}{\partial\psi_\mu}\,d\tau - \frac{\partial H_\theta'}{\partial\psi_\mu}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial\psi_\mu}\,d\bar\theta - \frac{\partial H_h'}{\partial\psi_\mu}\,dh - \frac{\partial H_{\bar h}'}{\partial\psi_\mu}\,d\bar h \\
&\quad - \frac{\partial H_\psi'}{\partial\psi_\mu}\,d\psi_\mu - \frac{\partial H_\chi'}{\partial\psi_\mu}\,d\chi - \frac{\partial H_e'}{\partial\psi_\mu}\,de \\
&= \frac{i}{e}\bigg\{eP^\mu - i\chi\Big(\psi^\mu - \bar h\gamma^\mu\theta\Big)\bigg\}\chi d\tau - \frac{i}{2}\bar h\gamma^\mu d\theta - \frac{i}{2}\gamma^\mu\theta d\bar h \\
&\quad - \frac{i}{2}\gamma^\mu\theta d\psi_\mu, \qquad (2.108)
\end{aligned}
$$

$$
\begin{aligned}
dP_\chi &= -\frac{\partial H'_0}{\partial \chi}\, d\tau - \frac{\partial H'_\theta}{\partial \chi}\, d\theta - \frac{\partial H'_{\bar\theta}}{\partial \chi}\, d\bar\theta - \frac{\partial H'_h}{\partial \chi}\, dh - \frac{\partial H'_{\bar h}}{\partial \chi}\, d\bar h \\
&\quad - \frac{\partial H'_\psi}{\partial \chi}\, d\psi_\mu - \frac{\partial H'_\chi}{\partial \chi}\, d\chi - \frac{\partial H'_e}{\partial \chi}\, de \\
&= \frac{i}{e}\Big\{eP_\mu - i\chi\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)\Big\}\Big(\psi^\mu - \bar h\gamma^\mu\theta\Big)d\tau, \qquad (2.109)
\end{aligned}
$$

$$
\begin{aligned}
dP_e &= -\frac{\partial H'_0}{\partial e}\, d\tau - \frac{\partial H'_\theta}{\partial e}\, d\theta - \frac{\partial H'_{\bar\theta}}{\partial e} d\bar\theta - \frac{\partial H'_h}{\partial e}\, dh - \frac{\partial H'_{\bar h}}{\partial e}\, d\bar h \\
&\quad - \frac{\partial H'_\psi}{\partial e}\, d\psi_\mu - \frac{\partial H'_\chi}{\partial e}\, d\chi - \frac{\partial H'_e}{\partial e}\, de \\
&= \Big\{\frac{1}{2}P^2 + \frac{1}{e^2}\chi^2\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)^2 + P_\mu\Big(\bar h\gamma^\mu h\Big)\Big\}d\tau. \qquad (2.110)
\end{aligned}
$$

Using the integrability conditions (1.13) and (1.14) we obtain the variations of the constraints (2.93) - (2.100),

$$dH'_0 = 0, \qquad (2.111)$$

$$dH'_{\bar\theta} = 0, \qquad (2.112)$$

are identically zero, whereas the variation of

$$dH'_\theta = -\Big\{\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\chi\gamma^\mu\theta\Big\}d\tau + iP_\mu\gamma^\mu d\theta \equiv H''_\theta d\tau, \qquad (2.113)$$

$$dH'_h = \Big(- iP_\mu\bar h\gamma^\mu\Big)d\tau \equiv H''_h d\tau, \qquad (2.114)$$

$$dH'_{\bar h} = -\Big\{\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu\Big)\chi\gamma^\mu\theta + eP_\mu\gamma^\mu h\Big\}d\tau \equiv H''_{\bar h} d\tau, \qquad (2.115)$$

$$dH'_\psi = \Big\{\frac{i}{e}\Big(eP_\mu - i\chi\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)\Big)\chi\Big\}d\tau \equiv H''_\psi d\tau, \qquad (2.116)$$

$$dH'_\chi = \Big\{\frac{i}{e}\Big(eP_\mu - i\chi\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)\Big)\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)\Big\}d\tau \equiv H''_\chi d\tau, \qquad (2.117)$$

$$dH'_e = \Big\{\frac{1}{2}P^2 + \frac{1}{e^2}\chi^2\Big(\psi_\mu - \bar h\gamma_\mu\theta\Big)^2 + P_\mu\Big(\bar h\gamma^\mu h\Big)\Big\}d\tau \equiv H''_e d\tau. \qquad (2.118)$$

are not. Therefore we obtain the following set of additional constraints;

$$H''_\theta = \left\{ -\frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu \Big) \bar{h}\gamma^\mu\chi \right\} = 0, \tag{2.119}$$

$$H''_h = \Big(-iP_\mu \bar{h}\gamma^\mu \Big) = 0, \tag{2.120}$$

$$H''_{\bar{h}} = -\left\{ \frac{i}{e}\Big(eP_\mu - i\chi\psi_\mu \Big)\chi\gamma^\mu\theta + eP_\mu\gamma^\mu h \right\} = 0, \tag{2.121}$$

$$H''_\psi = \left\{ \frac{i}{e}\Big(eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big)\chi \right\} = 0, \tag{2.122}$$

$$H''_\chi = \left\{ \frac{i}{e}\Big(eP_\mu - i\chi\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)\Big)\Big(\psi^\mu - \bar{h}\gamma^\mu\theta\Big) \right\} = 0, \tag{2.123}$$

and

$$H''_e = \left\{ \frac{1}{2}P^2 + \frac{1}{e^2}\chi^2\Big(\psi_\mu - \bar{h}\gamma_\mu\theta\Big)^2 + P_\mu\Big(\bar{h}\gamma_\mu h\Big) \right\} = 0. \tag{2.124}$$

One notices that the total differential of H''_θ, H''_h, $H''_{\bar{h}}$, H''_ψ, H''_χ and H''_e vanish identically, *i.e.*

$$dH''_\theta = 0, \tag{2.125}$$

$$dH''_h = 0, \tag{2.126}$$

$$dH''_{\bar{h}} = 0, \tag{2.127}$$

$$dH''_\psi = 0, \tag{2.128}$$

$$dH''_\chi = 0, \tag{2.129}$$

and

$$dH''_e = 0. \tag{2.130}$$

Thus the equations of motion (2.101) - (2.110) and the new constraints (2.119) - (2.124) represent an integrable system. Since the equations of motion are integrable, the action can be written as

$$
\begin{aligned}
dZ &= -H_0 d\tau - H_\theta d\theta - H_{\bar{\theta}} d\bar{\theta} - H_h dh - H_{\bar{h}} d\bar{h} \\
&\quad - H_\psi d\psi_\mu - H_\chi d\chi - H_e de + P_\mu dx_\mu \\
&= \Big\{ \frac{1}{2} e P^2 + \frac{1}{2e} \chi^2 \Big(\psi_\mu - \bar{h}\gamma_\mu \theta \Big)^2 - \frac{i}{2} \psi_\mu \Big(\bar{h}\gamma^\mu \dot{\theta} - \dot{\bar{h}}\gamma^\mu \theta \Big) \\
&\quad + \frac{i}{2} \Big(\psi^\mu - \bar{h}\gamma^\mu \theta \Big) \dot{\psi}_\mu \Big\} d\tau. \qquad (2.131)
\end{aligned}
$$

We now present a phase-space action for the spinning superparticle,

$$
\begin{aligned}
S = \int \Big\{ & \frac{1}{2} e P^2 + \frac{1}{2e} \chi^2 \Big(\psi_\mu - \bar{h}\gamma_\mu \theta \Big)^2 - \frac{i}{2} \psi_\mu \Big(\bar{h}\gamma^\mu \dot{\theta} - \dot{\bar{h}}\gamma^\mu \theta \Big) \\
& + \frac{i}{2} \Big(\psi^\mu - \bar{h}\gamma^\mu \theta \Big) \dot{\psi}_\mu \Big\} d\tau. \qquad (2.132)
\end{aligned}
$$

It is no surprise that the above result is simply the result of the spinning particle [28], added to anticommuting sectors of the superspace, as in the action (2.132).

In the case of superparticle, there are first and second-class constraints, the occurrence of second-class constraints arises from the fact that the Grassmann momenta, conjugate to the fermionic variables, are non-independent phase-space variables. The Hamilton-Jacobi formalism has an advantage that there is no difference between first and second class constraints and we do not need a gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations.

Chapter 3

Massive Superparticle

This chapter is devoted to study the superparticle with extended supersymmetric (*i.e.*, those containing more than one spinor generator) model, we apply the Hamilton-Jacobi formulation on the simplest extended $N = 2$ supersymmetric massive superparticle and we examine this model when the minimal coupling with an external $N = 2$ superpotential is introduced.

3.1 $N = 2$ Massive Superparticle Model

The action, which reproduces the desired massive superparticle in $D = 4$, $N = 2$ target superspace looks as follows: [29]

$$S = \int d\tau \left\{ \frac{1}{2}\left(\frac{\omega^2}{g} - gm^2 \right) + m\left(\theta^{\alpha}{}_{i}\dot{\theta}^{i}_{\alpha} + \bar{\theta}_{\dot{\alpha}i}\dot{\bar{\theta}}^{\dot{\alpha}i} \right) \right\}. \tag{3.1}$$

where $\omega^{\mu} = \dot{x}^{\mu} + i\theta^{\alpha}{}_{i}\sigma^{\mu}{}_{\alpha\dot{\alpha}}\dot{\bar{\theta}}^{\dot{\alpha}i} - i\dot{\theta}^{\alpha}_{i}\sigma^{\mu}{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i}$, $i = 1, 2, ..., N$. Here g is the worldline einbein.

The corresponding Lagrangian is written as

$$L = \frac{1}{2}\left(\frac{\omega^2}{g} - gm^2\right) + m\left(\theta^{\alpha}{}_{i}\dot{\theta}^{i}_{\alpha} + \bar{\theta}_{\dot{\alpha}i}\dot{\bar{\theta}}^{\dot{\alpha}i}\right). \tag{3.2}$$

The Hamiltonian analysis begin with the introduction of the momenta variables according to equations (1.2) and (1.3), which read as

$$P_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}} = \frac{1}{g}\left\{\dot{x}_{\mu} + i\left(\theta_{\alpha}{}^{i}\sigma_{\mu}{}^{\alpha\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}i}\right) - \left(i\dot{\theta}^{i}_{\alpha}\sigma_{\mu}{}^{\alpha\dot{\alpha}}\bar{\theta}_{\dot{\alpha}i}\right)\right\} = \frac{\omega_{\mu}}{g}, \tag{3.3}$$

$$\pi^{i}_{\alpha} = \frac{\partial_r L}{\partial \dot{\theta}^{\alpha}_{i}} = -iP_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i} + m\theta^{i}{}_{\alpha} = -H_{\theta^{i}_{\alpha}}, \tag{3.4}$$

$$\bar{\pi}_{\dot{\alpha}i} = \frac{\partial_r L}{\partial \dot{\bar{\theta}}^{\dot{\alpha}i}} = iP_{\mu}\theta^{\alpha}_{i}\sigma^{\mu}_{\alpha\dot{\alpha}} + m\bar{\theta}_{\dot{\alpha}i} = -H_{\bar{\theta}_{\dot{\alpha}i}}, \tag{3.5}$$

$$p_g = \frac{\partial L}{\partial \dot{g}} = 0 = -H_g. \tag{3.6}$$

We can solve (3.3) for $\dot{x}_{\mu}$ in terms of P_{μ} and other coordinates to obtain

$$\dot{x}_{\mu} = gP_{\mu} - i\left(\theta_{\alpha}{}^{i}\sigma_{\mu}{}^{\alpha\dot{\alpha}}\dot{\bar{\theta}}_{\dot{\alpha}i}\right) + \left(i\dot{\theta}^{i}_{\alpha}\sigma_{\mu}{}^{\alpha\dot{\alpha}}\bar{\theta}_{\dot{\alpha}i}\right). \tag{3.7}$$

The canonical Hamiltonian H_0 is

$$\begin{aligned} H_0 &= -L + P_{\mu}\dot{x^{\mu}} + \pi^{i}_{\alpha}\dot{\theta}^{\alpha}_{i} + \bar{\pi}_{\dot{\alpha}i}\dot{\bar{\theta}}^{\dot{\alpha}i} + p_g\dot{g} \\ &= \frac{1}{2}g\left(P^2 + m^2\right). \end{aligned} \tag{3.8}$$

Following the Hamilton-Jacobi formalism we obtain the set of HJPDE's,

$$H'_0 = P_0 + \frac{1}{2}g\left(P^2 + m^2\right), \tag{3.9}$$

$$H'_{\theta^{i}_{\alpha}} = \pi^{i}_{\alpha} + iP_{\mu}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i} - m\theta^{i}{}_{\alpha}, \tag{3.10}$$

$$H'_{\bar{\theta}_{\dot{\alpha}i}} = \bar{\pi}_{\dot{\alpha}i} - iP_{\mu}\theta^{\alpha}_{i}\sigma^{\mu}_{\alpha\dot{\alpha}} - m\bar{\theta}_{\dot{\alpha}i}, \tag{3.11}$$

$$H'_g = p_g. \tag{3.12}$$

The equations of motion read as

$$dx^\mu = \frac{\partial H'_0}{\partial P_\mu} d\tau + \frac{\partial H'_{\theta^i_\alpha}}{\partial P_\mu} d\theta^\alpha_i + \frac{\partial H'_{\bar\theta_{\dot\alpha i}}}{\partial P_\mu} d\bar\theta^{\dot\alpha i} + \frac{\partial H'_g}{\partial P_\mu} dg$$
$$= gP^\mu d\tau + i\left(\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\right) d\theta^\alpha_i - i\left(\theta^\alpha_i \sigma^\mu_{\alpha\dot\alpha}\right) d\bar\theta^{\dot\alpha i}, \tag{3.13}$$

$$dP_0 = -\frac{\partial H'_0}{\partial \tau} d\tau - \frac{\partial H'_{\theta^i_\alpha}}{\partial \tau} d\theta^\alpha_i - \frac{\partial H'_{\bar\theta_{\dot\alpha i}}}{\partial \tau} d\bar\theta^{\dot\alpha i} - \frac{\partial H'_g}{\partial \tau} dg$$
$$= 0, \tag{3.14}$$

$$dP_\mu = -\frac{\partial H'_0}{\partial x^\mu} d\tau - \frac{\partial H'_{\theta^i_\alpha}}{\partial x^\mu} d\theta^\alpha_i - \frac{\partial H'_{\bar\theta_{\dot\alpha i}}}{\partial x^\mu} d\bar\theta^{\dot\alpha i} - \frac{\partial H'_g}{\partial x^\mu} dg$$
$$= 0, \tag{3.15}$$

$$d\pi^i_\alpha = -\frac{\partial H'_0}{\partial \theta^\alpha_i} d\tau - \frac{\partial H'_{\theta^j_\beta}}{\partial \theta^\alpha_i} d\theta^\beta_j - \frac{\partial H'_{\bar\theta_{\dot\beta j}}}{\partial \theta^\alpha_i} d\bar\theta^{\dot\beta j} - \frac{\partial H'_g}{\partial \theta^\alpha_i} dg$$
$$= (iP_\mu \sigma^\mu_{\alpha\dot\alpha}) d\bar\theta^{\dot\alpha i} + m d\theta^i_\alpha, \tag{3.16}$$

$$d\bar\pi_{\dot\alpha i} = -\frac{\partial H'_0}{\partial \bar\theta^{\dot\alpha i}} d\tau - \frac{\partial H'_{\theta^j_\beta}}{\partial \bar\theta^{\dot\alpha i}} d\theta^\beta_j - \frac{\partial H'_{\bar\theta_{\dot\beta j}}}{\partial \bar\theta^{\dot\alpha i}} d\bar\theta^{\dot\beta j} - \frac{\partial H'_g}{\partial \bar\theta^{\dot\alpha i}} dg$$
$$= \left(-iP_\mu \sigma^\mu_{\alpha\dot\alpha}\right) d\theta^\alpha_i + m d\bar\theta_{\dot\alpha i}, \tag{3.17}$$

and

$$dp_g = -\frac{\partial H'_0}{\partial g} d\tau - \frac{\partial H'_{\theta^i_\alpha}}{\partial g} d\theta^\alpha_i - \frac{\partial H'_{\bar\theta_{\dot\alpha i}}}{\partial g} d\bar\theta^{\dot\alpha i} - \frac{\partial H'_g}{\partial g} dg$$
$$= \frac{1}{2}\left(P^2 + m^2\right) d\tau. \tag{3.18}$$

The next step is the exploration of the set of (3.13) to (3.18) are integrable or not. The variation of H'_0, H'_θ, and $H'_{\bar\theta}$,

$$dH'_0 = 0, \tag{3.19}$$

$$dH'_{\theta^i_\alpha} = 0, \tag{3.20}$$

and

$$dH'_{\bar{\theta}_{\dot{\alpha}i}} = 0, \tag{3.21}$$

are identically zero, whereas

$$dH'_g = \frac{1}{2}\Big(P^2 + m^2\Big)d\tau \equiv H''_g d\tau. \tag{3.22}$$

are not, with

$$H''_g = \frac{1}{2}\Big(P^2 + m^2\Big) = 0. \tag{3.23}$$

is a new constraint. We notice that the total differential of H''_g vanish identically, *i.e.*

$$dH''_g = P_\mu dP^\mu = 0. \tag{3.24}$$

Thus the equations of motion (3.13) - (3.18) and the new constraint (3.23) represent an integrable system. According to (1.12) the action can be written as

$$\begin{aligned} dZ &= -H_0 d\tau - H_{\theta^i_\alpha} d\theta^\alpha_i - H_{\bar{\theta}_{\dot{\alpha}i}} d\bar{\theta}^{\dot{\alpha}i} - H_g dg + P_\mu dx^\mu \\ &= \Bigg\{-\frac{1}{2}g\Big(P^2 + m^2\Big) - ip_\mu \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}i}\dot{\theta}^\alpha_i + m\theta^i{}_\alpha\dot{\theta}^\alpha_i \\ &\quad + iP_\mu\theta^\alpha_i\sigma^\mu_{\alpha\dot{\alpha}}\dot{\bar{\theta}}^{\dot{\alpha}i} - m\bar{\theta}_{\dot{\alpha}i}\dot{\bar{\theta}}^{\dot{\alpha}i} + p_\mu\dot{x}^\mu\Bigg\}d\tau. \end{aligned} \tag{3.25}$$

The phase-space action integral can be written as

$$S = \int\Bigg\{-\frac{1}{2}g\Big(P^2 - m^2\Big) + m\Big(\theta^\alpha{}_i\dot{\theta}^i_\alpha + \bar{\theta}_{\dot{\alpha}i}\dot{\bar{\theta}}^{\dot{\alpha}i}\Big)\Bigg\}d\tau. \tag{3.26}$$

3.2 $N = 2$ Massive Superparticle Coupled to External Superpotential

We start from the following action of $N = 2$ massive charged superparticle with charge e, coupled to an external superpotenital; superpotential is a gauge invariant function in superspace. [29]

$$S^{(e)} = \int d\tau \Big\{ \frac{1}{2}\Big(\frac{\omega^\mu \omega_\mu}{g} - gm^2 \Big) + m\Big(\theta^\alpha{}_i \dot{\theta}^i_\alpha + \bar{\theta}_{\dot{\alpha}i} \dot{\bar{\theta}}^{\dot{\alpha}i} \Big) \Big\}$$
$$+ ie \int d\tau \Big(\omega^\mu A_\mu + \dot{\theta}^\alpha_i A^i_\alpha + \dot{\bar{\theta}}_{\dot{\alpha}i} \bar{A}^{\dot{\alpha}i} \Big). \tag{3.27}$$

Here we restrict ourselves by the electromagnetic $U(1)$ group case, with gauge superfields $A_M(x^\mu, \theta, \bar{\theta}) = (A_\mu, A^i{}_\alpha, \bar{A}^i_{\dot{\alpha}})$. The form of the Lagrangian function, is obtained from (3.27)

$$L = \frac{1}{2}\Big(\frac{\omega^\mu \omega_\mu}{g} - gm^2 \Big) + m\Big(\theta^\alpha{}_i \dot{\theta}^i_\alpha + \bar{\theta}_{\dot{\alpha}i} \dot{\bar{\theta}}^{\dot{\alpha}i} \Big)$$
$$+ ie\Big(\omega^\mu A_\mu + \dot{\theta}^\alpha_i A^i_\alpha + \dot{\bar{\theta}}_{\dot{\alpha}i} \bar{A}^{\dot{\alpha}i} \Big). \tag{3.28}$$

To find the canonical Hamiltonian H_0, one typically begins by finding the conjugate momenta, which read as

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{g}\Big\{ \dot{x}_\mu + i\Big(\theta_\alpha{}^i \sigma_\mu{}^{\alpha\dot{\alpha}} \dot{\bar{\theta}}_{\dot{\alpha}i} \Big) - \Big(i\dot{\theta}^i_\alpha \sigma_\mu{}^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}i} \Big) \Big\} + ieA_\mu, \tag{3.29}$$

$$\pi^i_\alpha = \frac{\partial_r L}{\partial \dot{\theta}^\alpha_i} = -iP_\mu \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} + m\theta^i_\alpha + ieA^i_\alpha = -H_{\theta^i_\alpha}, \tag{3.30}$$

$$\bar{\pi}_{\dot{\alpha}i} = \frac{\partial_r L}{\partial \dot{\bar{\theta}}^{\dot{\alpha}i}} = iP_\mu \theta^\alpha_i \sigma^\mu_{\alpha\dot{\alpha}} + m\bar{\theta}_{\dot{\alpha}i} + ie\bar{A}_{\dot{\alpha}i} = -H_{\bar{\theta}_{\dot{\alpha}i}}, \tag{3.31}$$

$$p_g = \frac{\partial L}{\partial \dot{g}} = 0 = -H_g, \tag{3.32}$$

$$p_{A_\mu} = \frac{\partial L}{\partial \dot{A}_\mu} = 0 = -H_{A_\mu}, \tag{3.33}$$

$$p_{A^i_\alpha} = \frac{\partial L}{\partial \dot{A}^i_\alpha} = 0 = -H_{A^i_\alpha}, \tag{3.34}$$

$$p_{\bar{A}_{\dot{\alpha}i}} = \frac{\partial L}{\partial \dot{\bar{A}}_{\dot{\alpha}i}} = 0 = -H_{\bar{A}_{\dot{\alpha}i}}. \tag{3.35}$$

We can solve (3.29) for $\dot{x}_\mu$ in terms of P_μ and other coordinates to get

$$\dot{x}_\mu = gP_\mu - i\Big(\theta^i_\alpha \sigma^{\alpha\dot{\alpha}}_\mu \dot{\bar{\theta}}_{\dot{\alpha}i}\Big) + \Big(i\dot{\theta}^i_\alpha \sigma_\mu{}^{\alpha\dot{\alpha}} \bar{\theta}_{\dot{\alpha}i}\Big) - iegA_\mu. \tag{3.36}$$

Then the canonical Hamiltonian H_0 is obtained as

$$\begin{aligned} H_0 &= -L + P_\mu \dot{x^\mu} + \pi^i_\alpha \dot{\theta}^\alpha_i + \bar{\pi}_{\dot{\alpha}i} \dot{\bar{\theta}}^{\dot{\alpha}i} \\ &\quad + p_g \dot{g} + p_{A_\mu} \dot{A}_\mu + p_{A^i_\alpha} \dot{A}^i_\alpha + p_{\bar{A}_{\dot{\alpha}i}} \dot{\bar{A}}_{\dot{\alpha}i} \\ &= \frac{1}{2} g \Big\{ \Big(P_\mu - ieA_\mu\Big)^2 + m^2 \Big\}. \end{aligned} \tag{3.37}$$

Following the Hamilton-Jacobi formalism we obtain the set of HJPDE's,

$$H'_0 = P_0 + \frac{1}{2} g \Big\{ \Big(P_\mu - ieA_\mu\Big)^2 + m^2 \Big\}, \tag{3.38}$$

$$H'_{\theta^i_\alpha} = \pi^i_\alpha + iP_\mu \sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} - m\theta^i{}_\alpha - ieA^i_\alpha, \tag{3.39}$$

$$H'_{\bar{\theta}_{\dot{\alpha}i}} = \bar{\pi}_{\dot{\alpha}i} - iP_\mu \theta^\alpha_i \sigma^\mu_{\alpha\dot{\alpha}} - m\bar{\theta}_{\dot{\alpha}i} - ie\bar{A}_{\dot{\alpha}i}, \tag{3.40}$$

$$H'_g = p_g, \tag{3.41}$$

$$H'_{A_\mu} = p_{A_\mu}, \tag{3.42}$$

$$H'_{A^i_\alpha} = p_{A^i_\alpha}, \tag{3.43}$$

and

$$H'_{\bar{A}_{\dot{\alpha}i}} = p_{\bar{A}_{\dot{\alpha}i}}. \tag{3.44}$$

The equations of motion read as

$$
\begin{aligned}
dx^\mu &= \frac{\partial H_0'}{\partial P_\mu}\, d\tau + \frac{\partial H'_{\theta^i_\alpha}}{\partial P_\mu}\, d\theta^\alpha_i + \frac{\partial H'_{\bar\theta_{\dot\alpha i}}}{\partial P_\mu}\, d\bar\theta^{\dot\alpha i} + \frac{\partial H_g'}{\partial P_\mu} dg \\
&\quad + \frac{\partial H'_{A^\nu}}{\partial P_\mu}\, dA_\nu + \frac{\partial H'_{A^i_\alpha}}{\partial P_\mu}\, dA^\alpha_i + \frac{\partial H'_{\bar A_{\dot\alpha i}}}{\partial P_\mu}\, d\bar A^{\dot\alpha i} \\
&= g\Big(P^\mu - ieA^\mu\Big)\, d\tau + i\Big(\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha i}\Big)\, d\theta^\alpha_i \\
&\quad - i\Big(\theta^\alpha_i \sigma^\mu_{\alpha\dot\alpha}\Big)\, d\bar\theta^{\dot\alpha i},
\end{aligned} \tag{3.45}
$$

$$
\begin{aligned}
dP_0 &= -\frac{\partial H_0'}{\partial \tau}\, d\tau - \frac{\partial H'_{\theta^i_\alpha}}{\partial \tau}\, d\theta^\alpha_i - \frac{\partial H'_{\bar\theta_{\dot\alpha i}}}{\partial \tau}\, d\bar\theta^{\dot\alpha i} - \frac{\partial H_g'}{\partial \tau}\, dg \\
&\quad - \frac{\partial H'_{A_\mu}}{\partial \tau}\, dA_\mu - \frac{\partial H'_{A^i_\alpha}}{\partial \tau}\, dA^\alpha_i - \frac{\partial H'_{\bar A_{\dot\alpha i}}}{\partial \tau}\, d\bar A^{\dot\alpha i} \\
&= 0,
\end{aligned} \tag{3.46}
$$

$$
\begin{aligned}
dP_\mu &= -\frac{\partial H_0'}{\partial x^\mu}\, d\tau - \frac{\partial H'_{\theta^i_\alpha}}{\partial x^\mu}\, d\theta^\alpha_i - \frac{\partial H'_{\bar\theta_{\dot\alpha i}}}{\partial x^\mu}\, d\bar\theta^{\dot\alpha i} - \frac{\partial H_g'}{\partial x^\mu}\, dg \\
&\quad - \frac{\partial H'_{A^\nu}}{\partial x^\mu}\, dA_\nu - \frac{\partial H'_{A^i_\alpha}}{\partial x^\mu}\, dA^\alpha_i - \frac{\partial H'_{\bar A_{\dot\alpha i}}}{\partial x^\mu}\, d\bar A^{\dot\alpha i} \\
&= 0,
\end{aligned} \tag{3.47}
$$

$$
\begin{aligned}
d\pi^i_\alpha &= -\frac{\partial H_0'}{\partial \theta^\alpha_i}\, d\tau - \frac{\partial H'_{\theta^\beta_j}}{\partial \theta^\alpha_i}\, d\theta^j_\beta - \frac{\partial H'_{\bar\theta_{\dot\beta j}}}{\partial \theta^\alpha_i}\, d\bar\theta^{\dot\beta j} - \frac{\partial H_g'}{\partial \theta^\alpha_i}\, dg \\
&\quad - \frac{\partial H'_{A_\mu}}{\partial \theta^\alpha_i}\, dA_\mu - \frac{\partial H'_{A^\beta_j}}{\partial \theta^\alpha_i}\, dA^j_\beta - \frac{\partial H'_{\bar A_{\dot\beta j}}}{\partial \theta^\alpha_i}\, d\bar A^{\dot\beta j} \\
&= \Big(iP_\mu \sigma^\mu_{\alpha\dot\alpha}\Big)\, d\bar\theta^{\dot\alpha i} + m\, d\theta^i_\alpha,
\end{aligned} \tag{3.48}
$$

$$
\begin{aligned}
d\bar\pi_{\dot\alpha i} &= -\frac{\partial H_0'}{\partial \bar\theta^{\dot\alpha i}} d\tau - \frac{\partial H'_{\theta^\beta_j}}{\partial \bar\theta^{\dot\alpha i}} d\theta^j_\beta - \frac{\partial H'_{\bar\theta_{\dot\beta j}}}{\partial \bar\theta^{\dot\alpha i}} d\bar\theta^{\dot\beta j} - \frac{\partial H_g'}{\partial \bar\theta^{\dot\alpha i}} dg \\
&\quad - \frac{\partial H'_{A_\mu}}{\partial \bar\theta^{\dot\alpha i}} dA_\mu - \frac{\partial H'_{A^\beta_j}}{\partial \bar\theta^{\dot\alpha i}} dA^j_\beta - \frac{\partial H'_{\bar A_{\dot\beta j}}}{\partial \bar\theta^{\dot\alpha i}} d\bar A^{\dot\beta j} \\
&= \Big(-iP_\mu \sigma^\mu_{\alpha\dot\alpha}\Big)\, d\theta^\alpha_i + m\, d\bar\theta_{\dot\alpha i},
\end{aligned} \tag{3.49}
$$

$$
\begin{aligned}
dp_g &= -\frac{\partial H_0'}{\partial g}\,d\tau - \frac{\partial H_{\theta^i_\alpha}'}{\partial g}\,d\theta^\alpha_i - \frac{\partial H_{\bar\theta_{\dot\beta j}}'}{\partial g}\,d\bar\theta^{\dot\alpha i} - \frac{\partial H_g'}{\partial g}\,dg \\
&\quad - \frac{\partial H_{A_\mu}'}{\partial g}\,dA_\mu - \frac{\partial H_{A^i_\alpha}'}{\partial g}\,dA^\alpha_i - \frac{\partial H_{\bar A_{\dot\alpha i}}'}{\partial g}\,d\bar A^{\dot\alpha i} \\
&= -\frac{1}{2}\left\{\left(P^\mu - ie\,A^\mu\right)^2 + m^2\right\}d\tau, \qquad (3.50)
\end{aligned}
$$

$$
\begin{aligned}
dp_{A_\mu} &= -\frac{\partial H_0'}{\partial A_\mu}\,d\tau - \frac{\partial H_{\theta^i_\alpha}'}{\partial A_\mu}\,d\theta^\alpha_i - \frac{\partial H_{\bar\theta_{\dot\beta j}}'}{\partial A_\mu}\,d\bar\theta^{\dot\alpha i} - \frac{\partial H_g'}{\partial A_\mu}\,dg \\
&\quad - \frac{\partial H_{A_\nu}'}{\partial A_\mu}\,dA_\nu - \frac{\partial H_{A^i_\alpha}'}{\partial A_\mu}\,dA^\alpha_i - \frac{\partial H_{\bar A_{\dot\alpha i}}'}{\partial A_\mu}\,d\bar A^{\dot\alpha i} \\
&= -ie\,g\left(P^\mu - ieA^\mu\right)d\tau, \qquad (3.51)
\end{aligned}
$$

$$
\begin{aligned}
dp_{A^i_\alpha} &= -\frac{\partial H_0'}{\partial A^i_\alpha}\,d\tau - \frac{\partial H_{\theta^\beta_j}'}{\partial A^i_\alpha}\,d\theta^j_\beta - \frac{\partial H_{\bar\theta_{\dot\beta j}}'}{\partial A^i_\alpha}\,d\bar\theta^{\dot\beta j} - \frac{\partial H_g'}{\partial A^i_\alpha}\,dg \\
&\quad - \frac{\partial H_{A_\mu}'}{\partial A^i_\alpha}\,dA_\mu - \frac{\partial H_{A^\beta_j}'}{\partial A^i_\alpha}\,dA^j_\beta - \frac{\partial H_{\bar A_{\dot\beta j}}'}{\partial A^i_\alpha}\,d\bar A^{\dot\beta j} \\
&= ie\,d\theta^\alpha_i, \qquad (3.52)
\end{aligned}
$$

$$
\begin{aligned}
dp_{\bar A_{\dot\alpha i}} &= -\frac{\partial H_0'}{\partial \bar A_{\dot\alpha i}}\,d\tau - \frac{\partial H_{\theta^\beta_j}'}{\partial \bar A_{\dot\alpha i}}\,d\theta^j_\beta - \frac{\partial H_{\bar\theta_{\dot\beta j}}'}{\partial \bar A_{\dot\alpha i}}\,d\bar\theta^{\dot\beta j} - \frac{\partial H_g'}{\partial \bar A_{\dot\alpha i}}\,dg \\
&\quad - \frac{\partial H_{A_\mu}'}{\partial \bar A_{\dot\alpha i}}\,dA_\mu - \frac{\partial H_{A^\beta_j}'}{\partial \bar A_{\dot\alpha i}}\,dA^j_\beta - \frac{\partial H_{\bar A_{\dot\beta j}}'}{\partial \bar A_{\dot\alpha i}}\,d\bar A^{\dot\beta j} \\
&= ie\,d\bar\theta^{\dot\alpha i}. \qquad (3.53)
\end{aligned}
$$

Now we shall test whether of the set of equation from (3.38) to (3.44) is integrable or not. The total variation of

$$dH_0' = 0, \qquad (3.54)$$

$$dH_{\theta^i_\alpha}' = 0, \qquad (3.55)$$

$$dH_{\bar\theta_{\dot\alpha i}}' = 0, \qquad (3.56)$$

$$dH'_{A^i_\alpha} = 0, \tag{3.57}$$

and

$$dH'_{\bar{A}_{\dot{\alpha}i}} = 0. \tag{3.58}$$

are identically zero, whereas the variation of

$$dH'_g = -\frac{1}{2}\left\{\left(P^\mu - ieA^\mu\right)^2 + m^2\right\} d\tau \equiv H''_g d\tau, \tag{3.59}$$

$$dH'_{A_\mu} = -\left\{ie\, g\left(P^\mu - ieA^\mu\right)\right\} d\tau \equiv H''_{A_\mu} d\tau. \tag{3.60}$$

are not identically zero, with

$$H''_g = \frac{1}{2}\left(P^2 + m^2\right) = 0, \tag{3.61}$$

and

$$H''_{A_\mu} = \left\{ie\, g\left(P^\mu - ieA^\mu\right)\right\} = 0, \tag{3.62}$$

are new constraints. We notice that the total differential of H''_g and H''_{A_μ} vanish identically, *i.e.*

$$dH''_g = 0, \tag{3.63}$$

$$dH''_{A_\mu} = 0. \tag{3.64}$$

Thus the equations of motion (3.45) - (3.53) and the new constraints (3.61) and (3.62) represent an integrable system. According to (1.12) the action can be written as

$$dZ = -H_0 d\tau - H_{\theta^i_\alpha} d\theta^\alpha_i - H_{\bar{\theta}_{\dot{\alpha}i}} d\bar{\theta}^{\dot{\alpha}i} - H_g dg$$
$$- H_{A_\mu} dA_\mu - H_{A^i_\alpha} dA^i_\alpha - H_{\bar{A}_{\dot{\alpha}i}} d\bar{A}_{\dot{\alpha}i} + P_\mu dx^\mu \tag{3.65}$$

The canonical action integral can be written as

$$S^{(e)} = \int \Big\{ -\frac{1}{2} g \Big(P^2 + e^2 A^\mu A_\mu - m^2 \Big) + m \Big(\theta^\alpha{}_i \dot{\theta}^i_\alpha + \bar{\theta}_{\dot{\alpha} i} \dot{\bar{\theta}}^{\dot{\alpha} i} \Big)$$

$$+ ie \Big(\dot{\theta}^\alpha_i A^i_\alpha + \dot{\bar{\theta}}_{\dot{\alpha} i} \bar{A}^{\dot{\alpha} i} \Big) \Big\} \, d\tau. \tag{3.66}$$

Chapter 4

Hamilton-Jacobi Formulation of Strings

In this chapter the supersymmetry structure of superstring are treated as singular system, to relate this structure with the constraint structure in the Hamilton-Jacobi formulation. Different attempts to get the full set of superstring constraints proceeding from the consideration of the superparticle background were proposed in references [30]- [35].

4.1 Hamilton-Jacobi Formulation of Superstring

The model essentially consists of 26 vector bosons of an open string in which there are four bosonic coordinates of four dimensions and there are forty four Majorana fermions representing the remaining 22 bosonic coordinates, we divide them into four groups. They are labeled by $\mu = 0, 1, 2, 3$ and each group contains 11 fermions. These 11 fermions are again divided into two groups, one containing six and the other five. For convenience, in one group we have $j = 1, 2, 3, 4, 5, 6$, and in the other, $k = 1, 2, 3, 4, 5$. The string action is [36]

$$S = -\frac{1}{2\pi} \int d^2\sigma \Big\{ \partial^\alpha X^\mu \, \partial_\alpha X_\mu \, - \, i\bar{\psi}^{\mu,j} \, \rho^\alpha \, \partial_\alpha \psi_{\mu,j} \, - \, i\bar{\phi}^{\mu,k} \, \rho^\alpha \, \partial_\alpha \phi_{\mu,k} \Big\}. \quad (4.1)$$

X^μ , $\mu = 0, 1, 2, 3$ are coordinates for a string that is propagating in 4 space-time dimensions, which are related to the Lorentz Majorana spinors $\psi^{\mu,j}$, $(j = 1, ..., 6)$ and $\phi^{\mu,k}$, $(k = 1, ..., 5)$ in a world-sheet supersymmetry, with ρ^α is $2d$ Dirac matrices, and $\alpha = 0, 1$ are used to refer to τ and σ respectively.
The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2\pi} \Big\{ \partial^\alpha X^\mu \, \partial_\alpha X_\mu \, - \, i\bar{\psi}^{\mu,j} \, \rho^\alpha \, \partial_\alpha \psi_{\mu,j} \, - \, i\bar{\phi}^{\mu,k} \, \rho^\alpha \, \partial_\alpha \phi_{\mu,k} \Big\}. \quad (4.2)$$

The canonical momenta world-sheet vector density is obtained as

$$\mathcal{P}_\mu = \frac{\partial \mathcal{L}}{\partial\big(\partial^0 X^\mu\big)} = -\frac{1}{\pi} \Big(\partial_0 X_\mu \Big), \quad (4.3)$$

$$\pi_\psi = \frac{\partial_r \mathcal{L}}{\partial\big(\partial_0 \psi_{\mu,j}\big)} = \frac{i}{2\pi} \bar{\psi}^{\mu,j} \rho^0 = -\mathcal{H}_\psi, \quad (4.4)$$

$$\bar{\pi}_{\bar{\psi}} = \frac{\partial_r L}{\partial\big(\partial^0 \bar{\psi}_{\mu,j}\big)} = 0 = -H_{\bar{\psi}}, \quad (4.5)$$

$$\pi_\phi = \frac{\partial_r \mathcal{L}}{\partial\big(\partial_0 \phi_{\mu,k}\big)} = \frac{i}{2\pi} \, \bar{\phi}^{\mu,k} \rho^0 = -\mathcal{H}_\phi, \quad (4.6)$$

$$\bar{\pi}_{\bar{\phi}} = \frac{\partial_r L}{\partial\big(\partial^0 \bar{\phi}_{\mu,k}\big)} = 0 = -H_{\bar{\phi}}, \quad (4.7)$$

We can solve (4.3) for $\dot{X}_\mu$ in terms of $\mathcal{P}_\mu$ as

$$\dot{X}_\mu \equiv \partial_0 X_\mu = -\pi \mathcal{P}_\mu. \tag{4.8}$$

The canonical Hamiltonian density is given by

$$\begin{aligned}
\mathcal{H}_0 &= \mathcal{P}_\mu\big(\partial^0 X^\mu\big) + \pi_\psi\big(\partial_0 \psi_{\mu,j}\big) + \bar{\pi}_{\bar{\psi}}\big(\partial^0 \bar{\psi}_{\mu,j}\big) + \pi_\phi\big(\partial_0 \phi_{\mu,k}\big) \\
&\quad + \bar{\pi}_{\bar{\phi}}\big(\partial^0 \bar{\phi}_{\mu,k}\big) - \mathcal{L} \\
&= \frac{1}{2\pi}\Big\{-\pi^2\mathcal{P}^2 + \big(\partial^1 X^\mu\big)\big(\partial_1 X_\mu\big) - \, i\bar{\psi}^{\mu,j}\,\rho^1\,\partial_1\psi_{\mu,j} \\
&\quad - \, i\bar{\phi}^{\mu,k}\,\rho^1\,\partial_1\phi_{\mu,k}\Big\}.
\end{aligned} \tag{4.9}$$

and the canonical Hamiltonian may be written as

$$\begin{aligned}
H_0 &= \int d\sigma \mathcal{H}_0 \\
&= \frac{1}{2\pi}\int d\sigma\Big\{-\pi^2\mathcal{P}^2 + (X')^2 - \, i\bar{\psi}^{\mu,j}\,\rho^1\,\partial_1\psi_{\mu,j} \\
&\quad - \, i\bar{\phi}^{\mu,k}\,\rho^1\,\partial_1\phi_{\mu,k}\Big\}.
\end{aligned} \tag{4.10}$$

where $X'_\mu \equiv \partial^1 X^\mu = \frac{\partial X}{\partial \sigma}$. Making use of (1.7), the set of (HJPDE)'s reads

$$\begin{aligned}
\mathcal{H}'_0 &= \mathcal{P}_0 + \frac{1}{2\pi}\Big\{-\pi^2\mathcal{P}^2 + (X')^2 - \, i\bar{\psi}^{\mu,j}\,\rho^1\,\partial_1\psi_{\mu,j} \\
&\quad - \, i\bar{\phi}^{\mu,k}\,\rho^1\,\partial_1\phi_{\mu,k}\Big\},
\end{aligned} \tag{4.11}$$

$$\mathcal{H}'_\psi = \pi_\psi - \frac{i}{2\pi}\,\bar{\psi}^{\mu,j}\rho^0 = 0, \tag{4.12}$$

$$\mathcal{H}'_{\bar{\psi}} = \bar{\pi}_{\bar{\psi}} = 0, \tag{4.13}$$

$$\mathcal{H}'_\phi = \pi_\phi - \frac{i}{2\pi}\,\bar{\phi}^{\mu,k}\rho^0 = 0, \tag{4.14}$$

$$\mathcal{H}'_{\bar{\phi}} = \bar{\pi}_{\bar{\phi}} = 0. \tag{4.15}$$

The equations of motion corresponding to (1.10) and (1.11) have the following expressions:

$$
\begin{aligned}
dX_\mu &= \frac{\partial \mathcal{H}_0'}{\partial \mathcal{P}_\mu}\, d\tau + \frac{\partial \mathcal{H}'_\psi \nu, j}{\partial \mathcal{P}_\mu}\, d\psi^{\nu,j} + \frac{\partial \mathcal{H}'_{\bar\psi^{\nu,j}}}{\partial \mathcal{P}_\mu}\, d\bar\psi^{\nu,j} \\
&\quad + \frac{\partial \mathcal{H}'_\phi}{\partial \mathcal{P}_\mu}\, d\phi^{\nu,k} + \frac{\partial \mathcal{H}'_{\bar\phi^{\nu,k}}}{\partial \mathcal{P}_\mu}\, d\bar\phi^{\nu,k} \\
&= \Big(- \pi \mathcal{P}_\mu \Big) d\tau,
\end{aligned} \tag{4.16}
$$

$$
\begin{aligned}
d\mathcal{P}_\mu &= -\frac{\partial \mathcal{H}_0'}{\partial X_\mu}\, d\tau - \frac{\partial \mathcal{H}'_{\psi^{\nu,j}}}{\partial X_\mu}\, d\psi^{\nu,j} - \frac{\partial \mathcal{H}'_{\bar\psi^{\nu,l}}}{\partial X_\mu}\, d\bar\psi^{\nu,j} \\
&\quad - \frac{\partial \mathcal{H}'_{\phi^{\nu,k}}}{\partial X_\mu}\, d\phi^{\nu,k} - \frac{\partial \mathcal{H}'_{\bar\phi^{\nu,k}}}{\partial X_\mu}\, d\bar\phi^{\nu,k} = 0,
\end{aligned} \tag{4.17}
$$

$$
\begin{aligned}
d\pi_{\psi^{\mu,j}} &= -\frac{\partial \mathcal{H}_0'}{\partial \psi_{\mu,j}}\, d\tau - \frac{\partial \mathcal{H}'_{\psi^{\nu,l}}}{\partial \psi_{\mu,j}}\, d\psi^{\nu,l} - \frac{\partial \mathcal{H}'_{\bar\psi^{\nu,l}}}{\partial \psi_{\mu,j}}\, d\bar\psi^{\nu,l} \\
&\quad - \frac{\partial \mathcal{H}'_{\phi^{\nu,k}}}{\partial \psi_{\mu,j}}\, d\phi^{\nu,k} - \frac{\partial \mathcal{H}'_{\bar\phi^{\nu,k}}}{\partial \psi_{\mu,j}}\, d\bar\phi^{\nu,k} = 0,
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
d\bar\pi_{\bar\psi^{\mu,j}} &= -\frac{\partial \mathcal{H}_0'}{\partial \bar\psi^{\mu,j}}\, d\tau - \frac{\partial \mathcal{H}'_{\psi^{\nu,l}}}{\partial \bar\psi^{\mu,j}}\, d\psi^{\nu,l} - \frac{\partial \mathcal{H}'_{\bar\psi^{\nu,l}}}{\partial \bar\psi^{\mu,j}}\, d\bar\psi^{\nu,l} \\
&\quad - \frac{\partial \mathcal{H}'_{\phi^{\nu,k}}}{\partial \bar\psi^{\mu,j}}\, d\phi^{\nu,k} - \frac{\partial \mathcal{H}'_{\bar\phi^{\nu,k}}}{\partial \bar\psi^{\mu,j}}\, d\bar\phi^{\nu,k} \\
&= \left(\frac{i}{2\pi}\, \rho^1\, \partial_1 \psi_{\mu,j} \right) d\tau + \frac{i}{2\pi}\, \rho^0\, d\psi_{\mu,j},
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
d\pi_{\phi^{\mu,k}} &= -\frac{\partial \mathcal{H}_0'}{\partial \phi_{\mu,k}}\, d\tau - \frac{\partial \mathcal{H}'_{\psi^{\nu,j}}}{\partial \phi_{\mu,k}}\, d\psi^{\nu,j} - \frac{\partial \mathcal{H}'_{\bar\psi^{\nu,j}}}{\partial \phi_{\mu,k}}\, d\bar\psi^{\nu,j} \\
&\quad - \frac{\partial \mathcal{H}'_{\phi^{\nu,n}}}{\partial \phi_{\mu,k}}\, d\phi^{\nu,n} - \frac{\partial \mathcal{H}'_{\bar\phi^{\nu,n}}}{\partial \phi_{\mu,k}}\, d\bar\phi^{\nu,n} = 0,
\end{aligned} \tag{4.20}
$$

and

$$
\begin{aligned}
d\bar{\pi}_{\bar{\phi}^{\mu,k}} &= -\frac{\partial \mathcal{H}'_0}{\partial \bar{\phi}^{\mu,k}}\, d\tau - \frac{\partial \mathcal{H}'_{\phi^{\nu,j}}}{\partial \bar{\phi}^{\mu,k}}\, d\psi^{\nu,j} - \frac{\partial \mathcal{H}'_{\bar{\psi}^{\nu,j}}}{\partial \bar{\phi}^{\mu,k}}\, d\bar{\psi}^{\nu,j} \\
&\quad - \frac{\partial \mathcal{H}'_{\phi^{\nu,n}}}{\partial \bar{\phi}^{\mu,k}}\, d\phi^{\nu,n} - \frac{\partial \mathcal{H}'_{\bar{\phi}^{\nu,n}}}{\partial \bar{\phi}^{\mu,k}}\, d\bar{\phi}^{\nu,n} \\
&= \left(\frac{i}{2\pi}\, \rho^1\, \partial_1 \phi_{\mu,k} \right) d\tau + \frac{i}{2\pi}\, \rho^0\, d\phi_{\mu,k}. \qquad (4.21)
\end{aligned}
$$

The next step is to impose the integrability conditions and to make the above system to be integrable. The integrability conditions (1.13) and (1.14) read as

$$d\mathcal{H}'_0 = \frac{1}{\pi} X'_\mu dX'^\mu, \qquad (4.22)$$

$$d\mathcal{H}'_\psi = 0, \qquad (4.23)$$

$$d\mathcal{H}'_{\bar{\psi}} = \left(\frac{i}{2\pi} \rho^1 \partial_1 \psi_{\mu,j} \right) d\tau + \left(\frac{i}{2\pi} \rho^0 \right) d\psi_{\mu,j} \equiv H''_{\bar{\psi}} d\tau, \qquad (4.24)$$

$$d\mathcal{H}'_\phi = 0, \qquad (4.25)$$

$$d\mathcal{H}'_{\bar{\phi}} = \left(\frac{i}{2\pi} \rho^1 \partial_1 \phi_{\mu,k} \right) d\tau + \left(\frac{i}{2\pi} \rho^0 \right) d\phi_{\mu,k} \equiv H''_{\bar{\phi}} d\tau. \qquad (4.26)$$

By using (4.16), relation (4.26) can be written as

$$d\mathcal{H}'_0 = \Big(- X'_\mu \partial_1 \mathcal{P}^\mu \Big) d\tau \equiv \mathcal{H}''_0 d\tau \qquad (4.27)$$

Since the variation of $\mathcal{H}'_0$, $\mathcal{H}'_{\bar{\psi}}$ and $\mathcal{H}'_{\bar{\phi}}$ are not identically zero, we consider them as new constraints,

$$\mathcal{H}''_0 = -X'_\mu \partial_1 \mathcal{P}^\mu = 0, \qquad (4.28)$$

$$\mathcal{H}''_{\bar{\psi}} = \frac{i}{2\pi} \rho^1 \partial_1 \psi_{\mu,j} = 0, \qquad (4.29)$$

$$\mathcal{H}''_{\bar\phi} = \frac{i}{2\pi}\rho^1\partial_1\phi_{\mu,k} = 0, \tag{4.30}$$

and one should consider the total variations of them,

$$d\mathcal{H}''_0 = d\Big(-X'_\mu\partial^1\mathcal{P}^\mu\Big) = -\Big\{dX'_\mu\Big(\partial^1\mathcal{P}^\mu\Big) + X'_\mu d\Big(\partial_1\mathcal{P}^\mu\Big)\Big\}, \tag{4.31}$$

$$d\mathcal{H}''_{\bar\psi} = 0, \tag{4.32}$$

$$d\mathcal{H}''_{\bar\phi} = 0. \tag{4.33}$$

Relation (4.31) can be written as

$$d\mathcal{H}''_0 = -\Big\{-\pi\Big(\partial_1\mathcal{P}_\mu\Big)\Big(\partial^1\mathcal{P}^\mu\Big)\,d\tau + X'_\mu\partial^1\Big(d\mathcal{P}^\mu\Big)\Big\} = 0. \tag{4.34}$$

Using (4.17), relation (4.34) becomes

$$d\mathcal{H}''_0 = -\pi\Big\{\Big(\partial_1\mathcal{P}_\mu\Big)\Big(\partial^1\mathcal{P}^\mu\Big)\Big\}d\tau \equiv \mathcal{H}'''_0 d\tau \tag{4.35}$$

where $\mathcal{H}'''_0$ is a new constraint,

$$\mathcal{H}'''_0 = \pi\Big\{\Big(\partial_1\mathcal{P}_\mu\Big)\Big(\partial^1\mathcal{P}^\mu\Big)\Big\} = 0 \tag{4.36}$$

and one should consider the total variations of $\mathcal{H}'''_0$

$$\begin{aligned} d\mathcal{H}'''_0 &= \pi d\Big\{\Big(\partial_1\mathcal{P}_\mu\Big)\Big(\partial^1\mathcal{P}^\mu\Big)\Big\} = 2\pi\Big(\partial_1\mathcal{P}_\mu\Big)\,d\Big(\partial^1\mathcal{P}^\mu\Big) \\ &= 2\pi\Big(\partial_1\mathcal{P}_\mu\Big)\partial^1\Big(d\mathcal{P}^\mu\Big) = 0, \end{aligned} \tag{4.37}$$

which is identically zero. The set of equations (4.16) - (4.21) with relations (4.28),(4.29),(4.30) and (4.36) represent an integrable system, and the action in phase space can be written as

$$\begin{aligned} S = -\frac{1}{2\pi}\int\Big\{&\pi^2\mathcal{P}^2 + \Big(X'\Big)^2 - i\bar\psi^{\mu,j}\rho^0\partial_0\psi_{\mu,j} - i\bar\psi^{\mu,j}\rho^1\partial_1\psi_{\mu,j} \\ &- i\bar\phi^{\mu,k}\rho^0\partial_0\phi_{\mu,k} - i\bar\phi^{\mu,k}\rho^1\partial_1\phi_{\mu,k}\Big\}d\sigma d\tau. \end{aligned} \tag{4.38}$$

4.2 Hamilton-Jacobi Formulation of Green-Schwarz Superstring

A form of the action describing the motion of a superstring in *11*-dimensional Minkowski space-time has been proposed in [32],

$$S = \int d^2\sigma \Big\{ \frac{-g^{ab}}{2\sqrt{-g}} \Pi_a^\mu \Pi_{b\mu} - i\varepsilon^{ab}\Big(\partial_a x^\mu - \frac{i}{2}\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_a \theta\Big)\Big(\bar{\theta}\Gamma^\mu \partial_b \theta\Big) - \varepsilon^{ab}\xi_a \Big(n_\mu \Pi_b^\mu\Big) - n_\mu \varepsilon^{ab}\partial_a A_b^\mu - \phi\Big(n^2+1\Big)\Big\}. \tag{4.39}$$

where $\Pi_a^\mu \equiv \partial_a x^\mu - i\Big(\bar{\theta}\Gamma^{\nu\mu} n_\nu \partial_a \theta\Big)$

The quantities which appear in the Green- Schwarz action are the two-dimensional metric g^{ab}, a ten dimensional position x^μ , θ is a *32* -component anticommuting Majorana spinor of $SO(2,9)$; which is a rotation group, $n^\mu(\sigma,\tau)$ is $D11$ vector, ξ_a is a $d=2$ vector, $A_a^\mu(\sigma,\tau)$ is $D11$ vector, and $\phi(\sigma,\tau)$ is a scalar. As for the Γ-matrices, we employ the 32-dimensional Majorana representation and denote them by $\Gamma^{\mu\nu}$. In (4.39) we have set $\varepsilon^{ab} = -\varepsilon^{ba}$, $\varepsilon^{01} = -1$.

The Lagrangian density is

$$\mathcal{L} = \Big\{ \frac{-g^{ab}}{2\sqrt{-g}} \Pi_a^\mu \Pi_{b\mu} - i\varepsilon^{ab}\Big(\partial_a x^\mu - \frac{i}{2}\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_a \theta\Big)\Big(\bar{\theta}\Gamma^\mu \partial_b \theta\Big) - \varepsilon^{ab}\xi_a \Big(n_\mu \Pi_b^\mu\Big) - n_\mu \varepsilon^{ab}\partial_a A_b^\mu - \phi\Big(n^2+1\Big)\Big\}. \tag{4.40}$$

In more explicit form, the Lagrangian density (4.40) takes the form

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2\sqrt{-g}}\Bigg\{ g^{00}\Bigg((\partial_0 x^{\mu}\partial_0 x_{\mu}) - 2i\left(\bar{\theta}\Gamma^{\nu\mu}n_{\nu}\partial_0\theta\right)(\partial_0 x_{\mu})\Bigg) \\
&+ 2g^{01}\Bigg(\partial_0 x^{\mu}\partial_1 x_{\mu} - i\left(\bar{\theta}\Gamma^{\nu\mu}n_{\nu}\partial_0\theta\right)(\partial_1 x_{\mu}) - i\left(\bar{\theta}\Gamma^{\nu\mu}n_{\nu}\partial_1\theta\right)(\partial_0 x_{\mu})\Bigg) \\
&+ g^{11}\Bigg((\partial_1 x^{\mu}\partial_1 x_{\mu}) - 2i\left(\bar{\theta}\Gamma^{\nu\mu}n_{\nu}\partial_1\theta\right)(\partial_1 x_{\mu})\Bigg)\Bigg\} + i\left(\bar{\theta}\Gamma_{\mu}\partial_1\theta\right)(\partial_0 x^{\mu}) \\
&- i\left(\bar{\theta}\Gamma_{\mu}\partial_0\theta\right)(\partial_1 x^{\mu}) + \xi_0 n_{\mu}\Bigg((\partial_1 x^{\mu}) - i\left(\bar{\theta}\Gamma^{\nu\mu}n_{\nu}\partial_1\theta\right)\Bigg) \\
&- \xi_1 n_{\mu}\Bigg((\partial_0 x^{\mu}) - i(\bar{\theta}\Gamma^{\nu\mu}n_{\nu}\partial_0\theta)\Bigg) + n_{\mu}\partial_0 A_1^{\mu} - n_{\mu}\partial_1 A_0^{\mu} - \phi\left(n^2+1\right).
\end{aligned}
\tag{4.41}
$$

Now let us going to demonstrate the dynamics of physical variables in the action. The momenta variables according to equations (1.2) and (1.3) are

$$
\begin{aligned}
\mathcal{P}_{\mu} = \frac{\partial\mathcal{L}}{\partial\left(\partial^0 x^{\mu}\right)} = &-\frac{g^{00}}{\sqrt{-g}}\Big\{(\partial_0 x_{\mu}) - i(\bar{\theta}\Gamma^{\nu}_{\mu}n_{\nu}\partial_0\theta)\Big\} \\
&- \frac{g^{01}}{\sqrt{-g}}\Big\{(\partial_1 x_{\mu}) - i\left(\bar{\theta}\Gamma^{\nu}_{\mu}n_{\nu}\partial_1\theta\right)\Big\} + i\bar{\theta}\Gamma_{\mu}\partial_1\theta - \xi_1 n_{\mu},
\end{aligned}
\tag{4.42}
$$

$$
\pi_{\theta} = \frac{\partial_r\mathcal{L}}{\partial\left(\partial_0\theta\right)} = -\Big\{\mathcal{P}_{\mu}\left(i\bar{\theta}\Gamma^{\nu\mu}n_{\nu}\right) + \bar{\theta}\Gamma_{\mu}\left(\partial_1 x^{\mu}\right)\Big\} = -\mathcal{H}_{\theta}, \tag{4.43}
$$

$$
\bar{\pi}_{\bar{\theta}} = \frac{\partial_r L}{\partial\left(\partial^0\bar{\theta}\right)} = 0 = -H_{\bar{\theta}}, \tag{4.44}
$$

$$
\pi_{g_{ab}} = \frac{\partial\mathcal{L}}{\partial\left(\partial_0 g^{ab}\right)} = 0 = -\mathcal{H}_{g_{ab}}, \tag{4.45}
$$

$$
\mathcal{P}_0{}^{\mu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_0 A_0^{\mu}\right)} = 0 = -H_{A_0^{\mu}}, \tag{4.46}
$$

$$
\mathcal{P}_1{}^{\mu} = \frac{\partial\mathcal{L}}{\partial\left(\partial_0 A_1^{\mu}\right)} = n_{\mu} = -H_{A_1^{\mu}}, \tag{4.47}
$$

$$\mathcal{P}_{\xi_0} = \frac{\partial \mathcal{L}}{\partial\Big(\partial_0 \xi_0\Big)} = 0 = -H_{\xi_0}, \tag{4.48}$$

$$\mathcal{P}_{\xi_1} = \frac{\partial \mathcal{L}}{\partial\Big(\partial_0 \xi_1\Big)} = 0 = -H_{\xi_1}, \tag{4.49}$$

$$\mathcal{P}_n^{\mu} = \frac{\partial \mathcal{L}}{\partial\Big(\partial_0 n_\mu\Big)} = 0 = -H_{n^\mu}, \tag{4.50}$$

and

$$\pi_\phi = \frac{\partial \mathcal{L}}{\partial\Big(\partial_0 \phi\Big)} = 0 = -H_\phi. \tag{4.51}$$

We can solve (4.42) for $\dot{x}_\mu$ in terms of $\mathcal{P}_\mu$ and other coordinates

$$\begin{aligned}\dot{x}_\mu \equiv \partial_0 x_\mu = -\frac{\sqrt{-g}}{g^{00}} \Bigg\{ \mathcal{P}_\mu + \frac{g^{01}}{\sqrt{-g}} \Bigg\{ \left(\partial_1 x_\mu\right) - i(\bar{\theta}\Gamma^{\nu\mu} n_\nu \partial_1 \theta) \Bigg\} \\ - i\bar{\theta}\Gamma_\mu \partial_1 \theta + \xi_1 n_\mu \Bigg\} + i\left(\bar{\theta}\Gamma^{\nu\mu} n_\nu \partial_0 \theta\right).\end{aligned} \tag{4.52}$$

The canonical Hamiltonian density is given by

$$\begin{aligned}\mathcal{H}_0 &= \mathcal{P}_\mu \left(\partial^0 x^\mu\right) + \pi_\theta \left(\partial_0 \theta\right) + \bar{\pi}_{\bar{\theta}} \left(\partial_0 \bar{\theta}\right) + \pi_{g_{ab}} \left(\partial_0 g^{ab}\right) \\ &\quad + \mathcal{P}_0{}^\mu \left(\partial_0 A_0^\mu\right) + \mathcal{P}_1{}^\mu \left(\partial_0 A_1^\mu\right) + \mathcal{P}_{\xi_0} \left(\partial_0 \xi_0\right) \\ &\quad + \mathcal{P}_{\xi_1} \left(\partial_0 \xi_1\right) + \mathcal{P}^\mu{}_n \left(\partial_0 n^\mu\right) + \pi_\phi \left(\partial_0 \phi\right) - \mathcal{L} \\ &= \left(-\frac{\sqrt{-g}}{2g^{00}}\right) \Bigg\{ \mathcal{P}_\mu + \left(\frac{g^{01}}{\sqrt{-g}}\right) \Big(\left(\partial_1 x_\mu\right) - i\left(\bar{\theta}\Gamma^\nu_\mu n_\nu \partial_1 \theta\right) \Big) - i\bar{\theta}\Gamma_\mu \partial_1 \theta \\ &\quad + \xi_1 n_\mu \Bigg\}^2 + \left(-\frac{g^{11}}{2\sqrt{-g}}\right) \Big(\left(\partial_1 x^\mu\right)^2 - 2i\left(\bar{\theta}\Gamma^\nu_\mu n_\nu \partial_1 \theta\right)\left(\partial_1 x_\mu\right) \Big) \\ &\quad - \xi_0 n_\mu \Big(\partial_1 x^\mu - i\left(\bar{\theta}\Gamma^{\nu\mu} n_\nu \partial_1 \theta\right) \Big) + n_\mu \partial_1 A_0^\mu + \phi\left(n^2 + 1\right).\end{aligned} \tag{4.53}$$

The two dimensional metric is

$$g^{ab} = \begin{pmatrix} g^{00} & g^{01} \\ g^{10} & g^{11} \end{pmatrix}, \tag{4.54}$$

with $\left|g^{ab}\right| = g = g^{00}\ g^{11} - (g^{01})^2$, and consider $N = \frac{\sqrt{-g}}{g^{00}}$ and $N_1 = \frac{g^{01}}{g^{00}}$, then the canonical Hamiltonian density can be written as

$$\begin{aligned}\mathcal{H}_0 = &-\left(\frac{N}{2}\right)\left\{\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)^2 - \Pi_{1\mu}\Pi_1^\mu\right\} \\ &- N_1\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)\Pi_{1\mu} - \xi_0 n_\mu\left(\partial_1 x^\mu\right) \\ &- i\xi_0 n_\mu\left(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta\right) + n_\mu\partial_1 A_0^\mu + \phi\left(n^2+1\right).\end{aligned} \tag{4.55}$$

The canonical Hamiltonian may be written as

$$\begin{aligned}H_0 &= \int d\sigma\mathcal{H}_0 \\ &= \int d\sigma\Big\{-\frac{N}{2}\Big\{\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)^2 - \Pi_{1\mu}\Pi_1^\mu\Big\} \\ &\quad - N_1\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)\Pi_{1\mu} - \xi_0 n_\mu\left(\partial_1 x^\mu\right) \\ &\quad - i\xi_0 n_\mu\left(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta\right) + n_\mu\partial_1 A_0^\mu + \phi\left(n^2+1\right)\Big\}.\end{aligned} \tag{4.56}$$

The complete system of constraints can be presented in the form of the set of HJPDE's

$$\begin{aligned}\mathcal{H}'_0 = \mathcal{P}_0 &- \frac{N}{2}\Big\{\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)^2 - \Pi_{1\mu}\Pi_1^\mu\Big\} \\ &- N_1\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)\Pi_{1\mu} - \xi_0 n_\mu\left(\partial_1 x^\mu\right) \\ &- i\xi_0 n_\mu\left(\bar{\theta}\Gamma^{\nu\mu}n_\nu\partial_1\theta\right) + n_\mu\partial_1 A_0^\mu + \phi\left(n^2+1\right),\end{aligned} \tag{4.57}$$

$$\mathcal{H}'_\theta = \pi_\theta + \left(\mathcal{P}_\mu(i\bar{\theta}\Gamma^{\nu\mu}n_\nu) + (\bar{\theta}\Gamma_\mu\left(\partial_1 x^\mu\right)\right) = 0, \tag{4.58}$$

$$\mathcal{H}'_{\bar{\theta}} = \bar{\pi}_{\bar{\theta}} = 0, \tag{4.59}$$

$$\mathcal{H}'_{g^{ab}} = \pi_{g^{ab}} = 0, \tag{4.60}$$

$$\mathcal{H}'_{A_0^\mu} = \mathcal{P}_0{}^\mu = 0, \tag{4.61}$$

$$\mathcal{H}'_{A_1^\mu} = \mathcal{P}_1{}^\mu - n_\mu = 0, \tag{4.62}$$

$$\mathcal{H}'_{\xi_0} = \mathcal{P}_{\xi_0} = 0, \tag{4.63}$$

$$\mathcal{H}'_{\xi_1} = \mathcal{P}_{\xi_1} = 0, \tag{4.64}$$

$$\mathcal{H}'_{n^\mu} = \mathcal{P}_n^\mu = 0, \tag{4.65}$$

and

$$\mathcal{H}'_\phi = \pi_\phi = 0. \tag{4.66}$$

Now one can see that the dynamics of the variables is governed by the equations of motion of the form (1.10) and (1.11) such as

$$\begin{aligned} dx_\mu &= \frac{\partial \mathcal{H}'_0}{\partial \mathcal{P}_\mu}\, d\tau + \frac{\partial \mathcal{H}'_\theta}{\partial \mathcal{P}_\mu}\, d\theta + \frac{\partial \mathcal{H}'_{\bar{\theta}}}{\partial \mathcal{P}_\mu}\, d\bar{\theta} + \frac{\partial \mathcal{H}'_{g^{ab}}}{\partial \mathcal{P}_\mu}\, dg^{ab} + \frac{\partial \mathcal{H}'_{A_\nu^0}}{\partial \mathcal{P}_\mu}\, dA_0^\nu \\ &\quad + \frac{\partial \mathcal{H}'_{A_1^\nu}}{\partial \mathcal{P}_\mu}\, dA_\nu^1 + \frac{\partial \mathcal{H}'_{\xi_0}}{\partial \mathcal{P}_\mu}\, d\xi_0 + \frac{\partial \mathcal{H}'_{\xi_1}}{\partial \mathcal{P}_\mu}\, d\xi_1 + \frac{\partial \mathcal{H}'_{n^\nu}}{\partial \mathcal{P}_\mu}\, dn_\nu + \frac{\partial \mathcal{H}'_\phi}{\partial \mathcal{P}_\mu}\, d\phi \\ &= \left\{ -N\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu \partial_1 \theta + \xi_1 n_\mu\right) - N_1 \Pi_{1\mu} \right\} d\tau + i\left(\bar{\theta}\Gamma^\nu_\mu n_\nu\right) d\theta, \end{aligned} \tag{4.67}$$

$$\begin{aligned} d\mathcal{P}_\mu &= -\frac{\partial \mathcal{H}'_0}{\partial x_\mu}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial x_\mu}\, d\theta - \frac{\partial \mathcal{H}'_{\bar{\theta}}}{\partial x_\mu}\, d\bar{\theta} - \frac{\partial \mathcal{H}'_{g^{ab}}}{\partial x_\mu}\, dg^{ab} - \frac{\partial \mathcal{H}'_{A_0^\nu}}{\partial x_\mu}\, dA_\nu^0 \\ &\quad - \frac{\partial \mathcal{H}'_{A_1^\nu}}{\partial x_\mu}\, dA_\nu^1 - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial x_\mu}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial x_\mu}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\nu}}{\partial x_\mu}\, dn_\nu - \frac{\partial \mathcal{H}'_\phi}{\partial x_\mu}\, d\phi \\ &= 0, \end{aligned} \tag{4.68}$$

$$
\begin{aligned}
d\pi_\theta = &-\frac{\partial \mathcal{H}'_0}{\partial \theta}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial \theta}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial \theta}\, d\bar\theta - \frac{\partial {\mathcal{H}'}_{g^{ab}}}{\partial \theta}\, dg^{ab} - \frac{\partial {\mathcal{H}'}_{A_0^\mu}}{\partial \theta}\, dA^0_\mu \\
&- \frac{\partial {\mathcal{H}'}_{A_1^\mu}}{\partial \theta}\, dA^1_\mu - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial \theta}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial \theta}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\mu}}{\partial \theta}\, dn_\mu - \frac{\partial \mathcal{H}'_\phi}{\partial \theta}\, d\phi \\
= &\ 0,
\end{aligned}
\tag{4.69}
$$

$$
\begin{aligned}
d\bar\pi_{\bar\theta} = &-\frac{\partial \mathcal{H}'_0}{\partial \bar\theta}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial \theta}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial \bar\theta}\, d\bar\theta - \frac{\partial {\mathcal{H}'}_{g^{ab}}}{\partial \bar\theta}\, dg^{ab} - \frac{\partial {\mathcal{H}'}_{A_0^\mu}}{\partial \bar\theta}\, dA^\mu_0 \\
&- \frac{\partial {\mathcal{H}'}_{A_1^\mu}}{\partial \bar\theta}\, dA^\mu_1 - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial \theta}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial \bar\theta}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\mu}}{\partial \bar\theta}\, dn_\mu - \frac{\partial \mathcal{H}'_\phi}{\partial \bar\theta}\, d\phi \\
= &-\bigg\{ N\Big(\mathcal{P}_\mu\ + \xi_1 n_\mu\Big)\Big(i\Gamma^\mu\, \partial_1\theta\Big) + N\Big(\partial_1 x_\mu\Big)\Big(i\Gamma^{\mu\nu} n_\nu \partial_1\theta\Big) \\
&- N_1\Big(\mathcal{P}_\mu\ + \xi_1 n_\mu\Big)\Big(i\Gamma^{\mu\nu} n_\nu \partial_1\theta\Big) + N_1\Big(\partial_1 x_\mu\Big)\Big(i\Gamma^\mu \partial_1\theta\Big) \\
&- \xi_0 n_\mu\Big(i\Gamma^{\mu\nu} n_\nu \partial_1\theta\Big)\bigg\} d\tau - \bigg\{\mathcal{P}_\mu\Big(i\Gamma^{\mu\nu} n_\nu\Big) + \Big(\partial_1 x_\mu\Big)\Big(i\Gamma^\mu\Big)\bigg\} d\theta,
\end{aligned}
\tag{4.70}
$$

$$
\begin{aligned}
d\pi_{g^{ab}} = &-\frac{\partial \mathcal{H}'_0}{\partial g^{ab}}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial g^{ab}}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial g^{ab}}\, d\bar\theta - \frac{\partial {\mathcal{H}'}_{g^{ab}}}{\partial g^{ab}}\, dg^{ab} - \frac{\partial {\mathcal{H}'}_{A_0^\mu}}{\partial g^{ab}}\, dA^\mu_0 \\
&- \frac{\partial {\mathcal{H}'}_{A_1^\mu}}{\partial g^{ab}}\, dA^\mu_1 - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial g^{ab}}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial g^{ab}}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\mu}}{\partial g^{ab}}\, dn_\mu - \frac{\partial \mathcal{H}'_\phi}{\partial g^{ab}}\, d\phi \\
= &\ \frac{1}{2}\bigg\{\bigg[\Big(\mathcal{P}_\mu - i\bar\theta\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\Big)^2 + \Pi_{1\mu}\Pi_1^\mu\bigg]\bigg(\frac{\partial N}{\partial g^{ab}}\bigg) \\
&+ \Big(\mathcal{P}_\mu - i\bar\theta\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\Big)\Pi_1^\mu\bigg(\frac{\partial N_1}{\partial g^{ab}}\bigg)\bigg\} d\tau,
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
d\mathcal{P}^\mu_0 = &-\frac{\partial \mathcal{H}'_0}{\partial A^\mu_0}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial A^\mu_0}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial A^\mu_0}\, d\bar\theta - \frac{\partial {\mathcal{H}'}_{g^{ab}}}{\partial A^\mu_0}\, dg^{ab} - \frac{\partial {\mathcal{H}'}_{A_0^\nu}}{\partial A^\mu_0}\, dA^0_\nu \\
&- \frac{\partial {\mathcal{H}'}_{A_1^\nu}}{\partial A^\mu_0}\, dA^1_\nu - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial A^\mu_0}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial A^\mu_0}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\nu}}{\partial A^\mu_0}\, dn_\nu - \frac{\partial \mathcal{H}'_\phi}{\partial A^\mu_0}\, d\phi \\
= &\ 0,
\end{aligned}
\tag{4.72}
$$

$$
\begin{aligned}
d\mathcal{P}_1^\mu &= -\frac{\partial \mathcal{H}'_0}{\partial A_1^\mu}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial A_1^\mu}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial A_1^\mu}\, d\bar\theta - \frac{\partial \mathcal{H}'_{g^{ab}}}{\partial A_1^\mu}\, dg^{ab} - \frac{\partial \mathcal{H}'_{A_0^\nu}}{\partial A_1^\mu}\, dA_\nu^0 \\
&\quad - \frac{\partial \mathcal{H}'_{A_1^\nu}}{\partial A_1^\mu}\, dA_\nu^1 - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial A_1^\mu}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial A_1^\mu}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\nu}}{\partial A_1^\nu}\, dn_\mu - \frac{\partial \mathcal{H}'_\phi}{\partial A_1^\mu}\, d\phi \\
&= 0, \qquad (4.73)
\end{aligned}
$$

$$
\begin{aligned}
d\mathcal{P}_{\xi_0} &= -\frac{\partial \mathcal{H}'_0}{\partial \xi_0}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial \xi_0}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial \xi_0}\, d\bar\theta - \frac{\partial \mathcal{H}'_{g^{ab}}}{\partial \xi_0}\, dg^{ab} - \frac{\partial \mathcal{H}'_{A_0^\mu}}{\partial \xi_0}\, dA_0^\mu \\
&\quad - \frac{\partial \mathcal{H}'_{A_1^\mu}}{\partial \xi_0}\, dA_1^\mu - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial \xi_0}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial \xi_0}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\mu}}{\partial \xi_0}\, dn_\mu - \frac{\partial \mathcal{H}'_\phi}{\partial A_1^\mu}\, d\phi \\
&= \left\{ n_\mu \left(\partial_1 x_\mu + i\bar\theta\Gamma^{\mu\nu} n_\nu \partial_1\theta \right) \right\} d\tau, \qquad (4.74)
\end{aligned}
$$

$$
\begin{aligned}
d\mathcal{P}_{\xi_1} &= -\frac{\partial \mathcal{H}'_0}{\partial \xi_1}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial \xi_1}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial \xi_1}\, d\bar\theta - \frac{\partial \mathcal{H}'_{g^{ab}}}{\partial \xi_1}\, dg^{ab} - \frac{\partial \mathcal{H}'_{A_0^\mu}}{\partial \xi_1}\, dA_0^\mu \\
&\quad - \frac{\partial \mathcal{H}'_{A_1^\mu}}{\partial \xi_1}\, dA_1^\mu - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial \xi_1}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial \xi_1}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\mu}}{\partial \xi_1}\, dn_\mu - \frac{\partial \mathcal{H}'_\phi}{\partial A_1^\mu}\, d\phi \\
&= \left\{ N \left(\mathcal{P}_\mu - i\bar\theta\Gamma^\mu \partial_1\theta + \xi_1 n_\mu \right) n_\mu + N_1 \Pi_{1\mu} n^\mu \right\} d\tau, \qquad (4.75)
\end{aligned}
$$

$$
\begin{aligned}
d\mathcal{P}_n^\mu &= -\frac{\partial \mathcal{H}'_0}{\partial n_\mu}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial n_\mu}\, d\theta - \frac{\partial \mathcal{H}'_{\bar\theta}}{\partial n_\mu}\, d\bar\theta - \frac{\partial \mathcal{H}'_{g^{ab}}}{\partial n_\mu}\, dg^{ab} - \frac{\partial \mathcal{H}'_{A_0^\nu}}{\partial n_\mu}\, dA_\nu^0 \\
&\quad - \frac{\partial \mathcal{H}'_{A_1^\nu}}{\partial n_\mu}\, dA_\nu^1 - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial n_\mu}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial n_\mu}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\nu}}{\partial n_\mu}\, dn_\nu - \frac{\partial \mathcal{H}'_\phi}{\partial A_1^\mu}\, d\phi \\
&= \Big\{ N \left(\mathcal{P}_\mu - i\bar\theta\Gamma^\mu \partial_1\theta + \xi_1 n_\mu \right) \xi_1 - N \partial_1 x_\mu \left(i\bar\theta\Gamma^\mu \partial_1\theta \right) \\
&\quad - N_1 \xi_1 n_\mu \left(i\bar\theta\Gamma^{\mu\nu} \partial_1\theta \right) + N_1 \xi_1 \left(\partial_1 x_\mu - i\bar\theta\Gamma^{\mu\nu} n_\nu \partial_1\theta \right) \\
&\quad + \xi_0 \left(\partial_1 x_\mu + i\bar\theta\Gamma^{\mu\nu} n_\nu \partial_1\theta \right) + \xi_0 n_\mu \left(i\bar\theta\Gamma^{\mu\nu} \partial_1\theta \right) \\
&\quad - \partial_1 A_0^\mu - 2 n_\mu \phi \Big\} d\tau + dA_1^\mu, \qquad (4.76)
\end{aligned}
$$

and

$$
\begin{aligned}
d\pi_\phi = &-\frac{\partial \mathcal{H}'_0}{\partial \phi}\, d\tau - \frac{\partial \mathcal{H}'_\theta}{\partial \phi}\, d\theta - \frac{\partial \mathcal{H}'_{\bar{\theta}}}{\partial \phi}\, d\bar{\theta} - \frac{\partial \mathcal{H}'_{g^{ab}}}{\partial \phi}\, dg^{ab} - \frac{\partial \mathcal{H}'_{A_0^\mu}}{\partial \phi}\, dA_0^\mu \\
&- \frac{\partial \mathcal{H}'_{A_1^\mu}}{\partial \phi}\, dA_1^\mu - \frac{\partial \mathcal{H}'_{\xi_0}}{\partial \phi}\, d\xi_0 - \frac{\partial \mathcal{H}'_{\xi_1}}{\partial \phi}\, d\xi_1 - \frac{\partial \mathcal{H}'_{n^\mu}}{\partial \phi}\, dn_\mu - \frac{\partial \mathcal{H}'_\phi}{\partial A_1^\mu}\, d\phi \\
= &-\left(n^2+1\right) d\tau. \qquad (4.77)
\end{aligned}
$$

To check whether the set of equations (4.67) to (4.77) is integrable or not, let us consider the total variations of the set of (HJPDE)'s. The variations of relations (4.57)-(4.66) is listed as follows:

$$
\begin{aligned}
d\mathcal{H}'_0 = &\Big\{ N\left(\partial_1 x_\mu - i\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_1\theta\right) - N_1\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma^\mu\partial_1\theta + \xi_1 n_\mu\right) \\
&- \xi_0 n_\mu \Big\} d\left(\partial_1 x_\mu\right), \qquad (4.78)
\end{aligned}
$$

$$
d\mathcal{H}'_\theta = \left(i\bar{\theta}\Gamma^\mu\right) d\left(\partial_1 x_\mu\right), \qquad (4.79)
$$

$$
\begin{aligned}
d\mathcal{H}'_{\bar{\theta}} = &-\Big\{ N\Big[\mathcal{P}_\mu\left(i\Gamma^\mu\partial_1\theta\right) + \xi_1 n_\mu\left(i\Gamma^\mu\partial_1\theta\right) + \left(\partial_1 x_\mu\right)\left(i\Gamma^{\mu\nu} n_\nu\partial_1\theta\right)\Big] \\
&- N_1\Big[\mathcal{P}_\mu\left(i\Gamma^{\mu\nu} n_\nu\partial_1\theta\right) + \xi_1 n_\mu\left(i\Gamma^{\mu\nu} n_\nu\partial_1\theta\right) - \left(\partial_1 x_\mu\right)\left(i\Gamma^\mu\partial_1\theta\right)\Big] \\
&- \xi_0 n_\mu\left(i\Gamma^{\mu\nu} n_\nu\partial_1\theta\right)\Big\} d\tau \equiv \mathcal{H}''_{\bar{\theta}} d\tau, \qquad (4.80)
\end{aligned}
$$

$$
\begin{aligned}
d\mathcal{H}'_{g^{ab}} = &\frac{1}{2}\Big\{\Big[\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)^2 + \Pi_{1\mu}\Pi_1^\mu\Big]\left(\frac{\partial N}{\partial g^{ab}}\right) \\
&+ \Big[\left(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\right)\Pi_{1\mu}\Big]\left(\frac{\partial N_1}{\partial g^{ab}}\right)\Big\} d\tau \\
\equiv &\, \mathcal{H}''_{g^{ab}} d\tau, \qquad (4.81)
\end{aligned}
$$

$$
d\mathcal{H}'_{\xi_0} = \Big\{ n_\mu\left(\partial_1 x_\mu + i\bar{\theta}\Gamma^{\mu\nu} n_\nu\partial_1\theta\right)\Big\} d\tau \equiv \mathcal{H}''_{\xi_0} d\tau \qquad (4.82)
$$

$$
\begin{aligned}
d\mathcal{H}'_{\xi_1} &= \Big\{ N\Big(\mathcal{P}_\mu - i\bar{\theta}\Gamma^\mu\partial_1\theta + \xi_1 n_\mu\Big)n_\mu + N_1\Pi_{1\mu}n^\mu \Big\} d\tau \\
&\equiv \mathcal{H}''_{\xi_1} d\tau, &(4.83) \\
d\mathcal{H}'_{A_0^\mu} &= 0, &(4.84) \\
d\mathcal{H}'_{A_1^\mu} &= 0, &(4.85) \\
d\mathcal{H}'_{n^\mu} &= \Big\{ N\Big(\mathcal{P}_\mu - i\bar{\theta}\Gamma^\mu\partial_1\theta + \xi_1 n_\mu\Big)\xi_1 - N\partial_1 x_\mu\Big(i\bar{\theta}\Gamma^\mu\partial_1\theta\Big) \\
&- N_1\xi_1 n_\mu\Big(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta\Big) + N_1\xi_1\Big(\partial_1 x_\mu - i\bar{\theta}\Gamma^{\mu\nu}n_\nu\partial_1\theta\Big) \\
&+ \xi_0\Big(\partial_1 x_\mu + i\bar{\theta}\Gamma^{\mu\nu}n_\nu\partial_1\theta\Big) + \xi_0 n_\mu\Big(i\bar{\theta}\Gamma^{\mu\nu}\partial_1\theta\Big) \\
&- \partial_1 A_0^\mu - 2n_\mu\phi \Big\} d\tau \equiv \mathcal{H}''_{n^\mu} d\tau, &(4.86) \\
d\mathcal{H}'_\phi &= -\big(n^2 + 1\big) d\tau \equiv \mathcal{H}''_\phi d\tau. &(4.87)
\end{aligned}
$$

Further analysis gives the constraints $\mathcal{H}''_{\bar{\theta}}$, $\mathcal{H}''_{g^{ab}}$, $\mathcal{H}''_{\xi_0}$, $\mathcal{H}''_{\xi_1}$, $\mathcal{H}''_{n^\mu}$ and $\mathcal{H}''_\phi$, where

$$
\begin{aligned}
\mathcal{H}''_{\bar{\theta}} = -\Big\{ & N\Big[\mathcal{P}_\mu\Big(i\Gamma^\mu\partial_1\theta\Big) + \xi_1 n_\mu\Big(i\Gamma^\mu\partial_1\theta\Big) + \Big(\partial_1 x_\mu\Big)\Big(i\Gamma^{\mu\nu}n_\nu\partial_1\theta\Big)\Big] \\
& - N_1\Big[\mathcal{P}_\mu\Big(i\Gamma^{\mu\nu}n_\nu\partial_1\theta\Big) + \xi_1 n_\mu\Big(i\Gamma^{\mu\nu}n_\nu\partial_1\theta\Big) - \Big(\partial_1 x_\mu\Big)\Big(i\Gamma^\mu\partial_1\theta\Big)\Big] \\
& - \xi_0 n_\mu\Big(i\Gamma^{\mu\nu}n_\nu\partial_1\theta\Big)\Big\}, &(4.88) \\
\mathcal{H}''_{g^{ab}} = \frac{1}{2}\Big\{ & \Big[\Big(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\Big)^2 + \Pi_{1\mu}\Pi_1^\mu\Big]\Big(\frac{\partial N}{\partial g^{ab}}\Big) \\
& + \Big[\Big(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu\partial_1\theta + \xi_1 n_\mu\Big)\Pi_{1\mu}\Big(\frac{\partial N_1}{\partial g^{ab}}\Big)\Big]\Big\}, &(4.89)
\end{aligned}
$$

$$\mathcal{H}''_{\xi_0} = \left\{ n_\mu \Big(\partial_1 x_\mu + i\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_1 \theta \Big) \right\} = 0, \tag{4.90}$$

$$\mathcal{H}''_{\xi_1} = \left\{ N \Big(\mathcal{P}_\mu - i\bar{\theta}\Gamma^\mu \partial_1 \theta + \xi_1 n_\mu \Big) n_\mu + N_1 \Pi_{1\mu} n^\mu \right\} = 0, \tag{4.91}$$

$$\begin{aligned} \mathcal{H}''_{n^\mu} = \Big\{ & N \Big(\mathcal{P}_\mu - i\bar{\theta}\Gamma^\mu \partial_1 \theta + \xi_1 n_\mu \Big) \xi_1 - N \partial_1 x_\mu \Big(i\bar{\theta}\Gamma^\mu \partial_1 \theta \Big) \\ & - N_1 \xi_1 n_\mu \Big(i\bar{\theta}\Gamma^{\mu\nu} \partial_1 \theta \Big) + N_1 \xi_1 \Big(\partial_1 x_\mu - i\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_1 \theta \Big) \\ & + \xi_0 \Big(\partial_1 x_\mu + i\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_1 \theta \Big) + \xi_0 n_\mu \Big(i\bar{\theta}\Gamma^{\mu\nu} \partial_1 \theta \Big) \\ & - \partial_1 A_0^\mu - 2 n_\mu \phi \Big\} = 0, \end{aligned} \tag{4.92}$$

and

$$\mathcal{H}''_\phi = -\big(n^2 + 1\big) = 0. \tag{4.93}$$

However, a variation of the total derivatives of (4.88)-(4.93) are

$$d\mathcal{H}''_{\bar{\theta}} = -\left\{ N \Big(i\Gamma^{\mu\nu} n_\nu \partial_1 \theta \Big) - N_1 \Big(i\Gamma^\mu \partial_1 \theta \Big) \right\} d\Big(\partial_1 x_\mu \Big), \tag{4.94}$$

$$\begin{aligned} d\mathcal{H}''_{g^{ab}} = \Big\{ & \Big(\partial_1 x_\mu + i\bar{\theta}\Gamma^{\mu\nu} n_\nu \partial_1 \theta \Big) \left(\frac{\partial N}{\partial g^{ab}} \right) \\ & + \Big(\mathcal{P}_\mu - i\bar{\theta}\Gamma_\mu \partial_1 \theta + \xi_1 n_\mu \Big) \left(\frac{\partial N_1}{\partial g^{ab}} \right) \Big\} d\Big(\partial_1 x_\mu \Big), \end{aligned} \tag{4.95}$$

$$d\mathcal{H}''_{\xi_0} = n_\mu d\Big(\partial_1 x_\mu \Big), \tag{4.96}$$

$$d\mathcal{H}''_{\xi_1} = N_1 n_\mu d\Big(\partial_1 x_\mu \Big), \tag{4.97}$$

$$d\mathcal{H}''_{n^\mu} = \left\{ N \Big(i\bar{\theta}\Gamma_{\mu\nu} \partial_1 \theta \Big) - N_1 \xi_1 - \xi_0 \right\} d\Big(\partial_1 x_\mu \Big), \tag{4.98}$$

and

$$d\mathcal{H}''_\phi = 0. \tag{4.99}$$

However, the variations of constraints $\mathcal{H}'_0$, $\mathcal{H}'_\theta$, $\mathcal{H}''_{\bar{\theta}}$,$\mathcal{H}''_{g^{ab}}$,$\mathcal{H}''_{\xi_0}$,$\mathcal{H}''_{\xi_1}$ and $\mathcal{H}''_{n^\mu}$ do not vanish identically, a new set of constraints arises. According to the theory, this new set of constraints is added to the equations of motion of the system.

Using the open string boundary conditions [31], $x'^\mu = \partial_1 x_\mu = 0$, for $\sigma = 0$ and $\sigma = \pi$ the variations of the new constraints (4.88)-(4.93) are identically zero, and the system is integrable.

4.3 Hamilton-Jacobi Formulation of $D = 11$ Superstring

In this section we study the superstring problem in a more simple framework of mechanical model. Our starting point is the following Lagrangian action [32],

$$S = \int d\tau \left\{ \frac{1}{2e} \Pi^\mu \Pi_\mu + n^\mu \dot{z}^\mu - \frac{1}{2} \phi \big(n^2 + 1 \big) \right\}. \tag{4.100}$$

with

$$\Pi^\mu \equiv \dot{x}^\mu - i \big(\bar{\theta} \Gamma^{\nu\mu} \dot{\theta} \big) n_\nu - \xi n^\mu. \tag{4.101}$$

where x^μ, n^μ, e, ϕ and ξ are Grassmann even variables and θ^α are Grassmann odd variables, dependent on the evolution parameter τ . The singularity of the Lagrangian

$$L = \frac{1}{2e} \Pi^\mu \Pi_\mu + n_\mu \dot{z}^\mu - \phi \big(n^2 + 1 \big), \tag{4.102}$$

follows from the fact that the rank of the Hess matrix A_{ij} is one.

The Hamiltonian analysis begin with the introduction of the momenta variables according to equations (1.2) and (1.3)

$$P^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{e}\left\{\dot{x}^\mu - i\Big(\bar{\theta}\Gamma^{\mu\nu}\dot{\theta}\Big)n_\nu - \xi n^\mu\right\}, \tag{4.103}$$

$$\pi_\theta = \frac{\partial_r L}{\partial\dot{\theta}} = -i\Big(\bar{\theta}\Gamma^{\mu\nu}\Big)n_\nu P_\mu = -H_\theta, \tag{4.104}$$

$$\pi_{\bar{\theta}} = \frac{\partial_r L}{\partial\dot{\bar{\theta}}} = 0 = -H_{\bar{\theta}}, \tag{4.105}$$

$$P_n^\mu = \frac{\partial L}{\partial \dot{n}^\mu} = 0 = -H_n^\mu, \tag{4.106}$$

$$P_\xi = \frac{\partial L}{\partial\dot{\xi}} = 0 = -H_\xi, \tag{4.107}$$

$$P_e = \frac{\partial L}{\partial \dot{e}} = 0 = -H_e, \tag{4.108}$$

$$P_z^\mu = \frac{\partial L}{\partial \dot{z}^\mu} = n^\mu = -H_z^\mu, \tag{4.109}$$

and

$$\pi_\phi = \frac{\partial L}{\partial\dot{\phi}} = 0 = -H_\phi. \tag{4.110}$$

Since the rank of the Hess matrix is one, we can solve (4.103) for $\dot{x}^\mu$ in terms of P^μ and other coordinates

$$\dot{x}^\mu = eP^\mu + i\Big(\bar{\theta}\Gamma^{\mu\nu}\dot{\theta}\Big)n_\nu + \xi n^\mu. \tag{4.111}$$

A straightforward calculation shows that the canonical Hamiltonian H_0 is obtained as

$$H_0 = \frac{1}{2}eP^2 + \phi\big(n^2+1\big) + P^\mu \xi n_\mu. \tag{4.112}$$

Following the Hamilton-Jacobi formalism we obtain the set of (HJPDE)'s,

$$H_0' = P_0 + \frac{1}{2}eP^2 + \phi\big(n^2 + 1\big) + P^\mu \xi n_\mu, \tag{4.113}$$

$$H_\theta' = \pi_\theta + i\mu\bar{\theta}\Gamma^{\mu\nu} n_\nu P_\mu, \tag{4.114}$$

$$H_{\bar{\theta}}' = \pi_{\bar{\theta}}, \tag{4.115}$$

$$H_n'^{\,\mu} = P_n^\mu, \tag{4.116}$$

$$H_\xi' = P_\xi, \tag{4.117}$$

$$H_e' = P_e, \tag{4.118}$$

$$H_z'^{\,\mu} = P_z^\mu - n^\mu, \tag{4.119}$$

and

$$H_\phi' = \pi_\phi. \tag{4.120}$$

Therefore, the total differential equations for the characteristic equations (1.10) and (1.11) read as

$$\begin{aligned} dx^\mu &= \frac{\partial H_0'}{\partial P_\mu}\, d\tau + \frac{\partial H_\theta'}{\partial P_\mu}\, d\theta_i^\alpha + \frac{\partial H_{\bar{\theta}}'}{\partial P_\mu}\, d\bar{\theta}^{\dot{\alpha}i} + \frac{\partial H_{n_\nu}'}{\partial P_\mu}\, dn^\nu \\ &\quad + \frac{\partial H_\xi'}{\partial P_\mu}\, d\xi + \frac{\partial H_\xi'}{\partial P_\mu}\, de + \frac{\partial H_{z_\nu}'}{\partial P_\mu}\, dz^\nu + \frac{\partial H_\phi'}{\partial P_\mu}\, d\phi \\ &= \Big(eP^\mu + \xi n^\mu\Big)d\tau + i\Big(\bar{\theta}\Gamma^{\mu\nu} n_\nu\Big)d\theta, \end{aligned} \tag{4.121}$$

$$\begin{aligned} dP_0 &= -\frac{\partial H_0'}{\partial \tau}\, d\tau - \frac{\partial H_\theta'}{\partial \tau}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial \tau}\, d\bar{\theta} - \frac{\partial H_{n_\nu}'}{\partial \tau}\, dn^\nu \\ &\quad - \frac{\partial H_\xi'}{\partial \tau}\, d\xi - \frac{\partial H_\xi'}{\partial \tau}\, de - \frac{\partial H_{z_\nu}'}{\partial \tau}\, dz^\nu - \frac{\partial H_\phi'}{\partial \tau}\, d\phi \\ &= 0, \end{aligned} \tag{4.122}$$

$$\begin{aligned}
dP^\mu &= -\frac{\partial H_0'}{\partial x_\mu}\,d\tau - \frac{\partial H_\theta'}{\partial x_\mu}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial x_\mu}\,d\bar\theta - \frac{\partial H_{n_\nu}'}{\partial x_\mu}\,dn^\nu \\
&\quad - \frac{\partial H_\xi'}{\partial x_\mu}\,d\xi - \frac{\partial H_\xi'}{\partial x_\mu}\,de - \frac{\partial H_{z_\nu}'}{\partial x_\mu}\,dz^\nu - \frac{\partial H_\phi'}{\partial x_\mu}\,d\phi \\
&= 0,
\end{aligned} \tag{4.123}$$

$$\begin{aligned}
d\pi_\theta &= -\frac{\partial H_0'}{\partial \theta}\,d\tau - \frac{\partial H_\theta'}{\partial \theta}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial \theta}\,d\bar\theta - \frac{\partial H_{n^\mu}'}{\partial \theta}\,dn^\mu \\
&\quad - \frac{\partial H_\xi'}{\partial \theta}\,d\xi - \frac{\partial H_\xi'}{\partial \theta}\,de - \frac{\partial H_{z^\mu}'}{\partial \theta}\,dz^\mu - \frac{\partial H_\phi'}{\partial \theta}\,d\phi \\
&= 0,
\end{aligned} \tag{4.124}$$

$$\begin{aligned}
d\pi_{\bar\theta} &= -\frac{\partial H_0'}{\partial \bar\theta}\,d\tau - \frac{\partial H_\theta'}{\partial \bar\theta}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial \bar\theta}\,d\bar\theta - \frac{\partial H_{n^\mu}'}{\partial \bar\theta}\,dn^\mu \\
&\quad - \frac{\partial H_\xi'}{\partial \bar\theta}\,d\xi - \frac{\partial H_\xi'}{\partial \bar\theta}\,de - \frac{\partial H_{z^\mu}'}{\partial \bar\theta}\,dz^\mu - \frac{\partial H_\phi'}{\partial \bar\theta}\,d\phi \\
&= \Big(iP_\mu \Gamma^{\mu\nu} n_\nu\Big)\,d\theta,
\end{aligned} \tag{4.125}$$

$$\begin{aligned}
dP_n^\mu &= -\frac{\partial H_0'}{\partial n^\mu}\,d\tau - \frac{\partial H_\theta'}{\partial n^\mu}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial n^\mu}\,d\bar\theta - \frac{\partial H_{n^\mu}'}{\partial n_\nu}\,dn^\nu \\
&\quad - \frac{\partial H_\xi'}{\partial n^\mu}\,d\xi - \frac{\partial H_\xi'}{\partial n^\mu}\,de - \frac{\partial H_{z_\nu}'}{\partial n^\mu}\,dz^\nu - \frac{\partial H_\phi'}{\partial n^\mu}\,d\phi \\
&= -\Big(2\phi n_\mu + \xi P_\mu\Big)d\tau - \Big(iP_\mu \Gamma^{\mu\nu}\Big)d\theta + dz^\mu,
\end{aligned} \tag{4.126}$$

$$\begin{aligned}
dP_\xi &= -\frac{\partial H_0'}{\partial \xi}\,d\tau - \frac{\partial H_\theta'}{\partial \xi}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial \xi}\,d\bar\theta - \frac{\partial H_{n^\mu}'}{\partial \xi}\,dn^\mu \\
&\quad - \frac{\partial H_\xi'}{\partial \xi}\,d\xi - \frac{\partial H_\xi'}{\partial \xi}\,de - \frac{\partial H_{z^\mu}'}{\partial \xi}\,dz^\mu - \frac{\partial H_\phi'}{\partial \xi}\,d\phi \\
&= -\Big(P_\mu n^\mu\Big)d\tau,
\end{aligned} \tag{4.127}$$

$$\begin{aligned}
dP_e &= -\frac{\partial H_0'}{\partial e}\,d\tau - \frac{\partial H_\theta'}{\partial e}\,d\theta - \frac{\partial H_{\bar\theta}'}{\partial e}\,d\bar\theta - \frac{\partial H_{n^\mu}'}{\partial e}\,dn^\mu \\
&\quad - \frac{\partial H_\xi'}{\partial e}\,d\xi - \frac{\partial H_\xi'}{\partial e}\,de - \frac{\partial H_{z^\mu}'}{\partial \xi}\,dz^\mu - \frac{\partial H_\phi'}{\partial e}\,d\phi \\
&= -\Big(\frac{1}{2}\,P^2\Big)d\tau,
\end{aligned} \tag{4.128}$$

$$
\begin{aligned}
dP_z^{\mu} &= -\frac{\partial H_0'}{\partial z^{\mu}}\, d\tau - \frac{\partial H_{\theta}'}{\partial z^{\mu}}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial z^{\mu}}\, d\bar{\theta} - \frac{\partial H_{n_{\nu}}'}{\partial z^{\mu}}\, dn^{\nu} \\
&\quad - \frac{\partial H_{\xi}'}{\partial z^{\mu}}\, d\xi - \frac{\partial H_{\xi}'}{\partial z^{\mu}}\, de - \frac{\partial H_{z_{\nu}}'}{\partial z^{\mu}}\, dz^{\nu} - \frac{\partial H_{\phi}'}{\partial z^{\mu}}\, d\phi \\
&= 0,
\end{aligned} \tag{4.129}
$$

and

$$
\begin{aligned}
dP_{\phi} &= -\frac{\partial H_0'}{\partial \phi}\, d\tau - \frac{\partial H_{\theta}'}{\partial \phi}\, d\theta - \frac{\partial H_{\bar{\theta}}'}{\partial \phi}\, d\bar{\theta} - \frac{\partial H_{n^{\mu}}'}{\partial z^{\mu}} dn^{\mu} \\
&\quad - \frac{\partial H_{\xi}'}{\partial \phi}\, d\xi - \frac{\partial H_{\xi}'}{\partial \phi}\, de - \frac{\partial H_{z^{\mu}}'}{\partial \phi}\, dz^{\mu} - \frac{\partial H_{\phi}'}{\partial \phi}\, d\phi \\
&= -\Big(n^2 + 1\Big) d\tau.
\end{aligned} \tag{4.130}
$$

To check whether the set of equations (4.121) to (4.130) are integrable or not, let us consider the total variations of the set of (HJPDE)'s. The variations of

$$dH_0' = 0, \tag{4.131}$$

$$dH_{\theta}' = 0, \tag{4.132}$$

$$dH_{\bar{\theta}}' = 0, \tag{4.133}$$

and

$$dH_z'^{\,\mu} = 0, \tag{4.134}$$

are identically zero, whereas the variations of

$$dH_n'^{\,\mu} = -\Big(2\phi n_{\mu} + \xi P_{\mu}\Big) d\tau - \Big(i P_{\mu} \Gamma^{\mu\nu}\Big) d\theta + dz^{\mu} \equiv H_n''^{\mu} d\tau, \tag{4.135}$$

$$dH_{\xi}' = -\Big(P_{\mu} n^{\mu}\Big) d\tau \equiv H_{\xi}'' d\tau, \tag{4.136}$$

$$dH_e' = -\Big(\frac{1}{2} P^2\Big) d\tau \equiv H_e'' d\tau, \tag{4.137}$$

and

$$dH'_\phi = -\Big(n^2 + 1\Big)d\tau \equiv H''_\phi d\tau. \tag{4.138}$$

are not. Therefore we obtain the following set of additional constraints:

$$H''^{\mu}_n = -\Big(2\phi n_\mu + \xi P_\mu\Big), \tag{4.139}$$

$$H''_\xi = -\Big(P_\mu n^\mu\Big), \tag{4.140}$$

$$H''_7 = -\Big(\frac{1}{2}P^2\Big), \tag{4.141}$$

and

$$H''_9 = -\Big(n^2 + 1\Big). \tag{4.142}$$

One notices that the total differentials of H''^{μ}_n, H''_ξ, H''_e and H''_ϕ vanish identically, *i.e.*

$$dH''^{\mu}_n = 0, \tag{4.143}$$

$$dH''_\xi = 0, \tag{4.144}$$

$$dH''_e = 0, \tag{4.145}$$

and

$$dH''_\phi = 0. \tag{4.146}$$

Thus the equations of motion (4.121) to (4.130) and the new constraints (4.139) to (4.142) represent an integrable system. Since the equations of motion are integrable, the action can be written as

$$S = \int d\tau \Big\{\frac{1}{2}eP^2 + n_\mu \dot{z}^\mu - \phi\Big(n^2 + 1\Big)\Big\}. \tag{4.147}$$

Chapter 5

Discussion and Conclusion

The main motivation of this study is to make a formal generalization of Hamilton-Jacobi formalism for constrained systems to be applicable to new areas in physics, such as supersymmetric systems which including different examples of the superparticles and superstring. In studying the superparticle and superstring theories in flat superspace, the dynamical systems subject to mixed fermionic first and second class constraints in an arbitrary space-time dimension, which are difficult to be separated by Dirac's method, so we must introduce a gauge transforation, which is difficult to be specified. The advantage of the Hamilton-Jacobi formalism is that we have no difference between first and second class constraints and we do not need gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equations.

Chapter *2* presents several examples of superparticle, such as massive, massless and spinning superparticle. The Hamilton-Jacobi formalism of the Brink-Schwarz superparticle present obstacles related to the mixing of first and second class constraints, the equations of motion are obtained as total differential equations in many variables, and the integrability conditions are examined to be identically zero. In the same manner, we introduced the equations of motion of a massless Siegel superparticle and the result of our work in this convention in the light of Dirac's method [25]. In the last example, a spinning particle mechanics was enlarged by rigid space-time supersymmetry and some spinning superparticle models with interesting features were studied, and the equations of motion are obtained in form of the spinning particle added to the superparticle sectors with a common bosonic part.

The electromagnetic interaction of massive superparticles with $N = 2$ external superpotential was studied in Chapter *3*. The equations of motion are obtained for the constrained system by applying the Hamilton-Jacobi formalism. One notices that our formalism does not depend on the N-extended supersymmetric which introduces extra degrees of freedom. In Hamilton-jacobi formalism of the classical constraints, no need for gauge fixing of first-class constraints, no need to eliminate second-class constraints, such as in Dirac method [37].

In chapter *4*, superstring is investigated by using the Hamilton-Jacobi formalism. The classical mechanics of $4D$ superstring is introduced, and the total variations of (HJPDE's) are checked. The seconde example is

Green-Schwarz Superstring. The manifest supersymmetric formulation of Green-Schwarz superstring is very difficult to quantize because of the entanglement between first- and second-class constraints and the infinitely gauge symmetry[38]. But by applying the Hamilton-Jacobi formulation which does not differentiate between the first and second class constraints; consequently, there is no difficulty in treating the constraints, and we do not need any gauge fixing terms. The system is completely integrable if and only if we use the boundary condition of the string as a constraint on such system. Finally, we present $D = 11$ action for mechanical system, the constraints of a dynamical system are expressed as the set of (HJPDE)'s, then the equations of motion are obtained. The integrability conditions are examined until a complete system is obtained.

Appendix A

Grassmann Variables

Grassmannian variables refer to anticommuting variables θ_α (α is a spinor indices), that is, the variables obeying the relations

$$\theta_\alpha \, \theta_\beta = - \theta_\beta \, \theta_\alpha; \qquad \alpha \neq \beta, \tag{A.1}$$

$$\bar{\theta}_{\dot{\alpha}} \, \bar{\theta}_{\dot{\beta}} = - \bar{\theta}_{\dot{\beta}} \, \bar{\theta}_{\dot{\alpha}}; \qquad \dot{\alpha} \neq \dot{\beta}, \tag{A.2}$$

$$(\theta_\alpha)^2 = (\bar{\theta}_{\dot{\alpha}})^2 = 0. \tag{A.3}$$

The generators of a n-dimensional Grassmann algebra are anticommuting classical variables θ_α with ($\alpha = 1, 2, ..., n$),

$$\left\{\theta_\alpha, \, \theta_\beta\right\} = \theta_\alpha \, \theta_\beta + \theta_\beta \, \theta_\alpha = 0 \tag{A.4}$$

A general element of a Grassmann algebra is defined as a power series of the generators. Since $(\theta_\alpha)^2 = 0$, the power series has only a finite number of elements.

$$f(\theta) = f_0 + \sum_{\alpha} f_\alpha \, \theta_\alpha + \sum_{\alpha\beta} f_{\alpha\beta} \, \theta_\alpha \theta_\beta + ... + f_{12...n} \theta_1 \theta_2 ... \theta_n. \tag{A.5}$$

The $f_{\alpha\beta...n}$ are ordinary complex numbers, which are antisymmetric in $\alpha, \beta, ..., n$. The integration rules for Grassmann variables are

$$\int d\theta_\alpha = 0, \quad \int d\theta_\alpha \, \theta_\alpha = 1 = -\int \theta_\alpha d\theta_\alpha. \tag{A.6}$$

and the differentiation is defined by

$$\frac{\partial}{\partial \theta_\alpha} K = 0; \qquad K \text{ is constant}, \tag{A.7}$$

$$\frac{\partial}{\partial \theta_\alpha} \theta_\alpha = 1, \tag{A.8}$$

$$\frac{\partial}{\partial \theta_\alpha} (\theta_\beta \theta_\gamma) = \delta_{\alpha\beta} \theta_\gamma - \delta_{\alpha\gamma} \theta_\beta. \tag{A.9}$$

Also we shall introduced a number of the relations of left and right derivatives

$$\frac{\partial_{l,r}}{\partial \theta_\alpha} \frac{\partial_{l,r}}{\partial \theta_{\alpha'}} (\Omega) = (-1)^{n_\alpha n_{\alpha'}} \frac{\partial_{l,r}}{\partial \theta_{\alpha'}} \frac{\partial_{l,r}}{\partial \theta_\alpha} (\Omega). \tag{A.10}$$

$$\frac{\partial_l}{\partial \theta_\alpha} \frac{\partial_r}{\partial \theta_{\alpha'}} (\Omega) = \frac{\partial_r}{\partial \theta_{\alpha'}} \frac{\partial_l}{\partial \theta_\alpha} (\Omega). \tag{A.11}$$

$$\frac{\partial_l}{\partial \theta_\alpha} (\Omega_1 \Omega_2) = \left(\frac{\partial_l}{\partial \psi_\alpha} \Omega_1 \right) \Omega_2 + (-1)^{n_{\Omega_1} n_\alpha} \Omega_1 \left(\frac{\partial_l}{\partial \psi_\alpha} \Omega_2 \right). \tag{A.12}$$

$$\frac{\partial_r}{\partial \theta_\alpha} (\Omega_1 \Omega_2) = (-1)^{n_{\Omega_2} n_\alpha} \left(\frac{\partial_r}{\partial \psi_\alpha} \Omega_1 \right) \Omega_2 + \Omega_1 \left(\frac{\partial_r}{\partial \psi_\alpha} \Omega_2 \right). \tag{A.13}$$

where $\Omega, \Omega 1$ and Ω_2 are elements of definite parity. One notices that the derivative of an even (odd) element is odd (even).

Bibliography

[1] Dirac P. A. M., *Gereralized Hamiltonian Dynamics*, Can. J. Math. Phys. **2**, (1950) 129.

[2] Dirac P. A. M., *Lectures on Quantum Mechanics*, Yashiva University, New York, (1964).

[3] Sundermeyer K., *Lecture notes in physics*, Spring-Verlag, Berlin, (1982).

[4] Jan Govaerts, *Hamiltonian Quantisation and Constrsined Dynamics*, Volume 4, Leuven University Press, (1991).

[5] Gitman D. M. and Tyutin I. V., *Quantization of Fields with Constraints* (Spring-Verlag, Berlin, Heidelbag) (1990).

[6] Güler Y., Nuovo Cimento **B107**, (1992) 1389.

[7] Güler Y., Nuovo Cimento **B107**, (1992) 1143.

[8] Carathéodory C., *Calculus of Variations and Partial Differential Equations of First Order*, Part II (Holden- Day) (1967).

[9] Farahat N. I. and Güler Y., Nuovo Cimento **B111**, (1996) 513.

[10] Rabei E.M., and Tawfiq S., Hadronic J, **20**, (1997) 2399.

[11] Muslih S. I., General Relativity and Gravitation, **34**, (2002) 1059.

[12] Farahat N. I. and Nassar Z., Islamic University Journal, **13**, (2005) 239.

[13] Farahat N. I. and Nassar Z., Hadronic Journal **25**, (2002) 239.

[14] Muslih S. I. and Güler Y., Nuovo Cimento **B113**, (1998) 277.

[15] Muslih S. I. and Güler Y., Nuovo Cimento **B112**,(1997) 97.

[16] Muslih S. I., Nuovo Cimento **B115**, (2000)1.

[17] Muslih S. I., Nuovo Cimento **B115**, (2000)7.

[18] Muslih S. I., Nuovo Cimento **B117**, (2002)4.

[19] Muslih S. I., Nuovo Cimento **B118** (2003) 505.

[20] Muslih S. I., S. I. Muslih, Modern Physics Letter **A**, **19**, (2004) 863.

[21] Weinberg S. , *The Quantum Theory of Fields* volume III, Cambridge, (2000).

[22] Polchinski J. , *An Introduction to the Bosonic String* volume 1 : *Superstring Theory and Beyond*volume 2, Cambridge, (1998).

[23] Zwiebach B., *A First Course in String Theory*, Cambridge (2004).

[24] Brink L. and Schwarz J. H. , Phys. Lett. **100B**, (1981) 310.

[25] Galajinsky A.V. and Gitman D.M., Nucl. Phys. **B536**, (1998) 435.

[26] Farahat N. I. and Elegla H. A., Turkish Journal of Physics ,**30**, (2006) 473.

[27] Kowalski-Glikman J., van Holten J. W., Aoyama S. and Lukierski J., Phys. Lett. **B201**, (1988) 487.

[28] Nassar Z., Canonical Formulation of Singular Systems, M. Sc. Thesis, Islamic University, (2002).

[29] Uvarov D. V. and Zheltukhin A. A.,Phys. Rev. **D 61**, (1999)15004.

[30] Deriglazov A. A. and Galajinsky A. V.,Phys. Rev. **D 54**, (1996) 5195.

[31] Green M. B. , Schwarz J. H., and Witten E., *Superstring theory*, volume 1, (Cambridge Monographs on Mathematical Physics), (1987).

[32] Gitman D.M. and Deriglazov A. A., International Journal of Modern Physics **A 14**, (1999) 2769.

[33] Gitman D.M. and Deriglazov A. A., Int. J. Mod. Phys. **A 13**, (1998)2559.

[34] Deriglazov A. A. , Galajinsky A. V., Modern Physics Letters **A 12**, (1997) 2993.

[35] Deriglazov A. A. , Galajinsky A. V. , and Gitman D. M. , Phys. Rev. **D 59**, (1999) 48902.

[36] Deo B. B., Phys. Lett. **B 557**, (2003) 115.

[37] Uvarov D. V., Modern Physics Letters **A 18**, 2003) 1611.

[38] Gaona A. and García J. A., High Energy Phys. **09** (2005) 083.