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Methods in The Treatment of Singular Lagrangian

طرق في معالجات اللاجرانج الاحادي

by

Theeb Y. Alsultan

Supervised by

Prof. Dr. Nasser I. Farahat

Professor of Physics

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إقــــــرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Methods in The Treatment of Singular Lagrangian

طرق في معالجات اللاجرانج الاحادي

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بناءً على موافقة عمادة البحث العلمي والدراسات العليا بالجامعة الإسلامية بغزة على تشكيل لجنة الحكم على أطروحة الباحث/ **ذيب ياسر ذيب السلطان** لنيل درجة الماجستير في **كلية *العلوم* قسم <u>الفيزياء</u>** وموضوعها:

طرق في معالجات اللاجرانج الآحادي

Methods in The Treatments of Singular Lagrangian

وبعد المناقشة العلنية التي تمت اليوم الأربعاء 07 محرم 1439هـ، الموافق 2017/09/27م الساعة الواحدة والنصف ظهراً في قاعة مؤتمرات مبنى القدس، اجتمعت لجنة الحكم على الأطروحة والمكونة من:

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وبعد المداولة أوصت اللجنة بمنح الباحث درجة الماجستير في **كلية *العلوم*/ قسم <u>الفيزياء</u>.**

واللجنة إذ تمنحه هذه الدرجة فإنها توصيه بتقوى الله ولزوم طاعته وأن يسخر علمه في خدمة دينه ووطنه.

والله ولي التوفيق،،،

عمادة البحث العلمي والدراسات العليا

أ.د. مازن اسماعيل هنية

To My Parents, wife,

daughter"Tala"

Familly,,,, to

Those who tought me

Abstract

In this thesis, the singular Lagrangian is studied by three approaches. The Hamiltonian formalism is treated using both Dirac's method and Güler method (Hamilton-Jacobi method). The third approach is to treat the singular Lagrangian as field (or continuous) system.

In Dirac's method, one introduces a primary constraints to the first - class constraints which have vanishing poisson brackets. The equation of motion are obtained as total derivatives in terms of poisson brackets.

In Hamilton-Jacobi formulation, which developed by Güler, the equations of motion are written as total differential equations in many variables. These equations must satisfy the integrability conditions.

The third approach is the treating of the singular Lagrangian as field (or continuous) system, We mixed both Lagrangian formulation and Hamilton-Jacobi method to obtain a solvable partial differential equation of second order. The solution of these equations are obtained easily. These solution satisfied the equations of motion.

In these three approach, the equation of motion are built for several physical models and integrability conditions of these equations of motion are discussed. A comparison between the results of these approaches is done and it is shown that the results are the same.

ملخص

تعتمد هذه الاطروحة علي دراسة اللاجرانج الاحادي من خلال ثلاث طرق. في البداية تم معالجة صيغة الهاميلتون باستخدام طريقتين ديراك وجولر ، اما الطريقة الثالثه فتتمثل بمعاملة اللاجرانج الأحادي لنظام متصل (مجال) .

في طريقة ديراك ، نستخدم أحد القيود الاساسية بحيث يتم تلاشي(أقواس بوسون)، ويتم الحصول علي معادلات الحركة من المشتقات الكليه لإقواس بوسون.

في طريقة هاميلتون ـ جاكوبي، التي طورها العالم جولر، تكتب معادلات الحركة كمعادلات تفاضلية بمتغيرات متعددة والتي يجب ان تحقق شروط التكامل.

في طريقة المجال المتصل، تم دمج طريقتى اللاجرانج وطريقة هاميلتون جاكوبي للحصول علي معادلات تفاضلية جزيئة من الدرجة الثانية، ويتم حل هذه المعادلات للحصول علي معادلات حركة.

في الثلاث طرق توصلنا الي بناء معادلات الحركة لعدة نماذج فيزيائية وتم دراسة شروط التكامل لمعادلات الحركة لكل نظام فيزيائي، وتمت المقارنة بين نتائج هذه الطرق وتبين أنً النتائج متطابقة.

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Introduction

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Introduction

1.1 Historical Background

The Hamiltonian formulation of singular systems is usually made through the formalism developed by Dirac (1; 2). In this formalism, the constraints caused by the singularity of Hess matrix $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$, $\quad i, j = 1, ..., n$, are added to the canonical Hamiltonian, and then the consistency conditions are worked out, being possible to eliminate some degrees of freedom of the system. Dirac also showed that the gauge freedom is caused by the presence of first class constraints. This formalism has a wide range of applications in field theory and it is still the main tool for the analysis of singular systems (3; 5). Despite the success of Dirac's method, it is always interesting to apply different formalisms to the analysis of singular systems.

The study of another formalisms for singular lagrangians systems may provide new tools to investigate these systems. In classical dynamics, different formalisms (Lagrangian, Hamiltonian, Hamilton-Jacobi) provide different approaches to the problems, each formalism has advantages and disadvantages in the study of some features of the systems and being equivalent among themselves. In the same way, different formalisms provide different views of the features of singular systems, which justify the interest in their study.

Also in this thesis, we generalize the Hamilton-Jacobi formalism that was developed by Güler (6; 7). This approach based on Carathéodory's equivalent Lagrangian method (8) to write down the Hamilton-Jacobi equations for the system and make use of its singularity to write the equations of motion as total differential equations in many variables . The advantage of the Hamilton-Jacobi formalism is that we have no difference between the first and the second constraints and we do not need gauge-fixing term because the gauge variables are separated in the processes of constructing an integrable system of total differential equation.

In this work we will investigate several models using Dirac and Hamilton-Jacobi (Güler) approaches. Further more, we will treate these models using Lagrangian formalism as field systems. In the following three sections we will give brief review of these formalisms.

1.2 Singular Systems

The singular Lagrangian system represents a special case of a more general dynamics called constrained system (2). The dynamics of the physical system is encoded by the Lagrangian, a function of positions and velocities of all degrees of freedoms which comprise the system . The singular Lagrangian can be achieved by two formulations, the Lagrangian and the Hamiltonian formulations.

This section serves as an initiation to the concept of singularities in the Lagrange formalism. We will introduce some basic notions such as constraints arising due to the singularities and the definition of the canonical momenta. We will start our discussion of Singular Lagrangian systems with the principle of least action. Any physical system can be described by a function L depending on the positions and velocities (9)

$$L = L(q_i(t), \dot{q}_i(t)), \qquad\qquad i = 1, ..., n \tag{1.2.1}$$

We assume, for the sake of simplicity, that this Lagrange function exhibits no explicit time dependence. The abbreviations $q(t)$ and $\dot{q}(t)$ stand for the set of all positions $q_i(t) = q_i(t)$ and velocities $\dot{q}_i(t) = \dot{q}_i(t)$, respectively, with $i = 1, ..., n$. The system motion proceeds in a way that the action integral

$$A = \int_{t_1}^{t_2} d\tau L(q_i(t), \dot{q}_i(t)), \tag{1.2.2}$$

becomes stationary under infinitesimal variations $\delta q_i(t)$. Assuming that the end points are fixed during the variation, i.e. $\delta q_i(t_1) = \delta q_i(t_2) = 0$, yields the equations of motion for the classical path, which is called Euler-Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{1.2.3}$$

Executing the total time derivative gives

$$\ddot{q}_j \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} = \frac{\partial L}{\partial q_i} - \dot{q}_j \frac{\partial^2 L}{\partial q_i \partial q_j}, \tag{1.2.4}$$

In this form we recognize that the accelerations $\ddot{q}_i$ can be uniquely expressed by the position and the velocities $\dot{q}_i$ if and only if the Hess matrix

$$w_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad\qquad i, j = 1, ..., n, \tag{1.2.5}$$

is invertible. In other words its determinant must not vanish.

$$\det w_{ij} \neq 0, \tag{1.2.6}$$

Since we are interested in the Hamiltonian formulation, we have to perform a Legendre transformation from the velocities to the momenta. The latter are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{1.2.7}$$

In the case that the determinant vanishes, the Lagrangian (1.2.1) is singular and some of the accelerations are not determined by the velocities and positions. This means that some of the variables are not independent from each other. The singularity of the Hessian is equivalent to the noninvertibility of (1.2.5). As a consequence, in a singular system we are not able to display the velocities as functions of the momenta and the positions. This gives rise to the existence of relations between the positions and momenta

$$\phi_m(p_i, q_i) = 0, \tag{1.2.8}$$

these relations are called primary constraints in Dirac′s approach. They follow directly from the structure of the Lagrangian and the definition of the momenta (1.2.7). The interesting point is that these functions are real restrictions on the phase space.

1.3 Dirac's Method

The standard methods of classical mechanics can't be applied directly to the singular Lagrangian theories. However, the basic idea of the classical treatment and the quantization of such systems were presented along time by Dirac (1; 2). And is now widely used in investigating the theoretical models in a contemporary elementary particle physics and applied in high energy physics, especially in the gauge theories(5). This is because the first-class constraints are generators of gauge transformation which lead to the gauge freedom (14) Let us consider a system which is described by the Lagrangian (1.2.1) is singular if the rank of the Hess matrix $A_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, ..., n,$ is $r = n - m, m < n$.otherwise, the Lagrangian will be regular The singular system characterized by the fact that all velocities $\dot{q}_i$ are not uniquely determined in terms of the coordinates and momenta only. In other words, not all momenta are independent, and there must exist a certain set of relations among them of the form (1.2.8) The generalized momenta corresponding to the generalized coordinates q_i are defined as

$$p_a = \frac{\partial L}{\partial \dot{q_a}}, \qquad a = 1, ..., n-r, \tag{1.3.1}$$

$$p_\mu = \frac{\partial L}{\partial \dot{q_\mu}}, \qquad \mu = n-r+1, ..., n, \tag{1.3.2}$$

where $\dot{q}_i$ stands for the total derivative with respect to t. The Relations (1.3.2) enable us to write the primary constraint as (1; 2)

$$H'_\mu = P_\mu + H_\mu = 0. \tag{1.3.3}$$

In this formulation the total Hamiltonian is defined as

$$H_T = H_0 + \lambda_\mu H'_\mu, \tag{1.3.4}$$

where the canonical Hamiltonian H_0 is defined as

$$H_0 = p_i \dot{q}_i - L, \qquad i = 1, ..., n, \tag{1.3.5}$$

and λ_μ are arbitrary functions. (Throughout this thesis, we use Einstein's summation rule which means that the repeating of indices indicate to summation). The equations of motion are obtained in term of Poisson brackets as

$$\dot{q}_i = \{q_i, H_T\} = \{q_i, H_\circ\} + \lambda_\mu \{q_i, H'_\mu\}, \tag{1.3.6}$$

$$\dot{p}_i = \{p_i, H_T\} = \{p_i, H_\circ\} + \lambda_\mu \{p_i, H'_\mu\}. \tag{1.3.7}$$

The consistency conditions, which means that the total time derivative of the primary constrains should be identically zero are given as

$$H'_\mu = \{H'_\mu, H_T\} = \{H'_\mu, H_\circ\} + \lambda_\mu \{H'_\mu, H'_\nu\} \approx 0, \tag{1.3.8}$$

where $\mu, \nu = 1, ..., r$. Equations (1.3.6, 1.3.7, 1.3.8) may be identically satisfied for the singular system with primary constraints. These equations may be reduced to 0 = 0, where it is identically satisfied as a result of primary constraints, else they will be lead to new conditions which are called secondary constraints. Repeating this procedure as many times as needed, one arrives at a final set of constraints or/and specifies some of λ_μ. Such constraints are classified into two types, a) Firstclass constraints which have vanishing Poisson brackets with all other constraints. b) Second-class constraints which have nonvanishing Poisson brackets. The second-class constraints could be used to eliminate conjugated pairs of the p's and q's from the theory by expressing them as functions of the remaining p's and q's . The total Hamiltonian for the remaining variable is then the canonical Hamiltonian plus the primary constraints H'_μ of the first type as in Eq. (1.3.4), where H'_μ are all the independent remaining first-class constraints.

1.4 Hamilton-Jacobi Approach (Güler Method)

The aim is to obtain a valid and consistent Hamilton-Jacobi theory of singular systems. The main point of the method is to define the equivalent Lagrangian (variational principle) and then pass to the phase space. This formulation leads us to a set of Hamilton-Jacobi partial differential equation (6), (7) and (8).

1.4.1 Construction of Phase Space

The starting point of the Hamilton - Jacobi method is to consider the Lagrangian $L = L(q_i, \dot{q}_i, t)$ with the Hess matrix (1.2.5) of rank $(n - r), r < n$. Then we can solve (1.3.1) for $\dot{q_a}$ in term of $q_i, \dot{x_\mu}, p_a$ and t as

$$\dot{q_a} = \dot{q_a}(q_i, \dot{x_\mu}, p_b; t). \tag{1.4.1}$$

Substituting (1.3.1) into (1.3.2), we get

$$p_\mu = \frac{\partial L}{\partial \dot{q_\mu}} = -H_\mu(q_i, \dot{x_\mu}, p_a; t). \tag{1.4.2}$$

Relations (1.4.2) indicate the fact that the generalized momenta p_μ are not all independent which is a natural result of the singular nature of the Lagrangian. Although, it seems that H_μ are functions of $\dot{x_\mu}$, it is a task to show that they do not depend on it explicitly. The fundamental equations of the equivalent Lagrangian method read as

$$p_0 = \frac{\partial S}{\partial t} = -H_0(q_i, \dot{x_\mu}, p_a; t); \qquad p_a = \frac{\partial S}{\partial q_a}, \qquad p_\mu = \frac{\partial S}{\partial q_\mu} \equiv -H_\mu, \tag{1.4.3}$$

where the function $S \equiv S(q_i, t)$ is the action. The Hamiltonian H_0 reads as

$$H_0 = p_i \dot{q_i} + p_\mu \dot{x_\mu}|_{p\nu=-H\nu} - L(t, q_i, \dot{x_\nu} \dot{q_a}), \qquad \mu, \nu = n-r+1, ..., n. \tag{1.4.4}$$

Like the functions H_μ , the Hamiltonian H_0 is also not an explicit function of $\dot{x_\mu}$. Therefore, the function $S \equiv S(q_i, t)$ should satisfy the following set of Hamilton Jacobi partial differential equation (HJPDEs) which is expressed as

$$H'_0\left(t, x_\mu, q_a, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0, \tag{1.4.5}$$

$$H'_\mu\left(t, x_\mu, q_a, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0, \tag{1.4.6}$$

where

$$H'_0 = p_0 + H_\circ, \qquad H'_\mu = p_\mu + H_\mu. \tag{1.4.7}$$

Equations (1.4.5) and (1.4.6) may be expressed in a compact form as

$$H'_\alpha\left(t_\beta, q_a, p_i = \frac{\partial S}{\partial q_i}, p_0 = \frac{\partial S}{\partial t}\right) = 0, \qquad \alpha, \beta = 0, n-r+1, ..., n, \qquad a = 1, ..., n-r, \tag{1.4.8}$$

where

$$H'_\alpha = p_\alpha + H_\alpha. \tag{1.4.9}$$

The equations of motion are written as total differential equations in many variables t_β as follows (7)

$$dq_i = \frac{\partial H'_\alpha}{\partial p_i} dt_\alpha, \qquad i = 0, 1, ...n, \tag{1.4.10}$$

$$dp_a = -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha, \qquad a = 1, ...n - r, \tag{1.4.11}$$

$$dp_\mu = -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \qquad \alpha = 0, n - r + 1, ..., n. \tag{1.4.12}$$

We define

$$Z = S(t_\alpha, q_a) \tag{1.4.13}$$

and making use of Eq.(1.4.8) and definitions of generalized momenta (1.4.10, 1.4.11, 1.4.12, 1.4.13) we obtain

$$dZ = \frac{\partial S}{\partial t_\alpha} dt_\alpha + \frac{\partial S}{\partial q_a} dt_a = (-H_\alpha dt_\alpha + p_a dq_a) = \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha. \tag{1.4.14}$$

Equations (1.4.10-1.4.12) and (1.4.14) are called the total differential equations for the characteristics. If these equations form a completely integrable set, the simultaneous solutions of them determine the function $S(t_\alpha, q_a)$ uniquely by the prescribed initial conditions. The set of Equations (1.26-1.28) is integrable if and only if the variations of H'_0 *and* H'_μ vanish identically (6; 17; 18) that is

$$dH'_0 = 0, \tag{1.4.15}$$

$$dH'_\mu = 0, \qquad \mu = n - r + 1, ..., n. \tag{1.4.16}$$

If condition (1.4.15) and (1.4.16) are not satisfied identically, one considers them as new constraints and again testes the integrability conditions. Hence , the canonical formulation leads to obtain the set of canonical phase space coordinates q_α and p_a

as functions of t_a , besides the canonical action integral is obtained in terms of the canonical coordinates. The Hamiltonians H_α are considered as the infinitesimal generators of canonical transformations given by parameters t_α respectively (6), (7) and (8).

1.5 Mixture of Lagrangian and Hamiltonian Formulation of Constrained System

1.5.1 Singular Lagrangian as Field System

Singular Lagrangian as field system has been studied in Ref [9]. As a natural extension of the Hamiltonian formulation we would like to study the Lagrangian approach of a constrained system. The usual way to pass from the Hamiltonian to the Lagrangian approach is to use Eqs. (1.4.10-1.4.12) Since there are additional constraints, Eq (1.4.7) given in the phase space, they should also appear as constraints in the configuration space. As we have stated before, Eqs. (1.4.10-1.4.12) and Eq.(1.4.7) allow us to treat the system as a continuous or field system. Thus, we propose that the Euler-Lagrange equations of a constrained system are in the form (field system)

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial L'}{\partial(\partial_\mu q_a)}\right) - \frac{\partial L'}{\partial q_a} = 0, \tag{1.5.1}$$

with constraints

$$dG_\mu = -\frac{\partial L\prime}{\partial x_\mu}dt, \tag{1.5.2}$$

where

$$L'(x_\mu, \partial\mu q_a, \dot{x_\nu}, q_a) \equiv L[q_a, x_\mu, \dot{q_a} = (\partial\mu q_a)\dot{x}], \qquad \dot{x_\nu} = \frac{dx_\nu}{dt}, \tag{1.5.3}$$

and

$$G_\mu = H_\mu\left(q_a, x_\mu, p_a = \frac{\partial L}{\partial \dot{q}_a}\right) \tag{1.5.4}$$

where H_μ is obtained form the Hamilton-Jacobi formalism (1.4.7).
One should notic that variations of constraints should be considered in order to have a consistent theory. Many physical models has been investigated by Hamilton-Jacobi approach,(9) -(16). The validity of this methods need further physical applications.

The thesis is arranged as follows: in chapter two models of Singular Lagrangians are studied using Dirac's method. In chapter three also the same models are investigated using Hamilton-Jacobi (Güler) method. The treatment of singular Lagrangian systems as field (continuous) systems is discussed in chapter four. Chapter five is devoted to the conclusion of the our study.

Chapter 2

On singular Lagrangian and Dirac's Method

Chapter 2

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In this chapter, we study some singular Lagrangians from the classical mechanics of particles and apply Dirac′s method for building the equations of motion. We will construct the total Hamiltonian H_T of the systems and obtain the equations of motion . The consistency conditions will be discussed.

2.1 The First singular Lagrangian

The First singular Lagrangian is given as (20)

$$L = \frac{m}{2}({\dot{q}_1}^2 + {\dot{q}_2}^2 + l^2{\dot{q}_3}^2 + 2l\dot{q}_1\dot{q}_3\cos q_3 + 2l\dot{q}_2\dot{q}_3\sin q_3) + V(q_1, q_2, q_3) \tag{2.1.1}$$

The Lagrangian (2.1.1) is singular since the Hess matrix (1.2.5) is of rank two.
The generalized momenta,(1.3.1) read as

$$p_1 = m\dot{q}_1 + ml\dot{q}_3\cos q_3, \tag{2.1.2}$$

$$p_2 = m\dot{q}_2 + ml\dot{q}_3\sin q_3, \tag{2.1.3}$$

$$p_3 = ml^2\dot{q}_3 + ml(\dot{q}_1 \cos q_3 + \dot{q}_2 \sin q_3). \tag{2.1.4}$$

Multiplying (2.1.2) by $\cos q_3$ and (2.1.3) by $\sin q_3$ and then subtracting the sum of the result from (2.1.5), one gets the primary constraints (1.2.8) according to Dirac as

$$\phi_1 = p_3 - lp_1 \cos q_3 - lp_2 \sin q_3 = 0. \tag{2.1.5}$$

Now, let us rewrite (2.1.2) and (2.1.3) as

$$p_1 - m\dot{q}_1 = ml\dot{q}_3 \cos q_3, \tag{2.1.6}$$

$$p_2 - m\dot{q}_2 = ml\dot{q}_3 \sin q_3. \tag{2.1.7}$$

From Eqs (2.1.6) and (2.1.7) one gets

$$p_1\dot{q}_1 + p_2\dot{q}_2 = \frac{{p_1}^2 + {p_2}^2}{2m} + \frac{m}{2}({\dot{q}_1}^2 + {\dot{q}_2}^2) - ml^2{\dot{q}_3}^2. \tag{2.1.8}$$

The usual Hamiltonian (1.3.5) is

$$H_0 = p_1\dot{q}_1 + p_2\dot{q}_2 + p_3\dot{q}_3 - L. \tag{2.1.9}$$

Using (2.1.2) , (2.1.3) and (2.1.4) , (2.1.9) takes the form

$$H_0 = \frac{{p_1}^2 + {p_2}^2}{2m} - V, \tag{2.1.10}$$

and using (1.3.4) , the total Hamiltonian is

$$H_T = \frac{{p_1}^2 + {p_2}^2}{2m} - V + \nu(p_3 - lp_1 \cos q_3 - lp_2 \sin q_3). \tag{2.1.11}$$

Now, the consistency condition reads as

$$\dot{\phi}_1 = [\phi_1, H_T] = V_{,3} - l \cos q_3 V_{,1} - l \sin q_3 V_{,2} \tag{2.1.12}$$

where,

$$V_{,i} = \frac{\partial V}{\partial q_i}, \qquad i = 1, 2, 3 \tag{2.1.13}$$

Relation (2.1.12) leads to the secondary constraint which is a relation between the coordinates

$$\phi_2 = V_{,3} - l\cos q_3 V_{,1} - l\sin q_3 V_{,2}, \tag{2.1.14}$$

so one can write the secondary constraint in the form

$$\phi_2 = q_3 - F(q_1, q_2). \tag{2.1.15}$$

Now let us evaluate $\dot{\phi}_2$

$$\dot{\phi}_2 = [\phi_2, H_T] = F_{,1}\frac{P_1}{m} + F_{,2}\frac{P_2}{m} - \nu(1 + l\cos q_3 F_{,1} + l\sin q_3 F_{,2}) \equiv 0, \tag{2.1.16}$$

from which we get the multiplier ν as

$$\nu = \frac{p_2 F_{,1} + p_2 F_{,2}}{m(1 + l\cos q_3 F_{,1} + l\sin q_3 F_{,2})}. \tag{2.1.17}$$

The equations of motion (1.3.6) and (1.3.7) read as

$$\dot{q_1} = \frac{p_1}{m} - \nu l\cos q_3, \tag{2.1.18}$$

$$\dot{q_2} = \frac{p_2}{m} - \nu l\sin q_3, \tag{2.1.19}$$

$$\dot{q_3} = \nu, \tag{2.1.20}$$

$$\dot{p_1} = \frac{\partial V}{\partial q_1}, \tag{2.1.21}$$

$$\dot{p_2} = \frac{\partial V}{\partial q_2}, \tag{2.1.22}$$

$$\dot{p_3} = \frac{\partial V}{\partial q_3} - \nu(lp_1\sin q_3 - lp_2\cos q_3). \tag{2.1.23}$$

The set of equations (2.1.18-2.1.23) with (2.1.17) represent a consistent set of ordinary differential equations.

2.2 Mittelstaedt's Lagrangian

The second model is Mittelstaedt's Lagrangian model (20), which is given as

$$L = \frac{1}{2m}(\dot{q_1} + \dot{q_2})^2 + \frac{1}{2\mu}\dot{q_3}^2 + V(q_1, q_2, q_3). \tag{2.2.1}$$

As the Hess matrix of the above Lagrangian is of rank two, this Lagrangian is Singular.

We start with obtaining the momenta (1.2.7), which are given as

$$p_1 = p_2 = \frac{1}{m}(\dot{q_1} + \dot{q_2}), \qquad\qquad p_3 = \frac{1}{\mu}\dot{q_3}, \tag{2.2.2}$$

The primary constraint is then

$$\phi_1 = p_2 - p_1 = 0. \tag{2.2.3}$$

The original Hamiltonian becomes

$$H_0 = \frac{m}{2}p_1^2 + \frac{\mu}{2}p_3^2 - V. \tag{2.2.4}$$

The total Hamiltonian is then

$$H_T = \frac{m}{2}p_1^2 + \frac{\mu}{2}p_3^2 - V + \nu(p_2 - p_1). \tag{2.2.5}$$

The consistency condition $\dot{\phi}_1 = [\phi_1, H_T]$ leads to the constraint

$$\phi_2 = \frac{\partial V}{\partial q_1} - \frac{\partial V}{\partial q_2} = 0. \tag{2.2.6}$$

This is a relation between $q_1, q_2 \quad and \quad q_3$ which briefly, is written as

$$\phi_2 = q_2 - F(q_1, q_3) = 0. \tag{2.2.7}$$

We then build the consistency condition $\dot{\phi}_2$ as

$$\dot{\phi}_2 = [\phi_2, H_T], \tag{2.2.8}$$

from which we find

$$\nu(1 + F_{,1}) - mP_1F_{,1} - \mu P_3F_{,3} = 0, \qquad \left(F_{,i} = \frac{\partial F}{\partial q_i}\right) \tag{2.2.9}$$

From Eq (2.31) one allows fixing varible ν as

$$\nu = \frac{mp_1F_{,1} + \mu p_3F_{,3}}{1 + F_{,1}}. \tag{2.2.10}$$

We can now write the canonical equations of motion (1.3.6) and (1.3.7) as

$$\dot{q_1} = mp_1 - \nu, \qquad \dot{q_2} = \nu, \qquad \dot{q_3} = \mu p_3, \tag{2.2.11}$$

$$\dot{p_1} = V_{,1}, \qquad \dot{p_2} = V_{,1}, \qquad \dot{p_3} = V_{,3}. \tag{2.2.12}$$

Eqs. (2.2.11) and (2.2.12) with (2.2.10) represent a set of consistent differential equations.

2.3 Deriglazov Lagrangian

The third model is Deriglazov Lagrangian which is given as (20)

$$L = q_2^2\dot{q_1}^2 + q_1^2\dot{q_2}^2 + 2q_1q_2\dot{q_1}\dot{q_2} + V(q_1, q_2). \tag{2.3.1}$$

This Lagrangian is singular since the Hess matrix is of rank one. The momenta (1.3.1) read as

$$p_1 = 2q_2^2\dot{q_1} + 2q_1q_2\dot{q_2}, \tag{2.3.2}$$

$$p_2 = 2q_1^2\dot{q_2} + 2q_1q_2\dot{q_1}. \tag{2.3.3}$$

Here the momenta $p_1 \quad and \quad p_1$ are not independent. The primary constraint is then

$$\dot{\phi_1} = q_1p_1 - q_2p_2 = 0. \tag{2.3.4}$$

The original Hamiltonian takes the form

$$H_0 = \frac{p_1^2}{4q_2^2} - V(q_1, q_2), \tag{2.3.5}$$

and the total Hamiltonian is then

$$H_T = \frac{p_1^2}{4q_2^2} - V(q_1, q_2) + \nu(q_1 P_1 - q_2 p_2). \tag{2.3.6}$$

The consistency condition is

$$\dot{\phi}_1 = [\phi_1, H_T] = q_1 V_{,1} - q_2 V_{,2} \equiv \phi_2 = 0, \tag{2.3.7}$$

where ϕ_2 is the secondary constraint.
Again $\dot{\phi}_2$ is

$$\dot{\phi}_2 = [\phi_2, H_T] = 0, \tag{2.3.8}$$

from which we find

$$-F_{,1}\frac{p_1}{2q_2^2} - q_1 F_{,1}\nu - q_2\nu = 0. \tag{2.3.9}$$

From Eq.(2.3.9) one allows fixing variable ν as

$$\nu = -\frac{p_1}{2F^2(F + q_1 F_{,1})} F_{,1}, \tag{2.3.10}$$

with

$$q_2 = F(q_1). \tag{2.3.11}$$

Therefore, the canonical equations of motion are

$$\dot{q}_1 = \frac{p_1}{2q_2^2} + q_1\nu \tag{2.3.12}$$

Substituting Eq.(2.3.10) in (2.3.12) we get

$$\dot{q}_1 = \frac{p_1}{2F(F + q_1 F_{,1})}. \tag{2.3.13}$$

The other equation of motion is

$$\dot{p_1} = V_{,1} + \frac{p_1^2}{2F^2(F + q_1 F_{,1})} F_{,1}. \tag{2.3.14}$$

One notices that Newton's equations of motion can be obtained if we let $V(x, y) = x^2 + y^2 \quad and \quad F(x) = \pm x$ as

$$2F(F + q_1 F_{,1})\ddot{q_1} + 2F(2F_{,1} + q_1 F_{,11})\dot{q_1}^2 - V_{,1} = 0 \tag{2.3.15}$$

Chapter 3

Hamilton-Jacobi Method

Chapter 3

Hamilton-Jacobi Method

In this chapter we study some singular Lagrangian systems from the classical mechanics of particles and apply Hamilton-Jacobi Method to construct Hamilton-Jacobi Partial Differential Equations (HJPDE),and then we write the equations of motion.

3.1 Charged Particle Moving in a constant Magnetic Field

The motion of charged particle in a plane is described by the singular Lagrangian (21).

$$L = \frac{1}{2}(\dot{q}_1 - q_3 q_2)^2 + \frac{1}{2}(\dot{q}_2 + q_3 q_1)^2. \tag{3.1.1}$$

The rank of the Hess matrix (1.2.5) is two. Then the singularity of the Lagrangian enables us to write the generalized momenta (1.3.1) and (1.3.2) as

$$p_1 = \dot{q}_1 - q_3 q_2, \tag{3.1.2}$$

$$p_2 = \dot{q}_2 + q_3 q_1, \tag{3.1.3}$$

$$p_3 = 0 \equiv -H_3. \tag{3.1.4}$$

We solve (3.1.2) and (3.1.3) for $\dot{q_1}$ and $\dot{q_2}$ in terms of p_1 and p_2 as

$$\dot{q_1} = p_1 + q_3 q_2 \equiv \omega_1 \tag{3.1.5}$$

$$\dot{q_2} = p_2 - q_3 q_1 \equiv \omega_2 \tag{3.1.6}$$

The canonical Hamiltonian H_0 (1.3.5) is then

$$\begin{aligned} H_0 &= p_a \dot{q_a} + p_\mu \dot{q_\mu} - L|_{\dot{q_a} \equiv \omega_a} \\ &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + p_1 q_3 q_2 - p_2 q_3 q_1. \end{aligned} \tag{3.1.7}$$

The set of (HJPDE) according to Eq.(1.4.7) is

$$H_0' = p_0 + \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + p_1 q_3 q_2 - p_2 q_3 q_1 = 0, \tag{3.1.8}$$

and

$$H_3' = p_3 + H_3 = p_3 = 0 \tag{3.1.9}$$

Relation (3.1.8) and (3.1.9) are the constraints that restrict the system. The total differential equations of motion (1.4.10) , (1.4.11) and (1.4.12) are

$$dq_1 = \frac{\partial H_0'}{\partial p_1} d\tau + \frac{\partial H_3'}{\partial p_1} dq_3, \tag{3.1.10}$$

$$dq_2 = \frac{\partial H_0'}{\partial p_2} d\tau + \frac{\partial H_3'}{\partial p_2} dq_3, \tag{3.1.11}$$

$$dq_3 = \frac{\partial H_0'}{\partial p_3} d\tau + \frac{\partial H_3'}{\partial p_3} dq_3, \tag{3.1.12}$$

$$dp_1 = -\frac{\partial H_0'}{\partial q_1} d\tau - \frac{\partial H_3'}{\partial q_1} dq_3, \tag{3.1.13}$$

$$dp_2 = -\frac{\partial H_0'}{\partial q_2} d\tau - \frac{\partial H_3'}{\partial q_2} dq_3, \tag{3.1.14}$$

$$dp_3 = -\frac{\partial H_0'}{\partial q_3} d\tau - \frac{\partial H_3'}{\partial q_3} dq_3. \tag{3.1.15}$$

Substituting Eqs.(3.1.8) and (3.1.9) in eqs.(3.1.10-3.1.15), we obtain the total differential equations of motion as

$$dq_1 = (p_1 + q_3 q_2)d\tau, \tag{3.1.16}$$

$$dq_2 = (p_2 - q_3 q_1)d\tau, \tag{3.1.17}$$

$$dq_3 = dq_3, \tag{3.1.18}$$

$$dp_1 = p_2 q_3 d\tau, \tag{3.1.19}$$

$$dp_2 = -p_1 q_3 d\tau, \tag{3.1.20}$$

$$dp_3 = -(p_1 q_2 - p_2 q_1)d\tau. \tag{3.1.21}$$

To check whether the above set of equations is integrable or not, let us consider the total variations of $H_0' \quad and \quad H_3'$. If fact

$$dH_0' = 0, \tag{3.1.22}$$

$$dH_3' = dp_3 = (-p_1 q_2 + p_2 q_1)d\tau, \tag{3.1.23}$$

since dH_3' is not identically zero. we have a new constraint H_4',

$$H_4' = (p_1 q_2 - p_2 q_1) \equiv 0. \tag{3.1.24}$$

Thus for a valid theory , the total differential of H_4' is identically zero,

$$dH_4' = p_1 dq_2 + q_2 dp_1 - p_2 dq_1 - q_1 dp_2 = 0, \tag{3.1.25}$$

so the system of Eqs.(3.1.16-3.1.21) together with Eq.(3.1.25) is integrable.

3.2 The second singular Lagrangian

As a second model, let us consider the singular Lagrangian,

$$L = \frac{m}{2}(\dot{q_1}^2 + \dot{q_2}^2 + l^2\dot{q_3}^2 + 2l\dot{q_1}\dot{q_3}\cos q_3 + 2l\dot{q_2}\dot{q_3}\sin q_3) + V(q_1, q_2, q_3) \tag{3.2.1}$$

where l and m are constants,

The singularity of the Lagrangian enables us to write the generalized momenta in chapter two Eqs.(2.1.2),(2.1.3)and(2.1.4)

$$p_1 = m\dot{q_1} + ml\dot{q_3}\cos q_3, \tag{3.2.2}$$

$$p_2 = m\dot{q_2} + ml\dot{q_3}\sin q_3, \tag{3.2.3}$$

$$p_3 = ml^2\dot{q_3} + ml(\dot{q_1}\cos q_3 + \dot{q_2}\sin q_3). \tag{3.2.4}$$

Multiplying equation (3.2.2) by $l\cos q_3$ and (3.2.3) by $l\sin q_3$ one gets the constraint relation

$$H_3 = p_3 - lp_1\cos q_3 - lp_2\sin q_3 = 0 \tag{3.2.5}$$

The canonical Hamiltonian H_0 (1.3.5) is then

$$\begin{aligned} H_0 &= p_a\dot{q_a} + p_\mu\dot{q_\mu} - L|_{\dot{q_a}\equiv\omega_a} \\ &= \frac{{p_1}^2 + {p_2}^2}{2m} - V(q_1, q_2, q_3). \end{aligned} \tag{3.2.6}$$

The set of (HJPBE) according to Eq(1.4.7) is

$$H_0' = p_0 + \frac{{p_1}^2 + {p_2}^2}{2m} - V(q_1, q_2, q_3) = 0, \tag{3.2.7}$$

$$H_3' = p_3 - lp_1\cos q_3 - lp_2\sin q_3 = 0. \tag{3.2.8}$$

The total differential equations of motion (1.4.10) , (1.4.11) and (1.4.12) are

$$dq_1 = \frac{\partial H_0'}{\partial p_1} d\tau + \frac{\partial H_3'}{\partial p_1} dq_3, \tag{3.2.9}$$

$$dq_2 = \frac{\partial H_0'}{\partial p_2} d\tau + \frac{\partial H_3'}{\partial p_2} dq_3, \tag{3.2.10}$$

$$dq_3 = \frac{\partial H_0'}{\partial p_3} d\tau + \frac{\partial H_3'}{\partial p_3} dq_3, \tag{3.2.11}$$

$$dp_1 = -\frac{\partial H_0'}{\partial q_1} d\tau - \frac{\partial H_3'}{\partial q_1} dq_3, \tag{3.2.12}$$

$$dp_2 = -\frac{\partial H_0'}{\partial q_2} d\tau - \frac{\partial H_3'}{\partial q_2} dq_3, \tag{3.2.13}$$

$$dp_3 = -\frac{\partial H_0'}{\partial q_3} d\tau - \frac{\partial H_3'}{\partial q_3} dq_3. \tag{3.2.14}$$

Substituting Eqs.(3.2.7) and (3.2.8) in Eqs.(3.2.9-3.2.14), we obtain the total differential equations of motion as

$$dq_1 = \frac{p_1}{m} dt - l \cos q_3 dq_3, \tag{3.2.15}$$

$$dq_2 = \frac{p_2}{m} dt - l \sin q_3 dq_3, \tag{3.2.16}$$

$$dq_3 = dq_3, \tag{3.2.17}$$

$$dp_1 = V_{,1} dt, \tag{3.2.18}$$

$$dp_2 = V_{,2} dt, \tag{3.2.19}$$

$$dp_3 = V_{,3} dt - (l p_1 \sin q_3 - l p_2 \cos q_3) dq_3. \tag{3.2.20}$$

From Eq. (3.2.17), one conduces that $q_3 = constant$. The set of equation of motion (3.2.15-3.2.20) are integrable if the variations of (3.2.7) and (3.2.8) are identically satisfied, that is

$$dH_0' = (V_{,1} l \cos q_3 + V_{,2} l \sin q_3 - V_{,3}) dq_3, \tag{3.2.21}$$

Similarly the variation of H_3' takes the form

$$dH_3' = (V_{,1}l\cos q_3 + V_{,2}l\sin q_3 - V_{,3})dt. \tag{3.2.22}$$

To be identically satisfied we should choose $V(q_1, q_2, q_3)$ such that

$$V_{,3} = V_{,1}l\cos q_3 + V_{,2}l\sin q_3 \tag{3.2.23}$$

3.3 The Mittelstaedt,s Lagrangian

As a third model, let us consider the Mittelstaedt,s singular Lagrangian (20)

$$L = \frac{1}{2m}(\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2\mu}\dot{q}_3{}^2 + V(q_1, q_2, q_3), \tag{3.3.1}$$

where m and μ are constants.

The singularity of the Lagrangian enables us to write generalized momenta (1.3.1) and (1.3.2) as

$$p_1 = p_2 = \frac{1}{m}\dot{q}_2 + \dot{q}_2, \tag{3.3.2}$$

and

$$p_3 = \frac{1}{\mu}\dot{q}_3. \tag{3.3.3}$$

we solve (3.3.2) and (3.3.3) for $\dot{q}_1, \quad \dot{q}_2 \quad and \quad \dot{q}_3$ interns of $p_1, \quad p_2 \quad and \quad p_3$ as

$$\dot{q}_2 + \dot{q}_2 = mp_1 - mp_2, \tag{3.3.4}$$

$$\dot{q}_3 = \mu p_3. \tag{3.3.5}$$

The constraint relation is

$$H_2 = p_1 - p_2 = 0 \tag{3.3.6}$$

The canonical Hamiltonian H_0 takes the form

$$\begin{aligned} H_0 &= p_a\dot{q_a} + p_\mu\dot{q_\mu} - L|_{\dot{q_a}\equiv\omega_a} \\ &= \frac{m}{2}p_1^2 + \frac{\mu}{2}p_3^2 - V(q_1, q_2, q_3). \end{aligned} \tag{3.3.7}$$

The set of (HJPBE) according to Eq(1.4.7) is

$$H_0' = p_0 + \frac{m}{2}p_1^2 + \frac{\mu}{2}p_3^2 - V(q_1, q_2, q_3) = 0, \tag{3.3.8}$$

and

$$H_2' = p_1 - p_2 = 0. \tag{3.3.9}$$

The total differential equations of motion (1.4.10) , (1.4.11) and (1.4.12) read as

$$dq_1 = \frac{\partial H_0'}{\partial p_1}d\tau + \frac{\partial H_2'}{\partial p_1}dq_2, \tag{3.3.10}$$

$$dq_2 = \frac{\partial H_0'}{\partial p_2}d\tau + \frac{\partial H_2'}{\partial p_2}dq_2, \tag{3.3.11}$$

$$dq_3 = \frac{\partial H_0'}{\partial p_3}d\tau + \frac{\partial H_2'}{\partial p_3}dq_2, \tag{3.3.12}$$

$$dp_1 = -\frac{\partial H_0'}{\partial q_1}d\tau - \frac{\partial H_2'}{\partial q_1}dq_2, \tag{3.3.13}$$

$$dp_2 = -\frac{\partial H_0'}{\partial q_2}d\tau - \frac{\partial H_2'}{\partial q_2}dq_2, \tag{3.3.14}$$

$$dp_3 = -\frac{\partial H_0'}{\partial q_3}d\tau - \frac{\partial H_2'}{\partial q_3}dq_2. \tag{3.3.15}$$

Substituting Eqs. (3.3.8) and (3.3.9) in Eqs.(3.3.10-3.3.15), we obtain the total differential equations of motion as

$$dq_1 = mp_1dt - dq_2, \tag{3.3.16}$$

$$dq_2 = dq_2, \tag{3.3.17}$$

$$dq_3 = \mu p_3dt, \tag{3.3.18}$$

$$dp_1 = V_{,1}dt, \tag{3.3.19}$$

$$dp_2 = V_{,2}dt, \tag{3.3.20}$$

$$dp_3 = V_{,3}dt. \tag{3.3.21}$$

The set of equation of motion (3.3.16-3.3.21) are integrable if only if the variations of (3.3.8) and (3.3.9) are identically satisfied. The variation of H_0' is

$$dH_0' = (V_{,1} - V_{,2})dq_2, \tag{3.3.22}$$

which is identically zero since q_2 is constant.

$$\begin{aligned} dH' &= dp_2 - dp_1 \\ &= (V_{,2} - V_{,1})dt. \end{aligned} \tag{3.3.23}$$

In order to obtain an integrable system $V_{,1}$ must be equal to $V_{,2}$

3.4 The Deriglazov Lagrangian

The last model is the Deriglazov singular Lagrangian (20)

$$L = q_2^2\dot{q_1}^2 + q_1^2\dot{q_2}^2 + 2q_1q_2\dot{q}_1\dot{q}_2 + V(q_1, q_2). \tag{3.4.1}$$

This Lagrangian is singular since the Hess matrix is of rank one, and the generalized momenta (1.3.1) and (1.3.2) read as

$$p_1 = 2q_2^2\dot{q}_1 + 2q_1q_2\dot{q}_2, \tag{3.4.2}$$

$$p_2 = 2q_1^2\dot{q}_2 + 2q_1q_2\dot{q}_1. \tag{3.4.3}$$

Here $p_1 \quad and \quad p_2$ are dependent. Multiplying Eq. (3.4.2) in q_1 and Eq.(3.4.3) in q_2 and solving for p_1, we get becomes constraint equation are

$$p_1 = \frac{q_2p_2}{q_1}. \tag{3.4.4}$$

using (3.4.4), the canonical Hamiltonian H_0 (1.3.5) is then

$$\begin{aligned} H_0 &= p_1\dot{q}_1 + p_2\dot{q}_2 - L \\ &= \frac{p_1^2}{4q_2^2} - V(q_1, q_2) \end{aligned} \tag{3.4.5}$$

The Set of (HJPDE) according to Eq (1.4.7) is

$$H_0' = p_0 + \frac{p_1^2}{4q_2^2} - V(q_1, q_2) = 0, \tag{3.4.6}$$

The above equation are the constraints restricting the system.

The total differential equations of motion (1.4.10) , (1.4.11) and (1.4.12) are

$$dq_1 = \frac{\partial H_0'}{\partial p_1} d\tau, \tag{3.4.7}$$

$$dq_2 = \frac{\partial H_0'}{\partial p_2} d\tau, \tag{3.4.8}$$

$$dp_1 = -\frac{\partial H_0'}{\partial q_1} d\tau, \tag{3.4.9}$$

$$dp_2 = -\frac{\partial H_0'}{\partial q_2} d\tau. \tag{3.4.10}$$

Substituting Eq. (3.4.7) in Eqs.(3.4.7-3.4.10), we obtain the total differential equations of motion as

$$dq_1 = \frac{p_1}{2q_2^2} dt, \tag{3.4.11}$$

$$dq_2 = 0, \tag{3.4.12}$$

$$dp_1 = V_{,1} dt, \tag{3.4.13}$$

$$dp_2 = V_{,2} dt. \tag{3.4.14}$$

The set of equation of motion (3.4.11-3.4.14) are integrable if the variations of (3.4.6) is identically satisfied, we notice that the variation

$$dH_0' = (q_2 V_{,2} - q_1 V_{,1}) dq_1 \equiv H_0'' dq_1 \tag{3.4.15}$$

is identically satisfied for a choice $V(q_1, q_2) = q_1 q_2$

Chapter 4

Singular Lagrangian as Field Systems

Chapter 4

Singular Lagrangian As Field Systems

The link between the treatment of singular Lagrangian as field system and the general Hamiltonian approach is studied. It is shown that the singular Lagrangian as field system are always in exact aggrement with the general approaches (9). The equations of motion in this treatment are second order partial differential equation .

4.1 Preliminaries

The Euler-Lagrange equations for field system is given as

$$\frac{\partial}{\partial x_\mu}\left(\frac{\partial L'}{\partial(\partial \mu q_a)}\right) - \frac{\partial L'}{\partial q_a} = 0, \tag{4.1.1}$$

and the constraints relation is defined as

$$dG_\mu = -\frac{\partial L\prime}{\partial x_\mu} dt, \tag{4.1.2}$$

where the modified Lagrangian L' is defined as

$$L'(x_\mu, \partial_\mu q_a, \dot{x_\nu}, q_a) \equiv L(q_a, x_\mu, \dot{q_a} = (\partial \mu q_a)\dot{x_\nu}), \qquad \dot{x_\nu} = \frac{dx_\nu}{dt} \tag{4.1.3}$$

and the constraint relations are

$$G_\mu = H_\mu \left(q_a, x_\mu, p_a = \frac{\partial L}{\partial \dot{q_a}} \right). \tag{4.1.4}$$

4.2 Examples

In this section , the singular Lagrangians which are investigated in chapter two and chapter three will be studied. As field systems.(or continuous system).

4.2.1 Example One

As a first example we consider the singular Lagrangian

$$L = \frac{1}{2}\dot{q}_1^2 - \frac{1}{4}(\dot{q}_2^2 - 2\dot{q}_2\dot{q}_3 + \dot{q}_3^2) + (q_1 + q_3)\dot{q}_2 - q_1 - q_2 - q_3^2. \tag{4.2.1}$$

Since the rank of the Hess matrix is two, this system can be treated as a continuous system in the form

$$q_1 = q_1(t, q_2), \qquad q_3 = q_3(t, q_2) \tag{4.2.2}$$

Now , let us write $\dot{q}_1 \quad and \quad \dot{q}_3$ as

$$\dot{q}_1 = \frac{dq_1}{dt} = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2, \qquad \dot{q}_3 = \frac{dq_3}{dt} = \frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2. \tag{4.2.3}$$

Substituting (4.2.3) into (4.2.1) ,we get the modified Lagrangian L' as

$$L' = \frac{1}{2}\left(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2\right)^2 - \frac{1}{4}\dot{q_2}^2 + \frac{1}{2}\dot{q}_2\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right) - \frac{1}{4}\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right)^2$$
$$+ (q_1 + q_3)\dot{q}_2 - q_1 - q_2 - q_3^2 \tag{4.2.4}$$

The Euler-Lagrange equations (4.1.1)read as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_1)}\right)+\frac{\partial}{\partial q_2}\left(\frac{\partial L'}{\partial(\partial_2 q_1)}\right)-\frac{\partial L'}{\partial q_1}=0,\\
\mathit{and}\qquad &\frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_3)}\right)+\frac{\partial}{\partial q_2}\left(\frac{\partial L'}{\partial(\partial_2 q_3)}\right)-\frac{\partial L'}{\partial q_3}=0.
\end{aligned}\tag{4.2.5}$$

where $x_0 \equiv t, \qquad x_2 \equiv q_2.$

More explicitly, the second order partial differential equations are

$$\frac{\partial^2 q_1}{\partial t^2}+2\dot{q}_2\frac{\partial^2 q_1}{\partial t\partial q_2}+\dot{q_2}^2\frac{\partial^2 q_1}{\partial {q_2}^2}+\ddot{q}_2\frac{\partial q_1}{\partial q_2}-\dot{q}_2+1=0,\tag{4.2.6}$$

$$\frac{\partial^2 q_3}{\partial t^2}+2\dot{q}_2\frac{\partial^2 q_3}{\partial t\partial q_2}+\dot{q_2}^2\frac{\partial^2 q_1}{\partial {q_2}^2}+\ddot{q}_2\frac{\partial q_3}{\partial q_2}-\ddot{q}_2+2\dot{q}_2-4q_3=0,\tag{4.2.7}$$

Now, we have to check the validity of constraint (4.1.2). As they were defined in (9). The usual Hamiltonian and the constraint relation are

$$H_0=\frac{1}{2}(p_1^2-2p_3^2)+q_1+q_2+q_3^2,\tag{4.2.8}$$

and

$$H_2=p_3-q_1-q_3.\tag{4.2.9}$$

Hence,

$$\begin{aligned}
G_0&=H_0\Bigg(q_a,x_\mu,p_a=\frac{\partial L}{\partial \dot{q_a}}\Bigg)\\
&=\frac{1}{2}\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_2}\dot{q}_2\right)^2-\frac{1}{4}\dot{q_2}^2-\frac{1}{4}\left(\frac{\partial q_3}{\partial t}+\frac{\partial q_3}{\partial q_2}\dot{q}_2\right)^2\\
&+\frac{1}{2}\dot{q}_2\left(\frac{\partial q_3}{\partial t}+\frac{\partial q_3}{\partial q_2}\dot{q}_2\right)^2+q_1+q_2+{q_3}^2,
\end{aligned}\tag{4.2.10}$$

and

$$\begin{aligned}
G_2&=H_2\Bigg(q_a,x_\mu,p_a=\frac{\partial L}{\partial \dot{q_a}}\Bigg)\\
&=-\frac{1}{2}\dot{q}_2+\frac{1}{2}\left(\frac{\partial q_3}{\partial t}+\frac{\partial q_3}{\partial q_2}\dot{q}_2\right)+q_1+q_3
\end{aligned}\tag{4.2.11}$$

Now, we are ready to test whether (4.1.2) are satisfied or not. In fact (4.1.2) for $\mu = 0$ is

$$dG_0 = -\frac{\partial L'}{\partial t}dt = 0. \tag{4.2.12}$$

Explicitly

$$\begin{aligned} dG_0 = [&\left(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2\right)\left(\frac{\partial^2 q_1}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_1}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_1}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_1}{\partial q_2}\right) - \frac{1}{2}\dot{q}_2\ddot{q}_2 \\ &- \frac{1}{2}\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right)\left(\frac{\partial^2 q_3}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_3}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_3}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_3}{\partial q_2}\right) \\ &+ \frac{1}{2}\dot{q}_2\left(\frac{\partial^2 q_3}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_3}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_3}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_3}{\partial q_2}\right) + \frac{1}{2}\ddot{q}_2\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right) \\ &+ \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2\dot{q}_2 + 2q_3\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right)]dt = 0. \end{aligned} \tag{4.2.13}$$

Replacing the expressions in the parentheses from Egs.(4.2.6) and (4.2.7) one gets

$$dG_0 = (\dot{q}_2 F_1)dt = 0, \tag{4.2.14}$$

where

$$F_1 = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2 + \frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2 - \dot{q}_2 + 2q_3 + 1 = 0. \tag{4.2.15}$$

Since F_1 is not identically zero, we consider it as a new constraint . Thus for a valid theory, variation of F_1 should be zero. Thus one gets

$$dF_1 = F_2 dt = 0, \tag{4.2.16}$$

where

$$F_2 = \dot{q}_2 - 3 - 2\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right) = 0, \tag{4.2.17}$$

Again, since F_2 is not identically zero, it is an additional constraint and its variation is

$$dF_2 = \left[\ddot{q}_2 - 2\left(\frac{\partial^2 q_1}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_1}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_1}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_1}{\partial q_2}\right)\right] dt = 0 \tag{4.2.18}$$

But due to Eq(4.2.6), the expression in parentheses is $\dot{q}_2 - 1$. So (4.2.18) leads us to the following differential equation for q_2 :

$$dF_2 = \ddot{q}_2 - 2\dot{q}_2 + 2, \tag{4.2.19}$$

which has the following solution :

$$q_2(t) = 2Ae^{2t} + t + c_1. \tag{4.2.20}$$

Besides, (4.1.2) for $\mu = 2$ is

$$dG_2 = \frac{\partial L}{\partial q_2} dt = -dt. \tag{4.2.21}$$

Hence,

$$\begin{aligned} dG_2 = [-\frac{1}{2}\ddot{q}_2 + \frac{1}{2}\left(\frac{\partial^2 q_3}{\partial t^2} + 2\dot{q}_2 \frac{\partial^2 q_3}{\partial t \partial q_2} + {\dot{q}_2}^2 \frac{\partial^2 q_3}{\partial {q_2}^2} + \ddot{q}_2 \frac{\partial q_3}{\partial q_2}\right) \\ + \dot{q}_2 \frac{\partial q_1}{\partial q_2} + \dot{q}_2 \frac{\partial q_3}{\partial q_2} + \frac{\partial q_1}{\partial t} + \frac{\partial q_3}{\partial t}]dt = -dt. \end{aligned} \tag{4.2.22}$$

Again the expression in the inner parentheses is replaced by $4q_3 + \ddot{q}_2 - 2\dot{q}_2$, from (4.2.7). Then (4.2.22) becomes

$$dG_2 = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2 + \frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2 - \dot{q}_2 + 2q_3 + 1 = F_1 dt \tag{4.2.23}$$

However, the constraint (4.2.23) is the same as (4.2.15). Thus, it does not give an additional constraint. Now, our problem is reduced to solving partial differential equations (4.2.6) and (4.2.7) with independent constraints (4.2.15) and (4.2.17). Making use of these constraints, one gets

$$\frac{\partial^2 q_1}{\partial^2 q_2} = 0, \tag{4.2.24}$$

which may have a solution in the form

$$q_1(t, q_2) = K(t)q_2 + T(t), \tag{4.2.25}$$

where K(t) and T(t) are functions to be determined. Some simple calculations lead us to the expressions

$$K(t) = constant, \quad T(t) = Ae^{2t} - Bt + D. \tag{4.2.26}$$

Since q_2 is determined as a function of t, the expression (4.2.25) can be written as

$$q_1(t) = Q'e^{2t} - t + D'. \tag{4.2.27}$$

Applying the same procedure to the variable q_3 we arrive at the differential equation

$$\frac{\partial^2 q_3}{\partial t \partial q_2} = 0, \tag{4.2.28}$$

which has the general solution

$$q_3(t, q_2) = C_1(t) + C_2(q_2). \tag{4.2.29}$$

However, further calculations give

$$C_1 = A''e^{-2t} + B'C_2(q_2) = 0. \tag{4.2.30}$$

Thus, we have

$$q_3(t) = Ae^{2t} + Be^{-2t} + C. \tag{4.2.31}$$

Eqs (4.2.27), and (4.2.31) are the solution of the system in the phase space $q_1, q_2 \quad and \quad q_3$.

4.2.2 Example Two

As a second example we consider the singular Lagrangian

$$L = \frac{1}{2}a_{ij}(t, q_k)\dot{q}_i\dot{q}_j + b_i(t, q_k)\dot{q}_i - c(t, q_k), \qquad i, j, k = 1, ..., 6, \tag{4.2.32}$$

where a_{ij} is a 6×6 symmetric matrix of rank 2, with matrix elements

$$a_{11} = a_{22} = 1, \tag{4.2.33}$$

$$a_{12} = a_{21} = 2, \tag{4.2.34}$$

$$a_{1\mu} = a_{\mu 1} = \alpha_\mu + 2\alpha'_\mu, \tag{4.2.35}$$

$$a_{2\mu} = a_{\mu 2} = 2\alpha_\mu + \alpha'_\mu, \tag{4.2.36}$$

$$a_{\mu\nu} = a_{\nu\mu} = \alpha_\mu\alpha_\nu + 2(\alpha_\mu\alpha'_\nu + \alpha'_\mu\alpha_\nu) + \alpha'_\nu\alpha'_\mu, \qquad \mu, \nu = 3, 4, 5, 6. \tag{4.2.37}$$

Here $\alpha_\mu and \quad \alpha'_\mu$ are constants and the functions $b_i and \quad c$ are

$$b_1 = q_2 + \alpha'_\mu q_\mu, \tag{4.2.38}$$

$$b_2 = q_2 - q_1 - (\alpha_\mu - \alpha')q_\mu, \tag{4.2.39}$$

$$b_\mu = \alpha_\mu b_1 + \alpha'_\mu b_2, \tag{4.2.40}$$

$$c = q_1 - 2q_2 + (\alpha_\mu - 2\alpha'_\mu)q_\mu. \tag{4.2.41}$$

As in the previous example this system can be treated as a continuous system in the form

$$q_1 = q_1(t, q_\mu), \qquad\qquad q_2 = q_2(t, q_\mu). \tag{4.2.42}$$

Thus,

$$\dot{q_1} = \frac{dq_1}{dt} = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_\mu}\dot{q_\mu}, \qquad \dot{q_2} = \frac{dq_2}{dt} = \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_\mu}\dot{q_\mu}. \tag{4.2.43}$$

Relation (4.2.43) can be replaced in (4.2.32) to obtain the following modified Lagrangian L':

$$\begin{aligned}
L' = &\frac{1}{2}\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_\mu}\dot{q}_\mu\right)^2+\frac{1}{2}\left(\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_\mu}\dot{q}_\mu\right)^2\\
&+2\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_\mu}\dot{q}_\mu\right)\left(\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_\mu}\dot{q}_\mu\right)+(\alpha_\mu+2\alpha'_\mu)\dot{q}_\mu\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_\mu}\dot{q}_\mu\right)\\
&+(2\alpha_\mu+\alpha'_\mu)\dot{q}_\mu\left(\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_\mu}\dot{q}_\mu\right)+(q_2-q_1-(\alpha_\mu-\alpha'_\mu)q_\mu)\left(\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_\mu}\dot{q}_\mu\right)\\
&+\frac{1}{2}a_{\mu\nu}\dot{q}_\mu\dot{q}_\nu+(q_2+\alpha'_\mu q_\mu)\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_\mu}\dot{q}_\mu\right)+(\alpha_\mu(q_2+\alpha'_\mu q_\mu)\\
&+\alpha'_\mu(q_2-q_1-(\alpha_\mu-\alpha'_\mu)q_\mu))\dot{q}_\mu-q_1+2q_2-(\alpha_\mu-2\alpha'_\mu)q_\mu.
\end{aligned} \tag{4.2.44}$$

The Euler-Lagrange equations (4.1.1) read as

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_1)}\right)+\frac{\partial}{\partial q_\mu}\left(\frac{\partial L'}{\partial(\partial_\mu q_1)}\right)-\frac{\partial L'}{\partial q_1}=0,\\
\textit{and}\qquad &\frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_2)}\right)+\frac{\partial}{\partial q_\mu}\left(\frac{\partial L'}{\partial(\partial_\mu q_2)}\right)-\frac{\partial L'}{\partial q_2}=0.
\end{aligned} \tag{4.2.45}$$

where $x_0 \equiv t, \qquad x_\mu \equiv q_\mu, \qquad \mu = 3,4,5,6.,$

More explicitly, the equations of motion (4.2.45) are

$$\begin{aligned}
&\frac{\partial^2 q_1}{\partial t^2}+2\dot{q}_\mu\frac{\partial^2 q_1}{\partial t\partial q_\mu}+\dot{q}_\mu\frac{\partial^2 q_1}{\partial q_\mu\partial q_\nu}\dot{q}_\nu+2\frac{\partial^2 q_2}{\partial t^2}+4\dot{q}_\mu\frac{\partial^2 q_2}{\partial t\partial q_\mu}\\
&+2\dot{q}_\mu\frac{\partial^2 q_2}{\partial q_\mu\partial q_\nu}\dot{q}_\nu+2\frac{\partial q_2}{\partial t}+\ddot{q}_\mu\frac{\partial q_1}{\partial q_\mu}+(2\ddot{q}_\mu+2\dot{q}_\mu)\frac{\partial q_2}{\partial q_\mu}\\
&+(\alpha_\mu+2\alpha'_\mu)\ddot{q}_\mu+2\alpha'_\mu\dot{q}_\mu+1=0.
\end{aligned} \tag{4.2.46}$$

and

$$\begin{aligned}
&2\frac{\partial^2 q_1}{\partial t^2}+4\dot{q}_\mu\frac{\partial^2 q_1}{\partial t\partial q_\mu}+2\dot{q}_\mu\frac{\partial^2 q_1}{\partial q_\mu\partial q_\nu}\dot{q}_\nu+\frac{\partial^2 q_2}{\partial t^2}+2\dot{q}_\mu\frac{\partial^2 q_2}{\partial t\partial q_\mu}\\
&+\dot{q}_\mu\frac{\partial^2 q_2}{\partial q_\mu\partial q_\nu}\dot{q}_\nu-2\frac{\partial q_1}{\partial t}+\ddot{q}_\mu\frac{\partial q_2}{\partial q_\mu}+(2\ddot{q}_\mu-2\dot{q}_\mu)\frac{\partial q_1}{\partial q_\mu}\\
&+(2\alpha_\mu+2\alpha'_\mu)\ddot{q}_\mu-2\alpha_\mu\dot{q}_\mu-2=0.
\end{aligned} \tag{4.2.47}$$

To solve the equations of motion, let us consider the following transformation: $q_\mu \longrightarrow \mu_\mu, \qquad t \longrightarrow t.$

Hence, Eqs. (4.2.46) and (4.2.47) are reduced to the following canonical forms (9)

$$\frac{\partial^2 q_1}{\partial t^2} = -2\frac{\partial^2 q_2}{\partial t^2} - 2\frac{\partial q_2}{\partial t} - 1, \tag{4.2.48}$$

and

$$2\frac{\partial^2 q_1}{\partial t^2} = -\frac{\partial^2 q_2}{\partial t^2} + 2\frac{\partial q_1}{\partial t} + 2. \tag{4.2.49}$$

Integrating (4.2.48) we get

$$\frac{\partial q_1}{\partial t} = -2\frac{\partial q_2}{\partial t} - 2q_2 - t. \tag{4.2.50}$$

Substituting (4.2.48) and (4.2.50) in (4.2.49) we obtain

$$\frac{\partial^2 q_2}{\partial t^2} - \frac{4}{3}q_2 = -\frac{4}{3} + \frac{2}{3}t. \tag{4.2.51}$$

The general solution of the homogeneous equation is

$$q_2(t) = Ae^{2t\sqrt{3}} + Be^{-2t\sqrt{3}}. \tag{4.2.52}$$

Choosing a particular solution as

$$q_2^{part} = -\frac{1}{2}t + 1, \tag{4.2.53}$$

and inserting it in (4.2.50) ,we obtain

$$q_2(t) = Ae^{2t\sqrt{3}} + Be^{-2t\sqrt{3}} - \frac{1}{2}t + 1. \tag{4.2.54}$$

Substituting (4.2.54) in (4.2.50) and integrating the result, q_1 is determined as

$$q_1(t) = Ke^{2t\sqrt{3}} + Le^{-2t\sqrt{3}} - Mt. \tag{4.2.55}$$

4.2.3 Example Three

Let us consider the singular Lagrangian discussed in chapter two and chapter three

$$L = \frac{m}{2}(\dot{q_1}^2 + \dot{q_2}^2 + l^2\dot{q_3}^2 + 2l\dot{q}_1\dot{q}_3 \cos q_3 + 2l\dot{q}_2\dot{q}_3 \sin q_3) + V(q_1, q_2, q_3), \qquad (4.2.56)$$

where l and m are constants. Since the rank of the Hess matrix is two,

this system can be be treated as a continuous system in the form

$$q_1 = q_1(t, q_3), \qquad\qquad q_2 = q_2(t, q_3), \qquad (4.2.57)$$

Now , let us write $\dot{q}_1 \quad and \quad \dot{q}_3$ as

$$\dot{q}_1 = \frac{dq_1}{dt} = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3}\dot{q}_3, \qquad\qquad \dot{q}_2 = \frac{dq_2}{dt} = \frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3}\dot{q}_3. \qquad (4.2.58)$$

Relations (4.2.58) can be replaced in (4.2.56) to obtain the following "modified Lagrangian" L' as

$$\begin{aligned} L' = {} & \frac{m}{2}\left(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3}\dot{q}_3\right)^2 + \frac{m}{2}\left(\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3}\dot{q}_3\right)^2 + \frac{m}{2}l^2\dot{q_3}^2 \\ & + ml\dot{q}_3\cos q_3 \left(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3}\dot{q}_3\right) + ml\dot{q}_3\sin q_3 \left(\frac{\partial q_2}{\partial t} + \frac{\partial q_2}{\partial q_3}\dot{q}_3\right) + V'(q_1, q_2) \end{aligned} \qquad (4.2.59)$$

The Euler-Lagrange equations (4.1.1) read as

$$\begin{aligned} & \frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_1)}\right) + \frac{\partial}{\partial q_3}\left(\frac{\partial L'}{\partial(\partial_3 q_1)}\right) - \frac{\partial L'}{\partial q_1} = 0, \\ and \qquad & \frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_2)}\right) + \frac{\partial}{\partial q_3}\left(\frac{\partial L'}{\partial(\partial_3 q_2)}\right) - \frac{\partial L'}{\partial q_2} = 0. \end{aligned} \qquad (4.2.60)$$

where $x_0 \equiv t, \qquad x_2 \equiv q_2$. More explicitly, the second order partial differential equations are

$$\begin{aligned} & m\frac{\partial^2 q_1}{\partial t^2} + 2m\dot{q}_3\frac{\partial^2 q_1}{\partial t \partial q_3} + m\dot{q_2}^2\frac{\partial^2 q_1}{\partial {q_2}^2} + m\ddot{q}_2\frac{\partial q_1}{\partial q_3} \\ & \qquad + ml\ddot{q}_3\cos q_3 - 2ml\dot{q}_3\sin q_3 - V_{,1} = 0, \end{aligned} \qquad (4.2.61)$$

and

$$m\frac{\partial^2 q_2}{\partial t^2}+2m\dot{q}_3\frac{\partial^2 q_2}{\partial t\partial q_3}+m{\dot{q}_2}^2\frac{\partial^2 q_2}{\partial {q_2}^2}+m\ddot{q}_2\frac{\partial q_2}{\partial q_3}$$
$$+ml\ddot{q}_3\sin q_3+2ml{\dot{q}_3}^2\cos q_3-V_{,2}=0, \tag{4.2.62}$$

The usual Hamiltonian H_0 and the constraint relation H_3 are given as

$$H_0'=p_0+\frac{{p_1}^2+{p_2}^2}{2m}-V=0, \tag{4.2.63}$$

and

$$H_3=-ml^2{\dot{q}_3}^2-ml(\dot{q}_1\cos q_3+\dot{q}_2\sin q_3). \tag{4.2.64}$$

Hence , $G_0 \quad and \quad G_3$ defined in (4.1.2) respecify are

$$\begin{aligned}
G_0 &= H_0\left(q_a,x_\mu,p_a=\frac{\partial L}{\partial \dot{q}_a}\right)\\
&=\frac{m}{2}\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}\dot{q}_3\right)^2+\frac{m}{2}\left(\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_3}\dot{q}_3\right)^2+\frac{m}{2}l^2{\dot{q}_3}^2\\
&\quad+ml\dot{q}_3\cos q_3\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}\dot{q}_3\right)+ml\dot{q}_3\sin q_3\left(\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_3}\dot{q}_3\right)\\
&\quad+V'(q_1,q_2),
\end{aligned} \tag{4.2.65}$$

and

$$\begin{aligned}
G_3 &= H_3\left(q_a,x_\mu,p_a=\frac{\partial L}{\partial \dot{q}_a}\right)\\
&=-ml^2\dot{q}_3-ml\cos q_3\left(\frac{\partial q_1}{\partial t}+\frac{\partial q_1}{\partial q_3}\dot{q}_3\right)-ml\cos q_3\left(\frac{\partial q_2}{\partial t}+\frac{\partial q_2}{\partial q_3}\dot{q}_3\right).
\end{aligned} \tag{4.2.66}$$

Now we are ready to test whether (4.1.2) are satisfied or not. In Fact (4.1.2) for $\mu=0$ is

$$dG_0=-\frac{\partial L'}{\partial t}dt=0. \tag{4.2.67}$$

Explicitly,

$$ml\dot{q}_1{\dot{q}_3}^2\sin q_3-ml\dot{q}_2{\dot{q}_3}^2\cos q_3+l\dot{q}_3\cos q_3V_{,1}+l\dot{q}_3\sin q_3V_{,2}=0. \tag{4.2.68}$$

Equation (4.1.2) for $\mu = 3$ is

$$\frac{dG_3}{dt} = 0, \tag{4.2.69}$$

Explicitly,

$$ml\dot{q}_1\dot{q}_3{}^2\sin q_3 - ml\dot{q}_2\dot{q}_3{}^2\cos q_3 - l\dot{q}_3\cos q_3 V_{,1} - l\dot{q}_3\sin q_3 V_{,2} = 0, \tag{4.2.70}$$

Subtracting equation (4.2.68) and (4.2.70) we get

$$2l\dot{q}_3\cos q_3 V_{,1} + 2l\dot{q}_3\sin q_3 V_{,2} = 0. \tag{4.2.71}$$

Choosing $V = q_1 \cos q_3 + q_2 \sin q_3$ equation (4.2.71) becomes

$$2l\dot{q}_3(\cos^2 q_3 + \sin^2 q_3) = 0 \tag{4.2.72}$$

The solution of equation (4.2.72) is

$$q_3 = constant \tag{4.2.73}$$

Substituting Eq (4.2.73) in Eq (4.2.61), we get

$$m\frac{\partial^2 q_1}{\partial^2 t} - V_{,1} = 0. \tag{4.2.74}$$

The solution of Eq. (4.2.74) is obtain as

$$q_1 = \frac{t^2}{2m} + At + B, \tag{4.2.75}$$

where A and B are constants.

Substituting Eq. (4.2.75) in Eq. (4.2.62), becomes

$$q_2 = \frac{t^2}{2m} + Ct + D, \tag{4.2.76}$$

where C and D are constants.

4.2.4 Example Four

As a last example let us consider the Mittelstaedt's Lagrangian model (20), which is given as

$$L = \frac{1}{2m}(\dot{q}_1 + \dot{q}_2)^2 + \frac{1}{2\mu}\dot{q}_3{}^2 + V(q_1, q_2, q_3) \tag{4.2.77}$$

This Lagrangian is singular since the Hess matrix is of rank two.

This system can be be treated as a continuous system in the form

$$q_1 = q_1(t, q_2), \qquad\qquad q_3 = q_3(t, q_2), \tag{4.2.78}$$

$$\dot{q}_1 = \frac{dq_1}{dt} = \frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2, \qquad \dot{q}_3 = \frac{dq_3}{dt} = \frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2. \tag{4.2.79}$$

Relations (4.2.79) with can be replaced in (4.2.77) to obtain following the "modified Lagrangian" L' as:

$$L' = \frac{1}{2m}\left[\left(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_3}\dot{q}_3\right) + \dot{q}_2\right]^2 + \frac{1}{2\mu}\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right)^2 + V(q_1, q_3). \tag{4.2.80}$$

The Euler-Lagrange equations (4.1)read as

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_1)}\right) + \frac{\partial}{\partial q_2}\left(\frac{\partial L'}{\partial(\partial_2 q_1)}\right) - \frac{\partial L'}{\partial q_1} = 0, \\ and \qquad &\frac{\partial}{\partial t}\left(\frac{\partial L'}{\partial(\partial_0 q_3)}\right) + \frac{\partial}{\partial q_2}\left(\frac{\partial L'}{\partial(\partial_2 q_3)}\right) - \frac{\partial L'}{\partial q_3} = 0. \end{aligned} \tag{4.2.81}$$

where $x_0 \equiv t, \qquad\qquad x_2 \equiv q_2.$

More explicitly, the second order partial differential equations are

$$\frac{\partial^2 q_1}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_1}{\partial t \partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_1}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_1}{\partial q_2} - \ddot{q}_2 - mV_1 = 0, \tag{4.2.82}$$

$$\frac{\partial^2 q_3}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_3}{\partial t \partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_1}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_3}{\partial q_2} - \mu V_{,1} = 0, \tag{4.2.83}$$

The quantities $H_0 \quad and \quad H_3$ are

$$H_0 = \frac{m}{2}p_1^2 + \frac{\mu}{2}p_3^2 - V(q_1, q_3), \tag{4.2.84}$$

$$H_3 = p_3 = \frac{1}{\mu}\dot{q}_3. \tag{4.2.85}$$

Hence,

$$\begin{aligned} G_0 &= H_0\left(q_a, x_\mu, p_a = \frac{\partial L}{\partial \dot{q}_a}\right) \\ &= \frac{1}{2m}\left[\left(\frac{\partial q_1}{\partial t} + \frac{\partial q_1}{\partial q_2}\dot{q}_2\right) + \dot{q}_2\right]^2 + \frac{1}{2\mu}\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right)^2 - V(q_1, q_3), \end{aligned} \tag{4.2.86}$$

$$\begin{aligned} G_3 &= H_3\left(q_a, x_\mu, p_a = \frac{\partial L}{\partial \dot{q}_a}\right) \\ &= \frac{1}{\mu}\left(\frac{\partial q_3}{\partial t} + \frac{\partial q_3}{\partial q_2}\dot{q}_2\right). \end{aligned} \tag{4.2.87}$$

Now we are ready to test whether (4.1.2) are satisfied or not . In fact (4.1.2) for $\mu = 0$ is

$$dG_0 = -\frac{\partial L'}{\partial t}dt = 0. \tag{4.2.88}$$

$$dG_0 = \left(\dot{q}_2\frac{\partial V}{\partial q_1}\right)dt = 0. \tag{4.2.89}$$

for $\mu = 3$ is

$$dG_3 = -\frac{\partial L'}{\partial q_3}dt = \left(\frac{\partial V}{\partial q_3}\right)dt = 0. \tag{4.2.90}$$

The Eqs. (4.2.82) and (4.2.83) becomes,

$$\frac{\partial^2 q_1}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_1}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_1}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_1}{\partial q_2} + \ddot{q}_2 = 0, \tag{4.2.91}$$

$$\frac{\partial^2 q_3}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_3}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_1}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_3}{\partial q_2} = 0. \tag{4.2.92}$$

Eqs. (4.2.91) and (4.2.92) are second order partial differential equations. From equations (4.2.79) the second derivatives of $q_1 \quad and \quad q_3$ are

$$\ddot{q}_1 = \frac{\partial^2 q_1}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_1}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_1}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_1}{\partial q_2}, \tag{4.2.93}$$

$$\ddot{q}_3 = \frac{\partial^2 q_3}{\partial t^2} + 2\dot{q}_2\frac{\partial^2 q_3}{\partial t\partial q_2} + \dot{q}_2{}^2\frac{\partial^2 q_3}{\partial q_2{}^2} + \ddot{q}_2\frac{\partial q_3}{\partial q_2}. \tag{4.2.94}$$

Therefore Eq. (4.2.91) becomes

$$\ddot{q}_1 + \ddot{q}_2 = 0. \tag{4.2.95}$$

The solution of Eq. (4.2.95) is given as

$$q_1(t) = -q_2 + At + B, \tag{4.2.96}$$

where A and B are constants

The Eq. (4.2.92) becomes

$$\ddot{q}_3 = 0, \tag{4.2.97}$$

with solution given as

$$q_3(t) = Ct + D. \tag{4.2.98}$$

where C and D are constants.

Chapter 5

Conclusion

Chapter 5

Conclusion

The Hamiltonian and Lagrangian formulations of singular Lagrangian systems are used to investigate some models of physical systems study to compare these techniques of these formulation.
In the Hamiltonian formulation both Dirac's method and Hamilton-Jacobi method (Güler's method) are used. In the Lagrangian formulation. The technique of treatment the singular Lagrangian as field (continuous) system was used. Besides, the Hamilton-Jacobi method is unified with Lagrangian formulation.

In Dirac's method, one introduces primary constraints to construct the total Hamiltonian, which consists of the primary constraints multiplied by the Lagrange multipliers added to canonical (usual) Hamiltonian.
The first - class constraints have vanishing poisson brackets. The equation of motion are obtained as total derivatives interms of poisson brackets.

In Hamilton-Jacobi formulation, which developed by Güler, the equations of motion are written as total differential equations in many variables. These equations must satisfy the integrability conditions. If the integrability conditions are not identically satisfied, then these will be continued until we obtain a complete system. Three models of physical system are discussed using these two method. The result are in exact agreement. In Hamilton-Jacobi method, we did not need to introduce an unknown multipliers as in Dirac's method.

In chapter four, the same physical models are discussed as field (continuous) systems in Lagrangian formulation. We mixed both Lagrangian formulation and Hamilton-Jacobi method to obtain a solvable partial differential equations of second order. The Euler - Lagrange equations of motion for field system are used to obtain the equations of motion.

Simultaneous solution of Euler - Lagrange equations with the constraints equations gives us the solutions of the dynamical systems.

These constraints equations are obtained from Hamilton-Jacobi approach.

These solutions satisfied the equations of motion that obtained in both Dirac's method and Hamilton-Jacobi method.

In fact, this comparison study needs more applications in physical systems in classical mechanics and field theory.

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