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Target Space Duality with Dilaton and Tachyon Field

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By

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Target Space Duality
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We study the target space duality of classical two dimensional sigma models. The models with dilaton and tachyon field are analyzed. As a motivating example the historical electric-magnetic duality is presented. We review the construction of the duality transformation and the integrability conditions for the nonlinear sigma models with target spaces described by general metrics and antisymmetric two-forms. We generalize the formalism for the models whose actions contain the dilaton and tachyon field. For the dilaton field case it is required that the duality is a property solely of the target manifolds, independent of the world-sheet geometry. For both cases the duality transformation is established and the integrability conditions are calculated. The set of restrictions on geometrical data describing the models is obtained, the previously calculated condition on connections on target spaces is maintained in both cases.

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Chapter 1

Fundamental Concepts

1.1 Introduction

The studies [3, 5, 6] of the classical target space duality with two mathematically non-equivalent nonlinear sigma models reveal many interesting results, the most remarkable is the existence of duality transformation between the models for which the target manifolds are symmetric spaces with opposite curvature. Here we generalize the previous results, in the classical action of the non-linear sigma model we will include two terms required by the renormalization theory: the tachyon and the dilaton field. Both cases are treated separately in interest of simplicity in Chapter 2 and Chapter 3 respectively.

The first steps we have to take in order to develop our construction is to specify the sigma models and to define the duality giving a meaning to the statement "...both models are physically equivalent". Before addressing these issues it would be instructive to review the first known historical example of duality transformation in field theory, the electric-magnetic duality transformation, which is presented in Section 1.2. In Section 1.3 we introduce the classical non-linear sigma model. Following [3] we will define the duality transformation in Section 1.4 for the basic case (i.e. for the action without the dilaton and tachyon terms). In Section 1.5 we present another approach defining pseudoduality and calculate the integrability conditions. These results are generalized in Chapters 2 and 3.

1.2 Electric-Magnetic Duality Transformation

Constructing the formalism in which the electric and the magnetic field appear in a symmetric way might have been Maxwell's motivation to include in the equations the "displacement current" $\frac{\partial \mathbf{E}}{\partial t}$ term, unknown at that time from the experimental observations [1]. The set of sourceless Maxwell equations is invariant under the simultaneous replacement

$$\mathbf{E} \rightarrow -\mathbf{B}, \qquad \mathbf{B} \rightarrow -\mathbf{E}.$$

This constitutes a $\mathbb{Z}_2$ discrete symmetry, often referred as a symmetry under discrete *electric-magnetic duality transformation*. In order to maintain this symmetry in presence of sources (electric charge density ρ_e and electric current $\mathbf{j}_e$) we have to include the presence of unobserved magnetic charge density ρ_m and magnetic current $\mathbf{j}_m$ in the set of equations

$$\begin{aligned} \operatorname{rot} \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} &= -\frac{4\pi}{c}\mathbf{j}_m, & \operatorname{div} \mathbf{B} &= 4\pi\rho_m, \\ \operatorname{rot} \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} &= \frac{4\pi}{c}\mathbf{j}_e, & \operatorname{div} \mathbf{E} &= 4\pi\rho_e. \end{aligned} \tag{1.2.1}$$

The $\mathbb{Z}_2$ symmetry may be generalized to $SO(2)$ continuous symmetry by substituting

$$\widetilde{\mathbf{E}} = \mathbf{E}\cos\beta - \mathbf{B}\sin\beta\,, \qquad \widetilde{\mathbf{B}} = \mathbf{E}\sin\beta + \mathbf{B}\cos\beta \tag{1.2.2}$$

and correspondingly[1] for charges and the currents

$$\widetilde{\rho}_e = \rho_e \cos\beta - \rho_m \sin\beta\,, \qquad \widetilde{\rho}_m = \rho_e \sin\beta + \rho_m \cos\beta,$$

$$\widetilde{\mathbf{j}}_e = \mathbf{j}_e \cos\beta - \mathbf{j}_m \sin\beta\,, \qquad \widetilde{\mathbf{j}}_m = \mathbf{j}_e \sin\beta + \mathbf{j}_m \cos\beta.$$

In covariant notation the equations (1.2.1) are

$$\nabla_\nu F^{\mu\nu} = -4\pi j_e^\mu\,, \tag{1.2.3}$$

$$\nabla_\nu\, {}^*F^{\mu\nu} = -4\pi j_m^\mu\,, \tag{1.2.4}$$

where the 2-form $^*\mathbf{F}$ is a Hodge dual to the field strength 2-form $\mathbf{F}$ defined by

$$\begin{aligned} \mathbf{F} =& B_z\, dx \wedge dy - B_y\, dx \wedge dz + B_x\, dy \wedge dz \\ &+ E_x\, dx \wedge dt + E_y\, dy \wedge dt + E_z\, dz \wedge dt. \end{aligned} \tag{1.2.5}$$

The duality transformation of the fields (1.2.2) may be written now as

$$\begin{pmatrix} \widetilde{\mathbf{F}} \\ {}^*\widetilde{\mathbf{F}} \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \mathbf{F} \\ {}^*\mathbf{F} \end{pmatrix}. \tag{1.2.6}$$

The action of the electromagnetic field in vacuum in terms of components of $\mathbf{F}$ is given by

$$S_0[F^{\mu\nu}] = \frac{1}{4\pi} \int d^4x F^{\mu\nu} F_{\mu\nu}. \tag{1.2.7}$$

[1] At the classical level there is a correspondence between the electric-magnetic duality and the non-existence of the magnetic monopole. If we assume that for all the particles the ratio $q_{\text{electric}}/q_{\text{magnetic}}$ is the same then we may always find an angle β for which $\widetilde{\mathbf{B}}$ is divergenceless. At the quantum level however we have to take into account the Dirac quantization condition.

Using (1.2.6) and remembering that $F^{\mu\nu}F_{\mu\nu} = {}^*F^{\mu\nu}\,{}^*F_{\mu\nu}$ we see immediately that under the duality transformation the action changes only by a scaling factor, i.e.

$$(\cos^2\beta - \sin^2\beta)\, S_0[F^{\mu\nu}] = \frac{1}{4\pi}\int d^4x \widetilde{F}^{\mu\nu}\widetilde{F}_{\mu\nu} \equiv \widetilde{S}_0[\widetilde{F}^{\mu\nu}]. \tag{1.2.8}$$

Note that for $\beta = (1/4 + n/2)\,\pi$ the action $\widetilde{S}_0$ vanishes.

The symmetric electromagnetic energy-momentum tensor is

$$T^{\mu\nu} = -\frac{1}{8\pi} g_{\alpha\beta}\left(F^{\mu\alpha}\, F^{\nu\beta} +^* F^{\mu\alpha}\, {}^*F^{\nu\beta}\right)\ , \tag{1.2.9}$$

clearly the duality transformation "preserves" the form of this tensor, by "preserving" we mean

$$T^{\mu\nu}(\mathbf{F}) = \widetilde{T}^{\mu\nu}(\widetilde{\mathbf{F}}). \tag{1.2.10}$$

The detailed studies of other aspects of electric-magnetic duality transformation is presented by Donev [2].

1.3 Nonlinear Sigma Model

The mapping $x : \Sigma \rightarrow M$ from the *world sheet* (2-dimensional space Σ with metric having the signature $(-, +)$ onto the *target space* (n-dimensional manifold M) may be viewed as a two dimensional model of n bosonic fields or as a string, 1-dimensional object evolving in time on manifold M. The target space will be parametrized by coordinates x^i, the world sheet by space-time coordinates $\vec{\sigma} = (\sigma, \tau)$ or by the light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$. We will study only local issues here. In order to specify Σ and M we need to prescribe the metrics $h_{\alpha\beta}$ and g_{ij} respectively, we will use torsion-free Riemannian connection. Now we are at the stage to introduce the action S which is required to be both Diff(Σ) and Diff(M) equivariant. The basic expression satisfying this requirement is

$$S = -\frac{1}{2} \int d^2\sigma \{ \sqrt{-h} h^{\alpha\beta} g_{ij}(x) \partial_\alpha x^i \partial_\beta x^j - \epsilon^{\alpha\beta} \partial_\alpha x^i \partial_\beta x^j B_{ij}(x) \} \tag{1.3.1}$$

where B_{ij} are the components of an antisymmetric 2-form B. The renormalization

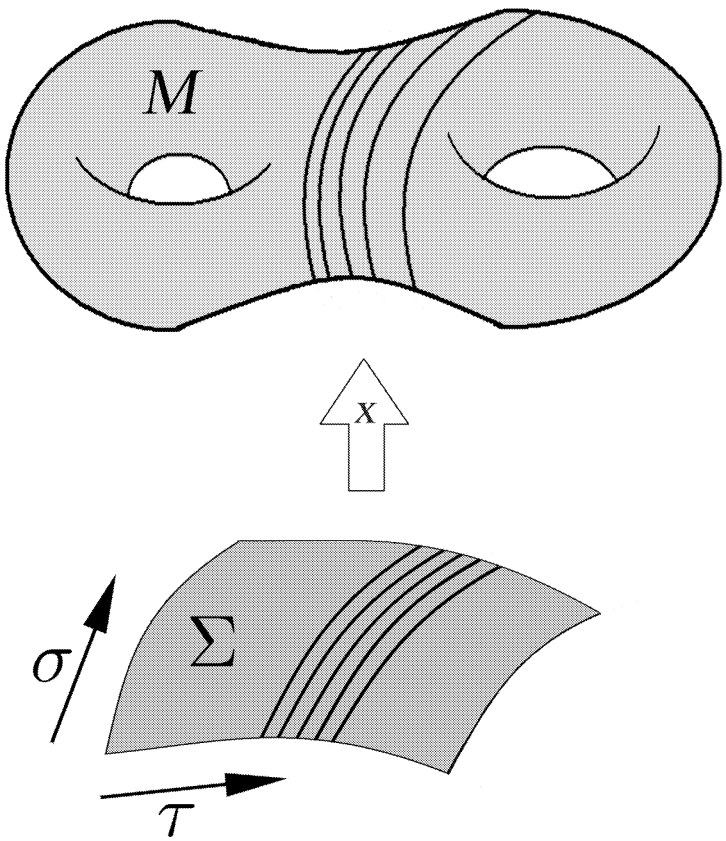

Figure 1.1: $x : \Sigma \to M$ viewed as a string moving on n-dimensional target space M.

theory requires considering more general action with the following terms:

$$S_V = \int d^2\sigma \sqrt{-h} V(x) \,, \tag{1.3.2}$$

$$S_\Phi = \int d^2\sigma \sqrt{-h} \Phi(x) R^{(2)}(\vec{\sigma}) \,. \tag{1.3.3}$$

We will refer to $V(x)$ appearing in the first term as the *potential function* or *tachyon field*. In the second term we have the *dilaton field* $\Phi(x)$ coupled to the world-sheet curvature scalar $R^{(2)}(\vec{\sigma})$. Both cases of nonlinear sigma model with potential function and dilaton field will be considered separately.

As shown in [3], studying the duality transformations can be simplified by using the local orthonormal coframing θ^i on M. We denote the torsion free Riemannian connection 1-forms as ω_{ij} which satisfy the Cartan structural equations:

$$0 = \omega_{ij} + \omega_{j\,i}\,, \tag{1.3.4}$$

$$d\theta^i = -\,\omega_{ij} \wedge \theta^j \tag{1.3.5}$$

$$d\omega_{ij} = -\,\omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\, R_{ijkl}\, \theta^k \wedge \theta^l\,. \tag{1.3.6}$$

The derivatives $x^i{}_\alpha$ are defined by the pullbacks to Σ:

$$x^*\theta^i = x^i{}_\alpha\, d\sigma^\alpha\,. \tag{1.3.7}$$

The covariant second derivatives are given by

$$x^i{}_{\mu;\,\nu} d\sigma^\nu \equiv dx^i{}_\mu + x^*\omega_{ij} x^j{}_\mu\,, \tag{1.3.8}$$

where x^* is the pullback. The connection coefficients $\omega_{ij\,\mu}$ are obtained from

$$x^*\omega_{ij} = \omega_{ij\mu}\, d\sigma^\mu\,. \tag{1.3.9}$$

1.4 Definition of Duality. Basic Developments

In our set-up we will have two non-equivalent nonlinear sigma models with target spaces M and $\widetilde{M}$. Any quantities which refer to the model with the target space $\widetilde{M}$ will be denoted by the same letter as the quantities which refer to the model with the target space M, but they will be distinguished by the tilde sign, e.g. the derivatives for both models are defined by

$$x^*\theta^i = x^i_\alpha \, d\sigma^\alpha \quad \text{and} \quad \tilde{x}^*\tilde{\theta}^i = \tilde{x}^i_\alpha \, d\sigma^\alpha \, . \tag{1.4.1}$$

Constructing the duality transformation for the cases of interest (i.e. the action with the dilaton or tachyon field terms) requires individual treatment. Here we will review the developments of the formalism for the action described by (1.3.1), the generalization is presented in Chapters 2 and 3.

The first step is to define when two non-linear sigma models are dual, see [8]. We will restrict our attention only to the local[2] and define:

Two nonlinear sigma models with target manifolds M and $\widetilde{M}$ are said to be dual to each other if there exists a canonical transformation between their respective phase spaces T^*PM and $T^*P\widetilde{M}$ that preserves the hamiltonian density, i.e. maps $\mathcal{H}$ onto $\widetilde{\mathcal{H}}$

In a classical mechanical hamiltonian system we may change the coordinates on the phase space from q and p to $\tilde{q} = \tilde{q}(q, p, t)$ and $\tilde{p} = \tilde{p}(q, p, t)$. This transformation is canonical if the new equations still have the form

$$\dot{\tilde{q}} = \frac{\partial \widetilde{H}}{\partial \tilde{p}} \, , \qquad \dot{\tilde{p}} = -\frac{\partial \widetilde{H}}{\partial \tilde{q}} \tag{1.4.2}$$

with another $\widetilde{H}$. Obviously the equations of motion do not retain their form. The Hamilton's equations may be derived by the principle of the stationary action

$$\delta \int (p \, dq - H \, dt) = 0 \, . \tag{1.4.3}$$

[2]Studying the global questions even for the most straightforward example, the circular target spaces, we find that the definition needs further refinement. The global issues for the system of partial differential equations, which are the duality equations, is a very difficult matter.

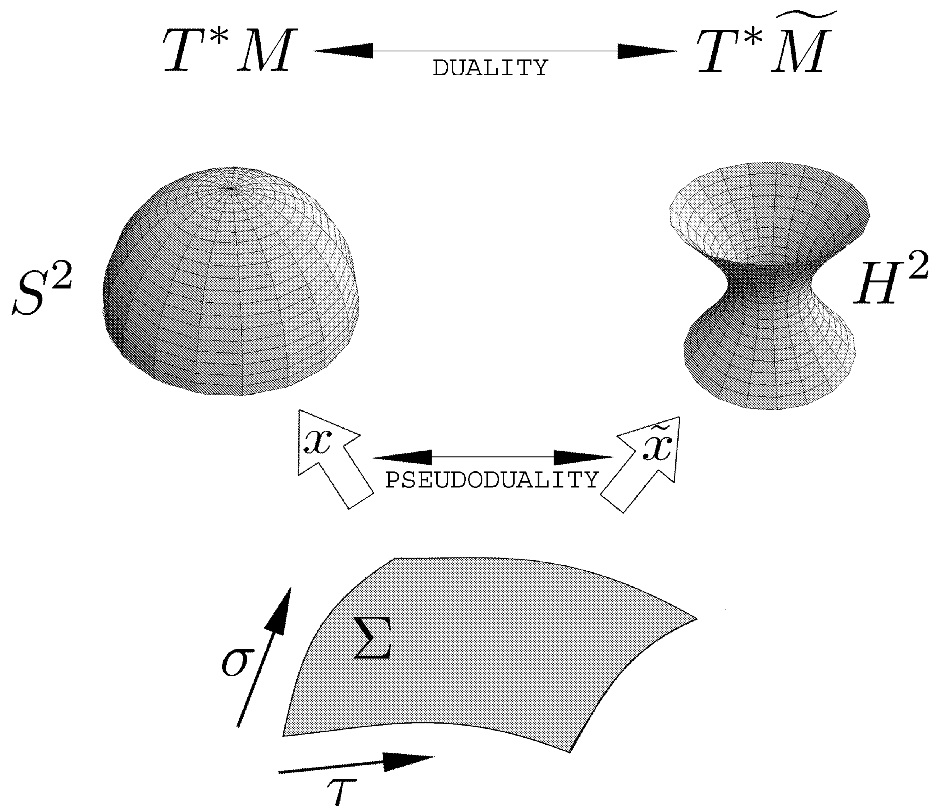

Figure 1.2: Duality versus pseudoduality. For definitions and explanations see Section 1.4 and 1.5 respectively. The targets spaces are sphere S^2 and hyperbolic surface H^2 with constant curvature. One should not take the drawing of the hyperboloid literally as the 2-dimensional space with constant negative curvature, there is no global 3-dimensional euclidean embedding.

Describing the system in terms of new variables we expect that

$$\delta \int \left(\tilde{p}\, d\tilde{q} - \widetilde{H}\, dt\right) = 0\,. \tag{1.4.4}$$

Two integrals are the same if their integrands differ by the total differential of some function F, called the generating function,

$$\tilde{p}\, d\tilde{q} - \widetilde{H}\, dt = p\, dq - H\, dt + dF\,. \tag{1.4.5}$$

For time-independent $F(q,\tilde{q})$ we have

$$\tilde{p} = \frac{\partial F}{\partial \tilde{q}}\,, \quad p = -\frac{\partial F}{\partial q}\,, \quad H = \widetilde{H}\,. \tag{1.4.6}$$

Our case differs from the classical mechanics analogy in two ways. First, in the field theory we have an infinite-dimensional phase space with local coordinates $(x(\sigma), \pi(\sigma))$, where the canonical momentum density is defined by

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = g_{ij}\dot{x}^j + B_{ij}\frac{\partial x^j}{\partial \sigma}\,. \tag{1.4.7}$$

Instead of the generating function we have to use the generating functional, we obtain the canonical transformation by the variations

$$\tilde{\pi}_i = \frac{\delta F}{\delta \tilde{x}^i}\,, \quad \pi_i = -\frac{\delta F}{\delta x^i}\,. \tag{1.4.8}$$

The second issue is that the arbitrary generating functional may lead to a model whose metric $\tilde{g}$ and antisymmetric form $\widetilde{B}$ might not be functions of only $\tilde{x}$, it may happen that $\tilde{g}$ or $\widetilde{B}$ depends on $\tilde{\pi}$ which is undesirable here. The general form of the generating functional that leads to the linear relationship between (x_σ, π) and $(\tilde{x}_\sigma, \tilde{\pi})$ may be written as

$$F\left[x(\sigma), \tilde{x}(\sigma)\right] = \int \alpha = \int d\sigma \left(\alpha_i(x(\sigma), \tilde{x}(\sigma))\frac{dx^i}{d\sigma} + (\tilde{\alpha}_i(x(\sigma), \tilde{x}(\sigma))\frac{d\tilde{x}^i}{d\sigma} \right)\,, \tag{1.4.9}$$

where 1-form α is defined on $M \times \widetilde{M}$ and the local components are given by

$$\alpha = \alpha_i(x,\tilde{x})dx^i + \tilde{\alpha}_i(x,\tilde{x})d\tilde{x}^i\,. \tag{1.4.10}$$

To calculate the variations (1.4.8) we may represent the deformation of the path $(x(\vec{\sigma}), \tilde{x}(\vec{\sigma})) \in M \times \widetilde{M}$ by the vector field v, the variation is given by the Lie derivative

$$\delta_v F = \int \mathcal{L}_v \alpha \,. \tag{1.4.11}$$

The Cartan's formula for the Lie derivative acting on an exterior form is

$$\mathcal{L}_v = \iota_v \circ d + d \circ \iota_v \,. \tag{1.4.12}$$

The variation of F is therefore

$$\delta_v F = \int \{\iota_v \, d\alpha + d\,(\iota_v \, \alpha)\} \,. \tag{1.4.13}$$

The vector field has compact support, subsequently the exact term $d\,(\iota_v \, \alpha)$ can be omitted. In local coordinate systems on M and $\widetilde{M}$ we introduce the components $l_{ij}(x, \tilde{x})$, $\tilde{l}_{ij}(x, \tilde{x})$ and $m_{ij}(x, \tilde{x})$ of $d\alpha$ as

$$d\alpha = -\frac{1}{2} \; l_{ij}(x, \tilde{x}) \, dx^i \wedge dx^j + -\frac{1}{2} \; \tilde{l}_{ij}(x, \tilde{x}) \, d\tilde{x}^i \wedge d\tilde{x}^j + m_{ij}(x, \tilde{x}) \, d\tilde{x}^i \wedge dx^j \tag{1.4.14}$$

where $l_{ij} = -l_{ji}$ and $\tilde{l}_{ij} = -\tilde{l}_{ji}$. Now we have to calculate $\iota_v \, d\alpha$. The interior product ι_v is an antiderivation, for one-forms α and β it holds that

$$\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta - \alpha \wedge (\iota_v \beta) \,. \tag{1.4.15}$$

If we choose for v

$$v = \delta x^k \frac{\partial}{\partial x^k} \tag{1.4.16}$$

and use (1.4.15) then the variation of F is

$$\delta_v F = -\int \delta x^k \, d\sigma \left(l_{kj} \frac{dx^j}{d\sigma} + m_{jk} \frac{d\tilde{x}^j}{d\sigma} \right) \,. \tag{1.4.17}$$

Similarly we may choose

$$v = \delta\tilde{x}^k \frac{\partial}{\partial\tilde{x}^k} \tag{1.4.18}$$

and compute

$$\delta_v F = \int \delta x^k \, d\sigma \left(m_{kj} \frac{dx^j}{d\sigma} + \tilde{l}_{kj} \frac{d\tilde{x}^j}{d\sigma} \right) . \tag{1.4.19}$$

From (1.4.17) and (1.4.19) we read that the canonical transformation is

$$\begin{aligned} \pi_k &= l_{kj} \, \frac{dx^j}{d\sigma} + m_{jk} \frac{d\tilde{x}^j}{d\sigma} \, , \\ \tilde{\pi}_k &= m_{kj} \frac{dx^j}{d\sigma} + \tilde{l}_{kj} \frac{d\tilde{x}^j}{d\sigma} \, . \end{aligned} \tag{1.4.20}$$

In matrix notation the above transformation is

$$\begin{pmatrix} m^t & 0 \\ -\tilde{l} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \tilde{x}_\sigma \\ \tilde{\pi} \end{pmatrix} = \begin{pmatrix} -l & \mathbb{1} \\ m & 0 \end{pmatrix} \begin{pmatrix} x_\sigma \\ \pi \end{pmatrix} . \tag{1.4.21}$$

The hamiltonian density and the momentum density for the model described by the action (1.3.1) are given respectively by

$$\mathcal{H} = \frac{1}{2} \, g^{ij}(x) \left(\pi_i - B_{ik} \frac{dx^k}{d\sigma} \right) \left(\pi_j - B_{j\,l} \frac{dx^l}{d\sigma} \right) + \frac{1}{2} \, g_{ij}(x) \, \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \, , \tag{1.4.22}$$

$$\mathcal{P} = \pi_i \frac{dx^i}{d\sigma} \tag{1.4.23}$$

and by analogous expressions for $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{P}}$. We will subsequently see why it is convenient to choose the orthonormal coframes on M and $\widetilde{M}$. We define Ψ and $\widetilde{\Psi}$ as

$$\Psi \equiv \begin{pmatrix} x^1_{\ \sigma} \\ x^2_{\ \sigma} \\ \vdots \\ \pi_1 - B_{1\,k}\, x^k_{\ \sigma} \\ \pi_2 - B_{2\,k}\, x^k_{\ \sigma} \\ \vdots \end{pmatrix} \qquad \text{and} \qquad \widetilde{\Psi} \equiv \begin{pmatrix} \tilde{x}^1_{\ \sigma} \\ \tilde{x}^2_{\ \sigma} \\ \vdots \\ \tilde{\pi}_1 - \widetilde{B}_{1\,k}\, \tilde{x}^k_{\ \sigma} \\ \tilde{\pi}_2 - \widetilde{B}_{2\,k}\, \tilde{x}^k_{\ \sigma} \\ \vdots \end{pmatrix} . \tag{1.4.24}$$

We will also define the following quadratic forms:

$$I_{2n} \equiv \begin{pmatrix} \mathbb{1}_{n\times n} & 0 \\ 0 & \mathbb{1}_{n\times n} \end{pmatrix} \tag{1.4.25}$$

and

$$Q_{2n} \equiv \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ \mathbb{1}_{n\times n} & 0 \end{pmatrix} . \tag{1.4.26}$$

If we indeed use the orthonormal coframes then both the hamiltonian and the momentum density may be expressed as

$$\mathcal{H} = \frac{1}{2}\, \Psi^t I_{2n} \Psi\, , \qquad \widetilde{\mathcal{H}} = \frac{1}{2}\, \widetilde{\Psi}^t I_{2n} \widetilde{\Psi} \tag{1.4.27}$$

and

$$\mathcal{P} = \frac{1}{2}\, \Psi^t Q_{2n} \Psi\, , \qquad \widetilde{\mathcal{P}} = \frac{1}{2}\, \widetilde{\Psi}^t Q_{2n} \widetilde{\Psi} \tag{1.4.28}$$

where in order to obtain (1.4.28) we used the antisymmetry of B and $\widetilde{B}$. We would like to express the canonical transformation written in terms of Ψ and $\widetilde{\Psi}$ in a form symbolically similar to (1.4.21). In order to do it we define

$$
\begin{aligned}
n\,(x,\tilde{x}) &:= l\,(x,\tilde{x}) - B\,(x,\tilde{x})\,, \\
\tilde{n}\,(x,\tilde{x}) &:= \tilde{l}\,(x,\tilde{x}) - \widetilde{B}\,(x,\tilde{x})\,.
\end{aligned}
\tag{1.4.29}
$$

With the above definitions we have

$$
\begin{pmatrix} m^t & 0 \\ -\tilde{n} & \mathbb{1} \end{pmatrix} \widetilde{\Psi} = \begin{pmatrix} -n & \mathbb{1} \\ m & 0 \end{pmatrix} \Psi\,. \tag{1.4.30}
$$

The canonical transformation (1.4.30) may be written as a linear transformation between Ψ and $\widetilde{\Psi}$ in a form

$$
\widetilde{\Psi} = \mathbf{M}\,\Psi \tag{1.4.31}
$$

with

$$
\mathbf{M} := \begin{pmatrix} -(m^t)^{-1}\,n & (m^t)^{-1} \\ -\tilde{n}\,(m^t)^{-1}\,n + m & \tilde{n}\,(m^t)^{-1} \end{pmatrix}. \tag{1.4.32}
$$

The condition that the linear transformation $\mathbf{M}$ preserves the hamiltonian density is the requirement that it preserves the quadratic form I_{2n}, i.e. the transformation $\mathbf{M}$ has to be in $\mathrm{O}(2n)$. We may consider also an additional requirement that it preserves the momentum density which means that the transformation belongs to $\mathrm{O}_Q(n,n)$, a group of matrices preserving the quadratic form Q_{2n}. We will write $\mathbf{M}$ in terms of $n \times n$ block matrices as

$$
\mathbf{M} = \begin{pmatrix} A & Z \\ C & D \end{pmatrix}. \tag{1.4.33}
$$

For $\mathbf{M} \in \mathrm{O}(2n)$ we have $\mathbf{M}^t\mathbf{M} = \mathbb{1}$ which it terms of block matrices is

$$
A^tA + C^tC = \mathbb{1}\,, \quad Z^tZ + D^tD = \mathbb{1}\,, \quad A^tZ + C^tD = 0\,. \tag{1.4.34}
$$

For $\mathbf{M} \in \mathrm{O}_q(n,n)$ we have $\mathbf{M}^t Q_{2n} \mathbf{M} = Q_{2n}$ which means that

$$A^t C + C^t A = 0\,, \quad Z^t D + D^t Z = 0\,, \quad A^t D + C^t Z = \mathbb{1}\,. \tag{1.4.35}$$

Let us first check what restrictions on $\mathbf{M}$ we have by requiring that the canonical transformation preserves the momentum density. Inspecting the conditions (1.4.35) for (1.4.32) we find a surprising result, all of these conditions are automatically satisfied, it means that $\mathbf{M}$ already belongs to $\mathrm{O}_Q(n,n)$ without necessity to fulfill any further requirements. The conditions that the canonical transformation $\mathbf{M}$ preserves the hamiltonian density are satisfied if

$$\begin{aligned} m m^t &= I + \tilde{n}^t \tilde{n}\,, \\ m^t m &= I + n^t n\,, \\ -mn &= \tilde{n} m\,. \end{aligned} \tag{1.4.36}$$

By defining $\tilde{m} = m^t$ and using the above relations we may simplify the form of $\mathbf{M}$ to

$$\mathbf{M} = \begin{pmatrix} -\tilde{m}^{-1}\, n & \tilde{m}^{-1} \\ \tilde{m}^{-1} & -\tilde{m}^{-1}\, n \end{pmatrix}. \tag{1.4.37}$$

Having satisfied (1.4.36) we have therefore $\mathbf{M} \in \mathrm{O}(2n) \cap \mathrm{O}_Q(n,n)$.

The coordinates on the respective phase spaces are (x^i_σ, π_σ) and $(\tilde{x}^i_\sigma, \tilde{\pi}_\sigma)$. Switching to the Lagrangian formalism we may formally write down the canonical transformation in terms of (x^i_σ, x^i_τ) and $(\tilde{x}^i{}_\sigma, \tilde{x}^i{}_\tau)$. Furthermore if we conjugate the transformation (1.4.31) with the operator

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \mathbb{1}_n & -\mathbb{1}_n \\ \mathbb{1}_n & \mathbb{1}_n \end{pmatrix}, \tag{1.4.38}$$

the duality transformation becomes

$$\begin{pmatrix} \tilde{x}_\sigma + \tilde{x}_\tau \\ \tilde{x}_\sigma - \tilde{x}_\tau \end{pmatrix} = \begin{pmatrix} \tilde{m}^{-1}(\mathbb{1} - n) & 0 \\ 0 & -\tilde{m}^{-1}(\mathbb{1} + n) \end{pmatrix} \begin{pmatrix} x_\sigma + x_\tau \\ x_\sigma - x_\tau \end{pmatrix} . \qquad (1.4.39)$$

In the above equation the matrix of transformation is block-diagonal since in the light-cone coordinates $\sigma^\pm = \tau \pm \sigma$ the solutions of the equations of motion are independently left and right moving traveling waves. We may introduce $\mathbf{T}_\pm = \tilde{m}^{-1}(\mathbb{1} \mp n) \in \mathrm{O}(n)$ and write

$$\tilde{x}^i_\pm = \pm(\mathbf{T}_\pm)^i_{\ j}\, x^j_\pm \, . \qquad (1.4.40)$$

1.5 Pseudoduality. Integrability Conditions

The less restrictive approach is to study "on-shell duality". Instead of constructing a transformation between the phase spaces we require only the existence of a transformation called *pseudoduality*[3] which is a map between solutions of the equations of motion of one model and solutions of the equations of motion of another model $\widetilde{M}$, see Fig.1.2. As we talk about the solutions of equations of motion, the central issue for this construction is integrability.

As a basic example we will take both sigma models with flat world-sheet and the action of the form (1.3.1) and we will put $B_{ij} = \widetilde{B}_{ij} = 0$ for both of them, namely

$$S = -\frac{1}{2} \int d^2\sigma \{ g_{ij}(x) \eta^{\alpha\beta} \partial_\alpha x^i \partial_\beta x^j \} \qquad (1.5.1)$$

and an analogous expression for $\widetilde{M}(\tilde{g})$ model. We will verify here a pseudoduality transformation by inspecting the stress-energy tensor in analogy to (1.2.10). The

[3]This term was introduced first in [7]

non-vanishing components of the stress-energy tensor for our models are:

$$T_{++} = g_{ij}(x)\,\partial_+ x^i \partial_+ x^j \qquad \widetilde{T}_{++} = \tilde{g}_{ij}(\tilde{x})\,\partial_+ \tilde{x}^i \partial_+ \tilde{x}^j$$

and

$$T_{--} = g_{ij}(x)\,\partial_- x^i \partial_- x^j \qquad \widetilde{T}_{--} = \tilde{g}_{ij}(\tilde{x})\,\partial_- \tilde{x}^i \partial_- \tilde{x}^j. \tag{1.5.2}$$

We may ask what is a general form of a linear transformation which preserves the form of the stress-energy tensor, i.e.

$$T_{++} = \widetilde{T}_{++} \qquad \text{and} \qquad T_{--} = \widetilde{T}_{--}\,. \tag{1.5.3}$$

The general transformation satisfying this requirement is written as

$$\begin{aligned} \tilde{x}_+(\vec{\sigma}) &= +\,\mathbf{T}_+(x,\tilde{x})\,x_+(\vec{\sigma})\,, \\ \tilde{x}_-(\vec{\sigma}) &= -\,\mathbf{T}_-(x,\tilde{x})\,x_-(\vec{\sigma})\,. \end{aligned} \tag{1.5.4}$$

Here $\mathbf{T}_+$ and $\mathbf{T}_-$ are orthogonal matrix valued functions $\mathbf{T}_\pm : \Sigma \to \mathrm{SO}(n)$. There is a temptation to study an algebraically simple example by restricting to the case $\mathbf{T}_+ = \mathbf{T}_- = \mathbb{1}$. This is however wrong, as shown by [6] we try to impose too restrictive constraint what results in the transformation which is not integrable.

We will try to construct a pseudoduality transformation with

$$\mathbf{T}_+ = \mathbf{T}_- \equiv \mathbf{T}. \tag{1.5.5}$$

In a geometrical, coordinate-free notation we may express (1.5.4) as

$$\tilde{x}^* \,\tilde{\theta} = x^* \star_\Sigma (\mathbf{T}\theta)\,, \tag{1.5.6}$$

where $\star_\Sigma$ operation denotes the Hodge duality on the world-sheet Σ. We may write

(1.5.6) in an equivalent yet simpler form by promoting the pull-backs to be not from the manifolds but from SO(M), the bundle of orthonormal coframes with global coframing $\{\omega^i, \omega_{jk}\}$. Symbolically it has the same form as the case with $\mathbf{T} = \mathbb{1}$, namely

$$\widetilde{X}^* \, \tilde{\omega}^i = X^* \, \star_\Sigma \, \omega^i \, . \tag{1.5.7}$$

Here however we have the lifts $X : \Sigma \to \mathrm{SO}(M)$ and $\widetilde{X} : \Sigma \to \mathrm{SO}(M)$. Once we have any expression containing the quantities defined with reference to orthonormal coframe bundles whenever we need we may rewrite this expression in terms of quantities defined with respect to a local coframe ω_U by choosing some neighborhood $U \subset M$. The coframe bundle SO(M) is locally diffeomorphic to $U \times \mathrm{SO}(n)$. The local coframe one-forms ω_U^i are related to the globally defined canonical one-forms ω_i by $\omega^i = (\mathbf{R}_U)^i{}_j \, \omega_U^j$ where the matrix $\mathbf{R}_U \in \mathrm{SO}(n)$ is defined over the neighborhood U. The global connection ω_{ij} and the Riemannian connection one-forms $(\omega_U)_{ij}$ for the local coframe ω_U^i are related over the neighbourhood U by

$$\omega_{ij} = \mathbf{R}_U (\omega_U)_{ij} \mathbf{R}_U^{-1} - d\mathbf{R}_U \mathbf{R}_U^{-1} \, . \tag{1.5.8}$$

Hereafter we will follow the convention in which the pullbacks are implicit and usually not explicitly written. In terms of the components[4] equation (1.5.6) is

$$\tilde{x}^i_{\pm} = \pm x^i_{\pm}. \tag{1.5.9}$$

We need to study what are the integrability conditions for (1.5.9) . Taking the exterior derivative of the above equation we have

$$-\tilde{\omega}^i_j \, \tilde{x}^j_{\pm} + \tilde{x}^i_{\pm;\,\mu} d\sigma^\mu = \mp \omega^i_j \, x^j_{\pm} \pm x^i_{\pm;\,\mu} d\sigma^\mu \, . \tag{1.5.10}$$

[4]Here $x^i{}_\mu$'s (and analogously $\tilde{x}^i{}_\mu$'s) are defined by the pull-back from the coframe bundle, $X^* \omega^i = x^i{}_\mu \, d\sigma^\mu$.

Now we collect the terms and wedge the above equation with $d\sigma^{\pm}$ obtaining

$$\tilde{x}_{\pm\mp}\, d\sigma^{\mp} \wedge d\sigma^{\pm} \mp x_{\pm\mp}\, d\sigma^{\mp} \wedge d\sigma^{\pm} = \pm x_{\pm}\, (\tilde{\omega} - \omega) \wedge d\sigma^{\pm}\,. \tag{1.5.11}$$

The equations of motion on the bundle may be written as $x^i_{\pm} \equiv x^i_{\mp} = 0$ and $\tilde{x}^i_{\pm} \equiv \tilde{x}^i_{\mp} = 0$. Using them we have

$$(\tilde{\omega} - \omega) \wedge d\sigma^{+} = (\tilde{\omega} - \omega) \wedge d\sigma^{-} = 0\,. \tag{1.5.12}$$

This gives us a condition on connection on the coframe bundles

$$\tilde{\omega}_{ij} - \omega_{ij} = 0\,. \tag{1.5.13}$$

To understand what it means in a local coframing we need to come back to $M \times \widetilde{M}$ using (1.5.8). Symbolically, the condition on the local connections $(\omega_U)_{ij}$ and $(\tilde{\omega}_{\widetilde{U}})_{ij}$ over the neighbourhoods $U \times \widetilde{U}$ is therefore

$$\mathbf{R}_U\, \omega_U\, \mathbf{R}_U^{-1} - d\mathbf{R}_U\, \mathbf{R}_U^{-1} = \widetilde{\mathbf{R}}_{\widetilde{U}}\;\, \tilde{\omega}_{\widetilde{U}}\, \widetilde{\mathbf{R}}_{\widetilde{U}}^{-1} - d\widetilde{\mathbf{R}}_{\widetilde{U}}\, \widetilde{\mathbf{R}}_{\widetilde{U}}^{-1}\,. \tag{1.5.14}$$

Here we will define $\mathbf{T} : M \times \widetilde{M} \to \mathrm{SO}(n)$ as

$$\begin{aligned} \mathbf{T} &\equiv \widetilde{\mathbf{R}}_{\widetilde{U}}^{-1} \mathbf{R}_U \;\;, \\ d\mathbf{T} &= -\, \widetilde{\mathbf{R}}_{\widetilde{U}}^{-1}\, d\widetilde{\mathbf{R}}_{\widetilde{U}}\;\; \widetilde{\mathbf{R}}_{\widetilde{U}}^{-1}\, \mathbf{R}_U \;\; + \widetilde{\mathbf{R}}_{\widetilde{U}}^{-1} d\mathbf{R}_U \;\; . \end{aligned} \tag{1.5.15}$$

Multiplying (1.5.14) by $\widetilde{\mathbf{R}}_{\widetilde{U}}^{-1}$ on the L.H.S, and by $\mathbf{R}_U$ on the R.H.S and using (1.5.15) we obtain

$$d\mathbf{T} - \mathbf{T}\omega_U + \tilde{\omega}_U \mathbf{T} = 0\,. \tag{1.5.16}$$

As expected both $\mathbf{R}_U$ and $\widetilde{\mathbf{R}}_{\widetilde{U}}$ were eliminated in favour of $\mathbf{T}$, instead of having the rotations $\mathrm{SO}(n) \times \mathrm{SO}(n)$ in the condition on connections we have only $\mathrm{SO}(n)$.

Equation (1.5.16) is a statement that the covariant derivative of $\mathbf{T}$ vanishes, having in mind the implicit pull-backs to Σ. We take the exterior derivative of (1.5.16) and use (1.5.16) again in order to replace $d\mathbf{T}$ terms obtaining

$$\left(\mathbf{T}^i{}_l\,\omega^l{}_k - \mathbf{T}^l{}_k\,\tilde{\omega}^i{}_l\right)\wedge\omega^k_j - \left(\mathbf{T}^k_l\,\omega^k_j - \mathbf{T}^l{}_j\,\tilde{\omega}^k_l\right)\wedge\tilde{\omega}^i{}_k + \mathbf{T}^i{}_k\,d\omega^k_j - \mathbf{T}^k_j\,d\tilde{\omega}^i{}_k = 0\,. \quad (1.5.17)$$

Noticing that $\omega^i{}_l\wedge\tilde{\omega}^k_j$ terms cancel out and grouping the remaining ones we have

$$\mathbf{T}^i{}_l\left(\omega^l{}_k\wedge\omega^k_j + d\omega^k_j\right) - \mathbf{T}^l{}_j\left(\tilde{\omega}^i{}_k\wedge\tilde{\omega}^k_l + d\tilde{\omega}^i{}_l\right) = 0\,. \quad (1.5.18)$$

The expressions in parenthesis may be recognized as curvature one-forms, indeed using Cartan structural equations (1.3.6) we have

$$\mathbf{T}^i{}_k R_{kjlm}\,\theta^l\wedge\theta^m + \mathbf{T}^k_j\widetilde{R}_{iklm}\,\tilde{\theta}^l\wedge\tilde{\theta}^m = 0\,. \quad (1.5.19)$$

Now we may substitute (1.4.1) together with the pseudoduality equation $\tilde{x}^i_\pm = \pm\mathbf{T}^i{}_j\,x^j_\pm$ which gives us

$$\mathbf{T}^i{}_k\mathbf{T}^j{}_l R_{klmn} = -\mathbf{T}^k_m\mathbf{T}^l{}_n R_{ijkl}\,. \quad (1.5.20)$$

The presence of the negative sign in (1.5.20) will have a particular importance. We will refer to (1.5.20) as the "opposite curvature" condition. Taking again the exterior derivative of (1.5.20) and using (1.5.16) in order to replace $d\mathbf{T}^i{}_k$ we have

$$\begin{aligned}&\mathbf{T}^i{}_k\mathbf{T}^j{}_l\left(dR_{klmn} - R_{plmn}\omega^p_k - R_{kpmn}\omega^p_l\right) - R_{klmn}\left(\mathbf{T}^p_k\mathbf{T}^j{}_l\tilde{\omega}^i_p + \mathbf{T}^i{}_k\mathbf{T}^p{}_l\tilde{\omega}^j_p\right)\\ &= \mathbf{T}^k_m\mathbf{T}^l{}_n\left(d\widetilde{R}_{ijkl} - \widetilde{R}_{ijpl}\tilde{\omega}^p_k + \widetilde{R}_{ijkp}\tilde{\omega}^p_l\right) - \widetilde{R}_{ijkl}\left(\mathbf{T}^p{}_m\mathbf{T}^l{}_n\omega^k_p + \mathbf{T}^k{}_m\mathbf{T}^p{}_n\omega^p_n\right)\,. \quad (1.5.21)\end{aligned}$$

The last terms on both sides of (1.5.21) may be rewritten using (1.5.20) in order to eliminate the mixed terms $R\tilde{\omega}$ and $\widetilde{R}\omega$ in favour of $R\omega$ and $\widetilde{R}\tilde{\omega}$ what results in

$$\begin{aligned} &\mathbf{T}^i{}_k\mathbf{T}^j{}_l\,(dR_{klmn} - R_{plmn}\omega^p{}_k - R_{kpmn}\omega^p{}_l - R_{klpn}\omega^p{}_m - R_{klmp}\omega^p{}_n) \\ &= -\mathbf{T}^k{}_m\mathbf{T}^l{}_n\left(dR_{ijkl} - R_{ijpl}\omega^p{}_k - R_{ijkp}\omega^p{}_l - R_{ipkl}\omega^p{}_j - R_{pjkl}\omega^p{}_i\right)\,. \end{aligned} \tag{1.5.22}$$

We immediately recognize the expressions in parenthesis as the covariant derivative of the Riemann tensor, $DR_{ijkl} = R_{ijkl;m}\,\theta^m$ and similarly for $\widetilde{R}$. Therefore we have[5]

$$\mathbf{T}^i{}_k\mathbf{T}^j{}_l\,R_{klmn;p}\,\theta^p = -\mathbf{T}^k{}_m\mathbf{T}^l{}_n\,\widetilde{R}_{ijkl;p}\,\tilde{\theta}^p \tag{1.5.23}$$

or

$$\mathbf{T}^i{}_k\mathbf{T}^j{}_l\,R_{klmn;p}\,x^p{}_\mu\,d\sigma^\mu = -\mathbf{T}^k{}_m\mathbf{T}^l{}_n\,\widetilde{R}_{ijkl;p}\,\tilde{x}^p{}_\mu\,d\sigma^\mu \tag{1.5.24}$$

if we use (1.4.1). Substituting the pseudoduality equation $\tilde{x}^i_\pm = \pm\mathbf{T}^i{}_j\,x^j_\pm$ we have two independent components of (1.5.24):

$$\begin{aligned} \mathbf{T}^i{}_k\mathbf{T}^j{}_l\,R_{klmn;q} &= -\,\mathbf{T}^k{}_m\mathbf{T}^l{}_n\,\widetilde{R}_{ijkl;p}\,\mathbf{T}^p{}_q\,, \\ \mathbf{T}^i{}_k\mathbf{T}^j{}_l\,R_{klmn;q} &= +\,\mathbf{T}^k{}_m\mathbf{T}^l{}_n\,\widetilde{R}_{ijkl;p}\,\mathbf{T}^p{}_q\,. \end{aligned} \tag{1.5.25}$$

We therefore obtain

$$R_{klmn;q} = \widetilde{R}_{ijkl;p} = 0\,. \tag{1.5.26}$$

This condition tells us that both manifolds M and $\widetilde{M}$ have to be *locally symmetric spaces*. From (1.5.26) we see that once we have the value of the Riemann tensor at any point we may have it for the whole manifold M by the parallel transport. The pairs of locally symmetric spaces with opposite curvature are called *dual symmetric spaces*. The simplest example is a panir consisting of n-dimensional sphere S^n and

[5] We can derive (1.5.23) in a faster way. By (1.5.16) the covariant derivative of $\mathbf{T}$ vanishes. If we choose at some point the Riemann normal coordinates the ordinary derivative of $\mathbf{T}$ vanishes as well, therefore differentiating (1.5.20) we have only the derivatives of R and $\widetilde{R}$, we may generalize this result to any point, demanding the general covariance we have (1.5.23)

hyperbolic space H^n with the metrics

$$g_{ij} = \frac{\delta_{ij}}{(1 + k\delta_{lm}x^l\, x^m})^2\,, \quad \tilde{g}_{ij} = \frac{\delta_{ij}}{(1 + \tilde{k}\delta_{lm}\tilde{x}^l\, \tilde{x}^m})^2\,, \tag{1.5.27}$$

where $k > 0$ and $\tilde{k} < 0$. Their geometry is characterized entirely by the value of the curvature scalar which for them is constant, the equation (1.5.20) becomes simply $k = -\tilde{k}$.

Chapter 2

The Dilaton Field

2.1 The Curved World-Sheet. Stress-Energy Tensor

In a series of previous papers [3, 4, 5, 6] a framework was developed for studying classical target space duality between nonlinear sigma models in two dimensional Minkowski space. References to previous original literature are given in those papers. Here we generalize to the case of a curved world sheet Σ and introduce a dilaton field Φ that couples to the world sheet curvature scalar $R^{(2)}(\vec{\sigma})$. The classical action for the sigma model is $S = S_0 + S_B + S_\Phi$ where

$$S_0 + S_B = -\frac{1}{2}\int d^2\sigma\{\sqrt{-h}h^{\alpha\beta}g_{ij}(x)\partial_\alpha x^i\partial_\beta x^j - \epsilon^{\alpha\beta}\partial_\alpha x^i\partial_\beta x^j B_{ij}(x)\}, \qquad (2.1.1)$$

$$S_\Phi = \int d^2\sigma\sqrt{-h}\Phi(x)R^{(2)}(\vec{\sigma}). \qquad (2.1.2)$$

The target manifolds with their respective geometrical data are denoted by $M(g, B, \Phi)$ and $\widetilde{M}(\tilde{g}, \widetilde{B}, \widetilde{\Phi})$. The Greek indices refer to the world sheet Σ with metric $h_{\alpha\beta}$ and coordinates $\vec{\sigma} = (\tau, \sigma)$. In two dimensions it is always possible to find a coordinate transformation that locally puts the metric in conformal form

$$h_{\alpha\beta} = e^{2\mu(\vec{\sigma})}\eta_{\alpha\beta}, \tag{2.1.3}$$

where $\eta_{\alpha\beta}$ is the flat world sheet metric with the signature $(-,+)$ and $\mu(\vec{\sigma})$ is the conformal factor. Introducing light-cone coordinates on the world sheet by $\sigma^{\pm} = \tau \pm \sigma$ we see that the curvature scalar is

$$R^{(2)} = 8e^{-2\mu(\vec{\sigma})}\,\partial^2_{+-}\mu(\vec{\sigma}). \tag{2.1.4}$$

A possible multiplicative constant for S_Φ can be absorbed into the definition of dilaton field Φ. Defining the closed 3-form H by $H = dB$ and the derivatives $x^i_{\pm}$ by pulling back an orthonormal coframe from the target space[1], $\theta^i = x^i{}_\alpha\, d\sigma^\alpha$ gives the classical equations of motion

$$x^i{}_{+-} + \frac{1}{2}\,H^i{}_{jk}(x)\,x^j_+ x^k_- + 2\,\Phi'_i(x)\,\partial^2_{+-}\mu(\vec{\sigma}) = 0 \tag{2.1.5}$$

where $d\Phi = \Phi'_i\,\theta^i$ and $x^i{}_{+-}$ is the second covariant derivative. There is an analogous expression for the $\widetilde{M}$ model. Any transformation of the metric of the form

$$\mu(\vec{\sigma}) \to \mu(\vec{\sigma}) + \xi(\sigma^+) + \eta(\sigma^-) \tag{2.1.6}$$

with arbitrary functions of one variable ξ and η leaves the form of equations of motion invariant. We note that the first two terms in the action $S_0 + S_B$ are manifestly independent of the choice of the conformal factor but the term S_Φ is not[2].

The $\Phi = 0$ models are trivially classically conformally invariant. Therefore at the classical level, it was sufficient to study those models on a flat world sheet because all

[1] See [3, 5] for notation.

[2] It is still invariant under simultaneous global scaling of the conformal factor $\mu(\vec{\sigma}) \to a\,\mu(\vec{\sigma})$ and the dilaton field $\Phi(x) \to \Phi(x)/a$.

dependence on the conformal factor was absent. Consequently, the study of classical target space duality in these conformally invariant models reduced to studying models with a flat world sheet. The classical conformal invariance is manifestly broken by the presence of a generic non-zero dilaton field. This means that the classical behavior of strings propagating on a target space M depends on local metrical properties of the world sheet Σ. In this chapter we ask the following question.

> Is it possible to have a classical duality transformations between strings propagating on target spaces M and $\widetilde{M}$ such that the duality transformation is only a property of the target spaces and it is independent of the metrical geometry of the world sheet Σ, *i.e.*, independent of the conformal factor μ?

Note that we are not requiring that the sigma models be conformally invariant but that the duality transformation be true on all world sheets.

The world sheet stress-energy tensor $T^{\alpha\beta}$ is defined variationally by

$$\delta S = \frac{1}{2} \int d^2\sigma \sqrt{-h} T^{\alpha\beta} \delta h_{\alpha\beta}. \tag{2.1.7}$$

The contribution from S_0 is

$$T^{\alpha\beta}_{(0)} = 4e^{-4\mu(\vec{\sigma})} \begin{pmatrix} x_- \, x_- & 0 \\ 0 & x_+ \, x_+ \end{pmatrix}, \tag{2.1.8}$$

the term S_B does not contribute to the stress-energy tensor although it contribute to the equations of motion. Using the equations of motion in the calculation of $\nabla_\alpha T^{\alpha\beta}$ you find terms such as $H_{ijk} x^i_- x^j_- x^k_+$ vanish due to the antisymmetry of H_{ijk}.

To calculate the contribution from S_Φ we have to integrate by parts twice and to observe that the Einstein tensor vanishes because the Hilbert-Einstein action is a topological invariant in 2 dimensions. After some algebra we arrive at the result that the contribution of the dilaton field to the stress-energy tensor is

$$T^{\alpha\beta}_{(\Phi)} = 2(h^{\gamma\alpha}h^{\delta\beta}\Phi_{\gamma;\delta} - h^{\alpha\beta}h^{\gamma\delta}\Phi_{\gamma;\delta}), \tag{2.1.9}$$

where $\Phi_\gamma := x^i_\gamma\,\partial_i\Phi(x)$ and $\Phi_{\gamma;\delta}$ is the second covariant derivative on the world sheet. The only non-vanishing connection coefficients for the metric (2.1.3) are $\Gamma^+_{++} = 2\partial_+\mu(\vec{\sigma})$ and $\Gamma^-_{--} = 2\partial_-\mu(\vec{\sigma})$. The explicit expression is

$$T^{\alpha\beta}_{(\Phi)} = 8e^{-4\mu}\begin{pmatrix} 2\,\partial_-\mu\;\Phi'_j\,x^j_- - \Phi'_{i;j}\,x^i_- x^j_- - \Phi'_i\,x^i_{--} & \Phi'_{i;j}\,x^i_+ x^j_- + \Phi'_i\,x^i_{+-} \\ \Phi'_{i;j}\,x^i_+ x^j_- + \Phi'_i\,x^i_{+-} & 2\,\partial_+\mu\;\Phi'_j\,x^j_+ - \Phi'_{i;j}\,x^i_+ x^j_+ - \Phi'_i\,x^i_{++} \end{pmatrix}. \tag{2.1.10}$$

Here, the derivatives Φ'_i are defined as components of the 1-form $d\Phi$ with respect to the orthonormal coframe of M and $\Phi'_{i;j}$ is the second covariant derivative on the target space. For the model $\widetilde{M}(\tilde{g},\widetilde{B},\widetilde{\Phi})$ these derivatives will be denoted as $\widetilde{\Phi}''_i$ and $\widetilde{\Phi}''_{i;j}$ respectively. Note that the stress-energy tensor does not vanish in the limiting case of a flat world sheet, $\mu(\vec{\sigma}) \to 0$, although in this limit both the action and the classical equations of motion are the same as for the flat world sheet case. In the limit $\mu(\vec{\sigma}) \to 0$, $T^{\alpha\beta}_{(\Phi)}$ has the property that its divergence vanishes identically, i.e. $\nabla_\beta T^{\alpha\beta}_{(\Phi)} = 0$ for any $x^i(\sigma^\mu)$ and not just for the solution of equation of motion.

In the case of [3, 4] where the action was $S_0 + S_B$, the transformation equation for the "on-shell" duality could have been written down by inspecting the stress-energy tensor. Here there are two difficulties. The stress-energy tensor contains the terms $\Phi'_i(x)$ and $\Phi'_{i;j}(x)$ which have dependence on x's and the duality transformation involves derivatives of x's, therefore integrability issues arise. The other difficulty is just the mentioned possibility that two equivalent expressions may differ by a contribution whose divergence vanishes identically and this contribution has to be clearly identified.

In light cone coordinates the trace of the on-shell classical stress-energy tensor is proportional to

$$\Phi'_{i;j}\, x_+^i x_-^j - \frac{1}{2}\, \Phi'_i H^i{}_{jk}\, x_+^j x_-^k - 2\Phi'_i \Phi'_i\, \partial^2_{+-}\mu(\vec{\sigma}).$$

From this we see that the dilaton field must be constant for generic conformal invariance. i.e., the S_Φ term is topological. At the quantum level, the Φ constant models have $O(\hbar)$ corrections that break the conformal symmetry. Putting these two pieces of information together we see that at the quantum level it is consistent to choose the dilaton term to be $O(\hbar)$ in order to look for conformally invariant theories. The condition for conformal invariance, the tracelessness of the energy-momentum tensor, is obtained by combining contributions (both classical and quantum) from the dilaton term with the quantum corrections from the other terms. This gives a set of restrictions on geometrical data describing a conformally invariant model, see e.g., [9, 10]. A goal of this chapter is to study the relationship between classical duality and the conformal invariance of the classical field theory.

2.2 A Toy Model

As a guideline consider a classical mechanics time-dependent Hamiltonian system. By prescribing a generating function $F(q, \tilde{q}, t)$ one obtains both the canonical transformation and the relation between the hamiltonians by

$$\tilde{p} = \frac{\partial F(q,\tilde{q},t)}{\partial \tilde{q}}, \qquad -p = \frac{\partial F(q,\tilde{q},t)}{\partial q}, \qquad \widetilde{H} - H = -\frac{\partial F(q,\tilde{q},t)}{\partial t}\,. \tag{2.2.1}$$

Here we consider the inverse problem. Both hamiltonians are given and we want to determine the conditions that have to be satisfied in order to establish a canonical transformation. We assume the Hamiltonians are of the form

$$H = \frac{1}{2}\,(p - A(q,t))^2 + V(q,t), \qquad \widetilde{H} = \frac{1}{2}\,(\tilde{p} - \widetilde{A}(\tilde{q},t))^2 + \widetilde{V}(\tilde{q},t). \tag{2.2.2}$$

Consider a generating function of the form

$$F(q,\tilde{q},t) = q\tilde{q} + f(t)(\widetilde{W}(\tilde{q}) - W(q)) \tag{2.2.3}$$

with $f(t), \widetilde{W}(\tilde{q}), W(q)$ to be determined. Using (2.2.1) we see that

$$q = \tilde{p} - f\widetilde{W}', \qquad \tilde{q} = -(p - fW') \tag{2.2.4}$$

and

$$\frac{1}{2}\,(\tilde{p} - \widetilde{A})^2 + \widetilde{V} - \left\{\frac{1}{2}\,(p - A)^2 + V\right\} = -\dot{f}(\widetilde{W} - W). \tag{2.2.5}$$

We rewrite (2.2.5) using (2.2.4) and group together terms according to their q and $\tilde{q}$ dependence:

$$\begin{aligned} 0 =& \frac{1}{2}\,q^2 - V - \frac{1}{2}\,(A - fW') - \dot{f}W \\ & - \left\{\frac{1}{2}\,\tilde{q}^2 - \widetilde{V} - \frac{1}{2}\,(\widetilde{A} - f\widetilde{W}') - \dot{f}\widetilde{W}\right\} \\ & - q(\widetilde{A} - f\widetilde{W}') - \tilde{q}(A - fW'). \end{aligned} \tag{2.2.6}$$

To eliminate the mixed q and $\tilde{q}$ dependence we require that the summands in the last line of the equation above satisfy

$$A - fW' = h(t)q, \qquad \widetilde{A} - f\widetilde{W}' = -h(t)\tilde{q}. \tag{2.2.7}$$

The remaining part of (2.2.6) gives immediately the conditions

$$V(q,t) = \frac{1}{2}\,(1-h(t))^2 q^2 - \dot{f}W,$$
$$\widetilde{V}(\tilde{q},t) = \frac{1}{2}\,(1-h(t))^2 \tilde{q}^2 - \dot{f}\widetilde{W}. \tag{2.2.8}$$

To make this example more similar to the sigma model case consider the special case where

$$A(q,t) = f(t)B(q), \qquad A(\tilde{q},t) = f(t)\widetilde{B}(\tilde{q}). \tag{2.2.9}$$

From (2.2.7) we see that

$$B - W' = q, \qquad \widetilde{B} - \widetilde{W}' = \tilde{q}, \qquad h(t) = f(t). \tag{2.2.10}$$

Integrating we obtain the generating function for this transformation

$$F(q,\tilde{q},t) = q\tilde{q} + h(t)\left(-\frac{1}{2}\,(\tilde{q}^2 - q^2) + \int_0^{\tilde{q}} d\tilde{q}'\,\widetilde{B}(\tilde{q}') - \int_0^{q} dq'\,B(q')\right). \tag{2.2.11}$$

2.3 Target Space Duality

In the field theory case we have to consider hamiltonian densities of the form

$$\mathcal{H} = \frac{1}{2}\,g^{ik}(\pi_i - B_{ij}x^j_\sigma)(\pi_k - B_{kl}x^l_\sigma) + g_{ik}\frac{1}{2}\,x^i_\sigma x^k_\sigma + 2\eta^{\alpha\beta}(\partial^2_{\alpha\beta}\mu)\Phi(x) \tag{2.3.1}$$

and the analogous expression for the $\widetilde{M}$ model. The explicit time dependence enters via the conformal factor. By analogy to (2.2.1) the imposed requirement is that the Hamiltonians of both models differ only by a time derivative of the generating functional.

$$\widetilde{H} - H = \int d\sigma\,(\widetilde{\mathcal{H}} - \mathcal{H}) = -\frac{\partial F}{\partial \tau} \tag{2.3.2}$$

The general form of the generating functional is taken to be of the form

$$F[x,\tilde{x}] = \int \alpha + \int (\partial_\sigma \mu \, Y + \partial_\tau \mu \, Z) d\sigma, \tag{2.3.3}$$

where $Y(x,\tilde{x})$ and $Z(x,\tilde{x})$ are functions on $M \times \widetilde{M}$ and $\alpha(x,\tilde{x})$ is a 1-form on $M \times \widetilde{M}$ written as

$$\alpha = \alpha_i(x,\tilde{x}) dx^i + \tilde{\alpha}_i(x,\tilde{x}) d\tilde{x}^i. \tag{2.3.4}$$

The generalization of the canonical transformation is then

$$\pi_i = m_{ji} \frac{d\tilde{x}^j}{d\sigma} + l_{ij} \frac{dx^j}{d\sigma} - \partial_\sigma \mu \frac{\partial Y}{\partial x^i} - \partial_\tau \mu \frac{\partial Z}{\partial x^i}, \tag{2.3.5}$$

$$\widetilde{\pi}_i = m_{ij} \frac{dx^j}{d\sigma} + \tilde{l}_{ij} \frac{d\tilde{x}^j}{d\sigma} + \partial_\sigma \mu \frac{\partial Y}{\partial \tilde{x}^i} + \partial_\tau \mu \frac{\partial Z}{\partial \tilde{x}^i}, \tag{2.3.6}$$

where m_{ij}, l_{ij} and $\tilde{l}_{ij}$ are given by

$$d\alpha = -\frac{1}{2} \, l_{ij}(x,\tilde{x}) \, dx^i \wedge dx^j + \frac{1}{2} \, \tilde{l}_{ij}(x,\tilde{x}) \, d\tilde{x}^i \wedge d\tilde{x}^j + m_{ij}(x,\tilde{x}) \, d\tilde{x}^i \wedge dx^j. \tag{2.3.7}$$

Here it would be desirable to maintain a symmetric formulation between tilded and untilded quantities. Therefore we introduce the following definitions:

$$n \equiv l - B, \qquad \tilde{n} \equiv \tilde{l} - \widetilde{B}, \tag{2.3.8}$$

$$\widetilde{m}_{ij} \equiv m_{ji}, \tag{2.3.9}$$

$$dY = Y_i' \, \theta^i - Y_i'' \, \widetilde{\theta}^i, \tag{2.3.10}$$

$$dZ = Z_i' \, \theta^i - Z_i'' \, \widetilde{\theta}^i, \tag{2.3.11}$$

where $(\theta,\widetilde{\theta})$ is an orthonormal coframe of $M \times \widetilde{M}$. By $\| \; \|$ and $\langle \; , \; \rangle$ we denote a norm and a scalar product on a target space, we also suppress target space indices $i, j \ldots$

hereafter. Using the form of the canonical transformations we see that the integrand of (2.3.2) is

$$\begin{aligned}\widetilde{\mathcal{H}} - \mathcal{H} = &\frac{1}{2}\,\|m x_\sigma + \tilde{n}\tilde{x}_\sigma - \mu_\sigma\, Y'' - \mu_\tau\, Z''\|^2 + \frac{1}{2}\,\|\tilde{x}_\sigma\|^2 + \\ &-\|\tilde{m}\tilde{x}_\sigma + n x_\sigma - \mu_\sigma\, Y' - \mu_\tau\, Z'\|^2 + \frac{1}{2}\,\|x_\sigma\|^2 + \\ &-2(-\mu_{\tau\tau} + \mu_{\sigma\sigma})(\widetilde{\Phi} - \Phi). \end{aligned} \tag{2.3.12}$$

The next step is to group together terms with different x and $\tilde{x}$ behavior:

$$\begin{aligned}\widetilde{\mathcal{H}} - \mathcal{H} =& \frac{1}{2}\,\langle x_\sigma, (m^t m - n^t n - I)\, x_\sigma\rangle - \frac{1}{2}\,\langle \tilde{x}_\sigma, (\tilde{m}^t\tilde{m} - \tilde{n}^t\tilde{n} - I)\,\tilde{x}_\sigma\rangle \\ &+ \langle \tilde{x}_\sigma, (\tilde{n}^t m - \tilde{m}^t n)\, x_\sigma\rangle \\ &+ \mu_\tau[-\langle x_\sigma, m^t\, Z'' - n^t\, Z'\rangle + \langle \tilde{x}_\sigma, \tilde{m}^t\, Z' - \tilde{n}^t\, Z''\rangle] \\ &+ \mu_\sigma[-\langle x_\sigma, m^t\, Y'' - n^t\, Y'\rangle + \langle \tilde{x}_\sigma, \tilde{m}^t\, Y' - \tilde{n}^t\, Y''\rangle] \\ &+ \frac{1}{2}\,\|\mu_\sigma\, Y'' + \mu_\tau\, Z''\|^2 - \frac{1}{2}\,\|\mu_\sigma\, Y' + \mu_\tau\, Z'\|^2 \\ &- 2(-\mu_{\tau\tau} + \mu_{\sigma\sigma})(\widetilde{\Phi} - \Phi). \end{aligned} \tag{2.3.13}$$

On the level of Hamiltonian densities the condition (2.3.2) is expressed as

$$\widetilde{\mathcal{H}} - \mathcal{H} = -(\mu_{\sigma\tau}\, Y + \mu_{\tau\tau}\, Z) + \frac{\partial}{\partial\sigma} h.$$

It will be convenient to have this condition written as

$$\begin{aligned}\widetilde{\mathcal{H}} - \mathcal{H} = &\mu_\tau(Y_i'\, x^i{}_\sigma - Y_i''\, \tilde{x}^i{}_\sigma) + \mu_\sigma(Z_i'\, x^i{}_\sigma - Z_i''\, \tilde{x}^i{}_\sigma) \\ &+ (-\mu_{\tau\tau} + \mu_{\sigma\sigma})\, Z + \frac{d}{d\sigma}\,(h - \mu_\tau\, Y - \mu_\sigma\, Z)\,. \end{aligned} \tag{2.3.14}$$

Looking at $x_\sigma x_\sigma$, $\tilde{x}_\sigma \tilde{x}_\sigma$, $\tilde{x}_\sigma x_\sigma$ terms in (2.3.13) we recover the relations

$$\tilde{m}^t \tilde{m} = I + \tilde{n}^t \tilde{n} = I - \tilde{n}^2 = m m^t \ , \tag{2.3.15}$$

$$m^t m = I + n^t n = I - n^2 \ , \tag{2.3.16}$$

$$-mn = \tilde{n} m, \tag{2.3.17}$$

which are the same as the ones calculated in [3]. Incorporating these in the remaining terms of (2.3.13) and using (2.3.14) we obtain

$$\begin{aligned} \frac{d}{d\sigma}(h - \mu_\tau Y - \mu_\sigma Z) = {} & \mu_\tau[-\langle x_\sigma, m^t Z'' + n Z' + Y' \rangle + \langle \tilde{x}_\sigma, \tilde{m}^t Z' + \tilde{n} Z'' + Y'' \rangle] \\ & + \mu_\sigma[-\langle x_\sigma, m^t Y'' + n Y' + Z' \rangle + \langle \tilde{x}_\sigma, \tilde{m}^t Y' + \tilde{n} Y'' + Z'' \rangle] \\ & + \frac{1}{2} \|\mu_\sigma Y'' + \mu_\tau Z''\|^2 - \frac{1}{2} \|\mu_\sigma Y' + \mu_\tau Z'\|^2 \\ & + (\mu_{\tau\tau} - \mu_{\sigma\sigma}) [2(\widetilde{\Phi} - \Phi) + Z]. \end{aligned} \tag{2.3.18}$$

Now we require that our construction be independent of the conformal factor μ. The terms linear in μ_τ and μ_σ should vanish which gives us the equations:

$$\begin{pmatrix} Y' \\ Y'' \end{pmatrix} = - \begin{pmatrix} n & m^t \\ \tilde{m}^t & \tilde{n} \end{pmatrix} \begin{pmatrix} Z' \\ Z'' \end{pmatrix}, \tag{2.3.19}$$

$$\begin{pmatrix} Z' \\ Z'' \end{pmatrix} = - \begin{pmatrix} n & m^t \\ \tilde{m}^t & \tilde{n} \end{pmatrix} \begin{pmatrix} Y' \\ Y'' \end{pmatrix}. \tag{2.3.20}$$

The above matrix equations are equivalent because

$$\begin{pmatrix} n & m^t \\ \tilde{m}^t & \tilde{n} \end{pmatrix}^2 = \mathbb{1}. \tag{2.3.21}$$

Next we concentrate on terms quadratic in first derivatives of the conformal factor.

They give us respectively the following equations:

$$\mu_\tau^2 : \qquad \|Z''\|^2 - \|Z'\|^2 = 0, \tag{2.3.22}$$

$$\mu_\sigma^2 : \qquad \|Y''\|^2 - \|Y'\|^2 = 0, \tag{2.3.23}$$

$$\mu_\tau \mu_\sigma : \qquad \langle Y'', Z'' \rangle - \langle Y', Z' \rangle = 0. \tag{2.3.24}$$

From the term linear in second-order derivatives we obtain that

$$Z = 2(\widetilde{\Phi} - \Phi). \tag{2.3.25}$$

This gives us immediately a condition

$$\|\Phi'\|^2 = \|\widetilde{\Phi}''\|^2. \tag{2.3.26}$$

The L.H.S. of (2.3.26) is only a function of x, the R.H.S. only of $\tilde{x}$ which means that it is in fact a restriction saying that the 1-forms $d\Phi$ and $d\widetilde{\Phi}$ have the same norm in their respective metrics.[3]

Now we are ready to write down the duality equation

$$\begin{pmatrix} \tilde{x}_\sigma + 2\mu_\sigma \widetilde{\Phi}'' \\ \tilde{x}_\tau + 2\mu_\tau \widetilde{\Phi}'' \end{pmatrix} = \begin{pmatrix} -(m^t)^{-1} n & (m^t)^{-1} \\ m - \tilde{n}(m^t)^{-1} n & \tilde{n}(m^t)^{-1} \end{pmatrix} \begin{pmatrix} x_\sigma + 2\mu_\sigma \Phi' \\ x_\tau + 2\mu_\tau \Phi' \end{pmatrix}, \tag{2.3.27}$$

$$= \begin{pmatrix} -(m^t)^{-1} n & (m^t)^{-1} \\ (m^t)^{-1} & \tilde{n}(m^t)^{-1} \end{pmatrix} \begin{pmatrix} x_\sigma + 2\mu_\sigma \Phi' \\ x_\tau + 2\mu_\tau \Phi' \end{pmatrix}. \tag{2.3.28}$$

In a light cone basis these equations become

$$\tilde{x}_\pm + 2\mu_\pm \widetilde{\Phi}'' = \pm T_\pm \; (x_\pm + 2\mu_\pm \Phi') \;, \tag{2.3.29}$$

[3]The other possibility that (2.3.26) maps directly M onto $\widetilde{M}$ is ruled out, such a map apart from some trivial cases will not lead to (2.3.6) and even it will not map one solution of equations of motion onto the other.

where $T_\pm$ are orthogonal matrices given by

$$T_\pm = (m^t)^{-1}(I \mp n)\,. \tag{2.3.30}$$

Specifying to the case $n = 0$ the above become

$$\tilde{x}_\pm + 2\mu_\pm \widetilde{\Phi}'' = \pm T\,(x_\pm + 2\mu_\pm \Phi') \tag{2.3.31}$$

with a single orthogonal matrix T.

A final curiosity is that h according to eq. (2.3.18) is the Hodge dual of the respective term in the generating function (2.3.3).

2.4 Integrability Conditions

Here we study the integrability conditions for the classical duality equations. It is instructive to study momentarily a more general duality equation than (2.3.31). Dimensional considerations impose the form

$$\tilde{x}^i_\pm + 2\mu_\pm \tilde{u}^i = \pm x^i_\pm \pm 2\mu_\pm u^i\,. \tag{2.4.1}$$

These equations are interpreted as equations on a bundle of orthonormal coframes as in references [5, 6]. The vector valued functions u^i and $\tilde{u}^i$ are functions on the same bundle. We denote by $'$ and $''$ we denote the derivatives with respect to x and $\tilde{x}$. Taking the derivative of (2.4.1) we have

$$\begin{aligned} &\tilde{x}^i_{+-} \mp x^i_{+-} - \tilde{\omega}^i{}_{j\mp}\tilde{x}^j_\pm \pm \omega^i{}_{j\mp} x^j_\pm = \\ &= 2\mu_\pm \left[(-\tilde{u}'^i{}_{;j} \pm u'^i{}_{;j})x^j_\mp + (-\tilde{u}''^i{}_{;j} \pm u''^i{}_{;j})\tilde{x}^j_\mp + \tilde{\omega}^i{}_{j\mp}\tilde{u}^j \mp \omega^i{}_{j\mp}u^j\right] + 2\partial^2_{+-}\mu\left[-\tilde{u}^i \pm u^i\right]\,. \end{aligned} \tag{2.4.2}$$

By the use of equations of motion (2.1.5) we may eliminate second derivatives on the L.H.S of (2.4.2) which now reads as

$$-2\partial^2_{+-}\mu(\widetilde{\Phi}''^i \mp \Phi'^i) + \frac{1}{2}\,(\mp\widetilde{H}^i{}_{jk}\tilde{x}^j_\pm\tilde{x}^k_\mp + H^i{}_{jk}x^j_\pm x^k_\mp) - \tilde{\omega}^i{}_{j\mp}\tilde{x}^j_\pm \pm \omega^i{}_{j\mp}x^j_\pm\,. \tag{2.4.3}$$

Now the strategy is to use the duality equation (2.4.1) in order to replace selectively[4] $\tilde{x}^i_\mu$ with x^i_μ. The L.H.S is thus

$$-\,2\partial^2_{+-}\mu(\widetilde{\Phi}''^i \mp \Phi'^i) + \frac{1}{2}\,(\pm\widetilde{H}^i{}_{jk} + H^i{}_{jk})x^j_\pm x^k_\mp \mp (\tilde{\omega}^i{}_{j\mp} - \omega^i{}_{j\mp})x^j_\pm - 4\mu_\pm\mu_\mp\widetilde{H}^i{}_{jk}\tilde{u}^j u^k \tag{2.4.4}$$

$$+\,\mu_\pm\widetilde{H}^i{}_{jk}(-\tilde{u}^j \pm u^j)x^k_\pm + \mu_\mp\widetilde{H}^i{}_{jk}(-\tilde{u}^j \pm u^j)x^k_\pm - 2\mu_\pm\omega^i{}_{j\mp}(-\tilde{u}^j \pm u^j)\,. \tag{2.4.5}$$

Here, we have to identify as before [4] the orthogonal groups in both coframes bundles by

$$\tilde{\omega}_{ij} + \frac{1}{2}\,H_{ijk}\tilde{\omega}^k = \omega_{ij} + \frac{1}{2}\,\widetilde{H}_{ijk}\omega^k\,. \tag{2.4.6}$$

This leads to some constraints on the curvature [6]. Using (2.4.6), then again (2.4.1) and collecting the terms we obtain

$$\begin{aligned}0 =& 2\partial^2_{+-}\mu(-\widetilde{\Phi}''^i \pm \Phi'^i + \tilde{u}^i \mp u^i) + 4\mu_\pm\mu_\mp(\tilde{u}''^i{}_{;j} \mp u''^i{}_{;j})(\tilde{u}^j \pm u^j)\\ &+ \mu_\pm\left(\widetilde{H}^i{}_{jk}\tilde{u}^k + H^i{}_{jk}u^k - 2(\tilde{u}'^i{}_{;j} \pm u'^i{}_{;j} \pm \tilde{u}''^i{}_{;j} - u''^i{}_{;j})\right)x^j_\mp\\ &+ \mu_\mp\left(\mp H^i{}_{jk}(-\tilde{u}^j \pm u^j) + \widetilde{H}^i{}_{jk}(-\tilde{u}^j \pm u^j)\right)x^k_\pm\,.\end{aligned} \tag{2.4.7}$$

From the first line of (2.4.7) we see that

$$u^i = \Phi'^i, \qquad \tilde{u}^i = \widetilde{\Phi}''^i. \tag{2.4.8}$$

[4]The subsequent equation may be not therefore explicitly "tilded-untilded" symmetric, the final result however has to be as neither M nor $\widetilde{M}$ is formally distinguished.

Requiring μ independence leads to the following results

$$\widetilde{\Phi}''_{i;j}\widetilde{\Phi}''_j = 0\,, \tag{2.4.9}$$

$$\widetilde{\Phi}''_{i;j}\Phi'_j = 0\,, \tag{2.4.10}$$

$$\Phi'_{i;j} + \widetilde{\Phi}''_{i;j} = 0\,, \tag{2.4.11}$$

$$\Phi'^i\, H^i{}_{jk} + \widetilde{\Phi}''^i\, \widetilde{H}^i{}_{jk} = 0\,, \tag{2.4.12}$$

$$\widetilde{\Phi}''^i\, H^i{}_{jk} + \Phi'^i\, \widetilde{H}^i{}_{jk} = 0\,. \tag{2.4.13}$$

From this we see that $\Phi'_{i;j}\Phi_j = 0$ and we conclude that $\|d\Phi\|$ and $\|d\widetilde{\Phi}\|$ are both constants.

The trace of the classical energy-momentum tensor is given by:

$$\mathrm{Tr}\,[T^{\alpha\beta}] = e^{2\mu}\,T^{+-} = 8e^{-2\mu}\left(\Phi'_{i;j}\,x^i_+x^j_- + \Phi'_i\,x^i_{+-}\right), \tag{2.4.14}$$

$$= 8e^{-2\mu}\left(\Phi'_{i;j}\,x^i_+x^j_- - \frac{1}{2}\,\Phi'_iH^i{}_{jk}\,x^j_+x^k_- - 2\Phi'_i\Phi'_i\,\partial^2_{+-}\mu(\vec{\sigma})\right). \tag{2.4.15}$$

If we use the duality transformations we can compare $\mathrm{Tr}\,[T^{\alpha\beta}]$ with $\mathrm{Tr}\,[\widetilde{T}^{\alpha\beta}]$ and find

$$\mathrm{Tr}\,[\widetilde{T}^{\alpha\beta}] - \mathrm{Tr}\,[T^{\alpha\beta}] = -16e^{-2\mu}\left(\widetilde{\Phi}''_i\widetilde{\Phi}''_i - \Phi'_i\Phi'_i\right)\partial^2_{+-}\mu\,. \tag{2.4.16}$$

2.5 Conclusions

In order to enforce[5] the duality equations, we have to impose various constraints on g, B and Φ. For example, the differentials of the dilaton fields are required to have constant norms in their respective target space dual metrics. A special and simple case is the linear dilaton field.

[5] We considered only the necessary condition

At the one-loop quantum level the condition[6] that the model has to satisfy in order to be conformally invariant involves the second derivatives of a dilaton field, *e.g.*, [9, 10]. It raises a question of whether it is possible to establish at the quantum level a more general form of duality transformations that leads to a preservation of the form of the beta functions and the role of classical duality within such a construction.

[6]By imposing the vanishing of beta function.

Chapter 3

The Tachyon Field

3.1 Preliminary Remarks

In two articles [3, 4] henceforth referred to as Paper I and Paper II respectively, a general theory was developed for irreducible target space duality[1] for classical sigma models characterized by a target space M, a Riemannian metric g and an antisymmetric tensor field B. By irreducible duality we mean that all fields participate in the duality transformation and that there are no spectator fields. This rules out the derivation of the important Buscher formulas [11] and also duality in WZW models [12] where the duality transformation is performed by gauging an anomaly free subgroup, e.g., see the discussions in [13, 14, 15, 16, 17, 18, 19, 20]. The latter remark requires some explanation. For example, consider a WZW model on a compact simple Lie group G where the diagonal subgroup G_D of the symmetry group $G \times G$ is the anomaly free subgroup that is gauged. Schematically, the prescription to construct the dual model is that the original model with fields g is augmented to an equivalent G_D gauge invariant model with fields g, A, λ where A are G_D gauge fields and λ are Lagrange multipliers that enforce the vanishing of the field strengths. In principle the idea is to eliminate the variables g and A in favor of the Lagrange multipliers λ.

[1] For a list of references see Papers I and II.

Naively the original model with variables g had $\dim G$ degrees of freedom. The dual model with variables λ would also have $\dim G$ degrees of freedom. Unfortunately this procedure does not work for a variety of reasons. If by brute force we attempt to eliminate the variables g, A then the action for the λ fields is nonlocal. We can try to finesse things by using the gauge invariance of the theory $g \to hgh^{-1}$. Unfortunately this does not allow us to gauge g to the identity element. The best we can do is gauge g to an element t of a maximal torus T and we have a residual T gauge invariance. This residual gauge invariance can be used to gauge $\operatorname{rank} G$ of the Lagrange multipliers to zero [16]. The A variables can be eliminated and we are left with a local action involving only t and the remaining Lagrange multipliers. We note that the "t" variables are spectators and thus the methods of Papers I and II do not apply. See the worked out example in [17]. Finally we mention why the results of Papers I and II suggest that it is impossible to eliminate the variables g, A in favor of a local action involving only the Lagrange multipliers λ. If $\mathfrak{g}$ denotes the Lie algebra of G then the Lagrange multipliers take values in $\mathfrak{g}^*$, the dual vector space. This strongly suggests that the duality transformation here is related to the cotangent bundle $T^*G = G \times \mathfrak{g}^*$. In Papers I and II we showed that duality associated with any cotangent bundle T^*M implied that M is a compact Lie group with the 3-form H being exact in a very specific way. The 3-form in the WZW model is topologically nontrivial.

3.2 Framework

The sigma model with target space M, metric g, 2-form B and potential function U will be denoted by (M, g, B, U) and has lagrangian density

$$\begin{aligned}\mathcal{L} = \frac{1}{2}\, g_{ij}(x) \left(\frac{\partial x^i}{\partial \tau}\frac{\partial x^j}{\partial \tau} - \frac{\partial x^i}{\partial \sigma}\frac{\partial x^j}{\partial \sigma} \right) + B_{ij}(x) \frac{\partial x^i}{\partial \tau}\frac{\partial x^j}{\partial \sigma} - U(x) \\ + A_i(x)\frac{\partial x^i}{\partial \tau} + C_i(x)\frac{\partial x^i}{\partial \sigma}. \end{aligned} \tag{3.2.1}$$

We will generally follow the notation and formalism introduced in [3, 4]. In the above we have introduced two background fields A_i and C_i that break worldsheet Lorentz invariance for the following reason. Assume we have a Lorentz invariant field theory with fields (x^i, y^a) but where we are not interested in irreducible duality. Namely, only the fields x^i participate in the duality transformation and the y^a fields are spectators. In this case we can regard the fields y^a as parameters and g, B and U depend on the y^a parametrically. In the full lagrangian density there may be a term of type $K_{ai}(x, y)\partial y^a \partial x^i$ and this will become a contribution to the second line of (3.2.1) when the fields y^a are held fixed. The canonical momentum density is

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = g_{ij}\dot{x}^j + B_{ij}x'^j + A_i(x)\ . \tag{3.2.2}$$

The hamiltonian density may be written as

$$\begin{aligned}\mathcal{H} = \frac{1}{2}\, g^{ij} \left(\pi_i - B_{ik}\frac{dx^k}{d\sigma} \right) \left(\pi_j - B_{jl}\frac{dx^l}{d\sigma} \right) + \frac{1}{2}\, g_{ij}\frac{dx^i}{d\sigma}\frac{dx^j}{d\sigma} \\ + U(x) + \frac{1}{2}\, g^{ij}A_iA_j - g^{ij}A_i \left(\pi_j - B_{jl}\frac{dx^l}{d\sigma} \right) - C_i\frac{dx^i}{d\sigma}\ . \end{aligned} \tag{3.2.3}$$

We are interested in studying duality between sigma models with lagrangian densities of the type (3.2.1). Here we consider a generalization of the canonical transformations considered in Paper I that still leads to a linear relationship between the respective π and $dx/d\sigma$ in the two models. The generating function F for the duality canonical transformation will be of the form

$$F[x(\sigma), \tilde{x}(\sigma)] = \int \alpha + \int u(x, \tilde{x}) d\sigma \,, \tag{3.2.4}$$

where α is a 1-form on $M \times \widetilde{M}$, see Paper I, and u a function on $M \times \widetilde{M}$. The canonical transformation is given by

$$\begin{aligned}
(\pi - Bx')_i &= m_{ji} \frac{d\tilde{x}^j}{d\sigma} + n_{ij} \frac{dx^j}{d\sigma} - \frac{\partial u}{\partial x^i} \,, \\
(\tilde{\pi} - \widetilde{B}\tilde{x}')_i &= m_{ij} \frac{dx^j}{d\sigma} + \tilde{n}_{ij} \frac{d\tilde{x}^j}{d\sigma} + \frac{\partial u}{\partial \tilde{x}^i} \,,
\end{aligned}$$

where m_{ij} and n_{kl} will be discussed shortly.

It is best now to go to orthonormal coframes on M and $\widetilde{M}$. Let $(\theta^1, \ldots, \theta^n)$ be a local orthonormal coframe[2] for M. The Cartan structural equations are

$$\begin{aligned}
d\theta^i &= -\omega_{ij} \wedge \theta^j \,, \\
d\omega_{ij} &= -\omega_{ik} \wedge \omega_{kj} + \frac{1}{2} R_{ijkl} \theta^k \wedge \theta^l \,,
\end{aligned}$$

where $\omega_{ij} = -\omega_{ji}$ is the unique torsion free Riemannian connection associated with the metric g. We remind the reader that the analog of $dx^i/d\sigma$ in an orthonormal coframe is ${x^i}_\sigma$ defined by $\theta^i = {x^i}_\sigma d\sigma$. There are similar definitions pertaining to $\widetilde{M}$.

Following the discussion in Paper I, the canonical transformation may be expressed in terms of a 2-form γ closely related to $d\alpha$ on $M \times \widetilde{M}$ and given by

$$\gamma = -\frac{1}{2} n_{ij}(x, \tilde{x}) \theta^i \wedge \theta^j + m_{ij}(x, \tilde{x}) \tilde{\theta}^i \wedge \theta^j + \frac{1}{2} \tilde{n}_{ij}(x, \tilde{x}) \tilde{\theta}^i \wedge \tilde{\theta}^j \,. \tag{3.2.5}$$

[2]Because we will be working in orthonormal frames we do not distinguish an upper index from a lower index in a tensor.

The 2-form γ is not closed but satisfies

$$d\gamma = H - \widetilde{H} \tag{3.2.6}$$

where $H = dB$ and $\widetilde{H} = d\widetilde{B}$. The derivatives of the function u are given in the orthonormal frame by[3]

$$du = u_i\theta^i - \tilde{u}_i\tilde{\theta}^i \,. \tag{3.2.7}$$

In terms of the orthonormal frame the canonical transformation may be written as

$$(\pi - Bx_\sigma)_i = m_{ji}\tilde{x}^j{}_\sigma + n_{ij}x^j{}_\sigma - u_i \,, \tag{3.2.8}$$

$$(\tilde{\pi} - \widetilde{B}\tilde{x}_\sigma)_i = m_{ij}x^j{}_\sigma + \tilde{n}_{ij}\tilde{x}^j{}_\sigma - \tilde{u}_i \,, \tag{3.2.9}$$

where we now interpret the components of $\pi, \tilde{\pi}, B, \widetilde{B}, m, n, \tilde{n}$ to be given with respect to the orthonormal frames.

We digress for a second and make a general observation. Assume we have equal dimensional vector spaces V and $\widetilde{V}$ where we use the notation $(\cdot,\cdot)$ for the inner product on either space. Let $L : V \to \widetilde{V}$ be an invertible linear transformation and let $Q(v) = \frac{1}{2}\,(v,v) + (a,v) + b$ be a real value quadratic function on V. There is a corresponding quadratic function $\widetilde{Q}$ on $\widetilde{V}$. Assume we are told that the affine transformation $\tilde{v} = Lv + \tilde{w}$ maps Q into $\widetilde{Q}$. A short computation shows that

$$\frac{1}{2}\,(Lv, Lv) + (\tilde{a} + \tilde{w}, Lv) + \tilde{b} + (\tilde{a}, \tilde{w}) + \frac{1}{2}\,(\tilde{w}, \tilde{w}) = \frac{1}{2}\,(v,v) + (a,v) + b\,.$$

Comparing both sides we would conclude that L is an isometry, $\tilde{a} + \tilde{w} = La$ and $b = \tilde{b} + (\tilde{a}, \tilde{w}) + \frac{1}{2}\,(\tilde{w}, \tilde{w})$.

In our case we require the canonical transformation to preserve the hamiltonian

[3] Note the unconventional negative sign in the definition above. This is introduced to make subsequent equations more symmetric.

densities up to a total derivative

$$\widetilde{\mathcal{H}} = \mathcal{H} + \frac{dh}{d\sigma} , \tag{3.2.10}$$

where h is a function on $M \times \widetilde{M}$. Our canonical transformation, given by (3.2.8) and (3.2.9), is an affine mapping of (x_σ, π) into $(\tilde{x}_\sigma, \tilde{\pi})$. We can use parts of the general argument about the transformation of quadratic functions given in the previous paragraph. We have to be careful because the derivative in (3.2.10) modifies some of the conclusions of the previous paragraph. The linear part of the transformation must be an isometry, a restriction studied in Paper I, where we found

$$mm^t = I - \tilde{n}^2 , \tag{3.2.11}$$

$$m^t m = I - n^2 , \tag{3.2.12}$$

$$-mn = \tilde{n}m . \tag{3.2.13}$$

If we write[4] $dh = h_i \theta^i - \tilde{h}_i \tilde{\theta}^i$ then $dh/d\sigma = h_i x^i{}_\sigma - \tilde{h}_i \tilde{x}^i{}_\sigma$. From this we learn that

$$\begin{aligned} h_i &= C_i - m_{ji}(\tilde{u}_j + \widetilde{A}_j) - n_{ij}(u_j + A_j) , \\ \tilde{h}_i &= \widetilde{C}_i - m_{ij}(u_j + A_j) - \tilde{n}_{ij}(\tilde{u}_j + \widetilde{A}_j) . \end{aligned}$$

The problem we have to solve is to find functions $u, h : M \times \widetilde{M} \to \mathbb{R}$ such that

$$du = u_i \theta^i - \tilde{u}_i \tilde{\theta}^i , \tag{3.2.14}$$

$$\begin{aligned} dh = &\Big(C_i - m_{ji}(\tilde{u}_j + \widetilde{A}_j) - n_{ij}(u_j + A_j) \Big) \theta^i \\ &- \Big(\widetilde{C}_i - m_{ij}(u_j + A_j) - \tilde{n}_{ij}(\tilde{u}_j + \widetilde{A}_j) \Big) \tilde{\theta}^i . \end{aligned} \tag{3.2.15}$$

The integrability equations for the system given above, $d^2 u = 0$ and $d^2 h = 0$, lead

[4] See footnote 3.

to hyperbolic PDEs for u. Finally we observe that there is one more relation that must be satisfied for (3.2.10) to hold:

$$\widetilde{U} + \frac{1}{2}\,(\tilde{u}_j + \widetilde{A}_j)(\tilde{u}_j + \widetilde{A}_j) = U + \frac{1}{2}\,(u_j + A_j)(u_j + A_j)\,. \tag{3.2.16}$$

Remember that U is a function on M and $\widetilde{U}$ is a functions on $\widetilde{M}$ so $dU = U_i\theta^i$ and $d\widetilde{U} = \widetilde{U}_i\tilde{\theta}^i$.

The generating function for canonical transformations is only locally defined. We could ask whether it is possible to give a more global formulation. We think this is possible. Notice that of primary interest to us is not the function u but its derivatives. For this reason it is convenient to "define"

$$v_i = u_i + A_i \quad \text{and} \quad \tilde{v}_j = \tilde{u}_j + \widetilde{A}_j\,. \tag{3.2.17}$$

Let F_A be the curvature $A = A_i\theta^i$ and $\widetilde{F}_{\widetilde{A}}$ be the curvature of $\widetilde{A} = \widetilde{A}_i\tilde{\theta}^i$. Consider a 1-form on $M \times \widetilde{M}$ defined by

$$\xi = v_i\theta^i - \tilde{v}_i\tilde{\theta}^i\,. \tag{3.2.18}$$

The equation $d^2u = 0$ is replaced by

$$d\xi = F_A - \widetilde{F}_{\widetilde{A}}\,. \tag{3.2.19}$$

In a similar fashion the equation $d^2h = 0$ is replaced by

$$d\left[-(m_{ji}\tilde{v}_j + n_{ij}v_j)\theta^i + (m_{ij}v_j + \tilde{n}_{ij}\tilde{v}_j)\tilde{\theta}^i\right] = -F_C + \widetilde{F}_{\widetilde{C}}\,, \tag{3.2.20}$$

where F_C is the curvature of $C = C_i\theta^i$ and $\widetilde{F}_{\widetilde{C}}$ is the curvature of $\widetilde{C} = \widetilde{C}_i\tilde{\theta}^i$. Similarly the equation for the potentials becomes

$$\widetilde{U} + \frac{1}{2}\,\tilde{v}_i\tilde{v}^i = U + \frac{1}{2}\,v_i v^i\,. \tag{3.2.21}$$

3.3 Pseudoduality

Here we switch to the framework where we consider the map between the paths on one manifold and the paths on the other. We use directly (3.2.8),(3.2.9) and (3.2.11) to (3.2.13), having in mind the rest of the discussion as a guideline. In this way we work directly with equations of motion what makes the calculations more straightforward; we have a system of PDE's for which we obtain the integrability conditions. Moreover, for the 2-dimensional space the Hodge duality transforms 1-forms into another 1-forms. We may use it to write the equation in geometrical, covariant fashion.

We restrict to the case $A = C = 0$. Introducing the lightcone coordinates $\sigma^{\pm} = \tau \pm \sigma$ the equations of motion for lagrangian (3.2.1) are

$$x^k_{+-} = -\frac{1}{2}\;H_{kij}x^i{}_+x^j{}_- - \frac{1}{4}\,U_k\,, \tag{3.3.1}$$

where $dU = U_k\theta^k$.

We rewrite the duality transformations (3.2.8) and (3.2.9) in terms of the velocities as

$$x^i{}_\tau = m_{ji}\tilde{x}^j{}_\sigma + n_{ij}x^j{}_\sigma - u_i\,, \tag{3.3.2}$$

$$\tilde{x}^i{}_\tau = m_{ij}x^j{}_\sigma + \tilde{n}_{ij}\tilde{x}^j{}_\sigma - \tilde{u}_i\,. \tag{3.3.3}$$

Mimicking the computations of Section 3 of [3] we find that

$$\begin{pmatrix} m^t & 0 \\ -\tilde{n} & I \end{pmatrix} \begin{pmatrix} \tilde{x}_\sigma \\ \tilde{x}_\tau + \tilde{u} \end{pmatrix} = \begin{pmatrix} -n & I \\ m & 0 \end{pmatrix} \begin{pmatrix} x_\sigma \\ x_\tau + u \end{pmatrix} . \tag{3.3.4}$$

We can now mimic the discussion in Section 1 of [5] and restrict ourselves to the special case $T_+ = T_-$. If we lift to the frame bundle as discussed in [6] the pseudoduality equations become

$$\tilde{x}^i_\pm + \frac{1}{2}\,\tilde{u}^i = \pm\left(x^i_\pm + \frac{1}{2}\,u^i\right) . \tag{3.3.5}$$

Using the notation from Section 7 of [6] we have the equations of motion may be written as

$$d(*\xi^i) + \xi_{ij} \wedge (*\xi^j) = \frac{1}{2}\,h_{ijk}\xi^j \wedge \xi^k - U_i\,d\sigma^0 \wedge d\sigma^1\,, \tag{3.3.6}$$

and the duality equations as

$$\tilde{\xi}^i + \tilde{u}^i\,d\sigma^0 = *\left(\xi^i + u^i\,d\sigma^0\right) . \tag{3.3.7}$$

More explicitly we have that

$$\tilde{\xi}^i = *\xi^i + u^i\,d\sigma^1 - \tilde{u}^i\,d\sigma^0\,, \tag{3.3.8}$$

$$\xi^i = *\tilde{\xi}^i + \tilde{u}^i\,d\sigma^1 - u^i\,d\sigma^0\,. \tag{3.3.9}$$

Next we define the covariant derivatives of u_i and $\tilde{u}_i$ by

$$\begin{aligned} du_i + \omega_{ij}u^j &= u'_{ij}\omega^j + u''_{ij}\tilde{\omega}^j\,, \\ d\tilde{u}_i + \tilde{\omega}_{ij}\tilde{u}^j &= \tilde{u}'_{ij}\omega^j + \tilde{u}''_{ij}\tilde{\omega}^j\,. \end{aligned} \tag{3.3.10}$$

From $d^2u = 0$ we see that

$$u'_{ij} = u'_{ji}\,, \tag{3.3.11}$$

$$\tilde{u}''_{ij} = \tilde{u}''_{ji}\,, \tag{3.3.12}$$

$$u''_{ij} = -\tilde{u}'_{ji}\,. \tag{3.3.13}$$

The reason for the unusual negative sign in (3.3.13) is the unconventional definition (3.2.7).

To determine conditions necessitated for the duality equations we study the integrability conditions on (3.3.8) by taking its exterior derivative

$$\begin{aligned} 0 = &-\frac{1}{2}h^i{}_{jk}\xi^j \wedge \xi^k - u'^i{}_j\xi^j \wedge d\sigma^1 + \tilde{u}'^i{}_j\xi^j \wedge d\sigma^0 \\ &- u''^i{}_j\tilde{\xi}^j \wedge d\sigma^1 + \tilde{u}''^i{}_j\tilde{\xi}^j \wedge d\sigma^0 \\ &- \tilde{\xi}^j \wedge \xi^i{}_j + \tilde{\xi}^j \wedge \tilde{\xi}^i{}_j \\ &- V^i d\sigma^1 \wedge d\sigma^0 - \tilde{u}_j d\sigma^0 \wedge \xi^{ij} + \tilde{u}_j d\sigma^0 \wedge \tilde{\xi}^{ij} \end{aligned}$$

As in [6] we identify the orthogonal groups in the two frame bundles by requiring that

$$\tilde{\omega}_{ij} + \frac{1}{2}\ H_{ijk}\tilde{\omega}^k = \omega_{ij} + \frac{1}{2}\ \widetilde{H}_{ijk}\omega^k\,. \tag{3.3.14}$$

Substituting this into the equation above leads to

$$\begin{aligned} 0 = &-\frac{1}{2}h^i{}_{jk}\xi^j \wedge \xi^k + \frac{1}{2}\tilde{h}^i{}_{jk}\xi^j \wedge \tilde{\xi}^k \\ &- u'^i{}_j\xi^j \wedge d\sigma^1 + \tilde{u}'^i{}_j\xi^j \wedge d\sigma^0 \\ &- \frac{1}{2}\tilde{h}^i{}_{jk}\tilde{u}^j\xi^k \wedge d\sigma^0 - \frac{1}{2}h^i{}_{jk}\tilde{\xi}^j \wedge \tilde{\xi}^k - u''^i{}_j\tilde{\xi}^j \wedge d\sigma^1 \\ &+ \tilde{u}''^i{}_j\tilde{\xi}^j \wedge d\sigma^0 + \frac{1}{2}h^i{}_{jk}\tilde{u}^j\tilde{\xi}^k \wedge d\sigma^0 - V^i d\sigma^1 \wedge d\sigma^0 \end{aligned}$$

Next we use the following Hodge duality relations

$$\xi^j \wedge \xi^k = -(*\xi^j \wedge *\xi^k)\,,$$
$$(*\tilde{\xi}^i) \wedge d\sigma^1 = -\tilde{\xi}^i \wedge d\sigma^0\,,$$
$$(*\tilde{\xi}^i) \wedge d\sigma^0 = -\tilde{\xi}^i \wedge d\sigma^1\,,$$
$$(*\tilde{\xi}^j) \wedge \tilde{\xi}^k = (*\tilde{\xi}^k) \wedge \tilde{\xi}^j\,,$$

that we substitute into the integrability conditions to obtain

$$\begin{aligned} 0 = &-u''^i{}_j\tilde{\xi}^j \wedge d\sigma^1 - \tilde{u}'^i{}_j\tilde{\xi}^j \wedge d\sigma^1 + h^i{}_{jk}u^j\tilde{\xi}^k \wedge d\sigma^1 \\ &+ u'^i{}_j\tilde{\xi}^j \wedge d\sigma^0 + \tilde{u}''^i{}_j\tilde{\xi}^j \wedge d\sigma^0 + \frac{1}{2}\tilde{h}^i{}_{jk}u^j\tilde{\xi}^k \wedge d\sigma^0 - \frac{1}{2}h^i{}_{jk}\tilde{u}^j\tilde{\xi}^k \wedge d\sigma^0 \\ &- u_j u'^{ij} d\sigma^1 \wedge d\sigma^0 - h^i{}_{jk}u^j\tilde{u}^k d\sigma^1 \wedge d\sigma^0 + \tilde{u}_j\tilde{u}'^{ij} d\sigma^1 \wedge d\sigma^0 - V^i d\sigma^1 \wedge d\sigma^0\,. \end{aligned}$$

Extracting the information contained in the equation above and the ones obtained by taking the exterior derivative of (3.3.9) we see that

$$u''_{ij} + \tilde{u}'_{ij} + h_{ijk}u^k = 0\,, \tag{3.3.15}$$
$$u'_{ij} + \tilde{u}''_{ij} - \frac{1}{2}\,\tilde{h}_{ijk}u^k + \frac{1}{2}\,h_{ijk}\tilde{u}^k = 0\,, \tag{3.3.16}$$
$$U_i + u'_{ij}u^j - \tilde{u}'_{ij}\tilde{u}^j + h_{ijk}u^j\tilde{u}^k = 0\,, \tag{3.3.17}$$
$$\tilde{u}'_{ij} + u''_{ij} + \tilde{h}_{ijk}\tilde{u}^k = 0\,, \tag{3.3.18}$$
$$\tilde{u}''_{ij} + u'_{ij} - \frac{1}{2}\,h_{ijk}\tilde{u}^k + \frac{1}{2}\,\tilde{h}_{ijk}u^k = 0\,, \tag{3.3.19}$$
$$\widetilde{U}_i + \tilde{u}''_{ij}\tilde{u}^j - u''_{ij}u^j + \tilde{h}_{ijk}\tilde{u}^j u^k = 0\,. \tag{3.3.20}$$

From the above we immediately learn that

$$h_{ijk} u^k = \tilde{h}_{ijk} \tilde{u}^k \,, \tag{3.3.21}$$

$$h_{ijk} \tilde{u}^k = \tilde{h}_{ijk} u^k \,, \tag{3.3.22}$$

$$h_{ijk} u^j \tilde{u}^k = 0 \,. \tag{3.3.23}$$

An important observation that follows from the above is that if we go back to $M \times \widetilde{M}$ then we expect that u should be a function of both sets of variables, *i.e.* a nontrivial function on $M \times \widetilde{M}$.

$$u'_{ij} - u'_{ji} = 0 \,, \tag{3.3.24}$$

$$u''_{ij} - u''_{ji} + h_{ijk} u^k = 0 \,, \tag{3.3.25}$$

$$u'_{ij} + \tilde{u}''_{ij} = 0 \,, \tag{3.3.26}$$

$$u''_{ij} + \tilde{u}'_{ji} = 0 \,, \tag{3.3.27}$$

$$\tilde{u}''_{ij} - \tilde{u}''_{ji} = 0 \,, \tag{3.3.28}$$

$$U_i + u'_{ij} u^j + u''_{ij} \tilde{u}^j = 0 \,, \tag{3.3.29}$$

$$\widetilde{U}_i + \tilde{u}'_{ij} u^j + \tilde{u}''_{ij} \tilde{u}^j = 0 \,. \tag{3.3.30}$$

The above is consistent with the condition that on $M \times \widetilde{M}$ we require that

$$\left(U + \frac{1}{2}\, u_i u^i \right) = \left(\widetilde{U} + \frac{1}{2}\, \tilde{u}_i \tilde{u}^i \right) + \text{constant}, \tag{3.3.31}$$

in agreement with (3.2.16)

3.4 Conclusions

We obtained a set of nonlinear algebraic equations in a sense they do not contain the derivatives of x^i or $\tilde{x}^i$. The geometric condition (3.3.14) on connections on M and $\widetilde{M}$ is unchanged by the presence of the potentials and the conclusions from [6] hold in the case discussed here. In addition we have equations (3.3.24) to (3.3.30) involving second derivatives of the generating function u and the derivatives of the potential. Using (3.3.24) to (3.3.28) we can integrate (3.3.30) and (3.3.31) obtaining (3.3.31) which has already appeared as a condition (3.2.21) for hamiltonian density to be preserved . This condition has not been used in the following derivation. Having the solution of equations of motion on M the condition (3.3.31) is a constraint for the solution on $\widetilde{M}$. There could be however a choice of generating function u by which the constraint is satisfied automatically. If with an appropriate choice of coordinate system we have r coordinates for which $u_i = \partial u / \partial x^i$ for $i = 1...r$ (see (3.2.7), the definition of u_i) using the following substitution for the generating function

$$u = f_1(x^1 + \tilde{x}^1) + g_1(x^1 - \tilde{x}^1) + ... + f_r(x^r + \tilde{x}^r) + g_r(x^r - \tilde{x}^r)$$

the condition (3.3.31) has the form

$$-(U(x) - \widetilde{U}(\tilde{x})) = f_1'(x^1 + \tilde{x}^1)g_1'(x^1 - \tilde{x}^1) + ... + f_r'(x^r + \tilde{x}^r)g_r'(x^r - \tilde{x}^r)$$

We are interested in the functions for which the combinations such as $f_i'(x^i + \tilde{x}^i)g_i'(x^i - \tilde{x}^i)$ separate into the sum of two terms, one being only a function of x^i and the other only of $\tilde{x}^i$. In general case we still have (3.3.24) to (3.3.28) which give a nontrivial constraint.

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