2012-06-06

Topics in Elementary Particle Physics

Xiang Jin
University of Miami, jinxiang2049@hotmail.com

UNIVERSITY OF MIAMI

TOPICS IN ELEMENTARY PARTICLE PHYSICS

By

Xiang Jin

A DISSERTATION

Submitted to the Faculty
of the University of Miami
in partial fulfillment of the requirement for
the degree of Doctor of Philosophy

Coral Gables, Florida

June 2012

UNIVERSITY OF MIAMI

A dissertation submitted in partial fulfillment of
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Xiang Jin

Approved:

_____	_____
Thomas L. Curtright, Ph.D. Professor of Physics	Dean of the Graduate School
_____	_____
Luca Mezincescu, Ph.D. Professor of Physics	Carolyne Van Vliet, Ph.D. Adjunct Professor of Physics
_____	_____
Fulin Zuo, Ph.D. Professor of Physics	Alexander Dvorsky, Ph.D. Associate Professor of Mathematics

JIN, XIANG (Ph.D., Physics)
Topics in Elementary Particle Physics (June 2012)

Abstract of a dissertation at the University of Miami

Dissertation supervised by Professor Thomas L. Curtright
No. of pages in text. (167)

The author of this thesis discusses two topics in elementary particle physics: n-ary algebras and their applications to M-theory (Part I), and functional evolution and Renormalization Group flows (Part II).

In part I, Lie algebra is extended to four different n-ary algebraic structure: generalized Lie algebra, Filippov algebra, Nambu algebra and Nambu-Poisson tensor; though there are still many other n-ary algebras. A natural property of Generalized Lie algebras — the Bremner identity, is studied, and proved with a totally different method from its original version. We extend Bremner identity to n-bracket cases, where n is an arbitrary odd integer. Filippov algebras do not focus on associativity, and are defined by the Fundamental identity. We add associativity to Filippov algebras, and give examples of how to construct Filippov algebras from $su(2)$, bosonic oscillator, Virasoro algebra. We try to include fermionic charges into the ternary Virasoro-Witt algebra, but the attempt fails because fermionic charges keep generating new charges that make the algebra not closed. We also study the Bremner identity restriction on Nambu algebras and Nambu-Poisson tensors. So far, the only example 3-algebra being used in physics is the BLG model with 3-algebra A_4, describing two M2-branes interactions. Its extension with Nambu algebra, BLG-NB model, is believed to describe infinite M2-branes condensation. Also, there is another

propose for M2-brane interactions, the ABJM model, which is constructed by ordinary Lie algebra. We compare the symmetry properties between them, and discuss the possible approaches to include these three models into a grand unification theory.

In Part II, we give an approximate solution for Schroeder's equations, based on series and conjugation methods. We use the logistic map as an example, and demonstrate that this approximate solution converges to known analytical solutions around the fixed point, around which the approximate solution is constructed. Although the closed-form solutions for Schroeder's equations can not always be approached analytically, by fitting the approximation solutions, one can still obtain closed-form solutions sometimes. Based on Schroeder's theory, approximate solutions for trajectories, velocities and potentials can also be constructed. The approximate solution is significantly useful to calculate the beta function in renormalization group trajectory. By "wrapping" the series solutions with the conjugations from different inverse functions, we generate different branches of the trajectory, and construct a counterexample for a folk theorem about limited cycles.

Acknowledgements

I owe a great amount of thanks to my advisor Thomas Curtright, who has not only been a mentor for me but also a great role model of excellence in academia. I thank him for teaching me many things in theoretical physics, introducing to me many concepts, methods and techniques, as well as new problems. I also thank him for sharing with me his inspiring ideas and the insightful and unique explanations he patiently gave to all of my questions over the years. I have greatly benefited from the experience of working with him in many aspects

I am very grateful to my collaborator Luca Mezincescu, who shared with me hard work and also taught me plenty of physics.

I thank Professors O. Alvarez, S. Barnes, A. Dvorsky, R. Nepomechie, N. Saveliev, C. Van Vliet, from whom I learned advanced Physics and Mathematics courses, and the Directors of Graduate Studies, J. Cohn and F. Zuo for much academic advice and support.

I am also thankful for the academic support from the University of Miami and its Department of Physics.

I also thank many other people in elementary particle physics whom I met in various places for helpful discussions or conversations: I. Bandos, C. Bender, M. Giannotti, F. Rojas, C. Zachos, and many others.

I would also like to thank other professors or colleagues with whom I worked when I was a teaching assistant: J. Ashkenazi, G. Ghandour, J. Gundersen, P. Karavassilev, and many others.

I want to thank all other friends at the University of Miami

Finally, I am especially grateful to my family for constant support.

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Chapter 1

Introduction of Part I

In this introduction, we motivate and formulate the research questions to be addressed in Part I of this thesis. First, we introduce the origin of n-ary algebras and review the development of them within the last forty years. Then we give a short review of M2-brane and point out that one of the most important research problems of M-theory is the dynamics between multiple M2-branes and M5-branes. We then go on to explain the recent connection between 3-algebra and dynamics of multiple M2-branes through BLG model. We also mention that there is an alternative candidate for multiple M2-branes theory, and its possible relation to BLG model. These allow us to formulate the precise questions we want to address in Part I. Finally we go through the organization of Part I by chapters.

1.1 History of n-Ary Algebras

About forty years ago, Nambu[1] proposed an elegant generalization of classical Hamiltonian mechanics, which was an alternative to the canonical formulation. In Nambu's formalism, he replaced the usual canonical binary Poisson bracket by Nambu bracket with three entries. Since this ternary innovated algebraic structure is

Poissonlike, it is called the classical Nambu bracket, and it is a volume-element Jacobian determinant in a higher dimensional space. Nambu also considered imposing various properties on the classical Nambu bracket, but it was not clear at that time[1]. Later, Filippov[2], Takhtajan[3] and many others[4,5,6,7] systematically studied the original Nambu brackets and made many extensions.

In 80s, Filippov[2] quantized classical Nambu bracket in a conventional Hilbert space operator formalism, writing it in the form of 3- or n-linear fully antisymmetric commutator. He also imposed a property called Filippov identity (also known as Fundamental Identity) to the quantized Nambu brackets, formulating the Filippov algebras, which is the original and the first version of n-ary algebras. Fundamental identity is considered to be the Jacobi identity in 3- or n-bracket case, and it is the most critical property when defining Filippov algebras. In 1994, Takhtajan[3] defined a special class of Nambu brackets, called the Nambu-Poisson brackets. In his system, Nambu-Poisson bracket can be used to write down the equations of motion with the advantage that a larger symmetry is kept manifest compared with the canonical formulation, which requires a gauge fixing condition. Based on Filippov's work, many modifications have been made on Filippov algebras, and many other similar algebras have been generated, like generalized Lie algebra, n-Leibniz algebra, etc[7,8,9,10,11].

Because of its Lie algebra-like form, many researchers attempted to search Lie algebra-like properties of n-ary algebras[2,12,13,14]. However, after almost forty years, we still know little about this new algebraic structure by two reasons. One is

that there are not many n-ary algebras found so far; the Fundamental identity is so strong that no matter what kind of the modifications made on it, it still rules out most n-ary algebra candidates. Besides Fundamental identity, people rarely found other properties that can be applied to Nambu bracket and n-ary algebras until 1997. Bremner[15,16,17] found the universal property (Bremner Identity) for all quantized ternary Nambu bracket, and then Curtright, Jin and Mezincescu extended it to n-bracket case[18]. Another reason is that people have not found an effective representation for n-ary algebras.

1.2 What is an M2-brane?

String theory is considered as a candidate to unify all the fundamental interactions since the early 1970s. During the development of string theory, one natural question was always asked: can one extend the one-dimensional strings to higher dimensional objects, like 2-dimensional membranes or, in general d-dimensional objects? The first study of membranes was back in 1962, even before the borning of string theory. Dirac[19] proposed that the excited states of a brane-like object could correspond to the electron and the muon. However, because of the lack of spin in his theory, the brane idea was not accepted until superysymmetry and supergravity were studied.

The study of branes was in parallel with string theory, generalizing from the Green-Schwarz action. The attempts to quantize branes were not successful until 1986,

when supersymmetry was incorporated to 3-branes theory by Hughes, Liu and Polchinski[20]. Later, Bergshoe, Sezgin and Townsend[21] wrote down the Lagrangian describing a single supersymmetric membrane propagating in D dimensions.

On the other hand, eleven-dimensional supergravity was constructed in 1978[22]. Two different techniques, i.e. the doubled field approach and the method of nonlinear realization were combined and with their help, people could understand the equations of motion of the bosonic p-form fields. The eleven-dimensional theory has the advantage of uniqueness and simplicity. Only three different particles: the gravition, the Majorana gravitino, and a gauge potential (a third rank antisymmetric tensor), are contained.

The eleven-dimensional space in strong coupling limit was generalized from two of the string theories (type IIA and $E_8 \times E_8$ heterotic string). At the Strings '95 conference in the University of Southern California, Witten named it M-theory and proposed that M-theory to be an extension of string theory that would unify all five string theories via dualities, and the eleven-dimensional gravity is its low energy limit. In M-theory, the fundamental ingredients are the M2-branes and their solitonic electric-magnetic duals, M5-branes.

Maldacena's AdS/CFT[23] correspondence provides an approach to stacks of M2- and M5-branes; the gravity side of the correspondence, where these branes are defined, is a good starting point. In eleven-dimensional supergravity, a stack of

M2-branes has an $AdS_4 \times S^7$ geometry, and is dual to a three dimensional conformal field theory. The gauge theory is of Chern-Simons type and it contains $\mathrm{N} = 8$ maximal supersymmetry.

Nowadays, the dynamics of M2-branes and M5-branes is still under a long way of studying, and M-theory still remains mysterious.

1.3 3-Algebra and Multiple M2-branes

Before 2006, n-ary algebras had already been studied for more than three decades, but people were still not clear about how to apply it to physics successfully. While on the other hand, actions describing single M2-brane and single M5-brane had been known for a long time; however, actions for multiple M2-branes and multiple M5-branes dynamics were still mysterious. Fortunately, in 2006, a novel theory appeared and combined n-ary algebras and multiple M2-branes dynamics together. Bagger and Lamber[24,25,26], and Gustavsson[27] independently proposed an action describing a pair of M2-branes, based on Filippov 3-algebras, rather than Lie algebras. This model is called BLG model, and was inspired by the work of Basu and Harvey[28]. Actually, the equation of motion for multiple M2-branes came before the action for them. Basu-Harvey equation successfully describes the configuration of multiple M2-branes ending on an M5-brane, though it does not reply on specific realization of any 3-algebra like structure. Bagger, Lamber and Gustavsson then started from Baso-Harvey equation, proposed BLG model.

BLG model has 3-dimensional maximally supersymmetric gauge theories with $OSp(8|4)$ superconformal symmetry. This model has successfully described pairs of M2-branes on an M-fold, and has provided a relationship between M2-branes and Dp-branes. The Filippov 3-algebra utilized in it is 4-dimensional A_4, which is finite positive-definite. The Fundamental identity of A_4 plays a very important role in BLG model: it implies that BLG model has $\mathrm{N}=8$ supersymmetry. However, BLG still suffers several disappointing points: first, it describes the dynamics of a pair of M2-branes on an orbifold rather than a flat space, which does satisfy BPS state very well. Second, because of gauge symmetry A_4, its gauge group $SU(2)\times SU(2)$ only describes two M2-branes sitting on one another; the action describing low energy dynamics of an arbitrary finite number of M2-branes remains unsolved.

BLG model initiated a trend of studying the relation between 3-algebras (not just Filippov 3-algebras) and M2-branes[29,30,31,32]. It seems that the A_4 algebra is the only finite positive-definite metric Filippov 3-algebra, all others are either direct sums of A_4 or trivial one-dimensional 3-algebras. So it is almost impossible to modify BLG model by enriching finite positive-definite metric Filippov 3-algebras. The condition of "finite positive-definite metric" seems too strong to rule out all other 3-algebras to be applied to this model. To remedy this, "finite" condition can be abandoned. Filippov algebras are originated from classic Nambu bracket, and classic Nambu bracket combining with Fundamental identity would generate Nambu algebra. Nambu algebra is an infinite positive-definite metric algebra, and of course, it is also a special case of Filippov algebras.

Bagger and Lambert[26] first suggested to apply Nambu algebra in BLG model, and others[33,34,35,36] showed the success of this modification. Nambu algebra gives volume-preserving diffeomorphism group $SDiff(S^3)$ for the gauge group of this modified model. The new BLG-NB model contains $\mathrm{N}=8$ maximal supersymmetry also, and describes infinite M2-branes condensation. However, how to describe finite number of M2-branes still can not be answered by this model.

Besides BLG model and BLG-NB model, Aharony, Bergman, Jafferis and Maldacena[23] proposed another model that is believed to describe an arbitrary finite number of M2-branes, the ABJM model. ABJM model was constructed using a Lie algebra, and it has gauge group $SU(N)\times SU(N)$ and manifest $\mathrm{N}=6$ supersymmetry. It was shown that, ABJM model is equivalent to BLG model when $N=2$. However, it is not clear how to enhance the supersymmetry when N is greater than 2, and one can not expect to directly read off the degrees of freedom from the Lagrangian of ABJM model. If ABJM model is correct, it should be possible to take the large N limit to describe M2-branes condensation, which means ABJM model should be equivalent to BLG-NB model. The unification work between BLG, ABJM and BLG-NB is still under study.

1.4 Research Questions

In Part I of this thesis, we focus on the properties of n-ary algebras, and how they are applied to M-theory. The problems we want to confront are:

1. To classify different types of n-ary algebras, study the relations between them.
2. To study the restrictions on structure constants of different n-ary algebras given by Bremner identity.
3. To study the relations between Fundamental Identity and Bremner Identity.
4. To study BLG-NB model, and its gauge group $SDiff(S^3)$, as well as its relations to $SU(2)\times SU(2)$.
5. To study the connection of ABJM model to BLG model, study supersymmetry enhancement.
6. To study the gauge theory of ABJM model at large N limit.

1.5 Roadmap of Part I

In chapter 2, we first review the structure theory of regular Lie algebras and their representation theory. Then we classify n-ary algebras into generalized Lie algebras, Filippov algebras, Nambu algebras and Possion structure. Though there are more n-ary algebras than we list here, we believe that these four are most closed to physics and should be of particular interest of research. We attempt to enrich 3-algbras by calculating ternary versions of Virasoro-Witt algebra, superconformal algebra and Ladder operator, and find that only Virasoro-Witt algebra can generate 3-algebra under some certain conditions. We also show a non-computer involved approach of proving Bremner identity in quantum Nambu 3-bracket case, and extend it to n-bracket case.

In chapter 3, we study the formulism of BLG model. We review A_4 and see it plays an important role in supersymmetry invariance and M2-brane number description. We talk about the success and challenge for BLG model. Then we extend BLG model to BLG-NB model by substituting A_4 with Nambu algebra. The gauge group of BLG-NB model is $SDiff(S^3)$, a conjecture is to decompose the gauge theory of $SDiff(S^3)$ into the direct product of two $SDiff(S^2)$ gauge theory. We give a short review of constructing classical Yang-Mills theories at large N limit, and believe that this approach would give us the hint of connecting BLG and BLG-NB model.

In chapter 4, we introduce ABJM model and its connection to BLG model and BLG-NB model. At $N = 2$ case, ABJM model is believed to be equivalent to BLG model, but the supersymmetry enhancement is still a problem when $N \neq 2$. Also the gauge theory of ABJM model when N goes infinity is mysterious, and we discuss the possibility and approaches to unify ABJM model and BLG model.

In chapter 5, we make a conclusion of Part I.

Chapter 2

n-Ary Algebras

A Lie algebra is an algebraic structure based on Lie bracket operation involving two entries. The n-ary generalizations of Lie algebra are algebraic structures in which two-Lie bracket has been replaced by a bracket with n entries. Like Lie algebras, n-ary algebras also satisfiy a specific characteristic identity which plays the role of the Jacobi identity for Lie algebra. There are more than one ways to define n-ary algebras, and different definitions would emphasize different arithmatics and correspond to different identities. However, no matter how one defines a n-ary algebra, there are not as many of them as Lie algebras. The characteristic identities rule out most algebraic structures from being n-ary algebras.

In section 2.1, we briefly review Lie algebras and metric Lie algebras. In section 2.2, we introduce generalized Lie algebras, which emphasize on the associativity of n-brackets. The characteristic identity of generalized Lie algebras is generalized Jacobi identity (GJI). And besides GJI, generalized Lie algebras also satisfy Bremner identity and extended Bremner identity. In section 2.3, we study Filippov algebras with Fundamental identity, which do not emphasize on associativity. We give several examples of constructing Filippov 3-algebras from some well-known Lie algebras. Section 2.4 talks about Nambu-algebras, which is the infinite dimensional version of

Filippov algebra. Some Filippov 3-algebras given in last section are rewritten in Nambu-algebra version. We also give the condition at which Bremner identity can be applied to Nambu algebras. In section 2.5, the Nambu-Poisson tensor is discussed, and we figure out that Bremner identity would only generate trivial results.

2.1 Lie Algebras

In this section, we summarize some basic concepts of the theory of Lie algebras. Like many other algebraic structure, a Lie algebra is an object defined as a set in which one can do some algebraic operations that satisfy certain conditions. A Lie algebra structure is a vector space with an extra bilinear operation that assigns every pair of vectors to the third one.

2.1.1 General properties of Lie algebras

Definition 2.1: A Lie algebra is a vector space g together with a bilinear operation $g \times g \to g$, the bilinear operation, Lie bracket [,] : $(X,Y) \to [X,Y]$ that satisfies

1. Anticommutativity:

$$[X,Y] = -[Y,X], \tag{2.1}$$

2. bilinearity:

$$[aX+bY,Z] = a[X,Z]+b[Y,Z], \qquad [Z,aX+bY] = a[Z,X]+b[Z,Y], \tag{2.2}$$

3. and the Jacobi Identity

$$[X,[Y,Z]]+[Z,[Y,X]]+[Y,[X,Z]] = 0, \tag{2.3}$$

for all $x,y,z \in g$ and $a,b \in R$.

A finite example of Lie algebras is the linear transformations of a finite vector space, which is denoted the general linear Lie algebra gl; and an infinite-dimensional example is the Lie algebra of the vector fields on a manifold M.

If the bilinear operation is defined as associative, then the Lie bracket can be expressed as $[X,Y]=XY-YX$, and this yields the Jacobi Identity necessarily. The Jacobi Identity can be written as

$$[X,[Y,Z]]=[[X,Y],Z]+[Y,[X,Z]]. \tag{2.4}$$

This form is similar to the Leibniz rule of derivative of a product of two functions:

$$D(fg)=(Df)g+f(Dg). \tag{2.5}$$

Therefore, a linear transformation $D:\ g\to g$ of the Lie algebra is said to be a derivation of the Lie algebra. Consider the adjoint map $ad_X(Y):=[X,Y]$, then the Jacobi Identity can be written as

$$ad_X([Y,Z])=[ad_X(Y),Z]+[Y,ad_X(Z)]. \tag{2.6}$$

Hence, by definition, if g is a Lie algebra, the adjoin maps are derivations of it.

Like other algebraic structures, a Lie algebra has a basis $\{e_a\}$ spanning the vector space g. One can specify the structure constants f_{ab}^c connecting Lie bracket and the basis:

$$[e_a,e_b]=f_{ab}^c e_c, \tag{2.7}$$

The structure obeys

$$f_{ab}^c=-f_{ba}^c, \tag{2.8}$$

and the Jacobi identity can be written in the third form in terms of structure constants:

$$f^c_{ab} f^e_{cd} + f^c_{bd} f^e_{ca} + f^c_{da} f^e_{cb} = 0 . \quad (2.9)$$

2.1.2 Metric Lie algebras

On Lie algebras, we can define the metric and construct scalar products. A Lie algebra is *metric* when it is endowed with an invariant, symmetric and nondegenerate bilinear form $< , > : g \times g \to R$, which defines the scalar product in the g vector space:

$$[X, < Y, Z >] = < [X, Y], Z > + < Y, [X, Z] >= 0 , \quad (2.10)$$

for all $X, Y, Z \in g$.

The components of this bilinear form in terms of the basis $\{e_a\}$ are

$$< e_a, e_b >= g_{ab} \quad . \quad (2.11)$$

Once the metric is defined, one can raise and lower the indices of structure constants

$$f_{abc} = f^d_{ab} g_{dc} \quad . \quad (2.12)$$

Since our purpose of this section is to review some basic concepts of Lie algebras, we do not go further about the properties of metric Lie algebras, but only list some definitions and results here[37]:

Definition 2.2: A metric Lie algebra is said to be indecomposable if it can not be written as an orthogonal direct sum of metric Lie algebras with dimension greater than zero.

Definition 2.3: A subspace, h, of elements of a Lie algebra, g, is called a Lie subalgebra if it is closed under the Lie bracket:

$$[h_1, h_2] \in h, \tag{2.13}$$

for all $h_1, h_2 \in h \subset g$.

Definition 2.4: A subset, I, of elements of a Lie algebra, g, is called an ideal and denoted $I \lhd g$, if it is absorbent with respect to the Lie bracket, which means

$$[I, g] \subset I. \tag{2.14}$$

The total space g and the element 0 are always ideals of g. All other ideals of g are called proper.

Definition 2.5: A Lie algebra g is said to be simple if it is not one-dimensional and it does not contain any proper ideals.

Definition 2.6: A Lie algebra g is said to be semisimple if it is isomorphic to a direct sum of simple Lie algebras.

2.1.3 Representation theory of Lie algebras

The theory representing Lie algebras is always the most popular topic in Lie algebras' research. It contains so much subtopics and properties that it can easily cost hundreds of pages to talk about. In physics, we are interested in unitary representation most, because scalar Lagrangians are constructed based on it. Here, we give the definition of the representation of a Lie algebra:

Definition 2.7: A representation of a Lie algebra g is a linear map $\rho: g \to gl(V)$, where V is a vector space, that preserves the Lie bracket:

$$\rho([X,Y]) = [\rho(X), \rho(Y)], \tag{2.15}$$

for all $X, Y \in g$

Definition 2.8: A representation of a Lie algebra g is *unitary* if it preserves the scalar product of g:

$$< \rho(X), \rho(Y) >=< X, Y > . \tag{2.16}$$

2.2 Generalized Lie Algebras

2.2.1 n-ary algebras

Based on the Lie algebra concepts talked about above, we can extend regular Lie algebra structure to algebraic structure of brackets with $n > 2$ entries. Generally, n-ary bracket is constructed from a multilinear operation:

$$[\ ,\ ,\ \overbrace{\cdots}^{n}\ ,\ ,\ ]:\ G \times \overbrace{\cdots}^{n} \times G \to G,$$

where G is generic n-ary algebra.

For all n-ary bracket structure, we require it has similar properties to Lie bracket:

1. Anticommutativity:

$$[\cdots, X, \cdots, Y, \cdots] = -[\cdots, Y, \cdots, X, \cdots], \tag{2.17}$$

means if we switch the positions of an arbitrary pair of entries while keep the positions of other entries same, then total sign of the bracket will be changed. Therefore, n-ary bracket is totally antisymmetric to an arbitrary pair of entries.

2. Multilinearity:

$$[\cdots, aX + bY, \cdots] = a[\cdots, X, \cdots] + b[\cdots, Y, \cdots]. \qquad (2.18)$$

As to the Jacobi identity, there are two interpretations of extending it to n-ary case, and these lead to two main different generalizations of the n-ary algebras. One is the antisymmetrization of the nested n-ary brackets, by extending nexted Lie brackets $[\,,\,[\,,\,]\,]$; the other one is to extend the derivation character of Lie algebra to n-ary case. Generalized Lie algebra[8,9] is generated by the first case, and Filippov algebra[2] is by the second (we talk about it in next section).

2.2.2 Generalized Lie algebras

For generalized Lie algebras, the Jacobi identity is extended to generalized Jacobi identity:

For n even, the generalized Lie bracket satisfies the following identity

$$\sum_{\sigma \in S_{2n-1}} (-1)^{\pi(\sigma)} [[X_{\sigma(1)}, \cdots, X_{\sigma(n)}], X_{\sigma(n+1)}, \cdots, X_{\sigma(2n-1)}] = 0, \qquad (2.19)$$

where S_{2n-1} is $2n-1$ order symmetry group, and σ goes through all the elements of S_{2n-1}. Curtright and Zachos[4] proved that an arbitrary n(even)-bracket obeys (2.19), no matter if the entries compose an algebra or not.

For n odd, it is proved[4] that the right hand side is not zero. The totally antisymmetrized action of odd n-bracket on other odd n-bracket results in a $(2n-1)$-bracket. For example, 3-bracket acting on 3-bracket results:

$$\sum_{\sigma\in S_5}(-1)^{\pi(\sigma)}[[X_{\sigma(1)},\cdots,X_{\sigma(3)}],X_{\sigma(4)},X_{\sigma(5)}]=C[X_1,X_2,X_3,X_4,X_5] \quad (2.20)$$

where C is a constant coefficient.

Generalized Lie algebras emphasize on the associativity of the composition of the elements in their multibracket, so the multibracket of order n can be defined by the fully antisymmetrized product of its entries, making it similar to regular commutators

$$[X_{i_1},\cdots X_{i_n}]:=\sum_{\sigma\in S_n}(-1)^{\pi(\sigma)}X_{i_{\sigma(1)}}\cdots X_{i_{\sigma(n)}} \quad . \quad (2.21)$$

In regular Lie algebras, we can always use Levi-Civita symbols denoting Lie bracket $[X_{i_1},X_{i_2}]=\in_{i_1i_2}^{j_1j_2}X_{j_1}X_{j_2}$, and we can do the same to generalized Lie algebras:

$$[X_{i_1},\cdots X_{i_n}]=\in_{i_1\cdots i_2}^{j_1\cdots j_2}X_{j_1}\cdots X_{j_2} \, . \quad (2.22)$$

Since generalized Jacobi identity restricts that n must be even, the generalized Lie algebra is defined as:

Definition 2.9: the n order generalized Lie algebra[8,9] is a vector space G endowed with a fully antisymmetric bracket $G\times\overbrace{\cdots}^{n}\times G\to G$, $(X_1,\cdots,X_n)\mapsto[X_1\cdots,X_n]\in G$, such that the generalized Jacobi identity is fulfilled. And given a basis $\{X_i\}$ of G ($i=1,\cdots,d=\dim G$), the structure constants of

generalized Lie algebra is

$$[X_{i_1},\cdots X_{i_n}]=C_{i_1\cdots i_n}{}^{j}X_j. \tag{2.23}$$

2.2.3 Properties of quantum Nambu brackets

The totally antisymmetrized n-bracket is also known as quantum Nambu bracket or n-bracket. For even quantum Nambu bracket, (2.19) is the identity they all obey. And for odd quantum Nambu bracket, (2.20) could be considered as an identity, but it involves higher order bracket and makes the form complicated. Therefore, we do not consider (2.20) as an identity for odd quantum Nambu bracket, and there does exist an identity that can be applied to all odd quantum Nambu bracket—the Bremner identity.

Bremner[15] showed that

$$[[A,[b,c,d],e],f,g]=[[A,b,c],[d,e,f],g], \tag{2.24}$$

where A is fixed, but it is implicitly understood that lower case entries are totally antisymmetrized by summing over all $6!$ signed permutations of them.

When Bremner attempted to prove this identity, he used computer to count and calculate all the terms on both hand side of (2.24). Here, we give an alternative proof that does not involve computer. The Bremner Identity can be proved through a resolution of both left- and right-hand sides as a series of canonically ordered words. By direct calculation we find

$$\begin{aligned}[[A,[b,c,d],e],f,g]&=24Abcdefg-36bAcdefg+36bcAdefg-24bcdAefg\\&+36bcdeAfg-36bcdefAg+24bcdefgA\end{aligned}, \tag{2.25}$$

where all lower case entries are implicitly totally antisymmetrized. Precisely the same expansion holdes for $[[A,b,c],[d,e,f],g]$, again by direct calculation. Hence the identity is established.

That is to say, both $[[A,[b,c,d],e],f,g]$ and $[[A,b,c],[d,e,f],g]$ can be rendered as a 7-bracket plus another 3-bracket containing 3-brackets, when antisymmetrized over lower case entries.

$$[[A,[b,c,d],e],f,g]=\frac{1}{20}[A,b,c,d,e,f,g]-\frac{1}{6}[A,[b,c,d,],[e,f,g]]=[[A,b,c],[d,e,f],g] \tag{2.26}$$

Thus the Bremner identity amounts to the combinatorial statement, as written, that there are two distinct ways to present a 7-bracket in terms of nested 3-brackets.

With Bremner identity, one would think about extending it to higher order cases. For odd n-bracket, (2.19) produces the $(2n-1)$-bracket upon total antisymmetrization[4]. Apparently, the simplest identity obeyed by odd brackets of only one type, that does not introduce higher order brackets, requires that they act at least thrice. For any odd n, a valid relation is the immediate generalization of that found by Bremner for the case of 3-brackets. To show this, we present two easily established lemmas. Firstly,

Lemma 2.1:

$$[A,B_1,\cdots,B_J]=J!\sum_{j=0}^{J}(-1)^j B_1\cdots B_j A B_{j+1}\cdots B_J\ . \tag{2.27}$$

Total antisymmetrization of the B's is understood. Here we have also used the convention that an empty product equals 1. Explicitly, $B_1 \cdots B_0 = 1 = B_{J+1} \cdots B_J$, so that the first and last terms in the sum are $AB_1 \cdots B_J$ and $(-1)^J B_1 \cdots B_J A$, respectively.

Lemma 2.2:

$$[A, B_1, \cdots, B_J, Z] = J! \sum_{j=0}^{J} (-1)^j \sum_{k=0}^{J-j} (-1)^k B_1 \cdots B_k A B_{k+1} \cdots B_{J-j} Z B_{J-j+1} \cdots B_J$$
$$- J! \sum_{j=0}^{J} (-1)^j \sum_{k=0}^{J-j} (-1)^k B_1 \cdots B_k Z B_{k+1} \cdots B_{J-j} A B_{J-j+1} \cdots B_J$$

(2.28)

Finally, it is rather tedious but fairly straightforward to use both lemmas to prove

Theorem 2.1: For associative products, with implicit total antisymmetrization of the B's,

$$[[A, [B_1, \cdots, B_{2L+1}], B_{2L+2}, \cdots, B_{4L}], B_{4L+1}, \cdots, B_{6L}] = [[A, B_1, \cdots, B_{2L}], [B_{2L+1}, \cdots, B_{4L+1}], B_{4L+2}, \cdots, B_{6L}]$$

(2.29)

Proof of Theorem: The result follows from resolving the left- and right-hand sides into sums of canonically ordered words, as illustrated above for the case of 3-brackets. We have

$$[[A, B_1, \cdots, B_{2L}], [B_{2L+1}, \cdots, B_{4L+1}], B_{4L+2}, \cdots, B_{6L}] = \sum_{n=0}^{6L} (-1)^n m_n^{(1)} B_1 \cdots B_n A B_{n+1} \cdots B_{6L},$$

$$[[A, [B_1, \cdots, B_{2L+1}], B_{2L+2}, \cdots, B_{4L}], B_{4L+1}, \cdots, B_{6L}] == \sum_{n=0}^{6L} (-1)^n m_n^{(2)} B_1 \cdots B_n A B_{n+1} \cdots B_{6L}.$$

(2.30)

All the coefficients $m_n^{(1,2)}$ in these two resolutions are manifestly positive integers.

The theorem is established by showing that $m_n^{(1)} = m_n^{(2)}$ for all n.

By direct calculation, through the use of the two lemma, we find

$$m_n^{(1)} = m_n^{(2)} = (2L+1)!(2L-1)! \times c_n, \tag{2.31}$$

$$c_n = \begin{cases} (n+1)(4L-n)/2 & \text{for } 0 \le n \le 2L \\ 10L^2 - 6Ln + L + n^2 & \text{for } 2L+1 \le n \le 3L \\ c_{6L-n} & \text{for } 3L+1 \le n \le 6L \end{cases}. \tag{2.32}$$

The determination of the m_n's is just a matter of enumerating the ways to obtain a particular intercalation of A among the B's.

Consider in more detail some of the calculations involved. As a first step, with the implicit antisymmetrization, the internal brackets $[B_1, \cdots, B_{2L+1}]$ or $[B_{2L+1}, \cdots, B_{4L+1}]$ may be supplanted by products: $[B_1, \cdots, B_{2L+1}] = (2L+1)!(B_1 \cdots B_{2L+1})$ or $[B_{2L+1}, \cdots, B_{4L+1}] = (2L+1)!(B_{2L+1} \cdots B_{4L+1})$. Then we may write, on the one hand,

$$\begin{aligned} &[[A, B_1, \cdots, B_{2L}], [B_{2L+1}, \cdots, B_{4L+1}], B_{4L+2}, \cdots, B_{6L}] \\ &= -(2L+1)![[A, B_1, \cdots, B_{2L}], B_{4L+2}, \cdots, B_{6L}, (B_{2L+1} \cdots B_{4L+1})] \end{aligned}. \tag{2.33}$$

In this expression, we may now rename indices, bearing in mind the antisymmetrization.

$$\begin{aligned} &[[A, B_1, \cdots, B_{2L}], [B_{2L+1}, \cdots, B_{4L+1}], B_{4L+2}, \cdots, B_{6L}] \\ &= (2L+1)![[A, B_{2L}, \cdots, B_{4L-1}], B_1, \cdots, B_{2L-1}, (B_{4L} \cdots B_{6L})] \end{aligned}. \tag{2.34}$$

Next, we apply Lemma 2.2 for $J = 2L-1$, and identify $[A, B_{2L}, \cdots, B_{4L-1}]$ with A and $(B_{4L} \cdots B_{6L})$ with Z.

$$[A,B_1,\cdots,B_{2L-1},\mathrm{Z}]=(2L-1)!\sum_{j=0}^{2L-1}(-1)^j\sum_{k=0}^{2L-1-j}(-1)^k B_1\cdots B_k \mathrm{A} B_{k+1}\cdots B_{2L-1-j}\mathrm{Z}B_{2L-j}\cdots B_{2L-1}$$

(2.35a)

$$-(2L-1)!\sum_{j=0}^{2L-1}(-1)^j\sum_{k=0}^{2L-1-j}(-1)^k B_1\cdots B_k \mathrm{A} B_{k+1}\cdots B_{2L-1-j}\mathrm{Z}B_{2L-j}\cdots B_{2L-1}$$

(2.35b)

To continue, consider first the coefficients $m_n^{(1)}$ where $n\le 2L$.

For the determination of $m_{n\le 2L}^{(1)}$, since Z consists of $(2L+1)B$'s, it must be placed to the right of A in the application of Lemma 2.2. otherwise there would be too many B's to the left of A. Thus for $m_{n\le 2L}^{(1)}$ we need keep only the first line in the last relation, (2.35a). To place a total of n B's to the left of the A contained in $\mathrm{A}=[A,B_{2L},\cdots,B_{4L-1}]$, with k B's already to the left as in (2.35a), we then need only the terms in A with an additional $(n-k)$ B's to the left of A. That is to say, from Lemma 1.1, with $J=2L$ and all B indices shifted up by $2L-1$,

$$\mathrm{A}=[A,B_{2L},\cdots,B_{4L-1}]=(2L)!\sum_{l=0}^{2L}(-1)^n B_{2L}\cdots B_{l+2L-1}AB_{l+2L}\cdots B_{4L-1}\,, \tag{2.36}$$

and from this we need only the term with $l=n-k$. The net result for $m_{n\le 2L}^{(1)}$ is

$$m_{n\le 2L}^{(1)}=(2L+1)!(2L)!(2L-1)!\ \times c_{n\le 2L}\,, \tag{2.37}$$

$$c_{n\le 2L}=\sum_{j=0}^{2L-1}\sum_{k=0}^{2L-1-j}\sum_{l=0}^{2L}\delta_{l,n-k}\,|_{n\le 2L}=\sum_{j=0}^{2L-1}\sum_{k=0}^{\min(n,2L-1-j)}1=\frac{(n+1)(4L-n)}{2}\,. \tag{2.38}$$

On the other hand, with similar steps, we have

$$\begin{aligned}&[[A,[B_1,\cdots,B_{2L+1}],B_{2L+2},\cdots,B_{4L}],B_{4L+1},\cdots,B_{6L}]\\&=(2L+1)![[A,B_1,\cdots,B_{2L-1},(B_{2L}\cdots B_{4L})],B_{4L+1},\cdots,B_{6L}]\end{aligned}\,. \tag{2.39}$$

We again apply Lemma 2.2 for $J=2L-1$, but to $[A,B_1,\cdots,B_{2L-1},(B_{2L}\cdots B_{4L})]$, so

now we identify A with A, and $(B_{2L}\cdots B_{4L})$ with Z. As before, consider first only $m_n^{(2)}$ coefficients where $n \le 2L$. For the determination of $m_{n\le 2L}^{(2)}$, Z must once again be placed to the right of A, so we need keep only the line (2.35a). We pick up an additional $(n-k)$ B's by applying again Lemma 2.1, only this time to the remaining outside bracket in (2.39). The net result for $m_{n\le 2L}^{(2)}$ is

$$m_{n\le 2L}^{(2)} = (2L+1)!(2L)!(2L-1)! \times c_{n\le 2L}. \tag{2.40}$$

With exactly the same expression for $c_{n\le 2L}$ as before, (2.38). Thus we have shown $m_{n\le 2L}^{(1)} = m_{n\le 2L}^{(2)}$.

Next, consider the coefficients where $2L+1 \le n \le 3L$. There are still contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ still turn out to be the same. But in this case the sums contributing to c_n give

$$\sum_{j=0}^{2L-1}\sum_{k=0}^{2L-1-j}\sum_{l=0}^{2L}\delta_{l,n-k}\,|_{2L+1\le n\le 3L} = \sum_{j=0}^{2L-1-(n-2L)}\sum_{k=(n-2L)}^{2L-1-j} 1 = \frac{(4L+1-n)(4L-n)}{2}. \tag{2.41}$$

Moreover, from applying Lemma 2.2, there are now contributions to either $m_n^{(1)}$ or $m_n^{(2)}$ from the second, (2.35b), where the respective Z's are placed to the left of the A's. Following steps similar to those above, it is not difficult to see that these other terms contribute the same amount to either $m_n^{(1)}$ or $m_n^{(2)}$, for $2L+1 \le n \le 3L$. Namely, $(2L+1)!(2L)!(2L-1)! \times$

$$\sum_{j=0}^{2L-1}\sum_{k=0}^{2L-1-j}\sum_{l=0}^{2L}\delta_{l,n+j-4L}\,|_{2L+1\le n\le 3L} = \sum_{j=4L-n}^{2L-1}\sum_{k=0}^{2L-1-j} 1 = \frac{(n-2L+1)(n-2L)}{2}. \tag{2.42}$$

Thus the net result in $m_{2L+1\le n\le 3L}^{(1)} = m_{2L+1\le n\le 3L}^{(2)} = (2L+1)!(2L)!(2L-1)! \times c_{2L+1\le n\le 3L}$

with

$$c_{2L+1\le n\le 3L} = \frac{1}{2}(4L+1-n)(4L-n) + \frac{1}{2}(n-2L+1)(n-2L) = 10L^2 - 6Ln + L + n^2 , \quad (2.43)$$

Finally, consider the coefficients for $3L+1 \le n \le 6L$. These are given by an elementary reflection symmetry: $m_n^{(1)} = m_{6L-n}^{(1)}$ and $m_n^{(2)} = m_{6L-n}^{(2)}$. Thus

$$m_n^{(1)} = m_n^{(2)} = (2L+1)!(2L)!(2L-1)! \ \times c_{6L-n} \quad \text{for} \quad 3L+1 \le n \le 6L . \quad (2.44)$$

As a check, the coefficients must sum to give the number of generic terms that appear in three nested $(2L+1)$-brackets (i.e. in either $[[[\cdots]\cdots]\cdots]$ or $[[\cdots]\cdots[\cdots]]$). That is , $\sum_{n=0}^{6L} m_n = ((2L+1)!)^3$. Equivalently,

$$\sum_{n=0}^{6L} c_n = 2L(2L+1)^2 . \quad (2.45)$$

This condition is indeed satisfied by the c_n given in (2.32).

2.3 Filippov Algebras

2.3.1 Basic properties of Filippov algebras

Extending Jacobi identity with the derivation property generates Filippov algebras[2], and the corresponding Fundamental identity (also known as Filippov identity) distinguishes Filippov algebras from the generalized Lie algebras of the previous section.

The vector where Filippov algebras are defined is the same as the vector we

define the generalized Lie algebras. For Filippov algebras, we still have the fully antisymmetric multilinear operation $[\ ,\ ,\overbrace{\cdots}^{n},\ ,\ ]: \overbrace{G\times\cdots\times G}^{n}\to G$. $[X_1,\cdots,X_n]\in G$ is still called n-bracket, and we have adjoint map given by

$$ad_{X_1\cdots X_{n-1}}: Z\to[X_1,\cdots,X_{n-1},Z],\ \ \forall X_i,Z\in G, \tag{2.46}$$

which for $n=2$ reproduces the action of ad_X on a Lie algebra. $ad_{X_1\cdots X_{n-1}}$ is a derivation of the n-bracket, that the following definition holds:

Definition 2.10: $ad_{X_1\cdots X_{n-1}}$ is an inner derivation of the n-bracket i.e.,

$$\begin{aligned} ad_{X_1\cdots X_{n-1}}[Y_1,\cdots,Y_n] &= \sum_{i=1}^{n}[Y_1,\cdots,ad_{X_1\cdots X_{n-1}}Y_i,\cdots,Y_n] \\ &= \sum_{i=1}^{n}[Y_1,\cdots,[X_1,\cdots,X_{n-1},Y_i],\cdots,Y_n] \end{aligned}. \tag{2.47}$$

The left hand side of (2.47) can be written as $[X_1,\cdots,X_{n-1},[Y_1,\cdots,Y_n]]$, and we have the Fundamental identity:

$$[X_1,\cdots,X_{n-1},[Y_1,\cdots,Y_n]]=\sum_{i=1}^{n}[Y_1,\cdots,[X_1,\cdots,X_{n-1},Y_i],\cdots,Y_n]. \tag{2.48}$$

With Fundamental identity, we can easily define Filippov algebras,

Definition 2.11: A Filippov algebra G is a vector space together with a multilinear fully antisymmetric operation $[\ ,\ ,\overbrace{\cdots}^{n},\ ,\ ]: \overbrace{G\times\cdots\times G}^{n}\to G$, such that the n-bracket satisfies Fundamental identity (2.48).

Fundamental identity also can be written as right multiplication, which means

$$[[Y_1,\cdots,Y_n],X_1,\cdots,X_{n-1}]=\sum_{i=1}^{n}[Y_1,\cdots,[Y_i,X_1,\cdots,X_{n-1}],\cdots,Y_n]. \tag{2.49}$$

Once one chooses a basis $\{X_i\}$ of G $(i=1,\cdots,d=\dim G)$, the n-bracket can be defined by the structure constants,

$$[X_{i_1},\cdots X_{i_n}]=f_{i_1\cdots i_n}{}^{d}X_d \quad . \tag{2.50}$$

With structure constants, the Fundamental identity can be expressed as

$$f_{i_1\cdots i_n}{}^{l}f_{j_1\cdots j_{n-1}l}{}^{s}=\sum_{k=1}^{n}f_{j_1\cdots j_{n-1}i_k}{}^{l}f_{i_1\cdots i_{k-1}li_{k+1}\cdots i_n}{}^{s} \tag{2.51}$$

Here we note that the n-bracket in Filippov algebras do not emphasize on associativity, meaning the realization of the n-bracket may not necessarily take the form of (2.21).

The structure of the Filippov algebras was first developed by Filippov[2], and then followed by Kasymov[12,13] and others[38]. Here we do not introduce those structural properties because we still do not know how to apply them to physics, and the whole picture of the algebra is still not very clear.

2.3.2 Examples of Filippov algebras

Now we give some examples of Filippov algebras. Since Filippov algebras do not necessarily have realization in form of (2.21), there are so many different ways to define the operations of the n-bracket. [7] gives a way to construct a Filippov algebra with matrix operators.

Example 2.1: Let A,B,C be square matrices, define 3-bracket by

$$[A,B,C]=tr(A)[B,C]+tr(B)[C,A]+tr(C)[A,B]\,. \tag{2.52}$$

where tr takes the trace of the matrix, and [,] is quantum commutator. This

realization obeys antisymmetric condition and Fundamental identity. And it can be extended to the arbitrary n-case. There are other matrix realizations of 3-bracket satisfying antisymmetric condition and Fundamental identity, see e.g. [31,39,40,41].

Filippov gave an example of simple Filippov algebra[2] defined on a n-dimensional real Euclidean vector space, the Euclidean A_{n+1}.

Example 2.2: Define three-algebra A_4 on four-dimensional Euclidean vector space V. Let $\{e_i\}$ be the basis of V and $v_1^a, v_2^a, v_3^a (a = 1,2,3,4)$ be the coordinates of three vectors $v_1, v_2, v_3 \in V$. The 3-bracket is defined by the 'vector product' of v_1, v_2, v_3,

$$[v_1, v_2, v_3] = \begin{vmatrix} e_1 & e_2 & e_3 & e_4 \\ v_1^1 & v_1^2 & v_1^3 & v_1^4 \\ v_2^1 & v_2^2 & v_2^3 & v_2^4 \\ v_3^1 & v_3^2 & v_3^3 & v_3^4 \end{vmatrix}, \tag{2.53}$$

which is antisymmetric and satisfies Fundamental identity. We will come back and visit A_4 in the next chapter. The A_{n+1} algebra is constructed with the similar form, on the basis $\{e_i\}$ where $i = 1, \cdots, n+1$. The 'vector product' of n vectors $v_1, v_2, \cdots, v_n$, $v_p = v_p^q e_q$, is defined by the determinant

$$[v_1, v_2, \cdots, v_n] = \begin{vmatrix} e_1 & \cdots & e_n & e_{n+1} \\ v_1^1 & \cdots & v_1^n & v_1^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ v_n^1 & \cdots & v_n^n & v_n^{n+1} \end{vmatrix}. \tag{2.54}$$

2.3.3 Constructing Filippov algebras from Lie algebras

So far, besides A_4, other Filippov algebras have rarely been applied to physics,

most of them are more like a mathematical toys, but rather something we can use to solve physics problems. We attempted to construct Filippov algebras from some well known Lie algebras, which means we define the 3-bracket and FI by

$$[A,B,C] = ABC + BCA + BAC - ACB - BAC - CBA,$$

$$fi[D,E;A,B,C] = [D,E,[A,B,C]] - [[D,E,A],B,C] - [A,[D,E,B],C] - [A,B,[D,E,C]] \tag{2.55}$$

and hope these new algebras can be applied to physics someday soon. We call these algebras quantal ternary algebras. We give several examples in [41]:

Example 2.3: Nambu's $su(2)$

Consider Nambu's application to $su(2)$.

$$[L_x, L_y, L_z] \equiv L_x[L_y, L_z] + L_y[L_z, L_x] + L_z[L_x, L_y] = i(L_x^2 + L_y^2 + L_z^2). \tag{2.56}$$

To close this algebra, it is necessary to include the $su(2)$ Casimir. But, having done so, one may rescale by a fourth root of the Casimir

$$Q_x = \frac{L_x}{\sqrt[4]{L^2}}, \qquad Q_y = \frac{L_y}{\sqrt[4]{L^2}}, \qquad Q_z = \frac{L_z}{\sqrt[4]{L^2}}. \tag{2.57}$$

And define a fourth charge as that fourth root,

$$Q_t = \sqrt[4]{L^2}. \tag{2.58}$$

Then,

$$[Q_a, Q_b, Q_c] = i\varepsilon_{abcd} Q^d. \tag{2.59}$$

where $\varepsilon_{xyzt} = +1$ with a $[-1,-1,-1,+1]$ Lorentz signature. The usual $\varepsilon\varepsilon$ identities now imply that this example is special: The Fundamental identity holds for Nambu's $su(2)$. In fact, this is the only finite quantal ternary algebra that satisfies the Fundamental identity.

Toward the end, we note that Nambu's $su(2)$ has sub-3-algebras that close. They are easily found. For example,

$$Q_x, \qquad Q_y, \qquad Q_z \pm Q_t. \tag{2.60}$$

Moreover, each of these subalgebras can be realized in terms of the classical brackets, which we will talk about in the next section.

Example 2.4: The bosonic oscillator.

The usual four charges $1, a, a^+$, and $N = a^+ a$ give the quantal ternary algebra

$$[1, N, a] = -a, \ [1, N, a^+] = a^+, \ [1, a, a^+] = 1, \ [N, a, a^+] = -1 - N. \tag{2.61}$$

Three of these reduce to just commutators: $[1, N, a] = [N, a]$, $[1, N, a^+] = [N, a^+]$, and $[1, a, a^+] = [a, a^+]$. This would suggest that the same algebra might also be realized as (2.52), except for the fact that the operators at hand are not of trice class. In any case, the fourth relation in (2.61) is not so simple.

However, if we take linear combinations as

$$R_1 = N, \qquad R_2 = \frac{1}{\sqrt{2}}(a^+ + a), \qquad R_3 = \frac{1}{\sqrt{2}} i(a^+ - a), \qquad R_4 = N + 1, \tag{2.62}$$

then we are back to a variation on Nambu's theme for $su(2)$: In this case, $sl(2, R)$.

$$[R_a, R_b, Q_c] = i\varepsilon_{abcd} R^d \tag{2.63}$$

with $\in_{1234} = +1$, again with Lorentz metric to raise indices, $\eta_{ab} = [1,1,1,-1]$. So, what is new here?

There are two additional bilinears, a^2 and a^{+2}, whose 3-brackets give

oscillator trilinears.

$$[a,a^2,a^{+2}]=2a+2Na\,,\ \ [a^+,N,a^2]=-2a-Na\,,\ \ [a,N,a^2]=-a^3\,,$$

$$[a^+,a^2,a^{+2}]=2a^++2a^+N\,,\ \ [a,N,a^{+2}]=2a^++a^+N\,,\ \ [a^+,N,a^{+2}]=a^{+3}\,. \quad (2.64)$$

Therefore, upon closure, the ternary algebra becomes infinite, and the standard enveloping algebra for the oscillator is obtained. From Filippov's perspective, it is perhaps disappointing that the oscillator enveloping algebra does not satisfy the Fundamental identity. For example

$$\begin{aligned}&[[a^+,a^+a,a^{+2}],a,a^2]-[[a^+,a,a^2],a^+a,a^{+2}]-[a^+,[a^+a,a,a^2],a^{+2}]-[a^+,a^+a,[a^{+2},a,a^2]]\\&=-2\\&[[a^+a,a^{+2},a^2],a,a^{+2}]-[[a^+a,a,a^{+2}],a^{+2},a^2]-[a^+a,[a^{+2},a,a^{+2}],a^2]-[a^+a,a^{+2},[a^2,a,a^{+2}]]\\&=20a^+\end{aligned} \quad (2.65)$$

In any case, the Fundamental identity does not hold in this example. But, necessarily, the associative enveloping algebra does satisfy the Bremner identity.

Example 2.5: Virasoro-Witt 3-algebra

For the oscillator there is a familiar, infinite Lie algebra contained within the enveloping algebra[134]. Consider

$$\Gamma_n=-(a^+)^nN\,, \quad (2.66)$$

For $n\geq 0$. Commutators give the well-known Virasoro-Witt algebra.

$$[\Gamma_n,\Gamma_m]=(n-m)\Gamma_{n+m}\,, \quad (2.67)$$

For $m,n\geq 0$. It is less well-known that the corresponding quantal 3-brackets are

$$[\Gamma_n,\Gamma_m,\Gamma_k]=0\,. \quad (2.68)$$

Thus we have a null 3-algebra for an infinite set of non-trivial, non-commuting

oscillator charges. The Fundamental identity is trivially satisfied in this case, as is the Bremner identity.

More structure is evident if we slightly modify the oscillator realization of the Virasoro-Witt algebra. For parameters β and γ, define and compute,

$$L_n = -(a^+)^n(N+\gamma+n\beta), \qquad [L_n, L_m] = (n-m)L_{n+m}. \tag{2.69}$$

The parameter β is related to the $sl(2,R)$ Casimir, $C = \beta(1-\beta)$. Now we find a non-null quantal 3-bracket when $0 \neq \beta \neq 1$.

$$[L_n, L_m, L_k] = \beta(1-\beta)(n-m)(m-k)(n-k)M_{n+m+k}, \tag{2.70}$$

where a second sequence of charges has been defined by

$$M_n = (a^+)^n. \tag{2.71}$$

While the Lie algebra of the L's and M's is also well-known[42], their 3-algebra has been investigated only recently[43]. To close the 3-algebra, we must consider all additional 3-brackets involving the M's:

$$[L_k, L_m, M_n] = (k-m)(L_{k+m+n} + (1-2\beta)nM_{k+m+n}),$$

$$[L_k, M_m, M_n] = (n-m)M_{k+m+n},$$

$$[M_k, M_m, M_n] = 0. \tag{2.72}$$

While the calculation is involved, the Bremner identity may be confirmed to hold for this ternary algebra. This result follows from the use of only (2.72) and (2.70), and does not make explicit use of the oscillator realization employed to obtain the 3-algebra. So this algebra is consistent with an underlying associative operator product no matter how it is realized.

The modification of the oscillator realization to include the parameter β has led to a larger ternary algebra involving both L's and M's. But, so enlarged, the algebra as presented in (2.72) and (2.70) is cumbersome. It may be streamlined by a linear change of basis, effectively from L_n and M_n back to the original Γ_n and M_n, as in (2.66) and (2.69) . That is to say, let

$$\Gamma_n = L_n + (\gamma + n\beta)M_n . \tag{2.73}$$

Regardless of how the algebra is realized, this change of basis simplifies (2.70) and (2.72). We find a remarkably concise form for the ternary algebra.

$$[\Gamma_n, \Gamma_m, \Gamma_k] = 0 ,$$

$$[M_k, M_m, M_n] = 0 ,$$

$$[\Gamma_k, M_m, M_n] = (n-m)M_{k+m+n} ,$$

$$[\Gamma_k, \Gamma_m, M_n] = (k-m)(\Gamma_{k+m+n} + nM_{k+m+n}) . \tag{2.74}$$

All explicit β dependence is thereby removed from the 3-algebra in this basis, and all values of the $sl(2,R)$ Casimir. Now, what about the Fundamental identities?

The Fundamental identities fail sometimes. This was discussed in[43], in the original basis, but it is much more transparent in terms of (2.74). It is trivial to see that the Filippov condition is satisfied when only Γ's, or when only M's, are involved: $fi(\Gamma_p, \Gamma_q; \Gamma_k, \Gamma_m, \Gamma_n) = 0$ and $fi(M_p, M_q; M_k, M_m, M_n) = 0$. The condition is also satisfied when their two, three, or four M's mixing it up with Γ's. But when one M is entangled with four Γ's, the condition fails, in general:

$$fi(\Gamma_p, \Gamma_q; \Gamma_k, \Gamma_m, M_n) = (p-q)(k-m)(k+m-p-q)nM_{k+m+n+p+q} ,$$

$$fi(\Gamma_p, M_q; \Gamma_k, \Gamma_m, M_n) = (n-k)(k-m)(m-n)qM_{k+m+n+p+q} . \tag{2.75}$$

On the other hand, the BI is again seen to always hold. We stress that these results follow from the use of only (2.74) without explicit use of the oscillator realization.

The situation with the Fundamental identities can be remedied if we perform an Inonu-Wigner contraction[44]. This produces an algebra that satisfies the Fundamental identity in all case. The procedure is to rescale $\Gamma_k, M_k \mapsto \Re_k \equiv \lambda^{-1}\Gamma_k, \Im_k \equiv \lambda M_k$ and take the formal limit $\lambda \to \infty$. The result is just to discard the term kM_{k+m+n} in the last line of (2.74).

$$[\Re_k, \Re_m, \Re_n] = 0$$

$$[\Im_k, \Im_m, \Im_n] = 0\,,$$

$$[\Re_k, \Im_m, \Im_n] = (n-m)\Im_{k+m+n}\,,$$

$$[\Re_k, \Re_m, \Im_n] = (k-m)(\Re_{k+m+n} + n\Im_{k+m+n})\,. \tag{2.76}$$

Remarkably, the contracted 3 -algebra so obtained is invariant under the $O(2)$ transformation

$$\Re_k, \Im_k \mapsto \Re_k \cos\theta + \Im_k \sin\theta, \Im_k \cos\theta - \Re_k \sin\theta\,. \tag{2.77}$$

The results in (2.74)-(2.77) provide the whole story, so far as we know it, for the ternary VW algebra. However, for completeness, we also wish to make contact with various other results in [43]. By redefinition of the charges of the original basis, it was observed in [43] that a "classical limit" could be constructed, in which the $sl(2,R)$ Casimir went to infinity, in such a way that all FIs were OK. In fact, this also just amounts to a contraction of the ternary algebra. Rescaling

$$Q_k = \frac{1}{\sqrt[4]{\beta(1-\beta)}} L_k\,, \qquad R_k = \sqrt[4]{\beta(1-\beta)} M_k\,, \tag{2.78}$$

substituting into (2.70) and (2.72) above, and taking the limit $\beta \to \infty$, the resulting algebra is

$$[Q_n, Q_m, Q_k] = (n-m)(m-k)(n-k)R_{n+m+k},$$

$$[R_k, Q_m, Q_n]|_{\beta\to\infty} = (m-n)(Q_{k+m+n} + 2ikR_{k+m+n}),$$

$$[Q_k, R_m, R_n] = (n-m)R_{k+m+n},$$

$$[R_k, R_m, R_n] = 0. \qquad (2.79)$$

For finite β, there would be an additional R_{k+m+n} term in the second relation. Again, the contracted algebra obeys the FI's in all cases.

2.3.4 Attempt to construct ternary super-Virasoro algebras

In last section, we successfully construct ternary Virasoro-Witt algebra by rescaling technique. Virasoro algebras play a very important role in string theory research, and the generators are bosonic charges. Here, we would like to include some fermionic charge and extend Virasoro algebra to ternary super-Virasoro algebra. The first step is to construct super-Virasoro algebra. Define fermionic operators by:

$$\{b, b^+\} = 1, \ \{b, b\} = \{b^+, b^+\} = 0, b^+ b = \tilde{N} \ . \qquad (2.80)$$

We need to add fermionic terms to L's and add two more fermion-like charges. The generator of this super-Virasoro algebra is constructed as

$$L_n = a^{+n}(-N - \frac{1}{2}n\tilde{N} + \alpha n),$$

$$F_n = \frac{1}{\sqrt{2}}a^{+n}(b - b^+ N + 2\alpha n b^+),$$

$$D_n = \frac{1}{\sqrt{2}}a^{+n}(b + b^+ N - 2\alpha n b^+). \qquad (2.81)$$

Before defining the ternary algebra, we need to make sure the Lie algebra generated by these generators is closed. Since we have both bosonic charge and fermion charge, two types of Lie brackets have to be defined：(a) the commutator $[A,B]=AB-BA$, where at least one of A or B must be bosonic charge; (b) the anticommutator $\{A,B\}=AB+BA$, where both A and B have to be fermionic charges. The commutation and anticommutation relations involving these three charges are as following:

$$[L_m,L_n]=(m-n)L_{m+n},\qquad [L_m,F_n]=(\frac{m}{2}-n)F_{m+n},\qquad \{F_m,F_n\}=L_{m+n},$$

$$[L_m,D_n]=(\frac{m}{2}-n)D_{m+n},\quad \{D_m,D_n\}=-L_{m+n},\quad \{F_m,D_n\}=(m-n)(a^+)^{m+n}(\alpha+\frac{1}{2}\tilde{N}) \tag{2.82}$$

Besides (2.80), we want to include three more charges, they are

$$N_n=a^{+n}(\alpha+\frac{1}{2}\tilde{N}),$$

$$E_n=\frac{1}{\sqrt{2}}a^{+n}b^+,$$

$$M_n=a^{+n}, \tag{2.83}$$

where N's and M's are bosonic charges and E's are fermionic charges. Together with (2.84), the whole commutation and anticommutation relations between these charges are

$$[L_m,L_n]=(m-n)L_{m+n},\quad [L_m,M_n]=-nM_{m+n},\qquad [L_m,N_n]=-nN_{m+n},$$

$$[M_m,M_n]=0,\qquad [N_m,M_n]=0,\qquad [N_m,N_n]=0,$$

$$\{F_m,F_n\}=L_{m+n},\qquad \{D_m,D_n\}=-L_{m+n},\qquad \{F_m,D_n\}=(m-n)N_{m+n},$$

$$\{F_m, E_n\} = \frac{1}{2} M_{m+n}, \qquad \{D_m, E_n\} = \frac{1}{2} M_{m+n}, \qquad \{E_m, E_n\} = 0,$$

$$[L_m, F_n] = (\frac{m}{2} - n) F_{m+n}, \quad [L_m, D_n] = (\frac{m}{2} - n) D_{m+n}, \quad [L_m, E_n] = -(\frac{m}{2} + n) E_{m+n},$$

$$[N_m, F_n] = -\frac{1}{2} D_{m+n}, \qquad [N_m, D_n] = -\frac{1}{2} F_{m+n}, \qquad [N_m, E_n] = \frac{1}{2} E_{m+n},$$

$$[M_m, F_n] = mE_{m+n}, \qquad [M_m, D_n] = -mE_{m+n}, \qquad [M_m, E_n] = 0. \tag{2.84}$$

Now, the first step is success: the super-algebra is constructed; the second step is to define the 3-brackets and check if the ternary algebra is closed as well as if the FIs hold. There are three different ways to define the 3-brackets, depending on the entries: (a) if at least two of the three entries are bosonic, then we have bosonic 3-bracket $[A, B, C] = A[B, C] + B[C, A] + C[A, B]$; (b) if two of the three entries are fermionic, then we have mixed 3-bracket (suppose A, B are fermionic) $\{A, B; C] = A[B, C] + B[C, A] + C\{A, B\}$; (c) if all three entries are fermionic, then we have fermionic 3-bracket $\{A, B, C\} = A\{B, C\} + B\{C, A\} + C\{A, B\}$.

(a) Bosonic 3-bracket

By noticing that the idempotent $\tilde{N}^k = \tilde{N}$ is invariant under the action of all bosonic operators, i.e. $[\tilde{N}, N] = [\tilde{N}, M_n] = 0$, we can look $\tilde{N}$ as effectively a constant in bosonic algebra. Make a change by $\beta = 1 + \alpha - \tilde{N}/2$, where β is a constant in the purely bosonic cases, which changes the realization of bosonic charges to

$$L_n = a^{+n}(-N + (\beta - 1)n),$$

$$N_n = (1 + 2\alpha - \beta) M_n,$$

$$M_n = a^{+n}, \tag{2.85}$$

where N_n is just M_n times a constant, it is the same as M_n. So in the purely bosonic cases, there are only two charges L_n and M_n. Then we have:

$$[L_k, L_m, L_n] = (\beta^2 - \beta)(k-m)(m-n)(n-k)M_{k+m+n},$$

$$[L_k, L_m, M_n] = (k-m)[L_{k+m+n} + (1-2\beta)nM_{k+m+n}],$$

$$[L_k, M_m, M_n] = (n-m)M_{k+m+n},$$

$$[M_k, M_m, M_n] = 0. \tag{2.86}$$

Therefore the ternary is closed in purely bosonic case. (2.86) allows us to check FI's, and except for the four L's and one M cases, all other FI's hold. These exceptional cases give

$$[L_i, L_j, [L_k, L_m, M_n]] - [[L_i, L_j, L_k], L_m, M_n] - [L_k, [L_i, L_j, L_m], M_n] - [L_k, L_m, [L_i, L_j, M_n]]$$
$$= -[4(\beta^2 - \beta) - (1-2\beta)^2]n(k-m)(i-j)(k+m-i-j)M_{i+j+k+m+n} \tag{2.87}$$

and

$$[M_i, L_j, [L_k, L_m, L_n]] - [[M_i, L_j, L_k], L_m, L_n] - [L_k, [M_i, L_j, L_m], L_n] - [L_k, L_m, [M_i, L_j, L_n]]$$
$$= [4(\beta^2 - \beta) - (1-2\beta)^2]i(k-m)(m-n)(n-k)M_{i+j+k+m+n} \tag{2.88}$$

This is the situation studied in [43], and by applying the same technique discussed there (rescale and take $\beta \to \pm\infty$), we can remedy it. So for purely bosonic case, this algebra satisfies FI's.

(b) Fermonic 3-bracket

In purely fermionic case, it is more complicated than the bosonic case. The 3-brackets involving E s always give closed form

$$\{E_k, E_m, E_n\} = 0, \qquad \{E_k, E_m, F_n\} = E_{k+m+n}, \qquad \{E_k, E_m, D_n\} = E_{k+m+n},$$

$$\{E_k, F_m, F_n\} = \frac{3}{2}F_{k+m+n} - \frac{1}{2}D_{k+m+n} - \frac{1}{2}(1+4\alpha)(2k+m+n)E_{k+m+n},$$

$$\{E_k, D_m, D_n\} = \frac{3}{2}D_{k+m+n} - \frac{1}{2}F_{k+m+n} + \frac{1}{2}(1+4\alpha)(2k+m+n)E_{k+m+n},$$

$$\{E_k, F_m, D_n\} = \frac{1}{2}D_{k+m+n} + \frac{1}{2}F_{k+m+n} + \frac{1}{2}(1+4\alpha)(m-n)E_{k+m+n}. \qquad (2.89)$$

But those involving only F 's or D 's always generate extra charges, as

$$\begin{aligned}
\{F_k, F_m, F_n\} &= \frac{1}{\sqrt{2}}(a^+)^{k+m+n}[-3Nb + 3b^+a^2a^{+2} + (2\alpha-1)(k+m+n)b + [2(1-2\alpha)(k+m+n)+3]b^+N, \\
&\quad + [4\alpha^2(mn+nk+km) - \alpha[(n+k)^2 + (k+m)^2 + (m+n)^2]]b^+] \\
\{D_k, D_m, D_n\} &= -\frac{1}{\sqrt{2}}(a^+)^{k+m+n}[-3Nb - 3b^+a^2a^{+2} + (2\alpha-1)(k+m+n)b - [2(1-2\alpha)(k+m+n)+3]b^+N, \\
&\quad - [4\alpha^2(mn+nk+km) - \alpha[(n+k)^2 + (k+m)^2 + (m+n)^2]]b^+] \\
\{F_k, D_m, D_n\} &= \frac{1}{\sqrt{2}}(a^+)^{k+m+n}[Nb - b^+a^2a^{+2} + [k+2\alpha(k-m-n)]b + (4\alpha k - m - n - 1)b^+N, \\
&\quad + [4\alpha^2(mn-nk-km) + 2\alpha(mn+k^2)]b^+] \\
\{D_k, F_m, F_n\} &= \frac{1}{\sqrt{2}}(a^+)^{k+m+n}[-Nb - b^+a^2a^{+2} - [k+2\alpha(k-m-n)]b + (4\alpha k - m - n - 1)b^+N \\
&\quad + [4\alpha^2(mn-nk-km) + 2\alpha(mn+k^2)]b^+]
\end{aligned} \qquad (2.90)$$

From (2.89), Nb and b^+N are two extra charges. If we include them to close the algebra, then the 3-brackets involving them would create more extra charges. Therefore, the ternary algebra is not closed, and it is obvious that not all FI's would be satisfied.

(c) Mixed 3-bracket

In the mixed case, we need to consider totally 35 different combinations between bosonic charges and fermionic charges, and nine of them generate extra charges. The

closed relations are

$$[M_k, M_m, E_n] = 0 \quad , \quad [M_k, M_m, F_n] = (m-k)E_{k+m+n} \quad ,$$

$$[M_k, M_m, D_n] = -(m-k)E_{k+m+n} ,$$

$$[M_k, N_m, E_n] = \frac{1}{2}E_{k+m+n} , \quad [M_k, N_m, F_n] = -\frac{1}{2}D_{k+m+n} - \frac{1}{2}(1+4\alpha)kE_{k+m+n} ,$$

$$[M_k, N_m, D_n] = -\frac{1}{2}F_{k+m+n} + \frac{1}{2}(1+4\alpha)kE_{k+m+n} , \quad [N_k, N_m, E_n] = 0 ,$$

$$[N_k, N_m, F_n] = \frac{1}{2}(\alpha + 2\alpha^2)(m-k)E_{k+m+n} ,$$

$$[N_k, N_m, D_n] = -\frac{1}{2}(\alpha + 2\alpha^2)(m-k)E_{k+m+n} ,$$

$$[L_k, M_m, E_n] = (\frac{k}{2} + n - m)E_{k+m+n} ,$$

$$[L_k, M_m, F_n] = (n - \frac{m+k}{2})F_{k+m+n} - \frac{m}{2}D_{k+m+n} - (1+4\alpha)(mn - \frac{1}{2}mk)E_{k+m+n} ,$$

$$[L_k, M_m, D_n] = (n - \frac{m+k}{2})D_{k+m+n} - \frac{m}{2}F_{k+m+n} + (1+4\alpha)(mn - \frac{1}{2}mk)E_{k+m+n} ,$$

$$[L_k, L_m, E_n] = -\frac{3}{4}(m-k)F_{k+m+n} + \frac{3}{4}(m-k)D_{k+m+n} - (1+4\alpha)(m-k)(n + \frac{m+k}{2})E_{k+m+n}$$

$$[L_k, N_m, E_n] = \frac{1}{4}F_{k+m+n} - \frac{1}{4}D_{k+m+n} - \frac{1}{2}(1+4\alpha)mE_{k+m+n} ,$$

$$[M_k; E_m, E_n\} = 0 , \quad [M_k; E_m, F_n\} = \frac{1}{2}M_{k+m+n} , \quad [M_k; E_m, D_n\} = \frac{1}{2}M_{k+m+n} ,$$

$$[M_k; F_m, F_n\} = L_{k+m+n} + 3kN_{k+m+n} - k(1+4\alpha)M_{k+m+n} ,$$

$$[M_k; D_m, D_n\} = -L_{k+m+n} - 3kN_{k+m+n} + k(1+4\alpha)M_{k+m+n} ,$$

$$[M_k; F_m, D_n\} = (m-n)N_{k+m+n} , \quad [N_k; E_m, E_n\} = 0 ,$$

$$[N_k; E_m, F_n\} = \frac{3}{2}N_{k+m+n} - \frac{1}{4}(1+4\alpha)M_{k+m+n} ,$$

$$[N_k; E_m, D_n\} = \frac{3}{2} N_{k+m+n} - \frac{1}{4}(1+4\alpha) M_{k+m+n},$$

$$[N_k; F_m, D_n\} = (1+4\alpha)(m-n) N_{k+m+n} - 3(\alpha^2 + \frac{1}{2}\alpha)(m-n) M_{k+m+n},$$

$$[L_k; E_m, E_n\} = 0,$$

$$[L_k; E_m, F_n\} = \frac{1}{2} L_{k+m+n} + \frac{1}{2}(4n-k) N_{k+m+n} + \frac{1}{2}[(\frac{k}{2} - n) - \alpha(3n-k-m)] M_{k+m+n},$$

$$[L_k; E_m, D_n\} = \frac{1}{2} L_{k+m+n} + \frac{1}{2}(4n-k) N_{k+m+n} + \frac{1}{2}[(\frac{k}{2} - n) - \alpha(3n-k-m)] M_{k+m+n}.$$

(2.91)

The non-closed relations are:

$$[L_k, L_m, F_n] = \frac{1}{\sqrt{2}}(m-k)(a^+)^{k+m+n}[\frac{1}{2} Nb - \frac{1}{2} b^+ a^{+2} a^2 + [\frac{n}{2} + \alpha(n-k-m)]b - [\frac{m+k+1}{2} - 2\alpha n] b^+ N + [\alpha(2n^2 + \frac{km}{2}) + \alpha^2(km + 2n^2 - 2nk - 2mn)] b^+],$$

$$[L_k, L_m, D_n] = \frac{1}{\sqrt{2}}(m-k)(a^+)^{k+m+n}[\frac{1}{2} Nb + \frac{1}{2} b^+ a^{+2} a^2 + [\frac{n}{2} + \alpha(n-k-m)]b + [\frac{m+k+1}{2} - 2\alpha n] b^+ N - [\alpha(2n^2 + \frac{km}{2}) + \alpha^2(km + 2n^2 - 2nk - 2mn)] b^+],$$

$$[L_k, N_m, F_n] = \frac{1}{\sqrt{2}}(a^+)^{k+m+n}[\frac{1}{2} Nb + \frac{1}{2} b^+ a^{+2} a^2 + [\frac{1}{2} n + \alpha(n-k-m)]b + (\frac{m+k+1}{2} - 2\alpha n) b^+ N + [\alpha(\frac{km-k^2}{2} - 2mn) + \alpha^2(km + 2kn - 2mn + 2n^2 - k^2)] b^+],$$

$$[L_k, N_m, D_n] = \frac{1}{\sqrt{2}}(a^+)^{k+m+n}[\frac{1}{2} Nb - \frac{1}{2} b^+ a^{+2} a^2 + [\frac{1}{2} n + \alpha(n-k-m)]b - (\frac{m+k+1}{2} - 2\alpha n) b^+ N - [\alpha(\frac{km-k^2}{2} - 2mn) + \alpha^2(km + 2kn - 2mn + 2n^2 - k^2)] b^+],$$

$$[N_k; F_m, F_n\} = (a^+)^{k+m+n}[(\frac{1}{2} - \alpha) N - \frac{3}{2} \tilde{N} N + [\alpha(k+m+n) - \frac{k+m+n}{2}] \tilde{N} - \alpha(k + \frac{m+n}{2}) + \alpha^2(m+n)],$$

$$[N_k; D_m, D_n\} = -(a^+)^{k+m+n}[(\frac{1}{2} - \alpha) N - \frac{3}{2} \tilde{N} N + [\alpha(k+m+n) - \frac{k+m+n}{2}] \tilde{N} - \alpha(k + \frac{m+n}{2}) + \alpha^2(m+n)],$$

$$[L_k; F_m, F_n\} = (a^+)^{k+m+n}[a^{+2}a^2 + [(k+\frac{m+n}{2}+1) - \alpha(k+m+n)]N + \frac{k+m+n}{2}\tilde{N}N + [\frac{1}{2}(k^2 + 2mn) + \alpha(k^2 - 2km - 2kn)]\tilde{N} + [\alpha^2 k(m+n) - \alpha(k^2 + 2mn - \frac{km+kn}{2})]],$$

$$[L_k; D_m, D_n\} = -(a^+)^{k+m+n}[a^{+2}a^2 + [(k+\frac{m+n}{2}+1) - \alpha(k+m+n)]N + \frac{k+m+n}{2}\tilde{N}N + [\frac{1}{2}(k^2 + 2mn) + \alpha(k^2 - 2km - 2kn)]\tilde{N} + [\alpha^2 k(m+n) - \alpha(k^2 + 2mn - \frac{km+kn}{2})]],$$

$$[L_k; F_m, D_n\} = (a^+)^{k+m+n}[(n-m)(\alpha+\frac{1}{2})N + \frac{m-n}{2}\tilde{N}N + \alpha(k+m+n)(n-m)]\tilde{N} + (\frac{\alpha}{2} + \alpha^2)k(m-n)].$$

(2.92)

Similar to the purely fermionic case, these extra charges would generate more and more extra charges if we try to include them. So super-Virasoro algebra can be extended to purely bosonic ternary algebra, but not involving any fermionic charges, and it is still not clear how to remedy this problem so far.

2.4 Nambu Algebras

2.4.1 Basic properties of Nambu algebras

The Nambu algebra is the infinite-dimensional version of Filippov algebras, where n-bracket is defined by the Jacobian determinant of n functions on a n-dimensional manifold, rather than the determinant 'vector product' of n vectors on a $n+1$-dimensional space.

Definition 2.12: Let $f_1, f_2, \cdots, f_n$ be functions on R^n with coordinates $\{x^i\}, i = 1, \cdots, n$. The n-bracket $\{f_1, f_2, \cdots, f_n\}$, or Nambu bracket, or classical

bracket is defined by the Jacobian determinant

$$\{f_1, f_2, \cdots, f_n\} := \in_{1\cdots n}^{i_1\cdots i_n} \partial_{i_1} f^1 \cdots \partial_{i_n} f^n = \left|\frac{\partial(f_1, \cdots, f_n)}{\partial(x^1, \cdots, x^2)}\right|, \tag{2.93}$$

where $f_i = f_i(x^1, \cdots, x^n)$. Nambu bracket can also be expressed as in terms of multivector

$$\{f_1, f_2, \cdots, f_n\} = \Lambda_{(n)}(df_1, \cdots, df_n) \tag{2.94}$$

where $\Lambda_{(n)} = \frac{\partial}{\partial x^1} \wedge \cdots \wedge \frac{\partial}{\partial x^n}$.

With Nambu bracket, we can define Nambu algebras on manifold

Definition 2.13: The Nambu algebra, or classical algebra is a set of function $f_1, f_2, \cdots, f_n$ defined on a n-dimensional manifold, together with Nambu brackets such that Nambu brackets satisfy FI's.

Unlike Filippov algebras, all the Nambu brackets satisfy FI's naturally, and this can be checked for $n = 3$ case explicitly by applying 'Schouten identities'[45].

2.4.2 Examples of $n = 3$ Nambu algebras

We give some example of Nambu algebras in this section.

Example 2.6: Classical 3-bracket algebra for exponentials

Consider the infinite set of exponentials,

$$E_a = \exp(a \cdot r), \tag{2.95}$$

and compute the classical bracket,

$$\{E_a, E_b, E_c\} = a \cdot (b \times c) E_{a+b+c} \,. \tag{2.96}$$

The indices here are 3-vectors, with $\bullet$ and $\times$ the usual dot and cross products. This infinite algebra does satisfy Bremner identity.

It is not known how to realize (2.96) as operator 3-brackets. Although, there is a quantal 4-bracket which gives this 3-bracket as a classical limit[46]. To see this, compute the operator 4-bracket $[\exp(a \cdot r), \exp(b \cdot r), \exp(c \cdot r), w]$ where we assume the exponentials do not involve w, and where we take w and x, and also y and z, to be independent canonically conjugate pairs of variables, i.e. $[w, x] = i\hbar, [y, z] = i\hbar$, but $[w, y] = 0$, etc. The result for the 4-bracket is then given directly by the commutator resolution[4][48].

$$\begin{aligned}[e^{a\cdot r}, e^{b\cdot r}, e^{c\cdot r}, w] = 4\hbar e^{(a+b+c)\cdot r} (a_x \sin(\frac{1}{2}\hbar b_\perp \times c_\perp) \cos(\frac{1}{2}\hbar(b_\perp + c_\perp) \times a_\perp) \\ + b_x \sin(\frac{1}{2}\hbar c_\perp \times a_\perp) \cos(\frac{1}{2}\hbar(c_\perp \times a_\perp) \times b_\perp) + c_x \sin(\frac{1}{2}\hbar a_\perp \times b_\perp) \cos(\frac{1}{2}\hbar(a_\perp + b_\perp) \times c_\perp))\end{aligned}, \tag{2.97}$$

where $a = (a_x, a_y, a_z)$, $a_\perp = (a_y, a_z)$, $a_\perp \times b_\perp = a_y b_z - b_y a_z$, etc. In the limit $\hbar \to 0$, this gives the anticipated classical 3-bracket,

$$\frac{1}{2\hbar^2}[\exp(a \cdot r), \exp(b \cdot r), \exp(c \cdot r), w] = a \cdot (b \times c) \exp(a + b + c) \cdot r + O(\hbar^2) \,. \tag{2.98}$$

Before the classical limit is taken, however, (2.97) does not satisfy the FI: There are violations at $O(\hbar^6)$ and beyond.

Example 2.8: Ternary Virasoro-Witt algebra as in classical bracket

The contracted 3-algebra obtained above (2.76) is invariant under the

$O(2)$ transformation (2.77). An interpretation of this symmetry, as well as the validity of the FI's, is obvious in the contracted algebra's realization as a Nambu algebra. That is

$$\{xe^{kz}, xe^{mz}, xe^{nz}\} = \{ye^{kz}, ye^{mz}, ye^{nz}\} = 0,$$

$$\{xe^{kz}, ye^{mz}, ye^{nz}\} = (n-m)ye^{(k+m+n)z},$$

$$\{ye^{kz}, xe^{mz}, xe^{nz}\} = (m-n)xe^{(k+m+n)z}. \qquad (2.99)$$

In this realization the $O(2)$ symmetry is nothing but a rotation about the z-axis.

We may also realize (2.79) in terms of Nambu brackets. Explicitly we find

$$\{(x-iky)e^{kz}, (x-imy)e^{mz}, (x-iny)e^{nz}\} = (k-m)(m-n)(k-n)ye^{z(k+m+n)},$$

$$\{ye^{kz}, (x-imy)e^{mz}, (x-iny)e^{nz}\} = (m-n)((x-i(k+m+n)y)e^{z(k+m+n)} + 2ikye^{z(k+m+n)}),$$

$$\{(x-iky)e^{kz}, ye^{mz}, ye^{nz}\} = (n-m)ye^{z(k+m+n)},$$

$$\{ye^{kz}, ye^{mz}, ye^{nz}\} = 0. \qquad (2.100)$$

But suppose we just transform back to the original linear combinations to recover the classical versions of the L's. What is the effect on the algebra? To answer this, let

$$\ell_n \equiv (x-(\gamma+n\beta)y)e^{nz}, \qquad p_n = ye^{nz}. \qquad (2.101)$$

We obtain

$$\{\ell_k, \ell_m, \ell_n\} = -\beta^2(k-m)(k-n)(m-n)p_{k+m+n},$$

$$\{\ell_k, \ell_m, p_n\} = (k-m)(l_{k+m+n} - 2\beta n p_{k+m+n}),$$

$$\{\ell_k, p_m, p_n\} = (n-m)p_{k+m+n},$$

$$\{p_k, p_m, p_n\} = 0. \qquad (2.102)$$

This differs from the original, uncontracted quantal algebra (2.70) and (2.72) only in

the β-dependent coefficient on the right hand side. Namely, $-\beta^2$ appears instead of $\beta(1-\beta)$ and -2β instead of $1-2\beta$. So, to repeat the observation made in [43], we may again identify this Nambu algebra with the infinite $sl(2,R)$ Casimir limit, $\beta \to \pm\infty$, of the quantal algebra.

Example 2.9: Classical Virasoro-Witt algebra is a subalgebra of (2.96)

The may be understood as follows. Clearly, from (2.96), any three exponentials with co-planar vectors will have a vanishing classical bracket. By representing all the $\mathfrak{R}$'s with a set of such co-planar exponentials, and all the $\mathfrak{I}$'s with another set of co-planar exponentials, the first two lines of (2.74) will be satisfied. In general then, there are two distinct planes: One for the $\mathfrak{R}$'s and one for the $\mathfrak{I}$'s. The remaining challenge, viewed geometrically, is to put these two distinct planes together so that the last two lines of (2.74) will also be satisfied. An obvious guess is that the two planes should intersect at right angles. Another, related guess is that the index appearing in (2.74) should correspond to modes along the line of intersection of the two planes.

Therefore, to play the role of the classical $\mathfrak{R}$'s, tale $l_k \equiv E_{\hat{x}+k\hat{z}} = \exp(x+kz)$, while for the $\mathfrak{I}_k$'s, take $p_k \equiv E_{\hat{y}+k\hat{z}} = \exp(y+kz)$. Some elementary algebra then give $(\hat{x}+m\hat{z})$. $((\hat{y}+n\hat{z})\times(\hat{y}+k\hat{z})) = k-n$ and $(\hat{y}+k\hat{z})\times((\hat{x}+m\hat{z})\times(\hat{x}+n\hat{z})) = m-n$ as well as

$$(\hat{x}+m\hat{z})+(\hat{y}+n\hat{z})+(\hat{y}+k\hat{z}) = (\hat{x}+\hat{y})+\hat{y}+(k+m+n)\hat{z}\,, \qquad (2.103)$$

and

$$(\hat{y}+k\hat{z})+(\hat{x}+m\hat{z})+(\hat{x}+n\hat{z}) = (\hat{x}+\hat{y})+\hat{x}+(k+m+n)\hat{z}\,. \qquad (2.104)$$

So, modulo the common spurious vector $(\hat{x}+\hat{y})$ we have just what we need to

obtain the contracted algebra from the classical brackets (2.96). Now, if we incorporate the inverse of this spurious term into the definition of a modified Nambu bracket, as a multiplicative factor,

$$\{A,B,C\}_{\text{mod}} \equiv \frac{\partial(A,B,C)}{\partial(x,y,z)} e^{-(\hat{x}+\hat{y})\cdot\vec{r}} = \frac{\partial(A,B,C)}{\partial(x,y,z)} e^{-x-y}, \qquad (2.105)$$

then we have realized on exponentials the classical, contracted ternary Virasoro-Witt algebra.

$$\{l_k, l_m, l_n\}_{\text{mod}} = 0, \qquad \{p_k, p_m, p_n\} = 0,$$

$$\{\ell_k, \ell_m, p_n\} = (k-m) l_{k+m+n}, \qquad \{\ell_k, p_m, p_n\} = (n-m) p_{k+m+n}. \qquad (2.106)$$

But what effect does the multiplicative factor have on FI's?

It cannot obviate the FI's, because we have already verified them for the contracted algebra. Another way to see this is to note the multiplicative factor is just the Jacobian for the variable change $(x,y,z) \mapsto (e^x, e^y, z)$. In terms of these new exponential variables the realization is

$$l_k = x e^{kz}, \qquad p_k = y e^{kz}, \qquad (2.107)$$

where these are to be acted on by unmodified Nambu brackets for the new x, y, z variables.

We may summarize either (2.99) or (2.102) as simply the closure of functions of the form $xf(z)$ and $yg(z)$ under Nambu brackets. A complementary algebra is given by the closure of the Nambu brackets for functions of the form $\sqrt{z} f(x,y)$. This may be expressed as a two-parameter algebra[47] if we choose $f(x,y) = \exp(ax+by)$.

Again, the FI is guaranteed to hold since only Nambu brackets are involved.

2.4.3 Bremner identity and Nambu algebras

Bremner identity is the essential ingredient for Filippov algebras, would it do the same to Nambu algebras? In last section, we stated that algebra (2.96) does satisfy Bremner identity; however, generally Bremner identity is not applicable to Nambu algebra. By the definition, Nambu bracket has the form (2.93), and for $n = 3$ case, Nambu bracket becomes

$$\{A, B, C\} \equiv \frac{\partial(A, B, C)}{\partial(x, y, z)} = \in^{xyz} \partial_x A \partial_y B \partial_z C , \tag{2.108}$$

where A, B, C are defined on 3-manifold, $\in^{xyz}$ is Levi-Civita symbol, and x, y, z are the coordinates. We find that each Nambu bracket would generate the products of the first order derivatives of A, B, C. While the Bremner identity for Nambu bracket takes the form of

$$\{\{A,\{B,C,D\},E\},F,G\} = \{\{A,B,C\},\{D,E,F\},G\} \tag{2.109}$$

where A is fixed, and we sum over all 6! signed antisymmetric permutations. The left hand side of (2.109) is

$$\{\{A,\{B,C,D\},E\},F,G\} = \in^{ghi} \in^{def} \in^{abc} \partial_g \partial_d (\partial_a B \partial_b C \partial_c D \partial_e A \partial_f E) \partial_h F \partial_i G , \tag{2.110}$$

and right hand side is

$$\{\{A,B,C\},\{D,E,F\},G\} = \in^{ghi} \in^{abc} \in^{def} \partial_g (\partial_a A \partial_b B \partial_c C) \partial_h (\partial_d D \partial_e E \partial_f F) \partial_i G , \tag{2.111}$$

There are non-trivial third order derivatives about B, C, D in (2.109) (the third order derivatives about A, E vanish because of antisymmetrization):

$$\begin{aligned} 6 \in^{ghi} \in^{def} \in^{abc} & (\partial_g \partial_d \partial_a B \partial_b C \partial_c D + \partial_c B \partial_g \partial_d \partial_a C \partial_b D \\ & + \partial_b B \partial_c C \partial_g \partial_d \partial_a D) \partial_e A \partial_f E \partial_h F \partial_i G \end{aligned} . \tag{2.112}$$

While (2.111) does not have third order derivatives contained in it. So for an arbitrary realization of Nambu bracket, Bremner identity does not hold. The realization we give in Example 1.6 have special property that the functions A, B defined on 3-manifold possess:

$$\text{(a)} \qquad \frac{\partial A}{\partial x} \propto \frac{\partial^n A}{\partial x^n}, \text{ where } n \text{ is an integer.} \tag{2.113}$$

$$\text{(b)} \qquad \frac{\partial A}{\partial x} \propto \frac{\partial B}{\partial y}, \text{ where } A \neq B . \tag{2.114}$$

Besides the realization with the conditions above, we have not found any other realizations satisfy Bremner identity. Therefore, differing from Quantal bracket, Nambu brackets do not have Bremner identity as their essential ingredient.

2.5 Nambu-Poisson Tensor

When we define Nambu bracket in (2.93), we assume that the antisymmetric tensors before the derivatives are Levi-Civita symbols that they are constant and independent of the coordinates. Actually, those antisymmetric tensors do not necessarily have to be constants, and they can be defined as variables of the coordinates.

2.5.1 Poisson tensor

First, we consider a general Poisson bracket involving an antisymmetric, but otherwise arbitrary, 2-tensor ω^{ab}, the Poisson tensor.

$$\{A, B\} = \omega^{ab} \partial_a A \partial_b B . \tag{2.115}$$

Repeated indices are implicitly summed from 1 to n. For any ω^{ab} this is obviously a derivation: $\{A,BC\}=\{A,B\}C+B\{A,C\}$. But it is more interesting for physics purposes that there are situations where the Poisson bracket realizes a Lie algebra. This is evident if we bracket $\{A,B\}$ with C to obtain two functionally independent terms.

$$\{C,\{A,B\}\}=\omega^{cd}\omega^{ab}\partial_d(\partial_c C\partial_a A\partial_b B)+(\partial_c C\partial_a A\partial_b B)\omega^{cd}\partial_d\omega^{ab}. \qquad (2.116)$$

The combination that constitutes the Lie-algebra mandated Jacobi 'identity' similarly gives two terms.

$$\{C,\{A,B\}\}-\{\{C,A\}B\}-\{A,\{C,B\}\}=\Omega^{abcd}\partial_d(\partial_a A\partial_b B\partial_c C)+\Omega^{abc}(\partial_a A\partial_b B\partial_c C), \qquad (2.117)$$

where we have defined 4- and 3-tensors

$$\Omega^{abcd}\equiv\omega^{cd}\omega^{ab}-\omega^{ac}\omega^{bd}+\omega^{ad}\omega^{bc},$$
$$\Omega^{abc}\equiv\omega^{cd}\partial_d\omega^{ab}-\omega^{ac}\partial_d\omega^{bd}+\omega^{ad}\partial_d\omega^{bc}. \qquad (2.118)$$

Now the first term on the right hand side of (2.117) always vanishes by symmetry: For any $\omega^{ab}=-\omega^{ba}$, the corresponding Ω^{abcd} is a totally antisymmetric 4-tensor, and hence $\Omega^{abcd}\partial_d(\partial_a A\partial_b B\partial_c C)$ is identically zero. So, for constant ω^{ab} the Jacobi identity $\{C,\{A,B\}\}-\{\{C,A\}B\}-\{A,\{C,B\}\}=0$ is indeed an identity for Poisson brackets.

But, in general, the second term on the right hand side of (2.117) does not vanish for non-constant 3-tensors. Hence there is a condition for the Jacobi identity to be satisfied: $\Omega^{abc}=0$. When true, we are dealing with a Lie algebra on a Poisson manifold. If in addition n is even and the 2-tensor has an inverse, such that

$\omega_{ab}\omega^{bc} = \delta_a^c$, then we have a symplectic manifold, and we can construct the 2-form $\omega = \omega_{ab} dx^a \wedge dx^b$. In this case the condition for the Jacobi identity to hold is easily rendered to be $0 = \partial_a \omega_{bc} + \partial_b \omega_{ca} + \partial_c \omega_{ab}$, or equivalently just that the 2-form is closed: $d\acute{E} = 0$. While this is at first sight a generalization from the constant 3-tensor case, this is somewhat illusory. For such closed 2-forms Darboux proved the existence of local coordinates on the manifold such that ω_{ab} is constant.

2.5.2 Nambu-Poisson tensor

Poisson tensor can be easily extended to Nambu-Poisson tensor if Poisson bracket is substituted by ternary Nambu bracket.

$$\{A, B, C\} = \omega^{abc} \partial_a A \partial_b B \partial_c C \,. \tag{2.119}$$

With this structure we encounter a few similarities with the Poisson bracket case, but more importantly, we also encounter some dramatic differences. Just like the Poisson bracket above, this is a derivation: $\{A, B, CD\} = \{A, B, C\}D + C\{A, B, D\}$. Also like the Poisson bracket case, the action of one 3-bracket on another produces two independent terms.

$$\begin{aligned} \{\{A, B, C\}, D, E\} = {} & \omega^{abc}\omega^{def}\partial_f(\partial_a A \partial_b B \partial_c C \partial_d D \partial_e E) \\ & + (\partial_a A \partial_b B \partial_c C \partial_d D \partial_e E)\omega^{def}\partial_f \omega^{abc} \end{aligned} \,. \tag{2.120}$$

But here the differences arise. What is the appropriate analogue of the Jacobi identity?

In general, there is no perfect analogue of the Jacobi identity involving the action of one 3-bracket on another, or a linear combination of such. To see this we need only consider the case of constant ω^{abc} and make use of some elementary group theory: The symmetrized product of two antisymmetric 3-tensors does not contain a

totally antisymmetric 6-tensor. Indeed, with the standard partition labeling of symmetric group representations, where the sequence of integers represents the number of boxes in the rows of a Young frame, we have

$$\{1,1,1\}^2_{symmetric} = \{2,1,1,1,1\} + \{2,2,2\}. \tag{2.121}$$

The antisymmetric 6-tensor is found instead in the antisymmetrized product,

$$\{1,1,1\}^2_{antisymmetric} = \{1,1,1,1,1,1\} + \{2,2,1,1\}. \tag{2.122}$$

Now, because it is the symmetrized tensor product of two ω's that appears in (2.120), these group properties imply that any linear combination, obtained by permuting the entries in one classical 3-bracket acting on another, cannot possibly vanish without imposing some condition on ω^{abc}, and/or on the number of variables n. Neither partial nor full antisymmetrizations of A, B, C, D, E in (2.120) can avoid both of the tensors on the right hand side of (2.120), and in general

$$\omega^{abcdef}_{\{2,1,1,1,1\}}\partial_f(\partial_a A\partial_b B\partial_c C\partial_d D\partial_e E) \neq 0 \neq \omega^{abcdef}_{\{2,2,2\}}\partial_f(\partial_a A\partial_b B\partial_c C\partial_d D\partial_e E). \tag{2.123}$$

We emphasize that this is different from the Poisson bracket case, where the group theory is

$$\{1,1\}^2_{symmetric} = \{1,1,1,1\} + \{2,2\} \tag{2.124}$$

and

$$\{1,1\}^2_{antisymmetric} = \{2,1,1,1\}, \tag{2.125}$$

and where the linear combination of brackets in the Jacobi identity serves to single out $\{1,1,1,1\}$ and thus eliminate the $\omega\omega$ term for all antisymmetric 2-tensors.

Admittedly, there is one very special case where group theory does not impose an impasse for classical 3-on-3-bracket identities, namely, $n = 3$. When there are only three independent variables, the $\{2,1,1,1,1\}$ representation is absent! In this special case we obtain the FI.

$$\{\{A,B,C\},D,E\}=\{\{A,D,E\},B,C\}+\{A,\{B,D,E\},C\}+\{A,B,\{C,D,E\}\}. \qquad (1.125)$$

Of course, for this case there is only one antisymmetric 3-tensor, namely, that of Kronecker, so $\omega^{abc}\propto\in^{abc}$. But, alas, FI for 3-brackets does not go willingly into higher dimensional manifolds.

More generally, if we define a Nambu N-bracket as

$$\{A_1,A_2,\cdots,A_N\}=\omega^{a_1a_2\cdots a_N}\partial_{a_1}A_1\partial_{a_2}A_2\cdots\partial_{a_N}A_N. \qquad (2.126)$$

Then the even N cases are like that of the Poisson bracket, while the odd N cases are like that for the 3-bracket. When N is even, we have $\{1^{2N}\}\subset\{1^N\}^2_{symmetric}$, so for general constant ω's the action of one N-bracket on another vanishes when totally antisymmetrized over the $2N-1$ entries in the double bracket. When N is odd, we have $\{1^{2N}\}\subset\{1^N\}^2_{antisymmetric}$, so for constant ω's the action of any permuted linear combination of one N-bracket on another does not vanish without imposing some condition on the ω's and/or on the number of variables.

Chapter 3

BLG and BLG-NB Model

Though n-ary algebras had been studied for almost forty years, people still knew little about their applications to physics until recently, Bagger, Lambert and Guastavsson[24,25,26,27] introduced Filippov 3-algebra A_4 to M2-branes interaction. 3-algebra plays an important role in the supersymmetric invariance and the gauge symmetry of their model. Their breakthrough theory achieves a big success, and leads a huge amount of following works[29, 30, 31, 32, 33, 34] . However, BLG model still suffers some problems, and motivates others to extend it by replacing A_4 with Nambu 3-algebra[34,35,36,54,55], which gives BLG-NB model.

In section 3.1, we introduce 3-algebra A_4, and some attempts to construct it. The Fundament -al identity is too strong that A_4 believed to be the only finite positive-definite 3-algebra for BLG model, all other 3-algebras would be just the trivial products of A_4. In section 3.2, we study the formulism of BLG model. We see that the motivation of BLG model is Basu-Harvey equation[54], and we also discuss the successes and challenges of BLG model. In section 3.3, we introduce BLG-NB model based on Nambu 3-algebra. BLG-NB model describes infinite number of M2-branes condensation. In section 3.4, we give a short review of constructing the

large N limit classical Yang-Mill theories, and hope it would provide the hint for us to connect BLG gauge theory with BLG-NB gauge theory.

3.1 3-Algebra A_4

In this section, we introduce the 3-algebra used in BLG model and give an example of how to construct it.

3.1.1 A_4

The 3-algebra used to construct BLG model is four dimensional Filippov algebra A_4 [24,25,26,27], with generator T^a, $a = 1,\cdots,4$, that is endowed with trilinear antisymmetric product

$$[T^a, T^b, T^c] = f^{abc}{}_d T^d, \tag{3.1}$$

and from which it is clear that $f^{abc}{}_d = f^{[abc]}{}_d$. A_4 is metric 3-algebra, and the metric is defined by

$$h^{ab} = Tr(T^a, T^b). \tag{3.2}$$

The metric is assumed to be positive definite. The trace-form is a bilinear map Tr: $A_4 \times A_4 \to C$ that is symmetric and invariant:

$$Tr(T^a, T^b) = Tr(T^b, T^a) \quad \text{and} \quad Tr(T^a \cdot T^b, T^c) = Tr(T^a, T^c \cdot T^b). \tag{3.3}$$

This allows us to raise and lower indices: $f^{abcd} = f^{abc}{}_e h^{ed}$.

We require two conditions on 3-bracket, the first is the Fundamental identity

$$[T^a,T^b,[T^c,T^d,T^e]]=[[T^a,T^b,T^c],T^d,T^e]+[T^c,[T^a,T^b,T^d],T^e] \\ +[T^c,T^d,[T^a,T^b,T^e]] \quad . \tag{3.4}$$

Fundamental identity can be written in terms of structure constants

$$f^{efg}{}_d f^{abc}{}_g = f^{efa}{}_g f^{bcg}{}_d + f^{efb}{}_g f^{cag}{}_d + f^{efc}{}_g f^{abg}{}_d \quad . \tag{3.5}$$

The second condition is

$$Tr(T^a,[T^b,T^c,T^d])=-Tr([T^a,T^b,T^c],T^d)\,, \tag{3.6}$$

for all $a,b=1,\cdots,4$, and this implies that the f^{abcd} are totally antisymmetric,

$$f^{abcd}=f^{[abcd]}. \tag{3.7}$$

As mentioned in section 2.3 that the Filippov algebra does not emphasize on the associativity, the A_4 discussed here is a nonassociative algebra. We introduce the associator

$$<A,B,C>=(A\cdot B)\cdot C-A\cdot(B\cdot C)\,, \tag{3.8}$$

which would vanishes in an associative algebra. And the 3-bracket is defined by

$$[A,B,C]=<A,B,C>+<B,C,A>+<C,A,B> \\ -<A,C,B>-<B,A,C>-<C,B,A>. \tag{3.9}$$

The associator also implies that

$$Tr(<A,B,C>,D)=Tr((A\cdot B)\cdot C,D)-Tr(A\cdot(B\cdot C),D) \\ =Tr(A\cdot B,C\cdot D)-Tr(A,(B\cdot C)\cdot D) \quad . \\ =-Tr(A,<B,C,D>) \tag{3.10}$$

By normalizing the generators, the trace-form of two generator can be reduced to Dirac δ, i.e. $Tr(T^a,T^b)\propto\delta^{ab}$, and the structure constant is reduced to Levi-Civita symbol times some constant

$$f^{abcd} = const \times \in^{abcd} . \tag{3.11}$$

This kind of structure constants generate the space of all 4×4 antisymmetric matrices, which form the Lie algebra $so(4)$. In [25], Bagger and Lambert gave an example of how to construct A_4 for the matrix form. They considered the 3-algebra of hermitian matrices that anticommute with a fixed Hermitian matrix G, with $G^2 = 1$. Define the product between two matrices

$$A \cdot B = QABQ, \tag{3.12}$$

Where $Q = (1 + iG)/\sqrt{2}$. Take trace-form $Tr(A, B) = trace(Q^{-1} A Q^{-1} B)$ where *trace* denotes the standard matrix trace. The associator turns out to be

$$< A, B, C >= 2GABC, \tag{3.13}$$

and hence

$$[A, B, C] = 2G(ABC + BCA + CAB - ACB - BAC - CBA). \tag{3.14}$$

Consider the Dirac γ matrices as a realization of A_4. Take $\gamma_i (i = 1, \cdots, 4)$ as generators, with $G = \gamma_5$ and $Q = (1 + i\gamma_5)/\sqrt{2}$. The product is then $\gamma^a \cdot \gamma^b = Q\gamma^a \gamma^b Q = \gamma_5 \gamma^a \gamma^b$, and one finds that

$$< \gamma^a, \gamma^b, \gamma^c >= 2\gamma_5 \gamma^a \gamma^b \gamma^c . \tag{3.15}$$

Thus

$$[\gamma^a, \gamma^b, \gamma^c] = 2 \cdot 3! \gamma_5 \gamma^{abc} = 2 \cdot 3! \in^{abc}{}_d \gamma^d , \tag{3.16}$$

And hence $f^{abcd} = 12 \in^{abcd}$. One can check explicitly that the structure constant like this satisfies the Fundamental identity.

3.1.2 Other attempts

The founders of BLG model give an example for 3-algebra, A_4; then, an intuitive question would be asked: do any other 3-algebras exist that can be applied to BLG model also? Since the day BLG model was born, the search for new 3-algebras has never stopped, however, we have never found anything that can substitute A_4 in BLG model. Here, we give an example of those attempts.

The realization of A_4 described above is based on a fixed Hermitian matrix G, but that does not seem to be compatible with a conventional Lie algebra interpretation. [29] gives an attempt to construct seven-dimensional positive-definite 3-algebra by dispensing with auxiliary matrix and applying the procedure described above to one of the familiar nonassociative algebras, namely the algebra of octonions.

The generators of the imaginary octonions are denoted by e_a with $a = 1,2,\cdots,7$, and they have nonassociative multiplication table

$$e_a e_b = t_{abc} e_c - \delta_{ab}. \tag{3.17}$$

The totally antisymmetric tensor t_{abc} has the following nonvanishing components

$$t_{124} = t_{235} = t_{346} = t_{457} = t_{561} = t_{672} = t_{713} = 1. \tag{3.18}$$

t_{abc} can be regarded as an invariant tensor describing the totally antisymmetric coupling of three seven dimensional representations of the Lie group G_2. Let T_{ab} denote a generator of an $SO(7)$ rotation in the ab plane. The $so(7)$ Lie algebra is

$$[T_{ab}, T_{cd}] = T_{ad}\delta_{bc} - T_{bd}\delta_{ac} - T_{ac}\delta_{bd} + T_{bc}\delta_{ad}. \tag{3.19}$$

The generators of G_2 can be described as a 14-dimensional subalgebra of this Lie

algebra. A possible choice of basis is given by

$$X_1 = T_{24} - T_{56} \quad \text{and} \quad Y_1 = T_{24} - T_{37} \tag{3.20}$$

and cyclic permutations of the indices. This gives 14 generators X_A consisting of X_a and $X_{a+7} = Y_a$. The seven-index epsilon symbol, which is an invariant tensor of $so(7)$, can be used to derive an antisymmetric fourth-rank tensor of G_2:

$$f^{abcd} = \frac{1}{6} \in_{abcdefg} t_{efg} . \tag{3.21}$$

This tensor has nonzero components

$$f_{7356} = f_{1467} = f_{2571} = f_{3612} = f_{4723} = f_{5134} = f_{6245} = 1 . \tag{3.22}$$

However, the problem of G_2 is, the structure constants f^{abcd} is not satisfied by Fundamental identity. Therefore this attempt is not a success.

3.2 BLG Model

The strong coupling limit of Type IIA string theory at the low energy limit of 11-dimensional supergravity gives M-theory[48][49]. M-theory admits a fully 32-supersymmetric solution with the geometry of $AdS_4 \times S^7$ and isometry group $OSp(8|4)$, and the action of the superconformal gauge theories that is constructed due to the AdS/CFT correspondence is expected to have worldvolume $d = 3$ [50]. In M-theory, the fundamental ingredients are the M2-branes (or membranes). M5-branes are their solitonic electric-magnetic dual. The action describing a single M2-brane [21] and the action for single M5-brane[51,52,53] have been known for a long time. However, the action for multiple M5-branes is still

unknown, and the action describing multiple M2-branes are proposed[24,25,26,27] only recently base on 3-algebra structure. BLG-model contain eight $d=3$ real scalar fields, coming from the eight transverse coordinates of the M2-brane plus 16 real $d=3$ Goldstone fermions (the other 16 being removed by κ-symmetry) and, since they have a $N=8$ supersymmetry, they present a natural $SO(8)$ R-symmetry. It was also proposed that the gauge fields should be Chern-Simons like.

3.2.1 BLG action

The starting point of BLG action is to provide an action that can satisfied Basu-Harvey equation[54]. This is contrary to most regular research process in particle physics—write down the action, then obtain the equation of motion. Generalizing Nahm's equation, which was used to describe the analogues configuration of D1-branes ending on D3-branes, the Basu-Harvey equation was proposed to describe multiple M2-branes ending on an M5-brane

$$\frac{dX^i}{ds}+i\frac{K}{3!}\in^{ijkl}[X^j,X^k,X^l]=0, \tag{3.23}$$

where $X^i(s)$'s represent spatial fluctuations of the M2-branes, and s is a worldvolume coordinate. This equation admits a funnel solution:

$$X^i(s)=f(s)R(T^i), \tag{3.24}$$

$$f(s)=\frac{1}{\sqrt{2Ks}}, \tag{3.25}$$

where T^i satisfies $SO(4)$-invariant algebra A_4, and $R(T^i)$ is any representation of A_4. A_4 describes a fuzzy three-sphere with radius r given by

$$r^2\equiv\sum_i(X^i)^2\propto f^2(s)\propto\frac{1}{Ks}. \tag{3.26}$$

Hence

$$r^2 = \frac{\alpha}{Ks} \tag{3.27}$$

for some constant α. The T^i's then represent the Cartesian coordinates of the fuzzy three-sphere.

The BLG model is given by the worldvolume Lagrangian density

$$\begin{aligned} L_{BLG} = &-\frac{1}{2} < D_\mu X^I, D^\mu X^I > + \frac{i}{2} < \overline{\psi}, \Gamma^\mu D_\mu \psi > - \frac{i}{4} g < [\overline{\psi}, X^I, X^J], \Gamma_{IJ} \psi > \\ &- \frac{g^2}{2 \cdot 3!} < [X^I, X^J, X^K], [X^I, X^J, X^K] > + \frac{1}{g} L_{CS} \end{aligned} , \tag{3.28}$$

where L_{CS} is Chern-Simons term with the form

$$L_{CS} = \frac{1}{2} \varepsilon^{\mu\upsilon\rho} (f_{abcd} A_\mu^{ab} \partial_\upsilon A_\rho^{cd} + \frac{2}{3} f_{cda}{}^g f_{efgb} A_\mu^{ab} A_\upsilon^{cd} A_\rho^{ef}), \tag{3.29}$$

and the corvariant derivative is $(D_\mu V)^a = \partial_\mu V^a - f_{cdb}{}^a A_\mu^{cd} V^b$, $I, J, K = 3, \cdots, 10$, $\mu, \upsilon, \rho = 0,1,2$. ψ is an 11-dimensional Majorana spinor satisfying the chirality condition

$$\Gamma_{012} \psi = -\psi . \tag{3.30}$$

The Chern-Simons term was called 'twisted' because it does not have the standard CS expression.

The corresponding action may be split into three terms, as

$$I_{BLG} = \int d^3 x L_{kin} + \int d^3 x L_{\text{int}} + \frac{1}{g} \int d^3 x L_{CS} , \tag{3.31}$$

where L_{kin} contains the first two terms on the right hand side of (3.28), and L_{int} contains the third and the fourth term of (3.28). This action is scale-invariant since the

gauge fields have length dimension $A = L^{-1}$ and the constant g is dimensionless. Then, the kinetic terms for the worldvolume matter fields are also scale-invariant with $[X] = L^{-1/2}$ and $[\psi] = L^{-1}$, the expected for a $d = 3$ theory with no dimensionful constants.

For the background with $\psi = A = 0$, a BPS condition should guarantee that[31]

$$(\partial_\mu X^I \Gamma^\mu \Gamma_I - \frac{1}{6}[X^I, X^J, X^K]\Gamma_{IJK})\varepsilon = 0, \tag{3.32}$$

for some constant spinor ε.

3.2.2 Gauge and supersymmetry transformations

The BLG action is invariant under both gauge symmetry and supersymmetry; it has $OSp(8|4)$ superconformal symmetry[29]. Further, the Noether currents associated with the BLG Lagrangian generate the $d = 3, \mathrm{N} = 8$ superPoincare algebra with central charges

$$\{Q_\alpha^p, Q_\beta^q\} = -2(\gamma^\mu \gamma^0)_{\alpha\beta}\delta^{pq} P_\mu + \varepsilon_{\alpha\beta} Z^{[pq]} + (\gamma^\mu \gamma^0)_{\alpha\beta} Z_\mu^{(pq)}, \tag{3.33}$$

where $\mu = 0,1,2; \alpha, \beta = 1,2;\ \ p, q = 1, \cdots, 8$. The symmetric central charge is traceless, $\delta_{pq} Z_\mu^{(pq)} = 0$, and this algebra has an obvious $SO(8)$ automorphism group under which the eight $d = 3$ two-component Majorana supercharges Q^q form a chiral $Spin(8)$ spinor.

The gauge transformations are given by

$$\delta X^{Ia} = \lambda^{cd} f_{cdb}{}^a X^{Ib},$$

$$\delta\psi^a = \lambda^{cd} f_{cdb}{}^a \psi^b ,$$

$$\delta(f_{cdb}{}^a A_\mu^{cd}) = \partial_\mu (f_{cdb}{}^a \lambda^{cd}) + 2 f_{cdb}{}^a f_{efg}{}^c \lambda^{dg} A_\mu^{ef} . \tag{3.34}$$

The gauge group algebra corresponding to these transformations is A_4, which can be decomposed into $A_4 = su(2) \oplus su(2)$. The gauge group for BLG model is thus $SU(2) \times SU(2)$.

The supersymmetry transformations are given by

$$\delta_\varepsilon X^I = i\bar{\varepsilon}\Gamma^I \psi ,$$

$$\delta_\varepsilon \psi = D_\mu X^I \Gamma^\mu \Gamma^I \varepsilon - \frac{g}{3!}[X^I, X^J, X^K]\Gamma^{IJK}\varepsilon ,$$

$$\delta_\varepsilon (f_{cdb}{}^a A_\mu^{cd}) = igf_{cdb}{}^a \bar{\varepsilon}\Gamma_\mu \Gamma^I X^{Ic} \psi^d , \tag{3.35}$$

where the supersymmetry parameter ε has standard dimensions $[\varepsilon] = L^{\frac{1}{2}}$. When one checks the supersymmetry invariance, the Fundamental identity of A_4 plays an important role to close the supersymmetric transformation algebras, and this is why 3-algebras are so important to generate $\mathrm{N} = 8$ supersymmetry.

3.2.3 Successes and challenges

As the first model describing the interaction of multiple M2-branes, BLG model have already achieved some successes, but also suffered some problems[54]. After integrating out auxiliary fields and field redefinitions, it was shown[55] that the BLG action precisely turns into the Super Yang-Mills action for D2-branes, and it can also give the SYM theory for Dp-branes. It provides an approach to the M2-brane worldvolume theory in a large constant field background[33,36]. However, it was

soon realized that, BLG model can only describe the interaction between two M2-branes on so called ‘M-fold’, because of its gauge group. And it is still not clear about the connection between algebraic properties of a 3-algebra and physical degrees of freedom of M2-branes.

Also there is an interesting question: what is the Higgs mass in BLG model? One can apply the Higgs mechanism to BLG potential and obtain the Higgs mass. [71] and [72] talked about how to use Higgs mechanism to determine the leading higher-derivative corrections to BLG field.

3.3 BLG-NB Model

The $SU(2)\times SU(2)$ gauge symmetry of BLG model restricts that it can only describe the interaction between two M2-branes ($N=2$). And in Chapter 2, we mentioned that ‘almost’ all the finite positive-definite Filippov 3-algebras are just direct products of A_4 ’s, therefore if one sticks to the condition ‘finite positive-definite’, there would be no new theories come out. The condition of ‘finite positive-definite’ and Fundamental identity preclude almost all other Filippov 3-algebras to this model. However, if we release the ‘finite’ condition, substituting Filippov 3-algebra with Nambu 3-algebra, we would have a similar model but describing the low energy limit of a ‘condensate’ of M2-branes.

3.3.1 Nambu 3-algebras

Nambu 3-algebra is defined[56] on a three-dimensional compact oriented manifold without boundary, and let $M_3 = S^3$ denote this manifold, hence the algebra is defined on a 3-sphere. We define Nambu 3-bracket by

$$\{\phi_1(y),\phi_2(y),\phi_3(y)\} = e^{-1}(y)\varepsilon^{ijk}\partial_i\phi_1(y)\partial_j\phi_2(y)\partial_k\phi_3(y), \tag{3.36}$$

where $\phi_1(y),\phi_2(y),\phi_3(y)$ are three functions defined on S^3; e is the scalar density; $y^i = (y^1, y^2, y^3)$ are local coordinates on S^3.

The generator of this algebra is $\{\mathrm{e}_a(y)\}$, where a denotes a set of discrete indices. And the scalar function can be expressed as

$$\phi(y) = \sum_a \phi^a \mathrm{e}_a(y), \tag{3.37}$$

which is completed with

$$< \mathrm{e}_a(y), \mathrm{e}_b(y) >= \delta_{ab}, \qquad \phi^a =< \phi(y), \mathrm{e}_a(y) >,$$

$$< \phi_1(y),\phi_2(y) >= \int_{S^3} \mu(y)\phi_1(y)\phi_2(y),$$

$$\sum_a \mathrm{e}_a(y)\mathrm{e}_a(y') = \delta^3(y,y'), \qquad \int_{S^3} \mu(y)\phi(y)\delta^3(y,y') = \phi(y'), \tag{3.38}$$

where $\mu(y) = e(y)dy^1 \wedge dy^2 \wedge dy^3$.

By (3.37), the 3-bracket can be written by

$$\{\phi_1(y),\phi_2(y),\phi_3(y)\} = \sum_{abc} \phi_1^a(y)\phi_2^b(y)\phi_3^c(y)\{\mathrm{e}_a(y),\mathrm{e}_b(y),\mathrm{e}_c(y)\}. \tag{3.39}$$

The structure constants of this Nambu 3-algebra , referred to the basis $\{\mathrm{e}_a(y)\}$ are, the coefficients $f_{abc}{}^d$ that appear in the expression

$$\{\mathrm{e}_{\mathrm{a}}(y),\mathrm{e}_{\mathrm{b}}(y),\mathrm{e}_{\mathrm{c}}(y)\}=\sum_{\mathrm{a}} f_{\mathrm{abc}}{}^{\mathrm{d}}\mathrm{e}_{\mathrm{d}}(y)\,. \tag{3.40}$$

There the structure constants can be expressed by

$$f_{\mathrm{abcd}}=<\{\mathrm{e}_{\mathrm{a}}(y),\mathrm{e}_{\mathrm{b}}(y),\mathrm{e}_{\mathrm{c}}(y)\},\mathrm{e}_{\mathrm{d}}(y)>=\int_{S^3}\mu(y)e^{-1}(y)\in^{ijk}\partial_i\mathrm{e}_{\mathrm{a}}(y)\partial_j\mathrm{e}_{\mathrm{b}}(y)\partial_k\mathrm{e}_{\mathrm{c}}(y)\mathrm{e}_{\mathrm{d}}(y) \tag{3.41}$$

3.3.2 BLG-NB model

BLG-NB model is a Nambu bracket extension of BLG model, the transition from BLG to BLG-NB will be achieved by replacing the A_4 3-bracket by Nambu 3-bracket. The matter fields appearing in BLG-NB model depend on three-dimensional worldvolume Minkowski coordinates $x^{\mu}=(x^0,x^1,x^2)$ as before but now include the Nambu 3-algebra basis index a. Therefore, these fields have the coordinate expansions

$$X^I(x,y)=\sum_{\mathrm{a}} X^{I\mathrm{a}}(x)\mathrm{e}_{\mathrm{a}}(y)\,,\qquad \psi^I(x,y)=\sum_{\mathrm{a}}\psi^{I\mathrm{a}}(x)\mathrm{e}_{\mathrm{a}}(y)\,, \tag{3.42}$$

in which the sum over the index a for A_4 has been replaced by a sum over the set of indices a. The potential field is given by $A_{\mu}^{\mathrm{ab}}(x)=-A_{\mu}^{\mathrm{ba}}(x)$.

The BLG-NB Lagrangian is given by

$$\begin{aligned}L_{BLG-NB}=&\int_{S^3} d^3y e(y)(-\frac{1}{2}D_{\mu}X^I(x,y)D^{\mu}X^I(x,y)+\frac{i}{2}\overline{\psi}(x,y)\Gamma^{\mu}D_{\mu}\psi(x,y)\\&-g\frac{i}{4}\{\overline{\psi}(x,y),X^I(x,y),X^J(x,y)\}\Gamma_{IJ}\psi(x,y)\\&-\frac{g^2}{2\cdot 3!}\{X^I(x,y),X^J(x,y),X^K(x,y)\}\{X^I(x,y),X^J(x,y),X^K(x,y)\}\\&+\frac{1}{g}L_{CS}\end{aligned}\,,$$

(3.43)

where the covariant derivative is given by

$$D_\mu \phi(x,y) = \partial_\mu \phi(x,y) + s_\mu^k(x,y)\partial_k \phi(x,y), \tag{3.44}$$

and

$$s_\mu^k(x,y) = -e^{-1}(y)\varepsilon^{ijk}\sum_{ab}\partial_i e_a(y)\partial_j e_b(y)A_\mu^{ab}(x). \tag{3.45}$$

The Chern-Simons term can be written in terms of $s_\mu^k(x,y)$

$$\begin{aligned} L_{CS}(x) &= \frac{1}{2}\sum_{abcd}(f_{abc}{}^d A^{ab}(x)\wedge dA^{cd}(x) + \frac{2}{3}\sum_{efg} f_{cda}{}^g f_{efgb}A^{ab}(x)\wedge A^{cd}(x)\wedge A^{ef}(x)) \\ &= -\frac{1}{2}\int_{S^3}\mu(y)(s^k(x,y)\wedge dA_k(x,y) - \frac{1}{3}\varepsilon_{ijk}s^i(x,y)\wedge s^j(x,y)\wedge s^k(x,y)) \end{aligned}, \tag{3.46}$$

where $A^{ab}(x) = A_\mu^{ab}(x)dx^\mu$, and $s^i(x,y) = s_\mu^i(x,y)dx^\mu$ are spacetime one-forms. This Chern-Simons term was called 'CS-like' because it is not entirely written in terms of the gauge field $s^i(x,y)$, it also requires the potential term $A_i(x,y)$.

3.3.3 Gauge and supersymmetry transformations

By using the structure constants, the gauge transformations are determined by local functions $\xi(x,y)$ on the S^3 [35,56],

$$\begin{aligned} \delta X^I(x,y) &= -\xi^k(x,y)\partial_k X^I(x,y), \\ \delta\psi(x,y) &= -\xi^k(x,y)\partial_k\psi(x,y), \\ \delta s^i(x,y) &= d\xi^i(x,y) - \xi^j(x,y)\partial_j s^j(x,y) + \partial_j\xi^j(x,y)s^j(x,y), \end{aligned} \tag{3.47}$$

with the condition that $\partial_k(e(y)\xi^k(x,y)) = 0$. The gauge group corresponding to this transfor -mation is called volume-preserving diffeomorphism group, denoted by

$SDiff(S^3)$. It is discussed in [56] that, if one defines the fundamental objects of Nambu algebra by an inner derivation, meaning the fundamental objects act on the elements of Nambu algebra by left multiplication, then different fundamental objects may induce the same element in Nambu algebra. This implies that $SDiff(S^3)$ is the only group satisfying this gauge algebra, all other groups generated by this Nambu algebra are diffeomorphic to $SDiff(S^3)$.

The supersymmetry transformations of matter fields are

$$\delta_\varepsilon X^I(x,y) = i\bar{\varepsilon}\Gamma^I\psi(x,y),$$

$$\delta_\varepsilon\psi(x,y) = D_\mu X^I(x,y)\Gamma^\mu\Gamma^I\varepsilon - \frac{g}{3!}\{X^I(x,y), X^J(x,y), X^K(x,y)]\Gamma^{IJK}\varepsilon,$$

$$\delta_\varepsilon s^k_\mu(x,y) = -ige^{-1}(y)\varepsilon^{ijk}\bar{\varepsilon}\Gamma_\mu\Gamma^I\partial_i X^I(x,y)\partial_j\psi(x,y), \tag{3.48}$$

which ensure manifest $\mathrm{N}=8$ supersymmetry.

3.4 Discussion

The biggest difference between BLG model and BLG-NB model is the gauge group. The gauge symmetry of BLG-NB model has turned to be the infinite-dimensional Lie algebra of the volume-preserving diffeomorphisms group $SDiff(S^3)$. The $SDiff(S^3)$ gauge theories are called 'exotic'[34] because they can not be obtained from an 'abstract' Yang-Mills theory, whereas it is possible for $SDiff(S^2)$ gauge theories. The transition from BLG model to BLG-NB model is

similar to the transition from classical $SU(N)$ Yang-Mills theories to their large N limits. In Hoppe's work [57] of study the canonical quantization of a relativistic spherical membrane of Dirac type in the light cone gauge, his showed that the $SU(N)$ Lie algebra, when $N \to \infty$, is isomorphic to the infinite-dimensional Lie algebra of area-preserving diffeomorphisms of the two-sphere S^2.

3.4.1 From $SU(N)$ to $SDiff(S^2)$

Here we give a short review[58] of this transition: suppose we have classical $SU(N)$ Yang-Mills fields

$$A_\mu(x) = A_\mu^a t_a, \tag{3.49}$$

where t_a's are generators of $SU(N)$, and $a = 1, \cdots, N^2 - 1$, $\mu = 0,1,2,3$, with gauge transformations

$$\delta A_\mu = \partial_\mu \omega + [A_\mu, \omega], \quad \text{where} \quad \omega = \omega^a t_a, \tag{3.50}$$

and

$$\delta F_{\mu\upsilon} = [F_{\mu\upsilon}, \omega], \quad \text{where} \quad F_{\mu\upsilon} = \delta_\mu A_\upsilon - \delta_\upsilon A_\mu + [A_\mu, A_\upsilon]. \tag{3.51}$$

Eventually, we want to replace (3.49)-(3.51) by

$$A_\mu(x,\theta,\varphi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} A_\mu^{lm}(x) Y_{lm}(\theta,\varphi), \tag{3.52}$$

$$\delta A_\mu(x,\theta,\varphi) = \partial_\mu \omega(x,\theta,\varphi) + \{A_\mu, \omega\}, \tag{3.53}$$

and $$\delta F_{\mu\upsilon} = \{F_{\mu\upsilon}, \omega\} \quad \text{with} \quad F_{\mu\upsilon} = \delta_\mu A_\upsilon - \delta_\upsilon A_\mu + \{A_\mu, A_\upsilon\}, \tag{3.54}$$

where Y_{lm} are the spherical harmonics on S^2, and the Poisson bracket of two functions on S^2 is defined as

$$\{f,g\}=\frac{\partial f}{\partial\cos\theta}\frac{\partial g}{\partial\varphi}-\frac{\partial f}{\partial\varphi}\frac{\partial g}{\partial\cos\theta}. \tag{3.56}$$

The commutator can be replace by Poisson bracket by

$$\lim_{N\to\infty} N[A_\mu, A_\upsilon]=\{A_\mu, A_\upsilon\}. \tag{3.57}$$

First, we construct the structure constants of $SDiff(S^2)$. It is known that the generators of symplectic transformations of a surface have the local form

$$L_f=\frac{\partial f}{\partial\sigma_2}\frac{\partial}{\partial\sigma_1}-\frac{\partial f}{\partial\sigma_1}\frac{\partial}{\partial\sigma_2}, \tag{3.58}$$

which satisfy Lie algebra

$$[L_f, L_g]=L_{\{f,g\}}. \tag{3.59}$$

In the case of S^2, the local coordinates are $\sigma_1=\varphi$ and $\sigma_2=\cos\theta$; and we can form a basis of generators by choosing $f=Y_{lm}(\theta,\varphi)$ to be the spherical harmonics

$$L_{lm}=\frac{\partial Y_{lm}}{\partial\cos\theta}\frac{\partial}{\partial\varphi}-\frac{\partial Y_{lm}}{\partial\varphi}\frac{\partial}{\partial\cos\theta}. \tag{3.60}$$

Then (3.59) implies that

$$[L_{lm}, L_{l'm'}]=f_{lm,l'm'}^{l''m''}L_{l''m''}, \tag{3.61}$$

where the structure constants $f_{lm,l'm'}^{l''m''}$ are defined by

$$\{Y_{lm}, Y_{l'm'}\}=f_{lm,l'm'}^{l''m''}Y_{l''m''}. \tag{3.62}$$

These structure constants have been calculated explicitly in [57] and [59]. If we let $l=1$ and $m=0,\pm1$, then we reduce (3.61) to the $SU(2)$ Lie group, which are usual angular momentum generators. The spherical harmonics $Y_{lm}(\theta,\varphi)$ are harmonic homogeneous polynomials of degree l in three Euclidean coordinates x_1, x_2, x_3. They

are given by

$$x_1 = \cos\varphi \sin\theta, \qquad x_2 = \sin\varphi \sin\theta, \qquad x_3 = \cos\theta, \tag{3.63}$$

and

$$Y_{lm}(\theta,\varphi) = \sum_{\substack{i_k=1,2,3 \\ k=1,\cdots,l}} a^{(m)}_{i_1\cdots i_l} x_{i_1} \cdots x_{i_l}, \tag{3.64}$$

where $a^{(m)}_{i_1\cdots i_l}$ is a symmetric and traceless tensor. Let S_1, S_2, S_3 be $N \times N$ hermitian matrices which form an representation of $SU(2)$: $[S_i, S_j] = i \in_{ijk} S^k$. The matrix polynomials

$$T^{(N)}_{lm} = \sum_{i_k=1,2,3} a^{(m)}_{i_1\cdots i_l} S_{i_1} \cdots S_{i_l} \tag{3.65}$$

can be used to construct a basis of $N^2 - 1$ matrices for the fundamental representation of $SU(N)$, with structure constants

$$[T^{(N)}_{lm}, T^{(N)}_{l'm'}] = if^{(N)l''m''}_{lm,l'm'} T^{(N)}_{lm}. \tag{3.66}$$

It is proved that, as N goes to infinity, we have

$$\lim_{N\to\infty} f^{(N)l''m''}_{lm,l'm'} = f^{l''m''}_{lm,l'm'}, \tag{3.66}$$

which coincides with the structure constants in (3.62).

3.4.2 From BLG to BLG-NB

The great improvement of BLG-NB model from BLG model is that, BLG-NB model is conjectured as describing infinite M2-branes' condensation[34]. The gauge group of BLG mode is Lie group $SU(2) \times SU(2)$, and it is believed that the action with gauge symmetry of gauge group $SU(N) \times SU(N)$ shall describe the dynamics of N M2-branes. Therefore, besides $SDiff(S^3)$, $SU(\infty) \times SU(\infty)$ shall also be a

gauge group for BLG-NB model, which means

$$SDiff(S^3) \text{ gauge theory} \approx SU(\infty)\times SU(\infty) \text{ gauge theory.} \quad (3.67)$$

But so far, it is still not clear about how to derive $SDiff(S^3)$ from $SU(\infty)\times SU(\infty)$. Also, by last subsection, we see that as N goes to infinity, $SU(N)$ gauge theory becomes to $SDiff(S^2)$ gauge theory. So we conjecture that the relation between $SDiff(S^3)$ gauge theory and $SDiff(S^2)$ gauge theory can be expressed as

$$SDiff(S^3) \text{ gauge theory} \approx SDiff(S^2)\times SDiff(S^2) \text{ gauge theory.} \quad (3.68)$$

We will come back to this relation in next chapter and discuss the possible approaches to prove this.

As BLG model, BLG-NB model does not give the action describing the interaction between finite number of M2-branes, it is only the large N limit of a more general theory. And it still suffers those challenges of BLG model.

Chapter 4

Multiple M2-Branes

BLG model is the first model describing the dynamics of multiple M2-branes, however it only tells us the interaction between two M2-branes. Its Nambu bracket extension—BLG-NB model, is proposed to describe infinite number of M2-branes' condensation. If there does exist a general theory describing multiple M2-branes dynamics, then both BLG model and BLG-NB model would just be the special cases for this theory. And this theory should be able to provide the action for finite number of M2-branes. So far, such theory has not emerged yet; however, there is a proposal that might help us to achieve this goal. Two years after BLG model first appeared, in 2008, Aharnony, Bergman, Jafferis and Maldacena[23] proposed another action to describe multiple M2-branes interaction, we call it the ABJM model. This ABJM model is believed to describe finite number of M2-branes, which can be considered to fill the gap between BLG model and BLG-NB model.

In section 4.1, we introduce ABJM Lagrangian and its symmetries. In section 4.2, we discuss the possibility of unifying BLG, BLG-NB and ABJM model in a general theory. The difficulties lie at the supersymmetry enhancement for ABJM model when $N > 2$, as well as the large N limit of gauge symmetry. So far, there is no authentic way for the supersymmetry enhancement. As to the large N limit, we propose that

using hyperspherical harmonics might be a possible way to construct the structure constants for the gauge group algebra.

4.1 ABJM Model

The ABJM model is constructed without 3-algebra structure, it describes the low energy physics of world-volume of multiple M2-branes in orbifold C^4/Z_k [23,60,61]. It gives up the full $\mathrm{N}=8$ manifest supersymmetry, and the superconformal Chern-Simons theories with gauge group $U(N)\times U(N)$ and $SU(N)\times SU(N)$ are defined on three-dimensional spacetime.

4.1.1 ABJM Lagrangian

The ABJM action is composed of four complex scalars $Z_A(A=1,2,3,4)$, which describe complexified eight coordinates transverse to M2-branes world-volume; and four 3-dimensional spinors ψ_A, and their adjoints $\overline{Z}_A$ and $\overline{\psi}_A$. It has two gauge fields A_μ and $\tilde{A}_\mu$ that have Chern-Simons kinetic term of level $(k,-k)$. The matter fields can be realized with $N\times N$ matrices, and transform as $(N,\overline{N})$ for Z_A and ψ_A, and $(\overline{N},N)$ for $\overline{Z}_A$ and $\overline{\psi}_A$.

Here we give the Lagrangian of ABJM model in the same convention of [60] except the Hermitian gauge fields

$$L = L_{CS} + L_{kin} + L_{Yukawa} + L_{potential}\,, \tag{4.1}$$

where

$$L_{CS}+L_{kin}=\frac{k}{4\pi}\in^{\mu\upsilon\rho}\ tr(A_\mu\partial_\upsilon A_\rho-i\frac{2}{3}A_\mu A_\upsilon A_\rho-\tilde{A}_\mu\partial_\upsilon\tilde{A}_\rho+i\frac{2}{3}\tilde{A}_\mu\tilde{A}_\upsilon\tilde{A}_\rho) \\ -tr(D_\mu\overline{Z}^A D^\mu Z_A-i\overline{\psi}_A\gamma_\mu D^\mu\psi^A\}, \tag{4.2}$$

$$L_{Yukawa}=-\frac{2\pi i}{k}tr(\overline{Z}^A Z_A\overline{\psi}_B\psi^B-Z_A\overline{Z}^A\psi^B\overline{\psi}_B) \\ -\frac{2\pi i}{k}tr(2\overline{Z}^A\psi^B Z_A\overline{\psi}_B-2Z_A\overline{\psi}_B\overline{Z}^A\psi^B) \\ -\frac{2\pi i}{k}\in^{ABCD}tr(Z_A\overline{\psi}_B Z_C\overline{\psi}_D)+\frac{2\pi i}{k}\in_{ABCD}tr(\overline{Z}^A\psi^B\overline{Z}^C\psi^D), \tag{4.3}$$

and

$$L_{potential}=\frac{4\pi^2}{3k^2}tr(Z_A\overline{Z}^A Z_B\overline{Z}^B Z_C\overline{Z}^C+\overline{Z}^A Z_A\overline{Z}^B Z_B\overline{Z}^C Z_C+ \\ 4Z_A\overline{Z}^C Z_B\overline{Z}^A Z_C\overline{Z}^B-6Z_A\overline{Z}^A Z_B\overline{Z}^C Z_C\overline{Z}^C), \tag{4.4}$$

where γ_μ is two dimensional Dirac matrices, k is Chern-Simons level and the trace is over $N\times N$ matrices of either gauge group and leaves the gauge invariant quantities. The covariant derivatives are given by

$$D_\mu X_A=\partial_\mu X_A-iA_\mu X_A+iX_A\tilde{A}_\mu$$

$$D_\mu\overline{X}^A=\partial_\mu\overline{X}^A-i\tilde{A}_\mu\overline{X}^A+i\overline{X}^A A_\mu, \tag{4.5}$$

where X_A denotes $Z_A,\overline{Z}_A,\psi_A$ and $\overline{\psi}_A$. The potential part of Lagrangian can be written in quadratic form as

$$V=\frac{2\pi^2}{3k^2}tr(W_{AB}^C\overline{W}_C^{AB}) \tag{4.6}$$

with

$$W_{AB}^C=(2Z_A\overline{Z}^C Z_B-\delta_B^C Z_A\overline{Z}^D Z_D-\delta_A^C Z_D\overline{Z}^D Z_B) \\ -(2Z_B\overline{Z}^C Z_A-\delta_A^C Z_B\overline{Z}^D Z_D-\delta_B^C Z_D\overline{Z}^D Z_A)$$

$$\begin{aligned}\overline{W}_C^{AB} &= (2\overline{Z}^A Z_C \overline{Z}^B - \delta_C^B \overline{Z}_A Z_D \overline{Z}^D - \delta_C^A \overline{Z}^D Z_D Z_B) \\ &\quad -(2\overline{Z}^B Z_C \overline{Z}^A - \delta_C^A \overline{Z}_B Z_D \overline{Z}^D - \delta_C^B \overline{Z}^D Z_D Z_A)\end{aligned}. \tag{4.7}$$

We can see there is no 3-algebra used in the definition, all are done with regular Lie algebras.

4.1.2 Supersymmetries and gauge symmetries

ABJM model gives up $N=8$ manifest supersymmetry, but it still has $N=6$ supersymmetry, whose transformation rules are

$$\delta A_\mu = \frac{2\pi i}{k}(\eta^{AB}\gamma_\mu Z_A \overline{\psi}_B + \eta_{AB}\gamma_\mu \psi^B \overline{Z}^A),$$

$$\delta \widetilde{A}_\mu = \frac{2\pi i}{k}(\eta^{AB}\gamma_\mu \overline{\psi}_B Z_A + \eta_{AB}\gamma_\mu \overline{Z}^A \psi^B),$$

$$\delta Z_A = -i\eta_{AB}\psi^B,$$

$$\delta\psi^A = [\gamma^\mu D_\mu Z_B - \frac{4\pi}{3k}(Z_{[C}\overline{Z}^C Z_{B]})]\eta^{BA} + \frac{8\pi}{3k}(Z_B\overline{Z}^A Z_C)\eta^{BC} - \frac{4\pi}{3k}\varepsilon^{ABCD}(Z_B\overline{Z}^E Z_C)\eta_{DE}$$

(4.8)

where η^{AB} is supersymmetry transformation parameter, with $\eta^{AB} = -\eta^{BA}$, $\eta_{AB} = (\eta^{AB})^* = \frac{1}{2}\in_{ABCD}\eta^{CD}$.

The infinitesimal gauge transformations are given by

$$\delta A_\mu = D_\mu \Lambda = \partial_\mu \Lambda + i[A_\mu, \Lambda],$$

$$\delta \widetilde{A}_\mu = D_\mu \widetilde{\Lambda} = \partial_\mu \widetilde{\Lambda} + i[\widetilde{A}_\mu, \widetilde{\Lambda}],$$

$$\delta X_A = -i\Lambda X_A + iX_A\Lambda, \tag{4.9}$$

with gauge group $G = U(N) \times U(N)$ and $SU(N) \times SU(N)$.

Also there is a global $SU(4)$ R-symmetry. The fields with lower index label the 4 representation of R-symmetry, and those with upper index label the $\overline{4}$ representation.

One can check that the vacuum moduli space of the ABJM model at the classical level, i.e., solutions of $V(\phi) = 0$ up to gauge transformations. Then (4.7) would lead to the equation for its minima

$$Z_A \overline{Z}^C Z_B = Z_B \overline{Z}^C Z_A \qquad \text{and} \qquad \overline{Z}^A Z_C \overline{Z}^B = \overline{Z}^B Z_C \overline{Z}^A . \tag{4.10}$$

This implies that the Hermitian matrices $Z_A \overline{Z}^C$, has to commute with each other, and similar for $\overline{Z}^A Z_C$. The vacuum solutions are thus given by diagonal Z_A up to gauge equivalence

$$Z_A = diag(z_A^1, \cdots, z_A^N) . \tag{4.11}$$

It means in that point of vacuum moduli space the gauge group $U(N) \times U(N)$ is broken to $U(N)^N$, and the gauges fields become to

$$A_\mu = diag(a_\mu^1, \cdots, a_\mu^N) \qquad \text{and} \qquad \widetilde{A}_\mu = diag(\widetilde{a}_\mu^1, \cdots, \widetilde{a}_\mu^N) . \tag{4.12}$$

4.2 A Unification Theory?

ABJM model and BLG model are two candidates for multiple M2-branes interaction based on different algebraic structure. ABJM uses regular Lie algebra

while BLG needs the Filippov 3-algebra. Also, there are many differences between the supersymmetry, gauge symmetry and R-symmetry. Here we list the symmetry features of ABJM, BLG and BLG-NB model:

Model / Symmetry	ABJM	BLG	BLG-NB
Supersymmetry	N = 6	N = 8	N = 8
Gauge Symmetry	$SU(N)\times SU(N)$	$SU(2)\times SU(2)$	$SDiff(S^3)$
R-Symmetry	$SU(4)\times U(1)$	$SO(8)$	$SO(8)$
Chern-Simons Level	Arbitrary	$k=1$	$k=1$
M2-brane Number	N (finite)	2	∞

Table 4.1

Based on the features of these three models, one would intuitively consider unifying them into one unification theory. It seems every time we facing multiple M2-branes dynamics, we shall use BLG if the M2-brane number is 2, use ABJM if $2 < N < \infty$, then BLG-NB if N goes to infinity. But before saying this approach is safe, we have to check if the transitions between these three models are possible and smooth. First, let us look the connection between ABJM and BLG.

4.2.1 ABJM to BLG

To transit ABJM to BLG, we need to check: 1. if the Lagrangian and Chern-Simons term are of the same form between these two; 2. how to enhance

$\mathrm{N}=6$ supersymmetry to $\mathrm{N}=8$ supersymmetry, and enhance $SU(4)\times U(1)=SO(6)$ R-symmetry to $SO(8)$ R-symmetry. At $N=2$ case, ABJM model is equivalent to BLG model[23] if the Chern-Simons level is equal to 1. Both theories have $SU(2)\times SU(2)$ gauge symmetry when $N=2$. ABJM has an $SU(4)$ flavor symmetry, with additional $SU(2)$ symmetry exchanging the scalar fields. These two symmetries combine to give an $SO(8)$ global R-symmetry, and this $SO(8)$ symmetry rotates all 8 real scalars in the (2,2) representation. This implies $\mathrm{N}=8$ superconformal symmetry. Now we checked that at $N=2$ case, the ABJM model has the same symmetries as BLG model.

Next, let us check the lagrangian and Chern-Simons term. There are two ways to do it: one is to rewrite BLG model in terms of regular Lie algebra, and the other is to rewrite ABJM in 3-algebra form, or using four-index structure constants. The first way can be achieved by rewriting BLG action in bifundamental representation[62]. For the second way, we have to modify the original Filippov 3-algebra, and different modifications generate different forms of the action. Bagger and Lambert[63] give an example using the complex 3-algebra which has the first two entries antisymmetric with each other:

$$[T^a,T^b;\overline{T}^{\bar{c}}]=f^{ab\bar{c}}{}_dT^d, \tag{4.13}$$

and require the $f^{ab\bar{c}}{}_d$ to satisfy the fundamental identity

$$f^{ef\bar{g}}{}_bf^{cb\bar{a}}{}_d-f^{fe\bar{a}}{}_bf^{cb\bar{g}}{}_d+f^{*\bar{g}\bar{a}f}{}_{\bar{b}}f^{ce\bar{b}}{}_d+f^{*\bar{a}\bar{g}e}{}_{\bar{b}}f^{cf\bar{b}}{}_d=0 \tag{4.14}$$

with inner product

$$h^{\bar{a}b}=Tr(\overline{T}^{\bar{a}},T^b). \tag{4.15}$$

Then the BLG version of the ABJM Lagrangian becomes

$$\begin{aligned} L = & -D^{\mu}\overline{Z}_{A}^{a}D_{\mu}Z_{a}^{A} - i\overline{\psi}^{Aa}\gamma^{\mu}D_{\mu}\psi_{Aa} - \frac{2}{3}W_{Bd}^{CD}\overline{W}_{CD}^{Bd} - if^{ab\bar{c}\bar{d}}\overline{\psi}_{\bar{d}}^{A}\psi_{Aa}Z_{b}^{B}\overline{Z}_{B\bar{c}} \\ & + 2if^{ab\bar{c}\bar{d}}\overline{\psi}_{\bar{d}}^{A}\psi_{Ba}Z_{b}^{B}\overline{Z}_{A\bar{c}} + \frac{i}{2}\in_{ABCD} f^{ab\bar{c}\bar{d}}\overline{\psi}_{\bar{d}}^{A}\overline{\psi}_{\bar{c}}^{B}Z_{a}^{C}Z_{b}^{D} \\ & - \frac{i}{2}\in^{ABCD} f^{cd\bar{a}\bar{b}}\overline{\psi}_{Ac}\psi_{Bd}\overline{Z}_{C\bar{a}}\overline{Z}_{D\bar{b}} + L_{CS} \end{aligned} \quad , (4.16)$$

where

$$W_{Bd}^{CD} = f^{ab\bar{c}}{}_{d}Z_{a}^{C}Z_{b}^{D}\overline{Z}_{B\bar{c}} - \frac{1}{2}\delta_{B}^{C}f^{ab\bar{c}}{}_{d}Z_{a}^{E}Z_{b}^{D}\overline{Z}_{\bar{E}\bar{c}} + \frac{1}{2}\delta_{B}^{D}f^{ab\bar{c}}{}_{d}Z_{a}^{E}Z_{b}^{C}\overline{Z}_{E\bar{c}}, \quad (4.17)$$

and

$$L_{CS} = \frac{1}{2}\in^{\mu\nu\lambda}(f^{ab\bar{c}\bar{d}}A_{\mu\bar{c}b}\partial A_{\lambda\bar{d}a} + \frac{2}{3}f^{ac\bar{d}}{}_{g}f^{ge\bar{f}\bar{b}}A_{\mu\bar{b}a}A_{\upsilon\bar{d}c}A_{\lambda\bar{f}e}). \quad (4.18)$$

This form is identical to the Lagrangian of BLG model (3.28) up to some constants. Therefore, at $N=2$ case, ABJM model is equivalent to BLG model. [64] gives another example using Jodan triple systems, and their rewritten ABJM action agrees with (3.28) also. So we believe that there are more than one forms of ABJM model in BLG version, depending on how we define the 3-algebra.

For $2 < N < \infty$ case, one can use the similar method provided by [63] and [64] to rewrite ABJM action, and it is straightforward to let $SU(2)\times SU(2)$ gauge symmetry go to higher level $SU(N)\times SU(N)$ gauge symmetry. But the problem happens here is we do not know how to enhance supersymmetry in this case. [23] conjectured that ABJM action shall have an $\mathrm{N}=8$ supersymmetry, but so far no one gives a rigorous proof yet. [65] gives a suggestion using Hermitian 3-algebra and monopole operator to identify new $\mathrm{N}=2$ supersymmetry, and they claim that the

ABJM potential written with Hermitian 3-algebra is $SO(8)$ invariant. Though there are still some relations in their work needed to be checked, we consider they give the best approach attacking this supersymmetry enhancement so far. As to the supersymmetry enhancement, one can also check[66, 67].

4.2.2 ABJM to BLG-NB

To connect these two models, we have to show that as N goes to infinity, ABJM model has the same action and symmetries as BLG-NB model does; or in other words, at large N limit,

$$\lim_{N\to\infty} \text{ABJM} \approx \text{BLG-NB}. \tag{4.19}$$

Though this conjecture looks similar to the case of 'ABJM to BLG', it is more difficult to achieve. Because first, we still do not know what kind of the Nambu 3-algebras we shall use to substitute (4.13), then we do not have the BLG-NB version of ABJM action or Chern-Simons term. Second, we still need to face the supersymmetry enhancement challenge, which probably can not be circumvented until one can enhance two extra supersymmetries for 'ABJM to BLG' case. Third, we have not figured out how to connect the gauge symmetries between these two models.

The gauge symmetry group of ABJM model is $SU(N)\times SU(N)$, while the one of BLG-NB is $SDiff(S^3)$. What we want to prove is (3.67)

$$\lim_{N\to\infty} SU(N)\times SU(N) \text{ gauge theory} \approx SDiff(S^3) \text{ gauge theory}. \tag{4.20}$$

It is well known that, the symmetry group of three-sphere is $SU(2)\times SU(2)$, which describe the direct product between two two-spheres. And we have also given the

transition from $SU(N)$ gauge theory to $SDiff(S^2)$ gauge theory, so we conjecture that the ABJM gauge theory can be related to BLG-NB gauge theory by (3.68), which is

$$SDiff(S^2)\times SDiff(S^2) \text{ gauge theory } \approx SDiff(S^3) \text{ gauge theory.} \quad (4.21)$$

Actually, the relation

$$\lim_{N\to\infty} SU(N)\times SU(N) \text{ gauge theory } \approx SDiff(S^2)\times SDiff(S^2) \text{ gauge theory} \quad (4.22)$$

is very straightforward by section 3.4.1, but (4.21) is still mysterious. So far, we have not been able to find the appropriate limiting formula for $SDiff(S^3)$ structure constants. We conjecture that the structure constants of $SDiff(S^3)$, might be calculated from the hyperspherical harmonics of three-sphere, which is similar to calculating the structure constants of $SDiff(S^2)$ from spherical harmonics. However, because the form of hyperspherical harmonics is much more complicated than spherical harmonics, it involves a large amount of calculations. And the choice of the auxiliary function is not clear, now this work is still in process.

The large N limit of ABJM model might involve taking two limits simultaneously, strong coupling and large N, and they might not commute with each other[34]. But logically, there is not contradiction between Nambu 3-algebra and ABJM model. After the discover of ABJM model, a BPS Wilson loop operator was constructed[68, 69, 70], and the perturbative calculation of the expectation value was performed. And the large N behavior of the expection value of the saddle point equation was studied in [70].

Chapter 5

Conclusion of Part I

In part I, we first briefly review regular Lie algebras and their properties. Then we extend Lie algebras to n-ary algebras by substituting Lie brackets by n-brackets. The Jacobi identity of Lie algebras is extended to generalized Jacobi identiy for generalized Lie algebras, which focus on the associativity of operators; and Fundamental identity for Filippov algebras, which do not emphasize on associativity. Besides generalized Jacobi identity and Fundamental identity, there is a third identity satisfied by all associative algebras, the Bremner Identity. We successfully extend the original Bremner identity from $n=3$ case to n equals to arbitrary odd integer number cases. We try to construct Filippov 3-algebras from associative algebras, and achieve some success. But generally, there are not as many 3-algebras as Lie algebras, because the characteristic identities are so strong, that they rule out most candidates. Nambu algebras are infinite dimensional Filippov algebras, they satisfy Fundamental identity naturally, but Bremner identity only for some special cases.

3-algebras play an important role in generating $\mathrm{N}=8$ supersymmetric multiple M2-brane model. BLG model utilizes A_4 as its gauge group algebra, which describes the interaction between two M2-branes. So far A_4 is the only finite positive-definite 3-algebra can be used in BLG model, all other finite

positive-definite 3-algebras are just direct products of it. One can extend BLG model to BLG-NB model by replacing A_4 with Nambu 3-algebra, which describes M2-brane condensation. The gauge group of BLG-NB model is volume-preserving diffeomorphism group $SDiff(S^3)$. Unfortunately, multiple M2-branes are not like D-branes, which can be derived from $\mathrm{N}=4$ Yang-Mills theory.

Besides BLG model, ABJM model is another proposal for multiple M2-brane theory. ABJM model does not involve any 3-algebraic structures, but it can be rewritten in terms of four index structure constants, and there are more than one way to do this. We consider to unify ABJM mode and BLG model, building a unification theory of multiple M2-branes. We try to build the connections in aspects of action, Chern-Simons term, gauge symmetry, supersymmetry and R-symmetry. The most difficult part is the supersymmetry enhancement, so far we have not figure any effective way to attack this problem. As to the gauge symmetry, we have to calculate the large N limit of ABJM model gauge group, which is $SU(N)\times SU(N)$. We propose to use hyperspherical harmonics to calculate the structure constants of $SDiff(S^3)$, and then compare the results with the results of large N limit.

BLG model is considered as the first example, which successfully applies n-ary algebras to physics. It motivates researches on both n-ary algebras and multiple M2-branes. We believe ultimately, the development of multiple M2-brane theory has to be connected to the development of n-ary algebras.

Chapter 6

Introduction of Part II

In this chapter, we motivate and formulate the research questions to be addressed in Part II of this thesis. First we review some history of functional equations, and one of its special cases – the Schroeder equation. Then we briefly review some basic concepts about renormalization group theory and the c-theorem. We then go on to explain the application of functional evolutional methods to renormalization group equations, and point out that there is a counterexample for the common belief about limit cycles. These allow us to formulate the research questions we want to address in Part II. Finally, we go through the organization of Part II by chapters.

6.1 History of Functional Equations and Schroeder's Equations

Functional equations is one of the most powerful and beautiful fields in mathematics, and have been widely utilized in natural, social and behavioral sciences. In mathematical modeling, models exhibit technical failures or inconsistencies on many occasions, which make them inadmissible. Then functional equations are a tool that prevents arbitrariness and allows model selection to be based on adequate constraints[73]. The theory of functional equations is very old, and the method had

been utilized a long time ago by Oresme, Napier, Kepler, and Galileo. It has been systematically studied since eighteenth century by d'Alembert, Euler, Cauchy, Abel, and Riemann[74]. In mathematics, a functional equation is any equation that specifies a function in implicit form. And often, the equation relates the value of a function at some point with its values at other points. Solving functional equations usually is very difficult; however by using some common methods, most functional equations are approachable[75]. The most common method of solving functional equations is substitution. Also there are many approximate methods which involve computer programming to solve functional equations numerically, and they are proven to be effect and fast with appropriate algorithms.

Schroeder's equation is a functional equation with one independent variable: find the function $\psi(x)$ such that $\psi(f(x)) = s\psi(x)$, where $f(x)$ is given. It is, of course, the eigenvalue equation for the composition operator C_f, defined by $C_f\psi = \psi \circ f$. Schroeder's equation was first studied by Ernst Schroeder[76] in the early 1870s, in his pioneering work on iteration of analytic functions. He tried to understand Newton's method in the complex plane, and obtained the idea of using iteration to find solutions of equations involving analytic functions[77]. He realized that each univalent solution ψ established a conformal "conjugation" between the action of ψ and the simpler mapping of multiplication by s. Schroeder also studied the solution of logistic maps in Schroeder equation, and obtained the analytical solution for $s = 2$ and $s = 4$ cases. He originated the systematic study of iteration as a means of solving analytic equations, and was the first to use conjugation as a fundamental tool to understand iteration near an attractive fixed point. Later, in 1884,

Koenigs published the fundamental existence-uniqueness theory for analytic solution of Schroeder's equation near an attractive fixed point[78]. A little more than one century after Schroeder's original papers on iteration, Gaughran and Schwartz established the connection between Schroeder's equation and the compactness problem for composition operators on Hardy spaces[79]. Originally, Schroeder only gave analytical solutions for two special cases of logistic map. The general solutions to Schroeder's equation can be difficult to approach, and so far there is not such a theory that one can follow to obtain the general solutions. However, Curtright and Zachos[80,81] invented an innovative method, the series and conjugation method, which can calculate the approximate solutions. And for those which have analytical solutions, the approximate solutions converge to the analytical solutions by taking the infinite limit. This method can be applied to many forms of the function ψ, and there always exists a solution approachable. For some other readable and informative accounts of Schroeder's equation, one can check[82,83].

6.2 Renormalization Group

In theoretical physics, the renormalization group (RG) is defined as a mathematical apparatus that investigate the changes of a system at different scales. In elementary particle physics, it reflects the changes in the quantum field theory (QFT) as the energy scale at which physical processes occur varies, energy/momentum and resolution distance scales being effectively conjugate under the uncertainty principle[84]. The renormalization group is related to two invariance: scale invariance

and conformal invariance, which correspond to scale transformation and conformal transformation respectively. The idea of scale invariance is dated back to Euclid and Galileo, and the scale transformation become popular again at the end of the 19th century because of the turbulence research[85,86]. In late 1940s, the regular formalism for eliminating the UV divergences in QFT was developed on the basis of covariant perturbation theory for the scattering S-matrix. This breakthrough is connected with the establishing of quantum electrodynamics (QED).

The first modern article introducing renormalization is by Stueckelberg and Petermann, in which[87] they anticipated the idea in quantum field theory, and noted that renormalization exhibits a group of transformations transferring quantities from the bare terms to the counterterms. They discovered a group of infinitesimal transformations related to finite arbitrariness arising the S-matrix elements upon elimination of the UV divergences. They also introduced a famous function in QED, which is later called beta function. At the same time, Bogoliubov developed a technique of supplementing the definition of products of singular Stuckelberg-Feynman propagators[88] and proved a theorem[89] on the finiteness and uniqueness of the S-matrix elements in any order of perturbation theory. In 1954, on the basis of Dyson's renormalization transformations[90] formulated in the regularized form, Gell-Mann and Low[91] discovered that the coupling parameter $g(\mu)$ at the energy scale μ is effectively given by the group equation

$$g(\mu) = G^{-1}((\mu / M)^d G(g(M)), \tag{6.1}$$

for some function G and constant d, in terms of the coupling at a reference scale M. They then realized that the effective scale can be taken arbitrarily, and can vary to

define the theory at any other scale

$$g(k) = G^{-1}((k/\mu)^d G(g(\mu)) = G^{-1}((k/M)^d G(g(M)). \tag{6.2}$$

On the basis of this group equation, Gell-Mann and Low then focused on infinitesimal transformations, and invented a computational method based on a mathematical flow function of the coupling parameter g, and determined the differential change of the coupling $g(\mu)$ with respect to a small change in energy scale μ through a differential equation, the renormalization group equation

$$\frac{\partial g}{\partial \ln(\mu)} = \psi(g) = \beta(g). \tag{6.3}$$

where $\psi(g)$ is flow function.

Perphaps, Gell-Mann and Low did not pay enough attention to the group character of the analysis, and in any chance they missed a chance to establish a connection between their results and QCD (Quantum Chromodynamics) perturbation theory. Bogoliubov and others[92] used the group properties of finite Dyson transformations for the coupling constant, fields and Green functions, and derived additional functional group equations for the propagators and vertices in QED in the general case. In 1970, Callan and Symanzik[93,94] reformulated the renormalization group theory in particle physics in more practical terms. They use the beta function to describe the “running of coupling” parameter with scale, and found it amount to the “canonical trace anomaly”, which represents the quantum-mechanical breaking of scale symmetry in a field theory.

Later, the conformal symmetry is associated with the vanishing of the beta

function. This can happen naturally if a coupling constant is attracted, by running towards a fixed point at which $\beta(g) = 0$. In QCD, the fixed point occurs at short distances where $g \to 0$ and is called an ultraviolet fixed point. In string theory, the conformal invariance of the string world-sheet is a fundamental symmetry. The beta function is a function of the geometry of the space-time in which the string moves. This determines the space-time dimensionality of the string theory and enforces Einstein's equations of general relativity on the geometry. The renormalization group is of fundamental importance to string theory and theories of grand unification.

6.3 Functional Equation in Renormalization Group

The renormalization group equation has been bonded with functional method since the day it was born. The functional renormalization group has in recent years been successfully applied to a wide variety of non-perturbative problems such as critical phenomena, fermionic systems, gauge theories, supersymmetry and quantum gravity[95,96,97,98,99]. Although these systems have different physical natures, they all share similar flow equations. The formulism established by Stueckelberg and Petermann, and Gell-Mann and Low has nice mathematical expression in terms of the functional conjugation methods of Schroeder's methods. The traditional way to solve finite renormalization group equation is to integrate a perturbative approximant to its algebra, the beta function, to obtain the full renormalization group trajectory for scale not equal to 1. The theory about Schroeder's equations gives us a different way to calculate and analyze the renormalization group trajectories, based on the theory of

functional conjugacy equation. In this conjugacy form, it is more apparent to see the global self-similar function structure of the renormalization group trajectory. Also, in conventional local relations, the interaction between continuous and discrete rescaling is often inaccessible while it can be illuminated in conjugacy form.

In renormalization theory, it has long been presumed that, under mild assumptions, scale invariance implies conformal invariance. In two-dimensional spacetime, Zamolodchikov[100] and Polchinski[101] argued and gave a proof that scale invariance implies conformal invariance for unitary quantum field theories with finite correlation functions. Zamolodchikov stated that there exists a positive real function c of coupling constant g in a two-dimensional renormalizable field theory which decreases monotonically under the influence of a renormalization group transformation, and this theorem is call c-Theorem. Then Cardy[102] extended c-Theorem in two-dimensional spacetime to so called a-Theorem in four-dimensional spacetime. It seems[103] both c-Theorem and a-Theorem point out that, based on monotonically evolving observables that the underlying couplings can not have renormalization group trajectories which are limit cycles[104] (closed curves that is invariant under renormalization). The limit cycle is characterized by a discrete scaling symmetry: the renormalization group flow executes a complete cycle around the curve every time the cutoff changes by a multiplicative factor λ called the discrete scaling factor. The discrete scaling symmetry is reflected in log-periodic behavior of physical observables as functions of the momentum scale[105]. However, the renormalization group scaling function ψ, obtained by Schroeder's method, is periodic and yields limit cycles even for a real coupling. Actually, back to 2003, the

"Russian Doll" superconductivity model given by [106,107], gave an example of the physics repeats itself cyclically in self-similar modules. This model identifies a two-dimensional field theory whose renormalization involves a renormalization group limit cycle in apparent contradiction to c -Theorem. Besides providing a counterexample for limit cycle in renormalization, functional conjugacy methods also indicate turning points for renormalization group trajectories, and generate multiple branches for renormalization group equation solutions.

6.4 Research Questions

In Part II, our focus is to establish the solutions to Schroeder's equations by series and conjugation method, investigate the convergence property of the solutions, and construct limit cycle example for renormalization group trajectories. The problems we want to confront are:

1. To establish an approximate solution to Schroeder's equations for arbitrary functional relation.
2. To investigate the asymptotic behavior of approximate solutions, comparing them with some well known analytical solutions.
3. To study the velocities and accelerations generated by the approximate solutions.
4. To study properties at fixed point and turning point of approximate solutions.
5. To understand the coefficient behaviors for approximate solutions.
6. To construct a counterexample for limit cycle claim.

6.5 Roadmap of Part II

In Chapter 7, we introduce Schroeder's equations and its analytical solutions for logistic map when the scale s equals to 2 and 4. Then we demonstrate an approximate solution through series and conjugation method, a solution for sine function is provided as an example. We derive approximate solutions for $s = 2,4$ cases, and calculate the error between them and analytical solutions. At small x values (x close to 0), approximate solutions agree with analytical very well, but when x approaches to 1, there is obvious difference between them. The method to deriving the approximate solution can help us obtain solution for other scale values. The solutions can be understood as trajectories of particles in classical mechanics, the geometric properties of the particles' trajectories are also of interest, the surfaces of trajectory curvatures seem to have chaotic properties.

In Chapter 8, the beta function is obtained by solving renormalization group equations approximately. We talk about different behaviors of beta function with different scales, and also show that the solutions are not unique. For each scale value, approximate solutions consist of infinite amount of branches, and turning points connect these branches together. Based on these different branches and turning points, we give a counterexample of limit cycle in renormalization group theory. The beta function is established by series and recursive methods, we investigate the coefficients for each order, and try to find a universal form for the coefficients.

In Chapter 9, we summarize our results for Part II.

Chapter 7

Approximate Methods to Solve Schroeder's Equations

7.1 Functional Equations and Schroeder's Equations

Mathematicians have been working with functional equations since the 14th century, though the strict discipline was built much later than that. Nowadays, functional equations form a modern branch of mathematics. Topics which are covered under functional equation include Cauchy equations, Jensen equations, Pexider's equations, Abel equations, and functional equations for distance measures[108]. Some advanced topics involve functional equations in abstract domains like semigroups, groups, and Banach spaces. When the focus of a function equation is on continuity of functions and a domain is specified, this becomes a question of topology, in particular this sometimes becomes questions about the group of homeomorphisms or diffeomorphisms of a set. In Part II, we do not go through the details of those common functional equations and their properties, but investigate a special functional equation with one independent variable, the Schroeder's equation.

7.1.1 Functional equations

In mathematics, a function equation is any equation that specifies a function in

implicit form[109]. Often the equation relates the value of a function at some points with its values at other points. In other words, functional equations are those in which a function is sought which is to satisfy certain relations among its values at all points. For function equations with one variable, they normally have the form like

$$F(x, f(x), f(G(x))) = 0, \tag{7.1}$$

where x is variable, $f(x)$ is the function we want to solve, and F and G are two given functions. Here we give the forms for some famous functional equations:

(1) $f(x+y) = f(x)f(y)$, satisfied by all exponential functions (7.2)

(2) $f(xy) = f(x) + f(y)$, satisfied by all algorithmic functions (7.3)

(3) $f(x+y) = f(x) + f(y)$, Cauchy equation (7.4)

(4) $f((x+y)/2) = (f(x) + f(y))/2$, Jensen equation (7.5)

(5) $f(h(x)) = f(x) + 1$, Abel equation (7.6)

(6) $f(x+y) + f(x-y) = 2f(x)f(y)$, d'Alembert equation (7.7)

In simple cases, a functional equation can be solved by introducing some substitutions to yield more information or additional equations. For example, if a functional equation has the form like

$$h(x)f(x) + g(x)f(a-x) = s(x), \tag{7.8}$$

where $f(x)$ is the function we want to solve; $h(x)$, $g(x)$, and $s(x)$ are known; a is a constant . Then by replacing x by $a-x$, we have

$$h(a-x)f(a-x) + g(a-x)f(x) = s(a-x). \tag{7.9}$$

Substituting this into (7.8) and solving for $f(x)$, we get the explicit form for $f(x)$.

However, like differential equations, solving functional equations for arbitrary

forms can be very difficult. Except for few well known cases, there does not exist a standard procedure to solve an arbitrary functional equation. For a general case, one may have to design an algorithm and use numerical methods.

7.1.2 Schroeder's equations

A Schroeder's equation[76] is a functional equation with a fixed point: given function $f(x)$, find function $\psi(x)$ such that

$$\psi(f(x)) = s\psi(x), \tag{7.10}$$

for some constant s. If x_0 is a fixed point of $f(x)$, meaning $f(x_0) = x_0$, then $\psi(x_0) = 0$ for some $s \neq 1$. Provided $\psi(x_0)$ is finite and $\psi'(x_0)$ does not vanish or diverge, the eigenvalue s is given by $s = f'(x_0)$. Let $\phi = \psi^{-1}$ be the conjugacy function of ψ, then we have a transpose form of Schroeder's equation

$$f(\phi(y)) = \phi(cy). \tag{7.11}$$

Schroeder's equation can be converted to other well known functional equations. By changing variable, let $a(x) = \log(\psi(x))/\log(s)$, Schroeder's equation becomes to Abel equation

$$a(h(x)) = a(x) + 1. \tag{7.12}$$

Similarly, the change of variables $\psi(x) = \log(\varphi(x))$ converts Schroeder's equation to Bottcher equation

$$\varphi(f(x)) = (\varphi(x))^s. \tag{7.13}$$

The solutions of Schroeder's equation depend on the form of the given function $f(x)$. In Schroeder's original paper[76], he gave a good number of particular

solutions. Koenigs[78] gave an analytical solution for Schroeder's equation if x_0 is an attracting fixed point, meaning $0 < |f'(x_0)| < 1$. In [110], Szekeres used the series expansion around a fixed point and studied the relevant convergence properties of the solution of the resulting orbit and analyticity properties. Here, we give two examples constructed in Schroeder's early paper[76]. The given function $f(x) = sx(1-x)$ in both examples are called logistic map, which is a polynomial mapping of degree 2.

Example 7.1, $f(x) = 2x(1-x)$. The solution is

$$\psi(x) = -\frac{1}{2}\ln(1-2x), \tag{7.14}$$

with its conjugate

$$\psi^{-1}(x) = \frac{1}{2}(1-e^{-2x}). \tag{7.15}$$

Example 7.2, $f(x) = 4x(1-x)$. The solution is

$$\psi(x) = (\arcsin\sqrt{x})^2, \tag{7.16}$$

with its conjugate

$$\psi^{-1}(x) = (\sin\sqrt{x})^2. \tag{7.17}$$

7.2 Conjugation and Evolution

One of the numerical ways solving Schroeder's equations is to use the iterating relations and generate the approximate solutions[80,81].

7.2.1 Iteration

First, let us consider an evolution trajectory $x(t)$ of a one-dimensional system, specified by a local, time-translation-invariant law

$$\frac{dx(t)}{dt} = v(x(t)). \tag{7.18}$$

By solving this differential equation, one can obtain the trajectory as a function of the time t and initial position $x(0)$

$$x(t) = f_t(x(0)). \tag{7.19}$$

Then we consider a series of time lattice with interval $\Delta t = 1$ between two consecutive points, so that

$$x(1) = f_1(x(0)), \tag{7.20}$$

$$x(t+1) = f_{t+1}(x(0)) = f_1(x(t)). \tag{7.21}$$

We have relation (7.21) because $x(t)$ can be considered as initial position of $x(t+1)$. It is straightforward to compute iterates of (7.20) at each time point $t = \cdots,-2,-1,0,1,2,3,\cdots$, which gives

$$\begin{aligned} x(2) &= f_1(f_1(x)) = f_2(x), \\ x(n) &= f_1(f_1 \cdots (f_1(x))) = f_n(x), \\ x(-1) &= f_1^{-1}(x) = f_{-1}(x). \end{aligned} \tag{7.22}$$

By assuming the domains for the various functions overlap properly, we have associative and commutative composition

$$\begin{aligned} x &= f_{-1}(f_1(x)) = f_1(f_{-1}(x)), \\ x(k+n) &= f_k(f_n(x)) = f_n(f_k(x)). \end{aligned} \tag{7.23}$$

Schroeder approached this problem by introducing auxiliary function $\psi(x)$ and solving Schroeder's equations. We take $x=0$ as a fixed point of $f_1(x)$, meaning $f_1(0)=0$. Construct an analytical $f_t(x)$ around it and the Schroeder's equation is

$$\psi(f_1(x)) = s\psi(x), \tag{7.24}$$

for some $s \neq 1$. At the fixed point, it gives $\psi(0)=0$, and if $\psi'(0) \neq 0$, then $s = f_1'(0)$. The inverse function satisfies Poincare's equation

$$\psi^{-1}(sx) = f_1(\psi^{-1}(x)). \tag{7.25}$$

Upon iteration of the functional equation, ψ acts upon the splinter of x to give

$$s^n\psi(x) = \psi(f_n(x)) = \psi(f_1(f_1 \cdots (f_1(x)))). \tag{7.26}$$

Then for arbitrary time t, Schroeder's equation becomes

$$s^t\psi(x) = \psi(f_t(x)), \tag{7.27}$$

and the trajectory can be expressed in terms of $\psi(x)$ and $\psi^{-1}(x)$

$$x(t) = f_t(x) = \psi^{-1}(s^t\psi(x)). \tag{7.28}$$

In a suitable domain, this gives the general iterate for any t (it can be fractional, negative, and infinitesimal), analytic around the fixed point $x=0$.

This solution manifestly satisfies the associative and abelian composition properties for all iterates and inverse iterates. That is to say, $f_{t_1+t_2}(x) = f_{t_1}(f_{t_2}(x))$, hence $x(t_1+t_2) = f_{t_1}(x(t_2))$. However, in the limit $s \to 1$, all iterates and inverse iterates lose their distinction and degenerate to the identity map, $f_0(x) = x$, and the method fails as written.

For a very elementary illustration of the technique, consider Schroeder's early example of a recursive evolution law,

$$f_1(x) = 2x(1+x)\,, \tag{7.29}$$

so $f_1(0) = 0$, and $s = f_1'(0) = 2$. Schroeder's equation is then solved by

$$\psi(x) = \frac{1}{2}\ln(1+2x)\,, \tag{7.30}$$

with its conjugate

$$\psi^{-1}(x) = \frac{1}{2}(e^{2x} - 1)\,, \tag{7.31}$$

giving

$$x(t) \equiv f_t(x) = \frac{1}{2}((1+2x)^{2^t} - 1)\,. \tag{7.32}$$

7.2.2 Approximate solution

In this subsection, we talk about how to construct approximate solution for functional equations[111]. As other numerical methods, we have to check the convergence behavior of the solutions. Functional equation (7.21) can be written as

$$x_t(x_k(x)) = x_k(x_t(x)) = x_t \circ x_k\,. \tag{7.33}$$

In this situation, a useful direct approach is to construct an N th-order formal series approximation for x_t around a fixed point of x_1, say at $x = 0$, by series analysis of the $k = 1$ case of (7.33), written as

$$x_t(x_1(x)) = x_1(x_t(x))\,. \tag{7.34}$$

With initial conditions corresponding to x_0 is identity map, the approximate solution has the form

$$x_t^{(Napprox)}(x) = \sum_{k=1}^{N} c_k(t) x^k \ , \tag{7.35}$$

where $c_1(0) = 1$, $c_{k>1}(0) = 0$, so $x_{t=0}^{(Napprox)}(x) = x$. Note that it is only necessary to construct accurate approximations to x_t for any unit interval in t, for then the composition rules $x_{1+t} = x_1 \circ x_t$ and $x_{-1+t} = x_{-1} \circ x_t$ can be used to reach higher or lower values of t.

Now, for a given N the series (7.35) may not produce accurate results for values of t, or for an interval of initial x, of interest. Quite possibly, this may be overcome if the series is convergent, by taking larger N, but if the series is only a formal one, perhaps asymptotic, with zero radius of convergence, larger N may not be a viable option. So what can be done then?

Formally, for any fixed N, with $x_n = \overbrace{x_1 \circ \cdots \circ x_1}^{n \text{ compositions}}$, it follows that

$$x_t = \lim_{n \to \pm\infty} x_n \circ x_t^{(Napprox)} \circ x_{-n} \tag{7.36}$$

is a solution to (7.34). This is a consequence of

$$x_1 \circ (x_n \circ x_t^{(Napprox)} \circ x_{-n}) = (x_{n+1} \circ x_t^{(Napprox)} \circ x_{-n-1}) \circ x_1 \, . \tag{7.37}$$

In the limit $n \to \pm\infty$, equation (7.34) is obtained, if either of those limits exist. For a specified class of problems it might be possible to give an elegant proof that either limit exists by using various fixed point theorems from functional analysis[112,113], but that is not our objective here.

The objective is to estimate the numerical accuracy obtained by conjugating (7.35)

a finite number of times, n, with the given, exactly known, 'finite step' function $x_{\pm 1}$. That is, our concern here is the relative error obtained as a function of n, prior to taking the limit $n \to \pm\infty$. In many cases, n-fold conjugation with the given x_1, and its inverse x_{-1}, dramatically improves the numerical accuracy of the series approximation, with the error vanishing exponentially in either $N \ln n$ or Nn. This behavior came to light in follow-up studies of earlier work[114,115,80,81]. We give two examples of how to construct approximate solutions and calculate relative errors in next two subsections.

7.2.3 A rational illustration

As a tractable example[76,91,110,116,117], consider

$$x_1(x) = \frac{x}{1-x}, \qquad x_{-1}(x) = \frac{x}{1+x}. \tag{7.38}$$

In this particular simple case, an exact solution to (7.33) is

$$x_t(x) = \frac{x}{1-xt}. \tag{7.39}$$

But to illustrate the series method, consider (7.34) given only x_1 in (7.38). Recursive analysis in powers of x immediately gives $c_k(t) = t^{k-1}$, where we have defined the form and scale of the t parameterization by the choice $c_2(t) = t$, to be in agreement with (7.39). For instance, with $x_t(x) = x + tx^2 + c_3(t)x^3 + O(x^4)$ we find

$$\frac{x_t(x)}{1-x_t(x)} - x_t\left(\frac{x}{1-x}\right) = (t^2 - c_3(t))x^4 + O(x^5). \tag{7.40}$$

Thus, (7.34) is satisfied if and only if $c_3(t) = t$. So it goes to higher orders in x, with each c_{k+1} determined by $c_{j\le k}$ to satisfy (7.34) —not just by direct expansion of

(7.39).

The result for the approximation is therefore

$$x_t^{(Napprox)}(x) = \sum_{k=1}^{N} t^{k-1} x^k = \frac{(1 - t^N x^N)x}{1 - tx}. \tag{7.41}$$

Then, by n-fold conjugation of this approximation with x_1 and its inverse, obtain

$$x_n \circ x_t^{(Napprox)} \circ x_{-n}(x) = x(1 - \left(\frac{tx}{1+nx}\right)^N)/(1 - tx + nx\left(\frac{tx}{1+nx}\right)^N). \tag{7.42}$$

This indeed converges to the exact result, (7.39), as $n \to \infty$ for any fixed $N > 1$. But what is the relative error for finite n?

Since we know the exact answer in this elementary case, this error is not difficult to compute. For any fixed N and finite n the relative error is

$$R_t(x, N, n) \overset{\text{defin.}}{=} \frac{x_t(x) - x_n \circ x_t^{(Napprox)} \circ x_{-n}(x)}{x_t(x)} = \frac{1 - tx + nx}{1 - tx + nx\left(\frac{tx}{1+nx}\right)^N}\left(\frac{tx}{1+nx}\right)^N. \tag{7.43}$$

For $N > 1$ this indeed vanishes as $n \to \infty$, for any fixed x, so long as $tx \neq 1$. However, more importantly, for n finite but large compared to both t and $1/x$, this error goes to zero as the $(N-1)$ st power of n,

$$R_t(x, N, n) \underset{n >> |t|, |1/x|}{\approx} \frac{t^N x}{1 - tx}\frac{1}{n^{N-1}}. \tag{7.44}$$

Therefore, in principle, one can obtain numerical results to any desired accuracy by repeated conjugation of the approximate series with the given step-function x_1. It is remarkable that it is only required to take any fixed $N \geq 2$ to produce such results, although in practice, as is manifested in (7.44), computational efficiencies can be

improved by taking larger N since the desired accuracy is then reached for smaller n.

The result for the relative error can be understood in terms of Schroeder theory[76] as it applies to this simple example. The continuous iterate x_t given in (7.39) may be constructed from a Schroeder function, ψ, and its inverse. An exact result for this particular example is given by

$$\psi(x) = \exp(-1/x), \qquad \psi^{-1}(x) = -1/\ln(x). \tag{7.45}$$

This result is for a particular choice (namely, $s = e$) of the 'multiplier' in Schroeder's equation, $s\psi(x) = \psi(x_1(x))$. The multiplier is undetermined when $d^n\psi(x)/dx^n = \exp(-1/x)|_{x=0}$ is zero, or undefined, for all n. In such cases, it is usually possible to choose s just as a matter of taste. In Schroeder's conjugacy framework, the general iterate is $x_t(x) = \psi^{-1}(e^t\psi(x))$, or,

$$\psi(x_t(x)) = e^t\psi(x). \tag{7.46}$$

Thus, $\omega \equiv \psi(x)$ is just the change of variable that reduces the effect of evolution in t to nothing but a multiplicative rescaling, $\omega_t = e^t\omega$.

But suppose that the exact x_t were supplanted by an approximation of the form

$$x_t^{(Napprox)} = x_t + \alpha x_t^{N+1} + O(x_t^{N+2}), \tag{7.47}$$

for some coefficient α. Then,

$$\psi(x_t^{(Napprox)}) = (1 + \alpha x_t^{N-1} + O(x_t^N))\exp(-1/x_t). \tag{7.48}$$

Alternatively, with $x_t = -1/\ln\psi(x_t) = -1/\ln\omega_t$,

$$\omega_t^{(Napprox)} \equiv \psi(x_t^{(Napprox)}[\omega_t]) = \left(1 + \frac{\alpha}{(-\ln \omega_t)^{N-1}} + O\left(\frac{1}{(-\ln \omega_t)^N}\right)\right)\omega_t . \quad (7.49)$$

Then, it follows from the multiplicative rescaling behavior of ω that under the variable change $x_t \to \omega_t$ the conjugated approximation $x_n \circ x_t^{approx} \circ x_{-n}$ presents itself as

$$\omega_n \circ \omega_t^{approx} \circ \omega_{-n} = \left(1 + \frac{\alpha}{(n - \ln \omega_t)^{N-1}} + O\left(\frac{1}{(n - \ln \omega_t)^N}\right)\right)\omega_t . \quad (7.50)$$

This gives a relative error with the same power-law asymptotics as (7.44), namely, $R \underset{n\to\infty}{\sim} \alpha / n^{N-1}$.

7.2.4 Sine function

We construct the approximate solution $\sin_t^{(approx)}(x)$ for $\sin_t(x)$ with $\sin_1(x) = \sin(x)$ and $\sin_{-1}(x) = \arcsin(x)$. In this notation, the Abelian functional composition equation is

$$\sin_t \circ \sin_s = \sin_{t+s} . \quad (7.51)$$

Specializing to $s = 1$, written as

$$\sin_t(\sin(x)) = \sin(\sin_t(x)) , \quad (7.52)$$

we find a formal series solution,

$$\sin_t^{(approx)}(x) = x - \frac{1}{3!}tx^3 + \frac{(5t-4)t}{5!}x^5 - \frac{(175t^2 - 336t + 164)t}{3\times 7!}x^7 + \frac{(25t-24)(8-7t)^2 t}{9!}x^9 + O(x^{11}) . \quad (7.53)$$

This is the approximate solution up to $O(x^{11})$, and we can calculate the solution up to arbitrary order. It is not obtained by just taking Taylor expansion of both hand sides of

(7.52). Assume the approximate solution has the form of

$$\sin_t^{(Napprox)}(x) = \sum_{k=1}^{N} c_k(t)x^k \,, \tag{7.54}$$

and by Taylor expansion of $\sin(x)$, we can predict that the series solution does not contain even order terms, then we assume

$$\sin_t^{(3approx)}(x) = x - c_3 t x^3 \,, \tag{7.55}$$

for the approximation up to 3 rd order. Here c_3 is the constant that we are trying to determine through (7.52). Plug (7.55) to (7.52), we have

$$\sin_t^{(3approx)}(\sin(x)) - \sin(\sin_t^{(3approx)}(x)) = O(x^7) \,. \tag{7.56}$$

For the left hand side of (7.56), take Taylor expansion of $\sin(x)$, then we have

$$O(x^7) = O(x^7) \,. \tag{7.57}$$

This means, no matter what kind of c_3 we have, approximate solution always satisfies (7.52) up to 7 th order. We choose $c_3 = 1/3!$, and have $\sin_t^{(5approx)}(x)$ as

$$\sin_t^{(5approx)}(x) = x - \frac{1}{3!} t x^3 + c_5 x^5 \,. \tag{7.58}$$

Then (7.52) would yield

$$\sin_t^{(5approx)}(\sin(x)) - \sin(\sin_t^{(5approx)}(x)) = O(x^9) \,, \tag{7.59}$$

and Taylor expansion would give

$$\frac{1}{360}(120c_5 + 4t - 5t^2)x^7 + O(x^9) = O(x^9) \,. \tag{7.60}$$

The only condition that we can make (7.60) true is to take $120c_5 + 4t - 5t^2 = 0$, and it gives $c_5 = (5t - 4)t/5!$. By repeating the same process, we can obtain c_7 and c_9 as listed in (7.53). This is how we find series solution for (7.52).

In this case, the series (7.53) is probably not convergent for almost all t [110,116,118]. Although it obviously is convergent for $t \in -$. Rather, for generic t the series appears to be asymptotic. For example, for $t = 1/2$, using Mathematica to extend the series(7.53) to $O(x^{81})$ or so, the smooth behavior of the series coefficients c_k for $k \le 61$ suggests a finite radius of convergence $\approx 4/3$, as estimated by $1/|c_k|^{1/k}$. But then less smooth behavior sets in for $c_{k>61}$ and numerical estimated of the radius begin to fall toward zero, as would be expected for an asymptotic series.

Also note in passing that (7.53) implies the corresponding Schroeder function has an essential singularity at $x = 0$, as given explicitly by

$$\psi(x) \overset{x\to 0}{\sim} x^{6/5} \exp\left(\frac{3}{x^2} + \frac{79}{1050} x^2 + \frac{29}{2625} x^4 + O(x^5) \right), \tag{7.61}$$

to be compared to (7.45). This follows from $\psi(x) = \exp(\int dx / v(x))$, using $v(x) = d \sin_t(x)/dt\,|_{t=0} \approx d \sin_t^{(approx)}(x)/dt\,|_{t=0}$.

Nonetheless, the conjugation method produces approximations $\sin_t \approx A_{n,t}$ which provide compelling numerical evidence for the existence of a limit, hence an exact result for $\sin_t$, as $n \to \infty$. But note it is important in this case to take the conjugations to be of the form

$$A_{n,t} = \sin_{-n} \circ \sin_t^{(approx)} \circ \sin_n , \tag{7.62}$$

The general rule being to act first with functions that are smaller in magnitude than the identity map, thereby leading to evaluation of the truncated series at points closer

to $x = 0$.

The improvement wrought by conjugation of the truncated ninth-order series in (7.53) is easily seen in following graphs, for the case $t = 1/2$. The series itself is not credible beyond $x = \pi/2$, but a single conjugation forces the expected periodicity in x and gives the more plausible green curve seen in Figure 7.1. Repeated conjugation does not produce any discernible differences with the green curve to the accuracy of Figure 7.1, but when the graph is magnified, as it is in Figure 7.2, numerical evidence for convergence of the sequence of conjugations is quite compelling.

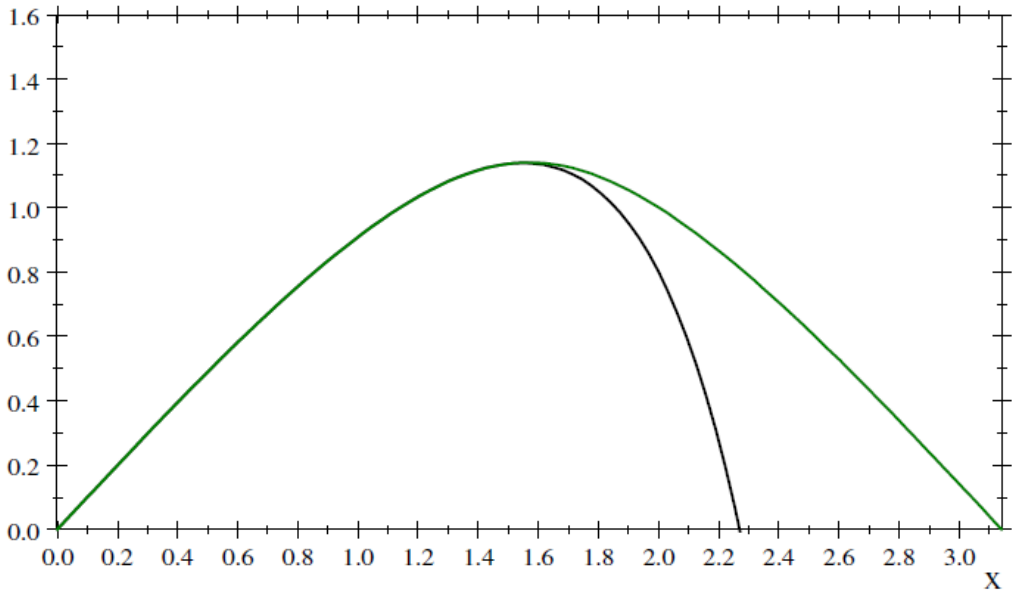


Figure 7.1. The ninth-order series (7.53) for $t = 1/2$, in black, along with its $n = 1$ conjugation (7.62), in green

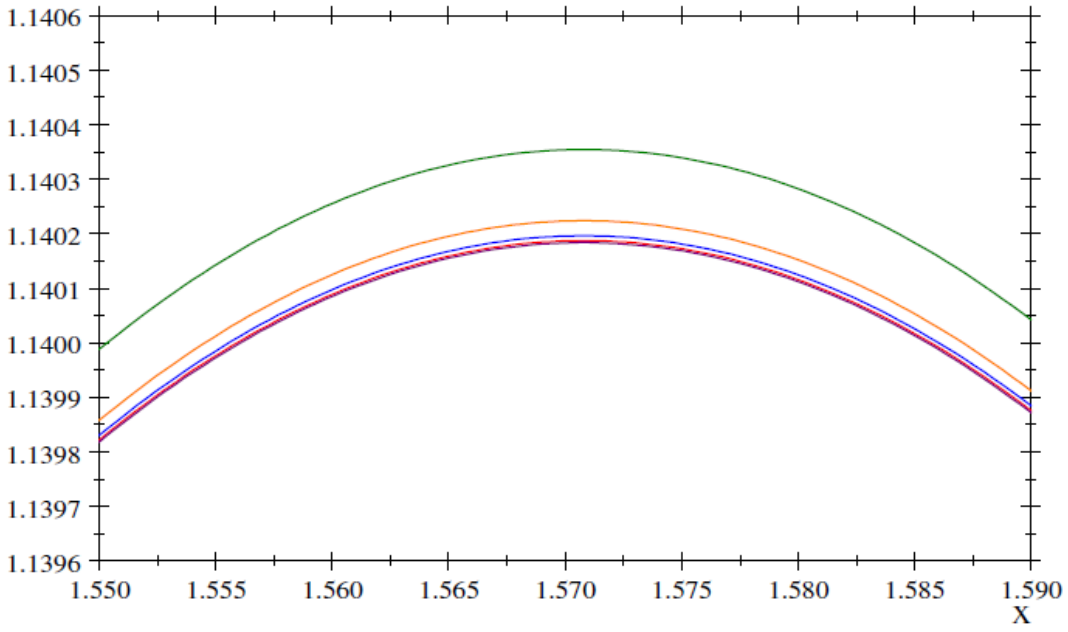


Figure 7.2. The $t = 1/2$, $n = 1,2,3,4$ and 5-fold conjugation (7.62), in green, orange, blue, red, and purple, respectively

Proceeding in this way leads to the set of curves for various values of t shown in Figure 7.3. Each of these curves results from the 5-fold conjugation $A_{5,t}$ of the truncated ninth-order series (7.53). Note once again, as previously remarked in a general context, it is only necessary to construct accurate approximations to $\sin_t$ for any unit interval in t, for then the composition rules $\sin_{1+t} = \sin \circ \sin_t$ and $\sin_{-1+t} = \arcsin \circ \sin_t$ can be used to construct the curves for higher or lower values of t. Also note from the numerics the obvious inference that $\sin_t(x)$ becomes the periodic triangular 'sawtooth' function as $t \to 0$, with

$$\lim_{t \to 0} \sin_t((2k+1)\pi/2) = (-1)^k \pi/2 . \tag{7.63}$$

As constructed, $\sin_t(x)$ for $t \geq 0$ is guaranteed to be real for all real values of

x, but certainly it is not obvious for generic t what numerical values are actually attained at the extrema for $x = \pi/2 \bmod \pi$. It suffices here for us to point out that the maxima are approximated by the simple expression

$$\sin_t(\pi/2) \approx \left(\frac{\pi}{2}\right)^{1-\sqrt{t}}. \tag{7.64}$$

At least, this is true for $0 \le t \le 1$, where the relative error between the exact (numerical) value of $\sin_t(\pi/2)$ and this approximation is less than about 3 parts per mille. The branch point at $t = 0$ exhibited in this approximate expression is perhaps the most direct numerical evidence that the iterates are not analytic in t.

The graph in Figure 7.2 give a sense of the overall relative error, but lacking closed-form expressions for either the iterates or the conjugations of the series approximations, closed-form results for the error are not available for generic t. For $t \in Z$, however, precise calculation is indeed possible since both exact results and convergent series are known. It suffices here to consider just one exact case, $t = 1$. Defining the relative error as before,

$$R_1(x,n) = \frac{\sin x - \sin_{-n} \circ \sin_1^{(approx)} \circ \sin_n(x)}{\sin x}, \tag{7.65}$$

we have computed numerically the error involved in various conjugations of the ninth-order series (see Figure 7.4). As previously remarked, conjugation by the sine guarantees that the approximations are periodic. The results are shown here.

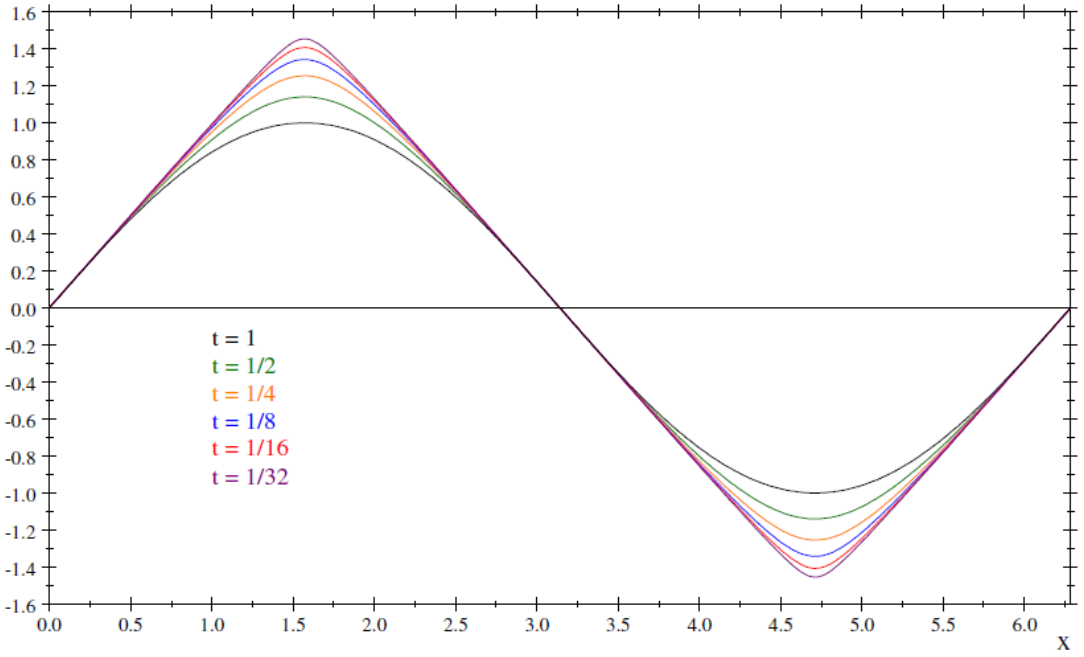


Figure 7.3. Various $\sin_t(x)$ **are given by 5-fold conjugation** $A_{5,t}(x)$ **of the Ninth-order series (7.53), for** $t \le 1$

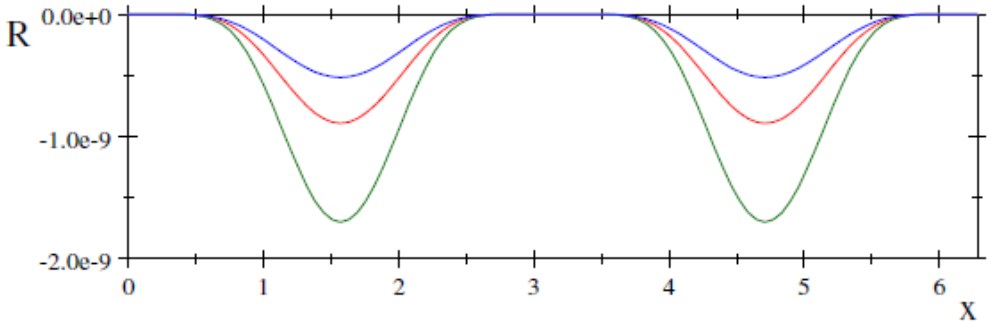


Figure 7.4. $R_1(x,n)$ **for** $n = 4,5$ **and** 6 **, in green, red and blue, respectively**

It would be interesting to compute relative errors for other, generic t, but at this stage it is only possible for us to compute the relative successive differences,

$$S_t(x,n) = \frac{\sin_{-n} \circ \sin_t^{(approx)} \circ \sin_n(x) - \sin_{-n+1} \circ \sin_t^{(approx)} \circ \sin_{n-1}(x)}{\sin_{-n} \circ \sin_t^{(approx)} \circ \sin_n(x)} . \qquad (7.66)$$

For example, consider $t = 1/2$ using the ninth-order series (see Figure 7.5)

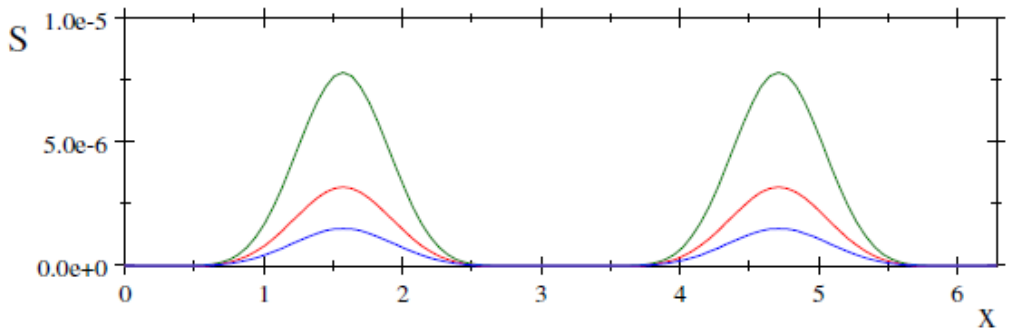


Figure 7.5. $S_{1/2}(x,n)$ **for** $n = 4,5$ **and** 6 **, in green, red and blue, respectively**

7.3 Approximate Solutions to Logistic Maps

As mentioned in last section, logistic maps are first examples studied by Schroeder in his early paper[76], they are also of special interests in Chaos theory. In general, the logistic map is defined as

$$x(x) = sx(1-x), \tag{7.67}$$

where s is scale constant. If we consider

$$x_1(x) = sx(1-x), \tag{7.68}$$

then (7.34) becomes

$$x_t(sx(1-x)) = sx_t(1-x_t), \tag{7.69}$$

with $x = 0$ as its fixed point. The exact solutions to Schroeder's equation (7.10) for $s = 2,4$ were given by Schroeder[76], and they provide us the opportunity to

calculate the relative error of our approximation solutions.

7.3.1 Theory of relative error

The sequence of conjugations converges more rapidly in situations where the underlying Schroeder function is analytic about the fixed point, in contrast to (7.45) and (7.61). Instead of power-law behavior, for such situations the relative error vanishes exponentially in n, the number of conjugations. The general theory is well illustrated by the logistic map (7.67), for $0 \le x \le s/4$ and $0 \le s \le 4$. The result of the theory is as follows

Theorem 7.1. Relative error after n-fold conjugation of the truncated series (7.35) is given by

$$\begin{aligned} R_t(x,s,N,n) &= \frac{x_t(x) - x_n(x_t^{(Napprox)}(x_{-n}(x)))}{x_t(x)} \\ &= \left(\frac{1}{s}\right)^{nN} r(x,t,s,N,n) \\ &= \left(\frac{1}{s}\right)^{nN} \lambda^{-t} c_{N+1}(t,s) x^N + O(x^{N+1}) \end{aligned} \quad . \tag{7.70}$$

Proof: To sketch a proof, and to see more clearly the assumptions involved, as well as to obtain expressions for $r(x,t,s,N,n)$, write the truncated series as

$$x_t^{(Napprox)}(x) = x_t(x) - x^{N+1}\delta_N(x,t), \tag{7.71}$$

Where $\delta_N(x,t)$ represents the exact difference, whose expansion in x begins $O(x^0)$. Thus, the conjugation gives exactly

$$x_n(x_t^{(Napprox)}(x_{-n}(x))) = x_n(x_t(x_{-n}(x)) - (x_{-n}(x))^{N+1}\delta_N(x_{-n}(x),t)). \tag{7.72}$$

Now expand the RHS in powers of $(x_{-n})^{N+1}\delta_N$,

$$x_n(x_t(x_{-n}(x)) - (x_{-n}(x))^{N+1}\delta_N(x_{-n}(x),t)) = x_n(x_t(x_{-n}(x))) \\ -(x_{-n}(x))^{N+1}\delta_N(x_{-n}(x),t)x'_n(x_t(x_{-n}(x))) + O((x_{-n}^2)^{N+1}\delta_N^2 . \quad (7.73)$$

Since it consists of exact trajectories which obey (7.33), the first term gives $x_n(x_t(x_{-n}(x))) = x_t(x)$, while the second term involves

$$x'_n(x_t(x_{-n}(x))) = \frac{1}{\frac{d}{dx}x_t(x_{-n}(x))}\frac{d}{dx}x_n(x_t(x_{-n}(x))) = \frac{1}{\frac{d}{dx}x_{t-n}(x)}\frac{d}{dx}x_t(x), \quad (7.74)$$

again using (7.33). Thus,

$$x_t(x) - x_t^{(Napprox[n])}(x) = (x_{-n}(x))^{N+1}\delta_N(x_{-n}(x),t)\frac{dx_t(x)/dx}{dx_{t-x}(x)/dx} + O((x_{-n}^2)^{N+1}\delta_N^2). \quad (7.75)$$

To proceed, we require that the unit step function is such that $x_{-n}(x)$ flows toward the fixed point at the origin for the point x under consideration (so we suppose $|s|>1$, but if not, just interchange x_1 and x_{-1}). We also suppose that n has been chosen large compared to t so that

$$x_{t-n}(x) \equiv s^{t-n}\varepsilon_n(x,t) \quad (7.75)$$

is small, where $\varepsilon_n(x,t) = x + O(x^2)$. If these requirements are met, then

$$x_t(x) - x_t^{(Napprox[n])}(x) = s^{-n(N+1)}\varepsilon_n^{N+1}(x,0)\delta_N(s^{-n}\varepsilon_n(x,0),t)\frac{dx_t(x)/dx}{s^{t-n}d\varepsilon_n(x,t)/dx}, \\ +O[s^{-2n(N+1)}\varepsilon_n^{2N+2}\delta_N^2] \quad (7.76)$$

where $d\varepsilon_n(x,t)/dx = 1 + O(x)$. The result for the relative error is therefore of the form in the statement of the theorem, with

$$r(x,t,s,N,n) = \frac{\varepsilon_n^{N+1}(x,0)\delta_N(s^{-n}\varepsilon_n(x,0),t)}{s^t d\varepsilon_n(x,0)/dx}\frac{dx_t(x)/dx}{x_t(x)} + O[s^{-n(N+2)}\varepsilon_n^{2N+2}\delta_N^2]. \quad (7.77)$$

For small x, we have

$$\varepsilon_n^{N+1}(x,0) = x^{N+1}(1+O(x)), \qquad d\varepsilon_n(x,t)/dx = 1 + O(x),$$

$$\delta_N(s^{-n}\varepsilon_n(x,0),t) = c_{N+1}(t,s) + O(x)\,, \qquad \frac{dx_t(x)/dx}{x_t(x)} = \frac{1}{x}(1+O(x))\,, \tag{7.78}$$

and therefore

$$r(x,t,s,N,n) = s^{-t}c_{N+1}(t,s)x^N + O(x^{N+1})\,, \tag{7.79}$$

again as previously stated.

The form given in (7.77) enables analysis of the size of the error as a function of x. Moreover, the form in (7.79) suggests an approximate scaling law for the error

$$R_t(x,s,N,n) \approx s^N R_t(x,s,N,n+1)\,, \tag{7.80}$$

at least for x near the origin. It is interesting to check whether this is true for larger x. In fact, it is.

7.3.2 $s = 2$ logistic map

Consider the case $s = 2$, which can be solved in closed form to obtain

$$\psi(x) = -\frac{1}{2}\ln(1-2x)\,, \qquad \psi^{-1}(x) = -\frac{1}{2}(1-e^{-2x})\,, \tag{7.81}$$

$$x_t(x) = \frac{1}{2}(1-(1-2x)^{2^t})\,, \qquad \frac{dx_t(x)}{dx} = 2^t(1-2x)^{-1+2^t}\,, \tag{7.82}$$

$$\varepsilon_n(x,t) = 2^{n-t-1}(1-(1-2x)^{2^{-n}})\,, \qquad \frac{d\varepsilon_n(x,t)}{dx} = (1-2x)^{-1+2^{t-n}}\,, \tag{7.83}$$

as well as

$$\varepsilon_n^{N+1}(x,0) = (2^n\frac{1}{2}(1-(1-2x)^{2^{-n}}))^{N+1}\,. \tag{7.84}$$

The approximation solution can be obtained by solving (7.34) recursively for $x_1(x) = 2x(1-x)$ up to $O(x^{N+1})$

$$x_t^{(Napprox)}(x) = (2^t x + \sum_{k=2}^{N} c_k(t))x^k \,, \tag{7.85}$$

where the first several coefficients $c_k(t)$ are

$$c_2(t) = -2^t(2^t - 1)\,, \qquad c_3(t) = \frac{2}{3}2^t(2^t - 1)(2^t - 2)\,,$$

$$c_4(t) = -\frac{1}{3}2^t(2^t - 1)(2^t - 2)(2^t - 3)\,, \quad c_5(t) = \frac{2}{15}2^t(2^t - 1)(2^t - 2)(2^t - 3)(2^t - 4)\,,$$

$$c_6(t) = -\frac{2}{45}2^t(2^t - 1)(2^t - 2)(2^t - 3)(2^t - 4)(2^t - 5)\,. \tag{7.86}$$

The general form for coefficients is

$$c_k(t) = \frac{(-2)^{k-1}}{k!}\prod_{j=0}^{k-1}(2^t - j)\,, \tag{7.87}$$

and fortunately, we can express the sum of the term after first N terms in terms of hypergeometric functions, which give us the δ_N by

$$\delta_N(x,t) = \left(\frac{(-2)^N}{(N+1)!}\prod_{j=0}^{N}(2^t - j)\right) \times \text{hypergeom}([1, N+1-2^t],[2+N],2x)\,, \tag{7.88}$$

with the definition

$$x_t^{(Napprox)}(x) = x_t(x) - x^{N+1}\delta_N(x,t)\,. \tag{7.89}$$

These $s = 2$ results are all well behaved enough for the steps in the proof of the theorem to be valid for $x \le 1/2$. However, at the upper end of the interval, $0 \le x \le 1/2$, some additional consideration is needed. Expression (7.75) is $1/2$ at $x = 1/2$, independent of t and n, and therefore not small. That is to say, at $x = 1/2$ the 2^{-nN} prefactor in $R_t(x,s,N,n)$ is not present to suppress the error. On the other hand, the ratio $(dx_t(x)/dx)/(d\varepsilon_n(x,t)/dx)$ in (7.76), and in (7.77), always vanishes at $x = 1/2$, for any t, so the leading contribution to the relative error is actually zero at

that point. Thus, the upper end of the x interval does not pose a problem after all. In fact, as is evident in the graphs to follow, or by direct calculation, the maximum magnitude of the leading contribution to R occurs for $x \le 1/2$, for which the 2^{-nN} prefactor is present.

Putting all this together for the $s = 2$ logistic map, the leading approximation to the relative error is

$$R_t(x,2,N,n) \approx \delta_N\left(\frac{1}{2}\left(1-(1-2x)^{2^{-n}}\right),t\right) \times \frac{2^{n-N}\left(1-(1-2x)^{2^{-n}}\right)^{N+1} \times (1-2x)^{2^t(1-2^{-n})}}{\left(1-(1-2x)^{2^t}\right)} . \tag{7.90}$$

For comparison purpose, we plot (in Figure 7.6) this last expression for various N and n, and selected t, especially to check (7.80) for $s = 2$. That scaling law is seen to be hold fairly well, even for $x \approx 1/2$.

Finally, we note there is no discernible difference, to the accuracy of this last plot, between the leading approximations and the exact results for the relative error as computed from (7.82) and the 5-, 6- and 7-fold conjugations of $x_t^{(5approx)}(x)$. The largest of these differences, between the exact and the leading approximation of the relative error for $N = 5, n = 5$, is shown in Figure 7.7. So the exact relative errors are essentially indistinguishable from their leading approximations, at least for $s = 2$ and $t = 1/2$ and $t = 3/4$, and the relative error after 7 conjugations of the fifth-order series is always less than 3 parts in 10^{12} for these two values of t. Other values of t taken from the unit interval $[0,1]$ are similarly well approximated by the combined series and conjugation methods.

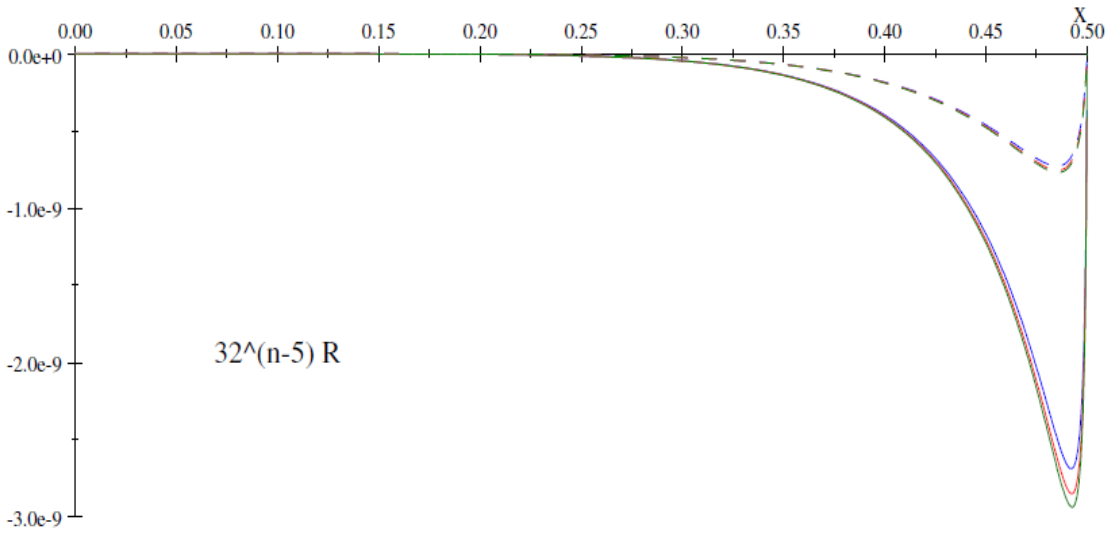


Figure 7.6. Leading approximations to $2^{5(n-5)} R_t(x, s=2, N=5, n)$ for $t=1/2$ (solid) and $t=3/4$ (dashed) with $n=5,6$ and 7, in blue, red and green, respectively

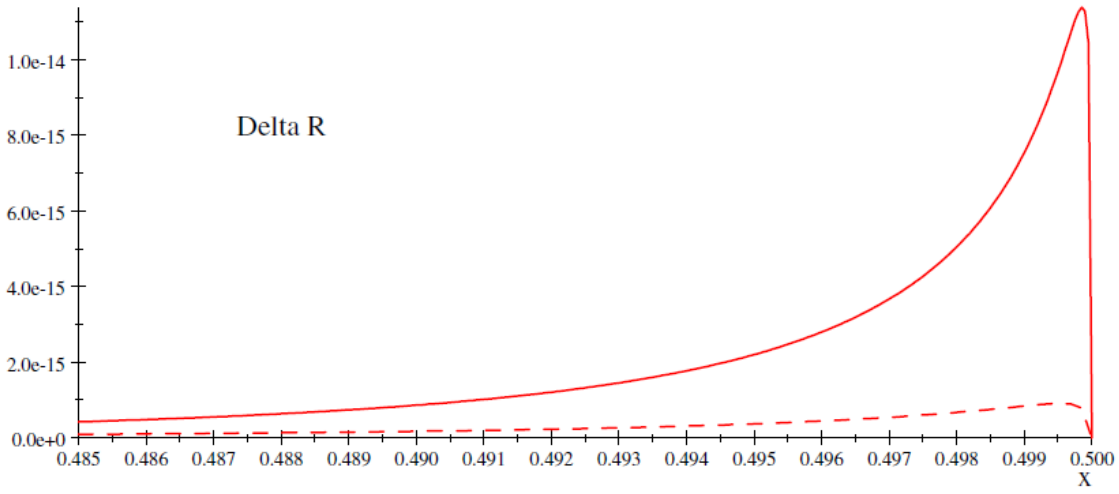


Figure 7.7. $\Delta R = R_t(x, s=2, N=5, n=5)\big|_{\text{exact}} - R_t(x, s=2, N=5, n=5)\big|_{\text{leading approx}}$ **for $t=1/2$ (solid) and $t=3/4$ (dashed)**

7.3.3 $s=4$ logistic map

$s=4$ case is similar to $s=2$ case, we use the same procedures, and it can be solved in closed form also.

$$x \mapsto x_1(x) = 4x(1-x), \tag{7.91}$$

$$\psi(x) = (\arcsin\sqrt{x})^2, \qquad \psi^{-1}(x) = (\sin\sqrt{x})^2, \tag{7.92}$$

$$x_t(x) = (\sin(2^t \arcsin\sqrt{x}))^2, \qquad \frac{dx_t(x)}{dx} = 2^t \frac{1}{\sqrt{1-x}} \sin(2^{t+1} \arcsin\sqrt{x}), \tag{7.93}$$

$$\varepsilon_n(x) = 4^{n-t}(\sin(2^{t-n} \arcsin\sqrt{x}))^2, \quad \frac{d\varepsilon_n(x)}{dx} = 2^{n-t} \frac{1}{\sqrt{1-x}} \sin(2^{n-t+1} \arcsin\sqrt{x}). \tag{7.94}$$

Let $z \equiv 2^t$, then the expansion of $x_t(x)$ is

$$(\sin(z \arcsin\sqrt{x}))^2 = \sum_{n=1}^{\infty} c(n,z) x^n, \tag{7.95}$$

with

$$c(n,z) = 2\frac{(-4)^{n-1}}{(2n)!}\left(\prod_{k=0}^{n-1}(z^2-k^2)\right). \tag{7.96}$$

Again, $x_t(x)$ can be expressed by using hypergeometric function

$$\begin{aligned}(\sin(z \arcsin\sqrt{x}))^2 = {} & 2\sum_{n=1}^{N} \frac{(-4)^{n-1}}{(2n)!}\left(\prod_{k=0}^{n-1}(z^2-k^2)\right)x^n \\ & + 2\frac{4^N \Gamma(N+1-z)\Gamma(N+1+z)}{\Gamma(2N+3)} \frac{z\sin\pi z}{\pi} x^{N+1} \\ & \times \text{hypergeom}([1, N+1+z, N+1-z], [2+N, N+\frac{3}{2}], x)\end{aligned}, \tag{7.97}$$

and

$$\begin{aligned}\delta_N(x,t) = {} & \left(-2\frac{4^N}{(2N+2)!}\prod_{j=0}^{N}(j^2-z^2)\right)\times \\ & \text{hypergeom}([1, N+1-z, N+1+z], [2+N, N+\frac{3}{2}], x)\end{aligned}. \tag{7.98}$$

The leading approximation to the relative error is calculated from (7.70), and Figure 7.8 shows the plot for $N=5$ with 3-, 4- and 5-fold conjugations of $x_t^{(5approx)}(x)$ when $t=1/2$. One can see that the difference is very small between different conjugations. Note that this Figure 7.8 shows a sudden dip (actually, a downward spike, if resolved in more details) at $x\approx 0.8$ as a consequence of R having a zero and changing sign at that point. Also in this Figure, we draw a light blue curve, which is the relative error for small x formula

$$R_t(x,s,N,n)\approx\left(\frac{1}{s}\right)^{nN} s^{-t}c_{N+1}(t,s)x^N, \tag{7.99}$$

so for $s=4$ map,

$$R_t(x,s=4,N,n)\approx 4^{-t-nN}\frac{2(-4x)^N}{(2N+2)!}\left(\prod_{k=0}^{N}(4^t-k^2)\right). \tag{7.100}$$

For $x\le 0.7$, it agrees with other curves very well, meaning it can substitute exact relative error at small x.

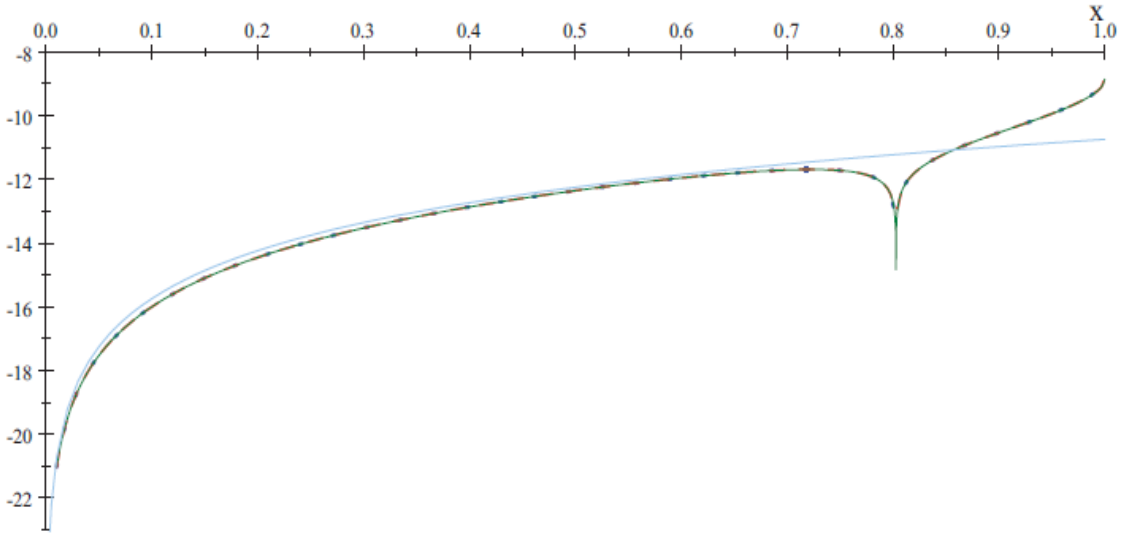


Figure 7.8. Leading approximations to $\log_{10}(4^{5(n-3)}\,|\,R_t(x,s=4,N=5,n)\,|)$ for $t=1/2$ with $n=3,4$ and 5, in blue, red, and green, respectively

7.4 Discussion

From last section, we see that the approximate solution we obtained agree with closed form solutions very well, and we believe series and conjugation method is a reliable way to solve (7.34). With series and conjugation method, we can calculate approximate solutions of logistic maps with general s, or other functions. Also, since $x_t(x)$ gives the trajectories as functions of both time t and position x, one would be interested in first, the trajectory surfaces of the motion; and second, the velocities and potentials based on these trajectories. The first one can be drawn by plotting three-dimensional surfaces, and one can calculate the points with maximum or minimum curvatures. As to the second, one can use approximate method to obtain approximate velocities and potentials directly, which we talk about in next chapter.

7.4.1 Logistic maps with general s

Our interests in logistic maps with other s are those for $0 < s < 4$. For each of them we put restriction $0 \le x \le s/4$, because we want both x_1 and x_{-1} be real. So far, for logistic maps with s other than 2 and 4, no one has solved them with a closed form, therefore the series and conjugation method seems the only way to approach them. For general s, finding the series approximation of solution to (7.34) is not easy, because series method is based on correctly predicting first two or three terms, and it is hard to do it for general s. Instead of solving (7.34) directly, we use Poincare's equation (7.11) to find the recursion relation between coefficients.

For any s consider a power series

$$\psi^{-1}(x,s) = x + x\sum_{n=1}^{\infty} x^n c_n(s)\,. \tag{7.101}$$

Then the Poincare equation leads to a recursion relation for the s-dependent coefficients.

$$c_{n+1}(s) = \frac{1}{1-s^{n+1}} \sum_{j=0}^{n} c_j c_{n-j}\,, \tag{7.102}$$

with $c_0(s) = 1$, $c_1(s) = \dfrac{1}{1-s}$, $c_2(s) = \dfrac{2}{(1-s)(1-s^2)}$, etc. The explicit coefficients are easily recognized for $s = 2$ and 4, and immediately yield the two closed-form cases. Similarly, let

$$\psi(x,s) = x + x\sum_{n=1}^{\infty} (-x)^n d_n(s)\,. \tag{7.103}$$

Then, as a consequence of Schroeder's equation, $d_1(s) = 1/(1-s)$, and for $n \geq 2$,

$$d_n(s) = \frac{1}{1-s^n} \sum_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \binom{n+1-k}{k} s^{n-k} d_{n-k}\,. \tag{7.104}$$

where $\lfloor \cdots \rfloor$ denotes the integer-valued floor function. In principle, these series solve (7.10) for any s, within their radii of convergence.

From extensive numerical studies, we believe the radius of convergence for (7.103) depends on s as follows

$$R_\psi = \begin{cases} \left|1 - \dfrac{1}{s}\right| & \text{if } 0 \leq s \leq 2 \\ \dfrac{s}{4} & \text{if } 2 \leq s \leq 4 \end{cases}. \tag{7.105}$$

The first few terms for ψ and ψ^{-1} for generic s are given explicitly by

$$\psi(x,s) = x + \frac{x^2}{s-1} + \frac{2s}{(s+1)}\frac{x^3}{(s-1)^2} + \frac{s(1+5s^2)}{(s^2+s+1)(s+1)}\frac{x^4}{(s-1)^3} + \frac{2s^3(3+2s+7s^3)}{(s^2+1)(s^2+s+1)(s+1)^2}\frac{x^5}{(s-1)^4} + \frac{2s^3(1+3s+14s^3+14s^4+7s^5+21s^7)}{(s^4+s^3+s^2+s+1)(s^2+1)(s^2+s+1)(s+1)^2}\frac{x^6}{(s-1)^5} + O(x^7) \quad ,(7.106)$$

$$\psi^{-1}(x,s) = x + \frac{x^2}{s-1} + \frac{2}{(s+1)}\frac{x^3}{(s-1)^2} + \frac{5+s}{(s^2+s+1)(s+1)}\frac{x^4}{(s-1)^3} + \frac{2(2s^2+3s+7)}{(s^2+1)(s^2+s+1)(s+1)^2}\frac{x^5}{(s-1)^4} + \frac{2(3s^4+8s^3+14s^2+14s+21)}{(s^4+s^3+s^2+s+1)(s^2+1)(s^2+s+1)(s+1)^2}\frac{x^6}{(s-1)^5} + O(x^7) \quad , \quad (7.107)$$

From which we infer that $\psi(x,s)/x$ and $\psi^{-1}(x,s)/x$ are actually series in $x/(1-s)$ with s-dependent coefficients that are analytic near $s=1$.

The trajectories interpolating the splinter (integer t) of the logistic map are then,

$$x(x) = \psi^{-1}(s^t\psi(x,s),s) = s^t x + \frac{s^t(1-s^t)}{s-1}x^2 + \frac{2s^t(1-s^t)(s-s^t)}{(s+1)(s-1)^2}x^3 + \frac{s^t(1-s^t)(s-s^t)(1+5s^2-(s+5)s^t)}{(s+1)(s^2+s+1)(s-1)^3}x^4 + \frac{2s^t(1-s^t)(s-s^t)(s^2-s^t)(7s^3+2s+3-s^t(2s^2+3s+7))}{(s+1)^2(s^2+1)(s^2+s+1)(s-1)^4}x^5 + O(x^6) \quad , \quad (7.108)$$

The trajectories are single-valued functions of the time so long as ψ^{-1} is single-valued.

7.4.2 $s=1$ logistic map

Here, $s=1$ is a special case because according to the definition of Schroeder's equation, $s \neq 1$. And from last subsection, (7.106), (7.107) and (7.108) would have singularity if one takes $s \to 1$. So we have to approach $s=1$ separately. Also, when

$0 \le x \le 1/4$, the shape of $x_1 = x(1-x)$ is similar to $x_1 = \sin x$, so it might provide us an alternative way to study $s = 1$ case by reusing the properties of $x_1 = \sin x$.

For $s = 1$, one has

$$x_1 = x(1-x), \qquad x_{-1} = \frac{1}{2}(1 \pm \sqrt{1-4x}), \tag{7.109}$$

and the first five terms of approximate solution shall have the form

$$\begin{aligned} x_t^{(5approx)}(x) = x - tx^2 + (-t+t^2)x^3 + \frac{1}{2}(-3t+5t^2-2t^3)x^4 \\ + \frac{1}{3}(-8t+18t^2-13t^3+3t^4)x^5 + O(x^6) \end{aligned} . \tag{7.110}$$

Therefore, it is of different form from (7.108).

7.4.3 Surface and curvature

Actually $x_t(x)$ denotes the surface of space-time, it would be straightforward to see the relation between position and time from the surface. Sometimes, by fixing the time (or position), we can guess or fit the curve of position (or time), writing the curve in a closed form; and then find the closed form of the surface from the curve. Taking $\sin_t(x)$ as an example, if we take 5-fold conjugation $\sin_t^{(5)}(x)$ and fix the position at $x = \pi/2$, then the curve at $x = \pi/2$ can be fitted by function $g(t) = (\pi/2)^{1-\sqrt{t}}$ when $0 \le t \le 1$, as shown in Figure 7.9.

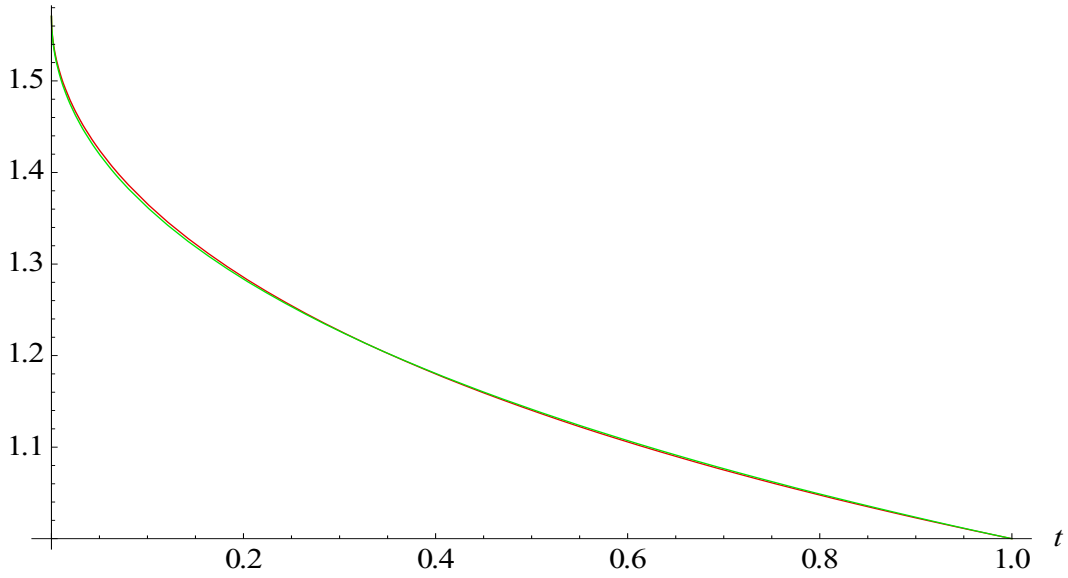


Figure 7.9. $\sin_t^5(\pi/2)$ **(red)** ***vs*** $g(t)$ **(green)**

For a more general case, consider using $g(x,t)=\sin(x)(x/\sin(x))^{1-\sqrt{t}}$ to fit $\sin_t^{(5)}(x)$. Three-dimensional graphs on region $0\le t\le 1$ and $0\le x\le \pi/2$ are like following. Figure 7.10 shows the top part of the graph, while Figure 7.11 shows the interpolation at $x=0.01,0.5,1,1.5$. One can see these two function agree with each other very well on this region. This example shows that, even if we can not find the analytical solution to (7.34) for $\sin(x)$, by drawing the graph of the approximate solution, we can still find an analytical function fitting the approximate solution very well. If we substitute $\sin_t(x)$ with $g(x,t)$ in (7.34), and compare the graphs of both hand sides, we would find $g(x,t)$ can be considered as the exact solution to (7.34), as shown in Figure 7.12.

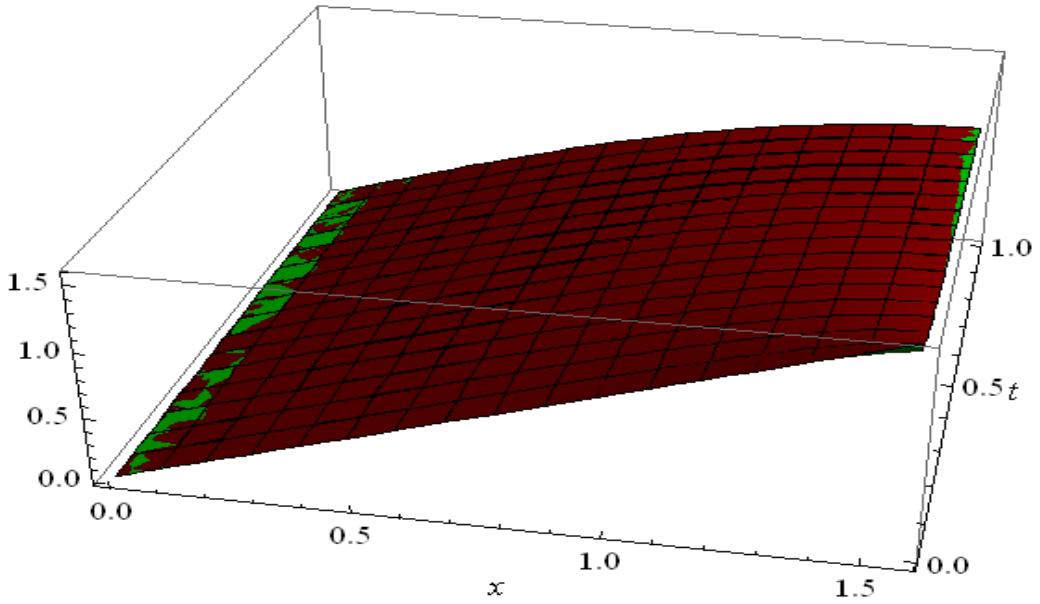
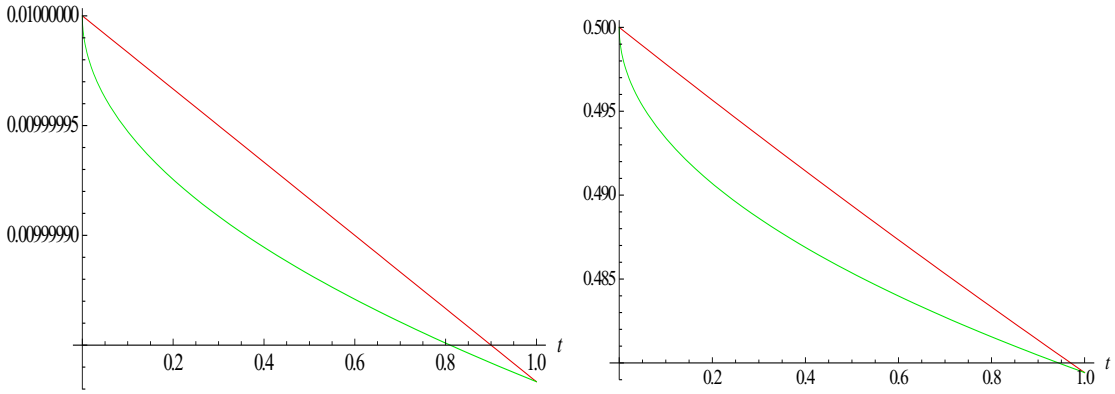
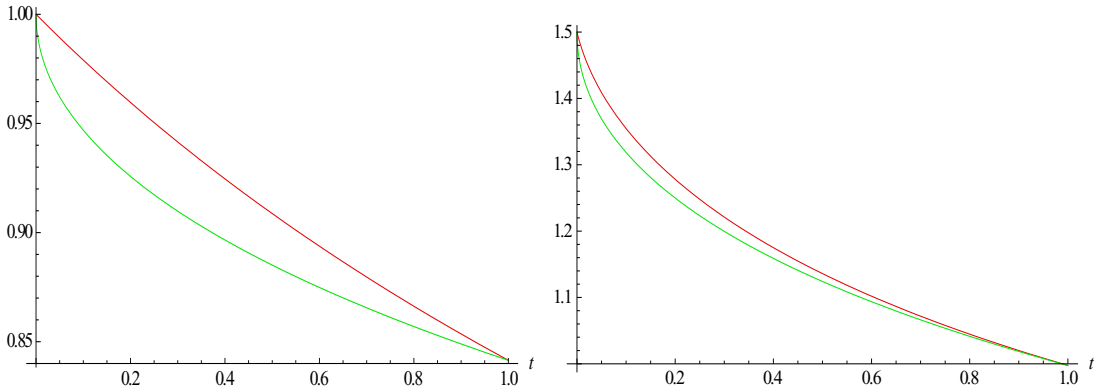


Figure 7.10. $\sin_t^{(5)}(x)$ **(red)** ***vs*** $g(x,t)$ **(green) from the top**



$x = 0.01$ $x = 0.5$



$x = 1$ $x = 1.5$

Figure 7.11. Interpolation of Figure 7.10

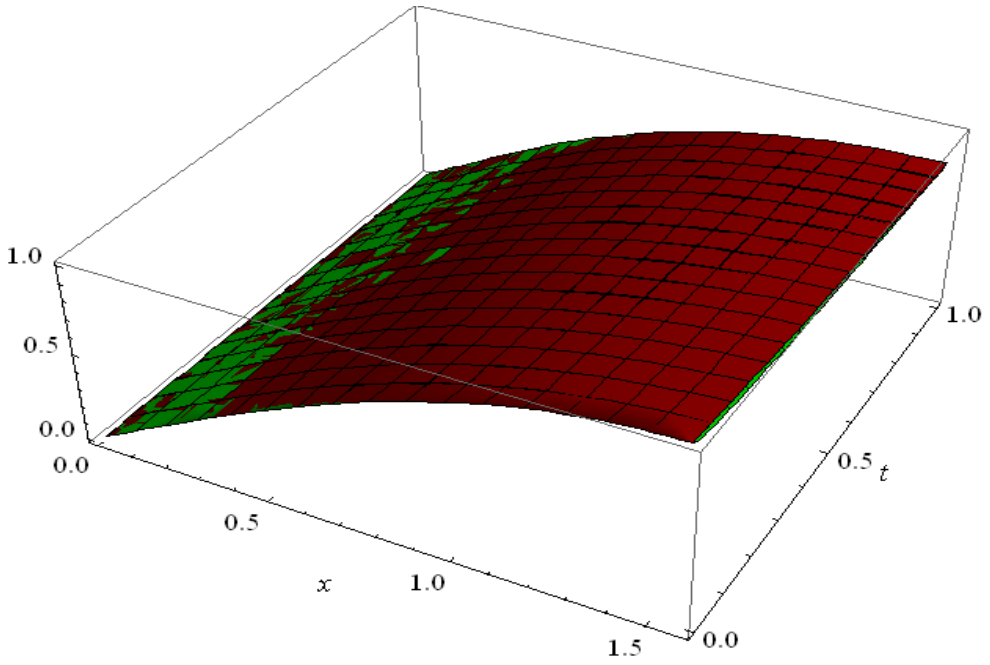


Figure 7.12. Left hand side (red) *vs* right hand side (green) of (7.34)

The second we want to give here is the $s = 1$ logistic map. Still take 5-fold conjugation $x_t^{(5)}(x)$ and fix the position at $x = 1/2$, then the curve there can be fitted by function $g(t) = 2^{-1-\sqrt{t}}$ when $0 \le t \le 1$. Plotting both $x_t^{(5)}(1/2)$ and $g(t) = 2^{-1-\sqrt{t}}$ gives Figure 7.13 The largest difference on this graph is less than 2%, so $g(t)$ is a good estimation. Here, please notice that, in (7.109), the inverse function of x_1 has two different branches. It is straightforward to see that different branches would generate different approximation solutions. We talk about the solutions generated by different branches in next chapter. The inverse function we used here is

$x_{-1}^{-} = (1-\sqrt{1-4x})/2$. As to $x_t^{(5)}(x)$, we guess the surface fitting it should be of the form

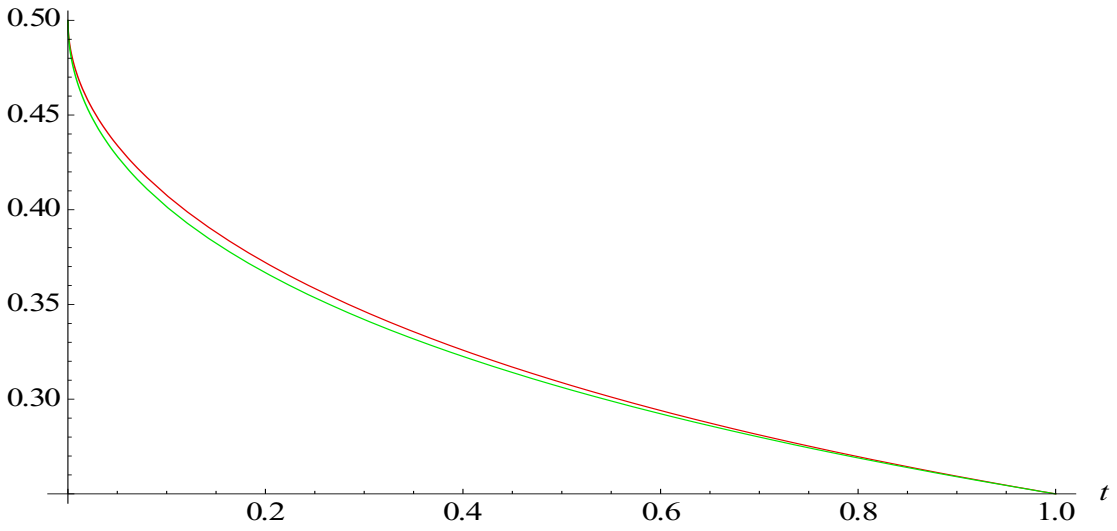


Figure 7.13. $x_t^{(5)}(1/2)$ **(red)** ***vs*** $g(t)$ **(green)**

$g(x,t) = x(1-x)^{t^{1-x}}$. However, not like the $\sin_t^{(5)}(x)$ case, there is a noticeable difference between $g(x,t)$ and $x_t^{(5)}(x)$ on the region $0 \le x \le 1/2$ when t approaches 0, because $t \to 0$ makes $g(x,t)$ approach to 1. Figure 7.14 shows this difference.

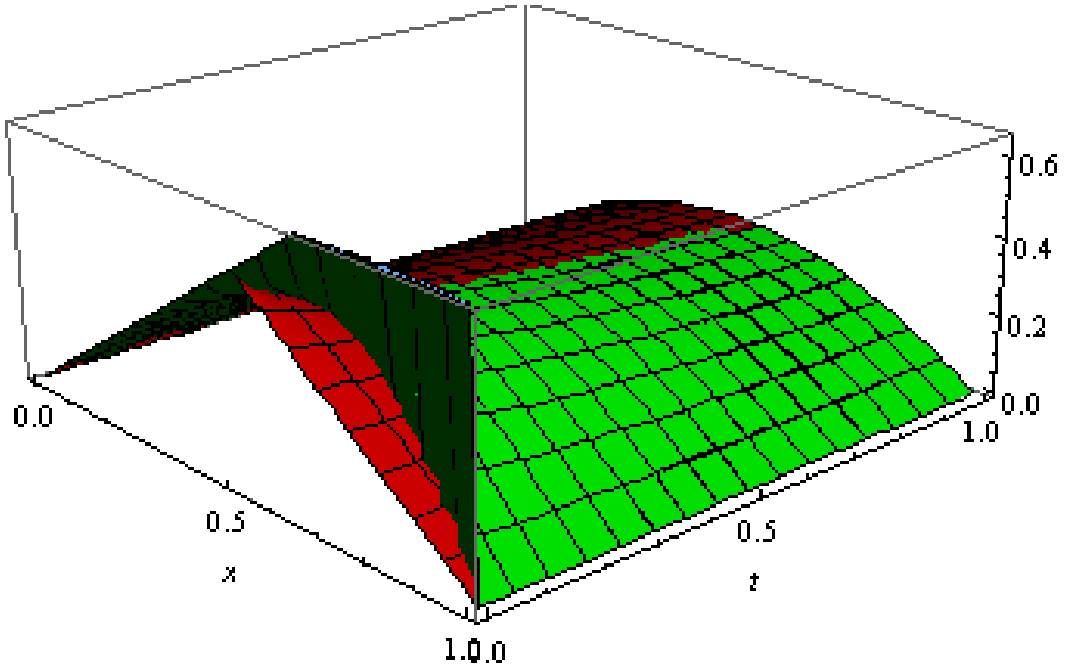


Figure 7.14. $x_t^{(5)}(x)$ **(red)** ***vs*** $g(x,t)$ **(green)**

If we take another branch of the inverse function $x_{-1}^{+} = (1+\sqrt{1-4x})/2$, we can obtain another approximate solution, which is $(x_{-1}^{+})^5 \circ x_t^{approx} \circ (x_1)^5(x)$, and put two branches together on region $-1 \le t \le 1$ and $0 \le x \le 1$ (Figure 7.15). For $(x_{-1}^{+})^5 \circ x_t^{approx} \circ (x_1)^5(x)$, we can fit it with a function similar to $g(x,t) = x(1-x)^{t^{1-x}}$, which would still have the noticeable different when $t = 0$. For each surface, if one can fit it by an analytical function, then the curvature at every point can be calculate exactly. For those which can not be fitted by analytical functions very well, the curvature calculation can be very complicated and tedious, because for just 5-fold conjugation, one has to calculate totally eleven functions acting on functions, sometimes it is even impossible to do with computers.

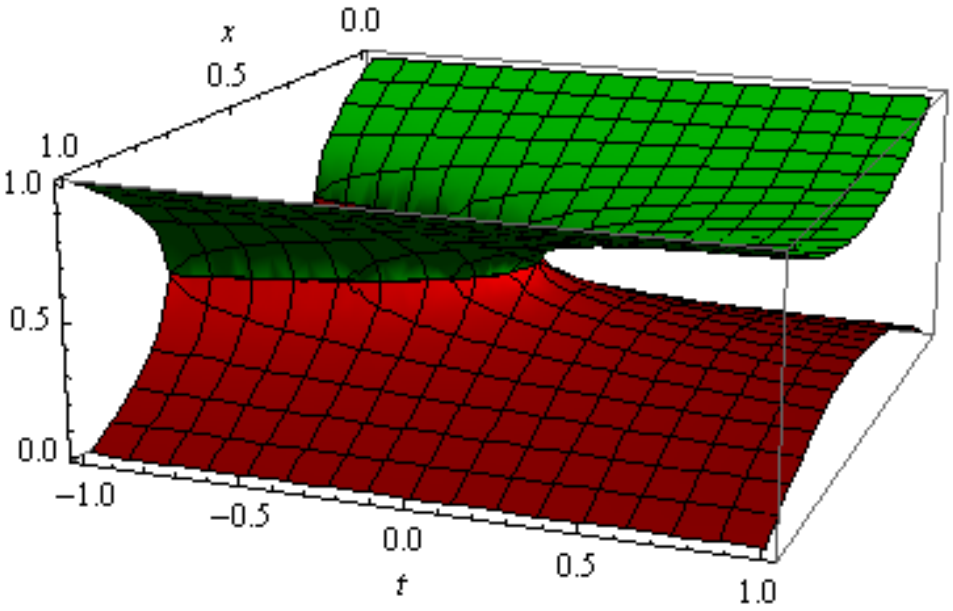


Figure 7.15. $(x_{-1}^{-})^5 \circ x_t^{approx} \circ (x_1)^5(x)$ **(red)** ***vs*** $(x_{-1}^{+})^5 \circ x_t^{approx} \circ (x_1)^5(x)$ **(green)**

Chapter 8

Renormalization Group Flows

8.1 Potentials, Velocities, and Actions

Before we start talking about the RG flows, let us study the relation between functional conjugation and some common physical quantities, such as potential, velocity and action. Originally, the solution to (7.34), $x_t(x)$, can be envisioned as the trajectory of a particle, evolving from initial position $x \equiv x(t)|_{t=0}$. The particle is moving under the influence of a potential according to Hamiltonian dynamics. Our objective is not only to get the trajectory, but also to find the potential, velocity, and the action of the particle. Since we have obtained the trajectories from Chapter 7, it is very straightforward to find the velocities just by differentiating the trajectories with respect to time t,

$$v_t(x) \equiv \frac{dx_t(x)}{dt} = (\ln s) s^t \psi(x) (\psi^{-1}(s^t \psi(x)))', \tag{8.1}$$

where $x_t(x)$ is given by (7.28).

8.1.1 The potentials

Here, we assume the motion of the particle is governed by a Lagrangian, with $L = mv^2/2 - V(x)$ type[80,81], and that the energy of the system is fixed. Therefore, the result for the velocity immediately gives the x-dependence of the corresponding

potential. Namely,

$$V(x) = -\frac{1}{2}mv^2(x) + \text{constant}\,. \tag{8.2}$$

Since the energy is fixed, we only need to take the initial velocity $v_t(x)|_{t=0}$, which gives

$$v(x) \equiv v_t(x)|_{t=0} = \frac{\ln s}{\dfrac{d}{dx}\ln\psi(x)}\,. \tag{8.3}$$

To simplify the calculate, we write the potential as $V(x) \equiv -v^2(x)$, then

$$V(x) = -(\ln s)^2\left(\frac{\psi(x)}{\psi'(x)}\right)^2. \tag{8.4}$$

(8.4) tells us how to determine the potential from the solutions to Schroeder's equations. Also, if the trajectories are already known, then one can find potential directly through

$$V(x) = -\left(\frac{dx_t(x)}{dt}\right)^2|_{t=0}\,. \tag{8.5}$$

On the other hand, the potential can also be determined directly from the functional equation[115] it inherits from ψ. That is

$$V(x_1(x,s),s) = \left(\frac{d}{dx}x_1(x,s)\right)^2 V(x,s)\,. \tag{8.6}$$

If the discrete map possesses a fixed point, we may attempt to solve this functional equation for V by series in x about that fixed point. Again, we take $s \neq 1$ logistic map as an example, then (8.6) becomes

$$V(sx(1-x),s) = s^2(1-2x)^2V(x,s)\,. \tag{8.7}$$

Series method gives the approximate solution to this one[114] about $x = 0$, with initial conditions $V(0,s) = 0$. The series solution can be expressed as

$$V(x,s) = -(\ln^2 s)x^2\left(1+\sum_{n=1}^{\infty} a_n(s)x^n\right), \tag{8.8}$$

where $a_1 = \frac{2}{1-s}$, $a_2 = \frac{5-3s}{(1-s)^2(s+1)}$, and

$$a_{n+2} = \frac{1}{(1-s^{n+2})}\left(4a_{n+1} - 4a_n + \sum_{j=1+\left\lfloor\frac{n-1}{2}\right\rfloor}^{n+1}(-1)^{n-j}a_j s^j \binom{j+2}{n+2-j}\right), \tag{8.9}$$

for $n>1$, where $\lfloor\cdots\rfloor$ is the floor function.

8.1.2 The velocities

Similar to the potential, if $x_t(x)$ is already known, then velocity can be determined very easily. However, sometimes it would be easier to obtain the velocity by series method directly, which means we need the functional equation for velocity first. We differentiate both hand sides of (7.34) with respect to t, and then take $t=0$ because for t with other values, they are similar to $t=0$ case. For $s\neq 1$ logistic map, functional equation for velocity when $t=0$ is

$$v(s(x(1-x)) = s(1-2x)v(x). \tag{8.10}$$

And the series solution is of the form

$$\begin{aligned} v^{(5approx)}(x) = x + \frac{1}{1-s}x^2 + \frac{2}{(1+s)(1-s)}x^3 - \frac{(4+5s)}{(1+s)(-1+s^3)}x^4 \\ -\frac{2(4+5s+7s^2)}{(1+s+s^2)(-1+s^4)}x^5 + O(x^6) \end{aligned}. \tag{8.11}$$

However, this approximate solution does not describe the real velocity very well. To see this, we draw the curve of (8.11) when $s=2$ and compare it with the analytical solution (Figure 8.1). The analytical solution of $s=2$ case can be derived from (7.82),

$$v(x)\big|_{s=2} = -\frac{1-2x}{2}\ln 2 \ln(1-2x). \tag{8.12}$$

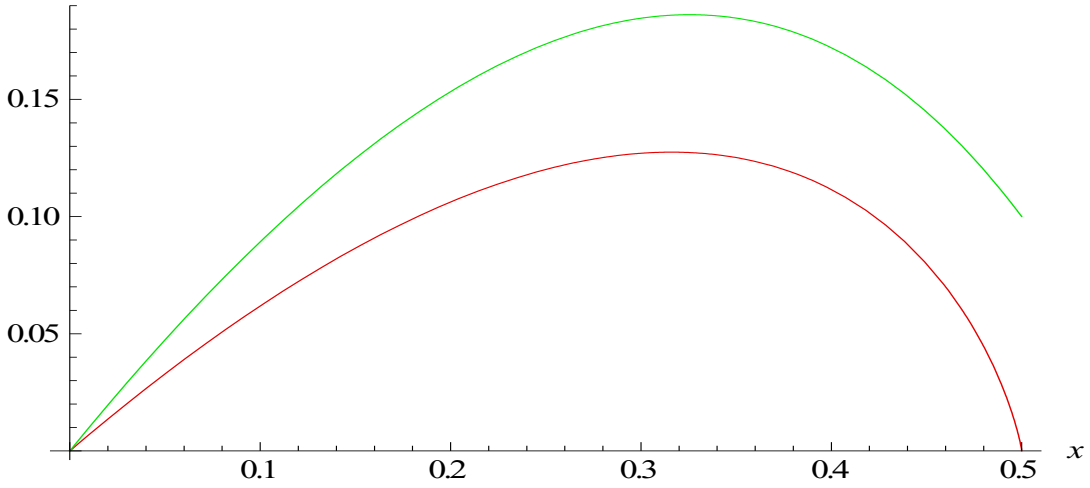


Figure 8.1. $v^{(5approx)}(x)$ **(green) vs** $v(x)\big|_{s=2}$ **(red)**

One can see that the two curves do not overlap each other, $v(x)\big|_{s=2}$ vanishes at $x=1/2$, which is the turning point for $s=2$ case; while $v^{(6approx)}(x)$ does not. We have to do some modifications to our series solutions to make sure they vanish at turning points.

Let $y = s(x(1-x))$, then the inverse is

$$x = \frac{1-\sqrt{1-4y/s}}{2}, \tag{8.13}$$

where we take one branch of the inverse. Plug (8.13) into (8.10), we get the equation about y, and then substitute all the y's with x's, giving

$$v(x) = s\sqrt{1-\frac{4x}{s}}\,v\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{s}}\right)\right). \tag{8.14}$$

Based on (8.14), we use $v^{(approx)}$ to substitute the v on the right hand side, and put a factor $\ln s$ in front the right hand side. Then the modified approximate solution is

$$v^{\mathrm{mod}}(x) = s \ln s \sqrt{1-\frac{4x}{s}}\, v^{(approx)}\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{s}}\right)\right). \qquad (8.15)$$

Now we compare $v^{\mathrm{mod}}(x)\big|_{s=2}$ and $v(x)\big|_{s=2}$, having graph like Figure 8.2.

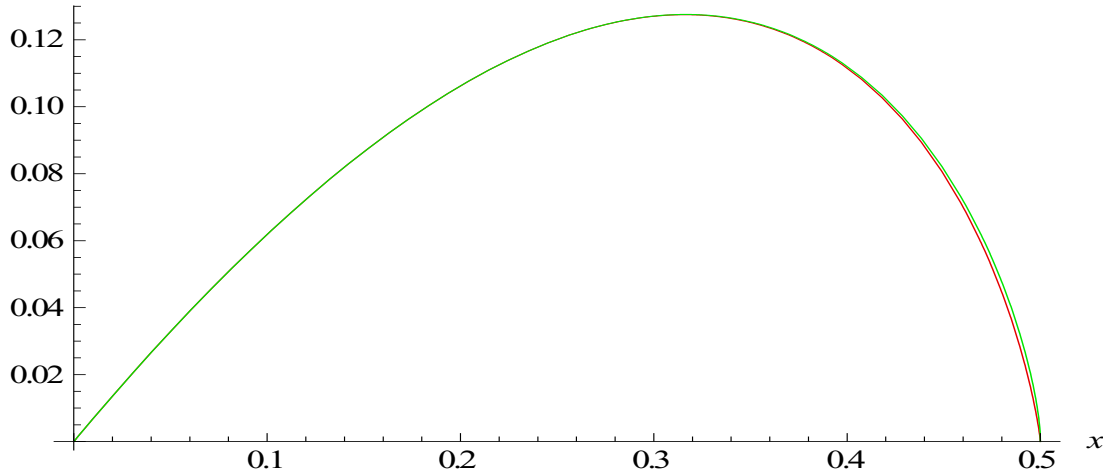


Figure 8.2. $v^{\mathrm{mod}}(x)\big|_{s=2}$ **(green) vs** $v(x)\big|_{s=2}$ **(red)**

Same thing happens for $s=4$ case (Figure 8.3)

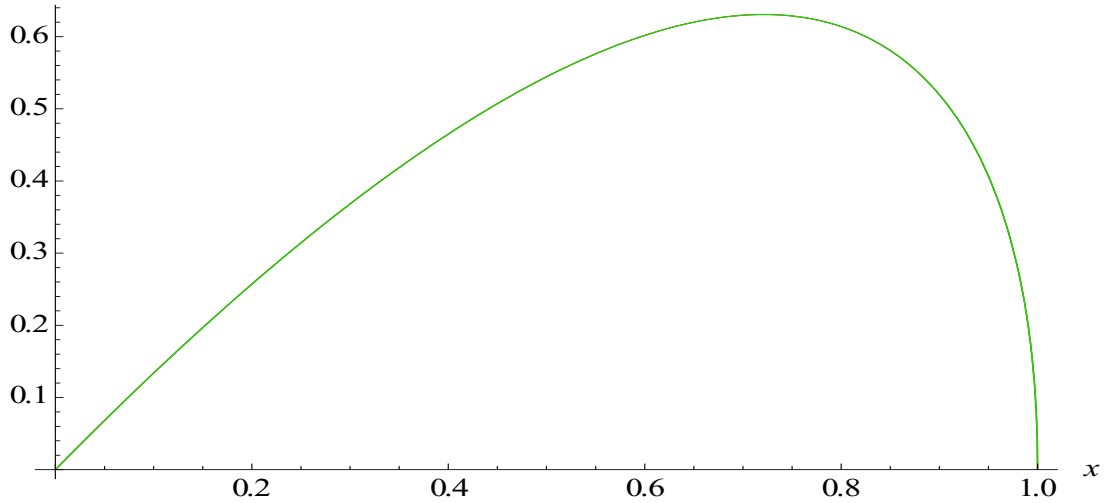


Figure 8.3. $v^{\mathrm{mod}}(x)\big|_{s=4}$ **(green) vs** $v(x)\big|_{s=4}$ **(red)**

One can see that the modified approximate solution fits better in $s=4$ than $s=2$ case,

because the green curve overlaps the red one completely. Therefore logistic map with other s, this method shall give a nice estimation of the real curve.

8.1.3 The actions

Once we have the velocity, the action becomes accessible. We could use the original definition of the action

$$S(x) = \int v^2 dt \,. \tag{8.16}$$

However, this method involving the integration of the velocity square, it is not integrable if the velocity takes a complicated form, especially for many cases we have to use modified series solution because analytical solution does not always exist. So instead of using (8.16), we change the expression a little to make the calculation easier,

$$S(x) = \int v^2 dt = \int v \frac{dx}{dt} dt = \int v dx \,, \tag{8.17}$$

which is just the integration of velocity. If one uses approximate solution for (8.17), then the velocity should be substituted by modified series solution (8.15). Figure 8.4 shows the comparison between analytical action derived from (8.12) and approximate solution derived from (8.15) when $s = 2$. One can see up to some constant, the two curves agree with each other very well, actually that is not a surprise because their velocity functions agree with each other well too.

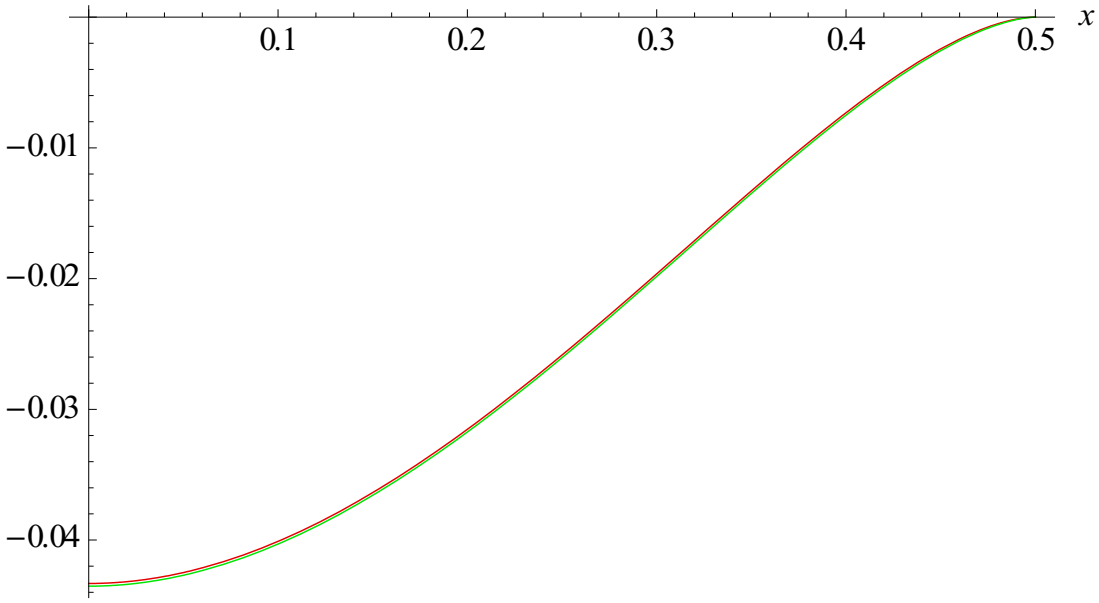


Figure 8.4. Exact action(red) *vs* approximate action when $s = 2$

8.2 β-Function

The approximate method talked above can be applied to calculate β-functions in RG theory. The β-function behaves like the velocity of the particle, so we can use (8.15) to obtain approximate β-function if the trajectory is of the logistic map type. Here, one may notice that when we derive (8.15), we just used one out of two branches of the inverse function, so what if we use the other branch? We would see, the other branch is also useful, and by combining different branches, we can construct some interesting behavior

8.2.1 Renormalization group

We start with Gell-Mann-Low finite renormalization group equation[91],

$$\psi(g(t)) = \lambda^{t-t_0}\psi(g), \tag{8.18}$$

where $t = \ln\mu$ is of the distance/energy reference point; λ sets its scale; $g \equiv g(t_0)$ and we can set $t_0 = 0$; ψ is the RG "scaling function", which "rectifies" group flow to linear flow. This equation is of the same form of (7.27), and is usually solved[119] by integrating a perturbative approximant to its algebra, the β-function,

$$\frac{dg}{dt} = \beta(g) \equiv (\ln\lambda)\psi(g)/\psi'(g), \tag{8.19}$$

in g, to obtain the full RG trajectory, for $\lambda \neq 1$,

$$g(t) = \psi^{-1}(\lambda^t\psi(g)). \tag{8.20}$$

However, there is a different way to calculate and analyze the RG trajectory, based on the theory of functional conjugate equation, which is discussed above. In a conjugacy form, the global self-similar functional structure of the RG trajectory is more apparent, and illuminates the interplay between continuous and discrete rescaling (step-scaling in lattice gauge theory or chaotic maps), often inaccessible to conventional local relations. For example, in lattice gauge theory, for a discrete leap $f(g) = g(1)$, (8.18) becomes

$$\psi(f(g)) = \lambda\psi(g). \tag{8.21}$$

We do an analytic interpolation between $g = g(0)$ and $f(g) = g(1)$ from boundary data without the benefit of a local propagation relation (8.20). Then we can infer the β-function and hence manufacture an underlying Hamiltonian dynamical system which yields the RG flow by conserving energy.

Here is our method: consider analytic $f_t(g)$ around a fixed point of $f(g)$. Without

loss of generality, take the fixed point to be $g=0$, $f(0)=0$, which implies that $\psi(0)=0$. And if $\psi'(0)\neq 0,\infty$, then $\lambda=f'(0)$. We solve for $\psi(g)$ in terms of $f(g)$, which needs recursion of the respective series coefficients. Finally, we invert $\psi(g)$ to obtain ψ^{-1}, and set $\lambda\to 1$ if the problem requites it. The final answer may be convergent even if ψ is diverged for $\lambda=1$. The group orbit found is thus analytic around the fixed point $g=0$. The β-function is then of an emergent feature

$$\beta(g)=\ln\lambda/(\ln\psi(g))'. \tag{8.22}$$

Moreover, for nonlocal relations, Julia equation can be obtained

$$\beta(g(1))=\frac{df}{dg}\beta(g), \tag{8.23}$$

which is of the same form of (8.10), and λ is equivalent to s.

8.2.2 β-function for $s=1$ logistic map

In last section, we have given the approximate solution to velocities for $s\neq 1$ logistic maps, which are the same as the β-functions for $s\neq 1$ logistic maps. In this subsection, we systematically talk about the β-function for $s=1$ logistic map. The functional equation is

$$\beta(x(1-x))=(1-2x)\beta(x). \tag{8.24}$$

The first several terms of series solution are

$$\beta(x)=x^2+x^3+\frac{3}{2}x^4+\frac{8}{3}x^5+\frac{31}{6}x^6+\frac{157}{15}x^7+\frac{649}{30}x^8+\frac{9427}{210}x^9+\cdots. \tag{8.25}$$

It is hard to guess the general expression of the coefficients just by observing (8.25), but the simple form of (8.24) gives us the opportunity to do it. One can find that the coefficient of n-th order term is determined by all the coefficients of the terms less than

n-th order. Let c_n denote the coefficient of x^n in (8.25), and we compare the x^n terms on both hand sides. On the right hand side, the coefficient in front of x^n is just $c_n - 2c_{n-1}$. While on the left hand side, it is a polynomial of $x(1-x)$, which are binomials with different orders. For binomial $(x-x^2)^k$, it contains x^n term if and only if $\lfloor (n+1)/2 \rfloor \le k \le n$. So the coefficients of term x^n on both hand sides satisfy equation

$$\sum_{k=\left\lfloor \frac{n+1}{2} \right\rfloor}^{n} (-1)^{n-k} c_k \binom{k}{n-k} = c_n - 2c_{n-1}. \tag{8.26}$$

If we take $k = n$ on left hand side of (8.26), we find that the c_n on each hand side just cancel each other, then how do we find c_n? Actually, if we just consider the coefficients of x^n term, it can give us coefficient c_{n-1}, but not c_n. After c_n's cancelling each other in (8.26), we take c_{n-1} term out of the summation, then (8.26) becomes

$$\sum_{k=\left\lfloor \frac{n+1}{2} \right\rfloor}^{n-2} (-1)^{n-k} c_k \binom{k}{n-k} - (n-1)c_{n-1} = -2c_{n-1}. \tag{8.27}$$

Next, we make a change $n \to n+1$ for (8.27), and solve the corresponding c_n, obtaining

$$c_n = \frac{1}{n-2} \sum_{k=\left\lfloor \frac{n+2}{2} \right\rfloor}^{n-1} (-1)^{n+1-k} c_k \binom{k}{n+1-k}. \tag{8.28}$$

One can check the first several c_n's are the same as the coefficients in (8.25).

By applying Mathematica 7.0, we can calculate as many c_n's as we want, and we are interested in the behavior of c_n's when n is large. One can see that, as n getting big, c_n goes up very quickly, and it is not easy for us to find the pattern of it. So instead

of studying c_n directly, we put a factor $F=(1700!/c_{1700})^{1/1700}$ in front of each c_n, and plot $F(c_n/n!)^{1/n}$, as shown in Figure 8.5.

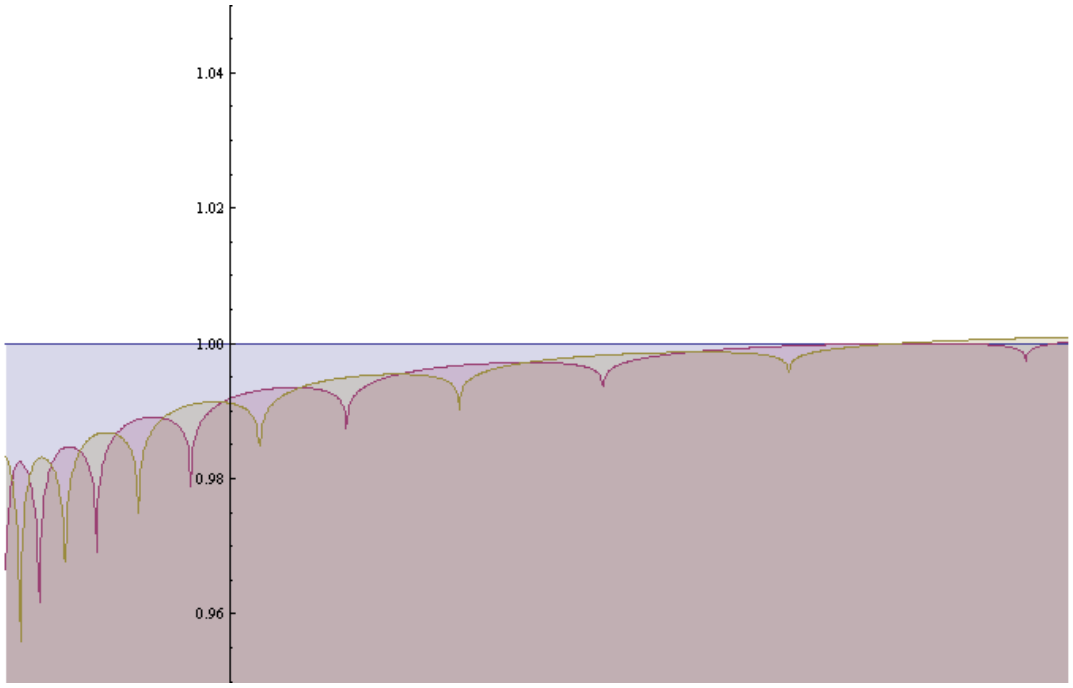


Figure 8.5. $F(c_n/n!)^{1/n}$ **(part)**

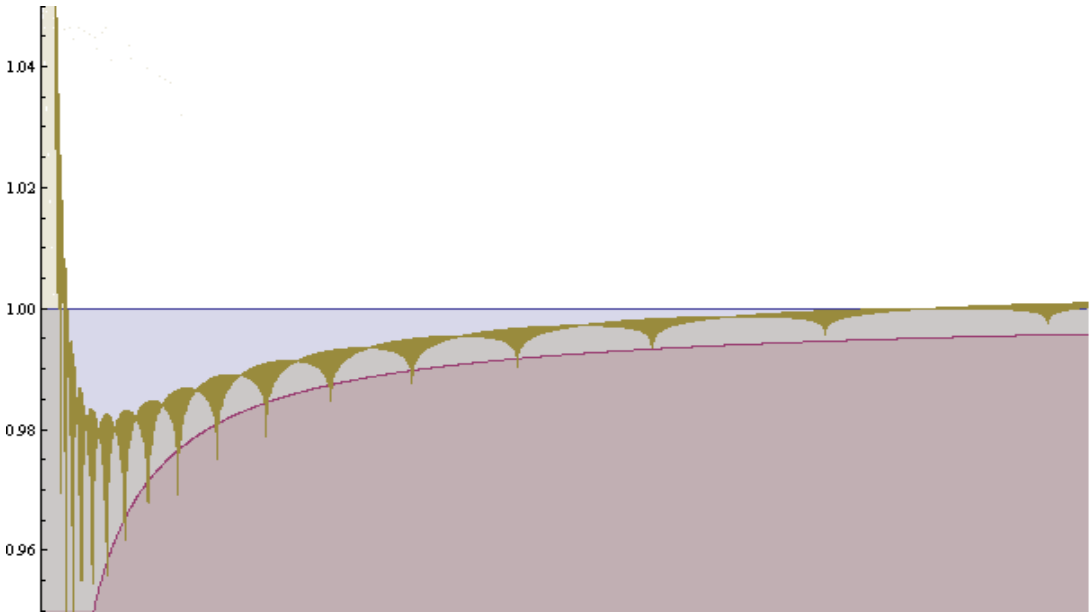


Figure 8.6. $F(c_n/n!)^{1/n}$ **(whole) with fitting curve**

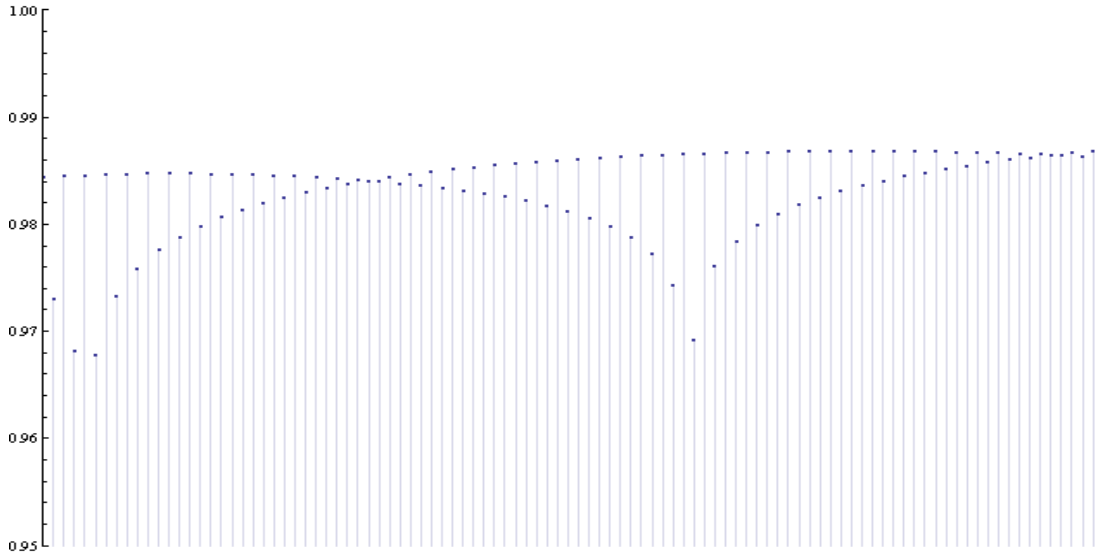


Figure 8.7. $F(c_n/n!)^{1/n}$ **(**$200 \le n \le 300$**)**

There are two curves in Figure 8.5, the red one denotes c_n when n is even, and the orange one denotes c_n when n is odd. Figure 8.5 only shows part of the whole draw (Figure 8.6), but it indicates that $F(c_n/n!)^{1/n}$ is monotonic but also oscillating. In Figure 8.7, one can see the oscillation between odd terms and even terms. Although the whole draw is a combination of monotonic up function and oscillating function, the oscillating amplitude tends to decrease as n getting bigger. We can predict that at large n, the oscillation would eventually disappear, and the whole draw would behave like a smooth simple curve. We try to fit this curve by interpolating at the minima from Figure 8.6, which is the red curve below the initial draw. The curve function is

$$f(x) = e^{-1.11257\frac{\log x}{x}}. \tag{8.29}$$

Also in [115,114], the authors talked about another estimation of nth root of c_n, which takes the form like

$$g(x)=\left(2^{-x/2}e^{-3x/2}\Gamma(x+1)\right)^{1/x}, \tag{8.30}$$

where $\Gamma(x)$ is the gamma function. Actually, this estimation predicts $(c_n)^{1/n}$ very accurately (see Figure 8.8). The blue horizontal line in Figure 8.8 is just constant 1, which is used as reference here. At a large scale, we can not tell the difference between $(c_n)^{1/n}$ and $g(x)$, though

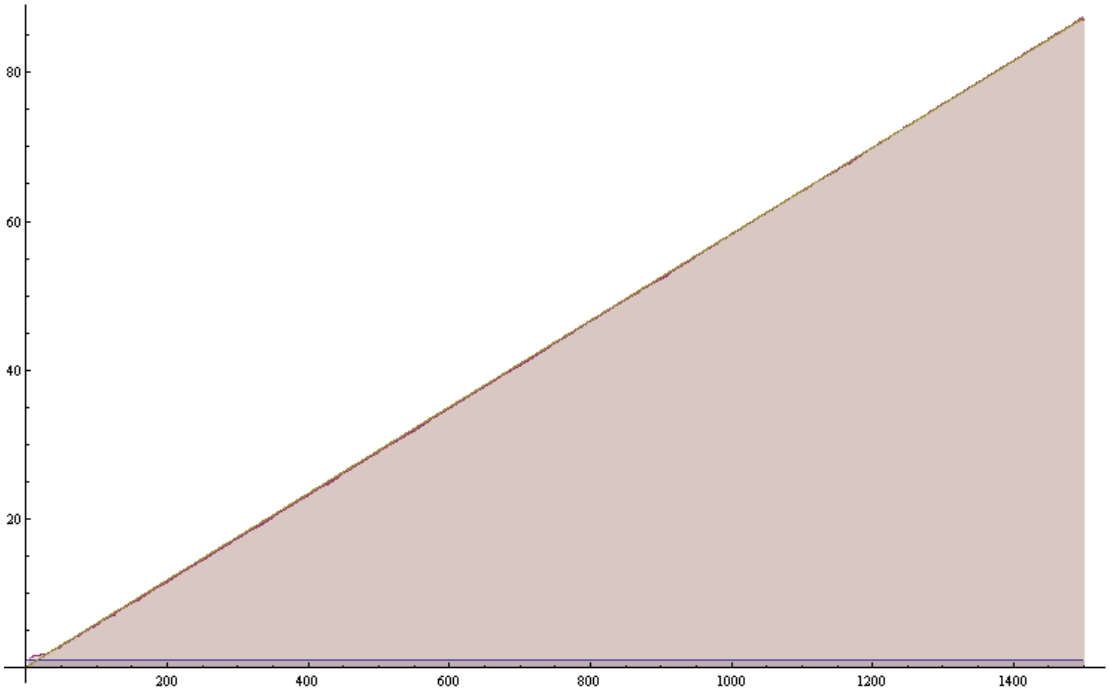


Figure 8.8. $\left(2^{-x/2}e^{-3x/2}\Gamma(x+1)\right)^{1/x}$ ***vs*** $(c_n)^{1/n}$

there is a difference when $n\le 50$. Readers who are interested in how this estimation works, can check [115,114].

8.2.3 β-function with different branches

For an inversible function, its inverse can have more than one branch. For example, the logistic map $f_1(x)=sx(1-x)$ has two branches of inverse:

$f_{-1}^{-}(x)=(1-\sqrt{1-4x/s})/2$ and $f_{-1}^{+}(x)=(1+\sqrt{1-4x/s})/2$. When we calculate the conjugation of series solution in Chapter 7 and modify the approximate velocity in this chapter, mostly we only use the first branch $f_{-1}^{-}(x)$, then the question is: why not use the other branch? If we use it, what would happen? At the end of Chapter 7, we compare the $s=1$ surface of approximate solution for both branches (Figure 7.15), which are symmetric to $x-t$ plane when the solution value is one half. When we modify the series solution of velocity or, the β-function, we can use the other branch and generate some interesting patterns. For example, we start with the series solution (8.11), and then use both branches to 'wrap' it, meaning: if we have the original approximate β-function

$$\beta_0(x)=s\sqrt{1-\frac{4x}{s}}v^{(approx)}\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{s}}\right)\right), \tag{8.31}$$

the next step is to use $\beta_0(x)$ to substitute the $v^{(approx)}$ in (8.31) but the $\beta_0(x)$ is the function of $f_{-1}^{+}(x)=(1+\sqrt{1-4x/s})/2$,

$$\beta_1(x)=-s\sqrt{1-\frac{4x}{s}}\beta_0\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right). \tag{8.32}$$

Then we repeat the step but alternate the sign to generate more branches of β-function, namely

$$\beta_2(x)=-s\sqrt{1-\frac{4x}{s}}\beta_1\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right),\quad \beta_3(x)=-s\sqrt{1-\frac{4x}{s}}\beta_2\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right),$$

$$\beta_4(x)=-s\sqrt{1-\frac{4x}{s}}\beta_3\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right),\quad \beta_5(x)=-s\sqrt{1-\frac{4x}{s}}\beta_4\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right),$$

$$\beta_6(x)=-s\sqrt{1-\frac{4x}{s}}\beta_5\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right),\quad \beta_7(x)=-s\sqrt{1-\frac{4x}{s}}\beta_6\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right)$$

$$\beta_8(x) = -s\sqrt{1-\frac{4x}{s}}\,\beta_7\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{s}}\right)\right). \tag{8.33}$$

Nine branches are listed here, but we can generate as many as we wish. These nine branches are drawn in Figure 8.9 when $s=3$, from which one can see that they are connected by turning points and they describe an oscillating behavior. The motion of the particle starts from the origin, then go through branch β_0 (red curve), reaching one of the turning point $(0.75,0)$; then go through branch β_1 (green curve), reaching the second turning point $(0.5625,0)$; then go through the next branch, etc. By observing Figure 8.9, we conclude that when $s=3$, distance between two consecutive turning points always decreases as branch number goes up. Also, we can separate the graph into upper part and lower part, and all the trajectories in the upper part go through point $(0.67,0.263)$, while all the trajectories in the lower part go through point $(0.67,-0.263)$. This is the special feature for $s=3$, describing a damped oscillation but the particle can reach the same speed $(v=0.263)$ at one certain point $(x=0.67)$. When $2<s<3$, the oscillation is still damped, which means the distance between two consecutive turning points decreases; but there is not such a point that different trajectories reach the same speed at that point (see Figure 8.10). And one can see the damping is more obvious.

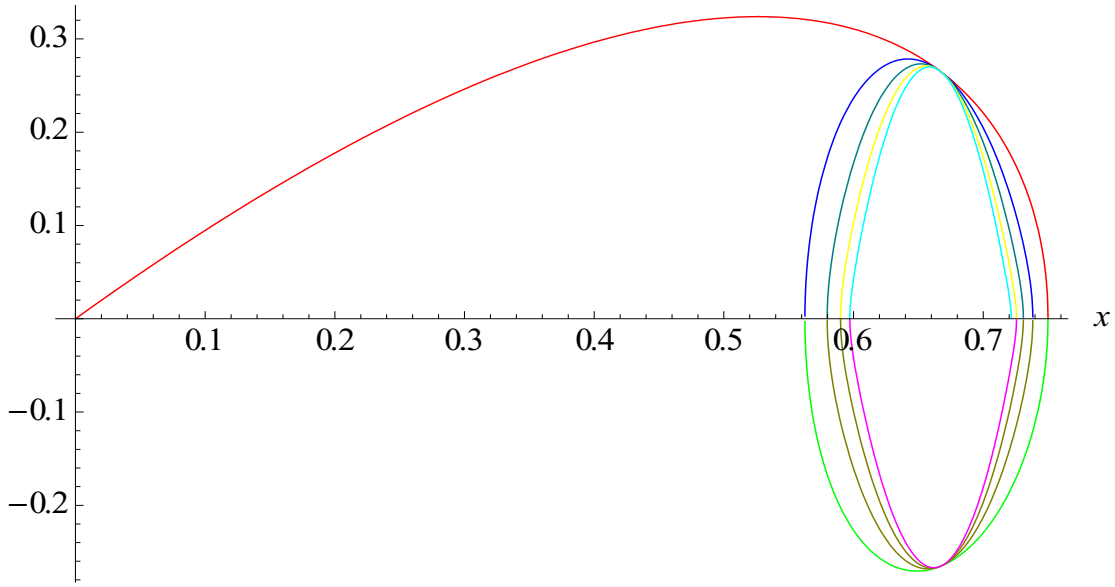


Figure 8.9. (8.31), (8.32) and (8.33) when $s = 3$

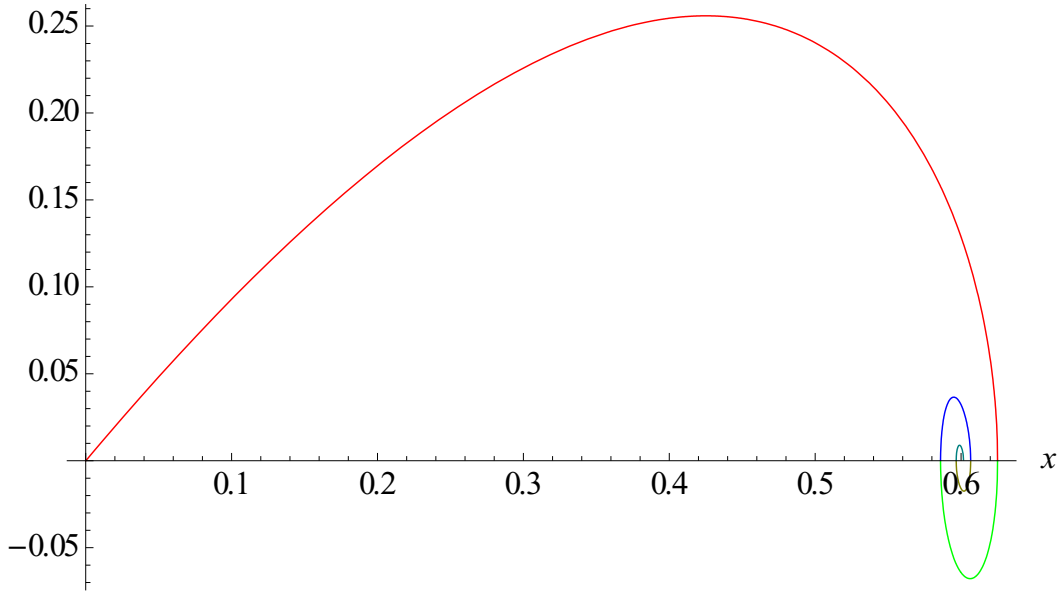


Figure 8.10. (8.31), (8.32) and (8.33) when $2 < s < 3$

When $s > 3$, the damping phenomenon disappears, but the two turning points are fixed, and the maximal speed for each oscillation all the way increases (see Figure 8.11).

When $s \le 2$, there is no oscillating motion (see Figure 8.12 and Figure 8.13).

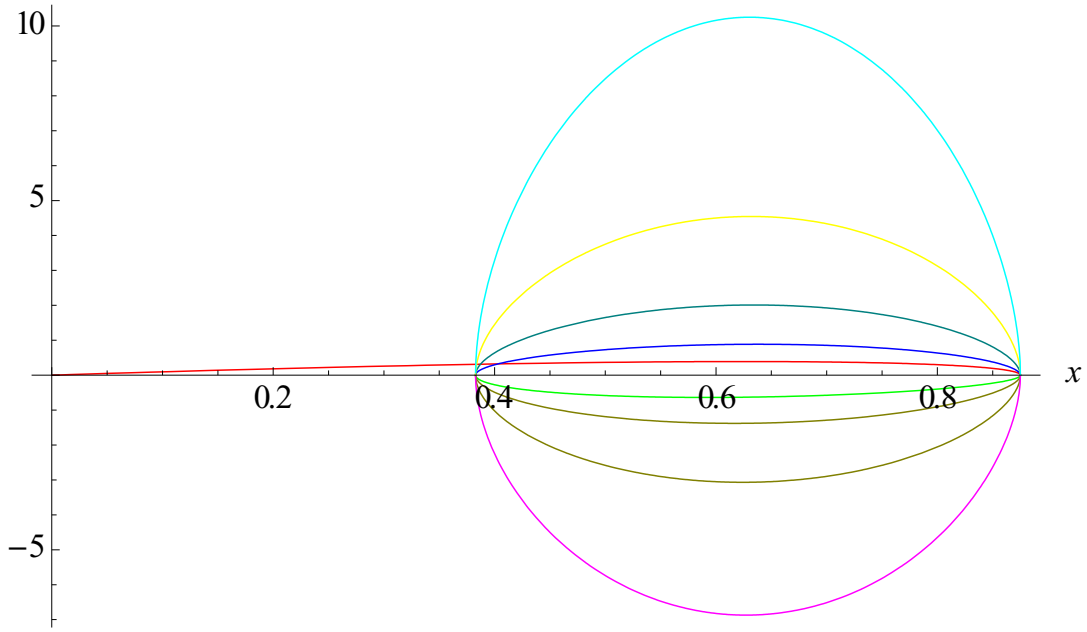


Figure 8.11. (8.31), (8.32) and (8.33) when $s = 3.5$

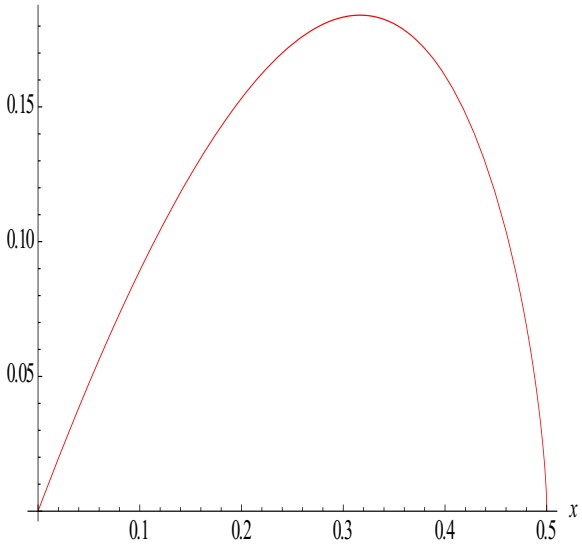


Figure 8.12. (8.31), (8.32) and (8.33) when $s = 2$

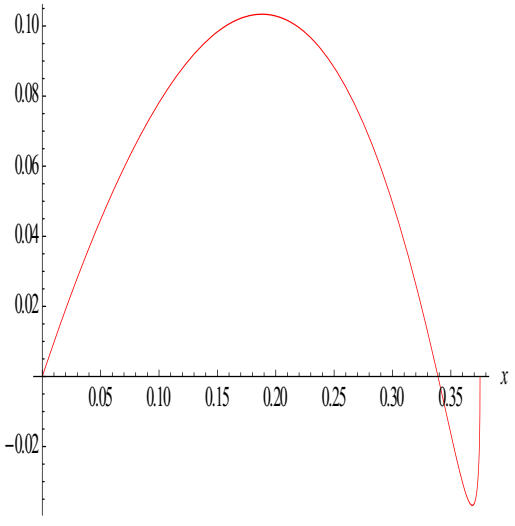


Figure 8.13. (8.31), (8.32) and (8.33) when $s = 1.5$

Figure 8.9 gives system that is quasi-Hamiltonian since the resulting trajectory is not single-valued. It is suggested that the underlying analytic potential for this kind of recursion is of fundamental[114]. [114] also describes the recursion for potentials, which we do not discuss here.

8.3 A Counter Example with a Limit Cycle

Monotonic renormalization group flows of the c- and a-functions are often cited as reasons why cyclic or chaotic coupling trajectories cannot occur. It is argued here, based on simple examples, that this is not necessarily true. Simultaneous monotonic and cyclic flows can be compatible if the flow-function is multi-valued in the couplings[120].

8.3.1 Setup

Exact general results for RG flows are important as they may provide physical insight for strongly coupled systems. The c-theorem for two-dimensional systems[100] and the a-theorem for four-dimensional systems[102,121] are two such results that have been established for every broad classes of models[122]. The c-theorem shows the existence of a monotonically decreasing function of the length scale, $c(L)$, which interpolates between two dimensional Virasoro central charges of theories at conformal fixed points, and thereby provides an intuitively correct count of system degrees of freedom – fewer in the infrared than in the ultraviolet. The a-theorem establishes similar monotonic flow for the induced coefficient of the Euler density, $a(L)$, for a four-dimensional theory in a curved spacetime background.

It is a common conclusion – a "folk theorem" – based on these monotonically evolving "observables" that the underlying couplings can not have RG trajectories which are limit cycles or undergo more exotic (e.g. chaotic) oscillations (e.g. see second bullet item under §6 in [103]). The point of here is to explain and illustrate with just one coupling, as simply as possible, why this conclusion is unwarranted. (Somewhat similar criticism of the monotonic folklore has been proffered in other contexts, involving degenerate Morse function counterexamples for models with vorticity in the flow of several couplings[123].)

In principle, we believe cyclic or perhaps even chaotic coupling trajectories are not ruled out by either the c- or a-theorems, nor are they necessarily excluded by other monotonic "potential flow-functions." To illustrate our reasoning, we begin with a very simple example based on a mechanical analogy. While this example does indeed exhibit both monotonic flow and a cycling trajectory, it has the peculiar feature – insofar as intuitively counting degrees of freedom is concerned – that the monotonic flow is unbounded both above and below. Nevertheless, we recall there is a field theory model that produces just such behavior[106]. We then exhibit another example where the monotonic flows in bounded below and the coupling trajectory is not only cyclic but, in fact, chaotic.

8.3.2 Monotonic flow with a cycling trajectory

The essential ideas, expressed for a single coupling $x(t)$, where $t = \ln L$, are given by general statements for a locally gradient RG flow

$$\frac{dx(t)}{dt} = \beta(x(t)) = -\frac{dC(x(t))}{dx(t)}, \tag{8.34}$$

$$\frac{dC(x(t))}{dx(t)} = \frac{dx}{dt}\frac{dC}{dx} = \beta\frac{dC}{dx} = -\left(\frac{dC}{dx}\right)^2, \tag{8.35}$$

And by a specific example of a flow-function, namely,

$$C_0(x) = -\frac{\pi}{4} - \frac{1}{2}\arcsin(x) - \frac{1}{2}x\sqrt{1-x^2}\,. \tag{8.36}$$

The corresponding β function is

$$\beta_0(x) = -\frac{d}{dx}C_0(x) = \sqrt{1-x^2}\,. \tag{8.37}$$

The RG flow is given by

$$\frac{dx}{dt} = \sqrt{1-x^2}\,, \tag{8.38}$$

which is easily recognized as a "right-moving" simple harmonic oscillator(SHO) started from rest at $x=-1$. This of course has a turning point, $x=+1$, reached in finite Δt, at which point the only way to continue the evolution is the change branches of the square root, $\sqrt{1-x^2} \to -\sqrt{1-x^2}$, to produce a "left-moving" SHO. When this procedure is repeated as turning points are encountered, the cyclic evolution emerges.

In addition, when the first turning point is encountered C switches to a second branch, given by

$$C_1(x) = -\frac{3\pi}{4} + \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2}\,. \tag{8.39}$$

This gives the expected switch between branches for the β function,

$$\frac{dx}{dt} = -\frac{d}{dx}C_1(x) = -\sqrt{1-x^2}\,. \tag{8.40}$$

More importantly, this C function continues to decrease monotonically as a function of t after switching branches.

This is easily understood for this simple example just because the monotonically changing C is nothing but the negative of the definite integral of "the oscillator's kinetic energy" $T = (dx/dt)^2$,

$$C = -\int \beta dx = -\int_{x(0)=-1}^{x(t)} \frac{dx}{dt} dx = -\int_0^t T dt\,, \tag{8.41}$$

where the integral is taken along the actual trajectory of the oscillator – a path that conserves total "energy," cf. RG invariants. (That is to say, C is just the reduced or abbreviated action of Euler, Maupertuis, and Lagrange, or perhaps more consistently with the notation, it is the characteristic function of Hamilton.)

In fact, to obtain the correct evolution for the continuous flow in question, it is absolutely necessary not only to switch between the two branches for $\beta(x) = \pm\sqrt{1-x^2}$, but also to switch among an infinite set of branches for the C-function, as successive turning points are encountered. Thus, as an analytic function, C involves a nontrivial Riemann sheet structure[124, 125]. With initial flow to the right, $dx/dt\,|_{t=0} > 0$, after N encounters with turning points, the evolution is given by

$$\frac{dx}{dt} = (-1)^N \sqrt{1-x^2} = -\frac{d}{dx} C_N(x)\,, \tag{8.42}$$

$$C_N(x) = -\frac{\pi}{4}(1+2N) - (-1)^N \left(\frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2}\right), \tag{8.43}$$

where arcsin is the principal branch of the inverse sine function. We plot a few branches of C in Figure 8.14. More directly, as a function of t,

$$C(t) = -\frac{1}{2}(t - \cos t \sin t)\,, \tag{8.44}$$

which is indeed monotonic in t, as shown in Figure 8.15.

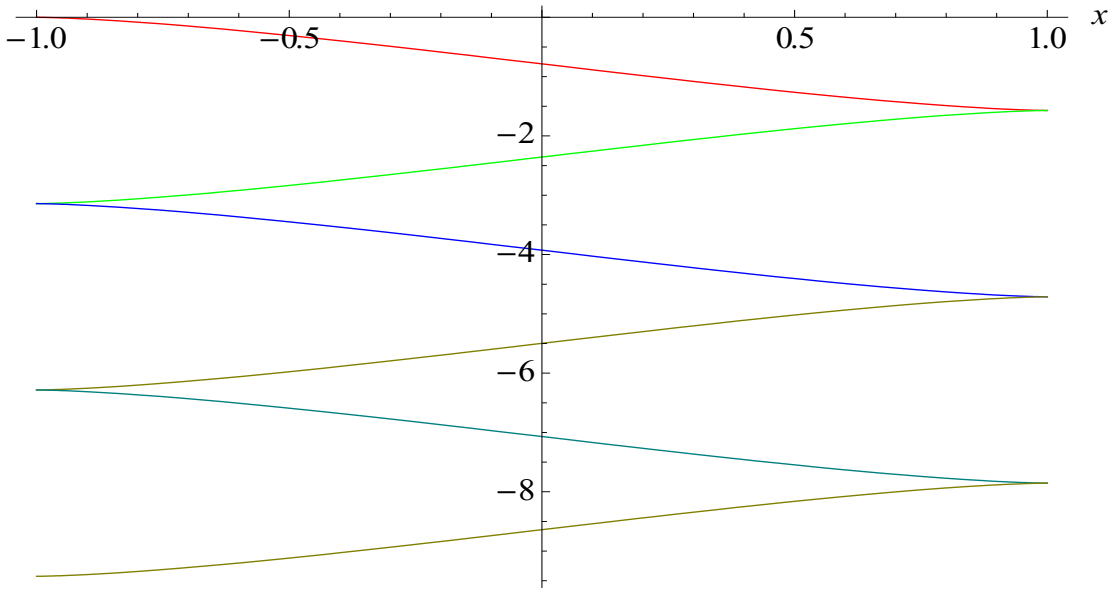


Figure 8.14. Six branches of the SHO $C(x)$ function

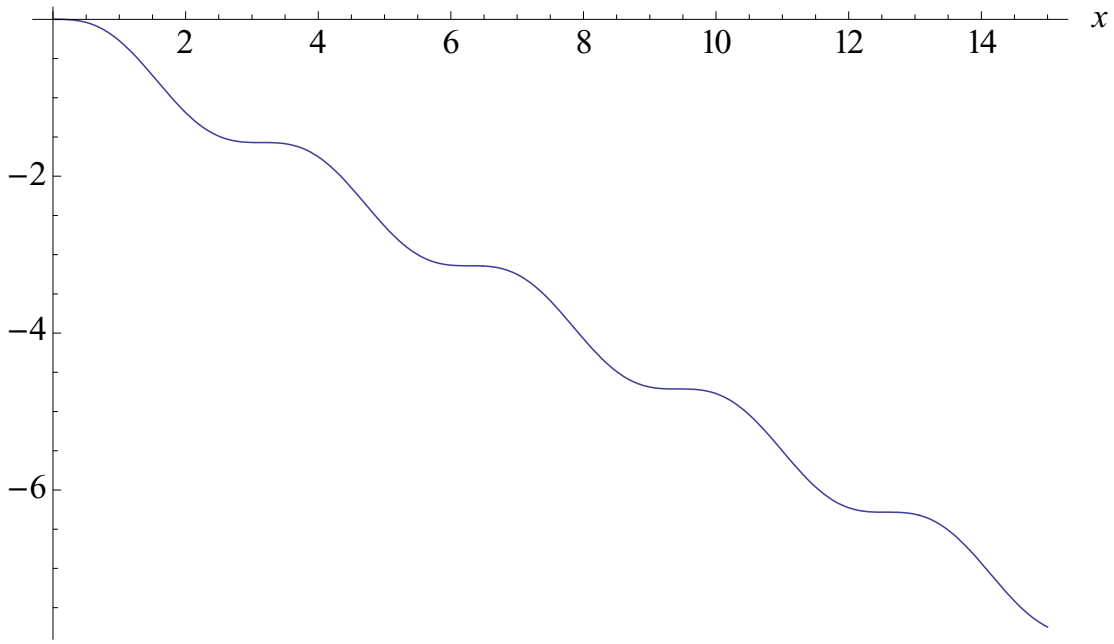


Figure 8.15. Monotonic flow for the SHO $C(t)$

The SHO example of simultaneous monotonic and cyclic flows, while certainly familiar, is perhaps disconcerting, not just because of the multi-valuedness of $C(x)$, but

also because $C(t)$ is unbounded both above the below. However, this same cyclic flow may also be observed by selecting different coordinates for the coupling, without changing the physics of the system. Indeed, the "Russian doll superconductivity model" of Leclair et al. [126,127] provides a single flowing coupling u that illustrates what we have in mind. For that model the RG β and corresponding C function are given by innocuous polynomials,

$$\frac{du}{dt} = \frac{1}{2}(1+u^2), \qquad C = -\frac{1}{2}u(1+\frac{1}{3}u^2). \tag{8.45}$$

Despite this uncomplicated local behavior, the global trajectories go through finite excursions in the course of their cyclic evolution:

$$u(t) = \tan\left(\frac{1}{2}t + \arctan u(0)\right). \tag{8.46}$$

Thus it is difficult to keep track of the monotonicity of C, if any, as it executes an infinite jump during the course of each cycle.

The system is perhaps easier to grasp upon being expressed in terms of a "dual" coupling, x,

$$u = \pm\sqrt{\frac{1+x}{1-x}}, \qquad \frac{dx}{dt} = \pm\sqrt{1-x^2}. \tag{8.49}$$

That is to say, the RG flow of the model is equivalent to the SHO as described earlier. Note the cyclic switching between the branches of $u(x)$ corresponding to right-moving and left-moving SHO motion, including an infinite jump upon reaching $x=1$, as shown in Figure 8.16. Similar analysis can be carried out for theories with several coupling constants. (For models with limit cycles in $4-\varepsilon$ dimensions, see [128,129].) We leave the study of these for another venue.

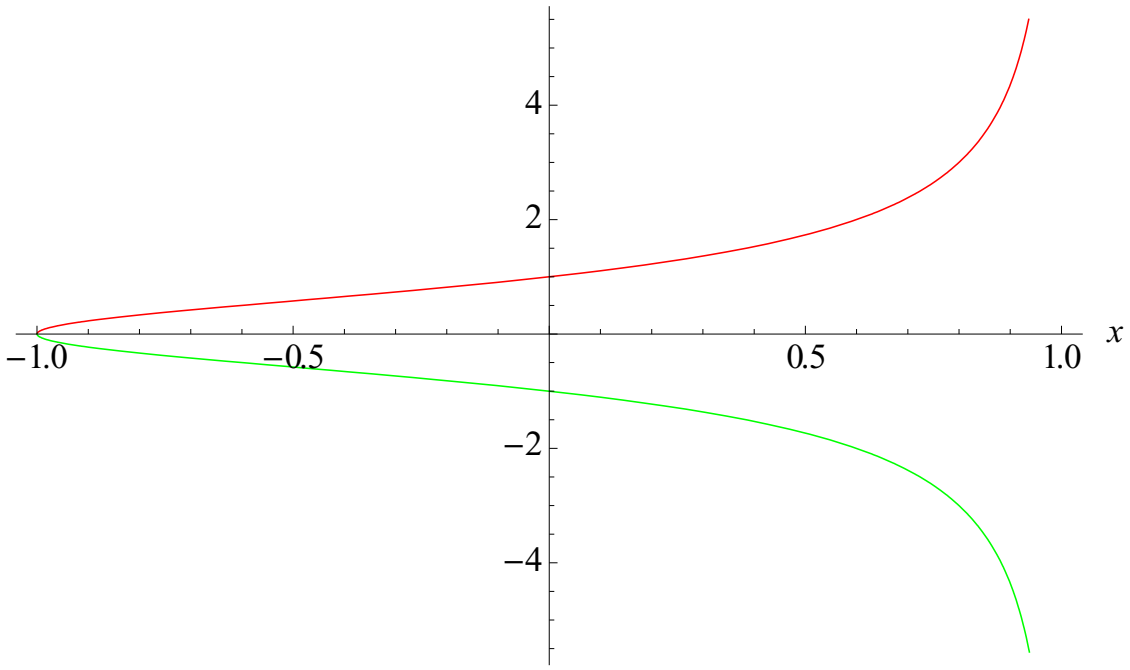


Figure 8.16. The Russian doll – SHO RG duality

8.3.3 Cyclic and chaotic trajectory

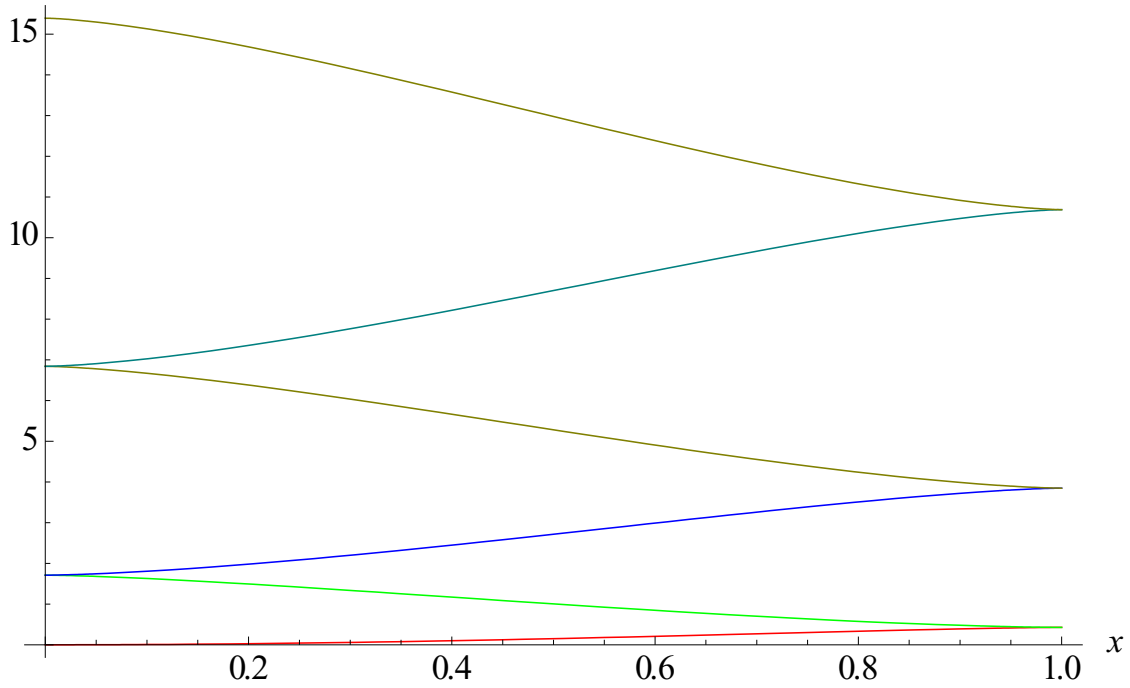
We consider a model with a cyclic but chaotic trajectory which also exhibits a monotonic flow-function. Again, a solvable example involving a single coupling is sufficient to make the point. Perhaps the simplest system with chaotic RG evolution is the Ising model with imaginary magnetic field, described by the special case of the logistic map with parameter 4 [130,131]. The exact trajectory and β function are given by

$$x(t) = (\sin(2^{-t} \arcsin\sqrt{x}))^2 , \tag{8.48}$$

$$\frac{dx(t)}{dt} = -(\ln 4)\sqrt{x(t)(1-x(t)}\arcsin\sqrt{x(t)} , \tag{8.49}$$

where the arcsin function in this last expression switches branches upon encountering turning points. Similarly, the corresponding C function, considered as a function of $x(t)$, also changes branches at turning points.

The direction of the flow in t is such that the origin is an attractive fixed point in the infrared, so $x \to 0$ as $L \,\&\, t = \ln L \to +\infty$. On the other hand, x becomes chaotic, exhibiting cycles of arbitrary length, as $L \to 0$ and $t \to -\infty$. That is to say, for any initial $x \in (0,1]$ the flow for $t > 0$ is monotonically toward the fixed point at $x = 0$, while for $t < 0$ the flow is toward a turning point at $x = 1$, where dx/dt reverses and the flow is toward a second turning point at $x = 0$ – the zero of β. As the evolution continues into the UV, with $t < 0$, the trajectory oscillates between the pair of turning points, $x = 0$ and $x = 1$, with increasing average "speed."



8.17. Six branches of the logistic $C(x)$ function

There are in infinite number of branches for both $\beta(x)$ and $C(x)$ in this case. Those branches are given by

$$\beta_N(x) = -(\ln 4)\sqrt{x(1-x)}\left\{(-1)^N\left\lfloor\frac{1+N}{2}\right\rfloor\pi + \arcsin\sqrt{x}\right\}, \qquad (8.50)$$

$$C_N(x)=\frac{1}{8}(\ln 4)\left\{4x^2(x-1)^2+\left(\sqrt{x(1-x)}(1-2x)-(-1)^N\left\lfloor\frac{1+N}{2}\right\rfloor\pi-\arcsin\sqrt{x}\right)^2\right\}. \tag{8.51}$$

Here arcsin is understood to be the principal branch, and N counts the number of encounters with the trajectory turning points at $x=0$ and $x=1$. The first six branches of $C(x)$ are shown in Figure 8.17. As $t\to\infty$, the flow is toward the origin, with $x(+\infty)=0$ and $C(+\infty)=0$, while as $t\to-\infty$, $C\to+\infty$. This is more clearly seen by plotting

$$C(t)=-\int_0^{x(t)}\beta(x)dx=\int_t^{\infty}(\beta(x(t)))^2\,dt\,, \tag{8.52}$$

for $0<x(t)|_{t=0}<1$. The flow of C is monotonic in t and bounded below, $C\geq 0$. This is shown in Figure 8.18 for $x(t)|_{t=0}=1/2$.

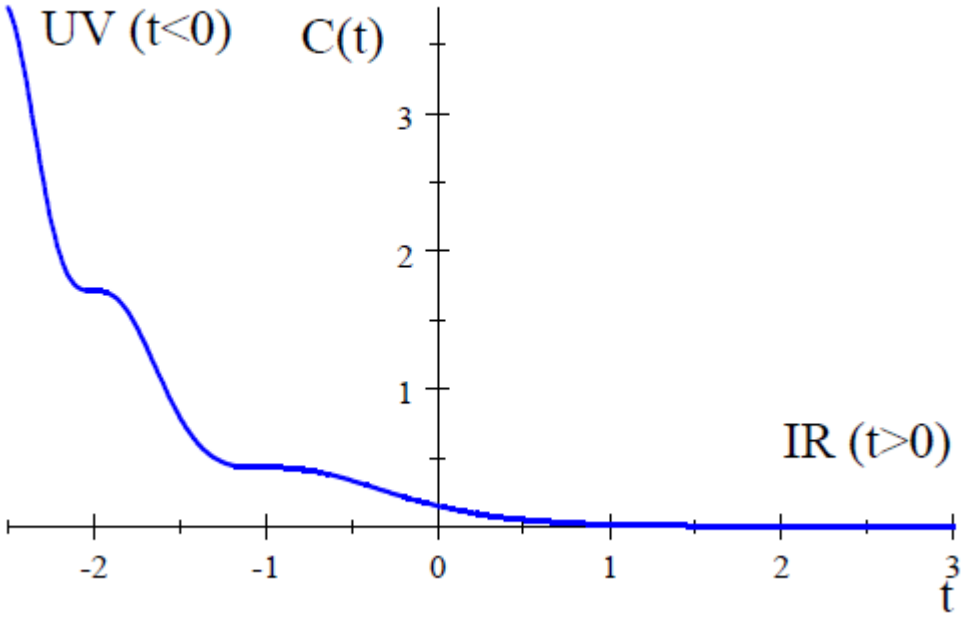


Figure 8.18. Monotonic flow for the logistic $C(t)$

A full discussion of Lagrangian models that realize this second example will have to be given elsewhere. Suffice it to say here that the chaotic RG trajectories have indeed appeared in spin-glass systems[132,133]. The point we wish to emphasize is that such behavior is not necessarily inconsistent with c- and a-theorems.

Chapter 9

Conclusion of Part II

In part II, we introduce Schroeder's equation, and derive functional equation (7.33) from it. We give an approximate solution to (7.33) around a fixed point of x_1 through series and conjugation method. It is believed that all the functional equations of form (7.33) can be solved approximate by this method, as long as one knows x_1. The journey of searching for series solution would take some time in the beginning, because one needs to construct the first two or three terms by guessing. But after the first several terms have been constructed, one can always write the solution up to as high order as he wants. For logistic map when $s \neq 1$, there exists a general form the series solutions, and there also may exist an analytical function that fits each approximate solution. Once an approximate solution has been constructed, the convergence is of concern. We prove a theorem about the relative error, and show that the error is very small around the fixed point.

We consider x_t in (7.34) as the trajectory of a particle's motion. Based on (7.34), one can also derive the functional equations about the velocity and potential. There is a general approximate solution for logistic map potential when $s \neq 1$. As to the velocity, we have to use recursion relation because the velocity is supposed to vanish at turning points. When $s = 1$, we give the general approximate solution for velocity,

and find that the coefficient before each order has monotonic but oscillating behavior.

The approximate solution we introduce here is significantly useful to calculate the β-function in RG theory, because β-function acts as velocities, and x_t is similar to RG trajectory. This series and conjugation method has many advantages: the global self-similar functional structure of the RG trajectory is more apparent, and it illuminates the interplay between continuous and discrete rescaling, which is inaccessible to conventional local relations. When we talk about the conjugation, we normally only use one branch of the inverse functions of x_1, but other branches are also useful. We give an example of using other branches for β-function recursion, and it predicts oscillation of particle when $s > 2$. If we make different branches combine with each other, we would have different oscillations, the reason is different combinations generate different potentials. It is commonly believed that monotonic RG trajectory can not have limit cycles. However, we give a counterexample that simultaneous monotonic and cyclic flows can be compatible if the flow-function is multi-valued in the couplings. And it seems these trajectories are not ruled out by either the c- and a-theorems.

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