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Supersymmetric Landau Models

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UNIVERSITY OF MIAMI

SUPERSYMMETRIC LANDAU MODELS

By

Andrey V. Beylin

A DISSERTATION

Submitted to the Faculty
of the University of Miami
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

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SUPERSYMMETRIC LANDAU MODELS

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Supersymmetric Landau Models

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This work is focused on the different supersymmetric extensions of the Landau model. We aim to fully solve each model and describe its energy levels, wavefunctions, Hilbert space and define a norm on it, as well as find symmetry generators and transformations with respect to them.

Several possible generalizations were considered before. It was found for Landau model on the so called Superflag manifold as well as planar Superflag and Superplane Landau models that standard norm on the Hilbert space is not positive definite. Later for planar cases it was found that it is possible to fix this by introducing a new norm which will be invariant and positive definite. Surprisingly this procedure brings up 'hidden' symmetries for the known super Landau models.

In the dissertation we apply the same procedure for Landau model on supersphere and Superflag manifolds. It turns out that superpspherical Landau model is equivalent to the Superflag model with one of the parameters fixed. Because the model on supersphere can be recovered from the Superflag we will do calculations of corrected norm only for the Superflag.

After this we develop a different generalization of the Superplane Landau model. Starting with Lagrangian in a superfield form we introduce two arbitrary functions of superfields $K(\Phi)$ and $V(\Phi)$ into the Lagrangian. We follow with the component form of Lagrangian. The quantization of the model is possible, and we will show that there is a reparametrization which turn equation of motion of the

first scheme into the second set. Standard metric is again non-positive definite and we apply already known procedure to correct it. It will not be possible to solve Schrodinger equations in general with undefined K and V , so we consider one specific case which give us Landau model on a sphere with $N = 2$ supersymmetry, which put it apart from the superspherical Landau model, which have a superpshere for a target space but do not possess supersymmetry.

For the benefit of all beings

Acknowledgments

During this five-year stay at the University of Miami I have encountered many people. Some of them helped me to stay on track in my research, some cheered me on with my work, others encouraged me to look for a new horizons. Let me mention a few of those people.

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Chapter 1

Introduction

Mathematical physics can be defined as "application of mathematics to problems in physics and the development of mathematical methods suitable for such applications and for the formulation of physical theories", as given by the Journal of Mathematical Physics [1]. As one of the most well-known branches of mathematical physics supersymmetry was proposed to be a possible solution for several questions in theoretical physics like hierarchy problem or unification theories. Introducing supersymmetry in the string theory was a fascinating step in the evolution and made it feasible for physicists to aim for solving more complicated problems. But every expansion make theory more complex and complexity is an unfortunate features of super string theory, so to better understand supersymmetry it was natural to turn to researching simpler models with anticommutative variables included.

With the importance of supersymmetry understood it has become interesting to learn how to introduce SuSy into well explored areas of theoretical physics, e.g. classical and quantum mechanics. Thus the research of the models in supersymmetric mechanics and supersymmetric quantum mechanics is not only interesting by itself but also give us hope to understand better more complicated areas like super strings, super Yang Mills, physics on the spaces with anticommutative coordinates. The purpose of this dissertation is a detailed investigation of the several supersymmetric quantum mechanical systems which stem from a so called Landau model.

In 1930 Landau proposed a quantum problem with a charged particle moving on the plane in the uniform magnetic field perpendicular to the surface and showed that particles energy is quantized [2]. The energy levels are usually called 'Landau levels'

with the ground level being the ‘Lowest Landau level’ abbreviated LLL. One of the first generalizations was proposed by Haldane [22] in 1983 and is a system with a particle moving on a sphere in \mathbb{E}^3 with magnetic monopole in its center. Intriguing fact is that the spherical Landau model also turned to be exactly solvable with energy levels quantized. In the LLL the coordinates on the sphere possess certain degree of non-commutativity turning it into fuzzy sphere. This property make it related to Quantum Hall Effect [23] and theories with noncommutative coordinates. One of the ideas which brought up the study of Super Landau models was introduction of a fuzzy superpshere. It is a supersymmetric case of a fuzzy sphere, so we can anticipate that there exist a supersymmetric version of the Landau model on a sphere. This was done in [5] with action being of SCSQM¹ on the $CP^{(n|m)}$ with a Kahler metric. The model is indeed solvable and possess fuzzy LLL, which lead us to a question about how to find similar models, what symmetries they can possess and what are their properties?

Research of Super Landau models² essentially started with an introduction of the Landau model on the Superflag manifold [6]. In the text of the dissertation we will consider five cases: Landau model on the superpshere and Superflag, their planar versions Superplane and Planar Superflag, and then superfield generalization of a Superplane model. In the next section we will discuss Superplane Landau model, which is the simplest one of all the cases and thus let us explore its properties in details. There are three different ways to construct Superplane Landau model. First way is a naive generalization of original model by modifying its Lagrangian with fermionic fields. We will start by writing fermionic Lagrangian similar to the original model and solving pure fermionic model. The model is trivial but already possess the most important feature — some of the eigenstates are not positive definite. It is known that when fermionic kinetic term in the Lagrangian contain second order

¹Supersymmetric Chern-Simons Quantum Mechanics

²There is another approach to the supersymmetrization of Landau model which bring us to a similar but not exactly the same models [14].

derivative then the model is not unitary, and this system is another confirmation to that fact. We will later observe how all Super Landau models have non-unitary metric and devise a method to fix it. After joining fermionic and bosonic Lagrangian to make a planar Super Landau model we will see how it can be solved and that eigenstates again do not all possess positive norm.

It is possible to create the same model by pure geometrical means and in the section 2.2 we will show how the Lagrangian of a planar model Super Landau can be found from invariant forms on $IU(1|1)/[U(1|1) \times \mathcal{Z}]$. In the next subsection we will describe a method to fix the states of the negative norm. In short the idea is to find a new metric different from a natural one such that we would keep the symmetries of the model but new Hermitian conjugation will make the model unitary. In the last part we will discuss how Superflag Landau model was originally constructed 2.4.

Chapter 3 is based on [4] and discusses in detail properties of Superflag and superpsphere Landau model. Structure of this section is straightforward – first we will solve Landau model on the superpsphere, find eigenvalues and describe energy levels, and after directly calculating norms of the eigenfunctions we will see that they are not positive definite. Next step is the same procedure for the Superflag model, which is in certain ways a little bit simpler then the superpsphere. After it we modify the norm of a Superflag model and show that for a specific choice of parameters Superflag is equivalent to superpsphere. It turns out that modified norm bring up some hidden symmetries of the system, so in the end it possess $SU(2|2)$ and not $SU(2|1)$ symmetry.

With four cases studied one of the questions is if it possible to generalize Landau models further. Chapter 4 will show how superfield approach can be used to express the Lagrangian of the Superplane Landau model [3]. With superfields we have a third way of setting up a model on the Superplane, and this method is the easiest

to generalize. We introduce arbitrary potentials into the Lagrangian and investigate new features of the system following the same course as before. We have considered two quantization schemes for these models but after calculations we will see that they are equivalent. These models as well possess states of negative norm and we can fix it by using the same method as before. In the last part of the section we will explore a particular case of the potentials which create a Landau model on a sphere, in fact it is an $N = 2$ supersymmetric extension of the model describing particle on the 2-sphere in the field of the magnetic monopole. This is different from the Superspherical Landau model which does not possess worldline supersymmetry but instead has odd coordinates in the target space.

We will summarize all important results in the end and give a short list of possible direction for a further research.

Chapter 2

Supersymmetric extensions of Landau models

We will start with Superplane Landau model, which is the simplest supersymmetric expansion of the original problem. It will be our main toy model. After that we will discuss construction of three other models Superflag, superpsphere and planar Superflag. At the end we will show the method of dealing with non-positive definite metrics.

2.1 Superplane Landau Model

To begin with the Superplane Landau model we have to write a Lagrangian for it. This is a simple case so there are several different ways to obtain this Lagrangian. We will show only the most straightforward approach in this section, other methods will be explored on the way.

Here by “Superplane” we mean the superspace $\mathbb{C}^{(1|1)}$ parametrized by complex coordinates (z, ζ) , where z is a complex number and ζ a complex anticommuting variable. Let us assume that Superplane Lagrangian can be written as

$$L_0 = L_b + L_f, \tag{2.1}$$

where

$$L_b = |\dot{z}|^2 - i\kappa(\dot{z}\bar{z} - \dot{\bar{z}}z) \tag{2.2}$$

is the Lagrangian for the original planar Landau model. L_b is purely bosonic, and L_f will be a fermionic part of the expression. Let's define it for anticommuting variables

ζ similar to the bosonic part

$$L_f = \dot{\zeta}\dot{\bar{\zeta}} - i\kappa \left(\dot{\zeta}\bar{\zeta} + \dot{\bar{\zeta}}\zeta \right). \quad (2.3)$$

Here 2κ is energy spacing(which we take to be positive) as in the standard Landau model.

This Lagrangian L_0 is quadratic and doesn't have any complex interaction terms. Later we will see that there are different ways to generalize this form; for example one can obtain planar Superflag this way. The potential term in L_f has different sign then L_b , because anticommuting variables have different complex conjugation properties and we need Lagrangian to be Hermitian. Another thing to note is a second derivative of the fermionic kinetic term, this is not standard and it will later create problems for us as a non-positive definite metric.

Let us first summarize Landau's results for the standard, "bosonic" Landau model. The equation of motion for classical system has the general solution

$$z = z_0 + (\dot{z}_0/\kappa)e^{-i\kappa t} \sin \kappa t, \quad (2.4)$$

so the motion is periodic with angular frequency 2κ . With p being the complex momentum conjugate to z we obtain Hamiltonian

$$L_b = \dot{z}p + \dot{\bar{z}}\bar{p} - H_b, \quad H_b = |p + i\kappa\bar{z}|^2, \quad (2.5)$$

and proceed with quantizations

$$p \rightarrow \hat{p} = -i\partial_z, \quad \bar{p} \rightarrow \hat{\bar{p}} = -i\partial_{\bar{z}}. \quad (2.6)$$

There is a trivial ordering ambiguity ($p\bar{p}$ or $\bar{p}p$) but the natural symmetric ordering

yields quantum Hamiltonian

$$\hat{H}_b = a^\dagger a + \kappa, \quad (2.7)$$

where a and a^\dagger are creation and annihilation operators

$$a = i(\partial_{\bar{z}} + \kappa z), \quad a^\dagger = i(\partial_z - \kappa \bar{z}), \quad [a, a^\dagger] = 2\kappa. \quad (2.8)$$

The ground states, which span the LLL, can be found easily, it is annihilated by a and has energy κ . Higher Landau levels created by acting with a^\dagger on the ground state, and the energy levels can be found to be $E = 2\kappa(N + 1/2)$ for non-negative integer N .

Fermionic model is extremely similar to the bosonic case. It has the general solution

$$\zeta = \zeta_0 + (\dot{\zeta}_0/\kappa)e^{-i\kappa t} \sin \kappa t, \quad (2.9)$$

and the motion is again periodic with period 2κ . Classical Hamiltonian is found to be

$$L_f = -i\dot{\zeta}\pi - i\dot{\bar{\zeta}}\bar{\pi} - H_f, \quad H_f = (\bar{\pi} - \kappa\zeta)(\pi - \kappa\bar{\zeta}), \quad (2.10)$$

where π is the momentum conjugate to ζ , and $\bar{\pi}$ is the complex conjugate of π . Quantization replacement for anticommuting momenta

$$\pi \rightarrow \hat{\pi} = \partial_\zeta, \quad \bar{\pi} \rightarrow \hat{\bar{\pi}} = \partial_{\bar{\zeta}}. \quad (2.11)$$

Once again we have trivial ordering ambiguity and in the fermionic case natural antisymmetric ordering yields the quantum Hamiltonian

$$\hat{H}_f = -\alpha^\dagger \alpha - \kappa, \quad (2.12)$$

where

$$\begin{aligned}\alpha &= (\partial_{\bar{\zeta}} - \kappa\zeta) , & \alpha^\dagger &= (\partial_{\zeta} - \kappa\bar{\zeta}) , \\ \{\alpha, \alpha^\dagger\} &= -2\kappa .\end{aligned}\tag{2.13}$$

Fermionic Hamiltonian (2.12) has four linear independent eigenfunctions. Two of them which have energy $-\kappa$ we denote collectively by Ψ_- , and the other two with energy $+\kappa$ correspondingly Ψ_+ . Ground level wavefunctions Ψ_- can be found from the requirement that they are annihilated by α . Then $\alpha^\dagger\Psi_-$ will give us Ψ_+ , and since α^\dagger is a fermionic operator then $\alpha^\dagger\alpha^\dagger = 0$, and we see that Ψ_+ is being annihilated by α^\dagger . Decomposing each of the function and solving for components it can be found

$$\begin{aligned}\Psi_- &= A_- (1 + \kappa\bar{\zeta}\zeta) + B_- \zeta , \\ \Psi_+ &= A_+ (1 - \kappa\bar{\zeta}\zeta) + B_+ \bar{\zeta} .\end{aligned}\tag{2.14}$$

the natural inner product for wavefunctions on the complex grassman plane is given up to an overall constant factor by

$$\langle \Psi_1, \Psi_2 \rangle = \partial_{\zeta} \partial_{\bar{\zeta}} (\Psi_1^* \Psi_2) .\tag{2.15}$$

Not that this expression is invariant under translations and phase rotations of ζ . It is straightforward to verify that wavefunctions with different energies are orthogonal with respect to this inner product, and that

$$\begin{aligned}\langle \Psi_-, \Psi_- \rangle &= 2\kappa \bar{A}_- A_- + \bar{B}_- B_- , \\ \langle \Psi_+, \Psi_+ \rangle &= -2\kappa \bar{A}_+ A_+ - \bar{B}_+ B_+ .\end{aligned}\tag{2.16}$$

Through the computations we never assigned specific parity to either Ψ_{\pm} or A_{\pm} , B_{\pm} . Usually we would take all coefficients A and B Grassman even in which case all the states of Ψ_- have positive norm and all the states of Ψ_+ have negative norm. But we can otherwise assume that Ψ_- and Ψ_+ have a definite Grassmann parity, then

either the A or the B coefficient is Grassmann-odd. In this case the states of Ψ_- have non-negative norm, but for Ψ_+ situation changes, that now it may have non-negative norm but one of the higher level states will always have negative norm. So whatever assumptions we make about parity of A, B only Ψ_- will always have states of non-negative norm.

We have to be careful about parameter κ , since L_b and L_f behave differently. Setting $\kappa = 0$ will give us simply a free particle on the plane for bosonic case. In the fermionic Landau model setting $\kappa = 0$ because the model becomes unphysical, quantum Hamiltonian \hat{H}_f becomes non-diagonalizable. In the simplest bosonic case this parameter is proportional for magnetic field, but this goes in contradiction with its behavior for fermionic model. So it is possible that κ may have some alternative meaning for a supersymmetric Landau models.

We are now prepared for the Landau model of a particle on the Superplane. The Hamiltonian form of the full Lagrangian L_0 is found by combining (2.5) and (2.10)

$$L_0 = \left(\dot{z}p - i\dot{\zeta}\pi \right) + c.c. - (H_b + H_f). \quad (2.17)$$

We quantize using rules (2.11) and (2.6) together and arrive to the following quantum Hamiltonian

$$H = \partial_{\bar{\zeta}}\partial_{\zeta} - \partial_z\partial_{\bar{z}} + \kappa \left(\bar{z}\partial_{\bar{z}} + \bar{\zeta}\partial_{\bar{\zeta}} - z\partial_z - \zeta\partial_{\zeta} \right) + \kappa^2 \left(z\bar{z} + \zeta\bar{\zeta} \right). \quad (2.18)$$

Recalling definitions of the boson and fermion creation and annihilation operators we express H in operator form

$$H = a^\dagger a - \alpha^\dagger \alpha. \quad (2.19)$$

The quantum Hamiltonian has energy levels $2\kappa N$ for non-negative integer N . Lowest Landau level (LLL) states have zero energy and are annihilated by both a and α

which bring us the following form

$$\psi^{(0)} = e^{-\kappa K_2} \psi_{an}^{(0)}(z, \zeta), \quad \text{where} \quad K_2 = |z|^2 + \zeta \bar{\zeta}. \quad (2.20)$$

Here $\psi_{an}^{(0)}(z, \zeta)$ is an analytic function, it is an arbitrary function of z, ζ and does not depend on $\bar{z}, \bar{\zeta}$. The first excited states will have energy 2κ , and are linear combinations of states $a^\dagger \psi^{(0)}$ and $\alpha^\dagger \psi^{(0)}$. The wavefunctions at higher Landau levels have energy $E_N = 2\kappa N$ and obtained by multiple action of a^\dagger on the first level states. Thus N -th level have wavefunction

$$\psi^{(N)} = (-ia^\dagger)^N e^{-\kappa K_2} \psi_+^{(N)}(z, \zeta) - N (-ia^\dagger)^{N-1} \alpha^\dagger e^{-\kappa K_2} \psi_-^{(N)}(z, \zeta), \quad (2.21)$$

where $\psi_\pm(z, \zeta)$ are two analytic functions of z and ζ ; we can write them as

$$\psi_\pm^{(N)}(z, \zeta) = A_\pm^{(N)}(z) + \zeta B_\pm^{(N)}(z). \quad (2.22)$$

Coefficients A and B are analytical functions of z which make manifest four-fold degeneracy of excited states. By analogy with pure fermionic model we can guess that excited states will have some states of negative norm and only ground level be purely positive definite. Let's show this explicitly. The natural invariant inner product is

$$\langle \phi | \psi \rangle = \int d\mu \overline{\phi(z, \bar{z}; \zeta, \bar{\zeta})} \psi(z, \bar{z}; \zeta, \bar{\zeta}), \quad \text{where} \quad d\mu = dz d\bar{z} d\zeta d\bar{\zeta}. \quad (2.23)$$

For the N -th level eigenfunction one can find

$$\langle \psi^{(N)} | \psi^{(N)} \rangle = (2\kappa)^N N! \left[-N \left\| \psi_-^{(N)} \right\|^2 + \left\| \psi_+^{(N)} \right\|^2 \right], \quad (2.24)$$

where we have defined

$$\left\| \phi_{an} \right\|^2 \equiv \int d\mu e^{-2\kappa K_2} \overline{\phi_{an}} \phi_{an} \quad (2.25)$$

for any analytic function, or superfield, $\phi_{an}(z, \zeta)$. Substituting component expansion of $\psi_{\pm}^{(N)}$ into the last integral and performing integration over grassman variables (we have to be careful with order as coefficients A and B may be odd, depending on parity of ψ), we arrive at

$$\left\| \psi_{\pm}^{(N)} \right\|^2 = \int dz d\bar{z} e^{-2\kappa|z|^2} \left(2\kappa \overline{A_{\pm}^{(N)}}(z) A_{\pm}^{(N)}(z) + \overline{B_{\pm}^{(N)}}(z) B_{\pm}^{(N)}(z) \right), \quad (2.26)$$

so the minus sign in (2.24) indeed implies an indefinite norm. This concludes detailed analysis of Superplane model with naive norm on the superspace.

2.2 Symmetries and geometrical interpretation of a Superplane model

With the Superplane Landau model in hand we can write down its symmetries, which are inherited from the Superplane. Our superspace is parameterized by two complex variables (z, ζ) so that there will be super-translations $(P, P^\dagger, \Pi, Pi^\dagger)$, $SU(1|1)$ super-rotations (Q, Q^\dagger, C) , and an independent $U(1)$ phase rotation J . Infinitesimal transformations for each of this operators are straightforward and follow from its definition. We will skip expressions for infinitesimal transformation of the phase space and present differential form of these operators. In the end we will write down Cartan form for the coset of symmetry group and will be able to recreate exactly the same Superplane Lagrangian from the geometrical point of view. For the more detailed explanations refer to [7].

Super-translations generated on the phase space by the operators

$$\begin{aligned} P &= -i(\partial_z + \kappa\bar{z}), & P^\dagger &= -i(\partial_{\bar{z}} - \kappa z) \\ \Pi &= \partial_\zeta + \kappa\bar{\zeta}, & \Pi^\dagger &= \partial_{\bar{\zeta}} + \kappa\zeta. \end{aligned} \quad (2.27)$$

Their non-zero (anti)commutation relations are

$$[P, P^\dagger] = 2\kappa, \quad \{\Pi^\dagger, \Pi\} = 2\kappa. \quad (2.28)$$

Here κ make its appearance as a central charge of our algebra. For the commutation relations we will later use notation $Z = \kappa$. The $SU(1|1)$ super-rotation transformations include two odd and one even generator

$$Q = z\partial_\zeta - \bar{\zeta}\partial_{\bar{z}}, \quad Q^\dagger = \bar{z}\partial_{\bar{\zeta}} + \zeta\partial_z, \quad (2.29)$$

$$C = z\partial_z + \zeta\partial_\zeta - \bar{z}\partial_{\bar{z}} - \bar{\zeta}\partial_{\bar{\zeta}}. \quad (2.30)$$

The only non-zero (anti)commutation relations of these generators is

$$\{Q, Q^\dagger\} = C. \quad (2.31)$$

Finally we have an independent $U(1)$ phase rotation which is generated by the Hermitian operator

$$J = \frac{1}{2} [z\partial_z - \zeta\partial_\zeta - \bar{z}\partial_{\bar{z}} + \bar{\zeta}\partial_{\bar{\zeta}}]. \quad (2.32)$$

The rest of non-zero commutation relations

$$[Q, P] = i\Pi, \quad \{Q^\dagger, \Pi\} = iP, \quad [C, P] = -P, \quad [C, \Pi] = -\Pi. \quad (2.33)$$

$$[J, Q] = Q, \quad [J, Q^\dagger] = -Q^\dagger, \quad [J, P] = -P, \quad [J, \Pi] = \Pi. \quad (2.34)$$

The supergroup generated by the five even charges (P, P^\dagger, C, J, Z) and the four odd charges $(\Pi, \Pi^\dagger, Q, Q^\dagger)$ with (anti)commutation relations (2.28), (2.31), (2.33), (2.34) will be called $IU(1|1)$. Superplane is parametrized by (z, ζ) which correspond to P, Π , so we can view Superplane as the coset superspace $IU(1|1)/[U(1|1) \times \mathcal{Z}]$. It is

important to note that $IU(1|1)$ is a contraction of $SU(2|1)$. This is a hint for possible generalization of a Superplane model on $IU(1|1)$ to a supersphere model on $SU(2|1)$.

We just discussed very straightforward and quantum mechanically transparent way of generalizing standard Landau model to a supersymmetric model on Superplane. But with the knowledge of the symmetries we can turn to a geometrical point of view and observe how the same Lagrangian can be reproduced from the differential properties of the supermanifold. This is an important example, because the same method is used to construct Landau model on the Superflag, but that case is computationally much harder.

Consider a coset superspace

$$\mathcal{K} = IU(1|1)/[U(1) \times U(1) \times \mathcal{Z}]. \quad (2.35)$$

Here $U(1) \times U(1)$ are generators C and J from above, and Z is a central charge. We parametrize coset space by one even and two odd coordinates (u, η^1, η^2) and define its exponential representation as

$$g = e^{\mathcal{A}_1} e^{\mathcal{A}_2}, \quad (2.36)$$

where

$$\mathcal{A}_1 = \eta^1 \Pi - \eta^2 Q + \bar{\eta}_1 \Pi^\dagger - \bar{\eta}_2 Q^\dagger, \quad \mathcal{A}_2 = -iuP - i\bar{u}P^\dagger. \quad (2.37)$$

Definition of the exponential can be changed, but in the end it will give the same result. In our case we define odd coordinates so that they will anticommute with odd generators and signs in (2.37) are chosen for later convenience. We also use different

set of coordinates (u, η^1, η^2) which is related to our old set (z, ζ, ξ) by

$$u = z - \frac{1}{2}\zeta\bar{\xi}, \quad \eta^1 = \zeta + z\xi - \frac{1}{3}\bar{\xi}\xi, \quad \eta^2 = \xi. \quad (2.38)$$

The left-covariant Cartan forms and the superconnections on the stability subgroup are defined by

$$g^{-1}dg = i\omega_P P + i\bar{\omega}_P P^\dagger + \omega^1 \Pi + \bar{\omega}_1 \Pi^\dagger - \omega^2 Q - \bar{\omega}_2 Q^\dagger + A_C C + A_{2\kappa} \kappa. \quad (2.39)$$

There appear no connection associated with J . Other connections and 1-forms are given by

$$\begin{aligned} \omega_P &= - \left(1 + \frac{1}{2}\bar{\xi}\xi \right) dz - \bar{\xi}d\zeta, & \omega^1 &= \xi dz + \left(1 - \frac{1}{2}\bar{\xi}\xi \right) d\zeta, & \omega^2 &= d\xi, \\ A_{2\kappa} &= - (\bar{z} dz - z d\bar{z} - \bar{\zeta} d\zeta - \zeta d\bar{\zeta}), & A_C &= \frac{1}{2} (\xi d\bar{\xi} + \bar{\xi} d\xi). \end{aligned} \quad (2.40)$$

We can use this expressions and combine them into Lagrangian. We would generally use square of 1-forms to make a kinetic term and connections to make a potential term of the Lagrangian. In particular our Superplane Lagrangian can be obtained in a manifestly invariant form in terms of pullbacks of the above Cartan forms and is given by

$$L_0 = |\hat{\omega}_P|^2 + \hat{\omega}^1 \hat{\omega}_1 + i\kappa \hat{A}_{2\kappa}. \quad (2.41)$$

Here the ‘‘hat’’ denotes a pullback. It can be easily checked that L_0 is independent of $\xi, \bar{\xi}$ variables. With a different choice of Lagrangian we can create planar Superflag Landau model. In fact it’s Lagrangian looks simpler in this form because it doesn’t have term $\hat{\omega}^1 \hat{\omega}_1$.

2.3 Alternative metric for Superplane model

With the complete analysis of a Superplane Landau model we now ask ourselves - what can be done about non-positive definite metric? Consider some abstract quantum model with a complete system of energy eigenvectors $|f_A\rangle$ for the Hamiltonian, H , which obey

$$\langle f_A | f_B \rangle = (-)^{g(A)} \delta_{AB}, \quad g(A) = \begin{cases} 0 & : A = a \\ 1 & : A = \alpha \end{cases}. \quad (2.42)$$

Here $g(A)$ is the grading which indicate positive norm states with $A = a$ and negative norm states with $A = \alpha$, thus it should not be confused with grassman parity. Hamiltonian of our system will be taken to be Hermitian $H = H^\dagger$, where operation (\dagger) is a naive hermitian conjugation with respect to original non-positive-definite inner product. In order to improve inner product we introduce a ‘metric operator’ G and define it in such a way so that it would take care of a minus sign where needed

$$G |f_A\rangle \equiv |Gf_A\rangle = (-)^{g(A)} |f_A\rangle, \quad G = G^\dagger. \quad (2.43)$$

It is easy to notice from this definition that $[H, G] = 0$. Now new ‘corrected’ inner product is defined by

$$\langle\langle f_A | f_B \rangle\rangle \equiv \langle Gf_A | f_B \rangle = \delta_{AB}. \quad (2.44)$$

We will use double dagger sign for the ‘improved’ hermitian conjugate \mathcal{O}^\ddagger , with respect to $\langle\langle \dots \rangle\rangle$. Thus for any operator \mathcal{O}

$$\begin{aligned} \langle\langle \mathcal{O}^\ddagger f_A | f_B \rangle\rangle &= \langle\langle f_A | \mathcal{O} | f_B \rangle\rangle = \langle Gf_A | \mathcal{O} | f_B \rangle = \\ &= \langle \mathcal{O}^\ddagger Gf_A | f_B \rangle = \langle G (G^{-1} \mathcal{O}^\ddagger G) f_A | f_B \rangle, \end{aligned} \quad (2.45)$$

where

$$\mathcal{O}^\ddagger \equiv G^{-1}\mathcal{O}^\dagger G = \mathcal{O}^\dagger + S_{\mathcal{O}}. \quad (2.46)$$

Here $S_{\mathcal{O}}$ we will call a “shift operator” for a given \mathcal{O} . It shows difference between conjugation properties with respect to naive and corrected metrics and it is given by

$$S_{\mathcal{O}} \equiv G^{-1} [\mathcal{O}^\dagger, G]. \quad (2.47)$$

Only operators which commute with H will have the same conjugation in each inner product. Not that G is hermitian $G = G^\dagger$, which implies $(\mathcal{O}^\ddagger)^\ddagger = \mathcal{O}$. Shift operator have convenient property

$$S_{\mathcal{O}^\ddagger} = -S_{\mathcal{O}}^\dagger. \quad (2.48)$$

As a consequence, we have

$$\tilde{\mathcal{O}} \equiv \mathcal{O} + \frac{1}{2}S_{\mathcal{O}}^\dagger, \text{ then } \tilde{\mathcal{O}}^\ddagger = \tilde{\mathcal{O}}^\dagger. \quad (2.49)$$

We will extensively use properties of G , $S_{\mathcal{O}}$ and $\tilde{\mathcal{O}}$ later on as well as a following idea

Lemma 1. *Since $[G, H] = 0$, the Hamiltonian H is hermitian in both inner products, $H = H^\dagger = H^\ddagger$. Moreover, if the operator \mathcal{O} is a constant of motion, then the corresponding shift operator is also a constant of motion. Indeed, from $[\mathcal{O}, H] = 0$ it follows that $[\mathcal{O}^\dagger, H] = 0$ and $[\mathcal{O}^\ddagger, H] = 0$. This is a signal that the algebra of operators which are in involution with the Hamiltonian may be larger than originally assumed: the system may have some ‘hidden’ symmetries.*

Let’s make use of this method in the Superplane model. In (2.24) only ψ_- component generated negative norm states. We are going to guess metric function G now. Now lets look at (2.21), and note how does each analytic function appear in the

expression. Let's see how \hat{H}_f from (2.12) acts on each of them

$$(-\alpha^\dagger\alpha - \kappa)\psi_+^{(0)} = -\kappa\psi_+^{(0)}, \quad (2.50)$$

$$\begin{aligned} (-\alpha^\dagger\alpha - \kappa)\alpha^\dagger\psi_-^{(0)} &= (-\alpha^\dagger\alpha\alpha^\dagger - \kappa\alpha^\dagger)\psi_-^{(0)} = \\ &= (-\alpha^\dagger\{\alpha, \alpha^\dagger\} - \kappa\alpha^\dagger)\psi_-^{(0)} = \kappa\alpha^\dagger\psi_-^{(0)}. \end{aligned} \quad (2.51)$$

Basically \hat{H}_f acts as $-\kappa$ on $\psi_+^{(N)}$ which is positive definite and as κ on $\psi_-^{(N)}$ which is negative definite. Thus let us postulate $G = -\kappa^{-1}H_f$ or in differential form

$$G = \frac{1}{\kappa} [\partial_\zeta\partial_{\bar{\zeta}} + \kappa^2\bar{\zeta}\zeta + \kappa(\zeta\partial_\zeta - \bar{\zeta}\partial_{\bar{\zeta}})]. \quad (2.52)$$

It is easy to see that G commutes with H, a, a^\dagger , but does not commute with α, α^\dagger . Note that $\hat{H}_f\hat{H}_f = \kappa^2$ which will give us $G^{-1} = G$. Using (2.46) we can easily find

$$\alpha^\ddagger = -\alpha^\dagger, \quad (2.53)$$

The Hamiltonian may now be written in the manifestly positive form

$$H = a^\dagger a + \alpha^\ddagger \alpha. \quad (2.54)$$

This far we just fixed inner product. But this operation brings out new symmetry properties. To investigate things further we need to look at the $ISU(1|1)$ symmetry generators, it turns out that metric operator commutes with all of them except Q, Q^\dagger . Thus modified hermitian conjugation

$$Q^\ddagger = Q^\dagger - \frac{i}{\kappa} S. \quad (2.55)$$

Shift operator could be found using (2.47) and is given by

$$S = i (\partial_z \partial_{\bar{z}} + \kappa^2 \bar{z} \zeta - \kappa \bar{z} \partial_{\bar{z}} - \kappa \zeta \partial_z). \quad (2.56)$$

This expression for S could be conveniently written through creation/destruction operators, and together with S^\dagger we obtain

$$S = a^\dagger \alpha, \quad S^\dagger = a \alpha^\dagger. \quad (2.57)$$

These operators follow anticommutations of $N = 2$ supersymmetry algebra

$$\{S, S^\dagger\} = 2\kappa H, \quad \{S, S\} = 0 = \{S^\dagger, S^\dagger\}, \quad (2.58)$$

Both S and S^\dagger annihilate LLL leaving worldline supersymmetry unbroken. This is an additional previously ‘hidden’ symmetry which commute with original symmetry group. We also have to modify $ISU(1|1)$, in order to make it consistent with new conjugation. Instead of Q and C we will now use

$$\tilde{Q} = Q - \frac{i}{2\kappa} S^\dagger, \quad \tilde{C} = C + \frac{1}{2\kappa} H. \quad (2.59)$$

This operator \tilde{Q} now commutes with G , thus $\tilde{Q}^\dagger = \tilde{Q}^\dagger$. Tis operators follow the same anticommuting properties as before

$$\{\tilde{Q}, \tilde{Q}^\dagger\} = \tilde{C}. \quad (2.60)$$

This is final result given for Landau model on Superplane. We was able to fully solve quantum mechanical problem, analyse its solutions and symmetries, modify norm so that all eigenvectors are positive definite. Last procedure brings out hidden $N = 2$ symmetry of the problem. Our system had this symmetry from the beginning, but

because of our choice of coordinates it was hidden.

We discussed Superplane Landau model with all details, so that it can serve as a toy model and step for the more complicated cases. For Landau model on the planar Superflag we will have very similar result with extra $N = 2$ supersymmetry just with more complicated expressions. It is also important to note that for planar Superflag case we will have additional parameter M and hidden supersymmetry manifests unbroken for $M < 0$. We will not go any more deeper in that case. All details on planar Superflag model can be found in [7] and [8].

2.4 Construction of the Superflag and superpsphere model

Now we turn to analysis of the Superflag Landau model. During discussion of the Landau model on the Superplane it was shown how we can obtain its Lagrangian using differential geometry on the coset space. We are going to use the same idea for the Superflag Landau model. The name Superflag describe $SU(2|1)/[U(1) \times U(1)]$ coset space similarly to purely bosonic Flag manifold $SU(3)/[U(1) \times U(1)]$. Model on the flag manifold was discussed in [21], and we are following the same steps in creating supersymmetric generalization of it. Main idea is to calculate Cartan forms and $U(1)$ connections, and then make an invariant Lagrangian from them.

The group $SU(2|1)$ acts linearly on vectors in a vector superspace of dimension $(1|2)$, for definition of the algebra refer to the appendix B. A simple choice of basis in this superspace is provided by the supermatrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -\xi^2 & 1 & 0 \\ -\xi^1 & z & 1 \end{pmatrix}, \quad (2.61)$$

where z is a complex variable and ξ^i ($i = 1, 2$) are complex anticommuting variables, with complex conjugates $\bar{\xi}_i$. Later we will change coordinates into ξ, ζ . Although this supermatrix is simple looking, before we proceed with calculations of Cartan forms we need to make it into unitary supermatrix with orthonormal columns. Applying Gramm-Schmidt procedure on the last expression we can find

$$U = \left(\frac{1}{K_1} \begin{bmatrix} 1 \\ -\xi^2 \\ 1 \\ -\xi^1 \end{bmatrix} \left(\frac{K_1}{K_2} \right)^{\frac{1}{2}} \begin{bmatrix} (\bar{\xi}_2 + z\bar{\xi}_1) / K_1^2 \\ 1 - \bar{\xi}_1 (\xi^1 - z\xi^2) \\ z + \bar{\xi}_2 (\xi^1 - z\xi^2) \end{bmatrix} \frac{1}{K_2^{\frac{1}{2}}} \begin{bmatrix} \bar{\xi}_1 - \bar{z}\bar{\xi}_2 \\ -\bar{z} \\ 1 \end{bmatrix} \right), \quad (2.62)$$

where

$$K_1 = 1 + \bar{\xi}_1 \xi^1 + \bar{\xi}_2 \xi^2, \quad K_2 = 1 + \bar{z}z + (\xi^1 - z\xi^2) (\bar{\xi}_1 - \bar{z}\bar{\xi}_2). \quad (2.63)$$

Last expression defines U such that $U \in SU(2|1)$ and it also provides parametrization of the coset superspace $SU(2|1)/[U(1) \times U(1)]$. Now to compute the Cartan forms and $U(1)$ connections

$$U^{-1}dU \equiv \Omega = \begin{pmatrix} 0 & \bar{E}_2 & \bar{E}_1 \\ -E^2 & 0 & -\bar{E}_+ \\ -E^1 & E^+ & 0 \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \mathcal{B} & 0 & 0 \\ 0 & \mathcal{B} - \mathcal{A} & 0 \\ 0 & 0 & \mathcal{A} \end{pmatrix}. \quad (2.64)$$

The Cartan 1-forms are $E^A = (E^+, E^1, E^2)$ and their complex conjugates are

$\bar{E}_A = (\bar{E}_+, \bar{E}_1, \bar{E}_2)$. One can find

$$\begin{aligned}
E^+ &= K_1^{-\frac{1}{2}} K_2^{-1} [dz - K_1^{-1} (d\xi^1 - zd\xi^2) (\bar{\xi}_2 + z\bar{\xi}_1)], \\
E^1 &= (K_1 K_2)^{-\frac{1}{2}} [d\xi^1 - zd\xi^2], \\
E^2 &= K_2^{-\frac{1}{2}} [d\xi^1 (\bar{z} - \xi^2 (\bar{\xi}_1 - \bar{z}\bar{\xi}_2)) + d\xi^2 (1 + \xi^1 (\bar{\xi}_1 - \bar{z}\bar{\xi}_2))] . \quad (2.65)
\end{aligned}$$

For the $U(1)$ connections \mathcal{A} and \mathcal{B} we have, similarly, that

$$\mathcal{A} = -idZ^M \partial_M \log K_2 + c.c., \quad \mathcal{B} = idZ^M \partial_M \log K_1 + c.c., \quad (2.66)$$

where $Z^M = (z, \xi^1, \xi^2)$ are the complex coordinates and $\bar{Z}_M = (\bar{z}, \bar{\xi}_1, \bar{\xi}_2)$ their complex conjugates. Now we rewrite 1-forms $(E^A, \mathcal{A}, \mathcal{B})$ as the corresponding 1-forms on the particle's worldline (i.e. go to dt from $dz, d\xi^1, d\xi^2$). Thus, we now have

$$E^A = dt \omega^A, \quad \omega^A \equiv \dot{z} E_z^A + \dot{\xi}^i E_i^A \quad (2.67)$$

and

$$\begin{aligned}
\mathcal{A} &= dt A, \quad A \equiv [\dot{z} \mathcal{A}_z + \dot{\xi}^i \mathcal{A}_i] + c.c., \\
\mathcal{B} &= dt B \quad B \equiv \dot{\xi}^i \mathcal{B}_i + c.c.. \quad (2.68)
\end{aligned}$$

In principle all the coefficients $\omega^A = (\omega^+, \omega^1, \omega^2)$ can be used to construct $SU(2|1)$ -invariant kinetic terms, but we can observe that in (2.65) E^i has $d\xi^i$ appear by itself without another grassman coordinate and in E^+ $d\xi^i$ appear only in fermion-bilinear terms. This means that terms quadratic in ω^i will be higher-derivative for fermion

variables. So we will use only ω^+ for the kinetic term

$$\omega^+ = \dot{z}\omega + \xi^i \omega_i, \quad (2.69)$$

where

$$\begin{aligned} \omega &= K_1^{-\frac{1}{2}} K_2^{-1}, \\ \omega_1 &= -K_1^{-\frac{3}{2}} K_2^{-1} (\bar{\xi}_2 + z\bar{\xi}_1), \\ \omega_2 &= K_1^{-\frac{3}{2}} K_2^{-1} z (\bar{\xi}_2 + z\bar{\xi}_1). \end{aligned} \quad (2.70)$$

In addition to the kinetic term, there are two possible WZ terms that we may construct from A and B .

$$\begin{aligned} \mathcal{A}_z &= -iK_2^{-1} [\bar{z} - \xi^2 (\bar{\xi}_1 - \bar{z}\bar{\xi}_2)], \\ \mathcal{A}_1 &= -iK_2^{-1} (\bar{\xi}_1 - \bar{z}\bar{\xi}_2), \\ \mathcal{A}_2 &= iK_2^{-1} z (\bar{\xi}_1 - \bar{z}\bar{\xi}_2), \end{aligned} \quad (2.71)$$

and

$$\mathcal{B}_i = -iK_1^{-1} \bar{\xi}_i. \quad (2.72)$$

Finally we can write Lagrangian as

$$L = |\omega^+|^2 + NA + MB, \quad (2.73)$$

where N and M are two constants. For more details please refer to [6]. In that paper Superflag Landau model was solved and negative-norm states were confirmed.

Chapter 3

Landau Models on the superpshere and Superflag manifolds

With Superplane Landau model fully developed we can turn to the more complicated cases. In this section we will solve Landau models on the superpshere $SU(2|1)/U(1|1)$ and Superflag $SU(2|1)/U(1) \times U(1)$ manifolds. As the negative norm states will appear, we will find ‘metric operator’ to make a new positive definite scalar product, in the similar way it was done to for the planar case.

We will start with construction of Lagrangian for the model on a superpshere, then we will find eigenfunctions and energy levels and calculate scalar product of eigenfunctions in naive metric. It will show clearly that scalar product is not positive definite. After this we will move to the model on the Superflag and go through the same steps. It turns out that norm of eigenfunction in the Superflag case can be made into norm on the superpshere case by fixing one parameter and a transformation of another. This make superpshere equivalent to the specific case of the Superflag, and for the rest of the chapter we will concentrate on Superflag.

Successfully finding metric operator and fixing norm we will recover some new symmetries. This ‘hidden’ symmetries are different from the Superplane case in the way that we don’t have a worldline supersymmetry anymore, instead we have our off-shell symmetry algebra expanded from $SU(2|1)$ to $SU(2|2)$.

3.1 Model on the superpsphere

The model

The Riemann superpsphere $\mathbb{C}\mathbb{P}^{(1|1)} \cong SU(2|1)/U(1|1)$ is a complex supermanifold with complex coordinates

$$Z^A = (Z^0, Z^1) = (z, \zeta), \quad \bar{Z}^{\bar{B}} = (\bar{Z}^0, \bar{Z}^1) = (\bar{z}, \bar{\zeta}), \quad (3.1)$$

where z is a complex coordinate of the Riemann sphere, with complex conjugate \bar{z} , and ζ is its anti-commuting partner, with complex conjugate $\bar{\zeta}$. For the Superflag later we will have set of three complex coordinates (z, ζ, ξ) . Note that this is different from the coordinates used before. This change is made for later convenience when we will switch to analytic subspace on Superflag. These two sets of coordinates are connected by

$$\xi^1 = \zeta + z\xi, \quad \xi^2 = \xi. \quad (3.2)$$

We are not giving commutation relations of $SU(2|1)$ yet. All of the properties of algebras used are collected in the appendix (B).

The Riemann superpsphere is not only a complex supermanifold but also a Kähler supermanifold, with Kähler 2-form

$$\mathcal{F} = 2i dZ^A \wedge d\bar{Z}^{\bar{B}} \partial_{\bar{B}} \partial_A \mathcal{K}, \quad (3.3)$$

where

$$\mathcal{K} = \log(1 + z\bar{z} + \zeta\bar{\zeta}) \quad (3.4)$$

is the Kähler potential, which is real because the usual convention for complex conju-

gation of products of anti-commuting variables implies that $(\partial_\zeta)^* = -\partial_{\bar{\zeta}}$, and hence that

$$(\partial_{\bar{B}}\partial_A\mathcal{K})^* = (-1)^{a+b}(\partial_{\bar{B}}\partial_A\mathcal{K}). \quad (3.5)$$

Here a is the Grassmann parity associated with the A or \bar{A} index; i.e. $a = 0$ for $A = 0$ and $\bar{A} = 0$, and $a = 1$ for $A = 1$ and $\bar{A} = 1$ (to avoid ambiguities with this simplified notation, one must arrange for all barred indices to have letters that differ from those of unbarred indices, but this restriction is easily accommodated).

The Kähler 2-form may be written locally as $\mathcal{F} = d\mathcal{A}$, where

$$\mathcal{A} = -i \left(dZ^A \partial_A - d\bar{Z}^{\bar{B}} \partial_{\bar{B}} \right) \mathcal{K} \equiv dZ^A \mathcal{A}_A + d\bar{Z}^{\bar{B}} \mathcal{A}_{\bar{B}} \quad (3.6)$$

is the Kähler connection. It turns out that connection is $SU(2|1)$ invariant. The Kähler metric of the Riemann supersphere is

$$dZ^A d\bar{Z}^{\bar{B}} g_{\bar{B}A} = dZ^A d\bar{Z}^{\bar{B}} \partial_{\bar{B}} \partial_A \mathcal{K}. \quad (3.7)$$

It is manifestly Kähler gauge invariant, and hence $SU(2|1)$ invariant. Before proceeding we record, for future use, the components of the metric and inverse metric. The metric components are

$$\begin{aligned} g_{\bar{z}z} &= \frac{1 + \zeta\bar{\zeta}}{(1 + z\bar{z} + \zeta\bar{\zeta})^2}, & g_{z\zeta} &= -\frac{z\bar{\zeta}}{(1 + z\bar{z})^2}, \\ g_{\bar{\zeta}z} &= \frac{\bar{z}\zeta}{(1 + z\bar{z})^2}, & g_{\bar{\zeta}\zeta} &= \frac{1}{1 + z\bar{z}}. \end{aligned} \quad (3.8)$$

The inverse metric components are

$$\begin{aligned} g^{z\bar{z}} &= (1 + z\bar{z})(1 + z\bar{z} + \zeta\bar{\zeta}), & g^{z\bar{\zeta}} &= (1 + z\bar{z})z\bar{\zeta}, \\ g^{\zeta\bar{z}} &= -(1 + z\bar{z})\bar{z}\zeta, & g^{\zeta\bar{\zeta}} &= 1 + z\bar{z}(1 - \zeta\bar{\zeta}). \end{aligned} \quad (3.9)$$

The metric $g_{\bar{B}A}$ and its inverse $g^{A\bar{B}}$ are related by the conditions

$$g^{A\bar{B}}g_{\bar{B}C} = \delta^A_C, \quad g_{\bar{B}C}g^{C\bar{A}} = \delta_{\bar{B}}^{\bar{A}}. \quad (3.10)$$

We are going to use metric to write kinetic term and connections for potential terms in the Lagrangian. In general the classical Lagrangian of the superspherical Landau model is

$$L = \dot{Z}^A \dot{\bar{Z}}^{\bar{B}} g_{\bar{B}A} + N \left(\dot{Z}^A \mathcal{A}_A + \dot{\bar{Z}}^{\bar{B}} \mathcal{A}_{\bar{B}} \right), \quad (3.11)$$

where the overdot indicates differentiation with respect to an independent variable, which we interpret as time. Observe that L is real as a consequence of (3.5). The $SU(2|1)$ variation of this Lagrangian is a total time derivative, for any real number N , so our model is really $SU(2|1)$ -invariant. Constant $2N$ can be interpreted as the particle's electric charge, after moving to the quantum theory we will see that it requires $2N$ to be an integer.

We are following the same procedure as before. From Lagrangian we derive Hamiltonian, then we quantize it and try to rewrite it in the form which will permit factorization. The last part will be solving for eigenvalues and eigenvectors of factorized Hamiltonian.

We will proceed directly to the Hamiltonian form of the Lagrangian,

$$L = \dot{Z}^A P_A + \dot{\bar{Z}}^{\bar{B}} P_{\bar{B}} - (P_A - N \mathcal{A}_A) g^{A\bar{B}} (P_{\bar{B}} - N \mathcal{A}_{\bar{B}}), \quad (3.12)$$

where the inverse metric is defined in (3.9), (3.10) and the conjugate momenta are

$$P_A = (p_z, -i\pi_\zeta), \quad P_{\bar{B}} = (p_{\bar{z}}, -i\pi_{\bar{\zeta}}). \quad (3.13)$$

Here, $p_{\bar{z}}$ is the complex conjugate of p_z and $\pi_{\bar{\zeta}}$ is the complex conjugate of π_ζ ; the

factors of $-i$ are needed for this as a consequence of the rule for complex conjugation of products of anti-commuting variables (A.6), so we can write for generalized momenta

$$(P_A)^* = (-1)^a P_{\bar{A}}. \quad (3.14)$$

Since the inverse metric behaves in the same way as the metric under complex conjugation, one sees that the new Lagrangian, in Hamiltonian form, is real, and one may verify that elimination of the momenta returns us to the Lagrangian (3.11). We may now read off the classical Hamiltonian, which we rewrite as

$$\begin{aligned} H_{class} &= (P_A - N\mathcal{A}_A) g^{A\bar{B}} (P_{\bar{B}} - N\mathcal{A}_{\bar{B}}) \\ &= (-1)^{a(a+b)} g^{A\bar{B}} (P_A - N\mathcal{A}_A) (P_{\bar{B}} - N\mathcal{A}_{\bar{B}}). \end{aligned} \quad (3.15)$$

To quantize, we make the replacements

$$p_z \rightarrow -i\partial_z, \quad p_{\bar{z}} \rightarrow -i\partial_{\bar{z}}, \quad \pi_\zeta \rightarrow \partial_\zeta, \quad \pi_{\bar{\zeta}} \rightarrow \partial_{\bar{\zeta}}, \quad (3.16)$$

which imply

$$P_A \rightarrow -i\partial_A, \quad P_{\bar{B}} \rightarrow -i\partial_{\bar{B}}. \quad (3.17)$$

This yields the quantum Hamiltonian

$$H = -(-1)^{a(a+b)} g^{A\bar{B}} \nabla_A^{(N)} \nabla_{\bar{B}}^{(N)}, \quad (3.18)$$

where

$$\nabla_A^{(N)} = \partial_A - N(\partial_A \mathcal{K}), \quad \nabla_{\bar{B}}^{(N)} = \partial_{\bar{B}} + N(\partial_{\bar{B}} \mathcal{K}). \quad (3.19)$$

These covariant derivatives have the super-commutator

$$\nabla_{\bar{B}}^{(N)} \nabla_A^{(\tilde{N})} - (-1)^{ab} \nabla_A^{(\tilde{N})} \nabla_{\bar{B}}^{(N)} = -\left(N + \tilde{N}\right) g_{\bar{B}A}, \quad (3.20)$$

with all other super-commutators equal to zero. For further use, we present here the explicit expressions for $\nabla_A^{(N)}, \nabla_B^{(N)}$

$$\begin{aligned}\nabla_z^{(N)} &= \partial_z - N \frac{\bar{z}}{1 + z\bar{z} + \zeta\bar{\zeta}}, & \nabla_{\bar{z}}^{(N)} &= \partial_{\bar{z}} + N \frac{z}{1 + z\bar{z} + \zeta\bar{\zeta}}, \\ \nabla_\zeta^{(N)} &= \partial_\zeta - N \frac{\bar{\zeta}}{1 + z\bar{z} + \zeta\bar{\zeta}}, & \nabla_{\bar{\zeta}}^{(N)} &= \partial_{\bar{\zeta}} - N \frac{\zeta}{1 + z\bar{z} + \zeta\bar{\zeta}}.\end{aligned}\quad (3.21)$$

The $SU(2|1)$ invariance of the model can be made manifest by writing the Hamiltonian (3.18) in terms of the Casimir operators. We substitute differential representations of algebra generators into Casimir operators and compare it with Hamiltonian operator. One finds that

$$H = C_2. \quad (3.22)$$

The spectrum

It is not clear from the beginning how to apply factorization method to supersphere. Moreover, the lowest Landau level (LLL) is known from earlier work [5]; in the present context, in which we have chosen an operator ordering such that the ground state energy is zero, the LLL wave functions are components of a superfield $\Psi_0^{(N)}$, satisfying the analyticity constraint

$$\nabla_B^{(N)} \Psi_0^{(N)} = 0, \quad (3.23)$$

and they carry an irreducible superspin N representation of $SU(2|1)$ that decomposes into the reducible $(N - 1/2) \oplus N$ representation of $SU(2)$.

At $\ell = 1$ we have the superfield wave function

$$\Psi_1^{(N)} = \nabla_C^{(N+1)} \Phi^C. \quad (3.24)$$

After acting with H on this wave function, we move the $\nabla_{\bar{B}}^{(N)}$ derivative to the right, where it annihilates Φ^C , but we pick up a super-commutator term, which we simplify using (3.20). The result is

$$H\Psi_1^{(N)} = (2N + 1)g^{A\bar{B}}\nabla_A^{(N)}g_{\bar{B}C}\Phi^C. \quad (3.25)$$

Now we use the identity

$$(-1)^{a(a+b)}g^{A\bar{B}}\nabla_A^{(N)}g_{\bar{B}C} = \nabla_C^{(N+1)}, \quad (3.26)$$

which itself is a consequence of the identity

$$(-1)^{a(a+b)}g^{A\bar{B}}(\partial_A g_{\bar{B}C}) = -\partial_C \mathcal{K}. \quad (3.27)$$

The result is that $\Psi_1^{(N)}$ is an eigenfunction of H with energy eigenvalue $(2N + 1)$. At $\ell = 2$ we have the superfield wave function

$$\Psi_2^{(N)} = \nabla_D^{(N+1)}\nabla_C^{(N+3)}\Phi^{CD}. \quad (3.28)$$

After acting with H on this superfield we again move $\nabla_{\bar{B}}^{(N)}$ to the right, where it annihilates the chiral superfield Φ , but we now pick up two super-commutator terms. Simplifying these with (3.20), we find that

$$\begin{aligned} H\Psi_{(+2)}^{(N)} &= (-1)^{a(a+b)}(2N + 1)g^{A\bar{B}}\nabla_A^{(N)}g_{\bar{B}D}\nabla_C^{(N+3)}\Phi^{CD} \\ &+ (-1)^{a(a+b)+bd}(2N + 3)g^{A\bar{B}}\nabla_A^{(N)}\nabla_D^{(N+1)}g_{\bar{B}C}\Phi^{CD}. \end{aligned} \quad (3.29)$$

Now we use the identity

$$(-1)^{bc} \nabla_{(C}^{(N+1)} g_{\bar{B}D)} \equiv g_{\bar{B}(C} \nabla_{D)}^{(N+3)}, \quad (3.30)$$

where the brackets indicate graded symmetrization in the unbarred indices, to rewrite (3.29) as

$$H \Psi_{(+2)}^{(N)} = (-1)^{a(a+b)} (4N+4) g^{A\bar{B}} \nabla_A^{(N)} g_{\bar{B}D} \nabla_C^{(N+3)} \Phi^{CD}. \quad (3.31)$$

Then, using (3.26), we confirm that $\Psi_2^{(N)}$ is an eigenfunction of H with energy eigenvalue $(4N+4)$. No new identities are needed to repeat these steps at higher levels, and the result for the ℓ th level may be obtained by induction. Energy eigenvalues are

$$E_\ell = C_2(\ell) = \ell(\ell + 2N) \quad (3.32)$$

for non-negative integer ℓ , and the states in the ℓ th Landau level, for $\ell > 0$, have superfield wave functions of the form

$$\Psi_\ell^{(N)} = \nabla_{A_1}^{(N+1)} \dots \nabla_{A_\ell}^{(N+2\ell-1)} \Phi^{A_\ell \dots A_1}, \quad (3.33)$$

where the superfield $\Phi^{A_\ell \dots A_1}$ is totally graded symmetric in its ℓ indices and satisfies the analyticity condition

$$\nabla_{\bar{B}}^{(N)} \Phi^{A_\ell \dots A_1} = 0. \quad (3.34)$$

The graded symmetry means that Φ has only two independent components, which we may take to be

$$\Phi^{z\dots z} \equiv \Phi_\ell^{(+)}, \quad \Phi^{z\dots\zeta} = \Phi_\ell^{(-)}. \quad (3.35)$$

It follows that

$$\Psi_\ell^{(N)} = \Psi_{(+)\ell}^{(N)} + \Psi_{(-)\ell}^{(N)}, \quad (3.36)$$

where the two independent superfields $\Psi_{(\pm)\ell}^{(N)}$ are given by

$$\Psi_{(+)\ell}^{(N)} = \nabla_z^{(N+1)} \dots \nabla_z^{(N+2\ell-1)} \Phi_\ell^{(+)} \quad (3.37)$$

and

$$\Psi_{(-)\ell}^{(N)} = \left[\sum_{p=1}^{\ell} \nabla_z^{(N+1)} \dots \nabla_\zeta^{(N+2p-1)} \dots \nabla_z^{(N+2\ell-1)} \right] \Phi_\ell^{(-)}. \quad (3.38)$$

The LLL is exceptional in that only the (+) component is defined, and this is the ground state wave function that we called $\Psi_0^{(N)}$. In general, both of the $\Psi_{(\pm)}$ components will carry an irreducible representation of $SU(2|1)$, so only the LLL has a representation carried by a single analytic superfield. We arrived at this result using insights gained from earlier studies of the planar limit, and by analogy with the \mathbb{CP}^2 Landau model.

Hilbert space norm

The Hilbert space has a natural $SU(2|1)$ -invariant norm, defined as the superspace integral³ [5]

$$\|\Psi\|^2 = \int d\mu_0 e^{-\mathcal{K}} \Psi^* \Psi, \quad (3.39)$$

where

$$d\mu_0 = dz d\bar{z} \partial_\zeta \partial_{\bar{\zeta}}. \quad (3.40)$$

For the ground state this norm reproduces the results in [5]. For the first excited state we may simplify the norm by means of the integration by parts identity

$$\int d\mu_0 e^{-\mathcal{K}} \left(\nabla_A^{(N)} \Phi^A \right)^* \Theta \equiv -(-1)^a \int d\mu_0 e^{-\mathcal{K}} (\Phi^A)^* \left(\nabla_A^{(N-1)} \Theta \right), \quad (3.41)$$

³For the definition of the integral look in the appendix A at the page 89.

valid for arbitrary superfield Θ . Using also the super-commutator identity (3.20) and the chirality condition on Φ^C , we find that

$$\|\Psi_1^{(N)}\|^2 = (-1)^a (2N + 1) \int d\mu_0 e^{-\mathcal{K}} (\Phi^B)^* g_{\bar{B}A} \Phi^A. \quad (3.42)$$

Similar steps may be used to simplify the norm of $\Psi_\ell^{(N)}$ for $\ell > 1$, but one now needs the identity, analogous to (3.30),

$$(-1)^{bc} \nabla_{(\bar{A}}^{(N+2)} g_{\bar{B})C} \equiv g_{(\bar{A}C} \nabla_{\bar{B})}^{(N)}, \quad (3.43)$$

where the brackets again indicate graded symmetrization, but now in the barred indices. The final result for ℓ -th level wave function could be obtained in the similar way with the repeated use of this identities, although the calculations are more cumbersome because of the plus and minus components (4.52). At the end we obtain

$$\|\Psi_\ell^{(N)}\|^2 = \sigma_\ell \frac{(2N + 2\ell - 1)! \ell!}{(2N + \ell - 1)!} \int d\mu_0 e^{-\mathcal{K}} (\Phi^{B_1 \dots B_\ell})^* g_{\bar{B}_1 A_1} \dots g_{\bar{B}_\ell A_\ell} \Phi^{A_\ell \dots A_1}, \quad (3.44)$$

where

$$\sigma_\ell = (-1)^{\sum_i b_i + \sum_i^{\ell-1} a_i b_{i+1}}. \quad (3.45)$$

In terms of the two independent chiral superfields $\Phi_\ell^{(\pm)}$, we have⁴

$$\begin{aligned} \|\Psi_\ell^{(N)}\|^2 &= \frac{(2N + 2\ell - 1)! \ell!}{(2N + \ell - 1)!} \int d\mu_0 e^{-\mathcal{K}} \left\{ (\Phi_\ell^{(+)})^* (g_{\bar{z}z})^\ell \Phi_\ell^{(+)} \right. \\ &\quad + \ell (\Phi_\ell^{(+)})^* (g_{\bar{z}z})^{\ell-1} g_{z\bar{\zeta}} \Phi_\ell^{(-)} - \ell (\Phi_\ell^{(-)})^* (g_{\bar{z}z})^{\ell-1} g_{\bar{\zeta}z} \Phi_\ell^{(+)} \\ &\quad \left. + (\Phi_\ell^{(-)})^* \left[-\ell (g_{\bar{z}z})^{\ell-1} g_{\bar{\zeta}\zeta} + \ell(\ell-1) (g_{\bar{z}z})^{\ell-2} g_{z\bar{\zeta}} g_{\bar{\zeta}z} \right] \Phi_\ell^{(-)} \right\}. \end{aligned} \quad (3.46)$$

⁴Although the $\ell = 0, 1$ cases are special, and need to be considered separately, this result for $\ell \geq 2$ is also correct for $\ell = 0, 1$. In particular, all terms involving $\Phi^{(-)}$ are absent for $\ell = 0$, as expected.

To proceed, we solve the analyticity constraint (4.2) on the $\Phi_\ell^{(\pm)}$ superfields by writing

$$\Phi_\ell^{(\pm)} = e^{-N\kappa} \varphi_\ell^{(\pm)}, \quad (3.47)$$

where $\varphi_\ell^{(\pm)}$ are unconstrained analytic superfields. We may expand $\varphi_\ell^{(\pm)}$ in component fields as follows

$$\varphi_\ell^{(-)} = A_\ell + \zeta\psi_\ell, \quad \varphi_\ell^{(+)} = \chi_\ell + \zeta F_\ell. \quad (3.48)$$

If (as the notation suggests) the component functions (χ, ψ) are assumed to be Grassmann odd, and the component functions (A, F) are assumed to be Grassmann even, then Ψ will be Grassmann odd. With the reverse Grassmann parity assignments to the component functions, Ψ will have even Grassmann parity. In either of these two cases the ‘Hilbert’ space is actually a supervector space rather than a vector space. If, instead, all component functions are assumed to be Grassmann even then Ψ will not have a definite Grassmann parity but the Hilbert space will be a standard Hilbert space. There is no need here to choose between these alternatives as long as we are careful not to perform any re-ordering that would require us to specify one of them. Substituting for $\Phi_\ell^{(\pm)}$ in (3.46) and performing the Berezin integration, we arrive at the result

$$\begin{aligned} \|\Psi_\ell^{(N)}\|^2 &= \frac{(2N + 2\ell - 1)! \ell!}{(2N + \ell - 1)!} \int \frac{dz d\bar{z}}{(1 + z\bar{z})^{2(N+\ell)+1}} \left[-\ell(2N + \ell) |A_\ell|^2 - \ell \bar{\psi}_\ell \psi_\ell \right. \\ &\quad \left. - \ell (\bar{\chi}_\ell + \bar{z} \bar{\psi}_\ell) (\chi_\ell + z \psi_\ell) + \frac{2(N + \ell) + 1}{1 + z\bar{z}} \bar{\chi}_\ell \chi_\ell + |F_\ell|^2 \right]. \end{aligned} \quad (3.49)$$

The above norm is $SU(2|1)$ invariant, by construction, but not positive definite, so the associated quantum theory is not unitary. However, there could be an alternative $SU(2|1)$ invariant norm that *is* positive-definite. Indeed there is, but we shall investigate this in the context of the more general Superflag model since we

may then specialize to $M = 0$ to get a unitary superspherical Landau model. Quite apart from the fact that we will then have the main result in the context of a more general model, another reason for this approach to the problem is that computations are easier for the Superflag model. This is because the additional anti-commuting variable of the classical theory becomes an additional superspace coordinate in the quantum theory, and expansion in this coordinate yields (\pm) pairs of superfields of the type that we have been considering. This simplification also allows the Superflag model to be solved exactly by a factorization trick.

3.2 Solving Superflag

We already introduced Superflag Lagrangian to be

$$L = |\omega^+|^2 + \left[\dot{Z}^M (N' \mathcal{A}_M + M \mathcal{B}_M) + c.c. \right], \quad (3.50)$$

We now turn to a Hamiltonian analysis of the general Superflag model. Generalized momenta are found to be

$$\begin{aligned} \mathcal{P}_\zeta &= \pi_\zeta - iN' \mathcal{A}_\zeta - iM \mathcal{B}_\zeta, & \mathcal{P}_z &= (p_z - N' \mathcal{A}_z - M \mathcal{B}_z), \\ \mathcal{P}_{\bar{\zeta}} &= \pi_{\bar{\zeta}} - iN' \mathcal{A}_{\bar{\zeta}} - iM \mathcal{B}_{\bar{\zeta}}, & \mathcal{P}_{\bar{z}} &= (p_{\bar{z}} - N' \mathcal{A}_{\bar{z}} - M \mathcal{B}_{\bar{z}}). \end{aligned} \quad (3.51)$$

Since we have auxiliary coordinates in Lagrangian there will appear constraints when we transition to Hamiltonian formalism. The model has four primary constraints, which occur in two complex conjugate pairs. One pair is

$$\varphi_\zeta = \mathcal{P}_\zeta + i(\bar{\xi} K_2 + \bar{\zeta} z) \mathcal{P}_z, \quad \varphi_{\bar{\zeta}} = \mathcal{P}_{\bar{\zeta}} - i(\xi K_2 + \zeta \bar{z}) \mathcal{P}_{\bar{z}}, \quad (3.52)$$

$$\varphi_\xi = \pi_\xi - iM \mathcal{B}_\xi, \quad \varphi_{\bar{\xi}} = \pi_{\bar{\xi}} - iM \mathcal{B}_{\bar{\xi}}. \quad (3.53)$$

After substitutions we can find Hamiltonian

$$H_0 = K_2^2 K_1^{-1} [1 + (\bar{\zeta} + \bar{z}\bar{\xi}) \zeta] [1 + \bar{\zeta} (\zeta + z\xi)] \mathcal{P}_z \mathcal{P}_{\bar{z}}, \quad (3.54)$$

where the subscript is a reminder that we may add any function on phase space that vanishes on the subspace specified by the primary constraints. A remarkable feature of this Hamiltonian is that it is independent of M . When we pass to the quantum theory, this means that the energy levels are independent of M but this does not mean that the parameter M is irrelevant because it can affect the norms of the quantum states.

Before proceeding to the quantum theory we have to address a minor difficulty. The Hamiltonian H_0 does not commute, even ‘weakly’, with the constraints. This difficulty can be circumvented by introducing the new variables

$$\xi^1 = \zeta + z\xi, \quad \xi^2 = \xi. \quad (3.55)$$

These were the variables used in [6], and the analog of H_0 found by using these variables commutes with the constraints. Alternatively, one can modify the Hamiltonian by adding terms proportional to the constraint functions such that the new Hamiltonian commutes, at least weakly, with the constraints. This second approach was the one adopted in [8] for the planar Superflag, and we will do the same here. Specifically, we take the new Hamiltonian to be

$$H = K_2^2 K_1 (\mathcal{P}_z + i\xi \mathcal{P}_\zeta) (\mathcal{P}_{\bar{z}} + i\xi \mathcal{P}_{\bar{\zeta}}). \quad (3.56)$$

It may be verified that H is weakly equivalent to H_0 but commutes (strongly) with the constraints.

To pass to the quantum theory we make the replacement $P_A \rightarrow -i\partial_A$, as in (3.17),

where $A = (z, \zeta)$, and we also make the replacement

$$\pi_\xi \rightarrow \partial_\xi, \quad \pi_{\bar{\xi}} \rightarrow \partial_{\bar{\xi}}, \quad (3.57)$$

which is needed only for the second pair of constraints (3.53). The resulting Hamiltonian operator⁵ is

$$H_{N'} = -K_2^2 K_1 \left(\nabla_z^{(N')} - \xi \nabla_\zeta^{(N')} \right) \left(\nabla_{\bar{z}}^{(N')} - \bar{\xi} \nabla_{\bar{\zeta}}^{(N')} \right), \quad (3.58)$$

where covariant derivatives are

$$\nabla_A^{(N')} = \partial_A - iN' \mathcal{A}_A, \quad \nabla_{\bar{A}}^{(N')} = \partial_{\bar{A}} - iN' \mathcal{A}_{\bar{A}}. \quad (3.59)$$

Here $\nabla_z^{(N')} - \xi \nabla_\zeta^{(N')}$ can be understood as creation operator and $\nabla_{\bar{z}}^{(N')} - \bar{\xi} \nabla_{\bar{\zeta}}^{(N')}$ as annihilation operator, there is an ordering ambiguity in the order of these operators which allows addition of a constant to the Hamiltonian. We just set this constant to zero. Because the analytic constraint operators commute, we may quantize *à la* Gupta-Bleuler by requiring physical states to be annihilated by these operators. The result is that ‘physical’ wave functions must take the form

$$\Psi = K_1^M K_2^{-N'} \Phi(z, \bar{z}_{sh}, \zeta, \xi), \quad (3.60)$$

where Φ is a ‘reduced’ wave function that depends on \bar{z} only through the ‘shifted’ coordinate

$$\bar{z}_{sh} = \bar{z} - \xi \bar{\zeta} - \bar{z} (\zeta + z\xi) \bar{\zeta}. \quad (3.61)$$

This system was already solved in [6], so here we are going to reproduce already known formulas for energy levels, eigenfunctions, but in new coordinates (z, ζ, ξ) .

⁵Operator ordering ambiguities allow the addition of a constant, which we have set to zero.

For $2N'$ an integer, which we may assume to be positive, the Hamiltonian may be diagonalized in the physical subspace, with energy eigenvalues

$$E_{N'} = \ell(2N' + \ell + 1), \quad \ell = 0, 1, 2, \dots \quad (3.62)$$

The wave functions for the LLL ($\ell = 0$) is

$$\Psi^{(0)} = K_1^M K_2^{-N'} \Phi_{an}^{(0)}(z, \zeta, \xi). \quad (3.63)$$

That is, the reduced LLL wave function is an *analytic* function. The reduced wave function at all higher levels may be expressed in terms of a level ℓ *analytic* function $\Phi_{an}^{(\ell)}$ according to the formula

$$\Phi^{(\ell)} = \mathcal{D}^{2(N'+1)} \dots \mathcal{D}^{2(N'+\ell)} \Phi_{an}^{(\ell)}(z, \zeta, \xi) \quad (\ell > 0), \quad (3.64)$$

where

$$\mathcal{D}^{2N'} \equiv \nabla_z^{2N'} - \xi \nabla_\zeta^{2N'} = \partial_z - \xi \partial_\zeta - \frac{2N' \bar{z}_{sh}}{1 + z \bar{z}_{sh}}. \quad (3.65)$$

As in the case of the superspherical Landau model, there is a natural $SU(2|1)$ invariant inner product on Hilbert space defined by a superspace integral, although the superspace now has an additional complex anti-commuting coordinate. As shown in [6], this inner product is

$$\langle \Upsilon | \Psi \rangle = \int dz d\bar{z} \partial_\zeta \partial_{\bar{\zeta}} \partial_\xi \partial_{\bar{\xi}} K_2^{-2} \Upsilon^* \Psi. \quad (3.66)$$

Performing the Berezin integration over *all* anti-commuting coordinates, we get an ordinary integral over the sphere with an integrand determined by the four analytic

functions $(A^{(\ell)}, \psi^{(\ell)}, \chi^{(\ell)}, F^{(\ell)})$ appearing in the (ζ, ξ) -expansion of $\Phi_{an}^{(\ell)}$:

$$\Phi_{an}^{(\ell)} = A^{(\ell)} + \zeta \left[\psi^{(\ell)} + \frac{\partial_z \chi^{(\ell)}}{(2N' + 2\ell + 1)} \right] + \xi \chi^{(\ell)} + \zeta \xi F^{(\ell)}. \quad (3.67)$$

The net result, after integrating by parts to remove all derivatives, is that wave functions at different levels are orthogonal, while

$$\begin{aligned} \|\Psi_{N'}^{(\ell)}\|^2 &\equiv \langle \Psi | \Psi \rangle = \ell! \frac{(2N' + \ell + 1)!}{(2N' + 1)!} \int \frac{dz d\bar{z}}{(1 + z\bar{z})^{2(N'+\ell+1)}} \times \\ &\quad \left\{ (2M - \ell)(2M + 2N' + \ell + 1) \bar{A}^{(\ell)} A^{(\ell)} + \bar{F}^{(\ell)} F^{(\ell)} \right. \\ &\quad + \frac{(N' + \ell + 1)(2N' + 2M + \ell + 1)}{(2N' + 2\ell + 1)(1 + z\bar{z})} \bar{\chi}^{(\ell)} \chi^{(\ell)} \\ &\quad \left. + (2M - \ell)(1 + z\bar{z}) \bar{\psi}^{(\ell)} \psi^{(\ell)} \right\}. \end{aligned} \quad (3.68)$$

This is a simplified form of the result given in [6]; the unusual expansion of (3.67) has led to a norm that is diagonal in the component functions.

With the above norm, the model has ghosts. For positive M (which was the only case considered in [6]) there are ghosts whenever $\ell > 2M$ and if $2M$ is a non-negative integer then there are zero-norm states for $\ell = 2M$. This means, in particular, that the model has ghosts in this ‘naive’ norm for any positive M . The same is true for negative M , and in this case there are zero norm states even for $\ell = 0$.

Of course, the sign of the norm has physical relevance only for Grassmann-even component functions, and either $A^{(\ell)}$ or $\psi^{(\ell)}$ would be Grassmann-odd if we were to assume (as in [6]) that wave functions are superfields (i.e. have definite Grassmann parity). However, even in this case the above statements concerning ghosts still apply. We have been careful to allow for (i) wave functions that are superfields, in which case the ‘Hilbert’ space is actually a vector superspace, and (ii) wave functions for which all component fields are ordinary functions (or bundle sections), in which case

the Hilbert space is a vector space.

3.3 Unitary norm

The $SU(2|1)$ symmetry of the Superflag model implies the existence of Noether charges, which become differential operators in the quantum theory, satisfying the (anti)commutation relations of $SU(2|1)$ given in Appendix B. These differential operators acting on the whole Superflag wave functions, determine a simpler set of differential operators that act on the analytic wave functions, and vice-versa since the full Noether charge operators can be recovered from the simpler ‘analytic’ operators that we now present. The even generators are

$$\begin{aligned}
J_- &= -i\partial_z, \\
J_+ &= -i[-2(N' + \ell)z + z^2\partial_z + z\zeta\partial_\zeta - (\zeta + z\xi)\partial_\xi], \\
J_3 &= -(N' + \ell) + z\partial_z + \frac{1}{2}(\zeta\partial_\zeta - \xi\partial_\xi), \\
F &= 2M + N' + \frac{1}{2}(\zeta\partial_\zeta + \xi\partial_\xi).
\end{aligned} \tag{3.69}$$

Note the ℓ -independence of B ; for the other generators one should view ℓ as an operator (later to be called L) that takes the value ℓ in the ℓ th level. The odd generators are

$$\Pi = \partial_\zeta, \quad Q = z\partial_\zeta - \partial_\xi \tag{3.70}$$

and

$$\begin{aligned}
\Pi^\dagger &= (2M + 2N' + \ell)\zeta - \zeta z\partial_z + \xi[(2M - \ell)z - \zeta\partial_\xi], \\
Q^\dagger &= \zeta\partial_z - (2M - \ell)\xi.
\end{aligned} \tag{3.71}$$

These results may be compared to the expressions (). In the present case, the full differential operators representing the generators $(J_+, \Pi^\dagger, Q^\dagger)$, which are determined by the simpler ‘analytic’ forms given above, are the Hermitian conjugates of the generators (J_-, Π, Q) in the ‘naive’ norm.

We are now in a position to work out the $SU(2|1)$ representation content at each Landau level. Let us first consider the $SU(2)$ content. We have

$$\begin{aligned}
J^2 &= J_- J_+ + J_3^2 + J_3 \\
&= (N' + \ell + 1)(N' + \ell) - \left(N' + \ell + \frac{1}{4}\right) \zeta \partial_\zeta \\
&\quad + \left[\zeta \partial_z + \left(N' + \ell + \frac{3}{4} - \frac{1}{2} \zeta \partial_\zeta\right) \xi \right] \partial_\xi.
\end{aligned} \tag{3.72}$$

Now we act with this operator on the analytic wave functions of (3.67), which we may rewrite as

$$\Phi_{an}^{(\ell)} = A^{(\ell)} + \zeta \psi^{(\ell)} + \left[\xi + \frac{\zeta \partial_z}{2N' + 2\ell + 1} \right] \chi^{(\ell)} + \zeta \xi F^{(\ell)}. \tag{3.73}$$

We find that

$$\begin{aligned}
J^2 \Phi_{an}^{(\ell)} &= (N' + \ell)(N' + \ell + 1) A^{(\ell)} + \left(N' + \ell - \frac{1}{2}\right) \left(N' + \ell + \frac{1}{2}\right) \zeta \psi^{(\ell)} \\
&\quad + \left(N' + \ell + \frac{1}{2}\right) \left(N' + \ell + \frac{3}{2}\right) \left[\xi + \frac{\zeta \partial_z}{2N' + 2\ell + 1} \right] \chi^{(\ell)} \\
&\quad + (N' + \ell)(N' + \ell + 1) \zeta \xi F^{(\ell)}.
\end{aligned} \tag{3.74}$$

One reads off from this result the eigenfunctions of J^2 and their eigenvalues. Acting

with J_3 on the J^2 eigenfunctions we get

$$\begin{aligned}
J_3 [A^{(\ell)}] &= (z\partial_z - N' - \ell) A^{(\ell)}, \\
J_3 [\zeta\psi^{(\ell)}] &= \zeta \left(z\partial_z - N' - \ell + \frac{1}{2} \right) \psi^{(\ell)}, \\
J_3 \left[\left(\xi + \frac{\zeta\partial_z}{2N' + 2\ell + 1} \right) \chi^{(\ell)} \right] &= \left(\xi + \frac{\zeta\partial_z}{2N' + 2\ell + 1} \right) \left(z\partial_z - N' - \ell - \frac{1}{2} \right) \chi^{(\ell)} \\
J_3 [\zeta\xi F^{(\ell)}] &= \zeta\xi (z\partial_z - N' - \ell) F^{(\ell)}. \tag{3.75}
\end{aligned}$$

Putting this all together we find the following sets of $(2s+1)$ spin- s joint eigenfunctions of J^2 and J_3 :

$$\begin{aligned}
s &= (N' + \ell) : z^n a_n, \quad n = 0, \dots, 2N' + 2\ell, \\
s &= \left(N' + \ell - \frac{1}{2} \right) : \zeta z^p \psi_p, \quad p = 0, \dots, 2N' + 2\ell - 1, \\
s &= \left(N' + \ell + \frac{1}{2} \right) : \left(\xi + \frac{(q+1)\zeta}{2N' + 2\ell + 1} \right) z^q \chi_q, \quad q = 0, \dots, 2N' + 2\ell + 1, \\
s &= (N' + \ell) : \zeta\xi z^m f_m, \quad m = 0, \dots, 2N' + 2\ell \tag{3.76}
\end{aligned}$$

for constants $(a_m, \psi_p, \chi_q, f_m)$.

As mentioned already, there are two separate cases in which the ‘naive’ norm considered so far has ghosts when $M < 0$. These are (i) $2M < -2N' - 1$, and (ii) $-2N' - 1 < 2M < 0$. Consider the operator

$$G_{an} = -1 + 2\xi\partial_\xi + \frac{2}{2N' + 2\ell + 1} \zeta\partial_z\partial_\xi. \tag{3.77}$$

This commutes with J^2 and J_3 , and hence with the Hamiltonian, as is clear from the alternative expression

$$G_{an} = \frac{1}{2N' + 2\ell + 1} \left[2J^2 + 2(F - 2M + \ell)^2 - (2N' + 2\ell + 1)^2 \right]. \tag{3.78}$$

It also has the property that

$$G_{an}^2 \equiv 1. \quad (3.79)$$

As was explained in [8], the same properties hold for the corresponding ‘full’ operator G , so each of the eigenstates listed above has a definite ‘ G -parity’. By inspection, one sees that for

$$-2N' - 1 < 2M < 0, \quad (3.80)$$

the positive (negative) norm eigenstates have positive (negative) G -parity, and therefore that the G is the ‘metric operator’ for M in the above range, in the sense that the new norm

$$|||\Psi|||^2 \equiv \langle \Psi | G \Psi \rangle \quad (3.81)$$

is positive definite; we refer to [8] for details of the formalism.

For $M = 0$ there are zero-norm states, but still no negative-norm states. Now consider the operator

$$\tilde{G}_{an} = 1 - 8(F - 2M - N') + 8(F - 2M - N')^2. \quad (3.82)$$

It is manifest that \tilde{G}_{an} commutes with the Hamiltonian, and hence the same is true of \tilde{G} . One may verify that $\tilde{G}_{an}^2 \equiv 1$, so that the eigenstates listed above also have a definite \tilde{G} -parity. Inspection shows that when $2M < -2N' - 1$ the states with positive (negative) norm have (positive) negative \tilde{G} -parity. The operator \tilde{G} is therefore a ‘metric’ operator for $2M < -2N' - 1$, which is a range that has no counterpart in the planar limit. The metric operator for $M > 0$ is a more-complicated ‘dynamical’ one, depending on the level. We skip the details of this case.

3.4 Hidden symmetries

We know that there is hidden worldline supersymmetry of the *planar* super-Landau models, for $M \leq 0$. This implies the existence of some enlarged supersymmetry algebra for the spherical super-Landau models, and we now aim to investigate this. For simplicity, we now place M in the range for which the metric operator defining the unitary models is the operator G defined by (3.77). As we have seen, this means that M should satisfy (3.80) but, as we have also seen, we may allow $M = 0$ too. In other words, we now restrict M such that

$$-2N' - 1 < 2M \leq 0. \quad (3.83)$$

Now, let \mathcal{O} be some operator that commutes with the Hamiltonian as in the Lemma 1. \mathcal{O}^\dagger and $\tilde{\mathcal{O}}$ where defined in (2.46) and (2.49). In addition to them we will use

$$\mathcal{O}_G \equiv [G, \mathcal{O}]. \quad (3.84)$$

Because \mathcal{O} commutes with Hamiltonian, \mathcal{O}_G also commutes with Hamiltonian. Note that

$$(\mathcal{O}_G)^\dagger = [G, \mathcal{O}^\dagger] = -[G, \mathcal{O}]^\dagger = -(\mathcal{O}_G)^\dagger \equiv -\mathcal{O}_G^\dagger. \quad (3.85)$$

Symmetry generators that do not commute with G thus generate, in general, additional symmetries that are ‘hidden’ in the sense that their existence was not built into the construction of the model. For the Superflag model, it is the odd generators that

fail to commute with G , and this leads to the following new symmetry generators

$$\begin{aligned}
\Pi_G &= -\frac{2}{2N' + 2\ell + 1} \partial_\xi \partial_z, \\
\Pi_G^\dagger &= \frac{4M - 2\ell}{2N' + 2\ell + 1} [\zeta (1 + z\partial_z) + (2N' + 2\ell + 1) z\xi - \zeta\xi\partial_\xi], \\
Q_G &= \frac{2}{2N' + 2\ell + 1} (2N' + 2\ell + 1 - z\partial_z - \zeta\partial_\zeta) \partial_\xi, \\
Q_G^\dagger &= -\frac{4M - 2\ell}{2N' + 2\ell + 1} [(2N' + 2\ell + 1) \xi + \zeta\partial_z].
\end{aligned} \tag{3.86}$$

The naive hermitian conjugate of a symmetry operator \mathcal{O} will not coincide with its new hermitian conjugate \mathcal{O}^\dagger unless \mathcal{O} commutes with G . For this reason, it is convenient to choose a basis in which the original $SU(2|1)$ symmetry operators \mathcal{O} are replaced by the operators $\tilde{\mathcal{O}}$ which commute with G even when \mathcal{O} does not, and $\tilde{\mathcal{O}}^\dagger = \tilde{\mathcal{O}}^\dagger$. In case that \mathcal{O} is hermitian with respect to the ‘naive’ Hilbert space metric, the operator $\tilde{\mathcal{O}}$ will also be hermitian with respect to the new Hilbert space norm.

When applied to the operators Π and Q , the definition (2.49) yields

$$\begin{aligned}
\tilde{\Pi} &= \Pi + \frac{1}{2}\Pi_G, & \tilde{\Pi}^\dagger &= \Pi^\dagger - \frac{1}{2}\Pi_G^\dagger, \\
\tilde{Q} &= Q + \frac{1}{2}Q_G, & \tilde{Q}^\dagger &= Q^\dagger - \frac{1}{2}Q_G^\dagger,
\end{aligned} \tag{3.87}$$

where we have used the remarkable identities

$$\Pi_G G = \Pi_G, \quad Q_G G = Q_G. \tag{3.88}$$

In terms of the rescaled odd charges

$$(\tilde{\Pi}', \tilde{Q}') = \sqrt{\frac{2N' + 2\ell + 1}{2M + 2N' + \ell + 1}} (\tilde{\Pi}, \tilde{Q}), \tag{3.89}$$

and the redefined $U(1)$ generator

$$F' = F - 2M + \ell, \quad (3.90)$$

one finds, after some computation, that the non-zero (anti)commutation relations of the odd charges $(\tilde{\Pi}', \tilde{Q}')$, and their hermitian conjugates, and the even $SU(2) \times U(1)$ charges (J_3, J_\pm, F') are precisely of the standard $SU(2|1)$ form given in appendix B.4. Thus, these charges provide an alternative basis for the $SU(2|1)$ symmetry algebra.

Now we turn to the ‘hidden’ symmetry charges. Their non-zero anticommutators are

$$\begin{aligned} \left\{ \Pi_G, \Pi_G^\dagger \right\} &= \frac{4(\ell - 2M)}{2N' + 2\ell + 1} (J_3 + \check{F}), & \left\{ Q_G, Q_G^\dagger \right\} &= \frac{4(\ell - 2M)}{2N' + 2\ell + 1} (-J_3 + \check{F}), \\ \left\{ \Pi_G, Q_G^\dagger \right\} &= -i \frac{4(\ell - 2M)}{2N' + 2\ell + 1} J_-, & \left\{ \Pi_G^\dagger, Q_G \right\} &= i \frac{4(\ell - 2M)}{2N' + 2\ell + 1} J_+, \end{aligned} \quad (3.91)$$

where

$$\check{F} = 2M + 2N' + \ell + 1 - F. \quad (3.92)$$

Notice that the coefficients are level-dependent. The ℓ -dependence in the denominators is easily removed by a level-dependent rescaling of the odd charges but the $(\ell - 2M)$ factor in the numerators is more problematic because when $M = 0$ this factor is zero for $\ell = 0$ but non-zero for $\ell > 0$. For this reason, we will discuss these two cases separately.

$$-2N' - 1 < 2M < 0$$

In this case we may define new odd charges by

$$\check{\Pi}_G = -\sqrt{\frac{2N' + 2\ell + 1}{4(\ell - 2M)}} Q_G^\dagger, \quad \check{Q}_G = \sqrt{\frac{2N' + 2\ell + 1}{4(\ell - 2M)}} \Pi_G^\dagger, \quad (3.93)$$

in terms of which the anti-commutation relations of (3.91) become

$$\begin{aligned} \{\check{\Pi}_G, \check{\Pi}_G^\dagger\} &= -J_3 + \check{F}, & \{\check{Q}_G, \check{Q}_G^\dagger\} &= J_3 + \check{F}, \\ \{\check{\Pi}_G, \check{Q}_G^\dagger\} &= iJ_-, & \{\check{\Pi}_G^\dagger, \check{Q}_G\} &= -iJ_+. \end{aligned} \quad (3.94)$$

To present the commutators of these new odd charges with the even charges of $SU(2|1)$ we need to give only the non-zero commutators with $(\check{\Pi}_G, \check{Q}_G)$ charges since the remainder are found by hermitian conjugation; these are

$$\begin{aligned} [\check{F}, \check{\Pi}_G] &= -\frac{1}{2}\check{\Pi}_G, & [\check{F}, \check{Q}_G] &= -\frac{1}{2}\check{Q}_G, \\ [J_3, \check{\Pi}_G] &= -\frac{1}{2}\check{\Pi}_G, & [J_3, \check{Q}_G] &= \frac{1}{2}\check{Q}_G, \\ [J_+, \check{\Pi}_G] &= i\check{Q}_G, & [J_-, \check{Q}_G] &= -i\check{\Pi}_G. \end{aligned} \quad (3.95)$$

This shows that the new odd symmetry charges transform as a charged doublet under the $U(2)$ subgroup of $SU(2|1)$. In fact, the operators $(\check{\Pi}_G^\dagger, \check{Q}_G^\dagger)$, together with their hermitian conjugates, and the even charges (J_3, J_\pm, \check{F}) , obey the (anti)commutation relations of $SU(2|1)$. The full symmetry group therefore contains two distinct $SU(2|1)$ superalgebras. As F' is the $U(1)$ charge of one of these superalgebras and \check{F} the $U(1)$ charge of the other one, the full symmetry group must contain

$$2Z = F' + \check{F} = 2N' + 2\ell + 1, \quad (3.96)$$

which is a level-dependent central charge. However, this level-dependence does not

present a problem; it just means that we have a central charge

$$Z = \frac{1}{2}(2L + 2N' + 1), \quad (3.97)$$

where L is the level operator .

The two $SU(2|1)$ superalgebras are non-commuting because there are non-zero anti-commutators of the odd charges from one with the odd charges from the other.

These are

$$\begin{aligned} \{\tilde{\Pi}', \check{\Pi}'_G\} &= \{\tilde{Q}', \check{Q}'_G\} = i\mathcal{J}_-, \\ \{\tilde{\Pi}'^\dagger, \check{\Pi}'_G\} &= \{\tilde{Q}'^\dagger, \check{Q}'_G\} = -i\mathcal{J}_+, \end{aligned} \quad (3.98)$$

where the analytic operators representing \mathcal{J}_\pm are

$$\begin{aligned} \mathcal{J}_+ &= i\sqrt{(\ell - 2M)(2M + 2N' + \ell + 1)}\xi\zeta, \\ \mathcal{J}_- &= \frac{i}{\sqrt{(\ell - 2M)(2M + 2N' + \ell + 1)}}\partial_\xi\partial_\zeta. \end{aligned} \quad (3.99)$$

These satisfy, together with

$$\mathcal{J}_3 = \frac{1}{2}(-1 + \xi\partial_\xi + \zeta\partial_\zeta), \quad (3.100)$$

the standard $su(2)$ commutation relations

$$[\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_3, \quad [\mathcal{J}_3, \mathcal{J}_\pm] = \pm\mathcal{J}_\pm. \quad (3.101)$$

Finally, the non-zero commutators of these new $SU(2)$ charges with the odd

charges are

$$\begin{aligned}
[\mathcal{J}_+, \tilde{\Pi}'] &= -i\check{\Pi}_G, & [\mathcal{J}_+, \tilde{Q}'] &= -i\check{Q}_G, \\
[\mathcal{J}_-, \check{\Pi}'_G] &= i\tilde{\Pi}', & [\mathcal{J}_-, \check{Q}_G] &= i\tilde{Q}', \\
[\mathcal{J}_3, \tilde{\Pi}'] &= -\frac{1}{2}\tilde{\Pi}', & [\mathcal{J}_3, \tilde{Q}'] &= -\frac{1}{2}\tilde{Q}', \\
[\mathcal{J}_3, \check{\Pi}'_G] &= \frac{1}{2}\check{\Pi}'_G, & [\mathcal{J}_3, \check{Q}_G] &= \frac{1}{2}\check{Q}_G,
\end{aligned} \tag{3.102}$$

and hermitian conjugates. These commutation relations show that $(\tilde{\Pi}', \check{\Pi}'_G)$ and $(\tilde{Q}', \check{Q}_G)$ are doublets of the $SU(2)$ group generated by $(\mathcal{J}_\pm, \mathcal{J}_3)$.

We have now shown that the charges

$$\{J_\pm, J_3, \mathcal{J}_\pm, \mathcal{J}_3, Z; \tilde{\Pi}', \tilde{Q}'; \check{\Pi}'_G, \check{Q}_G\} \tag{3.103}$$

span a Lie superalgebra, with structure constants that are level independent. We have therefore found a finite-dimensional ‘enlarged’ symmetry algebra. The brackets where the central charge Z defined in (3.97) contributes, are:

$$\begin{aligned}
\{\check{\Pi}'_G, \tilde{\Pi}'_G^\dagger\} &= -J_3 - \mathcal{J}_3 + Z, & \{\check{Q}_G, \tilde{Q}'_G^\dagger\} &= J_3 - \mathcal{J}_3 + Z, \\
\{\tilde{\Pi}', \check{\Pi}'_G^\dagger\} &= -J_3 + \mathcal{J}_3 + Z, & \{\tilde{Q}', \check{Q}'_G^\dagger\} &= J_3 + \mathcal{J}_3 + Z.
\end{aligned} \tag{3.104}$$

Its even subalgebra is that of $SU(2) \times SU(2) \times U(1)$, where the $U(1)$ charge is central, and its four complex odd generators transform as the $(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$ of $SU(2) \times SU(2)$. This uniquely fixes the full symmetry algebra to be that of $SU(2|2)$; recall that the groups $SU(p|q)$ have even subgroup $SU(p) \times SU(q) \times U(1)$ with the $U(1)$ charge being central when $p = q$.

Chapter 4

Generalized Landau Model

We were able to deeply analyze the properties of Landau models on the Superplane and Superflag manifolds . Together with previously studied planar Landau models it gives us a lot of information about supersymmetrization of the original model. For the next step we would like to work out a more general approach to the problem. Most straightforward way to deal with a supersymmetric theory is when one starts with action for superfields. In this way supersymmetry is guaranteed by construction and even if superfield action have simple form its component version maybe highly nonlinear and nontrivial. Simplest model we had considered so far was the Landau model on the Superplane, which had its worldline $N = 2$ supersymmetry hidden. We showed two different ways of constructing its Lagrangian, directly starting from original Landau model in section 2.1 and from Cartan forms in 2.2. We will start by presenting yet another method for constructing Superplane model which will make its $N = 2$ symmetry manifest from the beginning.

4.1 Superplane from superfields

We start with the necessary definitions. The basic objects are two $\mathcal{N}=2$, $d=1$ chiral bosonic and fermionic superfields Φ and Ψ of the same dimension.

The real $\mathcal{N}=2$, $d=1$ superspace is parametrized as

$$(\tau, \theta, \bar{\theta}). \tag{4.1}$$

The left and right chiral superspaces are defined by

$$(t_L, \theta), \quad (t_R, \bar{\theta}), \quad t_L = \tau - i\theta\bar{\theta}, \quad t_R = \tau + i\theta\bar{\theta} = t_L + 2i\theta\bar{\theta}. \quad (4.2)$$

It will be convenient to work in the left (chiral) basis, so for brevity we will use the notation $t_L \equiv t$, $t_R = t + 2i\theta\bar{\theta}$. In this basis, the $\mathcal{N}=2$ covariant derivatives are defined by

$$\bar{D} = -\frac{\partial}{\partial\bar{\theta}}, \quad D = \frac{\partial}{\partial\theta} - 2i\bar{\theta}\partial_t, \quad \{D, \bar{D}\} = 2i\partial_t, \quad D^2 = \bar{D}^2 = 0. \quad (4.3)$$

The chiral superfields Φ and Ψ obey the conditions

$$\bar{D}\Phi = \bar{D}\Psi = 0 \quad (4.4)$$

and in the left-chiral basis have the following component field contents

$$\Phi(t, \theta) = z(t) + \theta\chi(t), \quad \Psi(t, \theta) = \psi(t) + \theta h(t), \quad (4.5)$$

where the complex fields $z(t), h(t)$ are bosonic and $\chi(t), \psi(t)$ are fermionic. The conjugated superfields, in the same left-chiral basis, have the following θ -expansions

$$\bar{\Phi} = \bar{z} - \bar{\theta}\bar{\chi} + 2i\theta\bar{\theta}\dot{\bar{z}}, \quad \bar{\Psi} = \bar{\psi} + \bar{\theta}\bar{h} + 2i\theta\bar{\theta}\dot{\bar{\psi}}. \quad (4.6)$$

Also, we shall need the component structure of the following superfields

$$\begin{aligned} D\Phi &= \chi - 2i\bar{\theta}\dot{z} + 2i\theta\bar{\theta}\dot{\chi}, & \bar{D}\bar{\Phi} &= (D\Phi)^\dagger = \bar{\chi} + 2i\theta\dot{\bar{z}}, \\ D\Psi &= h - 2i\bar{\theta}\dot{\psi} + 2i\theta\bar{\theta}\dot{h}, & \bar{D}\bar{\Psi} &= -(D\Psi)^\dagger = -\bar{h} + 2i\theta\dot{\bar{\psi}}. \end{aligned} \quad (4.7)$$

Consider the following action with real parameter ρ

$$S = - \int dt d^2\theta \{ \Phi \bar{\Phi} + \Psi \bar{\Psi} + \rho [\Phi D \Psi - \bar{\Phi} \bar{D} \bar{\Psi}] \} \equiv \int dt \{ \mathcal{L}_1 + \rho \mathcal{L}_2 \} . \quad (4.8)$$

After component expansion of superfields and taking the Berezin integral, we find

$$\begin{aligned} \mathcal{L}_1 &\Rightarrow -2i \left(z \dot{\bar{z}} + \psi \dot{\bar{\psi}} \right) - (\chi \bar{\chi} + h \bar{h}) , \\ \mathcal{L}_2 &\Rightarrow -2i \left(z \dot{h} + \chi \dot{\psi} + \dot{\bar{z}} \bar{h} + \bar{\chi} \dot{\bar{\psi}} \right) . \end{aligned} \quad (4.9)$$

The fields h and χ are auxiliary and they can be eliminated by their equations of motion

$$\chi = 2i\rho \dot{\bar{\psi}} , \quad h = -2i\rho \dot{\bar{z}} . \quad (4.10)$$

Upon substituting this into the sum $\mathcal{L} \equiv \mathcal{L}_1 + \rho \mathcal{L}_2$, the latter becomes

$$\mathcal{L} \Rightarrow -2i \left(z \dot{\bar{z}} + \psi \dot{\bar{\psi}} \right) + 4\rho^2 \left(\dot{z} \dot{\bar{z}} + \dot{\psi} \dot{\bar{\psi}} \right) . \quad (4.11)$$

After redefining

$$\bar{\psi} = \zeta , \quad \psi = \bar{\zeta} , \quad 4\rho^2 \equiv \frac{1}{\kappa} , \quad (4.12)$$

and integrating by parts, the Lagrangian (4.11) takes the form

$$\mathcal{L} = -i \left(z \dot{\bar{z}} - \bar{z} \dot{z} + \zeta \dot{\bar{\zeta}} - \bar{\zeta} \dot{\zeta} \right) + \frac{1}{\kappa} \left(\dot{z} \dot{\bar{z}} + \dot{\zeta} \dot{\bar{\zeta}} \right) . \quad (4.13)$$

This expression is equivalent to the Superplane model Lagrangian (2.1), after reversing of the time, $t \rightarrow -t$ together with an overall factor κ . By construction, the superfield action (4.8) is manifestly $\mathcal{N}=2$ supersymmetric. The $\mathcal{N}=2$ transformations

of the component fields can be found from

$$\delta\Phi = - [\epsilon Q - \bar{\epsilon}\bar{Q}] \Phi, \quad \delta\Psi = - [\epsilon Q - \bar{\epsilon}\bar{Q}] \Psi. \quad (4.14)$$

These are general expression for supersymmetry transformations in the superfield form. We have in the left-chiral basis,

$$Q = \frac{\partial}{\partial\theta}, \quad \bar{Q} = -\frac{\partial}{\partial\bar{\theta}} - 2i\theta\partial_t, \quad \{Q, \bar{Q}\} = -2i\partial_t = 2P_0. \quad (4.15)$$

From this we can find transformations of the component fields under supersymmetry. It follows from (4.14), (4.15) that off-shell

$$\delta z = -\epsilon\chi, \quad \delta\chi = 2i\bar{\epsilon}\dot{z}, \quad \delta\psi = -\epsilon h, \quad \delta h = 2i\bar{\epsilon}\dot{\psi}. \quad (4.16)$$

With the on-shell values (4.10) for the auxiliary fields and with the relabelling (4.12), these transformations become

$$\delta z = -\frac{i}{\sqrt{\kappa}} \epsilon\dot{\zeta}, \quad \delta\zeta = -\frac{i}{\sqrt{\kappa}} \bar{\epsilon}\dot{z}. \quad (4.17)$$

4.2 Generalized action with interaction

Now we are in a convenient situation. We have a well studied toy model with a simple looking action defined on chiral superfields. Natural question now is what if we make this action more complicated? Or to be more precise - how we can to make generalize this action to make more interesting and rich but still solvable model? Let us consider the following action

$$S = - \int dt d^2\theta \{K(\Phi, \bar{\Phi}) + V(\Phi, \bar{\Phi})\Psi\bar{\Psi} + \rho [\Phi D\Psi - \bar{\Phi}\bar{D}\bar{\Psi}]\} \equiv \int dt d^2\theta \mathcal{L}. \quad (4.18)$$

Here, like in (4.8), ρ is a real parameter. In general, the potentials K and V are arbitrary real functions of the chiral and antichiral scalar superfields $\Phi, \bar{\Phi}$. In principle, in the third term in (4.18) we could also replace Φ and $\bar{\Phi}$ by arbitrary mutually conjugate potentials. However, in the case of generic dependence of such potentials on Φ and $\bar{\Phi}$, the component action can be shown to be non-polynomial in the time derivatives of z and \bar{z} . Such an exotic feature does not show up if these potentials are, respectively, holomorphic and antiholomorphic. In this case the action can be reduced to the form (4.18) through a field redefinition. Thus we take (4.18) as the starting point. $\mathcal{N}=2$ supersymmetry also admit superfield terms $A(\Phi)\bar{D}\bar{\Phi}$ and $B(\bar{\Phi})\Psi$ in the action, but after breaking down to components they would induce some new potential-like terms without derivatives, as well as a modification of other terms. As far as we are interested in a generalization of the Superplane model action (4.8), we ignore this possibility.

Our purpose is to find a quantum formulation of this system. Also, we wish to learn which K 's and V 's permit the stationary Schrödinger equation for this system to be solved, that is, in which case the eigenfunctions and eigenvalues of the relevant Hamiltonian can be fully determined.

To begin, we rewrite the Lagrangian density in terms of the component fields

$$\begin{aligned} \mathcal{L}_{comp} = & i(\dot{z}K_z - \dot{\bar{z}}K_{\bar{z}}) - \chi\bar{\chi}K_{z\bar{z}} - iV(\psi\dot{\bar{\psi}} - \dot{\psi}\bar{\psi}) - h\bar{h}V \\ & - \chi\psi\bar{h}V_z + \bar{\chi}\bar{\psi}hV_{\bar{z}} - \chi\bar{\chi}\psi\bar{\psi}V_{z\bar{z}} + i\psi\bar{\psi}(\dot{z}V_z - \dot{\bar{z}}V_{\bar{z}}) \\ & + 2i\rho(\dot{z}h - \dot{\bar{z}}\bar{h} - \chi\dot{\psi} - \bar{\chi}\dot{\bar{\psi}}), \end{aligned} \quad (4.19)$$

where $K_z \equiv \partial_z K$, etc.⁶ It is worthwhile to remark that (4.19) is immediately put in the Hamiltonian form, since it is linear in the time derivatives of the dynamical fields

⁶Herewith, the lower-case indices $z, \bar{z}, \psi, \bar{\psi}$ denote derivatives, as well as mark the relevant momenta P and connections \mathcal{A} (see below).

$z, \bar{z}, \psi, \bar{\psi}$. We will use for them the following notation $A, B = (z, \psi)$, $\bar{A}, \bar{B} = (\bar{z}, \bar{\psi})$. And the usual definition of momenta

$$P_A = \frac{\partial L}{\partial \dot{Z}^A}, \quad P_{\bar{B}} = \frac{\partial L}{\partial \dot{\bar{Z}}^{\bar{B}}}. \quad (4.20)$$

To get rid of the auxiliary fields $h, \chi, \bar{h}, \bar{\chi}$ lets compute momenta from (4.20) and (4.19), and use those equations to express auxiliary fields. We will obtain

$$\begin{aligned} h &= \frac{1}{2i\rho} (P_z - iK_z - i\psi\bar{\psi}V_z), & \bar{h} &= -\frac{1}{2i\rho} (P_{\bar{z}} + iK_{\bar{z}} + i\psi\bar{\psi}V_{\bar{z}}), \\ \chi &= \frac{1}{2i\rho} (P_\psi - iV\bar{\psi}), & \bar{\chi} &= \frac{1}{2i\rho} (P_{\bar{\psi}} - iV\psi), \end{aligned} \quad (4.21)$$

Hamiltonian form of (4.19) can be easily written down

$$\mathcal{L}_{comp} = \dot{Z}^A P_A + \dot{\bar{Z}}^{\bar{B}} P_{\bar{B}} - H_{class} (Z^A, P_C, \bar{Z}^{\bar{B}}, P_{\bar{C}}). \quad (4.22)$$

The classical Hamiltonian can now be expressed as

$$H_{class} = \mathcal{P}_A g^{A\bar{B}} \mathcal{P}_{\bar{B}}, \quad (4.23)$$

where we introduced the supermetric $g^{A\bar{B}}$ and classical ‘‘covariant derivatives’’

$$\mathcal{P}_A = P_A - \mathcal{A}_A, \quad \mathcal{P}_{\bar{A}} = P_{\bar{A}} - \mathcal{A}_{\bar{A}}. \quad (4.24)$$

The entries of the supermetric are given by

$$g^{z\bar{z}} = \frac{V}{4\rho^2}, \quad g^{\psi\bar{\psi}} = -\frac{1}{4\rho^2} (K_{z\bar{z}} + \psi\bar{\psi}V_{z\bar{z}}), \quad g^{z\bar{\psi}} = -\frac{V_{\bar{z}}\bar{\psi}}{4\rho^2}, \quad g^{\psi\bar{z}} = \frac{V_z\psi}{4\rho^2}, \quad (4.25)$$

while the gauge superconnections are defined by

$$\mathcal{A} = i \left(dZ^A \partial_A - d\bar{Z}^{\bar{B}} \partial_{\bar{B}} \right) \mathcal{K} \equiv dZ^A \mathcal{A}_A + d\bar{Z}^{\bar{B}} \mathcal{A}_{\bar{B}}, \quad (4.26)$$

where

$$\mathcal{K} = (K + \psi \bar{\psi} V). \quad (4.27)$$

The explicit form of (4.26) is

$$\mathcal{A}_z = i(K_z + \psi \bar{\psi} V_z), \quad \mathcal{A}_{\bar{z}} = -i(K_{\bar{z}} + \psi \bar{\psi} V_{\bar{z}}), \quad \mathcal{A}_\psi = iV \bar{\psi}, \quad \mathcal{A}_{\bar{\psi}} = iV \psi. \quad (4.28)$$

Though the connections have a nice Kähler form, the generic supermetric (see (4.34) below) cannot be expressed through \mathcal{K} or any other Kähler-like potential, so the supermanifold we deal with is not super Kähler (as distinct e.g. from the supersphere model with Lagrangian (3.11)).

Instead of calculating auxiliary fields through momenta we can always use standard approach of solving equation of motions. Varying (4.19) with respect to the non-propagating fields $h, \bar{h}, \chi, \bar{\chi}$, we obtain for them the following expressions

$$\begin{aligned} h &= -\chi \psi V^{-1} V_z - 2i\rho V^{-1} \dot{z}, & \bar{h} &= \bar{\chi} \bar{\psi} V^{-1} V_{\bar{z}} + 2i\rho V^{-1} \dot{\bar{z}}, \\ \chi &= 2i\rho A^{-1} [1 - \psi \bar{\psi} B A^{-1}] \nabla \bar{\psi}, & \bar{\chi} &= -2i\rho A^{-1} [1 - \psi \bar{\psi} B A^{-1}] \nabla \psi, \end{aligned} \quad (4.29)$$

where

$$A \equiv K_{z\bar{z}}, \quad B \equiv V_{z\bar{z}} - V^{-1} V_z V_{\bar{z}}, \quad (4.30)$$

and

$$\nabla \psi \equiv \dot{\psi} + \dot{z} V^{-1} V_z \psi, \quad \nabla \bar{\psi} \equiv \dot{\bar{\psi}} + \dot{\bar{z}} V^{-1} V_{\bar{z}} \bar{\psi}. \quad (4.31)$$

After substituting these expressions back into the off-shell Lagrangian (4.19), we

obtain its on-shell form

$$\begin{aligned} \mathcal{L}_{comp} = & 4\rho^2 V^{-1} \dot{z}\dot{\bar{z}} - 4\rho^2 A^{-1} [1 - \psi\bar{\psi} BA^{-1}] \nabla\psi\nabla\bar{\psi} \\ & + i(\dot{z}K_z - \dot{\bar{z}}K_{\bar{z}}) + i\psi\bar{\psi}(\dot{z}V_z - \dot{\bar{z}}V_{\bar{z}}) - iV(\dot{\psi}\bar{\psi} - \dot{\bar{\psi}}\psi). \end{aligned} \quad (4.32)$$

This Lagrangian can be written as

$$\mathcal{L} = \dot{Z}^A \dot{\bar{Z}}^{\bar{B}} g_{\bar{B}A} + \left(\dot{Z}^A \mathcal{A}_A + \dot{\bar{Z}}^{\bar{B}} \mathcal{A}_{\bar{B}} \right), \quad (4.33)$$

where

$$g_{\bar{z}z} = \frac{4\rho^2}{V} \left(1 - \psi\bar{\psi} \frac{V_z V_{\bar{z}}}{AV} \right), \quad g_{\bar{\psi}\psi} = -\frac{4\rho^2}{A} \left(1 - \psi\bar{\psi} \frac{B}{A} \right),$$

$$g_{\bar{z}\psi} = -4\rho^2 \frac{V_{\bar{z}}}{AV} \bar{\psi}, \quad g_{\bar{\psi}z} = 4\rho^2 \frac{V_z}{AV} \psi, \quad (4.34)$$

while the connection terms are given by (4.28). It is easy to check that $g_{\bar{B}A}$ is indeed the inverse of (4.25). So (4.23) is a Hamiltonian which correspond to component Lagrangian (4.32).

4.3 Quantization scheme 1

In order to step up to the quantum level we have to quantize our Hamiltonian. Because of the presence of arbitrary functions in the model, we will encounter operator ordering ambiguities in this process. During quantization procedure we want to keep supersymmetry intact, which means that we need $[Q, \bar{Q}] = H$ for quantum Hamiltonian and generators of supersymmetry. It is easier to quantize supercharges then

Hamiltonian, so first we will find the supercharges of the classical system and then define their quantum versions in such a way that the involution $Q \rightarrow \bar{Q}$ of the $\mathcal{N} = 2$ supersymmetry algebra becomes hermitian conjugation of the quantum system. The quantum Hamiltonian can be read off from the anticommutator of the corresponding supercharges. In this Hamiltonian, the coefficients of the terms having the second order in the derivatives with respect to the target space variables should be identical with the coefficients of the terms bilinear in semi-classical momenta in (4.23).

The Lagrangian (4.19) transforms into a total derivative under the transformations (4.16)

$$\delta\mathcal{L}_{comp} = \frac{d}{dt} [-i\epsilon (\chi K_z + V_z \chi \psi \bar{\psi} + V h \bar{\psi} + 2\rho \chi h) + c.c.]. \quad (4.35)$$

Classical supercharges are found as a Noether invariants. After calculations we get

$$\begin{aligned} Q &= \chi (P_z - iK_z - iV_z \psi \bar{\psi} - 2i\rho h) + h (P_\psi - iV \bar{\psi}), \\ \bar{Q} &= \bar{\chi} (P_{\bar{z}} + iK_{\bar{z}} + iV_{\bar{z}} \psi \bar{\psi} + 2i\rho \bar{h}) - \bar{h} (P_{\bar{\psi}} - iV \psi), \end{aligned} \quad (4.36)$$

where $\chi, \bar{\chi}, h$ and \bar{h} are given by the expressions (4.21). After substituting auxiliary fields and some work, these supercharges can be rewritten as

$$Q = \frac{1}{2i\rho} \mathcal{P}_z \mathcal{P}_\psi, \quad \bar{Q} = \frac{1}{2i\rho} \mathcal{P}_{\bar{\psi}} \mathcal{P}_{\bar{z}}, \quad (4.37)$$

where the classical covariant derivatives $\mathcal{P}_A, \mathcal{P}_{\bar{A}}$ were defined in (4.24). So the following Poisson brackets will be useful

$$\{z, P_z\}_{PB} = 1, \quad \{\bar{z}, P_{\bar{z}}\}_{PB} = 1, \quad \{\psi, P_\psi\}_{PB} = -1, \quad \{\bar{\psi}, P_{\bar{\psi}}\}_{PB} = -1, \quad (4.38)$$

Under these brackets, the covariant derivatives obey the relations

$$\{\mathcal{P}_A, \mathcal{P}_{\bar{B}}\}_{PB} = -2i\partial_A\partial_{\bar{B}}\mathcal{K}, \quad \{\mathcal{P}_A, \mathcal{P}_B\}_{PB} = 0. \quad (4.39)$$

where the potential \mathcal{K} is defined in (4.27). Evaluating the brackets between the supercharges according to the usual rules we obtain

$$\begin{aligned} \{Q, \bar{Q}\}_{PB} &= -\frac{1}{4\rho^2}[\mathcal{P}_z\{\mathcal{P}_\psi, P_{\bar{\psi}}\}_{PB}\mathcal{P}_{\bar{z}} - \mathcal{P}_z\mathcal{P}_{\bar{\psi}}\{\mathcal{P}_\psi, P_{\bar{z}}\}_{PB} \\ &\quad - \{\mathcal{P}_z, P_{\bar{\psi}}\}_{PB}\mathcal{P}_{\bar{z}}\mathcal{P}_\psi - \mathcal{P}_{\bar{\psi}}\{\mathcal{P}_z, P_{\bar{z}}\}_{PB}\mathcal{P}_\psi] = -2iH_{class}, \end{aligned} \quad (4.40)$$

where H_{class} was defined in (4.23)⁷ and

$$\{Q, Q\}_{PB} = \{\bar{Q}, \bar{Q}\}_{PB} = 0. \quad (4.41)$$

We are going to pursue the quantization of this system in the following way. We first replace

$$P_A \rightarrow -i\partial_A, \quad P_{\bar{B}} \rightarrow -i\partial_{\bar{B}}. \quad (4.42)$$

We tackle the quantization ordering ambiguities by focusing on the definition of Q given by (4.37): As expressed in terms of \mathcal{P}_A , the supercharge Q does not exhibit any ordering ambiguities. Then we are led to introduce a general inner product on the target superspace $(z, \bar{z}, \psi, \bar{\psi})$ with a general measure, and to define Q^\dagger as an Hermitian conjugate of Q with respect to this inner product.

⁷It is curious that the inverse supermetric $g^{A\bar{B}}$ entering (4.23) is related by (4.40) through (4.39) to the second derivatives of the potential \mathcal{K} :

$$\partial_A\partial_{\bar{B}}\mathcal{K} = -4\rho^2\epsilon_{AB}\epsilon_{\bar{B}\bar{C}}g^{B\bar{C}},$$

where $\epsilon_{AB}, \epsilon_{\bar{A}\bar{B}}$ are *symmetric* constant tensors with the only non-zero entries $\epsilon_{z\psi} = \epsilon_{\psi z} = 1$ and $\epsilon_{\bar{z}\bar{\psi}} = \epsilon_{\bar{\psi}\bar{z}} = 1$, respectively. This is an indication that the underlying geometry of our general $\mathcal{N} = 2$ super LM is an interesting modification of the super Kähler geometry, such that it is the inverse metric which is expressed through second derivatives of some scalar potential, not the standard metric as in the (super)Kähler case.

The inner product is defined as

$$\langle f, g \rangle = \int dz d\bar{z} d\psi d\bar{\psi} F \overline{(f(z, \bar{z}, \psi, \bar{\psi}))} g(z, \bar{z}, \psi, \bar{\psi}), \quad (4.43)$$

where the measure F is assumed to have the following $\psi, \bar{\psi}$ expansion

$$F = F_0(z, \bar{z}) + \bar{\psi}\psi F_3(z, \bar{z}), \quad (4.44)$$

with the real functions F_0 and F_3 to be determined⁸. The superfunction f has the generic $\psi, \bar{\psi}$ expansion

$$f(z, \bar{z}, \psi, \bar{\psi}) = f_0(z, \bar{z}) + \psi f_1(z, \bar{z}) + \bar{\psi} f_2(z, \bar{z}) + \bar{\psi}\psi f_3(z, \bar{z}), \quad (4.45)$$

and similarly for g .

Now we wish to compute the hermitian conjugates of the basic operators with respect to this general inner product. We note that the anticommuting variables are always standing on the left, so to compute the component norms we will never need to ascribe a definite Grassmann parity to the component fields. With this in mind, we derive

$$(\partial_\psi)^\dagger = \partial_{\bar{\psi}} + \psi \frac{F_3}{F_0}, \quad (\partial_z)^\dagger = -\partial_{\bar{z}} - \frac{\partial_{\bar{z}} F_0}{F_0} - \bar{\psi}\psi \frac{F_3}{F_0} \left(\frac{\partial_{\bar{z}} F_3}{F_3} - \frac{\partial_{\bar{z}} F_0}{F_0} \right). \quad (4.46)$$

The quantum version of $\mathcal{P}_{\bar{B}}$ can be obtained in a similar way, i.e. through hermitian conjugation of \mathcal{P}_B with respect to the above inner product. As a result, the quantum version of Q^\dagger will be expressed in terms of the quantum versions of $\mathcal{P}_{\bar{B}}$ according to eq. (4.24), but with the properly modified connection terms (due to

⁸Measure F must be even that is why we do not have components F_1 and F_2 .

(4.46)). This modification will change the quantum version of eq. (4.39). Then the quantum version of equation (4.40) will imply constraints on the measure F , so as to preserve the form of “kinetic” terms in the quantum Hamiltonian (i.e., terms bilinear in the partial derivatives) because the ordering procedure cannot modify the coefficients of these highest-order terms. These coefficients are specified by the quantum version of the relations (4.39), and (4.40). Requiring them to coincide with those in the classical Hamiltonian implies the measure to be trivial,

$$F_3 = 0, \quad F_0 = \omega(z) \bar{\omega}(\bar{z}), \quad (4.47)$$

where $\omega(z)$ is an arbitrary holomorphic function. In the inner product (4.43), the holomorphic and antiholomorphic factors in F_0 can always be absorbed into the redefinition of the superfunctions f and g , and so, without loss of generality, we can choose $F_0 = 1$.

Having such a constant measure comes as both a bonus and a surprise. It is a bonus because with such a measure both Q and Q^\dagger are naturally on the same footing. Otherwise, it would be difficult to explain why we start by quantizing Q and then define Q^\dagger , and not the other way around. It is a surprise because both the Lagrangian and the Hamiltonian in the general case involve a non-trivial supermetric. In the quantum models associated with homogeneous superspaces as the targets, the integration measure in the inner product can be naturally constructed with the help of the supervolume form, by requiring this measure to be invariant under the group of super-isometries of the target space (see e.g. [4]). In our case the only prerequisite symmetry of the Lagrangian and Hamiltonian is $\mathcal{N} = 2$ supersymmetry, the transformations of which involve the momenta (time derivatives of the coordinates). No isometry acting only on the coordinates is assumed a priori. This invalidates the usual arguments for the construction of the invariant measure through the standard

supervolume form. In principle, using the ordering ambiguities, one can arrange the quantum theory in such a way that the measure will involve a non-trivial factor (see Section 5). However, the final answers will be the same as in the present case.

The quantum version of our covariant derivatives $\mathcal{P}_A, \mathcal{P}_{\bar{B}}$ will be

$$\begin{aligned}\mathcal{P}_z &= -i(\partial_z + K_z + \psi\bar{\psi}V_z), & \mathcal{P}_{\bar{z}} &= -i(\partial_{\bar{z}} - K_{\bar{z}} - \psi\bar{\psi}V_{\bar{z}}), \\ \mathcal{P}_\psi &= -i(\partial_\psi + \bar{\psi}V), & \mathcal{P}_{\bar{\psi}} &= -i(\partial_{\bar{\psi}} + \psi V),\end{aligned}\tag{4.48}$$

and, correspondingly, the non-vanishing relations in (4.39) become

$$\begin{aligned}[\mathcal{P}_z, \mathcal{P}_{\bar{z}}] &= 2(K_{z\bar{z}} + \psi\bar{\psi}V_{z\bar{z}}), & \{\mathcal{P}_\psi, \mathcal{P}_{\bar{\psi}}\} &= -2V, \\ [\mathcal{P}_z, \mathcal{P}_{\bar{\psi}}] &= -2\psi V_z, & [\mathcal{P}_\psi, \mathcal{P}_{\bar{z}}] &= 2\bar{\psi}V_{\bar{z}}.\end{aligned}\tag{4.49}$$

Now it is straightforward to compute the quantum Hamiltonian

$$H_q = \frac{1}{4\rho^2}[\mathcal{P}_z V \mathcal{P}_{\bar{z}} + \mathcal{P}_z \mathcal{P}_{\bar{\psi}} V_{\bar{z}} \bar{\psi} - V_z \psi \mathcal{P}_{\bar{z}} \mathcal{P}_\psi + \mathcal{P}_{\bar{\psi}}(K_{z\bar{z}} + \psi\bar{\psi}V_{z\bar{z}})\mathcal{P}_\psi].\tag{4.50}$$

With this hermitian Hamiltonian at hand we turn to the study of the eigenvalue equation

$$H_q \Psi(z, \bar{z}, \psi, \bar{\psi}) = \lambda \Psi(z, \bar{z}, \psi, \bar{\psi}),\tag{4.51}$$

where Ψ is assumed to have general $\psi, \bar{\psi}$ expansion (4.45),

$$\Psi(z, \bar{z}, \psi, \bar{\psi}) = f_0(z, \bar{z}) + \psi f_1(z, \bar{z}) + \bar{\psi} f_2(z, \bar{z}) + \bar{\psi} \psi f_3(z, \bar{z}).\tag{4.52}$$

An important property of this Hamiltonian is that it does not mix any components of $\Psi(z, \bar{z}, \psi, \bar{\psi})$ which are linear in $\psi, \bar{\psi}$, i.e.

$$H_q \psi f_1(z, \bar{z}) = \lambda_1 \psi f_1(z, \bar{z}), \quad H_q \bar{\psi} f_2(z, \bar{z}) = \lambda_2 \bar{\psi} f_2(z, \bar{z}).\tag{4.53}$$

Or, in the component form,

$$-\frac{1}{4\rho^2}(\partial_{\bar{z}} - K_{\bar{z}})V(\partial_z + K_z)f_1 = \lambda_1 f_1, \quad (4.54)$$

and

$$-\frac{1}{4\rho^2}(\partial_z + K_z)V(\partial_{\bar{z}} - K_{\bar{z}})f_2 = \lambda_2 f_2. \quad (4.55)$$

In other words, the corresponding subspaces are invariant subspaces of H_q . We can also find another pair of invariant subspaces of H_q which consist of components of the wave superfunction $\Psi(z, \bar{z}, \psi, \bar{\psi})$ which are even in $\psi, \bar{\psi}$. We represent Ψ in (4.52) as a sum

$$\Psi = \Psi^L + \Psi^H, \quad (4.56)$$

where

$$\Psi^L = f_0^L + \bar{\psi}\psi V f_0^L + \bar{\psi}f_2 \equiv \Psi_{even}^L + \bar{\psi}f_2, \quad \Psi^H = f_0^H - \bar{\psi}\psi V f_0^H + \psi f_1 \equiv \Psi_{even}^H + \psi f_1. \quad (4.57)$$

This corresponds just to rearranging the component fields in (4.52) as

$$f_0 = f_0^L + f_0^H, \quad f_3 = V(f_0^L - f_0^H). \quad (4.58)$$

The superfunctions Ψ_{even}^L and Ψ_{even}^H also prove to be invariant subspaces under the action of H_q ,

$$H_q \Psi_{even}^L = \lambda_3 \Psi_{even}^L, \quad H_q \Psi_{even}^H = \lambda_4 \Psi_{even}^H. \quad (4.59)$$

This gives rise to the other two eigenvalue equations completing (4.54) and (4.55)

$$-\frac{1}{4\rho^2}(\partial_z + K_z)(\partial_{\bar{z}} - K_{\bar{z}})Vf_0^L = \lambda_3 f_0^L \quad (4.60)$$

$$-\frac{1}{4\rho^2}(\partial_{\bar{z}} - K_{\bar{z}})(\partial_z + K_z)Vf_0^H = \lambda_4 f_0^H. \quad (4.61)$$

Thus, passing to the parametrization (4.56), (4.57) of the general wave superfunction reduces the diagonalization of the Hamiltonian H_q to two ordinary eigenvalue problems.

Indeed, by the factorization lemma which states that the non-zero eigenvalues of the operators \mathcal{BC} and \mathcal{CB} are the same (appendix C), it can be easily seen that the non-zero eigenvalues of the operators in (4.54) and (4.60) coincide. The same is true for the operators in (4.55) and (4.61). This is a consequence of the fact that these states are transformed into each other by the $\mathcal{N} = 2$ supersymmetry transformations (see below).

With $F = 1$, the inner product (4.43) of the component functions, in terms of the invariant states of the Hamiltonian H_q described above, is as follows

$$\begin{aligned} \langle f, g \rangle &= \int dz d\bar{z} d\psi d\bar{\psi} (\bar{\Psi}^L \Psi^L + \bar{\Psi}^H \Psi^H) \\ &= \int dz d\bar{z} (\bar{f}_1 g_1 - \bar{f}_2 g_2 + 2V \bar{f}_0^L g_0^L - 2V \bar{f}_0^H g_0^H). \end{aligned} \quad (4.62)$$

The corresponding norm, $\langle f, f \rangle$ is diagonal and, evidently, the norms of states corresponding to f_0^H and f_2 appear with the wrong sign. Therefore, like in the previous cases [4, 8], in order to restore the positive definiteness we are led to introduce the metric operator

$$G = \frac{[\mathcal{P}_{\bar{\psi}}, \mathcal{P}_{\psi}]}{2V} + 2 \left(\frac{\psi \partial \bar{\partial} \psi}{-} \bar{\psi} \frac{\partial}{\partial \bar{\psi}} \right). \quad (4.63)$$

This metric operator commutes with Q and Q^\dagger ,

$$[G, Q] = [G, Q^\dagger] = 0, \quad (4.64)$$

and it is a constant of motion by itself. Under the new inner product

$$\langle\langle f, g \rangle\rangle = \langle Gf, g \rangle, \quad (4.65)$$

the operators appearing in formulas (4.54), (4.55), (4.60) and (4.61) are hermitian positive-definite operators. It follows that their eigenvalues must be ≥ 0 , and the possible zero modes (specifying the ground state wave functions) are related to solutions of the equations

$$(\partial_z + K_z)g = 0, \quad (\partial_{\bar{z}} - K_{\bar{z}})h = 0. \quad (4.66)$$

Notice that the superwave functions Ψ_{even}^L , $\bar{\psi}f_2$, Ψ_{even}^H , ψf_1 corresponding to the invariant subspaces of H_q are mutually orthogonal with respect to (4.62) and (4.65), as should be. Actually, the only effect of passing to the new inner product is the change of the minus signs to the plus signs in the component expression (4.62), i.e. the change of the relative sign between terms related to each of the two irreducible $\mathcal{N} = 2$ multiplets (the signs between products or norms of the fields belonging to the same multiplet cannot alter because the metric operator G commutes with the $\mathcal{N} = 2$ supersymmetry generators).

As the last topic of this Section, we shall study the action of the supersymmetry generators on the invariant subspaces of the Hamiltonian which we described above.

We have

$$Q\Psi^L = 0,$$

$$Q^\dagger\Psi^L = \frac{i}{2\rho}[(\partial_{\bar{z}} - K_{\bar{z}})f_2 + 2\psi(\partial_{\bar{z}} - K_{\bar{z}})Vf_0^L - \bar{\psi}\psi V(\partial_{\bar{z}} - K_{\bar{z}})f_2], \quad (4.67)$$

and

$$Q^\dagger\Psi^H = 0,$$

$$Q\Psi^H = \frac{i}{2\rho}[(\partial_z + K_z)f_1 + 2\bar{\psi}(\partial_z + K_z)Vf_0^H + \bar{\psi}\psi V(\partial_z + K_z)f_1]. \quad (4.68)$$

Now it is easy to see that the general superfunction Ψ contains two irreducible $\mathcal{N} = 2$ multiplets (f_1, f_0^L) and (f_2, f_0^H) , which, before the redefinition of the norm, have positive and negative norms, respectively. Defining the $\mathcal{N} = 2$ supersymmetry transformation of the general wave function $\Psi = \Psi^L + \Psi^H$ as

$$\delta\Psi = (\epsilon Q + \bar{\epsilon} Q^\dagger)\Psi, \quad (4.69)$$

we find from (4.67) and (4.68)

$$\begin{aligned} \delta f_0^L &= \frac{i}{2\rho} \epsilon (\partial_z + K_z) f_1, & \delta f_1 &= -\frac{i}{\rho} \bar{\epsilon} (\partial_{\bar{z}} - K_{\bar{z}}) V f_0^L, \\ \delta f_0^H &= \frac{i}{2\rho} \bar{\epsilon} (\partial_{\bar{z}} - K_{\bar{z}}) f_2, & \delta f_2 &= -\frac{i}{\rho} \epsilon (\partial_z + K_z) V f_0^H. \end{aligned} \quad (4.70)$$

The ground state wave superfunctions $\Psi_{vac}^L, \Psi_{vac}^H$ are defined as zero eigenvalues of H_q . The corresponding wave functions are solutions of eqs. (4.66), so $\Psi_{vac}^L, \Psi_{vac}^H$ automatically obey the conditions

$$Q\Psi_{vac}^L = Q^\dagger\Psi_{vac}^L = Q\Psi_{vac}^H = Q^\dagger\Psi_{vac}^H = 0, \quad (4.71)$$

as a consequence of the relations (4.67), (4.68). The set of ground states is spanned by two holomorphic and two antiholomorphic functions

$$\begin{aligned} (f_2)_{vac} &= e^K \tilde{f}_2(z), \quad (f_0^L)_{vac} = V^{-1} e^K \tilde{f}_0^L(z), \\ (f_1)_{vac} &= e^{-K} \tilde{f}_1(\bar{z}), \quad (f_0^H)_{vac} = V^{-1} e^{-K} \tilde{f}_0^H(\bar{z}). \end{aligned} \quad (4.72)$$

Using the transformation properties (4.70), it is straightforward to check that the functions (4.72) are indeed singlets under $\mathcal{N} = 2$ supersymmetry.

Finally, we notice that the obvious requirement of finiteness for the z, \bar{z} integrals present in the definition of the inner products (4.62) and (4.65), and of the corresponding norms, imposes rather severe restrictions on the asymptotic behavior of the admissible class of wave functions f_0^L, f_0^H, f_1 and f_2 as $z, \bar{z} \rightarrow \infty$, as well as on the admissible choice of the potentials $K(z, \bar{z})$ and $V(z, \bar{z})$. This issue is difficult to analyze in general. We shall discuss it on a specific example in section 4.5.

4.4 Quantization scheme 2

Lets discuss quantization procedure once again. Our classical Hamiltonian is fixed, but there may be different quantum versions of it depending on quantization. Nevertheless we have to demand the identity of the coefficients of the terms quadratic in the momenta in the quantum and classical versions of the Hamiltonian. We observed in previous case how this imposes very strong conditions on the hermitian adjoint properties of the covariant derivatives. More specifically after we obtain quantum operator \mathcal{P}_A^\dagger computed within the natural inner product, we must demand it to be consistent with classical momentum $\mathcal{P}_{\bar{A}}$. In (4.47) this condition forced the integration measure in the inner product to be almost constant.

Now we will consider equivalent classical form of the supersymmetry charges Q

and \bar{Q} , in order to avoid this troublesome constraint. Indeed, consider the following expressions for the supersymmetry generators

$$Q' = \frac{1}{2i\rho} \mathcal{P}'_z \mathcal{P}_\psi, \quad \bar{Q}' = \frac{1}{2i\rho} \mathcal{P}'_{\bar{\psi}} \mathcal{P}'_{\bar{z}}, \quad (4.73)$$

where

$$\begin{aligned} \mathcal{P}'_z &= (P_z - iK_z - \frac{V_z}{V} \psi P_\psi), & \mathcal{P}'_{\bar{z}} &= (P_{\bar{z}} + iK_{\bar{z}} - \frac{V_{\bar{z}}}{V} \bar{\psi} P_{\bar{\psi}}), \\ \mathcal{P}_\psi &= (P_\psi - i\bar{\psi} V), & \mathcal{P}_{\bar{\psi}} &= (P_{\bar{\psi}} - i\psi V). \end{aligned} \quad (4.74)$$

It is easy to see that $\mathcal{P}_z - \mathcal{P}'_z \sim \psi P_\psi$, because of this $\mathcal{P}'_z \mathcal{P}_\psi = \mathcal{P}_z \mathcal{P}_\psi$, and therefore the classical supercharge $Q = Q'$ is not modified, and a similar argument is valid for conjugated supercharge $\bar{Q} = \bar{Q}'$. The corresponding classical brackets among these “new covariant derivatives” can be easily obtained from (4.38), (4.39).

We are going to use the same quantization rules (4.42), and $1/V$ as a measure in (4.43). In the first quantization scheme we defined quantum operator Q and a measure which gave us conjugation rules and operator Q^\dagger , here we start by defining both Q and Q^\dagger that is why we get a different measure $1/V$ for the second case. We are also going to use the following quantum ordering prescription in the definitions

$$\begin{aligned} \mathcal{P}'_z &= -i(\partial_z + K_z - \frac{V_z}{V} \psi \partial_\psi + a \frac{V_z}{V}), & \mathcal{P}'_{\bar{z}} &= -i(\partial_{\bar{z}} - K_{\bar{z}} - \frac{V_{\bar{z}}}{V} \bar{\psi} \partial_{\bar{\psi}} - a \frac{V_{\bar{z}}}{V}), \\ \mathcal{P}_\psi &= -i(\partial_\psi + \bar{\psi} V), & \mathcal{P}_{\bar{\psi}} &= -i(\partial_{\bar{\psi}} + \psi V), \end{aligned} \quad (4.75)$$

where the extra terms with a real constant a in the expression of the \mathcal{P}' 's reflects the ordering ambiguity in the products ψP_ψ and $\bar{\psi} P_{\bar{\psi}}$. We have freedom to fix value of a later to insure consistence of quantum and classical versions of a Hamiltonian. It can be checked that this definitions of quantum momentum and the choice of measure

together bring correct conjugation properties

$$\mathcal{P}'_z{}^\dagger = \mathcal{P}'_{\bar{z}}, \quad \mathcal{P}'_\psi{}^\dagger = -\mathcal{P}'_{\bar{\psi}}. \quad (4.76)$$

Therefore, if the supercharge Q' is ordered as in (4.73), and Q'^\dagger is defined as the hermitian conjugate of Q' with respect to the inner product with the measure $1/V$, we can again implement the involution of the abstract $\mathcal{N} = 2$ superalgebra as the hermitian conjugation of the quantum operators. It remains to check that the coefficients of the terms quadratic in the momenta of the quantum and classical Hamiltonians are equal. The algebra of the new covariant derivatives is

$$\begin{aligned} \{\mathcal{P}_\psi, \mathcal{P}_{\bar{\psi}}\} &= -2V, \quad [\mathcal{P}'_z, \mathcal{P}'_{\bar{z}}] = 2K_{z\bar{z}} + (\partial_z \partial_{\bar{z}} \ln V) (\bar{\psi} \partial_{\bar{\psi}} - \psi \partial_\psi + 2a), \\ [\mathcal{P}'_z, \mathcal{P}_{\bar{\psi}}] &= 0, \quad [\mathcal{P}'_{\bar{z}}, \mathcal{P}_\psi] = 0, \quad \mathcal{P}_\psi \mathcal{P}_\psi = \mathcal{P}_{\bar{\psi}} \mathcal{P}_{\bar{\psi}} = 0, \\ [\mathcal{P}'_z, \mathcal{P}_\psi] &= -i(\partial_z \ln V) \mathcal{P}_\psi, \quad [\mathcal{P}'_{\bar{z}}, \mathcal{P}_{\bar{\psi}}] = -i(\partial_{\bar{z}} \ln V) \mathcal{P}_{\bar{\psi}}. \end{aligned} \quad (4.77)$$

Then the quantum Hamiltonian reads

$$\begin{aligned} 2\tilde{H}_q &= \{Q', Q'^\dagger\} = \\ &= \frac{1}{2\rho^2} \left[\mathcal{P}'_z V \mathcal{P}'_{\bar{z}} + \mathcal{P}_{\bar{\psi}} \left(K_{z\bar{z}} + \frac{1}{2} \partial_z \partial_{\bar{z}} \ln V (\bar{\psi} \partial_{\bar{\psi}} - \psi \partial_\psi + 2a) \right) \mathcal{P}_\psi \right] \end{aligned} \quad (4.78)$$

An inspection of this expression reveals that it contains terms which formally appear as having three odd derivatives, viz. $\mathcal{P}_{\bar{\psi}} (\bar{\psi} \partial_{\bar{\psi}} - \psi \partial_\psi) \mathcal{P}_\psi$. Upon rewriting them in detail, because of the ordering, these terms generate an additional term in the product of the two odd momenta. By choosing $a = -\frac{1}{2}$ one can cancel the additional contribution to ensure that the coefficients of the momenta-squared terms in the quantum Hamiltonian are identical to those in the classical Hamiltonian. Should we

have chosen another ordering instead of the one in (4.73), for example the symmetrical (Weyl) prescription $\frac{1}{2}(\mathcal{P}'_z \mathcal{P}_\psi + \mathcal{P}'_\psi \mathcal{P}_z)$ with the corresponding definition of Q^\dagger , it can be shown (with the help of (4.77)) to require a different value of a . In what follows we are going to pursue the consequences of the ordering chosen in (4.73).

Now, proceeding as we did in the previous section, we obtain the same invariant subspaces of the new quantum Hamiltonian. Using the same expansion for the relevant wave superfunction, we derive

$$\tilde{H}\psi f_1 = -\frac{1}{4\rho^2}V \left(\partial_{\bar{z}} - K_{\bar{z}} + \frac{1}{2}\partial_{\bar{z}} \ln V \right) \left(\partial_z + K_z - \frac{1}{2}\partial_z \ln V \right) \psi f_1 = \lambda_1 \psi f_1, \quad (4.79)$$

and

$$\tilde{H}\bar{\psi} f_2 = -\frac{1}{4\rho^2}V \left(\partial_z + K_z + \frac{1}{2}\partial_z \ln V \right) \left(\partial_{\bar{z}} - K_{\bar{z}} - \frac{1}{2}\partial_{\bar{z}} \ln V \right) \bar{\psi} f_2 = \lambda_2 \bar{\psi} f_2. \quad (4.80)$$

Then, using (4.57), we obtain the other set of invariant subspaces

$$\tilde{H}f_0^H = -\frac{1}{4\rho^2} \left(\partial_z - K_z - \frac{1}{2}\partial_z \ln V \right) V \left(\partial_{\bar{z}} + K_{\bar{z}} + \frac{1}{2}\partial_{\bar{z}} \ln V \right) f_0^H = \lambda_3 f_0^H, \quad (4.81)$$

and

$$\tilde{H}f_0^L = -\frac{1}{4\rho^2} \left(\partial_z + K_z - \frac{1}{2}\partial_z \ln V \right) V \left(\partial_{\bar{z}} - K_{\bar{z}} + \frac{1}{2}\partial_{\bar{z}} \ln V \right) f_0^L = \lambda_4 f_0^L. \quad (4.82)$$

The inner product (4.43) of the component functions, with the measure V^{-1} , in terms of the invariant states of the Hamiltonian H_q described above, is given by the integral

$$\langle f, g \rangle = \int \frac{dz d\bar{z}}{V} (\bar{f}_1 g_1 - \bar{f}_2 g_2 + 2V \bar{f}_0^L g_0^L - 2V \bar{f}_0^H g_0^H). \quad (4.83)$$

At this stage it is easy to see that, changing the functions in (4.83) by

$$f_0^L \rightarrow V^{\frac{1}{2}} f_0^L, \quad f_0^H \rightarrow V^{\frac{1}{2}} f_0^H, \quad f_i \rightarrow V^{\frac{1}{2}} f_i, \quad (i = 1, 2), \quad (4.84)$$

we come back to (4.62), while the equations (4.79) - (4.82) are converted into the previous set (4.54), (4.55), (4.61), (4.60). The supercharges of the different quantization schemes are connected by the relation

$$V^{-\frac{1}{2}} Q' V^{\frac{1}{2}} = Q, \quad V^{-\frac{1}{2}} Q'^{\dagger} V^{\frac{1}{2}} = Q^{\dagger}. \quad (4.85)$$

It is also easy to find the explicit relation between the Hamiltonians H_q and \tilde{H}_q defined by eqs. (4.50) and (4.78) with $a = -\frac{1}{2}$

$$\tilde{H}_q = H_q + \frac{1}{8\rho^2} \left[V_{z\bar{z}} + \frac{1}{2} \frac{V_z V_{\bar{z}}}{V} + i(V_z \mathcal{P}_{\bar{z}} + V_{\bar{z}} \mathcal{P}_z) - i \frac{V_z V_{\bar{z}}}{V} (\psi \mathcal{P}_{\psi} + \bar{\psi} \mathcal{P}_{\bar{\psi}}) \right]. \quad (4.86)$$

This relation can be rewritten as the following simple similarity transformation,

$$V^{-\frac{1}{2}} \tilde{H}_q V^{\frac{1}{2}} = H_q, \quad (4.87)$$

which agrees with (4.85) and proves the equivalence of the two quantization schemes.⁹

Now, using (4.57), we have

$$Q\Psi^L = 0, \quad Q^{\dagger}\Psi^L = \frac{i}{2\rho} \left[(\partial_{\bar{z}} - K_{\bar{z}} - \frac{1}{2} \partial_{\bar{z}} \ln V) f_2 + 2\psi V (\partial_{\bar{z}} - K_{\bar{z}} + \frac{1}{2} \partial_{\bar{z}} \ln V) f_0^L \right. \\ \left. - \bar{\psi} \psi V (\partial_{\bar{z}} - K_{\bar{z}} - \frac{1}{2} \partial_{\bar{z}} \ln V) f_2 \right], \quad (4.88)$$

⁹A similar equivalence transformation between various quantization schemes in the conventional supersymmetric quantum mechanics and its relation to different definitions of the inner product were discussed many years ago in [12].

and

$$Q^\dagger \Psi^H = 0, Q \Psi^H = \frac{i}{2\rho} [(\partial_z + K_z - \frac{1}{2} \partial_z \ln V) f_1 + 2\bar{\psi} V (\partial_z + K_z + \frac{1}{2} \partial_z \ln V) f_0^H + \bar{\psi} \psi V (\partial_z + K_z - \frac{1}{2} \partial_z \ln V) f_1]. \quad (4.89)$$

As follows from (4.79) - (4.82), the ground state wave functions corresponding to zero eigenvalues of \tilde{H} are defined by the equations

$$\begin{aligned} (\partial_z + K_z - \frac{1}{2} \partial_z \ln V) (f_1)_{vac} &= (\partial_{\bar{z}} - K_{\bar{z}} + \frac{1}{2} \partial_{\bar{z}} \ln V) (f_0^L)_{vac} = 0, \\ (\partial_{\bar{z}} - K_{\bar{z}} - \frac{1}{2} \partial_{\bar{z}} \ln V) (f_2)_{vac} &= (\partial_z + K_z + \frac{1}{2} \partial_z \ln V) (f_0^H)_{vac} = 0, \end{aligned} \quad (4.90)$$

which imply that the ground state wave superfunctions $\Psi_{vac}^L, \Psi_{vac}^H$ are singlets of $\mathcal{N} = 2$ supersymmetry, like in the first quantization scheme (eqs. (4.71)).

Finally, we note that the passing to the positive-definite inner product from (4.83) in this quantization scheme is accomplished by the same operator G as in (4.63), but now it should be transformed on the pattern of (4.85) and (4.87). Since this quantization scheme is found to be equivalent to the first one there is no need to give excessive details on the solution of the quantum model as it can be effectively constructed from the known expressions in the previous subsection.

4.5 Particle on the superpshere

We are unable to proceed further into analysis of quantum model without specific choice of potentials K and V . There is no recipe on what choice to make in order have solvable Schrodinger equations. One insight is to have extra symmetry introduced into our system. Lets have our potentials realize a symmetry for a particle on a sphere $\mathbb{C}\mathbb{P}^1 \sim SU(2)/U(1)$. Consider the $SU(2)$ invariant subclass of our original superfield

action (4.18), with the following potentials

$$K(\Phi, \bar{\Phi}) = -N \ln(1 + \Phi\bar{\Phi}), \quad V(\Phi, \bar{\Phi}) = (1 + \Phi\bar{\Phi})^2. \quad (4.91)$$

It is easy to check that under this choice (4.18) is invariant with respect to the standard \mathbb{CP}^1 realization of the $SU(2)$ transformations

$$\delta\Phi = \varepsilon + i\beta\Phi + \bar{\varepsilon}\Phi^2, \quad \delta\Psi = -(i\beta + 2\bar{\varepsilon}\Phi)\Psi. \quad (4.92)$$

Thus the superfields Φ and $\bar{\Phi}$ can be interpreted as the complex coordinates of $\mathbb{CP}^1 \sim SU(2)/U(1)$, with $K(\Phi, \bar{\Phi})$ being related to the Kähler potential. To be more precise we will have components z and \bar{z} to be complex coordinates of \mathbb{CP}^1 , with $K(z, \bar{z}) = -N \ln(1 + z\bar{z})$ being the Kähler potential. Because of this we deal with the dynamics of a particle on the sphere in a magnetic field — the field of a Dirac monopole located at the center. For this particular case the on-shell Lagrangian (4.32) (up to a renormalization factor) reads

$$\begin{aligned} \mathcal{L}_{su(2)} = & \frac{\dot{z}\dot{\bar{z}}}{(1 + z\bar{z})^2} + N^{-1} (1 + z\bar{z})^2 [1 + 2N^{-1}\psi\bar{\psi}(1 + z\bar{z})^2] \nabla\psi\nabla\bar{\psi} \\ & - i \left[\frac{N - 2\psi\bar{\psi}(1 + z\bar{z})^2}{1 + z\bar{z}} (\dot{z}\bar{z} - \dot{\bar{z}}z) - (1 + z\bar{z})^2 (\dot{\psi}\bar{\psi} - \psi\dot{\bar{\psi}}) \right], \end{aligned} \quad (4.93)$$

where

$$\nabla\psi = \dot{\psi} + 2\frac{\dot{z}\bar{z}}{1 + z\bar{z}}\psi, \quad \nabla\bar{\psi} = \dot{\bar{\psi}} + 2\frac{\dot{\bar{z}}z}{1 + z\bar{z}}\bar{\psi}. \quad (4.94)$$

This Lagrangian can be rewritten as

$$\mathcal{L}_{su(2)} = \dot{Z}^A \dot{\bar{Z}}^{\bar{B}} g_{\bar{B}A} + \left(\dot{Z}^B \mathcal{A}_B + \dot{\bar{Z}}^{\bar{B}} \mathcal{A}_{\bar{B}} \right), \quad (4.95)$$

with

$$g_{z\bar{z}} = \frac{1}{(1+z\bar{z})^2} + \frac{4z\bar{z}}{N} \psi\bar{\psi}, \quad g_{\bar{\psi}\psi} = \frac{(1+z\bar{z})^2}{N} \left[1 + \frac{2\psi\bar{\psi}(1+z\bar{z})^2}{N} \right],$$

$$g_{z\psi} = \frac{2(1+z\bar{z})}{N} z\bar{\psi}, \quad g_{\bar{\psi}z} = -\frac{2(1+z\bar{z})}{N} \bar{z}\psi, \quad (4.96)$$

$$\mathcal{A}_z = i\bar{z} \frac{-N + 2\psi\bar{\psi}(1+z\bar{z})^2}{1+z\bar{z}}, \quad \mathcal{A}_{\bar{z}} = iz \frac{N - 2\psi\bar{\psi}(1+z\bar{z})^2}{1+z\bar{z}}, \quad (4.97)$$

$$\mathcal{A}_\psi = i(1+z\bar{z})^2 \bar{\psi}, \quad \mathcal{A}_{\bar{\psi}} = i\psi(1+z\bar{z})^2. \quad (4.98)$$

The entries of the inverse target space metric are given by

$$g^{z\bar{z}} = (1+z\bar{z})^2, \quad g^{\psi\bar{\psi}} = \frac{1}{(1+z\bar{z})^2} [N - 2\psi\bar{\psi}(1+z\bar{z})^2(1+2z\bar{z})],$$

$$g^{\psi\bar{z}} = 2(1+z\bar{z})\bar{z}\psi, \quad g^{z\bar{\psi}} = -2(1+z\bar{z})z\bar{\psi}. \quad (4.99)$$

The action corresponding to the Lagrangian (4.93) is invariant under the $\mathcal{N} = 2$ supersymmetry transformations (4.16), with the auxiliary fields h and χ being expressed by the general formulas (4.29), and under $SU(2)$ transformations (4.92)

$$\delta z = \varepsilon + i\beta z + \bar{\varepsilon} z^2, \quad \delta \psi = -(i\beta + 2\bar{\varepsilon} z) \psi. \quad (4.100)$$

These invariances are the only symmetries of the considered model. The Lagrangian (4.93) presents an $\mathcal{N} = 2$ supersymmetric extension of the $SU(2)$ invariant bosonic Lagrangian describing a particle in the background of a Dirac monopole placed at the center of the 2-sphere $S^2 \sim \mathbb{CP}^1$ (and so underlying a LM on the 2-sphere S^2 [22]).

Actually, like in the bosonic case (Landau model on the sphere [22]), we deal with a group of models parametrized by the parameter N . The quantization of these models follows the general pattern, and we will specialize the general results obtained in the preceding Sections. Working within the alternative quantization scheme, the

corresponding eigenvalue equations are

$$\begin{aligned} -V\nabla_{\bar{z}}^{(N+1)}\nabla_z^{(N+1)}f_1 &= \lambda_1 f_1, & -V\nabla_z^{(N-1)}\nabla_{\bar{z}}^{(N-1)}f_2 &= \lambda_2 f_2, \\ -\nabla_{\bar{z}}^{(N-1)}V\nabla_z^{(N-1)}f_0^H &= \lambda_3 f_0^H, & -\nabla_z^{(N+1)}V\nabla_{\bar{z}}^{(N+1)}f_0^L &= \lambda_4 f_0^L, \end{aligned} \quad (4.101)$$

where

$$\nabla_z^{(N)} = \partial_z - N\frac{\bar{z}}{1+\bar{z}z}, \quad \nabla_{\bar{z}}^{(N)} = \partial_{\bar{z}} + N\frac{z}{1+\bar{z}z}. \quad (4.102)$$

One more advantage of the alternative quantization scheme in the present case is that the integration measure in the inner product (4.83) is just the $SU(2)$ invariant integration measure over \mathbb{CP}^1 , $dz d\bar{z}/(1+z\bar{z})^2$, so requiring the relevant wave functions to be normalizable actually amounts to the standard demand of their square-integrability on $\mathbb{CP}^1 \sim S^2$, under which the function proves to be globally defined on S^2 . In turn, this implies that the normalizable wave functions should encompass irreducible unitary representations of $SU(2)$. It is useful to know the $SU(2)/U(1)$ transformations of the wave functions f_1, f_2, f_0^L, f_0^H which leave invariant the inner product (4.83) in the model under consideration

$$\begin{aligned} \delta f_1 &= -[(N+1)(\varepsilon\bar{z} - \bar{\varepsilon}z) + \delta z\partial_z + \delta\bar{z}\partial_{\bar{z}}]f_1, \\ \delta f_2 &= -[(N-1)(\varepsilon\bar{z} - \bar{\varepsilon}z) + \delta z\partial_z + \delta\bar{z}\partial_{\bar{z}}]f_1, \\ \delta f_0^{L,H} &= -\left[\varepsilon\bar{z} + \bar{\varepsilon}z + N(\varepsilon\bar{z} - \bar{\varepsilon}z) + \delta z\partial_z + \delta\bar{z}\partial_{\bar{z}}\right]f_0^{L,H}. \end{aligned} \quad (4.103)$$

Now we shall analyze the structure of the wave functions as solutions of (4.101) - (4.102). It turns out that this structure essentially depends on the value of $N \in (\mathbb{N}, \mathbb{N} + \frac{1}{2})$. The normalizability requirement imposes rather severe restrictions on the admissible choice of the wave functions.

Ground states

We start our analysis with the ground states. From the point of view of the underlying bosonic Landau model on $S^2 \sim \mathbb{CP}^1$ [22], they correspond to the lowest Landau level (LLL). The LLL wave functions are defined by the equations (4.90) specialized to the case under consideration

$$\nabla_z^{(N+1)} f_1 = \nabla_{\bar{z}}^{(N-1)} f_2 = \nabla_z^{(N-1)} f_0^H = \nabla_{\bar{z}}^{(N+1)} f_0^L = 0. \quad (4.104)$$

The first of eqs. (4.104) has solutions of the form $f_1 \sim (1 + z\bar{z})^{N+1}$ which are not normalizable for any choice of $N \geq 0$. The other equations, depending on the value of N , yield the following non-trivial ground-state wave functions.

- For $N = 0$, one has two normalizable singlet ground states:

$$f_0^{H,0}(z, \bar{z}) = \frac{f_0^{H,0}}{1 + \bar{z}z}, \quad f_0^{L,0}(z, \bar{z}) = \frac{f_0^{L,0}}{1 + \bar{z}z}. \quad (4.105)$$

where $f_0^{H,0}$ and $f_0^{L,0}$ are constants. Thus in this case the ground states are $SU(2)$ singlets.

- For $N = \frac{1}{2}$, one has normalizable doublet ground states

$$f_0^{L,0}(z, \bar{z}) = \frac{A + Bz}{(1 + \bar{z}z)^{\frac{3}{2}}}, \quad (4.106)$$

the constants A and B thus forming spin 1/2 multiplet of $SU(2)$.

- For $N \geq 1$, one has the following set of the ground states

$$f_2^0(z, \bar{z}) = \frac{f_2^0(z)}{(1 + \bar{z}z)^{N-1}}, \quad N_{max} = 2(N-1), \quad f_0^{L,0}(z, \bar{z}) = \frac{f_0^{L,0}(z)}{(1 + \bar{z}z)^{N+1}}, \quad N_{max} = 2N. \quad (4.107)$$

Here, $f_2^0(z)$ and $f_0^{L,0}(z)$ are polynomials in z of the maximum degree N_{max} , thus implying that the ground states carry spins $N-1$ and N (the coefficients of the

z monomials are just the components of the corresponding $SU(2)$ multiplets, like in (4.106)).¹⁰

In accord with the general relations (4.71), all ground states are singlets under the $\mathcal{N} = 2$ SUSY transformations, which can be directly checked using eqs. (4.88), (4.89) adapted to the case at hand.

Higher LL states

The non-zero eigenvalues for supersymmetric partners, f_1 and f_0^L , go by the standard pattern, and for $N \geq 0$ one has

$$E_\ell = \ell(\ell + 2N + 1), \quad \ell = 1, 2, \dots, \quad (4.108)$$

$$f_1^1 = \tilde{f}_1^1, \quad f_1^\ell = \nabla_z^{(N+3)} \dots \nabla_z^{(N+2\ell-1)} \tilde{f}_1^\ell, \quad \ell > 1, \quad (4.109)$$

$$\nabla_{\bar{z}}^{(N+1)} \tilde{f}_1^\ell = 0 \Rightarrow \tilde{f}_1^\ell = \frac{\hat{f}_1^\ell(z)}{(1 + \bar{z}z)^{N+1}}, \quad f_0^{L,\ell} = \nabla_z^{(N+1)} \hat{f}_1^\ell, \quad \ell \geq 1, \quad (4.110)$$

where $\hat{f}_1^\ell(z, \bar{z})$ is expressed in terms of an analytic function $\hat{f}_1^\ell(z)$ in precisely the same way as $f_1^\ell(z, \bar{z})$ is in terms of $\tilde{f}_1^\ell(z)$, in (4.109). From the computation of the norm of f_1^ℓ and \hat{f}_1^ℓ , it follows that the polynomials $\tilde{f}_1^\ell(z)$ and $\hat{f}_1^\ell(z)$ have the maximum degree $N_{max} = 2(N + \ell)$. The convergence of the norm of $f_0^{L,\ell}$ is then guaranteed by that of the norm of \hat{f}_1^ℓ , upon performing an integration by parts. Thus the LL states with $\ell \geq 1$ are spanned by two independent $SU(2)$ multiplets of spin $N + \ell$ encoded in the wave functions $\tilde{f}_1^\ell(z)$ and $\hat{f}_1^\ell(z)$. This additional two-fold degeneracy of the spectrum is of course a consequence of $\mathcal{N} = 2$ supersymmetry which transforms $\tilde{f}_1^\ell(z)$ and $\hat{f}_1^\ell(z)$ into each other and commutes with $SU(2)$.

¹⁰Under $SU(2)$, the polynomial $f(z)$ of the maximal degree N_{max} transform as $\delta f(z) = N_{max} \bar{z} f(z) - \delta z f'(z)$. This generic transformation law agrees with the laws (4.103).

This sequence of eigenvectors and eigenvalues can be extended to include the ground (LLL) states for $f_0^{L,0}$ from (4.105) - (4.107) and correspondingly, to admit $\ell = 0$ in the eigenvalues (4.108). Since $f_0^{L,0}$ is a singlet of $\mathcal{N} = 2$ supersymmetry, no two-fold degeneracy comes out at $\ell = 0$. The completed set of eigenvalues is given by

$$E_{\ell'}^L = \ell'(\ell' + 2N + 1), \quad \ell' = 0, 1 \dots, \text{ for } N \geq 0. \quad (4.111)$$

Now we shall focus on the second $\mathcal{N} = 2$ multiplet of wave functions. The non-zero eigenvalues of supersymmetric partners f_2 and f_0^H must be split according to $0 \leq N < 1$ and $N \geq 1$, as implied by the eigenvalue equation for f_2 , which demands that for $0 \leq N < 1$ we should work on the subspace of anti-analytic functions.

For $N \geq 1$ one has

$$E_{\ell} = \ell(\ell + 2N - 1), \quad \ell = 1 \dots, \quad (4.112)$$

$$f_2^{\ell} = \nabla_z^{(N+1)} \dots \nabla_z^{(N+2\ell-1)} \tilde{f}_2^{\ell}, \quad \nabla_{\bar{z}}^{(N-1)} \tilde{f}_2^{\ell} = 0 \Rightarrow \tilde{f}_2^{\ell} = \frac{\tilde{f}_2^{\ell}(z)}{(1 + \bar{z}z)^{N-1}}, \quad (4.113)$$

$$f_0^{H,\ell} = \nabla_{\bar{z}}^{(N-1)} \hat{f}_2^{\ell}, \quad (4.114)$$

where \hat{f}_2^{ℓ} is expressed through an analytic function \tilde{f}_2^{ℓ} in the same way as f_2^{ℓ} through \tilde{f}_2^{ℓ} . From the computation of the norm of f_2^{ℓ} and \hat{f}_2^{ℓ} , it follows that the polynomials $\tilde{f}_2^{\ell}(z)$ and \hat{f}_2^{ℓ} have the maximal degree $N_{max} = 2(N + \ell - 1)$. The convergence of the norm of $f_0^{H,\ell}$ is then guaranteed by that of the norm of \hat{f}_2^{ℓ} . Thus, like in the previous case, we observe two-fold degeneracy of the energy spectrum due to $\mathcal{N} = 2$ supersymmetry, having two irreducible $SU(2)$ multiplets with spin $N + \ell - 1$. Extending the range of ℓ to include 0 for the ground state vectors $f_2^0(z, \bar{z})$ from (4.107), one eventually obtains the full second sequence of eigenvectors corresponding to

$$E_{\ell}^H = \ell(\ell + 2N - 1), \quad \ell = 0, 1 \dots, \text{ for } N \geq 1. \quad (4.115)$$

Once again, no two-fold degeneracy occurs at $\ell = 0$ because $f_2^0(z, \bar{z})$ are singlets of $\mathcal{N} = 2$ supersymmetry.

To summarize the above discussion, for $N \geq 1$ the eigenvalues and eigenfunctions are split into two sequences corresponding to two super monopole systems, one with the charge $2N$ and the other with the charge $2(N - 1)$. The first sequence extends to the entire range of $N \geq 0$.

It remains to analyze the case $0 \leq N < 1$ for the multiplet (f_2, f_0^H) . We have the following non-zero eigenvalues,

$$E_\ell = (\ell + 1)(\ell - 2N + 2), \quad \ell = 0, 1, \dots, \quad \text{for } 0 \leq N < 1. \quad (4.116)$$

$$\begin{aligned} f_2^0 &= \tilde{f}_2^0; & f_2^\ell &= \nabla_{\bar{z}}^{(N-3)} \dots \nabla_{\bar{z}}^{(N-2\ell-1)} \tilde{f}_2^\ell, \quad \ell > 0; \\ \nabla_z^{(N-1)} \tilde{f}_2^\ell &= 0 \Rightarrow \tilde{f}_2^\ell &= \frac{\hat{f}_2^\ell(\bar{z})}{(1 + \bar{z}z)^{1-N}}, & f_0^{H,\ell} &= \nabla_{\bar{z}}^{(N-1)} \hat{f}_2^\ell, \end{aligned} \quad (4.117)$$

where \hat{f}_2^ℓ is related to an anti-analytic function \tilde{f}_2^ℓ as f_2^ℓ is to \tilde{f}_2^ℓ . From the computation of the norms of f_2^ℓ and \hat{f}_2^ℓ , it follows that the polynomials $\tilde{f}_2^\ell(\bar{z})$ and $\hat{f}_2^\ell(\bar{z})$ have the maximal degree $N_{max} = 2(-N + \ell + 1)$ and, hence, encompass two independent $SU(2)$ multiplets with spin $1 - N + \ell$, revealing the same two-fold degeneracy as in the previous cases. The convergence of the norm of $f_0^{H,\ell}$ is then guaranteed by that of the norm of \hat{f}_2^ℓ .

For $N = 0$, one can make the shift $\ell' = \ell + 1$ and append the value $\ell' = 0$ associated with the ground-state function $f_0^{H,0}$ from (4.105), obtaining in this way the completed set of eigenvalues as

$$E_{\ell'}^{(N=0)} = \ell'(\ell' + 1), \quad \ell' = 0, 1, \dots \quad (4.118)$$

This set for $\ell > 0$ is clearly degenerate with the corresponding $N = 0$ set from (4.111). Therefore, in this case the system acquires an extra degeneracy: excited levels built on the corresponding $\mathcal{N} = 2$ singlet ground states possess the same energy. So in this case the system reveals a four-fold degeneracy (like in the Superplane Landau model [8]).

For $N = \frac{1}{2}$, there is no match for the singlet ground state (4.106) in the above sequence, so in this sector $\mathcal{N} = 2$ supersymmetry appears as spontaneously broken, even though for the whole system it is not, because for the other supermultiplet f_1, f_0^L , in the range $N \geq 0$, there is always an $\mathcal{N} = 2$ supersymmetric singlet ground state. Finally, let us note that, should we have chosen $N \leq 0$, we would expect that the role of f_1, f_0^L and analyticity will be replaced by f_2, f_0^H , and anti-analyticity (and vice-versa).

Chapter 5

Summary

Let us summarize the important results and then outline several possible routes for a further research. We just discussed a collection of models which are all generalizations of the one simple quantum mechanical problem. Each of these models is fully solvable analytically and have energy levels quantized¹¹. Unfortunately by construction all of these models have a second order fermionic kinetic term in the Lagrangian, and because of this all of them are originally non-unitary with some energy states having negative norm with respect to the natural inner product. There are multiple similarities and differences between these models but before going into details lets point out that this work is largely divided into two parts. The outcome of the research on the Superflag and superpsphere Landau model which was started in [5], [6] is discussed in the section 3 which was the first part of our investigation and the next step to further generalizing this whole approach is done in chapter 4. Our conclusions will be discussed in the same order.

The research of the supersymmetric Landau models started with a study of the lowest Landau level for a particle on the superpsphere $\mathbb{C}\mathbb{P}^{(1|1)} \cong SU(2|1)/U(1|1)$. One may take a limit in which only the lowest Landau level survives and in this limit the model provides a ‘quantum superspace’ description of the fuzzy superspheres [5]. The quantum states of the lowest Landau level all have positive norm with respect to an $SU(2|1)$ -invariant inner product that is naturally defined as a superspace integral, but this inner product implies the existence of negative norm states, or ‘ghosts’, in all

¹¹Note that for generalized Superplane Landau model we are able to solve it only for particular choice of the potentials, and can not say anything about energy levels in the general case

higher levels. Superflag Landau model involve an additional anti-commuting variable and an additional parameter M , which has no effect on the energy levels but does have an effect on the norms of states [6]. For positive M it was found that the first $[2M] + 1$ Landau levels are ghost-free, in the natural superspace norm, although there are still ghosts in higher Landau levels. For $M < 0$ all levels will contain ghosts. Another unusual feature of the Superflag is the appearance of the zero-norm states when the parameter $2M$ is a non-negative integer.

One surprising aspect of the Superplane Landau model is that the energy spectrum is precisely that of a model of supersymmetric quantum mechanics, at least if one quantizes in such a way that the state space is a conventional Hilbert space and not a vector superspace. This feature implies the existence of an alternative positive norm, with respect to which the Superplane Landau model is both unitary and ‘worldline’ supersymmetric but it is not obvious that a positive norm will preserve the original ‘internal’ supersymmetry¹². It was found in [8] that ‘internal’ supersymmetry permits a new alternative norm which is positive for a Superplane model, but for planar Superflag it is positive only when $M \leq 0$. For $M > 0$ a ‘dynamical’ combination of both the original and alternative norm is needed. A redefinition of the norm also changes the definition of hermitian conjugation, such that the new hermitian conjugates are ‘shifted’ by operators that generate ‘hidden’ symmetries. Remarkably, the non-zero ‘shift’ operators were found to be the odd generators of a hidden worldline supersymmetry for a Superplane model and a planar Superflag with $M \leq 0$. This supersymmetry is spontaneously broken for $M < 0$ but unbroken for $M = 0$.

We have carried out a similar analysis for the superspherical Landau model, and for the associated Superflag Landau models. The first important result of our analysis is the proof of a quantum equivalence between the $M = 0$ Superflag Landau model

¹²Internal supersymmetry is defined by the coset $IU(1|1)/[U(1|1) \times \mathcal{Z}]$ and discussed in chapter 2.2 .

with charge $2N' = 2N - 1$ and the superspherical Landau model with charge $2N$. Classically, there is an equivalence between these models for the same charge provided the energy is non-zero¹³. Another of our results is a proof that all Superflag Landau models admit a positive $SU(2|1)$ invariant Hilbert space norm. In general this norm is a ‘dynamical’ combination of the ‘naive’ superspace norm and an ‘alternative’ norm that involves a non-trivial Hilbert space ‘metric operator’. This alternative norm leads, by itself, to a unitary model when $-2N < 2M \leq 0$. We have ‘solved’ these unitary models for all N : that is to say, we have found the complete $SU(2|1)$ representation content at each Landau level. This part of investigation was carried out for the Superflag case, and the superspherical model could be obtained from a restriction $M = 0$. Although Superflag models possess additional potential and fermion variable they are in a some respect simpler then the superpsphere model.

One of our objectives was to see whether the hidden worldline supersymmetry of the planar Super Landau models is inherited from some analogous symmetry of spherical super-Landau models. The introduction of a non-trivial ‘metric operator’, implies the redefinition of some hermitian conjugates by ‘shift’ operators that are guaranteed by the formalism to be new ‘hidden’ symmetry generators. For $-2N < 2M < 0$ it was found that ‘hidden’ symmetries close to yield a finite-dimensional enlarged symmetry algebra. In these cases the manifest $SU(2|1)$ symmetry is a subgroup of an $SU(2|2)$ symmetry with a central charge that is linear in the ‘level operator’. The $M = 0$ case is similar in many respects but the lowest Landau level is now special and this prevents any simple construction of a finite basis of charges with level-independent (anti)commutation relations; it thus seems likely that any symmetry group of the superspherical Landau model that contains $SU(2|1)$ but has higher dimension will have infinite dimension.

¹³For the planar models there is no shift, quantum Superplane is equivalent to quantum planar Superflag with $M = 0$ and $N' = N$.

Since ‘hidden’ symmetries for the supersphere and Superflag does not manifest any worldline supersymmetry in contrast with the planar cases, one of the questions is if it possible to construct supersphere Landau model with a worldline supersymmetry. We attempt to generalize the Superplane Landau model by using superfield approach and introducing arbitrary superpotentials in the action. We have shown that the worldline $\mathcal{N} = 2$ supersymmetry is strong enough to define unambiguously a rather general family of quantum models. The naive definition of the inner product has been easily modified, so that the states with a negative norm become proper, positive-normed states in the redefined models. The fact that we have used systems for which kinetic terms of the odd variables were quadratic in time derivatives, has led to general wave functions containing reducible representations of supersymmetry, as can be seen by contemplating (4.56), (4.67), and (4.68). The target superspace has a built-in $N = 2$ supersymmetry as compared to the Superplane and planar Superflag Landau models where such property manifested only after the alternative norm was introduced. Also because of the (4.64) there are no hidden symmetries in this case.

The geometry of the general model has certain interesting features (see footnote 7), but it was not possible to solve equations of motion in the general case. We have found wave functions and energy levels for the special case of constant Gauss curvature and constant magnetic fields. The wave function has two components, each of them belonging to a representation of $SU(2)$ and transforming into each other under $\mathcal{N} = 2$ supersymmetry. In our non-minimal model the four-component wave function contains two SUSY lowest weights of different charges $2N$ and $2(N - 1)$. If $N \geq 1$ there are two families of zero-energy solutions annihilated by the supercharges (see formula (4.107)). For the case of $N = 0$, the system acquires an additional degeneracy.

Finally, let us set up several questions for a further study. First, it is interesting to inquire whether the general model (4.18) admits some non-trivial super-isometries

for special choices of the potentials K and V , like its Superplane prototype (4.8), which is known to respect $ISU(1|1)$ super-isometry [7]. Another task is to construct super Landau models with $\mathcal{N} = 4$ and higher \mathcal{N} world-line supersymmetries. Such models are not known even in the planar limit. They could bear a close relation to the Landau-type models on higher dimensional spaces, e.g. to the LM on R^4 [24]. One more question come from the closer analysis of the hidden symmetries of the Superflag Landau model. It turns out that the extended $SU(2|2)$ symmetry algebra has atypical representation on the each energy level. It would be instructive to find connection with the known atypical representations of this superalgebra and possibly to see if we can construct other atypical representations in this way.

The End.

Appendix A

Grassman algebra overview

Here is short collection of the definitions, which may be useful to somebody who is new to supersymmetry. It is based on paragraphs 1.9 - 1.11 from [20].

Grassman algebra is an associative algebra with unit generated by a set of linearly independent elements ζ^i which anticommute with each other

$$\zeta^i \zeta^j + \zeta^j \zeta^i = 0, \quad \text{in particular } (\zeta^i)^2 = 0. \quad (\text{A.1})$$

Any element of the algebra a can be represented as

$$a = \alpha + \sum_{k=1}^N \frac{1}{k!} C_{i_1 i_2 \dots i_k} \zeta^{i_1} \zeta^{i_2} \dots \zeta^{i_k}. \quad (\text{A.2})$$

Here N is a dimension of the set of generators. It can be infinite as well as finite and is usually considered to be infinite in a general case. Coefficients of this decomposition are complex numbers: $\alpha, C_{i_1 i_2 \dots i_k} \in \mathbb{C}$. Any supernumber can be split in its 'body' and 'soul', where body is purely non-grassman part (as α in (A.2)) and soul is the rest of the expression which involve grassman generators.

We are going to consider each supernumber z as a sum of its odd z_a and even z_c parts

$$\begin{aligned} z &= z_c + z_a, \\ z_c &= z_B + \sum_{k=1}^{\infty} \frac{1}{(2k)!} C_{i_1 i_2 \dots i_{2k}} \zeta^{i_1} \zeta^{i_2} \dots \zeta^{i_{2k}}, \\ z_a &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} C_{i_1 i_2 \dots i_{2k+1}} \zeta^{i_1} \zeta^{i_2} \dots \zeta^{i_{2k+1}}. \end{aligned} \quad (\text{A.3})$$

So z_c always has even number of generators ζ^i in decomposition and z_a always has odd number. If $z_a = 0$ we will say that z is a c-number and write $z \in \mathbb{C}_c$, if $z_c = 0$ we will call z an a-number and write $z \in \mathbb{C}_a$. This a,c-numbers are also called pure supernumbers, for them we can assign 'parity' $\varepsilon(z)$ by the rules

$$\varepsilon(z_c) = 0, \quad \varepsilon(z_a) = 1. \quad (\text{A.4})$$

Basically c-numbers behave similar to usual numbers and a-numbers similar to grassman generators. It can be expressed with the help of parity in the following way

$$\begin{aligned} \varepsilon(z \cdot w) &= \varepsilon(z) + \varepsilon(w) \pmod{2}, \\ z \cdot w &= (-1)^{\varepsilon(z)\varepsilon(w)} w \cdot z \end{aligned} \quad (\text{A.5})$$

Since we defined our superalgebra over the complex numbers we can define operation of conjugation on supernumbers. Complex conjugation has the following properties

$$\begin{aligned} (\zeta^i)^* &= \zeta^i, \quad i \in \mathbb{N}, \\ (\alpha z)^* &= \alpha^* z^*, \quad \alpha \in \mathbb{C}, \\ (z + w)^* &= z^* + w^*, \quad (zw)^* = w^* z^*. \end{aligned} \quad (\text{A.6})$$

Here the last property is especially important to us, because in any computations involving supernumbers or superfunctions order of objects is essential. With this we have enough definitions to go one step higher and discuss supervector spaces and supermatrices. While we are not using supermatrices directly, it is important to note this part of theory because we can not define superalgebras and supergroups without the use of supermatrices.

We can construct supervector spaces the same way as we do vector spaces. They are linear spaces in the usual sense supplied with operations of multiplication by supernumbers. We will usually consider supervectors of specific parity which give rise to even and odd supervector spaces.

Lets define supermatrix as a matrix with elements being supernumbers. We can decompose it in its body and soul $M = M_B + M_S$, then 'rank' of M is defined to be the rank of its body M_B . An $n \times n$ supermatrix is said to be nonsingular if its body is nonsingular. Now we can rewrite matrix as $M = M_B(\mathbb{I}_n + M_B^{-1}M_S)$, then the inverse matrix can be seen as a series

$$M^{-1} = M_B^{-1} + \sum_{k=1}^{\infty} (-1)^k (M_B^{-1}M_S)^k M_B^{-1}. \quad (\text{A.7})$$

It is clear that square supermatrix has an inverse if and only if its body is nonsingular.

Superspaces are more complicated then normal vector spaces. For example not every subspace of a finite dimensional superspace has a definite dimension. But for now we are going to stick with a good examples. Given a d -dimensional superspace \mathcal{L} we can always choose a basis consisting of pure supervectors only. Let p be the number of even basis vectors $\{\vec{e}_m\}$, and q be the number of the odd ones $\{\vec{e}_\mu\}$. Then any c-type vector $\vec{x} \in \mathcal{L}$ can now be written as

$$\vec{x} = x^m \vec{e}_m + \theta^\mu \vec{e}_\mu, \quad \text{where} \quad x^m \in \mathbb{C}_c \quad \theta^\mu \in \mathbb{C}_a. \quad (\text{A.8})$$

The set of c-type supervectors from \mathcal{L} can be identified as a space $\mathbb{C}^{p|q} \equiv \mathbb{C}_c^p \times \mathbb{C}_a^q$, which is called complex superspace of dimension $(p|q)$

$$\mathbb{C}^{p|q} = \{(x^1, x^2, \dots, x^p, \theta^1, \theta^2, \dots, \theta^q), x^m \in \mathbb{C}_c, \theta^\mu \in \mathbb{C}_a\}. \quad (\text{A.9})$$

We can obtain a real subspace $\mathbb{R}^{p|q}$ of the complex superspace by demanding that coefficients of vector decomposition are real supernumbers $x^m \in \mathbb{R}_c$ and $\theta^\mu \in \mathbb{R}_a$. In physics people usually consider even smaller spaces, instead of dealing with c-numbers we take complex numbers $x^m \in \mathbb{C}$ and instead a-numbers we take grassman algebra generators $\theta^\mu = \zeta^m u$. Actually this is not much of a simplification but it makes it easier to imagine the supervector space and work with it.

Lets consider supernumber valued function of the real superspace $\mathbb{R}^{p|q}$ with coordinates $z^M = (x^m, \theta^\mu)$. Without going into much details we are going to consider only nice functions, i.e. superanalytic and supersmooth functions which can be expressed by its series expansion in odd coordinates

$$f(x^1, \dots, x^p, \theta^1, \dots, \theta^q) = f_0(x^1, \dots, x^p) + \sum_{k=1}^q \frac{1}{k!} f_{[\mu_1 \mu_2 \dots \mu_k]}(x^1, \dots, x^p) \theta^{\mu_1} \theta^{\mu_2} \dots \theta^{\mu_k}. \quad (\text{A.10})$$

Here f_0 and $f_{[\dots]}$ are supersmooth and superanalytic functions on \mathbb{R}_c^p (or smooth and analytic on \mathbb{R}^p). Instead of complete antisymmetrisation in indices $f_{[\mu_1 \mu_2 \dots \mu_k]}$ we will often consider specific order, e.g. $f_{[12 \dots k]}$. Each superfunction can be decomposed into its even and odd part $f = f_c + f_a$ thus we can assign parity $\varepsilon(f)$ to the superfunction. If $\varepsilon(f) = 0$ we will often use bosonic instead of even and for $\varepsilon(f) = 1$ we use fermionic along with odd.

We want to define a derivative associated to a a-number. Because of anticommutativity order is important not only to the supernumbers but to the derivatives as well. Because of the order it is possible to define left and right derivative with

$$\begin{aligned} \text{left derivative} \quad \overrightarrow{\frac{\partial}{\partial \theta}} \theta &= 1, \\ \text{right derivative} \quad \theta \overleftarrow{\frac{\partial}{\partial \theta}} &= 1. \end{aligned} \quad (\text{A.11})$$

From now on we will only use left derivative $\frac{\partial}{\partial\theta} = \overrightarrow{\partial}$. It is convenient to assign parity to the superderivatives as well. In generalized notation $\partial_M = (\partial_m, \partial_\mu) \equiv \frac{\partial}{\partial z^M} = (\frac{\partial}{\partial z^m}, \frac{\partial}{\partial z^\mu})$. Now we can define $\varepsilon(\partial_m) = \varepsilon_m = 0$ and $\varepsilon(\partial_\mu) = \varepsilon_\mu = 1$. Further properties of partial derivatives such as commuting properties, generalized Leibniz rule, conjugation properties are collected here

$$\partial_M z^N = \delta_M^N, \quad (\text{A.12})$$

$$\partial_M \partial_N = (-1)^{\varepsilon_M \varepsilon_N} \partial_N \partial_M, \quad (\text{A.13})$$

$$\partial_M (fg) = (\partial_M f)g + (-1)^{\varepsilon_M \varepsilon(f)} f(\partial_M g), \quad (\text{A.14})$$

$$(\partial_M f)^* = (-1)^{\varepsilon_M(1+\varepsilon(f))} \partial_M f^*. \quad (\text{A.15})$$

Our next goal is to define integration on $\mathbb{R}^{p|q}$. The main question is how to define integral over a-number $\int f(\theta)d\theta$. Note that we want definite integral, but since we can not use $\theta_1, \theta_2 \in \mathbb{R}_a$ as limits of integration, the only way we can define limits is to make our integral over all the space \mathbb{R}_a . But since this space is limitless we must have

$$\int d\theta = 0. \quad (\text{A.16})$$

Once again — this is not an indefinite integral. Next, since $d\theta$ is a-type object then $\int \theta d\theta$ will be a c-number, normalization constant of the integral. So in addition to the obvious properties of linearity our integral will have

$$\int \frac{d}{d\theta} f(\theta) d\theta = 0, \quad (\text{A.17})$$

$$\int \theta d\theta = - \int d\theta \theta = -1. \quad (\text{A.18})$$

If we use these two properties on the series decomposition of a superfunction we can

easily find that integration over \mathbb{R}_a actually equivalent to the differentiation

$$\int d\theta f(\theta) = \frac{d}{d\theta} f(\theta). \quad (\text{A.19})$$

This will be all grassman algebra we will need for the purpose of this dissertation work.

Appendix B

Superalgebra definition

There are several different superalgebras appearing in the text but all of them belong to the same class of unitary superalgebras. We will define and give main properties of the $A(m|n)$ type superalgebras and then show commutation properties of algebras we encountered during discussion of super Landau models. More information can be found for example in [18].

B.1 $A(m|n)$

The unitary superalgebra $A(m-1|n-1)$ also denoted as $sl(m|n)$ with $m \neq n$ defined for $m > n \geq 0$ has as even part the Lie algebra $sl(m) \oplus sl(n) \oplus U(1)$, odd part is a representation of the even part. It has rank $m+n-1$ and dimension $(m+n)^2-1$. This superalgebra can be generated by the supermatrices of dimension $(m+n) \times (m+n)$ with a vanishing supertrace

$$M = \begin{pmatrix} X_{mm} & T_{mn} \\ T_{nm} & X_{nn} \end{pmatrix} \quad \text{where} \quad str(M) = tr(X_{mm}) - tr(X_{nn}) = 0 \quad (\text{B.1})$$

To construct basis we start with $(m+n)^2$ elementary matrices e_{IJ} of order $m+n$, such that element in k -th column and l -th row is given by $(e_{IJ})_{kl} = \delta_{Il}\delta_{jK}$. With this

we can define generators

$$\begin{aligned} E_{ij} &= e_{ij} - \frac{1}{m-n} \delta_{ij} (e_{kk} + e_{\bar{k}\bar{k}}), & E_{i\bar{j}} &= e_{i\bar{j}}, \\ E_{\bar{i}\bar{j}} &= e_{\bar{i}\bar{j}} + \frac{1}{m-n} \delta_{\bar{i}\bar{j}} (e_{kk} + e_{\bar{k}\bar{k}}), & E_{i\bar{j}} &= e_{i\bar{j}}, \end{aligned} \quad (\text{B.2})$$

where i, j run from 1 to m and \bar{i}, \bar{j} run from $m+1$ to $m+n$, in these expressions summation over repeated indicies is assumed. Then $Z = E_{kk} = E_{\bar{k}\bar{k}}$ generate $U(1)$ part, $E_{ij} - \frac{1}{m} \delta_{ij} Z$ generate $sl(m)$ part, $E_{\bar{i}\bar{j}} + \frac{1}{n} \delta_{\bar{i}\bar{j}} Z$ generate $sl(n)$ part. These are the generators of the bosonic subalgebra $sl(m) \oplus sl(n) \oplus U(1)$, and $E_{i\bar{j}}$ and $E_{\bar{i}j}$ transform as (\bar{m}, n) and (m, \bar{n}) representations of it.

Generators of the Cartan subalgebra are given by

$$H_i = E_{ii} - E_{i+1, i+1} \quad \text{for } 1 \leq i \leq m-1 \quad (\text{B.3})$$

$$H_{\bar{i}} = E_{\bar{i}\bar{i}} - E_{\bar{i}+1, \bar{i}+1} \quad \text{for } m+1 \leq \bar{i} \leq m+n-1 \quad (\text{B.4})$$

$$H_m = E_{mm} + E_{m+1, m+1} \quad (\text{B.5})$$

General commutation relation could be easily recovered from the defenition of the generators. It is also possible to write down general expression for the Casimir operators of this algebras. However in our case it is better to give expressions relevant to those cases we considred in this text.

The case when dimension of even part equal the dimension of the odd part $m = n$ is in a way more involved. Because $sl(n|n)$ contains one-dimensional ideal \mathcal{I} generated by \mathbb{I}_{2n} , th superalgebra $A(n-1, n-1)$ is defined by $A(n-1, n-1) \equiv sl(n|n)/\mathcal{I}$. It has rank $2n-2$ and dimension $4n^2-2$ for $n > 1$. Its even part is the Lie algebra $sl(n) \oplus sl(n)$.

B.2 $SU(1|1)$

This algebra consist of three generators, one even C and two odd Q, Q^\dagger . The only nonzero (anti)commutation relation is

$$\{Q, Q^\dagger\} = C \quad (\text{B.6})$$

B.3 $IU(1|1)$

This superalgebra is generated by the even charges (P, P^\dagger, C, J, Z) and the odd charges ($\Pi, \Pi^\dagger, Q, Q^\dagger$). Commutation properties

$$\begin{aligned} [P, P^\dagger] &= 2\kappa, & \{\Pi^\dagger, \Pi\} &= 2\kappa & \{Q, Q^\dagger\} &= C, \\ [Q, P] &= i\Pi, & \{Q^\dagger, \Pi\} &= iP, & [C, P] &= -P, & [C, \Pi] &= -\Pi, \\ [J, Q] &= Q, & [J, Q^\dagger] &= -Q^\dagger, & [J, P] &= -P, & [J, \Pi] &= \Pi. \end{aligned} \quad (\text{B.7})$$

B.4 $SU(2|1)$

Lie superalgebra $su(2|1)$ is spanned by four even generators F, J_3, J_\pm , and four odd generators which form doublet Π, Q and its complex conjugate Π^\dagger, Q^\dagger .

Commutation relations

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm. \quad (\text{B.8})$$

$$\begin{aligned}
[J_+, \Pi] &= iQ, & [J_-, Q] &= -i\Pi, \\
[J_3, \Pi] &= -\frac{1}{2}\Pi, & [J_3, Q] &= \frac{1}{2}Q, \\
[F, \Pi] &= -\frac{1}{2}\Pi, & [F, Q] &= -\frac{1}{2}Q
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
[J_-, \Pi^\dagger] &= iQ^\dagger, & [J_+, Q^\dagger] &= -i\Pi^\dagger, \\
[J_3, \Pi^\dagger] &= \frac{1}{2}\Pi^\dagger, & [J_3, Q^\dagger] &= -\frac{1}{2}Q^\dagger, \\
[F, \Pi^\dagger] &= \frac{1}{2}\Pi^\dagger, & [F, Q^\dagger] &= \frac{1}{2}Q^\dagger,
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
\{\Pi, \Pi^\dagger\} &= -J_3 + F, & \{Q, Q^\dagger\} &= J_3 + F, \\
\{\Pi, Q^\dagger\} &= iJ_-, & \{\Pi^\dagger, Q\} &= -iJ_+.
\end{aligned} \tag{B.11}$$

The quadratic Casimir operator

$$C_2 = \frac{1}{2} \{J_+, J_-\} + J_3^2 - F^2 - \frac{1}{2} [\Pi, \Pi^\dagger] - \frac{1}{2} [Q, Q^\dagger]. \tag{B.12}$$

Cubic Casimir operator

$$\begin{aligned}
C_3 &= \frac{i}{2} J_+ [Q^\dagger, \Pi] - \frac{i}{2} [\Pi^\dagger, Q] J_- + \frac{1}{2} J_3 ([Q, Q^\dagger] - [\Pi, \Pi^\dagger]) \\
&\quad - \frac{1}{2} F ([\Pi, \Pi^\dagger] + [Q, Q^\dagger]) + 2C_2 F - \Pi^\dagger \Pi - Q Q^\dagger.
\end{aligned} \tag{B.13}$$

B.5 $SU(2|2)$

We encounter as an enlarged symmetry of the Superflag Landau model. For convenience I will use the same symbols as in the text, even though they are not the simplest ones, I think that consistency with the text will provide additional convenience. Here are the generators of the algebra $\{J_\pm, J_3, \mathcal{J}_\pm, \mathcal{J}_3, Z; \tilde{\Pi}', \tilde{Q}'; \check{\Pi}_G, \check{Q}_G\}$,

where Z is a central charge.

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \quad (\text{B.14})$$

$$[\mathcal{J}_+, \mathcal{J}_-] = 2\mathcal{J}_3, \quad [\mathcal{J}_3, \mathcal{J}_\pm] = \pm \mathcal{J}_\pm. \quad (\text{B.15})$$

$$[J_3, \check{\Pi}_G] = -\frac{1}{2}\check{\Pi}_G, \quad [J_3, \check{Q}_G] = \frac{1}{2}\check{Q}_G, \quad (\text{B.16})$$

$$[J_+, \check{\Pi}_G] = i\check{Q}_G, \quad [J_-, \check{Q}_G] = -i\check{\Pi}_G.$$

$$(\text{ same for } \tilde{\Pi}', \tilde{Q}') \quad (\text{B.17})$$

$$\begin{aligned} [\mathcal{J}_+, \tilde{\Pi}'] &= -i\tilde{\Pi}_G, & [\mathcal{J}_+, \tilde{Q}'] &= -i\tilde{Q}_G, \\ [\mathcal{J}_-, \tilde{\Pi}_G] &= i\tilde{\Pi}', & [\mathcal{J}_-, \tilde{Q}_G] &= i\tilde{Q}', \\ [\mathcal{J}_3, \tilde{\Pi}'] &= -\frac{1}{2}\tilde{\Pi}', & [\mathcal{J}_3, \tilde{Q}'] &= -\frac{1}{2}\tilde{Q}', \\ [\mathcal{J}_3, \tilde{\Pi}_G] &= \frac{1}{2}\tilde{\Pi}_G, & [\mathcal{J}_3, \tilde{Q}_G] &= \frac{1}{2}\tilde{Q}_G, \end{aligned} \quad (\text{B.18})$$

$$\begin{aligned} \{\tilde{\Pi}', \tilde{\Pi}_G^\dagger\} &= \{\tilde{Q}', \tilde{Q}_G^\dagger\} = i\mathcal{J}_-, \\ \{\tilde{\Pi}_G^\dagger, \tilde{\Pi}'\} &= \{\tilde{Q}_G^\dagger, \tilde{Q}'\} = -i\mathcal{J}_+, \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} \{\check{\Pi}_G, \check{\Pi}_G^\dagger\} &= -J_3 - \mathcal{J}_3 + Z, \quad \{\check{Q}_G, \check{Q}_G^\dagger\} = J_3 - \mathcal{J}_3 + Z, \\ \{\tilde{\Pi}', \tilde{\Pi}'^\dagger\} &= -J_3 + \mathcal{J}_3 + Z, \quad \{\tilde{Q}', \tilde{Q}'^\dagger\} = J_3 + \mathcal{J}_3 + Z. \end{aligned} \quad (\text{B.20})$$

Appendix C

Factorization method

Based on the explanation from [10] here is brief explanation of a method we used for solving Schrodinger equations of a different super Landau models. Factorization method is a generalization of a how we treat harmonic oscillator in quantum mechanics. There are three important steps involved. First, Hamiltonian of the system must be written in a factorized way as product of two operators

$$H = AB. \tag{C.1}$$

Next, we consider reverse Hamiltonian $\tilde{H} = BA$, which have a form similar to H but operators in an opposite order. It turns out that H and \tilde{H} have the same non zero eigenvalues. Indeed, if $s \neq 0$ is an eigenvalue of \tilde{H}

$$\tilde{H}|\tilde{\Psi}_s \rangle = BA|\tilde{\Psi}_s \rangle = s|\tilde{\Psi}_s \rangle, \tag{C.2}$$

then if we define new wavefunction

$$|\Psi_s \rangle = A|\tilde{\Psi}_s \rangle, \tag{C.3}$$

we can show that this is an eigenfunction of H

$$H|\Psi_s \rangle = ABA|\tilde{\Psi}_s \rangle = A(BA|\tilde{\Psi}_s \rangle) = sA|\tilde{\Psi}_s \rangle = s|\Psi_s \rangle. \tag{C.4}$$

Finally, for the last step we need some kind of recurrence procedure that yields all

eigenfunctions and eigenvectors. For the harmonic oscillator we will have creation and annihilation operators a and a^\dagger instead of A, B . Recurrence procedure easily stems from the commutator $[a, a^\dagger] = 1$. But in more involved cases Hamiltonian (C.1) may have c-number factor in addition to the operators, which make iteration procedure more complicated. For example refer to the model on the supersphere. Factorized Hamiltonian (3.18) contain additional function from metric. Super-commutator between covariant derivatives which serve as creation, annihilation operators is given in (3.20). But as you can see super-commutator is not enough to get from $\Psi_0^{(N)}$ to $\Psi_1^{(N)}$, then to $\Psi_2^{(N)}$ and so on, we will need to use identities (3.26), (3.27), (3.30). For a Superflag and superspherical case of a generalized model situation is similar, but actual calculation and recurrence procedure is actually simpler.

Appendix D

Anticommuting calculation in the Mathematica 7.0

Since calculations in physics are often long and cumbersome, we thought of the possibility to check at least some of result on a computer. The problem is that common programs which can do analytical computations (Mathematica, Maple) can't deal with anticommuting objects very well. Still the complexity of dealing with supersymmetry calculations often comes from the fact that one has to keep track of the order of operators and functions. This kind of task is something, which can be simplified a lot with a computer. Mathematica has powerful methods of working with lists of objects plus it has noncommuting multiplication build in, written like `a**b`. There are no functions in Mathematica using noncommuting multiplication, so the rest of the program we had to build ourselves.

There were two goals motivating writing of this program. First we wanted to be able to do differentiation with respect to odd variables, it can be useful for example when from action in superfunction one moves to component action. There one has to differentiate (due to (A.19)) to obtain components of superfields. Another big part of calculation is dealing with supersymmetric differential operators, for example generators of symmetry algebra. Of course it is not easy to make computer program to 'understand' abstract operators. What is possible to do is to act with operators on the general superfield and check the result. For example one can check Casimir operators this way by seeing if the resulting superfield is proportional to the one acted upon.

In this section we will present the structure of the program which we used to check some important results in [3], [4]. Main idea is that any expression in Mathematica

can be disassembled in a set of embedded lists. This lists come with a type of operation on the elements. Here is an example of step by step disassembling of the supersymmetric expression¹⁴

$$f_0 + f_1\theta_1 + f_3\theta_1\theta_2 \quad \text{in Mathematica it will be} \quad f_0 + \theta_1 * f_1 + \theta_1 ** \theta_2 * f_3$$

$$\text{Sum}\{f_0, \text{Times}\{\theta_1, f_1\}, \text{Times}\{\text{NonCommutativeMultiply}\{\theta_1, \theta_2\}, f_3\}\} \quad (\text{D.1})$$

$$\text{without Heads} \quad \{f_0, \{\theta_1, f_1\}, \{\{\theta_1, \theta_2\}, f_3\}\}$$

First group of functions in the program is used to ‘check’ expressions, more specifically they order elements, collect terms with the same odd parts and simplify their bosonic parts, then get rid of terms with θ^2 or vanishing bosonic part. Now we will present parts of the code and then briefly explain meaning of each function. Basic definitions we start with

```
coordinates={{t},{h1,h2}};oddfunctions={ϵ,bϵ,f1[t],f2[t]};
p[x_]:=If[MemberQ[coordinates[[2]],x]||MemberQ[oddfunctions,x],1,0];
```

Here function `p[x_]` acts on symbol and return its parity, and first two list are used to set up the list of odd variables which will be used in the specific problem. Without it some functions will not know how to treat variables. Here is the first group of functions

```
gsplitm[e]:=Switch[Head[e],Times,Module[{l1={},l2={}},Map[
If[Head[#]==NonCommutativeMultiply,l2=Append[l2,#],l1=Append[l1,#]]&,
{Delete[e,0]};Map[If[p[#]==1,l2={#};l1=DeleteCases[l1,#]]&,l1];
{If[Length[l1]>1,Times@@l1,l1[[1]]}],Which[Length[l2]==0,Null,
lg[l2[[1]]]==1,l2[[1]],lg[l2[[1]]]>1,NonCommutativeMultiply@@l2]],
```

¹⁴Note that anticommuting multiplication `**` is prerogative to the regular multiplication `*`.

```

NonCommutativeMultiply, {1, e}, _, If [p[e]==0, {e,}, {1, e}]]];

gsplit[e_]:=Switch[
Head[e], Plus, Map[gsplitm[#]&, List[Delete[e, 0]]], _, {gsplitm[e]};

gcompile[l_List]:=Plus@@Map[Apply[Times, #]&, DeleteCases[l, Null, {2}]];

body[x_]:=Plus@@Select[gsplit[x], #[[2]]==Null&][[All, 1]];

soul[x_]:=g0[x-body[x]];

lg[x_]:=
If [Head[x]==NonCommutativeMultiply, Length[x], If [x===Null, 0, 1]];

gdim[x_]:=Module[{l=gsplit[g0[x]], n=0},
Do [n=Max[n, lg[l[[i]][[2]]]], {i, Length[l]}]; n];

pcheck[x_]:=Equal@@Map[Mod[lg[#], 2]&, Transpose[gsplit[g0[x]]][[2]]];

parity[x_]:=If [pcheck[x]==True, Mod[lg[gsplit[x][[1]][[2]]], 2],
Message[parity::undifine_p]];

g0[x_]:=If [
Length[gsplit[x]]<=1, x, gcompile[g2[g3[g2[g4[gsplit[Expand[x]]]]]]];

g2[l_List]:=DeleteCases[l, {0, _}];

g3[l_List]:=If [l==={ }, { }, Map[{Simplify[#[[1]]], #[[2]]}&,
Module[{answer={l[[1]]}}, Do [If [l[[i]][[2]]===l[[i+1]][[2]],
answer=ReplacePart [answer, answer[[-1]][[1]]+l[[i+1]][[1]], {-1, 1}],

```

```

answer=Append[answer,l[[i+1]]],{i,Length[l]-1};answer]]];

g4[l_List]:=Sort[Map[{If[Head#[[2]]===NonCommutativeMultiply,
Signature#[[2]],1]*#[[1]],If[Head#[[2]]===NonCommutativeMultiply,
Sort#[[2]],#[[2]]]&,1],If[And[#1[[2]]!=Null,#2[[2]]!=Null],
OrderedQ[{#1[[2]],#2[[2]]}],
If[#1[[2]]!=Null&&#2[[2]]===Null,False,True]]&];

```

Here `gsplit` used to turn expression in a set of embedded lists like in the last line of (D.1) and `gsplitm` work is to turn one term of a sum into the lists¹⁵. `gsplitm` is long because one have to take care of different orderings of commuting and noncommuting variables. But we had to assume that one will put all odd variables together, i.e. $\theta_1 f_3 \theta_2$ will not work. Next function `gcompile` is used to go back from representation as a list to actual mathematical expression usable by Mathematica. Next are several supplementary functions: `body`, `soul` which return 'body' and 'soul' of the expression, `lg` and `gdim` which return number of fermion variables in the single term and whole expression correspondingly. Also `pcheck` tells if expression can be assigned specific parity, then `parity` returns its value.

Now we have `g2`, `g3`, `g4`, where `g2` delete terms with squared fermions, next `g3` collect terms with the same fermions together and simplify their bosonic part, and `g4` put odd variables in the same order in all terms as well as order summands. Finally `g0` just apply all this functions together in order to brush up expression. Since all this functions do is move objects around it works very fast, that is why we use it excessively just in case.

Lets discuss only the most important functions, which are the multiplication and

¹⁵Using notations from (D.1) even parts like f_0, f_1, f_3 can actually be complicated expressions and contain other functions and operation. For example you can have $f_3 = g_0 - \frac{g_1 g_2}{g_3}$

differentiation. First one is easy

```
gmshort[l1_,l2_]:=Which[l1[[2]]===Null,l1[[1]]*l2[[1]],l2[[2]],
l2[[2]]===Null,{l1[[1]]*l2[[1]],l1[[2]]},True,
{Signature[l1[[2]]**l2[[2]]]*l1[[1]]*l2[[1]],Sort[l1[[2]]**l2[[2]]]}];
```

```
gm[x_]:=x;
```

```
gm[x_,y_]:=gcompile[g2[g3[g2[g4[Module[{lx=gsplit[Expand[x]],
ly=gsplit[Expand[y]],answer={}},Do[Do[answer=
Append[answer,gmshort[lx[[j]],ly[[i]]],
{j,Length[lx]}],{i,Length[ly]}];answer]]]]];
```

Here `gmshort` acts on expression with each having only one term, no summation. And `gm` use previous function to multiply two general expressions. We have also defined power and exponent of the supersymmetric expression. As exponent defined by its series, it is possible to have more general series of a supersymmetric function.

Lastly differentiation was the hardest to define. For a derivative with respect to c-number standard function from Mathematica works almost fine, just need to run `g0` once. But for a derivative with respect to a-number what happens is that $\frac{d}{d\theta_1}\theta_1**\theta_2$ becomes $1**\theta_2$ and then we need manually substitute last expression with θ_2 and take care of the signs. Here are two functions which do most of it. `gprojection[x,h]` is a supplementary function which return expression `x` where one of the variables `h` (odd or even) was set to zero, and `gDf[x,h]` differentiate supersymmetric expression `x` with respect to odd variable `h`

```
gprojection[x_,h_]:=x//.{h->0,h**_>0,_**h->0};
```



```

gDf[x_,h_]:=Module[{xxx,l=gsplit[g0[x]]},
g0[gprojection[gcompile[Map[If[Position#[[2]],1]==={},#,
{(-1)^(Position#[[2]]**xxx,1)[[1]][[1]]-1)*#[[1]],#[[2]]}]&,
g2[Map[gRefine[#]&,Join[MapAt[Expand[D[#,h]]&,l,
Table[{i,1},{i,Length[l]}]],MapAt[Expand[D[#,h]]&,l,
Table[{i,2},{i,Length[l]}]]/.{___**0**__:>0}]]],h]/.
{___**0**__:>0,A_**1**B_:>A**B,A_**1:>A,1**A_:>A}]16

```

These are core function of this program. We also defined transformations of a super-expression and a bracket like Poisson or Dirac bracket. To do this it is needed to define a table of substitutions which can take a dozen or two lines, but once done it is simple to use. It is possible to use this program for operators of algebra and check operator equations. We used it to check most of the formulas from sections 3.3, 3.4 after doing manual calculations.

¹⁶Long underlines are actually triple underline, which in Mathematica can stand for any sequence of zero or more expressions.

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