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Target Space Pseudoduality in Supersymmetric Sigma Models on Symmetric Spaces

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TARGET SPACE PSEUDODUALITY IN SUPERSYMMETRIC SIGMA MODELS ON SYMMETRIC SPACES

By
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We discuss the target space pseudoduality in supersymmetric sigma models on symmetric spaces. We first consider the case where sigma models based on real compact connected Lie groups of the same dimensionality and give examples using three dimensional models on target spaces. We show explicit construction of nonlocal conserved currents on the pseudodual manifold. We then switch the Lie group valued pseudoduality equations to Lie algebra valued ones, which leads to an infinite number of pseudoduality equations. We obtain an infinite number of conserved currents on the tangent bundle of the pseudodual manifold. Since pseudoduality imposes the condition that sigma models pseudodual to each other are based on symmetric spaces with opposite curvatures (i.e. dual symmetric spaces), we investigate pseudoduality transformation on the symmetric space sigma models in the third chapter. We see that there can be mixing of decomposed spaces with each other, which leads to mixings of the following expressions. We obtain the pseudodual conserved currents which are viewed as the orthonormal frame on the pullback bundle of the tangent space of $\tilde{G}$ which is the Lie group on which the pseudodual model based. Hence we obtain the mixing forms of curvature relations and one loop renormalization group beta function by means of these currents. In chapter four, we generalize the classical construction of pseudoduality transformation to supersymmetric case. We perform this both by component expansion method on manifold M and by orthonormal coframe method on manifold

$SO(M)$. The component method produces the result that pseudoduality tranformation is not invertible at all points and occurs from all points on one manifold to only one point where riemann normal coordinates valid on the second manifold. Torsion of the sigma model on M must vanish while it is nonvanishing on $\tilde{M}$, and curvatures of the manifolds must be constant and the same because of anticommuting grassmann numbers. We obtain the similar results with the classical case in orthonormal coframe method. In case of super WZW sigma models pseudoduality equations result in three different pseudoduality conditions; flat space, chiral and antichiral pseudoduality. Finally we study the pseudoduality tansformations on symmetric spaces using two different methods again. These two methods yield similar results to the classical cases with the exception that commuting bracket relations in classical case turns out to be anticommuting ones because of the appearance of grassmann numbers. It is understood that constraint relations in case of non-mixing pseudoduality are the remnants of mixing pseudoduality. Once mixing terms are included in the pseudoduality the constraint relations disappear.

To my parents

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Contents

Chapter 1

Introduction

The term duality has numerous meanings, and is an important concept in physics. When two different systems turn out to be equivalent we say that there is a duality between these systems. In string theory people use the term "target space duality" if there is a canonical transformation between target spaces in which strings move. This transformation preserves the hamiltonian. The simplest example is the standard abelian duality transformations [1, 2],

$$\partial_+ \tilde{\varphi} = +\partial_+ \varphi \tag{1.0.1}$$

$$\partial_- \tilde{\varphi} = -\partial_- \varphi \tag{1.0.2}$$

where φ is the massless free scalar field satisfying the wave equation $\partial^2_{+-}\varphi = 0$ in light cone coordinates. By means of duality transformations (1.0.1) and (1.0.2) we may construct $\tilde{\varphi}$ satisfying the wave equation $\partial^2_{+-}\tilde{\varphi} = 0$, and hence understand that these two theories are equivalent and dual to each other. A more general case is the pseudochiral models introduced by Zakharov and Mikhailov [3]. We consider a standard sigma model with target space a Lie Group G, which has equations of motion $\partial^\mu(g^{-1}\partial_\mu g) = 0$. If we look for a dual model with a Lie algebra valued field ϕ, then the duality transformation can be written as

$$g^{-1}\partial_\mu g = -\epsilon_\mu^\nu \partial_\nu \phi \tag{1.0.3}$$

But these dual models are not quantum mechanically equivalent to each other because it is realized that duality transformations should be canonical transformations [1] [4, 5], but this is not [6]. Subsequently people developed an extensive literature on nonabelian duality, and Poisson-Lie duality motivated by string theory.

There is an interesting duality transformation proposed by Alvarez [2, 7], which is called "pseudoduality" [2]. By contrast with usual duality transformations this "on shell duality" transformation is not canonical,and maps solutions of the equations of motion of the "pseudodual" models. We will use the term pseudodual when there is a pseudoduality transformation between different models. It is pointed out that this transformation preserves the stress energy tensor [7].

It was shown [1, 7] that pseudoduality transformation in sigma models provides that curvatures of dual models are constants, and have opposite signs, which restricts the condition that pseudoduality exists between sigma models only if they are based on symmetric spaces. It is also shown in this work that pseudoduality gives rise to infinite number of nonlocal conserved currents associated with pseudodual model.

1.1 Pseudoduality in Sigma Models

We take spacetime Σ to be two dimensional Minkowski space, and $\sigma^\pm = \tau \pm \sigma$ throughout thesis. The sigma model with target space M, metric g and two-form B is denoted by

[1] This is sometimes called as "off shell" duality

[2] This term was introduced first by Curtright and Zachos [4].

(M, g, B) and has the lagrangian [7, 8]

$$\begin{aligned}\mathcal{L} =& \frac{1}{2} g_{ij}(x)\partial_\mu x^i \partial_\mu x^j + \frac{1}{2} B_{ij}(x)\epsilon^{\mu\nu}\partial_\mu x^i \partial_\nu x^j \\ =& \frac{1}{2} g_{ij}(x)(\dot{x}^i \dot{x}^j - x'^i x'^j) + B_{ij}(x)\dot{x}^i x'^j\end{aligned} \tag{1.1.1}$$

where $x : \Sigma \to M$ and the closed 3-form H is defined by $H = dB$. This theory is classically conformally invariant. The stress energy tensor is given by

$$\Theta_{\pm\mp} = 0 \qquad \Theta_{\pm\pm} = g_{ij}(x)\partial_\pm x^i \partial_\pm x^j \tag{1.1.2}$$

We wish to construct the pseudoduality transformation between the solutions of the equations of motion of the sigma model (M, g, B) and that of a different sigma model $(\tilde{M}, \tilde{g}, \tilde{B})$. It is instructive to start with the sigma models based on riemannian manifolds. The equations of motion for the sigma model based on M is

$$x^i_{+-} = -\frac{1}{2} H^i_{jk} x^j_+ x^k_- \tag{1.1.3}$$

Duality transformation is best formulated in the bundle of orthonormal coframes, $SO(M)$. Since all local descriptions can be extended to global ones [7], we will use global expressions. We pick globally defined orientable orthonormal coframes w^i and $\tilde{w}^i$ in the coframe bundle $SO(M)$ and $SO(\tilde{M})$ respectively, and given in terms of σ^a derivatives by

$$w^i = x^i_a d\sigma^a \qquad \tilde{w}^i = \tilde{x}^i_a d\sigma^a \tag{1.1.4}$$

and the corresponding torsion free antisymmetric riemannian connection 1-forms w^i_j and $\tilde{w}^i_j$. We notice that w^i and w^i_j are linearly independent coframings on $SO(M)$, similarly for $SO(\tilde{M})$. These global coframings satisfy the Cartan structural equations

$$dw^i = -w^i_j \wedge w^j \tag{1.1.5}$$

$$dw^i_j = -w^i_k \wedge w^k_j + \frac{1}{2} R^i_{jkl} w^k \wedge w^l \tag{1.1.6}$$

where R^i_{jkl} is the Riemann curvature tensor on M. Similar equations can be written on $\tilde{M}$. We will define the pseudoduality transformation on the spacetime Σ, and hence we consider the pullbacks

$$X^* w^i = x^i_a d\sigma^a \qquad X^* w^i_j = w^i_{ja} d\sigma^a \tag{1.1.7}$$

where the lift from Σ to $SO(M)$ is defined, $X : \Sigma \to SO(M)$. Therefore afore mentioned pseudoduality equations in standard lightcone coordinates can be written as

$$\tilde{x}_+(\sigma) = +T_+(x, \tilde{x}) x_+(\sigma) \tag{1.1.8}$$

$$\tilde{x}_-(\sigma) = -T_-(x, \tilde{x}) x_-(\sigma) \tag{1.1.9}$$

where the orthogonal matrix valued functions $T_\pm : M \times \tilde{M} \to SO(n)$ are given in [9], and dim $M =$ dim $\tilde{M} = n$. In this thesis we will consider the case $T_\pm : \Sigma \to SO(n)$ and $T_+ = T_-$ = T. Thus pseudoduality equations will be

$$\tilde{x}_\pm(\sigma) = \pm T(\sigma) x_\pm(\sigma) \tag{1.1.10}$$

The covariant derivative of x^i_a (similarly for $\tilde{x}^i_a$) are written as

$$dx^i_a + w^i_j x^j_a = x^i_{ab} d\sigma^b \tag{1.1.11}$$

Since we want to extract more information about pseudoduality transformations, and find required conditions, we take exterior derivative of (1.1.10)

$$d\tilde{x}^i_{\pm} = \pm(dT^i_j)x^j_{\pm} \pm T^i_j dx^j_{\pm}$$

the covariant derivative (1.1.11) leads to

$$\tilde{x}^i_{\pm b}d\sigma^b - \tilde{w}^i_j\tilde{x}^j_{\pm} = \pm(dT^i_j)x^j_{\pm} \pm T^i_j x^j_{\pm b}d\sigma^b \mp T^i_j w^j_k x^k_{\pm}$$

We now insert the pseudoduality equations (1.1.10) back in this equation and arrange the terms to get

$$\tilde{x}^i_{\pm b}d\sigma^b = \pm(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k)x^k_{\pm} \pm T^i_j x^j_{\pm b}d\sigma^b \tag{1.1.12}$$

Since we would like to use the equations of motion to reveal the integrability conditions, we wedge by $d\sigma^{\pm}$ to get

$$\tilde{x}^i_{\pm\mp}d\sigma^{\mp} \wedge d\sigma^{\pm} = \pm(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k)x^k_{\pm} \wedge d\sigma^{\pm} \pm T^i_j x^j_{\pm\mp}d\sigma^{\mp} \wedge d\sigma^{\pm} \tag{1.1.13}$$

This can be split into the following two equations

$$\tilde{x}^i_{+-}d\sigma^- \wedge d\sigma^+ = +(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k)x^k_+ \wedge d\sigma^+ + T^i_j x^j_{+-}d\sigma^- \wedge d\sigma^+ \tag{1.1.14}$$

$$\tilde{x}^i_{-+}d\sigma^+ \wedge d\sigma^- = -(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k)x^k_- \wedge d\sigma^- - T^i_j x^j_{-+}d\sigma^+ \wedge d\sigma^- \tag{1.1.15}$$

Since we know that $x_{+-} = x_{-+}$ (also with tilde $\tilde{x}_{+-} = \tilde{x}_{-+}$), and $d\sigma^- \wedge d\sigma^+ = -d\sigma^+ \wedge d\sigma^-$, the left-hand sides of these equations can be equated each other to yield

$$(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k) x^k_+ \wedge d\sigma^+ - (dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k) x^k_- \wedge d\sigma^-$$
$$+ 2T^i_j x^j_{+-} d\sigma^- \wedge d\sigma^+ = 0$$

We may now use the equations of motion (1.1.3) to obtain

$$(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k) x^k_+ \wedge d\sigma^+ - (dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k) x^k_- \wedge d\sigma^-$$
$$- T^i_j H^j_{kl} x^k_+ x^l_- d\sigma^- \wedge d\sigma^+ = 0 \tag{1.1.16}$$

We split the last term into two terms by changing the orders of $(-)$ and $(+)$, and use $w = x_+ d\sigma^+ + x_- d\sigma^-$ to get

$$(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k - \frac{1}{2} T^i_j H^j_{kl} w^l) x^k_+ \wedge d\sigma^+$$
$$-(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k + \frac{1}{2} T^i_j H^j_{kl} w^l) x^k_- \wedge d\sigma^- = 0 \tag{1.1.17}$$

We define the following tensors in order to understand this equation better

$$U^i_{k-} d\sigma^- = dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k - \frac{1}{2} T^i_j H^j_{kl} w^l \tag{1.1.18}$$
$$U^i_{k+} d\sigma^+ = dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k + \frac{1}{2} T^i_j H^j_{kl} w^l \tag{1.1.19}$$

Hence equation (1.1.17) can be written as

$$U^i_{k-} x^k_+ d\sigma^- \wedge d\sigma^+ - U^i_{k+} x^k_- d\sigma^+ \wedge d\sigma^- = 0 \tag{1.1.20}$$

Besides, equations (1.1.18) and (1.1.19) yield that

$$U^i_{k+}d\sigma^+ + U^i_{k-}d\sigma^- = 2(dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k) \tag{1.1.21}$$

$$U^i_{k+}d\sigma^+ - U^i_{k-}d\sigma^- = T^i_j H^j_{kl} w^l \tag{1.1.22}$$

Finally we may easily get the following from (1.1.22)

$$U^i_{k+}d\sigma^+ \wedge d\sigma^- = +T^i_j H^j_{kl} w^l \wedge d\sigma^- \tag{1.1.23}$$

$$U^i_{k-}d\sigma^- \wedge d\sigma^+ = -T^i_j H^j_{kl} w^l \wedge d\sigma^+ \tag{1.1.24}$$

If these results are substituted in (1.1.20) one gets

$$T^i_j H^j_{kl} x^k_+ w^l \wedge d\sigma^+ + T^i_j H^j_{kl} x^k_- w^l \wedge d\sigma^- = 0$$

using the definition of w one obtains

$$T^i_j H^j_{kl} x^k_+ x^l_- d\sigma^+ \wedge d\sigma^- = 0 \tag{1.1.25}$$

Since we may choose x^k_+ and x^l_- arbitrarily at any σ we conclude that $H = 0$. This leads the tensors U^i_{k+} and U^i_{k-} to vanish by means of equations (1.1.23) and (1.1.24). Therefore we obtain the result from (1.1.18) (or (1.1.19))

$$dT^i_k + \tilde{w}^i_j T^j_k - T^i_j w^j_k = 0 \tag{1.1.26}$$

It is also evident from (1.1.14) (or (1.1.15)) that $\tilde{H} = 0$ on the pseudodual manifold $\tilde{M}$. This shows why Ivanov [10] uses the case $\tilde{H} = 0$ in his method. We would like to bring out more conditions on pseudoduality transformations from the equation (1.1.26), and so we search for the integrability conditions of this equation, and see what we can obtain. We take the exterior derivative and use again (1.1.26) to obtain

$$T^i_j R^j_{kmn} w^m \wedge w^n = \tilde{R}^i_{jmn} T^j_k \tilde{w}^m \wedge \tilde{w}^n \tag{1.1.27}$$

where we used the Cartan's second structural equation (1.1.6). If we use the definitions for w and $\tilde{w}$ followed by pseudoduality equations, we get the curvature relations

$$T^i_j R^j_{kmn} = -\tilde{R}^i_{jlp} T^j_k T^l_m T^p_n \tag{1.1.28}$$

Thus we get another condition for pseudoduality, curvatures of pseudodual manifolds M and $\tilde{M}$ must have opposite signs. To discover more conditions we keep searching for the integrability conditions of (1.1.28), and take exterior derivative and use again (1.1.26) to get

$$T^i_j R^j_{kmn;q} w^q = -\tilde{R}^i_{jlp;q} T^j_k T^l_m T^p_n \tilde{w}^q$$

where the covariant derivative of R is defined as $DR^i_{kmn} = R^i_{kmn;q} w^q = dR^i_{kmn} + R^q_{kmn} w^i_q - R^i_{qmn} w^q_k - R^i_{kqn} w^q_m - R^i_{kmq} w^q_n$, and similarly for $\tilde{R}$. This equation can be split into the following two independent equations

$$T^i_j R^j_{kmn;q} = -\tilde{R}^i_{jlp;q} T^j_k T^l_m T^p_n$$
$$T^i_j R^j_{kmn;q} = +\tilde{R}^i_{jlp;q} T^j_k T^l_m T^p_n$$

which give the solutions $R^j_{kmn;q} = \tilde{R}^i_{jlp;q} = 0$. Therefore we find that the manifolds M and $\tilde{M}$ must be locally symmetric spaces with the opposite curvatures. We finally conclude that pseudoduality exists between two sigma models based on the riemannian manifolds M and $\tilde{M}$ only if 3-forms H and $\tilde{H}$ vanish, and target spaces are the symmetric spaces with the opposite curvatures.

Chapter 2

WZW Models and Conserved Currents

In this section we will consider the pseudoduality between two sigma models with target space M a real connected compact Lie group G with an $Ad(G)$-invariant metric [17]. The orthonormal frame bundle is $SO(G) = G \times SO(n)$, and w^i and w^i_j can be chosen globally as above. The 3-form H^i_{jk} will be proportional to the structure constants f^i_{jk}, $H^i_{jk} = af^i_{jk}$ and the constant $a \in \mathbb{R}$. We especially specialize to the classical strict WZW model [11] which is the case $a = \pm 1$ so that the action is normalized to make the path integral well defined. The strict WZW model is the model with the Wess-Zumino term normalized so that the canonical equations of motion are given by $\partial_-(g^{-1}\partial_+ g) = 0$, where g is a function on spacetime taking values in some compact Lie group G. We will show that the WZW model on G is pseudodual to the WZW model on $\tilde{G}$ for any two compact n-dimensional Lie groups. Let **g** be the Lie algebra of G, $\tilde{\mathbf{g}}$ be Lie algebra of $\tilde{G}$. If the Isom(**g**, $\tilde{\mathbf{g}}$) is the vector space isometries from **g** to $\tilde{\mathbf{g}}$, we may write the pseudoduality mapping $*_\Sigma(\tilde{g}^{-1}d\tilde{g})(\sigma) = T(g^{-1}dg)(\sigma)$, where $T : \Sigma \to$ Isom(**g**, $\tilde{\mathbf{g}}$), and $*_\Sigma$ is the Hodge duality operator on Σ. We will also investigate some properties of conserved currents on target space manifolds that are pseudodual to each other, following a method discussed in [7] we find an infinite number of conservation laws in the pseudodual manifold. We work out the currents in case of pseudodualities [4, 7] between the sigma model on an abelian group and a strict WZW sigma model [11] on a compact Lie group [12, 13] of the same dimensionality. We specialize to the case of the abelian group $U(1) \times U(1) \times U(1)$, and of the Lie group $SU(2)$. We will afterwards work out WZW models with target spaces a general real connected compact Lie groups, and find solutions for the transformation matrix and pseudodual expressions in the next section.

We know [7, 10] that if we are given a sigma model on an abelian group, and a strict WZW sigma model on a compact Lie group, there is a duality transformation between these two manifolds that maps solutions of the equations of motion of the first manifold into the solutions of the equations of motion of the second manifold. Solutions of the equations of motions allow us to construct holomorphic [12] nonlocal conserved currents on these manifolds. Pseudoduality relations provide a way to form pseudodual currents, and we show that these currents are conserved.

Let $M = G$ be a compact Lie group of dimension n with an $Ad(G)$-invariant metric, and $g : \Sigma \to G$. We define the basic nonlocal conserved currents $J_+^{(L)} = (g^{-1}\partial_+ g)$ and $J_-^{(R)} = (\partial_- g)g^{-1}$ on the tangent bundle of G. What we demonstrate is that we can take these currents, and using the pseudoduality relations we obtain currents on G (not $\tilde{G}$) and these currents are conserved.

We would like to search for infinitely many conservation laws[14, 15, 16] on pseudodual manifolds. We first concentrate on a simple case, where $M = G = U(1) \times U(1) \times U(1)$ is an abelian group and $\tilde{M} = \tilde{G}$ is $SU(2)$. We show that infinite number of conservation laws of free scalar currents on G enable us to construct infinite number of pseudodual current conservation on $\tilde{G}$ by means of isometry preserving orthogonal map T between tangent bundles of these manifolds. We next focus our attention on a more complicated case, where $M = G$ is the Lie group $SU(2)$ and $\tilde{M} = \tilde{G}$ is $U(1) \times U(1) \times U(1)$. We find nonlocal conserved currents on G and construct pseudodual free currents on $\tilde{G}$ using pseudoduality relations. We show that pseudodual free scalar currents on $\tilde{G}$ gives us infinite number of conservation laws.

2.1 Pseudodual Currents : Simple Case

We take M as an abelian group, and the equations of motion become $\partial^2_{+-}\phi^i = 0$, where ϕ is free massless scalar field. Currents on the tangent bundle of M are hence given by $J^{(L)}_+ = (\partial_+\phi^i)X_i$ and $J^{(R)}_- = (\partial_-\phi^i)X_i$, where $\{X_i\}$ is a basis for the abelian Lie algebra. We notice that these currents are individually conserved, $\partial_- J^{(L)}_+ = \partial_+ J^{(R)}_- = 0$. Now we take $\tilde{M}$ as a compact Lie group of the same dimensionality with an $Ad(G)$-invariant metric. $\{\tilde{X}_i\}$ is the orthonormal basis for the Lie algebra of $\tilde{G}$ with bracket relations $[\tilde{X}_i, \tilde{X}_j]_{\tilde{G}} = \tilde{f}^k_{ij}\tilde{X}_k$, where the structure constants $\tilde{f}_{ijk}$ are totaly antisymmetric in ijk. Using the map $\tilde{g} : \Sigma \to \tilde{M}$ we may write equations of motion as $\partial_-(\tilde{g}^{-1}\partial_+\tilde{g}) = 0$. Currents on this manifold are defined by $\tilde{J}^{(L)}_+ = (\tilde{g}^{-1}\partial_+\tilde{g})^i\tilde{X}_i$ and $\tilde{J}^{(R)}_- = [(\partial_-\tilde{g})\tilde{g}^{-1}]^i\tilde{X}_i$. Again, by virtue of equations of motion we observe that these currents are conserved, $\partial_-\tilde{J}^{(L)}_+ = \partial_+\tilde{J}^{(R)}_- = 0$.

To construct pseudodual currents on the manifold M we make use of the pseudoduality conditions. The pseudoduality relations between the sigma model on an abelian group and a strict WZW sigma model on a compact Lie group of the same dimension are

$$(\tilde{g}^{-1}\partial_+\tilde{g})^i = +T^i_j\partial_+\phi^j \tag{2.1.1}$$

$$(\tilde{g}^{-1}\partial_-\tilde{g})^i = -T^i_j\partial_-\phi^j \tag{2.1.2}$$

where T is an orthogonal matrix and $\tilde{g}^{-1}d\tilde{g} = (\tilde{g}^{-1}d\tilde{g})^i\tilde{X}_i$.

Taking ∂_- of the first equation (2.1.1) we conclude that T is a function of σ^+ only. Taking ∂_+ of the second equation (2.1.2) gives us the differential equation for T

$$[(\partial_+T)T^{-1}]^i_j = -\tilde{f}^i_{kj}T^k_l\partial_+\phi^l \tag{2.1.3}$$

where we used the antisymmetricity of $\tilde{f}_{ikj}$ at right hand side of equation.

To get pseudodual currents on the manifold M, we first solve this differential equation for T, and then plug this into pseudoduality equations with an initially given $\partial_\pm \phi^i$ and from the pseudodual currents we find that these currents are conserved.

2.1.1 An Example

We consider the sigma model based on the product group $U(1) \times U(1) \times U(1)$ for M and a strict WZW model based on group $SU(2)$ for $\tilde{M}$. We may write a point on the sigma model to M as $\phi^i X_i$, where $i = 1, 2, 3$ and $\{X_i\}$ are basis. Equations of motions are $\partial^2_{+-}\phi^i = 0$. Currents may be written as $J_+^{(L)} = (\partial_+\phi^i)X_i$ and $J_-^{(R)} = (\partial_-\phi^i)X_i$. We learn from equations of motions that these currents are conserved.

We denote any element in $\tilde{G}$ as $\tilde{g} = e^{i\tilde{\theta}^k \tilde{X}_k}$, where $\{\tilde{\theta}^k\} = (\tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3)$ and $\{\tilde{X}_k\} = (-i\frac{\sigma_1}{2}, -i\frac{\sigma_2}{2}, -i\frac{\sigma_3}{2})$ is a basis for the Lie algebra of $SU(2)$. Structure constants are ϵ_{ijk}. Equations of motion for the strict WZW model are $\partial_-(\tilde{g}^{-1}\partial_+\tilde{g}) = 0$, where $\tilde{g}^{-1}d\tilde{g} = (\tilde{g}^{-1}d\tilde{g})^k\tilde{X}_k$. Currents for the Lie algebra are $\tilde{J}_+^{(L)} = (\tilde{g}^{-1}\partial_+\tilde{g})^k\tilde{X}_k$ and $\tilde{J}_-^{(R)} = [(\partial_-\tilde{g})\tilde{g}^{-1}]^k\tilde{X}_k$. Again equations of motion ensure that these currents are conserved.

We first solve the ordinary differential equation for T to find the pseudodual currents. Multiplying (2.1.3) by T_n^j from right we get

$$\partial_+ T_n^i = -\tilde{f}^i_{kj} T_l^k T_n^j \partial_+\phi^l \tag{2.1.4}$$

We put in an order parameter ε to look for a perturbation solution,

$$\partial_+ T_n^i = -\varepsilon \tilde{f}^i_{kj} T_l^k T_n^j \partial_+\phi^l \tag{2.1.5}$$

Presumably the solution is in the form $T = e^{\varepsilon\alpha_1} e^{\frac{1}{2}\varepsilon^2\alpha_2}(I + \mathcal{O}(\varepsilon^3))$, where α_1 and α_2 are antisymmetric matrices. Since we know that T is only a function of σ^+, α_1 and α_2 are also functions of σ^+. If we expand T

$$T = I + \varepsilon\alpha_1 + \frac{1}{2}\varepsilon^2(\alpha_2 + \alpha_1^2) + \mathcal{O}(\varepsilon^3) \tag{2.1.6}$$

then taking ∂_+ we end up with

$$\partial_+ T = \varepsilon\partial_+\alpha_1 + \frac{1}{2}\varepsilon^2[\alpha_1(\partial_+\alpha_1) + (\partial_+\alpha_1)\alpha_1 + \partial_+\alpha_2] + \mathcal{O}(\varepsilon^3) \tag{2.1.7}$$

If we compare (2.1.5) to (2.1.7), the latter may be written in tensor product form as

$$\begin{aligned}\partial_+ T &= -\varepsilon\tilde{f} \otimes (T\partial_+\phi) \otimes T = -\varepsilon\tilde{f} \otimes [(I + \varepsilon\alpha_1)\partial_+\phi] \otimes (I + \varepsilon\alpha_1) \\ &= -\varepsilon\tilde{f} \otimes \partial_+\phi \otimes I - \varepsilon^2\tilde{f} \otimes \alpha_1\partial_+\phi \otimes I - \varepsilon^2\tilde{f} \otimes \partial_+\phi \otimes \alpha_1\end{aligned} \tag{2.1.8}$$

Therefore we find

$$\partial_+\alpha_1 = -\tilde{f} \otimes \partial_+\phi \otimes I \tag{2.1.9}$$

$$\frac{1}{2}[\partial_+\alpha_2 + \alpha_1(\partial_+\alpha_1) + (\partial_+\alpha_1)\alpha_1] = -\tilde{f} \otimes \alpha_1\partial_+\phi \otimes I - \tilde{f} \otimes \partial_+\phi \otimes \alpha_1 \tag{2.1.10}$$

Solving (2.1.9) we get α_1 as follows

$$(\alpha_1)^i_n = -\int_0^{\sigma^+} \tilde{f}^i_{kn}\partial_+\phi^k \, d\sigma'^+ = -\tilde{f}^i_{kn}(\phi^k + C^k) \tag{2.1.11}$$

where C^k is a constant, and we choose it to be zero. Since α_1 is a function of σ^+ only, ϕ^k in the expression of α_1 should involve σ^+, not σ^-. From this we understand that we need to separate ϕ as right moving wave $\phi_R(\sigma^-)$ and left moving wave $\phi_L(\sigma^+)$, i.e. $\phi =$

$\phi_L(\sigma^+) + \phi_R(\sigma^-)$. Hence $(\alpha_1)^i_n = -\tilde{f}^i_{kn}\phi^k_L$, from which we find

$$\alpha_1 = \begin{pmatrix} 0 & \phi^3_L & -\phi^2_L \\ -\phi^3_L & 0 & \phi^1_L \\ \phi^2_L & -\phi^1_L & 0 \end{pmatrix} \tag{2.1.12}$$

Solving (2.1.10) we obtain

$$\begin{aligned}(\alpha_2)^i_n = &-\int_0^{\sigma^+} (\alpha_1\partial_+\alpha_1)^i_n \, d\sigma'^+ - \int_0^{\sigma^+} [(\partial_+\alpha_1)\alpha_1]^i_n \, d\sigma'^+ \\ &-2\int_0^{\sigma^+} \tilde{f}^i_{kn}(\alpha_1)^k_l \partial_+\phi^l_L \, d\sigma'^+ - 2\int_0^{\sigma^+} \tilde{f}^i_{mr}\partial_+\phi^m_L(\alpha_1)^r_n \, d\sigma'^+ \\ = &-\int_0^{\sigma^+} (\phi^i_L\partial_+\phi^n_L - \phi^n_L\partial_+\phi^i_L) \, d\sigma'^+ \end{aligned} \tag{2.1.13}$$

which gives us the following entries of α_2 with the help of (2.1.12)

$$\begin{aligned}
(\alpha_2)^1_1 &= 0 \\
(\alpha_2)^1_2 &= \int_0^{\sigma^+} [\phi^2_L(\partial_+\phi^1_L) - \phi^1_L(\partial_+\phi^2_L)] \, d\sigma'^+ \\
(\alpha_2)^1_3 &= \int_0^{\sigma^+} [\phi^3_L(\partial_+\phi^1_L) - \phi^1_L(\partial_+\phi^3_L)] \, d\sigma'^+ \\
(\alpha_2)^2_1 &= \int_0^{\sigma^+} [\phi^1_L(\partial_+\phi^2_L) - \phi^2_L(\partial_+\phi^1_L)] \, d\sigma'^+ \\
(\alpha_2)^2_2 &= 0 \\
(\alpha_2)^2_3 &= \int_0^{\sigma^+} [\phi^3_L(\partial_+\phi^2_L) - \phi^2_L(\partial_+\phi^3_L)] \, d\sigma'^+ \\
(\alpha_2)^3_1 &= \int_0^{\sigma^+} [\phi^1_L(\partial_+\phi^3_L) - \phi^3_L(\partial_+\phi^1_L)] \, d\sigma'^+ \\
(\alpha_2)^3_2 &= \int_0^{\sigma^+} [\phi^2_L(\partial_+\phi^3_L) - \phi^3_L(\partial_+\phi^2_L)] \, d\sigma'^+ \\
(\alpha_2)^3_3 &= 0
\end{aligned}$$

Plugging α_1 and α_2 into T and setting $\varepsilon = 1$ gives us

$$
\begin{aligned}
T^i_j &= \delta^i_j + (\alpha_1)^i_j + \frac{1}{2}[(\alpha_2)^i_j + (\alpha_1^2)^i_j] + \mathcal{O}(\phi^3) \\
&\approx \delta^i_j - \tilde{f}^i_{kj}\phi^k_L - \frac{1}{2}[\int_0^{\sigma^+} (\phi^i_L\partial_+\phi^j_L - \phi^j_L\partial_+\phi^i_L)\, d\sigma'^+ - \tilde{f}^i_{km}\tilde{f}^m_{nj}\phi^k_L\phi^n_L]
\end{aligned}
\tag{2.1.14}
$$

so the entries of T becomes

$$
\begin{aligned}
T^1_1 &= 1 - \frac{1}{2}[\phi^2_L\phi^2_L + \phi^3_L\phi^3_L] \\
T^1_2 &= \phi^3_L + \int_0^{\sigma^+} \phi^2_L(\partial_+\phi^1_L)\, d\sigma'^+ \\
T^1_3 &= -\phi^2_L + \int_0^{\sigma^+} \phi^3_L(\partial_+\phi^1_L)\, d\sigma'^+ \\
T^2_1 &= -\phi^3_L + \int_0^{\sigma^+} \phi^1_L(\partial_+\phi^2_L)\, d\sigma'^+ \\
T^2_2 &= 1 - \frac{1}{2}[\phi^1_L\phi^1_L + \phi^3_L\phi^3_L] \\
T^2_3 &= \phi^1_L + \int_0^{\sigma^+} \phi^3_L(\partial_+\phi^2_L)\, d\sigma'^+ \\
T^3_1 &= \phi^2_L + \int_0^{\sigma^+} \phi^1_L(\partial_+\phi^3_L)\, d\sigma'^+ \\
T^3_2 &= -\phi^1_L + \int_0^{\sigma^+} \phi^2_L(\partial_+\phi^3_L)\, d\sigma'^+ \\
T^3_3 &= 1 - \frac{1}{2}[\phi^1_L\phi^1_L + \phi^2_L\phi^2_L]
\end{aligned}
$$

We note that T is an orthogonal matrix. The type of the field $\phi(\sigma^+, \sigma^-) = \phi_L(\sigma^+) + \phi_R(\sigma^-)$ puts pseudoduality relations into the forms

$$
(\tilde{g}^{-1}\partial_+\tilde{g})^i = +T^i_j\partial_+\phi^j_L \tag{2.1.15}
$$

$$
(\tilde{g}^{-1}\partial_-\tilde{g})^i = -T^i_j\partial_-\phi^j_R \tag{2.1.16}
$$

We note that equation (2.1.15) has an invariance under $\tilde{g}(\sigma^+,\sigma^-) \longrightarrow h(\sigma^-)\tilde{g}(\sigma^+,\sigma^-)$. From this we can look for solution $\tilde{g}(\sigma^+,\sigma^-) = \tilde{g}_R(\sigma^-)\tilde{g}_L(\sigma^+)$, so first pseudoduality relation is reduced to $(\tilde{g}_L^{-1}\partial_+\tilde{g}_L)^i = +T^i_j\partial_+\phi^j_L$. This equation gives us the left current. Next we have to find $\tilde{g}_R(\sigma^-)$ using second pseudoduality equation to construct right current. Plugging $\tilde{g}(\sigma^+,\sigma^-) = \tilde{g}_R(\sigma^-)\tilde{g}_L(\sigma^+)$ into (2.1.16) and arranging terms we obtain

$$\tilde{g}_R^{-1}(\sigma^-)\partial_-\tilde{g}_R(\sigma^-) = -\tilde{g}_L(\sigma^+)(\tilde{X}_i T^i_j \partial_-\phi^j_R)\tilde{g}_L^{-1}(\sigma^+) \tag{2.1.17}$$

where $\{\tilde{X}_i\}$ are the Lie algebra basis of $\tilde{\mathbf{g}}$, and $(\tilde{X}_i)_{jk} = \epsilon_{jik}$. Since we want to construct pseudodual currents in the order of ϕ^n, we need $T(\sigma^+)$ to the order of ϕ^{n-1} to get $\tilde{J}^{(L)}_+(\sigma^+)$ to the order of ϕ^n. From equation (2.1.17) we see that the knowledge of T to $\mathcal{O}(\phi^{n-1})$ and $\tilde{g}_L$ to $\mathcal{O}(\phi^{n-1})$ allows us to construct $\tilde{g}_R$ to $\mathcal{O}(\phi^n)$, so we can construct $\tilde{J}^{(R)}_-(\sigma^-)$ to $\mathcal{O}(\phi^n)$.

First we construct $\tilde{J}^{(L)}_+(\sigma^+)$ to the order of ϕ^2, so we need T to $\mathcal{O}(\phi)$

$$T^i_j = \delta^i_j - \tilde{f}^i_{kj}\phi^k_L + \mathcal{O}(\phi^2) \tag{2.1.18}$$

Therefore, using first pseudoduality relation (2.1.15)

$$\tilde{J}^{(L)}_+(\sigma^+) = \tilde{g}_L^{-1}\partial_+\tilde{g}_L = \tilde{X}_i\partial_+\phi^i_L - \tilde{X}_i\tilde{f}^i_{kj}\phi^k_L\partial_+\phi^j_L \tag{2.1.19}$$

All we need is $\tilde{g}_L$ to the order of ϕ, so we need to solve $\tilde{g}_L^{-1}\partial_+\tilde{g}_L = \tilde{X}_i\partial_+\phi^i_L$ for $\tilde{g}_L(\sigma^+)$. Choosing initial condition as $\tilde{g}_L(\sigma^+ = 0) = I$, we get

$$\tilde{g}_L(\sigma^+) = I + \tilde{X}_i\phi^i_L + \mathcal{O}(\phi^2) \tag{2.1.20}$$

Its inverse is

$$\tilde{g}_L^{-1}(\sigma^+) = I - \tilde{X}_i\phi^i_L + \mathcal{O}(\phi^2) \tag{2.1.21}$$

Plugging these into (2.1.17) we find

$$\begin{aligned}\tilde{g}_R^{-1}\partial_-\tilde{g}_R &= -(I+\tilde{X}_l\phi_L^l)\tilde{X}_i(\delta_j^i - \tilde{f}_{kj}^i\phi_L^k)\partial_-\phi_R^j(I-\tilde{X}_k\phi_L^k)\\ &= -\tilde{X}_i\partial_-\phi_R^i + \mathcal{O}(\phi^3)\end{aligned} \tag{2.1.22}$$

We notice that the order of ϕ^2 terms are cancelled, and $\tilde{g}_R$ is a function of σ^- only. We let $\tilde{g}_R = e^{-\phi_R^i\tilde{X}_i}e^{\xi^k\tilde{X}_k}$, where ξ represents $\mathcal{O}(\phi^2)$. Expanding $\tilde{g}_R$

$$\begin{aligned}\tilde{g}_R &= (I-\phi_R^i\tilde{X}_i + \frac{1}{2}\phi_R^i\phi_R^j\tilde{X}_i\tilde{X}_j)(I+\xi^k\tilde{X}_k)\\ &= I-\phi_R^i\tilde{X}_i + \frac{1}{2}\phi_R^i\phi_R^j\tilde{X}_i\tilde{X}_j + \xi^k\tilde{X}_k + \mathcal{O}(\phi^3)\end{aligned} \tag{2.1.23}$$

the inverse $\tilde{g}_R^{-1}$ can be found from $\tilde{g}_R = e^{-\xi^k\tilde{X}_k}e^{\phi_R^i\tilde{X}_i}$

$$\begin{aligned}\tilde{g}_R^{-1} &= (I-\xi^k\tilde{X}_k)(I+\phi_R^i\tilde{X}_i + \frac{1}{2}\phi_R^i\phi_R^j\tilde{X}_i\tilde{X}_j)\\ &= I+\phi_R^i\tilde{X}_i + \frac{1}{2}\phi_R^i\phi_R^j\tilde{X}_i\tilde{X}_j - \xi^k\tilde{X}_k + \mathcal{O}(\phi^3)\end{aligned} \tag{2.1.24}$$

It follows then that equations (2.1.23) and (2.1.24) lead to

$$\tilde{g}_R^{-1}\partial_-\tilde{g}_R = -\partial_-\phi_R^i\tilde{X}_i + \partial_-\xi^k\tilde{X}_k + \mathcal{O}(\phi^3) \tag{2.1.25}$$

and comparison with (2.1.22) evaluates $\partial_-\xi^k = 0$, so ξ^k is constant and we choose it to be zero. Therefore, right current can be constructed using (2.1.23) and (2.1.24) as

$$\tilde{J}_-^{(R)}(\sigma^-) = (\partial_-\tilde{g}_R)\tilde{g}_R^{-1} = -(\partial_-\phi_R^i)\tilde{X}_i \tag{2.1.26}$$

we see that order of ϕ^2 disappears in the expression of right current. If we explicitly write pseudodual currents on the manifold M up to the order of ϕ^3 using equations (2.1.19) and (2.1.26) we get the following

$$\tilde{J}_{+}^{(L)}(\sigma^{+}) = \tilde{X}_i[\partial_+\phi_L^i - \tilde{f}^i_{kj}\phi_L^k\partial_+\phi_L^j - \frac{1}{2}[\int_0^{\sigma^+}(\phi_L^i\partial_+\phi_L^j - \phi_L^j\partial_+\phi_L^i)\,d\sigma'^+ \\ - \tilde{f}^i_{km}\tilde{f}^m_{nj}\phi_L^k\phi_L^n]\partial_+\phi_L^j]$$

$$\tilde{J}_{-}^{(R)}(\sigma^{-}) = -\tilde{X}_i(\partial_-\phi_R^i)$$

Therefore, our currents can be written as

$$\tilde{J}^{(\mu)} = \tilde{J}^{(\mu)}_{[0]} + \tilde{J}^{(\mu)}_{[1]} + \tilde{J}^{(\mu)}_{[2]} + \mathcal{O}(\phi^3) \tag{2.1.27}$$

where $\{\mu\} = (R, L)$. We can organize all these terms as

$$\tilde{J}^{(\mu)}(\phi) = \sum_0^\infty \tilde{J}^{(\mu)}_{[n]}(\phi) \tag{2.1.28}$$

It is easy to see that these currents are conserved, i.e. $\partial_+\tilde{J}_-^{(R)} = \partial_-\tilde{J}_+^{(L)} = 0$, by means of the equations of motion $\partial^2_{+-}\phi^i = 0$. Since each term satisfies $\partial_+\tilde{J}^{(R)}_{[n]} = \partial_-\tilde{J}^{(L)}_{[n]} = 0$ for all n separately, we have infinite number of conservation laws for each order of ϕ as pointed out in [7].

2.2 Pseudodual Currents : Complicated Case

In this case we consider the pseudoduality between two strict WZW models based on compact Lie groups of dimension n with Ad-invariant metrics. If $\{X_i\}$ are the orthonormal basis for the Lie algebra of G with commutation relations $[X_i, X_j]_G = f_{ij}^k X_k$, where f_{ijk} are totally antisymmetric in ijk, and $g : \Sigma \to M$ is the map to the target space, we may write equations of motion on G as $\partial_-(g^{-1}\partial_+ g) = 0$. Therefore, currents become

$J_+^{(L)} = (g^{-1}\partial_+ g)^i X_i$ and $J_-^{(R)} = [(\partial_- g)g^{-1}]^i X_i$. These currents are conserved. We make similar assumptions for the Lie group $\tilde{G}$. The pseudoduality equations are

$$(\tilde{g}^{-1}\partial_+\tilde{g})^i = +T^i_j(g^{-1}\partial_+ g)^j \tag{2.2.1}$$

$$(\tilde{g}^{-1}\partial_-\tilde{g})^i = -T^i_j(g^{-1}\partial_- g)^j \tag{2.2.2}$$

where T is an orthogonal matrix. Taking ∂_- of the first equation (2.2.1) we learn that T is a function of σ^+ only. Taking ∂_+ of the second equation (2.2.2) we get the differential equation for T

$$[(\partial_+ T)T^{-1}]^i_j = -\tilde{f}^i_{kj}T^l_k(g^{-1}\partial_+ g)^l + f^k_{ml}T^i_k T^j_l(g^{-1}\partial_+ g)^m \tag{2.2.3}$$

We follow the same method as we did in the previous part to find pseudodual currents. We first solve differential equation (2.2.3) for T, then replace this into the pseudoduality relations, and finally build pseudodual currents. We will see that these currents are conserved.

2.2.1 An Example

To illustrate all these steps in an example, we consider a strict WZW model based on Lie group $SU(2)$ for G, and a sigma model based on abelian group $U(1) \times U(1) \times U(1)$ for $\tilde{G}$. Using the map $g : \Sigma \to G$, we may represent any element in G by $g = e^{i\phi^k X_k}$, where $\{\phi^k\} = (\phi^1, \phi^2, \phi^3)$ are commuting fields and $\{X_k\}$ are the orthonormal basis for the Lie algebra of G, and $\{X_k\} = (-\frac{i}{2}\sigma_1, -\frac{i}{2}\sigma_2, -\frac{i}{2}\sigma_3)$ for the case of $SU(2)$. Structure constants are ϵ^i_{jk}, and commutation relations are the familiar form of Pauli matrices, $[-i\frac{\sigma_i}{2}, -i\frac{\sigma_j}{2}] = \epsilon^k_{ij}(-i\frac{\sigma_k}{2})$. Equations of motion are $\partial_-(g^{-1}\partial_+ g) = 0$. Nonlocal currents for the Lie algebra of $SU(2)$ are $J_+^{(L)} = (g^{-1}\partial_+ g)^k X_k$ and $J_-^{(R)} = [(\partial_- g)g^{-1}]^k X_k$. We want to construct currents up to the order of ϕ^2. If we consider infinitesimal coefficients $\{\phi^k\}$, keeping up to second orders we may expand g as

$$g = 1 + i\phi^k X_k - \frac{1}{2}(\phi^k \phi^l)(X_k X_l) + ... \tag{2.2.4}$$

Since we are looking for $J_+^{(L)}$ and $J_-^{(R)}$ up to the order of ϕ^2, we need g to the order of ϕ, hence

$$g = 1 + i\phi^k X_k \tag{2.2.5}$$

$$g^{-1} = 1 - i\phi^k X_k \tag{2.2.6}$$

To this order the solution to equations of motion $\partial_-(g^{-1}\partial_+ g) = 0$ is $g = g_R(\sigma^-)g_L(\sigma^+)$, which leads to $\phi(\sigma^+, \sigma^-) = \phi_R(\sigma^-) + \phi_L(\sigma^+)$. Thus equation (2.2.5) can be written as

$$g_L = 1 + i\phi_L^k X_k \tag{2.2.7}$$

$$g_R = 1 + i\phi_R^k X_k \tag{2.2.8}$$

and hence left and right currents can readily be obtained as

$$J_+^{(L)} = g_L^{-1}\partial_+ g_L = i\partial_+ \phi_L^m X_m + \frac{1}{2} f_{kl}^m \phi_L^k \partial_+ \phi_L^l X_m \tag{2.2.9}$$

$$J_-^{(R)} = (\partial_- g_R) g_R^{-1} = i\partial_- \phi_R^m X_m + \frac{1}{2} f_{kl}^m \phi_R^l \partial_- \phi_R^k X_m \tag{2.2.10}$$

Therefore, we conclude that $\partial_- J_+^{(L)} = \partial_+ J_-^{(R)} = 0$, i.e, currents are conserved on G. We first solve equation (2.2.3) to figure out the pseudodual currents. Since $\tilde{f}_{kj}^i = 0$, we have

$$[(\partial_+ T)T^{-1}]_j^i = f_{ml}^k T_k^i T_l^j (g_L^{-1}\partial_+ g_L)^m \tag{2.2.11}$$

this may be reduced to

$$[(\partial_+ T)]^i_n = f^k_{mn} T^i_k (g_L^{-1}\partial_+ g_L)^m \tag{2.2.12}$$

and putting in an order parameter ε we get

$$[(\partial_+ T)]^i_n = \varepsilon f^k_{mn} T^i_k (g_L^{-1}\partial_+ g_L)^m \tag{2.2.13}$$

We adapt to an exponential solution $T = e^{\varepsilon\alpha_1} e^{\frac{1}{2}\varepsilon^2\alpha_2}(I + \mathcal{O}(\varepsilon^3))$, where α_1 and α_2 are antisymmmetric matrices, and expanding this solution we get

$$T = I + \varepsilon\alpha_1 + \frac{1}{2}\varepsilon^2(\alpha_2 + \alpha_1^2) + \mathcal{O}(\varepsilon^3) \tag{2.2.14}$$

taking ∂_+ of T leads to

$$\partial_+ T = \varepsilon\partial_+\alpha_1 + \frac{1}{2}\varepsilon^2[\alpha_1(\partial_+\alpha_1) + (\partial_+\alpha_1)\alpha_1 + \partial_+\alpha_2] + \mathcal{O}(\varepsilon^3) \tag{2.2.15}$$

expressing (2.2.13) in tensor product form

$$\begin{aligned}\partial_+ T &= \varepsilon f(g_L^{-1}\partial_+ g_L) \otimes T = \varepsilon f(g_L^{-1}\partial_+ g_L) \otimes (I + \varepsilon\alpha_1) \\ &= \varepsilon f(g_L^{-1}\partial_+ g_L) + \varepsilon^2 f(g_L^{-1}\partial_+ g_L) \otimes \alpha_1 \end{aligned} \tag{2.2.16}$$

and comparing this with (2.2.15) we obtain α_1

$$\begin{aligned}(\alpha_1)^i_n &= \int_0^{\sigma^+} f^i_{mn}(g_L^{-1}\partial_+ g_L)^m d\sigma'^+ \\ &= i f^i_{mn}\phi^m_L + \frac{1}{2} f^i_{mn} f^m_{kl} \int_0^{\sigma^+} \phi^k_L \partial_+ \phi^l_L \, d\sigma'^+ \\ &= i\epsilon^i_{mn}\phi^m_L + \frac{1}{2}\int_0^{\sigma^+} (\phi^n_L\partial_+\phi^i_L - \phi^i_L\partial_+\phi^n_L)\, d\sigma'^+ \end{aligned} \tag{2.2.17}$$

this expression leads to the following entries

$$
\begin{aligned}
&(\alpha_1)^1_1 = 0 \\
&(\alpha_1)^1_2 = -i\phi^3_L - \frac{1}{2}\int_0^{\sigma^+} [\phi^1_L(\partial_+\phi^2_L) - \phi^2_L(\partial_+\phi^1_L)]\, d\sigma'^+ \\
&(\alpha_1)^1_3 = i\phi^2_L + \frac{1}{2}\int_0^{\sigma^+} [\phi^3_L(\partial_+\phi^1_L) - \phi^1_L(\partial_+\phi^3_L)]\, d\sigma'^+ \\
&(\alpha_1)^2_1 = i\phi^3_L + \frac{1}{2}\int_0^{\sigma^+} [\phi^1_L(\partial_+\phi^2_L) - \phi^2_L(\partial_+\phi^1_L)]\, d\sigma'^+ \\
&(\alpha_1)^2_2 = 0 \\
&(\alpha_1)^2_3 = -i\phi^1_L - \frac{1}{2}\int_0^{\sigma^+} [\phi^2_L(\partial_+\phi^3_L) - \phi^3_L(\partial_+\phi^2_L)]\, d\sigma'^+ \\
&(\alpha_1)^3_1 = -i\phi^2_L - \frac{1}{2}\int_0^{\sigma^+} [\phi^3_L(\partial_+\phi^1_L) - \phi^1_L(\partial_+\phi^3_L)]\, d\sigma'^+ \\
&(\alpha_1)^3_2 = i\phi^1_L + \frac{1}{2}\int_0^{\sigma^+} [\phi^2_L(\partial_+\phi^3_L) - \phi^3_L(\partial_+\phi^2_L)]\, d\sigma'^+ \\
&(\alpha_1)^3_3 = 0
\end{aligned}
$$

and

$$\partial_+\alpha_2 = 2f(g_L^{-1}\partial_+ g_L) \otimes \alpha_1 - [\alpha_1(\partial_+\alpha_1) + (\partial_+\alpha_1)\alpha_1] \tag{2.2.18}$$

Hence, α_2 is obtained as

$$(\alpha_2)^i_n = \int_0^{\sigma^+} (\phi^i_L\partial_+\phi^n_L - \phi^n_L\partial_+\phi^i_L)\, d\sigma'^+ \tag{2.2.19}$$

We see that this is equivalent to (2.1.13), and entries are the same as the negative of above results. Therefore, we can find T by means of (2.2.17) and (2.2.19), and setting $\varepsilon = 1$

$$T^i_n = \delta^i_n + i\epsilon^i_{mn}\phi^m_L + \frac{1}{2}(\delta^i_n\phi^m_L\phi^m_L - \phi^n_L\phi^i_L) \tag{2.2.20}$$

Again we note that T is an orthogonal matrix. Now using pseudoduality equations (2.2.1) and (2.2.2)

$$\partial_+\tilde{\phi_L}^i = +T^i_j(g^{-1}\partial_+ g)^j = +T^i_j(g_L^{-1}\partial_+ g_L)^j \tag{2.2.21}$$

$$\partial_-\tilde{\phi_R}^i = -T^i_j(g^{-1}\partial_- g)^j = -T^i_j(g_L^{-1}g_R^{-1}(\partial_- g_R)g_L)^j \tag{2.2.22}$$

since we are trying to find $\partial_+\tilde{\phi_L}^i$ and $\partial_-\tilde{\phi_R}^i$ up to the order of ϕ^2, we need T to the order of ϕ, hence using

$$T^i_n = \delta^i_j + i\epsilon^i_{mj}\phi^m_L + \mathcal{O}(\phi^2) \tag{2.2.23}$$

we may find

$$\begin{aligned}
\partial_+\tilde{\phi_L}^i &= T^i_j(g_L^{-1}\partial_+ g_L)^j \\
&= [\delta^i_j + i\epsilon^i_{mj}\phi^m_L][i\partial_+\phi^j_L + \frac{1}{2}\epsilon^j_{kl}\phi^k_L\partial_+\phi^l_L] \\
&= i\partial_+\phi^i_L - \frac{1}{2}\epsilon^i_{kl}\phi^k_L\partial_+\phi^l_L + \mathcal{O}(\phi^3)
\end{aligned} \tag{2.2.24}$$

$$\begin{aligned}
\partial_-\tilde{\phi_R}^i &= -T^i_j(g^{-1}\partial_- g)^j = -T^i_j[g_L^{-1}(g_R^{-1}\partial_- g_R)g_L]^j \\
&= -[\delta^i_j + i\epsilon^i_{mj}\phi^m_L][(1 - i\phi^k_L X_k)(1 - \phi^k_R X_k)(i\partial_-\phi^k_R X_k - \frac{1}{2}\phi^m_R\partial_-\phi^l_R X_l X_m \\
&- \frac{1}{2}\phi^l_R\partial_-\phi^m_R X_l X_m)(1 + i\phi^k_L X_k)]^j \\
&= -[\delta^i_j + i\epsilon^i_{mj}\phi^m_L][i\partial_-\phi^j_R + \epsilon^j_{mn}\phi^m_L\partial_-\phi^n_R + \frac{1}{2}\epsilon^j_{lm}\phi^l_R\partial_-\phi^m_R] \\
&= -i\partial_-\phi^i_R - \frac{1}{2}\epsilon^i_{lm}\phi^l_R\partial_-\phi^m_R
\end{aligned} \tag{2.2.25}$$

We see that these are free scalar currents on the tangent bundle to the pseudodual manifold $\tilde{G}$. Since $\partial_+\tilde{\phi_L}^i$ depends only on σ^+, and $\partial_-\tilde{\phi_R}^i$ only on σ^-, these pseudodual free scalar currents are conserved provided that equations of motion for free scalar fields hold. We go back to equations of motion to see that these pseudodual tangent bundle components take us to pseudodual conserved currents. Equations of motion, $\partial_-(g^{-1}\partial_+ g)^i = 0$, imply that $\partial^2_{+-}\phi^i = 0$. Obviously we find out the pseudodual conservation laws $\partial^2_{\pm\mp}\tilde{\phi}^i = 0$ in all ϕ-orders using these conditions.

2.3 Conclusion

We observed that nonlinear character of WZW models results in an infinite number of terms in the mapping T, which in turn leads to construct infinite number of nonlocal currents in pseudodual manifold. Calculations were motivated by the fact that sigma models have Lie group structures, and $T \in SO(n)$. Hence structure of Lie groups together with perturbation calculations reflects the nonlinear characteristic of sigma models. It is obvious that pseudoduality transformation leads to the pseudodual conserved currents in our cases where one model based on an abelian group $U(1)\times U(1)\times U(1)$ in two cases we discussed. However, One can consider general Lie group valued fields for both models, and see that this would also yield conserved currents on pseudodual model. We considered three dimensional models for simplicity but this can be extended to any dimension. Calculation of these currents gives us curvatures by means of Cartan structural equations (1.1.5) and (1.1.6), where $w^i = J^i$ and $w^i_j = \frac{1}{2} f^i_{kj} J^k$ is the antisymmetric connection, and J stands for both $J^{(L)}_+$ and $J^{(R)}_-$. These currents form an orthonormal frame on pullback bundle $g^*(TG)$ [1]. Since we considered abelian models, and hence obtained scalar currents, it is easily noted that curvatures are zero. In general case where sigma models based on general Lie groups, curvatures will be constant and opposite. This shows that sigma models are based on symmetric spaces as pointed out in [2]. The calculations and results of this section can be applied to pseudoduality relations between symmetric space sigma models to construct nonlocal currents and curvatures relations. We will discuss this in the next section.

[1] TG is the tangent bundle of G, i.e. $TG = G \times \mathbf{g}$.

Chapter 3

Pseudoduality Between Symmetric Space Sigma Models

In this section we present the general solution of the pseudoduality equations (2.2.1) and (2.2.2) between two symmetric space sigma models, and construct the pseudodual currents by means of these equations. We will do our calculations regarding G as a symmetric space $G \times G/G$, and then extend our construction using Cartan's decomposition of symmetric spaces. We will use the references [18, 19, 20, 21] for the symmetric space construction, and utilize the literature [15, 16, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33] on various applications to sigma models. Since we defined pseudoduality on spacetime coordinates, and we said above that pseudoduality is best done on the orthonormal coframes bundle $SO(M)$, we leave this construction to later. In this section we will do our calculations on the pullback bundle of target space M. Hence pulling structures back to spacetime is implicit, and not emphasized. We will see that this construction will give us complicated expressions for T as opposed to the simplified form (identity) on $SO(M)$.

3.1 Pseudoduality Between WZW Models : H = I

We consider a strict WZW sigma model based on a compact Lie group of dimension n. Lagrangian [11, 24, 31, 32, 33] for this model is defined by

$$\mathcal{L} = \frac{1}{2}Tr(g^{-1}\partial_\mu g g^{-1}\partial^\mu g) + \Gamma \tag{3.1.1}$$

where Γ represents the WZ term, and the field g is given by the map $g : \Sigma \rightarrow G$. We take Σ to be two dimensional Minkowski space, and $\sigma^{\pm} = \tau \pm \sigma$ is the standard lightcone coordinates as above. There is a global continuous symmetry $G_L \times G_R$ which gives us the conserved currents $J_+^{(L)} = g_L^{-1}\partial_+ g_L$ and $J_-^{(R)} = (\partial_- g_R)g_R^{-1}$ taking values in the Lie algebra of G, and $g = g_R(\sigma^-)g_L(\sigma^+)$ is the solution giving the invariance of these currents. The equations of motion following from (3.1.1) correspond to the conservation of these currents:

$$\partial_-(g_L^{-1}\partial_+ g_L) = \partial_+[(\partial_- g_R)g_R^{-1}] = 0 \tag{3.1.2}$$

Let $\tilde{G}$ be compact Lie group of the same dimension as G, and $\tilde{g} : \Sigma \rightarrow \tilde{G}$. Equations of motion are given by

$$\partial_-(\tilde{g_L}^{-1}\partial_+ \tilde{g_L}) = \partial_+[(\partial_- \tilde{g_R})\tilde{g_R}^{-1}] = 0 \tag{3.1.3}$$

Solutions of equations of motion for both models can be combined in pseudoduality equations (2.2.1) and (2.2.2) as

$$(\tilde{g}^{-1}\partial_+\tilde{g})^i = T^i_j(g^{-1}\partial_+ g)^j \tag{3.1.4}$$

$$(\tilde{g}^{-1}\partial_-\tilde{g})^i = -T^i_j(g^{-1}\partial_- g)^j \tag{3.1.5}$$

where T is an orthogonal matrix connecting target space elements $g^{-1}dg$ and $\tilde{g}^{-1}d\tilde{g}$.

Taking ∂_- of first equation (3.1.4) with the help of equations of motions (3.1.2) and (3.1.3) shows that T is a function of σ^+ only. Taking ∂_+ of second equation (3.1.5) gives us the following differential equation

$$[(\partial_+T)T^{-1}]^i_j = f^k_{ml}T^i_kT^j_l(g_L^{-1}\partial_+g_L)^m - \tilde{f}^i_{kj}T^k_l(g_L^{-1}\partial_+g_L)^l \tag{3.1.6}$$

We suggest an exponential solution [1] $T = e^X$, and use the result [16, 18, 22]

$$(\partial_+T)T^{-1} = -\frac{1-e^{adX}}{adX}\partial_+X = \sum_{n=0}^{\infty}\frac{1}{(n+1)!}[X, ..., [X, \partial_+X]] \tag{3.1.7}$$

where adX : $\mathbf{g} \to \mathbf{g}$, the adjoint representation of X, and $adX(Y) = [X, Y]$ $\forall Y \epsilon$ $\mathbf{g}$. We let $X \to \varepsilon X$ and look for a perturbation solution, and hence the left-hand side of equation (3.1.6) is

$$[(\partial_+T)T^{-1}]^i_j = \varepsilon(\partial_+X)^i_j + \frac{\varepsilon^2}{2}[X, \partial_+X]^i_j + \frac{\varepsilon^3}{3!}[X, [X, \partial_+X]] + ... \tag{3.1.8}$$

We insert an order parameter ε to the right-hand side of (3.1.6), and get

$$\begin{aligned}
[(\partial_+T)T^{-1}]^i_j =& \varepsilon f^k_{ml}T^i_kT^j_l(g_L^{-1}\partial_+g_L)^m - \varepsilon\tilde{f}^i_{kj}T^k_l(g_L^{-1}\partial_+g_L)^l \qquad (3.1.9)\\
=& \varepsilon f^k_{ml}(1+\varepsilon X)^i_k(1+\varepsilon X)^j_l(g_L^{-1}\partial_+g_L)^m - \varepsilon\tilde{f}^i_{kj}(1+\varepsilon X)^k_l(g_L^{-1}\partial_+g_L)^m\\
=& \varepsilon f^i_{mj}(g_L^{-1}\partial_+g_L)^m - \varepsilon\tilde{f}^i_{kj}(g_L^{-1}\partial_+g_L)^k + \varepsilon^2 f^i_{ml}X^j_l(g_L^{-1}\partial_+g_L)^m\\
&+ \varepsilon^2 f^k_{mj}X^i_k(g_L^{-1}\partial_+g_L)^m - \varepsilon^2\tilde{f}^i_{kj}X^k_l(g_L^{-1}\partial_+g_L)^l + \mathcal{O}(\varepsilon^3)
\end{aligned}$$

Comparing (3.1.8) and (3.1.9) in the first order of ε gives us

$$(\partial_+X)^i_j = (f^i_{kj} - \tilde{f}^i_{kj})(g_L^{-1}\partial_+g_L)^k \tag{3.1.10}$$

[1] We notice that $X \in so(n)$, lie algebra of $SO(n)$

This leads to the solution

$$X^i_j = X(0)^i_j + (f^i_{kj} - \tilde{f}^i_{kj}) \int_0^{\sigma^+} (g_L^{-1}\partial_+ g_L)^k d\sigma'^+ \tag{3.1.11}$$

Hence the matrix T may be written as

$$T^i_j = \delta^i_j + X(0)^i_j + (f^i_{kj} - \tilde{f}^i_{kj}) \int_0^{\sigma^+} (g_L^{-1}\partial_+ g_L)^k d\sigma'^+ \tag{3.1.12}$$

We see that if both sigma models based on the same groups, i.e $G = \tilde{G}$, target space of transformed model will be globally shifted as determined by the tangent space of unit element of T. We set $X(0)^i_j$ equal to zero.

Now we plug this in the pseudoduality equations (3.1.4) and (3.1.5) to find fields $\tilde{g}^{-1}\partial_+\tilde{g}$ and $\tilde{g}^{-1}\partial_-\tilde{g}$ which lead us to construct the pseudodual currents. We switch from Lie group-valued fields to the lie algebra-valued fields, and we let [2] $g = e^Y$ and $\tilde{g} = e^{\tilde{Y}}$. Using the result [16, 18, 22]

$$e^{-X}\partial_\mu e^X = \frac{1 - e^{-adX}}{adX}\partial_\mu X = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!}[X, ..., [X, \partial_\mu X]] \tag{3.1.13}$$

we can write the following

$$g_L^{-1}\partial_+ g_L = \partial_+ Y_L - \frac{1}{2!}[Y_L, \partial_+ Y_L] + \frac{1}{3!}[Y_L, [Y_L, \partial_+ Y_L]] + ... \tag{3.1.14}$$

$$\begin{aligned} g^{-1}\partial_- g =& \partial_- Y_R - [Y_L, \partial_- Y_R] - \frac{1}{2}[Y_R, \partial_- Y_R] + \frac{1}{2}[Y_L, [Y_R, \partial_- Y_R]] \\ &+ \frac{1}{2}[Y_L, [Y_L, \partial_- Y_R]] + \frac{1}{6}[Y_R, [Y_R, \partial_- Y_R]]... \end{aligned} \tag{3.1.15}$$

[2] Y is the lie algebra of g, $Y \in \mathbf{g}$.

and the equations of motion for the left and right currents will be

$$\partial_-(g_L^{-1}\partial_+ g_L) = \partial^2_{+-}Y_L - \frac{1}{2!}\partial_-[Y_L, \partial_+ Y_L] + \frac{1}{3!}\partial_-[Y_L, [Y_L, \partial_+ Y_L]] + ... = 0 \quad (3.1.16)$$

$$\partial_+[(\partial_- g_R)g_R^{-1}] = \partial^2_{+-}Y_R + \frac{1}{2!}\partial_+[Y_R, \partial_- Y_R] + \frac{1}{3!}\partial_+[Y_R, [Y_R, \partial_- Y_R]] + ... = 0 \quad (3.1.17)$$

where $g_{L/R} = e^{Y_{L/R}}$, and we used equation (3.1.7). We may write similar equations with tilde ($\tilde{}$). Hence transformation matrix T (3.1.12) will be

$$T^i_j = \delta^i_j + (f^i_{kj} - \tilde{f}^i_{kj})Y^k_L - \frac{1}{2!}(f^i_{kj} - \tilde{f}^i_{kj}) \int_0^{\sigma^+} [Y_L, \partial_+ Y_L]^k d\sigma'^+ \quad (3.1.18)$$

We impose a solution $Y = \sum_{n=1}^{\infty} \varepsilon^n y_n$ to determine the nonlinear parts of the equations (3.1.14) and (3.1.15) in terms of ε, where ε is a small parameter. Thus transformation matrix (3.1.18) becomes

$$T^i_j = \delta^i_j + \varepsilon(f^i_{kj} - \tilde{f}^i_{kj})y^k_{L1} + \varepsilon^2(f^i_{kj} - \tilde{f}^i_{kj})[y^k_{L2} - \frac{1}{2}\int_0^{\sigma^+} [y_{L1}, \partial_+ y_{L1}]^k d\sigma'^+] + \mathcal{O}(\varepsilon^3) \quad (3.1.19)$$

and we have the following expressions for (3.1.14) and (3.1.15)

$$\begin{aligned} g_L^{-1}\partial_+ g_L &= \varepsilon\partial_+ y_{L1} + \varepsilon^2(\partial_+ y_{L2} - \frac{1}{2}[y_{L1}, \partial_+ y_{L1}]) \quad (3.1.20) \\ &+ \varepsilon^3(\partial_+ y_{L3} - \frac{1}{2}[y_{L1}, \partial_+ y_{L2}] - \frac{1}{2}[y_{L2}, \partial_+ y_{L1}] + \frac{1}{6}[y_{L1}, [y_{L1}, \partial_+ y_{L1}]]) + \mathcal{O}(\varepsilon^4) \end{aligned}$$

$$\begin{aligned} g^{-1}\partial_- g &= \varepsilon\partial_- y_{R1} + \varepsilon^2(\partial_- y_{R2} - [y_{L1}, \partial_- y_{R1}] - \frac{1}{2}[y_{R1}, \partial_- y_{R1}]) \quad (3.1.21) \\ &+ \varepsilon^3(\partial_- y_{R3} - [y_{L2}, \partial_- y_{R1}] - [y_{L1}, \partial_- y_{R2}] - \frac{1}{2}[y_{R2}, \partial_- y_{R1}] - \frac{1}{2}[y_{R1}, \partial_- y_{R2}] \\ &+ \frac{1}{2}[y_{L1}, [y_{R1}, \partial_- y_{R1}]] + \frac{1}{2}[y_{L1}, [y_{L1}, \partial_- y_{R1}]]) + H.O(\varepsilon) \end{aligned}$$

Therefore first pseudoduality equation (3.1.4) can be split into infinite number of equations, determined by each order of ε as follows,

$$
\begin{aligned}
&(1.i)\ \partial_+\tilde{y}^i_{L1} = \partial_+ y^i_{L1} \\
&(1.ii)\ \partial_+\tilde{y}^i_{L2} + \frac{1}{2}[\tilde{y}_{L1}, \partial_+\tilde{y}_{L1}]^i_{\tilde{G}} = \partial_+ y^i_{L2} + \frac{1}{2}[y_{L1}, \partial_+ y_{L1}]^i_G \\
&(1.iii)\ \partial_+\tilde{y}^i_3 - \frac{1}{2}[\tilde{y}_1, \partial_+\tilde{y}_2]^i_{\tilde{G}} - \frac{1}{2}[\tilde{y}_2, \partial_+\tilde{y}_1]^i_{\tilde{G}} + \frac{1}{6}[\tilde{y}_1, [\tilde{y}_1, \partial_+\tilde{y}_1]_{\tilde{G}}]^i_{\tilde{G}} = \partial_+ y^i_3 \\
&\ + \frac{1}{2}[y_1, \partial_+ y_2]^i_G + \frac{1}{2}[y_2, \partial_+ y_1]^i_G - [y_1, \partial_+ y_2]^i_{\tilde{G}} - [y_2, \partial_+ y_1]^i_{\tilde{G}} - \frac{1}{3}[y_1, [y_1, \partial_+ y_1]_G]^i_G \\
&\ + \frac{1}{2}[y_1, [y_1, \partial_+ y_1]_G]^i_{\tilde{G}} - \frac{1}{2}[\int_0^{\sigma^+}[y_1, \partial_+ y_1]_G\, d\sigma'^+, \partial_+ y_1]^i_G + \frac{1}{2}[\int_0^{\sigma^+}[y_1, \partial_+ y_1]_G\, d\sigma'^+, \partial_+ y_1]^i_{\tilde{G}} \\
&\ \cdots
\end{aligned}
\tag{3.1.22}
$$

where we used subindex G ($\tilde{G}$) to represent commutation relations for the sigma model based on Lie group G ($\tilde{G}$). (1.i) gives $\tilde{y}_{L1} = y_{L1} + C_{L1}$, where C_{L1} is a constant, and we set it equal to zero, and leads to (1.ii). Likewise second pseudoduality equation (3.1.5) gives the following infinite set of equations

$$
\begin{aligned}
&(2.i)\ \partial_-\tilde{y}^i_{R1} = -\partial_- y^i_{R1} \\
&(2.ii)\ \partial_-\tilde{y}^i_{R2} - \frac{1}{2}[\tilde{y}_{R1}, \partial_-\tilde{y}_{R1}]^i_{\tilde{G}} = -\partial_- y^i_{R2} + \frac{1}{2}[y_{R1}, \partial_- y_{R1}]^i_G \\
&(2.iii)\ \cdots
\end{aligned}
\tag{3.1.23}
$$

where we used (2.i) and (1.i) in (2.ii), and (2.i) leads to $\tilde{y}_{R1} = -y_{R1} + C_{R1}$, C_{R1} is a constant which is set to zero. We notice the fact that (3.1.22) only depends on σ^+, and

(3.1.23) on σ^- point out pseudodual conserved currents, which can be written as follows

$$\tilde{J}_+^L(\sigma^+) = \tilde{g}^{-1}\partial_+\tilde{g} = \sum_{n=1}^{\infty} \varepsilon^n \tilde{J}_+^{L[n]}(\sigma^+) \tag{3.1.24}$$

$$\tilde{J}_-^R(\sigma^-) = (\partial_-\tilde{g})\tilde{g}^{-1} = \sum_{n=1}^{\infty} \varepsilon^n \tilde{J}_-^{R[n]}(\sigma^-) \tag{3.1.25}$$

where each component is determined by the orders of ε's, which are given by expression (3.1.20) (with tilde). The nonlocal expressions of currents are determined with the help of (3.1.22) and (3.1.23)

$$\tilde{J}_+^{L[1]}(\sigma^+) = \partial_+\tilde{y}_{L1}^i = \partial_+ y_{L1}^i \tag{3.1.26}$$

$$\tilde{J}_+^{L[2]}(\sigma^+) = \partial_+\tilde{y}_{L2} - \frac{1}{2}[\tilde{y}_{L1}, \partial_+\tilde{y}_{L1}]_{\tilde{G}} = \partial_+ y_{L2}^i + \frac{1}{2}[y_{L1}, \partial_+ y_{L1}]_G^i - [y_{L1}, \partial_+ y_{L1}]_{\tilde{G}} \tag{3.1.27}$$

$$\cdots$$

$$\tilde{J}_-^{R[1]}(\sigma^-) = \partial_-\tilde{y}_{R1}^i = -\partial_- y_{R1}^i \tag{3.1.28}$$

$$\tilde{J}_-^{R[2]}(\sigma^-) = \partial_-\tilde{y}_{R2} + \frac{1}{2}[\tilde{y}_{R1}, \partial_-\tilde{y}_{R1}]_{\tilde{G}} = -\partial_- y_{R2}^i + \frac{1}{2}[y_{R1}, \partial_- y_{R1}]_G^i + [y_{R1}, \partial_- y_{R1}]_{\tilde{G}} \tag{3.1.29}$$

$$\cdots$$

We see that these currents are conserved, $\partial_-\tilde{J}_+^L = \partial_+\tilde{J}_-^R = 0$. It is observed that pseudodual currents are expressed as a nonlocal function of lie algebra valued fields on **g**. As a result we obtained a family of nonlocal conserved currents on the WZW model on G. This family is a consequence of infinite set of terms of T which is a function of lie algebra valued fields **g**.

3.1.1 An Example

We consider sigma models based on Lie groups $G = SO(n+1)$ and $\tilde{G} = SO(n,1)$. The corresponding lie algebra are given by

$$\textbf{so(n + 1)} = \begin{pmatrix} a & b \\ -b^t & c \end{pmatrix} \qquad \textbf{so(n, 1)} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b}^t & \tilde{c} \end{pmatrix} \qquad \begin{array}{c} a = \tilde{a} = n \times n \\ b = \tilde{b} = n \times 1 \\ c = \tilde{c} = 1 \times 1 \end{array} \tag{3.1.30}$$

Let $g = e^Y$ and $\tilde{g} = e^{\tilde{Y}}$, and fields $g_L^{-1}\partial_+ g_L$ and $\tilde{g_L}^{-1}\partial_+\tilde{g_L}$ are given by (3.1.14). We get the following expressions

$$Y_L = \begin{pmatrix} a_L & b_L \\ -b_L^t & c_L \end{pmatrix} \qquad \partial_+ Y_L = \begin{pmatrix} \partial_+ a_L & \partial_+ b_L \\ -\partial_+ b_L^t & \partial_+ c_L \end{pmatrix}$$

$$\tilde{Y_L} = \begin{pmatrix} \tilde{a_L} & \tilde{b_L} \\ \tilde{b_L}^t & \tilde{c_L} \end{pmatrix} \qquad \partial_+ \tilde{Y_L} = \begin{pmatrix} \partial_+ \tilde{a_L} & \partial_+ \tilde{b_L} \\ \partial_+ \tilde{b_L}^t & \partial_+ \tilde{c_L} \end{pmatrix}$$

$$[Y_L, \partial_+ Y_L] = \begin{pmatrix} 0 & a_L\partial_+ b_L + b_L\partial_+ c_L - (\partial_+ a_L)b_L - (\partial_+ b_L)c_L \\ -b_L^t(\partial_+ a_L) - c_L(\partial_+ b_L^t) + (\partial_+ b_L^t)a_L + (\partial_+ c_L)b_L^t & 0 \end{pmatrix}$$

$$[\tilde{Y}_L, \partial_+ \tilde{Y}_L] = \begin{pmatrix} 0 & \tilde{a}_L\partial_+ \tilde{b}_L + \tilde{b}_L\partial_+ \tilde{c}_L - (\partial_+ \tilde{a}_L)\tilde{b}_L - (\partial_+ \tilde{b}_L)\tilde{c}_L \\ \tilde{b}_L^t(\partial_+ \tilde{a}_L) + \tilde{c}_L(\partial_+ \tilde{b}_L^t) - (\partial_+ \tilde{b}_L^t)\tilde{a}_L - (\partial_+ \tilde{c}_L)\tilde{b}_L^t & 0 \end{pmatrix}$$

Hence up to the second order terms we get the expressions for the fields on the target space elements

$$g_L^{-1}\partial_+ g_L = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} + H.O \qquad \tilde{g_L}^{-1}\partial_+\tilde{g_L} = \begin{pmatrix} \tilde{X}_1 & \tilde{X}_2 \\ \tilde{X}_3 & \tilde{X}_4 \end{pmatrix} + H.O \tag{3.1.31}$$

where we defined the following

$$X_1 = \partial_+ a_L \qquad \tilde{X}_1 = \partial_+ \tilde{a}_L \quad X_4 = \partial_+ c_L \qquad \tilde{X}_4 = \partial_+ \tilde{c}_L$$

$$X_2 = \partial_+ b_L - \frac{a_L \partial_+ b_L + b_L \partial_+ c_L - (\partial_+ a_L) b_L - (\partial_+ b_L) c_L}{2}$$

$$\tilde{X}_2 = \partial_+ \tilde{b}_L - \frac{\tilde{a}_L \partial_+ \tilde{b}_L + \tilde{b}_L \partial_+ \tilde{c}_L - (\partial_+ \tilde{a}_L) \tilde{b}_L - (\partial_+ \tilde{b}_L) \tilde{c}_L}{2}$$

$$X_3 = -\partial_+ b_L^t - \frac{-b_L^t (\partial_+ a_L) - c_L (\partial_+ b_L^t) + (\partial_+ b_L^t) a_L + (\partial_+ c_L) b_L^t}{2}$$

$$\tilde{X}_3 = \partial_+ \tilde{b}_L^t - \frac{\tilde{b}_L^t (\partial_+ \tilde{a}_L) + \tilde{c}_L (\partial_+ \tilde{b}_L^t) - (\partial_+ \tilde{b}_L^t) \tilde{a}_L - (\partial_+ \tilde{c}_L) \tilde{b}_L^t}{2}$$

Likewise we get the following expressions related to fields $g^{-1}\partial_- g$ and $\tilde{g}^{-1}\partial_- \tilde{g}$ using (3.1.15)

$$[Y_L, \partial_- Y_R] = \begin{pmatrix} 0 & a_L \partial_- b_R + b_L \partial_- c_R - (\partial_- a_R) b_L - (\partial_- b_R) c_L \\ -b_L^t (\partial_- a_R) - c_L (\partial_- b_R^t) + (\partial_- b_R^t) a_L + (\partial_- c_R) b_L^t & 0 \end{pmatrix}$$

$$[Y_R, \partial_- Y_R] = \begin{pmatrix} 0 & a_R \partial_- b_R + b_R \partial_- c_R - (\partial_- a_R) b_R - (\partial_- b_R) c_R \\ -b_R^t (\partial_- a_R) - c_R (\partial_- b_R^t) + (\partial_- b_R^t) a_R + (\partial_- c_R) b_R^t & 0 \end{pmatrix}$$

$$[\tilde{Y}_L, \partial_- \tilde{Y}_R] = \begin{pmatrix} 0 & \tilde{a}_L \partial_- \tilde{b}_R + \tilde{b}_L \partial_- \tilde{c}_R - (\partial_- \tilde{a}_R) \tilde{b}_L - (\partial_- \tilde{b}_R) \tilde{c}_L \\ \tilde{b}_L^t (\partial_- \tilde{a}_R) + \tilde{c}_L (\partial_- \tilde{b}_R^t) - (\partial_- \tilde{b}_R^t) \tilde{a}_L - (\partial_- \tilde{c}_R) \tilde{b}_L^t & 0 \end{pmatrix}$$

$$[\tilde{Y}_R, \partial_- \tilde{Y}_R] = \begin{pmatrix} 0 & \tilde{a}_R \partial_- \tilde{b}_R + \tilde{b}_R \partial_- \tilde{c}_R - (\partial_- \tilde{a}_R) \tilde{b}_R - (\partial_- \tilde{b}_R) \tilde{c}_R \\ \tilde{b}_R^t (\partial_- \tilde{a}_R) + \tilde{c}_R (\partial_- \tilde{b}_R^t) - (\partial_- \tilde{b}_R^t) \tilde{a}_R - (\partial_- \tilde{c}_R) \tilde{b}_R^t & 0 \end{pmatrix}$$

$$g^{-1}\partial_- g = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} + H.O \qquad \tilde{g}^{-1}\partial_- \tilde{g} = \begin{pmatrix} \tilde{Z}_1 & \tilde{Z}_2 \\ \tilde{Z}_3 & \tilde{Z}_4 \end{pmatrix} + H.O \tag{3.1.32}$$

$$Z_1 = \partial_- a_R \qquad \tilde{Z}_1 = \partial_- \tilde{a}_R \qquad\qquad Z_4 = \partial_- c_R \qquad \tilde{Z}_4 = \partial_- \tilde{c}_R$$

$$Z_2 = \partial_- b_R - (a_L + \frac{a_R}{2}) \partial_- b_R - (b_L + \frac{b_R}{2}) \partial_- c_R + (\partial_- a_R)(b_L + \frac{b_R}{2}) + (\partial_- b_R)(c_L + \frac{c_R}{2})$$

$$\tilde{Z}_2 = \partial_- \tilde{b}_R - (\tilde{a}_L + \frac{\tilde{a}_R}{2}) \partial_- \tilde{b}_R - (\tilde{b}_L + \frac{\tilde{b}_R}{2}) \partial_- \tilde{c}_R + (\partial_- \tilde{a}_R)(\tilde{b}_L + \frac{\tilde{b}_R}{2}) + (\partial_- \tilde{b}_R)(\tilde{c}_L + \frac{\tilde{c}_R}{2})$$

$$Z_3 = -\partial_- b_R^t + (b_L^t + \frac{b_R^t}{2})\partial_- a_R + (c_L + \frac{c_R}{2})\partial_- b_R^t - (\partial_- b_R^t)(a_L + \frac{a_R}{2}) - (\partial_- c_R)(b_L^t + \frac{b_R^t}{2})$$

$$\tilde{Z}_3 = \partial_- \tilde{b}_R^t - (\tilde{b}_L^t + \frac{\tilde{b}_R^t}{2})\partial_- \tilde{a}_R - (\tilde{c}_L + \frac{\tilde{c}_R}{2})\partial_- \tilde{b}_R^t + (\partial_- \tilde{b}_R^t)(\tilde{a}_L + \frac{\tilde{a}_R}{2}) + (\partial_- \tilde{c}_R)(\tilde{b}_L^t + \frac{\tilde{b}_R^t}{2})$$

Obviously equations of motion are satisfied. Since we want to reduce constraints on the conservation laws and bring the nonlinear characters of conserved currents into the open we let $e = \sum_{n=1}^{\infty} \varepsilon^n e_n$, where e stands for the matrix components a, b and c. We may find solutions in the orders of ε's. But we need to find transformation matrix T first and foremost.

Trivial Case: T = I

Let us consider first a trivial solution where transformation matrix is identity. Pseudoduality equations will be

$$(\tilde{g_L}^{-1}\partial_+ \tilde{g_L})^i = (g_L^{-1}\partial_+ g_L)^i \tag{3.1.33}$$

$$(\tilde{g}^{-1}\partial_- \tilde{g})^i = -(g^{-1}\partial_- g)^i \tag{3.1.34}$$

Using (3.1.31) the first equation (3.1.33) leads to

$$\partial_+ \tilde{a}_{L1} = \partial_+ a_{L1} \qquad \partial_+ \tilde{a}_{L2} = \partial_+ a_{L2}$$

$$\partial_+ \tilde{c}_{L1} = \partial_+ c_{L1} \qquad \partial_+ \tilde{c}_{L2} = \partial_+ c_{L2}$$

$$\partial_+ \tilde{b}_{L1} = \partial_+ b_{L1} \qquad \partial_+ \tilde{b}_{L1}^t = -\partial_+ b_{L1}^t$$

$$\partial_+ \tilde{b}_{L2} = \partial_+ b_{L2} + \tfrac{1}{2}[A_{L1}(\partial_+ b_{L1}) + B_{L1}(\partial_+ c_{L1}) - (\partial_+ a_{L1})B_{L1} - (\partial_+ b_{L1})C_{L1}]$$

$$\partial_+ \tilde{b}_{L2}^t = -\partial_+ b_{L2}^t - \tfrac{1}{2}[B_{L1}^t(\partial_+ a_{L1}) + C_{L1}(\partial_+ b_{L1}^t) - (\partial_+ b_{L1}^t)A_{L1} - (\partial_+ c_{L1})B_{L1}^t]$$

where we used the solutions of first six equations in the last two lines as follows

$$\tilde{a}_{L1} = a_{L1} + A_{L1} \qquad \tilde{a}_{L2} = a_{L2} + A_{L2}$$
$$\tilde{c}_{L1} = c_{L1} + C_{L1} \qquad \tilde{c}_{L2} = c_{L2} + C_{L2}$$
$$\tilde{b}_{L1} = b_{L1} + B_{L1} \qquad \tilde{b}^t_{L1} = -b^t_{L1} - B^t_{L1}$$
$$\tilde{b_{L2}} = b_{L2} + \tfrac{1}{2}(A_{L1}b_{L1} + B_{L1}c_{L1} - a_{L1}B_{L1} - b_{L1}C_{L1}) + B_{L2}$$
$$\tilde{b}^t_{L2} = -b^t_{L2} - \tfrac{1}{2}(B^t_{L1}a_{L1} + C_{L1}(\partial_+ b^t_{L1}) - (\partial_+ b_{L1})A_{L1} - c_{L1}B^t_{L1}) - B^t_{L2}$$

where A_{L1}, A_{L2}, B_{L1}, B_{L2}, C_{L1} and C_{L2} are constants. Therefore pseudodual left current (3.1.31) up to the order of ε^2 in nonlocal expressions is

$$\tilde{g}_L^{-1}\partial_+\tilde{g}_L = \begin{pmatrix} \tilde{M}_1 & \tilde{M}_2 \\ \tilde{M}_3 & \tilde{M}_4 \end{pmatrix} + H.O \tag{3.1.35}$$

where we defined the following symbols for the entries of matrix

$$\tilde{M}_1 = \varepsilon\partial_+\tilde{a}_{L1} + \varepsilon^2\partial_+\tilde{a}_{L2} = \varepsilon\partial_+ a_{L1} + \varepsilon^2\partial_+ a_{L2}$$
$$\tilde{M}_4 = \varepsilon\partial_+\tilde{c}_{L1} + \varepsilon^2\partial_+\tilde{c}_{L2} = \varepsilon\partial_+ c_{L1} + \varepsilon^2\partial_+ c_{L2}$$

$$\begin{aligned}
\tilde{M}_2 &= \varepsilon\partial_+\tilde{b}_{L1} + \varepsilon^2[\partial_+\tilde{b}_{L2} - \frac{1}{2}(\tilde{a}_{L1}\partial_+\tilde{b}_{L1} + \tilde{b}_{L1}\partial_+\tilde{c}_{L1} - (\partial_+\tilde{a}_{L1})\tilde{b}_{L1} - (\partial_+\tilde{b}_{L1})\tilde{c}_{L1})] \\
&= \varepsilon\partial_+ b_{L1} + \varepsilon^2[\partial_+ b_{L2} - \frac{1}{2}[a_{L1}(\partial_+ b_{L1}) + b_{L1}(\partial_+ c_{L1}) - (\partial_+ a_{L1})b_{L1} - (\partial_+ b_{L1})c_{L1}]] \\
\tilde{M}_3 &= \varepsilon\partial_+\tilde{b}^t_{L1} + \varepsilon^2[\partial_+\tilde{b}^t_{L2} - \frac{1}{2}[\tilde{b}^t_{L1}(\partial_+\tilde{a}_{L1}) + \tilde{c}_{L1}(\partial_+\tilde{b}^t_{L1}) - (\partial_+\tilde{b}^t_{L1})\tilde{a}_{L1} - (\partial_+\tilde{c}_{L1})\tilde{b}^t_{L1}]] \\
&= -\varepsilon\partial_+ b^t_{L1} - \varepsilon^2[\partial_+ b^t_{L2} - \frac{1}{2}[b^t_{L1}(\partial_+ a_{L1}) + c_{L1}(\partial_+ b^t_{L1}) - (\partial_+ b^t_{L1})a_{L1} - (\partial_+ c_{L1})b^t_{L1}]]
\end{aligned}$$

Obviously this current is conserved. To find right current we use 2^{nd} pseudoduality equation (3.1.34) and we find the following expressions up to the order of ε^2

$$
\begin{gathered}
\partial_- \tilde{a}_{R1} = -\partial_- a_{R1} \qquad \partial_- \tilde{a}_{R2} = -\partial_- a_{R2} \\
\partial_- \tilde{c}_{R1} = -\partial_- c_{R1} \qquad \partial_- \tilde{c}_{R2} = -\partial_- c_{R2} \\
\partial_- \tilde{b}_{R1} = -\partial_- b_{R1} \qquad \partial_- \tilde{b}^t_{R1} = \partial_- b^t_{R1} \\
\partial_- \tilde{b_{R2}} = -\partial_- b_{R2} + (a_{R1} - A_{L1} + \tfrac{A_{R1}}{2})(\partial_- b_{R1}) + (b_{R1} - B_{L1} + \tfrac{B_{R1}}{2})(\partial_- c_{R1}) \\
-(\partial_- a_{R1})(b_{R1} - B_{L1} + \tfrac{B_{R1}}{2}) - (\partial_- b_{R1})(c_{R1} - C_{L1} + \tfrac{C_{R1}}{2}) \\
\partial_- \tilde{b}^t_{R2} = \partial_- b^t_{R2} - (-B^t_{L1} + b^t_{R1} + \tfrac{B^t_{R1}}{2})(\partial_- a_{R1}) - (-C_{L1} + c_{R1} + \tfrac{C_{R1}}{2})(\partial_- b^t_{R1}) \\
+(\partial_- b^t_{R1})(-A_{L1} + a_{R1} + \tfrac{A_{R1}}{2}) + (\partial_- c_{R1})(-B^t_{L1} + b^t_{R1} + \tfrac{B^t_{R1}}{2})
\end{gathered}
$$

where we used the solution of first six equations in the last two equations as

$$
\begin{gathered}
\tilde{a}_{R1} = -a_{R1} - A_{R1} \qquad \tilde{a}_{R2} = -a_{R2} - A_{R2} \\
\tilde{c}_{R1} = -c_{R1} - C_{R1} \qquad \tilde{c}_{R2} = -c_{R2} - C_{R2} \\
\tilde{b}_{R1} = -b_{R1} - B_{R1} \qquad \tilde{b}^t_{R1} = b^t_{R1} + B^t_{R1}
\end{gathered}
$$

where A_{R1}, A_{R2}, B_{R1}, C_{R1} and C_{R2} are constants. A brief computation yields the following expression for the right current

$$
(\partial_- \tilde{g}_R)\tilde{g}_R^{-1} = \begin{pmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_3 & \tilde{N}_4 \end{pmatrix} + H.O \tag{3.1.36}
$$

$$\tilde{N}_1 = \varepsilon\partial_-\tilde{a}_{R1} + \varepsilon^2\partial_-\tilde{a}_{R2} = -\varepsilon\partial_- a_{R1} - \varepsilon^2\partial_- a_{R2}$$

$$\tilde{N}_4 = \varepsilon\partial_-\tilde{c}_{R1} + \varepsilon^2\partial_-\tilde{c}_{R2} = -\varepsilon\partial_- c_{R1} - \varepsilon^2\partial_- c_{R2}$$

$$\begin{aligned}
\tilde{N}_2 &= \varepsilon\partial_-\tilde{b}_{R1} + \varepsilon^2[\partial_-\tilde{b}_{R2} + \tfrac{1}{2}(\tilde{a}_{R1}\partial_-\tilde{b}_{R1} + \tilde{b}_{R1}\partial_-\tilde{c}_{R1} - (\partial_-\tilde{a}_{R1})\tilde{b}_{R1} - (\partial_-\tilde{b}_{R1})\tilde{c}_{R1})] \\
&= -\varepsilon\partial_- b_{R1} + \varepsilon^2[-\partial_- b_{R2} + (\tfrac{3}{2}a_{R1} + A_{R1} - A_{L1})(\partial_- b_{R1}) + (\tfrac{3}{2}b_{R1} + B_{R1} - B_{L1})(\partial_- c_{R1}) \\
&\quad -(\partial_- a_{R1})(\tfrac{3}{2}b_{R1} + B_{R1} - B_{L1}) - (\partial_- b_{R1})(\tfrac{3}{2}c_{R1} + C_{R1} - C_{L1})]
\end{aligned}$$

$$\begin{aligned}
\tilde{N}_3 &= \varepsilon\partial_-\tilde{b}^t_{R1} + \varepsilon^2[\partial_-\tilde{b}^t_{R2} + \tfrac{1}{2}[\tilde{b}^t_{R1}(\partial_-\tilde{a}_{R1}) + \tilde{c}_{R1}(\partial_-\tilde{b}^t_{R1}) - (\partial_-\tilde{b}^t_{R1})\tilde{a}_{R1} - (\partial_-\tilde{c}_{R1})\tilde{b}^t_{R1}]] \\
&= \varepsilon\partial_- b^t_{R1} + \varepsilon^2[\partial_- b^t_{R2} - (\tfrac{3}{2}b^t_{R1} + B^t_{R1} - B^t_{L1})(\partial_- a_{R1}) - (\tfrac{3}{2}c_{R1} + C_{R1} - C_{L1})(\partial_- b^t_{R1}) \\
&\quad +(\partial_- b^t_{R1})(\tfrac{3}{2}a_{R1} + A_{R1} - A_{L1}) + (\partial_- c_{R1})(\tfrac{3}{2}b^t_{R1} + B^t_{R1} - B^t_{L1})]
\end{aligned}$$

We see that this current is also conserved.

Nontrivial Case: General T

In this case we use the general expression (3.1.19) of transformation matrix T. Pseudoduality equations are given by (3.1.4) and (3.1.5), and gave us the equations (3.1.22) and (3.1.23) which can be written as

$$\begin{aligned}
&\partial_+\tilde{a}_{L1} = \partial_+ a_{L1} \quad \partial_+\tilde{b}_{L1} = \partial_+ b_{L1} \quad \partial_+\tilde{b}^t_{L1} = -\partial_+ b^t_{L1} \quad \partial_+\tilde{c}_{L1} = \partial_+ c_{L1} \\
&\partial_-\tilde{a}_{R1} = -\partial_- a_{R1} \quad \partial_-\tilde{b}_{R1} = -\partial_- b_{R1} \quad \partial_-\tilde{b}^t_{R1} = \partial_- b^t_{R1} \quad \partial_-\tilde{c}_{R1} = -\partial_- c_{R1} \\
&\partial_+\tilde{a}_{L2} = \partial_+ a_{L2} \quad \partial_+\tilde{c}_{L2} = \partial_+ c_{L2} \quad \partial_-\tilde{a}_{R2} = -\partial_- a_{R2} \quad \partial_-\tilde{c}_{R2} = -\partial_- c_{R2} \\
&\partial_+\tilde{b}_{L2} = \partial_+ b_{L2} - \frac{1}{2}[A_{L1}(\partial_+ b_{L1}) + B_{L1}(\partial_+ c_{L1}) - (\partial_+ a_{L1})B_{L1} - (\partial_+ b_{L1})C_{L1}] \\
&\partial_+\tilde{b}^t_{L2} = -\partial_+ b^t_{L2} + \frac{1}{2}[B^t_{L1}(\partial_+ a_{L1}) + C_{L1}(\partial_+ b^t_{L1}) - (\partial_+ b^t_{L1})A_{L1} - (\partial_+ c_{L1})B^t_{L1}] \\
&\partial_-\tilde{b}_{R2} = -\partial_- b_{R2} + (a_{R1} + \frac{A_{R1}}{2})(\partial_- b_{R1}) + (b_{R1} + \frac{B_{R1}}{2})(\partial_- c_{R1}) \\
&\qquad - (\partial_- a_{R1})(b_{R1} + \frac{B_{R1}}{2}) - (\partial_- b_{R1})(c_{R1} + \frac{C_{R1}}{2}) \\
&\partial_-\tilde{b}^t_{R2} = \partial_- b^t_{R2} - (b^t_{R1} + \frac{B^t_{R1}}{2})(\partial_- a_{R1}) - (c_{R1} + \frac{C_{R1}}{2})(\partial_- b^t_{R1}) \\
&\qquad + (\partial_- b^t_{R1})(a_{R1} + \frac{A_{R1}}{2}) + (\partial_- c_{R1})(b^t_{R1} + \frac{B^t_{R1}}{2})
\end{aligned}$$

where we used the solutions of first three lines for the last four expressions. Solutions of these equations are

$$
\begin{aligned}
\tilde{a}_{L1} = a_{L1} + A_{L1} \quad \tilde{b}_{L1} = b_{L1} + B_{L1} \quad \tilde{b}^t_{L1} = -b^t_{L1} - B^t_{L1} \\
\tilde{c}_{L1} = c_{L1} + C_{L1} \quad \tilde{a}_{R1} = -a_{R1} - A_{R1} \quad \tilde{b}_{R1} = -b_{R1} - B_{R1} \\
\tilde{b}^t_{R1} = b^t_{R1} + B^t_{R1} \quad \tilde{c}_{R1} = -c_{R1} - C_{R1} \quad \tilde{a}_{L2} = a_{L2} + A_{L2} \\
\tilde{c}_{L2} = c_{L2} + C_{L2} \quad \tilde{a}_{R2} = -a_{R2} - A_{R2} \quad \tilde{c}_{R2} = -c_{R2} - C_{R2} \\
\tilde{b}_{L2} = b_{L2} + B_{L2} - \frac{1}{2}[A_{L1}b_{L1} + B_{L1}c_{L1} - a_{L1}B_{L1} - b_{L1}C_{L1}] \\
\tilde{b}^t_{L2} = -b^t_{L2} - B^t_{L2} + \frac{1}{2}[B^t_{L1}a_{L1} + C_{L1}b^t_{L1} - b^t_{L1}A_{L1} - c_{L1}B^t_{L1}]
\end{aligned}
$$

where A_{L1}, A_{R1}, B_{L1}, B_{R1}, C_{L1}, C_{R1}, and B_{L2} are constants. We did not find solutions of $\tilde{b}_{R2}$ and $\tilde{b}^t_{R2}$ because of their complicated forms and no need to use them. Hence pseudodual left current (3.1.24) will be

$$
\begin{aligned}
\tilde{J}^{(L)}_+ = \tilde{g}^{-1}\partial_+\tilde{g} &= \varepsilon\partial_+\tilde{y}_{L1} + \varepsilon^2\{\partial_+\tilde{y}_{L2} - \frac{1}{2}[\tilde{y}_{L1}, \partial_+\tilde{y}_{L1}]_{\tilde{G}}\} + H.O. \\
&= \begin{pmatrix} \tilde{M}_1 & \tilde{M}_2 \\ \tilde{M}_3 & \tilde{M}_4 \end{pmatrix} + H.O.
\end{aligned}
\tag{3.1.37}
$$

where

$$
\begin{aligned}
\tilde{M}_1 &= \varepsilon\partial_+\tilde{a}_{L1} + \varepsilon^2\partial_+\tilde{a}_{L2} = \varepsilon\partial_+ a_{L1} + \varepsilon^2\partial_+ a_{L2} \\
\tilde{M}_4 &= \varepsilon\partial_+\tilde{c}_{L1} + \varepsilon^2\partial_+\tilde{c}_{L2} = \varepsilon\partial_+ c_{L1} + \varepsilon^2\partial_+ c_{L2} \\
\tilde{M}_2 &= \varepsilon\partial_+\tilde{b}_{L1} + \varepsilon^2[\partial_+\tilde{b}_{L2} - \frac{1}{2}\{\tilde{a}_{L1}(\partial_+\tilde{b}_{L1}) + \tilde{b}_{L1}(\partial_+\tilde{c}_{L1}) - (\partial_+\tilde{a}_{L1})\tilde{b}_{L1} - (\partial_+\tilde{b}_{L1})\tilde{c}_{L1}\}] \\
&= \varepsilon\partial_+ b_{L1} + \varepsilon^2[\partial_+ b_{L2} - \frac{1}{2}\{a_{L1}(\partial_+ b_{L1}) + b_{L1}(\partial_+ c_{L1}) - (\partial_+ a_{L1})b_{L1} - (\partial_+ b_{L1})c_{L1}\}] \\
\tilde{M}_3 &= \varepsilon\partial_+\tilde{b}^t_{L1} + \varepsilon^2[\partial_+\tilde{b}^t_{L2} - \frac{1}{2}\{\tilde{b}^t_{L1}(\partial_+\tilde{a}_{L1}) + \tilde{c}_{L1}(\partial_+\tilde{b}^t_{L1}) - (\partial_+\tilde{b}^t_{L1})\tilde{a}_{L1} - (\partial_+\tilde{c}_{L1})\tilde{b}^t_{L1}\}] \\
&= -\varepsilon\partial_+ b^t_{L1} - \varepsilon^2[\partial_+ b^t_{L2} - (\frac{b^t_{L1}}{2} + B^t_{L1})(\partial_+ a_{L1}) - (\frac{c_{L1}}{2} + C_{L1})(\partial_+ b^t_{L1}) \\
&+ (\partial_+ b^t_{L1})(\frac{a_{L1}}{2} + A_{L1}) + (\partial_+ c_{L1})(\frac{b^t_{L1}}{2} + B^t_{L1})]
\end{aligned}
$$

Pseudodual right current (3.1.25) can be constructed as follows

$$
\begin{aligned}
\tilde{J}^{(R)}_- &= (\partial_-\tilde{g})\tilde{g}^{-1} = \varepsilon\partial_-\tilde{y}_{R1} + \varepsilon^2\{\partial_-\tilde{y}_{R2} + \frac{1}{2}[\tilde{y}_{R1}, \partial_-\tilde{y}_{R1}]_{\tilde{G}}\} + H.O. \\
&= \begin{pmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_3 & \tilde{N}_4 \end{pmatrix} + H.O. \qquad (3.1.38)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{N}_1 &= \varepsilon\partial_-\tilde{a}_{R1} + \varepsilon^2\partial_-\tilde{a}_{R2} = -\varepsilon\partial_- a_{R1} - \varepsilon^2\partial_- a_{R2} \\
\tilde{N}_4 &= \varepsilon\partial_-\tilde{c}_{R1} + \varepsilon^2\partial_-\tilde{c}_{R2} = -\varepsilon\partial_- c_{R1} - \varepsilon^2\partial_- c_{R2} \\
\tilde{N}_2 &= \varepsilon\partial_-\tilde{b}_{R1} + \varepsilon^2\{\partial_-\tilde{b}_{R2} + \frac{1}{2}[\tilde{a}_{R1}(\partial_-\tilde{b}_{R1}) + \tilde{b}_{R1}(\partial_-\tilde{c}_{R1}) - (\partial_-\tilde{a}_{R1})\tilde{b}_{R1} - (\partial_-\tilde{b}_{R1})\tilde{c}_{R1}]\} \\
&= -\varepsilon\partial_- b_{R1} - \varepsilon^2\{\partial_- b_{R2} - (\frac{3a_{R1}}{2} + A_{R1})(\partial_- b_{R1}) - (\frac{3b_{R1}}{2} + B_{R1})(\partial_- c_{R1}) \\
&+ (\partial_- a_{R1})(\frac{3b_{R1}}{2} + B_{R1}) + (\partial_- b_{R1})(\frac{3c_{R1}}{2} + C_{R1})\} \\
\tilde{N}_3 &= \varepsilon\partial_-\tilde{b}^t_{R1} + \varepsilon^2\{\partial_-\tilde{b}^t_{R2} + \frac{1}{2}[\tilde{b}^t_{R1}(\partial_-\tilde{a}_{R1}) + \tilde{c}_{R1}(\partial_-\tilde{b}^t_{R1}) - (\partial_-\tilde{b}^t_{R1})\tilde{a}_{R1} - (\partial_-\tilde{c}_{R1})\tilde{b}^t_{R1}]\} \\
&= \varepsilon\partial_- b^t_{R1} + \varepsilon^2\{\partial_- b^t_{R2} - (\frac{3b^t_{R1}}{2} + B^t_{R1})(\partial_- a_{R1}) - (\frac{3c_{R1}}{2} + C_{R1})(\partial_- b^t_{R1}) \\
&+ (\partial_- b^t_{R1})(\frac{3a_{R1}}{2} + A_{R1}) + (\partial_- c_{R1})(\frac{3b^t_{R1}}{2} + B^t_{R1})\}
\end{aligned}
$$

It is apparent that these currents are conserved.

3.2 Cartan Decomposition of Symmetric Spaces

We saw in the above example that symmetric spaces can be decomposed into two pieces, one piece remains invariant under transformation T though the other piece is transformed in such a way that it behaves like a new symmetric space. Let π be the projection $G \longrightarrow M$, sending each $g \in G$ to submersion M. We see that M is symmetric space after invariant parts of G are eliminated.

Let H be a closed subgroup of a connected Lie group G, and σ be an involutive automorphism of G such that $F_0 \subset H \subset F = Fix(\sigma)$. Symmetric space M is the coset space $M = G/H$. If **g** is the Lie algebra of G, **h** is the Lie algebra of H, and **m** is the Lie subspace (not the Lie algebra) of M, then $\mathbf{g} = \mathbf{m} \oplus \mathbf{h}$, where **h** is closed under brackets while **m** is $Ad(H)$-invariant subspace of **g**, i.e, $Ad_h(\mathbf{m}) \subset \mathbf{m}$ for all $h \in H$. If $X \in \mathbf{g}$, then

$X = X_h + X_m$, where $X_h \in$ **h**, and $X_m \in$ **m**. The involutive automorphism $d\sigma$ is such that $d\sigma(X_h) = X_h$ and $d\sigma(X_m) = -X_m$. Bracket relations for the symmetric space are defined by

$$[\mathbf{h}, \mathbf{h}] \subset \mathbf{h}, \quad [\mathbf{h}, \mathbf{m}] \subset \mathbf{m}, \quad [\mathbf{m}, \mathbf{m}] \subset \mathbf{h} \tag{3.2.1}$$

The currents $J_+^{(L)} = g^{-1}\partial_+ g$ and $J_-^{(R)} = (\partial_- g)g^{-1}$ on **g** can be split into the currents $J_m^{(L)} = g^{-1}D_+ g$ and $J_m^{(R)} = (D_- g)g^{-1}$ on **m** and $J_h^{(L)} = A_+$ and $J_h^{(R)} = gA_- g^{-1}$ on **h**, where $D_\pm$ is the covariant derivative acting on **m**, and $A_\pm$ is the gauge field defined on **h**.

If one defines indices $i, j, k, ...$ for the space elements of **g**, indices $a, b, c, ...$ for the space elements of **h**, and indices $\alpha, \beta, \gamma, ...$ for the space elements of **m**, then (3.2.1) allows only structure constants f^a_{bc}, $f^\alpha_{a\beta}$, $f^\alpha_{\beta a}$, and $f^a_{\alpha\beta}$. The other structure constants vanish. This leads to the following equations of motion,

$$k_+ = g^{-1}D_+ g \quad \Longrightarrow \quad D_- k_+ = 0 \tag{3.2.2}$$

$$k_- = g^{-1}D_- g \quad \Longrightarrow \quad D_+ k_- = [k_-, A_+] + [A_-, k_+] \tag{3.2.3}$$

$$A_+ = g^{-1}D'_+ g \quad \Longrightarrow \quad D'_- A_+ = 0 \tag{3.2.4}$$

$$A_- = g^{-1}D'_- g \quad \Longrightarrow \quad D'_+ A_- = [A_-, A_+] + [k_-, k_+] \tag{3.2.5}$$

where $k_\pm(A_\pm)$ belongs to **m**(**h**), and $D(D')$ is the covariant derivative acting on **m**(**h**).

It is natural to write down the Pseudoduality equations (3.1.4) and (3.1.5) in the most general split form on two spaces **m** and **h** as follows

$$\begin{aligned} \tilde{k}^\alpha_+ &= T^\alpha_\beta k^\beta_+ + T^\alpha_a A^a_+ \qquad & \tilde{A}^a_+ &= T^a_b A^b_+ + T^a_\alpha k^\alpha_+ \\ \tilde{k}^\alpha_- &= -T^\alpha_\beta k^\beta_- - T^\alpha_a A^a_- \qquad & \tilde{A}^a_- &= -T^a_b A^b_- - T^a_\alpha k^\alpha_- \end{aligned} \tag{3.2.6}$$

where

$$g^{-1}\partial_+ g = \begin{pmatrix} k_+ \\ A_+ \end{pmatrix} \begin{matrix} \text{on } \mathbf{m}-\text{space} \\ \text{on } \mathbf{h}-\text{space} \end{matrix} \tag{3.2.7}$$

$$g^{-1}\partial_- g = \begin{pmatrix} k_- \\ A_- \end{pmatrix} \begin{matrix} \text{on } \mathbf{m}-\text{space} \\ \text{on } \mathbf{h}-\text{space} \end{matrix} \tag{3.2.8}$$

and

$$T = \begin{pmatrix} T^{\alpha}_{\beta} & T^{\alpha}_{a} \\ T^{a}_{\beta} & T^{a}_{b} \end{pmatrix} \begin{matrix} \text{on } \mathbf{m}-\text{space} \\ \text{on } \mathbf{h}-\text{space} \end{matrix} \tag{3.2.9}$$

Apparently T^{α}_{a} and T^{a}_{β} represent the mixing components of the isometry preserving map T. Before considering this most general pseudoduality relations which lead to mixed expressions it is worth to analyze pseudoduality equations between pure symmetric spaces and their counter H-spaces without mixing parts.

3.2.1 Non-Mixing Pseudoduality

We set the mixing components T^{α}_{a} and T^{a}_{β} in equation (3.2.6) equal to zero, and consider the pseudoduality equations on **m** and **h**-spaces as follows

$$\tilde{k}^{\alpha}_{\pm} = \pm T^{\alpha}_{\beta} k^{\beta}_{\pm} \tag{3.2.10}$$

$$\tilde{A}^{a}_{\pm} = \pm T^{a}_{b} A^{b}_{\pm} \tag{3.2.11}$$

When we take D_- of (3.2.10), and D'_- of (3.2.11) ('+' equations only) followed by the equations of motion (3.2.2) and (3.2.4) we obtain the result that both T^{α}_{β} and T^{a}_{b} depend only on σ^+. Now let us take D_+ of '$-$' equation in (3.2.10), and use (3.2.3) to get

$$[\tilde{k}_-, \tilde{A}_+]^\alpha + [\tilde{A}_-, \tilde{k}_+]^\alpha = -(D_+ T^\alpha_\beta) k^\beta_- - T^\alpha_\beta [k_-, A_+]^\beta - T^\alpha_\beta [A_-, k_+]^\beta \tag{3.2.12}$$

Since k_- and A_- can be treated independently, this equation can be split into the following equations

$$\tilde{f}^\alpha_{\beta a} \tilde{k}^\beta_+ T^a_c = T^\alpha_\beta f^\beta_{\lambda c} k^\lambda_+ \tag{3.2.13}$$

$$\tilde{f}^\alpha_{a\beta} \tilde{A}^a_+ T^\beta_\lambda = -D_+ T^\alpha_\lambda + T^\alpha_\beta f^\beta_{a\lambda} A^a_+ \tag{3.2.14}$$

First equation (3.2.13) gives us a relation between structure constants, $\tilde{f}^\alpha_{\beta a} T^\beta_\lambda T^a_c = T^\alpha_\beta f^\beta_{\lambda c}$, which leads second equation to yield $D_+ T^\alpha_\lambda = 0$. Therefore we conclude that T^α_β has to be a constant, and we choose it to be identity. Similarly we take $D^{'}_+$ of '$-$' equation in (3.2.11), and use (3.2.5) to get

$$[\tilde{A}_-, \tilde{A}_+]^a + [\tilde{k}_-, \tilde{k}_+]^a = -(D^{'}_+ T^a_b) A^b_- - T^a_b [A_-, A_+]^b - T^a_b [k_-, k_+]^b \tag{3.2.15}$$

This equation yields the following results

$$\tilde{f}^a_{\alpha\beta} \tilde{k}^\alpha_+ T^\beta_\lambda = T^a_b f^b_{\beta\lambda} k^\beta_+ \tag{3.2.16}$$

$$\tilde{f}^a_{bc} \tilde{A}^b_+ T^c_d = -D^{'}_+ T^a_d + T^a_b f^b_{cd} A^c_+ \tag{3.2.17}$$

First equation (3.2.16) verifies the result above up to the permutation of indices, $\tilde{f}^a_{\alpha\beta} T^\alpha_\nu T^\beta_\lambda = T^a_b f^b_{\nu\lambda}$. Second equation (3.2.17) produces the following solution

$$T^a_b = T^a_b(0) + (f^a_{cb} - \tilde{f}^a_{cb}) \int_0^{\sigma^+} A^c_+ D^{'} {\sigma^{'}}^+ + H.O. \tag{3.2.18}$$

where we choose $T^a_b(0)$ to be identity. It is easy to see that these equations yield the following bracket relations

$$[\tilde{k}_+, \tilde{A}_-]^\alpha = -T^\alpha_\beta [k_+, A_-]^\beta \tag{3.2.19}$$

$$[\tilde{k}_-, \tilde{A}_+]^\alpha = -T^\alpha_\beta [k_-, A_+]^\beta \tag{3.2.20}$$

$$[\tilde{k}_+, \tilde{k}_-]^a = -T^a_b [k_+, k_-]^b \tag{3.2.21}$$

$$[\tilde{A}_+, \tilde{A}_-]^a = -T^a_b [A_+, A_-]^b + (D^{'}_+ T^a_b) A^b_- \tag{3.2.22}$$

that verifies the equations of motion on pseudodual space as pointed out above, $D_+\tilde{k}^\alpha_- = -T^\alpha_\beta D_+ k^\beta_-$ and $D_+\tilde{A}^a_- = -T^a_b D^{'}_+ A^b_- - (D^{'}_+ T^a_b) A^b_-$. We notice that if H and $\tilde{H}$ are the same for both manifolds, i.e., $f^a_{bc} = \tilde{f}^a_{bc}$, then T^a_b reduces to identity, and we recover the flat space pseudoduality relations on two manifolds. One can easily construct nonlocal field expressions using above solutions, which are

$$\tilde{k}_\pm = \pm k_\pm \tag{3.2.23}$$

$$\tilde{A}_\pm = \pm A_\pm \pm \int_0^{\sigma^+} ([A_+(\sigma^{'+}), A_\pm(\sigma^+)]_H - [A_+(\sigma^{'+}), A_\pm(\sigma^+)]_{\tilde{H}}) D^{'} \sigma^{'+} + H.O. \tag{3.2.24}$$

One may readily construct nonlocal expressions of the conserved pseudodual currents by means of these fields and following the method in section 2 (2).

3.2.2 Mixing Pseudoduality

We now consider mixing of **m** and **h**-spaces in pseudodual expressions. Pseudoduality equations can be written as in (3.2.6). We take ∂_- of first equation on **m**-space (3.2.6), and obtain

$$(\partial_- T^\alpha_\beta) k^\beta_+ + (\partial_- T^\alpha_a) A^a_+ = 0 \tag{3.2.25}$$

since **m** and **h**-spaces are independent, we get $\partial_- T^{\alpha}_{\beta} = \partial_- T^{\alpha}_{a} = 0$, so T^{α}_{β} and T^{α}_{a} don't depend on σ^-. Now we take ∂_+ of second equation on **m**-space (3.2.6) and see that

$$\begin{aligned}[\tilde{k}_-, \tilde{A}_+]^\alpha + [\tilde{A}_-, \tilde{k}_+]^\alpha =& -(\partial_+ T^{\alpha}_{\beta}) k^{\beta}_- - T^{\alpha}_{\beta}[k_-, A_+]^\beta - T^{\alpha}_{\beta}[A_-, k_+]^\beta \\ & - (\partial_+ T^{\alpha}_{a}) A^{a}_- - T^{\alpha}_{a}[A_-, A_+]^a - T^{\alpha}_{a}[k_-, k_+]^a \end{aligned} \tag{3.2.26}$$

We substitute the expressions for $\tilde{k}_-$ and $\tilde{A}_-$ into this equation, and compare the coefficients of k_- and A_- to get the following expressions

$$\partial_+ T^{\alpha}_{\lambda} = [f^{\beta}_{b\lambda} T^{\alpha}_{\beta} - \tilde{f}^{\alpha}_{a\beta}(T^{a}_{b} T^{\beta}_{\lambda} - T^{\beta}_{b} T^{a}_{\lambda})] A^{b}_+ + [f^{a}_{\beta\lambda} T^{\alpha}_{a} - \tilde{f}^{\alpha}_{a\nu}(T^{a}_{\beta} T^{\nu}_{\lambda} - T^{\nu}_{\beta} T^{a}_{\lambda})] k^{\beta}_+ \tag{3.2.27}$$

$$\partial_+ T^{\alpha}_{b} = [f^{\nu}_{\beta b} T^{\alpha}_{\nu} - \tilde{f}^{\alpha}_{a\nu}(T^{a}_{\beta} T^{\nu}_{b} - T^{\nu}_{\beta} T^{a}_{b})] k^{\beta}_+ + [f^{a}_{cb} T^{\alpha}_{a} - \tilde{f}^{\alpha}_{\beta a}(T^{\beta}_{c} T^{a}_{b} - T^{a}_{c} T^{\beta}_{b})] A^{c}_+ \tag{3.2.28}$$

Since we only need to find currents up to the second order terms, it suffices to find mapping tensors using only initial values

$$\begin{aligned} T^{\alpha}_{\lambda}(\sigma^+) =& T^{\alpha}_{\lambda}(0) + (f^{\alpha}_{b\lambda} - \tilde{f}^{\alpha}_{b\lambda} + \tilde{f}^{\alpha}_{a\beta} T^{\beta}_{b}(0) T^{a}_{\lambda}(0)) \int_0^{\sigma^+} A^{b}_+ D' \sigma'^+ \\ & + (f^{a}_{\beta\lambda} T^{\alpha}_{a}(0) - \tilde{f}^{\alpha}_{a\lambda} T^{a}_{\beta}(0) + \tilde{f}^{\alpha}_{a\beta} T^{a}_{\lambda}(0)) \int_0^{\sigma^+} k^{\beta}_+ D\sigma'^+ + H.O. \end{aligned} \tag{3.2.29}$$

$$\begin{aligned} T^{\alpha}_{b}(\sigma^+) =& T^{\alpha}_{b}(0) + (f^{\alpha}_{\beta b} + \tilde{f}^{\alpha}_{b\beta} - \tilde{f}^{\alpha}_{a\nu} T^{a}_{\beta}(0) T^{\nu}_{b}(0)) \int_0^{\sigma^+} k^{\beta}_+ D\sigma'^+ \\ & + (f^{a}_{cb} T^{\alpha}_{a}(0) - \tilde{f}^{\alpha}_{\beta b} T^{\beta}_{c}(0) + \tilde{f}^{\alpha}_{\beta c} T^{\beta}_{b}(0)) \int_0^{\sigma^+} A^{c}_+ D' \sigma'^+ + H.O. \end{aligned} \tag{3.2.30}$$

where all initial values are chosen to be identity. Therefore pseudodual nonlocal currents on $\tilde{\mathbf{m}}$ can be written as

$$\begin{aligned}\tilde{k}^{\alpha}_{+} =&k^{\alpha}_{+} + T^{\alpha}_{b}(0)A^{b}_{+} + (f^{a}_{\beta\lambda}T^{\alpha}_{a}(0) - \tilde{f}^{\alpha}_{a\lambda}T^{a}_{\beta}(0) + \tilde{f}^{\alpha}_{a\beta}T^{a}_{\lambda}(0))k^{\lambda}_{+}\int_{0}^{\sigma^{+}} k^{\beta}_{+}D\sigma'^{+}\\ &+ (f^{\alpha}_{b\beta} - \tilde{f}^{\alpha}_{b\beta} + \tilde{f}^{\alpha}_{a\nu}T^{\nu}_{b}(0)T^{a}_{\beta}(0))\int_{0}^{\sigma^{+}}(A^{b}_{+}(\sigma'^{+})k^{\beta}_{+}(\sigma^{+}) - k^{\beta}_{+}(\sigma'^{+})A^{b}_{+}(\sigma^{+}))d\sigma'^{+}\\ &+ (f^{a}_{cb}T^{\alpha}_{a}(0) - \tilde{f}^{\alpha}_{\beta b}T^{\beta}_{c}(0) + \tilde{f}^{\alpha}_{\beta c}T^{\beta}_{b}(0))A^{b}_{+}\int_{0}^{\sigma^{+}} A^{c}_{+}D'\sigma'^{+} + H.O. \end{aligned} \tag{3.2.31}$$

$$\begin{aligned}\tilde{k}^{\alpha}_{-} =& - k^{\alpha}_{-} - T^{\alpha}_{b}(0)A^{b}_{-} - (f^{a}_{\beta\lambda}T^{\alpha}_{a} - \tilde{f}^{\alpha}_{a\lambda}T^{a}_{\beta}(0) + \tilde{f}^{\alpha}_{a\beta}T^{a}_{\lambda}(0))k^{\lambda}_{-}\int_{0}^{\sigma^{+}} k^{\beta}_{+}D\sigma'^{+}\\ &+ (f^{\alpha}_{\beta b} + \tilde{f}^{\alpha}_{b\beta} - \tilde{f}^{\alpha}_{a\nu}T^{\nu}_{b}(0)T^{a}_{\beta}(0))\int_{0}^{\sigma^{+}}(A^{b}_{+}(\sigma'^{+})k^{\beta}_{-}(\sigma^{+}) - k^{\beta}_{+}(\sigma'^{+})A^{b}_{-}(\sigma^{+}))d\sigma'^{+}\\ &- (f^{a}_{cb}T^{\alpha}_{a}(0) - \tilde{f}^{\alpha}_{\beta b}T^{\beta}_{c}(0) + \tilde{f}^{\alpha}_{\beta c}T^{\beta}_{b}(0))A^{b}_{-}\int_{0}^{\sigma^{+}} A^{c}_{+}D'\sigma'^{+} + H.O. \end{aligned} \tag{3.2.32}$$

Conservation laws of these currents up to the second order terms are obvious. Now we consider pseudoduality equations on **h**-space (3.2.6). We take ∂_{-} of first equation, and we obtain

$$(\partial_{-}T^{a}_{b})A^{b}_{+} + (\partial_{-}T^{a}_{\alpha})k^{\alpha}_{+} = 0 \tag{3.2.33}$$

Hence we get $\partial_{-}T^{a}_{b} = \partial_{-}T^{a}_{\alpha} = 0$, which implies that T^{a}_{b} and T^{a}_{α} don't depend on σ^{-}. Taking ∂_{+} of second equation we get the following equation

$$\begin{aligned}[\tilde{A}_{-}, \tilde{A}_{+}]^{a} + [\tilde{k}_{-}, \tilde{k}_{+}]^{a} =& - (\partial_{+}T^{a}_{b})A^{b}_{-} - T^{a}_{b}[A_{-}, A_{+}]^{b} - T^{a}_{b}[k_{-}, k_{+}]^{b}\\ &- (\partial_{+}T^{a}_{\alpha})k^{\alpha}_{-} - T^{a}_{\alpha}[k_{-}, A_{+}]^{\alpha} - T^{a}_{\alpha}[A_{-}, k_{+}]^{\alpha} \end{aligned} \tag{3.2.34}$$

We replace $\tilde{A}_-$ and $\tilde{k}_-$ in this equation to obtain the following results

$$\partial_+ T^a_d = (T^a_b f^b_{ed} - \tilde{f}^a_{bc} T^b_e T^c_d - \tilde{f}^a_{\alpha\beta} T^\alpha_e T^\beta_d) A^e_+ + (T^a_\alpha f^\alpha_{\lambda d} - \tilde{f}^a_{bc} T^b_\lambda T^c_d - \tilde{f}^a_{\alpha\beta} T^\alpha_\lambda T^\beta_d) k^\lambda_+ \tag{3.2.35}$$

$$\partial_+ T^a_\nu = (T^a_b f^b_{\lambda\nu} - \tilde{f}^a_{bc} T^b_\lambda T^c_\nu - \tilde{f}^a_{\alpha\beta} T^\alpha_\lambda T^\beta_\nu) k^\lambda_+ + (T^a_\alpha f^\alpha_{d\nu} - \tilde{f}^a_{bc} T^b_d T^c_\nu - \tilde{f}^a_{\alpha\beta} T^\alpha_d T^\beta_\nu) A^d_+ \tag{3.2.36}$$

We again want to find solutions up to the second order terms, so we only use initial values to get

$$\begin{aligned} T^a_d(\sigma^+) =& T^a_d(0) + (f^a_{ed} - \tilde{f}^a_{ed} - \tilde{f}^a_{\alpha\beta} T^\alpha_e(0) T^\beta_d(0)) \int_0^{\sigma^+} A^e_+ D' \sigma'^+ \\ &+ (T^a_\alpha(0) f^\alpha_{\lambda d} - \tilde{f}^a_{bd} T^b_\lambda(0) - \tilde{f}^a_{\lambda\beta} T^\beta_d(0)) \int_0^{\sigma^+} k^\lambda_+ D\sigma'^+ + H.O. \end{aligned} \tag{3.2.37}$$

$$\begin{aligned} T^a_\nu(\sigma^+) =& T^a_\nu(0) + (f^a_{\lambda\nu} - \tilde{f}^a_{\lambda\nu} - \tilde{f}^a_{bc} T^b_\lambda(0) T^c_\nu(0)) \int_0^{\sigma^+} k^\lambda_+ D\sigma'^+ \\ &+ (T^a_\alpha(0) f^\alpha_{d\nu} - \tilde{f}^a_{dc} T^c_\nu(0) - \tilde{f}^a_{\alpha\nu} T^\alpha_d(0)) \int_0^{\sigma^+} A^d_+ D' \sigma'^+ + H.O. \end{aligned} \tag{3.2.38}$$

Thus pseudodual fields up to the second order terms on H space will be

$$\begin{aligned} \tilde{A}^a_+ =& A^a_+ + T^a_\lambda(0) k^\lambda_+ + (f^a_{ed} - \tilde{f}^a_{ed} - \tilde{f}^a_{\alpha\beta} T^\alpha_e(0) T^\beta_d(0)) A^d_+ \int_0^{\sigma^+} A^e_+ D' \sigma'^+ \\ &+ (T^a_\alpha(0) f^\alpha_{\lambda d} - \tilde{f}^a_{bd} T^b_\lambda(0) - \tilde{f}^a_{\lambda\beta} T^\beta_d(0)) \int_0^{\sigma^+} (k^\lambda_+(\sigma'^+) A^d_+(\sigma^+) - A^d_+(\sigma'^+) k^\lambda_+(\sigma^+)) d\sigma'^+ \\ &+ (f^a_{\lambda\nu} - \tilde{f}^a_{\lambda\nu} - \tilde{f}^a_{bc} T^b_\lambda(0) T^c_\nu(0)) k^\nu_+ \int_0^{\sigma^+} k^\lambda_+ D\sigma'^+ + H.O. \end{aligned} \tag{3.2.39}$$

$$\begin{aligned}\tilde{A}^a_- = &-A^a_- - T^a_\lambda(0)k^\lambda_- - (f^a_{ed} - \tilde{f}^a_{ed} - \tilde{f}^a_{\alpha\beta}T^\alpha_e(0)T^\beta_d(0))A^d_- \int_0^{\sigma^+} A^e_+ D'\sigma'^+ \\ &- (T^a_\alpha(0)f^\alpha_{\lambda d} - \tilde{f}^a_{bd}T^b_\lambda(0) - \tilde{f}^a_{\lambda\beta}T^\beta_d(0)) \int_0^{\sigma^+} (k^\lambda_+(\sigma'^+)A^d_-(\sigma^+) - A^d_+(\sigma'^+)k^\lambda_-(\sigma^+))d\sigma'^+ \\ &- (f^a_{\lambda\nu} - \tilde{f}^a_{\lambda\nu} - \tilde{f}^a_{bc}T^b_\lambda(0)T^c_\nu(0))k^\nu_- \int_0^{\sigma^+} k^\lambda_+ D\sigma'^+ + H.O. \end{aligned} \tag{3.2.40}$$

It is obvious that conservation laws (3.2.4) and (3.2.5) up to the second order terms are satisfied

$$\tilde{D}'_- \tilde{A}^a_+ = 0 \tag{3.2.41}$$

$$\begin{aligned}\tilde{D}'_+ \tilde{A}^a_- = &- [A_-, A_+]^a_{\tilde{G}} - [k_-, k_+]^a_{\tilde{G}} - [T(0)A_-, T(0)A_+]^a_{\tilde{G}} - [A_-, T(0)k_+]^a_{\tilde{G}} \\ &- [T(0)k_-, A_+]^a_{\tilde{G}} - [T(0)A_-, k_+]^a_{\tilde{G}} - [k_-, T(0)A_+]^a_{\tilde{G}} \\ &- [T(0)k_-, T(0)k_+]^a_{\tilde{G}} + H.O. \end{aligned} \tag{3.2.42}$$

3.2.3 Dual Symmetric Spaces and Further Constraints

It is well-known [2, 19] that two normal symmetric spaces are dual symmetric spaces if there exist

1. a Lie algebra isomorphism $S : \mathbf{h} \longrightarrow \tilde{\mathbf{h}}$ such that $\tilde{Q}(SV, SW) = -Q(V, W)$ for all $V, W \in \mathbf{h}$, and Q is inner product.

2. a linear isometry $T : \mathbf{m} \longrightarrow \tilde{\mathbf{m}}$ such that $[TX, TY] = -S[X, Y]$ for all $X, Y \in \mathbf{m}$.

Item (1) tells us that brackets in $\mathbf{h}$ and $\tilde{\mathbf{h}}$ are the same while item (2) tells us that inner products in $\mathbf{m}$ and $\tilde{\mathbf{m}}$ are the same. Item (1) yields the result $f^a_{cb} = \tilde{f}^a_{cb}$ for non-mixing pseudoduality, which leads T^a_b to be a constant. Hence pseudoduality transformations will simply be

$$\tilde{k}^{\alpha}_{\pm} = \pm k^{\alpha}_{\pm} \tag{3.2.43}$$

$$\tilde{A}^{a}_{\pm} = \pm A^{a}_{\pm} \tag{3.2.44}$$

with the bracket relations (3.2.19)-(3.2.22) given by

$$[\tilde{k}_{+}, \tilde{A}_{-}]^{\alpha} = -[k_{+}, A_{-}]^{\alpha} \tag{3.2.45}$$

$$[\tilde{k}_{-}, \tilde{A}_{+}]^{\alpha} = -[k_{-}, A_{+}]^{\alpha} \tag{3.2.46}$$

$$[\tilde{k}_{+}, \tilde{k}_{-}]^{a} = -[k_{+}, k_{-}]^{a} \tag{3.2.47}$$

$$[\tilde{A}_{+}, \tilde{A}_{-}]^{a} = -[A_{+}, A_{-}]^{a} \tag{3.2.48}$$

On the other hand one can write the following bracket relations between pseudodual target spaces for the mixing pseudoduality case

$$\begin{aligned} [\tilde{k}_{-}, \tilde{A}_{+}]^{\alpha} + [\tilde{A}_{-}, \tilde{k}_{+}]^{\alpha} = &- T^{\alpha}_{\beta}[k_{-}, A_{+}]^{\beta} - T^{\alpha}_{\beta}[A_{-}, k_{+}]^{\beta} \\ &- T^{\alpha}_{a}[A_{-}, A_{+}]^{a} - T^{\alpha}_{a}[k_{-}, k_{+}]^{a} \end{aligned} \tag{3.2.49}$$

$$\begin{aligned} [\tilde{A}_{-}, \tilde{A}_{+}]^{a} + [\tilde{k}_{-}, \tilde{k}_{+}]^{a} = &- T^{a}_{b}[A_{-}, A_{+}]^{b} - T^{a}_{b}[k_{-}, k_{+}]^{b} \\ &- T^{a}_{\alpha}[k_{-}, A_{+}]^{\alpha} - T^{a}_{\alpha}[A_{-}, k_{+}]^{\alpha} \end{aligned} \tag{3.2.50}$$

which in turn leads to relations of connection two-forms between symmetric and corresponding H-spaces, which is consistent with the result found in section 5 (5). These equations produce that all components of the pseudoduality map T must be constant, and we choose them to be identity. Hence pseudoduality equations will simply be

$$\tilde{k}^{\alpha}_{\pm} = \pm k^{\alpha}_{\pm} \pm T^{\alpha}_{a}(0) A^{a}_{\pm} \tag{3.2.51}$$

$$\tilde{A}^{a}_{\pm} = \pm A^{a}_{\pm} \pm T^{a}_{\alpha}(0) k^{\alpha}_{\pm} \tag{3.2.52}$$

3.2.4 An Example

We consider the Lie groups we used in the previous section. We saw that invariant subspace of $SO(n+1)$ is $1 \times SO(n)$. We pick H space as $SO(n)$. Hence our symmetric space is $M = \frac{SO(n+1)}{SO(n)}$. The Lie algebra **g** $= so(n+1)$ can be written as

$$so(n+1) = \begin{pmatrix} a & b \\ -b^t & c \end{pmatrix} \qquad \begin{array}{l} a = 1 \times 1 \\ b = 1 \times n \\ c = n \times n \end{array} \tag{3.2.53}$$

which can be split as

$$\begin{pmatrix} a & b \\ -b^t & c \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b^t & 0 \end{pmatrix} \qquad \mathbf{g} = \mathbf{h} \oplus \mathbf{m} \tag{3.2.54}$$

Let $Y \in$ **g**, $X \in$ **h**, and $Z \in$ **m**. Then, $D^{'}Z = 0$ and $DX = 0$. Using the expansions

(3.1.14) and (3.1.15), we may write the following expressions

$$k_+^\alpha = D_+ Z_L^\alpha - \frac{1}{2}[X_L, D_+ Z_L]^\alpha - \frac{1}{2}[Z_L, D_+^{'} X_L]^\alpha + H.O. \tag{3.2.55}$$

$$A_+^a = D_+^{'} X_L^a - \frac{1}{2}[X_L, D_+^{'} X_L]^a - \frac{1}{2}[Z_L, D_+ Z_L]^a + H.O. \tag{3.2.56}$$

$$k_-^\alpha = D_- Z_R^\alpha - [X_L, D_- Z_R]^\alpha - [Z_L, D_-^{'} X_R]^\alpha - \frac{1}{2}[X_R, D_- Z_R]^\alpha - \frac{1}{2}[Z_R, D_-^{'} X_R]^\alpha + H.O. \tag{3.2.57}$$

$$A_-^a = D_-^{'} X_R^a - [X_L, D_-^{'} X_R]^a - [Z_L, D_- Z_R]^a - \frac{1}{2}[X_R, D_-^{'} X_R]^a - \frac{1}{2}[Z_R, D_- Z_R]^a + H.O. \tag{3.2.58}$$

We describe solutions $X = \sum_{n=1}^{\infty} \varepsilon^n x_n$ and $Z = \sum_{n=1}^{\infty} \varepsilon^n z_n$, where ε is a small parameter. It is clear that equations of motion (3.2.2)-(3.2.5) for all orders of ε are satisfied. In the following calculations we are going to use expressions up to the order of ε^2 for simplicity.

Now we consider dual symmetric space $\tilde{M} = \frac{SO(n,1)}{SO(n)}$, where $\tilde{H} = SO(n)$. Lie algebra $\tilde{\mathbf{g}} = so(n,1)$ is written as

$$so(n,1) = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b}^t & \tilde{c} \end{pmatrix} \qquad \begin{array}{l} \tilde{a} = 1 \times 1 \\ \tilde{b} = 1 \times n \\ \tilde{c} = n \times n \end{array} \tag{3.2.59}$$

which is split as

$$\begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{b}^t & \tilde{c} \end{pmatrix} = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{c} \end{pmatrix} + \begin{pmatrix} 0 & \tilde{b} \\ \tilde{b}^t & 0 \end{pmatrix} \qquad \tilde{\mathbf{g}} = \tilde{\mathbf{h}} \oplus \tilde{\mathbf{m}} \tag{3.2.60}$$

Let $\tilde{Y} = \tilde{X} + \tilde{Z}$, where $\tilde{Y} \in \tilde{\mathbf{g}}$, $\tilde{X} \in \tilde{\mathbf{h}}$, and $\tilde{Z} \in \tilde{\mathbf{m}}$. We get the same fields as equations (3.2.55)-(3.2.58) with tilde. Equations of motion will be the same with tilde. We may now find pseudodual fields using our expressions found above. We note that because of the special form of our Lie groups, mixing components of the map T vanishes, and we simply get non-mixing pseudoduality condition.

We insert our expressions into equations (3.2.43) and (3.2.44) to get infinitely many pseudoduality relations. Up to the order of ε^2 terms equation (3.2.43) will be

$$\tilde{D}_+\tilde{z}^{\alpha}_{L1} = D_+ z^{\alpha}_{L1} \qquad \tilde{D}_-\tilde{z}^{\alpha}_{R1} = -D_- z^{\alpha}_{R1} \tag{3.2.61}$$

$$\tilde{D}_+\tilde{z}^{\alpha}_{L2} - \frac{1}{2}[\tilde{x}_{L1}, \tilde{D}_+\tilde{z}_{L1}]^{\alpha} - \frac{1}{2}[\tilde{z}_{L1}, \tilde{D}'_+\tilde{x}_{L1}]^{\alpha} = D_+ z^{\alpha}_{L2} - \frac{1}{2}[x_{L1}, D_+ z_{L1}]^{\alpha} - \frac{1}{2}[z_{L1}, D'_+ x_{L1}]^{\alpha}$$

$$\begin{aligned}
&\tilde{D}_-\tilde{z}^{\alpha}_{R2} - [\tilde{x}_{L1}, \tilde{D}_-\tilde{z}_{R1}]^{\alpha} - [\tilde{z}_{L1}, \tilde{D}'_-\tilde{x}_{R1}]^{\alpha} - \frac{1}{2}[\tilde{x}_{R1}, \tilde{D}_-\tilde{z}_{R1}]^{\alpha} - \frac{1}{2}[\tilde{z}_{R1}, \tilde{D}'_-\tilde{x}_{R1}]^{\alpha} = \\
&- D_- z^{\alpha}_{R2} + [x_{L1}, D_- z_{R1}]^{\alpha} + [z_{L1}, D'_- x_{R1}]^{\alpha} + \frac{1}{2}[x_{R1}, D_- z_{R1}]^{\alpha} + \frac{1}{2}[z_{R1}, D'_- x_{R1}]^{\alpha}
\end{aligned}$$

and equation (3.2.44) will be

$$\tilde{D}'_+\tilde{x}^{a}_{L1} = D'_+ x^{a}_{L1} \qquad \tilde{D}'_-\tilde{x}^{a}_{R1} = -D'_- x^{a}_{R1} \tag{3.2.62}$$

$$\tilde{D}'_+\tilde{x}^{a}_{L2} - \frac{1}{2}[\tilde{x}_{L1}, \tilde{D}'_+\tilde{x}_{L1}]^{a} - \frac{1}{2}[\tilde{z}_{L1}, \tilde{D}_+\tilde{z}_{L1}]^{a} = D'_+ x^{a}_{L2} - \frac{1}{2}[x_{L1}, D'_+ x_{L1}]^{a} - \frac{1}{2}[z_{L1}, D_+ z_{L1}]^{a}$$

$$\begin{aligned}
&\tilde{D}'_-\tilde{x}^{a}_{R2} - [\tilde{x}_{L1}, \tilde{D}'_-\tilde{x}_{R1}]^{a} - [\tilde{z}_{L1}, \tilde{D}_-\tilde{z}_{R1}]^{a} - \frac{1}{2}[\tilde{x}_{R1}, \tilde{D}'_-\tilde{x}_{R1}]^{a} - \frac{1}{2}[\tilde{z}_{R1}, \tilde{D}_-\tilde{z}_{R1}]^{a} = \\
&- D'_- x^{a}_{R2} + [x_{L1}, D'_- x_{R1}]^{a} + [z_{L1}, D_- z_{R1}]^{a} + \frac{1}{2}[x_{R1}, D'_- x_{R1}]^{a} + \frac{1}{2}[z_{R1}, D_- z_{R1}]^{a}
\end{aligned}$$

Since we know

$$D_{\pm}z_n = \begin{pmatrix} 0 & D_{\pm}b_n \\ -D_{\pm}b_n^t & 0 \end{pmatrix} \qquad D'_{\pm}x_n = \begin{pmatrix} D'_{\pm}a_n & 0 \\ 0 & D'_{\pm}c_n \end{pmatrix}$$

$$[x_1, D'_{\pm}x_1] = \begin{pmatrix} [a_1, D'_{\pm}a_1] & 0 \\ 0 & [c_1, D'_{\pm}c_1] \end{pmatrix}$$

$$[z_1, D_{\pm}z_1] = \begin{pmatrix} (D_{\pm}b_1)b_1^t - b_1(D_{\pm}b_1^t) & 0 \\ 0 & (D_{\pm}b_1^t)b_1 - b_1^t(D_{\pm}b_1) \end{pmatrix}$$

$$[x_1, D_{\pm}z_1] = \begin{pmatrix} 0 & a_1D_{\pm}b_1 - (D_{\pm}b_1)c_1 \\ -c_1(D_{\pm}b_1^t) + (D_{\pm}b_1^t)a_1 & 0 \end{pmatrix}$$

$$[z_1, D'_{\pm}x_1] = \begin{pmatrix} 0 & b_1D'_{\pm}c_1 - (D'_{\pm}a_1)b_1 \\ -b_1^t(D'_{\pm}a_1) + (D'_{\pm}c_1)b_1^t & 0 \end{pmatrix}$$

One can write similar expressions on the pseudodual space replacing each term with tilded terms. Only exception is that we switch b_n^t with $-\tilde{b}_n^t$ so that we get the convenient lie algebra on tilded space. Therefore pseudoduality equations above (3.2.61) and (3.2.62) will give the following expressions

$$\tilde{D}_+\tilde{b}_{L1} = D_+b_{L1} \qquad \tilde{D}_+\tilde{b}_{L1}^t = -D_+b_{L1}^t$$

$$\tilde{D}_-\tilde{b}_{R1} = -D_-b_{R1} \qquad \tilde{D}_-\tilde{b}_{R1}^t = D_-b_{R1}^t$$

$$\tilde{D}'_+\tilde{a}_{L1} = D'_+a_{L1} \qquad \tilde{D}'_+\tilde{c}_{L1} = D'_+c_{L1}$$

$$\tilde{D}'_-\tilde{a}_{R1} = -D'_-a_{R1} \qquad \tilde{D}'_-\tilde{c}_{R1} = -D'_-c_{R1}$$

$$\tilde{D}_+\tilde{b}_{L2} = D_+b_{L2} + \frac{1}{2}\{(\tilde{a}_{L1} - a_{L1})D_+b_{L1} - D_+b_{L1}(\tilde{c}_{L1} - c_{L1})\}$$
$$+ \frac{1}{2}\{(\tilde{b}_{L1} - b_{L1})D'_+c_{L1} - D'_+a_{L1}(\tilde{b}_{L1} - b_{L1})\}$$

$$\tilde{D}_+\tilde{b}^t_{L2} = -D_+b^t_{L2} - \frac{1}{2}\{(\tilde{c}_{L1} - c_{L1})D_+b^t_{L1} - D_+b^t_{L1}(\tilde{a}_{L1} - a_{L1})\}$$
$$+ \frac{1}{2}\{(\tilde{b}^t_{L1} + b^t_{L1})D'_+a_{L1} - D'_+c_{L1}(\tilde{b}^t_{L1} + b^t_{L1})\}$$

$$\tilde{D}'_+\tilde{a}_{L2} = D'_+a_{L2} + \frac{1}{2}[(\tilde{a}_{L1} - a_{L1}), D'_+a_{L1}]$$
$$- \frac{1}{2}\{D_+b_{L1}(b^t_{L1} + \tilde{b}^t_{L1}) - (b_{L1} - \tilde{b}_{L1})D_+b^t_{L1}\}$$

$$\tilde{D}'_+\tilde{c}_{L2} = D'_+c_{L2} + \frac{1}{2}[(\tilde{c}_{L1} - c_{L1}), D'_+c_{L1}]$$
$$- \frac{1}{2}\{D_+b^t_{L1}(b_{L1} - \tilde{b}_{L1}) - (b^t_{L1} + \tilde{b}^t_{L1})D_+b_{L1}\}$$

$$\tilde{D}_-\tilde{b}_{R2} = -D_-b_{R2} + \{(a_{L1} - \tilde{a}_{L1}) + \frac{1}{2}(a_{R1} - \tilde{a}_{R1})\}D_-b_{R1}$$
$$- D_-b_{R1}\{(c_{L1} - \tilde{c}_{L1}) + \frac{1}{2}(c_{R1} - \tilde{c}_{R1})\}$$
$$+ \{(b_{L1} - \tilde{b}_{L1}) + \frac{1}{2}(b_{R1} - \tilde{b}_{R1})\}D'_-c_{R1}$$
$$- D'_-a_{R1}\{(b_{L1} - \tilde{b}_{L1}) + \frac{1}{2}(b_{R1} - \tilde{b}_{R1})\}$$

$$\begin{aligned}\tilde{D}_{-}\tilde{b}^{t}_{R2} = D_{-}b^{t}_{R2} &- \{(c_{L1} - \tilde{c}_{L1}) + \frac{1}{2}(c_{R1} - \tilde{c}_{R1})\}D_{-}b^{t}_{R1} \\ &+ D_{-}b^{t}_{R1}\{(a_{L1} - \tilde{a}_{L1}) + \frac{1}{2}(a_{R1} - \tilde{a}_{R1})\} \\ &- \{(b^{t}_{L1} + \tilde{b}^{t}_{L1}) + \frac{1}{2}(b^{t}_{R1} + \tilde{b}^{t}_{R1})\}D^{'}_{-}a_{R1} \\ &+ D^{'}_{-}c_{R1}\{(b^{t}_{L1} + \tilde{b}^{t}_{L1}) + \frac{1}{2}(b^{t}_{R1} + \tilde{b}^{t}_{R1})\}\end{aligned}$$

$$\begin{aligned}\tilde{D}^{'}_{-}\tilde{a}_{R2} = -D^{'}_{-}a_{R2} &+ [(a_{L1} - \tilde{a}_{L1}) + \frac{1}{2}(a_{R1} - \tilde{a}_{R1}), D^{'}_{-}a_{R1}] \\ &+ D_{-}b_{R1}\{(b^{t}_{L1} + \tilde{b}^{t}_{L1}) + \frac{1}{2}(b^{t}_{R1} + \tilde{b}^{t}_{R1})\} \\ &- \{(b_{L1} - \tilde{b}_{L1}) + \frac{1}{2}(b_{R1} - \tilde{b}_{R1})\}D_{-}b^{t}_{R1}\end{aligned}$$

$$\begin{aligned}\tilde{D}^{'}_{-}\tilde{c}_{R2} = -D^{'}_{-}c_{R2} &+ [(c_{L1} - \tilde{c}_{L1}) + \frac{1}{2}(c_{R1} - \tilde{c}_{R1}), D^{'}_{-}c_{R1}] \\ &+ D_{-}b^{t}_{R1}\{(b_{L1} - \tilde{b}_{L1}) + \frac{1}{2}(b_{R1} + \tilde{b}_{R1})\} \\ &- \{(b^{t}_{L1} + \tilde{b}^{t}_{L1}) + \frac{1}{2}(b^{t}_{R1} + \tilde{b}^{t}_{R1})\}D_{-}b_{R1}\end{aligned}$$

where tilded terms on the right hand sides can be replaced by solving corresponding equations. One can obtain the conserved nonlocal currents using these terms.

3.3 Curvatures

3.3.1 Case I: Curvatures on g and g̃

Let us find the curvatures related to symmetric spaces, and see the relations between dual symmetric parts. We first consider the case where H = id. We may choose orthonormal frame $\{J\}$ on the pullback bundle $g^{*}(TG)$, where J stands for both $J^{(R)}$ and $J^{(L)}$. These currents satisfy the Maurer-Cartan equation

$$dJ^i + \frac{1}{2}f^i_{jk}J^j \wedge J^k = 0 \tag{3.3.1}$$

where $w^i = J^i$ and $w^i_k = \frac{1}{2}f^i_{jk}J^j$ is the antisymmetric riemannian connection. Curvature can be found using torsion free Cartan structural equations

$$dw^i + w^i_j \wedge w^j = 0 \tag{3.3.2}$$

$$dw^i_j + w^i_k \wedge w^k_j = \frac{1}{2}R^i_{jkl}w^k \wedge w^l \tag{3.3.3}$$

Substituting $w^i = J^i$ and $w^i_j = \frac{1}{2}f^i_{kj}J^k$ into first equation gives us the Maurer-Cartan equation (3.3.1). Curvature tensor associated with **g** can be found using second equation (3.3.3),

$$R^i_{jmn} = -\frac{1}{2}(f^i_{km}f^k_{nj} + f^i_{kj}f^k_{mn}) = \frac{1}{2}f^i_{kn}f^k_{jm} \tag{3.3.4}$$

where we used jacobi identity in the last equation, $f^i_{k[m}f^k_{nj]} = 0$. We may find similar relations for pseudodual space with tilde (just put $\tilde{}$ on each term). To relate curvature tensor on pseudodual space with regular space, we use nonlocal expressions (3.1.26)-(3.1.29). Since both currents yield the same result, we just use (3.1.26) and (3.1.27) for the final expression. We may write $\tilde{J}^i$ in nonlocal terms as

$$\tilde{J}^i = \varepsilon dy^i_1 + \varepsilon^2[dy^i_2 + \frac{1}{2}f^i_{jk}y^i_1 \wedge dy^k_1 - \tilde{f}^i_{jk}y^j_1 \wedge dy^k_1] + H.O. \tag{3.3.5}$$

Hence $\tilde{w}^i = \tilde{J}^i$, and $\tilde{w}^i_k$ can be written as

$$\begin{aligned}\tilde{w}^i_k &= \frac{1}{2}\tilde{f}^i_{jk}\tilde{J}^j \\ &= \frac{\varepsilon}{2}\tilde{f}^i_{jk}dy^j_1 + \frac{\varepsilon^2}{2}\tilde{f}^i_{jk}[dy^j_2 + \frac{1}{2}f^j_{mn}y^m_1 \wedge dy^n_1 - \tilde{f}^j_{mn}y^m_1 \wedge dy^n_1] + H.O.\end{aligned} \tag{3.3.6}$$

We plug $\tilde{w}^i$ and $\tilde{w}^i_k$ into the second Cartan structural equation on pseudodual space in the form

$$d\tilde{w}^i_j + \tilde{w}^i_k \wedge \tilde{w}^k_j = \frac{1}{2}\tilde{R}^i_{jkl}\tilde{w}^k \wedge \tilde{w}^l \tag{3.3.7}$$

to obtain the curvature expression

$$\tilde{R}^i_{jmn} = \frac{1}{2}\tilde{f}^i_{kj}f^k_{mn} - \tilde{f}^i_{kj}\tilde{f}^k_{mn} + \frac{1}{2}\tilde{f}^i_{mk}\tilde{f}^k_{nj} \tag{3.3.8}$$

Since by definition $\tilde{R}^i_{jmn}$ (3.3.4)can also be written as

$$\tilde{R}^i_{jmn} = \frac{1}{2}\tilde{f}^i_{kn}\tilde{f}^k_{jm} \tag{3.3.9}$$

we get a relation between structure constants on spaces **g** and $\tilde{\mathbf{g}}$

$$\frac{1}{2}\tilde{f}^i_{kj}f^k_{mn} = \frac{1}{2}\tilde{f}^i_{kj}\tilde{f}^k_{mn} \tag{3.3.10}$$

where we used the jacobi identity $\tilde{f}^i_{k[n}\tilde{f}^k_{jm]} = 0$. Though we do not set f^k_{mn} equal to $\tilde{f}^k_{mn}$, we may treat them on equal footing, and use one for another interchangeably in paired terms. Hence $\tilde{R}^i_{jmn}$ (3.3.8) can be written in nonlocal structure constants as

$$\tilde{R}^i_{jmn} = -\frac{1}{2}f^i_{kn}f^k_{jm} = -R^i_{jmn} \tag{3.3.11}$$

where we used $f^i_{k[j}f^k_{nm]} = 0$ after setting tilde terms with nontilde terms. We note that we obtained pseudodual space curvature as the negative regular space curvature. This shows that spaces are dual symmetric spaces as we expressed above.

3.3.2 Case II: Curvatures on Decomposed Spaces

Let us decompose the current as $J = J^{\alpha}t_{\alpha} + J^{a}t_{a}$, where we use indices $\alpha, \beta, \gamma, ...$ for **m** space and indices $a, b, c, ...$ for **h** space, and t_{α} and t_{a} are corresponding generators. We can write the commutation relations as

$$[t_a, t_b] = f^{c}_{ab}t_c \qquad [t_a, t_\beta] = f^{\alpha}_{a\beta}t_\alpha \qquad [t_\alpha, t_\beta] = f^{c}_{\alpha\beta}t_c \tag{3.3.12}$$

Maurer-Cartan equation (3.3.1) can be decomposed as

$$dJ^a + \frac{1}{2}f^a_{bc}J^b \wedge J^c + \frac{1}{2}f^a_{\alpha\beta}J^\alpha \wedge J^\beta = 0 \qquad on\ \mathbf{h} - space \tag{3.3.13}$$

$$dJ^\alpha + f^\alpha_{\beta a}J^\beta \wedge J^a = 0 \qquad on\ \mathbf{m} - space \tag{3.3.14}$$

We can also decompose Cartan structural equations. Decomposition of first structural equation gives us

$$dw^a + w^a_b \wedge w^b + w^a_\alpha \wedge w^\alpha = 0 \qquad on\ \mathbf{h} - space \tag{3.3.15}$$

$$dw^\alpha + w^\alpha_\beta \wedge w^\beta + w^\alpha_a \wedge w^a = 0 \qquad on\ \mathbf{m} - space \tag{3.3.16}$$

comparison of these equations with the Maurer-Cartan equations (3.3.13)-(3.3.14) gives us the following connections

$$w^a = J^a \quad w^a_c = \frac{1}{2}f^a_{bc}J^b \quad w^a_\beta = \frac{1}{2}f^a_{\alpha\beta}J^\alpha \tag{3.3.17}$$

$$w^\alpha = J^\alpha \quad w^\alpha_\beta = \frac{1}{2}f^\alpha_{a\beta}J^a \quad w^\alpha_a = \frac{1}{2}f^\alpha_{\beta a}J^\beta \tag{3.3.18}$$

Decomposition of second Cartan structural equation leads to the following equations

$$
\begin{aligned}
dw^a_b + w^a_c \wedge w^c_b + w^a_\lambda \wedge w^\lambda_b =& \frac{1}{2} R^a_{bcd} w^c \wedge w^d + \frac{1}{2} R^a_{bc\lambda} w^c \wedge w^\lambda \\
&+ \frac{1}{2} R^a_{b\lambda c} w^\lambda \wedge w^c + \frac{1}{2} R^a_{b\lambda\mu} w^\lambda \wedge w^\mu
\end{aligned} \tag{3.3.19}
$$

$$
\begin{aligned}
dw^a_\alpha + w^a_c \wedge w^c_\alpha + w^a_\lambda \wedge w^\lambda_\alpha =& \frac{1}{2} R^a_{\alpha bc} w^b \wedge w^c + \frac{1}{2} R^a_{\alpha b\beta} w^b \wedge w^\beta \\
&+ \frac{1}{2} R^a_{\alpha\beta b} w^\beta \wedge w^b + \frac{1}{2} R^a_{\alpha\lambda\mu} w^\lambda \wedge w^\mu
\end{aligned} \tag{3.3.20}
$$

$$
\begin{aligned}
dw^\alpha_\beta + w^\alpha_\gamma \wedge w^\gamma_\beta + w^\alpha_a \wedge w^a_\beta =& \frac{1}{2} R^\alpha_{\beta ab} w^a \wedge w^b + \frac{1}{2} R^\alpha_{\beta a\gamma} w^a \wedge w^\gamma \\
&+ \frac{1}{2} R^\alpha_{\beta\gamma a} w^\gamma \wedge w^a + \frac{1}{2} R^\alpha_{\beta\lambda\mu} w^\lambda \wedge w^\mu
\end{aligned} \tag{3.3.21}
$$

$$
\begin{aligned}
dw^\alpha_a + w^\alpha_\gamma \wedge w^\gamma_a + w^\alpha_b \wedge w^b_a =& \frac{1}{2} R^\alpha_{abc} w^b \wedge w^c + \frac{1}{2} R^\alpha_{ab\lambda} w^b \wedge w^\lambda \\
&+ \frac{1}{2} R^\alpha_{a\lambda b} w^\lambda \wedge w^b + \frac{1}{2} R^\alpha_{a\lambda\mu} w^\lambda \wedge w^\mu
\end{aligned} \tag{3.3.22}
$$

Inserting (3.3.17) and (3.3.18) into (3.3.19) gives the following curvature components

$$R^a_{bde} = \frac{1}{2}(f^a_{dc} f^c_{eb} - f^a_{cb} f^c_{de}) = \frac{1}{2} f^a_{ce} f^c_{bd} \tag{3.3.23}$$

$$R^a_{b\alpha\beta} = \frac{1}{2}(f^a_{\alpha\lambda} f^\lambda_{\beta b} - f^a_{cb} f^c_{\alpha\beta}) = \frac{1}{2} f^a_{\lambda\beta} f^\lambda_{b\alpha} \tag{3.3.24}$$

$$R^a_{bc\lambda} = R^a_{b\lambda c} = 0 \tag{3.3.25}$$

where we used the jacobi identity $f^a_{c[d} f^c_{be]} = 0$ in (3.3.23), and $f^a_{\lambda\alpha} f^\lambda_{b\beta} + f^a_{cb} f^c_{\beta\alpha} + f^a_{\lambda\beta} f^\lambda_{\alpha b}$ in (3.3.24). Likewise (3.3.20) gives the following curvature components

$$R^a_{\alpha c\lambda} = \frac{1}{2}(f^a_{cd} f^d_{\lambda\alpha} - f^a_{\beta\alpha} f^\beta_{c\lambda}) = \frac{1}{2} f^a_{\beta\lambda} f^\beta_{\alpha c} \tag{3.3.26}$$

$$R^a_{\alpha\lambda c} = \frac{1}{2}(f^a_{\lambda\beta} f^\beta_{c\alpha} - f^a_{\beta\alpha} f^\beta_{\lambda c}) = \frac{1}{2} f^a_{bc} f^b_{\alpha\lambda} \tag{3.3.27}$$

$$R^a_{\alpha bc} = R^a_{\alpha\lambda\mu} = 0 \tag{3.3.28}$$

where we used the jacobi identity $f^a_{dc}f^d_{\alpha\lambda} + f^a_{\beta\alpha}f^\beta_{\lambda c} + f^a_{\beta\lambda}f^\beta_{c\alpha} = 0$ in (3.3.26), and $f^a_{\beta\lambda}f^\beta_{\alpha c} + f^a_{\beta\alpha}f^\beta_{c\lambda} + f^a_{bc}f^b_{\lambda\alpha} = 0$ in (3.3.27). Equation (3.3.21) produces the following curvature components

$$R^\alpha_{\beta bc} = \frac{1}{2}(f^\alpha_{b\gamma}f^\gamma_{c\beta} - f^\alpha_{a\beta}f^a_{bc}) = \frac{1}{2}f^\alpha_{\gamma c}f^\gamma_{\beta b} \tag{3.3.29}$$

$$R^\alpha_{\beta\lambda\mu} = \frac{1}{2}(f^\alpha_{\lambda a}f^a_{\mu\beta} - f^\alpha_{a\beta}f^a_{\lambda\mu}) = \frac{1}{2}f^\alpha_{a\mu}f^a_{\beta\lambda} \tag{3.3.30}$$

$$R^\alpha_{\beta a\gamma} = R^\alpha_{\beta\gamma a} = 0 \tag{3.3.31}$$

where we used the jacobi identity $f^\alpha_{\gamma b}f^\gamma_{\beta c} + f^\alpha_{a\beta}f^a_{cb} + f^\alpha_{\gamma c}f^\gamma_{b\beta} = 0$ in (3.3.29), and $f^\alpha_{a\lambda}f^a_{\beta\mu} + f^\alpha_{a\beta}f^a_{\mu\lambda} + f^\alpha_{a\mu}f^a_{\lambda\beta} = 0$ in (3.3.30). Finally, equation (3.3.22) gives the following curvature components

$$R^\alpha_{ac\lambda} = \frac{1}{2}(f^\alpha_{c\beta}f^\beta_{\lambda a} - f^\alpha_{\beta a}f^\beta_{c\lambda}) = \frac{1}{2}f^\alpha_{b\lambda}f^b_{ac} \tag{3.3.32}$$

$$R^\alpha_{a\lambda c} = \frac{1}{2}(f^\alpha_{\lambda b}f^b_{ca} - f^\alpha_{\beta a}f^\beta_{\lambda c}) = \frac{1}{2}f^\alpha_{\beta c}f^\beta_{a\lambda} \tag{3.3.33}$$

$$R^\alpha_{abc} = R^\alpha_{a\lambda\mu} = 0 \tag{3.3.34}$$

where we used the jacobi identity $f^\alpha_{\beta c}f^\beta_{a\lambda} + f^\alpha_{\beta a}f^\beta_{\lambda c} + f^\alpha_{b\lambda}f^b_{ca} = 0$ in (3.3.32), and $f^\alpha_{b\lambda}f^b_{ac} + f^\alpha_{\beta a}f^\beta_{c\lambda} + f^\alpha_{\beta c}f^\beta_{\lambda a} = 0$ in (3.3.33). Obviously we can write similar equations with tilde.

We want to write down curvature relations between symmetric spaces ($\mathbf{m}$ and $\tilde{\mathbf{m}}$) and corresponding closed spaces ($\mathbf{h}$ and $\tilde{\mathbf{h}}$) on $\mathbf{g}$ and $\tilde{\mathbf{g}}$. To realize this objective we will use the bracket relations derived from pseudoduality equations. In case of non-mixing pseudoduality, we will make use of bracket relation (3.2.45)-(3.2.48). After eliminating A_- and k_-

terms we obtain the following relations between connection one forms

$$\tilde{w}^{\alpha}_{a} = w^{\alpha}_{a} \qquad \tilde{w}^{\alpha}_{\beta} = w^{\alpha}_{\beta} \tag{3.3.35}$$

$$\tilde{w}^{a}_{\beta} = w^{a}_{\beta} \qquad \tilde{w}^{a}_{b} = w^{a}_{b} \tag{3.3.36}$$

where we used the definitions (3.3.17) and (3.3.18) for the connection two forms. Taking exterior derivative of these connections we obtain the result

$$\tilde{R}^{A}_{BCD} = -R^{A}_{BCD} \tag{3.3.37}$$

where A, B, C and D represent indices corresponding to M or H-space elements depending on which equation is used. But curvature expressions found above restrict all curvature components to exist. Therefore we will only have curvatures whose all indices belongs to one space (**m** or **h**) or being shared equally, otherwise they do not exist. On the other hand when we consider mixing pseudoduality, we observe that curvature components mix. From the connection two-forms we obtain the relations

$$\tilde{w}^{\alpha}_{\beta} + \tilde{w}^{\alpha}_{a}T^{a}_{\beta}(0) = w^{\alpha}_{\beta} + T^{\alpha}_{a}(0)w^{a}_{\beta} \tag{3.3.38}$$

$$\tilde{w}^{\alpha}_{\beta}T^{\beta}_{b}(0) + \tilde{w}^{\alpha}_{b} = T^{\alpha}_{a}(0)w^{a}_{b} + w^{\alpha}_{b} \tag{3.3.39}$$

$$\tilde{w}^{a}_{b} + \tilde{w}^{a}_{\beta}T^{\beta}_{b}(0) = w^{a}_{b} + T^{a}_{\beta}(0)w^{\beta}_{b} \tag{3.3.40}$$

$$\tilde{w}^{a}_{b}T^{b}_{\beta}(0) + \tilde{w}^{a}_{\beta} = T^{a}_{\gamma}(0)w^{\gamma}_{\beta} + w^{a}_{\beta} \tag{3.3.41}$$

It is clear that once mixing isometries disappear we have (3.3.35) and (3.3.36). Therefore curvature relations will be

$$\hat{R}^A_{B\mu\nu} = -(\bar{\tilde{R}}^A_{B\mu\nu} + \bar{\tilde{R}}^A_{B\mu c}T^c_\nu(0) + \bar{\tilde{R}}^A_{Bc\nu}T^c_\mu(0) + \bar{\tilde{R}}^A_{Bcd}T^c_\mu(0)T^d_\nu(0)) \tag{3.3.42}$$

$$\hat{R}^A_{B\mu d} = -(\bar{\tilde{R}}^A_{B\mu d} + \bar{\tilde{R}}^A_{Bcd}T^c_\mu(0) + \bar{\tilde{R}}^A_{B\mu\nu}T^\nu_d(0) + \bar{\tilde{R}}^A_{Bc\nu}T^c_\mu(0)T^\nu_d(0)) \tag{3.3.43}$$

$$\hat{R}^A_{Bc\nu} = -(\bar{\tilde{R}}^A_{Bc\nu} + \bar{\tilde{R}}^A_{Bcd}T^d_\nu(0) + \bar{\tilde{R}}^A_{B\mu\nu}T^\mu_c(0) + \bar{\tilde{R}}^A_{B\mu d}T^\mu_c(0)T^d_\nu(0)) \tag{3.3.44}$$

$$\hat{R}^A_{Bcd} = -(\bar{\tilde{R}}^A_{Bcd} + \bar{\tilde{R}}^A_{B\mu d}T^\mu_c(0) + \bar{\tilde{R}}^A_{Bc\mu}T^\mu_d(0) + \bar{\tilde{R}}^A_{B\mu\nu}T^\mu_c(0)T^\nu_d(0)) \tag{3.3.45}$$

where we defined $\hat{R}^\alpha_{\lambda\mu\nu} \equiv R^\alpha_{\lambda\mu\nu} + T^\alpha_a(0)R^a_{\lambda\mu\nu}$ and $\bar{\tilde{R}}^\alpha_{\lambda\mu\nu} \equiv \tilde{R}^\alpha_{\lambda\mu\nu} + \tilde{R}^\alpha_{b\mu\nu}T^b_\lambda(0)$, and A, B represent indices for **m** or **h**-spaces. Obviously if all mixing parts are set to zero we obtain the simplest case (3.3.37).

3.4 One Loop Renormalization Group β-function

It is noted that renormalization group β-function to one-loop order [8, 34, 35] is given by

$$\beta_{mn} = \frac{R_{mn}}{2\pi} \tag{3.4.1}$$

where R_{mn} is Ricci curvature of connections w^i_j. On **g** it is written as

$$\beta_{ij} = \frac{1}{4\pi} f^k_{nj} f^n_{ik} \tag{3.4.2}$$

On decomposed spaces **h** and **m** one loop β-functions will be

$$\beta_{ab} = \frac{1}{4\pi}(f^\alpha_{\beta b} f^\beta_{a\alpha} + f^c_{db} f^d_{ac}) \tag{3.4.3}$$

$$\beta_{\alpha\gamma} = \frac{1}{4\pi}(f^a_{\lambda\gamma} f^\lambda_{\alpha a} + f^a_{\alpha\lambda} f^\lambda_{a\gamma}) \tag{3.4.4}$$

It is readily observed that $R_{a\alpha} = R_{\alpha a} = 0$. On pseudodual spaces one can write the following relations

$$\beta_{ij} = -\tilde{\beta}_{ij} \qquad \beta_{ab} = -\tilde{\beta}_{ab} \qquad \beta_{\alpha\gamma} = -\tilde{\beta}_{\alpha\gamma} \tag{3.4.5}$$

if there is a non-mixing pseudoduality. On the other hand if there is a mixing pseudoduality we have

$$\beta_{ab} = -\tilde{\beta}_{ab} - \tilde{\beta}_{a\nu}T^{\nu}_{b}(0) - \tilde{\beta}_{\nu b}T^{v}_{a}(0) - \tilde{\beta}_{\mu\nu}T^{\mu}_{a}(0)T^{\nu}_{b}(0) \tag{3.4.6}$$

$$\beta_{\mu\nu} = -\tilde{\beta}_{\mu\nu} - \tilde{\beta}_{d\nu}T^{d}_{\mu}(0) - \tilde{\beta}_{\mu d}T^{d}_{\nu}(0) - \tilde{\beta}_{ab}T^{a}_{\mu}(0)T^{b}_{\nu}(0) \tag{3.4.7}$$

where we defined $\tilde{\beta}_{\nu b} \equiv \frac{1}{2\pi}\{\tilde{R}^{c}_{\nu\mu b}T^{\mu}_{c}(0) + \tilde{R}^{\mu}_{\nu cb}T^{c}_{\mu}(0)\}$, $\tilde{\beta}_{a\nu} \equiv \frac{1}{2\pi}\{\tilde{R}^{c}_{a\mu\nu}T^{\mu}_{c}(0) + \tilde{R}^{\mu}_{ac\nu}T^{c}_{\mu}(0)\}$, $\tilde{\beta}_{d\nu} \equiv \frac{1}{2\pi}\{\tilde{R}^{c}_{d\lambda\nu}T^{\lambda}_{c}(0) + \tilde{R}^{\lambda}_{dc\nu}T^{c}_{\lambda}(0)\}$ and $\tilde{\beta}_{\mu d} \equiv \frac{1}{2\pi}\{\tilde{R}^{c}_{\mu\lambda d}T^{\lambda}_{c}(0) + \tilde{R}^{\lambda}_{\mu cd}T^{c}_{\lambda}(0)\}$ on the contrary to (3.4.1). We notice that if all mixing isometries vanish, then we get (3.4.5). We notice that we will also obtain additional mixing components of β-function, but we avoid to obtain them.

3.5 Discussion

In this section we were able to obtain infinite number of pseudoduality equations by switching from Lie group expressions to lie algebra expressions. We observed that pseudoduality transformation respects the conservation law of currents. To understand what currents imply for let us write pseudoduality equations as

$$\tilde{J}^{(L)}_{+} = +TJ^{(L)}_{+}$$
$$\tilde{J}^{(L)}_{-} = -TJ^{(L)}_{-}$$

where $J_{\pm}^{(L)} = g^{-1}\partial_{\pm}g$. First equation implies that T is a function of σ^{+} as above. Second equation is interesting and gives the information about currents. If we take ∂_{+} of second equation we obtain that

$$[\tilde{g}^{-1}\partial_{-}\tilde{g}, \tilde{g}^{-1}\partial_{-}\tilde{g}]_{\tilde{G}} = -(\partial_{+}T)(g^{-1}\partial_{-}g) - T[g^{-1}\partial_{-}g, g^{-1}\partial_{+}g]_{G}$$

We notice that $g^{-1}\partial_{\pm}g \in \mathbf{g}$, and if we use the definition $ad_{\mathbf{g}}(X)(Y) = [X, Y]_{G}$ this equation can be written as

$$ad_{\tilde{\mathbf{g}}}(\tilde{J}_{+}^{(L)})(\tilde{J}_{-}^{(L)}) = (\partial_{+}T)J_{-}^{(L)} + T ad_{\mathbf{g}}(J_{+}^{(L)})(J_{-}^{(L)})$$

If the second pseudoduality equation is inserted then one gets

$$-ad_{\tilde{\mathbf{g}}}(\tilde{J}_{+}^{(L)})T - T ad_{\mathbf{g}}(J_{+}^{(L)}) = (\partial_{+}T)$$

It is obvious that this is the lie algebra version of the $AdG \times Ad\tilde{G}$ action on T. $ad_{\mathbf{g}}(J_{+}^{(L)})$ is the orthogonal flat connection on $g^{*}TG$ as defined in section (3.3). One may find curvature relations using these connections as above. Thus another interpretation of pseudoduality is that since $J_{+}^{(L)}$ depends only on σ^{+}, so does T. Hence if we define a parallel transport $P(\sigma)$ from (0, 0) to $\sigma = (\sigma^{+}, \sigma^{-})$, pseudoduality equations may be written as

$$*_{\Sigma}(\tilde{P}(\sigma))^{-1}(\tilde{g}^{-1}d\tilde{g}) = T(0)(P(\sigma)^{-1}g^{-1}dg)$$

where $T(0) = \tilde{P}(\sigma)T(\sigma)P^{-1}(\sigma)$. This means that we start with $g^{-1}dg$, and parallel transport it to origin, and do the same on the dual model. We finally use the fixed isometry $T(0)$ to equate these two fields at the origins.

Chapter 4

Pseudoduality In Supersymmetric Sigma Models

This model consists of both bosons and fermions, and they are transformed into each other by supersymmetry transformation. It improves the short distance behaviour of quantum theories and gives a beautiful solution to the hierarchy problem. Supersymmetric sigma models have a rich geometrical structure. It has been shown that target space of $N = 1$ sigma models is a (pseudo-)Riemannian manifold, $N = 2$ is the Kähler manifold and $N = 4$ is the hyper-Kähler manifold. Sigma models based on manifolds with torsion [40] have chiral supersymmetry in which the number of left handed supersymmetries differs from the number of right handed supersymmetries. In [7], pseudoduality in classical sigma models was extensively discussed, and in this section we are going to analyze pseudoduality transformation of supersymmetric extension of classical sigma models. We will focus on (1,0) and (1, 1) real supersymmetric sigma models in two dimensions, and find the required conditions which supersymmetry constrains the target space and following results for pseudoduality. We will refer to references [8, 36, 37, 38, 39] about supersymmetry and superspace constructions.

We use the superspace coordinates $(\sigma^{\pm}, \theta^{\pm})$, where the bosonic coordinates $\sigma^{\pm} = \tau \pm \sigma$ are the usual lightcone coordinates in two-dimensional Minkowski space, and the fermionic coordinates $\theta^{\pm}$ are the Grassmann numbers. The supercovariant derivatives are

$$D_{\pm} = \partial_{\theta^{\pm}} + i\theta^{\pm}\partial_{\pm} \tag{4.0.1}$$

and the supercharges generating supersymmetry are

$$Q_{\pm} = \partial_{\theta^{\pm}} - i\theta^{\pm}\partial_{\pm} \tag{4.0.2}$$

and it follows that

$$Q_{\pm}^2 = -i\partial_{\pm} \qquad D_{\pm}^2 = i\partial_{\pm}. \tag{4.0.3}$$

and all other anticommutations vanish. The scalar superfields in components have the form

$$X(\sigma,\theta) = x(\sigma) + \theta^{+}\psi_{+}(\sigma) + \theta^{-}\psi_{-}(\sigma) + \theta^{+}\theta^{-}F(\sigma) \tag{4.0.4}$$

where $x : \Sigma \to M$, $\psi_{\pm}$ are the two dimensional Majorana spinor fields, and F is the auxiliary real scalar field.

4.1 Pseudoduality in Heterotic Sigma Models

This model [8, 40, 41, 42, 43, 44] is enlarging the spacetime Σ in the classical case to the superspace $\Xi^{1,0}$ by adding a Grassmann degree of freedom. Hence the sigma model is the map consisting of a scalar x and a fermion ψ_{+}. This case has one left-handed supercharge Q_{+}, and does not contain any right-handed supercharge Q_{-}. The supersymmetry algebra will be

$$\{Q_{+}, Q_{+}\} = 2iP_{+}$$

where $\{,\}$denotes anticommutation, and $P_+ = -\partial_+$ as can be checked from (4.0.3). The supersymmetry transformations generated by Q_+ will be

$$\delta_\epsilon x(\sigma) = \epsilon_- \psi_+(\sigma)$$

$$\delta_\epsilon \psi_+(\sigma) = i\epsilon_- \partial_+ x(\sigma)$$

Hence the fermion ψ_+ can be thought of as the superpartner of the boson x. In what follows we will examine pseudoduality transformations between supermanifolds $\mathbb{M}$ [1] and $\tilde{\mathbb{M}}$ using components first, and then probe how it behaves when lifted to orthonormal coframe bundles $SO(\mathbb{M})$ [2] and $SO(\tilde{\mathbb{M}})$. We again emphasize that pseudoduality is defined between superspaces z which are the pullbacks of the manifols $\mathbb{M}$ and $\tilde{\mathbb{M}}$ in case of components, and $SO(\mathbb{M})$ and $SO(\tilde{\mathbb{M}})$ in case of orthonormal coframe method. This is implicitly intended in our calculations.

4.1.1 Components

In this case the superfield X has the form

$$X = x(\sigma) + \theta^+ \psi_+(\sigma) \tag{4.1.1}$$

where $X : \Xi^{1,0} \to \mathbb{M}$, and $\Xi^{1,0} = (\sigma^+, \sigma^-, \theta^+)$. The real grassmann coordinate θ^+ is anticommuting and $(\theta^+)^2 = 0$. We will assume that target space has torsion H, which is introduced into the action by a Wess-Zumino term. Reparametrization invariant action defined on a Riemannian manifold $\mathbb{M}$ with metric G_{ij}, standard connection $\mathbf{\Gamma}^i_{jk}$ and anti-

[1] $\mathbb{M}$ is the target space in which supersymmetric sigma models is defined.
[2] $SO(\mathbb{M}) = \mathbb{M} \times SO(n)$.

symmetric two-form B_{ij} can be written as

$$S = \int d^2\sigma d\theta (G_{ij} + B_{ij}) D_+ X^i \partial_- X^j \tag{4.1.2}$$

We may write similar expressions for manifold $\tilde{\mathbb{M}}$ using expressions with tilde. Since we want to write down pseudoduality transformations between two manifolds, we need to find out the equations of motions from action (4.1.2). If we write this action in terms of bosonic coordinates of superspace only, we obtain our original classical action plus fermionic terms. After expanding G_{ij} and B_{ij} in the first order terms and integrating this action under $d\theta$ gives the following

$$S = \int d^2\sigma [i(g_{ij} + b_{ij}) \partial_+ x^i \partial_- x^j - g_{ij} \psi_+^i \nabla_-^{(-)} \psi_+^j] \tag{4.1.3}$$

where $\nabla_-^{(-)} \psi_+^j = \nabla_- \psi_+^j - H_{kl}^j \psi_+^k \partial_- x^l$ and $\nabla_- \psi_+^j = \partial_- \psi_+^j + \Gamma_{kl}^j \psi_+^k \partial_- x^l$, and $H_{ijk} = \frac{1}{2}(\partial_i b_{jk} + \partial_j b_{ki} + \partial_k b_{ij})$. Equations of motion following from the action (4.1.3) are

$$\nabla_-^{(-)} \psi_+^i = 0 \tag{4.1.4}$$

$$\Box x^k = i \hat{R}_{lij}^k \psi_+^i \psi_+^j \partial_- x^l \tag{4.1.5}$$

where $\Box x^k = \nabla_+^{(+)} \partial_- x^k + \nabla_-^{(-)} \partial_+ x^k$, and the generalized curvature is defined as

$$\hat{R}_{ijkl} = R_{ijkl} - D_k H_{ijl} + D_l H_{ijk} + H_{ikn} H_{lj}^n - H_{jkn} H_{li}^n \tag{4.1.6}$$

We can write the Pseudoduality transformations as follows

$$D_+ \tilde{X}^i = +\mathcal{T}_j^i D_+ X^i \tag{4.1.7}$$

$$\partial_- \tilde{X}^i = -\mathcal{T}_j^i \partial_- X^j \tag{4.1.8}$$

where $\mathcal{T}$ is the transformation matrix, and is a function of superfield X. Since superfield depends on σ and θ^+, we may say that $\mathcal{T}$ is a function of σ and θ^+. We let $\mathcal{T}(\sigma, \theta) = T(\sigma) + \theta^+ N(\sigma)$. Splitting pseudoduality equations into the fermionic and bosonic parts leads to the following set of equations

$$\tilde{\psi}^i_+(\sigma) = +T^i_j(\sigma)\psi^j_+(\sigma) \tag{4.1.9}$$

$$\partial_-\tilde{\psi}^i_+(\sigma) = -T^i_j(\sigma)\partial_-\psi^j_+(\sigma) - N^i_j(\sigma)\partial_- x^j(\sigma) \tag{4.1.10}$$

$$\partial_+\tilde{x}^i(\sigma) = +T^i_j(\sigma)\partial_+ x^j(\sigma) - iN^i_j(\sigma)\psi^j_+(\sigma) \tag{4.1.11}$$

$$\partial_-\tilde{x}^i(\sigma) = -T^i_j(\sigma)\partial_- x^j(\sigma) \tag{4.1.12}$$

We see that the component T is responsible for the classical transformation which does not change the type of field, while N contributes to the fermionic degree of transformation which transforms bosonic fields to fermionic ones, and vice versa. Before finding pseudodual expressions it is worth to obtain constraint relations. We take ∂_- of (4.1.9) and set equal to (4.1.10), and then use the equation of motion (4.1.4) to obtain

$$N^i_k = -[M^i_{lk} + 2T^i_j(H^j_{lk} - \Gamma^j_{lk})]\psi^l_+ \tag{4.1.13}$$

where we define $\partial_k T^i_l = M^i_{lk}$. Now taking ∂_+ of (4.1.12) and setting equal to ∂_- of (4.1.11) followed by using equations of motion (4.1.4) and (4.1.5) yields

$$\begin{aligned} &[2T^i_k(H^k_{mn} - \Gamma^k_{mn}) + 2M^i_{(mn)}]\partial_+ x^m \partial_- x^n + iT^i_k \hat{R}^k_{mij}\psi^i_+\psi^j_+\partial_- x^m \\ &= iN^i_k(H^k_{mn} - \Gamma^k_{mn})\psi^m_+\partial_- x^n + i(\partial_- N^i_k)\psi^k_+ \end{aligned} \tag{4.1.14}$$

where $M^i_{(mn)}$ represents the symmetric part of M^i_{mn}. Real part of this equation gives

$$T^i_k(H^k_{mn} - \Gamma^k_{mn}) + 2M^i_{(mn)} = 0 \tag{4.1.15}$$

which implies that

$$H^k_{mn} = 0, \tag{4.1.16}$$

$$M^i_{(mn)} = T^i_k \Gamma^k_{mn} \tag{4.1.17}$$

Substituting these results into (4.1.13) leads to

$$N^i_k = M^i_{km} \psi^m_+ \tag{4.1.18}$$

Complex part of (4.1.14) together with (4.1.16), (4.1.17) and (4.1.18) gives the following equation

$$\partial_n M^i_{[mj]} = T^i_k R^k_{njm} + 2M^i_{[kj]} \Gamma^k_{mn} \tag{4.1.19}$$

where $M^i_{[mj]}$ denotes the antisymmetric part of M^i_{mj}. Solution of this equation gives the result for T.

Riemann Normal Coordinates

Before we attempt to find the general (global) solution for the equation (4.1.19), it is interesting to find the special solution where Riemann Normal coordinates [45, 46, 47] are used in both models. In these coordinates solution is expanded around a point (call this point as p on M, and $\tilde{p}$ on $\tilde{M}$) which Christoffel's symbols vanish. Curvature tensor R is the curvature of the point p, and constant. (4.1.17) implies that $M^i_{jm} = -M^i_{mj}$, and hence, equation (4.1.19) is reduced to

$$\partial_n M^i_{mj} = T^i_k R^k_{njm}$$

After integration we get

$$M^i_{mj} = M^i_{mj}(0) + \int T^i_k R^k_{njm} dx^n$$

and since $T^i_m = T^i_m(0) + \int M^i_{mj} dx^j$, we finally obtain

$$T^i_m = T^i_m(0) + M^i_{mj}(0)x^j + T^i_k(0)R^k_{njm} \int x^n dx^j + M^i_{kl}(0)R^k_{njm} \int dx^j \int x^l dx^n + H.O.$$

and

$$M^i_{mj} = M^i_{mj}(0) + T^i_k(0)R^k_{njm}x^n + M^i_{kl}(0)R^k_{njm} \int x^l dx^n + H.O.$$

and also using (4.1.18) we find

$$N^i_k = M^i_{km}(0)\psi^m_+ + T^i_j(0)R^j_{nmk}\psi^m_+ x^n + M^i_{jl}(0)R^j_{nmk}\psi^m_+ \int x^l dx^n + H.O.$$

We choose the initial condition $T^i_m(0) = \delta^i_m$. Hence Pseudoduality relations (4.1.9) - (4.1.12) up to the second order in x can be written as

$$\tilde{\psi}^i_+ = \psi^i_+ + M^i_{jk}(0)\psi^j_+ x^k + R^i_{nkj}\psi^j_+ \int x^n dx^k + H.O. \tag{4.1.20}$$

$$\partial_-\tilde{\psi}^i_+ = -M^i_{jm}(0)\psi^m_+ \partial_- x^j - R^i_{nmj}\psi^m_+ x^n \partial_- x^j + H.O. \tag{4.1.21}$$

$$\begin{aligned}\partial_+\tilde{x}^i &= \partial_+ x^i + M^i_{jk}(0)x^k\partial_+ x^j - iM^i_{jm}(0)\psi^m_+\psi^j_+ - iR^i_{nmj}\psi^m_+\psi^j_+ x^n \\ &\quad + R^i_{nkj}\partial_+ x^j \int x^n dx^k - iM^i_{kl}(0)R^k_{nmj}\psi^m_+\psi^j_+ \int x^l dx^n + H.O.\end{aligned} \tag{4.1.22}$$

$$\partial_-\tilde{x}^i = -\partial_- x^i - M^i_{jk}(0)x^k\partial_- x^j - R^i_{nlj}\partial_- x^j \int x^n dx^l + H.O. \tag{4.1.23}$$

Using the equation of motion (4.1.4) for tilde, i.e. $\partial_-\tilde{\psi}^i_+ = \tilde{H}^i_{jk}\tilde{\psi}^j_+\partial_-\tilde{x}^k$, and combining with (4.1.20) and (4.1.23) we find

$$\partial_- \tilde{\psi}_+^i = -\tilde{H}^i_{mj}\psi_+^m \partial_- x^j - \tilde{H}^i_{mk} M^k_{jn}(0)\psi_+^m x^n \partial_- x^j - \tilde{H}^i_{kj} M^k_{mn}(0)\psi_+^m x^n \partial_- x^j + H.O. \tag{4.1.24}$$

A comparison of equation (4.1.21) with equation (4.1.24) gives

$$\tilde{H}^i_{mj} = M^i_{jm}(0) \tag{4.1.25}$$

$$R^i_{nmj} = M^i_{km}(0) M^k_{jn}(0) + M^i_{jk}(0) M^k_{mn}(0) \tag{4.1.26}$$

Now we see that equation (4.1.5) with tilde is written as $\partial^2_{+-}\tilde{x}^i = \tilde{H}^i_{jk}\partial_+\tilde{x}^j\partial_-\tilde{x}^k + \frac{i}{2}\hat{\tilde{R}}^i_{jkl}\tilde{\psi}_+^k\tilde{\psi}_+^l\partial_-\tilde{x}^j$. Inserting (4.1.20), (4.1.22) and (4.1.23) into this equation gives

$$\partial^2_{+-}\tilde{x}^i = -\tilde{H}^i_{jk}\partial_+ x^j \partial_- x^k + i\tilde{H}^i_{jk} M^j_{mn}(0)\psi_+^n\psi_+^m\partial_- x^k - \frac{i}{2}\hat{\tilde{R}}^i_{jkl}\psi_+^k\psi_+^l\partial_- x^j + H.O. \tag{4.1.27}$$

Likewise we can write a relation for $\partial^2_{+-}\tilde{x}^i$ using (4.1.22) or (4.1.23) as

$$\partial^2_{+-}\tilde{x}^i = -M^i_{kj}(0)\partial_+ x^j \partial_- x^k - \frac{i}{2} R^i_{jkl}\psi_+^k\psi_+^l\partial_- x^j + H.O. \tag{4.1.28}$$

A simple comparison of (4.1.27) with (4.1.28) gives the following

$$\tilde{H}^i_{jk} = M^i_{kj}(0) \tag{4.1.29}$$

$$-R^i_{jkl} = -\hat{\tilde{R}}^i_{jkl} + 2\tilde{H}^i_{nj}\tilde{H}^n_{kl} \tag{4.1.30}$$

We notice that (4.1.25) is the same as (4.1.29), and $-\hat{\tilde{R}}^i_{jkl} + 2\tilde{H}^i_{nj}\tilde{H}^n_{kl} = -\tilde{R}^i_{jkl}$. Therefore we obtain $R^i_{jkl} = \tilde{R}^i_{jkl}$. We see that curvatures of the points p and $\tilde{p}$ are constant and same. This implies that pseudoduality between two models based on Riemann normal coordinates must have same curvatures. We see from (4.1.25) and (4.1.26) that this transformation works in one way, and is not invertible in this special solution.

General Solution

Now we find the global solution to equation (4.1.19). We know that we can write M^i_{kj} as the sum of symmetric and antisymmetric parts as follows

$$M^i_{kj} = \frac{1}{2}(M^i_{kj} - M^i_{jk}) + \frac{1}{2}(M^i_{kj} + M^i_{jk})$$

Inserting antisymmetric part of this matrix into (4.1.19), and using the result (4.1.17) gives

$$\partial_n M^i_{mj} = T^i_k R^k_{njm} + 2M^i_{kj}\Gamma^k_{mn} - 2T_l \Gamma^l_{kj}\Gamma^k_{mn}$$

If this equation is integrated, the result will be

$$\begin{aligned} M^i_{mj} =& M^i_{mj}(0) + 2M^i_{kj}(0)\int \Gamma^k_{mn}dx^n + 4M^i_{lj}(0)\int \Gamma^k_{mn}dx^n \int \Gamma^l_{ka}dx^a \\ &+ \int T^i_k(R^k_{njm} - 2\Gamma^k_{lj}\Gamma^l_{mn})dx^n + H.O. \end{aligned}$$

and using $T^i_m = T^i_m(0) + \int M^i_{mj}dx^j$ we find T up to the third order terms as follows

$$\begin{aligned} T^i_m =& T^i_m(0) + M^i_{mj}(0)x^j + 2M^i_{kj}(0)\int dx^j \int \Gamma^k_{mn}dx^n \\ &+ 4M^i_{lj}(0)\int dx^j \int \Gamma^k_{mn}dx^n \int \Gamma^l_{ka}dx^a + T^i_k(0)\int dx^j \int (R^k_{njm} - 2\Gamma^k_{lj}\Gamma^l_{mn})dx^n \\ &+ M^i_{kb}(0)\int dx^j \int (R^k_{njm} - 2\Gamma^k_{lj}\Gamma^l_{mn})x^b dx^n + H.O. \end{aligned}$$

which immediately leads to a final result for M^i_{mj}

$$M^i_{mj} = M^i_{mj}(0) + 2M^i_{kj}(0)\int \Gamma^k_{mn}dx^n + 4M^i_{lj}(0)\int \Gamma^k_{mn}dx^n \int \Gamma^l_{ka}dx^a$$
$$+ T^i_k(0)\int (R^k_{njm} - 2\Gamma^k_{lj}\Gamma^l_{mn})dx^n + M^i_{ka}(0)\int (R^k_{njm} - 2\Gamma^k_{lj}\Gamma^l_{mn})x^a dx^n + H.O.$$

One may find torsion and curvature relations using these explicit solutions as in the previous section. Let us inquire solutions by expressing equations (4.1.9) - (4.1.12) in terms of T instead of finding explicit solutions.

If (4.1.17) is inserted in the pseudoduality equations (4.1.9)-(4.1.12) we get

$$\tilde{\psi}^i_+ = +T^i_j\psi^j_+ \tag{4.1.31}$$

$$\partial_-\tilde{\psi}^i_+ = -T^i_j\partial_-\psi^j_+ - M^i_{jm}\psi^m_+\partial_- x^j \tag{4.1.32}$$

$$\partial_+\tilde{x}^i = +T^i_j\partial_+ x^j - iM^i_{jm}\psi^m_+\psi^j_+ \tag{4.1.33}$$

$$\partial_-\tilde{x}^i = -T^i_j\partial_- x^j \tag{4.1.34}$$

Using equations of motion for $\partial_-\tilde{\psi}^i_+$ and $\partial_-\psi^j_+$ in (4.1.32), one finds

$$(\tilde{H}^i_{mn} - \tilde{\Gamma}^i_{mn})\tilde{\psi}^m_+\partial_-\tilde{x}^n = T^i_j\Gamma^j_{mn}\psi^m_+\partial_- x^n - M^i_{nm}\psi^m_+\partial_- x^n \tag{4.1.35}$$

and inserting (4.1.31) and (4.1.34) into (4.1.35) leads to the following result

$$(\tilde{H}^i_{mn} - \tilde{\Gamma}^i_{mn})T^m_a T^n_b = M^i_{ba} - T^i_j\Gamma^j_{ab} \tag{4.1.36}$$

Now taking ∂_- of (4.1.33) (or ∂_+ of 4.1.34) leads to

$$\partial^2_{+-}\tilde{x}^i = M^i_{jk}\partial_+ x^j\partial_- x^k + T^i_j\partial^2_{+-}x^j - i\partial_n M^i_{jm}\psi^m_+\psi^j_+\partial_- x^n - iM^i_{jm}\partial_-\psi^m_+\psi^j_+ - iM^i_{jm}\psi^m_+\partial_-\psi^j_+$$

We use the equation of motion for $\partial^2_{+-}\tilde{x}^i$, $\partial^2_{+-}x^j$ and $\partial_-\psi^m_+$, and use the result (4.1.19) to get

$$(\tilde{H}^i_{jk} - \tilde{\Gamma}^i_{jk})\partial_+\tilde{x}^j\partial_-\tilde{x}^k + \frac{i}{2}\hat{\tilde{R}}^i_{jkm}\tilde{\psi}^k_+\tilde{\psi}^m_+\partial_-\tilde{x}^j = (M^i_{mn} - T^i_j\Gamma^j_{mn})\partial_+x^m\partial_-x^n - \frac{i}{2}T^i_jR^j_{mnk}\psi^n_+\psi^k_+\partial_-x^m \tag{4.1.37}$$

now using (4.1.31), (4.1.33) and (4.1.34) in (4.1.37) leads to

$$\begin{aligned}&-(\tilde{H}^i_{jk} - \tilde{\Gamma}^i_{jk})T^j_mT^k_n\partial_+x^m\partial_-x^n + i(\tilde{H}^i_{jl} - \tilde{\Gamma}^i_{jl})M^j_{kn}T^l_m\psi^n_+\psi^k_+\partial_-x^m \\ &-\frac{i}{2}\hat{\tilde{R}}^i_{abc}T^b_nT^c_kT^a_m\psi^n_+\psi^k_+\partial_-x^m = (M^i_{mn} - T^i_j\Gamma^j_{mn})\partial_+x^m\partial_-x^n - \frac{i}{2}T^i_jR^j_{mnk}\psi^n_+\psi^k_+\partial_-x^m\end{aligned}$$

which can be split into the following equations

$$(\tilde{H}^i_{jk} - \tilde{\Gamma}^i_{jk})T^j_mT^k_n = -M^i_{mn} + T^i_j\Gamma^j_{mn} = M^i_{[nm]} \tag{4.1.38}$$

$$\frac{1}{2}T^i_jR^j_{mnk} = \frac{1}{2}\hat{\tilde{R}}^i_{abc}T^b_nT^c_kT^a_m - (\tilde{H}^i_{jl} - \tilde{\Gamma}^i_{jl})M^j_{[kn]}T^l_m \tag{4.1.39}$$

we see that (4.1.36) and (4.1.38) are the same equations (by means of equation (4.1.17)). It is evident that right hand side of equation (4.1.38) is equal to the antisymmetric part of M^i_{nm}, and therefore, $\tilde{\Gamma}^i_{jk} = 0$. Equation (4.1.39) can be written as

$$\frac{1}{2}T^i_jR^j_{mnk} = \frac{1}{2}(\hat{\tilde{R}}^i_{abc} - 2\tilde{H}^i_{ja}\tilde{H}^j_{bc})T^b_nT^c_kT^a_m \tag{4.1.40}$$

where we used (4.1.38). $\tilde{H}$ can be figured out by (4.1.38) using the initial values of T and M, hence it is easy to see that $\tilde{H}^i_{mn} = M^i_{[nm]}(0)$. Therefore, we can write $\hat{\tilde{R}}^i_{abc} - 2\tilde{H}^i_{ja}\tilde{H}^j_{bc} = \tilde{R}^i_{abc}$, which leads to $R^i_{mnk} = \tilde{R}^i_{mnk}$ by equation (4.1.40). This means that curvatures will be related to each other by the relation $R^i_{mnk} = \tilde{R}^i_{mnk}$ around the point p on M where the transformation is identity, and $\tilde{R}^i_{mnk}$ is the curvature at point $\tilde{p}$. In this case all the points on manifold M will be mapped to only one point $\tilde{p}$ on $\tilde{M}$ where riemann normal coordinates are used.

4.1.2 Orthonormal Coframes

In this case we will present pseudoduality equations on the orthonormal coframe $SO(\mathbb{M})$. Equations of motion following from the action (4.1.2) in terms of the superfields are

$$X^k_{+-} = X^k_{-+} = -[\mathbf{\Gamma}^k_{ij}(X) - \mathcal{H}^k_{ij}(X)]X^i_+ X^j_- \tag{4.1.41}$$

where superfield X has the form (4.1.1), $D_+X = X_+$ and $\partial_- X = X_-$. We choose an orthonormal frame $\{\Lambda^i\}$ with the riemannian connection Λ^i_j on the superspace. If the superspace coordinates are defined by $z = (\sigma^\pm, \theta^+)$, then one form is given by

$$\Lambda^i = dz^M X^i_M \tag{4.1.42}$$

Covariant derivatives of X_M and X_{MN} will be

$$dX^i_M + \Lambda^i_j X^j_M = dz^N X^i_{MN} \tag{4.1.43}$$

The Cartan structural equations are

$$d\Lambda^i = -\Lambda^i_j \wedge \Lambda^j \tag{4.1.44}$$

$$d\Lambda^i_j = -\Lambda^i_k \wedge \Lambda^k_j + \Omega^i_j \tag{4.1.45}$$

where $\Omega^i_j = \frac{1}{2}\mathcal{R}^i_{jkl}\Lambda^k \wedge \Lambda^l$ is the curvature two form. Pseudoduality equations (4.1.7) and (4.1.8) are

$$\tilde{X}^i_{\pm} = \pm \mathcal{T}^i_j X^j_{\pm} \tag{4.1.46}$$

where $\mathcal{T}$ depends on superfield X. Taking the exterior derivative of both sides yields

$$d\tilde{X}^i_{\pm} = \pm d\mathcal{T}^i_j X^j_{\pm} \pm \mathcal{T}^i_j dX^j_{\pm}$$

Inserting (4.1.43) in this equation gives

$$-\tilde{\Lambda}^i_j \tilde{X}^j_{\pm} + dz^N \tilde{X}^i_{\pm N} = \pm d\mathcal{T}^i_j X^j_{\pm} \mp \mathcal{T}^i_j \Lambda^j_k X^k_{\pm} \pm dz^N \mathcal{T}^i_j X^j_{\pm N}$$

We now substitute (4.1.46) and arrange the terms to get

$$dz^N \tilde{X}^i_{\pm N} = \pm(d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_{\pm} \pm dz^N \mathcal{T}^i_j X^j_{\pm N}$$

We wedge the plus equation (upper sign) by dz^+ and minus equation (lower sign) by dz^-, and find the following equations

$$dz^+ \wedge dz^- \tilde{X}^i_{+-} = dz^+ \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_+ + dz^+ \wedge dz^- \mathcal{T}^i_j X^j_{+-} \tag{4.1.47}$$

$$dz^- \wedge dz^+ \tilde{X}^i_{-+} = -dz^- \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_- - dz^- \wedge dz^+ \mathcal{T}^i_j X^j_{-+} \tag{4.1.48}$$

Since $X_{+-} = X_{-+}$ (also with tilde) and $dz^+ \wedge dz^- = dz^- \wedge dz^+$ we may find the constraint relations by equating left hand sides

$$\begin{aligned} 2dz^+ \wedge dz^- \mathcal{T}^i_k X^k_{+-} + dz^+ \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_+ \\ + dz^- \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_- = 0 \end{aligned} \tag{4.1.49}$$

we substitute the equations of motion (4.1.41)

$$\begin{aligned} & - 2dz^+ \wedge dz^- \mathcal{T}^i_k [\mathbf{\Gamma}^k_{mn} - \mathcal{H}^k_{mn}] X^m_+ X^n_- + dz^+ \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_+ \\ & + dz^- \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_- = 0 \end{aligned} \tag{4.1.50}$$

and we use $dz^\pm X^n_\pm = \Lambda^n - dz^\mp X^n_\mp$ to get

$$\begin{aligned} & - dz^+ \wedge \mathcal{T}^i_k (\mathbf{\Gamma}^k_{mn} - \mathcal{H}^k_{mn}) X^m_+ \Lambda^n - dz^- \wedge \mathcal{T}^i_k (\mathbf{\Gamma}^k_{mn} + \mathcal{H}^k_{mn}) X^m_- \Lambda^n \qquad (4.1.51) \\ & + dz^+ \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_+ + dz^- \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_- = 0 \end{aligned}$$

Now we define the following tensors

$$dz^- \mathcal{U}^i_{k-} = (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) - \mathcal{T}^i_j (\mathbf{\Gamma}^j_{kn} - \mathcal{H}^j_{kn}) \Lambda^n \tag{4.1.52}$$

$$dz^+ \mathcal{U}^i_{k+} = -(d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) + \mathcal{T}^i_j (\mathbf{\Gamma}^j_{kn} + \mathcal{H}^j_{kn}) \Lambda^n \tag{4.1.53}$$

which satisfies the equation (4.1.51)

$$dz^+ \wedge dz^- \mathcal{U}^i_{k-} X^k_+ - dz^- \wedge dz^+ \mathcal{U}^i_{k+} X^k_- = 0 \tag{4.1.54}$$

They also yield the result

$$dz^{-}\mathcal{U}^{i}_{k-} + dz^{+}\mathcal{U}^{i}_{k+} = 2\mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}\Lambda^{n} \tag{4.1.55}$$

which gives

$$dz^{+} \wedge dz^{-}\mathcal{U}^{i}_{k-} = 2dz^{+} \wedge \mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}\Lambda^{n} \tag{4.1.56}$$

$$dz^{-} \wedge dz^{+}\mathcal{U}^{i}_{k+} = 2dz^{-} \wedge \mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}\Lambda^{n} \tag{4.1.57}$$

If these equations are substituted into (4.1.54), one obtains

$$2dz^{+} \wedge \mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}\Lambda^{n}X^{k}_{+} - 2dz^{-} \wedge \mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}\Lambda^{n}X^{k}_{-} = 0 \tag{4.1.58}$$

and using (4.1.42) gives the final result

$$2dz^{+} \wedge dz^{-}\mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}X^{n}_{-}X^{k}_{+} - 2dz^{-} \wedge dz^{+}\mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}X^{n}_{+}X^{k}_{-} = 0 \tag{4.1.59}$$

which shows that

$$dz^{+} \wedge dz^{-}\mathcal{T}^{i}_{j}\mathcal{H}^{j}_{kn}X^{k}_{+}X^{n}_{-} = 0 \tag{4.1.60}$$

Therefore, we conclude that $\mathcal{H} = 0$, and $\mathcal{U}^{i}_{k-} = \mathcal{U}^{i}_{k+} = 0$ by equations (4.1.56) and (4.1.57). Finally equation (4.1.52) and (4.1.53) gives the following result

$$(d\mathcal{T}^{i}_{k} - \mathcal{T}^{i}_{j}\Lambda^{j}_{k} + \tilde{\Lambda}^{i}_{j}\mathcal{T}^{j}_{k}) = \mathcal{T}^{i}_{j}\mathbf{\Gamma}^{j}_{kn}\Lambda^{n} \tag{4.1.61}$$

If we insert the equations of motion into (4.1.47) and (4.1.48), we obtain

$$-dz^+ \wedge dz^-(\tilde{\mathbf{\Gamma}}^i_{jk} - \tilde{\mathcal{H}}^i_{jk})\tilde{X}^j_+\tilde{X}^k_- = dz^+ \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j\Lambda^j_k + \tilde{\Lambda}^i_j\mathcal{T}^j_k)X^k_+$$
$$-dz^+ \wedge dz^-\mathcal{T}^i_j\mathbf{\Gamma}^i_{mn}X^m_+X^n_- \tag{4.1.62}$$
$$-dz^- \wedge dz^+(\tilde{\mathbf{\Gamma}}^i_{jk} - \tilde{\mathcal{H}}^i_{jk})\tilde{X}^j_+\tilde{X}^k_- = -dz^- \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j\Lambda^j_k + \tilde{\Lambda}^i_j\mathcal{T}^j_k)X^k_-$$
$$+dz^- \wedge dz^+\mathcal{T}^i_j\mathbf{\Gamma}^i_{mn}X^m_+X^n_- \tag{4.1.63}$$

Inserting $dz^-X^k_- = \Lambda - dz^+X_+$ (also with tilde) and $\tilde{X}_+$ (4.1.46) into (4.1.62), and $dz^+X_+ = \Lambda - dz^-X_-$ (also with tilde) and $\tilde{X}_-$ (4.1.46) into (4.1.63) gives

$$-dz^+ \wedge (\tilde{\mathbf{\Gamma}}^i_{mn} - \tilde{\mathcal{H}}^i_{mn})T^m_k\tilde{\Lambda}^n = dz^+ \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j\Lambda^j_k + \tilde{\Lambda}^i_j\mathcal{T}^j_k) - dz^+ \wedge \mathcal{T}^i_j\Gamma^i_{kn}\Lambda^n \tag{4.1.64}$$

$$-dz^- \wedge (\tilde{\mathbf{\Gamma}}^i_{mn} + \tilde{\mathcal{H}}^i_{mn})T^m_k\tilde{\Lambda}^n = -dz^- \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j\Lambda^j_k + \tilde{\Lambda}^i_j\mathcal{T}^j_k) + dz^- \wedge \mathcal{T}^i_j\Gamma^i_{kn}\Lambda^n \tag{4.1.65}$$

where we cancelled out X^k_+ in (4.1.64) and X^k_- in (4.1.65). We notice that right-hand sides of these equations become zero by means of the constraint relation (4.1.61), and we are left with

$$(\tilde{\mathbf{\Gamma}}^i_{mn} - \tilde{\mathcal{H}}^i_{mn})\mathcal{T}^m_k\tilde{\Lambda}^n = 0 \tag{4.1.66}$$
$$(\tilde{\mathbf{\Gamma}}^i_{mn} + \tilde{\mathcal{H}}^i_{mn})\mathcal{T}^m_k\tilde{\Lambda}^n = 0 \tag{4.1.67}$$

This shows that on the transformed superspace we must have $\tilde{\mathbf{\Gamma}} = 0$ and $\tilde{\mathcal{H}}^i_{mn} = 0$. We may find the relation between curvatures of the spaces using (4.1.61). We may define the connection one form $\Lambda^j_k = \mathbf{\Gamma}^j_{kn}\Lambda^n$, and hence (4.1.61) is reduced to

$$(d\mathcal{T}^i_k - 2\mathcal{T}^i_j\Lambda^j_k + \tilde{\Lambda}^i_j\mathcal{T}^j_k) = 0 \tag{4.1.68}$$

Taking exterior derivative, and using again (4.1.68) together with (4.1.45) gives

$$\mathcal{T}^i_j \Omega^j_k = \tilde{\Omega}^i_j \mathcal{T}^j_k \tag{4.1.69}$$

where new orthonormal coframe is replaced by 2Λ with the same curvature two form Ω on the manifold $\mathbb{M}$. It is obvious that integrability condition of this equation followed by the use of (4.1.42) and (4.1.46) yields a curvature relation between two (1, 0) supersymmetric sigma models which tied together with pseudoduality, which can be reduced to the same results found in the previous section. The reason why we get a positive sign in curvature expression in component expansion method is because of anticommuting grassmann numbers. This gives that pseudoduality transformation can be performed only if two sigma models are based on symmetric spaces with opposite curvatures on target spaces $\mathbb{M}$ and $\tilde{\mathbb{M}}$.

4.2 Pseudoduality in (1, 1) Supersymmetric Sigma Models

In this case [8] the classical spacetime Σ can be enlarged to the superspace $\Xi^{1,1}$ by adding Grassmann coordinates of opposite chirality. We will have one left-handed supercharge Q_+, and one right-handed supercharge Q_- as given by (4.0.2). The supersymmetry algebra can be written as

$$\{Q_\pm, Q_\pm\} = 2iP_\pm \qquad \{Q_+, Q_-\} = 0$$

where $P_\pm = -\partial_\pm$. The supersymmetry transformations will be

$$\begin{aligned}
\delta_\epsilon x^i &= \epsilon^+ \psi^i_+ + \epsilon^- \psi^i_- \\
\delta_\epsilon \psi^i_+ &= i\epsilon^+ \partial_+ x^i + \epsilon^- (\Gamma^i_{jk} + H^i_{jk}) \psi^i_+ \psi^k_- \\
\delta_\epsilon \psi^i_- &= -\epsilon^+ (\Gamma^i_{jk} + H^i_{jk}) \psi^j_+ \psi^k_- + i\epsilon^- \partial_- x^i
\end{aligned}$$

where $\epsilon^{\pm}$ is the constant anticommuting parameter.

4.2.1 Components

The superfield is written as

$$X = x + \theta^{+}\psi_{+} + \theta^{-}\psi_{-} + \theta^{+}\theta^{-}F \tag{4.2.1}$$

where $X : \Xi^{1,1} \to M$. We define the $(1,1)$ superspace $\Xi^{1,1} = (\sigma^{+}, \sigma^{-}, \theta^{+}, \theta^{-})$, where (σ^{+}, σ^{-}) are the null coordinates, and (θ^{+}, θ^{-}) are the Grassman coordinates of opposite chirality. The action of the theory is

$$S = \int d^2\sigma d^2\theta (G_{ij} + B_{ij}) D_{+}X^{i} D_{-}X^{j} \tag{4.2.2}$$

where supercovariant derivatives are given in (4.0.1). Similar definitions can be written for pseudodual model with tilde. First order expansion of G_{ij} and B_{ij}, followed by the $d^2\theta$ integral gives

$$S = -\int d^2x[(g_{ij}+b_{ij})\partial_{+}x^{i}\partial_{-}x^{j} + ig_{ij}\psi_{+}^{i}\nabla_{-}^{(-)}\psi_{+}^{j} + ig_{ij}\psi_{-}^{i}\nabla_{+}^{(+)}\psi_{-}^{j} - \frac{1}{2}\hat{R}_{bnam}^{+}\psi_{+}^{m}\psi_{-}^{n}\psi_{+}^{a}\psi_{-}^{b}]$$

where $\nabla_{\pm}^{(\pm)}\psi_{\mp}^{j} = \nabla_{\pm}\psi_{\mp}^{j} \pm H_{mn}^{j}\psi_{\mp}^{m}\partial_{\pm}x^{n}$, and $\hat{R}_{bnam}^{\pm} = R_{bnam} \pm D_{a}H_{nmb} \mp D_{m}H_{nab} + H_{baj}H_{mn}^{j} - H_{naj}H_{mb}^{j}$.

Equations of motion following from this action will be

$$F^i = (\Gamma^i_{jk} - H^i_{jk})\psi^j_+\psi^k_- \tag{4.2.3}$$

$$\nabla^{(-)}_-\psi^i_+ = \frac{i}{2}(\hat{R}^+)^i_{jmn}\psi^n_-\psi^j_+\psi^m_- \tag{4.2.4}$$

$$\nabla^{(+)}_+\psi^i_- = \frac{i}{2}(\hat{R}^+)^i_{jmn}\psi^n_+\psi^j_-\psi^m_+ \tag{4.2.5}$$

$$\begin{aligned}\Box x^k =& i(\hat{R}^-)^k_{nim}\psi^i_+\psi^m_+\partial_- x^n + i(\hat{R}^+)^k_{nim}\psi^i_-\psi^m_-\partial_+ x^n \\ &- (\hat{D}^k\hat{R}^+_{bnam})\psi^m_+\psi^n_-\psi^a_+\psi^b_-\end{aligned} \tag{4.2.6}$$

where $\hat{D}^k\hat{R}^+_{bnam} = D^k\hat{R}^+_{bnam} + H^k_{jn}(\hat{R}^+)^j_{bam} - H^k_{jb}(\hat{R}^+)^j_{nam} + H^k_{ja}(\hat{R}^+)^j_{mbn} - H^k_{jm}(\hat{R}^+)^j_{abn}$.

Pseudoduality transformations are

$$D_+\tilde{X}^i = +\mathcal{T}^i_j D_+ X^j \tag{4.2.7}$$

$$D_-\tilde{X}^i = -\mathcal{T}^i_j D_- X^j \tag{4.2.8}$$

where $\mathcal{T}$ is a function of superfield (4.2.1). Transformation matrix $\mathcal{T}$ can be expanded as $\mathcal{T}(X) = T(x) + \theta^+\psi^k_+\partial_k T(x) + \theta^-\psi^k_-\partial_k T(x) + \theta^+\theta^- F^k\partial_k T(x) - \theta^+\theta^-\psi^k_+\psi^l_-\partial_k\partial_l T(x)$. If pseudoduality transformations are written in components, first equation (4.2.7) yields the following set of equations

$$\tilde{\psi}^i_+ = T^i_j\psi^j_+ \tag{4.2.9}$$

$$\tilde{F}^i = T^i_j F^j - M^i_{jk}\psi^j_+\psi^k_- \tag{4.2.10}$$

$$\partial_+\tilde{x}^i = T^i_j\partial_+ x^j + iM^i_{jk}\psi^j_+\psi^k_+ \tag{4.2.11}$$

$$\partial_+\tilde{\psi}^i_- = T^i_j\partial_+\psi^j_- - 2iM^i_{[jk]}\psi^j_+ F^k + M^i_{kj}\psi^j_-\partial_+ x^k + i\partial_l M^i_{[jk]}\psi^k_+\psi^l_-\psi^j_+ \tag{4.2.12}$$

where $M^i_{jk} = \partial_k T^i_j$, and $M^i_{[jk]}$ represents the antisymmetric part of M^i_{jk}. Second equation

(4.2.8) will produce

$$\tilde{\psi}^i_- = -T^i_j \psi^j_- \tag{4.2.13}$$

$$\tilde{F}^i = -T^i_j F^j + M^i_{kj} \psi^j_+ \psi^k_- \tag{4.2.14}$$

$$\partial_- \tilde{x}^i = -T^i_j \partial_- x^j - i M^i_{jk} \psi^j_- \psi^k_- \tag{4.2.15}$$

$$\partial_- \tilde{\psi}^i_+ = -T^i_j \partial_- \psi^j_+ - 2i M^i_{[jk]} \psi^j_- F^k - M^i_{kj} \psi^j_+ \partial_- x^k - i \partial_l M^i_{[jk]} \psi^k_- \psi^l_+ \psi^j_- \tag{4.2.16}$$

We can find constraint relations using these equations. If (4.2.10) is set equal to (4.2.14), and equation of motion (4.2.3) is used, the result follows

$$T^i_j (\Gamma^i_{mn} - H^j_{mn}) = M^i_{(mn)} \tag{4.2.17}$$

where $M^i_{(mn)}$ is the symmetric part of M^i_{mn}. We immediately notice that $H^j_{mn} = 0$, and we are left with

$$T^i_j \Gamma^j_{mn} = M^i_{(mn)} \tag{4.2.18}$$

We next take ∂_- of (4.2.9)and set equal to (4.2.16) followed by the equations of motion (4.2.3) and (4.2.4) to obtain

$$[2M^i_{(mn)} - 2T^i_j \Gamma^j_{mn}] \psi^m_+ \partial_- x^n = -i[\partial_a M^i_{[bc]} + 2M^i_{[ck]} \Gamma^k_{ab} + T^i_j R^j_{abc}] \psi^c_- \psi^a_+ \psi^b_- \tag{4.2.19}$$

Real part of this equation is simply (4.2.18), and complex part will produce

$$\partial_a M^i_{[bc]} + 2M^i_{[ck]} \Gamma^k_{ab} + T^i_j R^j_{abc} = 0 \tag{4.2.20}$$

We now take ∂_+ of (4.2.13) and set equal to (4.2.12) followed by the equations of motion (4.2.3) and (4.2.5) to get

$$[2M^i_{(mn)} - 2T^i_j\Gamma^j_{mn}]\psi^m_-\partial_+x^n = -i[\partial_a M^i_{[bc]} + 2M^i_{[ck]}\Gamma^k_{ab} + T^i_j R^j_{abc}]\psi^c_+\psi^a_-\psi^b_+ \tag{4.2.21}$$

This equation is similar to (4.2.19), and we again notice that real part of this equation is equal to (4.2.18), and complex part is (4.2.20). We finally take ∂_- of (4.2.11), ∂_+ of (4.2.15), and set them equal to each other to find out the remaining constraints

$$\begin{aligned}2M^i_{(jk)}\partial_+x^j\partial_-x^k + 2T^i_j\partial^2_{+-}x^j &= 2iM^i_{[kj]}\psi^j_+\partial_-\psi^k_+ + i\partial_n M_{[kj]}\psi^j_+\psi^k_+\partial_-x^n \\ &+ 2iM^i_{[kj]}\psi^j_-\partial_+\psi^k_- + i\partial_n M_{[kj]}\psi^j_-\psi^k_-\partial_+x^n \end{aligned}\tag{4.2.22}$$

using equations of motion for $\partial^2_{+-}x^j$ (4.2.6), $\partial_-\psi^j_+$ (4.2.4) and $\partial_+\psi^j_-$ (4.2.5) yields

$$\begin{aligned}&(2M^i_{(mn)} - 2T^i_j\Gamma^j_{mn})\partial_+x^m\partial_-x^n + i(T^i_jR^j_{abc} + 2M^i_{[kb]}\Gamma^k_{ca} + \partial_a M^i_{[bc]})\psi^b_+\psi^c_+\partial_-x^a \\ &+ i(T^i_jR^j_{abc} + 2M^i_{[kb]}\Gamma^k_{ca} + \partial_a M^i_{[bc]})\psi^b_-\psi^c_-\partial_+x^a \\ &- (T^i_jD^jR_{abcd} + M^i_{[dk]}R^k_{cab} + M^i_{[bk]}R^k_{acd})\psi^d_+\psi^b_-\psi^c_+\psi^a_- = 0\end{aligned}\tag{4.2.23}$$

If this equation is split into real and complex parts the following results are obtained

$$(2M^i_{(mn)} - 2T^i_j\Gamma^j_{mn})\partial_+x^m\partial_-x^n = (T^i_jD^jR_{abcd} + M^i_{[dk]}R^k_{cab} + M^i_{[bk]}R^k_{acd})\psi^d_+\psi^b_-\psi^c_+\psi^a_-$$

$$\begin{aligned}&(T^i_jR^j_{abc} + 2M^i_{[kb]}\Gamma^k_{ca} + \partial_a M^i_{[bc]})\psi^b_+\psi^c_+\partial_-x^a \\ &+ (T^i_jR^j_{abc} + 2M^i_{[kb]}\Gamma^k_{ca} + \partial_a M^i_{[bc]})\psi^b_-\psi^c_-\partial_+x^a = 0\end{aligned}$$

First equation leads to the following results

$$M^i_{(mn)} = T^i_j \Gamma^j_{mn} \tag{4.2.24}$$

$$T^i_j D^j R_{abcd} + M^i_{[dk]} R^k_{cab} + M^i_{[bk]} R^k_{acd} = 0 \tag{4.2.25}$$

where (4.2.24) is the same as (4.2.18). Second equation gives

$$T^i_j R^j_{abc} + 2M^i_{[kb]} \Gamma^k_{ca} + \partial_a M^i_{[bc]} = 0 \tag{4.2.26}$$

which is the same equation as (4.2.20) with $b \leftrightarrow c$. Obviously we have three independent constraint relations, which are (4.2.18), (4.2.20), and (4.2.25).

Now we can find out pseudodual fields, and relations between two sigma models based on M and $\tilde{M}$ by means of pseudoduality equations. Using (4.2.10) or (4.2.14), and equation of motion (4.2.3) for F^j we get

$$\tilde{F}^i = M^i_{[nm]} \psi^m_+ \psi^n_- \tag{4.2.27}$$

Also definition of $\tilde{F}^i$ gives that

$$\begin{aligned} \tilde{F}^i &= (\tilde{\Gamma}^i_{jk} - \tilde{H}^i_{jk}) \tilde{\psi}^j_+ \tilde{\psi}^k_- \\ &= -(\tilde{\Gamma}^i_{jk} - \tilde{H}^i_{jk}) T^j_m T^k_n \psi^m_+ \psi^n_- \end{aligned} \tag{4.2.28}$$

where we used (4.2.9) and (4.2.13). Comparison of (4.2.27) with (4.2.28) gives that

$$(\tilde{\Gamma}^i_{jk} - \tilde{H}^i_{jk}) T^j_m T^k_n = M^i_{[mn]} \tag{4.2.29}$$

Hence we obtain that $\tilde{\Gamma}^i_{jk} = 0$. This means that pseudoduality transformation will be from any point on M to only one point where $\tilde{\Gamma}$ vanishes on $\tilde{M}$. We know that this is consistent with Riemann normal coordinates. We are left with

$$\tilde{H}^i_{jk} T^j_m T^k_n = M^i_{[nm]} \tag{4.2.30}$$

We next consider (4.2.12). Using equations of motion (4.2.3) and (4.2.5) we obtain

$$\partial_+ \tilde{\psi}^i_- = M^i_{[mn]} \psi^m_- \partial_+ x^n - \frac{i}{2} T^i_j R^j_{abc} \psi^c_+ \psi^a_- \psi^b_+ \tag{4.2.31}$$

where we used the constraint (4.2.20). On the other hand we can write the equation of motion (4.2.5) on $\tilde{M}$ as

$$\begin{aligned} \partial_+ \tilde{\psi}^i_- &= -\tilde{H}^i_{jk} \tilde{\psi}^j_- \partial_+ \tilde{x}^k + \frac{i}{2} (\hat{\tilde{R}}^+)^i_{jmn} \tilde{\psi}^n_+ \tilde{\psi}^j_- \tilde{\psi}^m_+ \\ &= \tilde{H}^i_{jk} T^j_m T^k_n \psi^m_- \partial_+ x^n + i(\tilde{H}^i_{jk} T^j_a M^k_{[bc]} - \frac{1}{2} (\hat{\tilde{R}}^+)^i_{jmn} T^n_c T^j_a T^m_b) \psi^c_+ \psi^a_- \psi^b_+ \end{aligned} \tag{4.2.32}$$

where we used (4.2.9), (4.2.11) and (4.2.13) in the first line of (4.2.32). If we compare (4.2.31) with (4.2.32) we see that

$$\tilde{H}^i_{jk} T^j_m T^k_n = M^i_{[mn]} \tag{4.2.33}$$

$$\frac{1}{2} T^i_j R^j_{abc} = \frac{1}{2} (\hat{\tilde{R}}^+)^i_{jmn} T^n_c T^j_a T^m_b - \tilde{H}^i_{jk} T^j_a M^k_{[bc]} \tag{4.2.34}$$

From (4.2.30) and (4.2.33) it is obvious that antisymmetric part of M^i_{mn} disappears, $M^i_{[mn]} = 0$, which leads to the result $\tilde{H}^i_{jk} = 0$. Hence (4.2.34) is reduced to

$$T^i_j R^j_{abc} = \tilde{R}^i_{jmn} T^n_c T^j_a T^m_b \tag{4.2.35}$$

We now simplify right hand side of (4.2.16). We use equations of motion (4.2.3) and

(4.2.4) and arrange the terms to get

$$\partial_- \tilde{\psi}^i_+ = \frac{i}{2} T^i_j R^j_{abc} \psi^c_- \psi^a_+ \psi^b_- \tag{4.2.36}$$

where we used the constraint (4.2.20). Also equation of motion for $\partial_- \tilde{\psi}^i_+$ on $\tilde{M}$ gives

$$\begin{aligned}\partial_- \tilde{\psi}^i_+ &= \frac{i}{2} \tilde{R}^i_{jmn} \tilde{\psi}^n_- \tilde{\psi}^j_+ \tilde{\psi}^m_- \\ &= \frac{i}{2} \tilde{R}^i_{jmn} T^n_a T^j_b T^m_c \psi^a_- \psi^b_+ \psi^c_-\end{aligned} \tag{4.2.37}$$

A comparison of (4.2.36) with (4.2.37) gives (4.2.35). When we take ∂_- of 4.2.11, and using relevant equations of motion together with the constraints (4.2.20) and (4.2.25) gives

$$\partial^2_{+-} \tilde{x}^i = \frac{i}{2} T^i_j R^j_{abc} \psi^b_- \psi^c_- \partial_+ x^a - \frac{i}{2} T^i_j R^j_{abc} \psi^b_+ \psi^c_+ \partial_- x^a \tag{4.2.38}$$

Likewise on $\tilde{M}$ we obtain

$$\begin{aligned}\partial^2_{+-} \tilde{x}^i =& \frac{i}{2} \tilde{R}^i_{abc} \tilde{\psi}^b_+ \tilde{\psi}^c_+ \partial_- \tilde{x}^a + \frac{i}{2} \tilde{R}^i_{abc} \tilde{\psi}^b_- \tilde{\psi}^c_- \partial_+ \tilde{x}^a - \frac{1}{2} \tilde{D}^i \tilde{R}_{abcd} \tilde{\psi}^d_+ \tilde{\psi}^b_- \tilde{\psi}^c_+ \tilde{\psi}^a_- \\ =& -\frac{i}{2} \tilde{R}^i_{mnk} T^n_b T^k_c T^m_a \psi^b_+ \psi^c_+ \partial_- x^a + \frac{i}{2} \tilde{R}^i_{mnk} T^n_b T^k_c T^m_a \psi^b_- \psi^c_- \partial_+ x^a \\ & - \frac{1}{2} \tilde{D}^i \tilde{R}_{jkmn} T^n_d T^k_b T^m_c T^j_a \psi^d_+ \psi^b_- \psi^c_+ \psi^a_-\end{aligned} \tag{4.2.39}$$

A quick comparison shows that we obtain equation (4.2.35), and $\tilde{D}^i \tilde{R}_{jkmn} = 0$. We notice that covariant derivatives of curvatures on both spaces vanish while curvatures are constants, and related to each other by (4.2.35). This obeys that both models are based on symmetric spaces.

4.2.2 Orthonormal Coframes

Equations of motion following from (4.2.2) are

$$X^k_{-+} = -[\mathbf{\Gamma}^k_{ij}(X) - \mathcal{H}^k_{ij}(X)]X^i_+ X^j_- \tag{4.2.40}$$

where $X_+ = D_+X$, $X_- = D_-X$ and $X_{-+} = D_+D_-X$. On the contrary to (1, 0) case, this time one writes that $X_{-+} = -X_{+-}$ and $\{X_+, X_-\} = 0$, where $\{,\}$ defines the anticommutation. Superspace coordinates are $z = (\sigma^{\pm}, \theta^{\pm})$, and orthonormal frame can be chosen as $\{\Lambda^i\}$ with connection one form $\{\Lambda^i_j\}$. Similar to (4.1.42) and (4.1.43) one form $\{\Lambda^i\}$ and covariant derivative of X_M can be written as

$$\Lambda^i = dz^M X^i_M \tag{4.2.41}$$

$$dX^i_M + \Lambda^i_j X^j_M = dz^N X^i_{MN} \tag{4.2.42}$$

Pseudoduality relations are

$$\tilde{X}^i_{\pm} = \pm \mathcal{T}^i_j X^j_{\pm} \tag{4.2.43}$$

We are going to mimic the calculations performed in (1, 0) case except notable differences $dz^+ \wedge dz^- = -dz^- \wedge dz^+$, $X_{+-} = -X_{-+}$, and $X_+X_- = -X_-X_+$. We take exterior derivative of (4.2.43), and then use (4.2.42) for both manifolds, and arrange the terms to get

$$dz^N \tilde{X}^i_{\pm N} = \pm(d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_{\pm} \pm dz^N \mathcal{T}^i_j X^j_{\pm N} \tag{4.2.44}$$

We wedge the plus equation by dz^+ and minus equation by dz^- to get

$$dz^+ \wedge dz^- \tilde{X}_{+-} = dz^+ \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_+ + dz^+ \wedge dz^- \mathcal{T}^i_j X^j_{+-} \tag{4.2.45}$$

$$dz^- \wedge dz^+ \tilde{X}_{-+} = -dz^- \wedge (d\mathcal{T}^i_k - \mathcal{T}^i_j \Lambda^j_k + \tilde{\Lambda}^i_j \mathcal{T}^j_k) X^k_- - dz^- \wedge dz^+ \mathcal{T}^i_j X^j_{-+} \tag{4.2.46}$$

we set left-hand sides equal to each other using $\tilde{X}_{+-} = -\tilde{X}_{-+}$ and $dz^+ \wedge dz^- = -dz^- \wedge dz^+$. We notice that we have symmetric expression which has antisymmetric terms in pairs. Therefore expressions from (4.1.49) to (4.1.69) can be repeated. This ends up with the same result, curvatures of the supersymmetric sigma models will be constant and opposite to each other, yielding the dual symmetric spaces.

4.3 Pseudoduality in Super WZW Models

At this point it is interesting to discuss the pseudoduality transformations on super WZW models [12]. The super WZW model has considerable interest in the context of conformal field theory. We use the superspace with coordinates (σ^+, σ^-, θ^+, θ^-) where $\sigma^\pm$ are the standard lightcone coordinates, and $\theta^\pm$ are the real Grassmann numbers, with supercharges $Q_\pm = \partial_{\theta^\pm} - i\theta^\pm \partial_\pm$ and supercovariant derivatives $D_\pm = \partial_{\theta^\pm} + i\theta^\pm \partial_\pm$. To define super WZW model we introduce the superfield $\mathcal{G}(\sigma, \theta)$ in G with components as expanded by

$$\mathcal{G}(\sigma, \theta) = g(\sigma)(1 + i\theta^+ \psi_+(\sigma) + i\theta^- \psi_-(\sigma) + i\theta^+ \theta^- \chi(\sigma)) \tag{4.3.1}$$

where the fermions $\psi_\pm(\sigma)$ take values in **g**, and are the superpartners of the group-valued fields $g(\sigma)$. The field $\chi(\sigma)$ is the auxiliary field. The lagrangian of the model can be written as

$$\mathcal{L} = \frac{1}{2} Tr(D_+ \mathcal{G}^{-1} D_- \mathcal{G}) + \mathbf{\Gamma} \tag{4.3.2}$$

where $\mathbf{\Gamma}$ represents the WZ term. Equations of motion following from this lagrangian are

$$D_-(\mathcal{G}^{-1}D_+\mathcal{G}) = 0 \tag{4.3.3}$$

$$D_+[(D_-\mathcal{G})\mathcal{G}^{-1}] = 0 \tag{4.3.4}$$

There is a global symmetry $G_L \times G_R$ which gives the conserved super currents $\mathcal{J}_+^L = \mathcal{G}^{-1}D_+\mathcal{G}$ and $\mathcal{J}_-^R = (D_-\mathcal{G})\mathcal{G}^{-1}$.

We can write similar expressions related to pseudodual WZW model with tilde. One can write the pseudoduality transformations using the similarity with bosonic case

$$\tilde{\mathcal{G}}^{-1}D_+\tilde{\mathcal{G}} = +\mathcal{T}(\sigma,\theta)\mathcal{G}^{-1}D_+\mathcal{G} \tag{4.3.5}$$

$$\tilde{\mathcal{G}}^{-1}D_-\tilde{\mathcal{G}} = -\mathcal{T}(\sigma,\theta)\mathcal{G}^{-1}D_-\mathcal{G} \tag{4.3.6}$$

Taking D_- of first equation (4.3.5) followed by (4.3.3) yields that $D_-\mathcal{T}(\sigma,\theta) = 0$. If $\mathcal{T}(\sigma,\theta)$ is expanded as $\mathcal{T}(\sigma,\theta) = T(\sigma) + \theta^+\lambda_+ + \theta^-\lambda_- + \theta^+\theta^-N(\sigma)$, then the condition $D_-\mathcal{T}(\sigma,\theta)$ implies that $\lambda_- = 0$, $N(\sigma) = 0$, $\partial_-T(\sigma) = 0$ and $\partial_-\lambda_+ = 0$. Hence $\mathcal{T}$ turns out to be

$$\mathcal{T}(\sigma,\theta) = T(\sigma^+) + \theta^+\lambda_+(\sigma^+) \tag{4.3.7}$$

Taking D_+ of second equation (4.3.6) gives the following equation

$$D_+\mathcal{T}_j^j(\sigma,\theta) = (\tilde{f}^i_{mn}\mathcal{T}_j^m\mathcal{T}_k^n - f^m_{jk}\mathcal{T}_m^i)(\mathcal{G}^{-1}D_+\mathcal{G})^k \tag{4.3.8}$$

Before going further to solve this equation, it is convenient to find out the values of some fields in terms of components. A brief computation shows that

$$\mathcal{G}^{-1}D_+\mathcal{G} = i\psi_+ + i\theta^+(g^{-1}\partial_+ g - i\psi_+^2) + i\theta^-(\chi - i\psi_-\psi_+) \tag{4.3.9}$$
$$- \theta^+\theta^-(\partial_+\psi_- + [g^{-1}\partial_+ g, \psi_-] + [\psi_+, \chi])$$
$$\mathcal{G}^- D_-\mathcal{G} = i\psi_- - i\theta^+(\chi - i\psi_-\psi_+) + i\theta^-(g^{-1}\partial_- g - i\psi_-^2) \tag{4.3.10}$$
$$+ \theta^+\theta^-(\partial_-\psi_+ + [g^{-1}\partial_- g, \psi_+] + [\chi, \psi_-])$$
$$(D_-\mathcal{G})\mathcal{G}^{-1} = g\{i\psi_- - i\theta^+(\chi - i\psi_-\psi_+) + i\theta^-(g^{-1}\partial_- g + i\psi_-^2) \tag{4.3.11}$$
$$+ \theta^+\theta^-(\partial_-\psi_+ + [\psi_-, \chi]\}g^{-1}$$

Hence, the equation of motion (4.3.3) produces the following equations

$$\chi = i\psi_-\psi_+ \tag{4.3.12}$$
$$\partial_-\psi_+ = 0 \tag{4.3.13}$$
$$\partial_-(g^{-1}\partial_+ g - i\psi_+^2) = 0 \tag{4.3.14}$$
$$\partial_+\psi_- = [\psi_-, g^{-1}\partial_+ g] + [\chi, \psi_+] \tag{4.3.15}$$

and (4.3.4) yields that

$$\chi = i\psi_-\psi_+ \tag{4.3.16}$$
$$\partial_-\psi_+ = [\chi, \psi_-] \tag{4.3.17}$$
$$\partial_+\psi_- = [\psi_-, g^{-1}\partial_+ g] \tag{4.3.18}$$
$$\partial_+(g^{-1}\partial_- g + i\psi_-^2) = [g^{-1}\partial_- g + i\psi_-^2, g^{-1}\partial_+ g] \tag{4.3.19}$$

We see that (4.3.12) and (4.3.16) are the same expressions, and determines the auxiliary field in terms of ψ_- and ψ_+. (4.3.13) implies that ψ_+ depends on σ^+ only, and (4.3.17) points out that χ commutes with ψ_- as expected. (4.3.14) gives us the bosonic left current conservation law by means of (4.3.13). Comparison of (4.3.15) with (4.3.18) shows that χ commutes with ψ_+, and (4.3.18) is the fermionic equation of motion for ψ_-, which leads (4.3.19) to the bosonic right current conservation law. Finally we may eliminate $\psi_\pm^2$ terms because these are fermionic fields and anticommute with each other.

Therefore the fields (4.3.9)-(4.3.11) can be written in simplified forms as

$$\mathcal{G}^{-1}D_+\mathcal{G} = i\psi_+ + i\theta^+ g^{-1}\partial_+ g \tag{4.3.20}$$

$$\mathcal{G}^{-1}D_-\mathcal{G} = i\psi_- + i\theta^- g^{-1}\partial_- g + \theta^+\theta^-[g^{-1}\partial_- g, \psi_+] \tag{4.3.21}$$

$$(D_-\mathcal{G})\mathcal{G}^{-1} = ig\psi_- g^{-1} + i\theta^-(\partial_- g)g^{-1} \tag{4.3.22}$$

We can now solve the equation (4.3.8) using (4.3.7) and (4.3.20). A little computation gives the components of $\mathcal{T}(\sigma,\theta)$ as

$$(\lambda_+)^i_j = i(\tilde{f}^i_{mn}T^m_j T^n_k - f^m_{jk}T^i_m)\psi^k_+ \tag{4.3.23}$$

$$\begin{aligned}(\partial_+ T)^i_j =& (\tilde{f}^i_{mn}(\lambda_+)^m_j T^n_k + \tilde{f}^i_{mn}T^m_j(\lambda_+)^n_k - f^m_{jk}(\lambda_+)^i_m)\psi^k_+ \\ &+ (\tilde{f}^i_{mn}T^m_j T^n_k - f^m_{jk}T^i_m)(g^{-1}\partial_+ g)^k\end{aligned} \tag{4.3.24}$$

If (4.3.23) is inserted in (4.3.24) the result follows

$$\begin{aligned}(\partial_+ T)^i_j =& (\tilde{f}^i_{mn}T^m_j T^n_k - f^m_{jk}T^i_m)(g^{-1}\partial_+ g)^k - i\tilde{f}^i_{kl}\tilde{f}^l_{mn}(T^m_j T^k_b - T^k_j T^m_b)T^n_a\psi^a_+\psi^b_+ \\ &- i\tilde{f}^i_{kl}f^m_{ba}T^k_j T^l_m\psi^a_+\psi^b_+ + if^m_{jb}f^n_{ma}T^i_n\psi^a_+\psi^b_+\end{aligned} \tag{4.3.25}$$

We want to find perturbation solution, and we notice that the order of the term $g^{-1}\partial_+ g$ is proportional to the order of the term $\psi\psi$. We find the following perturbative result up to the second order terms after integrating (4.3.25)

$$T^i_j(\sigma^+) = T^i_j(0) + A^i_{jk}\int_0^{\sigma^+}(g^{-1}\partial_+ g)^k d\sigma'^+ + B^i_{jab}\int_0^{\sigma^+}\psi^a_+\psi^b_+ d\sigma'^+ + H.O. \tag{4.3.26}$$

where $T^i_j(0) = \delta^i_j$, $A^i_{jk} = (\tilde{f}^i_{jk} - f^i_{jk})$, and $B^i_{jab} = i(\tilde{f}^i_{ak}\tilde{f}^k_{bj} + \tilde{f}^i_{jk}f^k_{ab} + f^i_{ak}f^k_{bj})$. Therefore λ_+ may be written as

$$\begin{aligned}(\lambda_+)^i_j =& iA^i_{jk}\psi^k_+ + C^i_{jkc}\psi^k_+\int_0^{\sigma^+}(g^{-1}\partial_+ g)^c d\sigma'^+ \\ &+ iD^i_{jkcd}\psi^k_+\int_0^{\sigma^+}\psi^c_+\psi^d_+ d\sigma'^+ + H.O.\end{aligned} \tag{4.3.27}$$

where constants C^i_{jkc} and D^i_{jkcd} are

$$C^i_{jkc} = \tilde{f}^i_{jn}A^n_{kc} + \tilde{f}^i_{nk}A^n_{jc} - f^n_{jk}A^i_{nc} = (\tilde{f}^i_{nc}\tilde{f}^n_{jk} - \tilde{f}^i_{n[j}f^n_{ck]} + f^i_{nc}f^n_{jk}) \tag{4.3.28}$$

$$\begin{aligned}D^i_{jkcd} =& \tilde{f}^i_{jn}B^n_{kcd} + \tilde{f}^i_{nk}B^n_{jcd} - f^n_{jk}B^i_{ncd} = i\tilde{f}^i_{jn}\tilde{f}^n_{cm}\tilde{f}^m_{kd} + i\tilde{f}^i_{nk}\tilde{f}^n_{cm}\tilde{f}^m_{jd} \\ &+ i\tilde{f}^i_{jn}\tilde{f}^n_{km}f^m_{da} + i\tilde{f}^i_{nk}\tilde{f}^n_{jm}f^m_{dc} - i\tilde{f}^i_{cm}\tilde{f}^m_{nd}f^n_{jk} + i\tilde{f}^i_{jn}f^n_{cm}f^m_{kd} \\ &+ i\tilde{f}^i_{nk}f^n_{cm}f^m_{jd} - i\tilde{f}^i_{nm}f^m_{dc}f^n_{jk} - if^i_{cm}f^m_{nd}f^n_{jk}\end{aligned} \tag{4.3.29}$$

As seen we have an expression for the transformation matrix (4.3.7) up to the third order terms. We notice that T represents even order terms while λ_+ represents odd order terms. Now we can proceed to find expressions on $\tilde{\mathbb{M}}$ using pseudoduality equations (4.3.5) and (4.3.6). If (4.3.7) and (4.3.20) are substituted in the first equation we obtain

$$\tilde{\psi}^i_+ = T^i_j\psi^j_+ \tag{4.3.30}$$

$$(\tilde{g}^{-1}\partial_+\tilde{g})^i = T^i_j(g^{-1}\partial_+ g)^j + (\lambda_+)^i_j\psi^j_+ \tag{4.3.31}$$

We notice that both of these equations depend only on σ^+. Likewise inserting (4.3.7) and

(4.3.21) into second equation (4.3.6) leads to

$$(\lambda_+)^i_j \psi^j_- = 0 \tag{4.3.32}$$

$$\tilde{\psi}^i_- = -T^i_j \psi^j_- \tag{4.3.33}$$

$$(\tilde{g}^{-1}\partial_-\tilde{g})^i = -T^i_j (g^{-1}\partial_- g)^j \tag{4.3.34}$$

$$[\tilde{g}^{-1}\partial_-\tilde{g}, \tilde{\psi}_+]^i = -T^i_j [g^{-1}\partial_- g, \psi_+]^j + i(\lambda_+)^i_j (g^{-1}\partial_- g)^j \tag{4.3.35}$$

These are the pseudoduality equations in components. We observe that if ψ_- and ψ_+ are set to zero we obtain bosonic case pseudoduality equations as pointed out in ([7]). We see that the term $(\lambda_+)^i_j \psi^j_+$ in equation (4.3.31) gives us $(\lambda_+)^i_j \psi^j_+ = -i[\tilde{\psi}_+, \tilde{\psi}_+]^i_{\tilde{G}} + iT^i_j [\psi_+, \psi_+]^j_G = 0$. The last equation (4.3.35) gives us the constraint (4.3.23). The equation (4.3.32) is interesting because it tells us that $[\tilde{\psi}_-, \tilde{\psi}_+]^i = -T^i_j [\psi_-, \psi_+]^j$, which gives us two choices. First choice is $\lambda_+ = 0$ which leads to either

$$\tilde{f}^i_{mn} T^m_k T^n_l = T^i_j f^j_{kl} \tag{4.3.36}$$

if $\psi_+ \neq 0$. This yields that $\partial_+ T = 0$ as can be seen from (4.3.24), and hence we get a trivial case, flat space pseudoduality equations as follows

$$\tilde{\psi}^i_\pm = \pm \psi^i_\pm \tag{4.3.37}$$

$$(\tilde{g}^{-1}\partial_\pm \tilde{g})^i = \pm (g^{-1}\partial_\pm g)^i \tag{4.3.38}$$

where we choose T to be identity. Therefore we obtain $\tilde{f}^i_{jk} = f^i_{jk}$ in (4.3.36). Or we set $\psi_+ = 0$, and hence last term in (4.3.26) will be eliminated, so pseudoduality relations will be

$$\begin{aligned}\tilde{\psi}^i_- =& - \psi^i_- - [\psi_-, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_{\tilde{G}} \\ &+ [\psi_-, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_G + H.O.\end{aligned} \tag{4.3.39}$$

$$\begin{aligned}(\tilde{g}^{-1}\partial_\pm \tilde{g})^i =& \pm (g^{-1}\partial_\pm g)^i \pm [g^{-1}\partial_\pm g, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_{\tilde{G}} \\ &\mp [g^{-1}\partial_\pm g, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_G + H.O.\end{aligned} \tag{4.3.40}$$

where we introduced the bracket $[\ ,\ ]_{G/\tilde{G}}$ to represent the commutations in $G/\tilde{G}$. Second choice will eliminate ψ_- and hence we get whole expressions (4.3.26) and (4.3.27) for T and λ_+. Therefore we obtain the following perturbation fields

$$\begin{aligned}\tilde{\psi}^i_+ =& \psi^i_+ + [\psi_+, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_{\tilde{G}} - [\psi_+, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_G \\ &+ i \int_0^{\sigma^+} [\psi_+(\sigma'^+), [\psi_+(\sigma'^+), \psi_+(\sigma^+)]_{\tilde{G}}]^i_{\tilde{G}} d\sigma'^+ \\ &+ i \int_0^{\sigma^+} [\psi_+(\sigma'^+), [\psi_+(\sigma'^+), \psi_+(\sigma^+)]_G]^i_G d\sigma'^+ + H.O.\end{aligned} \tag{4.3.41}$$

$$\begin{aligned}(\tilde{g}^{-1}\partial_\pm \tilde{g})^i =& \pm (g^{-1}\partial_\pm g)^i \pm [g^{-1}\partial_\pm g, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_{\tilde{G}} \mp [g^{-1}\partial_\pm g, \int_0^{\sigma^+} (g^{-1}\partial_+ g) d\sigma'^+]^i_G \\ &\pm i \int_0^{\sigma^+} [\psi_+(\sigma'^+), [\psi_+(\sigma'^+), (g^{-1}\partial_\pm g)(\sigma^+)]_{\tilde{G}}]^i_{\tilde{G}} d\sigma'^+ \\ &\pm i \int_0^{\sigma^+} [\psi_+(\sigma'^+), [\psi_+(\sigma'^+), (g^{-1}\partial_\pm g)(\sigma^+)]_G]^i_G d\sigma'^+ + H.O.\end{aligned} \tag{4.3.42}$$

where the cross terms $[\ ,[\ ,\ ]_G]_{\tilde{G}}$ vanish.

We have already derived our pseudoduality equations, conditions inducing pseudoduality, and finally the perturbative expressions of the pseudodual fields up to the third (fourth) order terms, leading to conserved currents on the pseudodual model. Using these fields it is possible to construct left and right super currents on pseudodual manifold $\tilde{\mathbb{M}}$. It is apparent from the expression (4.3.20) that we can easily construct right super currents belonging to special cases discussed above. To find left super currents we use the method we traced in third section (3).

4.3.1 Supercurrents in Flat Space Pseudoduality

In this case structure constants of both models are the same, $\tilde{f} = f$, and pseudoduality relations are given by (4.3.37) and (4.3.38). We let $g = e^{Y}$, where Y is the lie algebra. Using the expansion in the third section (3)

$$g^{-1}\partial_{\pm}g = \frac{1-e^{-adY}}{adY}\partial_{\pm} = \sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)!}[Y, ..., [Y, \partial_{\pm}Y]] \tag{4.3.43}$$

where adY is the adjoint representation of Y, and $adY(Z) = [Y, Z]$. We know that bosonic currents are invariant under $g \longrightarrow g_R(\sigma^-)g_L(\sigma^+)$, hence we obtain that $g^{-1}\partial_+ g \longrightarrow g_L^{-1}\partial_+ g_L$, which is

$$g_L^{-1}\partial_+ g_L = \partial_+ Y_L - \frac{1}{2!}[Y_L, \partial_+ Y_L] + \frac{1}{3!}[Y_L, [Y_L, \partial_+ Y_L]] + ... \tag{4.3.44}$$

Now we impose that $Y = \sum_0^{\infty} \varepsilon^n y_n$, where ε is a small parameter. Thus we get the following lie algebra valued field up to the third order terms

$$\begin{aligned} g_L^{-1}\partial_+ g_L =& \varepsilon\partial_+ y_{L1} + \varepsilon^2(\partial_+ y_{L2} - \frac{1}{2}[y_{L1}, \partial_+ y_{L1}]) \\ &+ \varepsilon^3(\partial_+ y_{L3} - \frac{1}{2}[y_{L1}, \partial_+ y_{L2}] - \frac{1}{2}[y_{L2}, \partial_{L1}] + \frac{1}{6}[y_{L1}, [y_{L1}, \partial_+ y_{L1}]]) + \mathcal{O}(\varepsilon^4) \end{aligned} \tag{4.3.45}$$

In a similar way one can find the expression for $g^{-1}\partial_- g$ [**?**]

$$
\begin{aligned}
g^{-1}\partial_- g =& \varepsilon\partial_- y_{R1} + \varepsilon^2(\partial_- y_{R2} - [y_{L1}, \partial_- y_{R1}] - \frac{1}{2}[y_{R1}, \partial_- y_{R1}]) \\
& + \varepsilon^3(\partial_- y_{R3} - [y_{L2}, \partial_- y_{R1}] - [y_{L1}, \partial_- y_{R2}] - \frac{1}{2}[y_{R2}, \partial_- y_{R1}] - \frac{1}{2}[y_{R1}, \partial_- y_{R2}]) \\
& + \frac{1}{2}[y_{L1}, [y_{R1}, \partial_- y_{R1}]] + \frac{1}{2}[y_{L1}, [y_{L1}, \partial_- y_{R1}]] + \mathcal{O}(\varepsilon^4)
\end{aligned} \tag{4.3.46}
$$

Since it works all the way up we are going to do all our calculations up to the second order of ε for simplicity and demonstration. We can write similar expressions for the manifold $\tilde{M}$. Pseudoduality equation (4.3.38) gives infinite number of sub-pseudoduality equations, from which we may write the following expressions coming from up to the second order of ε terms

$$\partial_+ \tilde{y}_{L1} = \partial_+ y_{L1} \tag{4.3.47}$$

$$\partial_- \tilde{y}_{R1} = -\partial_- y_{R1} \tag{4.3.48}$$

$$\partial_+ \tilde{y}_{L2} - \frac{1}{2}[\tilde{y}_{L1}, \partial_+ \tilde{y}_{L1}] = \partial_+ y_{L2} - \frac{1}{2}[y_{L1}, \partial_+ y_{L1}] \tag{4.3.49}$$

$$\partial_- \tilde{y}_{R2} - [\tilde{y}_{L1}, \partial_- \tilde{y}_{R1}] - \frac{1}{2}[\tilde{y}_{R1}, \partial_- \tilde{y}_{R1}] = -\partial_- y_{R2} + [y_{L1}, \partial_- y_{R1}] + \frac{1}{2}[y_{R1}, \partial_- y_{R1}] \tag{4.3.50}$$

First equation yields that $\tilde{y}_{L1} = y_{L1} + C_{L1}$, where C_{L1} is constant, and the second equation gives $\tilde{y}_{R1} = -y_{R1} - C_{R1}$, where C_{R1} is constant. Inserting these result into last equation gives

$$\partial_- \tilde{y}_{R2} + \frac{1}{2}[\tilde{y}_{R1}, \partial_- \tilde{y}_{R1}] = -\partial_- y_{R2} + \frac{3}{2}[y_{R1}, \partial_- y_{R1}] \tag{4.3.51}$$

where we used the equality of structure constants. We found this because we need this term in the expansion of bosonic right current [3], which is

$$(\partial_- g_R)g_R^{-1} = \varepsilon\partial_- y_{R1} + \varepsilon^2(\partial_- y_{R2} + \frac{1}{2}[y_{R1}, \partial_- y_{R1}]) + \mathcal{O}(\varepsilon^3) \tag{4.3.52}$$

Hence bosonic right and left currents on $\tilde{M}$ in terms of nonlocal expressions will be

$$\tilde{J}^L_+ = \tilde{g}_L^{-1}\partial_+\tilde{g}_L = \varepsilon\partial_+ y_{L1} + \varepsilon^2(\partial_+ y_{L2} - \frac{1}{2}[y_{L1}, \partial_+ y_{L1}]) + \mathcal{O}(\varepsilon^3) \tag{4.3.53}$$

$$\tilde{J}^R_- = (\partial_-\tilde{g}_R)\tilde{g}_R^{-1} = -\varepsilon\partial_- y_{R1} - \varepsilon^2(\partial_- y_{R2} - \frac{3}{2}[y_{R1}, \partial_- y_{R1}]) + \mathcal{O}(\varepsilon^3) \tag{4.3.54}$$

Obviously these currents are conserved by means of (4.3.14) and (4.3.19). Now we consider the fermionic components, and we let $\psi_\pm = \sum_{n=1}^\infty \varepsilon^n \psi_{n\pm}$. We denote $\psi_\pm$ as the sum of right and left components $\psi_\pm = \psi_{R\pm}(\sigma^-) + \psi_{L\pm}(\sigma^+)$. But from (4.3.13) we understand that ψ_+ includes ψ_{L+} only. Pseudoduality relations (4.3.37) again yields infinite number of subequations

$$\tilde{\psi}_{Ln+} = \psi_{Ln+} \tag{4.3.55}$$

$$\tilde{\psi}_{(L/R)n-} = -\psi_{(L/R)n-} \tag{4.3.56}$$

which hold true for each n. Thus left and right supercurrents on $\tilde{M}$ in nonlocal terms up to the second order of ε will be

$$\begin{aligned}\tilde{\mathcal{J}}^L_+ &= \tilde{\mathcal{G}}^{-1}D_+\tilde{\mathcal{G}} = i\tilde{\psi}_+ + i\theta^+(\tilde{g}^{-1}\partial_+\tilde{g}) \\ &= i\varepsilon(\psi_{L1+} + \theta^+\partial_+ y_{L1}) + i\varepsilon^2\{\psi_{L2+} + \theta^+(\partial_+ y_{L2} - \frac{1}{2}[y_{L1}, \partial_+ y_{L1}])\} + \mathcal{O}(\varepsilon^3)\end{aligned} \tag{4.3.57}$$

[3] see (3) [16, 18, 22] for details of this expansion

$$\tilde{\mathcal{J}}_-^R = (D_-\tilde{\mathcal{G}})\tilde{\mathcal{G}}^{-1} = i\tilde{g}\tilde{\psi}_-\tilde{g}^{-1} + i\theta^-(\partial_-\tilde{g})\tilde{g}^{-1} \tag{4.3.58}$$
$$= -i\varepsilon(\psi_{1-} + \theta^-\partial_- y_{R1}) - i\varepsilon^2\{\psi_{2-} + [y_{L1}, \psi_{1-}] - [y_{R1}, \psi_{1-}]$$
$$+ \theta^-(\partial_- y_{R2} - \frac{3}{2}[y_{R1}, \partial_- y_{R1}])\} + \mathcal{O}(\varepsilon^3)$$

It is obvious from the equations of motion that these currents in nonlocal expressions are conserved.

4.3.2 Supercurrents in Anti-chiral Pseudoduality

Now we consider our second case where ψ_+ vanishes. In this case we need to be careful when using bracket relations because structure constants are different. We have already found our nonlocal expressions in (4.3.39) and (4.3.40). We use the same expansions of lie algebra Y and fermionic field ψ_- in the powers of ε as used in the previous part. Therefore pseudoduality relations up to the second order of ε yield the following equations

$$\tilde{\psi}^i_{1-} = -\psi^i_{1-} \tag{4.3.59}$$
$$\tilde{\psi}^i_{2-} = -\psi^i_{2-} - [\psi_{1-}, y_{L1}]^i_{\tilde{G}} + [\psi_{1-}, y_{L1}]^i_G \tag{4.3.60}$$

$$\partial_+\tilde{y}^i_{L1} = \partial_+ y^i_{L1} \tag{4.3.61}$$
$$\partial_+\tilde{y}^i_{L2} - [\tilde{y}_{L1}, \partial_+\tilde{y}_{L1}]^i_{\tilde{G}} = \partial_+ y^i_{L2} + \frac{1}{2}[y_{L1}, \partial_+ y_{L1}]^i_G - [y_{L1}, \partial_+ y_{L1}]^i_{\tilde{G}} \tag{4.3.62}$$

$$\partial_-\tilde{y}^i_{R1} = -\partial_- y^i_{R1} \tag{4.3.63}$$
$$\partial_-\tilde{y}^i_{R2} + \frac{1}{2}[\tilde{y}_{R1}, \partial_-\tilde{y}_{R1}]^i_{\tilde{G}} = -\partial_- y^i_{R2} + \frac{1}{2}[y_{R1}, \partial_- y_{R1}]^i_G + [y_{R1}, \partial_- y_{R1}]^i_{\tilde{G}} \tag{4.3.64}$$

We may find out nonlocal supercurrents on the pseudodual manifold using these expressions

$$\tilde{\mathcal{J}}_{+}^{L} = i\varepsilon\theta^{+}\partial_{+}y_{L1} + i\varepsilon^{2}\theta^{+}\{\partial_{+}y_{L2} + \frac{1}{2}[y_{L1}, \partial_{+}y_{L1}]_{G}^{i} - [y_{L1}, \partial_{+}y_{L1}]_{\tilde{G}}^{i}\} + \mathcal{O}(\varepsilon^{3}) \quad (4.3.65)$$

$$\begin{aligned}\tilde{\mathcal{J}}_{-}^{R} &= -i\varepsilon(\psi_{1-} + \theta^{-}\partial_{-}y_{R1}) - i\varepsilon^{2}\{\psi_{2-} + [y_{L1}, \psi_{1-}]_{G} - [y_{R1}, \psi_{1-}]_{\tilde{G}} \quad (4.3.66)\\ &+ \theta^{-}(\partial_{-}y_{R2} - \frac{1}{2}[y_{R1}, \partial_{-}y_{R1}]_{G} - [y_{R1}, \partial_{-}y_{R1}]_{\tilde{G}})\} + \mathcal{O}(\varepsilon^{3})\end{aligned}$$

Obviously these currents in nonlocal expressions are conserved provided that equations of motion are satisfied.

4.3.3 Supercurrents in Chiral Pseudoduality

We consider our final case where ψ_{-} disappears. We notice that there is a contribution of chiral part in the isometry T which leads to third order terms in the field expressions on the target space of pseudodual manifold as can be seen from equations (4.3.41) and (4.3.42). Again we keep in our minds that structure constants are different. If the same conventions for Y and ψ_{+} are used as above, then pseudoduality relations up to the second order of ε can be calculated. Expressions for the fields $\tilde{g}^{-1}\partial_{\pm}\tilde{g}$ are the same as (4.3.61)-(4.3.64), and expression for the chiral field (4.3.41) gives that

$$\tilde{\psi}_{L1+}^{i} = \psi_{L1+}^{i} \quad (4.3.67)$$

$$\tilde{\psi}_{L2+}^{i} = \psi_{L2+}^{i} + [\psi_{L1+}, y_{L1}]_{\tilde{G}}^{i} - [\psi_{L1+}, y_{L1}]_{G}^{i} \quad (4.3.68)$$

Then nonlocal conserved supercurrents are found to be

$$\tilde{\mathcal{J}}_+^L = i\varepsilon(\psi_{L1+} + \theta^+\partial_+ y_{L1}) + i\varepsilon^2\{\psi_{L2+} + \theta^+(\partial_+ y_{L2} + \frac{1}{2}[y_{L1}, \partial_+ y_{L2}]_G \tag{4.3.69}$$
$$- [y_{L1}, \partial_+ y_{L1}]_{\tilde{G}})\} + \mathcal{O}(\varepsilon^3)$$
$$\tilde{\mathcal{J}}_-^R = -i\varepsilon\theta^-\partial_- y_{R1} - i\varepsilon^2\theta^-(\partial_- y_{R2} - \frac{1}{2}[y_{R1}, \partial_- y_{R1}]_G - [y_{R1}, \partial_- y_{R1}]_{\tilde{G}}) + \mathcal{O}(\varepsilon^3) \tag{4.3.70}$$

It is noted that all these supercurrents are the complements of each other, and special cases of a more general one. Under the limiting conditions they are equal to each other. If we denote the bosonic and fermionic components by $\tilde{J}_B$ and $\tilde{J}_F$ then they are written as

$$\tilde{\mathcal{J}}_\pm^{L/R} = \pm\tilde{J}_F^{L/R} \pm \theta^\pm \tilde{J}_B^{L/R} \tag{4.3.71}$$

Since these super currents serve as the orthonormal frame on the pullback bundle of the target space of G, we may find the corresponding bosonic and fermionic curvatures using them. If $L^i = \mathcal{J}^i$ is the left invariant Cartan one form which satisfies the Maurer-Cartan equation

$$d\mathcal{J}^i + \frac{1}{2}f^i_{jk}\mathcal{J}^j \wedge \mathcal{J}^k = 0 \tag{4.3.72}$$

and $L^i_k = \frac{1}{2}f^i_{jk}\mathcal{J}^j$ is the antisymmetric riemannian connection, then Cartan structural equations on superspace can be written as

$$dL^i + L^i_j \wedge L^j = 0 \tag{4.3.73}$$
$$dL^i_j + L^i_k \wedge L^k_j = \frac{1}{2}\mathcal{R}^i_{jkl}L^k \wedge L^l \tag{4.3.74}$$

where $\mathcal{R}^i_{jkl}$ is the curvature of superspace. If the calculations in the previous section is repeated using these equations in this case one can show that curvatures on $SO(M)$ are constants, and related to each other by $\tilde{\mathcal{R}}^i_{jkl} = -\mathcal{R}^i_{jkl}$, which shows that two superspaces are dual symmetric spaces. If this curvature relation is split into bosonic and fermionic parts, it is easy to see that fermionic part will yield a curvature relation which are opposite to each other, i.e. $(\tilde{\mathcal{R}}_F)^i_{jkl} = -(\mathcal{R}_F)^i_{jkl}$, while bosonic part will give that both curvatures will be the same, i.e. $(\tilde{\mathcal{R}}_B)^i_{jkl} = (\mathcal{R}_B)^i_{jkl}$, because of anticommuting numbers. This is consistent with the results found in the component expansion methods.

Chapter 5

Pseudoduality In Supersymmetric Sigma Models on Symmetric Spaces

5.1 Motivation

In the previous two works we studied target space pseudoduality between symmetric space sigma models for scalar fields, and supersymmetric sigma models. In this work we will analyse pseudoduality in G/H supersymmetric sigma models in two respects, on the orthonormal coframe first, and then using components.

We will work in superspace with coordinates $(\sigma^{\pm}, \theta^{\pm})$, where $\sigma^{\pm}$ are the standard lighcone coordinates on two dimensional Minkowski space and $\theta^{\pm}$ are the fermionic coordinates which are real Grassmann numbers. Supersymmetry generating charges and supercovariant derivatives are given respectively by

$$Q_{\pm} = \partial_{\theta^{\pm}} - i\theta^{\pm}\partial_{\pm} \tag{5.1.1}$$

$$D_{\pm} = \partial_{\theta^{\pm}} + i\theta^{\pm}\partial_{\pm} \tag{5.1.2}$$

which obey

$$Q_{\pm}^2 = -i\partial_{\pm} \qquad D_{\pm}^2 = i\partial_{\pm} \tag{5.1.3}$$

with all other anti-commutators vanishing. Lagrangian of the model is defined by

$$\mathcal{L}_G = \frac{1}{2}Tr(D_+\mathcal{G}^{-1}D_-\mathcal{G}) + \Gamma \quad (5.1.4)$$

with Γ representing WZ term. We introduced the superfield $\mathcal{G}(\sigma, \theta)$ taking values in a compact Lie group G, which can be expanded in components by

$$\mathcal{G}(\sigma, \theta) = g(\sigma)(1 + i\theta^+\psi_+(\sigma) + i\theta^-\psi_-(\sigma) + i\theta^+\theta^-\chi(\sigma)) \quad (5.1.5)$$

where $\psi_\pm$ take values in Lie algebra **g**, and χ is the auxiliary field. The lagrangian (5.1.4) has a global symmetry $G_L \times G_R$ acting on the superfield $\mathcal{G}$ by left and right multiplication, which produces the following equations of motion

$$D_-(\mathcal{G}^{-1}D_+\mathcal{G}) = 0 \quad (5.1.6)$$

$$D_+[(D_-\mathcal{G})\mathcal{G}^{-1}] = 0 \quad (5.1.7)$$

and yields the conserved super currents $\mathcal{J}_+^L = \mathcal{G}^{-1}D_+\mathcal{G}$ and $\mathcal{J}_-^R = (D_-\mathcal{G})\mathcal{G}^{-1}$ taking values in **g**. We may write similar expressions for the pseudodual sigma model using tilde. We were able to write pseudoduality relations in the previous section as

$$\tilde{\mathcal{G}}^{-1}D_\pm\tilde{\mathcal{G}} = \pm\mathcal{T}(\sigma, \theta)\mathcal{G}^{-1}D_\pm\mathcal{G} \quad (5.1.8)$$

where $\mathcal{T}(\sigma, \theta)$ is expanded as

$$\mathcal{T}(\sigma, \theta) = T(\sigma) + \theta^+\lambda_+ + \theta^-\lambda_- + \theta^+\theta^-N(\sigma)$$

Equations of motion (5.1.6) implies that $\lambda_- = 0$, $N = 0$, and $T(\sigma)$ and λ_+ depends on

σ^+. We saw (4.3) that component expansion of pseudoduality equations leads to three conditions; flat space pseudoduality which gives $\lambda_+ = 0$, $T(\sigma) = id$ and Lie groups have to be same, (Anti)chiral pseudoduality which gives vanishing (ψ_+) ψ_- in both models with distinct Lie groups. We saw that derived conserved super currents serve as the orthonormal frame on the pullback bundle of the target space, we derived curvature relations between two manifolds, which are constants and opposite to each other, implying that both superspaces are the dual symmetric spaces. Motivated by this result we examine pseudoduality conditions in super WZW models based on symmetric spaces. We begin with orthonormal coframe method, and then figure out component expansions.

5.2 Orthonormal Coframe Method

We consider a closed subgroup H of a connected Lie group G. We know that symmetric space M is the coset space $M = G/H$ such that Lie algebras **h** of H and **m** of M are the orthogonal complements of each other, and $\mathbf{g} = \mathbf{m} + \mathbf{h}$, where **h** is closed under brackets while **m** is $Ad(H)$-invariant subspace of **g**, $Ad_h(\mathbf{m}) \subset \mathbf{m}$ for all $h \in H$. Symmetric space conditions are given by the bracket relations

$$[\mathbf{h}, \mathbf{h}] \subset \mathbf{h} \qquad [\mathbf{h}, \mathbf{m}] \subset \mathbf{m} \qquad [\mathbf{m}, \mathbf{m}] \subset \mathbf{h} \tag{5.2.1}$$

To distinguish space elements of different Lie algebras we will use the indices $i, j, k...$ for the space elements of **g**, $\alpha, \beta, \gamma...$ for the space elements of **m**, and $a, b, c...$ for the space elements of **h**. Therefore (5.2.1) leads to the only allowed structure constants f_{abc} and $f_{a\alpha\beta}$ up to permutations of indices.

Let us first formulate G/H sigma model on superspace before embarking on pseudoduality. $\mathcal{G}(\sigma, \theta)$ was defined in (5.1.5), and $\mathcal{J}^L_\pm = \mathcal{G}^{-1} D_\pm \mathcal{G} \ \in \ \mathbf{g}$ can be split as

$$\mathcal{J}^L_\pm = \mathcal{K}_\pm + \mathcal{A}_\pm \tag{5.2.2}$$

where $\mathcal{K}_\pm \in \mathbf{m}$ and $\mathcal{A}_\pm \in \mathbf{h}$. The Lagrangian for the G/H sigma model is

$$\mathcal{L}_{G/H} = \frac{1}{2} Tr(\mathcal{K}_+ \mathcal{K}_-) + \Gamma_{G/H} \tag{5.2.3}$$

where $\Gamma_{G/H}$ represents the Wess-Zumino term for G/H supersymmetric sigma model. Equations of motion following from (5.1.6), (5.1.7) and (5.2.1) are

$$\mathcal{K}_{+-} = 0 \qquad \mathcal{K}_{-+} = [\mathcal{K}_-, \mathcal{A}_+] + [\mathcal{A}_-, \mathcal{K}_+] \tag{5.2.4}$$

$$\mathcal{A}_{+-} = 0 \qquad \mathcal{A}_{-+} = [\mathcal{A}_-, \mathcal{A}_+] + [\mathcal{K}_-, \mathcal{K}_+] \tag{5.2.5}$$

We choose an orthonormal coframe $\{L^i\}$ with the Riemannian connection L^i_j on the superspace G. L^i is the left invariant Cartan one form, which satisfies the Cartan structural equations

$$dL^i + L^i_j \wedge L^j = 0 \tag{5.2.6}$$

$$dL^i_j + L^i_k \wedge L^k_j = \frac{1}{2} \mathcal{R}^i_{jkl} L^k \wedge L^l \tag{5.2.7}$$

The Maurer-Cartan equation

$$dL^i + \frac{1}{2} f^i_{jk} L^j \wedge L^k = 0 \tag{5.2.8}$$

leads to $L^i_k = \frac{1}{2} f^i_{jk} L^j$. If the superspace coordinates are given by $z = (\sigma^\pm, \theta^\pm)$, and

$L^i = dz^M L^i_M$, the covariant derivative of L^i can be written as

$$dL^i_M + L^i_j L^j_M = dz^N L^i_{MN} \tag{5.2.9}$$

The pseudoduality equations (5.1.8) are written as

$$\tilde{L}^i_{\pm} = \pm \mathcal{T}^i_j L^j_{\pm} \tag{5.2.10}$$

We already know how to solve these equations from previous sections. Now let us construct the symmetric space M and its complement H-space formulations. We will use the same symbols as the supercurrents to define orthonormal coframes and corresponding connections on superspaces M and H. Let $\mathcal{K}^\alpha$ ($\mathcal{A}^a$) be the orthonormal coframe, and $\mathcal{K}^\alpha_\beta$ ($\mathcal{A}^a_b$) be the Riemannian connection on subspace M (H).

5.2.1 Setting up the Theory on M

We already found the equations of motion in (5.2.4), where $\mathcal{K}^\alpha = dz^M \mathcal{K}^\alpha_M$. The Maurer-Cartan equation (5.2.8) can be written as

$$d\mathcal{K}^\alpha + f^\alpha_{\beta a} \mathcal{K}^\beta \wedge \mathcal{A}^a = 0 \tag{5.2.11}$$

which leads to the following connections by comparison to (5.2.13)

$$\mathcal{K}^\alpha_\beta = \frac{1}{2} f^\alpha_{a\beta} \mathcal{A}^a \qquad \mathcal{K}^\alpha_a = \frac{1}{2} f^\alpha_{\beta a} \mathcal{K}^\beta \tag{5.2.12}$$

Cartan structural equations can be split on M as

$$d\mathcal{K}^{\alpha} + \mathcal{K}^{\alpha}_{\beta} \wedge \mathcal{K}^{\beta} + \mathcal{K}^{\alpha}_{a} \wedge \mathcal{A}^{a} = 0 \tag{5.2.13}$$

$$d\mathcal{K}^{\alpha}_{\beta} + \mathcal{K}^{\alpha}_{\gamma} \wedge \mathcal{K}^{\gamma}_{\beta} + \mathcal{K}^{\alpha}_{a} \wedge \mathcal{A}^{a}_{\beta} = \frac{1}{2}\mathcal{R}^{\alpha}_{\beta\lambda\mu}\mathcal{K}^{\lambda} \wedge \mathcal{K}^{\mu} + \frac{1}{2}\mathcal{R}^{\alpha}_{\beta ab}\mathcal{A}^{a} \wedge \mathcal{A}^{b} + \mathcal{R}^{\alpha}_{\beta\lambda a}\mathcal{K}^{\lambda} \wedge \mathcal{A}^{a} \tag{5.2.14}$$

$$d\mathcal{K}^{\alpha}_{a} + \mathcal{K}^{\alpha}_{\gamma} \wedge \mathcal{K}^{\gamma}_{a} + \mathcal{K}^{\alpha}_{b} \wedge \mathcal{A}^{b}_{a} = \frac{1}{2}\mathcal{R}^{\alpha}_{a\lambda\mu}\mathcal{K}^{\lambda} \wedge \mathcal{K}^{\mu} + \frac{1}{2}\mathcal{R}^{\alpha}_{abc}\mathcal{A}^{b} \wedge \mathcal{A}^{c} + \mathcal{R}^{\alpha}_{a\lambda b}\mathcal{K}^{\lambda} \wedge \mathcal{A}^{b} \tag{5.2.15}$$

The covariant derivative (5.2.9) is written

$$d\mathcal{K}^{\alpha}_{M} + \mathcal{K}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{M} + \mathcal{K}^{\alpha}_{a}\mathcal{A}^{a}_{M} = dz^{N}\mathcal{K}^{\alpha}_{MN} \tag{5.2.16}$$

We observe that all the fields on **m**-space have additional mixing components to **h**-space, which leads us to write down the pseudoduality equations on **m**-space in a predictable way

$$\tilde{\mathcal{K}}^{\alpha}_{\pm} = \pm\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\pm} \pm \mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\pm} \tag{5.2.17}$$

We take the exterior derivative, use (5.2.16) and (5.2.33), and arrange the terms to get

$$d\tilde{\mathcal{K}}^{\alpha}_{\pm} = \pm d\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\pm} \pm \mathcal{T}^{\alpha}_{\beta}d\mathcal{K}^{\beta}_{\pm} \pm d\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\pm} \pm \mathcal{T}^{\alpha}_{a}d\mathcal{A}^{a}_{\pm} \tag{5.2.18}$$

$$\begin{aligned} dz^{N}\tilde{\mathcal{K}}^{\alpha}_{\pm N} = &\pm (d\mathcal{T}^{\alpha}_{\lambda} + \tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{\lambda} + \tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{\lambda} - \mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\lambda} - \mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\lambda})\mathcal{K}^{\lambda}_{\pm} \\ &\pm (d\mathcal{T}^{\alpha}_{b} + \tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{b} + \tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{b} - \mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{b} - \mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{b})\mathcal{A}^{b}_{\pm} \\ &\pm dz^{N}\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\pm N} \pm dz^{N}\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\pm N} \end{aligned} \tag{5.2.19}$$

we now wedge this equation by $dz^{\pm}$ to see the effect of equations of motion

$$\begin{aligned} dz^{\pm}\wedge dz^{\mp}\tilde{\mathcal{K}}^{\alpha}_{\pm\mp} =& \pm dz^{\pm}\wedge(d\mathcal{T}^{\alpha}_{\lambda}+\tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{\lambda}+\tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{\lambda}-\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\lambda}-\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\lambda})\mathcal{K}^{\lambda}_{\pm} \\ &\pm dz^{\pm}\wedge(d\mathcal{T}^{\alpha}_{b}+\tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{b}+\tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{b}-\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{b}-\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{b})\mathcal{A}^{b}_{\pm} \\ &\pm dz^{\pm}\wedge dz^{\mp}\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\pm\mp}\pm dz^{\pm}\wedge dz^{\mp}\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\pm\mp} \end{aligned} \tag{5.2.20}$$

Equations of motion (5.2.4) and (5.2.5) provide some cancellations, and we obviously see that $(+)$ equation gives us the following constraint relations

$$d\mathcal{T}^{\alpha}_{\lambda}+\tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{\lambda}+\tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{\lambda}-\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\lambda}-\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\lambda}=0 \tag{5.2.21}$$

$$d\mathcal{T}^{\alpha}_{b}+\tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{b}+\tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{b}-\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{b}-\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{b}=0 \tag{5.2.22}$$

where we treated $\mathcal{K}^{\lambda}_{+}$ and $\mathcal{A}^{b}_{+}$ as independent components, and we set these equations equal to zero because $d\mathcal{T}$ is a one form. $(-)$ equation has pure contributions from the equations of motion

$$dz^{-}\wedge dz^{+}\tilde{\mathcal{K}}^{\alpha}_{-+}=-dz^{-}\wedge dz^{+}(\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{-+}+\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{-+}) \tag{5.2.23}$$

We use the corresponding equations of motions, and obtain the result

$$\begin{aligned} dz^{-}\wedge dz^{+}(\tilde{f}^{\alpha}_{a\beta}\tilde{\mathcal{A}}^{a}_{+}\tilde{\mathcal{K}}^{\beta}_{-}+\tilde{f}^{\alpha}_{\beta a}\tilde{\mathcal{K}}^{\beta}_{+}\tilde{\mathcal{A}}^{a}_{-} =& -\mathcal{T}^{\alpha}_{\beta}f^{\beta}_{a\lambda}\mathcal{A}^{a}_{+}\mathcal{K}^{\lambda}_{-}-\mathcal{T}^{\alpha}_{\beta}f^{\beta}_{\lambda a}\mathcal{K}^{\lambda}_{+}\mathcal{A}^{a}_{-} \\ &-\mathcal{T}^{\alpha}_{a}f^{a}_{bc}\mathcal{A}^{b}_{+}\mathcal{A}^{c}_{-}-\mathcal{T}^{\alpha}_{a}f^{a}_{\beta\lambda}\mathcal{K}^{\beta}_{+}\mathcal{K}^{\lambda}_{-}) \end{aligned} \tag{5.2.24}$$

If we use the expansions $\mathcal{K}^{\alpha}=dz^{M}\mathcal{K}^{\alpha}_{M}$ and $\mathcal{A}^{a}=dz^{M}\mathcal{A}^{a}_{M}$, and the connection one forms (5.2.12) and (5.2.32) the result follows

$$\tilde{\mathcal{K}}^{\alpha}_{\beta}\tilde{\mathcal{K}}^{\beta}_{-}+\tilde{\mathcal{K}}^{\alpha}_{a}\tilde{\mathcal{A}}^{a}_{-}=-\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\lambda}\mathcal{K}^{\lambda}_{-}-\mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{b}\mathcal{A}^{b}_{-}-\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{b}\mathcal{A}^{b}_{-}-\mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\lambda}\mathcal{K}^{\lambda}_{-} \tag{5.2.25}$$

Now we use pseudoduality equations (5.2.17) and (5.2.37) for $\tilde{\mathcal{K}}^{\alpha}_{-}$ and $\tilde{\mathcal{A}}^{a}_{-}$, and compare the coefficients of $\mathcal{K}^{\lambda}_{-}$ and $\mathcal{A}^{b}_{-}$ to obtain the results

$$\tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{\lambda} + \tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{\lambda} = \mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{\lambda} + \mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{\lambda} \tag{5.2.26}$$

$$\tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{b} + \tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{b} = \mathcal{T}^{\alpha}_{\beta}\mathcal{K}^{\beta}_{b} + \mathcal{T}^{\alpha}_{a}\mathcal{A}^{a}_{b} \tag{5.2.27}$$

we immediately notice that if these results are substituted into (5.2.21) and (5.2.22) we obtain $d\mathcal{T}^{\alpha}_{\lambda} = d\mathcal{T}^{\alpha}_{b} = 0$. Therefore we conclude that $\mathcal{T}^{\alpha}_{\lambda}$ and $\mathcal{T}^{\alpha}_{b}$ must be constant, and we choose them to be identity. Hence the pseudoduality relations between symmetric spaces will simply be

$$\tilde{\mathcal{K}}^{\alpha}_{\pm} = \pm\mathcal{K}^{\alpha}_{\pm} \pm \mathcal{T}^{\alpha}_{a}(0)\mathcal{A}^{a}_{\pm} \tag{5.2.28}$$

Here $\mathcal{T}^{\alpha}_{a}(0)$ is the identity mapping which provides the mixing of H-space to $\tilde{M}$. From the relations (5.2.26) and (5.2.27), which can simply be written as

$$\tilde{\mathcal{K}}^{\alpha}_{\lambda} + \tilde{\mathcal{K}}^{\alpha}_{a}\mathcal{T}^{a}_{\lambda}(0) = \mathcal{K}^{\alpha}_{\lambda} + \mathcal{T}^{\alpha}_{a}(0)\mathcal{A}^{a}_{\lambda} \tag{5.2.29}$$

$$\tilde{\mathcal{K}}^{\alpha}_{\beta}\mathcal{T}^{\beta}_{b}(0) + \tilde{\mathcal{K}}^{\alpha}_{b} = \mathcal{K}^{\alpha}_{b} + \mathcal{T}^{\alpha}_{a}(0)\mathcal{A}^{a}_{b} \tag{5.2.30}$$

we may find relations between curvatures by means of (5.2.14) and (5.2.15). Since these equations require H-space connections, before going further it is worth to analyze H-space pseudoduality.

5.2.2 Pseudoduality on H

One form is defined by $\mathcal{A}^{a} = dz^{M}\mathcal{A}^{a}_{M}$. The Maurer-Cartan equation (5.2.8) corresponding to H-space will be

$$dA^{a} + \frac{1}{2}f^{a}_{bc}A^{b}\wedge A^{c} + \frac{1}{2}f^{a}_{\alpha\beta}K^{\alpha}\wedge K^{\beta} = 0 \tag{5.2.31}$$

Cartan structural equations are split as

$$d\mathcal{A}^a + \mathcal{A}^a_b \wedge \mathcal{A}^b + \mathcal{A}^a_\beta \wedge \mathcal{K}^\beta = 0 \tag{5.2.32}$$

$$d\mathcal{A}^a_b + \mathcal{A}^a_c \wedge \mathcal{A}^c_b + \mathcal{A}^a_\lambda \wedge \mathcal{A}^\lambda_b = \frac{1}{2}\mathcal{R}^a_{bcd}\mathcal{A}^c \wedge \mathcal{A}^d + \frac{1}{2}\mathcal{R}^a_{b\lambda\mu}\mathcal{K}^\lambda \wedge \mathcal{K}^\mu + \mathcal{R}^a_{bc\lambda}\mathcal{A}^c \wedge \mathcal{K}^\lambda \tag{5.2.33}$$

$$d\mathcal{A}^a_\alpha + \mathcal{A}^a_c \wedge \mathcal{A}^c_\alpha + \mathcal{A}^a_\lambda \wedge \mathcal{K}^\lambda_\alpha = \frac{1}{2}\mathcal{R}^a_{\alpha bc}\mathcal{A}^b \wedge \mathcal{A}^c + \frac{1}{2}\mathcal{R}^a_{\alpha\lambda\mu}\mathcal{K}^\lambda \wedge \mathcal{K}^\mu + \mathcal{R}^a_{\alpha b\lambda}\mathcal{A}^b \wedge \mathcal{K}^\lambda \tag{5.2.34}$$

A comparison of (5.2.31) to (5.2.32) gives the following connections

$$\mathcal{A}^a_c = \frac{1}{2}f^a_{bc}\mathcal{A}^b \qquad \mathcal{A}^a_\beta = \frac{1}{2}f^a_{\alpha\beta}\mathcal{K}^\alpha \tag{5.2.35}$$

The covariant derivative of $\mathcal{A}^a$ is

$$d\mathcal{A}^a_M + \mathcal{A}^a_b\mathcal{A}^b_M + \mathcal{A}^a_\lambda\mathcal{K}^\lambda_M = dz^N\mathcal{A}^a_{MN} \tag{5.2.36}$$

Using the same reasoning above we may write the pseudoduality equations on H-space as

$$\tilde{\mathcal{A}}^a_\pm = \pm\mathcal{T}^a_b\mathcal{A}^b_\pm \pm \mathcal{T}^a_\beta\mathcal{K}^\beta_\pm \tag{5.2.37}$$

We take the exterior derivative

$$d\tilde{\mathcal{A}}^a_\pm = \pm d\mathcal{T}^a_b\mathcal{A}^b_\pm \pm \mathcal{T}^a_b d\mathcal{A}^b_\pm \pm d\mathcal{T}^a_\beta\mathcal{K}^\beta_\pm \pm \mathcal{T}^a_\beta d\mathcal{K}^\beta_\pm \tag{5.2.38}$$

and use the covariant derivatives (5.2.18) and 5.2.36 followed by the pseudoduality equations (5.2.28) and 5.2.37 to get

$$
\begin{aligned}
dz^N \tilde{\mathcal{A}}^a_{\pm N} = &\pm (d\mathcal{T}^a_c + \tilde{\mathcal{A}}^a_b \mathcal{T}^b_c + \tilde{\mathcal{A}}^a_\lambda \mathcal{T}^\lambda_c(0) - \mathcal{T}^a_b \mathcal{A}^b_c - \mathcal{T}^a_\beta \mathcal{K}^\beta_c)\mathcal{A}^c_\pm \\
&\pm (d\mathcal{T}^a_\lambda + \tilde{\mathcal{A}}^a_b \mathcal{T}^b_\lambda + \tilde{\mathcal{A}}^a_\lambda - \mathcal{T}^a_b \mathcal{A}^b_\lambda - \mathcal{T}^a_\beta \mathcal{K}^\beta_\lambda)\mathcal{K}^\lambda_\pm \\
&\pm dz^N \mathcal{T}^a_b \mathcal{A}^b_{\pm N} \pm dz^N \mathcal{T}^a_\beta \mathcal{K}^\beta_{\pm N}
\end{aligned}
\tag{5.2.39}
$$

If this equation is wedged by $dz^\pm$ one gets

$$
\begin{aligned}
dz^\pm \wedge dz^\mp \tilde{\mathcal{A}}^a_{\pm\mp} = &\pm dz^\pm \wedge (d\mathcal{T}^a_c + \tilde{\mathcal{A}}^a_b \mathcal{T}^b_c + \tilde{\mathcal{A}}^a_\lambda \mathcal{T}^\lambda_c(0) - \mathcal{T}^a_b \mathcal{A}^b_c - \mathcal{T}^a_\beta \mathcal{K}^\beta_c)\mathcal{A}^c_\pm \\
&\pm dz^\pm \wedge (d\mathcal{T}^a_\lambda + \tilde{\mathcal{A}}^a_b \mathcal{T}^b_\lambda + \tilde{\mathcal{A}}^a_\lambda - \mathcal{T}^a_b \mathcal{A}^b_\lambda - \mathcal{T}^a_\beta \mathcal{K}^\beta_\lambda)\mathcal{K}^\lambda_\pm \\
&\pm dz^\pm \wedge dz^\mp \mathcal{T}^a_b \mathcal{A}^b_{\pm\mp} \pm dz^\pm \wedge dz^\mp \mathcal{T}^a_\beta \mathcal{K}^\beta_{\pm\mp}
\end{aligned}
\tag{5.2.40}
$$

$(+)$ (upper) equation yields the following constraints

$$
d\mathcal{T}^a_c + \tilde{\mathcal{A}}^a_b \mathcal{T}^b_c + \tilde{\mathcal{A}}^a_\lambda \mathcal{T}^\lambda_c(0) - \mathcal{T}^a_b \mathcal{A}^b_c - \mathcal{T}^a_\beta \mathcal{K}^\beta_c = 0 \tag{5.2.41}
$$

$$
d\mathcal{T}^a_\lambda + \tilde{\mathcal{A}}^a_b \mathcal{T}^b_\lambda + \tilde{\mathcal{A}}^a_\lambda - \mathcal{T}^a_b \mathcal{A}^b_\lambda - \mathcal{T}^a_\beta \mathcal{K}^\beta_\lambda = 0 \tag{5.2.42}
$$

one finds out the following constraint relation between equations of motion from $(-)$ (lower) equation

$$
dz^- \wedge dz^+ \tilde{\mathcal{A}}^a_{-+} = -dz^- \wedge dz^+ (\mathcal{T}^a_b \mathcal{A}^b_{-+} + \mathcal{T}^a_\beta K^\beta_{-+}) \tag{5.2.43}
$$

We use the equations of motions and find that

$$dz^- \wedge dz^+ (\tilde{f}^a_{bc} \tilde{\mathcal{A}}^b_+ \tilde{\mathcal{A}}^c_- + \tilde{f}^a_{\beta\lambda} \tilde{\mathcal{K}}^\beta_+ \tilde{\mathcal{K}}^\lambda_- = - \mathcal{T}^a_b f^b_{cd} \mathcal{A}^c_+ \mathcal{A}^d_- - \mathcal{T}^a_b f^b_{\beta\lambda} \mathcal{K}^\beta_+ \mathcal{K}^\lambda_- \\ - \mathcal{T}^a_\beta f^\beta_{b\lambda} \mathcal{A}^b_+ \mathcal{K}^\lambda_- - \mathcal{T}^a_\beta f^\beta_{\lambda b} \mathcal{K}^\lambda_+ \mathcal{A}^b_-) \tag{5.2.44}$$

we again use $K_\alpha = dz^M K^\alpha_M$ and $A^a = dz^M A^a_M$ followed by connection forms (5.2.12) and (5.2.32) to obtain

$$\tilde{\mathcal{A}}^a_c \tilde{\mathcal{A}}^c_- + \tilde{\mathcal{A}}^a_\lambda \tilde{\mathcal{K}}^\lambda_- = -\mathcal{T}^a_b \mathcal{A}^b_c \mathcal{A}^c_- - \mathcal{T}^a_b \mathcal{A}^b_\lambda \mathcal{K}^\lambda_- - \mathcal{T}^a_\beta \mathcal{K}^\beta_\lambda \mathcal{K}^\lambda_- - \mathcal{T}^a_\beta \mathcal{K}^\beta_c \mathcal{A}^c_- \tag{5.2.45}$$

If the pseudoduality equations (5.2.28) and (5.2.37) for $\tilde{\mathcal{K}}^\lambda_-$ and $\tilde{\mathcal{A}}^c_-$ is inserted, one finds

$$\tilde{\mathcal{A}}^a_b \mathcal{T}^b_c + \tilde{\mathcal{A}}^a_\lambda \mathcal{T}^\lambda_c(0) = \mathcal{T}^a_b \mathcal{A}^b_c + \mathcal{T}^a_\beta \mathcal{K}^\beta_c \tag{5.2.46}$$

$$\tilde{\mathcal{A}}^a_b \mathcal{T}^b_\lambda + \tilde{\mathcal{A}}^a_\lambda = \mathcal{T}^a_b \mathcal{A}^b_\lambda + \mathcal{T}^a_\beta \mathcal{K}^\beta_\lambda \tag{5.2.47}$$

These equations together with constraint relations above yield that $d\mathcal{T}^a_c = d\mathcal{T}^a_\lambda = 0$, which shows that $\mathcal{T}^a_c$ and $\mathcal{T}^a_\lambda$ are constants, chosen to be identity as in the previous part. Therefore we are left with the pseudoduality equations in reduced form

$$\tilde{\mathcal{A}}^a_\pm = \pm \mathcal{A}^a_\pm \pm \mathcal{T}^a_\beta(0) \mathcal{K}^\beta_\pm \tag{5.2.48}$$

with corresponding constraint relations whose integrability conditions will give us the relations between curvatures

$$\tilde{\mathcal{A}}^a_c + \tilde{\mathcal{A}}^a_\lambda \mathcal{T}^\lambda_c(0) = \mathcal{A}^a_c + \mathcal{T}^a_\beta(0) \mathcal{K}^\beta_c \tag{5.2.49}$$

$$\tilde{\mathcal{A}}^a_b \mathcal{T}^b_\lambda(0) + \tilde{\mathcal{A}}^a_\lambda = \mathcal{A}^a_\lambda + \mathcal{T}^a_\beta(0) \mathcal{K}^\beta_\lambda \tag{5.2.50}$$

5.2.3 Integrability Conditions and Curvature Relations

We have already figured out relations between connection one forms, (5.2.29) and (5.2.30) for M-space, (5.2.49) and (5.2.50) for H-space, which leads to corresponding curvature relations via second Cartan structural equation. We start with taking exterior derivative of (5.2.29), and then insert in related Cartan's equations, and finally use the results (5.2.29), (5.2.30), (5.2.49) and (5.2.50) to obtain

$$\tilde{\Omega}^{\alpha}_{\lambda} + \tilde{\Omega}^{\alpha}_{b}\mathcal{T}^{b}_{\lambda}(0) = \Omega^{\alpha}_{\lambda} + \mathcal{T}^{\alpha}_{a}(0)\Omega^{a}_{\lambda} \tag{5.2.51}$$

where $\Omega^{\bullet}_{\bullet}$ is the curvature two form associated with the space whose indices are used. If we insert the expressions for curvature two forms, and use pseudoduality equations, one gets after some calculations

$$\hat{\mathcal{R}}^{\alpha}_{\lambda\mu\nu} = -(\bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu\nu} + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu c}\mathcal{T}^{c}_{\nu}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda c\nu}\mathcal{T}^{c}_{\mu}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda cd}\mathcal{T}^{c}_{\mu}(0)\mathcal{T}^{d}_{\nu}(0)) \tag{5.2.52}$$

$$\hat{\mathcal{R}}^{\alpha}_{\lambda\mu d} = -(\bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu d} + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda cd}\mathcal{T}^{c}_{\mu}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu\nu}\mathcal{T}^{\nu}_{d}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda c\nu}\mathcal{T}^{c}_{\mu}(0)\mathcal{T}^{\nu}_{d}(0)) \tag{5.2.53}$$

$$\hat{\mathcal{R}}^{\alpha}_{\lambda c\nu} = -(\bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda c\nu} + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda cd}\mathcal{T}^{d}_{\nu}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu\nu}\mathcal{T}^{\mu}_{c}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu d}\mathcal{T}^{\mu}_{c}(0)\mathcal{T}^{d}_{\nu}(0)) \tag{5.2.54}$$

$$\hat{\mathcal{R}}^{\alpha}_{\lambda cd} = -(\bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda cd} + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu d}\mathcal{T}^{\mu}_{c}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda c\mu}\mathcal{T}^{\mu}_{d}(0) + \bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu\nu}\mathcal{T}^{\mu}_{c}(0)\mathcal{T}^{\nu}_{d}(0)) \tag{5.2.55}$$

where we defined $\hat{\mathcal{R}}^{\alpha}_{\lambda\mu\nu} \equiv \mathcal{R}^{\alpha}_{\lambda\mu\nu} + \mathcal{T}^{\alpha}_{a}(0)\mathcal{R}^{a}_{\lambda\mu\nu}$ and $\bar{\tilde{\mathcal{R}}}^{\alpha}_{\lambda\mu\nu} \equiv \tilde{\mathcal{R}}^{\alpha}_{\lambda\mu\nu} + \tilde{\mathcal{R}}^{\alpha}_{b\mu\nu}\mathcal{T}^{b}_{\lambda}(0)$. It can readily be seen that if one identifies a pseudoduality transformations $M \longrightarrow \tilde{M}$ and $H \longrightarrow \tilde{H}$, then one simply has the expected relations $R^{\alpha}_{\lambda\mu\nu} = -\tilde{R}^{\alpha}_{\lambda\mu\nu}$ and so on. If we generalize this formation to remaining constraint equations above, and curvature relations followed by them, one can easily writes

$$\tilde{\Omega}^{A}_{B} + \tilde{\Omega}^{A}_{C}\mathcal{T}^{C}_{B}(0) = \Omega^{A}_{B} + \mathcal{T}^{A}_{C}(0)\Omega^{C}_{B} \tag{5.2.56}$$

where the indices A, B and C stands for the indices corresponding to M or H-space elements depending on which relation is used. Therefore, curvature relations will be

$$\hat{\mathcal{R}}^{A}_{B\mu\nu} = -(\bar{\tilde{\mathcal{R}}}^{A}_{B\mu\nu} + \bar{\tilde{\mathcal{R}}}^{A}_{B\mu c}\mathcal{T}^{c}_{\nu}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{Bc\nu}\mathcal{T}^{c}_{\mu}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{Bcd}\mathcal{T}^{c}_{\mu}(0)\mathcal{T}^{d}_{\nu}(0)) \tag{5.2.57}$$

$$\hat{\mathcal{R}}^{A}_{B\mu d} = -(\bar{\tilde{\mathcal{R}}}^{A}_{B\mu d} + \bar{\tilde{\mathcal{R}}}^{A}_{Bcd}\mathcal{T}^{c}_{\mu}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{B\mu\nu}\mathcal{T}^{\nu}_{d}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{Bc\nu}\mathcal{T}^{c}_{\mu}(0)\mathcal{T}^{\nu}_{d}(0)) \tag{5.2.58}$$

$$\hat{\mathcal{R}}^{A}_{Bc\nu} = -(\bar{\tilde{\mathcal{R}}}^{A}_{Bc\nu} + \bar{\tilde{\mathcal{R}}}^{A}_{Bcd}\mathcal{T}^{d}_{\nu}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{B\mu\nu}\mathcal{T}^{\mu}_{c}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{B\mu d}\mathcal{T}^{\mu}_{c}(0)\mathcal{T}^{d}_{\nu}(0)) \tag{5.2.59}$$

$$\hat{\mathcal{R}}^{A}_{Bcd} = -(\bar{\tilde{\mathcal{R}}}^{A}_{Bcd} + \bar{\tilde{\mathcal{R}}}^{A}_{B\mu d}\mathcal{T}^{\mu}_{c}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{Bc\mu}\mathcal{T}^{\mu}_{d}(0) + \bar{\tilde{\mathcal{R}}}^{A}_{B\mu\nu}\mathcal{T}^{\mu}_{c}(0)\mathcal{T}^{\nu}_{d}(0)) \tag{5.2.60}$$

5.3 Component Expansion Method

In this section we work out the pseudoduality by components. The superfield $\mathcal{G}(\sigma, \theta)$ is given by (5.1.5) in components. In the previous section (4.3) we saw that equations of motion (5.1.6) and (5.1.7) gave us the following results

$$\chi = i\psi_{-}\psi_{+} \tag{5.3.1}$$

$$\partial_{-}\psi_{+} = 0 \tag{5.3.2}$$

$$\partial_{+}\psi_{-} = [\psi_{-}, g^{-1}\partial_{+}g] \tag{5.3.3}$$

$$\partial_{+}(g^{-1}\partial_{-}g) = [g^{-1}\partial_{-}g, g^{-1}\partial_{+}g] \tag{5.3.4}$$

$$\partial_{-}(g^{-1}\partial_{+}g) = 0 \tag{5.3.5}$$

We offer the solutions $g = g_R(\sigma^-)g_L(\sigma^+)$ and $\psi_{\pm} = \psi_{\pm L}(\sigma^+) + \psi_{\pm R}(\sigma^-)$ in the right and left moving components. Hence we observe that $\psi_{+R} = 0$ from equation (5.3.2), ψ_{-R} commutes with g_L from equation (5.3.3), and equations (5.3.4) and (5.3.5) depend only on

σ^- and σ^+ respectively. Therefore we easily get the decomposition $\mathcal{G} = \mathcal{G}_R\mathcal{G}_L$, where

$$\mathcal{G}_R = g_R(1 + i\theta^-\psi_{-R}) \tag{5.3.6}$$

$$\mathcal{G}_L = g_L(1 + i\theta^+\psi_{+L} + i\theta^-\psi_{-L} - \theta^+\theta^-\psi_{-L}\psi_{+L}) \tag{5.3.7}$$

Using these relations one may get the following expressions which will be needed in constructing pseudoduality and conserved currents

$$\mathcal{G}_L^{-1}D_+\mathcal{G}_L = i\psi_{+L} + i\theta^+g_L^{-1}\partial_+g_L \tag{5.3.8}$$

$$(D_-\mathcal{G}_R)\mathcal{G}_R^{-1} = ig_R\psi_{-R}g_R^{-1} + i\theta^{-1}(\partial_-g_R)g_R^{-1} \tag{5.3.9}$$

$$\mathcal{G}^{-1}D_-\mathcal{G} = i\psi_{-R} + i\theta^-g^{-1}\partial_-g + \theta^+\theta^-[g^{-1}\partial_-g, \psi_{+L}] \tag{5.3.10}$$

We may decompose the fields $g^{-1}\partial_\pm g = k_\pm + A_\pm$ and $\psi_\pm = \phi_\pm + B_\pm$ on symmetric space, where $k_\pm, \phi_\pm \in$ **m** are the bosonic and fermionic symmetric space field components, and $A_\pm$, $B_\pm \in$ **h** are the corresponding gauge fields. If one indicates these fields in terms of right and left expressions, it is evident that $k_+ = k_{+L}$, $k_{+R} = 0$, $A_+ = A_{+L}$, $A_{+R} = 0$, $k_- = g_L^{-1}k_{-R}g_L$, $A_- = g_L^{-1}A_{-R}g_L$, $\phi_{+R} = B_{+R} = 0$. Hence one can write the superfield decompositions (5.2.2) as follows

$$\mathcal{K}_{+L} = i\phi_{+L} + i\theta^+k_{+L} \tag{5.3.11}$$

$$\mathcal{K}_- = i\phi_{-R} + i\theta^-k_- + \theta^+\theta^-([A_-, \phi_{+L}] + [k_-, B_{+L}]) \tag{5.3.12}$$

$$\mathcal{A}_{+L} = iB_{+L} + i\theta^+A_{+L} \tag{5.3.13}$$

$$\mathcal{A}_- = iB_{-R} + i\theta^-A_- + \theta^+\theta^-([k_-, \phi_{+L}] + [A_-, B_{+L}]) \tag{5.3.14}$$

where $\mathcal{K}_{+R} = \mathcal{A}_{+R} = 0$. Equations of motion in components following from (5.2.4) and

(5.2.5) will be

$$\phi_{+-R} = \phi_{+-L} = k_{+-R} = k_{+-L} = 0 \tag{5.3.15}$$

$$A_{+-R} = A_{+-L} = B_{+-R} = B_{+-L} = 0 \tag{5.3.16}$$

$$[B_{-R}, \phi_{+L}] = -[\phi_{-R}, B_{+L}] \tag{5.3.17}$$

$$\{B_{-R}, k_{+L}\} = -\{\phi_{-R}, A_{+L}\} \tag{5.3.18}$$

$$A_{-}\phi_{+L} = -k_{-}B_{+L} \tag{5.3.19}$$

$$\begin{aligned} k_{-+} = &-\{k_{-}, A_{+L}\} - \{A_{-}, k_{+L}\} - i[[A_{-}, \phi_{+L}], B_{+L}] \\ &- i[[k_{-}, B_{+L}], B_{+L}] - i[[k_{-}, \phi_{+L}], \phi_{+L}] - i[[A_{-}, B_{+L}], \phi_{+L}] \end{aligned} \tag{5.3.20}$$

$$[\phi_{-R}, \phi_{+L}] = -[B_{-R}, B_{+L}] \tag{5.3.21}$$

$$k_{-}\phi_{+L} = -A_{-}B_{+L} \tag{5.3.22}$$

$$\{B_{-R}, A_{+L}\} = -\{\phi_{-R}, k_{+L}\} \tag{5.3.23}$$

$$\begin{aligned} A_{-+} = &-\{A_{-}, A_{+L}\} - \{k_{-}, k_{+L}\} - i[[k_{-}, \phi_{+L}], B_{+L}] \\ &- i[[A_{-}, B_{+L}], B_{+L}] - i[[A_{-}, \phi_{+L}], \phi_{+L}] - i[[k_{-}, B_{+L}], \phi_{+L}] \end{aligned} \tag{5.3.24}$$

where [,] denotes commutation, and { , } denotes anticommutation relation. By means of (5.3.19) and (5.3.22), equations (5.3.20) and (5.3.24) can be simplified as follows

$$k_{-+} = -\{k_{-}, A_{+L}\} - \{A_{-}, k_{+L}\} - i\{B_{+L}, \phi_{+L}\}A_{-} \tag{5.3.25}$$

$$A_{-+} = -\{A_{-}, A_{+L}\} - \{k_{-}, k_{+L}\} - i\{B_{+L}, \phi_{+L}\}k_{-} \tag{5.3.26}$$

Similar expressions on pseudodual manifold can be written using tilde over each term. We may now establish the pseudoduality relations. We will first analyze non-mixing pseudoduality case which will lead mixing case to be well comprehended in turn.

5.3.1 Pseudoduality: Non-Mixing Case

Before considering the general case, we figure out the simplest case where mixing part of the pseudoduality map in (5.2.17) vanishes, $\mathcal{T}_a^\alpha = 0$. Let us first work out pseudoduality on symmetric space M, and then consider H-space since they are mutually dependent on each other. We think of $\mathcal{T}$ as a function of superfield X, and can be expanded as in the first section (5.1), $\mathcal{T}(\sigma,\theta) = T(\sigma^+) + \theta^+\lambda_+(\sigma^+)$. Consequently pseudoduality relations in components on M are written as

$$\tilde{\phi}^\alpha_{+L} = T^\alpha_\beta \phi^\beta_{+L} \tag{5.3.27}$$

$$\tilde{k}^\alpha_{+L} = T^\alpha_\beta k^\beta_{+L} + (\lambda_+)^\alpha_\beta \phi^\beta_{+L} \tag{5.3.28}$$

$$\tilde{\phi}^\alpha_{-R} = -T^\alpha_\beta \phi^\beta_{-R} \tag{5.3.29}$$

$$\tilde{k}^\alpha_{-} = -T^\alpha_\beta k^\beta_{-} \tag{5.3.30}$$

$$(\lambda_+)^\alpha_\beta \phi^\beta_{-R} = 0 \tag{5.3.31}$$

$$[\tilde{A}_-, \tilde{\phi}_{+L}]^\alpha + [\tilde{k}_-, \tilde{B}_{+L}]^\alpha = -T^\alpha_\beta([A_-, \phi_{+L}]^\beta + [k_-, B_{+L}]^\beta) + i(\lambda_+)^\alpha_\beta k^\beta_- \tag{5.3.32}$$

Likewise pseudoduality relations on H can be expanded in components as

$$\tilde{B}^a_{+L} = T^a_b B^b_{+L} \tag{5.3.33}$$

$$\tilde{A}^a_{+L} = T^a_b A^b_{+L} + (\lambda_+)^a_b B^b_{+L} \tag{5.3.34}$$

$$\tilde{B}^a_{-R} = -T^a_b B^b_{-R} \tag{5.3.35}$$

$$\tilde{A}^a_{-} = -T^a_b A^b_{-} \tag{5.3.36}$$

$$(\lambda_+)^a_b B^b_{-R} = 0 \tag{5.3.37}$$

$$[\tilde{k}_-, \tilde{\phi}_{+L}]^a + [\tilde{A}_-, \tilde{B}_{+L}]^a = -T^a_b([k_-, \phi_{+L}]^b + [A_-, B_{+L}]^b) + i(\lambda_+)^a_b A^b_- \tag{5.3.38}$$

When we take the corresponding $(+)$ covariant derivative of (5.3.29), we obtain that $(\mathfrak{D}_+ T^\alpha_\beta)\phi^\beta_{-R} = 0$, where $\mathfrak{D}$ is the covariant derivative acting on **m**-space. Together with equation (5.3.31) we are left with two options: First option is to consider that T^α_β is constant and $(\lambda_+)^\alpha_\beta$ is zero. This is consistent with the results we found in our previous work, which leads to flat space pseudoduality

$$\tilde{k}^\alpha_{+L} = k^\alpha_{+L} \qquad \tilde{k}^\alpha_- = -k^\alpha_- \tag{5.3.39}$$

$$\tilde{\phi}^\alpha_{+L} = \phi^\alpha_{+L} \qquad \tilde{\phi}^\alpha_{-R} = -\phi^\alpha_{-R} \tag{5.3.40}$$

with the corresponding bracket relations (5.3.32)

$$[\tilde{A}_-, \tilde{\phi}_{+L}]^\alpha = -[A_-, \phi_{+L}]^\alpha \qquad [\tilde{k}_-, \tilde{B}_{+L}]^\alpha = -[k_-, B_{+L}]^\alpha \tag{5.3.41}$$

Second option is to have $\phi_{-R} = 0$, which leads to $\tilde{\phi}_{-R} = 0$. In this case the isometry T^α_β can be found by taking $\mathfrak{D}_+$ of (5.3.30), which leads to

$$(\mathfrak{D}_+ T^\alpha_\beta) k^\beta_- = T^\alpha_\beta \{k_-, A_{+L}\}^\beta + \{\tilde{k}_-, \tilde{A}_{+L}\}^\alpha \tag{5.3.42}$$

with the constraint anti-commutation relation

$$\{\tilde{A}_-, \tilde{k}_{+L}\}^\alpha + i\tilde{f}^\alpha_{\beta a}\{\tilde{B}_{+L}, \tilde{\phi}_{+L}\}^\beta \tilde{A}^a_- = -T^\alpha_\beta\{A_-, k_{+L}\}^\beta - iT^\alpha_\beta f^\beta_{\nu a}\{B_{+L}, \phi_{+L}\}^\nu A^a_- \tag{5.3.43}$$

$\tilde{k}_-$ and $\tilde{A}_{+L}$ can be replaced using (5.3.30) and (5.3.34). Hence it is realized that T^α_β is a function of bosonic gauge field A_{+L}. On the other hand $(\lambda_+)^\alpha_\beta$ can be found by (5.3.32)

$$i(\lambda_+)^\alpha_\beta k^\beta_- = [\tilde{k}_-, \tilde{B}_{+L}]^\alpha + T^\alpha_\beta [k_-, B_{+L}]^\beta \tag{5.3.44}$$

with the bracket relation

$$[\tilde{A}_-, \tilde{\phi}_{+L}]^\alpha = -T^\alpha_\beta [A_-, \phi_{+L}]^\beta \tag{5.3.45}$$

where unknown tilded expressions can be substituted back using related equations above. It is observed that $(\lambda_+)^\alpha_\beta$ is given in terms of the fermionic gauge field B_{+L}.

Now we apply the same reasoning to H-space equations. We take $\mathfrak{D}'_+$ of (5.3.35), and have that $(\mathfrak{D}'_+ T^a_b) B^b_{-R} = 0$, where $\mathfrak{D}'$ is the covariant derivative acting on **h**-space. We again notice that we have two different options to satisfy this equation as well as (5.3.37). First option is to pick T^a_b to have a constant, and $(\lambda_+)^a_b$ vanishing value. This is compatible with the first option above and results in the previous work. This gives rise to the following flat space pseudoduality equations

$$\tilde{A}^a_{+L} = A^a_{+L} \qquad \tilde{A}^a_- = -A^a_- \tag{5.3.46}$$

$$\tilde{B}^a_{+L} = B^a_{+L} \qquad \tilde{B}^a_{-R} = -B^a_{-R} \tag{5.3.47}$$

along with the bracket relations

$$[\tilde{k}_-, \tilde{\phi}_{+L}]^a = -[k_-, \phi_{+L}]^a \qquad [\tilde{A}_-, \tilde{B}_{+L}]^a = -[A_-, B_{+L}]^a \tag{5.3.48}$$

Second option is to choose $B_{-R} = 0$, which will bring about $\tilde{B}_{-R} = 0$ respectively. In this case T^a_b can be found by taking $\mathfrak{D}'_+$ of (5.3.36), which will cause

$$(\mathfrak{D}'_+ T^a_b) A^b_- = T^a_b \{A_-, A_{+L}\}^b + \{\tilde{A}_-, \tilde{A}_{+L}\}^a \tag{5.3.49}$$

with the complemental equation

$$\{\tilde{k}_-, \tilde{k}_{+L}\}^a + i\tilde{f}^a_{\alpha\beta}\{\tilde{B}_{+L}, \tilde{\phi}_{+L}\}^\alpha \tilde{k}^\beta_- = -T^a_b\{k_-, k_{+L}\}^b - iT^a_b f^b_{\alpha\beta}\{B_{+L}, \phi_{+L}\}^\alpha k^\beta_- \tag{5.3.50}$$

where $\tilde{A}_-$ and $\tilde{A}_{+L}$ can be substituted with the relevant equations above. Consequently we are aware that T^a_b is a function of bosonic gauge field A_{+L} similar to T^α_β. $(\lambda_+)^a_b$ can be found using (5.3.38)

$$i(\lambda_+)^a_b A^b_- = [\tilde{A}_-, \tilde{B}_{+L}]^a + T^a_b[A_-, B_{+L}]^b \tag{5.3.51}$$

with the associated bracket relation

$$[\tilde{k}_-, \tilde{\phi}_{+L}]^a = -T^a_b[k_-, \phi_{+L}]^b \tag{5.3.52}$$

where $\tilde{A}_-$ and $\tilde{B}_{+L}$ can be replaced using related equations. We notice that $(\lambda_+)^a_b$ is a function of B_{+L} which is analogous to $(\lambda_+)^\alpha_\beta$. Although it seems that both **m** and **h**-space expressions are independent of each other, they are decomposed subspaces of **g**, and accordingly has to satisfy constraints arising from **g**. Because of this reason we will conclude that vanishing $(\lambda_+)^\alpha_\beta$ implies vanishing $(\lambda_+)^a_b$, likewise if ϕ_{-R} is set to zero, we have to consider $B_{-R} = 0$, which agrees with the result found in the previous work [**?**]. We know that commutation relations found above leads to the corresponding relations between connection two forms, which in turn give rise to relevant relations between curvatures.

5.3.2 Pseudoduality: Mixing Case

In this section we will consider the pseudoduality transformation that causes mixing of M and H-spaces by allowing mixing components of $\mathcal{T}$. Again the matrix $\mathcal{T}$ can be written in the form which has already been imposed by the constraints on G as $\mathcal{T} = T + \theta^+\lambda_+$. On

M-space pseudoduality equations will be

$$\tilde{\phi}^{\alpha}_{+L} = T^{\alpha}_{\beta}\phi^{\beta}_{+L} + T^{\alpha}_{a}B^{a}_{+L} \tag{5.3.53}$$

$$\tilde{k}^{\alpha}_{+L} = T^{\alpha}_{\beta}k^{\beta}_{+L} + T^{\alpha}_{a}A^{a}_{+L} + (\lambda_+)^{\alpha}_{\beta}\phi^{\beta}_{+L} + (\lambda_+)^{\alpha}_{a}B^{a}_{+L} \tag{5.3.54}$$

$$\tilde{\phi}^{\alpha}_{-R} = -T^{\alpha}_{\beta}\phi^{\beta}_{-R} - T^{\alpha}_{a}B^{a}_{-R} \tag{5.3.55}$$

$$\tilde{k}^{\alpha}_{-} = -T^{\alpha}_{\beta}k^{\beta}_{-} - T^{\alpha}_{a}A^{a}_{-} \tag{5.3.56}$$

$$0 = (\lambda_+)^{\alpha}_{\beta}\phi^{\beta}_{-R} + (\lambda_+)^{\alpha}_{a}B^{a}_{-R} \tag{5.3.57}$$

$$\begin{aligned}[\tilde{A}_-, \tilde{\phi}_{+L}]^{\alpha} + [\tilde{k}_-, \tilde{B}_{+L}]^{\alpha} &= -T^{\alpha}_{\beta}([A_-, \phi_{+L}]^{\beta} + [k_-, B_{+L}]^{\beta}) + i(\lambda_+)^{\alpha}_{\beta}k^{\beta}_{-} \\ &\quad - T^{\alpha}_{a}([k_-, \phi_{+L}]^{a} + [A_-, B_{+L}]^{a}) + i(\lambda_+)^{\alpha}_{a}A^{a}_{-}\end{aligned} \tag{5.3.58}$$

and on H-space we obtain the following pseudoduality equations

$$\tilde{B}^{a}_{+L} = T^{a}_{b}B^{b}_{+L} + T^{a}_{\beta}\phi^{\beta}_{+L} \tag{5.3.59}$$

$$\tilde{A}^{a}_{+L} = T^{a}_{b}A^{b}_{+L} + T^{a}_{\beta}k^{\beta}_{+L} + (\lambda_+)^{a}_{b}B^{b}_{+L} + (\lambda_+)^{a}_{\beta}\phi^{\beta}_{+L} \tag{5.3.60}$$

$$\tilde{B}^{a}_{-R} = -T^{a}_{b}B^{b}_{-R} - T^{a}_{\beta}\phi^{\beta}_{-R} \tag{5.3.61}$$

$$\tilde{A}^{a}_{-} = -T^{a}_{b}A^{b}_{-} - T^{a}_{\beta}k^{\beta}_{-} \tag{5.3.62}$$

$$0 = (\lambda_+)^{a}_{b}B^{b}_{-R} + (\lambda_+)^{a}_{\beta}\phi^{\beta}_{-R} \tag{5.3.63}$$

$$\begin{aligned}[\tilde{k}_-, \tilde{\phi}_{+L}]^{a} + [\tilde{A}_-, \tilde{B}_{+L}]^{a} &= -T^{a}_{b}([k_-, \phi_{+L}]^{b} + [A_-, B_{+L}]^{b}) + i(\lambda_+)^{a}_{b}A^{b}_{-} \\ &\quad - T^{a}_{\beta}([A_-, \phi_{+L}]^{\beta} + [k_-, B_{+L}]^{\beta}) + i(\lambda_+)^{a}_{\beta}k^{\beta}_{-}\end{aligned} \tag{5.3.64}$$

Let us find the constraint relations on pseudoduality transformations using the equations of motion. Hence we take (+) covariant derivative of (5.3.55), and obtain

$$(\mathfrak{D}_+T^{\alpha}_{\beta})\phi^{\beta}_{-R} + (\mathfrak{D}_+T^{\alpha}_{a})B^{a}_{-R} = 0 \tag{5.3.65}$$

If one deals with this equation together with (5.3.57), one can obtain two different conditions. First condition imposes that T^{α}_{β} and T^{α}_{a} are constants and chosen to be identity, and $(\lambda_+)^{\alpha}_{\beta}$ and $(\lambda_+)^{\alpha}_{a}$ vanish. Therefore one may obtain the pseudoduality equations

$$\tilde{k}^{\alpha}_{+L} = k^{\alpha}_{+L} + T^{\alpha}_{a}(0)A^{a}_{+L} \qquad \tilde{k}^{\alpha}_{-} = -k^{\alpha}_{-} - T^{\alpha}_{a}(0)A^{a}_{-} \tag{5.3.66}$$

$$\tilde{\phi}^{\alpha}_{+L} = \phi^{\alpha}_{+L} + T^{\alpha}_{a}(0)B^{a}_{+L} \qquad \tilde{\phi}^{\alpha}_{-R} = -\phi^{\alpha}_{-R} - T^{\alpha}_{a}(0)B^{a}_{-R} \tag{5.3.67}$$

with the constraint bracket relation

$$\begin{aligned}[\tilde{A}_-, \tilde{\phi}_{+L}]^{\alpha} + [\tilde{k}_-, \tilde{B}_{+L}]^{\alpha} = &- [A_-, \phi_{+L}]^{\alpha} - [k_-, B_{+L}]^{\alpha} \\ &- T^{\alpha}_{a}(0)([k_-, \phi_{+L}]^{a} + [A_-, B_{+L}]^{a})\end{aligned} \tag{5.3.68}$$

where $T^{\alpha}_{a}(0)$ represents the mixing component of T which is identity. We see that once we have the duality relations (5.3.66) and (5.3.67) we must have the bracket relation (5.3.68) on both spaces. We observe that mixings are included by means of gauge fields A and B.

Second condition on **m**-space is given by setting both ϕ_{-R} and B_{-R} equal to zero. We are careful at this point because we must have both fields vanishing. This is because these two fields form the ferminonic field ψ on space **g** which leads both fields to disappear simultaneously when split on **h** and **m**-spaces. Therefore we have $\tilde{\phi}_{-R} = 0$ from (5.3.55). To find T^{α}_{β} and T^{α}_{a} we take (+) covariant derivative of (5.3.56), which will lead to two independent equations

$$\begin{aligned}(\mathfrak{D}_+ T^{\alpha}_{\beta})k^{\beta}_{-} =& T^{\alpha}_{\beta}\{k_-, A_{+L}\}^{\beta}_{G} + T^{\alpha}_{a}\{k_-, k_{+L}\}^{a}_{G} + iT^{\alpha}_{a} f^{a}_{\beta\lambda}\{B_{+L}, \phi_{+L}\}^{\beta}_{G} k^{\lambda}_{-} \\ &- \{Tk_-, \tilde{A}_{+L}\}^{\alpha}_{\tilde{G}} - \{Tk_-, \tilde{k}_{+L}\}^{\alpha}_{\tilde{G}} - i\tilde{f}^{\alpha}_{\beta a}\{\tilde{B}_{+L}, \tilde{\phi}_{+L}\}^{\beta}_{\tilde{G}} T^{a}_{\lambda} k^{\lambda}_{-}\end{aligned} \tag{5.3.69}$$

$$\begin{aligned}(\mathfrak{D}_+ T^{\alpha}_{a})A^{a}_{-} =& T^{\alpha}_{\beta}\{A_-, k_{+L}\}^{\beta}_{G} + T^{\alpha}_{a}\{A_-, A_{+L}\}^{a}_{G} + iT^{\alpha}_{\beta} f^{\beta}_{\lambda a}\{B_{+L}, \phi_{+L}\}^{\beta}_{G} A^{a}_{-} \\ &- \{TA_-, \tilde{A}_{+L}\}^{\alpha}_{\tilde{G}} - \{TA_-, \tilde{k}_{+L}\}^{\alpha}_{\tilde{G}} - i\tilde{f}^{\alpha}_{\beta a}\{\tilde{B}_{+L}, \tilde{\phi}_{+L}\}^{\beta}_{\tilde{G}} T^{a}_{b} A^{b}_{-}\end{aligned} \tag{5.3.70}$$

where $\{\ ,\ \}_G$ represents anticommutation relation in G. We used the independence of k_- and A_- in deriving this equation, and they can be cancelled out to give transformation matrices. Terms with tilde can be replaced by nontilded ones using pseudoduality equations above, and hence giving T^α_β and T^α_a in terms of A_{+L}, k_{+L}, B_{+L} and ϕ_{+L}. These are coupled equations and can be solved perturbatively to yield terms up to the second order terms as we did in our previous works. In this case fermionic transformation matrices will be

$$i(\lambda_+)^\alpha_\beta k^\beta_- = T^\alpha_\beta [k_-, B_{+L}]^\beta_G + T^\alpha_a [k_-, \phi_{+L}]^a_G - [Tk_-, \tilde{\phi}_{+L}]^\alpha_{\tilde{G}} - [Tk_-, \tilde{B}_{+L}]^\alpha_{\tilde{G}} \tag{5.3.71}$$

$$i(\lambda_+)^\alpha_a A^a_- = T^\alpha_\beta [A_-, \phi_{+L}]^\beta_G + T^\alpha_a [A_-, B_{+L}]^a_G - [TA_-, \tilde{\phi}_{+L}]^\alpha_{\tilde{G}} - [TA_-, \tilde{B}_{+L}]^\alpha_{\tilde{G}} \tag{5.3.72}$$

which are functions of fermionic terms ϕ_{+L} and B_{+L} after cancelling k_- and A_- respectively. Again tilded terms can be replaced by nontilded ones using corresponding pseudoduality equations above. We notice that the constraint relations (5.3.50) and (5.3.52) found in nonmixing pseudoduality case turns out to be expressions for transformation matrices in mixing case. We understand that in the absence of mixing pseudoduality transformation imposes some constraints which correspond to mixing part of pseudoduality.

In a similar way one can figure out pseudoduality on H-space. We take (+) covariant derivative of (5.3.61)

$$(\mathfrak{D}'_+ T^a_b) B^b_{-R} + (\mathfrak{D}'_+ T^a_\beta) \phi^\beta_{-R} = 0 \tag{5.3.73}$$

When considered together with (5.3.63) one finds two conditions on pseudoduality. First condition is to pick T^a_b and T^a_β constant, and $(\lambda_+)^a_b$ and $(\lambda)^a_\beta$ vanishing. Of course these are dependent on conditions (5.3.66) and (5.3.67) on **m**-space and can not be independently set to zero. Therefore pseudoduality equations will be

$$\tilde{A}^a_{+L} = A^a_{+L} + T^a_\beta(0) k^\beta_{+L} \qquad \tilde{A}^a_- = -A^a_- - T^a_\beta(0) k^\beta_- \tag{5.3.74}$$

$$\tilde{B}^a_{+L} = B^a_{+L} + T^a_\beta(0) \phi^\beta_{+L} \qquad \tilde{B}^a_{-R} = -B^a_{-R} - T^a_\beta(0) \phi^\beta_{-R} \tag{5.3.75}$$

where we chose the constant matrices to be identity. These equations adopt the following constraint relation

$$\begin{aligned}[\tilde{k}_-, \tilde{\phi}_{+L}]^a + [\tilde{A}_-, \tilde{B}_{+L}]^a =& - [k_-, \phi_{+L}]^a + [A_-, B_{+L}]^a \\ & - T^a_\beta(0)([A_-, \phi_{+L}]^\beta + [k_-, B_{+L}]^\beta)\end{aligned} \tag{5.3.76}$$

Our second condition is to choose $B_{-R} = \phi_{-R} = 0$. This leads to $\tilde{B}_{-R} = 0$ on $\tilde{H}$. Transformation matrices can be found by taking (+) covariant derivative of (5.3.62) as

$$\begin{aligned}(\mathfrak{D}'_+ T^a_b) A^b_- =& T^a_b \{A_-, A_{+L}\}^b_G + T^a_\beta \{A_-, k_{+L}\}^\beta_G + i T^a_\beta f^\beta_{\lambda b} \{B_{+L}, \phi_{+L}\}^\lambda_G A^b_- \\ & - \{TA_-, \tilde{A}_{+L}\}^a_{\tilde{G}} - \{TA_-, \tilde{k}_{+L}\}^a_{\tilde{G}} - i \tilde{f}^a_{\alpha\beta} \{\tilde{B}_{+L}, \tilde{\phi}_{+L}\}^\alpha_{\tilde{G}} T^\beta_b A^b_-\end{aligned} \tag{5.3.77}$$

$$\begin{aligned}(\mathfrak{D}'_+ T^a_\beta) k^\beta_- =& T^a_b \{k_-, k_{+L}\}^b_G + T^a_\beta \{k_-, A_{+L}\}^\beta_G + i T^a_b f^b_{\alpha\beta} \{B_{+L}, \phi_{+L}\}^\alpha_G k^\beta_- \\ & - \{Tk_-, \tilde{A}_{+L}\}^a_{\tilde{G}} - \{Tk_-, \tilde{k}_{+L}\}^a_{\tilde{G}} - i \tilde{f}^a_{\alpha\beta} \{\tilde{B}_{+L}, \tilde{\phi}_{+L}\}^\alpha_{\tilde{G}} T^\beta_\lambda k^\lambda_-\end{aligned} \tag{5.3.78}$$

These are coupled differential equations, and can be solved perturbatively. It is obvious that T^a_b and T^a_β are functions of k_{+L}, ϕ_{+L}, A_{+L} and B_{+L}. Fermionic transformation matrices can be found by

$$i(\lambda_+)^a_b A^b_- = T^a_b [A_-, B_{+L}]^b_G + T^a_\beta [A_-, \phi_{+L}]^\beta_G - [TA_-, \tilde{\phi}_{+L}]^a_{\tilde{G}} - [TA_-, \tilde{B}_{+L}]^a_{\tilde{G}} \tag{5.3.79}$$

$$i(\lambda_+)^a_\beta k^\beta_- = T^a_b [k_-, \phi_{+L}]^b_G + T^a_\beta [k_-, B_{+L}]^\beta_G - [Tk_-, \tilde{\phi}_{+L}]^a_{\tilde{G}} - [Tk_-, \tilde{B}_{+L}]^a_{\tilde{G}} \tag{5.3.80}$$

which are functions of fermionic terms ϕ_{+L} and B_{+L}. Tilded terms on right-hand sides can be replaced using corresponding pseudoduality equations. Again these terms turn into constraint relations when mixing components of T vanish.

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