Aspects of higher-spin conformal field theories and their renormalization group flows

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Abstract

In this thesis, we study conformal field theories (CFTs) with higher-spin symmetry and the renormalization group flows of some models with interactions that weakly break the higher-spin symmetry.

When the higher-spin symmetry is exact, we will present CFT analogues of two classic results in quantum field theory: the Coleman-Mandula theorem, which is the subject of chapter 2, and the Weinberg-Witten theorem, which is the subject of chapter 3. Schematically, our Coleman-Mandula analogue states that a CFT that contains a symmetric conserved current of spin $s > 2$ in any dimension $d > 3$ is effectively free, and our Weinberg-Witten analogue states that the presence of certain short, higher-spin, "sufficiently asymmetric" representations of the conformal group is either inconsistent with conformal symmetry or leads to free theories in $d = 4$ dimensions. In both chapters, the basic strategy is to solve certain Ward identities in convenient kinematical limits and thereby show that the number of solutions is very limited. In the latter chapter, Hofman-Maldacena bounds, which constrain one-point functions of the stress tensor in general states, play a key role.

Then, in chapter 4, we will focus on the particular examples of the $O(N)$ and Gross-Neveu model in continuous dimensions. Using diagrammatic techniques, we explicitly calculate how the coefficients of the two-point function of a $U(1)$ current and the two-point function of the stress tensor (C_J and C_T, respectively) are renormalized in the $1/N$ and ϵ expansions. From the higher-spin perspective, these models are interesting since they are related via the AdS/CFT correspondence to Vasiliev gravity. In addition to checking and extending a number of previously-known results about C_T and C_J in these theories, we find that in certain dimensions, C_J and C_T are not monotonic along the renormalization group flow. Although it was already known that certain supersymmetric models do not satisfy a "C_J"- or "C_T"-theorem, this shows that such a theorem is unlikely to hold even under more restrictive assumptions.

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Mathematical physics represents the purest image that the view of nature may generate in the human mind; this image presents all the character of the product of art; it begets some unity, it is true and has the quality of sublimity; this image is to physical nature what music is to the thousand noises of which the air is full...

- Théophile de Donder

Contents

Chapter 1

Introduction

1.1 Why higher-spin conformal field theories?

The topic of this thesis is the classification of higher-spin conformal field theories and the study of their renormalization group flows. In this section, we will explain what these theories are, and why they are particularly interesting.

Conformal field theories (CFTs) are quantum field theories that are invariant under angle-preserving diffeomorphisms $g_{\mu\nu}(x) \to \Omega(x) g_{\mu\nu}(x)$. This implies that CFTs are *scale-invariant*. They appear naturally in many different fields of physics. They are, of course, of great interest in high energy theory: for example, the conformal invariance of the string worldsheet action is a critical ingredient of string theory, and the celebrated AdS/CFT correspondence [1] [2] [3] exactly relates CFTs to string theories in anti-de Sitter space. In statistical physics and condensed matter physics, CFTs describe quantum systems at criticality, as was famously demonstrated by Wilson and Fisher [4] in 1972, and have been conjectured as tools to study strongly coupled condensed matter systems via holographic dualities (see, e.g. [5] for a review). In mathematical physics, conformal field theory has motivated the construction of novel objects in algebra, number theory, and topology, such as vertex operator algebras and Borcherds algebras. Hence, on general grounds, CFTs are objects whose general properties are worth understanding carefully.

Higher-spin CFTs, or CFTs which have a conserved current of spin larger than 2, have been of particular interest for decades. Since the 1980's [6], it has been conjectured that strings at extremely high energies should behave as though they are tensionless. Since the masses of the higher-spin modes of string theory are proportional to the string tension, these higher-spin modes

would become massless in the tensionless limit, and with no other scale in the theory, one would thus expect the theory to become conformal at these energy scales, with the higher-spin symmetry being spontaneously broken at lower energies. This viewpoint has been tested by many authors, including in the context of the AdS/CFT correspondence (see, e.g. [7]). Hence, a classification of CFTs with higher-spin symmetry may contribute to our understanding of string theory at the highest energy scales and thereby provide a natural starting point for systematically studying corrections as the higher-spin symmetries are broken at lower energies. This mirrors how understanding free quantum field theories is the natural first step to systematically understanding interacting theories.

A more contemporary motivation arises from the study of Vasiliev's higher-spin gauge theories in anti de-Sitter space [8] [9] [10] in the context of the AdS/CFT correspondence. Under general principles of AdS/CFT, we expect that the conformal field theory duals to Vasiliev's theories (when given appropriate boundary conditions) should also have higher-spin symmetry. This turns out to be true; as we will review later in chapter 1, AdS/CFT relates the $O(N)$ [11] and Gross-Neveu models [12] [13] to Vasiliev theory. As in the string-theoretic example, it would be nice to be able to extend these dualities beyond the exactly conformal case. (Indeed, studying the $O(N)$ and Gross-Neveu models in continuous dimensions, where the interaction term becomes a relevant operator, is the topic of one chapter of this thesis.)

Higher-spin conformal field theories could be also relevant in other fields of physics besides high-energy theory. In condensed matter physics, the classification of critical phenomena is one of the overarching goals of the field as a whole, and one can imagine that there are universality classes where higher-spin symmetry plays an important role. More speculatively, in cosmology, there have been proposals for a "dS/CFT" correspondence [14] that relates theories in a de Sitter background to CFTs. Since it is widely speculated that the universe experienced a period of inflation during which the geometry of the universe was well-approximated by de Sitter space, such a duality might provide a setting where tools from CFT, and perhaps even higher-spin CFT, could provide insight. It is less clear in these contexts, however whether the higher-spin perspective generates any advantage compared to more conventional techniques.

Hence, there are many reasons why understanding the general behavior of higher-spin CFTs and their renormalization group flows is important. In the next two sections, we will provide more background on particular questions in that spirit that this thesis tackles, indicate in which chapter each question is addressed, and briefly summarize the results of the corresponding chapter.

1.2 Classical no-go theorems in quantum field theory

The baseline intuition for how higher-spin conformal field theories "should behave" is that the higher-spin symmetry is so constraining that it essentially disallows interactions. This intuition arises from the Coleman-Mandula and Weinberg-Witten theorems, two classic theorems in quantum field theory that essentially forbid interacting higher-spin theories in quantum systems that satisfy a certain number of technical assumptions. These assumptions, however, are not satisfied by conformal field theories, so one might worry that this intuition is incorrect. Chapters 2 and 3 answer this question in the negative; there do exist results similar to the Coleman-Mandula and Weinberg-Witten theorems in conformal field theory. In this section, we review both theorems, explaining why the situation in conformal field theory differs, and how these differences are addressed in chapters 2 and 3.

1.2.1 Coleman-Mandula theorem

The Coleman-Mandula theorem [15] states that, under a certain number of technical assumptions which we will elaborate shortly, all conserved charges besides those arising from the Poincare generators must be Lorentz scalars or else the theory is free in the sense that the S-matrix is trivial. A more concrete restatement of this result is that the only way to extend the Poincare group in an interacting quantum field theory is to add internal symmetries.

The relation to higher-spin theories is as follows: Recall that charges of a conserved current are generated by contracting the current with a Killing vector and then integrating over a hypersurface. The charges that a conserved current generates therefore have spin one less than the current. Hence, we have the familiar statement that scalar charges are generated by spin 1 currents and that the stress tensor generates momentum and angular momentum. So if we assume that a theory contains a unique spin-2 current (the stress tensor), violation of the Coleman-Mandula theorem would mean we have an interacting conserved current of spin 3 or greater[1].

There are a number of technical assumptions that this result relies on, but two in particular are important from the perspective of conformal field theory.

1. The theory has a mass gap.

2. The theory has an S-matrix.

[1] Actually, this statement is slightly false. The Coleman-Mandula theorem doesn't account for fermionic charges, and indeed such charges can exist. Haag, Lopuszanski, and Sohnius extended the Coleman-Mandula theorem to account for this possibility [16]. The theories they found are those containing supersymmetry. So, for instance, we can have an interacting supersymmetric theory containing a supercurrent of spin 3/2.

The first assumption obviously cannot hold in conformal field theory. Conformal field theories, by definition, are scale-invariant, so there can be no dimensionful scales in a CFT. This disallows a mass gap since a mass gap would be such a scale. Note that this doesn't mean that only "massless operators" can exist in a CFT. There are representations of the conformal group that have nonvanishing support on every mass shell. Conformal symmetry just implies that the spectral density of any such field as a function of mass has to be scale-invariant.

Then, recall that the matrix elements of the S-matrix are, by definition, transition amplitudes between states asymptotically "far in the past" and states asymptotically "far in the future". The second assumption cannot hold in conformal field theory because scale invariance forbids the construction of such asymptotic states, since there can be no scale that tells you when excitations are "sufficiently far apart" from each other. More concretely, conformal symmetry implies that the two-point function of any primary operator in a CFT exhibits power-law decay, not exponential decay. Thus, the scale in the exponential that would normally appear in two-point functions (which is the scale that would define when two excitations created by that operator are "far apart") is replaced by a dimensionless exponent, the scaling dimension. Another more qualitative picture of this point, is just that two local operators at any finite separation can be brought arbitrarily close together by acting with the dilatation operator. There is no meaningful sense in which "faraway operator insertions become free".

More broadly, the violation of the second assumption renders the entire proof of the Coleman-Mandula theorem, which is essentially a careful analysis of two-particle scattering amplitudes, incoherent in conformal field theory. An entirely new argument is required in order to constrain CFTs that contain higher-spin conserved currents. This project was first carried out by Maldacena and Zhiboedov [17] in $d = 3$ dimensions. In chapter 2 of this thesis, we generalize their argument to $d > 3$ dimensions as follows:

First, we show that the existence of a single symmetric higher-spin current implies the existence of an entire tower of higher-spin currents in conformal field theory. There are therefore an infinite number of Ward identities that the correlation functions of a higher-spin CFT must satisfy, and we explicitly find all the solutions assuming unitarity. The result is that all the correlation functions of symmetric operators in the theory must coincide with the correlation functions of either the theory of N free bosons, N free fermions, or N free $\frac{d-2}{2}$-form fields with N an integer. In particular, this means that all the stress tensor correlation functions have to agree with some free field result. This is a statement that interactions are essentially disallowed and therefore our result can indeed be viewed as an extension of the Coleman-Mandula theorem to conformal field theory.

1.2.2 Weinberg-Witten theorem

The Weinberg-Witten theorem [18] constrains the helicities of massless particles in a four-dimensional quantum field theory. It makes two statements:

1. If there exists a gauge-invariant current J_μ, there can be no charged massless particles of helicity larger than 1/2.

2. If there exists a conserved stress-energy tensor $T_{\mu\nu}$, there can be no massless particles of helicity larger than 1.

The proof follows from examining one-particle matrix elements of J and T in the limit of foreward scattering. The proof is instructive, so we briefly review it, starting with the first statement. If J_μ is a conserved current, it generates a charge Q. Q acts on one particle states by $Q|p,j\rangle = q|p,j\rangle$, where q is the charge carried by the massless one-particle state of momentum p and helicity j. Using the normalization of the states $\langle p',j|p,j\rangle = \delta^{(3)}(\vec{p}' - \vec{p})$, we then infer that $\langle p',j|Q|p,j\rangle = q\delta^{(3)}(\vec{p}' - \vec{p})$. But Q, by definition, is equal to $\int d^3x J^0(x)$, so

$$\langle p',j|Q|p,j\rangle = \int d^3x \langle p',j|J^0(t,\vec{x})|p,j\rangle \tag{1.2.1}$$

$$= \int d^3x \langle p',j|e^{i\mathbf{P}\cdot\vec{x}} J^0(t,0) e^{-i\mathbf{P}\cdot\vec{x}}|p,j\rangle \tag{1.2.2}$$

$$= \int d^3x e^{i(\mathbf{p}'-\mathbf{p})\cdot x} \langle p',j|J^0(t,0)|p,j\rangle \tag{1.2.3}$$

$$= (2\pi)^3 \delta^{(3)}(\vec{p}' - \vec{p}) \langle p',j|J^0(t,0)|p,j\rangle \tag{1.2.4}$$

from which it follows that

$$\langle p',j|J^0(t,0)|p,j\rangle = \frac{q}{(2\pi)^3} \tag{1.2.5}$$

In the forward scattering limit $p' \to p$, the only Lorentz vector that is present is p^μ, so we must have

$$\lim_{p'\to p} \langle p',j|J^\mu(t,0)|p,j\rangle = \frac{qp^\mu}{p^0(2\pi)^3} \tag{1.2.6}$$

So this object is *not zero*.

On the other hand, consider the center of mass frame where $\vec{p}$ points along the z direction so that $\vec{p}'$ points in the $-z$ direction (this is always possible for spacelike momentum transfer, and we can choose the direction along which the $p' \to p$ limit is taken). Then, perform a rotation by θ around

the z axis. The rotation generator acts on our massless states of helicity j as:

$$|p, j\rangle \to e^{ij\theta}|p, j\rangle \tag{1.2.7}$$

$$|p', j\rangle \to e^{-ij\theta}|p', j\rangle \tag{1.2.8}$$

Thus,

$$\langle p', j|J^\mu(t, 0)|p, j\rangle \to e^{2ij\theta}\langle p', j|J^\mu(t, 0)|p, j\rangle \tag{1.2.9}$$

But alternately, $J^\mu \to \Lambda^\mu_\nu J^\nu$, where Λ is the Lorentz transformation matrix that implements the rotation, so

$$\langle p', j|J^\mu(t, 0)|p, j\rangle \to \Lambda^\mu_\nu\langle p', j|J^\nu(t, 0)|p, j\rangle \tag{1.2.10}$$

Since the eigenvalues of Λ are $e^{\pm i\theta}$ and 1, we must have $j = 0$ or $j = \pm 1/2$ or else the matrix element would vanish, which we just established was nonzero. This finishes the proof for J_μ. The proof for the stress tensor proceeds analogously, except T transforms with two Λ matrices, so $e^{\pm 2i\theta}$ are also possible eigenvalues, which allows for $j = \pm 1$.

Sadly, this elegant and beautiful proof clearly cannot work in conformal field theory for two reasons. First, as mentioned in the context of the Coleman-Mandula discussion, there are no asymptotic states, and in particular, there are no one-particle states. One can try constructing ersatz one-particle states from some local CFT operator $\mathcal{O}(x)$ by a naive Fourier transformation $\int e^{ipx}\mathcal{O}(x)$, but in our investigations of such objects, one cannot easily constrain the corresponding three-point functions. Second, there is no notion of a field being "massless" in conformal field theory. The operator $\mathbf{P}^2$ isn't a Casimir element of the conformal group. In particular, it doesn't commute with the dilatation operator, so "mass" isn't a label for representations of the conformal group. This means that it is not even clear what the Weinberg-Witten theorem is supposed to say for conformal field theories. Which operators are we supposed to think are "sick"? In fact, we know that the naive reading where we just exclude all higher-spin content is clearly wrong since we know free field theories exist, and those at least contain symmetric higher spin operators. The Coleman-Mandula analogue we proved in chapter 2, however, seems to suggest that those are the only possibilities. So we conjecture that higher-spin operators that "behave like" free field higher-spin operators but live in a representation that does not appear in the spectrum of any free field theory are disallowed by conformal symmetry. To be precise, if we adopt the (A, B) notation for classifying representations of the Lorentz group[2], we make the following two conjectures:

[2]i.e. (A, B) fields have $2A$ undotted and $2B$ dotted indices in the van der Waerden notation.

1. A local operator $\mathcal{O}(x)$ of type $(k, 0)$ or $(0, k)$ that satisfies the Dirac equation $\partial\!\!\!/\mathcal{O} = 0$, saturates the unitarity bound, and has $k \geq 3/2$ cannot appear in a consistent unitary conformal field theory.

2. A local operator $\mathcal{O}(x)$ of type $(A, A + k)$ or $(A + k, A)$ that satisfies the conservation equation $\partial \cdot O = 0$, saturates the unitarity bound, and has $k \geq 5/2$ cannot appear in a consistent unitary conformal field theory.

To compare with free field theory: there are no $(0, k)$ free fields with $k \geq 3/2$ in $d = 4$, which motivates the first statement. As for the second statement, the most "imbalanced" conserved current that exists in free field theory is comprised of the 1-form $F_{\mu\nu}$, some derivatives, and another copy of $F_{\mu\nu}$. This has symmetry type $(A, A + 2)$ for some A that depends on how many derivatives we insert. So we conjecture that conserved currents of symmetry type $(A, A + k)$ with $k \geq 5/2$ and $A \neq 0$ are disallowed.

In chapter 3, we show our progress towards proving this statement. At the time of this writing, this work was in preparation for publication; in what appears, we have a complete proof of the first statement, and we will provide evidence in the special case when a conserved current in the $(3, 1/2)$ representation is present that the theory is free in some sense.

Our strategy is to enumerate the independent structures allowed by conformal symmetry for three point funtions of T and two copies of the field we would like to analyze. Then, we will impose the constraints that T is conserved, that the field satisfies an equation of motion or a conservation condition, and that the field satisfies the conformal Ward identities. This will be sufficient to prove statement 1, as we will see there are no structures consistent with these constraints remaining when the spin is too large. For statement 2, we need an additional step, since we will find that there are solutions for representations of the form $(j, 1/2)$, for all $j \geq 1$. To constrain them, we will compute certain energy one-point correlation functions, which are constrainted by the Hofman-Maldacena bounds [19] [20]. We will show that although the bounds can be satisfied, for certain polarizations of the current the one-point energy correlator vanishes, which strongly suggests, via an argument of Zhiboedov [21] that the theory is free.

1.3 The critical $O(N)$ and Gross-Neveu models in AdS/CFT

Since the previous results suggest that conformal field theories with exact higher-spin symmetry are essentially trivial, it makes sense to move towards interacting models where the higher-spin

symmetry is weakly broken. The critical $O(N)$ and Gross-Neveu models are paradigm examples. The goal of chapter 4 is to compute how certain two-point functions are renormalized in these two models in various dimensions.

The diverse motivations for studying those two-point functions is surveyed in section 1 of chapter 4, but in order to better contextualize that work within the paradigm of higher-spin conformal field theory, we will elaborate on one specific motivation for studying those two theories - namely, their connection to higher-spin AdS/CFT dualities. Before proceeding, it is important to emphasize again that AdS/CFT is far from the only motivation for studying the $O(N)$ and Gross-Neveu models. For instance, they may be used to test monotonicity theorems in new dimensions [22], to generate novel conformal fixed points [23], to generate models exhibiting emergent supersymmetry [24], and more.

The discussion in this section elliptically follows the excellent review [25], which should be referred to if the reader desires additional detail about the following.

The Vasiliev higher-spin gauge theories are quantum field theories that all enjoy the following basic features:

1. The equations of motion have a vacuum solution corresponding to AdS.

2. The field content consists of a scalar of mass $m^2 = -2$ (in unit where $\ell_{\text{AdS}} = 1$, and an infinite tower of higher-spin currents of all spins $s = 1, 2, \ldots, \infty$[3]. In particular, there is a graviton. It is a theory of gravity.

3. The theory is interacting. There are higher-derivative couplings that become singular in the flat-space limit, so one cannot make sense of the theory except in AdS.

Each of these features has an interpretation from the perspective of AdS/CFT. The first property, along with the fact that Vasiliev theory contains a graviton, suggests that a CFT dual should exist. The fact that it has a tower of higher-spin currents suggests that we should examine the singlet sector of an $O(N)/U(N)$-symmetric theory of N free fields, since free theories are precisely the theories that exhibit such higher-spin currents[4]. Schematically, if we suppress indices, they have the general form $J_s^{ij} \sim \sum_k c_{sk} \partial^k \phi^i \partial^{s-k} \phi^j$. Once we have that, the fact that the bulk theory is interacting squares away with our intuition, since, e.g. the three-point function of currents is nonzero in free theory, so there had better be a nonvanishing bulk three-point vertex.

[3] This doesn't violate Coleman-Mandula since the background is AdS, not flat space.

[4] The large-N expansion is needed, as always in AdS/CFT, to make the bulk perturbation theory well-defined. We need to take the singlet sector so that we can distinguish between single and multi-trace operators, i.e. so that the bulk theory has a coherent notion of single-particle and multi-particle states.

As it turns out, many entries of the AdS/CFT dictionary work perfectly, e.g. with the ansatz that the so-called "minimal bosonic" Vasiliev theory is dual to the singlet sector of the $O(N)$ vector model theory of free bosons, we find that:

1. The single trace operators all have a corresponding higher-spin gauge field in the bulk.

2. The spin-zero singlet on the CFT side $\phi^i\phi^i$ is dual to the scalar in the bulk with the right mass given by the AdS/CFT dictionary.

3. The three-point functions of the currents on the boundary match the three-point function of the corresponding gauge fields in the bulk [26] [27].

These pieces of evidence strongly motivate the conjecture that the singlet sector of the $O(N)$ vector model is dual to Vasiliev theory in AdS. This is a profound statement; it means that what is effectively a trivial, noninteracting CFT secretly contains the data of some model of quantum gravity and gives one the hope that such approaches can generate new, tractable, UV-finite models of quantum gravity.

Furthermore, there are a number of generalizations that allow one to relate Vasiliev gravity to interacting CFTs. This is where the critical $O(N)$ and Gross-Neveu models enter the picture. When one subjects the free $O(N)$ bosonic and free $U(N)$ fermionic models by the quartic double-trace deformations $(\phi_i\phi_i)^2$ and $(\bar{\psi}_i\psi_i)^2$ in continuous dimensions, the two theories flow from a free theory in the ultraviolet to an interacting IR fixed point. These two fixed points are the critical $O(N)$ and Gross-Neveu models, respectively. In the case of Vasiliev gravity in four dimensions, we will be interested in understanding this RG flow in three dimensions.

The AdS interpretation of these renormalization group flows can be inferred from how the dimensions of the operators change under the RG flow. For instance, in the bosonic theory, the scalar ϕ_i^2 flows from dimension $\Delta = d - 2 = 1$ to $\Delta = 2$. The AdS/CFT dictionary tells us, in units where $\ell_{\text{AdS}} = 1$, that a scalar of mass m^2 is dual to a CFT operator of dimension $\Delta_\pm = (d/2) \pm \sqrt{(d/2)^2 + m^2}$. As mentioned, in the Vasiliev theory we have $m^2 = -2$, so both $\Delta_+ = 2$ and $\Delta_- = d - \Delta_+ = 1$ are above the unitarity bound $\Delta \geq (d-2)/2 = 1/2$. The interpretation is that the Vasiliev theory is dual to both the UV and IR fixed points, but with different boundary conditions on the bulk fields.

The higher-spin currents can also be examined under the RG flow. One finds that the anomalous dimensions of the higher spin currents are of order $O(1/N)$, so they remain massless and conserved at leading order in $1/N$. This is the statement that the higher-spin symmetry is only weakly broken in these theories - one can only see the nonconservation in loop corrections. So in these theories,

one can consistently do perturbation theory around the exactly-conserved higher-spin theory, which realizes, in a simpler model than string theory, the dream of studying quantum gravity as a correction to a higher-spin theory.

In chapter 4, we study loop corrections to the two-point functions of the stress tensor and of the $U(1)$ conserved current in both the $O(N)$ and Gross-Neveu models in various dimensions. We do not make direct contact with particular holographic calculations in these theories, but the preceding motivate, on general grounds, a better understanding of the renormalization group flows of these theories.

1.4 Outline of this thesis

This thesis is organized as a series of papers. Chapter 2, the Coleman-Mandula analogue, is based on the paper [28], coauthored with Vasyl Alba, which extended the paper [29], also coauthored with Vasyl Alba. Chapter 3, the Weinberg-Witten analogue, is based on work with Clay Córdova and Thomas Dumistrescu in preparation for publication at the time of this writing. Chapter 4, the analysis of the two-point functions of the stress tensor and the $U(1)$ current in the $O(N)$ and Gross-Neveu models, is based on the paper [30] which was coauthored with Lin Fei, Simone Giombi, Igor Klebanov, and Grigory Tarnopolsky.

Chapter 2

Constraining conformal field theories with a higher spin symmetry in $d > 3$ dimensions

2.1 Introduction

Characterizing the theories dual to Vasiliev's higher-spin gauge theories in anti de-Sitter space [8] [9] [10] under the AdS/CFT correspondence [1] [2] [3] has been a topic of active research for over ten years, starting from the conjecture of Klebanov and Polyakov that Vasiliev's theory in four dimensions is dual to the critical $O(N)$ vector model in three dimensions [11] [13]. Under general principles of AdS/CFT, we expect that the conformal field theory duals to Vasiliev's theories (when given appropriate boundary conditions) should also have higher-spin symmetry, so it is natural to try to classify all higher-spin conformal field theories. In the case of CFT's in three dimensions, this task has already been accomplished by Maldacena and Zhiboedov [17], who showed that unitary conformal field theories with a unique stress tensor and a higher-spin current are essentially free in three dimensions. This can be viewed as an analogue of the Coleman-Mandula theorem [15] [16], which states that the maximum spacetime symmetry of theories with a nontrivial S-matrix is the super-Poincare group, along with any internal symmetries whose charges are Lorentz-invariant quantum numbers (i.e. are scalars with respect to the spacetime symmetry group).

In this chapter, we will prove an analogue of the Coleman-Mandula theorem for generic conformal

field theories in all dimensions greater than three. We will show that in any conformal field theory that (a) satisfies the unitary bound for operator dimensions, (b) satisfies the cluster decomposition axiom, (c) contains a symmetric conserved current of spin larger than 2, and (d) has a unique stress tensor in $d > 3$ dimensions, all correlation functions of symmetric currents of the theory are equal to the correlation functions of one of the following three theories - either the theory of n free bosons (for some integer n), a theory of n free fermions, or a theory of n free $\frac{d-2}{2}$-forms.

Note that in odd dimensions, the free $\frac{d-2}{2}$-form does not exist, and the status of our result is somewhat complicated. We do not show that there exists any solution to the conformal Ward identities that corresponds to this possibility in odd dimensions, although we do show that if one exists, it is unique. For every odd dimension $d \geq 7$, we know that an infinite tower of higher-spin currents must be present [31], but in $d = 5$, it may be the case that there are not infinitely many higher spin currents. Assuming that the solution exists and there are an infinite number of higher spin currents, we show that the correlation functions of the conserved currents of the theory may be understood as the analytic continuation of the correlation functions of the currents of the even-dimensional free $\frac{d-2}{2}$-form theory to odd dimensions. Then, even under all these assumptions, we do not show that there exists any conformal field theory that realizes this solution. That is, it is possible that this structure may have no good microscopic interpretation for other reasons. For example, in odd dimensions it could be possible that some correlation function of some operators is not consistent with the operator product expansion in the sense that it cannot be decomposed in a sum over conformal blocks with non-negative coefficients (i.e. consistent with unitarity[1]). Such questions are not explored in this work.

Furthermore, we note that a recent paper by Boulanger, Ponomarev, Skvortsov, and Taronna [31] strongly indicates that all the algebras of higher-spin charges that are consistent with conformal symmetry are not only Lie algebras but associative. Hence, they are all reproduced by the universal enveloping construction of [32] with the conclusion that any such algebra must contain a symmetric higher-spin current. This implies that our result should be true even after relaxing our assumption that the higher-spin current is symmetric. The argument is structured as follows:

In section 2.2, we will present the main technical tool of the chapter: we will define a particular limit of three-point functions of symmetric conserved currents called *lightcone limits*. We will show that such correlation functions behave essentially like correlation functions of a

[1]There is an example of this phenomenon. If one considers a theory of N scalar fields ϕ_i and computes the four-point function of the operator $\phi^2 = \sum_i \phi_i \phi_i$, it turns out that N should be greater then 1, otherwise the theory is nonunitary.

free theory in these limits, enabling us to translate complicated Ward identities of the full theory into simpler ones involving only free field correlators. We will also compute the Fourier transformation of these correlation functions; this will ultimately allow us to simplify certain Ward identities into easily-analyzed polynomial equations.

The rest of the chapter will then carry out proof of our main statement. The steps are as follows:

In section 2.3, we will solve the Ward identity arising from the action of the charge Q_s arising from a spin s current j_s on the correlator $\langle j_2 j_2 j_s \rangle$ in the lightcone limit, where j_2 is the stress tensor. We will show that the only possible solution is given by the free-field solution. This implies the existence of infinitely many conserved currents of arbitrarily high spin,[2] thereby giving rise to infinitely many charge conservation laws which powerfully constrain the theory.

In section 2.4, we will construct certain quasi-bilocal fields which roughly behave like products of free fields in the lightcone limit, yet are defined for any CFT. We will establish that all the higher-spin charges (whose existence was proven in the previous step) act on these quasi-bilocals in a particularly simple way.

In section 2.5, we will translate the action of the higher-spin charges on the quasi-bilocals into constraints on correlation functions of the quasi-bilocals. We will then show that these constraints are so powerful that they totally fix every correlation function of the quasi-bilocals to agree with the corresponding correlation function of a particular biprimary operator in free field theory on the lightcone.

In section 2.6, we show how the quasi-bilocal correlation functions can be used to prove that the three-point function of the stress tensor must be equal to the three-point function of either the free boson, the free fermion, or the free $\frac{d-2}{2}$-form, even away from the lightcone limit. This is then used to recursively constrain every correlation function of the CFT to be equal to the corresponding correlation function in the free theory, finishing the proof.

This strategy is similar to the argument in the three-dimensional case given in [17]. There are two main differences between the three-dimensional case and the higher-dimensional cases that we must account for:

[2]The fact that the existence of a higher-spin current implies the existence of infinitely many other higher-spin currents has been proven before in the four-dimensional case [33] under the additional assumptions that the theory flows to a theory with a well defined S-matrix in the infrared, that the correlation function $\langle j_2 j_2 j_s \rangle \neq 0$, and that the scattering amplitudes of the theory have a certain scaling behavior. This statement was also proven for $d \neq 4, 5$ in [31] by classifying all the higher-spin algebras in all dimensions other than 4 and 5. We give a proof for the sake of completeness, and also because our techniques differ from those two papers.

First, the Lorentz group in $d > 3$ admits asymmetric representations, but the three-dimensional Lorentz group does not. By asymmetric, we mean that a current $J_{\mu_1 \ldots \mu_n}$ is not invariant with respect to interchange of its indices. For example, in in the standard (j_1, j_2) classification of representations of the four-dimensional Lorentz group induced from the isomorphism of Lie algebras $\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$, these are the representations with $j_1 \neq j_2$. The existence of these representations means that many more structures are possible in $d > 3$ dimensions than in three dimensions (the asymmetric structures), and so many more coefficients have to be constrained in order to solve the Ward identities. We restrict our attention to Ward identities arising from the action of a symmetric charge to correlation functions of only symmetric currents; we will then show that asymmetric structures cannot appear in these Ward identities, making the exact solution of the identities possible.

Second, the space of possible correlation functions consistent with conformal symmetry is larger in $d > 3$ dimensions than in three dimensions. For example, consider the three-point function of the stress tensor $\langle j_2 j_2 j_2 \rangle$. It has long been known (see, e.g. [34] [35] [36] [37]) that this correlation function factorizes into three structures in $d > 3$ dimensions, as opposed to only two structures in three dimensions (ignoring a parity-violating structure which is eliminated in three dimensions by the higher-spin symmetry). These three structures correspond to the correlation functions that appear in the theories of free bosons, free fermions, and free $\frac{d-2}{2}$-forms. We will show that even though more structures are possible in four dimensions and higher, the Ward identities we need can still be solved.

We note that our work is related to a paper by Stanev [38], in which the four, five, and six-point correlation functions of the stress tensor were constrained in CFT's with a higher spin current in four dimensions. It was also shown that the pole structure of the general n-point function of the stress tensor coincides with that of a free field theory. Though this chapter reaches the same conclusions, we do not make the rationality assumption [39] of that paper.

This work is also related to [40], in which the authors showed that unitary "Cauchy conformal fields", which are fields that satisfy a certain first-order differential equation, are free in the sense that their correlation functions factorize on the 2-point function. Their result may be understood as establishing a similar result that applies even to certain fields which are not symmetric traceless, which we say nothing about.

2.2 Definition of the lightcone limits

The fundamental technical tool we need to extend into four dimensions and higher is the *lightcone limit*. In order to constrain the correlation functions of the theory to be equal to free field correlators, we will show that the three-point function of the $\langle j_2 j_2 j_2 \rangle$ must be equal to $\langle j_2 j_2 j_2 \rangle$ for a free boson, a free fermion, or a free $\frac{d-2}{2}$-form field - it cannot be some linear combination of these three structures. To this end, it will be helpful to split up the Ward identities of the theory into three different identities, each of which involves only one of the three structures separately. To do this, we will need to somehow project all the three-point functions of the theory into these three sectors. The lightcone limits accomplish this task.

Before defining the lightcone limits, we will set up some notation. As in [17], we are writing the flat space metric $ds^2 = dx^+ dx^- + d\vec{y}^2$ and contracting each current with lightline polarization vectors whose only nonzero component is in the minus direction: $j_s \equiv J_{\mu_1 \dots \mu_s} \epsilon^{\mu_1} \dots \epsilon^{\mu_s} = J_{--\dots-}$. We will also denote $\partial_1 \equiv \partial/\partial x_1^-$ and similarly for ∂_2 and ∂_3. Thus, in all expressions where indices are suppressed, those indices are taken to be minus indices. There are two things we will establish:

1. We need to define an appropriate limit for each of the three cases, which, when applied to a three-point function of conserved currents $\left\langle \underline{j_{s_1} j_{s_2}} j_{s_3} \right\rangle$, yields an expression proportional to an appropriate correlator of the free field theory. For example, in the bosonic case where all the currents are symmetric, we would like the lightcone limit to give us $\partial_1^{s_1} \partial_2^{s_2} \langle \phi \phi^* j_{s_3} \rangle_{\text{free}}$.

2. Second, we need to explicitly compute the free field correlator which we obtain from the lightcone limits. In the bosonic case where all currents are symmetric, this would mean that we need to compute the three-point function $\langle \phi \phi^* j_{s_3} \rangle$ in the free theory.

For the first task, we claim that the desired lightcone limits are:

$$\left\langle \underline{j_{s_1} j_{s_2}}_b j_{s_3} \right\rangle \equiv \lim_{|y_{12}| \to 0} |y_{12}|^{d-2} \lim_{x_{12}^+ \to 0} \langle j_{s_1} j_{s_2} j_{s_3} \rangle \propto \partial_1^{s_1} \partial_2^{s_2} \langle \phi \phi^* j_{s_3} \rangle_{\text{free}} \tag{2.2.1}$$

$$\left\langle \underline{j_{s_1} j_{s_2}}_f j_{s_3} \right\rangle \equiv \lim_{|y_{12}| \to 0} |y_{12}|^{d} \lim_{x_{12}^+ \to 0} \frac{1}{x_{12}^+} \langle j_{s_1} j_{s_2} j_{s_3} \rangle \propto \partial_1^{s_1 - 1} \partial_2^{s_2 - 1} \left\langle \psi \gamma_- \bar{\psi} j_{s_3} \right\rangle_{\text{free}} \tag{2.2.2}$$

$$\left\langle \underline{j_{s_1} j_{s_2}}_t j_{s_3} \right\rangle \equiv \lim_{|y_{12}| \to 0} |y_{12}|^{d+2} \lim_{x_{12}^+ \to 0} \frac{1}{(x_{12}^+)^2} \langle j_{s_1} j_{s_2} j_{s_3} \rangle \propto \partial_1^{s_1 - 2} \partial_2^{s_2 - 2} \left\langle F_{-\{\alpha\}} F_{-\{\alpha\}} j_{s_3} \right\rangle_{\text{free}} \tag{2.2.3}$$

Here, the subscript *b, f,* and t denote the bosonic, fermionic, and tensor lightcone limits. ϕ is a free boson, ψ is a free fermion, and F is the field tensor for a free $\frac{d-2}{2}$-form field; the repeated $\{\alpha\}$ indices indicate Einstein summation over all other indices. For example, in four dimensions, the "tensor" structure is just the ordinary free Maxwell field. For conciseness, we will often refer to

the free $\frac{d-2}{2}$-form field as simply the "tensor field" or the "tensor structure". Again, we emphasize that in odd dimensions, the free $\frac{d-2}{2}$-form field does not exist. In odd dimensions, our claim is that the only possible structure with the scaling behavior captured by the tensor lightcone limit is the one which coincides with the naive analytic continuation of the correlation functions of the free $\frac{d-2}{2}$ form field to odd d.

The justification for the first two equations comes from the generating functions obtained in [36] [37]; in those references, the three-point functions for correlation functions of conserved currents with y_{12} and x_{12}^+ dependence of those types was uniquely characterized, and so taking the limit of those expressions as indicated gives us the claimed result. In the tensor case, [37] did not find a unique structure, but rather, a one-parameter family of possible structures. Nevertheless, all possible structures actually coincide in the lightcone limit, as is proven in appendix B.

We note that parity-violating structures cannot appear after taking these lightcone limits. This is because the all-minus component of every parity violating structure allowed by conformal invariance in $d > 3$ dimensions is identically zero. To see this, observe that all parity-violating structures for three-point functions consistent with conformal symmetry must have exactly one $\epsilon_{\mu_1\mu_2\ldots\mu_d}$ tensor contracted with polarization vectors and differences in coordinates. Only two of these differences are independent of each other, and all polarization vectors in the all-minus components are set to be equal. Thus, there are only three unique objects that can be contracted with the ϵ tensor, but we need d unique objects to obtain a nonzero contraction. Thus, all parity-violating structures have all-minus components equal to zero.

Later in our argument, we will need expressions for the Fourier transformation of the lightcone-limit three point function of two free fields and a spin s current with respect to the variables x_1^- and x_2^- in the theories of a free boson, a free fermion, and a free $\frac{d-2}{2}$-form field. The computation for each of the three cases is straightforward and is given explicitly in appendix A. The results are as follows:

$$F_s^b \equiv \langle \underline{\phi\phi^*} j_s \rangle \propto (p_2^+)^s {}_2F_1\left(2 - \frac{d}{2} - s, -s, \frac{d}{2} - 1, p_1^+/p_2^+\right) \tag{2.2.4}$$

$$F_s^f \equiv \left\langle \underline{\psi\gamma_-\bar{\psi}} j_s \right\rangle \propto (p_2^+)^{s-1} {}_2F_1\left(1 - \frac{d}{2} - s, -s, \frac{d}{2}, p_1^+/p_2^+\right) \tag{2.2.5}$$

$$F_s^b \equiv \left\langle \underline{F_{-\{\alpha\}}F_{-\{\alpha\}}} j_s \right\rangle \propto (p_2^+)^{s-2} {}_2F_1\left(-\frac{d}{2} - s, -s, \frac{d}{2} + 1, p_1^+/p_2^+\right) \tag{2.2.6}$$

Here, ${}_2F_1$ is the hypergeometric function, and the proportionality sign in each formula indicates that we have omitted an overall nonsingular function which we are not interested in. That they are

indeed nonsingular is also proven in appendix A.

Before continuing, we emphasize that the three lightcone limits we have defined do not cover all possible lightcone behaviors which can be realized in a conformal field theory. We define only these three limits because one crucial step in our proof is to constrain the three-point function of the stress tensor $\langle j_2 j_2 j_2 \rangle$, which has only these three scaling behaviors.

Furthermore, though we have discussed only symmetric currents, one could hope that similar expressions could be generated for asymmetric currents - that is, lightcone limits of correlation functions of asymmetric currents are generated by one of the three free field theories discussed here. Unfortunately, running the same argument in [37] fails in the case of asymmetric currents in multiple ways. Consider the current $\langle j_2 j_s \bar{j}_s \rangle$, where j_s is some asymmetric current and $\bar{j}_s$ is its conjugate. To determine how such a correlator could behave the lightcone limit, one could write out all the allowed conformally invariant structures consistent with the spin of the fields, and seeing how each one behaves in the lightcone limits. Unlike the symmetric cases, one finds that in the lightcone limit many independent structures exist, and these structures behave differently depending on which pair of coordinates we take the lightcone limit. To put it another way, for a symmetric current s, one has the decomposition:

$$\langle j_2 j_s j_s \rangle = \sum_{j \in \{b,f,t\}} \langle j_2 j_s j_s \rangle_j \tag{2.2.7}$$

where the superscript j denotes the result after taking corresponding lightcone limit in any of the three pairs of coordinates (all of which yield the same result), and the corresponding structures can be understood as arising from some free theory. In the case of asymmetric j_s, this instead becomes a triple sum

$$\langle j_2 j_s \bar{j}_s \rangle = \sum_{j,k,l \in \{b,f,t\}} \langle j_2 j_s \bar{j}_s \rangle_{(j,k,l)} \tag{2.2.8}$$

where each sum corresponds to taking a lightcone limit in each of the three different pairs of coordinates, and we do not know how to interpret the independent structures in terms of a free field theory. This tells us that for asymmetric currents, the lightcone limit no longer achieves its original goal of helping us split up the Ward identities into three identities which can be analyzed independently; each independent structure could affect multiple different Ward identities. Again, we emphasize that this does not exclude the possibility of a different lightcone limit reducing the correlators of asymmetric currents to those of some other free theory. It simply means that our techniques are not sufficient to constrain correlation functions involving asymmetric currents, so we will restrict our attention to correlation functions that involve only symmetric currents.

2.3 Charge conservation identities

We will now use the results of the previous section to prove that every CFT with a higher-spin current contains infinitely many higher-spin currents of arbitrarily high (even) spin. We note that this result was proven in a different way in [31] for all dimensions other than $d = 4$ and $d = 5$, wherein they showed that there is a unique higher-spin algebra in $d \neq 4, 5$ and showed that they all infinitely many higher-spin currents. The discussion below is a different proof of this statement based on analysis of the constraints that conservation of the higher-spin charge imposes, and the techniques we develop here will be used later. As before, we treat the bosonic, fermionic, and tensor cases separately.

Before beginning, we will tabulate a few results about commutation relations that we will use freely throughout from this section onwards. Their proofs are identical to those in [17], and are therefore omitted:

1. If a current j' appears (possibly with some number of derivatives) in the commutator $[Q_s, j]$, then j appears in $[Q_s, j']$.

2. Three-point functions of a current with odd spin with two identical currents of even spin are zero: $\langle j_s j_s j_{s'} \rangle = 0$ if s is even and s' is odd.

3. The commutator of a symmetric current with a charge built from another symmetric current contains only symmetric currents and their derivatives:

$$[Q_s, j_{s'}] = \sum_{s''=\max[s'-s+1,0]}^{s'+s-1} \alpha_{s,s',s''} \partial^{s'+s-1-s''} j_{s''} \tag{2.3.1}$$

   The proof of this statement requires an additional step since one needs to exclude asymmetric currents contracted with invariant symbols like the ϵ tensor. For example, consider what structures could appear in $[Q_2, j_2]$ in four dimensions. In $SU(2)$ indices, this object has three dotted and three undotted spinor indices, so one could imagine that a structure like $\epsilon_{ab} j^{abcde\dot{c}\dot{d}\dot{e}}$ could appear in $[Q_2, j_2]$. However, $[Q_2, j_2]$ has conformal dimension 5, and the unitarity bound constrains the current j, which transforms in the $(5/2, 3/2)$ representation, to have conformal dimension at least $d - 2 + s = 6$, which is impossible. The proof for a general commutator $[Q_s, j_{s'}]$ follows in an identical manner.

4. $[Q_s, j_2]$ contains ∂j_s. This was actually proven for all dimensions in appendix A of [17]. Item 1 then implies that $[Q_s, j_s]$ contains $\partial^{2s-3} j_2$.

In these statements, we are implicitly ignoring the possibility of parity violating structures. For example, the three-point function $\langle 221 \rangle$, which is related to the $U(1)$ gravitational anomaly, may not be zero in a parity violating theory. As mentioned in section 2.2, however, the all-minus components of every parity-violating structure consistent with conformal symmetry is identically zero, so they will not appear in any of our identities here.

Let's start with the bosonic case. Consider the charge conservation identity arising from the action of Q_s on $\langle \underline{22}_b s \rangle$:

$$0 = \left\langle \underline{[Q_s, 2]2}_b s \right\rangle + \left\langle \underline{2[Q_s, 2]}_b s \right\rangle + \langle \underline{22}_b [Q_s, s] \rangle \tag{2.3.2}$$

If s is symmetric, we may use the general commutation relation (2.3.1) and the lightcone limit (2.2.1) to expand this equation out in terms of free field correlators:

$$0 = \partial_1^2 \partial_2^2 \left(\gamma(\partial_1^{s-1} + (-1)^s \partial_2^{s-1}) \left\langle \underline{\phi\phi^* s} \right\rangle_{free} + \sum_{2 \leq k < 2s-1 \text{ even}} \tilde{\alpha_k} \partial_3^{2s-1-k} \left\langle \underline{\phi\phi^* k} \right\rangle_{free} \right) \tag{2.3.3}$$

Note that the sum over k is restricted to even currents since $\langle 22k \rangle = 0$ for odd k. In addition, the fact that the coefficient in front of the ∂_2^{s-1} term is constrained to be $(-1)^s$ times the coefficient for the ∂_1^{s-1} term arises from the symmetry of $\langle \phi(x_1)\phi^*(x_2) j_s(x_3) \rangle$ under interchange of x_1 and x_2.

Now, we apply our Fourier space expressions for the three-point functions given in section 2.2. In the Fourier transformed variables, derivatives along the minus direction turn into multiplication by the momenta in the plus direction. After "cancelling out" the overall derivatives, which just yields an overall factor of $(p_1^+)^2 (p_2^+)^2$, the relevant equation is:

$$0 = \gamma((p_1^+)^{s-1} + (-1)^s (p_2^+)^{s-1}) F_s(p_1^+, p_2^+) + \sum_{2 \leq k < 2s-1 \text{ even}} \tilde{\alpha}_k (p_1^+ + p_2^+)^{2s-1-k} F_k(p_1^+, p_2^+) \tag{2.3.4}$$

The solution of (2.3.4) is not easy to obtain by direct calculation. We can make two helpful observations, however. First, not all coefficients can be zero. This is because we know 2 appears in $[Q_s, s]$, so at least $\tilde{\alpha}_2$ is not zero. Second, we know that the free boson exists (and is a CFT with higher spin symmetry), and therefore, the coefficients one obtains from that theory would exactly solve this equation. We will show that this solution is unique.

Suppose we have two sets of coefficients $(\gamma, \{\tilde{\alpha}_k\})$ and $(\gamma', \{\tilde{\beta}_k\})$ that solve this equation. First, suppose $\gamma \neq 0$ and $\gamma' \neq 0$. Then, we can normalize the coefficients so that $\gamma = \gamma'$ are equal for the two solutions. Then, subtract the two solutions from each other so that the γ terms vanish. If we

evaluate the result at some arbitrary nonzero value of p_2^+, we may absorb all overall p_2^+ factors into the coefficients and re-express the equation as a polynomial identity in a single variable $z \equiv p_1^+/p_2^+$:

$$0 = \sum_{2\leq k<2s-1 \text{ even}} \tilde{\delta}_k (1+z)^{2s-1-k} {}_2F_1(2-\frac{d}{2}-k,-k,\frac{d}{2}-1,-z) \tag{2.3.5}$$

Then, the entire right hand side is divisible by $1+z$ since s is even, so we may divide both sides by $1+z$. Setting $z=-1$, since ${}_2F_1(a,a,1,1) \neq 0$ for all negative half-integers a, we conclude that $\tilde{\delta}_{2s-2}=0$. Then, the entire right hand side is proportional to $(1+z)^2$, so we may divide it out. Then, setting $z=-1$ again, we find $\tilde{\delta}_{2s-4}=0$. Repeating this procedure, we conclude that all coefficients are zero, and therefore, that the two solutions are identical. On the other hand, suppose one of the solutions has $\gamma=0$. Then, the same argument establishes that all the coefficients $\tilde{\alpha}_k$ are zero. As noted earlier, however, the trivial solution is disallowed. Therefore, the solution is unique and coincide with one for free boson. Thus, we have infinitely many even conserved currents, as desired.

In the fermionic case, precisely the same analysis works. The action of Q_s on $\langle \underline{22}_f s\rangle$ for symmetric s leads to

$$0 = \partial_1^2\partial_2^2\Big(\gamma(\partial_1^{s-2}+(-1)^{s-1}\partial_2^{s-2})\langle\psi\bar{\psi}s\rangle + \sum_{2\leq k<2s-2 \text{ even}} \tilde{\alpha}^k \partial_3^{2s-2-k}\langle\psi\bar{\psi}k\rangle\Big), \tag{2.3.6}$$

Then, converting this expression to form factors and running the same analysis from the bosonic case verbatim establishes that the unique solution to this equation is the one arising in the theory of a free fermion.

In the tensor case, the argument again passes through exactly as before, except for two subtleties:

First, unlike in the bosonic and fermionic case, we do not have unique expressions for the three-point functions of currents with the tensor-type coordinate dependence, so this only demonstrates that the free-field solution is an admissible solution, but not necessarily the unique solution. Nevertheless, in the lightcone limit, all possible structures for three-point functions coincide with the free-field answer.[3] This was proven in appendix B.

Second, there may not exist a solution to the Ward identities in odd dimensions, because the free $\frac{d-2}{2}$-form does not exist in odd dimensions. However, if any solution exists, our argument shows that it is unique. In $d \geq 7$, it is known that there is a unique higher-spin algebra containing the tower of higher-spin currents described in the bosonic and fermionic cases [31]. In $d=5$, our technique shows

[3] Actually, we proved that correlators of the form $\langle 22s\rangle$ have a unique tensor structure even away from the lightcone limit. The proof, however, is very technical, and it is given in appendix C.

that if there is a solution for the Ward identity in the tensor lightcone limit, then it is unique. We do not prove, however, that there is an infinite tower of higher spin currents or that there is exactly one current of every spin. Finite dimensional representations would be inconsistent with unitarity. We do not explore this question further in this work. Henceforth, we assume that our theory does indeed contain the infinite tower of higher-spin currents necessary for our analysis.

2.4 Quasi-bilocal fields: basic properties

In this section, we will define a set of quasi-bilocal operators, one for each of the three lightcone limits, and characterize the charge conservation identities arising from the action of the higher-spin currents. As we will explain in section 2.5, these charge conservation identities will turn out to be so constraining that the correlation functions of the quasi-bilocal operators are totally fixed. This will then enable us to recursively generate all the correlation functions of the theory and prove that the three-point function of the stress tensor can exhibit only one of the three possible structures allowed by conformal symmetry. As in the three-dimensional case, we define the quasi-bilocal operators on the lightcone as operator product expansions of the stress tensor with derivatives "integrated out":

$$\underline{22}_b = \partial_1^2 \partial_2^2 B(\underline{x_1, x_2}) \tag{2.4.1}$$

$$\underline{22}_f = \partial_1 \partial_2 F_-(\underline{x_1, x_2}) \tag{2.4.2}$$

$$\underline{22}_t = V_{--}(\underline{x_1, x_2}) \tag{2.4.3}$$

The motivation behind these definitions can be understood by appealing to what these expressions look like in free field theory. There, they will be given by simple products of free fields:

$$B(x_1, x_2) \sim : \phi(x_1)\phi^*(x_2) : + : \phi(x_2)\phi^*(x_1) : \tag{2.4.4}$$

$$F_-(x_1, x_2) \sim : \bar{\psi}(x_1)\gamma_-\psi(x_2) : - : \bar{\psi}(x_2)\gamma_-\psi(x_1) : \tag{2.4.5}$$

$$V_{--} \sim : F_{-\{\alpha\}}(x_1)F_{-\{\alpha\}}(x_2) : \tag{2.4.6}$$

It is clear from the basic properties of our lightcone limits that when they are inserted into correlation functions with another conserved current j_s, they will be proportional to an appropriate free field

correlator. Since $\langle \underline{22}s\rangle = 0$ for odd s, only the correlation functions with even s will be nonzero:

$$\left\langle B(\underline{x_1, x_2})j_s\right\rangle \propto \langle \phi(x_1)\phi^*(x_2)j_s(x_3)\rangle_{\text{free}} \tag{2.4.7}$$

$$\left\langle F_-(\underline{x_1, x_2})j_s\right\rangle \propto \left\langle \psi(x_1)\gamma_-\bar{\psi}(x_2)j_s(x_3)\right\rangle_{\text{free}} \tag{2.4.8}$$

$$\left\langle V_{--}(\underline{x_1, x_2})j_s\right\rangle \propto \left\langle F_{-\{\alpha\}}(x_1)F_{-\{\alpha\}}(x_2)j_s(x_3)\right\rangle_{\text{free}} \tag{2.4.9}$$

Of course, away from the lightcone, things will not be so simple: we have not even defined the quasi-bilocal operators there, and their behavior there is the reason why they are not true bilocals. In fact, even on the lightcone, these expressions are not fully conformally invariant: the contractions of indices performed in equations 2.4.8 and 2.4.9 are only invariant under the action of the collinear subgroup of the conformal group defined by the line connecting x_1 and x_2. For now, however, the lightcone properties enumerated above are enough to establish the commutator of Q_s with the bilocals. As usual, we begin with the bosonic case:

Assume that $\langle \underline{22}_b 2\rangle \neq 0$. Our goal is to show that

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1})B(\underline{x_1, x_2}). \tag{2.4.10}$$

This can be shown using the same arguments as [17]. To begin, notice that the action of Q_s commutes with the lightcone limit. Thus,

$$\langle [Q_s, B]j_k\rangle = \left\langle \underline{[Q_s, j_2]j_2}j_k\right\rangle + \left\langle \underline{j_2[Q, j_2]}j_k\right\rangle = -\langle \underline{j_2 j_2}[Q_s, j_k]\rangle = \left\langle [Q_s, \underline{j_2 j_2}]j_k\right\rangle \tag{2.4.11}$$

This immediately leads to:

$$[Q_s, B(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1})\tilde{B}(\underline{x_1, x_2}) + (\partial_1^{s-1} - \partial_2^{s-1})B'(\underline{x_1, x_2}), \tag{2.4.12}$$

Here, $\tilde{B}$ is built from even currents, while B' is built from odd currents. This makes the whole expression symmetric. We would like to show that $B' = 0$. Therefore, suppose otherwise so that some current $j_{s'}$ has nontrivial overlap with B'. Then, the charge conservation identity $0 = \langle [Q_{s'}, B'j_2]\rangle$ yields

$$0 = \left\langle \left[Q_{s'}, B'(\underline{x_1, x_2})\right] j_2\right\rangle + \left\langle B'(\underline{x_1, x_2})\left[Q_{s'}, j_2\right]\right\rangle, \tag{2.4.13}$$

$$\Rightarrow 0 = \gamma\left(\partial_1^{s'-1} - \partial_2^{s'-1}\right)\left\langle \underline{\phi\bar{\phi}}j_2\right\rangle + \sum_{k=0}^{s'+1}\tilde{\alpha}_k\partial^{s'+1-k}\left\langle \underline{\phi\bar{\phi}}j_k\right\rangle. \tag{2.4.14}$$

Using the same techniques as the previous section, we obtain

$$0 = \gamma((p_1^+)^{s'-1} - (p_2^+)^{s'-1})F_2(p_1^+, p_2^+) + \sum_{k=0}^{s'+1} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s'+1-k} F_k(p_1^+, p_2^+). \tag{2.4.15}$$

In this sum, $\tilde{\alpha}_{s'} \neq 0$ because $j_{s'} \subset [Q_{s'}, 2]$. Therefore, we can use the same procedure as before to show that all $\tilde{\alpha}_k$ are nonzero if they are nonzero for the free field theory. In particular, since $\tilde{\alpha}_1$ is not zero for the complex free boson, the overlap between j_1 and B' is not zero. Now, let's consider

$$0 = \langle [Q_s, Bj_1] \rangle = \left(\partial_1^{s-1} - \partial_2^{s-1}\right) \langle B' j_1 \rangle + \langle B\,[Q_s, j_1] \rangle, \tag{2.4.16}$$

where Q_s is a charge corresponding to any even higher-spin current appearing in the operator product expansion of $\underline{j_2 j_2}_b$. We have shown the first term is not zero. We will prove that the second term must be equal to zero to get a contradiction. Specifically, we will show that there are no even currents in $[Q_s, j_1]$. Since B is proportional to $\underline{22}$, and since $\langle 22s \rangle = 0$ for all odd s, this yields the desired conclusion.

Consider the action of Q_s on $\langle 221 \rangle$. We obtain the now-familiar form:

$$0 = \gamma((p_1^+)^{s-1} - (p_2^+)^{s-1})F_1(p_1^+, p_2^+) + \sum_{k=0}^{s} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s-k} F_k(p_1^+, p_2^+) \tag{2.4.17}$$

We want to show that $\alpha_k = 0$ for even k. Recall the definition of F_k:

$$F_k = (p_2^+)^k {}_2F_1\left(2 - \frac{d}{2} - k, -k, \frac{d}{2} - 1, -\frac{p_1^+}{p_2^+}\right) \tag{2.4.18}$$

$$= \sum_{i=0}^{k} c_i^k (p_1^+)^i (p_2^+)^{s-i} \tag{2.4.19}$$

The hypergeometric coefficients c_i^k have the property that $c_i^k = (-1)^k c_{k-i}^k$. Now, we collect terms in equation (2.4.17) proportional to $(p_1^+)^s$ and $(p_2^+)^s$ - each sum must vanish separately for the entire polynomial to vanish. We obtain

$$\gamma + \sum_{0 \leq k \leq s \text{ odd}} \alpha_k u_k + \sum_{0 \leq k \leq s \text{ even}} \alpha_k v_k = 0 \tag{2.4.20}$$

$$-\gamma - \sum_{0 \leq k \leq s \text{ odd}} \alpha_k u_k + \sum_{0 \leq k \leq s \text{ even}} \alpha_k v_k = 0 \tag{2.4.21}$$

Here, u_k and v_k are sums of products of coefficients of the hypergeometric function and the binomial expansion of $(p_1^+ + p_2^+)^{s-k}$; we do not care about their properties except that, with the signs indicated

above, they are strictly positive, as can be verified by direct calculation. By adding and subtracting these equations, we obtain two separate equations that must be satisfied by the odd and even coefficients separately

$$\gamma + \sum_{0\leq k\leq s \text{ odd}} \alpha_k u_k = 0 \tag{2.4.22}$$

$$\sum_{0\leq k\leq s \text{ even}} \alpha_k v_k = 0 \tag{2.4.23}$$

Exactly analogously, we may do the same procedure to every other pair of monomials $(p+1^+)^a(p_2^+)^{s-a}$ and $(p_1^+)^{s-a}(p_2^+)^a$ to turn the constraints for the two monomials into constraints for the even and odd coefficients (where we're considering γ as an odd coefficient) separately. Hence, by multiplying each term by the monomial from which it was computed and then resumming, we find that the original identity (2.4.17) actually splits into two separate identities that must be satisfied. For the even terms, this identity is:

$$0 = \sum_{0\leq k\leq s \text{ even}} \alpha_k (p_1^+ + p_2^+)^{s-k}(p_2^+)^k {}_2F_1\left(2 - \frac{d}{2} - k, -k, \frac{d}{2} - 1, -\frac{p_1^+}{p_2^+}\right) \tag{2.4.24}$$

Then, we may again use the argument from section 2.3 to conclude that all $\alpha_k = 0$ for even k, which is what we wanted. Thus, $B' = 0$.

Now we would like to show that $B = \tilde{B}$. First of all we will show that $\tilde{B}$ is nonzero. Consider the charge conservation identity

$$0 = \langle [Q_s, Bj_2] \rangle = \left(\partial_1^{s-1} + \partial_2^{s-1}\right) \left\langle \tilde{B}2 \right\rangle + \langle B, [Q_s, 2] \rangle \tag{2.4.25}$$

Since $[Q_s, j_2] \supset \partial j_s$, and since $\langle Bs \rangle \neq 0$, the second term in that identity is nonzero, and so $\tilde{B}$ must be nonzero. Now we can normalize the currents in such a way that j_2 has the same overlap with $\tilde{B}$ and B. After normalization, we know that $B - \tilde{B}$ does not contain any spin 2 current because the stress tensor is unique, by hypothesis. Now, we will show that $B - \tilde{B}$ is zero by contradiction. Suppose $B - \tilde{B}$ is nonzero. Then, there is a current j_s whose overlap with $B - \tilde{B}$ is nonzero. Then, the charge conservation identity for the case $s > 2$ is

$$0 = \left\langle \left[Q_s, \left(B - \tilde{B}\right) j_2 \right] \right\rangle, \tag{2.4.26}$$

$$0 = \gamma \left(\left(p_1^+\right)^{s-1} + \left(p_2^+\right)^{s-1} \right) F_2(p_1^+, p_2^+) + \sum_{k=0}^{s+1} \tilde{\alpha}_k (p_1^+ + p_2^+)^{s+1-k} F_k\left(p_1^+, p_2^+\right), \tag{2.4.27}$$

where we assume that $\tilde{\alpha}_s \neq 0$. Then, we can again run the same analysis as section 2.3 to conclude that since $\tilde{\alpha}_s \neq 0$, we must have $\tilde{\alpha}_2 \neq 0$ - that is, j_2 has nonzero overlap with $B - \tilde{B}$, which is a contradiction. It means that $B - \tilde{B}$ has no overlap with any currents j_s for $s > 2$. The only possibility is to overlap only with spin zero currents. Suppose that there is a current j_0' that overlaps with $B - \tilde{B}$, where the prime distinguishes it from a spin 0 current j_0 that could appear in B. We first show that $\langle j_0 j_0' \rangle = 0$. Consider the charge conservation identity the action Q_4 on $\left\langle (B - \tilde{B}) j_0 \right\rangle$. The action of the charge is $[Q_4, 0] = \partial^3 j_0 + \partial j_2 + \ldots$, where the $\ldots$ represent terms that cannot overlap with $\underline{22}$ (from which B is constructed) or the even currents that appear in $\tilde{B}$. By hypothesis, $B - \tilde{B}$ has no overlap with j_2, so the identity simplifies to $\langle j_0 j_0' \rangle = 0$. Then, since j_0' is nonzero, it should have nontrivial overlap with some Q_s. Now, recall the fact that if a current j' appears (possibly with some number of derivatives) in the commutator of $[Q_s, j]$, then j appears in $[Q_s, j']$. Thus, there should be a current current of spin $s'' < s$ such that $[Q_s, j_{s''}] = j_0' + \ldots$. The action Q_s on $\left\langle \left(B - \tilde{B}\right) j_{s''} \right\rangle$ is

$$\left\langle \left[Q_s, \left(B - \tilde{B}\right) j_{s''} \right] \right\rangle = \partial_3^3 \left\langle \left(B - \tilde{B}\right) j_0' \right\rangle + \partial \left\langle \left(B - \tilde{B}\right) j_2 \right\rangle, \tag{2.4.28}$$

Here, we have used that the action of Q_s on B and $\tilde{B}$ is identical because $B' = 0$. Then, since the second term is zero, thus the first term is equal to zero as well. Thus, $B - \tilde{B}$ has no overlap with any currents and is equal to zero, as desired.

In the fermionic case, we can run almost the same argument as in the bosonic case, except there is no discussion of a possible j_0, since there is no conserved spin zero current in the free fermion theory. We obtain the action of the charge on the fermionic quasi-bilocal is

$$[Q_s, F_-(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1}) F_-(\underline{x_1, x_2}). \tag{2.4.29}$$

In the tensor case, we again can repeat the argument to obtain

$$[Q_s, V_{--}(\underline{x_1, x_2})] = (\partial_1^{s-1} + \partial_2^{s-1}) V_{--}(\underline{x_1, x_2}) \tag{2.4.30}$$

In this case, there is neither a conserved spin 0 or spin 1 current in the free tensor theory. The argument works the same way, however, if we consider j_3 instead of j_1 in the steps of the argument that require it.

2.5 Quasi-bilocal fields: correlation functions

In this section, we will discuss how to precisely define the quasi-bilocal operators in a way that makes their symmetries manifest. In particular, each of the three bilocals will be bi-primary operators in some sense. This will allow us to argue that the correlation functions of the bilocals should agree with an appropriate corresponding free-field result. We will then explore what this implies for the full theory in section 2.6.

2.5.1 Symmetries of the quasi-bilocal operators

Let us first consider the case of the bosonic bilocal operator $B(x_1, x_2)$. Recall that, on the lightcone, the bilocals should imitate products of the appropriate free fields. In the bosonic free-field theory, the operator product expansion of $\phi(x_1)\phi^*(x_2)$ is composed of all of the even-spin currents of the theory with appropriate numbers of derivatives and factors of $(x_1 - x_2)$ so that the expression has the correct conformal dimension. More explicitly, we may write:

$$\phi(x_1)\phi^*(x_2) = \sum_{\text{even } s\geq 0} b_s^{\text{free}}(x_1, x_2) \tag{2.5.1}$$

$$b_s^{\text{free}}(x_1, x_2) = \sum_{(k,l)|s+l-k=0} c_{skl}(x_1 - x_2)^k \partial^l j_s \left(\frac{x_1 + x_2}{2}\right) \tag{2.5.2}$$

All the coefficients c_{skl} may be computed exactly in the free theory just by Taylor expansion. We have shown that all the currents j_s exist in our theory for all even s. So we may define an analogous quantity in our theory as follows:

$$B(x_1, x_2) = \sum_{\text{even } s\geq 0} b_s(x_1, x_2) \tag{2.5.3}$$

$$b_s(x_1, x_2) = \sum_{(k,l)|s+l-k=0} c'_{skl}(x_1 - x_2)^k \partial^l j_s \left(\frac{x_1 + x_2}{2}\right) \tag{2.5.4}$$

Here, the c' are some other coefficients which are to be determined by demanding that this definition of B coincide with the definition given on the lightcone in the previous section, i.e. that $\partial_1^2 \partial_2^2 B(\underline{x_1, x_2}) = \underline{22}_b$. We claim that this can be accomplished by choosing the c' coefficients such that $\left\langle B(\underline{x_1, x_2}) j_s \right\rangle \propto \left\langle \underline{\phi(x_1)\phi^*(x_2)} j_s \right\rangle_{\text{free}}$. To see that there exists such a choice of c' which can achieve this condition, we explicitly compare $\langle B j_s \rangle$ and $\langle \phi\phi^* j_s \rangle_{\text{free}}$ term by term using 2.5.2 and 2.5.4. Each term in both of these correlation functions has the structure $(x_1 - x_2)^k \partial^l \langle j_{s'} j_s \rangle$ with coefficient $c_{s'kl}$ and $c'_{s'kl}$, respectively. Two-point functions of primary operators in CFT's are

determined up to a constant, so each term is identical up to a possible scaling, which can be eliminated by choosing the c' coefficient appropriately. Then, by applying $\partial_1^2\partial_2^2$ to both sides of $\left\langle B(\underline{x_1,x_2})j_s\right\rangle \propto \left\langle \underline{\phi(x_1)\phi^*(x_2)}j_s\right\rangle_{\text{free}}$, we find that our definition coincides on the lightcone, as desired. This construction works the same way for the fermionic and tensor quasi-bilocals with analogous results, except that the quasi-bilocals in those cases carry the appropriate spin structure.

Since the conformal transformation properties of a conserved current j_s is theory-independent in the sense that it is completely fixed by its spin and conformal dimension, it is manifest from this definition that the bosonic quasi-bilocal $B(\underline{x_1,x_2})$ has the same transformation properties under the full conformal group as a product of free bosons. That is, it is a scalar bi-primary field with a conformal dimension of 1 with respect to each argument.

On the other hand, consider the fermonic and tensor quasi-bilocals F_- and V_{--}. The same line of reasoning tells us that they will transform like products of free fields contracted in a particular way: F_- will transform like $:\psi\gamma_-\bar{\psi}:$ does in the free fermionic theory, and V_{--} will transform like $:F_{-\{\alpha\}}F_-^{\{\alpha\}}:$ does in the theory of a free $\frac{d-2}{2}$-form[4]. These contractions, however, are not preserved by the full conformal group - the special conformal transformations orthogonal to the $-$ direction will ruin the structure of the Lorentz contraction. Thus, even in the free theory, these objects are not preserved by the full conformal group. They are only preserved by the so-called collinear conformal group generated by K_-, P_+, J_{+-}, and D, where K, P, J, and D are the generators of special conformal transformations, translations, boosts, and dilatations, respectively. It is clear from the structure of the conformal algebra that the commutation relations of this subset of conformal generators closes, so it forms a proper sub-algebra. Thus, what we are allowed to conclude is that F_- and V_{--} are bi-primary operators with respect to this collinear subgroup, not the conformal group. Nevertheless, this will still be enough symmetry for our purposes.

The key fact which is still true for this more restricted set of symmetries is that under K_-, the special conformal transformation in the $-$ direction, the n-point function of fermionic and tensor quasi bi-primaries should scale separately in each variable. That is, under K_-, if $x \to x'$ and $g_{\mu\nu}(x) \to \Omega^2(x)g_{\mu\nu}(x)$, we have

$$\left\langle F_-(\underline{x_1',x_2'}),\cdots,F_-(\underline{x_{2n-1}',x_{2n}'})\right\rangle = \Omega(x_1)^{d/2-1}\ldots\Omega(x_{2n})^{d/2-1}\left\langle F_-(\underline{x_1,x_2}),\cdots,F_-(\underline{x_{2n-1},x_{2n}})\right\rangle \quad (2.5.5)$$

[4]Technically, the argument given above for the symmetries of the bosonic quasi-bilocal only works for even dimensions in the tensorial case since the free $\frac{d-2}{2}$-form exists only in even dimensions, so the matching procedure can't be carried out naively in odd dimensions. On the other hand, it is evident from the definition 2.4.3 that it has at least the collinear conformal symmetry since there are no derivatives to be "integrated out."

and

$$\Big\langle V_{--}(\underline{x_1', x_2'}), \cdots, V_{--}(\underline{x_{2n-1}', x_{2n}'})\Big\rangle$$
$$= \Omega(x_1)^{d/2-1} \ldots \Omega(x_n)^{d/2-1} \Big\langle V_{--}(\underline{x_1, x_2}), \cdots, V_{--}(\underline{x_{2n-1}, x_{2n}})\Big\rangle \quad (2.5.6)$$

The proof of these two statements is given in appendix D.

2.5.2 Correlation functions of the bosonic quasi-bilocal

Now we will discuss the structure of the n-point functions of the quasi-bilocals. Again, let's begin with the bosonic case. We wish to constrain $\Big\langle B(\underline{x_1, x_2}) \ldots B(\underline{x_{2n-1}, x_{2n}})\Big\rangle$. We established that $B(x_1, x_2)$ has the transformation properties of a product of two free fields under the full conformal group - i.e. it is a bi-primary field with dimension $\frac{d-2}{2}$ in each variable. That means that the n-point function can only depend on distances between coordinates d_{ij} and have conformal dimension $\frac{d-2}{2}$ with respect to each variable. Since x_1 and x_2 are lightlike separated, d_{12} cannot appear, and similarly for every pair of arguments of the same bilocal. There is also a permutation symmetry: B is symmetric in its two arguments, and the n point function must be symmetric under interchange of any pair of the identical B's. Finally, there is the higher-spin symmetry. In the bosonic case, the charge conservation identity (2.4.10) imposes the simple relation

$$\sum_{i=1}^{2n} \partial_i^{s-1} \Big\langle B(\underline{x_1, x_2}) \ldots B(\underline{x_{2n-1} x_{2n}})\Big\rangle, \quad \text{for all even s} \quad (2.5.7)$$

As shown in appendix E of [17], this fixes the x^- dependence of the n-point function to have the particular form:

$$\sum_{\sigma \in S^{2n}} g_\sigma \left(x^-_{\sigma(1)} - x^-_{\sigma(2)}, x^-_{\sigma(3)} - x^-_{\sigma(4)}, \ldots, x^-_{\sigma(2n-1)} - x^-_{\sigma(2n)}\right) \quad (2.5.8)$$

where S^{2n} is the set of permutations of $2n$ elements. The point is that the x_i^- dependence of the n-point function is constrained such that, for each g_σ, x_i^- can only appear in a difference with one and only one other coordinate. This is a very strong constraint. The conformal symmetry tells us that each g_σ in the above series can be written as a product of a dimensionful function of distances with the correct dimension in each variable times a smooth, dimensionless function of conformal cross-ratios. The constraint on the functional form of g_σ, however, forbids all such functions except the trivial function 1, because each cross ratio separately violates the constraint.

Putting it all together, we conclude that the n-point function has to be proportional to a sum of terms with equal coefficients, each of which is a product $\prod d_{ij}^{-(d-2)}$, where the product has n terms corresponding to some partition of the $2n$ points into pairs where no pair contains two arguments of the same bilocal. For example, the two-point function is:

$$\left\langle B(\underline{x_1, x_2})B(\underline{x_3, x_4})\right\rangle = \tilde{N}_b \left(\frac{1}{d_{13}^{d-2}d_{24}^{d-2}} + \frac{1}{d_{14}^{d-2}d_{23}^{d-2}}\right), \tag{2.5.9}$$

where $\tilde{N}_b$ is a constant of proportionality. One immediately notes that the expressions one obtains this way for all n-point functions of the quasi-bilocals are proportional to the n-point function of $:\phi(x_1)\phi(x_2):$ in a theory of free bosons.

2.5.3 Correlation functions of the fermionic and tensor quasi-bilocal

In the fermionic and tensor cases, we claim that the correlation functions of the quasi-bilocals still coincide with the correlation functions of the corresponding free field theories, despite the fact that the fermionic and tensor quasi-bilocals have less symmetry than the bosonic quasi-bilocal. The argument, however, is somewhat more complicated due to the reduced amount of symmetry. The proof is essentially the same for both the fermionic and tensor cases, so we will only present the argument for the tensor case. Our general strategy will be to compare the constraints that one obtains from the definition of V_{--} as the lightcone limit $\underline{22}_t$ with the constraints one obtains from the symmetries of V_{--} as established by its definition away from the lightcone given at the beginning of this section. In the bosonic case, we only used the latter, but in the fermionic case and tensor case, we will need the former as well.

First, we consider what the $2n$-point function of T_{--} is away from the lightcone. We know from the definition of $V_{--} = \underline{22}_t$ that if we take n lightcone limits of this $2n$ point function in each pair of adjacent arguments $(x_1, x_2), (x_3, x_4), \ldots (x_{2n-1}, x_{2n})$, we will obtain the n point function of $V_{--}(\underline{x_1, x_2})$. It may be the case that the definition of V_{--} given earlier as a sum of currents and descendants (with appropriate derivatives and powers of x) will yield a different result away from the lightcone, but nevertheless, it must agree with the $2n$-point function of T_{--} in the lightcone limit.

Generically, the $2n$ point function of T_{--} with arguments in arbitrary locations can be decomposed as a polynomial in some basis of conformally invariant structures. One convenient basis is the

$\{H_{ij}, V_i\}$ space defined in [41]. In this basis, we may write

$$\langle T_{--}(x_1)\cdots T_{--}(x_{2n})\rangle = \frac{\langle\langle T_{--}(x_1)\cdots T_{--}(x_{2n})\rangle\rangle}{d_{12}^{d-2}d_{23}^{d-2}\dots d_{2n-1,2n}^{d-2}d_{2n,1}^{d-2}} \tag{2.5.10}$$

where

$$\langle\langle T_{--}(x_1)\cdots T_{--}(x_{2n})\rangle\rangle = \sum_i f_i(\{u_j\})\left(\prod_{k<l} H_{kl}^{h_i^{(k,l)}}\right)\left(\prod_{k<l<m} V_{k,lm}^{v_i^{(k,lm)}}\right) \tag{2.5.11}$$

where $f_i(\{u_j\})$ is an arbitrary function of cross-ratios $\{u_j\}$, the h_{kl} and v_i coefficients satisfy

$$\sum_{l,m|k<l<m} v_i^{(k,lm)} + \sum_{n|k<n} h_i^{(k,n)} = 2 \quad \text{for all } i,k \tag{2.5.12}$$

and the conformal invariants are

$$V_{k,lm} = \frac{x_{kl}^+}{d_{kl}^2} + \frac{x_{km}^+}{d_{km}^2} \tag{2.5.13}$$

$$H_{kl} = \frac{-2(x_{kl}^+)^2}{d_{kl}^4} \tag{2.5.14}$$

Note that this decomposition omits structures which contain the epsilon tensor, which all vanish in our formalism because we contract all free indices with the same polarization vector in the $-$ direction.

We would like to understand the properties of this decomposition under the tensor lightcone limit 2.2.3. First, note that the universal dimensionful factor of distances that is factored out of $\langle\langle T\dots T\rangle\rangle$ in 2.5.10 is conventional. In principle, one could choose it to be something different and compensate by appropriate redefinitions of f_i. We have chosen it as shown in order to simplify the structure of this function under the lightcone limit. More precisely, the distances corresponding to pairs of points that become lightlike separated d_{12}, d_{34}, $\dots$, $d_{2n-1,2n}$ vanish in the lightcone limit, so they cannot explicitly appear in the correlation function, and we have chosen the universal factor so that this property is manifest. To see this, note that when we take the lightcone limit 2.2.3 of this general structure, the part of this universal factor corresponding to the distances between points that become lightlike separated - i.e. $d_{12}^{-d+2}d_{34}^{-d+2}\dots d_{2n-1,2n}^{-d+2}$ - becomes $d_{12}^4 d_{34}^4 \dots d_{2n-1,2n}^4$. This residual factor is exactly cancelled out by the V and H terms corresponding to the x^+ factors stripped away in 2.2.3. To see this, recall that the light-cone limits of correlation functions are well-defined and

non-divergent[5], so any structure consistent with conformal symmetry needs to appear with enough V's and H's with appropriate indices to cancel out the factors of $(x_{12}^+)^{-2}, (x_{34}^+)^{-2}, \ldots, (x_{2n-1,2n}^+)^{-2}$ that appear in the lightcone limit. As noted earlier, these factors of V's and H's come with exactly two powers each of $d_{12}^2, \ldots, d_{2n-1,2n}^2$, which is exactly what is needed to cancel out the residual term.

Thus, after the lightcone limit, the most general structure that can appear in the n-point function of V_{--} is:

$$\left\langle V_{--}(\underline{x_1, x_2}) \ldots V_{--}(\underline{x_{2n-1}, x_{2n}}) \right\rangle = \left\langle \underline{T_{--}(x_1)T_{--}(x_2)}_t \cdots \underline{T_{--}(x_{2n-1})T_{--}(x_{2n})}_t \right\rangle \tag{2.5.15}$$

$$= \sum_i \frac{f_i(\{u_j\})}{d_{23}^{d-2} \ldots d_{2n,1}^{d-2}} \prod_{k,l} \left(\frac{x_{kl}^+}{d_{kl}^2} \right)^{c_{kl}} \tag{2.5.16}$$

where the product over k and l is understood to be restricted to pairs (k, l) not corresponding to x_k, x_l lightlike separated, and $\sum c_{kl} = 2n$.

We can determine which terms of this form are consistent with the symmetries of V_{--}. Consider the n-point correlation function of V_{--}. Its transformation properties under Lorentz transformations and dilatations tell us that we must have $2n$ + indices in the numerator of the correlation function, and that the overall scaling dimension of the n-point function should be $2n \times d/2 = dn$. Then, as mentioned before, since V_{--} is a bi-primary under the collinear conformal group, the n-point function should scale appropriately in each variable separately after acting with K_- according to 2.5.6. In order to satisfy this constraint, for each independent structure appearing in the correlation function and each index i, we must have 2 factors of x_{ij}^+ in the numerator (not necessarily the same j for each of the 2 factors) and $d + 2$ powers of d_{ik} in the denominator for some k (again, not necessarily the same k for each of the $d + 2$ factors). Once we have picked such a denominator, there is still some ambiguity since conformally invariant functions f_i can still appear after imposing this constraint (since they are fixed by K_-), and such functions can change the denominator. What is tightly constrained here is the numerator - i.e. the spin structure. "Imbalanced" structures with that would otherwise be allowed by Lorentz symmetry, scaling symmetry, and permutation symmetry cannot appear. For example, for the two-point function $\langle V_{--}V_{--} \rangle$ in four dimensions, structures such as $\frac{(x_{13}^+)^4}{d_{13}^{12}} + \frac{(x_{24}^+)^4}{d_{24}^{12}}$ do not satisfy 2.5.6. Note that the numerators which are allowed by this constraint are precisely the ones that appear in free-field correlation functions (i.e. the ones arising from Wick contractions of free fields) and no others.

Now, let's impose the higher-spin constraint, which stipulates that the correlation function must

[5] As we remarked before, this is only true a priori if we subtract off the bosonic and fermionic pieces, but we will show in section 2.6 that if any one of the three lightcone limits are nonzero, it follows that the other two are zero, so this subtraction procedure is not actually necessary.

be a sum of terms g_σ which have the functional form given by 2.5.8. Since that constraint only involves the dependence in the x^- direction, it does not constrain the numerator, which involves only terms involving the x_i^+ variables. However, it does restrict the denominator to only have each index i involved in a power d_{ik} for one specific k since d_{ik} does depend nontrivially on x_{ik}^-. That is, the denominator is built out of terms like d_{ik}^{d+2}. This constrains the f_i powerfully. Since each cross ratio separately violates the higher spin constraint, the only f_i that can appear are the ones whose product with a denominator satisfying the higher-spin constraint is another denominator satisfying that constraint. That is, once we have picked a denominator, the f_i can only be very specific kinds of rational functions. We can still generate terms that don't appear in the free-field result, however, because the spin structure in the numerator doesn't have to match the index structure of the denominator. For example, the following structure could in principle appear in the four-point function of V_{--}, but obviously this structure is not generated in the free theory:

$$\frac{(x_{14}^+)^2(x_{27}^+)^2(x_{36}^+)^2(x_{58}^+)^2}{(d_{13}d_{24}d_{57}d_{68})^{d+2}} \tag{2.5.17}$$

This structure has a numerator which is consistent with free field theory but a denominator that does not match the result one would obtain from the free field propagator. Another possibility is to write a structure where the numerator corresponds to the connected part of the free-field correlator - i.e. the two factors of x_{ij}^+ appear with different j for some i.

$$\frac{x_{13}^+x_{32}^+x_{28}^+x_{86}^+x_{67}^+x_{75}^+x_{54}^+x_{41}^+}{(d_{13}d_{24}d_{57}d_{68})^{d+2}} \tag{2.5.18}$$

Purely on symmetry considerations, these terms are consistent with the general structure 2.5.16. Indeed, one can set 2.5.17 and 2.5.18 equal to 2.5.16 to explicitly solve for the function $f_i(\{u_j\})$ that generates it, and one can check that this f_i is indeed conformally invariant, as required. These structures are inconsistent, however, with cluster decomposition. To see this, we examine the tensor analogue of 2.5.4:

$$V_{--}(x_1,x_2) = \sum_{\text{even } s\geq 2} v_{--}^s(x_1,x_2) \tag{2.5.19}$$

$$v_{--}^s(x_1,x_2) = \sum_{k,l} c_{kl}(x_1-x_2)^k\partial^l j_s\left(\frac{x_1+x_2}{2}\right) \tag{2.5.20}$$

Comparing the conformal dimension of the left and right hand side yields the constraint that $s+l-k=2$. Hence, by setting $x_1=x_2$, we extract the $k=0$ piece, forcing $l=0$ and $s=2$ (since $s=1$

is not realized in the tensor sector). That is, $V_{--}(x,x) = T_{--}(x)$. By performing this projection on each factor of V_{--} in the correlation function (i.e. setting $x_1 = x_2$, $x_3 = x_4$, etc.), we obtain an expression for the n-point function of T_{--}, which we know must satisfy cluster decomposition since T is a local operator. Then, by taking the points to be separated very far apart from each other, we obtain constraints on how the structures must simplify. For example, we know that if we take x_1 and x_3 to be very far from all the other points, we must have that

$$\langle T_{--}(x_1)T_{--}(x_3)\dots T_{--}(x_{2n-1})\rangle \Longrightarrow \langle T_{--}(x_1)T_{--}(x_3)\rangle\langle T_{--}(x_5)\dots T_{--}(x_{2n-1})\rangle \tag{2.5.21}$$

This factorization property is not satisfied by the structure 2.5.17, for example. Indeed, the only way to satisfy all such constraints arising from cluster decomposition is to have all powers of x_{ij}^+ appear with the corresponding factor of d_{ij}^{d-2} in the denominator, modulo trivial equalities such as $x_{13}^+ = x_{14}^+$ (which arise since points which are taken to be $-$ separated in the lightcone limit have the same difference in the $+$ direction). If it appears with the wrong d_{ij} factor in the denominator (again, modulo the trivial relabelings of the spin structure), it cannot satisfy the cluster decomposition identity arising from taking the two points appearing in that factor to be very far from all the other points. The spin structure required by the factorization will simply not be present.

Hence, the only allowed terms are the ones that are built from free-field propagators $(x_{ij}^+)^2/d_{ij}^{d+2}$. Permutation symmetry implies that the coefficients of all the structures that can appear are the same up to disconnected terms which are fixed, as before, by cluster decomposition. This implies that the n-point function of bilocals V_{--} are exactly the same as the n-point function of stress tensors in free field theory up to a possible overall constant.

Clearly, this entire argument works for the fermionic case as well with only minor modifications - the projection procedure that isolates the contribution from the stress tensor is slightly more complicated since it appears at first order, not zeroth order, in x_{12} in the fermionic analogue of 2.5.20, and the correlation function is permutation anti-symmetric instead of symmetric because fermions anticommute. All other steps are the same, and we conclude that in the fermionic case, the n-point functions of bilocals are also given by the free field result. For example, the two-point functions of fermionic and tensor quasi-bilocals are given by

$$\left\langle F_-(\underline{x_1,x_2})F_-(\underline{x_3,x_4})\right\rangle = \tilde{N}_f\left(\frac{x_{13}^+x_{24}^+}{d_{13}^d d_{24}^d} - \frac{x_{14}^+x_{23}^+}{d_{14}^d d_{23}^d}\right) \tag{2.5.22}$$

$$\left\langle V_{--}(\underline{x_1,x_2})V_{--}(\underline{x_3,x_4})\right\rangle = \tilde{N}_t\left(\frac{(x_{13}^+)^2(x_{24}^+)^2}{d_{13}^{d+2}d_{24}^{d+2}} + \frac{(x_{14}^+)^2(x_{23}^+)^2}{d_{14}^{d+2}d_{23}^{d+2}}\right) \tag{2.5.23}$$

where $\tilde{N}_f$ and $\tilde{N}_t$ are overall constants that we will presently analyze.

2.5.4 Normalization of the quasi-bilocal correlation functions

Now, let's fix the the overall constants $\tilde{N}_b$, $\tilde{N}_f$, and $\tilde{N}_t$ in front of each n-point function. We claim that they all are fixed by the normalization of the two-point function of the bilocals. This can be seen by considering how one can obtain the n-point function of quasi-bilocals from the $n-1$ point function. We know the n-point function of some quasi-bilocal $\mathcal{A}$ is:

$$\underbrace{\langle \mathcal{A} \dots \mathcal{A} \rangle}_{n \text{ copies of } \mathcal{A}} = \tilde{N}_n g(d_{ij}) \tag{2.5.24}$$

where g is some known function which agrees with the result for the n-point function of the corresponding free theory bilocal. Each bilocal contains the stress tensor j_2 in its OPE, so we can consider acting on both sides with the projector P which isolates the contribution of j_2 from the first bilocal. We have already seen, for example, that for the tensor bilocal, this projector just sets $x_1 = x_2$. Then, we can integrate over the coordinate x_1. This yields the action of the dilatation operator on the $n-1$ point function, whose eigenvalue will be some multiple of the conformal dimension of the appropriate free field. So by this procedure, we can fix the coefficient in front of the n-point function in terms of the $n-1$ point function. So by recursion, all the coefficients of the correlation functions are fixed by the coefficient $\tilde{N}$ appearing in front of the two-point function.

2.6 Constraining all the correlation functions

We have shown now that the n-point functions of all the quasi-bilocal fields exactly coincide with the corresponding free-field result for a theory of N free fields of appropriate spin for some N (which we will show later must be an integer). Now, we will explain how to use these facts to constrain all the other correlation functions of the theory. We will start by proving that the three point function $\langle 222 \rangle$ must be either equal to the result for a free boson, a free fermion, or a free $\frac{d-2}{2}$ form. That is, if we write the most general possible form:

$$\langle 222 \rangle = c_b \langle 222 \rangle_{\text{free boson}} + c_f \langle 222 \rangle_{\text{free fermion}} + c_t \langle 222 \rangle_{\text{free tensor}}, \tag{2.6.1}$$

then the result will be consistent with higher-spin symmetry only if $(c_b, c_f, c_t) \propto (1,0,0)$ or $(0,1,0)$ or $(0,0,1)$.

We first show that if $\langle \underline{22}_b 2\rangle \neq 0$ then $\left\langle \underline{22}_f 2\right\rangle = 0 = \left\langle \underline{22}_t 2\right\rangle$. Consider the action of Q_4 on $\langle \underline{22}_b 2\rangle$. By exactly the same analysis as the charge conservation identities of section 2.3, we obtain exactly the same expression as equation (2.3.3), except the summation starts from $j = 0$. Thus, the existence of the spin 4 current implies the existence of a spin 0 current with $\langle \underline{22}_b 0\rangle \neq 0$. The action of charge Q_4 on j_0 is

$$[Q_4, j_0] = \partial^3 j_0 + \partial j_2 + \text{no overlap with } \underline{22}_b \tag{2.6.2}$$

Now consider the charge conservation identities arising from the action of Q_4 on $\left\langle \underline{22}_f 0\right\rangle$ and $\langle \underline{22}_t 0\rangle$. Since $\left\langle \underline{22}_f 0\right\rangle = 0 = \langle \underline{22}_t 0\rangle$, we conclude $\left\langle \underline{22}_f 2\right\rangle = 0 = \langle \underline{22}_t 2\rangle$, as desired.

Now, assume that $\langle \underline{22}_b 2\rangle = 0$. It suffices to show that if $\langle \underline{22}_t 2\rangle \neq 0$, then $\left\langle \underline{22}_f 2\right\rangle = 0$. In this case, by hypothesis, the quasi-bilocal V_{--} is nonzero. The results of the previous section tell us that the three point function of the tensor quasi-bilocal is proportional to:

$$\left\langle V_{--}(\underline{x_1, x_2})V_{--}(\underline{x_3, x_4})V_{--}(\underline{x_5, x_6})\right\rangle \propto \frac{\left(x_{13}^+\right)^2\left(x_{25}^+\right)^2\left(x_{46}^+\right)^2}{d_{13}^{\frac{d}{2}+2}d_{25}^{\frac{d}{2}+2}d_{46}^{\frac{d}{2}+2}} + perm. \tag{2.6.3}$$

and this precisely coincides with the three-point function of the free field operator $v_{--}(x_1, x_2) =: F_{-\{\alpha\}}(x_1)F_{-\{\alpha\}}(x_2) :$

$$\langle V_{--}(x_1, x_2)V_{--}(x_3, x_4)V_{--}(x_5, x_6)\rangle \propto \langle v_{--}(x_1, x_2)v_{--}(x_3, x_4)v_{--}(x_5, x_6)\rangle \tag{2.6.4}$$

Now, take x_1 and x_2 very close together and expand both sides of this equation in powers of $(x_1 - x_2)$. The zeroth order term of v is clearly the normal ordered product $: F_{-\alpha}(\frac{x_1+x_2}{2})F_{-\alpha}(\frac{x_1+x_2}{2}) :$ - this is precisely the free field stress tensor. On the other hand, we know from the previous section that the term in V_{--} which is zeroth order in $(x_1 - x_2)$ - i.e. the term that arises from setting $x_1 = x_2$, is just the stress tensor of the theory T_{--}. Repeating the same procedure for the pairs of coordinates (x_3, x_4) and (x_5, x_6), we obtain the desired result:

$$\langle 222\rangle = \langle 222\rangle_{\text{free tensor}} \tag{2.6.5}$$

$$\Rightarrow \left\langle \underline{22}_f 2\right\rangle = \langle \underline{22}_b 2\rangle = 0 \tag{2.6.6}$$

as required. Therefore, since the stress-energy tensor is unique,

$$\langle 222\rangle_b \neq 0 \Rightarrow \quad \langle 222\rangle_f = 0, \quad \langle 222\rangle_t = 0, \quad \underline{j_2 j_2}_b = \sum_{k=0}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_f = 0, \quad \underline{j_2 j_2}_t = 0, \quad (2.6.7)$$

$$\langle 222\rangle_f \neq 0 \Rightarrow \quad \langle 222\rangle_b = 0, \quad \langle 222\rangle_t = 0, \quad \underline{j_2 j_2}_f = \sum_{k=1}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_b = 0, \quad \underline{j_2 j_2}_t = 0, \quad (2.6.8)$$

$$\langle 222\rangle_t \neq 0 \Rightarrow \quad \langle 222\rangle_b = 0, \quad \langle 222\rangle_f = 0, \quad \underline{j_2 j_2}_t = \sum_{k=1}^{\infty} [j_{2k}], \quad \underline{j_2 j_2}_b = 0, \quad \underline{j_2 j_2}_f = 0, \quad (2.6.9)$$

where square brackets denotes currents and their descendants. This establishes the claim that the three-point function of the stress tensor coincides with the answer for some free theory.

At this point, we would like to stress that the factorization property we have proven here holds only for conformal field theories that satisfy the unitarity bound for the dimensions of operators. Clearly, all unitary CFT's have this property, but it is possible to conceive of non-unitary CFT's which also satisfy it. Without the unitarity bound's constraint on operator dimesions, however, various operators we have not considered could appear in all the charge conservation identities we have written. These operators make it possible to construct theories where the three-point function of the stress tensor decomposes as a nontrivial superpositions of the bosonic, fermionic, and tensor sectors. For example, we show in appendix F that the free five-dimensional Maxwell field is a non-unitary conformal field theory whose stress tensor decomposes into a superposition of all three sectors.

Returning to the main argument, we may now obtain all the other correlation functions, we may expand equation (2.6.4) to higher orders in $x_1 - x_2$, and use the correlation functions obtained at lower orders to fix the ones that appear at higher orders. For example, at second order in $x_1 - x_2$, we have:

$$v_{--} = (x_1 - x_2)^2 \left(: \partial^2 F_{-\alpha}\Big(\frac{x_1+x_2}{2}\Big) F_{-\alpha}\left(\frac{x_1+x_2}{2}\right) : \right.$$
$$\left. + : \partial F_{-\alpha}\left(\frac{x_1+x_2}{2}\right) \partial F_{-\alpha}\left(\frac{x_1+x_2}{2}\right) : \right), \quad (2.6.10)$$

and V_{--} contains terms involving only the spin 2, 3, and 4 currents. Using our answers for $\langle 222\rangle$ and our knowledge that $\langle 223\rangle = 0$, we can then fix $\langle 224\rangle$ to agree with the free field theory. This procedure recursively fixes all the correlators in the free tensor sector. The argument flows identically for the free bosonic and free fermionic sectors, except that the zeroth order term will not fix $\langle 222\rangle$, but some lower-order current. For example, in the bosonic theory, the zeroth order term will fix

$\langle 000 \rangle$, and one will need to carry out the power series expansion to higher orders in order to fix the correlators of the higher-spin conserved currents.

Then, one could consider correlation functions that have indices set to values other than minus. This works in exactly the same way, since the operator product expansion of two currents with minus indices will contain currents with other indices. This has the effect of doubling the number of bilocals required to build a correlation function, since we need to take an extra OPE to fix the index structure. Thus, an n-point function with non-minus indices can be fixed from $2n$ bilocals. Thus, we have fixed every correlation function from currents at appear in successive OPE's of two stress tensors, including those of every higher-spin current.

The last thing we will argue is that the normalization of the correlation functions matches the normalization for some free theory. For example, in the theory of N free bosons, the two-point function of $\sum_{i=1}^{N} : \phi_i \phi_i^* :$ will have overall coefficient N. The same is true for the fermionic and tensor cases. One might wonder if the overall coefficient $\tilde{N}$ of the quasi-bilocal could be non-integer, which would imply that it could not coincide with any theory of N free bosons. We will now argue that this is not possible. We start with the bosonic case, which works similarly to the argument presented in [17]:

In a theory of N free bosons, consider the operator

$$\mathcal{O}_{q,free} = \delta^{[i_1,\ldots,i_q]}_{[j_1,\ldots,j_q]} (\phi^{i_1} \partial \phi^{i_2} \ldots \partial^{q-1} \phi^{i_q})(\phi^{j_1} \partial \phi^{j_2} \ldots \partial^{q-1} \phi^{j_q}) \tag{2.6.11}$$

Here, δ is the totally antisymmetric delta function that arises from a partial contraction of ϵ symbols:

$$\delta^{[i_1,\ldots,i_q]}_{[j_1,\ldots,j_q]} \propto \epsilon^{i_1 \ldots, i_q, i_{q+1} \ldots i_N} \epsilon_{j_1 \ldots, j_q, i_{q+1} \ldots i_N} \tag{2.6.12}$$

We claim that in the full theory, there exists an operator $\mathcal{O}_q$ in the full theory whose correlation functions coincide with the correlation functions of $\mathcal{O}_{q,free}$ in the free theory. The proof of this is given in appendix E.

Consider the norm of the state that $\mathcal{O}_q$ generates. This is computed by the two point function $\langle \mathcal{O}_q \mathcal{O}_q \rangle$. It is obvious from the definition of $\mathcal{O}_q$ that it arises from the contraction of q bilocal fields, so this correlator is a polynomial in N of order q. The antisymmetry of the totally antisymmetric function in the definition of $\mathcal{O}_{q,free}$ enforces that the correlation function vanishes at $q > N$. So we know all the roots of the polynomial, and hence the correlation function is proportional to $N(N-1)\ldots(N-(q-1))$. Now, consider an analytic continuation of this correlator to non-integer

$\tilde{N}$. By taking $q = \lfloor N \rfloor + 2$, we find that this product is negative, which is impossible for the norm of a state. Since the correlators of $\mathcal{O}_q$ are forced to agree with the correlators of some operator in the full CFT, we conclude that the normalization $\tilde{N}$ of the scalar quasi-bilocals must be an integer.

The same argument can be ran in the tensor case for an operator defined similarly:

$$\mathcal{O}_q = \delta^{[i_1,\ldots,i_q]}_{[j_1,\ldots,j_q]}(F^{i_1}_{-\{\alpha_1\}}\partial F^{i_2}_{-\{\alpha_2\}}\ldots\partial^{q-1}F^{i_q}_{-\{\alpha_q\}})(F^{j_1}_{-\{\alpha_1\}}\partial F^{j_2}_{-\{\alpha_2\}}\ldots\partial^{q-1}F^{j_q}_{-\{\alpha_q\}}) \tag{2.6.13}$$

We again conclude that the normalization constant $\tilde{N}$ must be an integer.

The construction in the fermionic case is somewhat simpler. We know j_2 appears in F_-, and we can define an operator $\mathcal{O}_q = (j_2)^q$ by extracting the term in the operator product expansion of q copies of j_2 whose correlation functions coincide with the free fermion operator $(j_2)^q_{\text{free}}$. In the theory of N free fermions, $j_2 = \sum_i(\partial\psi_i)\gamma_-\bar{\psi}_i - \psi_i\gamma_-(\partial\bar{\psi}_i)$, where here i is the flavor index for the N fermions. By antisymmetry of the fermions, we know that $\mathcal{O}_q$ will be zero if $q \geq N$. Then, as in the bosonic case, we can consider the norm of the state that $\mathcal{O}_q$ generates, which is computed by $\langle\mathcal{O}_q\mathcal{O}_q\rangle$, and the rest of the argument runs as before. Thus, the normalization $\tilde{N}$ of the fermionic bilocals must be an integer.

It is worth noting the relationship between this result and one of the primary motivations for studying higher-spin CFT's - holographic dualities involving Vasiliev gravity in an anti-de Sitter space. As mentioned earlier, it has been conjectured that Vasiliev gravity is conjectured to be dual to a theory of N free scalar fields in the $O(N)$ singlet sector. This implies a relationship between the vacuum energy of Vasiliev gravity at tree-level and the free energy of a scalar field, namely, that $F_{\text{Vasiliev}}/G_N \sim NF_{\text{scalar}}$, where G_N is the Newton constant. Our result implies that this normalization constant N, and therefore, the Newton constant G_N is quantized in the Vasiliev theory in any dimension.

It must be noted, however, that we cannot claim that this quantization can be seen perturbatively in N. Recent work of Giombi and Klebanov [42] have shown that the one-loop correction to the vacuum energy of minimally coupled type A Vasiliev gravity in anti-de Sitter background does not vanish as expected. This was interpreted as a shift of $N \to N - 1$ in the tree-level calculation of the vacuum free energy. Our result cannot predict such a shift or any other $1/N$ corrections that appear in higher orders in perturbation theory. We claim only that the exact result, after summing all loop corrections, must be quantized.

2.7 Discussion and conclusions

In this chapter, we have shown that in a unitary conformal field theory in $d > 3$ dimensions with a unique stress tensor and a symmetric conserved current of spin higher than 2, the three-point function of the stress tensor must coincide with the three-point function of the stress tensor in either a theory of free bosons, a theory of free fermions, or a theory of free $\frac{d-2}{2}$-forms. This implies that all the correlation functions of symmetric currents of the theory coincide with the those in the corresponding free field theory.

Our technique was to use a set of appropriate lightcone limits to transform the data of certain key Ward identities into simple polynomial equations. Even though we could not directly solve for the coefficients in these identities like in three dimensions, we were nevertheless able to show that the only solution these Ward identities admit is the one furnished by the appropriate free field theory. This was the key step that allowed us to defined bilocal operators which were used to show that the three-point function of the stress tensor must agree with a free field theory. This in turn fixed all the other correlators of the theory to agree with those in the same free field theory. These results can be understood as an extension of the techniques and conclusions of [17] from three dimensions to all dimensions higher than three.

We stress that our classification into the bosonic, fermionic, and tensor free field theories depends somewhat sharply on our assumption that a unique stress tensor exists. Other free field theories with higher spin symmetry exist in $d > 3$ dimensions, such as a theory of free gravitons. This theory, however, does not have a stress tensor, and we make no statement about how the correlation functions of such theories are constrained, and analogously for theories with many stress tensors. On the other hand, we may consider the possibility of multiple stress tensors. It was argued in [17] that the result holds if there are two stress tensors instead of just one. This argument carries over to our result totally unchanged, and so our result also holds in the case of two stress tensors. We do not comment on the possibility of more than two stress tensors.

Moreover, we have not computed correlation functions or commutators for asymmetric currents and charges. In [32], it was shown that if one considers the possible algebras of charges in theories that contain asymmetric currents in four dimensions, a one-parameter family of algebras exists. This may suggest the existence of nontrivial higher-spin theories, though our result indicates that at least the subalgebra generated by the symmetric currents must agree with free field theory.

We also stress that the tensor structure is not well understood in all dimensions. In even dimensions, it corresponds to the theory of a free $\frac{d-2}{2}$-form field, which does not exist in odd dimensions.

In odd dimensions, the structure may not exist, and even it does, there may not exist a conformal field theory which realizes it. Our argument only tells us that if there is a solution of the conformal and higher-spin Ward identities corresponding to this structure, then it is unique. If the structure exists, we only know for a fact that it contains an infinite tower of higher-spin currents for $d \geq 7$ and in this case, the theory, if it exists, has the correlation functions we claimed. In $d = 5$, it is not known if all the higher-spin currents must be present. Assuming they are present, our results also flow through in $d = 5$. Even then, the tensor structure in odd dimensions could fail to have a good microscopic interpretation for many other reasons. For example, the four-point function of the stress tensor in this sector may not be consistent with the operator product expansion in the sense that it may not be decomposable as a sum over conformal blocks - i.e. it may be possible to continue all the correlation functions to odd dimensions, but not the blocks. We have not explored this question.

2.8 Appendix A: Form factors as Fourier transforms of correlation functions

In this appendix, we will explicitly calculate the Fourier-transformed, lightcone-limit three-point functions F_s^b, F_s^f, and F_s^v cited in section 2.2. Let's start with the bosonic case. We want to compute the relevant Fourier transformation of the three-point function $\langle \phi(x_1)\phi^*(x_2)j_s(x_3)\rangle$. The explicit form of $j_s(x_3)$ is given in [43] as:

$$j_s = \sum_{k=0}^{s} c_k \partial^k \phi \partial^{s-k}\phi^*, \quad c_k = \frac{(-1)^k}{2} \frac{\binom{s}{k}\binom{s+d-4}{k+\frac{d}{2}-2}}{\binom{s+d-4}{\frac{d}{2}-2}} \tag{2.8.1}$$

Wick's theorem and translation invariance of the correlatiors yields that::

$$\langle \phi(x_1)\phi^*(x_2)j_s(x_3)\rangle = \sum c_i (\partial_3^i \langle \phi(x_1)\phi^*(x_3)\rangle)(\partial_3^{s-i}\langle \phi(x_3)\phi^*(x_2)\rangle) \tag{2.8.2}$$

$$= \sum c_i (\partial_1^i \langle \phi(x_1)\phi^*(x_3)\rangle)(\partial_2^{s-i}\langle \phi(x_3)\phi^*(x_2)\rangle) \tag{2.8.3}$$

Then, we may Fourier transform term by term with respect to x_1^- and x_2^-. Recalling that the propagator of a scalar field is $(x^2)^{\frac{2-d}{2}}$ and that in the lightcone limit, $x_1^+ = x_2^+$ and $\vec{y}_1 = \vec{y}_2$, we

obtain:

$$\partial_1^{s-i}\partial_2^i\langle\phi(x_1)\phi^*(x_3)\rangle\langle\phi(x_3)\phi^*(x_2)\rangle$$

$$\longrightarrow i^s(p_1^+)^{s-i}(p_2^+)^i\int dx_1^- dx_2^- e^{ip_1^+x_1^-}e^{ip_2^+x_2^-}\frac{1}{\left(x_{13}^+x_{13}^- + \vec{y}_{13}^2\right)^{\frac{d-2}{2}}}\frac{1}{\left(x_{23}^+x_{23}^- + \vec{y}_{23}^2\right)^{\frac{d-2}{2}}} \tag{2.8.4}$$

$$= \frac{i^s(p_1^+)^{s-i}(p_2^+)^i}{(x_{13}^+)^{d-2}}\int dx_1^- dx_2^- e^{ip_1^+x_1^-}e^{ip_2^+x_2^-}\frac{1}{\left(x_{13}^- + \frac{\vec{y}_{13}^2}{x_{13}^+}\right)^{\frac{d-2}{2}}}\frac{1}{\left(x_{23}^- + \frac{\vec{y}_{13}^2}{x_{13}^+}\right)^{\frac{d-2}{2}}} \tag{2.8.5}$$

$$= \frac{i^s(p_1^+)^{s-i}(p_2^+)^i}{(x_{13}^+)^{d-2}}\left(\int dx_1^- e^{ip_1^+x_1^-}\frac{1}{\left(x_1^- - \bar{x}\right)^{\frac{d-2}{2}}}\right)\left(\int dx_2^- e^{ip_2^+x_2^-}\frac{1}{\left(x_2^- - \bar{x}\right)^{\frac{d-2}{2}}}\right) \tag{2.8.6}$$

Here, we have defined $\bar{x} = x_3^- - \frac{\vec{y}_{13}^2}{x_{13}^+}$. Depending on the parity of d, each integral has either a pole of order $\frac{d-2}{2}$ at $\bar{x}$ or a branch point at $\bar{x}$. Our prescription for evaluating this integral is as follows: First, we shift x_1^- and x_2^- by $\bar{x}$ so that the singularity is at 0, and then we will move move the singularity from 0 to $\text{sign}(p)i\epsilon$. Then, the integral can be evaluated by Schwinger parameterization. For example, suppose p_1^+ and p_2^+ are positive. Following our procedure, the x_1 integral becomes:

$$\int_{-\infty}^{\infty} dx_1^- e^{ip_1^+x_1^-}\frac{1}{\left(x_1^- - \bar{x}\right)^{\frac{d-2}{2}}} = e^{ip_1^+\bar{x}+p_1^+\epsilon}\int_{-\infty}^{\infty} dy e^{ip_1^+y}\frac{1}{(y-i\epsilon)^{\frac{d-2}{2}}} \tag{2.8.7}$$

$$= e^{ip_1^+\bar{x}+p_1^+\epsilon}\int_{-\infty}^{\infty} dy\int_0^{\infty} ds\frac{i}{\Gamma(\frac{d-2}{2})}e^{ip_1^+y}s^{\frac{d-4}{2}}e^{-is(y-i\epsilon)} \tag{2.8.8}$$

$$= \frac{ie^{ip_1^+\bar{x}+p_1^+\epsilon}}{\Gamma(\frac{d-2}{2})}\int_0^{\infty} ds 2\pi\delta(s-p_1^+)e^{ip_1^+y}s^{\frac{d-4}{2}}e^{-s\epsilon} \tag{2.8.9}$$

$$= \frac{2\pi ie^{ip_1^+\bar{x}}}{\Gamma(\frac{d-2}{2})}(p_1^+)^{\frac{d-4}{2}} \tag{2.8.10}$$

This function is indeed nonsingular, as required. The x_2 integral has exactly the same form, and so gives the same answer. Hence, we obtain that the Fourier transform of $\langle\phi\phi^* j_s\rangle$ is indeed proportional to $\sum c_i(p_1^+)^i(p_2^+)^{s-i}$, where the proportionality factor is a nonsingular function. The, noting that the coefficients c_i are the coefficients for the hypergeometric function with appropriate arguments, we obtain the answer cited in the text:

$$F_s^b \equiv \langle\underline{\phi\phi^*}j_s\rangle \propto (p_2^+)^s{}_2F_1\left(2-\frac{d}{2}-s,-s,\frac{d}{2}-1,p_1^+/p_2^+\right) \tag{2.8.11}$$

The fermionic and tensor cases can be tackled in exactly the same way. There are only two differences. First, the propagator in the free fermion and free tensor theories are $(x^2)^{\frac{1-d}{2}}$ and $(x^2)^{\frac{-d}{2}}$, respectively, as compared with the free scalar propagator $(x^2)^{\frac{2-d}{2}}$. Second, the coefficients in the expression for

j_s are different, as can be checked from the expressions in [44] [45] or in [43]. The end result is that the arguments of the hypergeometric function are different in the way claimed in the text.

2.9 Appendix B: Uniqueness of three-point functions in the tensor lightcone limit

Our goal in this section is to show that the free tensor solution for the lightcone limit of three-point functions explained in section 2.2 is indeed unique, at least in the lightcone limit.

Note that Lorentz symmetry constrains the propagator of spin j field to be of the form

$$\left\langle \psi_{-j}(x)\bar{\psi}_{-j}(0)\right\rangle \propto (x^+)^{2j}. \tag{2.9.1}$$

Generically, according to [37], the most generic conformally invariant expression one can write down for a three-point function of symmetric conserved currents with tensor-type coordinate dependence is:

$$\begin{aligned}\langle j_{s_1} j_{s_2} j_{s_3}\rangle = \frac{1}{x_{12}^{d-2}x_{23}^{d-2}x_{13}^{d-2}} \sum_{a,b,c} \Big(&\left(\Lambda_1^2 \alpha_{a,b,c} + \Lambda_2 \beta_{a,b,c}\right) \left(P_{12}P_{21}\right)^{\,a} Q_1^b \\ &\left(P_{23}P_{32}\right)^{\,c} \left(P_{13}P_{31}\right)^{\,-a-b+s_1} Q_2^{-a-c+s_2} Q_3^{a+b-c-s_1+s_3}\Big)\end{aligned} \tag{2.9.2}$$

where the $\alpha_{a,b,c}$ and $\beta_{a,b,c}$ are free coefficients, and the Λ_i are defined as:

$$\Lambda_1 = Q_1Q_2Q_3 + \left[Q_1P_{23}P_{32} + Q_2P_{13}P_{31} + Q_3P_{12}P_{21}\right], \tag{2.9.3}$$

$$\Lambda_2 = 8P_{12}P_{21}P_{23}P_{32}P_{13}P_{31}. \tag{2.9.4}$$

Here, the P and Q invariants are defined as in [46] and [47]. However, for the choice of polarization vector $\epsilon^\mu = \epsilon^-$ there is a nontrivial relation:

$$\Lambda_2\big|_{\epsilon_i^\mu=\epsilon^-} = -2\Lambda_1^2\big|_{\epsilon_i^\mu=\epsilon^-}, \quad \Lambda_1\big|_{\epsilon_i^\mu=\epsilon^-} = \frac{1}{4}\frac{x_{12}^+x_{23}^+x_{13}^+}{x_{12}^2x_{23}^2x_{13}^2}(\epsilon^-)^3. \tag{2.9.5}$$

Therefore, in the case $\epsilon^\mu = \epsilon^-$ the expression for this three-point function greatly simplifies. Instead of having two sets of undetermined coefficients c_a and d_a, one can combine the Λ_i's into a single prefactor $\alpha_1\Lambda_1^2 + \alpha_2\Lambda_2$, where the α_i are arbitrary and can be chosen to be convenient; to produce exact agreement with the canonically normalized free-tensor theory, we will choose $\alpha_1 = 1$ and

$\alpha_2 = \frac{1}{2(d-2)}$. Now, we take the lightcone limit, which corresponds to the point where

$$P_{23}P_{32} = 0, \quad Q_1 = -\left(\frac{P_{13}P_{31}}{Q_3} + \frac{P_{12}P_{21}}{Q_2}\right) \tag{2.9.6}$$

in P_{ij}, Q_i space. Then, the three-point function reduces to

$$\left\langle j_{s_1}\underline{j_{s_2}j_{s_3}}_t \right\rangle = \frac{\Lambda_1^2 + \Lambda_2/(2(d-2))}{x_{12}^{d-2}x_{23}^{d-2}x_{13}^{d-2}} \sum_{a=0}^{s_1-2} c_a \left(P_{12}P_{21}\right)^a \left(P_{13}P_{31}\right)^{s_1-2-a} Q_2^{s_2-a} Q_3^{S_3-s_1+a}, \tag{2.9.7}$$

Now, the c_a can be fixed demanding that all currents are conserved. The result is given by the following recurrence relation, with $c_0 = 1$:

$$\frac{c(a+1)}{c(a)} = \frac{(s_1-2-a)(s_1+\frac{d-4}{2}-a)(s_2+a+\frac{d-2}{2})}{(a+1)(a+\frac{d-2}{2}+2)(s_1+s_3+\frac{d-4}{2}-2-a)}$$

This solution exactly coincides with the free tensor solution, as required.

2.10 Appendix C: Uniqueness of $\langle s22\rangle$ for $s \geq 4$

Define

$$\langle j_{s_1}j_{s_2}j_{s_3}\rangle = \frac{\langle\langle j_{s_1}j_{s_2}j_{s_3}\rangle\rangle}{{x_{12}}^{d-2}{x_{23}}^{d-2}{x_{13}}^{d-2}}. \tag{2.10.1}$$

Using the previous defined V and H conformal invariants, we can write the most general expression for a conformally invariant correlation function as follows:

$$\begin{aligned}\langle\langle j_s j_2 j_2\rangle\rangle = V_1^{s-4}\Big[&a_1 H_{1,2}^2 H_{1,3}^2 + a_2\left(V_1V_2H_{1,2}H_{1,3}^2 + V_1V_3H_{1,2}^2H_{1,3}\right) + a_3V_1^2H_{1,2}H_{1,3}H_{2,3} + \\ &+ a_4\left(V_1^2V_3^2H_{1,2}^2 + V_1^2V_2^2H_{1,3}^2\right) + a_5V_1^2V_2V_3H_{1,2}H_{1,3} + \\ &+ a_6\left(V_1^3V_2H_{1,3}H_{2,3} + V_1^3V_3H_{1,2}H_{2,3}\right) + a_7\left(V_1^3V_2V_3^2H_{1,2} + V_1^3V_2^2V_3H_{1,3}\right) + \\ &a_8V_1^4H_{2,3}^2 + a_9V_1^4V_2V_3H_{2,3} + a_{10}V_1^4V_2^2V_3^2\Big].\end{aligned} \tag{2.10.2}$$

The coefficients can be solved by imposing charge conservation. For example, in $d = 4$ we obtain:

$$a_1 = -\frac{a_7(s-3)(s-1)(s-2)^2}{32(s+1)(s+4)} + \frac{a_4(s-5)(s-3)s(s-2)}{8(s+1)(s+4)} + \frac{a_5(s-3)(s-2)}{8(s+4)}, \tag{2.10.3}$$

$$a_2 = -\frac{a_4(s-2)^2}{s+4} + \frac{a_7(s-1)(s-2)}{4(s+4)} - \frac{a_5(s-2)}{2(s+4)}, \tag{2.10.4}$$

$$a_3 = -\frac{8a_4\left(s^2-3s-1\right)}{(s+1)(s+4)} + \frac{a_5(s-8)}{2(s+4)} + \frac{a_7(s-1)(2s-1)}{(s+1)(s+4)}, \tag{2.10.5}$$

$$a_6 = \frac{12a_4(s-2)}{(s-1)(s+4)} + \frac{6a_5}{(s-1)(s+4)} + \frac{a_7(s-2)}{2(s+4)}, \tag{2.10.6}$$

$$a_8 = \frac{a_7(s-2)\left(s^2+11s-2\right)}{4s(s+1)(s+4)} - \frac{6a_4(s-5)}{(s+1)(s+4)} + \frac{a_5(s-2)}{s(s+4)}, \tag{2.10.7}$$

$$a_9 = \frac{a_7\left(s^2+8s-8\right)}{s(s+4)} - \frac{24a_4(s-2)}{(s-1)(s+4)} + \frac{4a_5(s-2)(s+2)}{(s-1)s(s+4)}, \tag{2.10.8}$$

$$a_{10} = \frac{a_7\left(s^2+8s+4\right)}{s(s+4)} - \frac{24a_4(s+1)}{(s-1)(s+4)} + \frac{4a_5(s+1)(s+2)}{(s-1)s(s+4)}. \tag{2.10.9}$$

Therefore, $\langle\langle j_s j_2 j_2\rangle\rangle_t$ depends only on three parameters. The bosonic light-cone limit of this function is zero if

$$a_5 = \frac{a_7(s-2)(s-1)}{4(s+1)} - \frac{a_4(s-5)s}{s+1}. \tag{2.10.10}$$

The fermionic light-cone limit of this function is also zero if

$$a_4 = \frac{a_7}{4}. \tag{2.10.11}$$

Therefore, $\langle\langle s22\rangle\rangle_t$ depends only on one parameter or in other words it is unique up to a rescaling[6]

$$\langle\langle j_s j_2 j_2\rangle\rangle_t \propto V_1^{s-2}\Big[H_{12}^2 V_3^2 + \left(H_{23}V_1 + V_2\left(H_{13}+2V_1V_3\right)\right)^2 + H_{12}\left(H_{13}+2V_1V_3\right)\left(H_{23}+2V_2V_3\right)\Big], \tag{2.10.12}$$

In arbitrary dimension $d > 3$, the full expression is:

$$\begin{aligned}\langle\langle j_s j_2 j_2\rangle\rangle_t &= V_1^{s-2}\left[\left(H_{23}V_1 + H_{13}V_2 + H_{12}V_3 + 2V_2V_3V_1\right)^2 + \frac{2}{(d-2)}H_{12}H_{13}H_{23}\right] \\ &= V_1^{s-2}\left[\Lambda_1^2 + \frac{1}{2(d-2)}\Lambda_2\right].\end{aligned} \tag{2.10.13}$$

This formula coincides with the expression that was proposed in [37], and we have proven that this structure is unique.

[6] In [36] it was proven that there are only three structures for $\langle\langle 22s\rangle\rangle$ in d=4.

2.11 Appendix D: Transformation properties of bilocal operators under K_-

In this appendix, we will prove 2.5.5 and 2.5.6 by computing the action of a finite conformal transformation on them. The same results can be proven using the infinitesimal transformations, e.g. by using equation (3) of [48] and supplying the correct representation matrices for the Lie algebra of the Lorentz group. One can then check that the two computations agree by expanding our results to first order in b (remembering that only b^- is nonzero for K_-).

2.11.1 Fermionic case

Consider a special conformal transformation

$$x^\mu \to y^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b\cdot x) + b^2 x^2} \tag{2.11.1}$$

Under K_-, the parameter $b^\mu = b^- \delta^\mu_-$. We know that F_- has the same transformation properties as the contraction of free fields $\bar\psi\gamma_-\psi$ on the lightcone. Since K_- sends the lightcone into the lightcone, V_{--} transforms the same way as $\bar\psi\gamma_-\psi$ under K_-. Using the well-known expression for the finite conformal transformation of a Dirac spinor (e.g. [49])

$$\psi(y) = \left|\frac{\partial y}{\partial x}\right|^{\Delta-1/2} (1 - b_\mu x_\nu \gamma^\mu\gamma^\nu)\psi(x) \tag{2.11.2}$$

$$\bar\psi(y) = \left|\frac{\partial y}{\partial x}\right|^{\Delta-1/2} \bar\psi(x)(1 - b_\mu x_\nu \gamma^\nu\gamma^\mu) \tag{2.11.3}$$

we may therefore compute:

$$F_-(\underline{y_1, y_2}) \sim \bar{\psi}(y_1)\gamma^+\psi(y_2) \tag{2.11.4}$$

$$= \left|\frac{\partial y_1}{\partial x_1}\right|^{\Delta-1/2} \left|\frac{\partial y_2}{\partial x_2}\right|^{\Delta-1/2} \bar{\psi}(x_1)(1 - b_\mu(x_1)_\nu\gamma^\nu\gamma^\mu)\gamma^+(1 - b_\mu(x_2)_\nu\gamma^\mu\gamma^\nu)\psi(x_2) \tag{2.11.5}$$

$$= \left|\frac{\partial y_1}{\partial x_1}\right|^{\Delta-1/2} \left|\frac{\partial y_2}{\partial x_2}\right|^{\Delta-1/2} \bar{\psi}(x_1)$$

$$\times [\gamma^+ - b_+(x_1)_\nu\gamma^\nu\gamma^+\gamma^+ - \gamma^+b_+(x_2)_\nu\gamma^+\gamma^\nu + b_+(x_1)_\nu\gamma^\nu\gamma^+\gamma^+b_+(x_2)_\mu\gamma^+\gamma^\mu]\psi(x_2) \tag{2.11.6}$$

$$= \left|\frac{\partial y_1}{\partial x_1}\right|^{\Delta-1/2} \left|\frac{\partial y_2}{\partial x_2}\right|^{\Delta-1/2} \bar{\psi}(x_1)\gamma^+\psi(x_2) \tag{2.11.7}$$

$$= \Omega^{d/2-1}(x_1)\Omega^{d/2-1}(x_2)F_-(\underline{x_1, x_2}) \tag{2.11.8}$$

The cancellations occur because $\gamma^+\gamma^+ = \eta^{++} = 0$. This is exactly equation 2.5.5.

2.11.2 Tensor case

We'll start with the four-dimensional case for ease of notation and then at the end, we'll describe how one can generalize the computation to all dimensions. Consider a special conformal transformation

$$x^\mu \to y^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2(b \cdot x) + b^2x^2} \tag{2.11.9}$$

Under K_-, the parameter $b^\mu = b^-\delta^\mu_-$. We know that V_{--} has the same transformation properties as the contraction of free fields $F_{-\mu}F^\mu_-$ on the lightcone. Since K_- sends the lightcone into the lightcone, V_{--} transforms the same way as $F_{-\mu}F^\mu_-$ under K_-. We therefore compute:

$$V_{--}(\underline{y_1, y_2}) = \left|\frac{\partial y_1}{\partial x_1}\right|^{-\tau_F/d} \left|\frac{\partial y_2}{\partial x_2}\right|^{-\tau_F/d} \frac{\partial x_1^\mu}{\partial y_1^-}\frac{\partial x_1^\nu}{\partial y_1^\alpha}\frac{\partial x_2^\lambda}{\partial y_2^-}\frac{\partial x_2^\rho}{\partial y_2^\beta}\eta^{\alpha\beta}F_{\mu\nu}(x_1)F_{\lambda\rho}(x_2) \tag{2.11.10}$$

$$= (1 - b^-x_1^+)^{\tau_F}(1 - b^-x_2^+)^{\tau_F}(1 - b^-x_1^+)^2\eta^{\alpha\beta}F_{-\alpha}(x_1)F_{-\beta}(x_2) \tag{2.11.11}$$

$$= (1 - b^-x_1^+)(1 - b^-x_2^+)V_{--}(\underline{x_1, x_2}) \tag{2.11.12}$$

$$= \Omega(x_1)\Omega(x_2)V_{--}(\underline{x_1, x_2}) \tag{2.11.13}$$

In the above manipulations, $\tau_F = \Delta - s = 0$ is the twist of F, and in the second to last line, we used that $x_1^+ = x_2^+$ (because the points x_1 and x_2 are $-$ separated by hypothesis). This immediately implies 2.5.6 in the four-dimensional case. In general dimensions, the twist of F will not be 0, but rather $\Delta - s = d/2 - s$, and we will have a corresponding number of extra factors of $\partial x/\partial y$ to

contract with the additional indices of F. This will make the exponent of the Ω factors equal to $\frac{d}{2}-1$ instead of 1.

2.12 Appendix E: Proof that $\mathcal{O}_q$ exists

In this appendix, we will prove that an operator $\mathcal{O}_q$ whose correlation functions agree with the corresponding free field operator $\mathcal{O}_{q,free}$ defined in 2.6.11 exists in the operator spectrum of every conformal field theory with higher-spin symmetry. As usual, we will consider the bosonic case, since the tensor case works almost in precisely the same way. To prove our statement, we will show that in the free theory, for any $q \leq N$

$$A_{q,N}(x_1, x_2, \ldots, x_{q+1}) \equiv \left\langle \underbrace{\phi^2\phi^2\ldots\phi^2}_{\text{q copies}} \mathcal{O}_{q,free} \right\rangle \neq 0 \tag{2.12.1}$$

Here, $\phi^2 = \sum_i \phi_i^2$, which is known to appear in the OPE of two stress tensors. Thus, if we prove 2.12.1, then we would know that $\mathcal{O}_{q,free}$ appears in the operator product expansion of $2q$ copies of the free field stress tensor j_2. Then, just as knowing the OPE structure of products of free field stress tensors allowed us to obtain conserved currents from products of the quasi-bilocal fields, we can obtain $\mathcal{O}_q$ in the full theory by defining it to be the operator appearing in the operator product expansion of $2q$ copies of j_2 in the full theory whose correlation functions coincide with the correlation functions of $\mathcal{O}_{q,free}$ in the free theory. Thus, it suffices to prove 2.12.1.

First, note that we can immediately reduce to the $q = N$ case. This follows from the structure of the Wick contractions in $A_{q,N}$. To see this, note that every term in $\mathcal{O}_{q,free}$ involves exactly q of the N bosons, each of which appears twice for a total of $2q$ fields. Since ϕ^2 is bilinear in the the fields, the product of q copies of ϕ^2 will also contain $2q$ fields. Hence, we will need all the ϕ^2 fields to be contracted with the $\mathcal{O}_{q,free}$ fields in order to obtain a nonzero answer. Thus, for each term in $\mathcal{O}_{q,free}$, none of the $N-q$ flavors not appearing in that term will contribute, and so we can partition the terms in $A_{q,N}$ according to which of the q flavors appear. Since the correlation function is manifestly symmetric under relabelings of the N ϕ_i fields, this implies that each group of terms in this partition will equally contribute to the total correlation function an amount exactly equal to $A_{q,q}$. Hence, $A_{q,N} = \binom{N}{q} A_{q,q}$, so it suffices to show $A_{q,q}$ is nonzero.

Then, note that since $\mathcal{O}_{q,free}$ contains exactly two copies of each of the q ϕ_i fields, each of the q factors of ϕ^2 must contribute a different ϕ_i field for the contraction to be nonzero. Since $\mathcal{O}_{q,free}$ is manifestly invariant under arbitrary relabelings of the ϕ_i fields, we may relabel each term so that

the first copy of ϕ^2 contributes ϕ_1^2, the second copy of ϕ^2 contributes ϕ_2^2 and so on. That is, we have

$$A_{q,q} = q!\langle \phi_1^2(x_1)\phi_2^2(x_2)\dots\phi_q^2(x_q)\mathcal{O}_{q,free}(x_{q+1})\rangle \tag{2.12.2}$$

The correlator on the right-hand side can be easily computed by direct evaluation of the Wick contractions. To illustrate, consider the result given by the term in $\mathcal{O}_{q,free}$ corresponding to setting the internal indices $i_k = j_k = k$ for all $k \in \{1, 2, \dots, q\}$. The contribution of this term is, up to a sign, given by:

$$\prod_{k=1}^{q} \partial_{q+1}^{k-1} x_{k,q+1}^{2-d} \tag{2.12.3}$$

This is a rational function whose numerator is an integer. All other terms in the correlation function will be generated by permuting the powers of the partial derivatives that appear. Hence, each term in the overall sum will depend differently only each x_i, and the overall sum cannot cancel because the numerators have no x_i dependence. Thus, the correlation we wanted to show is nonzero is indeed nonzero, completing the proof.

2.13 Appendix F: The free Maxwell field in five dimensions

Consider the theory of a free Maxwell field in d dimensions. The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2\xi}(\partial A)^2 \tag{2.13.1}$$

where $\xi = \frac{d}{d-4}$. As was noted in [50], this theory is a conformal field theory with higher spin symmetry, but it is non-unitary in dimension $d > 4$. We claim that this theory is an example of a conformal, non-unitary theory where the three-point function of the stress tensor does not coincide with one of the three free structures described in the body of the chapter. This can be checked by explicit calculation. The canonical stress energy tensor is not trace-free, and it may be improved using the procedure of [51]. The result is

$$\begin{aligned} T^{--} = {} & 4\partial_+ A^\rho \partial_+ A_\rho + \partial^\rho A^- \partial_\rho A^- - 4\partial_+ A^\rho \partial_\rho A^- + 4\frac{(d-4)}{d} A^- \partial_+(\partial A) + \\ & + \frac{1}{(d-2)}\Big[4a(\partial A)\partial_+ A^- + 4a\ A^-\partial_+(\partial A) + 4a\partial_+ A^\rho \partial_\rho A^- + 4aA^\rho \partial_\rho \partial_+ A^- + \\ & + 16bA_\rho \partial_+^2 A^\rho + 16b\partial_+ A_\rho \partial_+ A^\rho - 2aA^-\partial^2 A^- - 2a\partial^\rho A^- \partial_\rho A^-\Big] - \\ & - 2\frac{(d-4)}{(d-1)}\Big[\partial_+ A_\rho \partial_+ A^\rho + A_\rho \partial_+^2 A^\rho\Big], \end{aligned} \tag{2.13.2}$$

where $a = 2 - d/2, b = d/4 - 1$. Now, the three point function $\langle T_{--}T_{--}T_{--}\rangle$ can be evaluated by Wick contraction, and the result can be decomposed as follows:

$$\langle T_{--}T_{--}T_{--}\rangle = c_s\langle T_{--}T_{--}T_{--}\rangle_s + c_f\langle T_{--}T_{--}T_{--}\rangle_f + c_t\langle T_{--}T_{--}T_{--}\rangle_t, \tag{2.13.3}$$

where $c_s = \frac{12125}{576}, c_f = -\frac{1000}{9}, c_t = \frac{54179}{576}$. This demonstrates that unitarity is a necessary assumption for our result; the three-point function of the stress tensor is not the same as the result for an appropriate free field theory. It is a superposition of the three possible structures.

Chapter 3

Towards a Weinberg-Witten theorem for conformal field theories in $d = 4$

3.1 Introduction

In this work we explore the general relationship between conformal field theory and conserved currents in four spacetime dimensions. We focus on asymmetric currents whose transformation properties under the Lorentz group have a net chirality. We demonstrate that a class of such currents automatically saturate the average null energy condition independent of any tuning of OPE coefficients. We comment on possible implications of these results.

3.1.1 Free conformal fields

The most well understood quantum field theories are those which are free. In conformal field theory we can abstractly state the signature of free theory in terms of the existence of certain conserved currents in the spectrum of local operators. Every conformal field theory contains a traceless conserved energy momentum tensor whose associated conserved charges generate the conformal symmetry. In a free field theory there are also conserved currents that carry more spin then the energy momentum

tensor. For instance in the theory of a free scalar field these take the form

$$J_{A_1 A_2 \cdots A_n} = \phi \overset{\leftrightarrow}{\partial}_{A_1} \overset{\leftrightarrow}{\partial}_{A_2} \cdots \overset{\leftrightarrow}{\partial}_{A_n} \phi - \text{traces} \ . \tag{3.1.1}$$

The charge algebra generated by these currents enlarges the conformal algebra, and the full system of resulting Ward identities then imply that the resulting system has energy momentum tensor correlators coinciding with those of a free field theory [17] [38] [29] [40] [28].

3.1.2 Elementary free fields

A parallel way of understanding free conformal field theories is that their spectrum of local operators can be constructed from elementary building blocks: the free fields themselves.[1] Again conformal symmetry greatly constrains the possibilities.

In general in conformal field theory, the state operator correspondence implies that local operators are in unitary representations of the conformal group. Free fields are the special class of these representations whose support in Fourier space is on the light-cone, i.e. those operators annihilated by ∂^2. A complete list of such fields may be constructed [52] [53]. Aside from the free scalar, there are spinning free fields $h_{\alpha_1 \cdots \alpha_j}$ which are totally symmetric on their spinor indices and obey a generalized Dirac equation

$$\partial^{\dot{\beta}\beta} h_{\beta \alpha_2 \cdots \alpha_j} = 0 \ . \tag{3.1.2}$$

In addition, there are complex conjugate fields with dotted as opposed to undotted indices. The dimension of these fields is fixed in terms of their spin from the conformal unitarity bound

$$\Delta(h_{\alpha_1 \cdots \alpha_j}) = \frac{j}{2} + 1 \ . \tag{3.1.3}$$

Note that the fields $h_{\alpha_1, \cdots \alpha_j}$ are gauge invariant local operators. Thus, in a Lagrangian model, it may be more appropriate to view them as free field strengths. For instance the ordinary free gauge field is given by its self-dual field strength $F_{\alpha\beta}$ (as well as its complex conjugate). The remaining operators should be viewed as field strengths for higher-spin free fields.

From the list of conformal free fields we may construct a plethora of higher spin conserved currents. For instance, the expected higher spin currents transforming as traceless symmetric tensors as in (3.1.1) are easily found and take the form $\bar{h} \partial^n h$ with all spinor indices symmetrized.

[1] In some applications, the actual operator spectrum may be the singlet sector of a global symmetry of the larger operator spectrum described here.

More curious however is the presence of currents transforming in chiral representations of the Lorentz group. In general for any $n, m \geq 1$, the representation theory of the conformal group permits conserved current operators transforming under $sl(2) \times sl(2)$ as symmetric tensors on n and m chiral and antichiral spinor indices. They obey the conservation equation

$$\partial^{\dot{\beta}\beta} J_{\beta\alpha_2\cdots\alpha_n\dot{\beta}\dot{\alpha}_2\cdots\dot{\alpha}_m} = 0 \ , \tag{3.1.4}$$

and their scaling dimensions are fixed in terms of their spin as

$$\Delta(J_{\alpha_1\cdots\alpha_n\dot{\alpha}_1\cdots\dot{\alpha}_m}) = \frac{1}{2}(n+m) + 2 \ . \tag{3.1.5}$$

The more familiar symmetric currents are those for which $n = m$. Henceforth we refer to such currents by their representation (n, m). If $n \neq m$ we sometimes refer to these currents as asymmetric or chiral, and call $|n - m|$ the chirality.

The charge algebra generated by the zero modes of an asymmetric current contains charges transforming in chiral representations of the Lorentz group. The most familiar and important example of this is the supercurrent $(2, 1)$ (as well as complex conjugate $(1, 2)$). Other examples are more exotic.

In the context of free field theory it is easy to construct asymmetric currents. The unitarity bounds (3.1.3) and (3.1.5) imply that any primary operator which is a bilinear constructed out of conformal free fields is conserved current. Therefore using a single higher spin conformal free field we can construct a series of currents

$$J_{\alpha_1\cdots\alpha_{n+2j}\dot{\alpha}_1\cdots\dot{\alpha}_n} = \mathrm{Sym}_{\{\alpha_i\},\{\dot{\alpha}_i\}} \left[h_{\alpha_1\cdots\alpha_j} \partial_{\alpha_{j+1}\dot{\alpha}_1} \partial_{\alpha_{j+2}\dot{\alpha}_2} \cdots \partial_{\alpha_{j+n}\dot{\alpha}_n} h_{\alpha_{j+n+1}\cdots\alpha_{n+2j}} \right] \ , \tag{3.1.6}$$

where the Sym notation means that we symmetrize over the α_i, the $\dot{\alpha}_i$ indices. Thus we see that the signature of higher-spin free fields of spin j is a tower of higher spin currents of chirality $2j$.

3.1.3 The Weinberg-Witten theorem

Although the representation theory of the conformal group permits free fields of arbitrary spin $(j, 0)$. There is a crucial physical requirement that bounds the spin. This is assumption of current algebra. This means that the generators of the conformal algebra must be expressed in terms of the integrals of a local conserved energy momentum tensor. The Weinberg-Witten theorem [18] states that that

this assumption is incompatible with the existence of free fields of spin $j > 2$. This leaves only the familiar scalar ($j = 0$), Weyl spinor ($j = 1$), and vector ($j = 2$) as viable free fields.

It is instructive to reproduce this result in the language of conformal field theory. We work in the operator product limit and consider those terms in the OPE of the energy-momentum tensor T and a higher-spin free field h that may contribute to the Ward identities of the conformal algebra. There are exactly three such terms

$$\begin{aligned} T_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}(x)h_{\beta_1\cdots\beta_j}(0) &\sim \mathrm{Sym}_{\{\alpha_i\},\{\dot{\alpha}_i\},\{\beta_k\}}\left[\frac{1}{x^6}\left(A\delta^{\gamma_1}_{\beta_1}\delta^{\gamma_2}_{\beta_2}x_{\alpha_1\dot{\alpha}_1}x_{\alpha_2\dot{\alpha}_2} + B\delta^{\gamma_2}_{\beta_2}\epsilon^{\zeta_1\gamma_1}\epsilon_{\alpha_1\beta_1}x_{\zeta_1\dot{\alpha}_1}x_{\alpha_2\dot{\alpha}_2}\right.\right. \\ &+ \left.\left. C\epsilon^{\zeta_1\gamma_1}\epsilon^{\zeta_2\gamma_2}\epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}x_{\zeta_1\dot{\alpha}_1}x_{\zeta_2\dot{\alpha}_2}\right)h_{\gamma_1\gamma_2\beta_3\cdots\beta_j}(0)\right] , \end{aligned} \tag{3.1.7}$$

where in the above, A, B, C are coefficients, and the Sym notation means that we symmetrize over the α_i, the $\dot{\alpha}_i$ and the β_k indices independently to match the properties of the left hand side.

The expression (3.1.7) takes into account only the general Lorentz transformation and scaling of the operators but not the more powerful constraints of conservation of T and the equation of motion of h. These are differential constraints which must be satisfied on each of the structures appearing in the OPE limit by adjusting the coefficients A, B, C. A straightforward calculation shows that for $j > 2$ no solution exists aside from the trivial choice $A = B = C = 0$. For small spins, nontrivial solutions exist. Specifically:

- For the free vector, $j = 2$, there are two linear constraints on A, B, C; there is a solution.
- For the free spinor, $j = 1$, the structure C does not exist and there one linear constraint on A and B; there is a solution.
- For the free scalar, $j = 0$, the structures B and C do not exist and the constraints are automatically satisfied.

We can now conclude the argument forbidding free conformal fields of spin $j > 2$. Indeed for these fields, we find that their OPE with the energy momentum tensor does not contain the correct singular terms to reproduce the conformal Ward identities. This is a contradiction and implies that such higher-spin free fields do not exist in any conformal field theory that contains an energy-momentum tensor.

3.1.4 Constraining conserved currents and summary of results

The fact that the spin of free fields is bounded in conformal field theory places strong restrictions on the chirality of conserved currents that may occur in free field theory. Specifically, since one may

not utilize the higher-spin free fields with $j > 2$ it follows that the net chirality of any conserved current in free field theory is bounded by twice this value, namely four:

$$J_{\alpha_1 \cdots \alpha_n \dot{\alpha}_1 \cdots \dot{\alpha}_m} \in \text{Free Theory Operator Spectrum} \Rightarrow |n - m| \leq 4 \ . \tag{3.1.8}$$

This leaves the status of general asymmetric conserved currents unclear. If higher-spin currents with chirality $|n - m| > 4$ do not occur in free field theory, do they occur in any theory? If they do occur one would like to exhibit such a field theory. If they do not occur, one would like to understand a general argument.

In this work we address these questions using only the general constraints of conformal invariance, causality and unitarity. Specifically we focus on a series of currents in Lorentz representation $(n, 1)$, working up from the familiar vector current $(1, 1)$ and supercurrent $(2, 1)$ to the more exotic conserved currents of increasing chirality.

In Section 3.2 we first construct the most general three point function $\langle TJ\bar{J} \rangle$ where J is a conserved current in this series and T is an energy momentum tensor. Starting from the most general conformally invariant expression for operators with these spins, we systematically impose the constraints of conservation of all currents and the conformal Ward identities on T. We find that for all such currents admit solutions satisfying these constraints. Unlike the higher spin free fields, chiral currents are compatible with the current algebra generated by the energy-momentum tensor.

In Section 3.3 we move on to study the compatibility of these correlators with the Average Null Energy Condition [54] [55]. This constraint on conformal field theory follows from causality. The version that we will use follows the pioneering analysis of Hofman and Maldacena [19] [20]: we view $\varepsilon \cdot J$ as creating a state by acting on the vacuum, where ε is a polarization tensor. Then we measure the expectation value energy, $\mathcal{E}(\theta, \phi)$, on the sphere at null infinity in this state. The average null energy condition states that this average energy is point-wise non-negative:

$$\langle \mathcal{E}(\theta, \phi) \rangle \geq 0 \ . \tag{3.1.9}$$

In our application, these expectation values are determined in terms of the three point function and $\langle \bar{J}TJ \rangle$ and imply constraints on the OPE coefficient. Strikingly, we find that as the spin is increased we do not find more constraints on the OPE coefficients, rather we find that as soon as $n > 2$ (i.e. beyond the supercurrent) the additional polarizations lead to energy correlators which vanish at a point on the sphere. This is in particular true of the extreme polarizations carrying

maximal angular momentum:

$$\langle J_{--\ldots--}|\mathcal{E}(0,0)|\bar{J}_{++\ldots+}\rangle = 0 \ , \tag{3.1.10}$$

provided $n > 2$. In other words: *asymmetric currents* $(n,1)$ *with* $n > 2$ *automatically saturate the bound* (3.1.9) *as a consequence only of conservation conditions and conformal invariance. No tuning of OPE coefficients is needed to saturate the bound.*

Saturation of the average null energy condition (3.1.9) is a strong indication that the theory is free. For instance in [21] it was argued that theories that saturate this bound when the state is created by the energy-momentum tensor implies free energy correlators. We expect that similar conclusions follow from our results. This gives a simple direct argument that conserved currents of large chirality do not occur in any conformal field theory. The existence of such a current implies that the theory is free, however according to (3.1.8) these currents are absent from valid free field theory spectrum.

Further conclusions and ongoing directions of investigation are described in Section 3.4.

3.2 Building and constraining three point functions

We begin with the first task described above - to construct the most general expression consistent with conformal symmetry for the three point function of a stress tensor T, a conserved current J, and its complex conjugate $\bar{J}$. Then, we will impose conservation of T, conservation of J, and the conformal Ward identities.

3.2.1 Conformal building blocks for three point functions

Three point functions in conformal field theory are completely fixed by conformal symmetry up to a set of constants because there are no cross-ratios one can write with only three points. In four dimensions, one can write the three point function of generic operators $\mathcal{O}_i$ as the product of (a) a scalar kinematical factor $\mathcal{K}$, and (b) a linear combination of independent tensors $\mathcal{T}_i$ that depend only on the spins of the operators. The only freedom is in the coefficients c_i that multiply the $\mathcal{T}_i$.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \mathcal{K}(x_1,x_2,x_3)\sum c_i\mathcal{T}_i(x_1,x_2,x_3) \tag{3.2.1}$$

Roughly, the "tensor structures" $\mathcal{T}_i$ span the possible irreducible Lorentz representations that can appear in the tensor product of the three operators, although it is not usually convenient to enumerate them in such a way that the representations are manifest.

The task of determining $\mathcal{K}$ and the possible $\mathcal{T}_i$ for generic three point functions was carried out by Elkhidir, et. al. in [56]. In that paper, they used the embedding formalism to lift three point functions from four to six dimensions, where the conformal group acts linearly as the six-dimensional Lorentz group. With this simplification, they computed $\mathcal{K}$ for a generic three point function and found that each tensor structure $\mathcal{T}_i$ can be written as a product of certain elementary "building blocks". We summarize these results now:

Let $\mathcal{O}_i$, $i \in \{1,2,3\}$, be primary operators of conformal dimension Δ_i and spin s_i. The $\mathcal{O}_i$ live in irreducible representations of the Lorentz group, which we label with a pair of integers (a_i, b_i) to denote the representation that has a_i completely symmetric undotted indices and b_i completely symmetric dotted indices. The a_i and b_i are related to the spins s_i of the representation by $(a_i+b_i)/2$. Let us write the index structure of the three operators as follows:

$$(\mathcal{O}_1)_{\alpha_1\ldots\alpha_{a_1}\dot{\alpha}_1\ldots\dot{\alpha}_{b_1}} \qquad (\mathcal{O}_2)_{\beta_1\ldots\beta_{a_2}\dot{\beta}_1\ldots\dot{\beta}_{b_2}} \qquad (\mathcal{O}_3)_{\gamma_1\ldots\gamma_{a_3}\dot{\gamma}_1\ldots\dot{\gamma}_{b_3}} \tag{3.2.2}$$

Then, the kinematical factor is:

$$\mathcal{K} = \frac{1}{x_{12}^{(\Delta_1+s_1)+(\Delta_2+s_2)-(\Delta_3+s_3)} x_{13}^{(\Delta_1+s_1)+(\Delta_3+s_3)-(\Delta_2+s_2)} x_{23}^{(\Delta_2+s_2)+(\Delta_3+s_3)-(\Delta_1+s_1)}} \tag{3.2.3}$$

In our problem, we are not interested in the most general operators. We will ultimately wish to take $\mathcal{O}_1$ to be the stress tensor $T_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}$, $\mathcal{O}_2$ to be a conserved current $J_{\beta_1\ldots\beta_k\dot{\beta}}$ which transforms in the $(k,1)$ representation for $k \geq 6$, and $\mathcal{O}_3$ to be its complex conjugate $\bar{J}_{\gamma\dot{\gamma}_1\ldots\dot{\gamma}_k}$, which transforms in the $(1,k)$ representation. Then, $\Delta_1 = 4$, $s_1 = 2$, $\Delta_2 = \Delta_3 = 2 + \frac{k+1}{2}$, and $s_2 = s_3 = (k+1)/2$. The kinematical factor then reduces to:

$$\mathcal{K} = \frac{1}{x_{12}^6 x_{13}^6 x_{23}^{2k}} \tag{3.2.4}$$

A similar simplification occurs for the tensor structures $\mathcal{T}_i$ of such correlators. Only a subset of the most general set of building blocks are relevant. This subset is defined as follows[2]

[2]Relative to [56], we have defined the I_{ij} and J_i tensors to be the values one obtains after projection from six to four dimensions instead of the six-dimensional expression. Also, our definition of the J_i differs by a factor of 1/2.

$$I_{12} = (x_{12})_{\beta\dot{\alpha}} \qquad I_{21} = -(x_{12})_{\alpha\dot{\beta}}$$

$$I_{13} = (x_{13})_{\gamma\dot{\alpha}} \qquad I_{31} = -(x_{13})_{\alpha\dot{\gamma}}$$

$$I_{23} = (x_{23})_{\gamma\dot{\beta}} \qquad I_{32} = -(x_{23})_{\beta\dot{\gamma}}$$

$$J_{1,23} = \frac{x_{12}^2 x_{13}^2}{x_{23}^2}\left(\frac{(x_{12})_{\alpha\dot{\alpha}}}{x_{12}^2} - \frac{(x_{13})_{\alpha\dot{\alpha}}}{x_{13}^2}\right) \qquad J_{1,32} = -J_{1,23}$$

$$J_{2,31} = \frac{x_{12}^2 x_{13}^2}{x_{23}^2}\left(\frac{(x_{12})_{\beta\dot{\beta}}}{x_{12}^2} - \frac{(x_{13})_{\beta\dot{\beta}}}{x_{13}^2}\right) \qquad J_{2,13} = -J_{2,31}$$

$$J_{3,12} = \frac{x_{12}^2 x_{13}^2}{x_{23}^2}\left(\frac{(x_{12})_{\gamma\dot{\gamma}}}{x_{12}^2} - \frac{(x_{13})_{\gamma\dot{\gamma}}}{x_{13}^2}\right) \qquad J_{3,21} = -J_{3,12}$$

In the above expression, one should consider the names of the indices on the right-hand side to correspond to the indices with the same names in 3.2.2. (The subscripts don't matter since ultimately we will be symmetrizing all the indices of the same type.) Then, every possible tensor structure $\mathcal{T}_i$ can be written as a product of these building blocks such that the right number of $\alpha, \dot{\alpha}, \beta, \dot{\beta}, \gamma, \dot{\gamma}$ indices appear, as exhibited in 3.2.2. Then one symmetrizes all subsets of indices which were symmetric in the original three point function. That is, one should symmetrize all the α_i, all the $\dot{\alpha}_i$, etc. So the task of writing a general three point function is reduced to enumerating all possible ways of combining the structures above appropriately.

To perform this enumeration properly (i.e. without including redundant structures), one has to account for the fact that these building blocks are not automatically independent. There are two relations. First, as we have explicitly pointed out above, the $J_{i,jk}$ are antisymmetric in the last two indices, i.e. $J_{i,jk}$ and $J_{i,kj}$ are not independent. Thus, in the following, we will suppress the last two indices. J_i is understood to be the structure in the left column of the table above. Second, there is a cubic relation that reduces $J_1 J_2 J_3$ to sums of products of I_{ij} and J_i tensors where no term contains all three J's. This means that we should not write $\mathcal{T}_i$ that have all three J's in it.

Before proceeding, we emphasize that these are not all the building blocks required to construct general three point functions. There are more building blocks $K_{i,jk}$ and $\bar{K}_{i,jk}$ which we have not defined. Each K carries two undotted indices, and each $\bar{K}$ carries two dotted indices. It turns out that there is a relation that reduces the product of any K with any $\bar{K}$ to a sum of products of I_{ij} and J_i tensors. This means that each tensor structure can be written in such a way that it contains

either K's or $\bar{K}$'s, but never both. Since the I_{ij} and J_i each have one dotted and one undotted index, this implies that neither K nor $\bar{K}$ can appear in a three point function that has an equal number of total undotted indices and dotted indices (i.e. $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$). As mentioned, all the correlation functions we study are of this type. If one wanted to study correlation functions that did not have this property, however, one would need to account for tensor structures that involve K or $\bar{K}$ tensors.

To illustrate this procedure, we give some examples. Consider the three point function $\langle TVV \rangle$ of the stress tensor and two conserved $U(1)$ currents:

$$\left\langle T_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}(x_1)V_{\beta\dot{\beta}}(x_2)V_{\gamma\dot{\gamma}}(x_3)\right\rangle = \mathcal{K}\sum c_i\mathcal{T}_i \tag{3.2.5}$$

The kinematical factor is $\mathcal{K} = x_{12}^{-6}x_{13}^{-6}x_{23}^{-2}$. The possible $\mathcal{T}_i$ are:

$$\mathcal{T}_1 = I_{12}I_{13}I_{21}I_{31} \tag{3.2.6}$$

$$\mathcal{T}_2 = I_{13}I_{31}J_1J_2 \tag{3.2.7}$$

$$\mathcal{T}_3 = I_{12}I_{23}I_{31}J_1 \tag{3.2.8}$$

$$\mathcal{T}_4 = I_{13}I_{21}I_{32}J_1 \tag{3.2.9}$$

$$\mathcal{T}_5 = I_{23}I_{32}J_1^2 \tag{3.2.10}$$

$$\mathcal{T}_6 = I_{12}I_{21}J_1J_3 \tag{3.2.11}$$

These were determined by programming a computer to exhaustively enumerate all possibilities for this index structure modulo the relations mentioned above. As one can verify by using the definitions, each of these structures contains the correct number of indices of each type. For instance, we can expand the first structure as follows:

$$\mathcal{T}_1 = I_{12}I_{13}I_{21}I_{31} = (x_{12})_{\beta\dot{\alpha}_1}(x_{13})_{\gamma\dot{\alpha}_2}(-x_{12})_{\alpha_1\dot{\beta}}(-x_{13})_{\alpha_2\dot{\gamma}} \tag{3.2.12}$$

In the above, the symmetrizations on the α_i and $\dot{\alpha}_i$ are implicit. These symmetrizations must be imposed by hand.

The c_i appearing in 3.2.5 will be constrained by demanding that T is conserved, V is conserved, and the conformal Ward identities are satisfied. We will describe how this can be done shortly.

We give another, more involved, example. Consider the three point function of the stress tensor T, a current transforming in the $(1, 6)$ representation, and its complex conjugate, which transforms

in the $(6,1)$ representation:

$$\left\langle T_{\alpha_1\alpha_2\dot\alpha_1\dot\alpha_2}(x_1)V_{\beta_1\ldots\beta_6\dot\beta}(x_2)V_{\gamma\dot\gamma_1\ldots\dot\gamma_6}(x_3)\right\rangle = \mathcal{K}\sum c_i\mathcal{T}_i \tag{3.2.13}$$

Now the kinematical factor is $\mathcal{K} = x_{12}^{-6}x_{13}^{-6}x_{23}^{-12}$. The possible $\mathcal{T}_i$ are:

$$\mathcal{T}_1 = I_{12}I_{13}I_{31}^2I_{32}^4J_2 \tag{3.2.14}$$

$$\mathcal{T}_2 = I_{12}^2I_{23}I_{31}^2I_{32}^4 \tag{3.2.15}$$

$$\mathcal{T}_3 = I_{12}I_{13}I_{21}I_{31}I_{32}^5 \tag{3.2.16}$$

$$\mathcal{T}_4 = I_{13}I_{31}I_{32}^5J_1J_2 \tag{3.2.17}$$

$$\mathcal{T}_5 = I_{12}I_{23}I_{31}I_{32}^5J_1 \tag{3.2.18}$$

$$\mathcal{T}_6 = I_{13}I_{21}I_{32}^6J_1 \tag{3.2.19}$$

$$\mathcal{T}_7 = I_{23}I_{32}^6J_1^2 \tag{3.2.20}$$

$$\mathcal{T}_8 = I_{12}I_{21}I_{32}^5J_1J_3 \tag{3.2.21}$$

$$\mathcal{T}_9 = I_{12}^2I_{21}I_{31}I_{32}^4J_3 \tag{3.2.22}$$

$$\mathcal{T}_{10} = I_{12}^2I_{31}^2I_{32}^3J_2J_3 \tag{3.2.23}$$

Again, the $\mathcal{T}_i$ have the right index structure to match the left-hand side of 3.2.13 and the c_i will be constrained by conservation of T, conservation of J, and the conformal Ward identities.

Later on, it will also be helpful to have expressions for the two point functions $\left\langle J(x_1)\bar J(x_2)\right\rangle$. In this case, the only allowed building blocks are I_{12} and I_{21}. Hence, we see that any two-point function $\langle\mathcal{O}_1\mathcal{O}_2\rangle$ vanishes unless the $\mathcal{O}_i$ transform in conjugate representations. Dimensional analysis then fixes the kinematical part. Generically, for an operator J that transforms in the (a,b) representation, we have

$$\left\langle J(x_1)\bar J(x_2)\right\rangle = \frac{C_J}{x_{12}^{2\Delta+a+b}}I_{12}^bI_{21}^a \tag{3.2.24}$$

where C_J is a constant. If we rewrite this equation as follows:

$$\left\langle J_{\beta_1\ldots\beta_m\dot\beta_1\ldots\dot\beta_n}(x_1)\bar J_{\gamma_1\ldots\gamma_n\dot\gamma_1\ldots\dot\gamma_m}(x_2)\right\rangle = \frac{C'_J}{x_{12}^{2\Delta+a+b}}\left(\prod_{i=1}^m x_{\beta_i\dot\gamma_i}\right)\left(\prod_{i=1}^n x_{\gamma_i\dot\beta_i}\right) \tag{3.2.25}$$

where C'_J is related to C_J by a sign that may be inferred from the preceding two equations, then actually it is known that unitarity implies that $(-i)^{m+n}C'_J > 0$ [57].

3.2.2 Working in the OPE limit

Conceptually, imposing the conservation conditions and the conformal Ward identities is straightforward. Requiring conservation of T means that we impose the equation

$$\partial_1^{\dot{\alpha}_1\alpha_1}\langle T_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}(x_1)\ldots\rangle = 0 \tag{3.2.26}$$

This will generate a number of linear relations among the c_i. Requiring conservation of J means that we impose the equation

$$\partial_2^{\dot{\beta}\beta_1}\left\langle \ldots J_{\beta_1\ldots\beta_k\dot{\beta}}(x_2)\ldots\right\rangle = 0 \tag{3.2.27}$$

This will generate additional linear relations among the c_i. Requiring that the three-point function satisfies the conformal Ward identities means that we contract T with all possible conformal Killing vectors $\xi_{\alpha\dot{\alpha}}$ and integrate x_1 around a small sphere surrounding one of the operators and then demand that the result generates the action of the corresponding charge Q_ξ on that operator [58]. For instance, if we integrate around the point where J is inserted, we obtain:

$$\int_{S^3_\epsilon}\left\langle \xi\cdot T J\bar{J}\right\rangle = \left\langle i[Q_\xi, J]\bar{J}\right\rangle \tag{3.2.28}$$

This also generates linear relations among the c_i.

In practice, performing these calculations with the full three point functions constructed in the previous section is cumbersome. When imposing the conservation equations, there are a large number of components that one must check. When imposing the conformal Ward identities, the relevant integrals are difficult to calculate.

Fortunately, one can leverage the conformal symmetry to simplify the calculation by working in the limit where T and J are very close to each other, and $\bar{J}$ is taken to infinity. No information is lost in this limit, as there is enough freedom in the conformal group to send the three positions in any three point function to arbitrary locations. One can physically understand this limit as taking the operator product expansion (OPE) of T and J and extracting just the component that is proportional to J, which is the only component that could contribute to the resulting two-point function with $\bar{J}$. In fact, since the configuration where x_3 is strictly taken to infinity is in the orbit of any configuration of three points under the action of the conformal group, we do not have to work beyond leading order in this OPE expansion.

If we expand to lowest nonvanishing order in x_{12}, the building blocks reduce to the following

expressions:

$$I_{12} \mapsto (x_{12})_{\beta\dot{\alpha}} \tag{3.2.29}$$

$$I_{23} \mapsto (x_{23})_{\gamma\dot{\beta}} \tag{3.2.30}$$

$$I_{13} \mapsto (x_{13})_{\gamma\dot{\alpha}} \tag{3.2.31}$$

$$J_1 \mapsto (x_{12})_{\alpha\dot{\alpha}} \tag{3.2.32}$$

$$J_2 \mapsto (x_{12})_{\beta\dot{\beta}} \tag{3.2.33}$$

$$J_3 \mapsto -\frac{1}{x_{12}^2}((x_{13})_{\chi\dot{\gamma}}(x_{13})_{\gamma\dot{\chi}}(x_{12})^{\dot{\chi}\chi}) \tag{3.2.34}$$

The expression for J_3 is more complicated than the others because at leading order in x_{12}, it is zero. One has to work to the next-to-leading-order to obtain a nonzero expression. The expression quoted above is the result after applying a rearrangement identity to the subleading term, which somewhat simplifies subsequent expressions.

We would like to extract the part of the OPE between T and J that is proportional to J. If we take the $x_1 \to x_2$ limit of the full tensor structures, we will obtain expressions where the two point function between J and $\bar{J}$ has been evaluated. We wish to "factor out" this two point function to make manifest the exact form of the OPE. Luckily, this is a simple task, since in the $x_1 \to x_2$ limit, $x_{13} \approx x_{23}$. This allows us to read the two point function directly by extracting any piece that involves x_3. For example, consider the structure $\mathcal{T}_1$ that contributes to the $\langle TVV \rangle$ correlator 3.2.5. Using the dictionary above, we find that

$$\mathcal{K}c_1\mathcal{T}_1 \equiv \frac{c_1}{x_{12}^6 x_{13}^6 x_{23}^2} I_{12}I_{13}I_{21}I_{31} \tag{3.2.35}$$

$$\xrightarrow{x_1 \to x_2} \frac{c_1}{x_{12}^6 x_{23}^8}(x_{12})_{\beta\dot{\alpha}_1}(x_{23})_{\gamma\dot{\alpha}_2}(x_{12})_{\alpha_1\dot{\beta}}(x_{23})_{\alpha_2\dot{\gamma}} \tag{3.2.36}$$

Again, we emphasize the right-hand size of the expression above does not indicate the symmetrizations, which must be imposed by hand. The two point function of V is given by 3.2.25:

$$\langle V_{\chi\dot{\chi}}(x_2)V_{\rho\dot{\rho}}(x_3)\rangle = \frac{C_V}{x_{23}^8}(x_{23})_{\rho\dot{\chi}}(-x_{23})_{\chi\dot{\rho}} \tag{3.2.37}$$

Comparing the two point function to the OPE limit of $\mathcal{K}\mathcal{T}_1$, we can easily identify the two point

function in the latter expression:

$$\mathcal{K}c_1\mathcal{T}_1 \xrightarrow{x_1\to x_2} -\frac{c_1/C_V}{x_{12}^6}(x_{12})_{\beta\dot\alpha_1}(x_{12})_{\alpha_1\dot\beta}\langle V_{\alpha_2\dot\alpha_2}(x_2)V_{\gamma\dot\gamma}(x_3)\rangle \tag{3.2.38}$$

This implies that in the OPE of T with J, the following term appears at leading order in the $x_1 \to x_2$ limit, which we will define to be $\bar{\mathcal{T}}_1$:

$$T_{\alpha_1\alpha_2\dot\alpha_1\dot\alpha_2}(x_1)V_{\beta\dot\beta}(x_2) \xrightarrow{x_1\to x_2} -\frac{c_1/C_V}{x_{12}^6}(x_{12})_{\beta\dot\alpha_1}(x_{12})_{\alpha_1\dot\beta}V_{\alpha_2\dot\alpha_2}(x_2)+\dots \tag{3.2.39}$$

$$\equiv (-c_1/C_V)\bar{\mathcal{T}}_1+\dots \tag{3.2.40}$$

We give another example involving J_3 to illustrate how that structure appears. Consider the structure $\mathcal{T}_6$ the contributes to 3.2.5. In the OPE limit, it becomes:

$$\mathcal{K}c_6\mathcal{T}_6 \equiv \frac{c_6}{x_{12}^6x_{13}^6x_{23}^2}I_{12}I_{21}J_1J_3 \tag{3.2.41}$$

$$\xrightarrow{x_1\to x_2}\frac{c_6}{x_{12}^8x_{23}^8}(x_{12})_{\beta\dot\alpha_1}(-x_{12})_{\alpha_1\dot\beta}(x_{12})_{\alpha_2\dot\alpha_2}(-(x_{23})_{\chi\dot\gamma}(x_{23})_{\gamma\dot\chi}(x_{12})^{\dot\chi\chi}) \tag{3.2.42}$$

$$= \frac{-c_6/C_V}{x_{12}^8}(x_{12})_{\beta\dot\alpha_1}(x_{12})_{\alpha_1\dot\beta}(x_{12})_{\alpha_2\dot\alpha_2}(x_{12})^{\dot\chi\chi}\langle V_{\chi\dot\chi}(x_2)V_{\gamma\dot\gamma}(x_3)\rangle \tag{3.2.43}$$

This implies that in the OPE of T with J, the following term appears at leading order in the $x_1 \to x_2$ limit:

$$T_{\alpha_1\alpha_2\dot\alpha_1\dot\alpha_2}(x_1)V_{\beta\dot\beta}(x_2) \xrightarrow{x_1\to x_2} -\frac{c_6/C_V}{x_{12}^8}(x_{12})_{\beta\dot\alpha_1}(x_{12})_{\alpha_1\dot\beta}(x_{12})_{\alpha_2\dot\alpha_2}(x_{12})^{\dot\chi\chi}V_{\chi\dot\chi}(x_2)+\dots \tag{3.2.44}$$

$$\equiv (-c_6/C_V)\bar{\mathcal{T}}_6 \tag{3.2.45}$$

To simplify the notation, in the $x_1 \to x_2$ limit of $\langle TJ\bar{J}\rangle$, we define rescaled coefficients $k_i = \pm c_i/C_J$, where the sign depends on whether the OPE limit of the corresponding $\mathcal{T}_i$ appears with a minus sign when all x_{ij} are placed in canonical order and the two point function is factored out. So for instance, in the $\langle TVV\rangle$ correlator we have $k_1 = -c_1/c_V$ and $k_6 = -c_6/C_V$ but $k_2 = c_2/C_V$, as one can check explicitly. Hence, after taking the $x_1 \to x_2$ limit, we are left with a sum of simple OPE structures

$$T(x_1)J(x_2) \xrightarrow{x_1\to x_2} \sum k_i\bar{\mathcal{T}}_i \tag{3.2.46}$$

where the $\bar{\mathcal{T}}_i$ are the OPE limits of the tensor structures.

We now make a few comments about the general structure of the $\bar{\mathcal{T}}_i$ for three point functions of

the form $\langle TJ\bar{J}\rangle$ which will be invoked in subsequent sections.

First, there are always an even number of $(x_{12})_{\chi\dot{\chi}}$'s in the numerator of the $\bar{\mathcal{T}}_i$. This follows from a counting argument on the number of J_3. If there are no J_3, then the expansions of the building blocks tell us that every index is free. So the four indices of T plus the $a+b$ indices of J have to be distributed between exactly two x_{12}'s and J. If there is one J_3, we can see that the number of free indices of J is reduced by 2, but those two indices are contracted with an x_{12}. The extra two free indices that are now left over, in addition to the four from T, leave 6 indices that cannot be carried by J, which are therefore distributed over 3 x_{12}. With the x_{12} with dummy indices, this makes 4 x_{12}'s in the numerator. The generalization to arbitrary numbers of J_3 is apparent.

Second, every $\bar{\mathcal{T}}_i$ scales like x_{12}^{-4} regardless of the spin or dimension of J. This is obvious since we are extracting the terms in the TJ OPE proportional to J, and hence we need the dimension of what is left over to match the dimension of T. It can also be seen from a counting argument based on the structure of the kinematical factor and the fact that every building block has conformal dimension -1.

Third, the $\bar{\mathcal{T}}_i$ are manifestly independent, as can be checked by taking definite values for all the indices for any generic three point function. We know a priori that nothing should be lost at leading order in the OPE if all we are interested in is the three point function $\langle TJ\bar{J}\rangle$, but this is an explicit check of that argument.

Fourth, the OPE limit is reversible. Again, we knew on general grounds that nothing was lost in the leading order, so it is comforting that one can reconstruct the full three-point function from explicit expressions in the OPE limit just by inspection, e.g. a factor of $(x_{12})_{\alpha\dot{\alpha}}$ can only from a J_1, structures that involve contractions between x_{12} and J are given by J_3, and so on.

3.2.3 Constraints in the OPE limit

The procedure in the previous section essentially "decoupled" all the γ and $\dot{\gamma}$ indices, although it is now less manifest how to impose the various constraints on the three point functions. Now we will explain how to do so.

Conservation of T is still straightforward to impose in the OPE limit. We simply compute:

$$\partial_1^{\dot{\alpha}_1\alpha_1} T_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}(x_1)J(x_2) \to \sum k_i \partial_1^{\dot{\alpha}_1\alpha_1}\bar{\mathcal{T}}_i \tag{3.2.47}$$

and demand that every component vanish. This is most easily accomplished with a computer, as expanding out all the implicit symmetrizations and checking various components is difficult to do

by hand. There are no derivatives on J here, so this procedure essentially amounts to just taking derivatives of the x_{ij} in spinor indices, which is a completely straightforward task. One potential pitfall are various signs and factors of 2 that can easily be neglected if one is unfamiliar with doing calculus in spinor indices. For instance, we have the following relations:

$$\partial^{\dot{\alpha}\alpha} x_{\beta\dot{\beta}} = -2\delta^{\dot{\alpha}}_{\dot{\beta}}\delta^{\alpha}_{\beta} \qquad v^{\dot{\alpha}\alpha} w_{\alpha\dot{\alpha}} = -2v^{\mu}w_{\mu} \qquad x_{\alpha\dot{\alpha}}x^{\dot{\alpha}\beta} = -x^{\mu}x_{\mu}\delta^{\beta}_{\alpha} \qquad x_{\alpha\dot{\alpha}}x^{\dot{\beta}\alpha} = -x^{\mu}x_{\mu}\delta^{\dot{\beta}}_{\dot{\alpha}} \tag{3.2.48}$$

and so on.

Conservation of J is not much harder. We compute

$$\partial_2^{\dot{\beta}\beta_1} T_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2}(x_1)J(x_2) \to \sum k_i \partial_2^{\dot{\beta}\beta_1}\bar{\mathcal{T}}_i \tag{3.2.49}$$

Now one might worry about derivatives on J since J does depend on x_2, but these terms are irrelevant since they will be subleading in x_{12}; recall that this OPE is ultimately to be inserted into a correlation function with $\bar{J}(x_3)$, so derivatives on $J(x_2)$ act only on factors of x_{23}.

Imposing the conformal Ward identities requires a little more setup. As mentioned earlier, we would like to contract the stress tensor $T_{\mu\nu}(x_1)$ with a conformal Killing vector ξ^{ν} and integrate x_1 over a little sphere surrounding x_2. If we write $x_{12} \sim x$ as a shorthand (i.e. suppress the position subscripts), the conformal Killing vectors are:

$$\text{Lorentz transformations:} \qquad (\xi_{\sigma\rho})^{\nu}T_{\mu\nu} = x_{\rho}T_{\mu\sigma} - x_{\sigma}T_{\mu\rho} \tag{3.2.50}$$

$$\text{Translations:} \qquad (\xi_{\rho})^{\nu}T_{\mu\nu} = -T_{\mu\rho} \tag{3.2.51}$$

$$\text{Special conformal transformations:} \qquad (\xi_{\rho})^{\nu}T_{\mu\nu} = 2x_{\rho}x^{\nu}T_{\mu\nu} - x^2 T_{\mu\rho} \tag{3.2.52}$$

$$\text{Dilatations:} \qquad \xi^{\nu}T_{\mu\nu} = x^{\nu}T_{\mu\nu} \tag{3.2.53}$$

We wish to parameterize $x_{12}^{\mu} = \epsilon n^{\mu}$, where n is a unit normal vector that will vary over the surface of the sphere. So the different generators actually scale differently with ϵ.

The measure on the surface of the S^3 is

$$\int_{S^3_{\epsilon}} d\Sigma^{\mu} = \epsilon^3 \int_{S^3_{\epsilon}} d\Omega n^{\mu} \tag{3.2.54}$$

where n is the unit normal vector.

Putting this all together, we would like to evaluate

$$\epsilon^3 \int_{S^3_\epsilon} d\Omega n^\mu [(\xi_{...})^\nu T_{\mu\nu}] J(x_2) \tag{3.2.55}$$

Immediately, we can see that neither translations nor special conformal transformations are going to impose any constraints at leading order. Every $\bar{\mathcal{T}}_i$ has an even number of x_{12} in the numerator, and therefore an even number of n^μ, and there is one n^μ from the measure. Then, translations and special conformal transformations contribute 0 and 2 copies of n^μ, respectively, so for those generators, there are an odd number of n^μ in the integrand. Since the integral over the sphere of an odd number of unit normals n^μ is zero, these do not contribute. The Lorentz transformations and dilatations do contribute, however, and the charges that correpond to them are familiar:

$$\text{Lorentz transformations:} \quad i[Q_\xi, J_{\beta_1 \ldots \beta_k \dot{\beta}_1 \ldots \dot{\beta}_l}](x_2) = \frac{1}{2} \left(\sum_{i=1}^{k} (\sigma_{\mu\nu})^{\chi_i}_{\beta_i} + \sum_{j=1}^{l} (\bar{\sigma}_{\mu\nu})^{\dot{\chi}_j}_{\dot{\beta}_j} \right) J_{\chi_1 \ldots \chi_k \dot{\chi}_1 \ldots \dot{\chi}_l}(x_2) \tag{3.2.56}$$

$$\text{Dilatations:} \quad i[Q_\xi, J](x_2) = \Delta_J J(x_2) \tag{3.2.57}$$

So now, all that has to be done is to evaluate the integrals corresponding to Lorentz transformations and dilatations and demand that they evaluate to the right hand side of the above equations. Doing this can be a little subtle, especially when imposing the Ward identity for Lorentz transformations to fields of high spin. Sometimes it is hard to tell without substituting explicit values of all the indices if two expressions are equal in spinor indices since Schouten identities can relate two expressions that superficially look unequal. It is typically helpful after integration to convert all Lorentz indices into spinor indices and then use various Schouten identities to shuffle all the spinor indices into a canonical order, e.g. one can minimize the number of β and $\dot{\beta}$ indices appear on J. This task is best accomplished on a computer.

3.2.4 An example: the free fermion

At this point, it may be helpful to illustrate the entire procedure, from start to finish, with a simple example from free field theory that can be worked out by hand, and where the answers are already known, e.g. from [35]. We will consider the theory of a single free fermion ψ_β. It has dimension $\Delta = 3/2$, which saturates the unitarity bound.

The kinematical factor for $\langle T\psi\bar{\psi}\rangle$ is given by plugging in $\Delta_1 = 1$, $\Delta_2 = \Delta_3 = 3/2$, $s_1 = 4$, and

$s_2 = s_3 = 1/2$ into equation 3.2.3. The result is

$$\mathcal{K} = \frac{1}{x_{12}^6 x_{13}^6 x_{23}^{-2}} \tag{3.2.58}$$

The possible tensor structures $\mathcal{T}_i$ are easily found:

$$\mathcal{T}_1 = J_1^2 I_{32} \tag{3.2.59}$$

$$\mathcal{T}_2 = J_1 I_{12} I_{31} \tag{3.2.60}$$

In the OPE limit, these become:

$$\mathcal{K} c_1 \mathcal{T}_1 \to k_1 \bar{\mathcal{T}}_1 = k_1 x_{12}^{-6} (x_{12})_{\alpha_1 \dot{\alpha}_1} (x_{12})_{\alpha_2 \dot{\alpha}_2} \psi_\beta \tag{3.2.61}$$

$$\mathcal{K} c_2 \mathcal{T}_2 \to k_2 \bar{\mathcal{T}}_2 = k_2 x_{12}^{-6} (x_{12})_{\alpha_1 \dot{\alpha}_1} (x_{12})_{\beta \dot{\alpha}_2} \psi_{\alpha_1} \tag{3.2.62}$$

Again, remember that here the necessary symmetrizations are implied. These have to be performed by hand.

Imposing conservation of T imposes no constraints on the k_i - i.e. both $\bar{\mathcal{T}}_i$ are identically conserved. Imposing the Dirac equation $\partial_2^{\dot{\beta}\beta} \psi_\beta(x_2)$ sets $k_1 = 0$. That is, one finds that $\bar{\mathcal{T}}_2$ is identically satisfies the equation of motion but $\bar{\mathcal{T}}_1$ does not. Precisely, one finds:

$$\begin{aligned} \partial_2^{\dot{\beta}\beta} k_1 \bar{\mathcal{T}}_1 = -\frac{k_1}{(x_{12})^8} \Bigg(& 3\psi_\beta (x_{12})_{\alpha_1 \dot{\alpha}_2} (x_{12})_{\alpha_2 \dot{\alpha}_1} (x_{12})^{\dot{\beta}\beta} \\ & + x_{12}^2 \delta_{\dot{\alpha}_2}^{\dot{\beta}} \psi_{\alpha_1} (x_{12})_{\alpha_2 \dot{\alpha}_1} + x_{12}^2 \delta_{\dot{\alpha}_1}^{\dot{\beta}} \psi_{\alpha_1} (x_{12})_{\alpha_2 \dot{\alpha}_2} \Bigg) + (\alpha_1 \leftrightarrow \alpha_2) \end{aligned} \tag{3.2.63}$$

One can be convinced that this is really not zero by taking some definite values for x_{12} and some definite indices. Then one imposes the Ward identities for dilatation and Lorentz transformations. Now there is only one structure. We start with dilatation. We would like to integrate:

$$\epsilon^3 \int_{S_\epsilon^3} d\Omega n^\mu \xi^\nu T_{\mu\nu} \psi_\beta = \epsilon^4 \int_{S_\epsilon^3} d\Omega n^\mu n^\nu T_{\mu\nu} \psi_\beta \tag{3.2.64}$$

where ξ^ν here is the conformal Killing vector for dilatations. Since only the second structure survives, we have, after converting everything back to spinor indices and substituting $x_{12} = \epsilon n$:

$$\frac{1}{4} \int_{S_\epsilon^3} d\Omega k_2 n^{\dot{\alpha}_1 \alpha_1} n^{\dot{\alpha}_2 \alpha_2} \psi_{\alpha_2}(x_2) n_{\alpha_1 \dot{\alpha}_1} n_{\beta \dot{\alpha}_2} = \frac{1}{4} \int_{S_\epsilon^3} d\Omega \frac{3}{2} k_2 J_\beta(x_2) = \frac{3}{4} \pi^2 k_2 J_\beta(x_2) \tag{3.2.65}$$

The overall factor of $1/4$ is because there are two contractions in vector indices which we have converted to contractions in spinor indices. These two are different by a factor of -2 as we see in the second entry of 3.2.48. On the right hand side, the factor of $3/2$ is a result of expanding the implied symmetrizations in the lower indices before contracting with the n's with upper indices. The last equality is trivial since the integrand is constant as a function of x_1, so we can just multiply by the surface area of the 3-sphere, $2\pi^2$. Note that all factors of ϵ have canceled out. Then, the Ward identity reads:

$$\frac{3}{4}\pi^2 k_2 J_\beta = \Delta_\psi \psi_\beta(x_2) = \frac{3}{2}\psi_\beta(x_2) \Longrightarrow k_2 = \frac{2}{\pi^2} \tag{3.2.66}$$

With no freedom left, we expect the Lorentz transformations to be identically satisfied, and they are. Again, paying attention to the factor of -2 induced by conversion from vector to spinor contractions, we find that, cf. 3.2.50:

$$\epsilon^3 \int_{S^3_\epsilon} d\Omega n^\mu (x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho})\psi_\beta = -\frac{1}{2}\epsilon^4 \int_{S^3_\epsilon} d\Omega n^{\dot{\alpha}_1\alpha_1} \left(n_{\chi\dot{\chi}} T_{\alpha_1\alpha_2\dot{\alpha}_1\dot{\alpha}_2} - n_{\alpha_2\dot{\alpha}_2} T_{\alpha_1\chi\dot{\alpha}_1\dot{\chi}}\right)\psi_\beta \tag{3.2.67}$$

$$= -\frac{1}{\pi^2}\epsilon^4 \int_{S^3_\epsilon} d\Omega n^{\dot{\alpha}_1\alpha_1} \left(n_{\chi\dot{\chi}}\psi_{\alpha_2}(x_2) n_{\alpha_1\dot{\alpha}_1} n_{\beta\dot{\alpha}_2} - n_{\alpha_2\dot{\alpha}_2}\psi_\chi(x_2) n_{\alpha_1\dot{\alpha}_1} n_{\beta\dot{\chi}}\right) \tag{3.2.68}$$

After expanding out the symmetrizations and performing all contractions, we will need the integral:

$$\int_{S^3_\epsilon} n_{\alpha\dot{\alpha}} n_{\beta\dot{\beta}} = -\pi^2 \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} \tag{3.2.69}$$

where ϵ is the antisymmetric tensor in two indices. Applying this integral and simplifying, we obtain the final result:

$$\epsilon^3 \int_{S^3_\epsilon} d\Omega n^\mu (x_\rho T_{\mu\sigma} - x_\sigma T_{\mu\rho})\psi_\beta = \epsilon_{\chi\beta}\epsilon_{\dot{\alpha}_2\dot{\chi}}\psi_{\alpha_2} + \epsilon_{\alpha_2\beta}\epsilon_{\dot{\alpha}_2\dot{\chi}}\psi_\chi \tag{3.2.70}$$

This is indeed what we want, as a bit of algebra shows that the generator of Lorentz transformations indeed acts this way:

$$i[Q_{\xi_{\sigma\rho}}, \psi_\beta] = (\sigma^\sigma)_{\alpha_2\dot{\alpha}_2}(\sigma^\rho)_{\chi\dot{\chi}} \frac{1}{2}(\sigma_{\sigma\rho})^\lambda_\beta \psi_\lambda = \epsilon_{\chi\beta}\epsilon_{\dot{\alpha}_2\dot{\chi}}\psi_{\alpha_2} + \epsilon_{\alpha_2\beta}\epsilon_{\dot{\alpha}_2\dot{\chi}}\psi_\chi \tag{3.2.71}$$

Although the free fermion example is very simple, all the essential features of the calculation for much larger representations is present. The only difference is that there will be more structures and integrals that involve more copies of n, but these are not serious obstacles for a computer.

3.3 Imposing Hofman-Maldacena bounds

When we carry out our procedure for currents of high chirality, we actually find there are solutions. Conservation of T, J, and the Ward identities is not enough to exclude these operators from the spectrum of a conformal field theory. We therefore seek an additional, independent constraint on the $\langle TJ\bar{J}\rangle$ correlator. The Hofman-Maldacena bounds [19] [20] are one such constraint. In this section, we will review the statement of these bounds and describe how we implement them.

Roughly, the Hofman-Maldacena bounds express the intuitive fact that the expectation value of the energy (suitably defined) in any state should be positive. We will begin by giving some general theoretical background about the bounds, then explain how we extend Hofman and Maldacena's calculation to higher-spin fields. We will find that for certain polarizations of the $(6,1)$ current, the one-point function of the energy vanishes identically. We argue, based on [21], that this strongly suggests that theories containing such a current are free. An interesting observation about this finding is that the vanishing of the eigenvalues actually depends only on the conservation of T and J. The conformal Ward identities are not necessary to see the vanishing of these eigenvalues.

3.3.1 Background

The Hofman-Maldacena bounds are a quantum-mechanical generalization of the classical Null Energy Condition, which states that T_{--} is always positive in classical field theory. In quantum field theory, however, such naive bounds are violated by vacuum fluctuations, and so one can only hope for a "less local" version of the Null Energy Condition. The Hofman-Maldacena bounds are a statement of this form.

First, we are instructed to consider the integrated energy flux per unit angle at infinity in some direction $\hat{n}$:

$$\mathcal{E}(t, r\hat{n}) = \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} n^i T_{0i}(t, r\hat{n}) \tag{3.3.1}$$

If we change to lightcone coordinates $y^{\pm} = t \pm z$ and take $\hat{n} = \hat{z}$, this expression can be rewritten as:

$$\mathcal{E} = \lim_{y^+\to\infty} \left(\frac{y^+ + y^-}{2}\right)^2 \int_{-\infty}^{\infty} dy^- T_{--}(y^+, y^-, \vec{y}) \tag{3.3.2}$$

The simplest version of the bounds posit that the expectation value of $\mathcal{E}$ in any state $|\psi\rangle$ is positive. The physical interpretation of this construction is that one puts a calorimeter that measures T_{--} very far away from the origin and then creates an excitation localized around the origin with some operator. The integral over y^- represents the fact that the calorimeter is collecting energy for the

entire history of the spacetime. We expect the calorimeter to record a non-negative energy.

In the context of this work, we would like to use these bounds to constrain the three-point function $\langle TJ\bar{J}\rangle$ that we have been studying. The natural idea is to consider a state created by J, say, a state of definite energy, and then compute the expectation value of $\mathcal{E}$ in that state. We will demand positivity of this expectation value *for every possible polarization of J*. In the case of the $(6,1)$ current, there are many possible polarizations but only one degree of freedom left after imposing all constraints in the previous section. It therefore seems that there is not enough freedom to satisfy the Hofman-Maldacena bounds. We will find that the way out is that for many of the polarizations, the expectation value of $\mathcal{E}$ actually vanishes. As we will elaborate, this strongly suggests that the theory is free. Before carrying out this calculation, we make two comments:

First, we point out that the Hofman-Maldacena bounds are a particular realization of the Averaged Null Energy Condition [54] [55], which states that one actually does not have to take y^+ to infinity for the expectation value of $\mathcal{E}$ to be positive on any state. That statement is itself a special case of the Quantum Null Energy Condition [59] [60] [61], which pertains to integrals of T_{--} over more general curves. None of these more sophisticated statements will be explored in this work, although they are of considerable interest in other settings.

Second, note that one can bring the calorimeter to future null infinity in any direction, not just $y^+ = t + z$. We only made that choice since we will only study one point functions of $\mathcal{E}$ and so we have the freedom to rotate our coordinates into that direction. In principle if one wanted to study multi-point functions of $\mathcal{E}$, one would need to keep the angles generic.

3.3.2 Implementing the bounds

Calculating the expectation value of $\mathcal{E}$ is a bit subtle because it requires one to compute an out-of-time-order correlation function. As mentioned, we would like to calculate the one point function of $\mathcal{E}$ in a state of definite positive energy $q > 0$ created by our current J, since that is the quantity which is related to the three-point functions we've been trying to constrain. Explicitly, we want to compute $\langle J(-q)\mathcal{E}\bar{J}(q)\rangle$, where the operators must appear in that specific order so that we are actually computing the expectation value of $\mathcal{E}$ in the state $\bar{J}(q)|0\rangle$ and therefore can invoke the Hofman-Maldacena bound. If we expand out the Fourier transforms and rearrange, we find this correlation function is related to the position-space correlators we have computed earlier by:

$$\langle J(-q)\mathcal{E}\bar{J}(q)\rangle = \int dy^- \int d^4x e^{iqx^0} \langle J(x)T_{--}(y^-, y^+ = \infty, \vec{y})J(0)\rangle \tag{3.3.3}$$

In the equation above, we have chosen the momentum to be totally timelike. This is allowable since there is a boost that sends any timelike momentum to a momentum of that form. After having fixed that, we may use the spatial rotations to put $\hat{n} = \hat{z}$ as mentioned earlier. We emphasize these choices are completely general; we do not lose any information about the one-point function by making them.

As mentioned, ensuring that the operators are in that particular order is essential. We enforce the ordering with a particular $i\epsilon$ prescription. The simplest way to see the correct prescription is by starting in Euclidean signature and Wick-rotating into Lorentzian signature. As is well-known, correlation functions in Euclidean signature are "automatically time-ordered" in the sense that (a) the Euclidean path integral automatically gives time-ordered correlation functions and (b) out-of-time-order Euclidean correlation functions don't make sense since they're formally infinite. The second statement can easily be seen from rewriting a general correlator as follows:

$$\langle 0|\mathcal{O}_1(t_1^E, \vec{x}_1)\dots\mathcal{O}_n(t_n^E, \vec{x}_n)|0\rangle = \langle 0|e^{Ht_1^E}\mathcal{O}_1(0, \vec{x}_1)e^{-Ht_1^E}\dots e^{Ht_n^E}\mathcal{O}_n(0, \vec{x}_n)e^{-Ht_n^E}|0\rangle \tag{3.3.4}$$

$$= \langle 0|\mathcal{O}_1(0, \vec{x}_1)e^{H(t_2^E - t_1^E)}\dots e^{H(t_{n-1}^E - t_n^E)}\mathcal{O}_n(0, \vec{x}_n)|0\rangle \tag{3.3.5}$$

If the operators are not in time order, some exponential factor in the middle appears with $t_i^E - t_{i+1}^E > 0$, which means that $e^{H(t_i^E - t_{i+1}^E)}$ is unbounded, and the correlator is formally infinite.

If we start with such a correlation function in time order, however, one can imagine giving each of these Euclidean times an imaginary part proportional to any desired Lorentzian time, $t_j^E \equiv \epsilon_j + it_j^L = i(t_j^L - i\epsilon_j)$. From the Lorentzian point of view, then, as long as $\epsilon_i > \epsilon_j$ for all $i < j$, the expression will be ordered as written regardless of the values of the t_j^L. By taking the $\epsilon_i \to 0$ after computing the correlation function, one obtains the out-of-time-order Lorentzian correlator.

Hence, we should take the expression for the three-point function $\left\langle T_{--}(y)J(x)\bar{J}(0)\right\rangle$ that we've developed and give the time coordinate of y a small negative imaginary part $-i\epsilon$ and the time coordinate of x a larger negative imaginary part $-2i\epsilon$. Then, we will perform the integrals in 3.3.3, and these $i\epsilon$ terms will tell us how to pick the appropriate contour. After integrating, we can set $\epsilon \to 0$ to obtain the desired out-of-time-order Lorentzian correlator.

The next step is to multiply T_{--} by $r^2 = ((y^+ - y^-)/2)^2$ and take $y^+ \to \infty$. When we write out the tensor structures for each correlator, we will find that this limit actually kills off many components of the three-point function which are otherwise nonvanishing at finite separation. At

this point, it is helpful to make some of the kinematical details more explicit. Let's write

$$x^\mu = (x^0 - 2i\epsilon, x^1, x^2, x^3) \tag{3.3.6}$$

$$y^\mu = (y^0 - i\epsilon, y^1, y^2, y^3) \tag{3.3.7}$$

so that the lightcone coordinates are:

$$y^\pm = (y^0 - i\epsilon) \pm y^3 \tag{3.3.8}$$

$$x^\pm = (x^0 - 2i\epsilon) \pm x^3 \tag{3.3.9}$$

The dictionary between vector and spinor indices is:

$$y_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} -y_0 + y_3 & y_1 - iy_2 \\ y_1 + iy_2 & -y_0 - y_3 \end{pmatrix} = \begin{pmatrix} y^0 + y^3 & y^1 - iy^2 \\ y^1 + iy^2 & y^0 - y^3 \end{pmatrix} \tag{3.3.10}$$

It will be helpful to rename our indices since the symmetry that is preserved by the integrations and Fourier transformations are the rotations in the $(1,2)$-plane, i.e. rotations in the plane transverse to the $(+,-)$ plane. A clockwise (positive) rotation by θ in the $(1,2)$ plane leaves $y_{1\dot{1}}$ and $y_{2\dot{2}}$ invariant but rotates $y_{1\dot{2}}$ by $e^{-i\theta}$ and $y_{2\dot{1}}$ by $e^{i\theta}$. So we will give the spinor components the following new names:

$$y_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \begin{pmatrix} y_{-\dot{+}} & y_{-\dot{-}} \\ y_{+\dot{+}} & y_{+\dot{-}} \end{pmatrix} \tag{3.3.11}$$

so, e.g. we have:

$$y_{1\dot{1}} \equiv y_{-\dot{+}} = -y_- = y^+ \tag{3.3.12}$$

$$y_{2\dot{2}} \equiv y_{+\dot{-}} = -y_+ = y^- \tag{3.3.13}$$

$$T_{11\dot{1}\dot{1}} \equiv T_{--\dot{+}\dot{+}} = T_{--} = T^{++} \tag{3.3.14}$$

and so on.

Now, we expand our correlation function $\left\langle T_{--\dot{+}\dot{+}}(y) J_{\beta_1 \ldots \beta_k \dot{\beta}}(x) \bar{J}_{\gamma\dot{\gamma}_1 \ldots \dot{\gamma}_k}(0) \right\rangle$ in our basis of tensor structures, which are expressed in terms of the buliding blocks I_{ij} and J_i. In the frame we're

considering,

$$I_{12} = (y-x)_{\beta\dot{\alpha}} \tag{3.3.15}$$

$$I_{23} = x_{\gamma\dot{\beta}} \tag{3.3.16}$$

$$I_{13} = y_{\gamma\dot{\alpha}} \tag{3.3.17}$$

$$J_{1,23} = \frac{(y-x)^2y^2}{x^2}\left(\frac{(y-x)_{\alpha\dot{\alpha}}}{(y-x)^2} - \frac{y_{\alpha\dot{\alpha}}}{y^2}\right) \tag{3.3.18}$$

$$J_{2,31} = \frac{(y-x)^2x^2}{y^2}\left(\frac{(y-x)_{\beta\dot{\beta}}}{(y-x)^2} + \frac{x_{\beta\dot{\beta}}}{x^2}\right) \tag{3.3.19}$$

$$J_{3,12} = \frac{x^2y^2}{(y-x)^2}\left(\frac{x_{\gamma\dot{\gamma}}}{x^2} - \frac{y_{\gamma\dot{\gamma}}}{y^2}\right) \tag{3.3.20}$$

It is easy to see that in the $y^+ \to \infty$ limit, we will obtain the following simplifications:

$$\lim_{y^+\to\infty}\frac{y_{-\dot{+}}}{y^2} = \lim_{y^+\to\infty}\frac{y^+}{y^2} = -\frac{1}{y^-}, \quad \lim_{y^+\to\infty}\frac{y_{-\dot{+}}}{(y-x)^2} = \lim_{y^+\to\infty}\frac{y^+}{(y-x)^2} = -\frac{1}{(y^- - x^-)} \tag{3.3.21}$$

$$\lim_{y^+\to\infty}\frac{(y-x)_{-\dot{+}}}{y^2} = \lim_{(y-x)^+\to\infty}\frac{y^+}{y^2} = -\frac{1}{y^-}, \quad \lim_{y^+\to\infty}\frac{(y-x)_{-\dot{+}}}{(y-x)^2} = \lim_{y^+\to\infty}\frac{(y-x)^+}{(y-x)^2} = -\frac{1}{(y^- - x^-)} \tag{3.3.22}$$

Note that if the numerator of any of these expressions had different indices, the limit would evaluate to zero. This enables us to make the following simplifications on the J_i:

$$J_{1,23} = -\frac{(y-x)^2y^2}{x^2}\left(\frac{x^-}{y^-(y^- - x^-)}\right) \tag{3.3.23}$$

$$J_{2,31} = \frac{(y-x)^2x^2}{y^2}\left(-\frac{\delta^-_\beta\delta^{\dot{+}}_{\dot{\beta}}}{y^- - x^-} + \frac{x_{\beta\dot{\beta}}}{x^2}\right) \tag{3.3.24}$$

$$J_{3,12} = \frac{x^2y^2}{(y-x)^2}\left(\frac{x_{\gamma\dot{\gamma}}}{x^2} + \frac{\delta^-_\gamma\delta^{\dot{+}}_{\dot{\gamma}}}{y^-}\right) \tag{3.3.25}$$

Note that J_1 always takes that form since it carries $\alpha\dot{\alpha}$ indices, which are fixed to be $-\dot{+}$ in this correlation function.

These equalities together enable one to write explicit expressions for the tensor structures in the $y^+ \to \infty$ limit that can readily be integrated. For instance, consider the tensor structure $\mathcal{T}_2$ for the

$(6,1)$ current 3.2.15. We compute, with the definite indices on T:

$$\lim_{y^+\to\infty}\left(\frac{y^+-y^-}{2}\right)^2 \mathcal{K} I_{12}^2 I_{23} I_{31}^2 I_{32}^4 \tag{3.3.26}$$

$$= \lim_{y^+\to\infty}\left(\frac{y^+-y^-}{2}\right)^2 \frac{1}{(y-x)^6 y^6 x^{12}} (y-x)_{\beta_1\dot{+}}(y-x)_{\beta_2\dot{+}} x_{\gamma\dot{\beta}}(-y)_{-\dot{\gamma}_1}(-y)_{-\dot{\gamma}_2}\prod_{i=3}^{6}(-x)_{\beta_i\dot{\gamma}_i} \tag{3.3.27}$$

$$= \frac{1}{4}\frac{1}{(y^- - x^-)^3 (y^-)^3 x^{12}}\delta^-_{\beta_1}\delta^-_{\beta_2}\delta^{\dot{+}}_{\dot{\gamma}_1}\delta^{\dot{+}}_{\dot{\gamma}_2}\prod_{i=3}^{6} x_{\beta_i\dot{\gamma}_i} \tag{3.3.28}$$

Restoring all the $i\epsilon$'s, one can perform the y^- integration using the residue theorem, and then evaluate the Fourier transform in x. This Fourier transformation can be performed using standard techniques for evaluating one-loop Feynman diagrams for any particular choice of indices. It is typically most convenient to write the Fourier factor $e^{iqx^0} = e^{iqx^+/2}e^{iqx^-/2}$, then integrate over x_1 and x_2 (the directions transverse to the $(+,-)$ plane) by, e.g., Schwinger parameterization[3] or Wick-rotating the corresponding Euclidean integral carefully, and finally evaluate the x^+ and x^- integrals using the residue theorem. In addition, one can infer from the residual $SO(2)$ invariance in the $(1,2)$-plane mentioned earlier that the integrals identically evaluate to zero for certain choices of indices. If the numerators do not have the same number of $-$ and $\dot{-}$ indices as $+$ and $\dot{+}$ indices, the integral must vanish since there are no $SO(2)$ invariants one can write with those indices. So numerators are always products of x^+, x^- and $x_{-\dot{-}}x_{+\dot{+}} = x_1^2 + x_2^2$, any combination of which can be simply handled using textbook methods.

With all the integrals in hand, one now has to impose that the three-point function is positive for any choice of polarization for J. For operators of low spin, the relevant inequalities are easy to work out by inspection, but for higher-spin fields it is convenient to organize the data a bit more systematically. Let λ and λ^* be polarization tensors for J and $\bar{J}$, not necessarily conjugate to each other. Due to current conservation, we may take them to be transverse to the timelike direction. We construct the matrix $M(\lambda,\lambda^*) = \langle \lambda\cdot J|\mathcal{E}|\lambda^*\cdot\bar{J}\rangle$ in a convenient basis of polarization tensors. Then, the Hofman-Maldacena bounds imply that all eigenvalues of this matrix are non-negative.

3.3.3 Results and interpretation

On general grounds, one expects the matrix M for the one-point correlator of $\mathcal{E}$ in a state created by the $(k,1)$ representations to have $k+2$ nonzero eigenvalues. The conserved current J which

[3] cf. appendix G of [62]

transforms in the $(k,1)$ representation has $2(k+1)$ possible polarizations, but k of them are not transverse to the momentum carried by J. We know such polarizations cannot contribute to the correlation function, since the conservation condition $\partial \cdot J$ implies that $p \cdot J = 0$. Our main finding is that one actually has fewer nonzero eigenvalues than anticipated. Starting with the $(2,1)$ representation (the supercurrent), M has only four nonzero eigenvalues for any $k \geq 2$. This is the result of delicate cancellations between various structures induced by the particular relations imposed by conservation of T and J.

To illustrate, consider the current $J_{\beta_1\beta_2\beta_3\dot{\beta}}$ that transforms in the $(3,1)$ representation. We expect five nonzero eigenvalues, but in fact, one is zero. Before imposing any constraints, there are ten structures, and our procedure yields that the polarization that sets all indices of J to carry negative $SO(2)$ charge (i.e. the polarization where all indices are set to $-$ and $\dot{-}$) is an eigenvector with the following eigenvalue:

$$\frac{\left\langle J_{----}\middle|\mathcal{E}\middle|\bar{J}_{+\dot{+}+\dot{+}}\right\rangle}{\left\langle J_{----}\middle|\bar{J}_{+\dot{+}+\dot{+}}\right\rangle} = \pi q\left(-\frac{3}{16}(k_2+k_{10}) - \frac{3}{32}k_7 - \frac{1}{8}k_8\right) \tag{3.3.29}$$

But the OPE computation of the constraints imposed by the conservation of T and the conservation of J imply that $k_7 = \frac{4}{3}(k_1 - k_5)$, $k_8 = 3(k_5 - k_1)$, and $k_{10} = -\frac{2}{3}k_1 - k_2$. Substituting these values back into the eigenvalue equation, we find that the eigenvalue is identically equal to zero. We emphasize here that we actually did not have to use the constraints implied by the conformal Ward identities here. Conservation of T and J was enough.

This effect persists and becomes increasingly dramatic as one moves up in spin, since more and more eigenvalues which are nonzero a priori have to experience such miraculous cancellations all at once. For instance, in the case of the current that transforms in the $(6,1)$ representation, four eigenvalues cancel in this way, with expressions similar to the above but with more complicated rational numbers multiplying all the coefficients. It turns out that this particular "extremal polarization" happens to be a zero eigenvalue of all the $(k,1)$ currents for $3 \leq k \leq 6$. The extra zero eigenvalues as k increases are the "next-to-extremal eigenvalues", where there is one $+$ index on the J, two $+$ indices, and so on.

A proof that this occurs for general k is still being worked out at the time of this writing, but qualitatively, such a result seems inevitable. The tensor structures for a $k > 4$ current are generated from the $k = 4$ structures by multiplying each structure by the appropriate power of I_{32} - i.e. the number of independent conformal structures saturates at ten for the $k = 4$ current, and in some sense all the tensor structures "look the same". Hence, one expects the various constraints to have the

same content, in some sense. This is visually apparent in some ways; for instance, when $3 \leq k \leq 6$, it always happens that conservation requires $k_9 = -k_1$.

When zero eigenvalues are present, we believe the theories are free. This inference is suggested by work by Zhiboedov, who argued in [21] that if the stress tensor takes a certain form, the vanishing of the energy one-point correlator implies that theory is effectively free in the sense that the two-point energy correlator $\langle \mathcal{E}(\hat{n}_1)\mathcal{E}(\hat{n}_2)\rangle \propto \delta(1 + \hat{n}_1 \cdot \hat{n}_2)$ and that all higher-point functions of energy correlators vanish identically. This is precisely what happens in free theories; qualitatively, these expressions physically mean that the stress tensor creates a state of two noninteracting particles, and hence the "S-matrix" that this correlation function suggests is trivial (of course, there is no actual S-matrix in a CFT). Our result therefore strongly suggests that the theory which contains these asymmetric currents are free. At the time of this writing, the details of this implication are still being worked out.

Another possibility our results motivate is that states created by symmetric higher-spin currents might also have vanishing eigenvalues. This would strongly suggest that the arguments of [29] and [28] might be drastically simplified. In those works, the Ward identities generated by the higher-spin currents on each other were used to constrain theories containing symmetric higher-spin currents. If we show that these currents have zero eigenvalues in this context, it would imply a similar result that relies only on the Ward identities that the stress tensor generates. At the time of this writing, we have not yet performed the computations that pertain to this conjecture.

Finally, we cannot yet comment on whether or not the these currents of large chirality actually imply that the theory is actually inconsistent with conformal symmetry. The fact that a zero eigenvalue strongly indicates that the theory is free leads one to believe that such a current cannot exist since there is no free theory that contains such a current, but this is merely intuition. Completing the analysis to determine whether this intuition holds is work in progress.

3.4 Conclusions

Although many tantalizing questions remain to be addressed, we have already demonstrated a number of interesting calculations. First, we found a simple proof that translates the Weinberg-Witten theorem to the setting of conformal field theory. Higher-spin free fields, i.e. operators in the $(k, 0)$ or $(0, k)$ representations that saturate the unitarity bound and therefore satisfy a Dirac equation, do not admit three-point functions with the stress tensor consistent with their equation of motion. Then, we turned our attention to conserved currents, which carry at least one dotted and one undotted

index. For these fields, we imposed the conservation conditions and the conformal Ward identities; this represents not only the effective application of recent results classifying the general structure of three-point functions of generic operators, but also demonstrated that, in opposition to our intuition from the Weinberg-Witten theorem, asymmetric higher-spin currents in conformal field theory can consistently couple to the stress tensor, at least at the level of the conformal Ward identities. Finally, our analysis of energy one-point functions in the context of the Hofman-Maldacena bounds suggest that these asymmetric currents can only live in free theories since certain polarizations which are transverse to the momentum nevertheless induce vanishing one-point energy correlators. Surprisingly, this is implied purely from the conservation conditions, and do not require the conformal Ward identities.

Going forward, it will be interesting to continue pursuing the Hofman-Maldacena calculations in a variety of ways. A more thorough analysis may lead us to exclude the asymmetric higher-spin currents altogether. Examining the Hofman-Maldacena bounds in states created by higher spin symmetric currents may simplify and shed additional physical intuition about existing results about such currents. Furthermore, the manifestations of the Average Null Energy Condition in other contexts provides endless directions for future work. The progress presented in this chapter represents only a promising first step; the techniques we have developed are extremely general and appear to have a lot of power in constraining the space of conformal field theories, and it will therefore be exciting to continue exploring and applying them in the near future.

Chapter 4

On C_T and C_J in the Gross-Neveu and $O(N)$ models

4.1 Introduction and summary

The essential data characterizing a d-dimensional conformal field theory (CFT) includes the scaling dimensions of conformal primary operators and their operator product coefficients [63,64]. In general, the normalizations of operators may be chosen arbitrarily; therefore, the normalizations of their two-point functions are not physical observables. Exceptions to this are provided by the conserved currents: their insertions into correlations functions of other operators are determined by the Ward identitites which fix the normalizations of the currents. Therefore, the coefficients of the two-point functions of conserved currents are physically meaningful. The most commonly encountered ones are C_J, which refers to the conserved spin-1 currents J^a_μ, $a = 1, \dots \dim(G)$, associated with a global symmetry of the theory with group G, and C_T, which refers to the stress-energy tensor $T_{\mu\nu}$ [35]:

$$\langle J^a_\mu(x_1) J^b_\nu(x_2)\rangle = C_J \frac{I_{\mu\nu}(x_{12})}{(x^2_{12})^{d-1}}\delta^{ab}\,, \tag{4.1.1}$$

$$\langle T_{\mu\nu}(x_1) T_{\lambda\rho}(x_2)\rangle = C_T \frac{I_{\mu\nu,\lambda\rho}(x_{12})}{(x^2_{12})^{d}}\,, \tag{4.1.2}$$

where

$$I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}\,,$$
$$I_{\mu\,\nu,\lambda\rho}(x) \equiv \frac{1}{2}(I_{\mu\lambda}(x)I_{\nu\rho}(x) + I_{\mu\rho}(x)I_{\nu\lambda}(x)) - \frac{1}{d}\delta_{\mu\nu}\delta_{\lambda\rho}\,. \tag{4.1.3}$$

These quantities have various applications: C_J determines the universal charge or spin conductivity [65, 66]; C_T appears in many contexts, including some properties of the Rényi and entanglement entropies [67, 68]. For example, C_T determines the leading response of the entanglement entropy across a sphere to small variations in its shape [68]; in particular, in $d = 3$ it determines its limiting behavior for entangling contours with cusps [69]. C_T is also one of the natural measures of the number of degrees of freedom, and in two dimensions it satisfies the famous Zamolodchikov theorem [70]. In higher dimensions there are counter-examples to the monotonicity of C_T [22, 71, 72], but it is still interesting to study its behavior under RG flow.

A number of results about C_J and C_T are available for CFTs in $d > 2$ [65, 66, 72–74]. Of special interest to us is the work by Petkou [74], who used large N methods and operator product expansions to determine the leading $1/N$ corrections to C_J and C_T for the critical scalar $O(N)$ model with quartic interaction $(\phi^i\phi^i)^2$. Defining

$$C_J = C_{J0}\Big(1 + \frac{C_{J1}}{N} + \frac{C_{J2}}{N^2} + \mathcal{O}(1/N^3)\Big)\,,$$
$$C_T = C_{T0}\Big(1 + \frac{C_{T1}}{N} + \frac{C_{T2}}{N^2} + \mathcal{O}(1/N^3)\Big)\,, \tag{4.1.4}$$

Petkou found [74]

$$C_{J1}^{\mathrm{O(N)}} = -\frac{8(d-1)}{d(d-2)}\eta_1^{\mathrm{O(N)}}, \tag{4.1.5}$$
$$C_{T1}^{\mathrm{O(N)}} = -2\left(\frac{2\mathcal{C}_{\mathrm{O(N)}}(d)}{d+2} + \frac{d^2+6d-8}{d\,(d^2-4)}\right)\eta_1^{\mathrm{O(N)}}\,. \tag{4.1.6}$$

Here

$$\eta_1^{\mathrm{O(N)}} = \frac{2\Gamma(d-2)\sin(\pi\frac{d}{2})}{\pi\Gamma(\frac{d}{2}-2)\Gamma(\frac{d}{2}+1)} \tag{4.1.7}$$

is the $1/N$ correction to the dimension of the fundamental scalar field ϕ^i, and

$$\mathcal{C}_{\mathrm{O(N)}}(d) = \psi(3-\frac{d}{2}) + \psi(d-1) - \psi(1) - \psi(\frac{d}{2})\,, \tag{4.1.8}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function. In $d = 3$, these results yield

$$\begin{aligned} C_J^{\mathrm{O(N)}}|_{d=3} &= C_{J0}^{\mathrm{O(N)}} \left(1 - \frac{64}{9\pi^2 N} + \mathcal{O}(1/N^2)\right) , \\ C_T^{\mathrm{O(N)}}|_{d=3} &= C_{T0}^{\mathrm{O(N)}} \left(1 - \frac{40}{9\pi^2 N} + \mathcal{O}(1/N^2)\right) . \end{aligned} \tag{4.1.9}$$

The critical $O(N)$ model with the quartic interaction $(\phi^i\phi^i)^2$ is weakly coupled in $4-\epsilon$ dimensions [75], and the results (4.1.5), (4.1.6) agree with the ϵ expansions found from conventional perturbation theory [72,76]. In recent works [22,23,77] it was shown that, for sufficiently large N, the $O(N)$ model has another weakly coupled description in $6 - \epsilon$ dimensions. It involves an additional scalar field σ with the action

$$\int d^dx \left(\frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}g_1\sigma\phi^i\phi_i + \frac{1}{6}g_2\sigma^3\right) . \tag{4.1.10}$$

In section 4.3 we will use this cubic $O(N)$ symmetric theory to develop the $6 - \epsilon$ expansion of C_J and C_T, providing additional checks of the large N results (4.1.5), (4.1.6). In particular, for $d = 6$ the large N result (4.1.6) yields [22]

$$C_{T1}^{\mathrm{O(N)}}|_{d=6} = 1 , \tag{4.1.11}$$

which precisely reproduces the contribution of a 6d canonical scalar field. More generally, in even dimensions d, generalizing the arguments leading to (4.1.10), we expect to find a (non-unitary) free theory of N canonical scalars ϕ^i and a $\Delta = 2$ scalar with local kinetic term $\sim \sigma(\partial^2)^{\frac{d}{2}-2}\sigma$. For instance, for $d = 8$ this was recently discussed in [78]. Here

$$C_{T1}^{\mathrm{O(N)}}|_{d=8} = -4 . \tag{4.1.12}$$

This implies that the ratio of the C_T of a free 4-derivative scalar to that of a canonical scalar is -4. The value of $C_{T1}^{\mathrm{O(N)}}$ for general even d is given in [79] and in eq. (4.3.54).

In section 4.4 we will derive formulae for C_J and C_T in the d-dimensional Gross-Neveu (GN) model [80], which has the action

$$S_{\mathrm{GN}} = -\int d^dx \left(\bar\psi_i\gamma^\mu\partial_\mu\psi^i + \frac{g}{2}(\bar\psi_i\psi^i)^2\right) . \tag{4.1.13}$$

We will take ψ^i with $i = 1, 2, \ldots \tilde N$ to be a collection of $\tilde N$ Dirac fermions, and we will denote $N = \tilde N \mathrm{Tr}\mathbf{1}$, where $\mathrm{Tr}\mathbf{1}$ is the trace of the identity operator on the vector space on which the Dirac matrices act. Since this factor can be absorbed into the expansion parameter N, one may keep it

arbitrary in intermediate steps of the calculation, and set it to the desired value at the end. For instance, for the case of $\tilde{N}$ 2-component Dirac fermions in $d = 3$, one should take $\mathrm{Tr}\mathbf{1} = 2$, i.e $N = 2\tilde{N}$. In $2 \leq d \leq 4$, it is natural to take ψ^i to be 4-component fermions, i.e. $N = \tilde{N}\mathrm{Tr}\mathbf{1} = 4\tilde{N}$. This allows us to smoothly connect to the GNY model in $d = 4-\epsilon$ described below. The 4-component fermion notation also appears naturally in $d = 3$ in the condensed matter applications of models involving fermions, see for instance [81–85].

The perturbing operator $O(x) = \frac{1}{2}(\bar{\psi}_i\psi^i)^2$ in (4.1.13) has dimension $\Delta = 2(d-1)$ in the free theory. In $d = 2$ the GN model is asymptotically free, while for $d > 2$ it is free in the IR and has an interacting UV fixed point (it is unitary for $2 < d < 4$). For this interacting CFT we will find, after lengthy calculations,[1]

$$C_{J1}^{\mathrm{GN}} = -\frac{8(d-1)}{d(d-2)}\eta_1^{\mathrm{GN}}\,, \tag{4.1.14}$$

$$C_{T1}^{\mathrm{GN}} = -4\eta_1^{\mathrm{GN}}\left(\frac{\mathcal{C}_{\mathrm{GN}}(d)}{d+2} + \frac{(d-2)}{d(d+2)(d-1)}\right)\,, \tag{4.1.15}$$

where

$$\eta_1^{\mathrm{GN}} = \frac{\Gamma(d-1)(d-2)^2}{4\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})^2} \tag{4.1.16}$$

is the $1/N$ correction to the dimension of the fundamental fermion field ψ^i, and

$$\mathcal{C}_{\mathrm{GN}}(d) = \psi(2-\frac{d}{2}) + \psi(d-1) - \psi(1) - \psi(\frac{d}{2})\,. \tag{4.1.17}$$

In $d = 3$, we find

$$\begin{aligned} C_J^{\mathrm{GN}}|_{d=3} &= C_{J0}^{\mathrm{GN}}\left(1 - \frac{64}{9\pi^2 N} + \mathcal{O}(1/N^2)\right)\,, \\ C_T^{\mathrm{GN}}|_{d=3} &= C_{T0}^{\mathrm{GN}}\left(1 + \frac{8}{9\pi^2 N} + \mathcal{O}(1/N^2)\right)\,. \end{aligned} \tag{4.1.18}$$

We will derive these results using a large N diagrammatic approach similar to that used in [65, 66, 89–92] (for a review, see [93]). We will also use the diagrammatic method to rederive the formulae (4.1.5), (4.1.6) for the scalar $O(N)$ model, finding complete agreement with the bootstrap method of [74]; these calculations are presented in section 4.3.3. The diagrammatic approach has also been used to calculate C_{J1} and C_{T1} in 3-dimensional QED [65, 66]. A paper [94], which is a follow-up to the present one, uses the diagrammatic approach to calculate the C_{J1} and C_{T1} in d-dimensional conformal QED and compare the results with the ϵ expansions. An important feature

[1] Besides their intrinsic interest, formulae (4.1.5), (4.1.6), (4.1.14), (4.1.15) may have applications to the higher-spin AdS/CFT dualities which relate the d-dimensional $O(N)$ [11] or Gross-Neveu models [12,13] to Vasiliev theories [86,87] in AdS_{d+1} (for a review, see [88]).

of the diagrammatic approach, which we will uncover, is the necessity of a divergent multiplicative "renormalization" Z_T for the stress-energy tensor (for the conserved current such a renormalization is not needed). Despite this renormalization, the anomalous dimension of the stress-tensor is, of course, exactly zero.

The interacting Gross-Neveu CFT has different perturbative ϵ expansions near 2 and 4 dimensions. In $2+\epsilon$ dimensions, where the theory has a weakly coupled UV fixed point, it involves the original GN formulation (4.1.13) with the quartic interaction. There is an alternate, Gross-Neveu-Yukawa (GNY) formulation of the theory [95, 96] which contains an additional real scalar field σ with a Yukawa coupling to the $\tilde{N}$ Dirac fermions:

$$S_{\rm GNY} = \int d^d x \left(-\bar{\psi}_i (\partial\!\!\!/ + g_1 \sigma)\psi^i + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_2}{24}\sigma^4 \right). \tag{4.1.19}$$

This theory, which may be regarded as the UV completion of the GN model, has a weakly coupled IR fixed point in $d = 4-\epsilon$. Using these tools, we develop the $2+\epsilon$ and $4-\epsilon$ expansions of C_T and C_J for the GN. In the large N limit these expansions agree with (4.1.14) and (4.1.15), providing their important perturbative checks. In particular, we see that for $d = 4$, the large N result (4.1.15) yields

$$C_{T1}^{\rm GN}|_{d=4} = \frac{2}{3}\,, \tag{4.1.20}$$

which precisely reproduces the contribution of a 4d free scalar field.[2] More generally, in even dimensions d, generalizing the arguments leading to (4.1.19), we expect to find a (non-unitary) free theory of $\tilde{N}$ Dirac fermions and a free scalar with $\Delta = 1$ and local kinetic term $\sim \sigma(\partial^2)^{\frac{d}{2}-1}\sigma$. For instance, in $d = 6$ we find

$$C_{T1}^{\rm GN}|_{d=6} = -2\,, \tag{4.1.21}$$

which implies that $C_T = -6/S_6^2$ for the 4-derivative scalar field in $d = 6$ (in units where $C_T = 6/(5S_6^2)$ for the ordinary 2-derivative scalar). The ratio of the C_T of a free $(d-2)$-derivative scalar to that of a canonical scalar is given in all even dimensions in eq. (4.4.28). Interestingly, it is always an integer.

Using the $2+\epsilon$ and $4-\epsilon$ expansions, in section 4.4.5 we carry out two-sided Padé extrapolations and find estimates for C_T and C_J in $d = 3$ for small values of $\tilde{N}$. The values of C_T we find are typically just $1-2\%$ above those for the theory of free fermions. Our estimates suggest that, as the

[2] Recall that in dimension d a free scalar has $C_T^{\rm sc} = \frac{d}{(d-1)S_d^2}$ and a free fermion $C_T^{\rm fer} = {\rm Tr}\mathbf{1}\frac{d}{2S_d^2}$ [35]. In $d = 4$, we then have $C_T^{\rm sc}/(\tilde{N}C_T^{\rm fer}) = \frac{2}{3\tilde{N}}$.

$d = 3$ theory flows from the interacting GN fixed point to the free fermion theory, C_T decreases for all $\tilde{N}$. There is a supersymmetric counter-example to the $d = 3$ "C_T-theorem" [71], but we find that the inequality $C_T^{\rm UV} > C_T^{\rm IR}$ applies both to the GN and the scalar $O(N)$ models in $d = 3$. However, as we discuss in section 4.4.2, for the GN model with large $\tilde{N}$ it is violated for $2 < d \lesssim 2.3$.

4.2 Change of C_J and C_T under double-trace perturbations

In this section we work out the general structure of the change in the C_J and C_T coefficients under RG flows in large N theories, which are induced by double-trace operators O^2. Both the critical scalar and the GN model are of this type, and in later sections we will carry out specific calculations for these models.

Before proceeding, let us introduce some useful notation that we will use in the rest of the chapter. To deal efficiently with the tensor structures in stress-energy tensor and current correlators, it is convenient to introduce an auxiliary null vector z^μ, satisfying

$$z^2 = z^\mu z^\nu \delta_{\mu\nu} = 0 \ . \tag{4.2.1}$$

We work in flat d-dimensional Euclidean space, so such a null vector is complex, but we will never need to specify an explicit form of z^μ. It is convenient to define the stress-energy tensor and current projected onto the auxiliary null vector

$$T(x) \equiv z^\mu z^\nu T_{\mu\nu} \,, \qquad J(x) \equiv z^\mu J_\mu \,. \tag{4.2.2}$$

From (4.1.2), we see that the two-point functions of T and J take the simple form

$$\begin{aligned} \langle T(x)T(0)\rangle &= \frac{4C_T}{(x^2)^d}\frac{x_z^4}{x^4} \,, \\ \langle J^a(x)J^b(0)\rangle &= \delta^{ab}\frac{-2C_J}{(x^2)^{d-1}}\frac{x_z^2}{x^2} \,, \end{aligned} \tag{4.2.3}$$

where we have introduced the notation $x_z \equiv z^\mu x_\mu$. Using the Fourier transform

$$\int \frac{d^dp}{(2\pi)^d}\frac{e^{ipx}}{(p^2)^\alpha} = \frac{\Gamma(\frac{d}{2}-\alpha)}{4^\alpha \pi^{\frac{d}{2}}\Gamma(\alpha)}\frac{1}{(x^2)^{\frac{d}{2}-\alpha}}, \tag{4.2.4}$$

$$\int d^dx \frac{e^{-ipx}}{(x^2)^\alpha} = \frac{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2}-\alpha)}{4^\alpha\Gamma(\alpha)}\frac{1}{(p^2)^{\frac{d}{2}-\alpha}} \,, \tag{4.2.5}$$

we find in momentum space

$$\begin{aligned}\langle T_{\mu\nu}(p)T_{\lambda\rho}(-p)\rangle &= C_T\frac{\pi^{\frac{d}{2}}\Gamma(1-\frac{d}{2})}{2^{d-2}\Gamma(d+2)}\,(p^2)^{\frac{d}{2}}\tilde{I}_{\mu\nu,\lambda\rho}(p)\,,\\ \langle J^a_\mu(p)J^b_\nu(-p)\rangle &= -C_J\frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-3}\Gamma(d)}\,(p^2)^{\frac{d}{2}-1}\Pi_{\mu\nu}(p)\delta^{ab}\,,\end{aligned} \tag{4.2.6}$$

where $\Pi_{\mu\nu}(p)=\delta_{\mu\nu}-p_\mu p_\nu/p^2$ and

$$\tilde{I}_{\mu\nu,\lambda\rho}(p)\equiv\frac{1}{2}\Pi_{\mu\nu}(p)\Pi_{\lambda\rho}(p)-\frac{d-1}{4}\big(\Pi_{\mu\lambda}(p)\Pi_{\nu\rho}(p)+\Pi_{\mu\rho}(p)\Pi_{\nu\lambda}(p)\big)\ . \tag{4.2.7}$$

Therefore,

$$\begin{aligned}\langle T(p)T(-p)\rangle &= C_T\frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-2}\Gamma(d+2)}\,\frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,,\\ \langle J^a(p)J^b(-p)\rangle &= C_J\frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-3}\Gamma(d)}\,\frac{p_z^2}{(p^2)^{2-\frac{d}{2}}}\delta^{ab}\ ,\end{aligned} \tag{4.2.8}$$

where $p_z\equiv z^\mu p_\mu$.

Let us consider a general CFT_0 in d Euclidean dimensions, and assume that it admits a large N expansion with the usual properties. Given a single trace operator $O(x)$ of dimension Δ_O in the spectrum of the CFT, we can consider the double-trace deformation

$$S_\lambda=S_{\mathrm{CFT}_0}+\lambda\int d^dx O(x)^2\,. \tag{4.2.9}$$

When $\Delta_O<d/2$, the deformation is relevant and there is a RG flow from CFT_0 to a new CFT where $\Delta_O^{\mathrm{IR}}=d-\Delta_O+\mathcal{O}(1/N)$ [97, 98]. When $\Delta_O>d/2$, the deformation is irrelevant, but one may show that there is a large N UV fixed point, where $\Delta_O^{\mathrm{UV}}=d-\Delta_O+\mathcal{O}(1/N)$, and the RG flow leads to CFT_0 in the IR. A well-known example of the IR fixed point is the scalar $O(N)$ model, i.e. the theory of N massless scalar fields ϕ^i perturbed by the $(\phi^i\phi^i)^2$ operator; we will discuss the calculation of C_T in this theory in section 4.3. A well-known example of the UV fixed point is the Gross-Neveu model (4.1.13); it will be discussed in section 4.4. To be definite when writing powers of N, we will assume below that the unperturbed CFT_0 is vector-like, i.e. $C_O\sim N$ and $\langle TT\rangle_0\sim N$.

The $1/N$ expansion in the perturbed CFT may be developed with the aid of a Hubbard-Stratonovich auxiliary field. We may rewrite the perturbed action as

$$S_\lambda=S_{\mathrm{CFT}_0}+\int d^dx\sigma O-\frac{1}{4\lambda}\int d^dx\sigma^2\,. \tag{4.2.10}$$

The equation of motion of σ imposes $\sigma = 2\lambda O$ and leads to the original action. However, by performing the path integral in the CFT_0, one may derive an effective action for σ. At large N, we have

$$\langle e^{-\int d^d x \sigma O} \rangle_0 \approx e^{\int d^d x d^d y \ \frac{1}{2}\sigma(x)\sigma(y)\langle O(x)O(y)\rangle_0 + \mathcal{O}(\sigma^3)} , \tag{4.2.11}$$

so the quadratic term in the σ effective action is

$$\begin{aligned} S[\sigma] &= -\frac{1}{2}\int d^d x d^d y \ \sigma(x)\sigma(y)\langle O(x)O(y)\rangle_0 - \frac{1}{4\lambda}\int d^d x \sigma^2 & (4.2.12) \\ &= -\frac{1}{2}\int \frac{d^d p}{(2\pi)^d}\sigma(p)\sigma(-p)\left(C_O \frac{(4\pi)^{d/2}\Gamma\left(d/2-\Delta_O\right)}{4^{\Delta_O}\Gamma\left(\Delta_O\right)}(p^2)^{\Delta_O - d/2} + \frac{1}{2\lambda}\right) , & (4.2.13) \end{aligned}$$

where we have used

$$\langle O(x)O(y)\rangle_0 = \frac{C_O}{|x-y|^{2\Delta_O}} = C_O \frac{(4\pi)^{d/2}\Gamma\left(d/2-\Delta_O\right)}{4^{\Delta_O}\Gamma\left(\Delta_O\right)}\int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)}(p^2)^{\Delta_O - d/2} . \tag{4.2.14}$$

When $\Delta_O < d/2$, we see that the second term in (4.2.13) can be dropped in the IR limit (and when $\Delta_O > d/2$, it can be dropped in the UV limit), and so at the perturbed fixed point we get the two-point function of σ, at leading order in $1/N$, to be

$$G_\sigma(p) = \langle \sigma(p)\sigma(-p)\rangle = -\frac{4^{\Delta_O}\Gamma\left(\Delta_O\right)}{C_O(4\pi)^{d/2}\Gamma\left(d/2-\Delta_O\right)}(p^2)^{d/2-\Delta_O} \equiv \tilde{C}_\sigma (p^2)^{d/2-\Delta_O} \tag{4.2.15}$$

or, in coordinate space,

$$G_\sigma(x,y) = \frac{(d/2-\Delta_O)\sin\left((d/2-\Delta_O)\pi\right)\Gamma\left(d-\Delta_O\right)\Gamma\left(\Delta_O\right)}{\pi^{d+1}C_O|x-y|^{2(d-\Delta_O)}} \equiv \frac{C_\sigma}{|x-y|^{2(d-\Delta_O)}} . \tag{4.2.16}$$

This shows that the scalar operator $\sigma \sim O$ now has dimension $d - \Delta_O + \mathcal{O}(1/N)$. At the perturbed fixed point, we may hence omit the last term in (4.2.10) and work with the action

$$S_{\mathrm{crit}} = S_{\mathrm{CFT}_0} + \int d^d x \sigma O . \tag{4.2.17}$$

A $1/N$ diagrammatic expansion can be obtained using this action and the effective σ propagator (4.2.16) (with the prescription that the planar bubble diagrams contributing to $\langle\sigma\sigma\rangle$ should not be included as they are already taken into account by the effective propagator).

The two-point function of the stress-energy tensor may be then computed as

$$\begin{aligned}
&\langle T(x)T(0)\rangle_{\text{crit}} = \int D\sigma \langle T(x)T(0)e^{-\int \sigma O}\rangle_0 \\
&= \langle T(x)T(0)\rangle_0 + \frac{1}{2}\int d^d z_1 d^d z_2 G_\sigma(z_1,z_2)\langle T(x)T(0)O(z_1)O(z_2)\rangle_0 \\
&+\frac{1}{2}\int d^d z_1 d^d z_2 d^d z_3 d^d z_4 G_\sigma(z_1,z_3)G_\sigma(z_2,z_4)\langle T(x)O(z_1)O(z_2)\rangle_0\langle T(0)O(z_3)O(z_4)\rangle_0 + \mathcal{O}\left(1/N\right) ,
\end{aligned} \tag{4.2.18}$$

where to obtain the "Aslamazov-Larkin term" [99] in the last line we have used the large N approximation to rewrite the 6-point function as a product of 3-point functions. Note that since $C_O \sim N$, both of the contributions above are of order N^0. By conformal invariance, we may write

$$\begin{aligned}
&\frac{1}{2}\int d^d z_1 d^d z_2 G_\sigma(z_1,z_2)\langle T(x)T(0)O(z_1)O(z_2)\rangle_0 = I_{\langle TTOO\rangle}\frac{(x_z)^4}{(x^2)^{d+2}} , \\
&\frac{1}{2}\int d^d z_1 \cdots d^d z_4 G_\sigma(z_1,z_3)G_\sigma(z_2,z_4)\langle T(x)O(z_1)O(z_2)\rangle_0\langle T(0)O(z_3)O(z_4)\rangle_0 = I_{\langle TOO\rangle^2}\frac{(x_z)^4}{(x^2)^{d+2}}
\end{aligned} \tag{4.2.19}$$

and so

$$\langle T(x)T(0)\rangle_{\text{crit}} = \left(4C_{T0} + I_{\langle TTOO\rangle} + I_{\langle TOO\rangle^2} + \mathcal{O}(1/N)\right)\frac{(x_z)^4}{(x^2)^{d+2}} . \tag{4.2.20}$$

Thus, we see that the change in C_T to leading order in $1/N$ receives contributions from both integrated 4-point and 3-point functions in the unperturbed CFT. While $\langle TOO\rangle$ has a universal form that only depends on Δ_O due to the conformal Ward identity, the 4-point function $\langle TTOO\rangle$ does not have a universal form. Therefore, unlike the sphere free energy [98,100], we do not expect a simple universal formula for the change in C_T that only depends on the dimension of the perturbing operator.

So far we have ignored the issues of regularization, but in fact the result (4.2.20) by itself is not well-defined, since the contributions $I_{\langle TTOO\rangle}$ and $I_{\langle TOO\rangle^2}$ are divergent and require regularization. The usual dimensional continuation does not work in this case, because the vertex in (4.2.17) is critical for all d within the $1/N$ expansion. One may use a simple momentum cutoff, however this makes the integrals hard to compute in general d. A regulator that is often employed, and which we will use in this chapter, is to formally shift the dimension of σ by a small parameter Δ that is taken to zero at the end of the calculation [89–91, 101]. Explicitly, we take the propagator in the regularized theory to be

$$G_\sigma(p) = \tilde{C}_\sigma (p^2)^{d/2-\Delta_O-\Delta} , \qquad \Delta \to 0 . \tag{4.2.21}$$

This makes the vertex dimensionful, $S_{\text{vertex}} = \mu^{\Delta} \int \sigma O$, where we introduced an arbitrary renormalization scale μ to compensate dimensions. Then, the integrals (4.2.19) in the regularized theory take the form

$$\begin{aligned} I_{\langle TTOO\rangle} &= \left(x^2\mu^2\right)^{\Delta} \left(\frac{1}{\Delta} I^{(1)}_{\langle TTOO\rangle} + I^{(0)}_{\langle TTOO\rangle} + \mathcal{O}(\Delta)\right) , \\ I_{\langle TOO\rangle^2} &= \left(x^2\mu^2\right)^{2\Delta} \left(\frac{1}{\Delta} I^{(1)}_{\langle TOO\rangle^2} + I^{(0)}_{\langle TOO\rangle^2} + \mathcal{O}(\Delta)\right) . \end{aligned} \tag{4.2.22}$$

Importantly, we see that the two contributions carry a different power of the renormalization scale, since they involve two and four vertices respectively. Then, we find

$$\begin{aligned} I_{\langle TTOO\rangle} + I_{\langle TOO\rangle^2} =& \frac{1}{\Delta} \left(I^{(1)}_{\langle TTOO\rangle} + I^{(1)}_{\langle TOO\rangle^2}\right) \\ &+ \log(\mu^2 x^2) \left(I^{(1)}_{\langle TTOO\rangle} + 2I^{(1)}_{\langle TOO\rangle^2}\right) + I^{(0)}_{\langle TTOO\rangle} + I^{(0)}_{\langle TOO\rangle^2} + \mathcal{O}(\Delta) . \end{aligned} \tag{4.2.23}$$

Absence of an anomalous dimension for T requires $I^{(1)}_{\langle TTOO\rangle} + 2I^{(1)}_{\langle TOO\rangle^2} = 0$, so that the logarithmic term vanishes. We will see in the explicit examples below that this is indeed the case, as expected. However, we see that the $1/\Delta$ pole cannot cancel by itself, since it involves a different combination of the coefficients (unless both contributions are finite by themselves, but in all examples we studied, this does not appear to be the case). A resolution of this issue is to allow for a divergent "Z-factor" renormalization of the stress tensor so that the poles are cancelled

$$T^{\text{ren}}(x) = Z_T \; T(x) , \qquad Z_T = 1 + \frac{1}{N} \left(\frac{Z_{T1}}{\Delta} + Z'_{T1} + \mathcal{O}(\Delta)\right) + \mathcal{O}(1/N^2) . \tag{4.2.24}$$

The pole coefficient Z_{T1} is fixed by cancellation of the $1/\Delta$ divergence in (4.2.23). In addition, we will find that a non-trivial finite shift Z'_{T1} is required in order for the conformal Ward identity to hold. This peculiar stress tensor "renormalization" is presumably due to the unusual features of the regularized $1/N$ perturbation theory, at least within the regularization scheme we employ. Putting everything together, one arrives at the following final answer for the shift in C_T to leading order at large N (recall that $C_{TO} \sim N$):

$$C_T = C_{T0} + \frac{1}{4} \left(I^{(0)}_{\langle TTOO\rangle} + I^{(0)}_{\langle TOO\rangle^2} + \frac{8}{N} C_{T0} Z'_{T1}\right) + \mathcal{O}(1/N) . \tag{4.2.25}$$

As we will see below, the shift proportional to Z'_{T1} is essential for reproducing the result of [102] for the scalar $O(N)$ model, and also for matching the $4-\epsilon$ and $2+\epsilon$ expansions for the GN model.

One may study in a similar way the current two point function $\langle JJ\rangle$. Assuming for simplicity that the perturbing operator is neutral under the symmetry generated by J, following analogous

steps as above, one ends up with

$$\begin{aligned}\langle J^a(x)J^b(0)\rangle_{\text{crit}} &= \int D\sigma \langle J^a(x)J^b(0)e^{-\int \sigma O}\rangle_0 \\ &= \langle J^a(x)J^b(0)\rangle_0 + \frac{\mu^{2\Delta}}{2}\int d^dz_1 d^dz_2 G_\sigma(z_1,z_2)\langle J^a(x)J^b(0)O(z_1)O(z_2)\rangle_0 + \mathcal{O}(1/N)\end{aligned} \tag{4.2.26}$$

This yields

$$\langle J^a(x)J^b(0)\rangle_{\text{crit}} = \delta^{ab}\left(-2C_{J0} + (x^2\mu^2)^\Delta\left(\frac{1}{\Delta}I^{(1)}_{\langle JJOO\rangle} + I^{(0)}_{\langle JJOO\rangle} + \mathcal{O}(\Delta)\right)\right)\frac{(x_z)^2}{(x^2)^d}\,. \tag{4.2.27}$$

In this case, since the only contribution is given by the integrated 4-point function, the absence of the anomalous dimension of J requires that $I^{(1)}_{\langle JJOO\rangle} = 0$. Therefore, no "$Z$-factor" is needed, at least to this order in the $1/N$ expansion (examining the Ward identities for J, we will find that a finite shift analogous to the one in (4.2.24) is not needed either).[3] Then, the final result is

$$C_J = C_{J0} - \frac{1}{2}I^{(0)}_{\langle JJOO\rangle} + \mathcal{O}(1/N)\,. \tag{4.2.28}$$

4.3 Scalar $O(N)$ model

4.3.1 Scalar with cubic interaction in $6-\epsilon$ dimensions

In this section, we will consider a theory of N scalar fields ϕ^i transforming under an internal $O(N)$ symmetry group and a scalar σ in $6-\epsilon$ dimensions described by the action (4.1.10). Dimensional analysis implies that the interactions are relevant for $d<6$, so we expect that there should exist a nontrivial infrared fixed point. We are interested in the case where $d=6-\epsilon$. For small ϵ and sufficiently large N, this fixed point indeed exists, and the coupling constants at that fixed point have been computed to ϵ^3 order by [22, 23, 77]. The answer they obtained at leading ϵ-order was:

$$g_{1\star} = \sqrt{\frac{6\epsilon(4\pi)^3}{(N-44)\zeta(N)^2+1}}\zeta(N), \qquad g_{2\star} = \sqrt{\frac{6\epsilon(4\pi)^3}{(N-44)\zeta(N)^2+1}}(1+6\zeta(N))\,, \tag{4.3.1}$$

where $\zeta(N)$ is the solution to the cubic equation

$$840\zeta^3 - (N-464)\zeta^2 + 84\zeta + 5 = 0\,, \tag{4.3.2}$$

[3] One may study a different model where double-trace perturbations include the product OO^* of an operator that is charged under the symmetry associated to J and its conjugate. In this case, an Aslamazov-Larkin contribution will be present, and one will need a "Z_J-factor" analogous to the Z_T discussed above.

which asymptotically tends to $\zeta = N/(840)+\ldots$ at large N.[4] Such a solution exists for $N > 1038$ [22].

The solution for the fixed point couplings (4.3.1) is valid for finite N, but its explicit form is somewhat cumbersome. Expanding in powers of $1/N$, one gets:

$$g_{1\star} = \sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{22}{N} + \frac{726}{N^2} + \ldots\right), \tag{4.3.3}$$

$$g_{2\star} = 6\sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{162}{N} + \frac{68766}{N^2} + \ldots\right). \tag{4.3.4}$$

Our goal is to compute the two-point function of the stress-energy tensor and of a conserved spin-1 current at order ϵ, and in particular compare with the large N results (4.1.5), (4.1.6) obtained in [102].

The spin-1 current corresponding to the global $O(N)$ symmetry of the model is given by

$$J^a_\mu(x) = \phi^i t^a_{ij} \partial_\mu \phi^j \,. \tag{4.3.5}$$

Here, the matrices t^a are the generators of the internal $O(N)$ symmetry group. Since the two point function of this current is proportional to δ_{ab}, we may as well pick a convenient generator. We will choose:

$$J(x) = z^\mu J_\mu(x) = z^\mu(\phi^1 \partial_\mu \phi^2 - \phi^2 \partial_\mu \phi^1)\,. \tag{4.3.6}$$

To the first non-trivial order in the ϵ-expansion, we find

$$\langle J(p) J(-p)\rangle = D_0 + D_1 + D_2 + \mathcal{O}(\epsilon^2)\,, \tag{4.3.7}$$

where the necessary diagrams are shown in Fig. 4.1. The solid lines here denote the ϕ propagators,

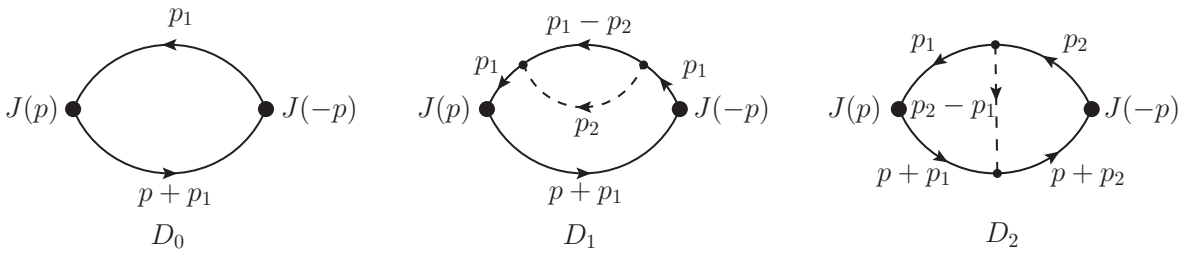


Figure 4.1: Diagrams for C_J up to order ϵ.

the dotted line the σ propagators, and the arrows here simply denote the flow of momentum. The

[4] The other roots correspond to fixed points with unstable directions (in the RG sense) that are not related to the $O(N)$ theory with $(\phi^i\phi^i)^2$ interaction.

explicit integrands for D_0, D_1, D_2 and the result of the integrations are given in Appendix E. After Fourier transforming to position space and dividing by the free field contribution D_0, [5] we obtain the result

$$\frac{C_J^{\mathrm{O(N)}}}{C_{J,\mathrm{free}}^{\mathrm{O(N)}}} = 1 + \frac{D_1 + D_2}{D_0} = 1 + \left(-\frac{5}{1152\pi^3} + \mathcal{O}(\epsilon) \right) g_{1\star}^2 = 1 + \epsilon \left(-\frac{5}{3N} - \frac{220}{3N^2} + \mathcal{O}\left(\frac{1}{N^3} \right) \right) + \mathcal{O}(\epsilon^2) \,, \tag{4.3.8}$$

where in the second step we have substituted the large N expansion (4.3.3) of the critical coupling. One may check that this precisely agrees with the $6 - \epsilon$ expansion (4.3.46) of the large N result (4.1.5) obtained in [102].

Let us now move to the calculation of C_T. The stress-energy tensor may be split into its ϕ and σ contributions, $T = z^\mu z^\nu T_{\mu\nu} = T_\phi + T_\sigma$, where

$$\begin{aligned} T_\phi =& z^\mu z^\nu \Big(\partial_\mu \phi^i \partial_\nu \phi^i - \frac{1}{4} \frac{d-2}{d-1} \partial_\mu \partial_\nu (\phi^i \phi^i) \Big) \,, \\ T_\sigma =& z^\mu z^\nu \Big(\partial_\mu \sigma \partial_\nu \sigma - \frac{1}{4} \frac{d-2}{d-1} \partial_\mu \partial_\nu (\sigma^2) \Big) \,. \end{aligned} \tag{4.3.9}$$

Here we have dropped terms proportional to $\delta_{\mu\nu}$ (including terms involving the interactions), since we work with the projected stress tensor along the null vector z^μ.

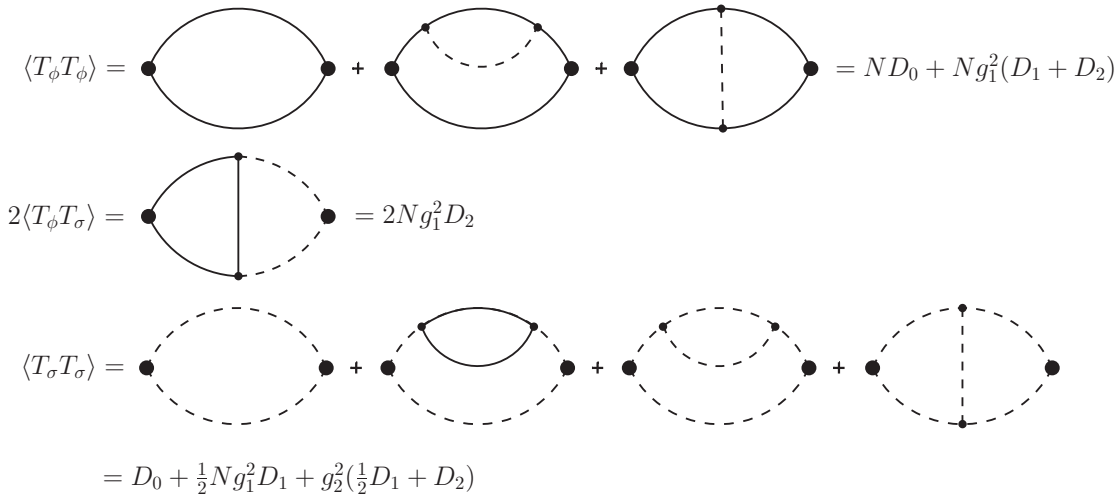


Figure 4.2: Diagrams for C_T up to order ϵ.

We may write $\langle T(p)T(-p)\rangle = \langle T_\phi(p)T_\phi(-p)\rangle + \langle T_\sigma(p)T_\sigma(-p)\rangle + 2\langle T_\phi(p)T_\sigma(-p)\rangle$, and the dia-

[5] It is important to divide by D_0 and take the $\epsilon \to 0$ *after* performing the Fourier transform. This is because the leading order behavior of the Γ functions arising from the Fourier transform (which are regularized by expanding in $d = 6 - \epsilon$) are proportional to $\epsilon/2$ for the second-order diagrams D_1 and D_2, but to ϵ for the one-loop diagram D_0. Effectively, this results in an "enhancement" of D_1 and D_2 by a factor of 2 relative to D_0.

grams contributing to each term are shown in figure 4.2. The explicit integrands and results are given in Appendix E. Putting everything together, the final result is:

$$\begin{aligned}\frac{C_T^{\mathrm{O(N)}}}{C_{T,\mathrm{free}}^{\mathrm{O(N)}}} &= 1+\frac{1}{N}+\left(\frac{\frac{1}{2}D_1+D_2}{ND_0}\right)(3Ng_{1\star}^2+g_{2\star}^2)\\ &= 1+\frac{1}{N}+\left(-\frac{7}{4608\pi^3}+\mathcal{O}(\epsilon)\right)\frac{3Ng_{1\star}^2+g_{2\star}^2}{N}\\ &= 1+\frac{1}{N}+\epsilon\left(-\frac{7}{4N}-\frac{98}{N^2}-\frac{10192}{N^3}+\mathcal{O}\left(\frac{1}{N^4}\right)\right)+\mathcal{O}(\epsilon^2)\,. \end{aligned} \tag{4.3.10}$$

Again, we find that this agrees with the $6-\epsilon$ expansion (4.3.53) of Petkou's result (4.1.6).

4.3.2 $1/N$ expansion

The $1/N$ expansion of the $O(N)$ model can be developed using the Hubbard-Stratonovich transformation, as reviewed in section 4.2. After introducing the Hubbard-Stratonovich auxiliary field and dropping the term quadratic in σ in the IR limit, we effectively have the following action, expressed in terms of bare fields:

$$S_{\text{crit scal}} = \frac{1}{2}\int d^dx\Big((\partial\phi_0^i)^2+\frac{1}{\sqrt{N}}\sigma_0\phi_0^i\phi_0^i\Big)\,. \tag{4.3.11}$$

The propagator of the ϕ_0^i field reads

$$\langle\phi_0^i(p)\phi_0^j(-p)\rangle_0 = \delta^{ij}/p^2. \tag{4.3.12}$$

After integrating over the fundamental fields ϕ_0^i, the auxiliary field σ_0 develops a non-local kinetic term with an effective propagator

$$\langle\sigma_0(p)\sigma_0(-p)\rangle_0 = \tilde{C}_{\sigma 0}/(p^2)^{\frac{d}{2}-2+\Delta}\,, \tag{4.3.13}$$

where

$$\tilde{C}_{\sigma 0} \equiv 2^{d+1}(4\pi)^{\frac{d-3}{2}}\Gamma\Big(\frac{d-1}{2}\Big)\sin\big(\frac{\pi d}{2}\big)\,, \tag{4.3.14}$$

and we have already introduced a regulator Δ [89–91,101], as described in section 4.2. This regulator essentially works analogously to ϵ in dimensional regularization, but there are some subtleties, which we will discuss in this section.

In order to cancel the divergences as $\Delta \to 0$ we have to renormalize the bare fields ϕ_0 and σ_0:

$$\phi = Z_\phi^{1/2}\phi_0, \quad \sigma = Z_\sigma^{1/2}\sigma_0\,, \tag{4.3.15}$$

where Z_ϕ and Z_σ have only poles in Δ (using a "minimal subtraction" scheme), and read

$$Z_\phi = 1 + \frac{1}{N}\frac{Z_{\phi 1}}{\Delta} + \mathcal{O}(1/N^2), \qquad Z_\sigma = 1 + \frac{1}{N}\frac{Z_{\sigma 1}}{\Delta} + \mathcal{O}(1/N^2)\,. \tag{4.3.16}$$

The full propagators of the renormalized fields in momentum space read

$$\langle \phi^i(p)\phi^j(-p)\rangle = \delta^{ij}\frac{\tilde{C}_\phi}{(p^2)^{\frac{d}{2}-\Delta_\phi}}, \qquad \langle \sigma(p)\sigma(-p)\rangle = \frac{\tilde{C}_\sigma}{(p^2)^{\frac{d}{2}-\Delta_\sigma}}\,, \tag{4.3.17}$$

where we introduced anomalous dimensions Δ_ϕ and Δ_σ and two point constants $\tilde{C}_\phi$ and $\tilde{C}_\sigma$ in the momentum space. All of them can be represented as series in $1/N$:

$$\Delta_\phi = \frac{d}{2} - 1 + \eta^{\mathrm{O(N)}}\,, \qquad \Delta_\sigma = 2 - \eta^{\mathrm{O(N)}} - \kappa^{\mathrm{O(N)}}\,, \tag{4.3.18}$$

where $\eta^{\mathrm{O(N)}} = \eta_1^{\mathrm{O(N)}}/N + \eta_2^{\mathrm{O(N)}}/N^2 + \mathcal{O}(1/N^3)$, $\kappa^{\mathrm{O(N)}} = \kappa_1^{\mathrm{O(N)}}/N + \kappa_2^{\mathrm{O(N)}}/N^2 + \mathcal{O}(1/N^3)$ and

$$\tilde{C}_\phi = 1 + \frac{\tilde{C}_{\phi 1}}{N} + \frac{\tilde{C}_{\phi 2}}{N^2} + \mathcal{O}(1/N^3)\,, \qquad \tilde{C}_\sigma = \tilde{C}_{\sigma 0} + \frac{\tilde{C}_{\sigma 1}}{N} + \frac{\tilde{C}_{\sigma 2}}{N^2} + \mathcal{O}(1/N^3)\,. \tag{4.3.19}$$

Recalling that we may drop all terms proportional to $\delta_{\mu\nu}$ since z^μ is null, the stress-energy tensor and the $O(N)$ current are:

$$\begin{aligned} T(x) &= z^\mu z^\nu \left(\partial_\mu\phi_0^i\partial_\nu\phi_0^i - \frac{1}{4}\frac{d-2}{d-1}\partial_\mu\partial_\nu(\phi_0^i\phi_0^i)\right)\,, \\ J^a(x) &= z^\mu\phi_0^i(t^a)^{ij}\partial_\mu\phi_0^j\,. \end{aligned} \tag{4.3.20}$$

In momentum space:

$$\begin{aligned} T(p) &= \frac{1}{2}\int\frac{d^dp_1}{(2\pi)^d}(2p_{1z}(p_{1z}+p_z) + cp_z^2)\phi_0^i(p+p_1)\phi_0^i(-p_1)\,, \\ J^a(p) &= \frac{1}{2}\int\frac{d^dp_1}{(2\pi)^d}i(2p_{1z}+p_z)\phi_0^i(-p_1)(t^a)^{ij}\phi_0^j(p+p_1)\,, \end{aligned} \tag{4.3.21}$$

where $c \equiv \frac{d-2}{2(d-1)}$.

For the Ward identity calculation performed below, we will first need to find $\tilde{C}_{\phi 1}$, η_1 and $Z_{\phi 1}$.

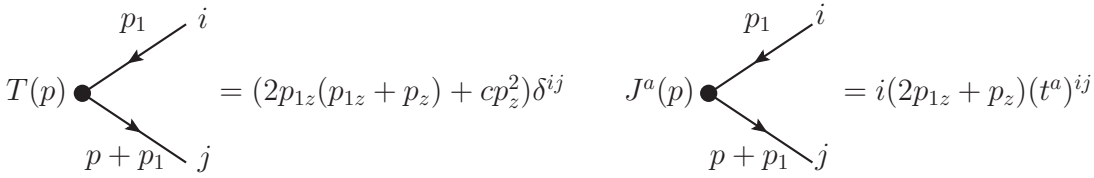


Figure 4.3: Momentum space Feynman rules for $T(p)$ and $J^a(p)$.

To compute them we have to consider the one loop diagram for the renormalization of the $\langle\phi\phi\rangle$-propagator, see figure 4.4.

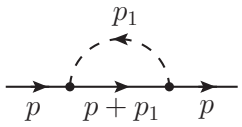


Figure 4.4: One loop correction to the $\langle\phi^i(p)\phi^j(-p)\rangle$ propagator.

Computing this diagram, we find the result (4.1.7), and

$$\tilde{C}_{\phi 1} = -\frac{1}{2}(3d^2-12d+8)\frac{\sin(\frac{\pi d}{2})\Gamma(d-2)}{\pi\Gamma(\frac{d}{2}+1)^2}\,. \tag{4.3.22}$$

As discussed in section 4.2, in order to cancel $1/\Delta$ poles in correlation functions involving T and J, one may introduce "Z_T" and "Z_J" factors as

$$T^{\rm ren}_{\mu\nu} = Z_T T_{\mu\nu}, \quad J^{{\rm ren},a}_{\mu} = Z_J J^a_\mu\,, \tag{4.3.23}$$

which admit the following decomposition:

$$Z_T = 1+\frac{1}{N}\Big(\frac{Z_{T1}}{\Delta}+Z'_{T1}\Big)+\mathcal{O}(1/N^2), \quad Z_J = 1+\frac{1}{N}\Big(\frac{Z_{J1}}{\Delta}+Z'_{J1}\Big)+\mathcal{O}(1/N^2)\,. \tag{4.3.24}$$

The explicit form of these factors can be obtained from Ward identities. Let us consider Z_T first. For this, we can examine the three point function $\langle T^{\rm ren}_{\mu\nu}\phi^i\phi^j\rangle$. Its structure is fixed by conformal symmetry and current conservation to be [35]

$$\langle T^{\rm ren}_{\mu\nu}(x_1)\phi^i(x_2)\phi^j(x_3)\rangle = \frac{-C_{T\phi\phi}}{(x_{12}^2x_{13}^2)^{\frac{d}{2}-1}(x_{23}^2)^{\Delta_\phi-\frac{d}{2}+1}}\Big((X_{23})_\mu(X_{23})_\nu-\frac{1}{d}\delta_{\mu\nu}(X_{23})^2\Big)\delta^{ij}\,, \tag{4.3.25}$$

where

$$(X_{23})_\nu = \frac{(x_{12})_\nu}{x_{12}^2} - \frac{(x_{13})_\nu}{x_{13}^2}\,. \tag{4.3.26}$$

The structure constant $C_{T\phi\phi}$ is not arbitrary and is related to C_ϕ by the Ward identity. To show this, we note that for the infinitesimal scaling transformation $\varepsilon_\nu = \varepsilon x_\nu$:

$$\langle \delta_\varepsilon \phi^i(x_2)\phi^j(x_3)\rangle = -\varepsilon \int d^d\Omega\, r^{d-2} r_\mu r_\nu \langle T^{\rm ren}_{\mu\nu}(x_1)\phi^i(x_2)\phi^j(x_3)\rangle, \tag{4.3.27}$$

where $r = |x_1 - x_2|$ and $\delta_\varepsilon\phi^i(x) = \varepsilon(\Delta_\phi + x_\mu\partial_\mu)\phi^i(x)$. Perfoming the integral in the limit $r \to 0$ we find

$$C_{T\phi\phi} = \frac{1}{S_d}\frac{d\Delta_\phi}{d-1} C_\phi\,, \tag{4.3.28}$$

where $S_d \equiv 2\pi^{d/2}/\Gamma(d/2)$ and C_ϕ is the two-point function constant in coordinate space; it is related to $\tilde{C}_\phi$ in momentum space (4.3.19) through the Fourier transform[6]. Taking the Fourier transform of (4.3.25) and using (4.3.28) we find[7]

$$\langle T^{\rm ren}(0)\phi(p)\phi(-p)\rangle = (d-2\Delta_\phi)\tilde{C}_\phi \frac{p_z^2}{(p^2)^{\frac{d}{2}-\Delta_\phi+1}}\,, \tag{4.3.29}$$

where we took the stress-energy tensor at zero momentum for simplicity. Now, to fix Z_T we compute (4.3.29) using a direct Feynman diagram calculation:

$$\langle T^{\rm ren}(0)\phi(p)\phi(-p)\rangle = Z_T Z_\phi \langle T(0)\phi_0(p)\phi_0(-p)\rangle\,. \tag{4.3.30}$$

To $1/N$ order we have four diagrams

$$\langle T(0)\phi_0(p)\phi_0(-p)\rangle = D_0 + D_1 + D_2 + D_3 + \mathcal{O}(1/N^2)\,, \tag{4.3.31}$$

which are shown in figure 4.5 and given explicitly in Appendix E.

[6]Notice that it is important that we define $\tilde{C}_{\sigma 0}$ in (4.3.14) in momentum space. Thus, C_σ in the coordinate space will depend on Δ. This dependence will affect the loop calculations in coordinate space.

[7]Here we fix some field, say $\phi = \phi^1$, and do not write the $O(N)$-index explicitly.

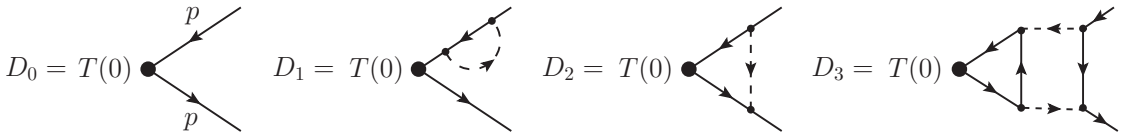


Figure 4.5: Diagrams contributing to $\langle T(0)\phi_0(p)\phi_0(-p)\rangle$ up to order $1/N$.

Computing these diagrams and using (4.3.29) and (4.3.30), we find

$$Z_{T1} = \frac{2\eta_1^{\mathrm{O(N)}}}{d+2}, \qquad Z'_{T1} = \frac{8\eta_1^{\mathrm{O(N)}}}{(d+2)(d-4)}\,, \tag{4.3.32}$$

where $\eta_1^{\mathrm{O(N)}}$ is given in (4.1.7). These renormalization constants will be of great importance for the C_T calculation.

To find Z_J, we again consider the three-point function $\langle J^a_\nu \phi^i \phi^j\rangle$, which is fixed by conformal invariance and current conservation [35]

$$\langle J^{\mathrm{ren},a}_\mu(x_1)\phi^i(x_2)\phi^j(x_3)\rangle = \frac{C_{J\phi\phi}}{(x_{12}^2 x_{13}^2)^{\frac{d}{2}-1}(x_{23}^2)^{\Delta_\phi-\frac{d}{2}+1}}(X_{23})_\mu (t^a)^{ij}\,, \tag{4.3.33}$$

and again the structure constant $C_{J\phi\phi}$ is exactly related to C_ϕ by the Ward identity. To show this, we perform an infinitesimal $O(N)$ rotation of fields $\delta_\varepsilon\phi^i = \varepsilon(t^a)^{ik}\phi^k$, and we get

$$\langle \delta_\varepsilon\phi^i(x_2)\phi^j(x_3)\rangle = \varepsilon\int d^d\Omega r^{d-2} r_\mu \langle J^{\mathrm{ren},a}_\mu(x_1)\phi^i(x_2)\phi^j(x_3)\rangle\,, \tag{4.3.34}$$

where $r = |x_1 - x_2|$. Using (4.3.33) and performing the integral in the limit $r \to 0$ we find

$$C_{J\phi\phi} = \frac{1}{S_d} C_\phi\,. \tag{4.3.35}$$

Taking the Fourier transform of (4.3.33) and using (4.3.35), we get

$$\langle J^{\mathrm{ren},a}(0)\phi^i(p)\phi^j(-p)\rangle = i(d-2\Delta_\phi)\tilde{C}_\phi \frac{p_z}{(p^2)^{\frac{d}{2}-\Delta_\phi+1}}(t^a)^{ij}\,, \tag{4.3.36}$$

where again we took the current at zero momentum to simplify the calculation. Now to fix Z_J, we can compute (4.3.36) by a direct perturbative calculation

$$\langle J^{\mathrm{ren},a}(0)\phi^i(p)\phi^j(-p)\rangle = Z_J Z_\phi \langle J^a(0)\phi_0^i(p)\phi_0^j(-p)\rangle\,, \tag{4.3.37}$$

and to $1/N$ order we have three diagrams

$$\langle J^a(0)\phi_0^i(p)\phi_0^j(-p)\rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2)\,, \tag{4.3.38}$$

which are shown in figure 4.6. Computing these diagrams and using (4.3.36) and (4.3.37), we find

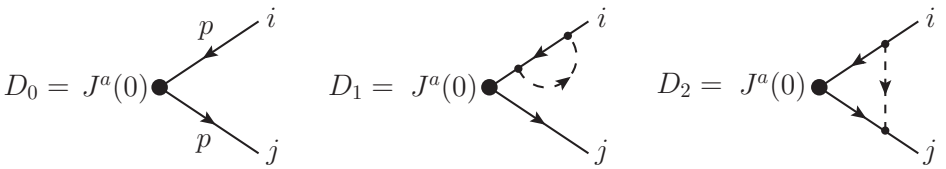


Figure 4.6: Diagrams contributing to $\langle J^a(0)\phi_0^i(p)\phi_0^j(-p)\rangle$ up to order $1/N$.

$$Z_J = 1 + \mathcal{O}(1/N^2)\,. \tag{4.3.39}$$

Therefore Z_J is trivial to order $1/N$ and will not affect the C_{J1} calculation.

4.3.3 Calculation of $C_{J1}^{\mathbf{O(N)}}$ and $C_{T1}^{\mathbf{O(N)}}$

There are three diagrams contributing to the $1/N$ correction to C_J, depicted in figure 4.7. The

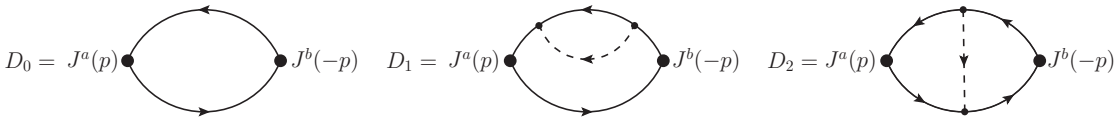


Figure 4.7: Diagrams contributing to $\langle J^a(p)J^b(-p)\rangle$ up to order $1/N$.

current two-point function up to order $1/N$ is then

$$\langle J^a(p)J^b(-p)\rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2)\,. \tag{4.3.40}$$

The sum of D_1 and D_2 corresponds to the contribution denoted $I_{\langle JJOO\rangle}$ in section 4.2. The explicit integrands and results for each diagram are given in Appendix E. To compute these diagrams, we use standard techniques to perform tensor reductions and partial fraction decompositions of the integrand, which are discussed in Appendix A. This results in a sum of simpler scalar integrals

which involve either the product of two elementary one-loop integrals of the form

$$\int \frac{d^d p_1}{(2\pi)^d} \frac{1}{p_1^{2\alpha}(p+p_1)^{2\beta}} = \frac{\Gamma(\frac{d}{2}-\alpha)\Gamma(\frac{d}{2}-\beta)\Gamma(\alpha+\beta-\frac{d}{2})}{(4\pi)^{d/2}\Gamma(\alpha)\Gamma(\beta)\Gamma(d-\alpha-\beta)}(p^2)^{d/2-\alpha-\beta} \equiv l(\alpha,\beta)(p^2)^{d/2-\alpha-\beta}\,, \tag{4.3.41}$$

or the two-loop "kite" diagram with the topology of D_2 and general power of the middle line

$$K(a) = \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{1}{p_1^2 (p+p_1)^2 p_2^2 (p+p_2)^2 (p_1-p_2)^{2a}}\,. \tag{4.3.42}$$

The result for this integral as a function of d and a can be obtained, for instance, by using the Gegenbauer polynomial technique [103, 104]. Putting all contributions together, the final result is

$$\langle J^a(p) J^b(-p)\rangle = \frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-3}\Gamma(d)} C_{J0}^{\mathrm{O(N)}} \left(1 - \frac{1}{N}\frac{8(d-1)}{d(d-2)}\eta_1^{\mathrm{O(N)}} + \mathcal{O}(1/N^2)\right) \frac{p_z^2}{(p^2)^{2-\frac{d}{2}}}\,, \tag{4.3.43}$$

where $\eta_1^{\mathrm{O(N)}}$ is given in (4.1.7) and

$$C_{J0}^{\mathrm{O(N)}} = -\frac{\mathrm{tr}(t^a t^b)}{(d-2)S_d^2}\,. \tag{4.3.44}$$

Using that in this case $Z_J = 1 + \mathcal{O}(1/N^2)$, we find

$$C_{J1}^{\mathrm{O(N)}} = -\frac{8(d-1)}{d(d-2)}\eta_1^{\mathrm{O(N)}}\,. \tag{4.3.45}$$

This agrees with the result of [102], who derived it using the conformal bootstrap technique. We can verify that $C_{J,1}$ is negative throughout the range $2 < d < 6$, as shown in figure 4.8. The value

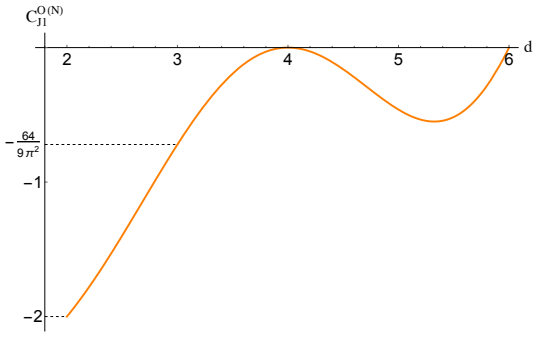


Figure 4.8: Plot of $C_{J1}^{\mathrm{O(N)}}$, which is negative throughout the range $2 < d < 6$.

in $d = 3$ is given in eq. (4.1.9), and from (4.3.45) one can also get

$$C_{J1}^{\mathrm{O(N)}}|_{d=2+\epsilon} = -2 + \epsilon + \frac{\epsilon^2}{2}, \quad C_{J1}^{\mathrm{O(N)}}|_{d=4-\epsilon} = -\frac{3\epsilon^2}{4} - \frac{\epsilon^3}{8}, \quad C_{J1}^{\mathrm{O(N)}}|_{d=6-\epsilon} = -\frac{5\epsilon}{3} + \frac{7\epsilon^2}{6} \tag{4.3.46}$$

We note that the $d = 6 - \epsilon$ expansion precisely agrees with the result (4.3.8) that we derived above from the cubic model.

Let us now turn to the calculation of C_T. There are four diagrams contributing to $\langle TT \rangle$ to order N^0

$$\langle T(p)T(-p)\rangle = D_0 + D_1 + D_2 + D_3 + \mathcal{O}(1/N)\,, \tag{4.3.47}$$

including the three-loop diagram of Aslamazov-Larkin type [99], which was not present in the calculation of C_J, as shown in figure 4.9. After tensor reductions, one obtains a large sum of scalar

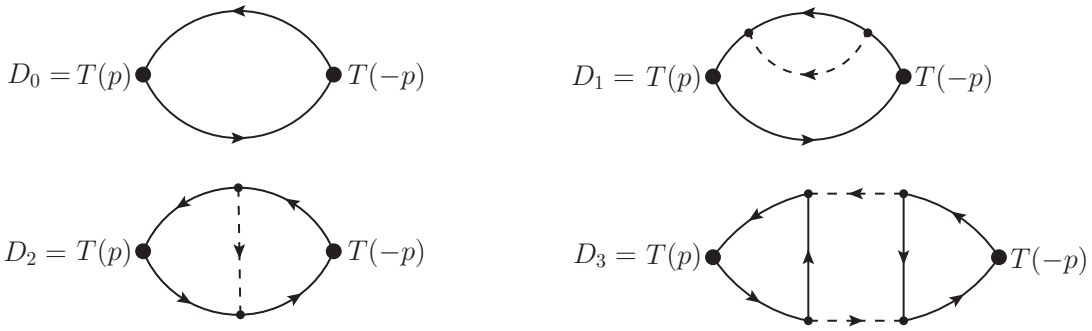


Figure 4.9: Diagrams contributing to $\langle T(p)T(-p)\rangle$ up to order N^0. The last one is the three-loop Aslamazov-Larkin diagram.

integrals that, in addition to (4.3.41) and (4.3.42), involve three-loop ladder scalar integrals with various powers of the propagator lines. The evaluation of this type of integrals is discussed in detail in Appendix 4.6, and the results for the individual diagrams are listed in Appendix E. After a very laborious computation, we obtain

$$\begin{aligned}\langle T(p)T(-p)\rangle &= \frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-2}\Gamma(d+2)}\times \\ &\times C_{T0}^{\mathrm{O(N)}}\bigg(1 - \frac{1}{N}\Big(\frac{1}{\Delta}\frac{4\eta_1^{\mathrm{O(N)}}}{(d+2)} + \eta_1^{\mathrm{O(N)}}\Big(\frac{4\mathcal{C}_{\mathrm{O(N)}}(d)}{d+2} + \frac{2\left(d^3+10d^2-48d+32\right)}{(d-4)(d-2)d(d+2)}\Big)\Big) + \mathcal{O}(1/N^2)\bigg)\frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,,\end{aligned} \tag{4.3.48}$$

where $\mathcal{C}_{\mathrm{O(N)}}(d) = \psi(3-\frac{d}{2}) + \psi(d-1) - \psi(1) - \psi(\frac{d}{2})$ and $\eta_1^{\mathrm{O(N)}}$ is given in (4.1.7), and

$$C_{T0}^{\mathrm{O(N)}} = \frac{Nd}{(d-1)S_d^2}\,. \tag{4.3.49}$$

As we have already discussed, the $1/\Delta$-pole is present, but there is no $\log(p^2/\mu^2)$ term, as expected since the stress-energy tensor is exactly conserved and cannot develop an anomalous dimension. In order to get an expression free of the $1/\Delta$ poles, we have to use "renormalized" stress-energy tensor $T_{\mu\nu}^{\mathrm{ren}} = Z_T T_{\mu\nu}$, where Z_T was derived above and given in (4.3.32). Therefore, we obtain

$$\begin{aligned}\langle T^{\mathrm{ren}}(p)T^{\mathrm{ren}}(-p)\rangle &= Z_T^2\langle T(p)T(-p)\rangle \\ &= \frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-2}\Gamma(d+2)}C_{T0}^{\mathrm{O(N)}}\left(1-\frac{\eta_1^{\mathrm{O(N)}}}{N}\Big(\frac{4\mathcal{C}_{\mathrm{O(N)}}(d)}{d+2}+\frac{2\left(d^2+6d-8\right)}{(d-2)d(d+2)}\Big)+\mathcal{O}(1/N^2)\right)\frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,.\end{aligned} \tag{4.3.50}$$

Note that, as desired, the $1/\Delta$ pole was cancelled. This is a non-trivial consistency check of our procedure, since the Z_T factor was obtained above from an independent Ward identity calculation. From (4.3.50), we thus find

$$C_{T1}^{\mathrm{O(N)}} = -2\eta_1^{\mathrm{O(N)}}\left(\frac{2\mathcal{C}_{\mathrm{O(N)}}(d)}{d+2}+\frac{d^2+6d-8}{(d-2)d(d+2)}\right)\,, \tag{4.3.51}$$

which exactly agrees with the result of [102]. We note that we may also write this result in a simpler form as

$$C_{T1}^{\mathrm{O(N)}} = -2\eta_1^{\mathrm{O(N)}}\left(\frac{2\Psi_{\mathrm{O(N)}}(d)}{d+2}+\frac{d+4}{d(d+2)}\right)\,, \tag{4.3.52}$$

where $\Psi_{\mathrm{O(N)}}(d) \equiv \psi(3-\frac{d}{2}) + \psi(d-1) - \psi(1) - \psi(\frac{d}{2}-1)$.

A plot of $C_{T1}^{\mathrm{O(N)}}$ in $2 < d < 6$ is given in figure 4.10. The value in $d = 3$ was already given in (4.1.9). From (4.3.51), one can also get

$$C_{T1}^{\mathrm{O(N)}}|_{d=2+\epsilon} = -1+\frac{3\epsilon^2}{4},\quad C_{T1}^{\mathrm{O(N)}}|_{d=4-\epsilon} = -\frac{5\epsilon^2}{12}-\frac{7\epsilon^3}{36},\quad C_{T1}^{\mathrm{O(N)}}|_{d=6-\epsilon} = 1-\frac{7\epsilon}{4}+\frac{23\epsilon^2}{288}\,. \tag{4.3.53}$$

We note that the result for $C_{T1}^{\mathrm{O(N)}}$ expanded in $d = 6-\epsilon$ precisely agrees with the the calculation in the cubic model, see (4.3.10). This constitutes a new perturbative check of the formula (4.3.51) for $C_{T1}^{\mathrm{O(N)}}$. Note that the leading term in $d = 6-\epsilon$ is just the contribution of the free scalar field σ in the cubic model. As discussed in the Introduction, for all even d, the critical $O(N)$ model is

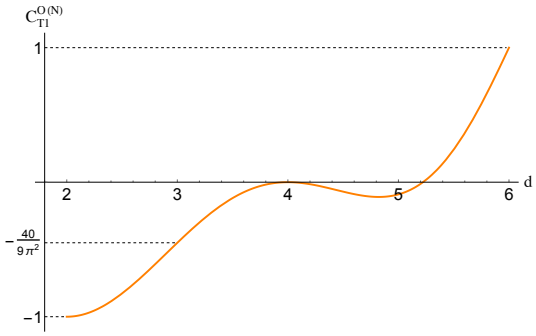


Figure 4.10: Plot of C_{T1}.

expected to reduce to a free theory of N ordinary conformal scalars, plus a $\Delta = 2$ scalar with kinetic term $\sim \sigma(\partial^2)^{\frac{d}{2}-2}\sigma$, see eq. (4.3.13). From (4.3.51) it follows that

$$C_{T1}^{\mathrm{O(N)}}|_{\text{even } d} = \frac{(-1)^{\frac{d}{2}+1}(d-4)(d-2)!}{(\frac{d}{2}+1)!(\frac{d}{2}-1)!} = (-1)^{\frac{d}{2}+1}\left[\binom{d-4}{\frac{d}{2}-3} - \binom{d-4}{\frac{d}{2}-5}\right] . \tag{4.3.54}$$

Interestingly, this is an integer for all even dimensions [79].[8] The formula (4.3.54) is the ratio of the C_T of a free $(d-4)$-derivative scalar to that of a canonical scalar. This means that

$$C_T^{(d-4)-\text{deriv. scalar}}|_{\text{even } d} = \frac{(-1)^{\frac{d}{2}+1}d(d-4)(d-2)!}{(d-1)(\frac{d}{2}+1)!(\frac{d}{2}-1)!S_d^2} . \tag{4.3.55}$$

It would be interesting to check this result via an explicit calculation using the action for a higher derivative scalar.

4.3.4 Padé approximations

For any quantity $f(d)$ known in the $\epsilon = 4-d$ and $\epsilon = d-2$ expansions up to a given order, we can construct a Padé approximant

$$\text{Padé}_{[m,n]}(d) = \frac{A_0 + A_1 d + A_2 d^2 + \ldots + A_m d^m}{1 + B_1 d + B_2 d^2 + \ldots + B_n d^n} , \tag{4.3.56}$$

where the coefficients A_i, B_i are fixed by requiring that the expansion of (4.3.56) agrees with the known terms in $f(4-\epsilon)$ and $f(2+\epsilon)$ obtained by perturbation theory. For the $O(N)$ model the $4-\epsilon$ expansion can be developed for any integer N using the weakly coupled Wilson-Fisher IR fixed

[8] In fact, we note that (4.3.54) appears to be equal (for $d > 4$) to $(-1)^{d/2+1}$ times the dimension of the irreducible representation of $Sp(d-4)$ labelled by the Young tableaux $[\underbrace{1,\ldots,1}_{d/2-3},0,\ldots,0]$.

point [75]. The $2+\epsilon$ expansion can be developed using standard perturbation theory only for $N > 2$, because this is when the $O(N)$ non-linear σ model has a weakly coupled UV fixed point [93,105,106].

For $C_J^{\mathrm{O(N)}}/C_{J,\mathrm{free}}^{\mathrm{O(N)}}$, the ϵ expansions read (the ϵ/N correction in $d = 2+\epsilon$ was guessed on the basis of the large N results and plausible assumptions, and the $d = 4-\epsilon$ expansion can be found in [76, 102]):

$$C_J^{\mathrm{O(N)}}/C_{J,\mathrm{free}}^{\mathrm{O(N)}}(d) = \begin{cases} \frac{N-2}{N} + \frac{\epsilon}{N} + \mathcal{O}(\epsilon^2) & \text{in} \quad d = 2+\epsilon\,, \\ 1 - \frac{3(N+2)\epsilon^2}{4(N+8)^2} + \mathcal{O}(\epsilon^3) & \text{in} \quad d = 4-\epsilon\,. \end{cases} \tag{4.3.57}$$

In this case we find that only the approximant Padé$_{[2,2]}$ is well-behaved, being free of poles and in good agreement at large N with the result (4.3.45) in $2 < d < 4$. We plot Padé$_{[2,2]}$ for different values of N in figure 4.11, and list a few of its numerical values in $d = 3$ in table 4.1.

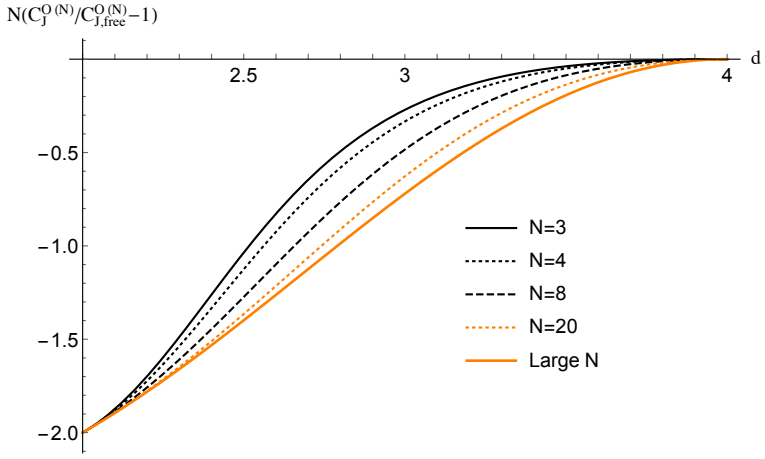


Figure 4.11: Plot of $N(C_J^{\mathrm{O(N)}}/C_{J,\mathrm{free}}^{\mathrm{O(N)}} - 1)$ for Padé$_{[2,2]}$.

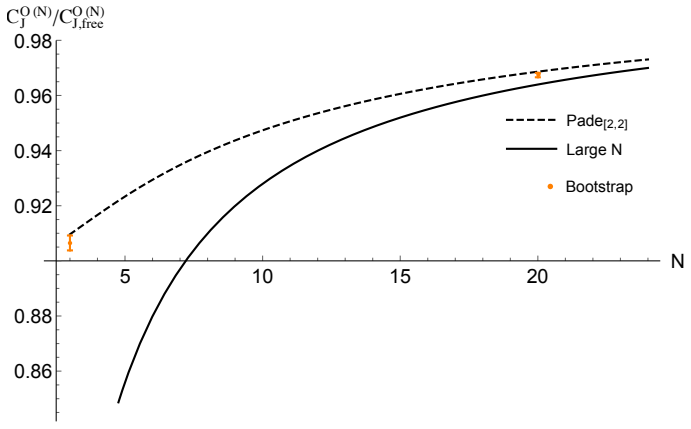


Figure 4.12: Plot of $C_J^{\mathrm{O(N)}}/C_{J,\mathrm{free}}^{\mathrm{O(N)}}$ in $d = 3$

We observe that the results we find are close to the C_J values obtained using the conformal

N	3	4	5	8	12	20	50
Padé$_{[2,2]}$	0.9096	0.9167	0.9234	0.9395	0.9535	0.9686	0.9860
$1-\frac{64}{9\pi^2 N}$	0.7598	0.8199	0.8559	0.9099	0.9400	0.9640	0.9856

Table 4.1: List of Padé$_{[2,2]}$ extrapolations for $C_J^{\mathrm{O(N)}}/C_{J,\mathrm{free}}^{\mathrm{O(N)}}$ for $d=3$. The second line corresponds to the large N result (4.3.45) in $d=3$.

bootstrap [107]. The quoted bootstrap value $C_J^{\mathrm{O(3)}}/C_{J,\mathrm{free}}^{\mathrm{O(3)}} = 0.9065(27)$ should be compared with our Padé$_{[2,2]}$ result 0.9096, and the bootstrap value $C_J^{\mathrm{O(20)}}/C_{J,\mathrm{free}}^{\mathrm{O(20)}} = 0.9674(8)$ with our Padé$_{[2,2]}$ result 0.9686.

For the $C_T^{\mathrm{O(N)}}/C_{T,\mathrm{free}}^{\mathrm{O(N)}}$ we use the following ϵ-expansions:

$$C_T^{\mathrm{O(N)}}/C_{T,\mathrm{free}}^{\mathrm{O(N)}}(d) = \begin{cases} 1-\frac{1}{N}+\frac{3(N-1)\epsilon^2}{4N(N-2)}+\mathcal{O}(\epsilon^3) & \text{in} \quad d=2+\epsilon\,, \\ 1-\frac{5(N+2)\epsilon^2}{12(N+8)^2}+\mathcal{O}(\epsilon^3) & \text{in} \quad d=4-\epsilon\,. \end{cases} \tag{4.3.58}$$

The leading correction in $d=4-\epsilon$ can be found in [72, 74, 76]. To determine the $2+\epsilon$ expansion we used the fact that there is a R^2_{abcd} correction to the central charge in the $d=2$ sigma model with general target space curvature [108, 109]. After specializing to the case of $N-1$ dimensional sphere, we find that this term $\sim (N-1)(N-2)g^2$. The $O(N)$ sigma model has a UV fixed point in $d=2+\epsilon$ for $N>2$ [93, 105, 106]. Setting the sigma model coupling g to its fixed point value $\sim \frac{\epsilon}{N-2}$, and using the large N result to normalize the correction, we find the result above.

The best approximant we find is Padé$_{[3,2]}$; it does not have poles and approaches the large N result (4.3.51) quite well. We plot Padé$_{[3,2]}$ for different N in figure 4.13. Also, we give the values of $C_T^{\mathrm{O(N)}}/C_{T,\mathrm{free}}^{\mathrm{O(N)}}$ for different N in $d=3$ in table 4.2.

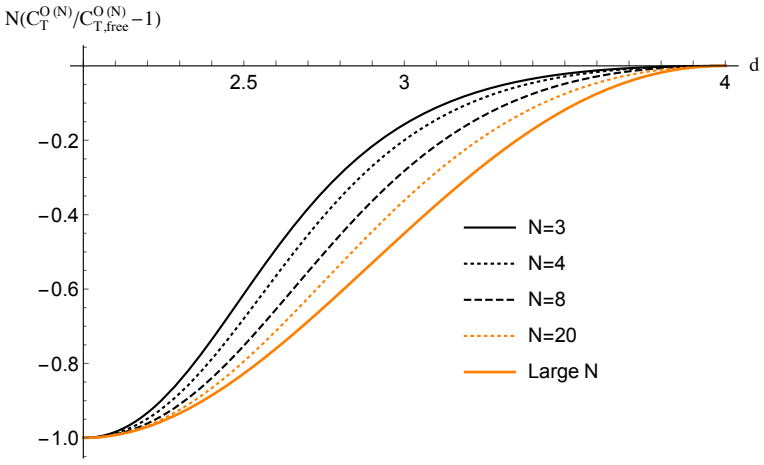


Figure 4.13: Plot of $N(C_T^{\mathrm{O(N)}}/C_{T,\mathrm{free}}^{\mathrm{O(N)}}-1)$ for Padé$_{[3,2]}$.

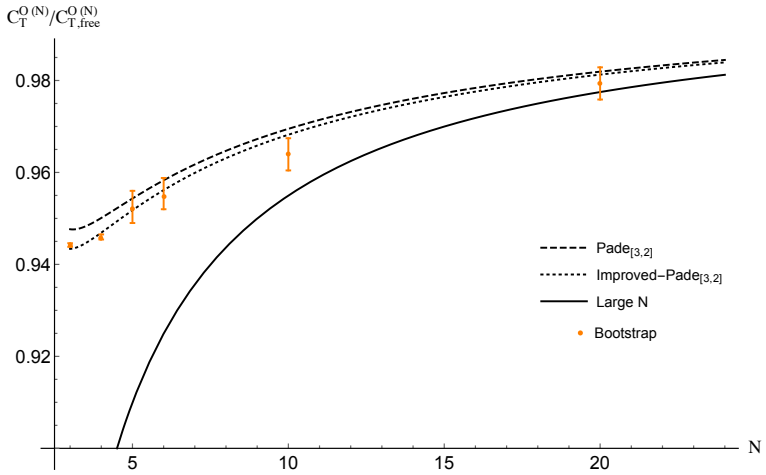


Figure 4.14: Plot of $C_T^{\mathrm{O(N)}}/C_{T,\mathrm{free}}^{\mathrm{O(N)}}$ in $d = 3$

N	3	4	5	8	12	20	50
Padé$_{[3,2]}$	0.9477	0.9501	0.9543	0.9647	0.9732	0.9819	0.9919
$1-\frac{40}{9\pi^2 N}$	0.8499	0.8874	0.9099	0.9437	0.9625	0.9775	0.9910

Table 4.2: List of Padé$_{[3,2]}$ extrapolations for $C_T^{\mathrm{O(N)}}/C_{T,\mathrm{free}}^{\mathrm{O(N)}}$ in $d = 3$. The second line is the large N result (4.3.51) in $d = 3$.

The results we find are close to the C_T values obtained using the conformal bootstrap [110]. The quoted bootstrap values (see Table 3 in [110]) are in good agreement with our Padé$_{[3,2]}$. This is shown in figure 4.14, where we also include the result of an "improved" Padé$_{[3,2]}$ approximant obtained by imposing exact agreement with the large N result (4.3.51) in $2 < d < 4$. Explicitly, this may be defined as

$$\text{Improved-Padé}(d,N) = \text{Padé}(d,N) + \frac{1}{N}\Big(C_{T1} - \lim_{N\to\infty}\big(N(\text{Padé}(d,N)-1)\big)\Big)\,, \tag{4.3.59}$$

which by construction exactly approaches the large N result when N goes to infinity. From figure 4.14, we see that it fits the bootstrap data even better than the regular Padé.

4.4 Gross-Neveu model

4.4.1 $1/N$ expansion

The Hubbard-Stratonovich analysis reviewed in Section 4.2 can be also applied to the Gross-Neveu model. Introducing the auxiliary field σ, and dropping the quadratic term in the critical limit, we

have the action

$$S_{\text{crit ferm}} = \int d^d x \Big(- \bar{\psi}_{0i} \partial\!\!\!/ \psi_0^i + \frac{1}{\sqrt{N}} \sigma_0 \bar{\psi}_{0i} \psi_0^i \Big) , \tag{4.4.1}$$

where $i = 1, \ldots, \tilde{N}$ and $N = \tilde{N} \mathrm{Tr} \mathbf{1}$. The propagator of the ψ_0^i field reads

$$\langle \psi_0^i(p) \bar{\psi}_{0j}(-p) \rangle_0 = \delta_j^i \frac{i p\!\!\!/}{p^2} . \tag{4.4.2}$$

The σ effective propagator obtained after integrating over the fundamental fields ψ_0^i reads

$$\langle \sigma_0(p) \sigma_0(-p) \rangle_0 = \tilde{C}_{\sigma 0} / (p^2)^{\frac{d}{2} - 1 + \Delta} , \tag{4.4.3}$$

where

$$\tilde{C}_{\sigma 0} \equiv -2^{d+1} (4\pi)^{\frac{d-3}{2}} \Gamma\Big(\frac{d-1}{2} \Big) \sin\big(\frac{\pi d}{2} \big) \tag{4.4.4}$$

and we have introduced the regulator Δ. Note that the power of p^2 in the propagator is $\frac{d}{2} - 1 + \Delta$ instead of $\frac{d}{2} - 2 + \Delta$ found in the scalar case. In order to cancel the divergences as $\Delta \to 0$ we have to renormalize the bare fields ψ_0 and σ_0:

$$\psi = Z_\psi^{1/2} \psi_0 , \qquad \sigma = Z_\sigma^{1/2} \sigma_0 , \tag{4.4.5}$$

where

$$Z_\psi = 1 + \frac{1}{N} \frac{Z_{\psi 1}}{\Delta} + \mathcal{O}(1/N^2) , \qquad Z_\sigma = 1 + \frac{1}{N} \frac{Z_{\sigma 1}}{\Delta} + \mathcal{O}(1/N^2) . \tag{4.4.6}$$

The full propagators of the renormalized fields read

$$\langle \psi^i(p) \bar{\psi}_j(-p) \rangle = \delta_j^i \tilde{C}_\psi \frac{i p\!\!\!/}{(p^2)^{\frac{d}{2} - \Delta_\psi + \frac{1}{2}}} , \qquad \langle \sigma(p) \sigma(-p) \rangle = \frac{\tilde{C}_\sigma}{(p^2)^{\frac{d}{2} - \Delta_\sigma}} , \tag{4.4.7}$$

where we introduced anomalous dimensions Δ_ψ and Δ_σ and two-point function normalizations $\tilde{C}_\psi$ and $\tilde{C}_\sigma$ in momentum space. Each of them may be represented as a series in $1/N$:

$$\Delta_\psi = \frac{d}{2} - \frac{1}{2} + \eta^{\mathrm{GN}} , \qquad \Delta_\sigma = 1 - \eta^{\mathrm{GN}} - \kappa^{\mathrm{GN}} , \tag{4.4.8}$$

where $\eta^{\rm GN} = \eta_1^{\rm GN}/N + \eta_2^{\rm GN}/N^2 + \mathcal{O}(1/N^3)$, $\kappa^{\rm GN} = \kappa_1^{\rm GN}/N + \kappa_2^{\rm GN}/N^2 + \mathcal{O}(1/N^3)$ and

$$\tilde{C}_\psi = 1 + \frac{\tilde{C}_{\psi 1}}{N} + \frac{\tilde{C}_{\psi 2}}{N^2} + \mathcal{O}(1/N^3)\,, \qquad \tilde{C}_\sigma = \tilde{C}_{\sigma 0} + \frac{\tilde{C}_{\sigma 1}}{N} + \frac{\tilde{C}_{\sigma 2}}{N^2} + \mathcal{O}(1/N^3)\,. \tag{4.4.9}$$

The stress-energy tensor and the current are

$$\begin{aligned} T &= -\frac{1}{2}(\bar{\psi}_{0i}\gamma_\mu \partial_\nu \psi_0^i - \partial_\mu \bar{\psi}_{0i}\gamma_\nu \psi_0^i) z^\mu z^\nu\,, \\ J^a &= -z^\mu \bar{\psi}_{0i} (t^a)^i_j \gamma_\mu \psi_0^j \end{aligned} \tag{4.4.10}$$

and in momentum space

$$\begin{aligned} T(p) &= -\frac{1}{2}\int \frac{d^d p_1}{(2\pi)^d} \bar{\psi}_{0i}(-p_1) i\gamma_z (2p_{1z} + p_z) \psi_0^i(p+p_1)\,, \\ J^a(p) &= -\int \frac{d^d p_1}{(2\pi)^d} \bar{\psi}_{0i}(-p_1)(t^a)^i_j \gamma_z \psi_0^j(p+p_1)\,. \end{aligned} \tag{4.4.11}$$

The diagrammatic representation is shown in figure 4.15.

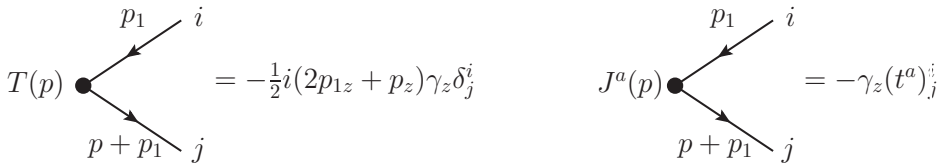


Figure 4.15: Momentum space Feynman rules for $T(p)$ and $J^a(p)$.

As in the scalar case, we define

$$T^{\rm ren}_{\mu\nu} = Z_T T_{\mu\nu}, \qquad J^{{\rm ren},a}_\mu = Z_J J^a_\mu\,, \tag{4.4.12}$$

where

$$Z_T = 1 + \frac{1}{N}\Big(\frac{Z_{T1}}{\Delta} + Z'_{T1}\Big) + \mathcal{O}(1/N^2), \quad Z_J = 1 + \frac{1}{N}\Big(\frac{Z_{J1}}{\Delta} + Z'_{J1}\Big) + \mathcal{O}(1/N^2)\,. \tag{4.4.13}$$

By a direct calculation presented in Appendices C and D, we show that Ward identities fix

$$Z_{T1} = \frac{2\eta_1^{\rm GN}}{d+2}, \qquad Z'_{T1} = \frac{8\eta_1^{\rm GN}}{(d+2)(d-2)}\,, \tag{4.4.14}$$

where $\eta_1^{\rm GN}$ is defined in (4.4.8) and reads

$$\eta_1^{\rm GN} = \frac{\Gamma(d-1)(\frac{d}{2}-1)^2}{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})^2}\,. \tag{4.4.15}$$

For the spin 1 current, we find $Z_J = 1 + \mathcal{O}(1/N^2)$, which means that it does not affect the C_{J1} calculation.

4.4.2 Calculation of $C_{J1}^{\rm GN}$ and $C_{T1}^{\rm GN}$

There are again three diagrams contributing to C_J/C_{J0} up to order $1/N$, given in figure 4.16. They are identical to the ones for the critical scalar, except the solid lines are fermionic instead of scalar.

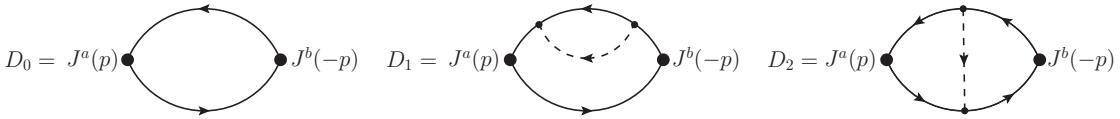


Figure 4.16: Diagrams contributing to $\langle J^a(p)J^b(-p)\rangle$ up to order $1/N$.

To compute the diagrams we use the same methods as for the case of the $O(N)$ model (see Appendices A, B). We find that the $1/\Delta$ divergence is canceled in the combination D_1+D_2, yielding the result (see Appendix E for the integrands and results for each diagram):

$$\begin{aligned}\langle J^a(p)J^b(-p)\rangle &= D_0 + D_1 + D_2 + \mathcal{O}(1/N^2)\\ &= \frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-3}\Gamma(d)} C_{J0}^{\rm GN}\left(1 - \frac{1}{N}\frac{8(d-1)}{d(d-2)}\eta_1^{\rm GN} + \mathcal{O}(1/N^2)\right)\frac{p_z^2}{(p^2)^{2-\frac{d}{2}}}\,,\end{aligned} \tag{4.4.16}$$

where $\eta_1^{\rm GN}$ is given in (4.4.15) and

$$C_{J0}^{\rm GN} = -{\rm tr}(t^a t^b){\rm Tr}\mathbf{1}\frac{1}{S_d^2}\,. \tag{4.4.17}$$

Therefore, we find the final result

$$C_{J1}^{\rm GN} = -\frac{8(d-1)}{d(d-2)}\eta_1^{\rm GN}\,. \tag{4.4.18}$$

We see that $C_{J1}^{\rm GN}$ for the critical fermion is always negative in the range $2 < d < 4$, thus a "C_J-theorem" inequality $C_J^{\rm UV} > C_J^{\rm IR}$ does not hold for the flow from the UV fixed point to the free fermions in the IR.

In $d = 3$, we obtain the value reported in eq. (4.1.18). In $d = 2+\epsilon$ and $d = 4-\epsilon$ dimensions, we

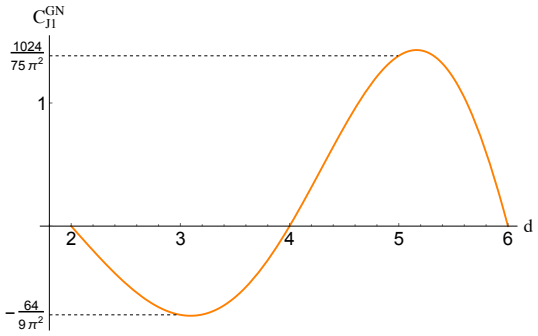


Figure 4.17: Plot of $C_{J1}^{\rm GN}$, which is negative throughout the range $2 < d < 4$.

find

$$C_{J,1}^{\rm GN}|_{d=2+\epsilon} = -\epsilon + \frac{\epsilon^3}{4} + \mathcal{O}(\epsilon^4)\,, \qquad C_{J,1}^{\rm GN}|_{d=4-\epsilon} = -\frac{3\epsilon}{2} + \frac{\epsilon^2}{2} + \frac{15\epsilon^3}{32} + \mathcal{O}(\epsilon^4)\,. \tag{4.4.19}$$

We will show that these values are in precise agreement with our C_J calculations for the GN and GNY models performed in sections 4.4.3 and 4.4.4 below.

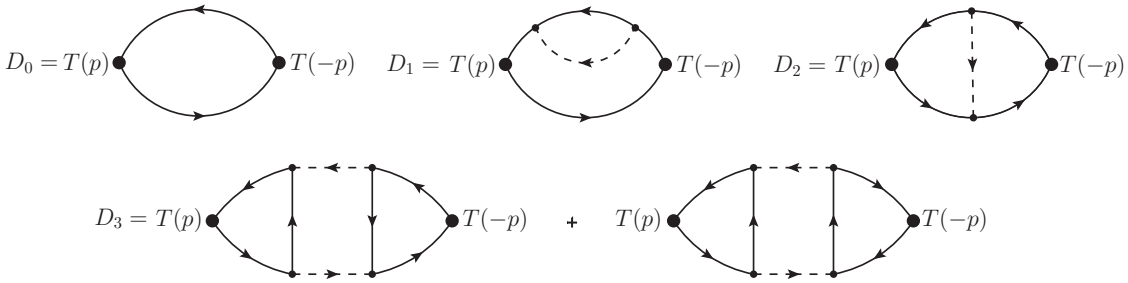


Figure 4.18: Diagrams contributing to $\langle T(p)T(-p)\rangle$ up to order N^0.

The diagrams contributing to the stress tensor two-point function

$$\langle T(p)T(-p)\rangle = D_0 + D_1 + D_2 + D_3 + \mathcal{O}(1/N^2)\,, \tag{4.4.20}$$

are shown in figure 4.18 (see Appendix E for the results). After a very laborious computation, the details of which are discussed in the Appendices, we obtain the final result

$$\begin{aligned}\langle T(p)T(-p)\rangle = {}& \frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-2}\Gamma(d+2)}\times \\ & \times C_{T0}^{\rm GN}\Big(1 - \frac{1}{N}\Big(\frac{1}{\Delta}\frac{4\eta_1^{\rm GN}}{(d+2)} + \eta_1^{\rm GN}\Big(\frac{4\mathcal{C}_{\rm GN}(d)}{d+2} + \frac{4\left(5d^2-8d+4\right)}{(d-2)(d-1)d(d+2)}\Big)\Big) + \mathcal{O}(1/N^2)\Big)\frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,,\end{aligned} \tag{4.4.21}$$

where $\mathcal{C}_{\rm GN}(d) \equiv \psi(2-\frac{d}{2})+\psi(d-1)-\psi(1)-\psi(\frac{d}{2})$, $\eta_1^{\rm GN}$ is given in (4.4.15) and

$$C_{T0}^{\rm GN} = \frac{Nd}{2S_d^2}\,. \tag{4.4.22}$$

As we already discussed, we see that $1/\Delta$-pole is present, but the $\log(p^2/\mu^2)$ term cancels out; this means that, as expected, the stress tensor does not have an anomalous dimension, because it is exactly conserved. In order to get a finite expression we have to use the renormalized stress-energy tensor $T_{\mu\nu}^{\rm ren} = Z_T T_{\mu\nu}$, where Z_T is given in (4.4.13) and (4.4.14). Therefore, we obtain

$$\begin{aligned}\langle T^{\rm ren}(p)T^{\rm ren}(-p)\rangle &= Z_T^2\langle T(p)T(-p)\rangle \\ &= \frac{\pi^{\frac{d}{2}}\Gamma(2-\frac{d}{2})}{2^{d-2}\Gamma(d+2)}C_{T0}^{\rm GN}\left(1-\frac{\eta_1^{\rm GN}}{N}\left(\frac{4\mathcal{C}_{\rm GN}(d)}{d+2}+\frac{4(d-2)}{(d-1)d(d+2)}\right)+\mathcal{O}(1/N^2)\right)\frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,.\end{aligned} \tag{4.4.23}$$

As in the scalar case discussed earlier, it is a non-trivial test of our procedure that the Z_T factor fixed by Ward identities has precisely the correct pole to cancel the $1/\Delta$ divergence in $\langle TT\rangle$. From (4.4.23), we then find one of our main results

$$C_{T1}^{\rm GN} = -4\eta_1^{\rm GN}\left(\frac{\mathcal{C}_{\rm GN}(d)}{d+2}+\frac{d-2}{(d-1)d(d+2)}\right)\,. \tag{4.4.24}$$

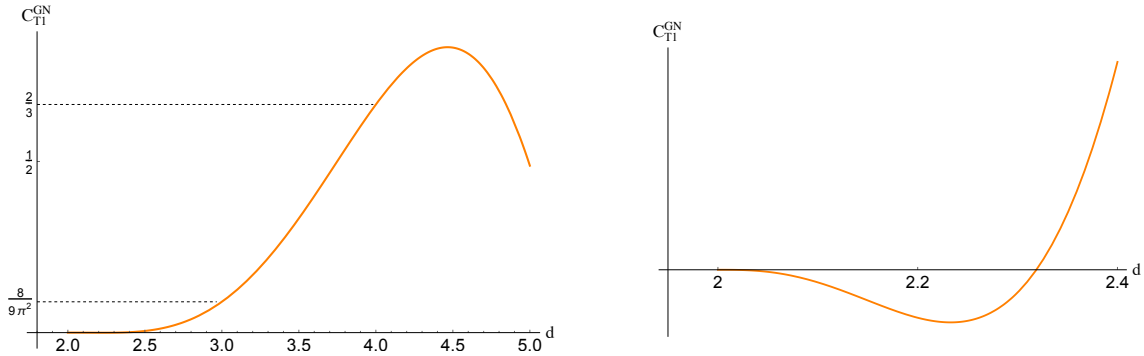


Figure 4.19: Plot of $C_{T1}^{\rm GN}$.

In $d=3$, we get the result quoted in eq. (4.1.18). It is interesting that $C_{T1}^{\rm GN} > 0$ in $d=3$. This means that the "C_T-theorem" inequality $C_T^{\rm UV} > C_T^{\rm IR}$ applies to the large N Gross-Neveu model in $d=3$. However, as plot 4.19 shows, this inequality is violated for $2 < d \lesssim 2.3$.

In $d=2+\epsilon$ and $d=4-\epsilon$, we find

$$C_{T1}^{\rm GN}|_{d=2+\epsilon} = -\frac{\epsilon^3}{8}+\mathcal{O}(\epsilon^4)\,, \qquad C_{T1}^{\rm GN}|_{d=4-\epsilon} = \frac{2}{3}-\frac{11\epsilon}{18}-\frac{17\epsilon^2}{54}+\mathcal{O}(\epsilon^3)\,. \tag{4.4.25}$$

As we show below, these precisely agree with the results obtained using the ϵ expansion in the GN and GNY models, respectively.

It is also interesting to look at general even dimensions d. In this case, the GN model is expected to be equivalent to a theory of $\tilde{N}$ free fermions plus a higher derivative scalar with local kinetic term $\sim \sigma(\partial^2)^{\frac{d}{2}-1}\sigma$ (see the form of the induced propagator (4.4.3)). The contribution to C_T of such a free scalar can be obtained from (4.4.24), which has a finite non-zero limit for all even $d > 2$

$$C_{T1}^{\rm GN}|_{\text{even } d} = \frac{(-1)^{\frac{d}{2}}(d-2)(d-2)!}{(\frac{d}{2}+1)!(\frac{d}{2}-1)!}\,, \tag{4.4.26}$$

From this, after multiplying by the overall free fermion factor (4.4.22), one may read off the C_T coefficient of the $(d-2)$-derivative scalar for all even d:

$$C_T^{(d-2)-\text{deriv. scalar}}|_{\text{even } d} = \frac{(-1)^{\frac{d}{2}}d(d-2)(d-2)!}{2(\frac{d}{2}+1)!(\frac{d}{2}-1)!S_d^2}\,. \tag{4.4.27}$$

Its ratio to C_T of a canonical scalar is

$$\frac{(-1)^{\frac{d}{2}}(d-1)(d-2)(d-2)!}{2(\frac{d}{2}+1)!(\frac{d}{2}-1)!} = (-1)^{\frac{d}{2}}\binom{d-1}{\frac{d}{2}-2}\,. \tag{4.4.28}$$

Interestingly, this is an integer; in $d = 6, 8, 10, \ldots$ we find $-5, 21, -84, \ldots$[9] It would be interesting to check the formula (4.4.28) by a direct calculation using the stress-energy tensor of the free $(d-2)$-derivative scalar.

4.4.3 Gross-Neveu-Yukawa model and $4-\epsilon$ expansions of C_J and C_T

In this section we consider the Gross-Neveu-Yukawa (GNY) model [95,96]. It is a theory of $\tilde{N}$ Dirac fermions ψ^i transforming under an internal $U(\tilde{N})$ symmetry group and a scalar field σ in $d = 4-\epsilon$ dimensions described by the action (4.1.19). As above, we define $N = \tilde{N}\mathrm{Tr}\mathbf{1}$, where $\mathbf{1}$ is the identity matrix for the Dirac representation. The model has a weakly coupled fixed point in $d = 4-\epsilon$, with the coupling constants given by, to leading order in ϵ [93],

$$g_{1\star} = \sqrt{\frac{16\pi^2\epsilon}{N+6}}\,, \tag{4.4.29}$$

$$g_{2\star} = 16\pi^2\epsilon\frac{24N}{(N+6)\left((N-6)+\sqrt{N^2+132N+36}\right)}\,. \tag{4.4.30}$$

[9] These correspond to $\pm$ the dimensions of the rank-$(d/2-2)$ totally antisymmetric representations of $SO(d-1)$.

As before, we will compute C_J and C_T up to two-loop level. We have not found such a calculation in the literature, so our results appear to be new.

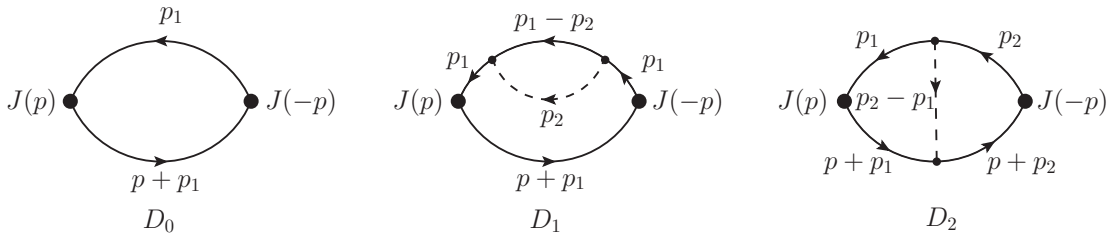


Figure 4.20: Diagrams for C_J to $1/N$ order

For simplicity, we will consider the two-point function of the $U(1)$ current

$$J = z^\mu \bar{\psi}_i \gamma_\mu \psi^i \,, \tag{4.4.31}$$

which, in the notation used above in eq. (4.4.10), just corresponds to a particular choice of generator of $U(N)$ (the one proportional to the identity). The diagrams contributing to

$$\langle J(p) J(-p) \rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2) \,. \tag{4.4.32}$$

are shown in figure 4.20 (see Appendix E for the integrands and results). The arrows are fermionic arrows, and we have defined our momenta in such a way that the flow of momentum coincides with the fermionic arrows. As before, the dashed line denotes the σ field.

After evaluating the integrals, Fourier transforming to position space, substituting the fixed-point values (4.4.29) and (4.4.30) of the coupling constants, and extracting the C_J coefficient from each term according to (4.2.3), we obtain:

$$C_J^{\text{GNY}} = \frac{1}{S_d^2} \left(N - \frac{3N\epsilon}{2(N+6)} + \mathcal{O}(\epsilon^2) \right) \,, \tag{4.4.33}$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the volume of the $(d-1)$-dimensional sphere (evaluated here in $d = 4-\epsilon$). Normalizing by the free field contribution, we find

$$\frac{C_J^{\text{GNY}}}{C_J^{\text{free}}} = 1 - \frac{3\epsilon}{2(N+6)} + \mathcal{O}(\epsilon^2) \,, \tag{4.4.34}$$

which precisely agrees, to leading order at large N, with the result (4.4.18) expanded in $d = 4 - \epsilon$, see eq. (4.4.19).

To study C_T we write $T = T_\psi + T_\sigma$, where

$$T_\psi = -\frac{1}{2}\left(\bar{\psi}_i \gamma_\mu \partial_\nu \psi^i - \partial_\mu \bar{\psi}_i \gamma_\nu \psi^i\right) z^\mu z^\nu \ , \tag{4.4.35}$$

and T_σ is given in (4.3.9). We have $\langle TT \rangle = \langle T_\psi T_\psi \rangle + 2\langle T_\psi T_\sigma \rangle + \langle T_\sigma T_\sigma \rangle$.

At leading order, $\langle T_\psi T_\sigma \rangle = 0$, while $\langle T_\sigma T_\sigma \rangle$ and $\langle T_\psi T_\psi \rangle$ are given by the free field one-loop integrals. At the next to leading order we have four diagrams, which we call D_1, D_2, D_3, and D_4; they are shown in figure 4.21 (see Appendix E for the explicit results).

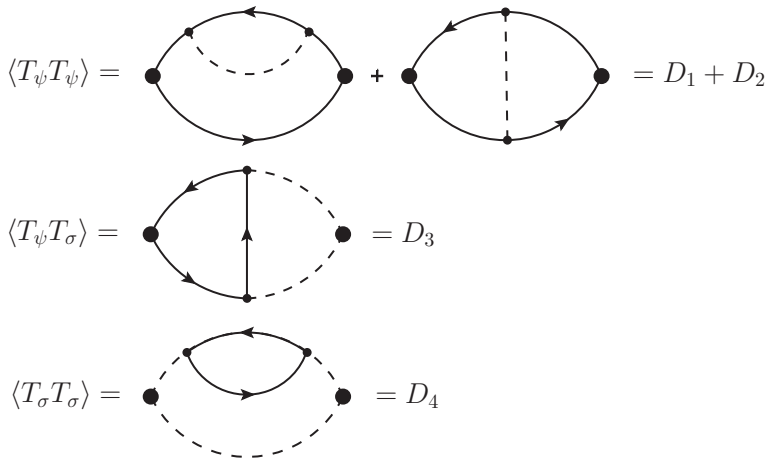


Figure 4.21: Diagrams for C_T in GNY model

After evaluating the integrals, Fourier transforming to position space, and plugging in the expression (4.4.29) for the coupling constant g_1 at the fixed point, we get

$$C_T^{\rm GNY} = \frac{d}{S_d^2}\left(\frac{N}{2} + \frac{1}{d-1} - \frac{5N\epsilon}{12(N+6)}\right) , \tag{4.4.36}$$

To compare to the large N calculation in the previous section, we should normalize this result by the contribution of $\tilde{N}$ free Dirac fermions. Using (4.4.22), we find

$$\frac{C_T^{\rm GNY}}{N C_{T0}^{\rm GN}} = 1 + \frac{2}{3N} - \frac{11N-24}{18N(N+6)}\epsilon + \mathcal{O}(\epsilon^2) \, . \tag{4.4.37}$$

Comparing with (4.4.25), we again find precise agreement with our large N result (4.4.24) expanded in $d = 4 - \epsilon$.

4.4.4 $2+\epsilon$ expansion of C_J and C_T

In this section, we will consider the Gross-Neveu model (4.1.13) in $d = 2+\epsilon$. The beta function and the critical value of g at the UV fixed point are [93]

$$\begin{aligned}\beta &= \epsilon g - (N-2)\frac{g^2}{2\pi} + (N-2)\frac{g^3}{4\pi^2} + (N-2)(N-7)\frac{g^4}{32\pi^3} + \mathcal{O}(g^5)\,,\\ g_* &= \frac{2\pi}{N-2}\epsilon + \frac{2\pi}{(N-2)^2}\epsilon^2 + \frac{(N+1)\pi}{2(N-2)^3}\epsilon^3 + \mathcal{O}(\epsilon^4)\,,\end{aligned} \tag{4.4.38}$$

where $N = \tilde{N}\mathrm{Tr}\mathbf{1}$. From the beta function we can also deduce the relation between the bare and renormalized couplings (here μ denotes the renormalization scale):

$$g_0 = \mu^{-\epsilon}\left(g + \frac{N-2}{2\pi}\frac{g^2}{\epsilon} - \frac{N-2}{8\pi^2}\frac{g^3}{\epsilon} + \frac{(N-2)^2}{4\pi^2}\frac{g^3}{\epsilon^2} + \mathcal{O}(g^4)\right)\,. \tag{4.4.39}$$

The UV fixed point of this model is related to the IR fixed point of the GNY model. One can check this by comparing the anomalous dimensions of the ψ and σ fields as in [93]. In this section we will derive C_J and C_T for the critical fermionic theory at to next-to-leading order.

To extract C_J, we may calculate the two-point function of the $U(1)$ current (4.4.31). The leading order contribution to C_J is the same diagram D_0 as in the GNY model, and the contribution of order g is depicted in figure 4.22. The diagrams contributing to g^2 order are shown in 4.23. [10]

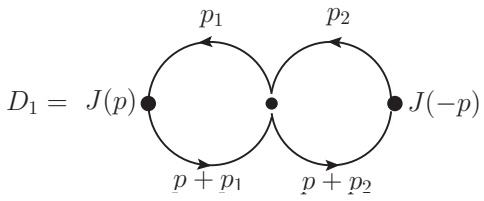


Figure 4.22: Two-loop diagram contributing to C_J to order g

There are three different topologies, and multiple ways of directing the fermion lines within each. As before, the arrows are fermionic arrows, and we have defined momenta in such a way that the flow of momentum coincides with the fermionic arrows. Notice that each insertion of J_μ carries a γ_μ, and we have omitted the diagrams that are zero due to having an odd number of γ's in the trace.

The explicit results for the diagrams $D_0, \ldots, D_4$ are collected in Appendix E. After plugging in

[10] We did not draw some of the diagrams with the D_4 topology because they cancel each other after using the formula $\mathrm{Tr}(\not{A}\not{B}\not{C}\not{D}) = \mathrm{Tr}(\not{A}\not{D}\not{C}\not{B})$, but diagrams with such a topology do appear in the $\langle TT\rangle$ computation. Also, the second diagram for D_4 in the figure has a partner with different orientation of the fermion line, but one can show that these diagrams are equal, therefore we have a factor of 2 for the integral of this diagram in formula (4.9.16).

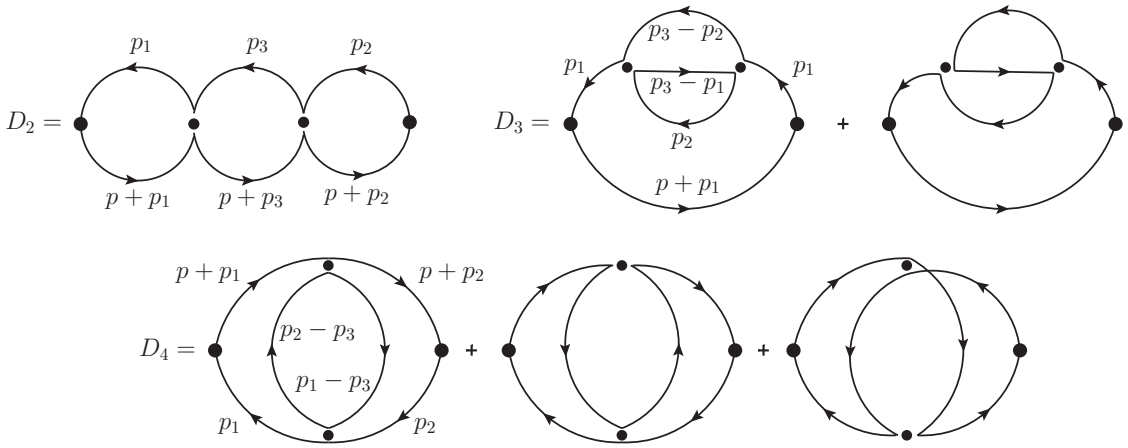


Figure 4.23: Diagrams contributing to C_J to order g^2.

the critical coupling from (4.4.38) and normalizing by the free field contribution, we find

$$C_J^{\rm GN}/C_{J,\rm free}^{\rm GN} = \frac{D_0+D_1+D_2+2D_3+D_4}{D_0} = 1-\frac{\epsilon}{N-2}-\frac{\epsilon^2}{2(N-2)^2}+\mathcal{O}(\epsilon^3)\,. \tag{4.4.40}$$

This agrees with our large-N formula (4.4.18) for $C_{J1}^{\rm GN}$ of the critical fermionic theory, expanded in $d=2+\epsilon$ to $\mathcal{O}(\epsilon^2)$.

The calculation of C_T proceeds similarly to the computation for C_J in the previous section. All the diagrams have identical topologies, with the difference that instead of J we insert the stress-energy tensor (4.4.35). The two-loop diagram D_1 with the same topology as the one in figure 4.22 actually vanishes; see eq. (4.9.19). Computing the three loop diagrams in figure 4.24 (see Appendix

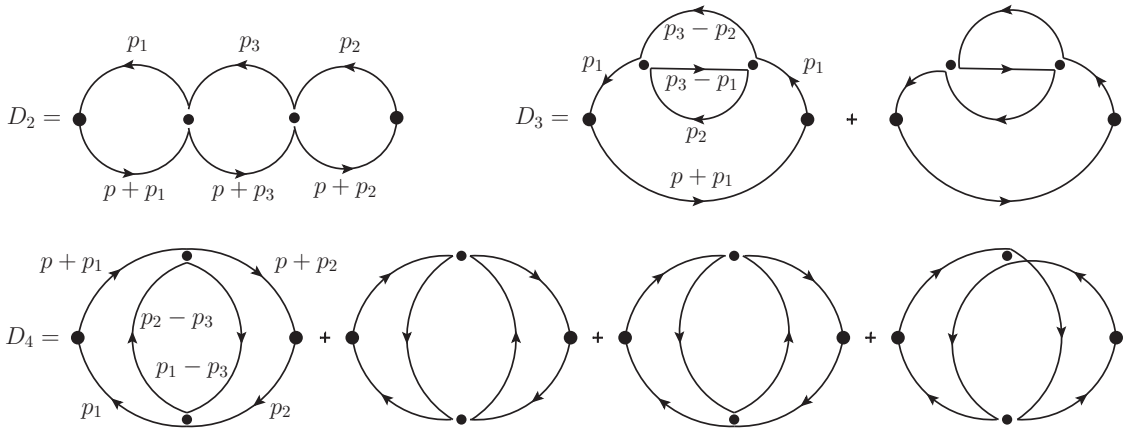


Figure 4.24: Diagrams contributing to C_T to order g^2.

E) and normalizing by the free field contribution, we find the following contribution to $C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}$:

$$\frac{D_0+D_2+2D_3+D_4}{D_0} = 1 + g^2\left(\frac{3(N-1)}{8(2\pi)^2}\epsilon + \mathcal{O}(\epsilon^2)\right). \tag{4.4.41}$$

Note that this $\mathcal{O}(g^2)$ term vanishes in $d=2$. Therefore, for $g=g_*$ the leading correction is of order ϵ^3; this is consistent with the vanishing of the $\mathcal{O}(\epsilon^2)$ term in our large-N result (4.4.25) for C_{T1}.

In order to determine the coefficient of the $\mathcal{O}(\epsilon^3)$ correction to $C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}$ at the critical point, we also need the g^3 term, which comes from four-loop Feynman diagrams. We will not perform this calculation directly, but rather use a shortcut involving the conformal perturbation theory in $d=2$. The GN-model involves the free Dirac fermions perturbed by a marginal operator $O = \frac{1}{2}(\bar\psi^i\psi_i)^2$ with the scaling dimension $\Delta_O = 2+\mathcal{O}(g)$

$$S = S_{\text{free ferm}} + g\int d^2x\, O(x)\,. \tag{4.4.42}$$

The Zamolodchikov c-function is defined as follows [70, 111]:

$$c(g) = C(g) + 4\beta(g)H(g) - 6\beta^2(g)G(g)\,, \tag{4.4.43}$$

where

$$\begin{aligned} C(g) &= 2w^4\langle T_{ww}(x)T_{ww}(0)\rangle|_{x^2=x_0^2}\,,\\ H(g) &= w^2x^2\langle T_{ww}(x)O(0)\rangle|_{x^2=x_0^2}\,,\\ G(g) &= x^4\langle O(x)O(0)\rangle|_{x^2=x_0^2}\,. \end{aligned} \tag{4.4.44}$$

Here $w = x^1+ix^2$, $T_{ww} = T_{11}-T_{22}-2iT_{12}$, and

$$\beta(g) = -(N-2)\frac{g^2}{2\pi} + \mathcal{O}(g^3)\,. \tag{4.4.45}$$

We notice that

$$C_T \propto C(g)\,. \tag{4.4.46}$$

Therefore, to find the g^3 term in C_T we have to find the central charge $c(g)$ to order g^3 and the

function $H(g)$ to order g. The term $\beta^2 G$ obviously does not contribute to this order. Thus, we have

$$C_T(g) \propto c(g) - 4\beta(g)H(g) + \mathcal{O}(g^4)\,. \tag{4.4.47}$$

Let us find $H(g)$ to order g. Using (4.4.42) we get

$$H(g) = gw^2x^2 \int d^2y \langle T_{ww}(x)O(0)O(y)\rangle|_{x^2=x_0^2} + \mathcal{O}(g^2)\,. \tag{4.4.48}$$

To compute this integral it is convenient to use dimensional regularization. We have

$$\langle T_{\mu\nu}(x)O(0)O(y)\rangle = \frac{-C_{TOO}}{(x^2(x-y)^2)^{\frac{d}{2}-1}(y^2)^{\Delta_O-\frac{d}{2}+1}}\Big(X_\mu X_\nu - \frac{1}{d}\delta_{\mu\nu}X^2\Big)\,, \tag{4.4.49}$$

where $X_\nu \equiv x_\nu/x^2 - (x-y)_\nu/(x-y)^2$. Therefore, we find

$$\int d^dy \langle T_{\mu\nu}(x)O(0)O(y)\rangle = -\frac{2C_{TOO}(d-\Delta_O)}{d(d-2\Delta_O)}\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}\frac{1}{(x^2)^{\Delta_O}}\Big(\delta_{\mu\nu} - d\frac{x_\mu x_\nu}{x^2}\Big). \tag{4.4.50}$$

In $d=2$ we obtain

$$\begin{aligned} H(g) &= gw^2x^2C_{TOO}\frac{\pi(2-\Delta_O)}{1-\Delta_O}\frac{\bar{w}^2}{(x^2)^{\Delta_O+1}} + \mathcal{O}(g^2) \\ &= gC_{TOO}\frac{\pi(2-\Delta_O)}{1-\Delta_O}\frac{1}{(x^2)^{\Delta_O-2}} + \mathcal{O}(g^2)\,. \end{aligned} \tag{4.4.51}$$

Since the operator O is marginal, $\Delta_O = 2 + \mathcal{O}(g)$, we have $H(g) = \mathcal{O}(g^2)$. This implies

$$C_T(g) \propto c(g) + \mathcal{O}(g^4)\,. \tag{4.4.52}$$

So we can write

$$C_T^{d=2}/C_T^{\text{free}} = 1 + (c(g) - c_{\text{free}})/c_{\text{free}} = 1 + \delta\tilde{F}/\tilde{F}_{\text{free}}\,, \tag{4.4.53}$$

where $\tilde{F} = -\sin(\pi d/2)F$, and F is the free energy on the d-dimensional sphere [112]. For a CFT in $d=2$ we have $\tilde{F} = \pi c/6$; therefore, $c(g) = 6\tilde{F}(g)/\pi$. For the free fermion free-energy in $d=2$ we have [112]

$$\tilde{F}_{\text{free}} = \frac{\pi N}{12} \tag{4.4.54}$$

corresponding to the standard value for the free fermion central charge, $c_{\rm free} = N/2$. For the change of the free-energy we find [113, 114]

$$\delta\tilde{F} = \pi^3 \int_0^g G(g)\beta(g)dg\,, \tag{4.4.55}$$

where from (4.4.44) we have $G(g) = N(N-1)/(2(2\pi)^4) + \mathcal{O}(g)$ and $\beta(g) = -(N-2)g^2/(2\pi) + \mathcal{O}(g^3)$. Therefore, we obtain for (4.4.53)

$$C_T^{d=2}/C_T^{\rm free} = 1 - \frac{\pi(N-1)(N-2)}{(2\pi)^4} g^3\,. \tag{4.4.56}$$

Thus, in $d = 2$ the leading correction is of order g^3. In $d = 2+\epsilon$ this term, evaluated at the fixed point $g^* = 2\pi\epsilon/(N-2)$, gives a correction of order ϵ^3. Adding this correction to the one coming from the order $g_*^2\epsilon$ term (4.4.41) we finally find

$$C_T^{\rm GN}/C_{T,\rm free}^{\rm GN} = 1 - \frac{(N-1)}{8(N-2)^2}\epsilon^3 + \mathcal{O}(\epsilon^4)\,. \tag{4.4.57}$$

In the large N limit this agrees with (4.4.25), providing a check of our large N calculation. The negative sign of the correction in (4.4.57) means that in $2+\epsilon$ dimensions the C_T theorem is violated for the GN model with all $N > 2$.

4.4.5 Padé approximations

We have the following ϵ expansions for $C_J^{\rm GN}/C_{J,\rm free}^{\rm GN}$:

$$C_J^{\rm GN}/C_{J,\rm free}^{\rm GN}(d) = \begin{cases} 1 - \frac{\epsilon}{N-2} - \frac{\epsilon^2}{2(N-2)^2} + \mathcal{O}(\epsilon^3) & \text{in} \quad d = 2+\epsilon\,, \\ 1 - \frac{3\epsilon}{2(N+6)} + \mathcal{O}(\epsilon^2) & \text{in} \quad d = 4-\epsilon\,. \end{cases} \tag{4.4.58}$$

In this case we find that only the approximant Padé$_{[2,2]}$ has no poles; it approaches the targe N result well. We plot Padé$_{[2,2]}$ for different N in figure 4.25. We also give the $d = 3$ values of $C_J^{\rm GN}/C_{J,\rm free}^{\rm GN}$ for different N in table 4.3. Note that N should be a multiple of 4, since when using the GNY description, we take $N = \tilde{N}{\rm tr}\mathbf{1} = 4\tilde{N}$.

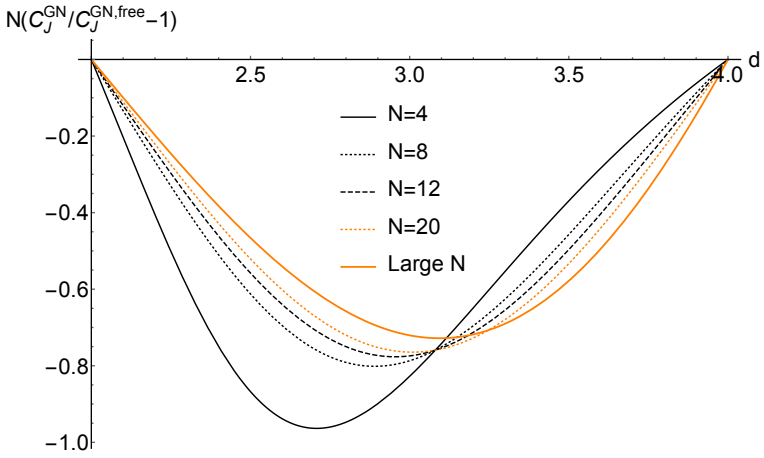


Figure 4.25: Plot of $N(C_J^{\rm GN}/C_{J,\rm free}^{\rm GN}-1)$ for Padé$_{[2,2]}$.

$N=\tilde{N}\mathrm{Tr}(\mathbb{1})$	4	8	12	16	20	24	100
Padé$_{[2,2]}$	0.7931	0.9016	0.9355	0.9520	0.9618	0.9683	0.9925
$1-\frac{64}{9\pi^2 N}$	0.8199	0.9099	0.9400	0.9550	0.9640	0.9700	0.9928

Table 4.3: List of Padé$_{[2,2]}$ extrapolations for $C_J^{\rm GN}/C_{J,\rm free}^{\rm GN}$ in $d=3$. The second line is the large N result (4.4.18) in $d=3$.

We have the following ϵ-expansions for $C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}(d)$

$$C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}(d)=\begin{cases}1-\frac{(N-1)}{8(N-2)^2}\epsilon^3+\mathcal{O}(\epsilon^4) & \text{in} \quad d=2+\epsilon\,,\\ 1+\frac{2}{3N}-\frac{11N-24}{18N(N+6)}\epsilon+\mathcal{O}(\epsilon^2) & \text{in} \quad d=4-\epsilon\,,\end{cases} \tag{4.4.59}$$

In this case we find that all two-sided Padé approximants have poles. One reason for this behavior is the non-monotonicity of the function we are trying to approximate. To make our approximation better, we apply instead the Padé procedure to the following combination

$$f(d)\equiv\Big(\frac{N}{2}\left(C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}(d)-1\right)-\frac{1}{d-1}\Big)/\Big(\frac{N}{2}+\frac{1}{d-1}\Big)\,. \tag{4.4.60}$$

This combination is natural from the point of view of the GNY model. It corresponds to writing $C_T^{GNY}=C_{T0}^{GNY}\,(1+f(d))$, where $C_{T0}^{GNY}=(\frac{Nd}{2}+\frac{d}{d-1})/S_d^2$ is the contribution of the $\tilde{N}$ free fermions and the single free scalar. We find that $f(d)$ is now a monotonic function at large N, and has the ϵ

expansions

$$f(d)=\begin{cases}-\frac{2}{N+2}+\frac{2N\epsilon}{(N+2)^2}-\frac{2N^2\epsilon^2}{(N+2)^3}+\frac{(15N^5-69N^4+58N^3+4N^2+8N)\epsilon^3}{8(N-2)^2(N+2)^4}+\mathcal{O}(\epsilon^4) & \text{in} \quad d=2+\epsilon\,,\\ -\frac{5N\epsilon}{2(N+6)(3N+2)}+\mathcal{O}(\epsilon^2) & \text{in} \quad d=4-\epsilon\,,\end{cases} \tag{4.4.61}$$

Applying Padé approximation to this function we find that Padé$_{[1,4]}$ and Padé$_{[4,1]}$ do not have poles for $N \geqslant 4$ and are in a good agreement with the large N result. Now we may return to the function $N(C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}(d)-1)$. We plot Padé$_{\rm aver} \equiv$ (Padé$_{[1,4]}$ + Padé$_{[4,1]}$)/2 for different N in figure 4.26. We also give the $d=3$ values of $C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}$ for different N in table 4.4. These values differ from

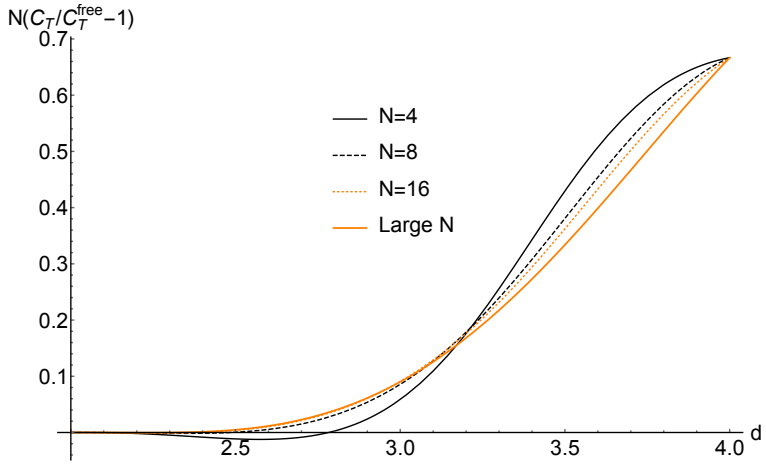


Figure 4.26: Plot of $N(C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}(d)-1)$ for Padé$_{\rm aver} \equiv$ (Padé$_{[1,4]}$ + Padé$_{[4,1]}$)/2.

one by only around a percent even for small N. We also note that the convergence to the large N limit appears to be very fast.

$N=\tilde{N}\mathrm{Tr}(\mathbb{1})$	4	8	12	16	20	24	100
Padé$_{\rm aver}$	1.0147	1.0107	1.0076	1.0057	1.0045	1.0037	1.0008
$1+\frac{8}{9\pi^2 N}$	1.0225	1.0113	1.0075	1.0056	1.0045	1.0038	1.0009

Table 4.4: List of Padé$_{\rm aver}$ extrapolations for $C_T^{\rm GN}/C_{T,\rm free}^{\rm GN}$ in $d=3$. The second line corresponds to the large N result (4.4.24) in $d=3$.

Note: After the first version of this work was published, we were informed by H. Osborn and A. Stergiou that, via a direct calculation, they obtained values of C_T for the higher-derivative scalar fields that agree with (4.3.55) and (4.4.27). The latter agreement provides additional evidence in favor of our results for the Gross-Neveu model.

4.5 Appendix A: Tensor reduction

In this appendix we describe the standard tensor reduction for Feynman integrals in general dimension (see for example [115]). We use this type of reduction because it doesn't change the dimension of the integrals, but unfortunately it sometimes adds new denominators to the integrals[11]. On the other hand, there is another type of tensor reduction called Davydychev recursion relations [116,117]. This method does not add new denominators to Feynman integrals, but it changes their dimension. This type of reduction was applied in the papers [65, 66] for a very similar computations in $d = 3$.

Let us first briefly review the main logic. Suppose we are trying to evaluate a m-loop Feynman integral with loop momenta p_i, where $i = 1, ..., m$, a single external momentum p, and n uncontracted Euclidean indices:

$$I^{\mu_1 \dots \mu_n}(p) = \int_{p_1,\dots,p_m} \frac{p_{i_1}^{\mu_1} \dots p_{i_n}^{\mu_n} (\text{Numer})}{(\text{Denom})}\,, \tag{4.5.1}$$

where $\int_p \equiv \int \frac{d^d p}{(2\pi)^d}$ and the (Numer) denotes some function of $(p_i \cdot p)$, $(p_i \cdot p_j)$ and p^2. We would like to convert this into a sum of scalar integrals only. First, we define the components of the loop momenta transverse to the external momentum as:

$$p_{i\perp}^{\mu} \equiv p_i^{\mu} - \frac{p_i \cdot p}{p^2} p^{\mu}\,. \tag{4.5.2}$$

Using this formula in (4.5.1), we get that the original integral $I^{\mu_1 \dots \mu_n}(p)$ is equal to a sum of integrals of the following form:

$$I_{\perp}^{\mu_1 \dots \mu_k}(p) = \int_{p_1,\dots,p_m} \frac{p_{j_1\perp}^{\mu_1} \dots p_{j_k\perp}^{\mu_k} (\text{Numer})}{(\text{Denom})}\,. \tag{4.5.3}$$

Now we notice that the tensor $I_{\perp}^{\mu_1 \dots \mu_k}$ is transverse with respect to all its indices:

$$p_{\mu_l} I_{\perp}^{\mu_1 \dots \mu_l \dots \mu_k}(p) = 0, \quad \text{for all} \quad l = 1, ..., k\,. \tag{4.5.4}$$

At the same time $I_{\perp}^{\mu_1 \dots \mu_k}$ can be expressed only from the external momentum p^{μ} and the Kronecker delta $\delta^{\mu\nu}$. Notice that if k is odd, then the integral is zero, because for instance there must be a term $p^{\mu_1} \delta^{\mu_2 \mu_3} ... \delta^{\mu_{k-1} \mu_k}$, and $I_{\perp}^{\mu_1 \dots \mu_k}$ cannot be made transverse to p^{μ_1}. Therefore, we can focus only on even k.

In this chapter we are dealing with the cases of $k = 2$ and $k = 4$. Let us start with the case of

[11] This is why in our Aslamazov-Larkin (ladder) type diagrams we have a_9 index (see (4.6.1)). In order to bring this index to zero we apply a complicated recursion relation, discussed in Appendix B.

$k = 2$, so we have

$$I_{\perp}^{\mu_1\mu_2}(p) = \int_{p_1,\ldots,p_m} \frac{p_{j_1\perp}^{\mu_1} p_{j_2\perp}^{\mu_2}(\text{Numer})}{(\text{Denom})} = (\delta^{\mu_1\mu_2} - p^{\mu_1}p^{\mu_2}/p^2)I(p)\,, \tag{4.5.5}$$

where $I(p)$ is some scalar function and j_1, j_2 can be $1, ..., m$. Now if we contract (4.5.5) with $\delta_{\mu_1\mu_2}$ we can easily find

$$I(p) = \frac{1}{d-1}\int_{p_1,\ldots,p_m} \frac{(p_{j_1\perp}\cdot p_{j_2\perp})(\text{Numer})}{(\text{Denom})}\,. \tag{4.5.6}$$

Further reduction to usual scalar integrals can be made by using:

$$p_{i\perp}\cdot p_{j\perp} = p_i\cdot p_j - \frac{1}{p^2}(p_i\cdot p)(p_j\cdot p), \quad (p_i\cdot p) = \frac{1}{2}((p+p_i)^2 - p^2 - p_i^2)\,. \tag{4.5.7}$$

Now consider the case of $k = 4$. We have

$$\begin{aligned} I_{\perp}^{\mu_1\mu_2\mu_3\mu_4}(p) =& \int_{p_1,\ldots,p_m} \frac{p_{j_1\perp}^{\mu_1} p_{j_2\perp}^{\mu_2} p_{j_3\perp}^{\mu_3} p_{j_4\perp}^{\mu_4}(\text{Numer})}{(\text{Denom})} \\ =& \Big(\delta^{\mu_1\mu_2}p^{\mu_3}p^{\mu_4} + \delta^{\mu_3\mu_4}p^{\mu_1}p^{\mu_2} - p^2\delta^{\mu_1\mu_2}\delta^{\mu_3\mu_4} - p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}/p^2\Big)I_1(p) \\ &+ \Big(\delta^{\mu_1\mu_3}p^{\mu_2}p^{\mu_4} + \delta^{\mu_2\mu_4}p^{\mu_1}p^{\mu_3} - p^2\delta^{\mu_1\mu_3}\delta^{\mu_2\mu_4} - p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}/p^2\Big)I_2(p) \\ &+ \Big(\delta^{\mu_1\mu_4}p^{\mu_2}p^{\mu_3} + \delta^{\mu_2\mu_3}p^{\mu_1}p^{\mu_4} - p^2\delta^{\mu_1\mu_4}\delta^{\mu_2\mu_3} - p^{\mu_1}p^{\mu_2}p^{\mu_3}p^{\mu_4}/p^2\Big)I_3(p)\,, \end{aligned} \tag{4.5.8}$$

where $j_1, ..., j_4$ can be $1, ..., m$ and I_1, I_2, I_3 are some scalar functions. The particular combination of tensor structures in front of them are fixed by the fact that they should vanish when contracted with p_{μ_1}(or p_{μ_2}) and p_{μ_3}(or p_{μ_4}). These are the only three structures with four Euclidean indices, constructed from p^μ and $\delta^{\mu\nu}$, and transverse with respect to all indices, so this decomposition is general.

Now, if we contract (4.5.8) with $\delta_{\mu_1\mu_2}\delta_{\mu_3\mu_4}$, $\delta_{\mu_1\mu_3}\delta_{\mu_2\mu_4}$, and $\delta_{\mu_1\mu_4}\delta_{\mu_2\mu_3}$, we get three equations, which have the solution

$$\begin{aligned} I_1 =& \frac{1}{(d^2-1)(d-2)p^2} \\ &\times \int\limits_{p_1,\ldots,p_m} \frac{\big((p_{j_1\perp}p_{j_3\perp})(p_{j_2\perp}p_{j_4\perp}) + (p_{j_1\perp}p_{j_4\perp})(p_{j_2\perp}p_{j_3\perp}) - d(p_{j_1\perp}p_{j_2\perp})(p_{j_3\perp}p_{j_4\perp})\big)(\text{Numer})}{(\text{Denom})} \end{aligned} \tag{4.5.9}$$

and I_2 and I_3 can be obtained from I_1 by replacements $j_2 \leftrightarrow j_3$ and $j_2 \leftrightarrow j_4$ correspondingly. Further reduction can be made by using formulas (4.5.7) and finally, everything reduces to scalar

integrals.

4.6 Appendix B: Recursion relations

The most difficult part of the calculation is the three-loop ladder (Aslamazov-Larkin) diagram with some non-trivial numerator. After the tensor reduction we are required to compute integrals of the form:

$$L\Big(\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_6 & a_7 & a_8 \end{array} \Big| \begin{array}{c} a_4 \\ a_5 \end{array} \Big| a_9\Big) = \\ = \int_{p_1,p_2,p_3} \frac{1}{p_1^{2a_1}(p+p_1)^{2a_2}(p_1-p_3)^{2a_3}p_3^{2a_4}(p+p_3)^{2a_5}p_2^{2a_6}(p+p_2)^{2a_7}(p_2-p_3)^{2a_8}(p_1-p_2)^{2a_9}}, \tag{4.6.1}$$

where $\int_p \equiv \int \frac{d^dp}{(2\pi)^d}$ and p is the external momentum. The indices a_1 to a_8 correspond to lines shown in Figure 4.27. Note that a_9, which corresponds to the momentum combination $p_1 - p_2$, does not appear in the figure. It is generated by tensor reductions and it can only be a negative integer in our calculation. It is not feasible to evaluate such a large number of diagrams individually. Therefore, we seek to reduce these into a small number of "master integrals" through integration by parts relations. However, programs such as FIRE [118] does not work well when multiple non-integer indices are included. We therefore need to implement our own reduction relations.

We would first like to use some recursion relation to reduce to $a_9 = 0$.

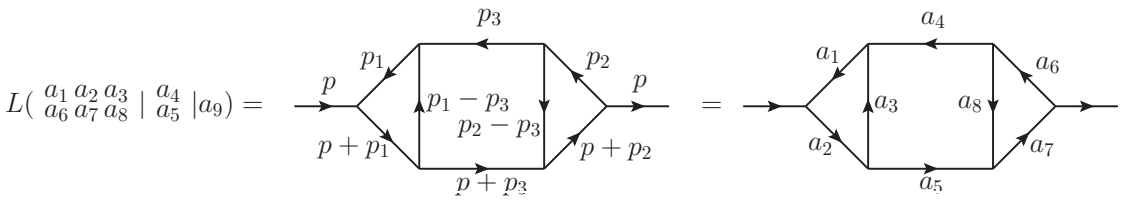


Figure 4.27: Example of a general ladder diagram ($p_1 - p_2$ and a_9 are not included)

The non-trivial general relation to reduce a_9 is:

$$
\begin{aligned}
L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3 & a_4 & \\ a_6\,a_7\,a_8 & a_5 & \end{array}\Big|a_9\Big) &= \frac{(d-a_{134689})(d-a_{235789})p^2}{(d-a_{1239}-1)(d-a_{6789}-1)} L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3 & a_4 & \\ a_6\,a_7\,a_8 & a_5 & \end{array}\Big|a_9+1\Big) \\
&- \frac{(d-a_{235789})(5d/2-a_{124567}-2a_{389}-1)}{(d-a_{1239}-1)(d-a_{6789}-1)} L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3 & a_4-1 & \\ a_6\,a_7\,a_8 & a_5 & \end{array}\Big|a_9+1\Big) \\
&- \frac{(d-a_{134689})(5d/2-a_{124567}-2a_{389}-1)}{(d-a_{1239}-1)(d-a_{6789}-1)} L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3 & a_4 & \\ a_6\,a_7\,a_8 & a_5-1 & \end{array}\Big|a_9+1\Big) \\
&+ \frac{3d/2-a_{123678}-2a_9-1}{d-a_{6789}-1} L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3-1 & a_4 & \\ a_6\,a_7\,a_8 & a_5 & \end{array}\Big|a_9+1\Big) \\
&+ \frac{3d/2-a_{123678}-2a_9-1}{d-a_{1239}-1} L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3 & a_4 & \\ a_6\,a_7\,a_8-1 & a_5 & \end{array}\Big|a_9+1\Big)\,, \qquad (4.6.2)
\end{aligned}
$$

where $a_{nml\ldots} \equiv a_n + a_m + a_l + \ldots$. The relation (4.6.2) is expected to hold for arbitrary indices. This relation can be used to reduce all integrals to have $a_9 = 0$. We will denote the ladder diagrams with $a_9 = 0$ as $L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3 & a_4 & \\ a_6\,a_7\,a_8 & a_5 & \end{array}\Big) \equiv L\Big(\begin{array}{c|c|c} a_1\,a_2\,a_3 & a_4 & \\ a_6\,a_7\,a_8 & a_5 & \end{array}\Big|0\Big)$. After the reduction of a_9 the majority of the integrals can be reduced to two-loop integrals of the form

$$
K(a_1,a_2,a_3,a_4,a_5) \equiv \int_{p_1,p_2} \frac{1}{p_1^{2a_1}(p+p_1)^{2a_4}(p_1-p_2)^{2a_5}p_2^{2a_2}(p+p_2)^{2a_3}}\,. \qquad (4.6.3)
$$

This integral is shown in figure 4.28.

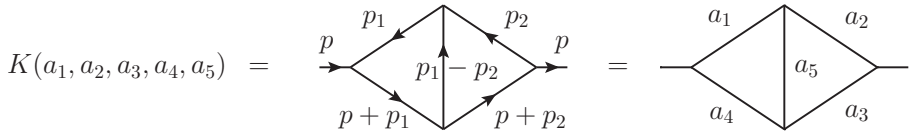


Figure 4.28: Diagrammatic representation of the integral $K(a_1,a_2,a_3,a_4,a_5)$.

There is an extensive literature about different methods for the computation of this type of integrals [91,92,103,104,119–122]. The other diagrams can be reduced to the diagram of type $L\Big(\begin{array}{c|c|c} 1\,1\,1 & \alpha & \\ 1\,1\,1 & \beta & \end{array}\Big)$, where

$$
\alpha = \frac{d}{2} - n + \Delta, \quad \beta = \frac{d}{2} - m + \Delta \qquad (4.6.4)
$$

and n and m are some integers. The diagram with $\alpha = \beta = \frac{d}{2} - 2 + \Delta$ was originally computed

in [91] and the result reads

$$L\Big(\begin{array}{c|c} 1\,1\,1 & d/2-2+\Delta \\ 1\,1\,1 & d/2-2+\Delta \end{array}\Big) = \big(p^2\big)^{\frac{d}{2}-2-2\Delta}\left(\frac{a(1)}{2^2\pi^{\frac{d}{2}}}\right)^6\left(\frac{a(\Delta+d/2-2)}{\pi^{\mu}2^{2(\Delta+d/2-2)}}\right)^2\frac{\pi^{\frac{d}{2}}2^{2(2\Delta-\frac{d}{2}+2)}}{a(2\Delta-\frac{d}{2}+2)}$$
$$\times\frac{\pi^{2d}a(2)^3a(\frac{d}{2}-1)^3a(d-3)}{\Gamma(\frac{d}{2})}\Big(\frac{1}{\Delta}+4B(2)-B(d-3)-3B(\frac{d}{2}-1)\Big)\,, \tag{4.6.5}$$

where

$$a(\alpha)=\Gamma(\frac{d}{2}-\alpha)/\Gamma(\alpha),\qquad B(x)=\psi(x)+\psi(\frac{d}{2}-x)\,. \tag{4.6.6}$$

We consider this integral as the master integral. All other diagrams of this type can be related to this master integral using a non-trivial recursion relation[12]:

$$L\Big(\begin{array}{c|c} 1\,1\,1 & \alpha \\ 1\,1\,1 & \beta \end{array}\Big) = \frac{(d-2-\alpha-\beta)(3d/2-4-\alpha-\beta)}{(d-3-\alpha)(d/2-1-\alpha)p^2}L\Big(\begin{array}{c|c} 1\,1\,1 & \alpha-1 \\ 1\,1\,1 & \beta \end{array}\Big)$$
$$-\frac{(d-2-\alpha-\beta)(d-3)}{(d-3-\alpha)(d/2-1-\alpha)p^2}L\Big(\begin{array}{c|c} 1\,1\,0 & \alpha \\ 1\,1\,1 & \beta \end{array}\Big)$$
$$+\frac{(2d-5-2\alpha-\beta)(d-3)}{(d-3-\alpha)(d/2-1-\alpha)p^2}L\Big(\begin{array}{c|c} 0\,1\,1 & \alpha \\ 1\,1\,1 & \beta \end{array}\Big)-\frac{(d-3)}{(d/2-1-\alpha)p^2}L\Big(\begin{array}{c|c} 1\,0\,1 & \alpha \\ 1\,1\,1 & \beta \end{array}\Big)\,, \tag{4.6.7}$$

where α, β can be arbitrary non-integer and the integrals of the type $L\Big(\begin{array}{c|c} 0\,1\,1 & \alpha \\ 1\,1\,1 & \beta \end{array}\Big)$ and etc can be reduced to the $K(a_1,...,a_5)$ integrals.

4.7 Appendix C: Z_T factor calculation for the critical fermion

In this appendix we present different methods for the computation of the Z_T factor for the stress-energy tensor in the Gross-Neveu model. For what follows, it is important for us to know $\tilde{C}_{\psi 1}$, $\tilde{C}_{\sigma 1}$ and $\eta_1^{\rm GN}$, $\kappa_1^{\rm GN}$ and $Z_{\psi 1}, Z_{\sigma 1}$ defined in (4.4.6), (4.4.8) and (4.4.9). To compute $\tilde{C}_{\psi 1}$, $\eta_1^{\rm GN}$ and $Z_{\psi 1}$ we have to consider the one loop diagram for the renormalization of the $\langle\psi\bar{\psi}\rangle$ propagator. The diagram is depicted in figure 4.29 and reads

[12]Notice that for some α and β in order to correctly apply this recursion relation one has to take into account $\mathcal{O}(\Delta)$ terms in the integrals.

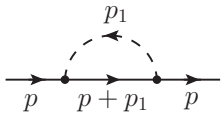


Figure 4.29: One loop correction to the $\langle \psi^i(p)\bar{\psi}_j(-p)\rangle$ propagator.

$$D_1 = \delta^i_j \frac{\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{\sigma 0}\, i(\not{p}+\not{p}_1)}{(p_1^2)^{\frac{d}{2}-1+\Delta}(p+p_1)^2}\,. \tag{4.7.1}$$

Using the integral (4.3.41) we find

$$\eta_1^{\rm GN} = Z_{\psi 1} = \frac{\Gamma(d-1)(\frac{d}{2}-1)^2}{\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}+1)\Gamma(\frac{d}{2})^2} \tag{4.7.2}$$

and

$$\tilde{C}_{\psi 1} = \frac{2^{d-1}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d+1}{2}\right)}{\pi^{3/2}(d/2)^2\Gamma(\frac{d}{2})}\,. \tag{4.7.3}$$

To find $\tilde{C}_\sigma$ and Δ_σ and Z_σ to the $1/N$ order we have to compute the diagrams for the $\langle\sigma_0(p)\sigma_0(-p)\rangle$ propagator represented in figure 4.30.

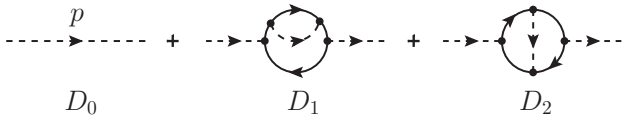


Figure 4.30: Diagrams contributing to $\langle\sigma_0(p)\sigma_0(-p)\rangle$ up to order $1/N$.

The expressions for the diagrams are[13]

$$\begin{aligned}
D_0 &= \frac{\tilde{C}_{\sigma 0}}{(p^2)^{\frac{d}{2}-1}}\,,\\
D_1 &= 2\Big(\frac{\tilde{C}_{\sigma 0}}{(p^2)^{\frac{d}{2}-1}}\Big)^2 \mu^{2\Delta}\int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{(-1)\mathrm{Tr}\big((\not{p}+\not{p}_1)\not{p}_1\not{p}_2\not{p}_1\big)\tilde{C}_{\sigma 0}}{(p+p_1)^2(p_1^2)^2(p_1-p_2)^{2(\frac{d}{2}-1+\Delta)}p_2^2}\,,\\
D_2 &= \Big(\frac{\tilde{C}_{\sigma 0}}{(p^2)^{\frac{d}{2}-1}}\Big)^2 \mu^{2\Delta}\int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{(-1)\mathrm{Tr}\big((\not{p}+\not{p}_1)(\not{p}+\not{p}_2)\not{p}_2\not{p}_1\big)\tilde{C}_{\sigma 0}}{(p+p_1)^2(p+p_2)^2p_1^2(p_1-p_2)^{2(\frac{d}{2}-1+\Delta)}p_2^2}
\end{aligned} \tag{4.7.4}$$

[13]Note that it is very important that we do not shift the power in the $\langle\sigma\sigma\rangle$-external lines by Δ! One can explain this by noticing that the Δ shift in a $\langle\sigma\sigma\rangle$-propagator under a Feynman integral is analogous to changing the dimension of the integral $d \to d' = d - 2\Delta$, while keeping intact the power of $\langle\sigma\sigma\rangle$-propagator [123, 124].

and

$$\langle\sigma(p)\sigma(-p)\rangle = Z_\sigma\langle\sigma_0(p)\sigma_0(-p)\rangle = Z_\sigma\big(D_0 + D_1 + D_2 + \mathcal{O}(1/N^2)\big)\,. \tag{4.7.5}$$

Computing these diagrams one finds

$$Z_{\sigma 1} = \frac{4^{\frac{d}{2}}\sin(\pi d/2)\Gamma\left(\frac{d+1}{2}\right)}{\pi^{3/2}\Gamma(\frac{d}{2}+1)}, \quad \Delta_\sigma = 1 + \frac{4^{\frac{d}{2}}\sin(\pi d/2)\Gamma\left(\frac{d+1}{2}\right)}{\pi^{3/2}\Gamma(\frac{d}{2}+1)}\frac{1}{N} + \mathcal{O}(1/N^2) \tag{4.7.6}$$

and

$$\tilde{C}_\sigma = \tilde{C}_{\sigma 0}\bigg(1 - \frac{1}{N}\eta_{\sigma 1}\Big(\mathcal{C}_{\rm GN}(d) + \frac{4(d-1)}{d(d-2)}\Big) + \mathcal{O}(1/N^2)\bigg)\,, \tag{4.7.7}$$

where $\Delta_\sigma = 1 + \eta_\sigma$ and $\eta_\sigma = \eta_{\sigma 1}/N + \mathcal{O}(1/N^2)$ and $\eta_\sigma = -\eta^{\rm GN} - \kappa^{\rm GN}$.

We recall that the "bare" stress-energy tensor $T_{\mu\nu}$ is related to "renormalized" one $T^{\rm ren}_{\mu\nu}$ as

$$T^{\rm ren}_{\mu\nu}(x) = Z_T T_{\mu\nu}(x)\,, \tag{4.7.8}$$

where $Z_T = 1+(Z_{T1}/\Delta+Z'_{T1})/N+\mathcal{O}(1/N^2)$. Let us first use the three-point function $\langle T^{\rm ren}_{\mu\nu}(x_1)\sigma(x_2)\sigma(x_3)\rangle$ to determine Z_T at $1/N$ order. Using conformal invariance and stress-energy tensor conservation one has the general expression for the three-point function

$$\langle T^{\rm ren}_{\mu\nu}(x_1)\sigma(x_2)\sigma(x_3)\rangle = \frac{-C_{T\sigma\sigma}}{(x_{12}^2x_{13}^2)^{\frac{d}{2}-1}(x_{23}^2)^{\Delta_\sigma-\frac{d}{2}+1}}\Big((X_{23})_\mu(X_{23})_\nu - \frac{1}{d}\delta_{\mu\nu}(X_{23})^2\Big)\,, \tag{4.7.9}$$

where

$$(X_{23})_\nu = \frac{(x_{12})_\nu}{x_{12}^2} - \frac{(x_{13})_\nu}{x_{13}^2} \tag{4.7.10}$$

and the Ward identity can be used to relate $C_{T\sigma\sigma}$ with C_σ

$$C_{T\sigma\sigma} = \frac{1}{S_d}\frac{d\Delta_\sigma}{d-1}C_\sigma\,, \tag{4.7.11}$$

where Δ_σ is the anomalous dimension of the filed σ and C_σ is the two-point constant of $\langle\sigma\sigma\rangle$-propagator in the coordinate space. Taking the Fourier transform of (4.7.9) and setting the momen-

tum of the stress-energy tensor to zero one finds, in terms of $T = z^\mu z^\nu T_{\mu\nu}$

$$\langle T^{\rm ren}(0)\sigma(p)\sigma(-p)\rangle = (d-2\Delta_\sigma)\tilde{C}_\sigma \frac{p_z^2}{(p^2)^{\frac{d}{2}-\Delta_\sigma+1}}\,, \tag{4.7.12}$$

where $\tilde{C}_\sigma$ is the normalization of the two-point function $\langle\sigma\sigma\rangle$ in momentum space. Now we can compute the three-point function $\langle T^{\rm ren}(0)\sigma(p)\sigma(-p)\rangle$ directly using Feynman diagrams. We write

$$\langle T^{\rm ren}(0)\sigma(p)\sigma(-p)\rangle = Z_T Z_\sigma \langle T(0)\sigma_0(p)\sigma_0(-p)\rangle \tag{4.7.13}$$

and the diagrams contributing to $\langle T(0)\sigma_0(p)\sigma_0(-p)\rangle$ up to order $1/N$ are shown in figure 4.31. Note that for some topologies we did not draw explicitly diagrams with the opposite fermion loop direction, but they have to be included.

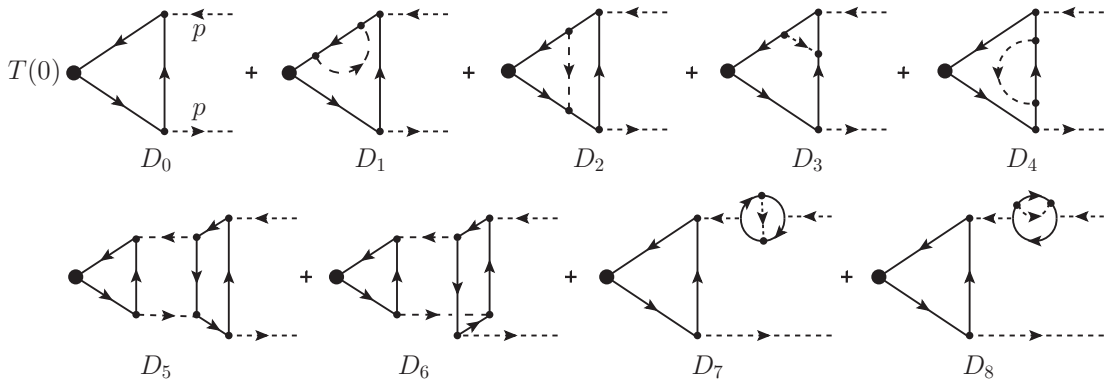


Figure 4.31: Diagrams contributing to $\langle T(0)\sigma_0(p)\sigma_0(-p)\rangle$ up to order $1/N$.

Computing these diagrams and equating the expression (4.7.12) with the diagrammatic result for the expression (4.7.13) we find

$$Z_{T1} = \frac{2\eta_1^{\rm GN}}{d+2}, \qquad Z'_{T1} = \frac{8\eta_1^{\rm GN}}{(d+2)(d-2)}\,, \tag{4.7.14}$$

where $\eta_1^{\rm GN}$ is given in (4.7.2).

Alternatively, we can consider the three-point function $\langle T^{\rm ren}_{\mu\nu}\psi^i\bar{\psi}_j\rangle$. Unfortunately, as far as we know, the general form of it in the coordinate space in general d is not known. But from general analysis and from our diagrammatic results we argue that in momentum space and setting T at zero

momentum, it has the form[14] :

$$\langle T^{\rm ren}(0)\psi(p)\bar\psi(-p)\rangle = i\tilde C_\psi\Big(\frac{\gamma_z p_z}{(p^2)^{\frac{d}{2}-\Delta_\psi+\frac12}} - (d-2\Delta_\psi+1)\frac{\not p\, p_z^2}{(p^2)^{\frac{d}{2}-\Delta_\psi+\frac32}}\Big)\,. \tag{4.7.15}$$

On the other hand we can compute $\langle T(0)\psi_0(p)\bar\psi_0(-p)\rangle$ directly by Feynman diagrams

$$\langle T^{\rm ren}(0)\psi(p)\bar\psi(-p)\rangle = Z_T Z_\psi \langle T(0)\psi_0(p)\bar\psi_0(-p)\rangle\,, \tag{4.7.16}$$

where the diagrams contributing to $\langle T(0)\psi_0(p)\psi_0(-p)\rangle = D_0 + D_1 + D_2 + D_3 + \mathcal{O}(1/N^2)$ are given in figure 4.32 and read

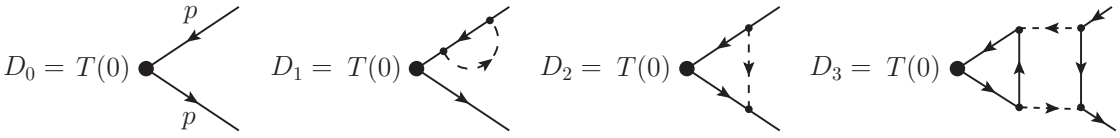


Figure 4.32: Diagrams contributing to $\langle T(0)\psi_0(p)\bar\psi_0(-p)\rangle$ up to order $1/N$.

$$\begin{aligned}
D_0 &= \frac{i\not p}{p^2}\frac{i}{2}(2p_z)\gamma_z\frac{i\not p}{p^2} = i\big(\frac{\gamma_z p_z}{p^2} - \frac{2\not p p_z^2}{(p^2)^2}\big)\,,\\
D_1 &= \frac{2(i)^5\mu^{2\Delta}}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{\not p\not p_1\not p p_z\gamma_z\not p\,\tilde C_{\sigma 0}}{(p^2)^3p_1^2(p-p_1)^{2(\frac{d}{2}-1+\Delta)}}\,,\\
D_2 &= \frac{(i)^5\mu^{2\Delta}}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{\not p\not p_1\, p_{1z}\gamma_z\not p_1\not p\,\tilde C_{\sigma 0}}{(p^2)^2(p_1^2)^2(p-p_1)^{2(\frac{d}{2}-1+\Delta)}}\,,\\
D_3 &= \frac{(-1)(i)^7\mu^{4\Delta}}{N}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\not p\,(\not p-\not p_2)\not p\,\tilde C_{\sigma 0}^2{\rm Tr}(\not p_1\,\gamma_z p_z\not p_1(\not p_1-\not p_2))}{(p^2)^2(p-p_2)^2(p_2^2)^{2(\frac{d}{2}-1+\Delta)}(p_1^2)^2(p_1-p_2)^2}\,.
\end{aligned} \tag{4.7.17}$$

Computing these diagrams and using (4.7.15) and (4.7.16), we find the same result (4.7.14) obtained above.

[14]Here we fix some field, say $\psi=\psi^1$ and don't write the flavor index explicitly.

4.8 Appendix D: Z_J factor calculation for the critical fermion

We can consider the three-point function $\langle J^a_\mu \psi^i \bar{\psi}_j \rangle$, which is fixed by conformal invariance and current conservation [125, 126]

$$\begin{aligned}&\langle J^a_\mu(x_1)\psi^i(x_2)\bar{\psi}_j(x_3)\rangle = \\ &\quad = -\left(C^{(1)}_{J\psi\bar{\psi}}\frac{(\not{x}_{12}\gamma_\mu \not{x}_{13})}{(x^2_{12})^{\frac{d}{2}}(x^2_{13})^{\frac{d}{2}}} + C^{(2)}_{J\psi\bar{\psi}}\frac{(X_{23})_\mu(\not{x}_{23})}{(x^2_{12}x^2_{13})^{\frac{d}{2}-1}x^2_{23}}\right)\frac{(t^a)^i_j}{(x^2_{23})^{\Delta_\psi+\frac{1}{2}-\frac{d}{2}}}\,. \end{aligned} \tag{4.8.1}$$

The Ward identity gives a relation between the structure constants $C^{(1)}_{J\psi\bar{\psi}}$ and $C^{(2)}_{J\psi\bar{\psi}}$

$$\langle \delta_\epsilon \psi^i(x_2)\bar{\psi}_j(x_3)\rangle = -\epsilon \int d^d\Omega r^{d-2} r_\mu \langle J^a_\mu(x_1)\psi^i(x_2)\bar{\psi}_j(x_3)\rangle, \tag{4.8.2}$$

where $r_\mu = (x_1 - x_2)_\mu$, $\int d^d\Omega = S_d = 2\pi^{d/2}/\Gamma(d/2)$ and $\delta_\epsilon \psi^i = \epsilon (t^a)^i_k \psi^k$. Performing the integral in the limit $r \to 0$ we find

$$C^{(1)}_{J\psi\bar{\psi}} + C^{(2)}_{J\psi\bar{\psi}} = \frac{C_\psi}{S_d}. \tag{4.8.3}$$

Taking the Fourier transform of (4.8.1) and using (4.8.3) we get for $J^{\text{ren},a}$ at zero momentum

$$\langle J^{\text{ren},a}(0)\psi^i(p)\bar{\psi}_j(-p)\rangle = \tilde{C}_\psi\Big((d-2\Delta_\psi+1)\frac{\not{p}p_z}{(p^2)^{\frac{d}{2}-\Delta_\psi+\frac{3}{2}}} - \frac{\gamma_z}{(p^2)^{\frac{d}{2}-\Delta_\psi+\frac{1}{2}}}\Big)(t^a)^i_j\,. \tag{4.8.4}$$

Now to fix Z_J we compute $\langle J^a \psi \bar{\psi}\rangle$ using Feynman diagrams

$$\langle J^{\text{ren},a}(0)\psi^i(p)\bar{\psi}_j(-p)\rangle = Z_J Z_\psi \langle J^a(0)\psi^i_0(p)\bar{\psi}_{0j}(-p)\rangle\,, \tag{4.8.5}$$

and to $1/N$ order we have three diagrams contibuting to (4.8.5), which are shown in figure 4.33. The diagrams read

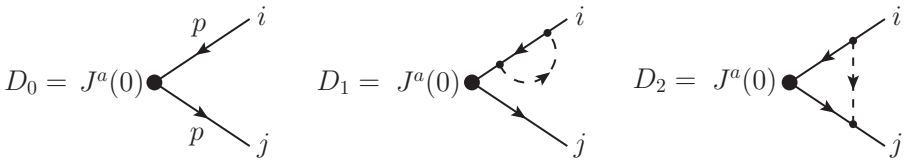


Figure 4.33: Diagrams contributing to $\langle J^a(0)\psi^i_0(p)\bar{\psi}_{0j}(-p)\rangle$ up to order $1/N$.

$$
\begin{aligned}
D_0 &= \frac{i\not{p}}{p^2}(-\gamma_z)\frac{i\not{p}}{p^2}(t^a)^i_j = \big(\frac{2\not{p}p_z}{(p^2)^2} - \frac{\gamma_z}{p^2}\big)(t^a)^i_j\,, \\
D_1 &= \frac{2(i)^4\mu^{2\Delta}}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{\not{p}\not{p}_1\not{p}\,(-\gamma_z)\not{p}\,\tilde{C}_{\sigma 0}}{(p^2)^3p_1^2(p-p_1)^{2(\frac{d}{2}-1+\Delta)}}(t^a)^i_j\,, \\
D_2 &= \frac{(i)^4\mu^{2\Delta}}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{\not{p}\not{p}_1\,(-\gamma_z)\not{p}_1\not{p}\,\tilde{C}_{\sigma 0}}{(p^2)^2(p_1^2)^2(p-p_1)^{2(\frac{d}{2}-1+\Delta)}}(t^a)^i_j
\end{aligned}
\tag{4.8.6}
$$

and

$$
\langle J^a(0)\psi_0^i(p)\bar{\psi}_{0j}(-p)\rangle = D_0 + D_1 + D_2 + \mathcal{O}(1/N^2)\,. \tag{4.8.7}
$$

Computing the diagrams and using (4.8.4) and (4.8.5) we find

$$
Z_J = 1 + \mathcal{O}(1/N^2)\,. \tag{4.8.8}
$$

4.9 Appendix E: Integrals and results

Integrals for C_J for the $O(N)$ scalar theory in $6-\epsilon$ (figure 4.1)

Explicitly, the diagrams are:

$$\begin{aligned}
D_0 &= \int \frac{d^d p_1}{(2\pi)^d} \frac{(p_{1z}+2p_z)^2}{p_1^2 (p+p_1)^2} \\
&= \frac{\pi^{-\frac{d}{2}} \Gamma(2-\frac{d}{2}) \Gamma(\frac{d}{2}-1)^2}{2^d (d-1)\Gamma(d-2)} \frac{p_z^2}{(p^2)^{2-\frac{d}{2}}} \,, \\
D_1 &= 2g_1^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{(2p_{1z}+p_z)^2}{(p_1^2)^2 (p+p_1)^2 (p_1-p_2)^2 p_2^2} \\
&= 2g_1^2 \frac{16-6d+d^2}{(d-6)(d-4)} \frac{p_z^2}{p^4} I_1 \,, \\
D_2 &= g_1^2 \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{(p_{1z}+2p_z)(p_{2z}+2p_z)}{p_1^2 p_2^2 (p+p_1)^2 (p+p_2)^2 (p_1-p_2)^2} \\
&= g_1^2 \frac{(32-60d+11d^2-2d^3) I_1 + (8-2d) p^2 I_2}{(d-4)^2 (d-1)} \frac{p_z^2}{p^4} \,.
\end{aligned} \tag{4.9.1}$$

We perform tensor reduction to get rid of the z indices, converting each integral into a sum of many scalar integrals with integer indices. Using FIRE [118], which implements integration by parts relations, we can reduce these into a small number of "master integrals". In the two loop case, the master integrals I_1, and I_2 can be easily evaluated:

$$I_1 = l(1,1) l(1, 2-\frac{d}{2}) \frac{1}{(p^2)^{3-d}} \,, \qquad I_2 = l(1,1) l(1,1) \frac{1}{(p^2)^{4-d}} \,, \tag{4.9.2}$$

where $l(\alpha, \beta)$ is the integral defined in (4.3.41).

Integrals for C_T for the $O(N)$ scalar theory in $d = 6 - \epsilon$ (figure 4.2)

Explicitly, the diagrams are:

$$\begin{aligned}
D_0 &= \int \frac{d^d p_1}{(2\pi)^d} \frac{(2p_{1z}(p_{1z}+p_z)+cp_z^2)^2}{(p_1)^2(p_1+p)^2} \\
&= \frac{(d-2)\pi^{-\frac{d}{2}}\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2}{2^{d+1}(d-1)^2(d+1)\Gamma(d-2)} \frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,, \\
D_1 &= 2\int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{(2p_{1z}(p_{1z}+p_z)+cp_z^2)^2}{(p_1+p)^2(p_1^2)^2(p_1-p_2)^2 p_2^2} \\
&= \frac{-768+864d-232d^2+36d^3-6d^4+d^5}{6(d-6)(d-4)(d-1)^2(3d-4)} \frac{p_z^4}{p^4} I_1\,, \\
D_2 &= \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{(2p_{1z}(p_{1z}+p_z)+cp_z^2)(2p_{2z}(p_{2z}+p_z)+cp_z^2)}{(p_1+p)^2 p_1^2 (p_2+p)^2 p_2^2 (p_1-p_2)^2} \\
&= \frac{(-768+608d+536d^2-700d^3+238d^4-29d^5+d^6)I_1 + (-192+192d-36d^2)p^2 I_2}{6(d-4)^2(d-1)^2(d+1)(3d-4)} \frac{p_z^4}{p^4}\,.
\end{aligned} \tag{4.9.3}$$

Integral for the anomalous dimension η of ϕ-field (figure 4.4)

The diagram reads

$$D_1 = \delta^{ij} \frac{\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{\sigma 0}}{(p_1^2)^{\frac{d}{2}-2+\Delta}(p+p_1)^2} \tag{4.9.4}$$

and can be easily computed using the integral (4.3.41).

Integrals for Z_T-factor for the critical scalar (figure 4.5)

$$\begin{aligned}
D_0 &= \frac{2p_z^2}{(p^2)^2}\,, \\
D_1 &= \frac{2\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{\sigma 0} 2p_z^2}{(p^2)^2 (p+p_1)^2 (p_1^2)^{\frac{d}{2}-2+\Delta}}\,, \\
D_2 &= \frac{\mu^{2\Delta}}{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{\tilde{C}_{\sigma 0} 2p_{1z}^2}{(p^2)^2 (p_1^2)^2 (p-p_1)^{2(\frac{d}{2}-2+\Delta)}}\,, \\
D_3 &= \frac{\mu^{4\Delta}}{N} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{\tilde{C}_{\sigma 0}^2 2p_{1z}^2}{(p^2)^2 (p-p_2)^2 (p_2^2)^{2(\frac{d}{2}-2+\Delta)} (p_2-p_1)^2 (p_1^2)^2}\,.
\end{aligned} \tag{4.9.5}$$

These diagrams can be easily calculated with the use of elementary integral (4.3.41).

Integrals for Z_J-factor for the critical scalar for (figure 4.6)

$$
\begin{aligned}
D_0 &= \frac{i2p_z}{(p^2)^2}(t^a)^{ij}\,,\\
D_1 &= \frac{2\mu^{2\Delta}}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{\tilde{C}_{\sigma 0}i2p_z}{(p^2)^2(p+p_1)^2(p_1^2)^{\frac{d}{2}-2+\Delta}}(t^a)^{ij}\,,\\
D_2 &= \frac{\mu^{2\Delta}}{N}\int\frac{d^dp_1}{(2\pi)^d}\frac{\tilde{C}_{\sigma 0}i2p_{1z}}{(p^2)^2(p_1^2)^2(p-p_1)^{2(\frac{d}{2}-2+\Delta)}}(t^a)^{ij}\,.
\end{aligned}
\tag{4.9.6}
$$

These diagrams can be easily calculated with the use of elementary integral (4.3.41).

Integrals for C_J for the critical scalar (figure 4.7)

Explicitly, the diagrams are

$$
\begin{aligned}
D_0 &= \frac{1}{2}\mathrm{tr}(t^at^b)(i)^2\int\frac{d^dp_1}{(2\pi)^d}\frac{(2p_{1z}+p_z)^2}{p_1^2(p_1+p)^2} = \frac{1}{2}\mathrm{tr}(t^at^b)\frac{4^{1-d}\pi^{\frac{3-d}{2}}}{\sin(\pi d/2)\Gamma\left(\frac{d+1}{2}\right)}\frac{p_z^2}{(p^2)^{2-\frac{d}{2}}}\,,\\
D_1 &= 2\cdot\frac{1}{2}\mathrm{tr}(t^at^b)(i)^2\frac{\mu^{2\Delta}}{N}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\tilde{C}_{\sigma 0}(2p_{1z}+p_z)^2}{(p_1+p)^2(p_1^2)^2(p_1-p_2)^2(p_2^2)^{\frac{d}{2}-2+\Delta}}\\
&= \frac{1}{N}\eta_1^{\mathrm{O(N)}}D_0\bigg(-2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)-\Big(2\mathcal{C}_{\ \mathrm{O(N)}}(d)+\frac{2\left(10d^3-47d^2+56d-16\right)}{(d-4)(d-2)(d-1)d}\Big)\bigg)\,,\\
D_2 &= \frac{1}{2}\mathrm{tr}(t^at^b)(i)^2\frac{\mu^{2\Delta}}{N}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\tilde{C}_{\sigma 0}(2p_{1z}+p_z)(2p_{2z}+p_z)}{(p_1+p)^2p_1^2(p_2+p)^2p_2^2(p_1-p_2)^{2(\frac{d}{2}-2+\Delta)}}\\
&= \frac{1}{N}\eta_1^{\mathrm{O(N)}}D_0\bigg(2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)+\Big(2\mathcal{C}_{\ \mathrm{O(N)}}(d)+\frac{2(2d-5)(3d-4)}{(d-4)(d-2)(d-1)}\Big)\bigg)\,.
\end{aligned}
\tag{4.9.7}
$$

Integrals for C_T for the critical scalar (figure 4.9)

Explicitly, the diagrams are:

$$
\begin{aligned}
D_0 &= \frac{N}{2}\int\frac{d^dp_1}{(2\pi)^d}\frac{(2p_{1z}(p_{1z}+p_z)+cp_z^2)^2}{p_1^2(p_1+p)^2}\\
&= N\frac{(\frac{d}{2}-1)\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2}{2(4\pi)^{\frac{d}{2}}(d-1)^2(d+1)\Gamma(d-2)}\frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,,\\
D_1 &= \mu^{2\Delta}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\tilde{C}_{\sigma 0}(2p_{1z}(p_{1z}+p_z)+cp_z^2)^2}{(p_1+p)^2(p_1^2)^2(p_1-p_2)^2(p_2^2)^{\frac{d}{2}-2+\Delta}}\\
&= \eta_1^{\mathrm{O(N)}}D_0\bigg(-2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)-2\Big(\mathcal{C}_{\ \mathrm{O(N)}}(d)+\frac{11d^4-45d^3+26d^2+36d-16}{(d-4)(d-2)(d-1)d(d+1)}\Big)\bigg)\,,\\
D_2 &= \frac{\mu^{2\Delta}}{2}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\tilde{C}_{\sigma 0}(2p_{1z}(p_{1z}+p_z)+cp_z^2)(2p_{2z}(p_{2z}+p_z)+cp_z^2)}{(p_1+p)^2p_1^2(p_2+p)^2p_2^2(p_1-p_2)^{2(\frac{d}{2}-2+\Delta)}}\\
&= \eta_1^{\mathrm{O(N)}}D_0\bigg(2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)\Big(\frac{d-2}{d+2}\Big)+2\Big(\frac{d-2}{d+2}\mathcal{C}_{\ \mathrm{O(N)}}(d)+\frac{3\left(3d^3-11d^2+4d+8\right)}{(d-4)(d-1)(d+1)(d+2)}\Big)\bigg)\,,
\end{aligned}
\tag{4.9.8}
$$

and

$$D_3 = \frac{\mu^{4\Delta}}{2}\int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \frac{\tilde{C}^2_{\sigma 0}(2p_{1z}(p_{1z}+p_z)+cp_z^2)(2p_{2z}(p_{2z}+p_z)+cp_z^2)}{p_1^2(p+p_1)^2(p_1-p_3)^2(p_3^2)^{\frac{d}{2}-2+\Delta}(p_3+p)^{2(\frac{d}{2}-2+\Delta)}p_2^2(p+p_2)^2(p_2-p_3)^2}$$
$$= \eta_1^{\mathrm{O(N)}} D_0\bigg(\Big(\frac{1}{\Delta}-2\log(p^2/\mu^2)\Big)\Big(\frac{4}{d+2}\Big)+\Big(\frac{4}{d+2}\mathcal{C}_{\mathrm{O(N)}}(d)+\frac{2\left(d^4+18d^3-93d^2+66d+56\right)}{(d-4)(d-2)(d-1)(d+1)(d+2)}\Big)\bigg),$$

where $c \equiv \frac{d-2}{2(d-1)}$.

Integrals for C_J for the critical fermion (figure 4.16)

The integrals are

$$D_0 = (-1)(i)^2\mathrm{tr}(t^at^b)\int\frac{d^dp_1}{(2\pi)^d}\frac{\mathrm{Tr}\big(\not{p}\gamma_z(\not{p}+\not{p}_1)\gamma_z\big)}{p^2(p+p_1)^2} = \frac{\mathrm{tr}(t^at^b)\mathrm{Tr}\mathbf{1}\pi^{1-\frac{d}{2}}\Gamma(\frac{d}{2})}{4^{\frac{d}{2}}(d-1)\Gamma(d-2)\sin(\pi d/2)}\frac{p_z^2}{(p^2)^{2-\frac{d}{2}}},$$

$$D_1 = \frac{2(-1)(i)^4\mathrm{tr}(t^at^b)\mu^{2\Delta}}{N}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\mathrm{Tr}\big(\gamma_z(\not{p}+\not{p}_1)\gamma_z\not{p}_1\not{p}_2\not{p}_1\big)\tilde{C}_{\sigma 0}}{(p+p_1)^2(p_1^2)^2(p_1-p_2)^{2(\frac{d}{2}-1+\Delta)}p_2^2}$$
$$= \frac{1}{N}\eta_1^{\mathrm{GN}}D_0\bigg(-2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)-2\Big(\mathcal{C}_{\mathrm{GN}}(d)+\frac{5d^2-10d+4}{(d-2)(d-1)d}\Big)\bigg), \tag{4.9.9}$$

$$D_2 = \frac{(-1)(i)^4\mathrm{tr}(t^at^b)\mu^{2\Delta}}{N}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\mathrm{Tr}\big(\gamma_z(\not{p}+\not{p}_1)(\not{p}+\not{p}_2)\gamma_z\not{p}_2\not{p}_1\big)\tilde{C}_{\sigma 0}}{(p+p_1)^2(p+p_2)^2p_1^2(p_1-p_2)^{2(\frac{d}{2}-1+\Delta)}p_2^2}$$
$$= \frac{1}{N}\eta_1^{\mathrm{GN}}D_0\bigg(2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)+2\Big(\mathcal{C}_{\mathrm{GN}}(d)+\frac{1}{d-1}\Big)\bigg).$$

Integrals for C_T for the critical fermion (figure 4.18)

The integrals are

$$D_0 = \frac{(-1)(i)^4\tilde{N}}{2^2}\int\frac{d^dp_1}{(2\pi)^d}\frac{(2p_{1z}+p_z)^2\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)\gamma_z\not{p}_1)}{p_1^2(p+p_1)^2}$$
$$= \frac{-N\pi^{1-\frac{d}{2}}\Gamma(\frac{d}{2})}{4^{\frac{d}{2}+1}(d-1)(d+1)\Gamma(d-2)\sin(\pi d/2)}\frac{p_z^4}{(p^2)^{2-\frac{d}{2}}},$$

$$D_1 = \frac{(-1)(i)^6\tilde{N}\mu^{2\Delta}}{2N}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{(2p_{1z}+p_z)^2\mathrm{Tr}\big(\gamma_z(\not{p}+\not{p}_1)\gamma_z\not{p}_1\not{p}_2\not{p}_1\big)\tilde{C}_{\sigma 0}}{(p+p_1)^2(p_1^2)^2(p_1-p_2)^{2(\frac{d}{2}-1+\Delta)}p_2^2}$$
$$= \eta_1^{\mathrm{GN}}D_0\bigg(-2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)-2\Big(\mathcal{C}_{\mathrm{GN}}(d)+\frac{2\left(3d^3-4d^2-2d+2\right)}{(d-2)(d-1)d(d+1)}\Big)\bigg), \tag{4.9.10}$$

$$D_2 = \frac{(-1)(i)^6\tilde{N}\mu^{2\Delta}}{2^2N}\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{(2p_{1z}+p_z)(2p_{2z}+p_z)\mathrm{Tr}\big(\gamma_z(\not{p}+\not{p}_1)(\not{p}+\not{p}_2)\gamma_z\not{p}_2\not{p}_1\big)\tilde{C}_{\sigma 0}}{(p+p_1)^2(p+p_2)^2p_1^2(p_1-p_2)^{2(\frac{d}{2}-1+\Delta)}p_2^2}$$
$$= \eta_1^{\mathrm{GN}}D_0\bigg(2\Big(\frac{1}{\Delta}-\log(p^2/\mu^2)\Big)\Big(\frac{d-2}{d+2}\Big)+2\Big(\frac{d-2}{d+2}\mathcal{C}_{\mathrm{GN}}(d)+\frac{2\left(2d^2-2d-1\right)}{(d-1)(d+1)(d+2)}\Big)\bigg).$$

The three-loop Aslamazov-Larkin contribution is [15]

$$\begin{aligned} D_3 =& \frac{(-1)^2(i)^8\tilde{N}^2\mu^{4\Delta}}{2N^2}\int\frac{d^dp_1d^dp_2d^dp_3}{(2\pi)^{3d}} \\ &\times\frac{(2p_{1z}+p_z)(2p_{2z}+p_z)\mathrm{Tr}\big(\gamma_z(\not{p}+\not{p}_1)(\not{p}_1-\not{p}_3)\not{p}_1)\mathrm{Tr}(\gamma_z\not{p}_2(\not{p}_2-\not{p}_3)(\not{p}+\not{p}_2)\big)\tilde{C}^2_{\sigma 0}}{p_1^2(p+p_1)^2(p_1-p_3)^2p_3^{2(\frac{d}{2}-1+\Delta)}(p+p_3)^{2(\frac{d}{2}-1+\Delta)}(p_2-p_3)^2(p+p_2)^2p_2^2} \\ =&\eta_1^{\mathrm{GN}}D_0\bigg(2\Big(\frac{1}{\Delta}-2\log(p^2/\mu^2)\Big)\Big(\frac{2}{d+2}\Big)+2\Big(\frac{2}{d+2}\mathcal{C}_{\mathrm{GN}}(d)+\frac{2d(d+5)}{(d-1)(d+1)(d+2)}\Big)\bigg). \end{aligned} \tag{4.9.11}$$

Integrals for C_J for the GNY model in $d=4-\epsilon$ (figure 4.20)

$$\begin{aligned} D_0 &= -\tilde{N}\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}(\not{p}_1\gamma_z(\not{p}+\not{p}_1)\gamma_z)}{p^2(p+p_1)^2} \\ &= N\frac{(d-2)\pi^{-\frac{d}{2}}\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)^2}{2^{d+1}(d-1)\Gamma(d-2)}\frac{p_z^2}{(p^2)^{2-\frac{d}{2}}}\,, \\ D_1 &= 2\tilde{N}g_1^2\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)\gamma_z\not{p}_1(\not{p}_1-\not{p}_2)\not{p}_1)}{(p+p_1)^2(p_1^2)^2(p_1-p_2)^2p_2^2} \\ &= -Ng_1^2\frac{4(d-3)I_1}{3(d-4)}\frac{p_z^2}{p^2}\,, \\ D_2 &= \tilde{N}g_1^2\int\frac{d^dp_1d^dp_2}{(2\pi)^{2d}}\frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)(\not{p}+\not{p}_2)\gamma_z\not{p}_2\not{p}_1)}{(p+p_1)^2(p+p_2)^2p_1^2(p_1-p_2)^2p_2^2} \\ &= Ng_1^2\frac{(d-3)(-4I_1+3p^2I_2)}{6(d-1)}\frac{p_z^2}{p^2}\,. \end{aligned} \tag{4.9.12}$$

[15] We used the fact that in D_3 the two diagrams with different orientation of the fermion loop are equal due to the identity $\mathrm{Tr}(\not{A}\not{B}\not{C}\not{D})=\mathrm{Tr}(\not{A}\not{D}\not{C}\not{B})=\mathrm{Tr}\mathbf{1}(A\cdot B\,C\cdot D+A\cdot D\,B\cdot C-A\cdot C\,B\cdot D)$.

Integrals for C_T for the GNY model in $d = 4 - \epsilon$ (figure 4.21)

These integrals are equal to:

$$\begin{aligned}
D_1 &= 2\tilde{N}g_1^2 \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\frac{1}{4}(2p_{1z}+p_z)^2 \mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)\gamma_z \not{p}_1 \not{p}_2 \not{p}_1)}{(p+p_1)^2(p_1^2)^2 p_2^2 (p_1-p_2)^2} \\
&= -N \frac{(d-3)(8-2d+d^2)I_1}{3(d-4)(3d-4)(3d-2)} \frac{p_z^4}{p^2}\,, \\
D_2 &= \tilde{N}g_1^2 \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\frac{1}{4}(2p_{1z}+p_z)(2p_{2z}+p_z) \mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)(\not{p}+\not{p}_2)\gamma_z \not{p}_2 \not{p}_1)}{(p+p_1)^2 p_1^2 p_2^2 (p_1-p_2)^2 p_1^2} \\
&= N \frac{(d-3)\left((-32+40d+12d^2-24d^3+4d^4)I_1 + (24-54d+27d^2)p^2 I_2\right)}{24(d-1)^2(d+1)(3d-4)(3d-2)} \frac{p_z^4}{p^2}\,, \\
D_3 &= 2\tilde{N}g_1^2 \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\frac{1}{2}(2p_{1z}(p_{1z}+p_z)+cp_z^2)(2p_{2z}+p_z) \mathrm{Tr}(\gamma_z \not{p}_2(-\not{p}_1+\not{p}_2)(\not{p}+\not{p}_2))}{(p+p_1)^2 p_1^2 (p_1-p_2)^2 (p+p_2)^2 p_1^2} \\
&= N \frac{\left((-48+76d+16d^2-57d^3+18d^4-d^5)I_1 + (24-54d+27d^2)p^2 I_2\right)}{12(d-1)^2(d+1)(3d-4)(3d-2)} \frac{p_z^4}{p^2}\,, \\
D_4 &= -\tilde{N}g_1^2 \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{(2p_{1z}(p_{1z}+p_z)+cp_z^2)^2 \mathrm{Tr}(\not{p}_2(\not{p}_1-\not{p}_2))}{(p+p_1)^2(p_1^2)^2 p_2^2 (p_1-p_2)^2} \\
&= -N \frac{(d-2)^2 d(d+2)(d+4)I_1}{24(d-4)(d-1)^2(3d-4)(3d-2)} \frac{p_z^4}{p^2}\,.
\end{aligned} \tag{4.9.13}$$

As before, we have a factor of 2 in the diagram D_1 to account for the fact that the loops may renormalize either the top or bottom line.

Integrals for C_J for the GN model in $d = 2 + \epsilon$ (figure 4.22 and 4.23)

We have:

$$\begin{aligned}
D_0 &= \tilde{N} \int \frac{d^dk}{(2\pi)^d} \frac{\mathrm{Tr}(\not{p}\gamma_z(\not{p}+\not{p}_1)\gamma_z)}{p^2(p+p_1)^2} = -N \frac{\pi^{1-\frac{d}{2}} \csc(\pi\frac{d}{2})\Gamma(\frac{d}{2})}{2^d(d-1)\Gamma(d-2)} \frac{p_z^2}{(p^2)^{2-\frac{d}{2}}}\,, \\
D_1 &= g\tilde{N} \int \frac{d^dp_1 d^dp_2}{(2\pi)^{2d}} \frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)(\not{p}+\not{p}_2)\gamma_z \not{p}_2 \not{p}_1)}{(p+p_1)^2(p+p_2)^2 p_1^2 p_2^2} = -gN \frac{\pi^{2-d}\csc^2(\pi\frac{d}{2})\Gamma(\frac{d}{2})^2}{4^d(1-d)^2\Gamma(d-2)^2} \frac{p_z^2}{(p^2)^{3-d}}\,.
\end{aligned} \tag{4.9.14}$$

$$\begin{aligned}
D_2 =& g^2\tilde{N} \int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)(\not{p}+\not{p}_3)(\not{p}+\not{p}_2)\gamma_z \not{p}_2 \not{p}_3 \not{p}_1)}{p_1^2(p_1+p)^2 p_2^2 (p_2+p)^2 p_3^2 (p_3+p)^2} = g^2 N \frac{(d-2)^3 p^4 p_z^2}{8(d-1)^3} M_2 \\
D_3 =& g^2\tilde{N}^2 \int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)\gamma_z \not{p}_1(\not{p}_3-\not{p}_2)\not{p}_1)\mathrm{Tr}(\not{p}_2(\not{p}_3-\not{p}_1))}{(p_1^2)^2(p_1+p)^2(p_1-p_3)^2 p_2^2 (p_2-p_3)^2} \\
& - g^2\tilde{N} \int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)\gamma_z \not{p}_1(\not{p}_3-\not{p}_2)(\not{p}_3-\not{p}_1)\not{p}_2 \not{p}_1)}{(p_1^2)^2(p_1+p)^2(p_1-p_3)^2 p_2^2 (p_2-p_3)^2} \\
=& g^2 \frac{N(N-1)(d-2)^2 p_z^2}{4(3d-4)(2d-3)} M_1
\end{aligned} \tag{4.9.15}$$

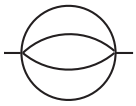
and

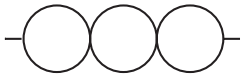
$$
\begin{aligned}
D_4 =& g^2\tilde{N}^2 \int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)(\not{p}+\not{p}_2)\gamma_z\not{p}_2\not{p}_1)\mathrm{Tr}((\not{p}_1-\not{p}_3)(\not{p}_2-\not{p}_3))}{p_1^2(p_1+p)^2(p_1-p_3)^2p_2^2(p_2+p)^2(p_2-p_3)^2} \\
&- 2g^2\tilde{N} \int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)(\not{p}_1-\not{p}_3)(\not{p}_2-\not{p}_3)(\not{p}+\not{p}_2)\gamma_z\not{p}_2\not{p}_1)}{p_1^2(p_1+p)^2(p_1-p_3)^2p_2^2(p_2+p)^2(p_2-p_3)^2} \\
&- g^2\tilde{N} \int \frac{d^dp_1 d^dp_2 d^dp_3}{(2\pi)^{3d}} \frac{\mathrm{Tr}(\gamma_z(\not{p}+\not{p}_1)(\not{p}_1-\not{p}_3)(-\not{p}_2)\gamma_z(-\not{p}-\not{p}_2)(-\not{p}_2+\not{p}_3)\not{p}_1)}{p_1^2(p_1+p)^2(p_1-p_3)^2p_2^2(p_2+p)^2(p_2-p_3)^2} \\
=& g^2 \frac{N(d-2)^2\left(\left(f_1(d)+Nf_2(d)\right)M_1 - 12p^4(d-3)^2(12-17d+6d^2)(11-9d+N(3d-3))M_3\right)p_z^2}{36(3d-4)^2(d-3)(d-1)^2(2d-3)(3d-8)}\,.
\end{aligned}
\tag{4.9.16}
$$

where:

$$
\begin{aligned}
f_1(d) &= 86112 - 260472d + 307525d^2 - 176601d^3 + 49203d^4 - 5319d^5 \\
f_2(d) &= 3(-7776 + 24912d - 30833d^2 + 18395d^3 - 5283d^4 + 585d^5)
\end{aligned}
\tag{4.9.17}
$$

After evaluating the traces and performing tensor reduction, each integral becomes a sum of many scalar integrals of the ladder-type with integer indices. Using FIRE [118] to apply integration by parts relations, we can convert all of them into a sum of three master integrals, M_1, M_2, and M_3 as shown in 4.34.

$M_1 =$  $= l(1,1)l(1,2-d/2)l(1,3-d)(p^2)^{3d/2-4}$

$M_2 =$  $= l(1,1)^3(p^2)^{3d/2-6}$

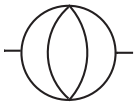
$M_3 =$  $= K(1,1,1,1,2-d/2)l(1,1)(p^2)^{3d/2-6}$

Figure 4.34: Master integrals

The first two master integrals are primitive and can be readily evaluated with the use of the integral (4.3.41). The integral $K(1,1,1,1,2-d/2)$ (defined in Appendix B (4.6.3)) in the third master integral can be evaluated using the Gegenbauer Polynomial technique [103, 104]. Its expansion in

$d = 2 + \epsilon$ is[16]:

$$K(1,1,1,1,2-d/2) = \\ = \frac{7}{6\pi^2\epsilon^2} + \frac{14(\gamma - \log(4\pi)) - 25}{12\pi^2\epsilon} + \frac{84(\gamma - \log(4\pi))^2 - 300(\gamma - \log(4\pi)) - 7\pi^2 - 228}{144\pi^2} + \mathcal{O}(\epsilon)\,. \tag{4.9.18}$$

Integrals for C_T for the GN model in $d = 2 + \epsilon$ (figure 4.24)

The integrals are

$$D_0 = \tilde{N} \int \frac{d^d p_1}{(2\pi)^d} \frac{1}{4} (2p_{1z} + p_z)^2 \frac{\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)\gamma_z \not{p}_1)}{p_1^2 (p_1 + p)^2} = -N \frac{\pi^{1-\frac{d}{2}} \csc(\pi \frac{d}{2}) \Gamma(\frac{d}{2})}{4^{\frac{d}{2}+1}(d^2-1)\Gamma(d-2)} \frac{p_z^4}{(p^2)^{2-\frac{d}{2}}}\,, \\ D_1 = -g\tilde{N} \int \frac{d^d p_1 d^d p_2}{(2\pi)^{2d}} \frac{1}{4} (2p_{1z} + p_z)(2p_{2z} + p_z) \frac{\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)(\not{p} + \not{p}_2)\gamma_z \not{p}_2 \not{p}_1)}{p_1^2 (p_1 + p)^2 p_2^2 (p_2 + p)^2} = 0\,. \tag{4.9.19}$$

and

$$\begin{aligned} D_2 =& g^2 \tilde{N} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} \frac{1}{4} (2p_{1z} + p_z)(2p_{2z} + p_z) \frac{\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)(\not{p} + \not{p}_3)(\not{p} + \not{p}_2)\gamma_z \not{p}_2 \not{p}_3 \not{p}_1)}{p_1^2 (p_1 + p)^2 p_2^2 (p_2 + p)^2 p_3^2 (p_3 + p)^2} = 0\,, \\ D_3 =& g^2 \tilde{N}^2 \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} \frac{1}{4} (2p_{1z} + p_z)^2 \frac{\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)\gamma_z \not{p}_1 (\not{p}_3 - \not{p}_2)\not{p}_1)\mathrm{Tr}(\not{p}_2(\not{p}_3 - \not{p}_1))}{(p_1^2)^2 (p_1 + p)^2 (p_1 - p_3)^2 p_2^2 (p_2 - p_3)^2} \\ & - g^2 \tilde{N} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} \frac{1}{4} (2p_{1z} + p_z)^2 \frac{\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)\gamma_z \not{p}_1 (\not{p}_3 - \not{p}_2)(\not{p}_3 - \not{p}_1)\not{p}_2 \not{p}_1)}{(p_1^2)^2 (p_1 + p)^2 (p_1 - p_3)^2 p_2^2 (p_2 - p_3)^2} \\ =& g^2 \frac{N(N-1)(d-2)^2(2 - 2d + d^2)}{32(d-1)(2d-3)(2d-1)(3d-4)} M_1 p_z^4 \end{aligned} \tag{4.9.20}$$

and

$$\begin{aligned} D_4 =& \frac{g^2 \tilde{N}^2}{4} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} \frac{(2p_{1z} + p_z)(2p_{2z} + p_z)\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)(\not{p} + \not{p}_2)\gamma_z \not{p}_2 \not{p}_1)\mathrm{Tr}((\not{p}_1 - \not{p}_3)(\not{p}_2 - \not{p}_3))}{p_1^2 (p_1 + p)^2 (p_1 - p_3)^2 p_2^2 (p_2 + p)^2 (p_2 - p_3)^2} \\ & + \frac{2g^2 \tilde{N}^2}{4} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} \frac{(2p_{1z} + p_z)(2p_{2z} + p_z)\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)(\not{p}_1 - \not{p}_3)\not{p}_1)\mathrm{Tr}(\gamma_z \not{p}_2 (\not{p}_2 - \not{p}_3)(\not{p} + \not{p}_2))}{p_1^2 (p_1 + p)^2 (p_1 - p_3)^2 p_2^2 (p_2 + p)^2 (p_2 - p_3)^2} \\ & - \frac{2g^2 \tilde{N}}{4} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} \frac{(2p_{1z} + p_z)(2p_{2z} + p_z)\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)(\not{p}_1 - \not{p}_3)(\not{p}_2 - \not{p}_3)(\not{p} + \not{p}_2)\gamma_z \not{p}_2 \not{p}_1)}{p_1^2 (p_1 + p)^2 (p_1 - p_3)^2 p_2^2 (p_2 + p)^2 (p_2 - p_3)^2} \\ & + \frac{g^2 \tilde{N}}{4} \int \frac{d^d p_1 d^d p_2 d^d p_3}{(2\pi)^{3d}} \frac{(2p_{1z} + p_z)(2p_{2z} + p_z)\mathrm{Tr}(\gamma_z(\not{p} + \not{p}_1)(\not{p}_1 - \not{p}_3)(-\not{p}_2)\gamma_z(-\not{p} - \not{p}_2)(-\not{p}_2 + \not{p}_3)\not{p}_1)}{p_1^2 (p_1 + p)^2 (p_1 - p_3)^2 p_2^2 (p_2 + p)^2 (p_2 - p_3)^2} \\ =& g^2 \frac{N(N-1)(d-2)^2 \Big(f_3(d) M_1 + 24 p^4 (d-3)^2 (24 - 118d + 203d^2 - 144d^3 + 36d^4) M_3 \Big)}{288(3d-4)^2 (d-3)(d-1)(d+1)(2d-3)(2d-1)(3d-8)(3d-2)} p_z^4\,, \end{aligned} \tag{4.9.21}$$

[16] Using the Gegenbauer Polynomial technique for the integral $K(1,1,1,1,2-d/2)$ we obtain an analytic expression for any d. This expression includes a hypergeometric function. To expand the hypergeometric function in $d = 2 + \epsilon$ we used the program HypExp [127].

where

$$f_3(d) = -28800 + 164832d - 368444d^2 + 406366d^3 - 234072d^4 + 67473d^5 - 7884d^6 + 81d^7 \quad (4.9.22)$$

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