Aspects of Holomorphic Sectors In Supersymmetric Theories

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A Dissertation
Presented to the Faculty
of Princeton University
in Candidacy for the Degree
of Doctor of Philosophy

Recommended for Acceptance
by the Department of
Physics
Adviser: Edward Witten

September 2016

Abstract

In this dissertation, we discuss some aspects of theories with extended supersymmetry that have interesting, exactly calculable holomorphic sectors. The two classes of theories we consider are $d = 4, \mathcal{N} = 2$ effective supergravities that describe Calabi-Yau compactifications of a Type IIA superstring, and two-dimensional theories with $\mathcal{N} = (0, 2)$ supersymmetry. In the first case, we study higher-derivative couplings in the 4d $\mathcal{N} = 2$ superpotential (as well as 2d $\mathcal{N} = (2, 2)$ superpotential in the presence of D4 branes), which is a holomorphic function of chiral superfields. It is described by the Gopakumar-Vafa formula in terms of BPS spectra of M-theory compactifications (and Ooguri-Vafa formula for the 2d $\mathcal{N} = (2, 2)$ case). In the second class of theories, we study another holomorphic object known as a chiral algebra, which emerges in the cohomology of one supercharge of a two-dimensional theory with $\mathcal{N} = (0, 2)$ supersymmetry.

In chapter two, we describe a detailed derivation of the Gopakumar-Vafa formula, as well as explain the Ooguri-Vafa formula at the end. The main idea of the derivation is to compute the effective superspace action on a properly chosen background due to BPS states winding the M-theory circle. A lot of technical and conceptual details, such as how supersymmetry of the background determines the action for BPS particles, why and in which limit the computation makes sense, are explained along the way.

In chapter three, we explore chiral algebras of $\mathcal{N} = (0, 2)$ theories. We explain why these objects are invariant along the RG flows and study some of their general properties. We give more details for theories known as $\mathcal{N} = (0, 2)$ Landau-Ginzburg (LG) models, and later we specialize to $\mathcal{N} = (2, 2)$ supersymmetry, for which we consider some concrete examples, such as LG models which flow to $\mathcal{N} = 2$ minimal models in the infrared.

Acknowledgements

I am grateful to my graduate adviser Edward Witten for his guidance and encouragement and all of the insightful advice he gave over the past four years. It was a great honor to be his student, and I have learned a lot from it.

I also want to thank Igor Klebanov and Peter Myers for serving on my dissertation committee, and Herman Verlinde for being a reader of this thesis.

I would like to thank the faculty, postdocs, students and staff of Princeton University and the Institute for Advanced Study. Over the past five years, it was a great pleasure to have discussions with: Chris Beem, Bruno Le Floch, Igor Klebanov, Aitor Lewkowycz-Zapirain, Victor Mikhailov, Vasily Pestun, Alexander Polyakov, Silviu Pufu, Grigory Tarnopolskiy, Joaquin Turiaci, Herman Verlinde, Itamar Yaakov, Ran Yakoby and Alexander Zhiboedov. I also want to thank Silviu Pufu and Ran Yakoby for their collaboration on another project (which is not part of this thesis).

I am grateful to my undergraduate adviser Andrei Losev for educating me on various issues in theoretical physics, as well as plotting my education at the Independent University of Moscow and the Math Department of the Higher School of Economics of Moscow. I am grateful to everybody at those two places for creating an absolutely excellent, intellectually challenging and at the same time amazingly flexible environment. I am thankful to everybody at the Science and Education Center at the Bogolyubov Institute of Theoretical Physics of Kiev for organizing student seminars which serve as a great way to learn theoretical physics for anyone interested in the subject. I am also grateful to the group of Quantum Field Theory at the Physics Department of the National Taras Shevchenko University of Kiev, especially to its head Stanislav Vilchinsky, for the personalized approach they take in educating students.

I am also deeply grateful to all my friends and family for their constant support.

In memory of my grandmother.

Contents

Chapter 1

Introduction

1.1 Supersymmetry

There are two principles in modern theoretical physics, fusion of which leads to the idea of supersymmetry. One is the fundamental principle of symmetry, which is deeply rooted in the way we experience and understand the world around us. Another is a more heuristic idea that whatever is not forbidden by fundamental laws can take place.

Symmetry has certainly been an extremely fruitful idea in physics since the earliest days of the subject. It arises both as a fundamentel principle in describing laws of Nature ("symmetries of equations") and as a simplifying device allowing to analyze properties of otherwise too complicated systems ("symmetries of solutions"). Noether's theorem tells us that "symmetries of equations" often lead to conserved charges. A specific symmetry called "Lorentz invariance" has been added to the list of fundamental laws of Nature after the discovery of Relativity – it has replaced an obsolete "Galilean invariance". It tells us that all quantities we use in physics should transform in a specific way under Lorentz transformations, i.e., are classified by representations of the Lorentz group, which in d dimensions is $SO(d-1,1)$. In particular, conserved charges themselves are classified by representations of this group, i.e., should be described as scalars, spinors, vectors etc.

We know examples of scalars: electric and color charges. Momentum is an example of vector charge, which corresponds to spacetime translations, while angular momentum provides an example of an anti-symmetric two-tensor charge, which corresponds to spacetime rotations. Can we go on and consider theories with charges transforming in arbitrary representations of the Lorentz group?

There exist no-go theorems in Quantum Field Theory, which provide answers to this kind of questions. One such theorem was proven by Coleman and Mandula [1], who demonstrated that in theories with non-trivial S-matrix and mass gap it is impossible to combine internal and space-time symmetries in a non-trivial way. In other words, any additional conserved charges should be Lorentz-scalars. In addition to the obvious fact that massless theories do not obey this theorem and can be conformal (which is a non-trivial extension of the Lorentz symmetry), this theorem had another loophole: it assumed that the algebra of symmetries was the usual Lie algebra. Generalizing to Lie superalgebras, Haag, Lopuszanski and Sohnius proved their more general result [2]. They found that in fact one can also have charges transforming in spinor representations. Such charges are fermionic objects referred to as supercharges. An improved result was that in the massive case, the symmetry algebra should be the direct sum of the super-Poincare algebra (possibly centrally extended) and the algebra of internal symmetries. In the massless case, conformal symmetry is possible, so the maximal symmetry is given by super-conformal algebra. In other words, all charges that are not in the super-Poincare or super-conformal algebra, should be scalars.[1]

So, it is natural to include charges transforming in spinorial representations of the Lorentz group, and moreover, no known principle forbids this. Such charges, as a

[1]In fact, this theorem also has loopholes which allow for the existence of theories with higher-spin symmetries.

result of spin-statistics relation, are fermionic, or Grassmann-odd objects, which of course makes them different from bosonic charges (e.g., they cannot have expectation values in any state). Finally, it is fairly obvious how to construct theories which actually have such odd charges: for every boson in such a theory there should be a fermion "superpartner" of the same mass and with the spin shifted by 1/2. Then the theory becomes invariant under Grassmann-odd rotations between bosons and fermions, provided that interactions are chosen in a specific way. These rotations are called supersymmetry, and their charges are fermionic spinorial charges that we were discussing above.

The algebra obeyed by supercharges is dictated again by representation theory of the Lorentz group. Since supercharges live in the spinor representation, their anti-commutator lives in the tensor product of two spinor representations. Decomposing this tensor product into various irreducible components, we get possible conserved charges that can appear on the right-hand side of the anticommutation relations. If we have supercharges Q_α and $Q_{\dot\alpha}$ of opposite chirality in 4 dimensions ($\alpha, \dot\alpha$ are spinor indices), then the simplest algebra is:

$$\{Q_\alpha, Q_{\dot\alpha}\} = \gamma^\mu_{\alpha\dot\alpha} P_\mu, \tag{1.1}$$

where P_μ is the momentum generator and γ^μ are gamma-matrices. In general, and in general dimension, there can be more terms on the right, such as various scalar and tensor central charges: $Z\varepsilon_{\alpha\beta}$, $Z_{\mu\nu}\gamma^{\mu\nu}_{\alpha\beta}$, etc. In some cases, Lorentz generators can also appear on the right-hand side, for example in various supersymmetric backgrounds. One example of such deformation of the supersymmetry algebra will be explored in the next Chapter of this thesis.

Theoretical discovery of supersymmetry dates back to early seventies (see [3–10]). Since then it has become an area of active research involving directly or indirectly a large part of high-energy theory community. A somewhat formal motivation for supersymmetry that we have just described is of course not the real explanation of its popularity. The actual reason is that theories with supersymmetry have various very nice properties which non-supersymmetric theories do not have.

One such property is an improved ultraviolet (UV) behavior. In supersymmetric theories, UV divergences are often either absent or much milder than in non-supersymmetric theories, and behavior of radiative corrections can therefore be quite different. This gives a framework for solving the fine-tuning problem of the Standard Model, that is stabilizing the otherwise short-distance sensitive mass of the Higgs boson (which was recently found to be at 125 MeV). If supersymmetry, which is not observed in the spectrum of real-world particles, is spontaneously broken around the weak scale, it can protect Higgs boson's mass from high radiative corrections, therefoe solving fine-tuning. Also, if supersymmetry is broken dynamically by small instanton effects in the underlying theory [11, 12], then the scale of this breaking is much smaller than the Planck scale, and this solves another famous problem of the Standard Model – the hierarchy problem. The symmetry associated to the small parameter (weak scale)/(Planck scale) naturally becomes the supersymmetry.

All this makes supersymmetry very attractive phenomenologically, and so a lot of work has been done in constructing realistic supersymmetric extensions of the Standard Model. Supersymmetry is also naturally incorporated in the String Theory, the most promising candidate for the "Theory of Everything".

Another reason for studying supersymmetric theories (which, in a sense, prevails in HEP-Th community) is that in theoretical physics, we like to have "theoretical laboratories", that is theories which do not necessarily describe any real-world systems but are relatively simple and have some characteristic properties of more realistic systems. Supersymmetric theories are a perfect example of such "theoretical laboratories": they are simple enough, so that a lot of questions can be addressed, yet complex enough to possess rich physical phenomena. For the same reason, supersymmetry has been a source for numerous connections between physics and math, leading to further insights into both fields. Such things happen when the subject is simple enough to be accessible for human mind, yet complex enough to be full of interesting and non-trivial phenomena.

1.2 Holomorphy and supersymmetry

One useful aspect of supersymmetric theories as "theoretical laboratories" is that quite often they have exactly solvable sectors. In other words, there exist quantities which can be computed exactly either to all orders in perturbation theory or even non-perturbatively. Very often it is related to the fact that certain objects in the theory are holomorphic (or meromorphic). Let us briefly review some examples of the usefulness of holomorphy, and more generally complex geometry in supersymmetry.

The most basic result along this lines is, of course, perturbative non-renormalization of 4d $\mathcal{N} = 1$ superpotential. It is determined by a single holomorphic function of chiral superfields, which is, in the superspace formalism, integrated only over the half of superspace. Using the superspace generalization of Feynman diagrams called supergraph formalism, one can see that perturbative corrections come only as integrals over the full superspace, thus not correcting the superpotential (see e.g., [13]).

Non-perturbative corrections due to instantons are nevertheless possible. However, their form is severely constrained by holomorphy. The powerful insight of Seiberg in [14] was to promote coupling constants λ_i to background fields, which was motivated by earlier ideas in string theory (there, coupling constant is an expectation value of the dilaton; its superpartner is an axion, and so Peccei-Quinn symmetry in combination with holomorphy implies non-renormalization of the superpotential). Then superpotential becomes holomorphic both in terms of original fields

ϕ and couplings λ_i. From this point of view, one also has more symmetries in the theory, some of which are broken spontaneously by the background values of λ_i.

Combination of these additional symmetries and holomorphy allows to constrain general form of the exact superpotential. Using the perturbation theory input at $\lambda_i \to 0$, this often leads to the complete answer. One simple example where this works is Wess-Zumino model with $W_{tree} = m\phi^2 + \lambda\phi^3$ (which does not necessarily make sense non-perturbatively though, because the theory is not asymptotically free). In this case, promoting m and λ to background fields, the most general form of effective superpotential allowed by holomorphy and symmetries is $W_{eff} = m\phi^2 f(\lambda\phi/m)$. Using input from perturbation theory, one then finds $W_{eff} = W_{tree}$, i.e., proves the non-renormalization theorem [14]. Similar considerations provide useful insights into the dynamics of gauge theories [14,15]. For gauge theories, while perturbatively superpotential receives no corrections, non-perturbative instanton corrections are possible and can lift vacua, sometimes causing dynamical supersymmetry breaking [16].

Non-renormalization theorems hold in various dimensions and with various amounts of supersymmetry. If we stay in 4d but increase the number of supercharges, the theory becomes more and more restricted. In $\mathcal{N} = 1$ theories with chiral multiplets, in addition to superpotential, there was also an independent Kahler potential determining the kinetic energy, which is subject to non-trivial renormalization. In gauge theories, there is also a running gauge coupling, which in a low-energy effective Lagrangian becomes a holomorphic function of chiral superfields.

As we move to $\mathcal{N} = 2$ theories, things become more restricted. Superpotential (which describes interactions of chiral mutliplets, if the theory is written in terms of $\mathcal{N} = 1$ superfields) is simply determined by the matter content of the theory. Perturbative beta-function becomes 1-loop exact, with the possibility of non-perturbative corrections [17]. Kahler geometry of chiral multiplets is now replaced by the hyper-Kahler geomtery of hypermultiplets in global supersymmetry [18] or by the quaternionic geometry of hypermultiplets in supergravity [19]. Vector multiplets Lagrangian (with up to two derivatives) is completely determined by a single holomorphic function (more precisely, a local holomorphic section of a certain bundle) called prepotential. The geometry of vector multiplets (i.e., target manifold for their scalars) is known as the special Kahler geometry [20–22] ("rigid" special Kahler geometry in global case and "local" or "projective" special Kahler geometry in supergravity case).

A class of "rigid" special Kahler geometries can be constructed as certain subspaces of the moduli spaces of Riemann surfaces (see e.g., [21]). This class of geometries is at the core of Seiberg-Witten theory, which provides a tool for studying the low-energy effective action in 4d $\mathcal{N} = 2$ gauge theories [23, 24]. The Seiberg-Witten solution incorporates all instanton corrections to the moduli space of vacua. A good example of supersymmetric theories being "theoretical laboratories" was the derivation of the Seiberg-Witten solution based on a different field-theoretic approach – the instanton counting [25, 26]. The dynamics of $\mathcal{N} = 2$ gauge theories is a rich area of research (see [27] for review), but we will not go into more details about it.

Moving to $\mathcal{N} = 4$ 4d theories, there is only one globally supersymmetric theory of this class, $\mathcal{N} = 4$ super Yang-Mills (with gauge group G), which has become a cornerstone of many theoretical studies over the past two decades. This theory has a lot of interesting properties: it is exactly finite with vanishing beta-function [17], so it is superconformal. It has a non-perturbative strong-weak coupling duality called S-duality, which inspired the discovery of its connection to Geometric Langlands program [28]. This theory was also the first and most successful example of the holographic principle, giving an AdS/CFT correspondence between itself and Type IIB superstring on $AdS_5 \times S^5$ [29–31]. Integrability at large number of colors [32] and remarkable properties of its amplitudes [33] has also been an active area of research.

In three dimensions, holomorphy and non-renormalization theorems start to appear in $\mathcal{N} = 2$ theories. In these theories one non-renormalization theorem states that superpotential depends only on chiral multiplets and cannot depend on linear multiplets, hence cannot depend on real masses or Fayet-Iliopoulos terms. Another theorem states that central charge cannot depend on chiral multiplets and is a linear combination of linear multiplets, and thus of real masses and FI terms [34].

Two dimensional supersymmetric theories are of particular interest, especially because of their relation to String Theory. Models with $\mathcal{N} = (2,2)$ supersymmetry, RG-fixed points of which may serve for compactifications of Type IIA and Type IIB superstrings, are quite rigid. Their superpotential is exactly not renormalized, and for theories which flow from $\mathcal{N} = (2,2)$ gauge theories, the only parameter which undergoes a non-trivial renormalization is an FI term, which receives a one-loop correction [35]. In 2d theories with $\mathcal{N} = (0,2)$ supersymmetry, perturbative non-renormalization of their superpotentials still holds, however, non-perturbative corrections due to instantons may exist, and may even render vacuum unstable [36,37]. This is crucial for constructing heterotic string compactifications and has been a subject of multiple studies, e.g., [38–43]. There has been some growth of interest in models with $\mathcal{N} = (0,2)$ supersymmetry recently, especially gauge theories (see e.g., [44–48]).

1.3 Overview of this thesis

In this thesis, we are going to discuss two quite unrelated projects, which however have one thing in common – both are about holomorphic sectors in supersymmetric theories. The first one is on Gopakumar-Vafa and Ooguri-Vafa formulas – results about certain F-terms in effective supergravity of Calabi-Yau compactifications of Type IIA superstring. F-terms are couplings that are supersymmetric, but can only be written as integrals over one half of superspace, as opposed to D-terms that are written as integrals over the full superspace.

The second one is about chiral algebra structures in the cohomology of one supercharge in two-dimensional theories with $\mathcal{N} = (0,2)$ supersymmetry. In this section we are giving a brief review of the rest of the thesis.

1.3.1 The Gopakumar-Vafa formula

Type IIA superstring theory compactified on a Calabi-Yau threefold Y is described by $d = 4, \mathcal{N} = 2$ effective supergravity. It has 8 supersymmetries and can be described in terms of $\mathcal{N} = 2$ superspace, which has four left-handed or negative chirality odd coordinates θ and four right-handed or positive chirality odd coordinates $\overline{\theta}$. While the so-called D-term interactions are written as integrals over the full superspace: $\int d^4x\, d^4\theta\, d^4\overline{\theta}(\dots)$, the F-terms involve integration only over a half of odd directions.

In this theory a series of F-terms are known to be exactly computable. In terms of $N = 2$ superspace, the interactions of interest are:

$$I_{\mathsf{g}} = -i \int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{F}_{\mathsf{g}}(\mathcal{X}^\Lambda)(\mathcal{W}_{AB}\mathcal{W}^{AB})^{\mathsf{g}}. \tag{1.2}$$

Here $\mathcal{F}_{\mathsf{g}}$ is a holomorphic function of $\mathcal{N} = 2$ chiral superfields $\mathcal{X}^\Lambda = X^\Lambda + \dots$ which describe vector multiplets, and $\mathcal{W}_{AB} = \mathcal{W}_{BA}$ is the so-called Weyl superfield – a chiral superfield whose bottom component is the anti-selfdual part of the graviphoton field strength (here $A, B = 1, 2$ are spinor indices of negative chirality). $\mathcal{W}_{AB}$ is related to $\mathcal{W}_{\mu\nu}$ with spacetime indices by contraction with gamma-matrices. The superfields used here naturally appear in the formulation of $d = 4, \mathcal{N} = 2$ supergravity, in which one first constructs superconformal gravity and then breaks the extra part of its gauge supergroup (dilatations, special conformal transformations, special supersymmetries and $SU(2) \times U(1)$ R-symmetry) by an explicitly choice of a certain gauge slice. The $\mathcal{N} = 2$ superspace is of course more subtle than the usual $\mathcal{N} = 1$ construction. In particular, the $\mathcal{N} = 2$ chiral superfields $\mathcal{X}$ and $\mathcal{W}$ used here, in addition to the usual chirality conditions $\overline{D}^i_A \mathcal{X} = \overline{D}^i_A \mathcal{W} = 0$, satisfy an extra constraint $(\epsilon_{ij} D^i \sigma_{ab} \overline{D}^j)^2 (\mathcal{X})^* = -96 \Delta \mathcal{X}$ (and the same for $\mathcal{W}$), and so are called reduced chiral superfields (see [49–51]). Since this constraint is not holomorphic in fields (and rather looks like a sort of reality condition), one implication of this is that both vector multiplet scalars X and their conjugates $\overline{X}$ appear in the component expansion of $\mathcal{X}$. This ensures, for example, that $\mathcal{F}_0(\mathcal{X})$ – the prepotential – generates kinetic terms for all vector multiplet fields. Some of the relevant concepts will be briefly reviewed later.

The interactions $\mathcal{F}_{\mathfrak{g}}$ have been the subject of multiple studies both in physical and mathematical literature. For each $\mathfrak{g}$, $\mathcal{F}_{\mathfrak{g}}$ receives contributions only from the $\mathfrak{g}$-loop order of superstring perturbation theory. One of the interesting physical applications of these higher-derivative F-terms is that they encode corrections to the area law for the macroscopic entropy of supersymmetric black holes [52–56]; such corrections are crucial for the match with the microscopic counting of states performed in string theory [57, 58]. The mathematical interest of these objects stems from the fact that they are identified with the topological string free energies (in the large-volume limit, to decouple the holomorphic anomaly), which encode the Gromov-Witten invariants.

These interactions were originally discovered from the topological string side [59] and identified as certain physical superstring amplitudes [60]. Later they were reinterpreted by Gopakumar and Vafa [61, 62] using the space-time effective theory and lifting to M-theory. The latter approach was recently reexamined in [63, 64].

Since M-theory at low energies is described by 11-dimensional effective supergravity, which upon dimensional reduction on a circle gives Type IIA superstring theory, one can as well study Calabi-Yau compactifications of M-theory. They give 5-dimensional $\mathcal{N} = 1$ effective supergravity, which after reduction on a circle of course coincides with the Calabi-Yau compactification of the Type IIA. The S^1, reduction on which connects M-theory with Type IIA, will always be referred to as "the M-theory circle".

The main result – the so-called Gopakumar-Vafa formula – gives an expression for $\mathcal{F}_g$ coefficients in terms of the spectrum of BPS states in M-theory compactified on $Y \times S^1$, where S^1 is the M-theory circle. This provides a remarkable bridge between the topological string and the M-theory, which can serve to transfer ideas in both directions. Mathematically, it reinterprets the non-integral Gromov-Witten invariants in terms of integral BPS invariants (for a recent paper discussing it in a more general context of symplectic geometry see [65]). Physically, it demonstrates that the BPS spectrum of the M-theory on Calabi-Yau can in principle be determined from the Gromov-Witten invariants. Direct computation of the BPS spectrum in M-theory is a hard problem – it involves finding the low-energy spectrum of M2-branes wrapped on holomorphic curves, which is a simple task only for a single membrane on a smooth curve, while for more general configurations, the membrane theory becomes strongly coupled.

The space-time derivation of the GV formula is based on computing the contribution to the Wilsonian effective action due to 5d BPS states winding the M-theory circle. Moreover, only trajectories with non-zero winding number have to be considered. Trajectories with zero winding number naively give an ultraviolet-divergent contribution, but as will be explained later, this contribution should be regarded as part of the 5d effective action and need not be calculated. Only a few terms in the 5d effective action are actually relevant to the GV formula, and these terms are known because of their relation to anomalies.

BPS states that are massive in five dimensions are more naturally treated as particles in deriving their contribution to the GV formula, while those that are massless (or anomalously light) in five dimensions are more naturally treated as fields. In this thesis we describe both particle and field theory computations, explaining numerous related subtleties.

The computation involves three basic steps:

1. Understand the proper background.
2. Make sure computation will make sense physically.
3. Compute the effective action.

The background

The particle computation and most of the field theory computation is based on turning on a constant graviphoton background, as suggested in [61, 62]. This background, from the 4d perspective, can be described as a flat $\mathbb{R}^4$ with a constant anti-selfdual graviphoton turned on (graviphoton is a $U(1)$ gauge field in the $\mathcal{N}=2$ supergravity multiplet) and with the vector multiplet scalars having some constant background values; all other fields are vanishing. The graviphoton field is particularly important for us simply because it appears as the lowest component of the superfield $\mathcal{W}_{AB}$ in interactions (1.2). Therefore, for sufficiently large $\mathfrak{g}$, the component expansions of these interactions are always proportional to some powers of the graviphoton. But there are more reasons to prefer this background.

The graviphoton background, despite its simplicity, has some interesting properties, which turn out to be crucial for our computation. In contrast to naive expectations, this background preserves all 8 supercharges, but the supersymmetry algebra gets deformed, with the deformation proportional to the vev of the graviphoton. Lifting this background to 5d gives a certain non-tivial solution of 5d $\mathcal{N}=1$ supergravity known in the literature as the "supersymmetric Gödel Universe" [86]. In pure 5d $\mathcal{N}=1$ supergravity, this background is described by the following metric:

$$\mathrm{d}s^2 = -(\mathrm{d}t - V_\mu \mathrm{d}x^\mu)^2 + \sum_{\mu=1}^{4} (\mathrm{d}x^\mu)^2, \tag{1.3}$$

where $V_\nu = \frac{1}{2}\mathsf{T}^-_{\mu\nu}x^\mu$, with $\mathsf{T}^-_{\mu\nu}$ being the vev of the 5d graviphoton, which is constant, antisymmetric, and anti-selfdual in the 4d sense. If we add vector multiplets, then additional gauge fields from vector multiplets should have vevs for their field strengths proportional to this $\mathsf{T}^-_{\mu\nu}$.

The fact that background preserves supersymmetry is important for the computation of F-terms. From a field theory point of view, one can compute the effective action in an expansion around any background, whether or not the background is a classical solution and whether or not it is supersymmetric. But one generally cannot learn anything about F-terms in a supersymmetric effective action by expanding around a background that is not supersymmetric. In such a background F-terms mix with D-terms and become hard to distinguish.

In field theory, one could compute F-terms by expanding around a background that is supersymmetric but is not necessarily a classical solution. However, this may lead to problems in string/M-theory because it does not have a satisfactory off-shell formulation. Despite its M-theoretic origin in the UV, our calculation will be simply a calculation in the low-energy effective field theory. From this point of view, it seems that whether or not the background solves classical equations of motion is not relevant for the calculation itself. But it is definitely reassuring to find out that the background actually is a solution. Also, the 5d massive BPS particles are given by M2 branes wrapping two-cycles inside of Y, and if we try to derive the worldline action for these particles starting from the M2-brane action, it is important for the background to be on-shell.

The graviphoton background is indeed a solution. This relies on two facts. First, anti-selfdual gauge field $F_{\mu\nu}$ has the vanishing Maxwell stress-energy tensor $T_{\mu\nu} = \frac{1}{2}\left(F_{\mu\alpha}F^\alpha_\nu - \frac{1}{4}\eta_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right)$. This ensures that it produces no gravitational back-reaction. Second, the graviphoton is a very special linear combination of elementary gauge fields entering the 4d action, with coefficients depending on the vector multiplet scalalrs. This linear combination has a property that it does not produce any backreaction on scalars either. So this is indeed a solution.

The deformed supersymmetry algebra of the graviphoton background allows to determine the relevant particle or field-theoretic actions of BPS multiplets, which we then integrate out in order to obtain the GV formula. As we mentioned above, we have one more technique for determining this action: by studying the world-volume action of the M2-brane wrapped on a two-cycle inside of Y. However, this only works well if a single M2-brane wraps a smooth curve. In general, the curve might not be smooth and there can be multiple M2-branes wrapping it: this leads to a strongly-coupled world-volume theory that cannot be easily accessed. On the other hand, if we fix the BPS particle action based on supersymmetry only, this will always work, no matter how complicated the detailed microscopic theory is. This turns out to be possible: the relevant superparticle action is quite simple, with the charge and the mass determined by the corresponding homology two-cycle in Y, while the deformed supersymmetry algebra of the graviphoton background helps to fix all magnetic moments.

We should note that there is a small subset of cases for which the graviphoton background is not helpful: to perform the field-theoretic computation of $\mathcal{F}_0$ and $\mathcal{F}_1$, we need another background. The reason is that the I_0 interaction does not depend on the graviphoton at all, while for I_1, the computation (which works for I_{g} with $\mathsf{g} \geq 2$) becomes divergent, as we will explain later. These two cases were treated separately in [64], and required special backgrounds. To compute $\mathcal{F}_0$, we use, from the 4d perspective, again the flat $\mathbb{R}^4$, but instead of turning on graviphoton, we turn on a small and slow spacetime-dependent perturbation for the vector multiplet scalars, and compute effective action for this perturbation. This effective action is then directly related to $\mathcal{F}_0$. To compute $\mathcal{F}_1$, we again take the $\mathbb{R}^4$ and turn on a small metric perturbation. The effective action for this perturbation is then proportional to $\mathcal{F}_1$.

Does the computation make sense?

The computation itself is not that complicated: in the particle case, it is a simple instanton computation in supersymmetric quantum mechanics, while in the field theory case, it is a supersymmetric version of Schwinger's famous derivation of the effective action in the background electromagnetic field. However, its interpretation, and in particular showing that the procedure makes sense physically, requires some effort. Let us review the relevant points.

There are two observations we need to make. First, the interactions $\mathcal{F}_{\mathfrak{g}}(\mathcal{X})$ that we are interested in appear at $\mathfrak{g}$ loops in the superstring perturbation theory, so their dependence on the four-dimensional string coupling constant g_{st} is known. Second, they are holomorphic quantities, so we expect that the power of holomorphy should play a certain role in our computation.

It turns out that the computation makes sense only in a certain limit in the parameter space, but due to these two observations, the answer in that limit determines $\mathcal{F}_{\mathfrak{g}}(\mathcal{X})$ completely. What is this limit?

In Type IIA string theory, $\mathcal{F}_{\mathfrak{g}}$ receives contributions from superstring worldsheets wrapping topologically nontrivial holomorphic curves $\Sigma \subset \mathbb{R}^4 \times Y$. They lift to M2-branes wrapping $\Sigma \times S^1 \subset \mathbb{R}^4 \times S^1 \times Y$, where S^1 is the M-theory circle. When this circle is large, an M2-brane on $\Sigma \times S^1$ can be thought of as a point particle with the worldline S^1. One can think of this as a dimensional reduction of the M2-brane worldvolume theory on Σ. As a result, there is an infinite set of massive particles (winding S^1) organized into multiplets. When $\Sigma \subset Y$ is holomorphic, the lightest of these multiplets becomes BPS-saturated and shortened. All long multiplets generate an effective interaction that is an integral over all of superspace, as explained in section 2.2.2. Therefore, if we are interested in F-terms, only BPS multiplets matter, and to compute the $I_{\mathfrak{g}}$'s, we must understand the contributions of 5d BPS states when the M-theory circle is large. Therefore, one relevant limit is the limit of the M-theory circle being large compared to the 11-dimensional Planck scale and the length scale of Y.

Another limit we should respect is the one where M-theory can be effectively described by the 11-dimensional supergravity. This corresponds, in Type IIA superstring theory, to the limit that the ten-dimensional string coupling constant g_{10} and the volume of Y are large. This will be explained in more details later, but the upshot is that going to the large-volume limit of Y is enough to determine the answer because of holomorphy, while the large-coupling limit is enough because the dependence on $g_{\rm st}$ is known and $g_{10} = e^{3\sigma/2} g_{\rm st}$, where e^{σ} is the radius of the M-theory circle in the 5d units. Moreover, this agrees neatly with this radius being very large.

M-theory structure still remains quite obscure, in particular its low energy effective action is largely unknown, except for a few terms of low dimension. Yet, we want to determine the interactions $I_{\mathfrak{g}}$ by a lift to M-theory, so an important question is: do we know enough about M-theory? Since the interactions $I_{\mathfrak{g}}$ are terms in a four-dimensional effective action, this question translates into the following question: what contributions to the $I_{\mathfrak{g}}$ can arise by classical dimensional reduction from five dimensions? Any such contribution is a potential source of a difficulty, because we do not know how to determine the complete effective action of M-theory in five dimensions. Luckily, with very limited exceptions, the $I_{\mathfrak{g}}$ do not come by classical dimensional reduction from five dimensions. As we explain in section 2.1.2, only certain very special terms arise this way (and only for $\mathfrak{g} = 0, 1$), and one knows just enough about the M-theory effective action to determine them.

Another subtlety about interpreting interactions in (1.2) is related to the holomorphic anomaly. The interactions written in (1.2) are of course holomorphic. However, it is known that they are related to topological string partition functions, which are not quite holomorphic, with the non-holomorphy governed by the holomorphic anomaly equations [59]. And it is known that the low-energy effective action, or 1PI effective action, of Type IIA compactified on Y actually involves those non-holomorphic quantities and therefore is not quite holomorphic [60]. From the 4d space-time point of view, non-holomorphy (and actually also non-locality) of the action is caused by the massless particles propagation; integrating out massless particles is what introduces non-locality and non-holomorphy [60]. However, the quantities written in (1.2) are terms in the Wilsonian effective action (more precisely, we integrate out all massive degrees of freedom only and write the resulting action for massless degrees of freedom), so they are local and holomorphic, and there is no contradiction.

If t^I are relevant Kahler moduli, the topological string partition functions are some non-holomorphic quantities $\mathcal{F}_{\mathsf{g}}(t, \bar{t})$. In order to extract holomorphic answers, one should treat $\bar{t}^I$ as an independent complex variable $\widetilde{t}^I$ and fix it at some value $\widetilde{t}_0^I$, the "base-point". The resulting function $\mathcal{F}_{\mathsf{g}}(t, \widetilde{t}_0)$ is holomorphic in t^I. The standard choice is to take the base-point at infinity, $\widetilde{t}_0^I = \infty$. This corresponds to the large-volume limit of the Calabi-Yau Y in Type IIA variables. Incidentally, this is precisely the regime at which we are doing our computation. So it comes as no surprise that the holomorphic answers we get match exactly with the topological amplitudes with the base-point at infinity.

Another question one should worry about is what spectrum of BPS states is relevant for our problem. The way Gopakumar-Vafa formula is derived suggests straightforwardly that the relevant BPS states are 5d BPS states. They wind the M-theory circle and generate effective couplings in 4d. One subtlety, which is not directly relevant for the computation, but was nevertheless discussed in [63], is that those are not the same as 4d BPS states. In general, 4d BPS states are not given by a simple Kaluza-Klein reduction of the 5d ones, some short multiplets can recombine into long multiplets and become non-BPS as the radius of the M-theory circle goes from infinity to some finite value.

The computation

The computation itself can be subdivided into several steps. The simplest and the most basic one is to understand the contribution of a 5d massive hypermultiplet. This multiplet can be conveniently described in the worldline formalism by a superparticle action whose non-relativistic limit is given by:

$$I = M \int \mathrm{d}t \left(-1 + \frac{1}{2}\dot{x}^{\mu}\dot{x}_{\mu} + \frac{i}{2}\varepsilon^{AB}\varepsilon^{ij}\psi_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}\psi_{Bj} + \mathsf{T}^{-}_{\mu\nu}x^{\mu}\dot{x}^{\nu} \right), \tag{1.4}$$

where ψ_{Ai} are worldline fermions, with index $i = 1,2$ corresponding to $\mathcal{N} = 2$ supersymmetry and $A = 1,2$ a chiral spinor index. It turns out that this non-relativistic supersymmetric quantum mechanical action is all we really need to know in order to calculate the hypermultiplet contribution (the answer being exact and of course relativistic). The reason is that the calculation we are doing is the instanton calculation. The instanton solution is described by this particle winding the M-theory circle. If we treat this circle as the "time" direction, then the solution looks like the particle is simply at rest. We do not need full relativistic formalism to compute the action of one particle at rest! How about the 1-loop determinant around this instanton solution? It is described by a quadratic action for fluctuations around the instanton solution, which in our case describes small deviations from the particle being at rest – exactly the non-relativistic approximation to the action. Is there any other contribution beyond this 1-loop determinant? We will argue later that the answer is no, and of course, secretly, what stands behind this 1-loop exactness is holomorphy of the quantity we are computing.

The next step of the computation is to generalize this to a massive BPS multiplet of an arbitrary spin. This is where the worldline formalism becomes especially useful: it is easy to consider arbitrary spins in this approach, while had we started with the field-theoretic (or "second-quantized") description, we would end up with a lot of technical troubles. For this part of the computation, extended supersymmetry of the graviphoton background also becomes very useful: it allows to fix all relevant terms of the worldline action. Leaving all the details for later sections, we just mention here that the only way spin modifies the answer is by introducing an additional factor. It takes the form of the trace over the spin space and describes the magnetic moment interaction of the BPS particle with the graviphoton background.

Having computed contributions due to massive BPS states of arbitrary spins, one could in principle obtain contributions of massless particles of any spin we need by simply taking the mass to zero limit. This is slightly unsatisfactory because the description we use for massive particles (non-relativistic limit of the worldline superparticle action) breaks down for massless states. In order to be more precise, we should use the field theory description for massless BPS states instead. There are three types of massless multiplets which appear in the effective 5d $\mathcal{N} = 1$ supergravity: hypermultiplets, vectormultiplets and one supergravity multiplet. Simple considerations explained later in this thesis show that it is enough to do the computation for a hypermultiplet only. The vectormultiplet contribution is simply equal to minus the hypermultiplet answer, and the supergravity multiplet contribution is minus twice the hypermultiplet answer. In the field theory description of the hypermultiplet, we can also turn on a non-zero mass. This way we can see that the massive answer agrees with the particle computation, and that the mass $\to 0$ limit actually exists (since the field theory description makes sense at zero mass).

The field-theoretic computation itself is a supersymmetric version of Schwinger's computation of an effective action in a background electromagnetic field. One detail is that due to supersymmetry, strictly speaking, the effective action on the graviphoton background is zero. To get a non-zero action, one actually has to perturb the background and slightly break the SUSY. There are several ways to do that. To compute $\mathcal{F}_{\mathfrak{g}}$ for $\mathfrak{g} \geq 2$, it is convenient to perturb the metric by turning on an anti-selfdual curvature background. So for these purposes, we put the 5d theory on $M_5 = S^1 \times M_4$, where M_4 is equipped with a hyper-Kahler metric of anti-selfdual curvature (which is assumed to be small enough). The relevant field-theoretic action is again determined by supersymmetry.

To find $\mathcal{F}_1$ and $\mathcal{F}_0$ in the field theory approach, we need to do more work, as it turns out that the Schwinger-like computation does not cover these two cases. To find $\mathcal{F}_0$, it is more convenient to turn on a small perturbation for the vector multiplet scalars. Then, by computing the two-point function of these perturbations, we find $\mathcal{F}_0$. To compute $\mathcal{F}_1$, it is still useful to turn on the anti-selfdual metric perturbation, but instead of doing the Schwinger-like computation, we calculate the two-point function of these perturbations, which boils down to calculating the two-point function of the stress-energy tensor in flat space. This determines the R^2 interaction in the effective action, where R is the Riemann tensor. Since the effective action has a term $\mathcal{F}_1(X)(R^-)^2$, where R^- is the anti-selfdual part of R, this directly determines the $\mathcal{F}_1$ coupling.

The formula

The point of everything written so far in this section is to review the proper derivation of the Gopakumar-Vafa formula. It would be incomplete to have this discussion without writing the formula itself. The formula that we will describe shortly gives an expression for the sum of all interactions I_{g} at once, namele for:

$$\mathcal{I} = \sum_{\mathsf{g}\geq 0} I_{\mathsf{g}} = -i \int \mathrm{d}^4x\, \mathrm{d}^4\theta \sum_{\mathsf{g}\geq 0} \mathcal{F}_{\mathsf{g}}(\mathcal{X}^\Lambda)(\mathcal{W}^2)^{\mathsf{g}}. \tag{1.5}$$

To describe the formula, we have to introduce some notions first. Every 5d BPS state can be characterized by its charges q_I with respect to the 5d gauge fields. BPS states correspond to M2-branes wrapping two-cycles in the Calabi-Yau Y, and therefore these charges are just coordinates in the second homology group $H_2(Y;\mathbb{Z})$. If ω_I is some basis in second cohomology $H^2(Y;Z)$, then for every two-cycle $Q \in H_2(Y;\mathbb{Z})$, we have simply $q_I = \int_Q \omega_I$. We collectively denote charges as $\vec{q} = (q_1, \ldots, q_{b_2})$, where b_2 is the second Betti number of Y.

Five-dimensional SUSY algebra has a real central charge ζ, and particles of charges $\vec{q}$ carry a non-trivial central charge $\zeta(\vec{q})$. In terms of vevs of scalars from the 5d vector multiplets h^I, the central charge is given by $\zeta(\vec{q}) = \sum_I q_I h^I$.

All BPS multiplets of charge $\vec{q}$ form a reducible representation of the 5d rotation group $SU(2)_\ell \times SU(2)_r$. Together, they are described by a Hilbert space $\widehat{\mathcal{H}}_{\vec{q}}$. The minimal such Hilbert space describes one hypermultiplet and is denoted by $\widehat{\mathcal{H}}_0$. One can think of $\widehat{\mathcal{H}}_0$ as the space of wavefunctions $L^2(\mathbb{R}^4)$ (for the spacetime motion of the particle) taking values in $(1/2,0) \oplus 2(0,0)$, which describes the spin degrees of freedom of the hypermultiplet. This combination, $(1/2,0) \oplus 2(0,0)$, is the most basic massive BPS multiplet. Any other massive BPS multiplet can be described as a tensor product $[(1/2,0) \oplus 2(0,0)] \otimes (j_1, j_2)$ with some representation of $SU(2)_\ell \times SU(2)_r$.

We then write $\widehat{\mathcal{H}}_{\vec{q}} = \widehat{\mathcal{H}}_0 \otimes \mathsf{V}_{\vec{q}}$, where $\mathsf{V}_{\vec{q}}$ is a vector space with an action of $SU(2)_\ell \times SU(2)_r$. This vector space encodes the spin content of BPS multiplets of charges $\vec{q}$, and the set of such vector spaces for every $\vec{q}$ describes the BPS spectrum of the theory. Let the generator of the rotation group acting on $\mathsf{V}_{\vec{q}}$ be denoted by $\mathcal{J}^\mu{}_\nu$. We also introduce an anti-selfdual rotation generator $\mathcal{J}_{\vec{q}} = \mathsf{W}^-_{\mu\nu} g^{E\nu\sigma} \mathcal{J}^\mu{}_\sigma$, where g^E stands for the Einstein frame metric in 4d, and $\mathsf{W}^-_{\mu\nu}$ is the 4d constant anti-selfdual graviphoton vev.

Another object we need to introduce is a superfield $\mathcal{Z}^I = \mathcal{X}^I/\mathcal{X}^0$. Now we can write the Gopakumar-Vafa formula:

$$\mathcal{I} = -\int \frac{d^4x d^4\theta}{(2\pi)^4} \sqrt{g^E} \sum_{\vec{q}|\zeta(\vec{q})\geq 0} \sum_{k=1}^{\infty} \frac{1}{k} \mathrm{Tr}_{\mathsf{V}_{\vec{q}}} \left[(-1)^F \exp(-i\pi k \mathcal{J}_{\vec{q}}/4\mathcal{X}^0) \right]$$
$$\times \exp\left(2\pi i k \sum_I q_I \mathcal{Z}^I\right) \frac{\frac{1}{64}\pi^2 \mathcal{W}^2}{\sin^2\left(\frac{\pi k \sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)}. \qquad (1.6)$$

The last two factors describe the hypermultiplet contribution, the trace factor describes the magnetic moment interaction of the BPS multiplets of arbitrary spin, the sum over k corresponds to particles winding the M-theory circle multiple times, and the first sum goes over the BPS spectrum of the theory.

1.3.2 The Ooguri-Vafa formula

In [66], Ooguri and Vafa proposed a generalization of the Gopakumar-Vafa story, where one can include D4-branes in the Type IIA setup while preserving four supersymmetries, which is a half of what one has without D4-branes. This was achieved by wrapping D4-branes on $\mathbb{R}^2 \times L$, where $\mathbb{R}^2 \subset \mathbb{R}^4$ is some plane in $\mathbb{R}^4$ and L is a real three-dimensional submanifold in Y. It is known from the work of [67] that the condition for this configuration to preserve half of SUSY is that L is a so-called special Lagrangian submanifold of Y. The submanifold L of a Calabi-Yau manifold Y is called special Lagrangian if it satisfies two conditions: 1) it is Lagrangian, i.e., it is an n-dimensional submanifold L of a $2n$-dimensional symplectic manifold Y, such that the symplectic form of Y restricts to zero on L; 2) the holomorphic volume form Ω of Y has the property that its imaginary part vanishes when pulled-back to L. Normalization of Ω is usually picked in such a way, that its real part, when pulled-back to L, is equal to the volume form of L. This situation is sometimes referred to by saying that $\mathrm{Re}\,\Omega$ is a calibration on Y, and L is a calibrated submanifold [68].

In a situation when we do not turn on any fluxes, we could slightly generalize this configuration by considering several special Lagrangian submanifolds L_i and several parallel planes $\mathbb{R}^2_i \subset \mathbb{R}^4$. Then wrapping N_i D4-branes on every $\mathbb{R}^2_i \times L_i$ would preserve the same four supercharges. However, in the OV story, we also turn on an anti-selfdual graviphoton background. As we will learn in Section 2.1.1, the anti-selfdual rotation $\mathcal{J} = W^{-\mu\nu}\mathcal{J}_{\mu\nu}$ then appears in the SUSY algebra, and it should be preserved by the brane configuration. This imposes additional restrictions. First of all, the planes $\mathbb{R}^2_i$ should coincide. Therefore, the most general brane configuration relevant for the OV story consists of N_i D4-branes supported on each $\mathbb{R}^2 \times L_i$, where the $\mathbb{R}^2$ factors are the same, while L_i are in general different special Lagrangian submanifolds of Y. Another restriction is that in order for it to be a symmetry, the rotation by $\mathcal{J}$ should not mix directions which are parallel to the brane with directions which are normal to it. For example, if $\mathbb{R}^4$ has coordinates x^1, x^2, x^3, x^4 and if $\mathbb{R}^2$ corresponds to $x^3 = x^4 = 0$, then $\mathcal{J}$ should be proportional to $\mathcal{J}_{12} - \mathcal{J}_{34}$. This means that while in the GV story the anti-selfdual graviphoton had three free parameters (because there are three anti-selfdual two-forms on $\mathbb{R}^4$), in the OV story there is only one parameter. The graviphoton background W^- should be proportional to $\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^3 \wedge \mathrm{d}x^4$.

In this situation, the effective action on $\mathbb{R}^4$ has two types of terms. First, there are 4-dimensional terms, which are represented as integrals over the $\mathbb{R}^4$ (these are just the effective action for the Type IIA superstring compactified on Y). Second, there are also two-dimensional terms which are supported on $\mathbb{R}^2$, i.e., they are written as integrals over the subspace $\mathbb{R}^2 \subset \mathbb{R}^4$ where the brane is supported. These two-dimensional terms describe degrees of freedom that propagate along the brane, as well as their interactions with the bulk supergravity. From the two-dimensional point of view, the bulk theory has $\mathcal{N} = (4,4)$ SUSY, and the effective action supported on $\mathbb{R}^2$ breaks it down to $\mathcal{N} = (2,2)$. The bulk supergravity action still has the F-terms described by the GV-formula, just as explained in the previous subsection.

The novelty here is that the two-dimensional action has similar terms, which are also F-terms from the 2d point of view, and which have a similar structure and a similar physical origin.

These terms are:

$$J_{\mathfrak{n}} = \int_{\mathbb{R}^2} \mathrm{d}^2 x\, \mathrm{d}^2\theta\, \mathcal{R}_{\mathfrak{n}}(\mathcal{X}^\Lambda;\mathcal{U}^\sigma)\mathcal{W}_{\parallel}^{\mathfrak{n}}. \quad \mathfrak{n} \geq 0. \tag{1.7}$$

Here the $\mathcal{X}^\Lambda$ are the same chiral superfields that appear in the GV formula, except that now, since the D4-branes explicitly break half of the supersymmetry, we integrate over only half as many θ's (the ones that correspond to the unbroken supersymmetry) and we restrict $\mathcal{X}^\Lambda$ to depend only on those θ's. Thus the $\mathcal{X}^\Lambda$ are now viewed as chiral superfields in a theory with $(2,2)$ supersymmetry on $\mathbb{R}^2$. For $\mathfrak{n} \geq 0$, $\mathcal{R}_{\mathfrak{n}}$ is a holomorphic function of the $\mathcal{X}^\Lambda$. Also, $\mathcal{W}_\parallel$ is the "parallel" component of the graviphoton superfield $\mathcal{W}_{AB}$ that appears in eqn. (1.2) in the following sense. As explained before, the graviphoton background should be proportional to $\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^3 \wedge \mathrm{d}x^4$ in order to preserve supersymmetry in the presence of D4 branes. The "parallel" component[2] is just the proportionality coefficient: $\mathcal{W} = \frac{\mathcal{W}_\parallel}{2}(\mathrm{d}x^1 \wedge \mathrm{d}x^2 - \mathrm{d}x^3 \wedge \mathrm{d}x^4)$. Finally, $\mathcal{U}^\sigma$, $\sigma = 1, \ldots, b_1(L)$, are chiral superfields associated to the moduli of L. These interactions are only generated by worldsheets with $2\mathfrak{g}+\mathfrak{h}-1 = \mathfrak{n}$, where $\mathfrak{g}$ is the genus, and $\mathfrak{h}$ is the number of boundary components.

The idea behind the OV formula is similar to what we had before. The Type IIA construction is lifted to M-theory on $\mathbb{R}^4 \times S^1 \times Y$. D4-branes on $\mathbb{R}^2 \times L_i$ become M5-branes on $\mathbb{R}^2 \times S^1 \times L_i$, while a string worldsheet Σ with possible boundaries on the D4-branes becomes an M2-brane worldvolume $S^1 \times \Sigma$ with possible boundaries on the M5-branes. We then compute the low energy effective action due to M2-brane states propagating around S^1. The radius of S^1 is taken to be large, so that the particle

[2] One could also define "perpendicular" components $\mathcal{W}_\perp$ and define chiral couplings that depend both on $\mathcal{W}_\parallel$ and $\mathcal{W}_\perp$, however the OV formula would not determine these couplings: it works only at $\mathcal{W}_\perp = 0$, as non-zero vevs for "perpendicular" components break the required SUSY.

approximation is valid and the actual computation is merely the "square root" of the computation done in the GV case. The final answer – the OV formula – is written in terms of the spectrum of BPS states in the three-dimensional theory living on the $S^1 \times \mathbb{R}^2$ part of the M5-brane worldvolume $S^1 \times \mathbb{R}^2 \times L_i$. For the purposes of finding this spectrum, one can think of $S^1 \times \mathbb{R}^2$ as just $\mathbb{R}^3$, as the radius of the circle is very large.

These BPS states correspond to holomorphic curves in Y with boundaries on L_i, and each such curve represents some class in the second relative homology $H_2(Y, L; \mathbb{Z})$. This means that charges of such BPS states are parametrized by $H_2(Y, L; \mathbb{Z})$, which of course includes bulk charges in $H_2(Y; \mathbb{Z})$, but also has boundary charges which depend on the topology of L_i and describe how the M2-brane ends on the M5-brane. This point will be discussed later. Denoting the boundary charges by r_ρ, we can write the OV formula describing the contribution of a single 3d BPS multiplet to the sum $\mathcal{J} = \sum_{\mathfrak{n}=0}^{\infty} J_{\mathfrak{n}}$ of interactions defined above:

$$i \int \frac{\mathrm{d}^2 x \mathrm{d}^2 \theta}{(2\pi)^2} \sqrt{g^E} \cdot \sum_{k=1}^{\infty} \frac{1}{k} \exp\left(2\pi i k \left(\sum_I q_I \mathcal{Z}^I + \sum_\rho r_\rho \mathcal{U}^\rho\right)\right) \\ \times \mathrm{Tr}_{\mathsf{V}_{\vec{q},\vec{r}}} \left[(-1)^F \exp\left(-i\pi k \mathfrak{J}_{\vec{q},\vec{r}}/4\mathcal{X}^0\right)\right] \frac{\pi \mathcal{W}_{\|}/8}{\sin(\pi k \mathcal{W}_{\|}/8\mathcal{X}^0)}. \quad (1.8)$$

This expression has to be summed over the BPS spectrum of the relevant 3d theory.

Even though everything looks just like the "square root" of the GV story (up to the presence of new charges r_ρ and new moduli $\mathcal{U}^\rho$), physical interpretation of this result is more involved, because in three dimensions, the infrared (IR) behavior of quantum field theories is quite different from that in five dimensions. A variety of IR problems with the OV formula were described in [63], but only the simplest case of this formula was studied, when all the IR problems were absent. We are not going to elaborate that point here. Instead, we will give a sampling of these IR issues here, and will review the derivation of the formula in the main text.

One problem appears if Y is compact. Macroscopically, an M5-brane is effectively supported on $\mathbb{R}^3 \subset \mathbb{R}^5$, which has a real codimension 2. It can behave as a vortex, producing a monodromy for a certain scalar field, resulting in long range effects, possibly relevant for the OV formula. This issue does not arise in the original example of Ooguri and Vafa [66], because their Y was non-compact.

Another problem appears if L is compact, even if Y is not, because of the two-form gauge field on the M5-brane. If $b_2(L)$ is positive, there are massless gauge fields living the $\mathbb{R}^3$ part of $\mathbb{R}^3 \times L$. Since we are in three dimensions, these force charged particles to confine in the IR. This problem can possibly be resolved by considering only BPS states that exist in the large-volume limit (see [63]). However, in the example of [66], L was also non-compact, it had two non-compact directions. This means that even though relevant BPS states propagate in only three dimensions, M5-brane gauge fields proparate in 5 dimensions, and there is no confinement issue.

For a compact L, even if $b_2(L) = 0$, IR questions still can arise if we attempt to generalize to the nonabelian case by placing $N \geq 2$ M5-branes on $\mathbb{R}^3 \times L$. At long distances, these would be described by a quantum field theory on $\mathbb{R}^3$, which in general is not IR-free. Again, this problem was avoided in the example of [66], since their L was topologically $\mathbb{R}^2 \times S^1$ and provided two more non-compact directions.

Generalizing beyond the example of [66] would require a careful analysis of the IR behavior of the theory, however, this is neither the subject of [63] nor the subject of the current thesis. There can exist further subtleties even if the 3d physics is infrared-free. It may be governed at long distances by a non-trivial topological field theory. In this case, the BPS states may be anyonic with a long range interaction of statistical nature. This would affect the contribution of BPS particles winding multiple times around the M-theory circle. Again, this is just another issue we are not going to address here. In [63] it was assumed that in the original example of Ooguri and Vafa this does not happen, though it was not entirely clear for what reasons.

1.3.3 Chiral algebras in $\mathcal{N} = (0, 2)$ theories

Another topic discussed in this thesis is on two-dimensional chiral algebras emerging in the cohomology of one of the supercharges in two-dimensional theories with $\mathcal{N} = (0, 2)$ supersymmetry.

Two-dimensional theories with $\mathcal{N} = (0, 2)$ supersymmetry have been attracting attention over the last couple of decades. A motivation largely came from their potential phenomenological relevance for heterotic string compactifications, which require the internal theory to be an $\mathcal{N} = (0, 2)$ SCFT. But these theories are interesting and rich quantum field theories by themselves, which makes them a good object to study and apply various physical ideas. Thinking in that direction, gauge theories are of course of particular importance in theoretical physics and deserve attention in various dimensions and with various amounts of supersymmetry. But besides that, $\mathcal{N} = (0, 2)$ gauged linear sigma models are known to be a useful tool to construct $\mathcal{N} = (0, 2)$ SCTFs, and hence heterotic string vacua, as infrared (IR) fixed points of the renormalization group (RG) flow (see [69, 70] or just [71] and references therein).

Recently, the dynamics of two-dimensional $\mathcal{N} = (0, 2)$ supersymmetric gauge theories, both abelian and non-abelian, have seen an increasing interest, especially due to developments in [72–74]. At the same time, more basic models of $\mathcal{N} = (0, 2)$ interacting matter without gauge fields, sometimes referred to as $\mathcal{N} = (0, 2)$ Landau-Ginzburg (LG) models, have been studied, some references being [70, 75, 76]. These models themselves flow to non-trivial SCFTs in the IR, but they also can be thought of as a step in constructing gauge theories, because one can start from an $\mathcal{N} = (0, 2)$ LG model with global flavor symmetries and then gauging these global symmetries to obtain an $\mathcal{N} = (0, 2)$ gauge theory. It is interesting to study what happens to various properties of the IR fixed point under gauging.

We only consider theories on flat spacetime here, and they are always assumed to have a conserved stress-energy tensor. These theories have two supercharges Q_+ and

$\overline{Q}_+$ obeying:

$$\{Q_+, \overline{Q}_+\} = 2P_{++}, \tag{1.9}$$

where we are using the light-cone notations: $x^{\pm\pm} = x^1 \pm x^0$. Since $\overline{Q}_+^2 = 0$, we can study the cohomology of $\overline{Q}_+$. Algebra (1.9) implies immediately the most basic property of the $\overline{Q}_+$-cohomology. It says that P_{++} is a $\overline{Q}_+$-exact operator, therefore it annihilates cohomology classes. Since $P_{++} \propto \partial_{++}$, it means that cohomology classes do not depend on x^{++}, only on x^{--}. After Wick rotation to Euclidean signature, this implies that cohomology depends on the spacetime point holomorphically. This observation was first made in [77] and then in [78] used to elucidate some properties of $\mathcal{N} = (2, 2)$ LG models and their IR fixed points. Then, part of the analysis from [78] was extended to $\mathcal{N} = (0, 2)$ gauge theories in [79].

In the presence of supersymmetry, the stress-energy tensor becomes a part of the so-called supercurrent multiplet, as discussed in details by Dumitrescu and Seiberg in [80]. Using the most general such multiplet for $\mathcal{N} = (0, 2)$ theories, it is easy to show that the exact quantum cohomology of $\overline{Q}_+$ is invariant along the renormalization group (RG) flow.[3] Detailed derivation will be discussed later, but the main idea is that using a stress-energy tensor, one can construct an operator which acts as a conserved charge for dilatations in the cohomology, therefore implying scaling symmetry there.

The $\mathcal{N} = (0, 2)$ supersymmetry algebra has a $U(1)$ automorphism – the R-symmetry, under which $\overline{Q}_+$ and Q_+ have opposite charges, while bosonic generators are neutral. If the theory flows to the $\mathcal{N} = (0, 2)$ SCFT in the IR, then R-symmetry always becomes a physical symmetry in the IR, in particular, it has a conserved current, which is part of the $\mathcal{N} = 2$ super-Virasoro algebra in the IR.

[3]In fact, even though the cohomology is invariant along the flow, it can happen that it jumps at the IR fixed point. This is somewhat related to the question of what space of observables should we work with. In this thesis we will be only studying observables which are polynomial in the elementary fields and their derivatives. We could have introduce a notion of convergence for observables and allow series of even more general observables. Of course, the cohomology and whether it jumps in the IR would depend on it. But then we would have to grapple with related analytic issues, which is not part of our plan. So we will not discuss such questions here.

Along the RG flow, the R-symmetry may or may not be a symmetry of the theory. In case it is not, it arises as an accidental symmetry in the IR SCFT. But even if it is a symmetry along the RG flow, it might mix with accidental global symmetries which show up only at the IR fixed point. Such accidents really occur in $(0,2)$ theories and have been studied in the literature [76]. As has been shown in [76], accidents are very mild in $\mathcal{N}=(2,2)$ theories and essentially can be excluded there. However, in $\mathcal{N}=(0,2)$ they are more subtle and can really obscure the relation between the UV and IR theories. Symmetry enhancement at the IR fixed point also means that $\overline{Q}_+$-cohomology can jump there, and therefore the cohomology computed in the UV might in principle have properties that would be impossible if this were the exact answer for the IR fixed point. We will return to this question later in this introduction.

In case the R-symmetry is an exact symmetry of the theory along the RG-flow, the most general $\mathcal{N}=(0,2)$ supercurrent multiplet becomes specialized to the so-called $\mathcal{R}$-multiplet, which includes R-symmetry current as one of its components. In this case the RG-invariance of the $\overline{Q}_+$-cohomology can be argued in a more familiar way.

Namely, the presence of the R-symmetry allows to twist the theory. This is done by shifting a stress-energy tensor of the original theory by a derivative of an R-current. The new stress-energy tensor has the property that its trace is $\overline{Q}_+$-exact, so this implies that trace vanishes at the level of cohomology. This means that there is actually an emergent conformal invariance present in cohomology. Another fact about the twisted stress-energy tensor is that its anti-holomorphic component is also $\overline{Q}_+$-exact, while the holomorphic component is only $\overline{Q}_+$-closed. So there is a holomorphic stress-energy tensor in cohomology.

It is trivial to observe that operator product expansion (OPE) of operators in the original theory induces a well-defined OPE of cohomology classes. Before we mentioned that cohomology classes depend on spacetime insertion points holomorphically (in Euclidean signature). This means that we obtain a well-defined holomorphic

OPE in the $\overline{Q}_+$-cohomology. This is the structure of a chiral algebra emerging in the cohomology. As we explained in the previous paragraph, there is a holomorphic stress-energy tensor in the cohomology. It acts on other cohomology classes through the OPE. Therefore there is a full holomorphic Virasoro algebra acting in the cohomology. Such chiral algebras are often referred to as W-algebras in the literature. However, the usual definition of W-algebras assumes that they are generated by the stress-energy tensor and a set of primaries, while chiral algebras arising in the $\overline{Q}_+$-cohomology of $\mathcal{N} = (0, 2)$ theories might be of a more general class in principle (we will see an example later), so we will not call them W-algebras.

Twisting by the R-symmetry of the $\mathcal{N} = (0, 2)$ theory is known as half-twisting in the literature, especially when one treats $\mathcal{N} = (2, 2)$ theories as $\mathcal{N} = (0, 2)$ models and only twists by the right-moving R-symmetry. This has been studied both in LG models and non-linear sigma models (NLSM). Chiral algebras of $\mathcal{N} = (0, 2)$ half-twisted sigma models were studied to some extent in the literature due to their connection to the theory of chiral differential operators. In particular, the perturbative approach was developed in [81] and [82], and some non-perturbative aspects were studied in [83] and [84].

If a theory does not have a conserved R-symmetry current along the RG flow, but nevertheless flows to the $\mathcal{N} = (0, 2)$ SCFT in the IR, we cannot tell for sure if there is a stress-energy tensor in the cohomology along the flow, we only now that it is there at the IR fixed point. It could be there along the flow, or it could jump into the cohomology only at the IR fixed point (as was explained in footnote 3, it depends on the analytic structure of the space of observables that we use and is not studied here). A simplification we have in theories with a conserved R-symmetry current present in the UV is that we can always explicitly construct a holomorphic stress-energy tensor in the cohomology. As we explained, this is done by twisting the physical stress-energy tensor by the correctly chosen R-symmetry current (correct

R-symmetry is picked though the c-maximization, or a related principle which will be explained later). In this thesis, we will concentrate only on theories which have a conserved R-symmetry current in the UV.

A class of theories we are interested in are Landau-Ginzburg models. They are constructed only from chiral and Fermi superfields, and the kinetic term is written in terms of the Euclidean metric (as opposed to more general Kahler metrics in the NLSM case). Chiral algebras of $\mathcal{N} = (0,2)$ LG models have not received too much attention in the literature. They were the main topic of [85] and will be discussed in this thesis.

All relevant technical details about 2d $(0,2)$ theories, such as the $\mathcal{N} = (0,2)$ superspace, supefields and Lagrangians will be reviewed at the beginning of Chapter 4. Then we will proceed to discuss the structure of the $\mathcal{N} = (0,2)$ supercurrent multiplet and use it to prove the RG-invariance of the $\overline{Q}_+$-cohomology. After that we will study general properties of chiral algebras in $(0,2)$ LG models with R-symmetry. We will also specialize to the $\mathcal{N} = (2,2)$ case and discuss LG models that flow to diagonal $\mathcal{N} = 2$ minimal models in the IR. After that we will briefly mention how gauging global symmetries of the LG model acts at the level of chiral algebras, which is a topic of an ongoing research.

One useful fact about chiral algebras is that the knowledge of the exact chiral algebra allows to perform some diagnostics of the theory in the IR, in particular sometimes it can be used to *prove* spontaneous supersymmetry breaking. When the chiral algebra that we find in the UV does not admit unitary representations (for example, if it has a current subalgebra of negative level), it means that something goes wrong in the IR. If for some complementary reasons we know that the IR CFT with normalizable vacuum exists, this means one of two things: either an accident happens, and the cohomology is enhanced in the IR in a way which fixes unitarity, or supersymmetry is spontaneously broken.

Accidental $U(1)$ symmetries in the IR can mix with other $U(1)$ symmetries and with the R-symmetry, thus possibly making all $U(1)$ current algebras consistent with unitarity (their levels should be non-negative, and currents should be primary operators). However, if we have non-abelian symmetries of negative levels in the cohomology, this cannot be fixed by accidents in the IR, and it really indicates supersymmetry breaking – for example, it can happen in Gadde-Gukov-Putrov theories [73, 74]. Another question which has to be addressed is whether the supersymmetry breaking is complete or partial. It is known that in $\mathcal{N} = (0, 2)$ theories, supersymmetry can be partially broken down to $\mathcal{N} = (0, 1)$, and it is diagnosed by the constant "space-filling brane current" in the SUSY algebra [80]. Such constant term in the SUSY algebra cannot be generated perturbatively and can come as an instanton effect in NLSMs or gauge theories. The presence of this term is therefore really determined by the theory in the UV, and partial breaking of SUSY is not a dynamical question about the theory at the IR fixed point, but rather about the whole RG flow. We assume that such a constant does not appear in LG theories we study, simply because they are topologically trivial in the UV and have no room for instantons (and of course, classically, they do not have such "space-filling branes").

Applying chiral algebras to SUSY breaking is complementary to the usual supersymmetric index approach. Non-vanishing of the index *proves* that supersymmetry is unbroken, while non-existence of unitary representation of the chiral algebra *proves* that something goes wrong in the IR, which sometimes implies SUSY breaking.

One more possibility for the IR behavior, which we excluded above by assuming that the IR CFT has a normalizable vacuum, is when such a vacuum is absent. This means that the spectrum of dimensions is not gapped in the IR but rather has a continuous branch, like the free boson. Absence of normalizable vacuum also allows for chiral algebras which do not admit unitary representations. We will see an $\mathcal{N} = (2, 2)$ example with this kind of behavior later.

Chapter 2

The Gopakumar-Vafa and Ooguri-Vafa formulas

In this chapter we give a detailed derivation of the Gopakumar-Vafa formula and in the last section review the Ooguri-Vafa generalization. It is based entirely on papers [63] and [64], the first of which is a joint work with Edward Witten. The motivation and the general picture were discussed in the Introduction, so here we start right away by analyzing the supersymmetric background relevant for the problem. Then we will describe the worldline and the field theoretic computations of the effective action on this background.

2.1 The Background And Its Supersymmetry

2.1.1 The Background In Five Dimensions

The Supersymmetric Gödel Solution

The bosonic fields of minimal supergravity in five dimensions are the metric tensor g and a $U(1)$ gauge field V, whose field strength is the 5d graviphoton $\mathsf{T} = \mathrm{d}V$. To describe the supersymmetric Gödel solution [86], we parametrize $\mathbb{R}^5$ with coordinates t

and x^μ, $\mu = 1, \ldots, 4$. The desired solution has the property that T has no component in the t direction, and its components in the x^μ directions are constant and anti-selfdual. We set $V_\nu = \frac{1}{2}\mathsf{T}^-_{\mu\nu}x^\mu$, where $\mathsf{T}^-_{\mu\nu}$ is constant (independent of t and the x^μ), antisymmetric, and anti-selfdual in the four-dimensional sense. We take the metric to be

$$\mathrm{d}s^2 = -(\mathrm{d}t - V_\mu \mathrm{d}x^\mu)^2 + \sum_{\mu=1}^{4}(\mathrm{d}x^\mu)^2. \tag{2.1}$$

For real T^-, this is a real and supersymmetric solution of 5d supergravity in Lorentz signature. It has the special property that the 5d graviphoton can also be viewed as the field strength of a "Kaluza-Klein" gauge field.

This is actually not a physically sensible solution, since a large circle in the hyperplane $t = 0$ can be a closed timelike curve. For our purposes, we would like to compactify the t direction to a circle, and moreover we want this circle to be spacelike, so that it can be interpreted as the M-theory circle. To make the circle spacelike, we will set t to be a multiple of $-iy$, where y will be a real variable of period 2π. To give the circle an arbitrary circumference $2\pi e^\sigma$, we take the relation between t and y to be $t = -iye^\sigma$. The solution (2.1) can then be written

$$\mathrm{d}s^2 = e^{2\sigma}\left(\mathrm{d}y + B_\mu \mathrm{d}x^\mu\right)^2 + \sum_\mu (\mathrm{d}x^\mu)^2, \tag{2.2}$$

where we have defined

$$B_\mu = -ie^{-\sigma}V_\mu. \tag{2.3}$$

This compactified solution can be generalized in an obvious way to depend on another real parameter: we give a constant expectation value to V_y, the component in the y direction of the gauge field V.

Clearly, to make the metric in eqn. (2.2) real, we have to take V_μ and T^- to be imaginary. This is not really troublesome, since a Schwinger-like calculation in a constant magnetic field still makes sense if the magnetic field is imaginary. (An imaginary magnetic field in Euclidean signature is somewhat analogous to a constant electric field in Lorentz signature, which was one of the original cases studied by Schwinger.)

The 4d interpretation of the 5d metric (2.2) requires some care. The 4d metric in Einstein frame is not $g_{\mu\nu} = \delta_{\mu\nu}$, which we would read off from (2.2), but rather is

$$g^E_{\mu\nu} = e^{\sigma}\delta_{\mu\nu}. \tag{2.4}$$

It is also convenient to define

$$\mathsf{W}^-_{\mu\nu} = 4e^{\sigma/2}\mathsf{T}^-_{\mu\nu}, \tag{2.5}$$

which turns out to be the 4d graviphoton. Thus

$$B_\nu = -i\frac{e^{-3\sigma/2}}{8}\mathsf{W}^-_{\mu\nu}x^\mu, \quad V_\nu = \frac{1}{2}\mathsf{T}^-_{\mu\nu}x^\mu. \tag{2.6}$$

We also write W^- as the curvature of a 4d gauge field

$$\mathsf{W}^-_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu, \;\; U_\mu = 4e^{\sigma/2}V_\mu. \tag{2.7}$$

T^-, W^-, V and U will be imaginary and B real. We define the 5d scalar quantity

$$(\mathsf{T}^-)^2 = \delta^{\mu\mu'}\delta^{\nu\nu'}\mathsf{T}^-_{\mu\mu'}\mathsf{T}^-_{\nu\nu'}, \tag{2.8}$$

raising and lowering indices using the 5d metric (2.2). But in defining a corresponding 4d scalar quantity $(\mathsf{W}^-)^2$, we raise and lower indices using the 4d Einstein frame metric:

$$(\mathsf{W}^-)^2 = g^{E\mu\mu'}g^{E\nu\nu'}\mathsf{W}_{\mu\mu'}\mathsf{W}_{\nu\nu'} = 16e^{-\sigma}(\mathsf{T}^-)^2. \tag{2.9}$$

We have described the basic five-dimensional solution that is used in the computation leading to the GV formula, along with its reduction to four dimensions. This solution has two properties that are important in deriving the GV formula: *(i)* it preserves all of the supersymmetry, not just half of it, as one might expect for a solution with an anti-selfdual graviphoton; *(ii)* it generalizes straightforwardly to the case that an arbitrary number of vector multiplets are included. We describe these two properties in sections 2.1.1 and 2.1.1.

Extended Supersymmetry

The supersymmetry algebra of the supersymmetric Gödel solution (2.1) can be described as follows. In describing spinors, we use the obvious orthonormal frame field

$$e^t = \mathrm{d}t - V_\mu \mathrm{d}x^\mu, \quad e^\mu = \mathrm{d}x^\mu, \quad \mu = 1, \dots, 4, \tag{2.10}$$

or the dual vector fields

$$v_t = \frac{\partial}{\partial t}, \quad v_\mu = \frac{\partial}{\partial x^\mu} + V_\mu \frac{\partial}{\partial t}. \tag{2.11}$$

The spinor representation of $SO(1,4)$ is four-dimensional and pseudoreal. Since it is pseudoreal, the supersymmetry generators in minimal 5d supergravity are actually a pair of spinors, which we denote $\epsilon^{\alpha i}$, where $\alpha = 1, \dots, 4$ is an $SO(1,4)$ spinor index and $i = 1, 2$ reflects the doubling needed to make the supersymmetry generator real (note that no symmetry acting on this index is assumed). Indices are raised and lowered using the $SO(1,4)$-invariant antisymmetric tensor $C_{\alpha\beta}$ (sometimes called the charge conjugation matrix) and a 2×2 antisymmetric tensor ε_{ij}. In five-dimensional Minkowski spacetime, the supersymmetry algebra is

$$\{Q_{\alpha i}, Q_{\beta j}\} = -i\Gamma^M_{\alpha\beta}\varepsilon_{ij}P_M + C_{\alpha\beta}\varepsilon_{ij}\zeta, \tag{2.12}$$

where P_M, $M = 0, \ldots, 4$ are the momentum generators, $(\Gamma^M)_\beta^\delta$ are Dirac gamma-matrices, $\Gamma^M_{\alpha\beta} = (\Gamma^M)_\beta^\delta C_{\delta\alpha}$, and we include the 5d central charge ζ.

Since the graviphoton field breaks $SO(1,4)$, it is convenient to write everything in terms of a $4+1$-dimensional split with coordinates x^μ, $\mu = 1, \ldots, 4$ and t. For this, we introduce four-dimensional gamma-matrices γ^μ with chirality matrix $\gamma_5 = -i\Gamma_0$, decompose $Q_{\alpha i}$ in terms of spinors Q_{Ai} and $Q_{\dot{A}i}$, $A, \dot{A} = 1, 2$ of negative and positive chirality, and we write the momentum generators as $H = -P_0$ and P_μ, $\mu = 1, \ldots, 4$. In 5d Minkowski spacetime, the supersymmetry algebra now reads

$$\begin{aligned} \{Q_{Ai}, Q_{Bj}\} &= \varepsilon_{AB}\varepsilon_{ij}(H + \zeta) \\ \{Q_{\dot{A}i}, Q_{\dot{B}j}\} &= \varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}(H - \zeta) \\ \{Q_{Ai}, Q_{\dot{B}j}\} &= -i\Gamma^\mu_{A\dot{B}}\varepsilon_{ij}P_\mu. \end{aligned} \tag{2.13}$$

Now let us discuss what happens to the supersymmetry algebra when the graviphoton field is turned on. The Killing spinor equation for a supersymmetry generator ϵ implies that it is independent of t and obeys the four-dimensional equation

$$\partial_\mu \epsilon - \frac{1}{4}\mathsf{T}^-_{\nu\rho}\gamma^{\nu\rho}\gamma_\mu\epsilon = 0. \tag{2.14}$$

Since γ_μ reverses the chirality and $\mathsf{T}^-_{\nu\rho}\gamma^{\nu\rho}$ annihilates spinors of positive chirality, this equation is trivially satisfied for any constant spinor η_{Ai} of negative chirality by

$$\epsilon_{Ai} = \eta_{Ai}, \quad \epsilon_{\dot{A}i} = 0. \tag{2.15}$$

This is enough to maintain half the supersymmetry. But somewhat less trivially, if $\eta_{\dot{A}i}$ is a constant spinor of positive chirality, the equation can also be solved by

$$\epsilon_{\dot{A}i} = \eta_{\dot{A}i}, \quad \epsilon_{Ai} = \mathsf{T}^-_{\mu\nu}x^\mu\gamma^\nu_{A\dot{A}}\eta_i^{\dot{A}}, \tag{2.16}$$

so a constant anti-selfdual graviphoton actually preserves all of the supersymmetry.

The extended supersymmetry is certainly surprising, but if one looks more closely, there is a surprise hidden even in the more trivial-looking supersymmetries (2.15). In gauge theory, a background with anti-selfdual field strength F_{AB} preserves the supersymmetries of positive chirality. (Anti-selfduality means that $F_{\dot{A}\dot{B}} = 0$, so the transformation of the gluino field λ associated to a positive chirality supersymmetry generator $\epsilon^{\dot{A}}$ is $\delta\lambda_{\dot{A}} = F_{\dot{A}\dot{B}}\epsilon^{\dot{B}} = 0$.) But the "trivial" supersymmetries in an anti-selfdual graviphoton background have *negative* chirality.

Since the anticommutator of two supersymmetries will be a bosonic symmetry, we have to understand the bosonic symmetries of this spacetime in order to understand the supersymmetry algebra. The Killing vector fields associated to the generators H and P_μ of translation symmetries are

$$\begin{aligned} h &= -\frac{\partial}{\partial t} \\ p_\mu &= \frac{\partial}{\partial x^\mu} - V_\mu \frac{\partial}{\partial t}. \end{aligned} \tag{2.17}$$

Note the contribution to p_μ that is proportional to V^μ; it reflects the fact that the graviphoton background is translation-invariant in the x^μ directions only up to a time translation. Because of this contribution, the translation generators do not commute:

$$\begin{aligned} [p_\mu, p_\nu] &= \mathsf{T}^-_{\mu\nu} h \\ [p_\mu, h] &= 0. \end{aligned} \tag{2.18}$$

(As discussed below, the commutator of the conserved charges P_μ corresponding to p_μ also contains a central term that is not seen in the commutator of the p_μ.)

We also must consider rotation symmetries. Without the graviphoton field, we would have a full action of $\mathrm{Spin}(4) \cong SU(2)_\ell \times SU(2)_r$, with $SU(2)_\ell$ rotating spinor indices A, B of negative chirality and $SU(2)_r$ rotating spinor indices $\dot{A}, \dot{B}$ of positive chirality. A constant anti-selfdual graviphoton field breaks $SU(2)_\ell \times SU(2)_r$ to $U(1)_\ell \times SU(2)_r$. The Killing vector fields that generate the unbroken rotation symmetries are unchanged from what they would be at $\mathsf{T}^- = 0$. The $SU(2)_r$ generators do not appear in the anticommutators of two supersymmetries (or of the other bosonic symmetry generators). Thus $SU(2)_r$ can be viewed as a group of outer automorphisms of the supersymmetry algebra. However, as we discuss momentarily, the $U(1)_\ell$ generator does appear on the right hand side of the supersymmetry algebra. The generator of $U(1)_\ell$ is associated to the Killing vector field

$$j = 4V^\mu \frac{\partial}{\partial x^\mu}. \tag{2.19}$$

We also express j in terms of standard angular momentum generators $j_{\mu\nu}$:

$$j = \mathsf{T}^{-\mu\nu} j_{\mu\nu}, \quad j_{\mu\nu} = x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\mu}. \tag{2.20}$$

It is convenient to write $J_{\mu\nu}$ for the conserved angular momentum corresponding to $j_{\mu\nu}$, and set

$$\mathsf{J} = \mathsf{T}^{-\,\mu\nu} J_{\mu\nu}. \tag{2.21}$$

We also want an analogous quantity for the theory compactified to four dimensions, but here we should be careful. In more detail $\mathsf{J} = \mathsf{T}^-{}_{\mu\nu} \left(\delta^{\nu\sigma} x^\mu \frac{\partial}{\partial x^\sigma} - \delta^{\mu\sigma} x^\nu \frac{\partial}{\partial x^\sigma} \right)$, where $\delta^{\nu\sigma}$ is the standard flat metric on $\mathbb{R}^4$. In four dimensions, we want to make a similar definition using the 4d graviphoton field W^- (eqn. (2.5)) and the Einstein metric g^E (eqn. (2.4)), so we define

$$\mathfrak{J} = \mathsf{W}^-_{\mu\nu} \left(g^{E\nu\sigma} x^\mu \frac{\partial}{\partial x^\sigma} - g^{E\mu\sigma} x^\nu \frac{\partial}{\partial x^\sigma} \right), \tag{2.22}$$

obeying

$$\mathsf{J} = \frac{e^{\sigma/2}}{4}\mathcal{J}. \tag{2.23}$$

In discussing the supersymmetry algebra, just as at $\mathsf{T}^- = 0$, we write Q_{Ai} and $Q_{\dot{A}i}$ for supersymmetries whose generators are parametrized by the negative and positive chirality spinors η_{Ai} and $\eta_{\dot{A}i}$ that appear in eqns. (2.15) and (2.16) above. Turning on the constant anti-selfdual graviphoton field modifies the supersymmetry algebra in two ways. The most obvious change is that because the generators (2.16) of positive chirality supersymmetries have a contribution linear in the x^μ, the $Q_{\dot{A}i}$ do not commute with the P_μ:

$$[P_\mu, Q_{\dot{A}i}] = \mathsf{T}^-_{\mu\nu}\Gamma^\nu_{\dot{A}A}Q^A_i. \tag{2.24}$$

Given this, there must be a correction to the anticommutator $\{Q_{\dot{A}i}, Q_{\dot{B}j}\}$, to avoid a problem with the $Q_{\dot{A}i}\cdot Q_{\dot{B}j}\cdot Q_{Ak}$ Jacobi identity. To compute what happens, all one has to know is that, in five-dimensional notation, if $\epsilon_{\alpha i}$ and $\epsilon'_{\beta j}$ are two Killing spinor fields, then, up to possible central terms, the anticommutator of the corresponding supersymmetries is associated to the Killing vector field[1] $u^m = \varepsilon^{ij}\overline{\epsilon}'_i\Gamma^m\epsilon_j$. The graviphoton field produces no correction to the anticommutator $\{Q_{Ai}, Q_{Bj}\}$ of negative chirality supersymmetries, since eqn. (2.15) asserts that (in the local Lorentz frame (2.10)), there is no T^--dependent contribution to the generators of these supersymmetries. In computing $\{Q_{Ai}, Q_{\dot{A}j}\}$, we do have to take into account the T^--dependent contribution to the generator of $Q_{\dot{A}j}$. But this just goes into building up the T^--dependent part of the Killing vector field p_μ (eqn. (2.17)), leaving no T^--dependent correction to the usual relation $\{Q_{Ai}, Q_{\dot{A}j}\} = -i\varepsilon_{ij}\sum_{\mu=1}^4 P_\mu\Gamma^\mu_{A\dot{A}}$. Where one does find a correction is in the anticommutator $\{Q_{\dot{A}i}, Q_{\dot{B}j}\}$, which acquires a term $\varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}\mathsf{J}$.

[1] If we use constant gamma-matrices Γ^m referred to the local Lorentz frame (2.10), this formula will give the components of the vector field u^m in the dual basis (2.11).

The central charge ζ that appears in the $\{Q, Q\}$ anticommutator (eqn. (2.13)) also appears in $[P_\mu, P_\nu]$. This should come as no surprise; it reflects the fact that in the presence of a constant magnetic field on $\mathbb{R}^4$ – in our case the graviphoton field – translations only commute up to a gauge transformation. To evaluate $[P_\mu, P_\nu]$, we use the $P_\mu \cdot Q_{Ai} \cdot Q_{\dot{B}j}$ Jacobi identity. Since

$$P_\nu = -\frac{i}{4}\Gamma_\nu^{\dot{A}A}\varepsilon^{ij}\{Q_{Ai}, Q_{\dot{A}j}\}, \tag{2.25}$$

the commutator $[P_\mu, P_\nu]$ can be simply computed using $[P_\mu, Q_{Ai}] = 0$, eqn. (2.24) for $[P_\mu, Q_{\dot{A}j}]$, and eqn. (2.13) for $\{Q_{Ai}, Q_{Bj}\}$. We find that $[P_\mu, P_\nu]$ is proportional to $H + \zeta$ (and not to H, as one might have supposed from eqn. (2.18) for $[p_\mu, p_\nu]$).

Putting the pieces together, the supersymmetry algebra is

$$\begin{aligned}
[P_\mu, P_\nu] &= -i\mathsf{T}^-_{\mu\nu}(H + \zeta) \\
[\mathsf{J}, P_\mu] &= 2i\mathsf{T}^-_{\mu\nu}P^\nu \\
[\mathsf{J}, Q_{Ai}] &= -\frac{i}{2}\mathsf{T}^-_{\mu\nu}\Gamma^{\mu\nu}_{AB}Q^B_i \\
[P_\mu, Q_{\dot{A}i}] &= \mathsf{T}^-_{\mu\nu}\Gamma^\nu_{\dot{A}B}Q^B_i, \\
\{Q_{Ai}, Q_{Bj}\} &= \varepsilon_{AB}\varepsilon_{ij}(H + \zeta) \\
\{Q_{Ai}, Q_{\dot{A}j}\} &= -i\Gamma^\mu_{A\dot{A}}\varepsilon_{ij}P_\mu \\
\{Q_{\dot{A}i}, Q_{\dot{B}j}\} &= \varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}(H - \zeta + \mathsf{J}),
\end{aligned} \tag{2.26}$$

with other commutators and anticommutators vanishing.

This 5d supersymmetry algebra will be our starting point in the particle-based computation in section 2.2. In section 2.3, we will perform a field theory computation that is conveniently expressed in terms of Kaluza-Klein reduction to four dimensions. For this, we will want a 4d version of the above algebra that arises after rotation to Euclidean time and compactifying the time direction on a circle of radius e^σ. We write the metric as in eqn. (2.2), and use a rescaled graviphoton field $\mathsf{W}^-_{\mu\nu}$ as in eqn. (2.5). In going to four dimensions, the gamma-matrices are scaled by $e^{\sigma/2}$ to refer them to the Einstein frame because of the e^{σ} in the definition of the 4d Einstein metric in (2.4), and accordingly to keep the supersymmetry algebra in a standard form, the supersymmetry generators must be scaled by $e^{\sigma/4}$. Thus, we introduce 4d supersymmetry generators $\mathcal{Q}_{Ai}$, $\mathcal{Q}_{\dot{A}j}$, defined by

$$\mathcal{Q}_{Ai} = e^{-\sigma/4} Q_{Ai}, \quad \mathcal{Q}_{\dot{A}j} = e^{-\sigma/4} Q_{\dot{A}j}. \tag{2.27}$$

Rotation to Euclidean time causes H to be accompanied by an extra factor of $-i$ (assuming one wishes H to remain hermitian). After compactification, H becomes a central charge in the 4d sense. The full 4d central charge is

$$\mathcal{Z} = e^{-\sigma/2}(\zeta + iH) \tag{2.28}$$

and the supersymmetry algebra in four dimensions is

$$\begin{aligned}
[P_\mu, P_\nu] &= -\frac{i}{4}\mathsf{W}^-_{\mu\nu}\overline{\mathcal{Z}} \\
[\mathcal{J}, P_\mu] &= 2i\mathsf{W}^-_{\mu\nu}P^\nu, \\
[\mathcal{J}, \mathcal{Q}_{Ai}] &= -\frac{i}{2}\mathsf{W}^-_{\mu\nu}\gamma^{\mu\nu}_{AB}\mathcal{Q}^B_i, \\
[P_\mu, \mathcal{Q}_{\dot{A}i}] &= \frac{1}{4}\mathsf{W}^-_{\mu\nu}\gamma^\nu_{\dot{A}B}\mathcal{Q}^B_i, \\
\{\mathcal{Q}_{Ai}, \mathcal{Q}_{Bj}\} &= \varepsilon_{AB}\varepsilon_{ij}\overline{\mathcal{Z}} \\
\{\mathcal{Q}_{Ai}, \mathcal{Q}_{\dot{A}j}\} &= -i\gamma^\mu_{A\dot{A}}\varepsilon_{ij}P_\mu \\
\{\mathcal{Q}_{\dot{A}i}, \mathcal{Q}_{\dot{B}j}\} &= -\varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}(\mathcal{Z} - \frac{1}{4}\mathcal{J})
\end{aligned} \tag{2.29}$$

As always, the moduli of the compactification do not appear explicitly in the algebra, but they affect the possible values of the central charge. For example, if the 5d gauge fields have holonomies around the M-theory circle, this affects the values of H and therefore of $\mathfrak{z}$.

Generalization To Any $b_2(Y)$

So far, we have considered the supersymmetric Gödel solution in pure supergravity, but we will need its generalization to include vector multiplets. For this, we could proceed abstractly, but it is convenient to consider the motivating example of compactification of M-theory to five dimensions on a Calabi-Yau manifold Y with second Betti number b_2. We introduce a basis ω_I, $I = 1, \ldots, b_2$ of $H^2(Y, \mathbb{Z})$, and define

$$\mathcal{C}_{IJK} = \frac{1}{6} \int_Y \omega_I \wedge \omega_J \wedge \omega_K. \tag{2.30}$$

The Kahler class ω of Y can be expanded as a linear combination of the ω_I:

$$\omega = \sum_{I=1}^{b_2} v^I \omega_I. \tag{2.31}$$

The v^I are interpreted as scalar fields in five dimensions (they take values in a certain Kahler cone) and their expectation values are moduli of the compactification. It turns out that only $b_2 - 1$ combinations of the b_2 fields v^I are in 5d vector multiplets. The volume of Y, which is

$$\mathcal{V} = \mathcal{C}_{IJK} v^I v^J v^K, \tag{2.32}$$

is part of a hypermultiplet (sometimes called the universal hypermultiplet). The remaining $b_2 - 1$ combinations of the v^I are in vector multiplets. It is convenient to define these combinations by setting

$$h^I = \frac{v^I}{v}, \quad v = \mathcal{V}^{1/3}, \tag{2.33}$$

so that

$$\mathcal{C}_{IJK}h^I h^J h^K = 1. \tag{2.34}$$

The h^I, with this constraint, parametrize the vector multiplet moduli space in five dimensions.

In five dimensions, a vector multiplet contains a real scalar field and a $U(1)$ gauge field. To find the gauge fields, we make a Kaluza-Klein expansion of the M-theory three-form field C:

$$C = \sum_I V^I \omega_I. \tag{2.35}$$

The V^I are abelian gauge fields in five dimensions, with field strengths $F^I = \mathrm{d}V^I$. One linear combination of the V^I, namely $V = \sum_I h_I V^I$, with $h_I = \mathcal{C}_{IJK}h^J h^K$, is in the supergravity multiplet. This is the the 5d graviphoton field. To be more exact, the graviphoton field strength is $\mathsf{T} = \sum_I h_I \mathrm{d}V^I$; $\mathrm{d}V$ is not gauge-invariant unless the h_I are constant. The orthogonal linear combinations of the V^I are in vector multiplets, together with the h^I. To describe orthogonal linear combinations of the V^I, it is useful to introduce vectors h^I_x, $x = 1, \ldots, b_2 - 1$ tangent to the hypersurface (2.34), i.e., obeying $h_I h^I_x = 0$. These can be defined as $h^I_x = \partial h^I/\partial\phi^x$, where ϕ^x are local coordinates on the hypersurface (2.34). The linear combinations of the V^I that are orthogonal to the graviphoton field[2] are $V_x = \sum_{IJK} \mathcal{C}_{IJK}h^I h^J_x V^K$. To be more precise, the gauge-invariant field strengths $F_x = \sum_{IJK} \mathcal{C}_{IJK}h^I h^J_x \mathrm{d}V^K$ are in vector multiplets.

The precise meaning of the statement that $\mathsf{T} = \sum_I h_I \mathrm{d}V^I$ is in the supergravity multiplet is that it appears in the supersymmetry transformation of the spin 3/2 gravitino field:

$$\delta\Psi_M = \nabla_M \epsilon + \frac{i}{8}\mathsf{T}_{NP}\left(\Gamma_M{}^{NP} - 4\delta^N_M \Gamma^P\right)\epsilon + \ldots, \tag{2.36}$$

[2] This orthogonality is in the natural metric $a_{IJ} = \frac{1}{4}\int_Y \omega_I \wedge *\omega_J$ on the Kahler cone. The hypersurface metric $g_{xy} = h^I_x h^J_y a_{IJ}$ is induced from this.

where the ellipsis represents fermionic terms. By contrast, the V_x appear along with derivatives of scalars in supersymmetry transformations of spin 1/2 fermi fields that are in vector multiplets. Let λ^x be fermionic fields related to the ϕ^x by supersymmetry. Then the precise meaning of the statement that F_x is in a vector multiplet is that

$$\delta\lambda^x = \frac{i}{2}\partial_M \phi^x \Gamma^M \epsilon + \frac{1}{4} F^x_{MN} \Gamma^{MN} \epsilon + \dots , \tag{2.37}$$

where again the ellipsis represents fermionic terms, and the index x in F^x was raised using the metric defined in footnote 2.

Now it should be clear how to embed the original supersymmetric Gödel solution (2.1), which corresponds to the case $b_2 = 1$ (no vector multiplets), in the theory with an arbitrary number of vector multiplets. Using the same V and the same metric as before, we simply take the h^I to be arbitrary constants, the V_x to vanish, and $V^I = h^I V$. This will ensure the vanishing of the right hand side of eqn. (2.37) and all desired properties are satisfied. Similarly, the compactified version of the solution is

$$\mathrm{d}s^2 = e^{2\sigma} \left(\mathrm{d}y + B_\mu \mathrm{d}x^\mu\right)^2 + e^{-\sigma} \sum_\mu (\mathrm{d}x^\mu)^2, \qquad B_\mu = -i\frac{e^{-\sigma}}{2} \mathsf{T}^-_{\nu\mu} x^\nu, \quad V^I_\mu = \frac{h^I}{2} \mathsf{T}^-_{\nu\mu} x^\nu, \tag{2.38}$$

again with

$$\mathsf{T}^-_{\mu\nu} = \frac{e^{-\sigma/2}}{4} \mathsf{W}^-_{\mu\nu}. \tag{2.39}$$

Again we take $\mathsf{T}^-_{\mu\nu}$ and V^I_μ to be imaginary and the metric to be real. If y is understood to be a periodic variable (with period 2π), we can slightly generalize this solution by giving nonzero constant values to V^I_y, the components of the fields V^I in the y direction.

Each gauge field V^I, for $I = 1, \dots, b_2$, couples to a conserved charge Q_I. The Q_I are components of the homology class of an M2-brane wrapped in Y. A wrapped

M2-brane with world volume Σ is an eigenstate of Q_I with eigenvalue

$$q_I = \int_\Sigma \omega_I. \tag{2.40}$$

The central charge in the 5d supersymmetry algebra is $\zeta = \sum_I h^I Q_I$ (see eqn. 2.75). Its values for a BPS particle with charges $Q_I = q_I$ is $\zeta(\vec{q}) = \sum_I h^I q_I$. A BPS particle with those charges couples to the linear combination $V(\vec{q}) = \sum_J q_J V^J$ of the V^I. In the background (2.38), the field strength of $V(\vec{q})$ is

$$F^{\{\vec{q}\}}_{\mu\nu} = \sum_I q_I h^I \mathsf{T}^-_{\mu\nu} = \zeta(\vec{q}) \mathsf{T}^-_{\mu\nu}. \tag{2.41}$$

So for each set of charges $\vec{q} = \{q_1, q_2, \ldots, q_{b_2}\}$, we will do a Schwinger calculation with background field $F^{\{\vec{q}\}}_{\mu\nu} = \zeta(\vec{q})\mathsf{T}^-_{\mu\nu}$. Part of the reason that a simple answer emerges is that the mass $m(\vec{q})$ of a BPS particle with charges $\vec{q}$ is also proportional to $\zeta(\vec{q})$, so that the ratio $F^{\vec{q}}/m(\vec{q})$ depends only on $\mathsf{T}^-_{\mu\nu}$ and not on the vector multiplet moduli.

2.1.2 The Background From A 4d Point Of View

Duality-Invariant Formalism

Here, we will describe the same background more fully from a 4d point of view.[3] We primarily work in Einstein frame (which is natural in supergravity) rather than the string frame. As we have already explained, M-theory compactification on a Calabi-Yau manifold Y gives a theory in five dimensions with $b_2(Y)$ abelian gauge fields, of which $b_2 - 1$ linear combinations are in vector multiplets and one is the graviphoton. Upon further compactification on a circle, we get one more vector multiplet, which comes from Kaluza-Klein reduction of the 5d metric on a circle.

[3] Some original supergravity references are [20, 49–51]. Our conventions are those of [87].

Thus in four dimensions, there are $b_2 + 1$ abelian gauge fields, of which b_2 linear combinations are in vector multiplets and one linear combination is the graviphoton.

In five dimensions, a vector multiplet contains a real scalar field; we described the scalars in $b_2 - 1$ vector multiplets via b_2 scalar fields h^I that obey a constraint (2.34). In four dimensions, a vector multiplet contains a *complex* scalar field. It is convenient to describe the scalar fields in b_2 vector multiplets via b_2+1 complex scalar fields X^Λ, $\Lambda = 0, \ldots, b_2$ that obey a gauge-invariance

$$X^\Lambda \to \lambda X^\Lambda, \quad \lambda \in U(1), \tag{2.42}$$

and a constraint

$$N_{\Lambda\Sigma} X^\Lambda \overline{X}^\Sigma = -1, \tag{2.43}$$

where $N_{\Lambda\Sigma}$ will be defined later. We will also eventually impose a condition to fix the $U(1)$ gauge-invariance. Alternatively, to emphasize the complex structure of the vector multiplet moduli space, one can replace the constraint (2.43) by an inequality $N_{\Lambda\Sigma} X^\Lambda \overline{X}^\Sigma < 0$ and replace the $U(1)$ gauge-invariance with a $\mathbb{C}^*$ gauge-invariance $X^\Lambda \to \lambda X^\lambda$, $\lambda \in \mathbb{C}^*$.

The X^Λ are the bottom components of superfields $\mathcal{X}^\Lambda$ that also contain fermion fields Ω^Λ and the field strengths $F^\Lambda_{\mu\nu}$ of the $U(1)$ gauge fields, which appear in the combinations:

$$\mathcal{F}^{\Lambda,-}_{\mu\nu} = F^{\Lambda,-}_{\mu\nu} - \frac{1}{2}\overline{X}^\Lambda \mathsf{W}^-_{\mu\nu} + \text{fermions}. \tag{2.44}$$

Here the 4d graviphoton field strength $\mathsf{W}^-_{\mu\nu}$ will be defined later and the fermionic terms are not important to us. Here and elsewhere in this paper, if $\mathcal{F}$ is a two-form then $\mathcal{F}^-$ is its anti-selfdual part. The superfields $\mathcal{X}^\Lambda$ have expansions

$$\mathcal{X}^\Lambda = X^\Lambda + \overline{\theta}^i \Omega^\Lambda_i + \frac{1}{2}\varepsilon_{ij}\overline{\theta}^i \sigma^{\mu\nu}\theta^j \mathcal{F}^{\Lambda,-}_{\mu\nu} + \cdots - \frac{1}{6}(\varepsilon_{ij}\overline{\theta}^i \sigma^{\mu\nu}\theta^j)^2 \Delta \overline{X}^\Lambda, \quad \Lambda = 0, \ldots, b_2, \tag{2.45}$$

where the θ's are superspace coordinates of negative chirality and $\Delta = D_\mu D^\mu$ is the Laplacian. Under the scaling (2.42), the field strengths $F^\Lambda_{\mu\nu}$ are invariant (Dirac quantization of magnetic flux gives a natural normalization of these field strengths, so they should not be rescaled), so the θ's transform as $\theta \to \lambda^{1/2}\theta$ and hence the chiral superspace measure transforms as

$$\mathrm{d}^4\theta \to \lambda^{-2}\mathrm{d}^4\theta. \tag{2.46}$$

The reader will note that although the X^Λ parametrize a complex manifold, as is manifest in the description in which they satisfy a $\mathbb{C}^*$ gauge invariance and an inequality $N_{\Lambda\Sigma}X^\Lambda\overline{X}^\Sigma < 0$, there are non-holomorphic terms in the expansion (2.45). Part of the reason for this is that the superfields $\mathcal{X}^\Lambda$ and their superspace derivatives obey a linear nonholomorphic constraint.

The kinetic energy of the vector multiplets comes from a holomorphic coupling

$$I_0 = -i\int \mathrm{d}^4x\mathrm{d}^4\theta\,\mathcal{F}_0(\mathcal{X}^\Lambda). \tag{2.47}$$

For consistency with the scaling (2.42) and (2.46), the function $\mathcal{F}_0$, which is called the prepotential, must be homogeneous in the $\mathcal{X}^\Lambda$ of degree 2. The interaction I_0 is in fact the case $\mathsf{g} = 0$ of the interactions I_g (defined in eqn. (1.2)) that are described by the GV formula.

The vector multiplets can be conveniently described in a T-duality invariant language. This makes some formulas we will need more transparent, even though, since there is no T-duality in M-theory, T-duality is not important in the derivation of the GV formula. One introduces the fields

$$\widehat{F}_\Lambda = \frac{\partial\mathcal{F}_0}{\partial X^\Lambda}, \tag{2.48}$$

which transform as $\widehat{F}_\Lambda \to \lambda \widehat{F}_\Lambda$, just like the X^Λ. One furthermore introduces the symplectic form $\Upsilon = \sum_\Lambda \mathrm{d}X^\Lambda \wedge \mathrm{d}\widehat{F}_\Lambda$ and the group $\mathrm{Sp}(2b_2+2,\mathbb{Z})$ of integer-valued linear transformations of the whole set of fields

$$\begin{pmatrix} X^\Lambda \\ \widehat{F}_\Lambda \end{pmatrix} \tag{2.49}$$

that preserve this symplectic form. From this point of view, the equation (2.48) can be described more symmetrically by saying that the vector multiplets parametrize (the quotient by $\mathbb{C}^*$ of) a $\mathbb{C}^*$-invariant Lagrangian submanifold of $\mathbb{C}^{2b_2+2}$, which we view as a complex symplectic manifold with holomorphic symplectic form Υ. One defines

$$\begin{aligned} N_{\Lambda\Sigma} &= 2\mathrm{Im}\,\widehat{F}_{\Lambda\Sigma}, \quad \widehat{F}_{\Lambda\Sigma} = \frac{\partial^2 \mathcal{F}_0}{\partial X^\Lambda \partial X^\Sigma} \\ \mathcal{N}_{\Lambda\Sigma} &= \overline{\widehat{F}}_{\Lambda\Sigma} + i\frac{(NX)_\Lambda (NX)_\Sigma}{(X,NX)}. \end{aligned} \tag{2.50}$$

Here $(NX)_\Lambda = N_{\Lambda\Sigma}X^\Sigma$ and $(X,NX) = N_{\Lambda\Sigma}X^\Lambda X^\Sigma$. These objects appear in the kinetic energy of the fields $\mathcal{X}^\Lambda$ after performing θ integrals. The constraint (2.43) can be written in a manifestly symplectic-invariant way:

$$i(X^\Lambda \overline{\widehat{F}_\Lambda} - \widehat{F}_\Lambda \overline{X}^\Lambda) = -1. \tag{2.51}$$

The action of $\mathrm{Sp}(2b_2+2,\mathbb{Z})$ on the fields $X^\Lambda, \widehat{F}_\Lambda$ has to be accompanied by linear transformations of the field strengths $F^\Lambda_{\mu\nu}$ and their duals. In fact, $\mathrm{Sp}(2b_2+2,\mathbb{Z})$ acts linearly on

$$\begin{pmatrix} F^{\Lambda+}_{\mu\nu} \\ G^+_{\Lambda\,\mu\nu} \end{pmatrix} \tag{2.52}$$

where the $G_{\Lambda\,\mu\nu}$ are the duals of $F^\Lambda_{\mu\nu}$ which can be defined by[4]:

$$G^+_{\Lambda\mu\nu} = \mathcal{N}_{\Lambda\Sigma} F^{\Sigma+}_{\mu\nu}. \tag{2.53}$$

One advantage of the redundant description via pairs of fields X^Λ, $\widehat{F}_\Lambda$ and also $F^\Lambda_{\mu\nu}$, $G_{\Lambda\,\mu\nu}$ is that this makes it possible to describe the graviphoton field in a manifestly holomorphic and duality-invariant fashion. The anti-selfdual part of the 4d graviphoton field is

$$\mathsf{W}^-_{\mu\nu} = 2(X^\Lambda G^-_{\Lambda\,\mu\nu} - \widehat{F}_\Lambda F^{\Lambda\,-}_{\mu\nu}). \tag{2.54}$$

In Lorentz signature, the selfdual part of the graviphoton field is the complex conjugate of this or

$$\mathsf{W}^+_{\mu\nu} = 2(\overline{X}^\Lambda G^+_{\Lambda\,\mu\nu} - \overline{\widehat{F}}_\Lambda F^{\Lambda\,+}_{\mu\nu}). \tag{2.55}$$

In Euclidean signature, $\mathsf{W}^+_{\mu\nu}$ is not the complex conjugate of $\mathsf{W}^-_{\mu\nu}$, but these formulas remain valid. Of course, we are interested in an anti-selfdual graviphoton background in which $\mathsf{W}^+_{\mu\nu} = 0$.

It is convenient to describe the anti-selfdual and selfdual parts of the graviphoton field using spinor indices, defined by

$$\mathsf{W}^-_{AB} = \gamma^{\mu\nu}_{AB} \mathsf{W}^-_{\mu\nu}, \quad \mathsf{W}^+_{\dot{A}\dot{B}} = \gamma^{\mu\nu}_{\dot{A}\dot{B}} \mathsf{W}^+_{\mu\nu}, \quad \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]. \tag{2.56}$$

Here $A, B = 1, 2$ and $\dot{A}, \dot{B} = 1, 2$ are respectively spinor indices of negative and positive chirality. Eqn. (2.54) shows that W^-_{AB} scales with degree 1 under the scaling $X^\Lambda \to \lambda X^\Lambda$. More precisely, W^-_{AB} is an anti-selfdual two-form valued in the pullback to spacetime of a holomorphic line bundle $\mathcal{L}$ over the vector multiplet moduli space; $\mathcal{L}$ is characterized by the fact that W^-_{AB} transforms with charge 1 under scaling. In terms of superfields, W^-_{AB} is the bottom component of a chiral superfield $\mathcal{W}_{AB}$ that

[4] If $\mathcal{L}$ is the Lagrangian density, one can define G_Λ by $G^{-\mu\nu}_\Lambda = 2i \frac{\partial \mathcal{L}}{\partial F^{\Lambda -}_{\mu\nu}}$.

likewise transforms with charge 1 under the equivalence (2.42), and $\mathcal{W}^2 = \mathcal{W}_{AB}\mathcal{W}^{AB}$ is similarly a chiral superfield of charge 2. To make the charges balance, in the interaction I_{g}

$$I_{\mathsf{g}} = -i\int_{\mathbb{R}^4} \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{F}_{\mathsf{g}}(\mathcal{X}^\Lambda)(\mathcal{W}_{AB}\mathcal{W}^{AB})^{\mathsf{g}} \tag{2.57}$$

that enters the GV formula, the functions $\mathcal{F}_{\mathsf{g}}$ must be homogeneous of degree $2-2\mathsf{g}$.

Background Gauge Fields In $d = 4$

Next we explain the 4d analogs of some observations that were made in section 2.1.1 for $d = 5$.

In the supersymmetric graviphoton background, the linear combinations of field strengths $\mathcal{F}^\Lambda_{\mu\nu}$ defined in (2.44) and appearing as components of the vector superfields $\mathcal{X}^\Lambda$ must vanish. This gives the very important relation

$$F^\Lambda_{\mu\nu} = \frac{1}{2}\overline{X}^\Lambda \mathsf{W}^-_{\mu\nu}, \tag{2.58}$$

which is the 4d analog of the corresponding 5d statement $F^I_{\mu\nu} = h^I \mathsf{T}^-_{\mu\nu}$.

A 4d particle with charges $\vec{q} = q_0, \ldots, q_{b_2}$ couples to the effective gauge field given by the linear combination $A_\mu(\vec{q}) = q_\Lambda A^\Lambda_\mu$. The field strength of $A_\mu(\vec{q})$ in the graviphoton background is

$$F_{\mu\nu}(\vec{q}) = \frac{1}{2}q_\Lambda \overline{X}^\Lambda \mathsf{W}^-_{\mu\nu} = \frac{1}{4}\overline{\mathcal{Z}}(\vec{q})\mathsf{W}^-_{\mu\nu}. \tag{2.59}$$

Thus, the effective field strength seen by such a particle in the graviphoton background is proportional to $\overline{\mathcal{Z}}(\vec{q})$, the complex conjugate of the central charge $\mathcal{Z}(\vec{q}) = 2\sum_\Lambda q_\Lambda X^\Lambda$. This statement is the generalization of the fact (eqn. (2.41)) that in $d = 5$, the effective field strength is proportional to the 5d central charge ζ.

Now, the mass of a BPS particle in $d = 4$ is $m(\vec{q}) = |\mathcal{Z}|$. This means that the dimensionless ratio $F(\vec{q})/m(\vec{q})^2$ that appears in the Schwinger calculation is proportional to $1/\mathcal{Z}$ and in particular is holomorphic. This is part of the mechanism by which a holomorphic answer emerges from the Schwinger calculation that we perform in terms of 4d fields in section 2.3, even though the particle masses are certainly not holomorphic in $\mathcal{Z}$.

Comparison To Perturbation Theory

Now let us explain why in Type IIA superstring perturbation theory, $\mathcal{F}_{\mathsf{g}}$ is generated only in genus g. We practice first with the case $\mathsf{g} = 0$, corresponding to the classical approximation. We have written the above formulas in 4d Einstein frame, in which the dilaton (which is in a hypermultiplet) does not couple directly to the $\mathcal{F}_{\mathsf{g}}$'s, which govern vector multiplets. To compare to string perturbation theory, we must transform to the string frame, which we do by a Weyl transformation of the metric $g_{\mu\nu} \to e^{-2\phi}\widehat{g}_{\mu\nu}$, where $\widehat{g}_{\mu\nu}$ is the metric in string frame, ϕ is the four-dimensional dilaton, and the string coupling constant is $g_{st} = e^{\phi}$. The Einstein-Hilbert action $\frac{1}{2\kappa_4^2}\int \mathrm{d}^4x\sqrt{g}R$ becomes

$$\frac{1}{2\kappa_4^2}\int \mathrm{d}^4x\sqrt{\widehat{g}}e^{-2\phi}R(\widehat{g}) \tag{2.60}$$

and is generated in string theory in genus 0. More generally, any genus g contribution to the effective action for external fields from the Neveu-Schwarz (NS) sector only is proportional to $g_{st}^{2\mathsf{g}-2} = \exp((2\mathsf{g}-2)\phi)$. But a genus g contribution to the effective action that has in addition k external Ramond-Ramond (RR) gauge field strengths (normalized in the standard way to satisfy ordinary Dirac quantization and standard Bianchi identities) is proportional to $g_{st}^{2\mathsf{g}-2+k} = \exp((2\mathsf{g}-2+k)\phi)$. Let us see how this works for $\mathcal{F}_0$. Performing the θ integrals in (2.47) gives a variety of terms, among them

$$I_0 = \int \mathrm{d}^4x \sqrt{\widehat{g}} \left(e^{-2\phi} \frac{\partial \mathcal{F}_0}{\partial X^\Lambda} \widehat{\Delta} \overline{X}^\Lambda + \frac{\partial^2 \mathcal{F}_0}{\partial X^\Lambda \partial X^\Sigma} \mathcal{F}^{\Lambda\,-}_{\mu\nu} \mathcal{F}^{\Sigma\,-\,\mu\nu} + \dots \right), \tag{2.61}$$

where $\widehat{\Delta} = \widehat{g}^{\mu\nu} \widehat{D}_\mu \widehat{D}_\nu$ is the Laplacian defined with the string metric. Bearing in mind that X^Λ is described by an NS-NS vertex operator and $\mathcal{F}^{\Lambda\,-} = F^{\Lambda\,-} - \frac{1}{2}\overline{X}^\Lambda \mathsf{W}^- + \dots$ is described by an RR vertex operator, we see that in Type IIA superstring theory, such interactions can be generated only in genus 0. The analog of this for $\mathcal{F}_\mathsf{g}$ is immediate. All we have to know is that $\mathsf{W}^-_{\mu\nu}\mathsf{W}^{-\,\mu\nu} = g^{\mu\mu'} g^{\nu\nu'} \mathsf{W}^-_{\mu\nu} \mathsf{W}^-_{\mu'\nu'}$ acquires a factor of $e^{4\phi}$ in transforming to the string frame and that $\mathsf{W}^-_{\mu\nu}$ comes from the Ramond-Ramond sector. Performing θ integrals in the definition of I_g gives a variety of terms such as

$$I_\mathsf{g} = \int \mathrm{d}^4x \sqrt{\widehat{g}} \left(\left(e^{-2\phi} \frac{\partial \mathcal{F}_\mathsf{g}}{\partial X^\Lambda} \widehat{\Delta} \overline{X}^\Lambda + \frac{\partial^2 \mathcal{F}_\mathsf{g}}{\partial X^\Lambda \partial X^\Sigma} \mathcal{F}^{\Lambda\,-}_{\lambda\sigma} \mathcal{F}^{\Sigma\,-\,\lambda\sigma} + \dots \right) e^{4\mathsf{g}\phi} (\mathsf{W}^{-\mu\nu} \mathsf{W}_{-\mu\nu})^\mathsf{g} + \dots \right), \tag{2.62}$$

and we see the expected scaling for interactions that in superstring perturbation theory are generated only in genus g.

Shift Symmetries

The full power of the duality-invariant formalism sketched in section 2.1.2 is not really needed for our problem. The reason is that reduction on a circle from five to four dimensions gives a natural duality frame, defined by the fields that arise in classical dimensional reduction from the classical gauge and gravitational fields in five dimensions. Moreover, among the $U(1)$ gauge fields in four dimensions, there is a distinguished one $A^0_\mu = -B_\mu$, which arises from the components $g_{\mu 5}$ of the five-dimensional metric. The other 4d gauge fields arise in Kaluza-Klein reduction of the

5d gauge fields V^I:

$$V^I = \sum_{\mu=1}^{4} A^I_\mu \mathrm{d}x^\mu + \alpha^I \left(\mathrm{d}y + B_\mu \mathrm{d}x^\mu\right), \quad I = 1, \ldots, b_2. \tag{2.63}$$

Here the A^I_μ are gauge fields in four dimensions, and the α^I are scalars. The holonomy of V^I around the Kaluza-Klein circle is $\exp(2\pi i \alpha^I)$, so we expect a symmetry

$$\alpha^I \to \alpha^I + n^I. \tag{2.64}$$

This shift in α^I is generated by a gauge transformation $\exp(in^I y)$ of A^I together with a redefinition of the gauge fields

$$A^I_\mu \to A^I_\mu + n^I A^0_\mu = A^I_\mu - n^I B_\mu, \quad n^I \in \mathbb{Z}. \tag{2.65}$$

Thus although the definition of A^0_μ is completely natural, A^I_μ is only well-defined up to an integer multiple of A^0_μ.

The gauge fields A^0_μ and A^I_μ, or rather their field strengths $\mathrm{d}A^0$ and $\mathrm{d}A^I$, appear in the superfields $\mathcal{X}^0 = X^0 + \cdots + \theta^2 \mathrm{d}A^0_\mu + \ldots$ and $\mathcal{X}^I = X^I + \cdots + \theta^2 \mathrm{d}A^I + \ldots$. The symmetries (2.65) extend to symmetries of the superfields

$$\mathcal{X}^I \to \mathcal{X}^I + n^I \mathcal{X}^0, \quad n^I \in \mathbb{Z}. \tag{2.66}$$

These transformations (accompanied by corresponding transformations of the derivatives $\partial \mathcal{F}_0 / \partial \mathcal{X}^\Lambda$) are the only $\mathrm{Sp}(2b_2{+}2, \mathbb{Z})$ duality transformations that are important in the derivation of the GV formula.

The ratios $\mathcal{Z}^I = \mathcal{X}^I / \mathcal{X}^0$ are invariant under scaling of the homogeneous coordinates $\mathcal{X}^\Lambda$ and parametrize the vector multiplet moduli space. They transform simply under (2.66):

$$\mathcal{Z}^I \to \mathcal{Z}^I + n^I, \quad n^I \in \mathbb{Z}. \tag{2.67}$$

Of course, these shifts really act only on the bottom components of the $\mathcal{Z}^I$, which we call Z^I:

$$Z^I \to Z^I + n^I, \quad n^I \in \mathbb{Z}. \tag{2.68}$$

The Z^I have a simple interpretation. Consider an M2-brane wrapped on $p \times S^1 \times \Sigma \subset \mathbb{R}^4 \times S^1 \times Y$, where p is a point in $\mathbb{R}^4$ and $\Sigma \subset Y$ is a holomorphic curve. Such an M2-brane is a supersymmetric cycle, and its action must be a holomorphic function of the vector multiplet moduli. The charges of the wrapped M2-brane are $q_I = \int_\Sigma \omega_I$ (the ω_I were introduced in eqn. (2.30)), and its mass is given[5] by the central charge $\zeta(\vec{q}) = \sum_I v^I q_I$ (which will be positive for a supersymmetric cycle). The real part of the action is simply the mass times the circumference of the Kaluza-Klein circle. If we write the metric of $\mathbb{R}^4 \times S^1 \times Y$ in the M-theory description as

$$\mathrm{d}s^2_{\mathrm{M}} = \mathrm{d}s^2_{10} + e^{2\gamma}(\mathrm{d}y + B_\mu dx^\mu)^2. \tag{2.69}$$

then the circumference is $2\pi e^\gamma$, so the real part of the M2-brane action is $2\pi e^\gamma \sum_I v^I q_I$. On the other hand, the imaginary part of the action is just the C-field period $-\int_{p\times S^1 \times \Sigma} C$. Recalling the definition (2.35) of the 5d gauge fields V^I, we see that this period is $2\pi\alpha^I q_I$. So the Euclidean action of the wrapped M2-brane is

$$S(\vec{q}) = 2\pi \sum_I q_I \left(e^\gamma v^I - i\alpha^I\right). \tag{2.70}$$

It is convenient to re-express this formula in terms of the circumference of the M-theory circle defined in 5d supergravity. The reason this is not the same as the circumference $2\pi e^\gamma$ measured in the 11d metric is that compactification from 11 di-

[5] We work in units in which the M2-brane tension is 1.

mensions to 5 dimensions on the Calabi-Yau manifold Y gives a 5d gravitational action $\int \mathrm{d}^5x\sqrt{g}_{\mathrm{M}}\mathcal{V}R(g_{\mathrm{M}})$, where $\mathcal{V}$ is the volume of Y in the eleven-dimensional description, g_{M} is the 5d metric in that description, and $R(g_{\mathrm{M}})$ is the Ricci scalar. In 5d supergravity, it is convenient and usual to make a Weyl transformation to Einstein frame, replacing g_{M} by a 5d-metric via $g_{\mathrm{M}} = \mathcal{V}^{-2/3}g_{5\mathrm{d}}$. The relation of e^γ to the radius e^σ of the circle in the 5d description is thus $e^\gamma = e^\sigma/v$, where $v = \mathcal{V}^{1/3}$. Thus we rewrite (2.70) in 5d terms:

$$S(\vec{q}) = 2\pi \sum_I q_I \left(e^\sigma h^I - i\alpha^I\right), \quad h^I = \frac{v^I}{v}. \tag{2.71}$$

The coefficients $e^\sigma h^I - i\alpha^I$ in $\mathcal{S}(\vec{q})$ must be holomorphic functions of the vector multiplet moduli, or in other words of the Z^I. Comparing the transformations (2.64) and (2.67), we find the relationship

$$Z^I = \alpha^I + ie^\sigma h^I, \tag{2.72}$$

which describes the background values of the superfield $\mathcal{Z}^I$. The action is thus $S(\vec{q}) = -2\pi i \sum_I q_I Z^I$. However, it is more convenient to introduce a superfield $\mathcal{S}(\vec{q})$ whose bottom component is $S(\vec{q})$:

$$\mathcal{S}(\vec{q}) = -2\pi i \sum_I q_I \mathcal{Z}^I. \tag{2.73}$$

This is more convenient because when we perform an actual computation in section 2.2, a particle propagating around the circle has fermionic as well as bosonic collective coordinates. Writing the action as a superfield is an easy way to incorporate the fermionic collective coordinates.

In the 5d description, we write the real part of the action as $2\pi e^\sigma m(\vec{q})$, with $m(\vec{q})$ the mass of the wrapped M2-brane in that description:

$$m(\vec{q}) = \sum_I q_I h^I. \tag{2.74}$$

Since the wrapped M2-brane is BPS, this mass also equals the central charge in the 5d description:

$$\zeta(\vec{q}) = \sum_I q_I h^I. \tag{2.75}$$

Eqn. (2.72) states in particular that $\mathrm{Im}\, Z^I = h^I e^\sigma$. Recalling the constraint (2.34), this implies the useful relation

$$\mathcal{C}_{IJK} \mathrm{Im}\, Z^I\, \mathrm{Im}\, Z^J\, \mathrm{Im}\, Z^K = e^{3\sigma}. \tag{2.76}$$

Validity Of The Calculation

In M-theory on $\mathbb{R}^4 \times S^1 \times Y$, we will perform a computation involving M2-branes wrapped on $S^1 \times \Sigma$ where Σ is a non-trivial cycle in Y. Our aim here is to describe the range of validity of the computation, and explain why this suffices to determine the full answer.

For M-theory to be a reasonable description, we would like Y not to be sub-Planckian, so we can ask for its volume $\mathcal{V}_{\mathrm{M}}$ in M-theory units not to be sub-Planckian. If we are not too close to a boundary of the Kahler cone of Y, then a wrapped M2-brane has a size of order $\mathcal{V}_{\mathrm{M}}^{1/6}$. To justify a calculation in which wrapped M2-branes propagating around S^1 are treated as elementary particles, we would like the S^1 to be much larger than the size of the M2-branes, which will be generically of order $\mathcal{V}_{\mathrm{M}}^{1/6}$. So we want

$$e^\gamma >> \mathcal{V}_{\mathrm{M}}^{1/6} \gtrsim 1, \tag{2.77}$$

where as in (2.69), e^γ is the radius of the M-theory circle in M-theory units.

When we relate M-theory on $\mathbb{R}^4 \times S^1 \times Y$ to Type IIA superstring theory on $\mathbb{R}^4 \times Y$, the ten-dimensional string coupling constant $g_{10} = e^\phi$ is related to γ by [88]

$$g_{10} = e^{3\gamma/2}. \tag{2.78}$$

Moreover, the metric of $\mathbb{R}^4 \times Y$ in the Type IIA description is

$$\mathrm{d}s^2_{\mathrm{IIA}} = e^\gamma \mathrm{d}s^2_{10}. \tag{2.79}$$

In particular, the volume of Y in the string theory description is (recall that $e^\gamma = e^\sigma/v$)

$$\mathcal{V}_{\mathrm{IIA}} = e^{3\gamma}\mathcal{V}_{\mathrm{M}} = e^{3\sigma}. \tag{2.80}$$

Eqns. (2.77), (2.78), and (2.80) show that in the region in which our computation is valid, g_{10} and V_{IIA} are both large. In particular, the fact that g_{10} is large means that, as expected, string perturbation theory is not useful in the region in which our calculation is valid. Moreover, as explained in section **??**, the fact that V_{IIA} is large means that we will not encounter the holomorphic anomaly. Notice from (2.78) and (2.80) that

$$g_{10} = e^{3\sigma/2}\frac{1}{\sqrt{\mathcal{V}_M}} = e^{3\sigma/2} g_{\mathrm{st}}, \tag{2.81}$$

where g_{st} is a 4-dimensional string coupling introduced in section 2.1.2.

The interactions $\mathcal{F}_{\mathsf{g}}(\mathcal{X})$, when expressed in string frame with Kahler moduli (and hence $\mathcal{V}_{\mathrm{IIA}}$) held fixed, have a known dependence on g_{st}, as explained in section 2.1.2. Due to (2.81), they have a known dependence on g_{10} as well. So a calculation that is only valid for $g_{10} >> 1$ can suffice to determine them.

Because of holomorphy, the same is true of a calculation that is only valid for large $\mathcal{V}_{\mathrm{IIA}}$. To explain this, we use the homogeneity of $\mathcal{F}_{\mathsf{g}}$ to write $\mathcal{F}_{\mathsf{g}}(\mathcal{X}) =$

$(\mathcal{X}^0)^{2-2\mathsf{g}}\Phi(\mathcal{Z}^1,\ldots,\mathcal{Z}^{b_2})$. To avoid clutter, in the following argument, we take $b_2 = 1$, so there is only one $\mathcal{Z}$. Also we write $\mathcal{Z} = \alpha + i\beta$ where β is defined in eqn. (2.72). The shift symmetry $\mathcal{Z} \to \mathcal{Z} + n$, $n \in \mathbb{Z}$, implies that the general form of $\Phi(\mathcal{Z})$ is $\Phi(\mathcal{Z}) = \sum_{n\in\mathbb{Z}} c_n \exp(2\pi i n \mathcal{Z})$, with constants c_n. (Moreover, these constants vanish for $n < 0$, since an exponential blowup for large volume would contradict what we know from supergravity.) We can write $\Phi(\mathcal{Z}) = \sum_{n=0}^{\infty} f_n(\beta)\exp(2\pi i n\alpha)$, where $f_n(\beta) = c_n \exp(-2\pi n\beta)$. Since each $f_n(\beta)$ has a known dependence on β, if we can compute these functions for large β, this will suffice to determine the whole function $\Phi(\mathcal{Z})$. But large β is precisely the large volume region in which the Schwinger-like computation is valid, so that computation can suffice to determine $\mathcal{F}_\mathsf{g}(\mathcal{X})$.

Classical Reduction From Five Dimensions

As explained in section **??** of the introduction, it is important to know to what extent the I_g's or equivalently the $\mathcal{F}_\mathsf{g}$'s can arise by classical dimensional reduction from five dimensions. We will describe the two contributions that are known and explain why they are the only ones.

After performing the θ integrals, I_0 contributes a term to the effective action with two derivatives, so a classical contribution to I_0 must come from the two-derivative part of the effective action in five dimension, or in other words from the minimal supergravity action with vector multiplets. This contributes the much-studied classical prepotential

$$\mathcal{F}_0^{\mathrm{cl}}(\mathcal{X}) = -\frac{1}{2}\sum_{IJK} \mathcal{C}_{IJK}\frac{\mathcal{X}^I\mathcal{X}^J\mathcal{X}^K}{\mathcal{X}^0}. \tag{2.82}$$

With this prepotential, and with the help of eqn. (2.76), one finds that the constraint (2.43) implies that $|X^0| = e^{-3\sigma/2}/2$. We choose the phase so that

$$X^0 = -\frac{i}{2}e^{-3\sigma/2}. \tag{2.83}$$

After performing θ integrals, the four-dimensional action that follows from $\mathcal{F}_0^{\rm cl}(\mathcal{X})$ includes kinetic terms for the gauge fields:

$$-\frac{i}{4\pi}\int \mathrm{d}^4x\sqrt{g}\mathcal{N}_{\Lambda\Sigma}g^{\mu\mu'}g^{\nu\nu'}F^{\Lambda+}_{\mu\nu}F^{\Sigma+}_{\mu'\nu'}+c.c. \tag{2.84}$$

Using (2.50) for $\mathcal{N}_{\Lambda\Sigma}$ and (2.72) for the ratios X^K/X^0, we find a parity-violating part of the kinetic term

$$\mathcal{I}^- = -\frac{3}{2\pi}\sum_{IJK}\mathcal{C}_{IJK}\int_{\mathbb{R}^4}\alpha^K(F^I-\alpha^I F^0)\wedge(F^J-\alpha^J F^0) \tag{2.85}$$

and a parity-conserving part

$$\mathcal{I}^+ = -\frac{1}{2\pi}\sum_{IJ}\int_{\mathbb{R}^4}e^{\sigma}a_{IJ}(F^I-\alpha^I F^0)\wedge\star(F^J-\alpha^J F^0)-\frac{1}{4\pi}\int_{\mathbb{R}^4}e^{3\sigma}F^0\wedge\star F^0, \tag{2.86}$$

where

$$a_{IJ} = -3\mathcal{C}_{IJK}h^K+\frac{9}{2}h_I h_J \tag{2.87}$$

is the metric on the Kahler cone defined in footnote 2. The parity-violating contribution descends from a Chern-Simons interaction

$$-\frac{1}{(2\pi)^2}\sum_{IJK}\mathcal{C}_{IJK}\int V^I\wedge \mathrm{d}V^J\wedge\mathrm{d}V^K \tag{2.88}$$

in five dimensions. Notice that although $\mathcal{I}^-$ is not left fixed by the shift symmetries $\alpha^I\to\alpha^I+n^I$, it changes only by a topological invariant, so its contribution to the classical equations of motion does respect the shift symmetries. As is usual in such problems, at the classical level, the shift symmetries are continuous symmetries with no restriction for the n^I to be integers. (Quantum mechanically, the shift symmetries are broken to discrete symmetries by M2-brane instanton effects that will be studied in section 2.2.) The parity-conserving term $\mathcal{I}^+$ descends from the gauge theory kinetic energy $-\frac{1}{2}\int a_{IJ}\mathrm{d}V^I\wedge\star\mathrm{d}V^J$ in five dimensions, along with the Einstein-Hilbert action, which contributes to the kinetic energy of A^0.

What about I_1? This interaction contributes four-derivative terms to the effective action in four dimensions, so a classical contribution to I_1 comes from a term in the five-dimensional effective action with four derivatives. In eleven-dimensional M-theory, essentially only one multi-derivative correction to the minimal two-derivative supergravity action is known. This is a term

$$\Delta I = \frac{1}{(2\pi)^4}\int C\wedge\left[\frac{1}{768}(\mathrm{Tr}\,R^2)^2 - \frac{1}{192}\mathrm{Tr}\,R^4\right] \tag{2.89}$$

(where R is the Riemann tensor, which is viewed as a matrix-valued two-form in defining the trace) that was originally found [89] by its role in anomaly cancellation in the field of an M5-brane.[6] In compactification on $M_5\times Y$ (where for us Y will be a Calabi-Yau three-fold and $M_5=\mathbb{R}^4\times S^1$), we consider a contribution to ΔI with two factors of R tangent to Y and the other two tangent to $\mathbb{R}^4\times S^1$. This contribution generates a Chern-Simons interaction in five dimensions $\frac{1}{16\cdot 24\pi^2}\sum_I c_{2,I}\int V^I\wedge\mathrm{Tr}\,R\wedge R$, where

$$c_{2,I} = \frac{1}{16\pi^2}\int_Y \omega_I \mathrm{Tr}\,R\wedge R \tag{2.90}$$

are the coefficients of the second Chern class $c_2(Y)$ in a basis dual to the ω_I. Upon reduction to four dimensions, that Chern-Simons coupling becomes

$$\frac{1}{16\cdot 12\pi}\sum_I c_{2,I}\int \alpha^I \mathrm{Tr}\,R\wedge R. \tag{2.91}$$

[6] The full supersymmetric completion of this coupling is not known. We normalize the C field so that the periods of its curvature $G=\mathrm{d}C$ differ by integer multiples of 2π. See [90, 91] for detailed formulas with other conventions.

Again, this possesses the shift symmetry, modulo a topological invariant. This four-dimensional interaction can be derived from

$$\mathcal{F}_1(\mathcal{X}) = -\frac{i}{64\cdot 12\pi}\sum_I c_{2,I}\frac{\mathcal{X}^I}{\mathcal{X}^0} = -\frac{i}{64\cdot 12\pi}\sum_I c_{2,I}\mathcal{Z}^I. \tag{2.92}$$

Could there be other classical contributions to $\mathcal{F}_1(\mathcal{X})$, apart from this known contribution? Since $\mathcal{F}_1(\mathcal{X})$ is invariant under scaling, it is a function of the ratios $\mathcal{Z}^I = \mathcal{X}^I/\mathcal{X}^0$. A term in $\mathcal{F}_1$ that is quadratic or higher order in the $\mathcal{Z}^I$ would violate the classical shift symmetries. We already know about the linear terms. What about a constant contribution? Depending on whether the constant is real or imaginary, it would contribute a parity-violating interaction $\int \mathrm{Tr}\, R\wedge R$ or a parity-conserving one $\int \mathrm{Tr}\, R\wedge\star R$. The parity-violating contribution must be absent, since M-theory conserves parity. To generate a parity-conserving R^2 interaction by classical dimensional reduction, we would have to start with $\int \mathrm{Tr}\, R\wedge\star R$ in five dimensions, but reduction of this to four dimensions gives $\int e^\sigma \mathrm{Tr}\, R\wedge\star R$, with an unwanted factor of e^σ. This factor is absent in the four-dimensional effective interaction associated to a constant $\mathcal{F}_1$. Thus, there is no way to generate classically a constant contribution to $\mathcal{F}_1$.

What about $\mathcal{F}_{\mathsf{g}}$ for $\mathsf{g} > 1$? No classical contributions are known and we claim that there are none. This actually almost follows from the shift symmetries. The shift symmetries imply that $\mathcal{F}_{\mathsf{g}}$ must be independent of the ratios $\mathcal{Z}^I$ and hence must be $d_{\mathsf{g}}(\mathcal{X}^0)^{2-2\mathsf{g}}$, for some constant d_{g}.

To show the vanishing of these constants, one may use a scaling argument similar to the one used above for $\mathcal{F}_1$. For this, one observes that for $\mathsf{g} > 1$, I_{g} generates among other things a 4d coupling $R^2F^{2\mathsf{g}-2}$ in which indices are contracted using only the metric tensor and not the 4d Levi-Civita tensor.[7] Such couplings can be lifted to $R^2F^{2\mathsf{g}-2}$ couplings in five dimensions, but as we found for $\mathsf{g} = 1$, the dimensional reduction of those couplings to four dimensions does not give the power of e^σ that is

[7] The case $\mathsf{g} = 1$ is exceptional partly because this statement fails for $\mathsf{g} = 1$ if $\mathcal{F}_1$ is a real constant.

needed to match I_{g}. Alternatively, one may argue as follows using the fact that the 4d couplings derived from I_{g} with $\mathsf{g} > 1$ also include terms that are of odd order in the Levi-Civita tensor $\varepsilon_{\mu\nu\alpha\beta}$, and depend only on the scalars X^Λ, the field strengths F^I, and the Riemann tensor R (and not their derivatives). We can see as follows that these terms do not arise by reduction of covariant, gauge-invariant couplings in five dimensions. To get such terms by reduction, the starting point should be 5d interactions that are local, generally covariant, and gauge-invariant, and odd order in the 5d Levi-Civita tensor (so that their reduction to $d = 4$ will be odd order in the 4d Levi-Civita tensor). A 5d interaction that is covariant and odd order in the Levi-Civita tensor cannot be the integral of a polynomial in the field strengths $F^I_{\mu\nu}$ and the Riemann tensor $R_{\mu\nu\alpha\beta}$; such a polynomial would have an even number of indices and there would be no way to contract them with the help of any number of copies of the metric tensor and an odd number of Levi-Civita tensors. We do not want to use covariant derivatives of F or R, since then we will get covariant derivatives in $d = 4$. So we have to start with a 5d interaction that is gauge-invariant and local (meaning that its variation is a gauge-invariant local function) but is not the integral of a gauge-invariant local density. The only such interactions are the standard Chern-Simons interactions.[8] We have already analyzed their contributions.

In short, only some very special and known contributions to $\mathcal{F}_0$ and $\mathcal{F}_1$ can arise by classical reduction from five dimensions. Everything else can be determined by a Schwinger-like calculation.

[8]This fact is widely used but not always explained. Let us say that a generally covariant interaction U has weight n if it scales as $U \to e^{n\lambda}U$ under a global Weyl transformation $g_{\mu\nu} \to e^{\lambda} g_{\mu\nu}$ (with constant λ). Any generally covariant interaction is a linear combination of interactions of definite weight. If U has non-zero weight n and its variation is a local gauge-invariant functional of the metric and other fields, then U is the integral of a gauge-invariant local density, since $U = (1/n)\int \mathrm{d}^5x\sqrt{g}\, g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}U$. Interactions of weight 0 are very few (in any odd dimension and in particular in dimension 5, they are all parity-odd and proportional to the Levi-Civita tensor). It is not hard to see by hand that the standard Chern-Simons functions are the only interactions U of weight 0 whose variations are local and gauge-invariant but that are not themselves integrals of locally-defined gauge-invariant densities. One can approach this last question by first classifying the possible variations of U; in five dimensions, these must be bilinear in the gauge field strength F or the Riemann tensor R.

2.2 The Schwinger Calculation With Particles

2.2.1 Overview

In this section, we come finally to the Schwinger-like calculation for massive BPS states in five dimensions. We view an M2-brane wrapped on $p \times S^1 \times \Sigma \subset \mathbb{R}^4 \times S^1 \times Y$ (where p is a point in $\mathbb{R}^4$ and Σ is a holomorphic curve in Y) as a supersymmetric instanton. We work in the regime that the radius of S^1 is large, so the M2-brane can be treated as a point particle coupled to 5d supergravity. The particle action is uniquely determined by supersymmetry modulo irrelevant terms of higher dimension (once the graviphoton is turned on, this assertion depends on the extended supersymmetry of the graviphoton background), and this makes our considerations simple. In appendix B, we show explicitly how the supersymmetric particle action emerges from an underlying M2-brane action.

In our computation, we consider only the leading order approximation to the particle action, ignoring all sorts of couplings of higher dimension, and we integrate over fluctuations around the classical particle orbit in a one-loop approximation. This computation gives a result with the correct dependence on the radius e^σ of the S^1 to contribute to the chiral interactions $\mathcal{F}_{\mathfrak{g}}(\mathcal{X})$. The terms we neglect all have extra powers of $e^{-\sigma}$ and so cannot contribute to those couplings.

In section 2.2.2, we consider the basic example of a massive BPS hypermultiplet that arises from wrapping an M2-brane on an isolated genus 0 holomorphic curve in Σ. We evaluate the contribution to the GV formula for the case that (as tacitly assumed above and in the introduction) the BPS state wraps just once around the S^1. In section 2.2.3, we explain what multiple winding means in this context and thereby get the full contribution of the hypermultiplet to the GV formula. In section 2.2.4, we evaluate the contribution of arbitrary massive BPS states to the GV formula.

This generalization is rather simple because of the extended supersymmetry of the graviphoton background.

As explained in the introduction, the treatment of BPS states that are *massless* in five dimensions requires a different approach, based on fields rather than particles. This is presented in section 2.3.

A Detail

We perform our calculation in terms of the variables Z^I and σ, and use the classical constraint $X^0 = -(i/2)e^{-3\sigma/2}$ (eqn. (2.83)) to express the results in a manifestly supersymmetric fashion in terms of superfields $\mathcal{X}^\Lambda$. But that classical constraint equation is derived from the classical prepotential $\mathcal{F}_0$. Since we will be computing in particular instanton corrections to $\mathcal{F}_0$, the constraint equation actually has instanton corrections. So supersymmetry actually implies that, when the effective action is expressed in terms of Z^I and σ, it has multi-instanton corrections and also instanton/anti-instanton corrections (that is, it has terms whose Z-dependence corresponds to effects of multiple BPS particles and/or antiparticles). These terms cannot be confused with 1-instanton or multi-instanton corrections to the $\mathcal{F}_{\mathsf{g}}$'s, because they have the wrong dependence on σ (they vanish too rapidly for large σ). But are there multi-instanton corrections to the $\mathcal{F}_{\mathsf{g}}$'s, as opposed to the 1-instanton contributions that we will evaluate? One expects the answer to this question to be "no," because the interactions among massive BPS particles are irrelevant at long distances, as we will explain for a related reason in section 2.2.3.

2.2.2 Massive Hypermultiplet

The Free Action

The central charge ζ is real in five dimensions. The mass M of a BPS particle is $M = |\zeta|$, and there are two types of massive BPS particles, with $\zeta > 0$ or $\zeta < 0$.

They arise from M2-branes and their antibranes wrapped on a holomorphic curve $\Sigma \subset Y$ (equivalently, they arise from M2-branes wrapped on Σ with positive or negative orientation). We will consider the case $\zeta > 0$, and we define what we mean by M2-branes (as opposed to antibranes) by saying that this case corresponds to wrapped M2-branes.

Let us consider a supermultiplet consisting of BPS particles of mass $M = \zeta$ at rest. The supersymmetry algebra reduces to

$$\{Q_{Ai}, Q_{Bj}\} = 2M\varepsilon_{ij}\varepsilon_{AB}, \quad \{Q_{\dot{A}i}, Q_{\dot{B}j}\} = 0 = \{Q_{Ai}, Q_{\dot{A}j}\}. \tag{2.93}$$

In a unitary theory, the vanishing of $\{Q_{\dot{A}i}, Q_{\dot{B}j}\}$ implies that the operators $Q_{\dot{A}i}$ annihilate the whole supermultiplet. On the other hand, the operators $Q_{Ai}/\sqrt{M}$ generate a Clifford algebra. The irreducible representation of this Clifford algebra consists of two bosonic states transforming as $(0,0)$ under $SU(2)_\ell \times SU(2)_r \cong \mathrm{Spin}(4)$ and two fermionic states transforming as $(1/2, 0)$. These four states make up a massive BPS hypermultiplet (of positive central charge). In M-theory compactified to five dimensions on Y, such a massive hypermultiplet arises from an M2-brane wrapped on an isolated genus 0 curve $\Sigma \subset Y$. If by "supermultiplet," we mean a set of states that provide an irreducible representation of the full superalgebra of spacetime symmetries (including the rotation group $\mathrm{Spin}(4) \cong SU(2)_\ell \times SU(2)_r$), then a general BPS supermultiplet at rest consists of the tensor product of the states of a massive hypermultiplet with some representation (j_ℓ, j_r) of $SU(2)_\ell \times SU(2)_r$. We consider a hypermultiplet here, and analyze the contribution to the GV formula of a general BPS supermultiplet in section 2.2.4.

We ultimately will perform a one-loop calculation involving small fluctuations around a particle trajectory of the form $p \times S^1 \subset \mathbb{R}^4 \times S^1$, for some $p \in \mathbb{R}^4$. In this one-loop approximation, the BPS particle is nearly at rest, meaning that it can be treated nonrelativistically. So we can approximate the Hamiltonian as $H = M + H'$, where $H' = \sum_{\mu=1}^{4} P_\mu^2/2M$ is the nonrelativistic Hamiltonian; here P_μ, $\mu = 1, \ldots, 4$ is the momentum. We replace the supersymmetry algebra (2.13) with its nonrelativistic limit

$$\begin{aligned} \{Q_{Ai}, Q_{Bj}\} &= 2M\varepsilon_{AB}\varepsilon_{ij} \\ \{Q_{\dot{A}i}, Q_{\dot{B}j}\} &= \varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}\frac{P^2}{2M} \\ \{Q_{Ai}, Q_{\dot{A}j}\} &= -i\Gamma^\mu_{A\dot{A}}\varepsilon_{ij}P_\mu. \end{aligned} \tag{2.94}$$

The momentum P_μ commutes with all supersymmetry generators, as does the nonrelativistic Hamiltonian H':

$$[H', Q_{Ai}] = [H', Q_{\dot{A}j}] = [H', P_\mu] = 0. \tag{2.95}$$

Eqn. (2.94) tells us that $\psi_{Ai} = Q_{Ai}/M\sqrt{2}$ (the normalization will be convenient) obeys fermion anticommutation relations $\{\psi_{Ai}, \psi_{Bj}\} = M^{-1}\varepsilon_{AB}\varepsilon_{ij}$. Eqn. (2.95) tells us further that the ψ_{Ai} commute with the Hamiltonian and thus obey $\dot{\psi}_{Ai} = 0$. To derive this equation of motion along with the anticommutation relations from an effective action, the action must be $\int \mathrm{d}t \frac{M}{2} i\varepsilon^{AB}\varepsilon^{ij}\psi_{Ai}\dot{\psi}_{Bj}$. As for the bosonic coordinates x^μ that represent the motion of the center of mass, the fact that the translation generators P_μ are conserved and that the Hamiltonian is $H' = P^2/2M$ tells us that up to an additive constant, the action is a free particle action $\frac{M}{2}\int \mathrm{d}t\, \dot{x}^2$. In Lorentz signature, the constant is minus the rest energy or $-M$ and thus the particle action in this approximation is

$$I = \int \mathrm{d}t \left(-M + \frac{M}{2}\dot{x}^\mu\dot{x}_\mu + \frac{iM}{2}\varepsilon^{AB}\varepsilon^{ij}\psi_{Ai}\dot{\psi}_{Bj} \right). \tag{2.96}$$

One can think of the first two terms $-M + \frac{1}{2}M\dot{x}^2$ as a nonrelativistic approximation to a covariant action

$$I_{\rm cov} = -M \int {\rm d}\tau \sqrt{-g_{MN}\frac{{\rm d}x^M}{{\rm d}\tau}\frac{{\rm d}x^N}{{\rm d}\tau}}, \tag{2.97}$$

where here τ is an arbitrary parameter along the particle path and g_{MN} is the full 5d metric. This action is valid for any particle orbit with a large radius of curvature.

This action (2.96) satisfies the expected supersymmetry algebra, with

$$Q_{Ai} = M\sqrt{2}\psi_{Ai}, \quad Q_{\dot{A}i} = -i\frac{M}{\sqrt{2}}\frac{{\rm d}x^\mu}{{\rm d}t}\Gamma_{\mu A\dot{A}}\psi_i^A, \quad P^\mu = M\frac{{\rm d}x^\mu}{{\rm d}t}, \quad H' = \frac{P^2}{2M}. \tag{2.98}$$

What really uniquely determines the action (2.96) is that it gives a minimal realization of the translation symmetries and supersymmetries that are spontaneously broken by the choice of superparticle trajectory. The conserved charges $Q_{Ai} = M\sqrt{2}\psi_{Ai}$ and $P^\mu = M\dot{x}^\mu$, being linear in ψ_{Ai} and $\dot{x}^\mu$, generate constant shifts of ψ_{Ai} and x^μ, which can be viewed as Goldstone fields for spontaneously broken symmetries. As usual, the low energy action for the Goldstone fields is uniquely determined.

Collective Coordinates

In the instanton calculation, the zero-modes of $x^\mu(\tau)$ and $\psi_{Ai}(\tau)$ will be collective coordinates that parametrize the choice of the superparticle orbit; we will denote those collective coordinates as x^μ and $\psi^{(0)}_{Ai}$. By integrating over all non-zero modes while keeping the zero-modes fixed, we will generate an effective action $\int {\rm d}^4x{\rm d}^4\psi^{(0)}(\dots)$. Up to an elementary factor that is computed shortly, the $\psi^{(0)}_{Ai}$ can be identified with the fermionic coordinates θ_{Ai} that are used in writing superspace effective actions in four dimensions, so $\int {\rm d}^4x{\rm d}^4\psi^{(0)}(\dots)$ is a chiral effective action $\int {\rm d}^4x{\rm d}^4\theta(\dots)$. Such an interaction is potentially non-trivial in the sense explained in section **??**; that is, it may not be possible to write it as a D-term. A 5d BPS particle with $\zeta < 0$ would have fermionic collective coordinates of opposite chirality and could similarly generate an anti-chiral interaction $\int \mathrm{d}^4x\mathrm{d}^4\overline{\theta}$. A superparticle that is not BPS spontaneously breaks all supersymmetries, so it can be described with eight fermionic collective coordinates and can only generate D-terms, that is non-chiral interactions $\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\overline{\theta}\, (\dots)$.

Let us determine the normalization of the measure for integration over the collective coordinates. First of all, this measure is independent of M. In fact, M can be removed from the action by absorbing a factor of $\sqrt{M}$ in both x^μ and ψ_{Ai}, as we will do later (eqn. (2.114)). Because the bosons x^μ and the fermions ψ_{Ai} both have four components, this rescaling affects neither the Gaussian integral for the non-zero modes nor the zero-mode measure $\mathrm{d}^4x\mathrm{d}^4\psi^{(0)}$.

However, it is fairly natural to factor the zero-mode measure as $M^2\mathrm{d}^4x\cdot M^{-2}\mathrm{d}^4\psi^{(0)}$. This is based on the following observation. With the action being proportional to M, the Gaussian integral over any non-zero bosonic mode gives a factor of $1/M^{1/2}$, and the integral over any non-zero fermionic mode gives a factor of $M^{1/2}$. To compensate for this, in defining the path integral measure, one includes a factor of $M^{1/2}$ for every bosonic mode and a factor $M^{-1/2}$ for every fermionic mode. So the bosonic and fermionic zero-mode measures, up to constants, are $M^2\mathrm{d}^4x$ and $M^{-2}\mathrm{d}^4\psi^{(0)}$.

To find the normalization of the measure for fermion zero-modes, we compare a matrix element computed by integrating over collective coordinates to the same matrix element computed in a Hamiltonian approach. Quantization of the four fermions ψ_{Ai} gives a four-dimensional Hilbert space $\mathcal{H}$, consisting of two spin 0 bosonic states and two fermionic states of spin $(1/2, 0)$. If $(-1)^F$ is the operator that distinguishes bosons from fermions, then the anticommutation relations can be used to show that

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F\psi_{A1}\psi_{B1}\psi_{C2}\psi_{D2} = \frac{1}{M^2}\varepsilon_{AB}\varepsilon_{CD}. \tag{2.99}$$

Now recall that such a trace can be computed by a path integral on a circle with periodic boundary conditions for the fermions. The integral $M^{-2}\int \mathrm{d}^4\psi^{(0)}\psi^{(0)}_{A1}\psi^{(0)}_{B1}\psi^{(0)}_{C2}\psi^{(0)}_{D2}$ over collective coordinates should reproduce the formula (2.99), so we want

$$\int \mathrm{d}^4\psi^{(0)}\,\psi^{(0)}_{A1}\psi^{(0)}_{B1}\psi^{(0)}_{C2}\psi^{(0)}_{D2} = \varepsilon_{AB}\varepsilon_{CD}. \tag{2.100}$$

Now we can compare the zero-mode measure $\mathrm{d}^4x\,\mathrm{d}^4\psi^{(0)}$ to the usual measure $\mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g^E}$ of a four-dimensional supersymmetric action. This comparison involves a few steps. First, with $Q_{Ai} = M\sqrt{2}\psi_{Ai}$ and $\{\psi_{Ai},\psi_{Bj}\} = M^{-1}\varepsilon_{AB}\varepsilon_{ij}$, we have $\{Q_{Ai},\psi_{Bj}\} = \sqrt{2}\varepsilon_{AB}\varepsilon_{ij}$. So Q_{Ai} acts on the fermionic collective coordinates as $\sqrt{2}\partial/\partial\psi^{(0)Ai}$. The 4d supersymmetry generators are $\mathcal{Q}_{Ai} = e^{-\sigma/4}Q_{Ai}$ (eqn. (**??**)) and the fermionic coordinates θ_{Ai} of superspace are usually normalized so that $\mathcal{Q}_{Ai}$ acts on them as $\partial/\partial\theta^{Ai}$. So we should set $\sqrt{2}\partial/\partial\psi^{(0)\,Ai} = e^{\sigma/4}\partial/\partial\theta^{Ai}$, or $\psi^{(0)}_{Ai} = \sqrt{2}e^{-\sigma/4}\theta_{Ai}$. Hence

$$\mathrm{d}^4\psi^{(0)} = \frac{e^{\sigma}}{4}\mathrm{d}^4\theta, \tag{2.101}$$

where $\mathrm{d}^4\theta$ is defined so that

$$\int \mathrm{d}^4\theta\,\theta_{A1}\theta_{B1}\theta_{C2}\theta_{D2} = \varepsilon_{AB}\varepsilon_{CD}. \tag{2.102}$$

We further write $\mathrm{d}^4x = \mathrm{d}^4x\sqrt{g^E}e^{-2\sigma}$, since $\sqrt{g^E} = e^{2\sigma}$ according to eqn. (2.4). So finally the zero-mode measure is

$$\mathrm{d}^4x\mathrm{d}^4\psi^{(0)} = \mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g^E}\frac{e^{-\sigma}}{4}. \tag{2.103}$$

Although we normalized the fermion zero-mode measure by comparison to a Hamiltonian calculation, we have not yet done the same for the bosons. This will be done in section 2.2.2, by comparing to a counting of quantum states.

The Action In A Graviphoton Background

Now let us turn on a graviphoton field. A particle of charge q couples to an abelian gauge field A with a coupling

$$\int \mathrm{d}\tau \, q A_M \frac{\mathrm{d}x^M}{\mathrm{d}\tau} \tag{2.104}$$

(and possible non-minimal couplings involving magnetic moments, etc.), where A_M has time component A_0 and spatial components A_μ. In the case of a superparticle coupled with charges $\vec{q}$ to the gauge fields of 5d supergravity, and assuming that the background gauge field is precisely the graviphoton T^-, we found in eqn. (2.41) that the effective magnetic field is $\zeta(\vec{q})\mathsf{T}^-$, where $\zeta(\vec{q})$ is the central charge. For a BPS superparticle of $M = \zeta(\vec{q})$, this means that we should replace qA_μ in eqn. (2.104) with MV_μ, where $\mathsf{T}^-_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. A convenient gauge choice is $V_\nu = \frac{1}{2}\mathsf{T}^-_{\mu\nu}x^\mu$. There is an important detail here, however. In a 5d covariant form, the action for a charged point particle coupled to V_μ, on an orbit with a large radius of curvature, is

$$-M \int \mathrm{d}\tau \sqrt{-g_{MN}\frac{\mathrm{d}x^M}{\mathrm{d}\tau}\frac{\mathrm{d}x^N}{\mathrm{d}\tau}} + \int \mathrm{d}\tau \, V_M \frac{\mathrm{d}x^M}{\mathrm{d}\tau}. \tag{2.105}$$

To apply this to the graviphoton background, we have to use the supersymmetric Gödel metric (2.1), which depends on V_μ. When we expand the square root taking this into account, the effect is to double the coefficient of the $V_\mu \dot{x}^\mu$ coupling. The action of the superparticle in the graviphoton background is thus, in some approximation,

$$I = M \int \mathrm{d}t \left(-1 + \frac{1}{2}\dot{x}^\mu \dot{x}_\mu + \frac{i}{2}\varepsilon^{AB}\varepsilon^{ij}\psi_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}\psi_{Bj} + \mathsf{T}^-_{\mu\nu}x^\mu \dot{x}^\nu \right). \tag{2.106}$$

Are there additional terms that should be included in this action? The spontaneously broken supersymmetries Q_{Ai} remain valid symmetries when the graviphoton field is turned on. So they commute with the exact Hamiltonian H' that describes the superparticle, and hence the fields $\psi_{Ai} = Q_{Ai}/M\sqrt{2}$ are time-independent. Hence we should not add to (2.106) a magnetic moment coupling

$$\int \mathrm{d}t\, \mathsf{T}^{-}_{AB} i\varepsilon^{ij}\psi_{Ai}\psi_{Bj}, \tag{2.107}$$

as this will give a time-dependence to ψ_{Ai}. (By contrast, we will encounter such magnetic moment couplings in section 2.2.4 for other fermion fields along the particle worldline.) Other interactions that might be added to (2.106) are irrelevant in the limit that the circumference of the circle is large. The precise scaling argument behind that statement is explained at the end of this section.[9]

The momentum conjugate to x^μ is $\pi_\mu = \delta I/\delta \dot{x}^\mu = M(\dot{x}_\mu - \mathsf{T}^{-}_{\mu\nu}x^\nu)$. Of course, it obeys $[\pi_\mu, \pi_\nu] = 0$, $[\pi_\mu, x^\nu] = -i\delta^\nu_\mu$. By contrast, the conserved momentum that generates spatial translations is

$$P^\mu = M\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}t} - 2\mathsf{T}^{-\,\mu\nu}x_\nu\right) = \pi^\mu - M\mathsf{T}^{-\,\mu\nu}x_\nu. \tag{2.108}$$

It obeys $[P_\mu, x^\nu] = -i\delta^\nu_\mu$ (so it generates spatial translations, just as π_μ does) but satisfies

$$[P_\mu, P_\nu] = -2iM\mathsf{T}^{-}_{\mu\nu}. \tag{2.109}$$

[9] In determining the effective action (2.106) for a BPS hypermultiplet, we have not needed the fact that the graviphoton background preserves eight supersymmetries; the four conserved supercharges Q_{Ai} were enough. In analyzing more general BPS supermultiplets in section 2.2.4, we will need the full supersymmetry algebra to determine magnetic moments.

This is part of the supersymmetry algebra. In fact, taking the nonrelativistic limit of eqn. (2.26), the nonrelativistic limit of the supersymmetry algebra is

$$\begin{aligned}
[P_\mu, P_\nu] &= -2iM\mathsf{T}^-_{\mu\nu} \\
[\mathsf{J}, P_\mu] &= 2i\mathsf{T}^-_{\mu\nu}P^\nu \\
[\mathsf{J}, Q_{Ai}] &= -\frac{i}{2}\mathsf{T}^-_{\mu\nu}\Gamma^{\mu\nu}_{AB}Q^B_i \\
[P_\mu, Q_{\dot{A}i}] &= \mathsf{T}^-_{\mu\nu}\Gamma^\nu_{\dot{A}B}Q^B_i, \\
\{Q_{Ai}, Q_{Bj}\} &= 2M\varepsilon_{AB}\varepsilon_{ij} \\
\{Q_{Ai}, Q_{\dot{A}j}\} &= -i\Gamma^\mu_{A\dot{A}}\varepsilon_{ij}P_\mu \\
\{Q_{\dot{A}i}, Q_{\dot{B}j}\} &= \varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}(H' + \mathsf{J}).
\end{aligned} \tag{2.110}$$

The nonrelativistic conserved Hamiltonian H' is not simply $P^2/2M$, which does not commute with the P_μ. H' can be conveniently written in terms of the conserved quantities P_μ and $\mathsf{T}^{-\mu\nu}L_{\mu\nu}$, where $L_{\mu\nu} = x_\mu\pi_\nu - x_\nu\pi_\mu$:

$$H' = \frac{P^2}{2M} - \mathsf{T}^{-\mu\nu}L_{\mu\nu}. \tag{2.111}$$

What are the conserved supersymmetries? Clearly – because we have not added the magnetic moment term (2.107) – the charges $Q_{Ai} = M\sqrt{2}\psi_{Ai}$ are conserved. To define conserved supercharges $Q_{\dot{A}j}$ of the opposite chirality, we must modify the definition in (2.98) by replacing $M\dot{x}^\mu$ with the conserved charge P^μ. Thus the supersymmetry generators are

$$Q_{Ai} = M\sqrt{2}\psi_{Ai}, \quad Q_{\dot{A}j} = -\frac{i}{\sqrt{2}}P^\mu\Gamma_{\mu A\dot{A}}\psi^A_j. \tag{2.112}$$

The nonrelativistic supersymmetry algebra (2.110) is satisfied.

The constant $-M$ in the Lorentz signature action contributes $+M$ to the energy. So after compactifying the time direction on a (Euclidean signature) circle of circumference $2\pi e^\sigma$, the constant term in the action contributes to the path integral a factor $\exp(-2\pi e^\sigma M) = \exp(-2\pi e^\sigma \sum_I q_I h^I)$. However, after compactifying, the gauge fields can have constant components α^I in the t direction and these give an imaginary contribution to the Euclidean action. As explained in eqns. (2.71) and (2.72), the effect of this is to replace $e^\sigma h^I$ by $-iZ^I$, and so to replace $\exp(-2\pi e^\sigma M)$ by $\exp(2\pi i \sum_I q_I Z^I)$. To take account of the fermionic collective coordinates of the particle orbit, we just have to extend the factor $\exp(2\pi i \sum_I q_I Z^I)$ to a superfield, namely

$$\exp\left(2\pi i \sum_I q_I \mathcal{Z}^I\right). \tag{2.113}$$

This factor, of course, must be multiplied by a one-loop determinant computed using the action (2.106). The product of boson and fermion determinants is independent of M, since M can be removed by rescaling x^μ and ψ_{Ai} by a common factor $1/\sqrt{M}$. (This scaling does not affect the path integral measure, or the zero-mode measure $\mathrm{d}^4x\,\mathrm{d}^4\psi^{(0)}$.) The one-loop computation can therefore be performed using the action

$$I' = \frac{1}{2}\int \mathrm{d}t\,\left(\dot{x}^\mu \dot{x}_\mu + i\varepsilon^{AB}\varepsilon^{ij}\psi_{Ai}\frac{\mathrm{d}}{\mathrm{d}t}\psi_{Bj} + 2\mathsf{T}^-_{\mu\nu}x^\mu\dot{x}^\nu\right). \tag{2.114}$$

The particle mass M has disappeared in eqn. (2.114), so the one-loop determinant depends only on the radius e^σ of the circle and on $\mathsf{T}^-_{\mu\nu}$. We can constrain this dependence using the scaling symmetry

$$t \to \lambda t, \quad x \to \lambda^{1/2}x, \quad \psi_{Ai} \to \psi_{Ai}, \quad \mathsf{T}^-_{\mu\nu} \to \lambda^{-1}\mathsf{T}^-_{\mu\nu}. \tag{2.115}$$

The measure $\mathrm{d}^4x\,\mathrm{d}^4\psi^{(0)}$ scales as λ^2, and the circumference $2\pi e^\sigma$ scales as λ, so scale-invariance implies that the one-loop determinant has the form $\mathrm{d}^4x\,\mathrm{d}^4\psi^{(0)}\,e^{-2\sigma}f(e^\sigma\mathsf{T}^-)$ for some function f. The meaning of the factor $\mathrm{d}^4x\,\mathrm{d}^4\psi^{(0)}$ is that we cannot integrate over the zero-modes or collective coordinates (the integral over the x^μ gives ∞ and the integral over $\psi^{(0)}$ gives 0), so we leave them unintegrated and interpret the result as a measure on the collective coordinate moduli space rather than a number. Lorentz invariance implies that f is really a function of $e^{2\sigma}\mathsf{T}^-_{\mu\nu}\mathsf{T}^{-\,\mu\nu}$, so the one-loop determinant has the form

$$\mathrm{d}^4x\,\mathrm{d}^4\psi^{(0)}\sum_{\mathsf{g}=0}^{\infty} c_{\mathsf{g}} e^{(2\mathsf{g}-2)\sigma}(\mathsf{T}^-_{\mu\nu}\mathsf{T}^{-\mu\nu})^{\mathsf{g}} \tag{2.116}$$

wth some constants c_{g}.

In this derivation, we used the metric $\mathrm{d}s^2 = -(\mathrm{d}t - V)^2 + \sum_\mu (\mathrm{d}x^\mu)^2$, as in eqn. (2.1), with a periodicity in imaginary time of $2\pi e^\sigma$. To write the effective action in conventional 4d variables, we use $\mathrm{d}^4x\mathrm{d}^4\psi^{(0)} = \mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g^E}\frac{e^{-\sigma}}{4}$ (eqn. (2.103)). Also, to express (2.116) in 4d terms, we should re-express $\mathsf{T}^-_{\mu\nu}\mathsf{T}^{-\mu\nu}$ in terms of the corresponding 4d quantity $\mathsf{W}^-_{\mu\nu}\mathsf{W}^{-\mu\nu} = 16e^{-\sigma}\mathsf{T}^-_{\mu\nu}\mathsf{T}^{-\mu\nu}$ (eqn. (2.9)). Actually, here we should replace W^- with the chiral superfield $\mathcal{W}$ whose bottom component is W^-. Setting $\mathcal{W}^2 = \mathcal{W}_{\mu\nu}\mathcal{W}^{\mu\nu}$, eqn. (2.116) becomes

$$\frac{1}{4}\mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g^E}\,\frac{e^{(3\mathsf{g}-3)\sigma}}{16^{\mathsf{g}}}(\mathcal{W}^2)^{\mathsf{g}}. \tag{2.117}$$

Now we recall that $\mathcal{X}^0 = -ie^{-3\sigma/2}/2$ (eqn. (2.83)). Having generalized the fields to superfields, we can integrate over the collective coordinates to get a contribution to the effective action:

$$-\int \mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g^E}(-64)^{-\mathsf{g}}\frac{(\mathcal{W}^2)^{\mathsf{g}}}{(\mathcal{X}^0)^{2\mathsf{g}-2}}. \tag{2.118}$$

This – and its generalization with a classical factor $\exp\left(2\pi i\sum_I q_I\mathcal{Z}^I\right)$ included – is a chiral interaction of the sort described in the GV formula and discussed throughout this paper. It is a non-trivial F-term in the sense explained in section **??**; it cannot be written as $\int \mathrm{d}^4x\mathrm{d}^8\theta(\dots)$. Suppose on the other hand that we add non-minimal terms to the action (2.106). Any translation-invariant and supersymmetric interaction that we might add that has not already been included in eqn. (2.106) would scale as a negative power of λ. Hence, a contribution to the superparticle path integral that depends on such interactions would be similar to (2.116) but with extra powers of $e^{-\sigma}$. Such interactions are trivial F-terms, and it is difficult to learn very much about them.

The Computation

Having come this far, the actual computation of the one-loop determinant using the action (2.114) is not difficult.

The one-loop path integral gives a zero-mode integral times $\sqrt{\det' \mathcal{D}_F / \det' \mathcal{D}_B}$, where $\mathcal{D}_B$, $\mathcal{D}_F$ are the bosonic and fermionic kinetic operators

$$\begin{aligned} \mathcal{D}_B &= -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\delta_{\mu\nu} + 2\mathsf{T}^-_{\mu\nu}\frac{\mathrm{d}}{\mathrm{d}t} \\ \mathcal{D}_F &= i\frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_{AB}\varepsilon_{ij} \end{aligned} \tag{2.119}$$

and $\det'$ is a determinant in the space orthogonal to the zero-modes. Moreover, the real symmetric operator $\mathcal{D}_B$ can be conveniently factored as a product of two imaginary, self-adjoint (and skew-symmetric) operators

$$\mathcal{D}_B = \mathcal{D}_1\mathcal{D}_2, \quad \mathcal{D}_1 = i\frac{\mathrm{d}}{\mathrm{d}t}\delta_{\mu\nu}, \quad \mathcal{D}_2 = i\left(\frac{\mathrm{d}}{\mathrm{d}t}\delta_{\mu\nu} - 2\mathsf{T}^-_{\mu\nu}\right). \tag{2.120}$$

So $\det' \mathcal{D}_B = \det' \mathcal{D}_1 \cdot \det \mathcal{D}_2$. Since $\mathcal{D}_1$ is conjugate to $\mathcal{D}_F$, the ratio $\det' \mathcal{D}_F / \det' \mathcal{D}_B$ actually equals $1/\det \mathcal{D}_2$. This determinant can be evaluated by writing down the eigenfunctions of $\mathcal{D}_2$ (which are simple exponentials) and computing the regularized product of the corresponding eigenvalues. This is a rather standard computation.

However, we will here take a shortcut. We use the fact that a path integral on a circle has a Hamiltonian interpretation; the path integral we want equals

$\int d^4x\, d^4\psi^{(0)}\, \mathrm{Tr}' \exp(-2\pi e^\sigma H')$, where Tr' is a trace with the zero-modes removed. We can pick coordinates on $\mathbb{R}^4$ in which T^- is the direct sum of two 2×2 blocks:

$$\mathsf{T}^- = \frac{1}{2}\begin{pmatrix} 0 & \mathsf{T} & & \\ -\mathsf{T} & 0 & & \\ & & 0 & -\mathsf{T} \\ & & \mathsf{T} & 0 \end{pmatrix}. \tag{2.121}$$

(We reversed the sign in the lower right block to make T^- anti-selfdual if the four coordinates are oriented in the standard way.) A factor of 1/2 was included in eqn. (2.121) so that

$$\mathsf{T}^2 = \mathsf{T}^{-\mu\nu}\mathsf{T}_{-\mu\nu}. \tag{2.122}$$

Let us compute the desired trace in the subspace corresponding to the upper block. The corresponding Hamiltonian describes a particle moving in two dimensions in a constant magnetic field T. The energy eigenstates are Landau bands[10] with energies $\left(\frac{1}{2}+m\right)\mathsf{T}$, $m = 0, 1, 2, \ldots$. The density of states per unit area in any one Landau band is $d^2x\,\mathsf{T}/2\pi$. So for the bosonic variables that describe motion in this plane, the one-loop path integral equals

$$\frac{d^2x}{2\pi}\mathsf{T}\sum_{m=0}^{\infty}\exp\left(-\pi e^\sigma \mathsf{T}\,(1+2m)\right) = \frac{d^2x}{2\pi}\frac{\mathsf{T}e^{-\pi e^\sigma \mathsf{T}}}{1-e^{-2\pi e^\sigma \mathsf{T}}} = \frac{d^2x}{4\pi}\frac{\mathsf{T}}{\sinh(\pi e^\sigma \mathsf{T})}. \tag{2.123}$$

Including an identical factor for the lower block in eqn. (2.121), and including the fermion zero-modes, the full one-loop path integral gives

$$\frac{d^4x d^4\psi^{(0)}}{(4\pi)^2}\frac{\mathsf{T}^2}{\sinh^2(\pi e^\sigma \mathsf{T})}. \tag{2.124}$$

[10] In speaking of Landau bands, we assume that T is real, while in the graviphoton background it is imaginary. The determinant that we are trying to compute is holomorphic in T, so it is determined for all T by what happens for T real.

Including also the classical factor that was described in eqn. (2.113), the contribution of the BPS hypermultiplet to the GV formula is

$$\frac{\mathrm{d}^4x\,\mathrm{d}^4\psi^{(0)}}{(4\pi)^2}\exp\left(2\pi i\sum_I q_I\mathcal{Z}^I\right)\frac{\mathsf{T}^2}{\sinh^2(\pi e^\sigma\mathsf{T})}. \tag{2.125}$$

Though we have derived this formula by a computation in the supersymmetric Gödel background, it gives part of the effective action in a more general background. Since the function $\mathsf{T}/\sinh(\pi e^\sigma\mathsf{T})$ is regular for real T, this contribution to the effective action is regular as long as the 5d graviphoton field is real. However, in the supersymmetric Gödel background, T is imaginary and the effective action has poles if T is large. This does not really affect our derivation. The GV formula governs couplings (1.2) that are each perturbative in T, and the computation we have performed can be understood as a convenient way to evaluate all such perturbative contributions together.

To express the result (2.123) in four-dimensional terms, we follow the same steps that led to eqn. (2.118). We write $\mathsf{T}=\sqrt{\mathsf{T}^{-\mu\nu}\mathsf{T}^-_{\mu\nu}}=\frac{e^{\sigma/2}}{4}\sqrt{(\mathsf{W}^-)^2}$, and interpret W^- as the bottom component of a superfield $\mathcal{W}$. We also use $e^{3\sigma/2}=-i/2\mathcal{X}^0$, and $\mathrm{d}^4x\mathrm{d}^4\psi^{(0)}=\frac{1}{4}\mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g^E}e^{-\sigma}$ (eqn. (2.103)). The resulting contribution to the 4d effective action is

$$-\int\frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g^E}\exp\left(2\pi i\sum_I q_I\mathcal{Z}^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi\sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)}. \tag{2.126}$$

2.2.3 Multiple Winding, Bubbling, And Comparison To String Theory

Following [61, 62], we will now explain the interpretation of this formula in string theory.

In perturbative string theory (either physical or topological string theory), we should distinguish the string worldsheet Σ^* from its image $\Sigma \subset Y$. Σ^* does not necessarily have the same genus as Σ. The map from Σ^* to Σ must be holomorphic if it is to contribute to the amplitudes $\mathcal{F}_{\mathsf{g}}$ in topological string theory (or in physical string theory, given the relation between the two), but is not necessarily an isomorphism.

In general, a non-constant holomorphic map $\varphi : \Sigma^* \to \Sigma$ may have any degree $k \geq 1$. The $\mathcal{Z}$-dependence of a contribution from a map of degree k is a factor $\exp(2\pi i k \sum_I q_I \mathcal{Z}^I)$. The formula (2.125) evidently corresponds to contributions with $k = 1$. This is not surprising, since in deriving the formula, we considered an M2-brane wrapped just once on Σ and assumed that the superparticle trajectory winds just once around the M-theory circle.

A string theory map $\varphi : \Sigma^* \to \Sigma$ of degree k will correspond in M-theory to a configuration in which, roughly speaking, an M2-brane worldvolume has a degree k map to $S^1 \times \Sigma$. There are two distinct effects in M-theory that combine to produce this result. First, the M2-brane may wrap k_1 times over Σ. Since multiple M2-branes cannot be treated semiclassically, the rigorous meaning of this statement is that a BPS state in M-theory may have an M2-brane charge that is k_1 times the homology class $[\Sigma]$ (in other words, k_1 times the charge of an M2-brane wrapped once on Σ). Second, regardless of what BPS state we consider and what its quantum numbers may be, when we use this BPS state to make an instanton in M-theory compactified on a circle, this state may wind k_2 times around the circle. The relation between the degree k measured in string theory, the charge k_1 of the BPS particle in units of $[\Sigma]$, and the number k_2 of times that the particle winds around the circle is $k = k_1 k_2$.

Thus what in string theory is the sum over the degree of the map φ is in the context of the GV formula a combination of two effects: a BPS particle may be multiply-charged and it may wind any number of times around the M-theory circle.

In this section, we describe the effect of multiple winding, and in section 2.2.4, we consider the effects of multiple charge.

It is easier to write down the formula that governs contributions with multiple winding than to explain properly what it means. So we will first write down the formula. If a point superparticle wraps k times around the M-theory circle, the effective circumference of the circle becomes $2\pi k e^{\sigma}$. The winding also multiplies the classical action $2\pi i \sum_I q_I \mathcal{Z}^I$ by k. To evaluate the contribution of a particle orbit of winding number k, we also have to divide by a factor of k to account for the cyclic symmetry between the k branches of the particle orbit. So the analog of eqn. (2.126) with k-fold wrapping is obtained by multiplying $\sum_I q_I \mathcal{Z}^I$ and e^{σ} by k, and dividing the whole formula by k. Summing over k gives the contribution of the given BPS state with any winding:

$$-\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left(2\pi i k\sum_I q_I\mathcal{Z}^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)}. \tag{2.127}$$

(In the denominator, the factor of k in $\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)$ comes from substituting $e^{\sigma}\mathsf{T}\to k e^{\sigma}\mathsf{T}$ in the denominator in (2.125).)

However, a careful reader may find this formula puzzling. A weakly coupled elementary point particle could wind k times around a circle, and such a contribution could be evaluated along the lines of the previous paragraph. Does this make sense for a wrapped M2-brane, whose self-interactions are not small? A multiply-wound M2-brane is not a concept that makes sense semiclassically, since a system of k parallel M2-branes for $k>1$ is actually strongly coupled. If parallel M2-branes are separated in a transverse direction, their interactions remain strong until the separation exceeds the eleven-dimensional Planck scale ϱ. (Beyond this scale, the long range forces due to graviton and C-field exchange cancel for nonrelativistic BPS particles.)

Ignoring the strong interactions between BPS particles in this derivation can be justified as follows. In nonrelativistic quantum mechanics in $D > 2$ spatial dimensions, a short-range interaction, no matter how strong, is irrelevant in the renormalization group sense and is unimportant at low energies, except for the possibility that it may generate a bound state. In explaining this, we take the Hamiltonian of a free particle to be $H_0' = P^2$, where P is the momentum; hence P has dimension $E^{1/2}$ (energy to the one-half power). A short-range interaction is equivalent to $H'' = c\delta^D(x)$, for some constant c, modulo less relevant couplings involving derivatives of a delta function. The constant c has dimension $E^{1-D/2}$ and so is an irrelevant coupling if $D > 2$. For application to the GV formula, we have $D = 4$, so the interactions are safely irrelevant except for possibly generating bound states. (Bound states are a short-range phenomenon that cannot be analyzed by renormalization group scaling.) Concretely, the irrelevance of a short-range coupling for $D > 2$ means the following. If a particle of mass M propagates a Euclidean distance L on a classical orbit (in our case, the classical orbit is a copy of the M-theory circle and $L = 2\pi e^{\sigma}$ is its circumference), the fluctuations in its position in the directions normal to the classical orbit are typically of order $\sqrt{L/M}$. No matter how large M may be, $\sqrt{L/M}$ is much greater than the interaction range ϱ if L is sufficiently large. Thus, at any given time, two branches of an orbit that wraps k times around the M-theory circle are unlikely to be within range of the interaction. The condition $D > 2$ ensures that this is unlikely to happen at any time along the orbit.

What we learn from this reasoning is that we can ignore the interactions among k parallel M2-branes except for the possibility that, when they are close together, they form a bound state. Since the M2-brane states under consideration are BPS states, such a bound state would be a bound state at threshold – a new BPS state with larger charges. In fact, a bound state of r BPS states that each have charges q_I would have charges $\widetilde{q}_I = rq_I$.

Because M2-branes are strongly coupled, it is not straightforward to determine if such bound states exist (and if so for what values of r and with what spin). If bound states exist, they are new BPS states that can themselves be treated as elementary superparticles, when they wrap around a sufficiently large M-theory circle. Their contribution can be evaluated by methods similar to what we have already described, with modifications to account for their spins; see section 2.2.4. The full GV formula involves a sum over all M-theory BPS states, possibly including bound states.

A further comment is called for. In eqn. (2.126), we consider only $k > 0$. Exchanging $k > 0$ with $k < 0$ amounts to a reflection of the M-theory circle. When combined with a reflection of $\mathbb{R}^4$, which reverses the four-dimensional chirality, this is a symmetry of M-theory. So orbits of $k < 0$ generate anti-chiral couplings, just as orbits of $k > 0$ generate chiral couplings. But what about $k = 0$? For $k = 0$, the BPS state has no net winding, so generically it does not propagate a macroscopic distance, even if the M-theory circle is large. Since we do not have a microscopic theory of M2-branes, we cannot make sense of a configuration in which an M2-brane propagates over a non-macroscopic distance. So we have no way to make sense of a $k = 0$ contribution. But intuitively, what we would want to say about such a contribution is as follows. In M-theory on $\mathbb{R}^4 \times S^1 \times Y$, an M2-brane wrapped on $\Sigma \subset Y$ and propagating a small distance in $\mathbb{R}^4 \times S^1$ is not, in leading order (in the inverse radius of S^1), affected by the compactification from $\mathbb{R}^5$ to $\mathbb{R}^4 \times S^1$. So whatever contribution it makes is part of the effective action for M-theory on $\mathbb{R}^5 \times Y$, compactified classically from $\mathbb{R}^5$ to $\mathbb{R}^4 \times S^1$. As we stressed in sections **??** and 2.1.2, an important input to the GV formula is that one knows the relevant effective action in five dimensions, before compactification. So there is no need to study the $k = 0$ contributions.

Now let us look more closely at what the formula (2.126) means in terms of perturbative string theory. Since

$$\frac{1}{\sin x} = \frac{1}{x}\left(1 + \frac{x^2}{6} + \dots\right), \tag{2.128}$$

we can expand eqn. (2.127) in a power series in $\mathcal{W}$:

$$-\int \frac{\mathrm{d}^4x\,\mathrm{d}^4\theta}{(2\pi)^4}(\mathcal{X}^0)^2 \sum_{k=1}^{\infty}\frac{1}{k^3}\exp\left(2\pi i k\sum_I q_I \mathcal{Z}^I\right)\left(1 + \frac{\pi^2 k^2 \mathcal{W}^2}{192(\mathcal{X}^0)^2} + \mathcal{O}(\mathcal{W}^4)\right). \tag{2.129}$$

In perturbative string theory, the contribution proportional to $\mathcal{W}^{2\mathfrak{g}}$ comes from worldsheets of genus $\mathfrak{g}$, as we have explained in section 2.1.2. Thus, the formula (2.129), even though it reflects a single wrapped M2-brane of genus 0 and degree 1, is interpreted in perturbative string theory as a sum of contributions with all values $k \geq 1$ and $\mathfrak{g} \geq 0$.

One might expect to compute these contributions in topological string theory (and therefore also in physical string theory, given their relationship) by counting degree k maps from a string worldsheet Σ^* of genus $\mathfrak{g}$ to a given holomorphic curve $\Sigma \subset Y$. However, in general this counting is not straightforward.

Let us look at a few cases. We can specialize to $\mathfrak{g} = 0$ by setting $\mathcal{W} = 0$ in eqn. (2.129). The $k = 1$ contribution is

$$-\int \frac{\mathrm{d}^4x\,\mathrm{d}^4\theta}{(2\pi)^4}(\mathcal{X}^0)^2 \cdot \exp\left(2\pi i\sum_I q_I \mathcal{Z}^I\right) \cdot 1. \tag{2.130}$$

This contribution is not hard to understand. A genus 0 worldsheet Σ^* with a holomorphic map $\Sigma^* \to \Sigma$ of degree 1 is unique up to isomorphism; it is isomorphic to Σ, with the map being the isomorphism. This uniqueness means that the contribution of genus 0 worldsheets singly wrapped on Σ to the topological string amplitude is precisely $\exp\left(2\pi i \sum_I q_I \mathcal{Z}^I\right) \cdot 1$. The occurrence of this factor in (2.130) is the most basic relation between topological string amplitudes and the physical string amplitudes that are described by the GV formula. The remaining factor $-\mathrm{d}^4x\,\mathrm{d}^4\theta\,(\mathcal{X}^0)^2/(2\pi)^4$ in (2.130) represents the embedding of the topological string amplitude in physical string theory.

Still with $\mathsf{g}=0$, we see in eqn. (2.129) that if we take $k>1$, in addition to the classical action being multiplied by k, the amplitude acquires a factor of $1/k^3$. This is not an integer, so this answer cannot come from a straightforward "counting" of holomorphic maps. The factor $1/k^3$ for a k-fold cover of a genus 0 curve was first discovered using mirror symmetry [92]. Its interpretation in topological string theory depends on the fact that for $k>1$, there is a nontrivial moduli space of degree k holomorphic maps from a genus 0 worldsheet Σ^* to Σ; this moduli space has orbifold singularities, because of which the "counting" does not give an integer. See [93] for a derivation along these lines. The GV formula has given this rather subtle factor of $1/k^3$ without much fuss [61, 62, 94].

One need not look far to find further subtleties that are nicely resolved by the GV formula. For example, for $\mathsf{g}>0$, assuming that Σ^* is smooth, and with Σ of genus 0, there does not exist a degree 1 holomorphic map $\Sigma^*\to\Sigma$. Thus naively perturbative string theory does not generate contributions to the chiral interactions $\mathcal{F}_{\mathsf{g}}$ with $k=1$ and $\mathsf{g}>0$. But such contributions are clearly visible in (2.129). As explained in [61,62], these contributions are interpreted in topological string theory in terms of contributions in which Σ^* is not smooth but is a union of various components Σ_i^* that are glued together at singularities. For $k=1$, $\mathsf{g}>0$, one of these components is of genus 0 and is mapped isomorphically onto Σ by a degree 1 map, and the others are mapped to Y by maps of degree 0 (thus, they are mapped to points in Y). Such a degeneration of Σ^* and its map to Y is sometimes called "bubbling" (fig. 2.1). By integration over the moduli of such bubbled configurations, one can compute topological string amplitudes with $k = 1$ and $\mathfrak{g} > 0$. More generally, such bubbled configurations contribute to a variety of topological string amplitudes.

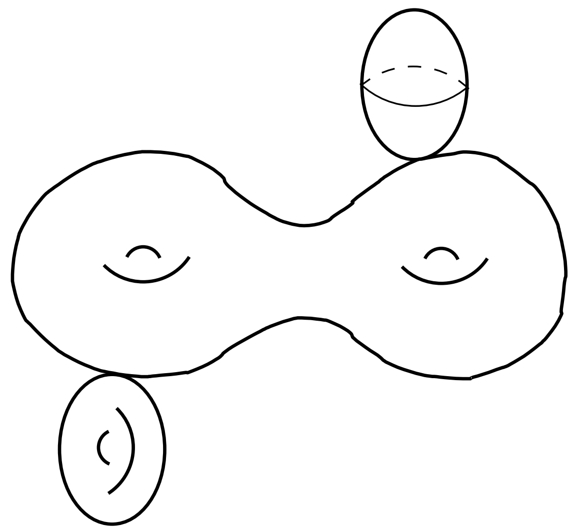

Figure 2.1: A Riemann surface Σ^* splits into several components Σ_i^*. In the context of topological string theory, ordinarily all components except one are mapped to points in Y. The splitting off from Σ^* of one or more components that are mapped to points in Y is called "bubbling." In the example shown, Σ^* has genus 3 and from top to bottom the components have genus 0, 2, and 1.

Thus [61, 62], the GV formula when interpreted in topological string theory describes a variety of multiply-wrapped and/or bubbled configurations associated to a given BPS state.

2.2.4 More General Massive BPS States

The General Answer

It takes only a few steps to generalize the hypermultiplet computation of section 2.2.2 to arbitrary massive BPS states. We give here a general description and explain some explicit formulas in section 2.2.4.

In general, it is inconvenient to describe a nonrelativistic action for an arbitrary BPS multiplet, but it is straightforward to describe a Hilbert space for this system, with an action of a Hamiltonian H', conserved momentum and angular momentum operators P^μ and $J_{\mu\nu}$, and supercharges Q_{Ai} and $Q_{\dot{A}j}$. We already know how to describe the hypermultiplet in this language. The appropriate Hilbert space[11] $\widehat{\mathcal{H}}_0$ is an irreducible representation of the action of free bosons x^μ and their canonical momenta π_μ as well as free fermions $\psi_{Ai} = Q_{Ai}/M\sqrt{2}$. The conserved momenta are $P^\mu = \pi^\mu$, the nonrelativistic Hamiltonian is

$$H' = \frac{P^2}{2M}, \tag{2.131}$$

the rotation generators act in the natural way, and the supercharges were defined in eqn. (2.112). These modes are needed to realize the spatial translations and spacetime supersymmetries, but a general set of BPS states may have additional degrees of freedom. So in general, the Hilbert space that describes BPS states with charge $\vec{q} = (q_1, \ldots, q_{b_2})$ and mass $M = \zeta(\vec{q})$ is $\widehat{\mathcal{H}}_{\vec{q}} = \widehat{\mathcal{H}}_0 \otimes \mathsf{V}_{\vec{q}}$, where $\widehat{\mathcal{H}}_0$ is the Hilbert space for a hypermultiplet and $\mathsf{V}_{\vec{q}}$ is a vector space with an action[12] of the rotation group $SU(2)_\ell \times SU(2)_r$. The action on $\widehat{\mathcal{H}}_{\vec{q}}$ of the supercharges, the momentum, and the Hamiltonian come entirely from their action on $\widehat{\mathcal{H}}_0$, but the rotation group acts also on $\mathsf{V}_{\vec{q}}$.

Now let us turn on the graviphoton field $\mathsf{T}^-_{\mu\nu}$. We can still define free fermions by $\psi_{Ai} = Q_{Ai}/M\sqrt{2}$. The momentum generators still commute with the Hamiltonian, but instead of commuting with each other they generate the Weyl algebra $[P_\mu, P_\nu] = -2iM\mathsf{T}^-_{\mu\nu}$. The minimal way to satisfy this is to deform the P^μ as in eqn. (2.108):

$$P^\mu = \pi^\mu - M\mathsf{T}^{-\,\mu\nu} x_\nu. \tag{2.132}$$

[11] Our notation will be as follows: a Hilbert space like $\widehat{\mathcal{H}}_0$ labeled with a hat describes the bosonic center of mass motion as well as the quantization of fermion zero-modes; an unhatted Hilbert space represents the quantized fermion zero-modes only.

[12] This action is not necessarily irreducible, as there may be several BPS multiplets of charge $\vec{q}$.

We may use this formula, since the representation of the canonical commutation relations $[P_\mu, P_\nu] = -2iM\mathsf{T}^-_{\mu\nu}$, $[P_\mu, x^\nu] = -i\delta^\mu_\nu$, $[x^\mu, x^\nu] = 0$ is unique up to isomorphism, implying that any further T^--dependent contributions that we might add to P^μ that preserve the commutation relations can be removed by a unitary transformation. So even for $\mathsf{T}^-_{\mu\nu} \neq 0$, there is a decomposition of the Hilbert space as $\widehat{\mathcal{H}}_{\vec{q}} = \widehat{\mathcal{H}}_0 \otimes \mathsf{V}_{\vec{q}}$ with the property that the supercharges $Q_{Ai} = M\psi_{Ai}$ and momentum generators P_μ act only on the first factor. The most obvious way to satisfy the supersymmetry algebra is to take $Q_{\dot{A}j}$ to similarly act only on $\widehat{\mathcal{H}}_0$ and to be given by the same formula as for the hypermultiplet:

$$Q_{\dot{A}j} = -\frac{i}{\sqrt{2}} P^\mu \Gamma_{\mu A\dot{A}} \psi^A_j. \tag{2.133}$$

To explain why eqn. (2.133) gives a sufficiently good approximation to $Q_{\dot{A}j}$, we use the scaling symmetry (2.115), which we extend to act on $\widehat{\mathcal{H}}_{\vec{q}}$ (and not just $\widehat{\mathcal{H}}_0$) by saying that $\mathsf{V}_{\vec{q}}$ is invariant under scaling. To compute the desired effective action in four dimensions, we have to evaluate the trace $\mathrm{Tr}'\,(-1)^F \exp(-2\pi e^\sigma H')$ (the symbol Tr' means that the trace is defined without an integral over collective coordinates). To do this computation after scaling the time by a large factor λ, since e^σ scales as λ, we are not interested in terms in H' that scale as a power of λ more negative than λ^{-1}. Since H' can be computed from the anticommutator $\{Q_{Ai}, Q_{\dot{A}j}\}$, this means that we are not interested in corrections to $Q_{\dot{A}j}$ that scale as a power more negative than $\lambda^{-1/2}$. In order for $Q_{\dot{A}j}$ to be conserved, it must be possible to write it with no explicit dependence on x^μ, just in terms of the conserved charges $\psi_{Ai} = Q_{Ai}/M\sqrt{2}$ and P^μ as well as matrices acting on $\mathsf{V}_{\vec{q}}$. (In particular, we cannot make use of the conserved angular momentum without spoiling the commutator $[Q_{\dot{A}i}, P_\mu]$, which comes out correctly if we use (2.133).) These requirements mean that no correction to (2.133) involving $\mathsf{T}^-_{\mu\nu}$ is possible: $\mathsf{T}^-_{\mu\nu}$ scales as λ^{-1} and the other possible ingredients in a hypothetical correction to the right hand side of (2.133) scale with nonpositive powers of λ (indeed, P^μ, ψ_{Ai}, and a matrix acting on $\mathsf{V}_{\vec{q}}$ scale respectively as $\lambda^{-1/2}$, 1, and 1).

Since the $Q_{\dot{A}j}$ act only on $\widehat{\mathcal{H}}_0$ and not on $\mathsf{V}_{\vec{q}}$, the same is true of

$$\{Q_{\dot{A}i}, Q_{\dot{B}j}\} = \varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}(H' + \mathsf{J}). \tag{2.134}$$

However, $\mathsf{J} = \mathsf{T}^-_{\mu\nu}J^{\mu\nu}$ is the sum of operators J_0 and $\mathsf{J}_{\vec{q}}$ that act on $\widehat{\mathcal{H}}_0$ and $\mathsf{V}_{\vec{q}}$, respectively. For the hypermultiplet, $\mathsf{V}_{\vec{q}}$ is trivial so $\mathsf{J}_{\vec{q}} = 0$ and direct evaluation of the left hand side of (2.134) using (2.133) leads to the formula for H' given in eqn. (2.111). In general, (2.134) implies that

$$H' = \frac{P^2}{2M} - \mathsf{T}^{-\mu\nu}L_{\mu\nu} - \mathsf{J}_{\vec{q}}. \tag{2.135}$$

The role of the $\mathsf{J}_{\vec{q}}$ term is to ensure that $H' + \mathsf{J}_{\vec{q}}$ acts only on $\widehat{\mathcal{H}}_0$ and not on $\mathsf{V}_{\vec{q}}$.

Now to evaluate the contribution of the BPS states of charge $\vec{q}$ to the GV formula, we need to evaluate $\mathrm{Tr}'_{\widehat{\mathcal{H}}_{\vec{q}}}(-1)^F \exp(-2\pi e^\sigma H')$. The trace factors as a trace in $\widehat{\mathcal{H}}_0$ times a trace in $\mathsf{V}_{\vec{q}}$. The trace in $\widehat{\mathcal{H}}_0$ is the one that we already evaluated in discussing the hypermultiplet. In acting on $\mathsf{V}_{\vec{q}}$, H' can be replaced by $-\mathsf{J}_{\vec{q}}$, so the trace in $\mathsf{V}_{\vec{q}}$ simply gives $\mathrm{Tr}_{\mathsf{V}_{\vec{q}}}(-1)^F \exp(2\pi e^\sigma \mathsf{J}_{\vec{q}})$. Using $\mathsf{J}_{\vec{q}} = \frac{e^{\sigma/2}}{4}\mathcal{J}_{\vec{q}}$ (eqn. (2.23)) and the usual formula $e^{3\sigma/2} = -i/2\mathcal{X}^0$, this trace is $\mathrm{Tr}_{\mathsf{V}_{\vec{q}}}(-1)^F \exp(-i\pi\mathcal{J}_{\vec{q}}/4\mathcal{X}^0)$.

The contribution of BPS states of charges $\vec{q}$ propagating once around the circle to the GV formula is obtained by just including this trace in (2.126):

$$-\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g^E}\mathrm{Tr}_{\mathsf{V}_{\vec{q}}}\left[(-1)^F\exp(-i\pi\mathcal{J}_{\vec{q}}/4\mathcal{X}^0)\right]\exp\left(2\pi i\sum_I q_I\mathcal{Z}^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi\sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)}. \tag{2.136}$$

This can be extended as before to include multiple windings:

$$-\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{Tr}_{\mathsf{V}_{\vec{q}}}\left[(-1)^F\exp(-i\pi k\mathcal{J}_{\vec{q}}/4\mathcal{X}^0)\right]$$
$$\times\exp\left(2\pi ik\sum_I q_I\mathcal{Z}^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)}. \tag{2.137}$$

To get the complete GV formula, we need to sum this formula over all possible charges $\vec{q}$. But states with $\zeta(\vec{q})<0$ do not contribute to the GV formula since they preserve the wrong supersymmetry. The complete GV formula is thus

$$-\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g^E}\sum_{q|\zeta(q)\geq 0}\sum_{k=1}^{\infty}\frac{1}{k}\mathrm{Tr}_{\mathsf{V}_{\vec{q}}}\left[(-1)^F\exp(-i\pi k\mathcal{J}_{\vec{q}}/4\mathcal{X}^0)\right]$$
$$\times\exp\left(2\pi ik\sum_I q_I\mathcal{Z}^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)}. \tag{2.138}$$

Actually, our derivation has assumed that $\zeta(\vec{q})>0$, not just $\zeta(\vec{q})\geq 0$, because we have assumed that the BPS states under discussion have a strictly positive mass in five dimensions. This means that our analysis does not apply to BPS states with $\vec{q}=0$, since such states are always massless in five dimensions. This case requires a different derivation, but with a suitable interpretation of what is meant by $\mathsf{V}_{\vec{q}}$ and with $\mathcal{J}_{\vec{q}}$ set to 0, the formula (2.138) also gives correctly the contribution of BPS states with $\vec{q}=0$, as we will learn in section 2.3. Our derivation also breaks down for BPS states with $\vec{q}\neq 0$ and $\zeta(\vec{q})=0$, but this case is nongeneric in the sense that it arises only if the Kahler moduli of Y are varied to approach a boundary of the Kahler cone.

Concrete Formulas

To make the formula (2.138) explicit, we need to know how to compute the space of BPS states of charge $\vec{q}$. M-theory in general and M2-branes in particular are not sufficiently well understood for it to be possible at present, for a given Calabi-Yau manifold Y, to give a full answer to this question. Potential complications include strong coupling of multiply-wrapped M2-branes, singularities in the moduli space of holomorphic curves in a Calabi-Yau manifold, bubbling, and the interplay of all of these. Luckily, there are favorable situations in which it is possible to explicitly determine the space of BPS states with charge $\vec{q}$ and show that the action of the supersymmetry algebra is as described above. Here, essentially following [62], we will just summarize a few highlights, leaving some further details for appendix B.

To quantize an M2-brane with worldvolume $\mathbb{R} \times \Sigma \subset \mathbb{R}^5 \times Y$, we have to quantize the fermions that live on the M2-brane. These transform as spinors on Σ with values in (positive chirality) spinors of the normal bundle to the M2-brane worldvolume. As is usual in Kaluza-Klein reduction, the modes that have to be included in the low energy description are the zero-modes along the compact manifold (in this case, the zero-modes of fields propagating on Σ). Half of the M2-brane fermions transform under $SU(2)_\ell \times SU(2)_r$ as $(1/2, 0)$ and half transform as $(0, 1/2)$. Zero-modes of the $(0, 1/2)$ fermions are related by supersymmetry to infinitesimal deformations of the complex submanifold $\Sigma \subset Y$. We say that Σ is "rigid" or "isolated" if there are no such fermion zero-modes, and we consider this case first.

For Σ rigid, the effective quantum mechanics is obtained just by quantizing the fermion zero-modes that transform as $(1/2, 0)$, along with the center of mass coordinates x^μ; there are no other bosonic or fermionic zero-modes. The M2-brane fermions that transform as $(1/2, 0)$ can be interpreted as differential forms on Σ. If Σ is of genus 0, its non-zero Betti numbers are $b_0 = b_2 = 1$. The corresponding zero-modes are precisely the fermionic collective coordinates ψ_{Ai} that we included in studying the hypermultiplet. However, for $\mathfrak{g} > 0$, one has $b_1 = 2\mathfrak{g}$, leading to additional zero-modes consisting of $2\mathfrak{g}$ copies of the $(1/2, 0)$ representation of the rotation group. In the effective quantum mechanical problem on $\mathbb{R} \times \Sigma$, where $\mathbb{R}$ parametrizes the time, these zero modes lead to $2\mathfrak{g}$ fields $\rho_{A\sigma}(t)$, $\sigma = 1, \ldots, \mathfrak{g}$ that correspond to $(1, 0)$-forms on Σ and $2\mathfrak{g}$ more fields $\widetilde{\rho}_{A\sigma}(t)$, $\sigma = 1, \ldots, \mathfrak{g}$ that correspond to $(0,1)$-forms. The action for these modes is

$$\mathcal{S}_\rho = \int \mathrm{d}t \sum_{\sigma=1}^{\mathfrak{g}} \left(i\widetilde{\rho}_{A\sigma} \frac{\mathrm{d}}{\mathrm{d}t} \rho^A_\sigma + \frac{i}{2} \mathsf{T}^-_{AB} \widetilde{\rho}^A_\sigma \rho^B_\sigma \right), \tag{2.139}$$

and the corresponding Hamiltonian is

$$H'_\rho = -\mathsf{J}_\rho = -\sum_{\sigma=1}^{\mathfrak{g}} \frac{i}{2} \mathsf{T}^-_{AB} \widetilde{\rho}^A_\sigma \rho^B_\sigma. \tag{2.140}$$

The problem of quantizing the four fermions $\widetilde{\rho}_{A\sigma}$, $\rho_{B\sigma}$, for $A, B = 1, 2$ and a fixed value of σ, is isomorphic to the problem of quantizing the four fermions ψ_{Ai} that appear already in the study of the hypermultiplet. Quantization of this system gives the familiar spin content $2(0,0) \oplus (1/2,0)$ of a massive BPS hypermultiplet, described by a four-dimensional Hilbert space $\mathcal{H}$. For this set of four states, $\mathrm{Tr}\,(-1)^F \exp(-i\pi\mathcal{J}/8\mathcal{X}^0) = \mathrm{Tr}\,(-1)^F \exp(2\pi e^\sigma \mathsf{J}) = -4\sin^2(\pi\sqrt{W^2}/8\mathcal{X}^0)$. The full set of fermions $\widetilde{\rho}_{A\sigma}$, $\rho_{B\sigma}$ consists of $\mathfrak{g}$ copies of this spectrum, leading to $\mathrm{Tr}\,(-1)^F \exp(2\pi e^\sigma \mathsf{J}) = (-1)^{\mathfrak{g}}(4\sin^2(\pi\sqrt{W^2}/8\mathcal{X}^0))^{\mathfrak{g}}$. From (2.137), it then follows that the contribution to the GV formula of BPS states that arise from an M2-brane wrapped on Σ is

$$\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4} \sum_{k=1}^{\infty} \frac{1}{k} (-1)^{\mathfrak{g}-1} \exp\left(2\pi i k \sum_I q_I \mathcal{Z}^I\right) \frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^{2-2\mathfrak{g}}(\pi k\sqrt{\mathcal{W}^2}/8\mathcal{X}^0)}. \tag{2.141}$$

We write $\mathcal{H}_{\mathfrak{g}}$ for the space obtained by quantizing $\widetilde{\rho}_{A\sigma}, \rho_{B\sigma}$, $\sigma = 1, \ldots, \mathfrak{g}$. Thus, $\mathcal{H}_{\mathfrak{g}}$ is the tensor product of $\mathfrak{g}$ copies of $\mathcal{H}$, that is $\mathfrak{g}$ copies of $2(0,0) \oplus (1/2,0)$.

This description of $\mathcal{H}_{\mathfrak{g}}$ makes manifest the action of $SU(2)_\ell$ (and the trivial action of $SU(2)_r$). However, as preparation for the case that Σ is not rigid, it is helpful to describe $\mathcal{H}_{\mathfrak{g}}$ in another way. Here, for each value of $A = 1, 2$, we combine together $\rho_{A\sigma}$, which is a $(1,0)$-form on Σ, with $\widetilde{\rho}_{A\sigma}$, which is a $(0,1)$-form on Σ, to make a field $\widehat{\rho}_{Ay}$, where $y = 1, \ldots, 2\mathsf{g}$ labels the choice of a harmonic 1-form on Σ. Let ε_{yz} be the intersection pairing on $H^1(\Sigma, \mathbb{Z})$. The canonical anticommutation relations of $\widehat{\rho}_{Ay}$ are $\{\widehat{\rho}_{Ay}, \widehat{\rho}_{Bz}\} = \varepsilon_{AB}\varepsilon_{yz}$. To make this look slightly more familiar, let us denote ρ_{Ay} as ρ_{+y} or ρ_{-y}, depending on the value of A. We also define $\rho_-^y = \varepsilon^{yz}\rho_{-z}$. Then the canonical anticommutators are

$$\begin{aligned} \{\rho_-^y, \rho_-^z\} = \{\rho_{+y}, \rho_{+z}\} &= 0 \\ \{\rho_-^y, \rho_{+z}\} &= \delta_z^y. \end{aligned} \tag{2.142}$$

We can regard ρ_{+z} as a set of fermion creation operators and ρ_-^y as the corresponding annihilation operators. The full space of states is a fermion Fock space, made by repeatedly acting with ρ_+ on a "ground state" that is annihilated by ρ_-. Since ρ_+ is an element of the first de Rham cohomology group $H^1(\Sigma)$, the one-particle states are a copy of $H^1(\Sigma)$. The space of k-fermion states is then the antisymmetric tensor product of k copies of $H^1(\Sigma)$; we denote this as $\wedge^k H^1(\Sigma)$. The space $\mathcal{H}_\mathsf{g}$ obtained by quantizing the fermions is obtained by summing this over k:

$$\mathcal{H}_\mathsf{g} = \oplus_{k=0}^{2\mathsf{g}} \wedge^k H^1(\Sigma). \tag{2.143}$$

This description of $\mathcal{H}_\mathsf{g}$ emphasizes its dependence on Σ, but hides the action of $SU(2)_\ell$. As explained in [62], to understand the $SU(2)_\ell$ action, it helps to recognize that $\mathcal{H}_\mathsf{g}$ can be interpreted as the cohomology of the Jacobian of Σ. Recall that the Jacobian of Σ is a Kahler manifold and that there is a natural Lefschetz $SU(2)$ action on the cohomology of any Kahler manifold (and more generally on the cohomology of a Kahler manifold with values in a flat vector bundle). The $SU(2)_\ell$ action on $\mathcal{H}_\mathsf{g}$ can be understood as the natural Lefschetz $SU(2)$ action on the cohomology of the Jacobian.

If Σ is not rigid, then in general there is a moduli space $\mathcal{M}$ that parametrizes the bosonic deformations of Σ. The only case in which it is straightforward to describe the BPS states that arise from an M2-brane simply-wrapped on the curves parametrized by $\mathcal{M}$ is the case that $\mathcal{M}$ is smooth[13] and compact and parametrizes a family of smooth curves $\Sigma \subset Y$. (It is quite exceptional for all these conditions to be satisfied.) In this case, the effective quantum mechanics problem that describes the degrees of freedom parametrized by $\mathcal{M}$ and takes into account the zero-modes of $(0, 1/2)$ fermions is the theory of differential forms on $\mathcal{M}$. However, remembering the $(1/2, 0)$ fermions, these are differential forms with values in $\mathcal{H}_{\mathfrak{g}}$. As Σ varies, $H^1(\Sigma)$ varies as the fiber of a flat vector bundle over $\mathcal{M}$. (We do not have to worry about Σ developing singularities because of our assumption that Σ is always smooth.) Since $\mathcal{H}_{\mathfrak{g}}$ is constructed from $H^1(\Sigma)$ as in (2.143), it also varies as the fiber of a flat vector bundle. The space of supersymmetric states in this situation is the de Rham cohomology of $\mathcal{M}$ with values in $\mathcal{H}_{\mathfrak{g}}$; we denote this as $H^*(\mathcal{M}; \mathcal{H}_{\mathfrak{g}})$. This is the contribution of Σ to $\mathsf{V}_{\vec{q}}$. The $SU(2)_r$ action is the natural Lefschetz $SU(2)$ action on $H^*(\mathcal{M}; \mathcal{H}_{\mathfrak{g}})$ (now making use of the fact that $\mathcal{M}$ is a Kahler manifold), and the $SU(2)_\ell$ action comes from the action of $SU(2)_\ell$ on $\mathcal{H}_{\mathfrak{g}}$.

It is explained in [62] that $H^*(\mathcal{M}; \mathcal{H}_{\mathfrak{g}})$ has a natural interpretation in terms of 4d BPS states in Type IIA superstring theory on Y. To determine the space of 4d BPS states, one has to quantize a suitable D-brane moduli space. Given the assumption that Σ is always smooth, one can argue that this quantization gives again $H^*(\mathcal{M}; \mathcal{H}_{\mathfrak{g}})$. (The argument involves describing the D-brane moduli space as a fiber bundle over $\mathcal{M}$ and computing its cohomology by a Leray spectral sequence.) Thus, under the hypothesis that Σ is always smooth, one expects that 4d BPS states always descend in a simple way from 5d BPS states (recall from section **??** that in general one does

[13] Technically, by saying that $\mathcal{M}$ is "smooth," we mean that the deformation theory of $\Sigma \subset Y$ is unobstructed so that in particular the infinitesimal deformations of Σ represent tangent vectors to $\mathcal{M}$.

not expect something as simple as this). The hypothesis that Σ is always smooth is almost never satisfied. However [95], there is a well-developed mathematical theory in which the hypothesis that Σ is always smooth is replaced with the hypothesis that Σ varies only in a Fano subvariety $W \subset Y$. Under this hypothesis, it is proved (with a precise set of mathematical definitions that hopefully match correctly the physics) that $H^*(\mathcal{M}, \mathcal{H}_\mathfrak{g})$, which is the space of 5d BPS states, agrees with what one would get in $d = 4$ by quantizing the D-brane moduli space. Thus this really does seem to give an interesting and perhaps surprising situation in which 4d BPS states descend simply from five dimensions.

The GV formula is often written in the following way. If one permits oneself to take formal sums and differences of vector spaces, then any $\mathbb{Z}_2$-graded representation of $SU(2)_\ell$ can be expanded as $\oplus_{\mathfrak{g}=0}^\infty A_\mathfrak{g} \otimes \mathcal{H}_\mathfrak{g}$, where $\mathcal{H}_\mathfrak{g}$ is the $SU(2)_\ell$ representation defined as the tensor product of $\mathfrak{g}$ copies of $2(0) \oplus (1/2)$ and $A_\mathfrak{g}$ is a $\mathbb{Z}_2$-graded vector space with trivial action of $SU(2)_\ell$. In particular, the space of BPS states of charge $\vec{q}$ is formally a sum $\oplus_{\mathfrak{g}=0}^\infty A_{\mathfrak{g},\vec{q}} \otimes \mathcal{H}_\mathfrak{g}$. Set $a_{\mathfrak{g},\vec{q}} = \mathrm{Tr}_{A_{\mathfrak{g},\vec{q}}}(-1)^F$. (Thus, a rigid curve $\Sigma \subset Y$ of genus $\mathfrak{g}$ and homology class $\vec{q}$ contributes 1 to $a_{\mathfrak{g},\vec{q}}$, and 0 to $a_{\mathfrak{g}',\vec{q}}$ for $\mathfrak{g}' \neq \mathfrak{g}$.) The complete GV formula can be written

$$\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4} \sum_{\vec{q}|\zeta(\vec{q})\geq 0} \sum_{\mathfrak{g}=0}^\infty \sum_{k=1}^\infty \frac{1}{k}(-1)^{\mathfrak{g}-1} a_{\mathfrak{g},\vec{q}} \exp\left(2\pi i k \sum_I q_I \mathcal{Z}^I\right) \frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^{2-2\mathfrak{g}}(\pi k\sqrt{\mathcal{W}^2}/8\mathcal{X}^0)}. \tag{2.144}$$

2.3 The Schwinger Calculation With Fields

2.3.1 Preliminary Reduction

The group of rotations that preserves the momentum vector of a massless particle in five dimensions is $SO(3)$, a subgroup of the corresponding group $SO(4)$ for a massive particle at rest. The spin of a five-dimensional massless particle is measured by a representation of the double cover of this $SO(3)$. This double cover is a diagonal subgroup $SU(2)_\Delta \subset SU(2)_\ell \times SU(2)_r$ of the group that measures the spin of a massive particle.

In a five-dimensional theory with minimal supersymmetry (eight supercharges), the states of a massless particle with specified momentum are annihilated by four of the supercharges and furnish a representation of the other four supersymmetries along with $SU(2)_\Delta$. The minimal such representation transforms under $SU(2)_\Delta$ as $W = 2(0) \oplus (1/2)$ (that is, two copies of spin 0 and one copy of spin 1/2). A general irreducible representation of supersymmetry and $SU(2)_\Delta$ is simply the tensor product of W with the spin j representation of $SU(2)_\Delta$, for some $j \in \frac{1}{2}\mathbb{Z}$. We write this tensor product as $R_j = (j) \otimes W$. In particular, $R_0 = (0) \otimes W \cong W$, since (0) is the trivial 1-dimensional representation of $SU(2)_\Delta$.

Taking CPT into account, R_j must appear in the spectrum an even number of times if j is an integer but may appear any integer number of times if j is a half-integer. The basic massless hypermultiplet H, vector multiplet V, and supergravity multiplet G are

$$\begin{aligned} H &= 2R_0 \\ V &= R_{1/2} \\ G &= R_{3/2}. \end{aligned} \tag{2.145}$$

We expect that a combination of massless supermultiplets that could be deformed in a supersymmetric fashion to a massive non-BPS multiplet does not contribute to the GV formula. (We have seen in section 2.2.2 that a massive non-BPS supermultiplet does not contribute to the GV formula.) For example, the combination $H \oplus V$ can be deformed by Higgsing to a massive non-BPS vector multiplet. To see this, notice that such a supermultiplet must realize eight supercharges, four of which transform as $(1/2,0)$ under $SU(2)_\ell \times SU(2)_r$ and four as $(0,1/2)$. The basic such representation is the massive vector multiplet $W_\ell \otimes W_r$, where W_ℓ admits the action of one set of four supersymmetries and W_r admits the action of the other set. (W_ℓ consists of four states transforming as $2(0,0) \oplus (1/2,0)$ while W_r consists of four states transforming as $2(0,0) \oplus (0,1/2)$.) When we restrict to $SU(2)_\Delta \subset SU(2)_\ell \otimes SU(2)_r$ and only four supersymmetries, we can identify both $SU(2)_\ell$ and $SU(2)_r$ with $SU(2)_\Delta$ and also ignore the supersymmetries that act on (say) W_ℓ. Then the massive non-BPS vector multiplet becomes $W \otimes W = (2(0) \oplus (1/2)) \otimes W = 2R_0 \oplus R_{1/2} = H \oplus V$. Accordingly, we expect that the combination $H \oplus V$ of massless supermultiplets does not contribute to the GV formula.

Since $W \otimes W$ can be deformed in a supersymmetric fashion to a massive non-BPS multiplet, the same is true, for any $j \in \frac{1}{2}\mathbb{Z}$, of $(j) \otimes W \otimes W$. For $j > 0$, this is the same as $(j) \otimes (2(0) \oplus (1/2)) \otimes W \cong R_{j+1/2} \oplus 2R_j \oplus R_{j-1/2}$. So we expect that any such combination does not contribute to the GV formula. For $j = 1$ or $j = 1/2$, we get the combinations $R_{3/2} \oplus 2R_1 \oplus R_{1/2}$ and $R_1 \oplus 2R_{1/2} \oplus R_0$. Taking linear combinations of these expressions and $H \oplus V = 2R_0 \oplus R_{1/2}$, we are led to expect that $R_{3/2} \oplus 4R_0 = G \oplus 2H$ does not contribute to the GV formula.

Granted this, the contribution of n_H hypermultiplets, n_V vector multiplets, and n_G supergravity multiplets is equivalent to the contribution of $n_H - n_V - 2n_G$ hypermultiplets. This combination has an interesting interpretation. The Betti numbers of a Calabi-Yau three-fold Y obey $b_0 = 1$, $b_1 = 0$, and $b_i = b_{6-i}$. Accordingly, the Euler characteristic of Y is $\chi(Y) = 2 + 2b_2 - b_3$. Generically (as long as one stays away from boundaries of the Kahler cone of Y), massless states in M-theory compactification on Y come entirely from classical dimensional reduction on Y of the eleven-dimensional supergravity multiplet. With this assumption, the number of vector multiplets is $n_V = b_2 - 1$, the number of hypermultiplets is $n_H = b_3/2$ (in six dimensions, b_3 is always even), and the number of supergravity multiplets is $n_G = 1$. Therefore,

$n_H - n_V - 2n_G = \frac{b_3}{2} - b_2 - 1 = -\frac{1}{2}\chi(Y)$. So the total contribution to the GV formula from massless states in five dimensions, away from boundaries of the Kahler cone, is $-\frac{1}{2}\chi(Y)$ times the contribution of a single massless hypermultiplet.

In section 2.3.2, we will calculate the contribution to the GV formula of a massless hypermultiplet. As explained in section **??**, this calculation cannot be naturally performed in the approach via 5d particles. But instead, since there is a natural field theory for a 5d massless hypermultiplet, there is no problem to perform the calculation in terms of fields. In fact, it is straightforward to generalize the field theory computation to a 5d *massive* BPS hypermultiplet, and we will do so. (As explained in section **??**, near a boundary of the Kahler cone of Y, there can be a massive BPS hypermultiplet that is light enough so that a description in 5d field theory makes sense.) Once one formulates the computation in 5d field theory, it is natural to make a Kaluza-Klein reduction to four dimensions and to write the answer as a sum over contributions of 4d mass eigenstates. There are some problems with $\mathcal{F}_0$ and $\mathcal{F}_1$, but this gives a representation of the answer for $\mathcal{F}_\mathfrak{g}$, $\mathfrak{g} \geq 2$ in terms of a sum over states of definite momentum around the M-theory circle, in contrast to the particle approach of section 2.2 that gives the answer as a sum over configurations of definite winding number. The two representations are related by a Poisson resummation. The winding number representation is usually more useful, moreover it holds for $\mathcal{F}_\mathfrak{g}$, $\mathfrak{g} \geq 0$.

The momentum representation breaks down for $\mathcal{F}_0$ and $\mathcal{F}_1$, and our field computation of $\mathcal{F}_\mathfrak{g}$, $\mathfrak{g} \geq 2$ simply does not work for these two cases. The reason why $\mathcal{F}_0$ is special is that it is a prepotential – this interaction simply does not talk to the graviphoton. At the background with anti-selfdual graviphoton turned on and supersymmetry slightly broken by the curvature (we will use this background to compute $\mathcal{F}_\mathfrak{g}$, $\mathfrak{g} \geq 2$), $\mathcal{F}_0$ still vanishes. The reason why $\mathcal{F}_1$ breaks down is more subtle. Schwinger calculation actually gives a finite answer for the contribution to $\mathcal{F}_1$ of a state with a given compact momentum. However, the sum over momentum eigenstates diverges, and there is no natural way to regularize it from the 4d point of view. One needs a different computation which would preserve the full 5d symmetry of the problem, only then it would be possible to obtaine a finite answer for $\mathcal{F}_1$. For that reason, we will treat $\mathcal{F}_0$ and $\mathcal{F}_1$ separately from $\mathcal{F}_\mathsf{g}$, $\mathsf{g} \geq 2$. Both will be computed directly in 5d field theory, without performing a KK reduction.

The upshot of the computation is to show that the contribution to the GV formula of a massless hypermultiplet is the obvious zero mass limit of the contribution of a massive hypermultiplet, which was computed in section 2.2. When the answer is stated this way, one may feel that one does not need to actually do the field theory computation for the hypermultiplet: the computation of section 2.2 is valid for a hypermultiplet of any non-zero mass, so could not we understand the zero mass case as a limit from non-zero mass? However, we find it instructive to do the explicit computation with 4d mass eigenstates. It is particularly illuminating to see how an answer emerges that is holomorphic in the 4d central charge $\mathcal{Z}$, even though naively a Schwinger-like calculation depends only on the particle mass $|\mathcal{Z}|$. Moreover, it turns out that there is a subtlety in the zero mass case, first identified in [56], that is best understood by performing a computation with 4d mass eigenstates.

We should stress that we consider the argument that was used to express the contribution of the supergravity multiplet as -2 times the contribution of a massless hypermultiplet to be somewhat heuristic. In the particle treatment of section 2.2, we had a very clear argument that a massive non-BPS superparticle cannot contribute to the GV formula. In general, we do not have an equally clear argument for the analogous statement in field theory. For the special case of $H \oplus V$, there is a clear argument, since the deformation to a massive non-BPS multiplet can be realized physically by Higgsing.

To interpret in the language of eqn. (2.138) the statements that $H \oplus V$ and $2H \oplus G$ do not contribute to the GV formula, we have to be slightly formal about what we mean by the contribution of a massless vector multiplet V or of the supergravity multiplet G to the vector space $\mathsf{V}_{\vec{q}}$ for $\vec{q} = 0$. To define $\mathsf{V}_{\vec{q}=0}$, we are supposed to write the space of BPS states of given momentum as the tensor product of the space of states for a hypermultiplet with some vector space $\mathsf{V}_{\vec{q}=0}$. Because of the fact that the hypermultiplet $H = 2R_0$ is two copies of R_0, while V and G are not divisible by 2, to define $\mathsf{V}_{\vec{q}=0}$ we would have to divide by 2 – an operation that does not make sense for vector spaces, though it makes sense for the trace that we actually need in eqn. (2.138). Since $H = 2R_0$ and $V = (1/2) \otimes R_0$, the contribution of V to $\mathsf{V}_{\vec{q}}$ is formally $\frac{1}{2}(1/2)$, that is one-half a copy of the spin 1/2 representation. This answer means that the contribution of V to $\mathrm{Tr}_{\mathsf{V}_{\vec{q}}}(-1)^F = \frac{1}{2}\mathrm{Tr}_{(1/2)}(-1)^F = -1$, so that the contribution of V to the GV formula is the same as the contribution of $-H$. Likewise, the contribution of G to $\mathsf{V}_{\vec{q}}$ is formally $\frac{1}{2}(3/2)$, meaning that its contribution to $\mathrm{Tr}_{\mathsf{V}_{\vec{q}}}(-1)^F$ is $\frac{1}{2}\mathrm{Tr}_{(3/2)}(-1)^F = -2$, reproducing the fact that G makes the same contribution to the GV formula as $-2H$. It is unappealing that V_0 does not exist and we must formally divide by 2. A possibly more natural approach is to place a factor of 1/2 in front of the $\vec{q} = 0$ contribution in (2.138). The intuition in doing this would be that since BPS states with $\zeta(\vec{q}) > 0$ contribute with weight 1 in (2.138) and those with $\zeta(\vec{q}) < 0$ contribute with weight 0, it is fairly natural to say that BPS states with $\vec{q} = 0$ and hence $\zeta(\vec{q}) = 0$ contribute with weight 1/2. If we do this, we would say that the contributions of H, V, and G to $\mathsf{V}_{\vec{q}=0}$ are respectively 2(0), (1/2) and (3/2).

2.3.2 Calculation For The Hypermultiplet

Preliminaries

We aim here to compute the one-loop effective action for a 4d BPS hypermultiplet in the graviphoton background. Actually, in a field theory calculation, it is not difficult to be more general, as long as the fields are slowly varying on a length scale set by the hypermultiplet mass (or by the graviphoton field strength if the hypermultiplet mass vanishes). There is actually a good reason to be more general. In contrast to the particle computation of section 2.2.2, it is difficult to do this computation in a manifestly supersymmetric fashion, because there is no convenient and simple superfield description of a hypermultiplet. Being able to perturb slightly around the graviphoton background will help in expressing the answer we get in a supersymmetric form. A simple perturbation will suffice for our purposes: rather than taking the four-manifold on which the hypermultiplet propagates to be flat (as in the reduction to four dimensions of the supersymmetric Gödel solution), we take this metric to be hyper-Kahler, with anti-selfdual Weyl tensor.

As remarked in footnote 9, the particle computation for a hypermultiplet, as opposed to a more general BPS multiplet, makes use only of the negative chirality supersymmetries Q_{Ai}. The same is true in the field theory computation: the negative chirality supersymmetries are enough to determine the action we will use. An anti-selfdual graviphoton preserves the Q_{Ai} even if its field strength is not constant,[14] but these supersymmetries are broken by anti-selfdual Riemannian curvature.

In our calculation, we will treat the scalars in vector multiplets as constants (so that the 5d and 4d central charges ζ and $\mathfrak{z}$ are constants). This means our calculation will not determine a contribution to the effective action that does not depend on $\mathcal{W}_{AB}$, that is a prepotential term $-i\int \mathrm{d}^4x\mathrm{d}^4\theta\mathcal{F}_0(\mathcal{X}^\Lambda)$. We will see that our calculation is also not powerful enough to fully understand $\mathcal{F}_1$.

We will assume that the minimal hypermultiplet action is sufficient to determine the quantum effective action modulo D-terms. For the particle computation, we proved the analogous statement in section 2.2.2 by a scaling argument.

[14]This is clear from eqn. (2.14). By contrast, a gauge field in a vector multiplet must be selfdual, not anti-selfdual, to preserve the Q_{Ai}. See the remarks following eqn. (2.16).

The Action

We start with a 5d metric on $M_5 = S^1 \times M_4$ of the form

$$\mathrm{d}s^2 = e^{2\sigma}(\mathrm{d}y + B)^2 + e^{-\sigma} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{2.146}$$

Here $g_{\mu\nu}$ is a hyper-Kahler metric with anti-selfdual Weyl curvature on the four-manifold M_4, and B_μ is a Kaluza-Klein gauge field. We require that $B_\mu = -i\frac{e^{-3\sigma/2}}{4}U_\mu$, where U_μ is the four-dimensional gauge field whose curvature is the 4d anti-selfdual graviphoton $\mathsf{W}^-_{\mu\nu}$. It is related to the 5d graviphoton $\mathsf{T}^-_{\mu\nu}$ by the usual equation (2.5). We also adjust curvatures of the 5d gauge fields to be $\mathrm{d}V^I = h^I \mathsf{T}^-$, just like in the graviphoton background. We will calculate the effective action in a region of M_4 in which the metric is very nearly flat and $\mathsf{W}^-_{\mu\nu}$ is very nearly constant – so that the background is very close to the standard graviphoton background. And this slightly curved background exactly preserves one of the useful features of the graviphoton background – the only nonzero gauge field is the graviphoton.

In general, in supergravity, hypermultiplets parametrize a quaternionic manifold $\mathcal{X}$. However, for a 1-loop computation, we can approximate $\mathcal{X}$ by a flat manifold, which in the case of a single hypermultiplet is just $\mathbb{R}^4$. We consider the general case that the hypermultiplet has charges q_I and hence a bare mass $M = \sum_I q_I h^I$ in five dimensions. The action is not just the obvious minimal coupling of bosons and fermions to a gravitational background, because in five dimensions, the fermions in a hypermultiplet have a non-minimal magnetic moment coupling to the graviphoton field.

We denote the scalars in the hypermultiplet as q^X, $X = 1, \ldots, 4$. The fermions are a pair of spinors $\xi^{\alpha p}$ where $\alpha = 1, \ldots, 4$ is a spinor index of $SO(1,4)$ and $p = 1, 2$.

The Lagrangian density is

$$\mathcal{L} = -\frac{1}{2} D_M q^X D^M q^X - \frac{1}{2} M^2 (q^X)^2 - 2M \bar{\xi}^1 \xi^2 + \bar{\xi}_p \not{D} \xi^p + \frac{3i}{8} h_I (\mathrm{d}V^I)_{MN} \bar{\xi}_p \Gamma^{MN} \xi^p. \tag{2.147}$$

For $M = 0$, this action can be found in [96] (along with its generalization to an arbitrary system of hypermultiplets). The mass terms can be generated by a coupling to a $U(1)$ vector multiplet and giving an expectation value to the scalar in the vector multiplet. Notice that the fermion mass term explicitly breaks an $SU(2)$ symmetry of the massless action (acting on the p index) down to $U(1)$.

The explicit magnetic moment term in (2.147) does not actually mean that the fermions have a magnetic moment. We have to recall that the minimal Dirac Lagrangian for a charged fermion describes a particle with a magnetic moment; one usually says that the particle has a g-factor of 2. Also, in reduction to $d = 4$, a contribution to the effective magnetic moment comes from the coupling of fermions to the 5d spin-connection in the action (2.147), which upon dimensional reduction with the metric (2.146) generates a magnetic moment coupling to the Kaluza-Klein gauge field B_μ and hence to the graviphoton. (This shifts the coefficient of the magnetic moment term in eqn. (2.149).) The net effect for $M \neq 0$ (the $M = 0$ case has some subtleties that will appear later) is that the effective magnetic moment vanishes in four dimensions. That must be so, at least for $M \neq 0$, since the particle description of section 2.2.2 made it clear that the fermions in a BPS hypermultiplet have no magnetic moment. Vanishing of the effective magnetic moment will be clear in eqn. (2.152).

The effective action generated by the hypermultiplet is simply the difference of the logarithms of boson and fermion determinants. A convenient way to calculate this difference is to reduce to four dimensions, expressing the answer as a sum of contributions of Kaluza-Klein modes of definite mass. This contrasts with the particle computation, where it is easier to consider orbits of definite winding number around the Kaluza-Klein circle. A Poisson resummation will be needed to convert the answer obtained as a sum over mass eigenstates to the answer expressed as a sum over orbits of definite winding.

The Kaluza-Klein mode with $-n$ units of momentum around the circle is a 4d hypermultiplet with a 4d central charge

$$\mathsf{z} = e^{-\sigma/2}(M - ie^{-\sigma}n - ie^{-\sigma}q_I\alpha^I), \tag{2.148}$$

where α^I are the constants that determine the holonomy around the circle of the gauge fields V^I. The mass of the hypermultiplet is $|\mathsf{z}|$. It will be convenient to simply think of the hypermultiplet as a pair of complex scalars ϕ^i, $i = 1, 2$, and a Dirac fermion ψ. We write ψ_L and ψ_R (or ψ_A and $\psi_{\dot{A}}$) for the components of ψ transforming with spin $(1/2, 0)$ or $(0, 1/2)$ under $SU(2)_\ell \times SU(2)_r$. After Kaluza-Klein reduction, the action density for the n^{th} mode is

$$\begin{aligned}\frac{\mathcal{L}_n}{2\pi} =& -\sum_{i=1}^{2}(|\nabla_\mu\phi^i|^2 + |\mathsf{z}\phi^i|^2) + 2\overline{\psi}^c_L \not{D}\psi_R + 2\overline{\psi}^c_R \not{D}\psi_L \\ & - 2\mathsf{z}\overline{\psi}^c_L\psi_L - 2\overline{\mathsf{z}}\,\overline{\psi}^c_R\psi_R + \frac{i}{4}\mathsf{W}^-_{\mu\nu}\overline{\psi}^c_L\gamma^{\mu\nu}\psi_L,\end{aligned} \tag{2.149}$$

where

$$\begin{aligned}\nabla_\mu\phi^i &= \partial_\mu\phi^i - i\frac{\overline{\mathsf{z}}}{4}U_\mu\phi^i, \\ D_\mu\psi &= \partial_\mu\psi + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\psi - i\frac{\overline{\mathsf{z}}}{4}U_\mu\psi.\end{aligned} \tag{2.150}$$

Here ω^{ab}_μ is the Levi-Civita connection on M_4, and U_μ is the gauge field whose curvature is $\mathsf{W}^-_{\mu\nu}$. Modulo possible terms of higher dimension, this action is actually determined by the 4d supersymmetry algebra (2.29), even if we consider only the Q_{Ai} supersymmetries and not $Q_{\dot{B}j}$.

The Computation

As long as $\mathfrak{z} \neq 0$, the problem of evaluating the bosons and fermion determinants in this problem can be simplified by integrating out ψ_R. (The case $\mathfrak{z} = 0$, which means that $M = q_I = 0$ and $n = 0$, needs special care and will be treated separately.) If we eliminate ψ_R classically by solving its equation of motion, the action density for ψ_L becomes

$$\frac{\mathcal{L}_n^{\psi_L}}{2\pi} = \frac{2}{\overline{\mathfrak{z}}}\overline{\psi}_L^c\left(\not{D}^2 + \frac{i}{8}\overline{\mathfrak{z}}\mathsf{W}^-_{\mu\nu}\gamma^{\mu\nu} - \overline{\mathfrak{z}}\mathfrak{z}\right)\psi_L. \tag{2.151}$$

Standard Dirac algebra gives $\not{D}^2 + \frac{i}{8}\overline{\mathfrak{z}}\mathsf{W}^-_{\mu\nu}\gamma^{\mu\nu} = D_\mu D^\mu$, showing the disappearance of the magnetic moment. Finally, by absorbing a factor of $\sqrt{\overline{\mathfrak{z}}}$ in ψ_L (and the same factor in $\overline{\psi}_L^c$), we eliminate the ugly factor of $1/\overline{\mathfrak{z}}$ in front of the kinetic enegy of ψ_L. After these manipulations the action becomes

$$\frac{\mathcal{L}_n^{\psi_L}}{2\pi} = 2\overline{\psi}_L^c\big(D_\mu D^\mu - \overline{\mathfrak{z}}\mathfrak{z}\big)\psi_L. \tag{2.152}$$

We have to be careful to include some constant factors generated by these manipulations. The Gaussian integral over ψ_R that is used to eliminate ψ_R generates a factor of $\overline{\mathfrak{z}}$ for every mode of ψ_R. The rescaling of ψ_L multiplies the path integral measure by a factor of $1/\overline{\mathfrak{z}}$ for every mode of ψ_L. Including these factors and also the determinants coming from functional integrals over ϕ^i and ψ_L, the path integral for the n^{th} Kaluza-Klein mode gives

$$(\overline{\mathfrak{z}})^{n_R - n_L}\frac{\det_L(-D^2 + |\mathfrak{z}|^2)}{\det^2(-\nabla^2 + |\mathfrak{z}|^2)}, \tag{2.153}$$

where $\det_L$ is the determinant in the space of left-handed fermions. Also, $n_R - n_L$ is formally the difference between the number of right- and left-handed fermion modes; we interpret this difference as the index of the Dirac operator, which we denote as $\mathcal{I}$.

In what follows, we write Tr_L for a trace in the space of left-handed fermions, and Tr for a trace in the space of scalar fields. Also, we drop the distinction between D and ∇ and write simply $-D^2 = -g^{\mu\nu} D_\mu D_\nu$ for the Laplacian acting on a field of any spin. The desired contribution to the effective action is minus the logarithm of (2.153) or

$$-\mathcal{I}\ln(\overline{\mathfrak{z}}) - \mathrm{Tr}_L \ln(|\mathfrak{z}|^2 - D^2) + 2\mathrm{Tr}\ln(|\mathfrak{z}|^2 - D^2). \tag{2.154}$$

With the help of

$$\ln A = \int_0^\infty \frac{\mathrm{d}s}{s}(e^{-s} - e^{-sA}), \tag{2.155}$$

we can rewrite (2.154) in the form

$$-\mathcal{I}\ln\overline{\mathfrak{z}} - \int_0^\infty \frac{\mathrm{d}s}{s}\left(2\mathrm{Tr} - \mathrm{Tr}_L\right)\left(e^{-s(|\mathfrak{z}|^2 - D^2)} - e^{-s}\right). \tag{2.156}$$

This formula is obtained by using the representation (2.155) of the logarithm for every mode. When we sum over all modes, the coefficient of e^{-s} in eqn. (2.156) is formally what we might call $n_L - 2n_0$, where n_0 is the total number of modes of spin 0. On a hyper-Kahler manifold M_4 with anti-selfdual Weyl curvature, the positive chirality spin bundle is simply a trivial bundle of rank 2, so n_R is the same as $2n_0$ and hence $n_L - 2n_0 = -\mathcal{I}$. So an equivalent formula is

$$-\mathcal{I}\,\ln\overline{\mathfrak{z}} - \int_0^\infty \frac{\mathrm{d}s}{s}\left[\left(2\mathrm{Tr} - \mathrm{Tr}_L\right)\left(e^{-s(|\mathfrak{z}|^2 - D^2)}\right) - \mathcal{I}e^{-s}\right], \tag{2.157}$$

Using (2.155) one more time, this is

$$-\int_0^\infty \frac{\mathrm{d}s}{s}\left[(2\mathrm{Tr}-\mathrm{Tr}_L)\left(e^{-s(|\mathcal{Z}|^2-D^2)}\right)-\mathcal{I}e^{-s\overline{\mathcal{Z}}}\right]. \tag{2.158}$$

A Hilbert space $\mathcal{H}$ consisting of two states transforming under $SU(2)_\ell \times SU(2)_r$ as $(0,0)$ and two transforming as $(1/2,0)$ was encountered in section 2.2.2. It arises upon quantizing a Clifford algebra generated by four fermions ψ_{Ai} with the familiar anticommutation relations $\{\psi_{Ai},\psi_{Bj}\}=\varepsilon_{AB}\varepsilon_{ij}$. Now we regard $\mathcal{H}$ as the fiber of a vector bundle over M_4 and write $\widehat{\mathcal{H}}$ for the space of sections of this bundle. Clearly, we can rewrite (2.158) in the form

$$-\int_0^\infty \frac{\mathrm{d}s}{s}\left(\mathrm{Tr}_{\widehat{\mathcal{H}}}(-1)^F e^{-s(|\mathcal{Z}|^2-D^2)}-\mathcal{I}e^{-s\overline{\mathcal{Z}}}\right). \tag{2.159}$$

We can interpret the operator $\exp(-s(|\mathcal{Z}|^2-D^2))$ acting on the space $\widehat{\mathcal{H}}$ as $\exp(-sH)$ where H is the Hamiltonian derived by quantizing the following superparticle action:

$$\mathsf{S}=\int \mathrm{d}t\left(-|\mathcal{Z}|^2+\frac{\dot{x}^2}{4}+\frac{\overline{\mathcal{Z}}}{4}U_\mu\dot{x}^\mu+\frac{i}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\nabla_t\psi_{Bj}\right). \tag{2.160}$$

Here x^μ are local coordinates for a point in M_4, so that $x^\mu(t)$ describes a particle orbit[15] in M_4; the ψ_{Ai} are fermi fields defined along the particle orbit; and $\nabla_t=\partial_t+\frac{1}{4}\dot{x}^\mu\omega_\mu^{ab}\gamma_{ab}$ is the pullback of the Levi-Civita connection of M_4 to the orbit. To compute $\mathrm{Tr}\,(-1)^F\exp(-sH)$, we perform a path integral on a circle of circumference s, with periodic boundary conditions for fermions, and using the Euclidean version of the above action:

$$\mathsf{S}_E=\int_0^\beta \mathrm{d}\tau\left(|\mathcal{Z}|^2+\frac{\dot{x}^2}{4}-i\frac{\overline{\mathcal{Z}}}{4}U_\mu\dot{x}^\mu+\frac{1}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\nabla_\tau\psi_{Bj}\right). \tag{2.161}$$

[15] We have normalized the kinetic energy of x^μ so that the bosonic Hamiltonian is P^2; if the kinetic energy were $\frac{1}{2}\dot{x}^2$, the Hamiltonian would be $P^2/2$.

Clearly, we have arrived at something very similar to what we had in the particle-based calculation. However, there are a few key differences. In section 2.2.2, we had to compute a Euclidean path integral on a circle of definite radius; this circle was effectively just the M-theory circle. Now the radius of the circle is an integration variable, the proper time s. Related to this, in section 2.2.2, we were computing the contribution of an orbit of definite winding number. Now we are computing the contribution of a particle of definite Kaluza-Klein momentum.

Also, the computation in 2.2.2 was performed in a manifestly supersymmetric framework. In our present computation, the starting point was not manifestly supersymmetric (because we lacked a convenient and manifestly supersymmetric description of the hypermultiplet). It is easy to guess from eqn. (2.161) how to express our present computation in a supersymmetric form. But to be sure, we will perturb slightly around the supersymmetric Gödel solution, allowing anti-selfdual Weyl curvature, and verify that the result can be expressed in terms of superfields in the expected way.

In doing this computation, we can assume that the radius of curvature is very large (on a scale set by the particle mass or the graviphoton field), and that the graviphoton field is nearly constant. This being so, the problem can be analyzed in a standard way, using the fact that if M_4 were flat and the graviphoton field exactly constant, the action would be quadratic and the path integral would be simple. The F-terms that are described by the GV formula have contributions that, when expressed in terms of ordinary fields (and taking the fermions to vanish and the scalars to be constants), take the form of R^2 times a function of W^- only, where $R_{\mu\nu}^{ab}$ is the Riemann tensor. So in evaluating the path integral, it suffices to work to quadratic order in R, and to ignore covariant derivatives of R or W^-.

We set $x^\mu(\tau) = x^\mu + z^\mu(\tau)$, where x^μ labels a point in M_4, and $z^\mu(0) = z^\mu(s) = 0$. The path integral over $x^\mu(\tau)$ splits as an integral over a field $z^\mu(\tau)$ that vanishes at

$\tau = 0$ and an ordinary integral over x^μ. Near x^μ, we use Riemann normal coordinates, which are Euclidean up to second order in z^μ. In these coordinates, the spin connection is

$$\omega_\mu^{ab}(z) = \frac{1}{2} z^\nu R_{\nu\mu}{}^{ab} + \mathcal{O}(z^2), \tag{2.162}$$

where the $\mathcal{O}(z^2)$ terms can be ignored as they are proportional to the covariant derivative of the Riemann tensor. Up to terms of order z^3, the part of the action that involves fermions is

$$\frac{1}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\dot{\psi}_{Bj} - \frac{1}{16}\dot{z}^\mu z^\nu R_{\nu\mu}{}^{ab}\varepsilon^{ij}\psi_{Ai}\gamma_{ab}^{AB}\psi_{Bj}. \tag{2.163}$$

The fermions $\psi_{Ai}(\tau)$ have four zero-modes $\psi_{Ai}^{(0)}$ – the modes that are independent of τ. The action (2.163) contains a coupling $R\psi^{(0)}\psi^{(0)}$, which is the only coupling that can saturate the fermion zero-modes. Using this coupling to saturate the zero-modes gives an explicit factor of R^2 in the path integral, and as we do not wish to compute terms of higher order in R, we can drop the coupling of R to other fermion modes. The action then reduces to

$$S_E = \int_0^\beta \mathrm{d}\tau \left[\frac{\dot{z}^2}{4} - i\frac{\overline{\mathsf{Z}}}{8}\left(\mathsf{W}_{\nu\mu}^- - \frac{i}{2\overline{\mathsf{Z}}}R_{\nu\mu}^-{}^{ab}\varepsilon^{ij}\psi_{Ai}^{(0)}\gamma_{ab}^{AB}\psi_{Bj}^{(0)}\right) z^\nu \dot{z}^\mu + \frac{1}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\dot{\psi}_{Bj} + |\mathsf{Z}|^2\right]. \tag{2.164}$$

Now we observe that replacing $iR\psi^{(0)}\psi^{(0)}/\overline{\mathsf{Z}}$ by $R\psi^{(0)}\psi^{(0)}$ has the effect of just multiplying the path integral by $-\overline{\mathsf{Z}}^2$. If we make this replacement, and also set $\psi_{Ai}^{(0)} = \sqrt{2}\theta_{Ai}$, and finally set $z^\mu = \sqrt{2}y^\mu$, then the action becomes

$$S_E = \int_0^s \mathrm{d}\tau \left[\frac{\dot{y}^2}{2} - i\frac{\overline{\mathsf{Z}}}{4}\mathcal{W}_{\mu\nu}^- y^\nu \dot{y}^\mu + \frac{1}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\dot{\psi}_{Bj} + |\mathsf{Z}|^2\right], \tag{2.165}$$

where

$$\mathcal{W}_{\mu\nu}(x,\theta) = \mathsf{W}_{\mu\nu}^-(x) + \cdots - R_{\mu\nu\lambda\rho}^-(x)\varepsilon_{ij}\overline{\theta}^i\sigma^{\lambda\rho}\theta^j + \ldots \tag{2.166}$$

is the superfield whose bottom component is $\mathsf{W}^-_{\mu\nu}$.

The constant term $|\mathfrak{z}|^2$ in the Lagrangian density just multiplies the path integral by $\exp(-s|\mathfrak{z}|^2)$. So

$$\mathrm{Tr}_{\widehat{\mathcal{H}}}(-1)^F \exp(-sH) = -\frac{e^{-s|\mathfrak{z}|^2}}{\overline{\mathfrak{z}}^2}\int \mathrm{d}^4y\mathrm{d}^4\theta\sqrt{g}$$
$$\int \mathcal{D}'y\,\mathcal{D}'\psi\exp\left(-\int_0^s \mathrm{d}\tau\left(\frac{\dot{y}^2}{2} - i\frac{\overline{\mathfrak{z}}}{4}\mathcal{W}^-_{\mu\nu}y^\nu\dot{y}^\mu + \frac{1}{2}\varepsilon^{ij}\varepsilon^{AB}\psi_{Ai}\dot{\psi}_{Bj}\right)\right), \quad (2.167)$$

where $\mathcal{D}'$ represents a path integral over non-zero modes only. Apart from the decoupled fermions ψ_{Ai}, the remaining path integral describes a particle in a constant magnetic field $\overline{\mathfrak{z}}\mathcal{W}$. This is a very standard path integral, and one way to evaluate it was described in section 2.2.2. We finally learn that

$$\mathrm{Tr}_{\widehat{\mathcal{H}}}(-1)^F \exp\left(-s(|\mathfrak{z}|^2 - D^2)\right) = -\frac{e^{-s|\mathfrak{z}|^2}}{(2\pi)^4}\int \mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g}\frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\overline{\mathfrak{z}}\sqrt{\mathcal{W}^2}}{8}}. \quad (2.168)$$

(The measure $\mathrm{d}^4\theta$ was defined in eqn. (2.102), and the same derivation applies here.)

When this is inserted in (2.158), we get

$$\int_0^\infty \frac{\mathrm{d}s}{s}\left(\frac{e^{-s|\mathfrak{z}|^2}}{(2\pi)^4}\int \mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g}\frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\overline{\mathfrak{z}}\sqrt{\mathcal{W}^2}}{8}} + \mathcal{I}e^{-s\overline{\mathfrak{z}}}\right). \quad (2.169)$$

To see that the integral converges near $s = 0$, we observe first that expanding the integrand gives a term proportional to $1/s^3$, but this term is independent of $\mathcal{W}$ and is annihilated by the $\mathrm{d}^4\theta$ integral. The next term in the expansion is proportional to $1/s$, but the index theorem for the Dirac operator ensures that this contribution cancels, so the integral converges for small s. In fact, with the help of the index theorem, (2.169) is equivalent to

$$\int_0^\infty \frac{\mathrm{d}s}{s}\left(\frac{1}{(2\pi)^4}\int \mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g}\left(e^{-s|\mathfrak{z}|^2}\frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\overline{\mathfrak{z}}\sqrt{\mathcal{W}^2}}{8}} + \frac{\pi^2\mathcal{W}^2}{3\cdot 64}e^{-s\overline{\mathfrak{z}}}\right)\right). \quad (2.170)$$

Since $\mathcal{Z}$ has non-negative real part, the integral also converges at large s.

To establish holomorphy in $\mathcal{Z}$, we simply rescale $s \to s/\overline{\mathcal{Z}}\mathcal{Z}$, to get:

$$\int_0^\infty \frac{\mathrm{d}s}{s}\left(\frac{1}{(2\pi)^4}\int \mathrm{d}^4x\mathrm{d}^4\theta\sqrt{g}\left(e^{-s}\frac{\pi^2\mathcal{W}^2/64}{\sinh^2\frac{s\sqrt{\mathcal{W}^2}}{8\mathcal{Z}}}+\frac{\pi^2\mathcal{W}^2}{3\cdot 64}e^{-s/\mathcal{Z}}\right)\right). \tag{2.171}$$

It is very satisfying to see holomorphy emerging even though the particle mass is certainly not holomorphic in $\mathcal{Z}$.

In this calculation, we have taken $\mathcal{Z}$ to be a complex constant, rather than a field. This means that we have not taken into account fluctuations in the scalar fields in vector multiplets. When such fluctuations are included, $\mathcal{Z}$ becomes a chiral superfield, and the effective action may have an additional contribution,[16] not determined in our computation, that depends only on $\mathcal{Z}$.

The Case $\mathcal{Z} = 0$

We recall that in this derivation, we assumed at the beginning that $\mathcal{Z} \neq 0$. Let us separately consider the case that $\mathcal{Z} = 0$. This case only arises if $M = q_I = 0$ and in addition the Kaluza-Klein momentum n vanishes. Looking back to the 4d action (2.149) with which we started, we see that for $\mathcal{Z} = n = 0$, the scalars ϕ^i do not couple to W^-, but the fermions have a magnetic moment coupling. (Thus, the case $\mathcal{Z} = 0$ is the only case in which the fermions have a magnetic moment.) This case is simple enough that we can get a very general answer, for arbitrary M_4.

We recall that a Dirac fermion is equivalent to a pair of Weyl (or Majorana) fermions. So in a notation slightly different from that in (2.149), we have two right-handed fermions ψ_R^1, ψ_R^2, and two left-handed fermions $\overline{\psi}_L^1$, $\overline{\psi}_L^2$. The fermion kinetic

[16] Since the action (2.149) depends on the 4d superfields $\mathcal{Z}^I$ only in the combination $\sum_I q_I\mathcal{Z}^I$ which appears in the central charge, any $\mathcal{W}$-independent function that we have not computed is a function of $\mathcal{Z}$ only. Even without any computation of terms that are independent of $\mathcal{W}$, eqn. (2.171) clearly needs some modification when $\mathcal{Z}$ is not constant, if only to ensure that it converges for $s \to 0$.

energy is

$$\int \mathrm{d}^4x\sqrt{g}\left(\overline{\psi}^1_L \not{D}\psi^1_R + \overline{\psi}^2_L \not{D}\psi^2_R\right). \tag{2.172}$$

Classically, there is a $U(1)$ symmetry under which ψ^1_R has charge 1, ψ^1_L has charge -1, and ψ^2_R, ψ^2_L are neutral. ($\overline{\psi}^1_L$ is just the transpose of ψ^1_L, with no complex conjugation involved, so it has charge -1 just like ψ^1_L, ensuring the invariance of the fermion kinetic energy.) However, this $U(1)$ symmetry is violated in a gravitational field. The net violation of the symmetry is given by the index $\mathcal{I}$ of the Dirac operator. Hence, on a four-manifold M_4 on which $\mathcal{I} \neq 0$, the fermion path integral vanishes when the graviphoton field vanishes. The graviphoton field W^- couples to a pair of left fermions

$$\int \mathrm{d}^4x\sqrt{g}\mathsf{W}^{-\mu\nu}\overline{\psi}_L{}^2\gamma_{\mu\nu}\psi^1_L. \tag{2.173}$$

Thus W^- effectively has charge 1 under the symmetry. If $\mathcal{I} < 0$, so that generically a left fermion has $|\mathcal{I}|$ zero-modes and a right fermion has none, then the insertion in the path integral of $|\mathcal{I}| = -\mathcal{I}$ copies of this interaction can give a nonzero result. The path integral is then proportional to $(\mathsf{W}^-)^{-\mathcal{I}}$. For $\mathcal{I} > 0$, the path integral vanishes, if the graviphoton is anti-selfdual as assumed in the above formulas for the action. (It would be inconvenient to restrict our discussion to the case that M_4 is hyper-Kahler with anti-selfdual Weyl curvature, as this forces $\mathcal{I} > 0$, while we have just seen that the more interesting case is $\mathcal{I} < 0$.)

For some purposes, one can describe this result by saying that the effective action contains a term $-\frac{\chi}{2}\mathcal{I}\log \mathsf{W}^-$, which in supersymmetric language could be derived from an F-term

$$\frac{\chi}{2}\int \mathrm{d}^4x\mathrm{d}^4\theta\mathcal{W}^2\log\sqrt{\mathcal{W}^2}. \tag{2.174}$$

(We recall from section 2.3.1 that the effective number of massless hypermultiplets is $-\chi/2$.) For the original calculation leading to a result along these lines, see [56].

However, this interpretation has some limitations. First, technically, the $\mathsf{W}^{-\mathcal{I}}$ behavior of the path integral arises only for $\mathcal{I} < 0$, not for $\mathcal{I} > 0$. For $\mathcal{I} > 0$, it is not possible to get a nonzero path integral by making a negative number of W^- insertions. Moreover, the path integral of the fermions under discussion on a four-manifold of $\mathcal{I} > 0$ does not blow up as $\mathsf{W}^{-\mathcal{I}}$ for $\mathsf{W}^- \to 0$. Rather, it vanishes identically for all W^-.

Furthermore, if one carries out the $\mathrm{d}^4\theta$ integral in (2.174), one gets, in addition to an $\mathcal{I} \log \mathsf{W}$ "coupling," a variety of interactions that are singular for $\mathsf{W} \to 0$ and look difficult to interpret.

Most fundamentally, the problem with trying to describe this effect by a term in the effective action such as (2.174) is that the effect is fundamentally non-local. Our derivation has shown that the effect comes entirely from integrating out particles that are massless in four dimensions, so one should not try to incorporate it into a 4d Wilsonian effective action.

It was observed in [56] that an F-term of the form $\mathcal{W}^2 \log \sqrt{\mathcal{W}^2}$ is not part of the relation between the $\mathcal{F}_{\mathsf{g}}$'s and the topological string. Indeed, such an effect is certainly not seen in the perturbative string theory calculation of [60]. The reason is clear from a low-energy point of view: perturbation theory with the interaction (2.173) will never generate a coupling of any number of gravitons to any (positive) number of graviphotons, since this is prevented by the $U(1)$ symmetry.

Comparison With The Particle-Based Calculation

Since we have taken z to be constant, we cannot compute the hypermultiplet contribution to $\mathcal{F}_0$. But we can compute its contribution to $\mathcal{F}_{\mathsf{g}}$, $\mathsf{g} \geq 1$.

Using the expansion

$$\frac{(x/2)^2}{\sinh^2(x/2)} = \sum_{n=0}^{\infty}(1-2n)B_{2n}\frac{x^{2n}}{(2n)!}, \tag{2.175}$$

where the B_{2n} are Bernoulli numbers (of alternating sign), and integrating over s term by term, we get

$$\begin{aligned}\mathcal{F}_1 &= -i\frac{\log \mathfrak{z}}{3(32\pi)^2}\\ \mathcal{F}_\mathfrak{g} &= -i\frac{1}{(16\pi)^2}\frac{B_{2\mathfrak{g}}}{2\mathfrak{g}(2\mathfrak{g}-2)}(4\mathfrak{z})^{2-2\mathfrak{g}}.\end{aligned} \tag{2.176}$$

Since $\mathrm{Re}\,\mathfrak{z} \geq 0$, the s integral converges and determines in $\mathcal{F}_1$ a definite branch of $\log \mathfrak{z}$, namely the one with $|\mathrm{Im} \log \mathfrak{z}| \leq \pi/2$.

Eqn. (2.176) determines the contribution to $\mathcal{F}_\mathfrak{g}$ of a 4d hypermultiplet of given $\mathfrak{z}$. To get the contribution of a 5d hypermultiplet of mass M, we have to sum over the Kaluza-Klein momentum n. This is particularly simple for $M = 0$, which only occurs for $q_I = 0$, in which case eqn. (3.82) for the central charge reduces to $\mathfrak{z} = -ine^{3\sigma/2} = 2n\mathcal{X}^0$. The sum over n can also be performed for $M \neq 0$ (see [61]), but this does not affect the qualitative point that we wish to make.

As discussed in section 2.3.2, for $M = 0$, we sum only over $n \neq 0$. The contribution of a massless 5d hypermultipet to $\mathcal{F}_\mathfrak{g}$ for $\mathfrak{g} \geq 2$ is

$$\mathcal{F}_\mathfrak{g}^{M=0} = -i\sum_{n\neq 0}\frac{1}{(16\pi)^2}\frac{B_{2\mathfrak{g}}}{2\mathfrak{g}(2\mathfrak{g}-2)}(8n\mathcal{X}^0)^{2-2\mathfrak{g}} = -\frac{i}{(16\pi)^2}\frac{B_{2\mathfrak{g}}}{\mathfrak{g}(2\mathfrak{g}-2)}(8\mathcal{X}^0)^{2-2\mathfrak{g}}\zeta(2\mathfrak{g}-2). \tag{2.177}$$

For $\mathfrak{g} = 1$, the sum over n is divergent. This should be interpreted as follows. The one-loop effective action in five dimensions is potentially ultraviolet divergent, but any such divergence is the integral of a gauge-invariant local expression. Such an integral cannot contribute to $\mathcal{F}_1$, as we explained in section 2.1.2. Similarly, although our knowledge of M-theory does not give us much insight about how to regularize the one-loop computation in five dimensions, any two regularizations that preserve 5d covariance will differ only by the integral of a gauge-invariant local expression, and will therefore give the same result for $\mathcal{F}_1$. Consequently, a computation that preserves

5d symmetry will give a finite and unambiguous answer for $\mathcal{F}_1$. The computation that we have performed was based on an expansion in Kaluza-Klein harmonics and did not preserve the 5d symmetry. This is why it does not give a satisfactory understanding of the contribution of a 5d field to $\mathcal{F}_1$.

We would like to compare the hypermultiplet contribution to the effective action as computed in the field-based approach to the earlier particle-based result (2.127). The field-based calculation involved a sum over states of definite momentum around the Kaluza-Klein circle, and the particle-based calculation involved a sum over orbits of definite winding number. As usual (and essentially as in [62]), to convert one to the other, one should perform a Poisson resummation. In doing this resummation, we should remember two facts, which turn out to be related. We cannot really compare the two computations for $\mathcal{F}_1$, because our field-based computation was not powerful enough to determine the sum over Kaluza-Klein momenta in $\mathcal{F}_1$. And in the particle-based computation, we do not want to include a contribution with winding number zero, because this contribution is not meaningful in the context of the particle-based computation.

Since we will not try to make a comparison for $\mathcal{F}_1$, we will ignore the $\mathcal{W}^2 e^{-s/\mathfrak{z}}$ term in eqn. (2.171), which only contributes to $\mathcal{F}_1$. To avoid having to worry about the potential divergence of the s integral for $s \to 0$, we simply remember that the result of the Poisson resummation should be expanded in powers of $\mathcal{W}$, keeping only terms of order ≥ 4. Also, in performing the Poisson resummation, we will discard by hand the contribution of winding number $k = 0$; it will be clear that this term only contributes to $\mathcal{F}_1$.

As in eqn. (2.71), we define $S(\vec{q}) = 2\pi(e^{\sigma}M - iq_I\alpha^I) = -2\pi i q_I Z^I$, so that the central charge of a particle of Kaluza-Klein momentum n is $\mathfrak{z} = e^{-3\sigma/2}(-in + S(\vec{q})/2\pi)$. We also rescale the Schwinger parameter by $s \to s\mathfrak{z}$ (this can be accompanied by a rotation of the integration contour in the complex plane, so that we still integrate over the positive s axis). The sum and integral to be performed are then

$$\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g}\sum_{n\in\mathbb{Z}}\int_0^\infty \frac{\mathrm{d}s}{s}\exp\left(-se^{-3\sigma/2}S(\vec{q})/2\pi\right)\exp(inse^{-3\sigma/2})\frac{\pi^2\mathcal{W}^2/64}{\sinh^2(s\sqrt{\mathcal{W}^2}/8)}. \tag{2.178}$$

Upon using $\sum_{n\in\mathbb{Z}} e^{in\theta} = 2\pi\sum_{k\in\mathbb{Z}}\delta(\theta-2\pi k)$, we get

$$\int \frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g}\sum_{k\in\mathbb{Z}}\int_0^\infty \frac{\mathrm{d}s}{s}\exp\left(-se^{-3\sigma/2}S(\vec{q})/2\pi\right)2\pi\delta(se^{-3\sigma/2}-2\pi k)\frac{\pi^2\mathcal{W}^2/64}{\sinh^2(s\sqrt{\mathcal{W}^2}/8)}. \tag{2.179}$$

We see that, as expected from the particle computation, there is no contribution from $k<0$, while $k=0$ formally makes only a contribution to $\mathcal{F}_1$, which we discard. Integrating over s with the help of the delta functions, promoting $S(\vec{q}) = -2\pi i q_I Z^I$ to a superfield $\mathcal{S}(\vec{q}) = -2\pi i q_I \mathcal{Z}^I$ (to get a formula that is valid even when the $\mathcal{Z}^I$ are not taken to be constants), and introducing again $\mathcal{X}_0 = -ie^{-3\sigma/2}/2$, we recover the familiar result

$$-\sum_{k=1}^{\infty}\frac{1}{k}\int\frac{\mathrm{d}^4x\mathrm{d}^4\theta}{(2\pi)^4}\sqrt{g}\exp\left(2\pi ik\sum_I q_I\mathcal{Z}^I\right)\frac{\frac{1}{64}\pi^2\mathcal{W}^2}{\sin^2\left(\frac{\pi k\sqrt{\mathcal{W}^2}}{8\mathcal{X}^0}\right)} \tag{2.180}$$

for the hypermultiplet contribution.

2.4 Field Theory Computations of $\mathcal{F}_0$ and $\mathcal{F}_1$

Here we will consider the field theory computations which work for $\mathcal{F}_0$ and $\mathcal{F}_1$. This case was omitted in our previous discussion and is required for completeness. First, we need to elaborate a little more on the 4d $\mathcal{N}=2$ supergravity and properties of interactions $\mathcal{F}_{\mathrm{g}}$.

2.4.1 Additional facts about supergravity

Here we review and recall some facts about $\mathcal{N}=2$ supergravity in four dimensions, which are required for the derivation in this section.

The formulation of 4d Poincare supergravity that we rely on is based on embedding in superconformal gravity, which is gauge-equivalent to Poincare supergravity in the sense that partial gauge fixing of the superconformal theory gives Poincare supergravity. This naturally comes with an $N=2$ superspace. Chiral superfields of weight 2 under dilations can be considered as possible F-terms in the superspace action of conformal supergravity. Given some superspace interaction, say an F-term $\int \mathrm{d}^4x\mathrm{d}^4\theta\, \Phi$, to find the corresponding terms in the Poincare supergravity action, one has to not only integrate over Grassmann coordinates θ, but also impose all gauge-fixing constraints that reduce the superconformal gauge group to the super Poincare.

Two superconformal matter multiplets, the compensators, disappear in this gauge-fixing. One usually chooses a vector multiplet and a hypermultiplet for this role (see [20] for details). Thus to build an $N=2$ Poincare supergravity coupled to n vector multiplets, one starts with $N=2$ superconformal gravity coupled to $n+1$ vector multiplets and 1 hypermultiplet.

As already discussed before, an $N=2$ vector multiplet in 4d contains a complex scalar, a vector and a doublet of spinors. Such multiplets are described by reduced chiral superfields $\mathcal{X}^\Lambda$, $\Lambda = 0\ldots n$ (see [49–51]), whose lowest components X^Λ are complex scalars, while the highest components are $-\frac{1}{6}(\varepsilon_{ij}\overline{\theta}^i\sigma^{\mu\nu}\theta^j)^2 D_\mu D^\mu \overline{X}^\Lambda$ and involve derivatives of complex conjugate scalars (because of the non-holomorphic constraint satisfied by reduced chiral superfields). Recall from section 2.1.2 that couplings of vector multiplets are described by the holomorphic prepotential $\mathcal{F}_0(\mathcal{X})$ (see [20]), which has to be homogeneous of degree 2 to define a term in the Lagrangian of conformal supergravity:

$$-i\int \mathrm{d}^4x\mathrm{d}^4\theta\, \mathcal{F}_0(\mathcal{X}) + c.c. \tag{2.181}$$

With the usual notations $\widehat{F}_\Lambda$, $\widehat{F}_{\Lambda\Sigma}$ etc. for derivatives of $\mathcal{F}_0$ (as introduced in Section 2.1.2) and $N_{\Lambda\Sigma} = 2\mathrm{Im}\,\widehat{F}_{\Lambda\Sigma}$, the superspace expression (2.181) implies the kinetic term for conformal scalars:

$$\int \mathrm{d}^4x\sqrt{g}N_{\Lambda\Sigma}D_\mu X^\Lambda D^\mu \overline{X}^\Sigma, \tag{2.182}$$

where the derivatives are covariant with respect to the superconformal gauge group. In order to get the kinetic energy for scalars of Poincare supergravity, one has to use a gauge condition which fixes dilatational symmetry of conformal supergravity. This usually has a form of some constraint on the superconformal scalars X^Λ. The freedom to perform local dilatations in conformal supergravity corresponds to the freedom to Weyl-rescale metric in Poincare supergravity. The standard gauge choice [20], which guarantees that the Poincare theory emerges written in the Einstein frame and which was used in Section 2.1.2, is

$$N_{\Lambda\Sigma}X^\Lambda\overline{X}^\Sigma = -1. \tag{2.183}$$

It is usually supplemented by the $U(1)$ R-symmetry gauge, which we picked as $iX^0 > 0$ when the expression (2.83) was written. A convenient choice of independent holomorphic scalars was:

$$Z^I = \frac{X^I}{X^0}, \quad I = 1\dots n. \tag{2.184}$$

The standard gauge choice (2.183) implies the following expression for $|X^0|^2$ in terms of other fields:

$$|X^0|^2 = \frac{1}{Y}, \quad Y = -N_{\Lambda\Sigma}Z^\Lambda\overline{Z}^\Sigma. \tag{2.185}$$

In this case, the kinetic energy for vector multiplet scalars (of Poincare supergravity) takes the form:

$$Y^{-1}\mathfrak{M}_{IJ}\partial_\mu Z^I\partial^\mu\overline{Z}^J, \quad \mathfrak{M}_{IJ} = N_{IJ} - (N_{I\Lambda}\overline{X}^\Lambda)(N_{J\Sigma}X^\Sigma), \tag{2.186}$$

and $Y^{-1}\mathfrak{M}_{IJ}$ is actually a Kahler metric:

$$Y^{-1}\mathfrak{M}_{IJ} = \frac{\partial}{\partial Z^I}\frac{\partial}{\partial \overline{Z}^J}\ln Y. \tag{2.187}$$

2.4.2 More properties of $\mathcal{F}_{\mathrm{g}}$

Recall that one of the conclusions of the section 2.1.2 was that only $\mathcal{F}_0$ and $\mathcal{F}_1$ receive contributions from the classical dimensional reduction of 5d supergravity on a circle. These classical parts, $\mathcal{F}_0^{\mathrm{cl}}$ and $\mathcal{F}_1^{\mathrm{cl}}$, were given by (2.82) and (2.92). Our task in this section is to compute the quantum corrections $\mathcal{F}_0^{\mathrm{q}}$ and $\mathcal{F}_1^{\mathrm{q}}$.

In section 2.1.2 we discussed shift symmetries. We can use them to obtain some restrictions on how the quantum corrections to $\mathcal{F}_{\mathrm{g}}$ depend on the scalars Z^I. We note that the only way the 5d BPS miultiplet action depends on α^I and h^I is through the linear combinations $q_I\alpha^I$ and $q_I h^I$ (as we will see in section 2.4.3). Thus, due to holomorphy, the quantum correction to $\mathcal{F}_{\mathrm{g}}$ should be a function of $q_I Z^I$. From shift symmetries, it actually should be a a function of $e^{2\pi i q_I Z^I}$. Thus we conclude that the general form of the contribution of one BPS multiplet to $\mathcal{F}_{\mathrm{g}}$, which we will usually denote by $\mathcal{F}_{\mathrm{g}}^{\mathrm{q}}$, is:

$$\mathcal{F}_{\mathrm{g}}^{\mathrm{q}} = \left(X^0\right)^{2-2\mathrm{g}}\sum_{k>0} c_{k,\mathrm{g}}e^{2\pi i k q_I Z^I}, \tag{2.188}$$

where we did not allow negative values of k, as the contribution $\propto e^{-2\pi k M}$, $M = q_I h^I > 0$ should decay faster for more massive particles, rather than exponentially grow (the $k<0$ terms would actually have Z^I replaced by $\overline{Z}^I$ and would contribute to the F-terms of the opposite chirality).

Constraints on $\mathcal{F}_{\mathrm{g}}$ from parity

M-theory has a discrete symmetry which is often called "parity" and is a combination of some orientation reversing diffeomorphism in 11d and a sign change of the 3-form gauge field C. This symmetry descends in an obvious way to the symmetry of the 5d action, and then to 4 dimensions as well. The fields A^I_μ and α^I, which originate from the 11d C-field, get an extra minus sign, while the field A^0_μ, which is a Kaluza-Klein gauge field, does not. So, to summarize, the 4d supergravity we obtain should be invariant under the parity defined as an orientation reversal combined with the following:

$$\begin{aligned} A^I_\mu &\to -A^I_\mu, \\ A^0_\mu &\to A^0_\mu, \\ \mathrm{Re}\, Z^I \equiv \alpha^I &\to -\alpha^I. \end{aligned} \tag{2.189}$$

How does it constrain the form of $\mathcal{F}_\mathsf{g}$? Since $d = 4, N = 2$ supergravity written in a given metric frame lifts in a unique way to the conformal supergravity, the parity symmetry also lifts there. It can then be extended to the symmetry of the superspace action. Since parity switches chiralities, we can conclude that the two terms of the form:

$$-i \int \mathrm{d}^4\theta\, \mathcal{F}_\mathsf{g}(\mathcal{X})\mathcal{W}^{2\mathsf{g}} + i \int \mathrm{d}^4\overline{\theta}\, \overline{\mathcal{F}}_\mathsf{g}(\overline{\mathcal{X}})\overline{\mathcal{W}}^{2\mathsf{g}} \tag{2.190}$$

are switched by parity, where the second term is the complex conjugate of the first and $\overline{\theta}$ are superspace coordinates of opposite chirality. This means, in particular, that for all $\mathsf{g} \geq 0$, $-i\mathcal{F}_\mathsf{g}(X)$ goes into $i\overline{\mathcal{F}}_\mathsf{g}(\overline{X})$ under parity. We are working in the gauge where $iX^0 > 0$, and so if we consider the non-homogeneous function $\widehat{\mathcal{F}}_\mathsf{g}(Z) = (X^0)^{2\mathsf{g}-2}\mathcal{F}_\mathsf{g}(X)$, we also find that parity complex conjugates $i\widehat{\mathcal{F}}_\mathsf{g}(Z)$, i.e., sends $\widehat{\mathcal{F}}_\mathsf{g}(Z)$ to $-\overline{\widehat{\mathcal{F}}_\mathsf{g}(Z)}$.

Since the only effect of parity on scalars Z^I is to multiply α^I by -1, it means that all terms $\widehat{\mathcal{F}}_\mathsf{g}(Z)$ in the GV formula should go to $-\overline{\widehat{\mathcal{F}}_\mathsf{g}(Z)}$ under $\alpha^I \to -\alpha^I$. This condition implies that $\widehat{\mathcal{F}}_{\mathsf{g}}(Z)$ should be imaginary at $\alpha^I = 0$.[17] In particular, $c_{k,\mathsf{g}}$ in (2.188) are imaginary. We will use it soon.

2.4.3 Computation of $\mathcal{F}_0$

Now we consider a light massive hypermultiplet coupled to the 5d supergravity which has enough scalars h^I (we will explain this requirement in Section 2.4.3). For the purposes of one-loop computation, the global geometry of space parametrized by scalars in the hypermultiplet is irrelevant. So, this multiplet can be described by a pair of complex scalars z^i, $i = 1, 2$ and a Dirac spinor Ψ in 5d. The quadratic action on the flat background with no gauge fields turned on is:

$$S_h = \int \mathrm{d}^5 x \left(\sum_{i=1}^{2} (-|\partial z^i|^2 - M^2 |z^i|^2) + \overline{\Psi}^c \partial\!\!\!/ \Psi - M \overline{\Psi}^c \Psi \right) . \tag{2.191}$$

We want to determine its contribution to the term $\mathcal{F}_0$ in the 4d $\mathcal{N} = 2$ effective superpotential. Our strategy is to determine first its contribution to the 4d Kahler metric on the vector multiplets moduli space, and then, since this metric is encoded in $\mathcal{F}_0$, to reconstruct the hypermultiplet contribution to $\mathcal{F}_0$.

To find the contribution to the Kahler metric, we need to compute the effective action governing fluctuations of vector multiplet scalars on the flat background $\mathbb{R}^{3,1} \times S^1$, which is the simplest possible background consistent with our problem.

Let us describe the precise setup. Note first that the expected answer has a known form (2.188), in which we only have to determine the constants $c_{k,\mathsf{g}}$. To do this, we can choose any convenient values for the background fields. One such field is the radius of the M-theory circle e^σ, and we should choose some value for it. Before, the computation was done in the large radius limit. This was the case because, in the particle computation performed earlier, one was integrating out particles that

[17] For analytic $\mathcal{F}_{\mathsf{g}}(Z)$, these two conditions are actually equivalent.

were not point-like. They were given by wrapped M2-branes and thus had some internal structure. But the computation was done in the approximation in which those particles were treated as point-like, which made sense only if the characteristic size of their trajectories – the radius of the M-theory circle which they wound – was much larger than the particle size.

In the current situation, we are doing the field theory computation, so the question of whether the particles are point-like or not becomes hidden behind the question of applicability of the field theory description. And, as was already noted before, we assume the field theory description to be valid for the massless or very light multiplets. Also, we know how the holomorphic answer (2.188) depends on the radius, and we know that the coefficients $c_{k,\mathrm{g}}$ do not depend on it. Therefore, once we have the action (2.191) and know what to compute, we are free to pick any convenient value for the radius. For simplicity, we set it equal to 1, that is $e^{\sigma} = 1$. We also do not switch on holonomies, $\alpha^I = 0$. We allow the 5d scalars h^I to depend on the point of $\mathbb{R}^{3,1}$, while they still should be invariant under translations along S^1. Since the mass of the BPS particle in 5d is expressed through its charges q_I as:

$$M = \sum_I q_I h^I, \tag{2.192}$$

$M(x)$ is allowed to fluctuate around its constant background value M, with fluctuations depending only on the point of $\mathbb{R}^{3,1}$. Now, to determine the Kahler metric deformation, we need to find a term in the effective action which is quadratic in $M(x)$ and has precisely two derivatives. Since the effective action is $S_{eff} = -i \ln \int \mathcal{D}z^i \mathcal{D}\Psi e^{iS_h}$, and we are looking for

$$\frac{\delta^2 S_{eff}}{\delta M(x) \delta M(y)}\Big|_{M=const}, \tag{2.193}$$

it is clear that all we need to compute is a connected two-point function of the mass terms:

$$-i\left\langle\left(2M\sum_i|z^i|^2+\overline{\Psi}^c\Psi\right)\left(2M\sum_i|z^i|^2+\overline{\Psi}^c\Psi\right)\right\rangle_{\text{conn}}, \tag{2.194}$$

and then, in the momentum space representation with an external momentum p, to extract the p^2-part of the answer. This will give the one-loop Kahler metric deformation due to the light hypermultipet.

After we calculate the Kahler metric deformation, we will have to reconstruct the prepotential deformation from it. We use notation cl and q to distinguish classical and one-loop parts, so for example the full prepotential is $\mathcal{F}_0=\mathcal{F}_0^{\text{cl}}+\mathcal{F}_0^{\text{q}}$, were the classical part is given by (2.82). The Kahler metric deformation is written in terms of the scalars $Z^I=X^I/X^0$ of Poincare supergravity. However, $\mathcal{F}_0(X)$ is a function of conformal scalars X^Λ, so to reconstruct it, we should know the expression of X^0 in terms of Z^I and $\overline{Z}^I$. Reconstructing $\mathcal{F}_0$ includes some subtleties, which we will discuss in detail later, after the two-point function computation.

The two-point function computation

In this subsection we compute (2.194). First of all, we need to know the relevant Green's functions on $\mathbb{R}^{3,1}\times S^1$. Let x^μ be coordinates on $\mathbb{R}^{3,1}$ and $y\in[0,2\pi]$ be a coordinate on S^1. If $G_0(x,y)$ and $D_0(x,y)$ are the Green's functions for bosons and fermions respectively on $\mathbb{R}^{4,1}$, i.e., they satisfy:

$$\begin{aligned}(\partial^2-M^2)G_0(x,y)&=\delta^{(4)}(x)\delta(y),\\ (\not{\partial}-M)D_0(x,y)&=\delta^{(4)}(x)\delta(y),\end{aligned} \tag{2.195}$$

then the Green's functions on $\mathbb{R}^{3,1} \times S^1$ are just:

$$G(x,y) = \sum_{k\in\mathbb{Z}} G_0(x, y + 2\pi k),$$
$$D(x,y) = \sum_{k\in\mathbb{Z}} D_0(x, y + 2\pi k). \tag{2.196}$$

Then (2.194) gives:

$$-8iM^2 G(x_1-x_2, y_1-y_2)G(x_2-x_1, y_2-y_1) + i\mathrm{Tr}\big[D(x_1-x_2, y_1-y_2)D(x_2-x_1, y_2-y_1)\big], \tag{2.197}$$

where (x_1, y_1) and (x_2, y_2) are the space-time points where the two mass terms are inserted. If $\mathcal{K}(x_1, y_1; x_2, y_2)$ denotes the expression (2.197), then the term in the effective action is

$$\int \mathrm{d}^4x_1 \mathrm{d}y_1 \mathrm{d}^4x_2 \mathrm{d}y_2\, \mathcal{K}(x_1, y_1; x_2, y_2) M(x_1)M(x_2). \tag{2.198}$$

We note that since $M(x)$ is independent of the circle direction y, we can integrate (2.197) over y_1 and y_2, or over $y \equiv y_1 - y_2$ and y_2. Another obvious step is to pass to the momentum representation for the $\mathbb{R}^{3,1}$ directions. Now we have

$$\int \mathrm{d}y_1 \mathrm{d}y_2\, \mathcal{K}(x_1, y_1; x_2, y_2) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\, \mathcal{K}(p) e^{ip(x_1-x_2)}, \tag{2.199}$$

and this $\mathcal{K}(p)$ is given by

$$\mathcal{K}(p) = -2\pi i \int_0^{2\pi} \mathrm{d}y \int \frac{\mathrm{d}^4q}{(2\pi)^4} \Big(8M^2 G(q,y)G(q-p,-y) - \mathrm{Tr}\big[D(q,y)D(q-p,-y)\big]\Big). \tag{2.200}$$

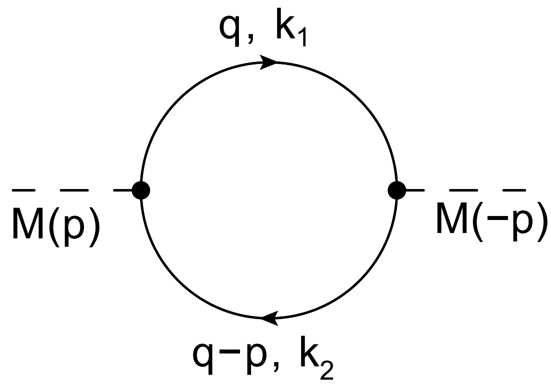

Figure 2.2: The two-point function of mass terms. Internal lines are labeled by the 4d momentum and the winding number.

If we substitute (2.196), this becomes:

$$\mathcal{K}(p) = -2\pi i \sum_{k_1,k_2} \int_0^{2\pi} \mathrm{d}y \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Big(8M^2 G_0(q, y-2\pi k_1) G_0(q-p, -y-2\pi k_2) \\ - \mathrm{Tr}\big[D_0(q, y-2\pi k_1) D_0(q-p, -y-2\pi k_2) \big] \Big). \tag{2.201}$$

This quantity is represented by the Feynman diagram on Figure 2.4.3, where scalars and bosons run inside the loop, and we label internal lines of the loop by the corresponding 4d momentum and the winding number k. It is clear from the picture that $k_1 + k_2$ plays the role of the total winding number of the particle as it circles the loop in the diagram. Another way to see it is to reintroduce non-zero constant holonomies α^I. These would just shift the momentum in the circle direction by $w \to w + q_I \alpha^I$ and contribute an overall factor $e^{-iq_I\alpha^I y}$ both in $G_0(p, y)$ and $D_0(p, y)$. Then, in the above expression for $\mathcal{K}(p)$, the only effect of holonomies would be to introduce an overall factor $e^{2\pi i(k_1+k_2)q_I\alpha^I}$, thus showing that $k_1 + k_2$ is indeed interpreted as the total winding number of the loop.

We need explicit expressions for G_0 and D_0 in a "mixed" representation, where momentum is used for the x^μ directions and position coordinate is used for the y

direction. It is easy to find that:

$$D_0(p,y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{2\pi} e^{iwy} \frac{M - i\not{p} - iw\Gamma^5}{p^2 + w^2 + M^2} = \frac{M - i\not{p}}{2\sqrt{p^2+M^2}} e^{-|y|\sqrt{p^2+M^2}} + \frac{\mathrm{Sign}(y)}{2}\Gamma^5 e^{-|y|\sqrt{p^2+M^2}},$$
$$G_0(p,y) = \int_{-\infty}^{\infty} \frac{\mathrm{d}w}{2\pi} e^{iwy} \frac{1}{p^2 + w^2 + M^2} = \frac{e^{-|y|\sqrt{p^2+M^2}}}{2\sqrt{p^2+M^2}}. \tag{2.202}$$

Substituting this into our expression for $\mathcal{K}(p)$, computing traces of gamma matrices and considering a given fixed $k = k_1 + k_2$, we get

$$-2\pi i \sum_{k_1+k_2=k} \int_0^{2\pi} \mathrm{d}y \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left(\frac{M^2 + q^2 - pq}{\sqrt{q^2+M^2}\sqrt{(q-p)^2+M^2}} + \mathrm{Sign}(y - 2\pi k_1)\mathrm{Sign}(y + 2\pi k_2) \right) e^{-|y-2\pi k_1|\sqrt{q^2+M^2} - |y+2\pi k_2|\sqrt{(q-p)^2+M^2}}. \tag{2.203}$$

For this computation and for the computation in the next section, we need the following two formulas:

$$\int_0^{2\pi} \mathrm{d}y \sum_{k_1+k_2=k} e^{-|y-2\pi k_1|A - |y+2\pi k_2|B} = \frac{e^{-2\pi|k|A} + e^{-2\pi|k|B}}{A+B} + \frac{e^{-2\pi|k|B} - e^{-2\pi|k|A}}{A-B}, \tag{2.204}$$

$$\int_0^{2\pi} \mathrm{d}y \sum_{k_1+k_2=k} e^{-|y-2\pi k_1|A - |y+2\pi k_2|B} \mathrm{Sign}(y - 2\pi k_1)\mathrm{Sign}(y + 2\pi k_2) = \frac{e^{-2\pi|k|A} + e^{-2\pi|k|B}}{A+B} - \frac{e^{-2\pi|k|B} - e^{-2\pi|k|A}}{A-B}. \tag{2.205}$$

Applying them to (2.203), we get:

$$
\begin{aligned}
&-2\pi i\int\frac{\mathrm{d}^4q}{(2\pi)^4}\left[\frac{M^2+q^2-pq}{\sqrt{q^2+M^2}\sqrt{(q-p)^2+M^2}}\times\right.\\
&\left(\frac{e^{-2\pi|k|\sqrt{(q-p)^2+M^2}}-e^{-2\pi|k|\sqrt{q^2+M^2}}}{\sqrt{q^2+M^2}-\sqrt{(q-p)^2+M^2}}+\frac{e^{-2\pi|k|\sqrt{q^2+M^2}}+e^{-2\pi|k|\sqrt{(q-p)^2+M^2}}}{\sqrt{q^2+M^2}+\sqrt{(q-p)^2+M^2}}\right)\\
&\left.-\frac{e^{-2\pi|k|\sqrt{(q-p)^2+M^2}}-e^{-2\pi|k|\sqrt{q^2+M^2}}}{\sqrt{q^2+M^2}-\sqrt{(q-p)^2+M^2}}+\frac{e^{-2\pi|k|\sqrt{q^2+M^2}}+e^{-2\pi|k|\sqrt{(q-p)^2+M^2}}}{\sqrt{q^2+M^2}+\sqrt{(q-p)^2+M^2}}\right].
\end{aligned}
\tag{2.206}
$$

This expression is perfectly convergent for $k\neq 0$ and we are going to compute it shortly, but first let us say a few words about $k=0$.

A digression about $k=0$. The case $k=0$ corresponds, in the particle language, to the contribution of closed trajectories that do not have any net winding number. Such trajectories in $\mathbb{R}^4\times S^1$ can be lifted to closed trajectories in $\mathbb{R}^5$. Thus the $k=0$ term should be understood as a contribution to the 5d effective action. It then contributes to the 4d effective action through the classical dimensional reduction. As was explained earlier, only two F-terms can receive contributions from the classical dimensional reduction. Those are precisely the $\mathcal{F}_0$ and $\mathcal{F}_1$ that are being studied in this section. The $\mathcal{F}_1$ term will be discussed in the next subsection, while for the prepotential $\mathcal{F}_0$, the only possible contributions from dimensional reduction originate from the 5d action (for supergravity with vector multiplets) with no more than 2 derivatives. Such an action in 5d is completely fixed by supersymmetry in terms of the coefficients $\mathcal{C}_{IJK}$ (see [97]). Dimensional reduction then gives the prepotential (2.82) in 4d depending on these coefficients. So the only possibility for the $k=0$ contribution to affect the $\mathcal{F}_0$ term in 4d is to shift the values of $\mathcal{C}_{IJK}$ in the 5d effective action. This does not happen. One way to see it is to note that the 5d action has a Chern-Simons term $C_{IJK}V^I\wedge \mathrm{d}V^J\wedge\mathrm{d}V^K$. It gives rise to the term $C_{IJK}\alpha^I F^J\wedge F^J$

in the 4d action, where $\alpha^I = V_y^I$ are holonomies along the circle. Any quantum computation will depend on holonomies through the combination $e^{2\pi i \alpha^I}$, and thus the term $C_{IJK}\alpha^I F^J \wedge F^J$ cannot be shifted.[18]

Back to the computation. Now, for $k \neq 0$, we want to Taylor expand the integrand in (2.206) and pick out the p^2-term in the expansion. Schematically, there will be two kinds of terms:

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[f_1(q^2) p^2 + f_2(q^2)(pq)^2 \right]. \tag{2.207}$$

In this type of integral one usually performs a Wick rotation $q^0 = -iq^4$, and then notes that, due to the spherical symmetry, $q_\mu q_\nu$ can be replaced by $\frac{q^2}{4}\eta_{\mu\nu}$. After that, we have:

$$-i \int \frac{\mathrm{d}^4 q_E}{(2\pi)^4} \left[f_1(q_E^2) + f_2(q_E^2) q_E^2/4 \right] p_E^2. \tag{2.208}$$

By going to spherical coordinates and recalling that the volume of the unit 3-sphere is $2\pi^2$, one has to compute:

$$-\frac{i\pi}{(2\pi)^3} \int_0^\infty q^3 \mathrm{d}q \left[f_1(q^2) + f_2(q^2) q^2/4 \right] p^2. \tag{2.209}$$

[18] From the string theory side, the values of $\mathcal{C}_{IJK}$ are given by the string three-point amplitudes on a sphere S^2 with one insertion of the NS-NS vertex operator corresponding to the scalar α^I and two inservions from the R-R sector corresponding to the field strengths $F^I_{\mu\nu}$ (see [98]).

We apply this to (2.206) (after Taylor expansion) and get the following expression at the p^2-order:

$$-\frac{\pi}{(2\pi)^2}p^2\int_0^\infty q^3\mathrm{d}q\left[\frac{e^{-2\pi|k|\sqrt{M^2+q^2}}q^2\left(3+4\pi^2k^2M^2+4\pi^2k^2q^2+6\pi|k|\sqrt{M^2+q^2}\right)}{8\left(M^2+q^2\right)^{5/2}}\right.$$
$$\left.-\frac{e^{-2\pi|k|\sqrt{M^2+q^2}}}{\left(M^2+q^2\right)^{3/2}}-\frac{2\pi|k|e^{-2\pi|k|\sqrt{M^2+q^2}}}{M^2+q^2}\right]. \tag{2.210}$$

By an obvious change of variables $x=\sqrt{M^2+q^2}$, this is transformed into:

$$-\frac{\pi}{(2\pi)^2}p^2\int_M^\infty \mathrm{d}x\, x(x^2-M^2)\left[\frac{e^{-2|k|\pi x}\left(x^2-M^2\right)\left(3+4\pi^2k^2M^2+6\pi|k|x+4\pi^2k^2\left(x^2-M^2\right)\right)}{8x^5}\right.$$
$$\left.-\frac{2\pi|k|e^{-2\pi|k|x}}{x^2}-\frac{e^{-2\pi|k|x}}{x^3}\right], \tag{2.211}$$

which gives:

$$\frac{\pi e^{-2\pi|k|M}}{(2\pi)^3|k|}p^2. \tag{2.212}$$

We sum this over $k\neq 0$ (k and $-k$ pair up) and get the corresponding kinetic term deformation in coordinate space:

$$-\frac{1}{2}\sum_{k=1}^\infty\frac{e^{-2\pi kM}}{(2\pi)^2k}\partial_\mu M(x)\partial^\mu M(x)=-\frac{1}{2}\sum_{k=1}^\infty\frac{e^{-2\pi kM}}{(2\pi)^2k}q_Iq_J\partial_\mu h^I\partial^\mu h^J. \tag{2.213}$$

Reconstructing $\mathcal{F}_0$

Now we aim to reconstruct the expression for $\mathcal{F}_0$ from the Kahler metric deformation we have computed. An important observation one should make first is that the one-loop quantum corrections also include contributions to the effective action that describe couplings of the vector multiplets scalars Z^I to the scalar curvature R. That is, effective supergravity emerges written in a non-Einstein frame. If we denote the corresponding one-loop contribution as $\frac{1}{2}\phi(Z,\overline{Z})R$, then the part of the Lagrangian density including also kinetic energy of scalars, written at the point with zero holonomies $\alpha^I = 0$, is:

$$\frac{1}{2}(1+\phi(Z,\overline{Z}))R + \frac{3}{2}\mathcal{C}_{IJK}h^I\partial_\mu h^J\partial^\mu h^K - \frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{-2\pi kM}}{(2\pi)^2 k}q_I q_J \partial_\mu h^I \partial^\mu h^J. \tag{2.214}$$

We could find this $\phi(Z,\overline{Z})$ by computing the two-point function of some scalar Z^I with the metric. This would require, similar to what we did before, a calculation of the two-point function of the mass term with the stress-energy tensor of the 5d hypermultiplet on the flat $\mathbb{R}^4\times S^1$ background. However, there is no need to do it as the structure of $N=2$ supergravity determines this function in terms of the quantities we have already calculated, as we will see soon.

We want to compare the deformed metric on scalars in (2.214) with the formulas from the Section 2.4.1, namely with the general expression for the Kahler metric (2.186) in the Einstein frame. However, since the action (2.214) is written in a non-Einstein frame, we have to rescale metric first, writing the action in the Einstein frame:

$$\frac{1}{2}R + \frac{3}{2}(1+\phi(Z,\overline{Z}))^{-1}\mathcal{C}_{IJK}h^I\partial_\mu h^J\partial^\mu h^K - \frac{1}{2}(1+\phi(Z,\overline{Z}))^{-1}\sum_{k=1}^{\infty}\frac{e^{-2\pi kM}}{(2\pi)^2 k}q_I q_J \partial_\mu h^I \partial^\mu h^J. \tag{2.215}$$

Keeping only the first order corrections, we find:

$$-\frac{3}{2}\phi(Z,\overline{Z})\mathcal{C}_{IJK}h^I\partial_\mu h^J\partial^\mu h^K - \frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{-2\pi kM}}{(2\pi)^2 k}q_I q_J \partial_\mu h^I \partial^\mu h^J, \tag{2.216}$$

which is the desired Kahler metric deformation. We want to compare it with the deformation of (2.186) under $\mathcal{F}_0 = \mathcal{F}_0^{\rm cl} + \mathcal{F}_0^{\rm q}$, where $\mathcal{F}_0^{\rm cl}$ is the classical prepotential

(2.82). Such a prepotential deformation results in $N_{\Lambda\Sigma} = N^{\rm cl}_{\Lambda\Sigma} + N^{\rm q}_{\Lambda\Sigma}$ and, through the gauge condition $N_{\Lambda\Sigma}X^\Lambda \overline{X}^\Sigma = -1$, in the deformation of the expression for X^0 in terms of other scalars. It is straightforward to find the first order correction of (2.186) at $\alpha^I = 0$ and $e^\sigma = 1$:

$$\frac{1}{4}N^{\rm q}_{IJ}\partial_\mu h^I \partial^\mu h^J + \frac{1}{4}(N^{\rm q}_{IJ}h^I h^J + N^{\rm q}_{00})\frac{3}{2}\mathcal{C}_{IJK}h^I\partial_\mu h^J \partial^\mu h^K. \tag{2.217}$$

Recalling the general expression for $\mathcal{F}^{\rm q}_0$ (2.188) deduced from shift symmetries, one can further write this as:

$$\begin{aligned}&-2\pi^2\partial_\mu M\partial^\mu M\sum_{k>0}k^2\,\mathrm{Im}\,(c_{k,0})e^{-2\pi kM}\\&+\left(2\pi M\sum_{k>0}k\,\mathrm{Im}\,(c_{k,0})e^{-2\pi kM}+\sum_{k>0}\mathrm{Im}\,(c_{k,0})e^{-2\pi kM}\right)\frac{3}{2}\mathcal{C}_{IJK}h^I\partial_\mu h^J\partial^\mu h^K,\end{aligned} \tag{2.218}$$

where we used $M = q_I h^I$. We now want to equate this to the result of the one-loop calculation given in (2.216). Also, it is useful to realize that at $\alpha^I = 0$, the function $\phi(Z, \overline{Z})$ is really a function $\phi(M)$ of $M = q_I h^I$ only, simply because it is a one-loop effect due to the particle of mass M. Equating (2.216) with (2.218) and slightly rearranging, we get:

$$\begin{aligned}&2\pi^2\partial_\mu M\partial^\mu M\sum_{k>0}k^2\left(\mathrm{Im}\,(c_{k,0})-\frac{1}{(2\pi)^4k^3}\right)e^{-2\pi kM}\\&=\left(\phi(M)+2\pi M\sum_{k>0}k\,\mathrm{Im}\,(c_{k,0})e^{-2\pi kM}+\sum_{k>0}\mathrm{Im}\,(c_{k,0})e^{-2\pi kM}\right)\frac{3}{2}\mathcal{C}_{IJK}h^I\partial_\mu h^J\partial^\mu h^K,\end{aligned} \tag{2.219}$$

which is the equation for the unknown coefficients $c_{k,0}$ and the unknown function $\phi(M)$. When written in such a way and if there are enough scalars h^I in the theory, one can show[19] that the only possible way to satisfy it is to set both sides to zero. Recalling that $c_{k,0}$ are imaginary due to parity, this gives:

$$
\begin{aligned}
c_{k,0} &= \frac{i}{(2\pi)^4 k^3}, \\
\phi(M) &= -\sum_{k>0} \frac{M}{(2\pi)^3 k^2} e^{-2\pi k M} - \sum_{k>0} \frac{1}{(2\pi)^4 k^3} e^{-2\pi k M}.
\end{aligned} \tag{2.220}
$$

With such values of $c_{k,0}$, we get:

$$
\mathcal{F}_0^{\mathrm{q}} = \frac{i}{(2\pi)^4} \left(X^0\right)^2 \sum_{k=1}^{\infty} \frac{1}{k^3} e^{2\pi i k q_I Z^I}, \tag{2.221}
$$

which agrees with the GV formula as claimed earlier in this thesis. It is now also obvious that for $\alpha^I \neq 0$, the expression for $\phi(Z, \overline{Z})$ is:

$$
\phi(Z, \overline{Z}) = -N^{\mathrm{q}}_{\Lambda\Sigma} X^\Lambda \overline{X}^\Sigma = -\frac{1}{4} e^{-3\sigma} N^{\mathrm{q}}_{\Lambda\Sigma} Z^\Lambda \overline{Z}^\Sigma. \tag{2.222}
$$

It would be an interesting consistency check to derive this $\phi(Z, \overline{Z})$ by computing the two-point function of the mass term with the stress-energy tensor in the 5d hypermultiplet theory.

---

[19] If there are enough scalars, one can find such a constant (i.e., independent of the space-time point) infinitesimal variation δh^I that $\mathcal{C}_{IJK} \delta h^I \partial_\mu h^J \partial^\mu h^K$ is non-zero, while $\delta M = q_I \delta h^I = 0$. Of course, constraint $\mathcal{C}_{IJK} h^I h^J h^K = 1$ defining the hypersurface $\mathcal{M}_h$ should be preserved too. Under such a variation in h^I, the equation (2.219) should be preserved. But since $\delta M = 0$, the only term whose variation is non-zero is $\mathcal{C}_{IJK} h^I \partial_\mu h^J \partial^\mu h^K$. Thus the expression in parenthesis by which it is multiplied should be zero for the equation to hold, which immediately implies (2.220). There are $b_2(Y)$ scalars $h^I, I = 1 \ldots b_2(Y)$, where $b_2(Y)$ is a second Betti number of Y. To have "enough scalars", we can take $b_2(Y) \geq 4$. To show this, put $\partial_\mu h^I = a_\mu \delta h^I$, i.e., assume that the gradient is parallel to the variation that we are seeking with some proportionality factor a_μ such that $a_\mu a^\mu \neq 0$ (we can obviously do that). The fact that $\mathcal{C}_{IJK} h^I h^J h^K = 1$ is preserved means that δh^I is tangent to $\mathcal{M}_h$. Also, as mentioned above, we have $q_I \delta h^I = 0$. Also, we want $\mathcal{C}_{IJK} \delta h^I \partial_\mu h^J \partial^\mu h^K = a_\mu a^\mu \mathcal{C}_{IJK} \delta h^I \delta h^J \delta h^K \neq 0$. When $b_2(Y) \geq 4$, the tangent space to $\mathcal{M}_h$ is at least three-dimensional, and $q_I \delta h^I = 0$ gives a subspace of dimension at least two. In such a space, we can clearly find such δh^I that a single condition $\mathcal{C}_{IJK} \delta h^I \delta h^J \delta h^K \neq 0$ is satisfied, and this is the variation we need, so $b_2(Y) \geq 4$ is enough. However, in Subsection 2.4.3 we will explain that the answer we get is valid for any $b_2(Y)$.

The case of arbitrary $b_2(Y)$

In the derivation of (2.220), we used the assumption that there are enough scalars, namely that $b_2(Y) \geq 4$, as explained in the footnote 19. However, there exist Calabi-Yau spaces with $b_2(Y) < 4$. For example, the quintic threefold has $b_2(Y) = 1$, which is the minimal possible value. In fact, the case of $b_2(Y) = 1$ seems to be even more problematic, because the 5d theory obtained by compactification on such a manifold has no vector multiplets and so no corresponding scalars. But our approach was to compute the Kahler metric for those scalars, so their existence was essential.

A possible way around is that the formula (2.221), describing the contribution of a single 5d hypermultiplet to $\mathcal{F}_0$, is universal and holds for any $b_2(Y)$. Once we know that the corresponding 5d BPS multiplet exists, this formula gives the answer irrespective of how big or small $b_2(Y)$ is. For $b_2(Y) \geq 4$, this already follows from our derivation, but for the cases of small $b_2(Y)$, one has to give a separate argument.

To do this, notice that we could set up a different computation of $\mathcal{F}_0$. Namely, we could use the kinetic energy of gauge fields. It has two good properties. One is that it is Weyl-invariant, so rescaling the metric into the Einstein frame would not affect the one-loop deformation of the kinetic term (unlike it was for scalars in (2.214)-(2.216)). Another is that the matrix of couplings $\mathcal{N}_{\Lambda\Sigma}$ defined in (2.50) does not depend on the dilatational gauge, i.e., on the expression for X^0, so that the gauge fields kinetic term deformation is directly related to $N^{\rm q}_{\Lambda\Sigma}$. So we could just compute the two-point function of 5d gauge fields (they exist for all $b_2(Y)$, unlike scalars), and get $\mathcal{F}_0^{\rm q}$ out of it directly. A disadvantage of such an approach is that it seems to be much more technically involved than what we have done here, and one would also need to know how to couple the minimal action (2.191) to gauge fields in a proper supersymmetric way. That is why we have chosen scalars for the computation. But such an alternative computation would clearly depend only on the properties of the

5d hypermultiplet, and not on $b_2(Y)$. Its existence establishes our claim that (2.221) provides the universal answer.

2.4.4 Computation of $\mathcal{F}_1$

In this subsection we consider the same light hypermultiplet as in (2.191), but here we determine its contribution to $\mathcal{F}_1$. The term $\mathcal{F}_1$ gives rise to a variety of interactions in the 4d effective action, and every one of them can potentially be used to set up a computation of $\mathcal{F}_1$. We find the following term:

$$(\mathrm{Im}\,\mathcal{F}_1)R^2 \equiv (\mathrm{Im}\,\mathcal{F}_1)R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho} \tag{2.223}$$

to be the most useful for this purpose. This term can be understood as a response to a small metric perturbation. Thus, it can be computed from the two-point function of the symmetric stress-energy tensor of the action (2.191). We consider a small metric perturbation around the flat 4d Minkowski background (the $\mathbb{R}^{3,1}$ part of $\mathbb{R}^{3,1} \times S^1$):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.224}$$

and we assume no metric perturbations in the circle direction. That is, the metric remains $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{d}y^2$. With the TT-gauge condition:

$$\begin{aligned} h^\mu_\mu &= 0 \\ \partial_\mu h^{\mu\nu} &= 0, \end{aligned} \tag{2.225}$$

we have: $R^2 = \partial_\lambda \partial_\sigma h^{\mu\nu} \partial^\lambda \partial^\sigma h_{\mu\nu} + O(h^3)$. So we will compute the following interaction:

$$8(\mathrm{Im}\,\mathcal{F}_1)h^{\mu\nu}(\partial^2)^2 h_{\mu\nu}. \tag{2.226}$$

If T_{MN} is the symmetric stress-energy tensor of (2.191), then for small perturbations $h_{\mu\nu}$ of the metric, the leading order contribution to $(\operatorname{Im}\mathcal{F}_1)R^2$ at one loop comes from:

$$\frac{1}{4}\int \mathrm{d}^5x\mathrm{d}^5y\langle T_{\mu\nu}(x)T_{\lambda\rho}(y)\rangle h^{\mu\nu}(x)h^{\lambda\rho}(y). \tag{2.227}$$

The useful relation to extract the one-loop answer $\mathcal{F}_1^{\mathrm{q}}$ is:

$$\int \mathrm{d}^5x_1\mathrm{d}^5x_2\langle T_{\mu\nu}(x_1)T_{\lambda\rho}(x_2)\rangle h^{\mu\nu}(x_1)h^{\lambda\rho}(x_2) = -64i\int \mathrm{d}^4x(\operatorname{Im}\mathcal{F}_1^{\mathrm{q}})h^{\mu\nu}(\partial^2)^2h_{\mu\nu}+\ldots, \tag{2.228}$$

where the ellipsis stands for terms with the wrong number of derivatives.

The two-point function computation

The symmetric stress-energy tensor is

$$T_{\mu\nu} = -2\sum_i \overline{z}^i\left(\overleftarrow{\partial}_{(\mu}\overrightarrow{\partial}_{\nu)}\right)z^i+\frac{1}{2}\overline{\Psi}^c\left(\gamma_{(\mu}\overrightarrow{\partial}_{\nu)}-\gamma_{(\mu}\overleftarrow{\partial}_{\nu)}\right)\Psi-\eta_{\mu\nu}\mathcal{L}. \tag{2.229}$$

Formula (2.228) implies that, due to the tracelessness of $h_{\mu\nu}$, the $\eta_{\mu\nu}\mathcal{L}$ term in the expression for $T_{\mu\nu}$ is unimportant.

Since the leading contribution to $R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}$ is proportional to $(\Box h)^2$, we need to find the $(p^2)^2$-order term of the $\langle T_{\mu\nu}T_{\lambda\rho}\rangle$ two-point function. We identify the contribution of bosons first:

$$8\times 2\pi\int_0^{2\pi}\mathrm{d}y\int\frac{\mathrm{d}^4p}{(2\pi)^4}h^{\mu\nu}(-p)h^{\lambda\rho}(p)\int\frac{d^4q}{(2\pi)^4}(q-p)_\mu q_\nu q_\lambda(q-p)_\rho G(q,y)G(q-p,-y). \tag{2.230}$$

Because of $\partial_\mu h^{\mu\nu}=0$, we have $p^\mu h_{\mu\nu}(p)=0$, and so the important part is:

$$8\times 2\pi\int_0^{2\pi}\mathrm{d}y\int\frac{\mathrm{d}^4p}{(2\pi)^4}h^{\mu\nu}(-p)h^{\lambda\rho}(p)\int\frac{d^4q}{(2\pi)^4}q_\mu q_\nu q_\lambda q_\rho G(q,y)G(q-p,-y). \tag{2.231}$$

The contribution of fermions is:

$$\frac{1}{4}\int \frac{\mathrm{d}^4p}{(2\pi)^4}h^{\mu\nu}(-p)h^{\lambda\rho}(p)\int \frac{d^4q}{(2\pi)^4}$$
$$\times \mathrm{Tr}\left\{(\gamma_{(\mu}q_{\nu)} - \gamma_{(\mu}(p-q)_{\nu)})D(q,y)(\gamma_{(\lambda}(q-p)_{\rho)} + \gamma_{(\lambda}q_{\rho)})D(q-p,-y)\right\}, \qquad (2.232)$$

And for the same reason, $p^\mu h_{\mu\nu} = 0$, the relevant part is:

$$\int \frac{\mathrm{d}^4p}{(2\pi)^4}h^{\mu\nu}(-p)h^{\lambda\rho}(p)\int \frac{d^4q}{(2\pi)^4}\mathrm{Tr}\left\{\gamma_\mu q_\nu D(q,y)\gamma_\lambda q_\rho D(q-p,-y)\right\}. \qquad (2.233)$$

So, we have to find the $(p^2)^2$-term of this expression:

$$2\pi\int_0^{2\pi}\mathrm{d}y\int\frac{\mathrm{d}^4q}{(2\pi)^4}\left[8q_\mu q_\nu q_\lambda q_\rho G(q,y)G(q-p,-y) + \mathrm{Tr}\left\{\gamma_{(\mu}q_{\nu)}D(q,y)\gamma_{(\lambda}q_{\rho)}D(q-p,-y)\right\}\right]. \qquad (2.234)$$

The following steps are as in the $\mathcal{F}_0$ case. We have:

$$2\pi\sum_{k_1,k_2}\int_0^{2\pi}\mathrm{d}y\int\frac{\mathrm{d}^4q}{(2\pi)^4}\Big[8q_\mu q_\nu q_\lambda q_\rho G_0(q,y-2\pi k_1)G_0(q-p,-y-2\pi k_2)$$
$$+\mathrm{Tr}\left\{\gamma_{(\mu}q_{\nu)}D_0(q,y-2\pi k_1)\gamma_{(\lambda}q_{\rho)}D_0(q-p,-y-2\pi k_2)\right\}\Big], \qquad (2.235)$$

and for given $k_1 + k_2 = k$, we get:

$$2\pi\sum_{k_1+k_2=k}\int_0^{2\pi}\mathrm{d}y\int\frac{\mathrm{d}^4q}{(2\pi)^4}\Bigg[\frac{2q_\mu q_\nu q_\lambda q_\rho}{\sqrt{q^2+M^2}\sqrt{(q-p)^2+M^2}}$$
$$+\frac{M^2 q_{(\nu}g_{\mu)(\lambda}q_{\rho)}}{\sqrt{q^2+M^2}\sqrt{(q-p)^2+M^2}} - \frac{q_\mu q_\nu (q-p)_{(\lambda}q_{\rho)} + (q-p)_{(\mu}q_{\nu)}q_\lambda q_\rho - q(q-p)q_{(\nu}g_{\mu)(\lambda}q_{\rho)}}{\sqrt{q^2+M^2}\sqrt{(q-p)^2+M^2}}$$
$$+ q_{(\nu}g_{\mu)(\lambda}q_{\rho)}\mathrm{Sign}(y-2\pi k_1)\mathrm{Sign}(y+2\pi k_2)\Bigg]e^{-|y-2\pi k_1|\sqrt{q^2+M^2}-|y+2\pi k_2|\sqrt{(q-p)^2+M^2}} \qquad (2.236)$$

We throw away terms proportional to p_μ, p_ν, p_λ or p_ρ, and get:

$$2\pi \sum_{k_1+k_2=k} \int_0^{2\pi} \mathrm{d}y \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Bigg[\frac{(M^2 + q(q-p))q_{(\nu} g_{\mu)(\lambda} q_{\rho)}}{\sqrt{q^2+M^2}\sqrt{(q-p)^2+M^2}} + q_{(\nu} g_{\mu)(\lambda} q_{\rho)} \mathrm{Sign}(y - 2\pi k_1) \mathrm{Sign}(y + 2\pi k_2) \Bigg] e^{-|y-2\pi k_1|\sqrt{q^2+M^2} - |y+2\pi k_2|\sqrt{(q-p)^2+M^2}}. \tag{2.237}$$

Computing the sums and integrating over y using the formulas (2.204) and (2.205), we find:

$$2\pi \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \Bigg[\frac{(M^2 + q(q-p))q_{(\nu} g_{\mu)(\lambda} q_{\rho)}}{\sqrt{q^2+M^2}\sqrt{(q-p)^2+M^2}} \times \left(\frac{e^{-2\pi|k|\sqrt{(q-p)^2+M^2}} - e^{-2\pi|k|\sqrt{q^2+M^2}}}{\sqrt{q^2+M^2} - \sqrt{(q-p)^2+M^2}} + \frac{e^{-2\pi|k|\sqrt{q^2+M^2}} + e^{-2\pi|k|\sqrt{(q-p)^2+M^2}}}{\sqrt{q^2+M^2} + \sqrt{(q-p)^2+M^2}} \right) + q_{(\nu} g_{\mu)(\lambda} q_{\rho)} \left(- \frac{e^{-2\pi|k|\sqrt{(q-p)^2+M^2}} - e^{-2\pi|k|\sqrt{q^2+M^2}}}{\sqrt{q^2+M^2} - \sqrt{(q-p)^2+M^2}} + \frac{e^{-2\pi|k|\sqrt{q^2+M^2}} + e^{-2\pi|k|\sqrt{(q-p)^2+M^2}}}{\sqrt{q^2+M^2} + \sqrt{(q-p)^2+M^2}} \right) \Bigg]. \tag{2.238}$$

Now we have to Taylor expand this to get an $O(p^4)$-order contribution. We then integrate over $\mathrm{d}^4 q$ at that order. We have to do the same tricks with Wick rotation and replacing products of q_μ by symmetric combinations of $\eta_{\mu\nu}$:

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \left[f_1(q^2) q_\mu q_\rho (p^2)^2 + f_2(q^2) q_\mu q_\rho p^2 (qp)^2 + f_3(q^2) q_\mu q_\rho (qp)^4 \right] \to -i \int \frac{\mathrm{d}^4 q_E}{(2\pi)^4} \left[f_1(q_E^2) \frac{q^2}{4} \eta_{\mu\rho} (p^2)^2 + f_2(q_E^2) \frac{q^4}{24} \eta_{\mu\rho} (p^2)^2 + f_3(q_E^2) \frac{q^6}{64} \eta_{\mu\rho} (p^2)^2 \right] + \dots \tag{2.239}$$

where the ellipsis represents terms that vanish upon contractions with $h^{\mu\nu} h^{\lambda\rho}$.

So, after Taylor expansion, we get:

$$
\begin{aligned}
-i\frac{(p^2)^2}{4\pi}\int_0^\infty q^3\mathrm{d}q\Big[&\frac{3e^{-2|k|\pi\sqrt{M^2+q^2}}M^4q^2}{16\left(M^2+q^2\right)^{9/2}}+\frac{e^{-2|k|\pi\sqrt{M^2+q^2}}k^2M^6\pi^2q^2}{4\left(M^2+q^2\right)^{9/2}}+\frac{e^{-2|k|\pi\sqrt{M^2+q^2}}M^2q^4}{6\left(M^2+q^2\right)^{9/2}}\\
&+\frac{5e^{-2|k|\pi\sqrt{M^2+q^2}}k^2M^4\pi^2q^4}{12\left(M^2+q^2\right)^{9/2}}+\frac{73e^{-2|k|\pi\sqrt{M^2+q^2}}q^6}{1536\left(M^2+q^2\right)^{9/2}}+\frac{77e^{-2|k|\pi\sqrt{M^2+q^2}}k^2M^2\pi^2q^6}{384\left(M^2+q^2\right)^{9/2}}\\
&+\frac{e^{-2|k|\pi\sqrt{M^2+q^2}}k^4M^4\pi^4q^6}{96\left(M^2+q^2\right)^{9/2}}+\frac{13e^{-2|k|\pi\sqrt{M^2+q^2}}k^2\pi^2q^8}{384\left(M^2+q^2\right)^{9/2}}+\frac{e^{-2|k|\pi\sqrt{M^2+q^2}}k^4M^2\pi^4q^8}{48\left(M^2+q^2\right)^{9/2}}\\
&+\frac{e^{-2|k|\pi\sqrt{M^2+q^2}}k^4\pi^4q^{10}}{96\left(M^2+q^2\right)^{9/2}}+\frac{3e^{-2|k|\pi\sqrt{M^2+q^2}}|k|M^4\pi q^2}{8\left(M^2+q^2\right)^{4}}+\frac{e^{-2|k|\pi\sqrt{M^2+q^2}}|k|M^2\pi q^4}{3\left(M^2+q^2\right)^{4}}\\
&-\frac{e^{-2|k|\pi\sqrt{M^2+q^2}}|k|^3M^4\pi^3q^4}{9\left(M^2+q^2\right)^{4}}+\frac{73e^{-2|k|\pi\sqrt{M^2+q^2}}|k|\pi q^6}{768\left(M^2+q^2\right)^{4}}-\frac{49e^{-2|k|\pi\sqrt{M^2+q^2}}|k|^3M^2\pi^3q^6}{288\left(M^2+q^2\right)^{4}}\\
&-\frac{17e^{-2|k|\pi\sqrt{M^2+q^2}}|k|^3\pi^3q^8}{288\left(M^2+q^2\right)^{4}}\Big].
\end{aligned}
\tag{2.240}
$$

Doing the same change of variables $x=\sqrt{M^2+q^2}$ as before and integrating, we get:

$$
-i\frac{(p^2)^2}{4\pi}\frac{e^{-2\pi|k|M}}{24\pi|k|}. \tag{2.241}
$$

Summing over $k\neq 0$ and using (2.228), we obtain:

$$
\operatorname{Im}\mathcal{F}_1^{\mathrm{q}}=\frac{1}{16\pi^2}\sum_{k=1}^{\infty}\frac{e^{-2\pi kM}}{64\times 3k}, \tag{2.242}
$$

so, using the fact that $\mathcal{F}_1$ is imaginary at $\alpha^I=0$ and then extending by holomorphy:

$$
\mathcal{F}_1^{\mathrm{q}}=\frac{1}{16\pi^2}\sum_{k=1}^{\infty}\frac{i}{64\times 3k}e^{2\pi ikq_IZ^I}. \tag{2.243}
$$

This is again compatible with our previous discussion.

A word about $k = 0$**.** Just as in the $\mathcal{F}_0$ case, the integral (2.240) is convergent only for $k \neq 0$. The $k = 0$ part is again interpreted as a term in the effective action in 5d. And this term then can or cannot contribute to $\mathcal{F}_1$ by the classical dimensional reduction. Before we argued that the only possible contribution to $\mathcal{F}_1$ from the classical dimensional reduction is of the form $c_{I,2}Z^I$ with real constants $c_{I,2}$. So the only remaining question one could ask here is whether the $k = 0$ part of the one-loop answer could contribute by shifting the values of these $c_{I,2}$.

The real part of $\mathcal{F}_1 \propto c_{I,2}Z^I$ enters the 4d interaction $\int c_{I,2}\alpha^I \mathrm{Tr}(R \wedge R)$, which comes from a Chern-Simons interaction in 5d of the form $\int c_{I,2}V^I \wedge \mathrm{Tr}(R \wedge R)$. The imaginary part of $\mathcal{F}_1$ corresponds to the 4d interaction $\int \sqrt{g}\mathrm{d}^4x\, c_{I,2}h^I R^2$, which apparently lifts to the 5d interaction of the form $\propto \int \sqrt{G}\mathrm{d}^5x\, c_{I,2}h^I R^2$. While the meaning of the latter term is not entirely clear, the 5d Chern-Simons term was discussed before. As explained in Section 2.1.2, it can be lifted even further, to the 11d action. Its 11d origin is an interaction $\frac{1}{(2\pi)^4}\int C \wedge \left[\frac{1}{768}(\mathrm{Tr}R^2)^2 - \frac{1}{192}\mathrm{Tr}R^4\right]$ (where the powers of R are with respect to the wedge product). This interaction was discovered in [89] due to its role in the anomaly cancelation in M-theory. This suggests that $c_{I,2}$ cannot be shifted. Another evidence that quantum corrections cannot shift $c_{I,2}$ appears if we turn on holonomies α^I. We know that they appear in a diagram computation only through the factors $e^{2\pi i k q_I \alpha^I}$, which means that the term $\int c_{I,2}\alpha^I \mathrm{Tr}(R \wedge R)$ (which has to be generated at $\alpha^I \neq 0$ background) cannot be shifted. Thus $c_{I,2}$ is not actually shifted by the $k = 0$ part of the one-loop answer, and it is enough to consider only $k \neq 0$ terms.

2.4.5 Some further remarks

We have computed the contribution of a single light hypermultiplet to $\mathcal{F}_0$ and $\mathcal{F}_1$. As was explained earlier, to get a contribution from all of the massless multiplets in the theory (that is, hypermultiplets, vector multiplets and the gravity multiplet), one just has to multiply the contribution of a massless hypermultiplet by $-\chi(Y)/2$, where $\chi(Y)$ is the Euler characteristic of the Calabi-Yau Y. The massless hypermultiplet contribution is a massless limit of what we have computed here.

Note that the superparticle description, which was advocated in Section 2.2 (and which is a perfect choice for massive BPS multiplets), does not have a sensible massless limit, even though in the answer one can formally take mass to zero. That is why the field theoretic description was essential for the complete picture. For $\mathfrak{g} \geq 2$, the field-theoretic computation of $\mathcal{F}_{\mathfrak{g}}$ was described in Section 2.3 of this thesis. The field-theoretic computation of $\mathcal{F}_0$ and $\mathcal{F}_1$ was presented in the current Section.

Finally, we note that the results of the one-loop computation are actually exact. This one-loop exactness follows in the usual way from holomorphy. If we go beyond quadratic order in the action and thus consider higher-loop corrections, they will be multiplied by extra powers of the mass $M = q_I h^I$, which will not be balanced by extra powers of holonomies $q_I \alpha^I$. This will violate holomorphy and therefore correspond to D-terms rather than F-terms.

2.5 The Ooguri-Vafa formula

In this section we review the derivation of the Ooguri-Vafa formula. A qualitative discussion was given in the Introduction, so here we will only discuss the central charges appearing in the OV formula and describe the computation. More details can be found in [63].

2.5.1 Central Charges

BPS states that contribute to the OV formula come in the UV from M2-branes (wrapping oriented curves) that end on the M5-brane supported on $\mathbb{R}^3 \times L$. Such M2-branes wrap cycles in the second relative homology $H_2(Y, L; \mathbb{Z})$. Just like in the

GV case $H_2(Y;\mathbb{Z})$ was the lattice of charges of possible BPS states, in the OV case, $H_2(Y,L;\mathbb{Z})$ is the lattice of corresponding charges. In the GV case, in order to write the action of a BPS particle, we needed the Kahler moduli $v^1,\ldots,v^{b_2}$ of M-theory on a Calabi-Yau manifold Y, where $b_2 = b_2(Y)$ is the second Betti number of Y, as well as the corresponding holonomies α^I around the M-theory circle. In the OV case, we need to add moduli w^ρ, $\rho = 1,\ldots,b_1(L)$ of the special Lagrangian submanifold L to the story, as well as holonomies β^ρ of the gauge fields living on the M5-brane. Let us describe all these objects in more detail.

An oriented two-dimensional surface $\Sigma \subset Y$ that may have a boundary on L has a homology class in the relative homology group $H_2(Y,L;\mathbb{Z})$. This group is related to $H_2(Y;\mathbb{Z})$ by an exact sequence that reads in part

$$\ldots H_2(L;\mathbb{Z}) \stackrel{\alpha}{\to} H_2(Y;\mathbb{Z}) \stackrel{\beta}{\to} H_2(Y,L;\mathbb{Z}) \stackrel{\gamma}{\to} H_1(L;\mathbb{Z}) \stackrel{\alpha'}{\to} H_1(Y;\mathbb{Z})\ldots \qquad (2.244)$$

The maps α and α' take a cycle in L and map it to Y using the embedding $L \subset Y$. The map β is defined using the fact that a cycle $\Sigma \subset Y$ that has no boundary is a special case of a cycle whose boundary is on L. And the map γ maps a cycle $\Sigma \subset Y$ whose boundary is in L to its boundary $\partial\Sigma \subset L$.

The physical meaning of the map α is as follows. Once we introduce an M5-brane, since an M2-brane can end on an M5-brane, some M2-brane charges might not be conserved any more. If $\Sigma \subset Y$ is homologous to a cycle in L, then an M2-brane wrapped on Σ can annihilate and disappear. Hence for the purpose of the OV formula, $H_2(Y;\mathbb{Z})$ should be replaced by the quotient $H_2(Y;\mathbb{Z})/\alpha(H_2(L;\mathbb{Z}))$, which parametrizes charges that are carried by M2-branes without boundary and are conserved in the presence of an M5-brane wrapped on $\mathbb{R}^3 \times L$. To keep our terminology familiar, we will assume in what follows that $\alpha = 0$. Otherwise, in all statements one replaces $H_2(Y;\mathbb{Z})$ by $H_2(Y;\mathbb{Z})/\alpha(H_2(L;\mathbb{Z}))$ and replaces $b_2(Y)$ by the rank of that group, which we might call $b_2' = b_2'(Y, L)$.

The interpretation of the map α' is as follows. If a 1-cycle in L is the boundary $\partial\Sigma$ of some $\Sigma \subset Y$ that represents a class in $H_2(Y, L;\mathbb{Z})$, this means by definition that $\partial\Sigma$, when embedded in Y, is a boundary (of Σ) and so vanishes in $H_1(Y;\mathbb{Z})$. So the image of $H_2(Y, L;\mathbb{Z})$ under γ is not all of $H_1(L;\mathbb{Z})$, but only the kernel of α'.

In any event, for a Calabi-Yau manifold Y, $H_1(Y;\mathbb{Z})$ is always a finite group. This means that we can set $\alpha' = 0$ if there is no torsion or we reduce modulo torsion. We ignore torsion in this thesis and consider only $\mathbb{Z}$-valued charges. At the end of this chapter we will comment shortly on the case with torsion and refer interested readers to [63]. With also α assumed to vanish, the long exact sequence (2.244) reduces to a short exact sequence

$$0 \to H_2(Y;\mathbb{Z}) \xrightarrow{\beta} H_2(Y, L;\mathbb{Z}) \xrightarrow{\gamma} H_1(L;\mathbb{Z}) \to 0. \tag{2.245}$$

This implies that the rank of $H_2(Y, L;\mathbb{Z})$ is the sum $b_2(Y) + b_1(L)$. That number (or $b_2' + b_1(L)$ if $\alpha \neq 0$) is the total number of $\mathbb{Z}$-valued charges of a BPS state in this situation.

By mapping the $H_2(Y, L;\mathbb{Z})$-valued charge of an M2-brane with boundary on L to $H_1(L;\mathbb{Z})$ via γ, we learn that such an M2-brane has a charge valued in $H_1(L;\mathbb{Z})$, or in other words that it carries $\mathbb{Z}$-valued charges $r_1, \ldots, r_{b_1}$ that are determined by its boundary. Concretely, these charges are dual to oriented circles $\ell^\rho \subset L$ that provide a basis of $H_1(L,\mathbb{Z})$ (modulo possible torsion). An M2-brane wrapped on Σ has charges r_ρ, $\rho = 1, \ldots, b_1(L)$ if its boundary $\partial\Sigma$ is homologous in $H_1(L;\mathbb{Z})$ to $\sum_\rho r_\rho \ell^\rho$.

Since there is no natural map from $H_2(Y, L;\mathbb{Z})$ to $H_2(Y;\mathbb{Z})$, there is no equally natural definition of the "bulk" charges of an M2-brane that is allowed to end on L. However, modulo torsion, we can always pick a splitting of the exact sequence

(2.245), and this enables us to define the bulk charges $q_1, \dots, q_{b_2}$. We will pick a fixed splitting in what follows, though one could proceed more intrinsically. If we use a different splitting, the q_I are shifted by integer linear combinations of the r_ρ.

A way to fix splitting, using the Splitting Lemma, is by providing a left inverse for β or a right inverse for γ. We chose to pick a right inverse for γ, which is a map $\delta : H_1(L;\mathbb{Z}) \to H_2(Y, L;\mathbb{Z})$ such that $\gamma \circ \delta = \text{id}$. We define this map by saying that for every oriented circle $\ell^\rho \subset L$ from the basis of $H_1(L;\mathbb{Z})$, we choose a surface $\sigma^\rho \subset Y$ to which it is mapped, such that $\partial\sigma^\rho = \ell^\rho$ (we assume that $H_1(Y;\mathbb{Z})$ is trivial). Then we say that an M2-brane wrapped on Σ has charges $q_1, \dots, q_{b_2(Y)}, r_1, \dots, r_{b_1(L)}$ if Σ is homologous in $H_2(Y, L;\mathbb{Z})$ to $\sum_\rho r_\rho \sigma^\rho + \sum_I q_I \beta(\omega^I)$, where ω^I is a basis of $H_2(Y;\mathbb{Z})$ dual to the basis of $H^2(Y;\mathbb{Z})$ we used before.

In M-theory, if L is compact, then just like the charges q_I that entered the GV formula, the new charges r_ρ also couple to abelian gauge fields. These are abelian gauge fields that only propagate along the support $\mathbb{R}^3 \times L$ of the M5-brane, so that macroscopically, they propagate along $\mathbb{R}^3 \subset \mathbb{R}^5$. These abelian gauge fields have a simple microscopic origin. Along the world-volume of an M5-brane, there propagates a two-form field (whose curvature is constrained to be selfdual). When we compactify the M5-brane on $\mathbb{R}^3 \times L$, the Kaluza-Klein expansion of the two-form field gives $b_1(L)$ abelian gauge fields on $\mathbb{R}^3$. As we have discussed in the introduction, states that are charged with respect to these gauge fields are actually confined. The derivation and interpretation of the OV formula are more straightforward if L does not admit any square-integrable harmonic 1-forms, either because L is compact with $b_1(L) = 0$ or because L is not compact and its geometry and topology ensure that harmonic 1-forms on L are not square-integrable. In this case, the symmetries associated to the moduli of L behave as global symmetries and the r_ρ are global charges that can contribute to the central charge of a BPS state.

With our choice of splitting provided by the right inverse of γ, we can also define a local coordinate system on the moduli space of special Lagrangian submanifold $\mathcal{M}(L)$. If ω is a Kahler form of Y, we define coordinates w^ρ on $\mathcal{M}(L)$ as:

$$w^\rho = \int_{\sigma^\rho} \omega. \tag{2.246}$$

Once we have picked the set of relative homology classes $[\sigma^1], \ldots, [\sigma^{b_1(L)}]$, these w^ρ are well-defined real numbers. To check this, suppose that we replace σ^ρ by some $\widetilde{\sigma}^\rho$ homologous to σ^ρ in $H_2(Y, L; \mathbb{Z})$. Consider the 2-chain $\sigma^\rho - \widetilde{\sigma}^\rho$. It represents a trivial class in $H_2(Y, L; \mathbb{Z})$, therefore the boundary $\partial\sigma^\rho - \partial\widetilde{\sigma}^\rho$ is a trivial class in $H_1(L; \mathbb{Z})$. This means that there exists a two-chain $c \subset L$, such that $\partial c = \partial\sigma^\rho - \partial\widetilde{\sigma}^\rho$. Since L is Lagrangian, $\omega|_L = 0$, so $\int_c \omega = 0$. Therefore we can write:

$$\int_{\sigma^\rho} \omega - \int_{\widetilde{\sigma}^\rho} \omega = \int_{\sigma^\rho} \omega - \int_{\widetilde{\sigma}^\rho} \omega - \int_c \omega = \int_{\sigma^\rho - \widetilde{\sigma}^\rho - c} \omega. \tag{2.247}$$

Now, by the definition of c, the 2-chain $\sigma^\rho - \widetilde{\sigma}^\rho - c$ is closed, moreover, since σ^ρ and $\widetilde{\sigma}^\rho$ are homologous in $H_2(Y, L; \mathbb{Z})$ and c is homologous to zero in relative homology, $\sigma^\rho - \widetilde{\sigma}^\rho - c$ is actually a boundary. Therefore, the above integral vanishes and we prove:

$$\int_{\sigma^\rho} \omega = \int_{\widetilde{\sigma}^\rho} \omega. \tag{2.248}$$

So w^ρ are well-defined, they do not change as we vary the representative σ^ρ. However, they change if we vary L itself – that is why they provide a coordinate system on $\mathcal{M}(L)$. Using a more formal theory of [99], it is not hard to prove that this is indeed a good coordinate system on $\mathcal{M}(L)$.

The area of a holomorphic curve $\Sigma \subset Y$ whose boundary is on L is determined by its homology class or in other words by the charges $q_1, \ldots, q_{b_2}$ and $r_1, \ldots, r_{b_1}$. This area in M-theory units is

$$A = \sum_I q_I v^I + \sum_\rho r_\rho w^\rho, \tag{2.249}$$

where v^I are the Kahler moduli of Y and w^ρ, $\rho = 1, \dots, b_1(L)$, are the moduli of L we have just defined. (If one changes the splitting that was used to define the q_I, then the w^ρ are shifted by integer linear combinations of the v^I.) To find the mass of an M2-brane wrapped on Σ measured in the 5d Einstein frame, we make a Weyl transformation to 5d variables

$$h^I = \frac{v^I}{v}, \quad k^\rho = \frac{w^\rho}{v}, \tag{2.250}$$

as in eqn. (2.33). Generalizing eqn. (2.74), the mass of a BPS particle with charges $\vec{q}, \vec{r}$ (in units in which the M2-brane tension is 1) is then

$$m(\vec{q}, \vec{r}) = \sum_I q_I h^I + \sum_\rho r_\rho k^\rho. \tag{2.251}$$

Assuming that Σ has a non-empty boundary on L, and that L is suitably noncompact, this particle propagates on $\mathbb{R}^3$ and is a BPS particle in a 3d theory that has $\mathcal{N} = 2$ supersymmetry (four superchanges). The 3d $\mathcal{N} = 2$ supersymmetry algebra has a real central charge ζ that equals the mass of a BPS particle, so it is given by the formula (2.251):

$$\zeta(\vec{q}, \vec{r}) = \sum_I q_I h^I + \sum_\rho r_\rho k^\rho. \tag{2.252}$$

Now let us compactify from $\mathbb{R}^5 \times Y$ to $\mathbb{R}^4 \times S^1 \times Y$, so that the M5-brane worldvolume becomes $\mathbb{R}^2 \times S^1 \times L$. As usual, we suppose that the S^1 has circumference $2\pi e^\sigma$. The real part of the action of a BPS particle of mass $m(\vec{q}, \vec{r})$ propagating around the S^1 is then $2\pi e^\sigma m(\vec{q}, \vec{r}) = 2\pi e^\sigma (\sum_I q_I h^I + \sum_\rho r_\rho k^\rho)$. However, just as in the derivation of the GV formula, the action also has an imaginary part that arises from the fact that when we compactify on a circle, the abelian gauge fields may have holonomies around the circle. As in our study of the GV formula, we write $\exp(2\pi i\alpha^I)$, $I = 1, \ldots, b_2(Y)$, for the holonomies of the gauge fields that arise from M-theory compactification on Y, and we similarly write $\exp(2\pi i\beta^\rho)$, $\rho = 1, \ldots, b_1(L)$ for the holonomies of the gauge fields that live on the M5-brane. (Again the definition of the β^ρ depends on a choice of splitting; in a change of splitting, they are shifted by integer linear combinations of the α^I.) Then a particle of charges $\vec{q}, \vec{r}$ propagating around the circle acquires a phase $\exp\left(2\pi i(\sum_I q_I\alpha^I + \sum_\rho r_\rho\beta^\rho)\right)$. As in eqn (2.71), we can interpret this to mean that the action for such a particle is

$$S(\vec{q}, \vec{r}) = 2\pi\left(\sum_I q_I(e^\sigma h^I - i\alpha^I) + \sum_\rho r_\rho(e^\sigma k^\rho - i\beta^\rho)\right) = -2\pi i\left(\sum_I q_I Z^I + \sum_\rho r_\rho U^\rho\right), \tag{2.253}$$

where Z^I was defined in eqn. (2.72) and similarly

$$U^\rho = \beta^\rho + ie^\sigma k^\rho. \tag{2.254}$$

Actually, Z^I and U^ρ are the bottom components of 2d chiral superfields $\mathcal{Z}^I$ and $\mathcal{U}^\rho$. As in the derivation of the GV formula, a BPS particle propagating around the circle in $\mathbb{R}^2 \times S^1$ has bosonic collective coordinates (its position along $\mathbb{R}^2$) and also fermionic collective coordinates. To take account of the fermionic collective coordinates, it is better to write the action as a superfield

$$\mathcal{S}(\vec{q}, \vec{r}) = -2\pi i\left(\sum_I q_I\mathcal{Z}^I + \sum_\rho r_\rho\mathcal{U}^\rho\right). \tag{2.255}$$

As in the derivation of the GV formula, the contribution of a BPS particle to the OV formula is given by $\exp(-\mathcal{S})$ multiplied by a product of one-loop determinants.

2.5.2 The Computation

The computation that we have to perform is not essentially new; in a sense, it is just the square root of the very simple computation that was already described in section 2.2.2. It is the interpretation that involves some difficulty.

We will only perform the particle computation. Even though, strictly speaking, the field theory computation is required to find the contribution of massless BPS states, we know from our experience with the GV formula that the massive answer has a well-defined zero mass limit.

As with the GV formula, we consider first a BPS superparticle that has only the two bosonic and two fermionic zero-modes that follow from supersymmetry. The basic example is an M2-brane wrapped on a holomorphic disc $\Sigma \subset Y$ whose boundary is on L. If Σ has no infinitesimal deformations, then the 3d BPS superparticle obtained by wrapping an M2-brane on Σ has only the minimal set of zero-modes. Ooguri and Vafa [66] give a useful example[20] in which L is topologically $S^1 \times \mathbb{R}^2$. The S^1 is the "equator" in a holomorphically embedded $\mathbb{CP}^1 \subset Y$, and taking Σ to be the upper or lower hemisphere of this $\mathbb{CP}^1$, one gets an example with only those bosonic or fermionic zero-modes that follow from the symmetries.

In our problem, this gives a superparticle that propagates on a two-plane $\mathbb{R}^2 \subset \mathbb{R}^4$. We take this to be the two-plane $x^3 = x^4 = 0$, parametrized by x^1 and x^2. The action that describes such a superparticle in the nonrelativistic limit is a simple truncation to $x^3 = x^4 = 0$ of the one that we used in deriving the GV formula. A way to make this truncation is to introduce the reflection $\boldsymbol{R}$ that acts by $x^3 \to -x^3$, $x^4 \to -x^4$ while leaving x^1, x^2 fixed. This reflection is a symmetry of the bosonic part of the action if the graviphoton field is $\boldsymbol{R}$-invariant, which is actually the case for the choice

[20] In their example, Y is the small resolution of the conifold, which can be described via a linear σ-model with gauge group $U(1)$ and chiral superfields u_1, u_2 of charge 1 and v_1, v_2 of charge -1. Thus Y is the quotient by $U(1)$ of the space $|u_1|^2 + |u_2|^2 - |v_1|^2 - |v_2|^2 = 1$. An embedding $\mathbb{CP}^1 \subset Y$ is defined by $v_1 = v_2 = 0$. L is defined by taking all u_i and v_j to be real. The symmetry $\boldsymbol{U}$ that is introduced below acts on Y by $v_2 \to -v_2$, leaving fixed u_1, u_2, and v_1.

that was already made in eqn. (2.121). We will extend $\boldsymbol{R}$ to a symmetry of the full action including the fermions, and then the $\boldsymbol{R}$-invariant part of eqn. (2.106) will serve as our superparticle action. The extension of $\boldsymbol{R}$ to the fermions is not completely trivial since on spinors $\boldsymbol{R}$ acts as $\sigma_{34} = \gamma_3\gamma_4$, and its square is -1, not $+1$. To get an operation that squares to $+1$, we have to combine the matrix σ_{34} acting on the spinor index A of a fermion field ψ_{Ai} with a matrix that acts on the additional index[21] i and also squares to -1. Let us call the combined operation $\boldsymbol{R}'$. Then the $\boldsymbol{R}'$-invariant part of the action (2.106) is the basic superparticle action relevant to the OV formula. It possesses the $\boldsymbol{R}'$-invariant part of the supersymmetry algebra of the action (2.106), and this is the appropriate symmetry for our problem.

To perform the path-integral for this problem, we simply proceed as follows. We have projected out half of the collective coordinates from (2.106), so the zero-mode measure will be $\mathrm{d}^2x\mathrm{d}^2\psi^{(0)}$ rather than $\mathrm{d}^4x\mathrm{d}^4\psi^{(0)}$. Also, we have to compute a bosonic determinant in just one of the 2×2 blocks in eqn. (2.121). This means that the one-loop path integral just gives, in a fairly obvious sense, the square root of the result in eqn. (2.124). Finally, in the classical action, we have to include the charges and moduli associated to the D4-brane and so replace $\sum_I q_I\mathcal{Z}^I$ with $\sum_I q_I\mathcal{Z}^I + \sum_\rho r_\rho\mathcal{U}^\rho$. Putting these statements together, the result for a BPS superparticle winding once around the circle is

$$\frac{\mathrm{d}^2x\mathrm{d}^2\psi^{(0)}}{2\pi}\exp\left(2\pi i\left(\sum_I q_I\mathcal{Z}^I + \sum_\rho r_\rho\mathcal{U}^\rho\right)\right)\frac{\mathsf{T}}{\sinh(\pi e^\sigma\mathsf{T})}. \tag{2.256}$$

To go from this formula to a Type IIA effective action, we follow much the same steps that were used to go from eqn. (2.125) to eqn. (2.126). We write $\mathsf{T} = \frac{e^{\sigma/2}}{4}\mathsf{W}_\parallel$, and interpret $\mathsf{W}_\parallel$ as the bottom component of a chiral superfield $\mathcal{W}_\parallel$. We also write

[21] Microscopically, the index $i = 1, 2$ distinguishes supersymmetries that originate from left-movers on the string worldsheet to those that originate from right-movers. Interaction with a D4-brane preserves a linear combination of the two types of symmetries. Which linear combination it is depends on the eigenvalue of σ_{34}, and that is why a reflection must be defined to act on the i index.

$\mathrm{d}^2x\mathrm{d}^2\psi^{(0)} = \frac{1}{2}\mathrm{d}^2x\mathrm{d}^2\theta\sqrt{g^E}e^{-\sigma/2}$ (where now g^E is the Einstein metric restricted to the brane) and use the usual formula $e^{3\sigma/2} = -i/2\mathcal{X}^0$. The resulting contribution to the effective action is

$$i\int\frac{\mathrm{d}^2x\mathrm{d}^2\theta}{(2\pi)^2}\sqrt{g^E}\exp\left(2\pi i\left(\sum_I q_I\mathcal{Z}^I + \sum_\rho r_\rho\mathcal{U}^\rho\right)\right)\frac{\pi\mathcal{W}_\|/8}{\sin(\pi\mathcal{W}_\|/8\mathcal{X}^0)}. \tag{2.257}$$

Before discussing the interpretation of this formula, we write its obvious extension, analogous to (2.127), to the case of a superparticle that wraps any number of times around the circle:

$$i\int\frac{\mathrm{d}^2x\mathrm{d}^2\theta}{(2\pi)^2}\sqrt{g^E}\sum_{k=1}^{\infty}\frac{1}{k}\exp\left(2\pi ik\left(\sum_I q_I\mathcal{Z}^I + \sum_\rho r_\rho\mathcal{U}^\rho\right)\right)\frac{\pi\mathcal{W}_\|/8}{\sin(\pi k\mathcal{W}_\|/8\mathcal{X}^0)}. \tag{2.258}$$

In deriving this formula, we have ignored superparticle interactions. It is not immediately obvious that this is right, since in general, as explained in section 2.2.3, short-range interactions are only irrelevant above $D = 2$ (short-range interactions in $D = 2$ have been studied, for example, in [100]). The justification for ignoring the interactions, under some assumptions, can be found in appendix C of [63].

The formulas (2.256) and (2.258) have been easy to write down, but it is a little more vexing to interpret them. Looking more closely at the spectrum of BPS particles and symmetries of the M-theory construction, it was noted in [63] that "semions", i.e., particles of fractional spin, could be present in the spectrum. This would imply long range interactions of statistical nature that, among other things, would surely interfere with the formula (2.258) for multiple windings. This was left as a puzzle there, and so we just move on to discuss the general case of BPS particles with arbitrary quantum numbers.

Luckily, we find no further trouble. In the general case, we write the Hilbert space describing BPS particles of charges $\vec{q},\vec{r}$ as $\widehat{\mathcal{H}}_{\vec{q},\vec{r}} = \widehat{\mathcal{H}}_0 \otimes \mathsf{V}_{\vec{q},\vec{r}}$, where $\widehat{\mathcal{H}}_0$ is the

Hilbert space described above that realizes the supersymmetry algebra in a minimal way and $\mathsf{V}_{\vec{q},\vec{r}}$ is some vector space. By arguments similar to those that we gave in the GV case, we can assume that the supersymmetry and translation generators act only in $\widehat{\mathcal{H}}_0$, but the rotation generators J_{12} and J_{34} and the Hamiltonian H' will also act in $\mathsf{V}_{\vec{q},\vec{r}}$. In fact, it is convenient to introduce the anti-selfdual and selfdual combinations of J_{12} and J_{34}: $J_- = J_{12} - J_{34}$ and $J_+ = J_{12} + J_{34}$. The rotation generator $\mathsf{J} = \frac{1}{2}\mathsf{T}^{-\mu\nu}J_{\mu\nu}$ that enters in the supersymmetry algebra is simply $\mathsf{J} = \mathsf{T}J_-$. We write $\mathsf{J}_{\vec{q},\vec{r}}$ for the matrix by which J acts on $\mathsf{V}_{\vec{q},\vec{r}}$. The same argument as in eqn. (2.134) shows that in acting on $\mathsf{V}_{\vec{q},\vec{r}}$, H' is equal to $-\mathsf{J}_{\vec{q},\vec{r}}$. Now we can repeat the reasoning that led to eqn. (2.136). Replacing $\widehat{\mathcal{H}}_0$ by $\widehat{\mathcal{H}}_0 \otimes \mathsf{V}_{\vec{q},\vec{r}}$ has the effect of multiplying the contribution of states of charges $\vec{q},\vec{r}$ propagating once around the circle by $\mathrm{Tr}_{\mathsf{V}_{\vec{q},\vec{r}}}(-1)^F \exp\left(2\pi e^{\sigma}\mathsf{J}_{\vec{q},\vec{r}}\right)$. Reasoning as in the derivation of eqn. (2.136), we can write this trace in two-dimensional terms as $\mathrm{Tr}_{\mathsf{V}_{\vec{q},\vec{r}}}(-1)^F \exp\left(-i\pi\mathcal{J}_{\vec{q},\vec{r}}/4\mathcal{X}^0\right)$, where $\mathcal{J} = \mathcal{W}_{\|}J_-$ acts as $\mathcal{J}_{\vec{q},\vec{r}}$ on $\mathsf{V}_{\vec{q},\vec{r}}$.

For k-fold winding, we have to multiply the exponent in this trace by k. The generalization of eqn. (2.258) is thus simply

$$i\int \frac{\mathrm{d}^2x\mathrm{d}^2\theta}{(2\pi)^2}\sqrt{g^E}\cdot\sum_{k=1}^{\infty}\frac{1}{k}\exp\left(2\pi i k\left(\sum_I q_I\mathcal{Z}^I + \sum_\rho r_\rho\mathcal{U}^\rho\right)\right)$$
$$\mathrm{Tr}_{\mathsf{V}_{\vec{q},\vec{r}}}\left[(-1)^F\exp\left(-i\pi k\mathcal{J}_{\vec{q},\vec{r}}/4\mathcal{X}^0\right)\right]\frac{\pi\mathcal{W}_{\|}/8}{\sin(\pi k\mathcal{W}_{\|}/8\mathcal{X}^0)}. \quad (2.259)$$

J_- and J_+ are generators of the two factors of $SU(2)_\ell \times SU(2)_r \sim SO(4)$, and the way they enter the OV formula is similar to the way $SU(2)_\ell$ and $SU(2)_r$ enter the GV formula. The OV and GV formulas depend respectively on the detailed J_- and $SU(2)_\ell$ quantum numbers of the BPS states, but J_+ and $SU(2)_r$ only enter to the extent that they affect the statistics of the states.

2.5.3 Further remarks

There are some further details about the OV and GV formulas and their interpretation that can be found in [63], though they somewhat deviate from the main topic of this thesis (which is on exactly calculable holomorphic sectors in supersymmetric theories) and therefore were omitted here.

One detail is related to understanding the role of holonomies α^I and β^ρ. From the Type IIA point of view, they are periods of the B-field and holonomies of the Chan-Paton bundle on the D4-brane respectively. They parametrize sheaf cohomology groups $H^2(Y;U(1))$ and $H^1(L;U(1))$ respectively, where $U(1)$ is a constant sheaf. They appear as parameters of the Calabi-Yau compactification.

From the M-theory point of view, they are not parameters of the Calabi-Yau compactification. Rather, they appear as holonomies of gauge fields when we compactify M-theory on a circle. This reflects the fact that the theory on $\mathbb{R}^5$ is a gauge theory with the gauge group $H^2(Y;U(1))$, and the theory on $\mathbb{R}^3$ – with the gauge group $H^1(L;U(1))$.

These remarks are related to another point discussed in [63] – inclusion of the torsion. In general, groups $H^2(Y;U(1))$ and $H^1(L;U(1))$ are not necessarily connected. They can have a torsion part, i.e., have several connected components. This means that M-theory compactified on Y has not only usual continuous gauge symmetries, but also discrete gauge symmetries. Both the GV and the OV formulas can be generalized to this case if we include an extra factor x^k in the trace in these formulas. This x is an element of the discrete group – the torsion. From the point of view of 4d or 2d effective field theories, this x can be thought of as an additional discrete parameter of the Calabi-Yau compactification.

Chapter 3

Chiral Algebras in $d = 2, \mathcal{N} = (0,2)$ theories

This chapter is based on paper [85].

3.1 $\mathcal{N} = (0,2)$ theories

In this section we discuss some general aspects of two-dimensional $(0,2)$-supersymmetric theories and their chiral algebras.

3.1.1 Conventions and some generalities

The two-dimensional theories with $(0,2)$ supersymmetry are characterized by the existence of two conserved supercharges Q_+ and $\overline{Q}_+$ of positive (or right-handed) chirality acting on the Hilbert space of the theory. They satisfy:

$$
\begin{aligned}
&Q_+^2 = \overline{Q}_+^2 = 0, \\
&\{Q_+, \overline{Q}_+\} = 2P_{++},
\end{aligned}
\tag{3.1}
$$

where $2P_{++} = P_0 + P_1$ is a light-cone momentum. The standard geometric realization of supersymmetry is to consider the superspace $\mathbb{R}^{2|2}$ with bosonic coordinates x^0, x^1 and fermionic coordinates θ^+ and $\overline{\theta}^+$. Superfields are distributions on this superspace taking values in operators acting on the Hilbert space. The supercharges Q_+ and $\overline{Q}_+$ act on operators (and therefore on superfields) by commutators, and the geometric realization of this action is through the differential operators:

$$\begin{aligned} \mathcal{Q}_+ &= \frac{\partial}{\partial\theta^+} + i\overline{\theta}^+ \frac{\partial}{\partial x^{++}}, \\ \overline{\mathcal{Q}}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} - i\theta^+ \frac{\partial}{\partial x^{++}}, \end{aligned} \tag{3.2}$$

so that for an arbitrary superfield F, we have $[Q_+, F]_\pm = \mathcal{Q}_+ F$, where $[\ldots]_\pm$ denotes a graded commutator. These operators obviously satisfy the required relation $\{\mathcal{Q}_+, \overline{\mathcal{Q}}_+\} = -2i\frac{\partial}{\partial x^+}$. We also have another pair of differential operators on $\mathbb{R}^{2|2}$, D_+ and $\overline{D}_+$, given by:

$$\begin{aligned} D_+ &= \frac{\partial}{\partial\theta^+} - i\overline{\theta}^+ \frac{\partial}{\partial x^{++}}, \\ \overline{D}_+ &= -\frac{\partial}{\partial\overline{\theta}^+} + i\theta^+ \frac{\partial}{\partial x^{++}}, \end{aligned} \tag{3.3}$$

for which the key property is that they anticommute with $\mathcal{Q}_+$ and $\overline{\mathcal{Q}}_+$ and hence can be used in constructing supersymmetric Lagrangians.

We also adopt the convention in which hermitian conjugation reverses the order of fermions, that is $(\theta_1\theta_2)^\dagger = \overline{\theta}_2\overline{\theta}_1$.

The basic superfields are

1) Chiral superfields satisfying $\overline{D}_+\Phi = 0$. The component expansion contains a complex scalar ϕ and a left spinor ψ_+:

$$\Phi = \phi + i\theta^+\psi_+ - i\theta^+\overline{\theta}^+\partial_{++}\phi \tag{3.4}$$

The antichiral superfield satisfies $D_+\overline{\Phi} = 0$ and is given by:

$$\overline{\Phi} = \overline{\phi} + i\overline{\theta}^+\overline{\psi}_+ + i\theta^+\overline{\theta}^+\partial_{++}\overline{\phi}. \tag{3.5}$$

2) Fermi superfields satisfying $\overline{D}_+\Lambda = E(\Phi)$, where $E(\Phi)$ is a chiral superfield constructed as a holomorphic function of basic chiral superfields. The component expansion contains a right-handed spinor λ and an auxiliary field G:

$$\Lambda = \lambda + \theta^+ G - i\theta^+\overline{\theta}^+\partial_{++}\lambda - \overline{\theta}^+ E(\Phi), \tag{3.6}$$

where E itself has to be expanded in components. The opposite chirality Fermi superfield satisfies $D_+\overline{\Lambda} = -\overline{E}(\overline{\Phi})$ and is given by:

$$\overline{\Lambda} = \overline{\lambda} + \overline{\theta}^+\overline{G} + i\theta^+\overline{\theta}^+\partial_{++}\overline{\lambda} - \theta^+\overline{E}(\overline{\Phi}). \tag{3.7}$$

3) Real superfields. These will appear in two contexts. One is in the description of the $\mathcal{N} = (0,2)$ supercurrent multiplet, which will be studied in details later. The more familiar one is in gauge theories, the real gauge superfield, which includes the left-moving component $2v_{--} = v_0 - v_1$ of the gauge field:

$$V = v_0 - v_1 - 2i\theta^+\overline{\chi}_- - 2i\overline{\theta}^+\chi_- + 2\theta^+\overline{\theta}^+ D. \tag{3.8}$$

For gauge theories, one also introduces covariant derivatives:

$$\begin{aligned}
\mathcal{D}_{++} &= \partial_{++} + \frac{i}{2}(v_0 + v_1), \\
2\mathcal{D}_{--} &= \partial_0 - \partial_1 + iV, \\
\mathcal{D}_{+} &= \frac{\partial}{\partial\theta^+} - i\overline{\theta}^+\mathcal{D}_{++}, \\
\overline{\mathcal{D}}_{+} &= -\frac{\partial}{\partial\overline{\theta}^+} + i\theta^+\mathcal{D}_{++}.
\end{aligned} \tag{3.9}$$

All derivatives involved in the definitions of Φ and Λ then become covariant.

The basic gauge covariant field strength is $\Upsilon = [\overline{\mathcal{D}}_+, \mathcal{D}_{--}]$, $\overline{\Upsilon} = -[\mathcal{D}_+, \mathcal{D}_{--}]$:

$$\Upsilon = -\chi_- + \theta^+(v_{--++} + iD) + i\theta^+\overline{\theta}^+\mathcal{D}_{++}\chi_-. \tag{3.10}$$

When we build gauge theories, fields V and Υ take values in the adjoint representation of the gauge group G, fields Φ are in the representation R_b, and fields Λ are in R_f.

If U is a real superfield, it can always be thought of as a real part of some chiral superfield (not necessarily a local one; also we will allow for superfields which are chiral only on-shell). We will denote the imaginary part of this chiral superfield by $\widetilde{U}$. Then $U + i\widetilde{U}$ is chiral on-shell and $U - i\widetilde{U}$ is antichiral. The relation between U and $\widetilde{U}$ is:

$$\begin{aligned}
\overline{D}_+\widetilde{U} &= i\overline{D}_+U, \\
D_+\widetilde{U} &= -iD_+U,
\end{aligned} \tag{3.11}$$

up to equations of motion. This $\widetilde{U}$ is defined up to a term which is constant on-shell. If the component expansion of U is

$$U = u + i\theta^+\chi_+ + i\overline{\theta}^+\overline{\chi}_+ + \theta^+\overline{\theta}^+\partial_{++}v, \tag{3.12}$$

where we wrote the highest component as a derivative of some function v, then the component expansion of $\widetilde{U}$ is:

$$\widetilde{U} = v + \theta^+\chi_+ - \overline{\theta}^+\overline{\chi}_+ - \theta^+\overline{\theta}^+\partial_{++}u, \tag{3.13}$$

again up to terms which vanish on equations of motion.

Note that if we want components of U and $\widetilde{U}$ to be local operators, then U cannot be an arbitrary local real superfield. Its highest component, written as $\partial_{++}v$ above, should be a derivative of a local field. Only in such a case v above is also local and hence $\widetilde{U}$ is also the local superfield.

3.1.2 Supercurrent multiplet and RG invariance

General case

The general $\mathcal{N} = (0,2)$ multiplet containing the stress-energy tensor and the supersymmetry current was described in [80]. It is referred to as the supercurrent multiplet. It consists of real superfields $\mathcal{S}_{++}, \mathcal{T}_{----}$ and a complex superfield $\mathcal{W}_-$ satisfying[1]:

$$\begin{aligned}
\partial_{--}\mathcal{S}_{++} &= D_+\mathcal{W}_- - \overline{D}_+\overline{\mathcal{W}}_-, \\
\overline{D}_+\mathcal{T}_{----} &= \partial_{--}\mathcal{W}_-, \\
D_+\mathcal{T}_{----} &= \partial_{--}\overline{\mathcal{W}}_-, \\
\overline{D}_+\mathcal{W}_- &= C,
\end{aligned} \tag{3.14}$$

[1]Our conventions are different from [80]

where C is a complex constant (a space-filling brane current). The component expansions which solve these constraints are:

$$\begin{aligned}
\mathcal{S}_{++} &= j_{++} - 2i\theta^+ S_{+++} - 2i\overline{\theta}^+ \overline{S}_{+++} - 2\theta^+\overline{\theta}^+ T_{++++}, \\
\mathcal{W}_- &= -\overline{S}_{+--} - i\theta^+ \left(T_{++--} + \frac{i}{2}\partial_{--} j_{++} \right) - \overline{\theta}^+ C + i\theta^+\overline{\theta}^+ \partial_{++}\overline{S}_{+--}, \\
\overline{\mathcal{W}}_- &= -S_{+--} + i\overline{\theta}^+ \left(T_{++--} - \frac{i}{2}\partial_{--} j_{++} \right) - \theta^+ \overline{C} - i\theta^+\overline{\theta}^+ \partial_{++} S_{+--}, \\
\mathcal{T}_{----} &= T_{----} - \theta^+\partial_{--} S_{+--} + \overline{\theta}^+ \partial_{--}\overline{S}_{+--} + \frac{1}{2}\theta^+\overline{\theta}^+ \partial_{--}^2 j_{++}.
\end{aligned} \tag{3.15}$$

Applying constraints (3.14) to these expansions implies conservation of S_+ (the supersymmetry current), conservation of T (the stress-energy tensor) and symmetry of T:

$$\begin{aligned}
\partial_{++} S_{+--} + \partial_{--} S_{+++} &= 0, \\
\partial_{++} T_{\pm\pm--} + \partial_{--} T_{\pm\pm++} &= 0, \\
T_{++--} - T_{--++} &= 0.
\end{aligned} \tag{3.16}$$

Quite naturally, constraints (3.14) do not determine the supercurrent multiplet uniquely. There are two types of ambiguities which preserve both the conservation laws and the form of equations (3.14). One ambiguity corresponds to improvement transformations:

$$\begin{aligned}
\mathcal{S}_{++} &\to \mathcal{S}_{++} + [D_+, \overline{D}_+] U, \\
\mathcal{W}_- &\to \mathcal{W}_- + \partial_{--}\overline{D}_+ U, \\
\overline{\mathcal{W}}_- &\to \overline{\mathcal{W}}_- + \partial_{--} D_+ U, \\
\mathcal{T}_{----} &\to \mathcal{T}_{----} + \partial_{--}^2 U,
\end{aligned} \tag{3.17}$$

where U is an arbitrary real scalar superfield. These transformations do not change conserved charges.

Another ambiguity corresponds to the possibility of modifying the supercurrent multiplet by another conserved current (say, corresponding to some flavor symmetry), satisfying an additional requirement of locality which will be explained in a moment. If we have another conserved superspace current $\mathcal{I}_{\pm\pm}$, that is a pair of real superfields satisfying:

$$\partial_{--}\mathcal{I}_{++} + \partial_{++}\mathcal{I}_{--} = 0, \tag{3.18}$$

then we can use it to shift the supercurrent multiplet, i.e., define a new multiplet:

$$\begin{aligned}
\mathcal{S}_{++} \to \widetilde{\mathcal{S}}_{++} &= \mathcal{S}_{++} + \mathcal{I}_{++}, \\
\mathcal{W}_{-} \to \widetilde{\mathcal{W}}_{-} &= \mathcal{W}_{-} + \frac{i}{2}\overline{D}_{+}\mathcal{I}_{--}, \\
\overline{\mathcal{W}}_{-} \to \widetilde{\overline{\mathcal{W}}}_{-} &= \overline{\mathcal{W}}_{-} - \frac{i}{2}D_{+}\mathcal{I}_{--}, \\
\mathcal{T}_{----} \to \widetilde{\mathcal{T}}_{----} &= \mathcal{T}_{----} + \frac{1}{2}\partial_{--}\widetilde{\mathcal{I}}_{--}.
\end{aligned} \tag{3.19}$$

Note that in the last equation we use $\widetilde{\mathcal{I}}_{--}$, a real superfield related to $\mathcal{I}_{--}$ as in (3.11). That is, $\widetilde{\mathcal{I}}_{--}$ is such that $\mathcal{I}_{--} + i\widetilde{\mathcal{I}}_{--}$ is chiral. The new superfields $\widetilde{\mathcal{S}}_{++}$, $\widetilde{\mathcal{W}}_{-}$ and $\widetilde{\mathcal{T}}_{----}$ will also satisfy the constraints (3.14). However, most conserved charges will be shifted by this transformation. Note that for the above transformation to make sense in a local QFT, both $\mathcal{I}_{\pm\pm}$ and $\widetilde{\mathcal{I}}_{--}$ have to be local, so there is an extra requirement on $\mathcal{I}_{\pm\pm}$ that not only it has to be a conserved local superspace current, but also $\widetilde{\mathcal{I}}_{--}$ has to be local. In the cases of interest for us, this will actually be the case.

One can easily read off the action of $\overline{Q}_+$ on various components of the supercurrent multiplet, and we are interested in the following:

$$\begin{aligned}
\{\overline{Q}_+, S_{+++}\} &= -i\left(T_{++++} + \frac{i}{2}\partial_{++}j_{++}\right), \\
\{\overline{Q}_+, S_{+--}\} &= i\left(T_{++--} - \frac{i}{2}\partial_{--}j_{++}\right), \\
\{\overline{Q}_+, T_{++++}\} &= \partial_{++}\overline{S}_{+++}, \\
\{\overline{Q}_+, T_{++--}\} &= -\partial_{++}\overline{S}_{+--}, \\
\{\overline{Q}_+, T_{----}\} &= -\partial_{--}\overline{S}_{+--}.
\end{aligned} \tag{3.20}$$

We see that neither component of the stress-energy tensor is annihilated by $\overline{Q}_+$, so components of T by itself do not represent any $\overline{Q}_+$-cohomology classes. However, certain relations hold in the cohomology, in particular $T_{++--} - \frac{i}{2}\partial_{--}j_{++}$ is $\overline{Q}_+$-exact. If we define the "virial current" V_μ as:

$$V_{--} = 0, \quad V_{++} = ij_{++}, \tag{3.21}$$

then we have:

$$T^\mu_\mu = \partial^\mu V_\mu - \{\overline{Q}_+, 4iS_{+--}\}, \tag{3.22}$$

which looks like condition for an effective scale-invariance [101], with the effective current for constant dilatations given by $d_\mu = x^\nu T_{\nu\mu} - V_\mu$. This current is "almost conserved":

$$\partial^\mu d_\mu = \{\overline{Q}_+, \dots\}. \tag{3.23}$$

The current d_μ itself is not $\overline{Q}_+$-closed. Even though d_μ is not precisely conserved, only up to $\overline{Q}_+$-exact terms, we still can try to define a "charge" D corresponding to this current. If we have a local operator $\mathcal{O}(0)$ inserted at the origin, we define the action of D on this operator as follows. Pick a contour C enclosing $\mathcal{O}(0)$ and define:

$$[D, \mathcal{O}(0)] = \oint_C \star d(x)\mathcal{O}(0) = \oint_C \mathrm{d}x^\mu \epsilon_{\mu\nu} d^\nu(x)\mathcal{O}(0). \tag{3.24}$$

This definition is clearly contour-dependent, since $d_\mu(x)$ is not conserved. As we deform the contour a bit, $[D, \mathcal{O}(0)]$ changes by $[\partial^\mu d_\mu(x), \mathcal{O}]$ integrated over the area swept by the deformation of the contour. But $\partial^\mu d_\mu(x)$ is $\overline{Q}_+$-exact, so if $\mathcal{O}(0)$ is $\overline{Q}_+$-closed, the change in $[D, \mathcal{O}(0)]$ under the contour deformation is $\overline{Q}_+$-exact. This means that $[D, \mathcal{O}(0)]$ is well-defined up to a $\overline{Q}_+$-exact piece when it acts on $\overline{Q}_+$-closed operators. Moreover, one can check that:

$$[D, \overline{Q}_+] = \overline{Q}_+, \tag{3.25}$$

which shows that D maps $\overline{Q}_+$-closed operators into $\overline{Q}_+$-closed operators. So we conclude that D is a well-defined operator in the cohomology. It generates scale-transformations there. Since D is not $\overline{Q}_+$-closed itself, we can say that scale transformations act as outer automorphisms in the cohomology.

Emergent conformal invariance in the cohomology

In the previous subsection we considered a general $\mathcal{N} = (0, 2)$ theory in 2d, which a priori did not have any R-symmetries. The lowest component j_{++} of the superfield $\mathcal{S}_{++}$ did not satisfy any conservation laws and, moreover, was not even accompanied by j_{--}. As was noted in [80], if we restrict to the case $C = 0$ and $\mathcal{W}_- = \frac{i}{2}\overline{D}_+\mathcal{R}_{--}$, where $\mathcal{R}_{--}$ is another real superfield (and also relabel $\mathcal{S}_{++}$ by $\mathcal{R}_{++}$), we get what is called an R-multiplet. The equation relating $\mathcal{S}_{++}$ and $\mathcal{W}_-$ becomes simply $\partial_{--}\mathcal{R}_{++} + \partial_{++}\mathcal{R}_{--} = 0$, so the lowest component j_{--} of $\mathcal{R}_{--}$ together with j_{++} form a conserved

R-current. So we have:

$$\mathcal{R}_{--} = j_{--} - 2i\theta^+ S_{+--} - 2i\overline{\theta}^+ \overline{S}_{+--} - 2\theta^+\overline{\theta}^+ T_{++--}, \tag{3.26}$$

with $\partial_{++}j_{--} + \partial_{--}j_{++} = 0$. In this situation, it becomes possible to define a new stress-energy tensor:

$$\begin{aligned} \widetilde{T}_{++++} &= T_{++++} + \frac{i}{2}\partial_{++}j_{++}, \\ \widetilde{T}_{++--} &= T_{++--} - \frac{i}{2}\partial_{--}j_{++}, \\ \widetilde{T}_{----} &= T_{----} - \frac{i}{2}\partial_{--}j_{--}, \end{aligned} \tag{3.27}$$

which is also symmetric and conserved (by virtue of the conservation of j), but also it satisfies:

$$\begin{aligned} &\widetilde{T}_{++++} = \{\overline{Q}_+, \dots\}, \quad \widetilde{T}_{++--} = \{\overline{Q}_+, \dots\}, \\ &\widetilde{T}_{----} \neq \{\overline{Q}_+, \dots\}, \quad \{\overline{Q}_+, \widetilde{T}_{----}\} = 0. \end{aligned} \tag{3.28}$$

This procedure for $\mathcal{N} = (0, 2)$ theories is known as a half-twisting. The above relations demonstrate that when it can be performed, one explicitly has the full 2d conformal invariance in the cohomology of $\overline{Q}_+$: the cohomology class represented by $\widetilde{T}_{----}$ plays the role of the holomorphic[2] stress-energy tensor. It is also consistent with the fact proven in the previous subsection – that the $\overline{Q}_+$-cohomology is invariant under the RG flow. The RG invariance of the chiral algebra implies that it carries a useful information about the IR fixed point.

Let us also take a closer look at the ambiguities of the supercurrent multiplet in the presence of R-symmetry. The improvement transformations are determined

[2]To be more precise, we should Wick rotate to the Euclidean signature in order to have holomorphy.

by a real superfield U though (3.17), which tells us how $\mathcal{R}_{++}, \mathcal{W}_{-}$ and $\mathcal{T}_{----}$ are improved. On $\mathcal{R}_{--}$ it acts by:

$$\mathcal{R}_{--} \to \mathcal{R}_{--} - 2\partial_{--}\widetilde{U}, \tag{3.29}$$

where the relation between U and $\widetilde{U}$ is as in (3.11), that is

$$\begin{aligned} U &= u + i\theta^{+}\chi_{+} + i\overline{\theta}^{+}\overline{\chi}_{+} + \theta^{+}\overline{\theta}^{+}\partial_{++}v, \\ \widetilde{U} &= v + \theta^{+}\chi_{+} - \overline{\theta}^{+}\overline{\chi}_{+} - \theta^{+}\overline{\theta}^{+}\partial_{++}u. \end{aligned} \tag{3.30}$$

For an improvement transformation of the R-multiplet to make sense we have to assume that both U and $\widetilde{U}$ are local superfields. In view of the comment we made before, this restricts the class of allowed U. While for a general supercurrent multiplet the improvement transformations were parametrized by an arbitrary local real superfield U, for the R-multiplet they are parametrized by such a local real superfields U that $\widetilde{U}$ is also local. Thus the R-multiplet allows a smaller class of improvements then a general supercurrent multiplet. This is not surprising after all. For the general supercurrent multiplet, only the stress-energy tensor and the supersymmetry currents are conserved, so improvements should only preserve their conservation. In the R-multiplet, on the other hand, we also have the conserved R-current, so preserving its conservation (and the R-charge value) restricts the class of allowed improvements.

In terms of component currents, the improvement transformation is:

$$
\begin{aligned}
&j_{++} \to j_{++} + 2\partial_{++}v, \quad j_{--} \to j_{--} - 2\partial_{--}v,\\
&T_{++++} \to T_{++++} + \partial_{++}^2 u, \quad T_{++--} \to T_{++--} - \partial_{++}\partial_{--}u, \quad T_{----} \to T_{----} + \partial_{--}^2 u,\\
&S_{+++} \to S_{+++} + i\partial_{++}\chi_+, \quad S_{+--} \to S_{+--} - i\partial_{--}\chi_+,\\
&\overline{S}_{+++} \to \overline{S}_{+++} - i\partial_{++}\overline{\chi}_+, \quad \overline{S}_{+--} \to \overline{S}_{+--} + i\partial_{--}\overline{\chi}_+.
\end{aligned}
\tag{3.31}
$$

As expected, this transformation does not spoil conservation of any of these currents. It does not shift values of any conserved charges either. Also, it is easy to check that components $\widetilde{T}_{++++}$ and $\widetilde{T}_{++--}$ of the half-twisted stress-energy tensor are shifted by $\overline{Q}_+$-exact terms. On the other hand, $\widetilde{T}_{----}$ is shifted by $\partial_{--}^2(u+iv)$, which is, being the lowest component of chiral superfield $U + i\widetilde{U}$, is $\overline{Q}_+$-closed but generally is not $\overline{Q}_+$-exact. Therefore, there is a family of possible holomorphic stress tensors in the $\overline{Q}_+$-cohomology, corresponding to different improvements.

Another ambiguity, namely shifting by the superspace current $\mathcal{I}_{\pm\pm}$, works in a straightforward way:

$$
\begin{aligned}
\mathcal{R}_{++} &\to \mathcal{R}_{++} + \mathcal{I}_{++},\\
\mathcal{R}_{--} &\to \mathcal{R}_{--} + \mathcal{I}_{--},\\
\mathcal{T}_{----} &\to \mathcal{T}_{----} + \frac{1}{2}\partial_{--}\widetilde{\mathcal{I}}_{--}.
\end{aligned}
\tag{3.32}
$$

If we denote the components of $\mathcal{I}_{\pm\pm}$ by:

$$
\mathcal{I}_{\pm\pm} = i_{\pm\pm} - 2i\theta^+ I_{+\pm\pm} - 2i\overline{\theta}^+ \overline{I}_{+\pm\pm} - 2\theta^+\overline{\theta}^+ H_{++\pm\pm},
\tag{3.33}
$$

and introduce a local operator h_{--} such that

$$\partial_{++} h_{--} = H_{++--}, \tag{3.34}$$

then the shifting transformation in components works as:

$$\begin{aligned} j_{\pm\pm} &\to j_{\pm\pm} + i_{\pm\pm}, \\ S_{+\pm\pm} &\to S_{+\pm\pm} + I_{+\pm\pm}, \\ \overline{S}_{+\pm\pm} &\to \overline{S}_{+\pm\pm} + \overline{I}_{+\pm\pm}, \\ T_{++\pm\pm} &\to T_{++\pm\pm} + H_{++\pm\pm}, \\ T_{----} &\to T_{----} - \partial_{--} h_{--}. \end{aligned} \tag{3.35}$$

This ambiguity will naturally arise in a later discussion.

OPE in the cohomology

If we have two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ representing nontrivial $\overline{Q}_+$-cohomology classes, we can consider their OPE. On very general grounds we have:

$$\mathcal{O}_1(x^{++}, x^{--})\mathcal{O}_2(0,0) = \sum_{n,m} (x^{++})^n (x^{--})^m \mathcal{O}_{n,m}(0,0). \tag{3.36}$$

Now recall that the operator ∂_{++} acts trivially in the cohomology, that is if $\mathcal{O}_1$ is $\overline{Q}_+$-closed, then $\partial_{++}\mathcal{O}_1$ is $\overline{Q}_+$-exact, and thus so is $\partial_{++}\mathcal{O}_1(x^{++}, x^{--})\mathcal{O}_2(0,0)$. Acting with ∂_{++} on the right-hand side then gives a $\overline{Q}_+$-exact answer, that is:

$$\sum_{n,m} n (x^{++})^{n-1} (x^{--})^m \mathcal{O}_{n,m}(0,0) = [\overline{Q}_+, \dots]. \tag{3.37}$$

This implies that all terms except those with $n = 0$ are $\overline{Q}_+$-exact. If the cohomology classes represented by $\mathcal{O}_i$ have scaling dimensions h_i, we can then write:

$$\mathcal{O}_1(x^{++}, x^{--})\mathcal{O}_2(0,0) = \sum_k \frac{1}{(x^{--})^{h_1+h_2-h_k}}\mathcal{O}_k(0,0) + [\overline{Q}_+, \ldots]. \tag{3.38}$$

Note also that, since in the cohomology we have left-movers only, scaling dimensions and spins coincide.[3] This, in particular, implies an obvious conclusion that no dimensionful constants can appear in the OPE of the cohomology classes. Any dimensionful constant will have non-trivial dimension but trivial spin, and therefore its appearance will either break scaling or Lorentz-invariance of the OPE. Indeed, if we have some dimensionful parameter μ, then in the expression:

$$\mathcal{O}_1(x^{++}, x^{--})\mathcal{O}_2(0,0) = \sum_k \frac{\mu^p}{(x^{--})^{\Delta}}\mathcal{O}_k(0,0) + [\overline{Q}_+, \ldots], \tag{3.39}$$

dimensional analysis implies $\Delta = h_1 + h_2 - h_k - h(\mu)p$, where $h(\mu)$ is the dimension of μ, while Lorentz invariance implies $\Delta = h_1 + h_2 - h_k$. This is possible only for $p = 0$, that is μ should not be there.

All dependence on dimensionful coupling constants of the original supersymmetric theory will therefore be hidden in the $\overline{Q}_+$-exact term. This simple observation will be helpful later. It will imply that one can turn off all dimensionful couplings for the OPE computation. In the models we are going to study this will mean that it is enough to compute OPE in the free theory.

3.1.3 Chiral algebras of superconformal theories

For superconformal theories, the $\mathcal{N} = (0,2), d = 2$ super-Poincare algebra of symmetries is enhanced to $\mathrm{Vir} \oplus \widetilde{\mathrm{SVir}}$, where Vir denotes the left-handed Virasoro al-

[3]Even if the operator representing the cohomology class in the full theory is not left-moving, the class it represents is left-moving. Since Lorentz-invariance of the full theory induces Lorentz-invariance in the cohomology, one indeed can use the argument made in the text.

gebra (generated by the holomorphic stress-energy tensor) and the $\widetilde{\text{SVir}}$ denotes the right-handed $\mathcal{N} = 2$ super-Virasoro algebra (generated by the corresponding anti-holomorphic currents). The left-handed algebra might be enlarged to the super-Virasoro as well (or even some larger W-algebra) if we have more symmetries on the left, but it graded-commutes with the $\mathcal{N} = 2$ Virasoro on the right in any case.

Let us restrict to the NS sector of the $\widetilde{\text{SVir}}$. The operators $\overline{Q}_+$ and Q_+ can be identified as $\widetilde{G}^+_{-1/2}$ and $\widetilde{G}^-_{-1/2}$ respectively – two of the fermionic generators of $\widetilde{\text{SVir}}$ (we put tildes on $\widetilde{\text{SVir}}$ and on its generators to emphasize that this is an anti-holomorphic algebra). In a conformal case, we have the radial quantization Hilbert space $\mathcal{H}$, and we assume that it has an inner product, such that $\widetilde{G}^-_{1/2} = \left(\widetilde{G}^+_{-1/2}\right)^\dagger$ is a special supersymmetry generator. Part of the super-Virasoro algebra relations are:

$$\begin{aligned} \{\widetilde{G}^-_{-1/2}, \widetilde{G}^+_{-1/2}\} &= 2\widetilde{L}_{-1}, \\ \{\widetilde{G}^+_{-1/2}, (\widetilde{G}^+_{-1/2})^\dagger\} \equiv \{\widetilde{G}^+_{-1/2}, \widetilde{G}^-_{1/2}\} &= 2\widetilde{L}_0 - \widetilde{J}_0. \end{aligned} \tag{3.40}$$

Recall that in conformal case we have a state-operator correspondence. Therefore, instead of computing the operator cohomology, we can equivalently ask for the cohomology of $\widetilde{G}^+_{-1/2}$ acting on the Hilbert space $\mathcal{H}$. The second equation in (3.40) shows that, by the standard Hodge theory argument, this cohomology can be identified with the kernel of $2\widetilde{L}_0 - \widetilde{J}_0$. Also, in a unitary theory, it shows that $2\widetilde{L}_0 - \widetilde{J}_0 \geq 0$.

Now, every state in the Hilbert space is built by acting with $\widetilde{L}_{-n}, \widetilde{J}_{-n}, \widetilde{G}^+_{-\alpha}, \widetilde{G}^-_{-\alpha}, n, \alpha > 0$ on a superconformal primary state. It is easy to see that all these operators except $\widetilde{G}^+_{-1/2}$ increase the eigenvalue of $2\widetilde{L}_0 - \widetilde{J}_0$, while $\widetilde{G}^+_{-1/2}$ does not change it. Therefore, if the primary state has $2\widetilde{L}_0 - \widetilde{J}_0 > 0$, then all states in its superconformal family have $2\widetilde{L}_0 - \widetilde{J}_0 > 0$ and thus do not contribute to the cohomology. On the other hand, if some primary state $|\Delta\rangle$ has zero eigenvalue of $2\widetilde{L}_0 - \widetilde{J}_0$, then so does $\widetilde{G}^+_{-1/2}|\Delta\rangle$, while other states in the same conformal family have $2\widetilde{L}_0 - \widetilde{J}_0 > 0$. But $(2\widetilde{L}_0 - \widetilde{J}_0)|\Delta\rangle = 0$

and (3.40) imply that $\widetilde{G}^+_{-1/2}|\Delta\rangle = 0$. Therefore, in such a case there is only one non-trivial state in the superconformal family which contributes to the cohomology – the primary state itself. This way we prove that in the NS sector of a unitary $\mathcal{N} = (0,2)$ superconformal theory there is an isomorphism:

$$H(\mathcal{H}, \widetilde{G}^+_{-1/2}) \simeq \{\text{Primaries of } \widetilde{\text{SVir}} \text{ with } 2\widetilde{L}_0 - \widetilde{J}_0 = 0\}$$
$$= \{|\psi\rangle \in \mathcal{H} : \widetilde{L}_n|\psi\rangle = \widetilde{J}_n|\psi\rangle = \widetilde{G}^+_{\alpha-1}|\psi\rangle = \widetilde{G}^-_\alpha|\psi\rangle = (2\widetilde{L}_0 - \widetilde{J}_0)|\psi\rangle = 0, n, \alpha > 0\}. \tag{3.41}$$

Notice that these are what is usually called the chiral primaries with respect to $\widetilde{\text{SVir}}$. In fact, this is essentially the construction of [102] applied to $\mathcal{N} = (0,2)$ theories. In the $\mathcal{N} = (2,2)$ case, [102] describe the chiral ring of the $\mathcal{N} = (2,2)$ model by studying the set of (anti)chiral primaries both with respect to the left- and the right-moving super-Virasoro algebras. For the $\mathcal{N} = (0,2)$ theories, we have in (3.41) only the chiral primary condition with respect to the right-moving super-Virasoro algebra. For that reason, the object we get is not just the chiral ring: it involves holomorphic OPEs as part of its structure and is usually referred to as the W-algebra, or also chiral algebra.

Another remark is that for $\mathcal{N} = (2,2)$ theories, the chiral algebra that we study encodes the (c,c) and (a,c) rings of [102] as a part of its structure. Indeed, by considering the subspace of $H(\mathcal{H}, \widetilde{G}^+_{-1/2})$ annihilated by $2L_0 - J_0$, where L_0 and J_0 are from the left-moving SVir algebra, we get the space $\{|\psi\rangle \in \mathcal{H} : (2L_0 - J_0)|\psi\rangle = (2\widetilde{L}_0 - \widetilde{J}_0)|\psi\rangle = 0\}$, which is the space of chiral primaries with respect to both SVir and $\widetilde{\text{SVir}}$, and therefore gives rise to the (c,c) ring under the OPE. Analogously, picking the subspace annihilated by $L_0 + J_0$, we get the (a,c) ring.

One consequence of this is that in $\mathcal{N} = (2,2)$ theories, the (anti)chiral primaries, which form the (c,c) or (a,c) rings of the theory, always show up in the chiral algebra as primaries of the left-moving $\widetilde{\text{SVir}}$. In the simplest cases they will generate the whole chiral algebra, but as we will see later, there might be other primary operators in the algebra, which are not simply elements of the (c, c) or (a, c) ring.

3.1.4 The operator cohomology and the superspace

Classical and quantum observables

In the models we are going to study later in this paper, the chiral algebra will turn out to be tree-level exact. As we will argue, no loop corrections will contribute to the cohomology. However, despite our usual intuition that "tree level" means "classical", it is important to understand that the quantum chiral algebra in the $\overline{Q}_+$-cohomology is not the same as the classical one. The distinction comes from the way we multiply operators.

In classical field theory, to multiply fields we use the usual point-wise multiplication of functions on space-time. In quantum theory, even at the tree level, we should subtract singularities which appear when different operators collide, which for example gives the usual notion of normal ordering in CFT.

It might happen (and it will happen in concrete examples) that the classical composite operator is $\overline{Q}_+$-closed, but the singular part we need to subtract to define the quantum operator is not $\overline{Q}_+$-closed. This subtlety should be taken into account when computing the chiral algebra of the theory. But still, as a step in this direction, it is useful to understand the structure of the classical cohomology first.

Classical observables and the cohomology

Let us introduce the space of classical observables $\mathcal{F}$ and the space of classical superobservables $\widehat{\mathcal{F}}$. We will sometimes refer to a generic field as ϕ and to a generic superfield as Φ. Both of these spaces classically carry the structures of supercommutative algebras.

Definition 2.1: $\mathcal{F}$ is a supercommutative algebra of polynomials of fields ϕ and their derivatives $\partial_{--}^{n}\partial_{++}^{m}\phi$ whose coefficients are analytic functions on a space-time, modulo classical equations of motion. In other words, $\mathcal{F} = C_\omega(M)[\ldots, \phi, \partial_{--}^{n}\partial_{++}^{m}\phi, \ldots]/\mathcal{I}$, where $C_\omega(M)$ denotes analytic functions on M, and $\mathcal{I}$ denotes an ideal generated by the equations of motion and all their derivatives.

If the classical equations of motion do not depend on space-time coordinates explicitly (only through the coordinate-dependence of the generating fields), we can introduce:

Definition 2.1': $\mathcal{F}_0$ is a subalgebra of $\mathcal{F}$ of observables which do not depend on a space-time point explicitly. In other words, it is generated by the same fields and their derivatives as $\mathcal{F}$ (and also modulo equations of motion), but the coefficients are taken to be just complex numbers rather than functions.

There are straightforward superspace analogs of these:

Definition 2.2: $\widehat{\mathcal{F}}$ is a supercommutative algebra of polynomials of superfields Φ and their bosonic and super-derivatives $\partial_{--}^{n}\partial_{++}^{m}D_{+}^{p}\overline{D}_{+}^{q}\Phi$ whose coefficients are analytic functions on superspace, modulo classical superspace equations of motion.

If the superspace equations of motion do not include any explicit dependence on a superspace point, i.e., if they have the form of a polynomial of generating fields $\partial_{--}^{n}\partial_{++}^{m}D_{+}^{p}\overline{D}_{+}^{q}\Phi$ with complex coefficients, we again can define a subalgebra:

Definition 2.2': $\widehat{\mathcal{F}}_0$ is a subalgebra of $\widehat{\mathcal{F}}$ of superobservables which do not depend on a space-time point explicitly. In other words, it is generated by the same superfields and their derivatives as $\widehat{\mathcal{F}}$ (and also modulo superspace equations of motion), but the coefficients are taken to be just complex numbers rather then functions.

Our goal is to compute the cohomology of $\overline{Q}_{+}$ acting on $\mathcal{F}$ in the situation when the equations of motion do not depend on the superspace point explicitly. The first observation is that the operator $\overline{Q}_{+}$ only acts on the generating fields of the algebra $\mathcal{F}$, it does not act on the c-number functions which can possibly multiply these fields.

This means that it is enough to compute the cohomology of $\overline{Q}_+$ acting on $\mathcal{F}_0$. To be more rigorous, we can introduce operators of multiplication by x^μ called $m(x^\mu)$:

$$\forall \mathcal{O} \in \mathcal{F},\ m(x^\mu)\mathcal{O} = x^\mu \mathcal{O}, \tag{3.42}$$

and notice that they commute with $\overline{Q}_+$. Then we can introduce a bigrading on $\mathcal{F}$ by saying that an explicit factor of $(x^0)^n(x^1)^m$ has degree (n, m). After this it becomes obvious that

$$H(\mathcal{F}) \simeq \overline{\bigoplus_{n,m\geq 0} H^{n,m}(\mathcal{F})}, \tag{3.43}$$

where the bar over the right hand side means that we should actually consider a completion of this space with respect to some norm, because we have to allow infinte sums (series) to account for the possibility of having analytic functions as coefficients.

As we mentioned, $\overline{Q}_+$ does not act in any way on x^μ, and because of that:

$$H^{n,m}(\mathcal{F}) \simeq H(\mathcal{F}_0). \tag{3.44}$$

Therefore, from now on we will only study the cohomology in $\mathcal{F}_0$, which of course only makes sense when the equations of motion do not depend on the superspace point explicitly.

The cohomology of $\overline{Q}_+$ in $\mathcal{F}_0$ and of $\overline{D}_+$ in $\widehat{\mathcal{F}}_0$

Take an arbitrary $\mathcal{A} \in \widehat{\mathcal{F}}_0$. $\mathcal{A}$ is some general superfield, and it can be expanded into components with respect to the Grassmann coordinates. The most basic property it satisfies is that the supersymmetry transformations of its components are encoded in the way differential operators $\mathcal{Q}_+$ and $\overline{\mathcal{Q}}_+$ act on it. This follows simply from the fact that this holds for the generating superfields from which $\mathcal{A}$ is constructed and the fact that we do not allow explicit dependence on the superspace coordinates in the algebra $\widehat{\mathcal{F}}_0$. So we have:

$$[Q_+, \mathcal{A}] = \mathcal{Q}_+\mathcal{A}, \tag{3.45}$$

and the same for $\overline{Q}_+$. Supersymmetry relates all components of $\mathcal{A}$ and it is straightforward to see that:

Proposition 2.1: If the lowest component $\mathcal{A}\big|$ of the superfield $\mathcal{A} \in \widehat{\mathcal{F}}_0$ vanishes, then $\mathcal{A} = 0$.

The algebras $\mathcal{F}_0$ and $\widehat{\mathcal{F}}_0$ are related in an obvious way: any element of $\mathcal{F}_0$ can be found as a component of some superfield in $\widehat{\mathcal{F}}_0$. In particular, we can always find a superfield $\mathcal{A}$ which contains a given element $a \in \mathcal{F}_0$ as its lowest component. Moreover, supersymmetry defines this $\mathcal{A}$ uniquely, so:

Proposition 2.2: For any $a \in \mathcal{F}_0$ there exists a unique $\mathcal{A} \in \widehat{\mathcal{F}}_0$ such that $a = \mathcal{A}\big|$.

The problem which we are addressing is to find the cohomology of $\overline{Q}_+$ in $\mathcal{F}_0$. That is, the classes of fields $a \in \mathcal{F}_0$ which satisfy $[\overline{Q}_+, a] = 0$, modulo those a for which $a = [\overline{Q}_+, b], b \in \mathcal{F}_0$. Now from the Proposition 2, we know that there exist $\mathcal{A}, \mathcal{B} \in \widehat{\mathcal{F}}_0$, such that $a = \mathcal{A}|$ and $b = \mathcal{B}|$. The equation $[\overline{Q}_+, a] = 0$ implies then $\overline{\mathcal{Q}}_+\mathcal{A}| = 0$.

There is a small subtlety here which shows why it is correct to look for the cohomology of $\overline{D}_+$ rather than $\overline{\mathcal{Q}}_+$: $\overline{D}_+$ acts on $\widehat{\mathcal{F}}_0$ by definition, while $\overline{\mathcal{Q}}_+ = \overline{D}_+ + 2i\theta^+\partial_+$ does not, as it introduces an explicit dependence on θ (therefore $\overline{\mathcal{Q}}_+$ acts from $\widehat{\mathcal{F}}_0$ to a bigger space $\widehat{\mathcal{F}}$). However, we can write: $\overline{D}_+\mathcal{A}| = \overline{\mathcal{Q}}_+\mathcal{A}| = 0$. But $\overline{D}_+\mathcal{A} \in \widehat{\mathcal{F}}_0$, so we can apply Proposition 1 and conclude that $\overline{D}_+\mathcal{A} = 0$. Analogously $a = [\overline{Q}_+, b]$ implies $\mathcal{A} = \overline{D}_+\mathcal{B}$. This proves the

Proposition 2.3: The cohomology of $\overline{Q}_+$ in $\mathcal{F}_0$ (denoted $H(\mathcal{F}_0)$) is isomorphic to the cohomology of $\overline{D}_+$ in $\widehat{\mathcal{F}}_0$ (denoted $H(\widehat{\mathcal{F}}_0)$). The isomorphism $H(\widehat{\mathcal{F}}_0) \to H(\mathcal{F}_0)$ is defined by taking the lowest component of the superfield.

This proposition shows why in the rest of the paper we are going to study the cohomology of $\overline{D}_+$.

3.2 Landau-Ginzburg models

The $\mathcal{N} = (0,2)$ Landau-Ginzburg (LG) model is described by a set of chiral superfields $\Phi^i, i = 1..n$ and Fermi superfields $\Lambda^a, a = 1..m$. The action is (we assume summation over repeated indices, even if they both appear upstairs or downstairs; sometimes we will write the sum sign explicitly to avoid possible confusion):

$$S = \frac{1}{\pi}\int \mathrm{d}^2x\mathrm{d}^2\theta \left\{-\frac{i}{2}\overline{\Phi}^i\partial_{--}\Phi^i - \frac{1}{2}\overline{\Lambda}^a\Lambda^a\right\} + \frac{1}{\pi}\int \mathrm{d}^2x\mathrm{d}\theta^+\Lambda^a J_a(\Phi)|_{\overline{\theta}^+=0} + h.c., \tag{3.46}$$

where in general $\overline{D}_+\Lambda^a = E^a(\Phi)$ and $\sum_a E^a(\Phi)J_a(\Phi) = 0$. The classical superspace equations of motion are:

$$\begin{aligned}
\overline{D}_+\partial_{--}\overline{\Phi}^i &= i\overline{\Lambda}^a\frac{\partial E^a}{\partial\Phi^i} - 2i\Lambda^a\frac{\partial J_a}{\partial\Phi^i},\\
D_+\partial_{--}\Phi^i &= -i\Lambda^a\frac{\partial\overline{E}^a}{\partial\overline{\Phi}^i} + 2i\overline{\Lambda}^a\frac{\partial\overline{J}_a}{\partial\overline{\Phi}^i},\\
\overline{D}_+\overline{\Lambda}^a &= -2J_a(\Phi),\\
D_+\Lambda^a &= 2\overline{J}_a(\overline{\Phi}).
\end{aligned} \tag{3.47}$$

The supersymmetry currents of this theory are:

$$\begin{aligned}
S_{+++} &= \frac{i}{2}\psi^i_+\partial_{++}\overline{\phi}^i, \quad \overline{S}_{+++} = -\frac{i}{2}\overline{\psi}^i_+\partial_{++}\phi^i,\\
S_{+--} &= \frac{i}{2}\lambda^a\overline{E}^a(\overline{\phi}) - i\overline{\lambda}^a\overline{J}_a(\overline{\phi}), \quad \overline{S}_{+--} = i\lambda^a J_a(\phi) - \frac{i}{2}\overline{\lambda}^a E^a(\phi).
\end{aligned} \tag{3.48}$$

It is not hard to find a superfield $\mathcal{S}_{++}$ such that $S_{+++} = \frac{i}{2}D_+\mathcal{S}_{++}|$ and $\overline{S}_{+++} = -\frac{i}{2}\overline{D}_+\mathcal{S}_{++}|$:

$$\mathcal{S}_{++} = \frac{1}{2}D_+\Phi^i\overline{D}_+\overline{\Phi}^i. \tag{3.49}$$

If we also introduce:

$$\mathcal{W}_- = \frac{i}{2}\overline{\Lambda}^a E^a - i\Lambda^a J_a,$$
$$\mathcal{T}_{----} = \partial_{--}\Phi^i\partial_{--}\overline{\Phi}^i + \frac{i}{2}\Lambda^a\partial_{--}\overline{\Lambda}^a - \frac{i}{2}\partial_{--}\Lambda^a\overline{\Lambda}^a, \tag{3.50}$$

then we find that:

$$\partial_{--}\mathcal{S}_{++} = D_+\mathcal{W}_- - \overline{D}_+\overline{\mathcal{W}}_-,$$
$$\overline{D}_+\mathcal{T}_{----} = \partial_{--}\mathcal{W}_-,$$
$$\overline{D}_+\mathcal{W}_- = 0. \tag{3.51}$$

We see that these are precisely the relations (3.14) of the $\mathcal{N} = (0,2)$ $d = 2$ supercurrent, and moreover, the component expansions of $\mathcal{S}_{++}$, $\mathcal{W}_-$ and $\mathcal{T}_{----}$, written as in (3.15), include the supersymmetry currents (3.48). Therefore, we have described the supercurrent multiplet of the theory (3.46). In a generic situation, it is not an R-multiplet, because there are no R-symmetries.

The algebra $\widehat{\mathcal{F}}_0$ is a supercommutative algebra freely generated by superfields $\Phi^i, \overline{\Phi}^i, \Lambda^a, \overline{\Lambda}^a$ and their derivatives (with respect to ∂_{--}, ∂_{++}, D_+ and $\overline{D}_+$ applied arbitrary number of times) modulo the relations. The relations are: the ones that follow from $\{D_+, \overline{D}_+\} = 2i\partial_{++}$ and $D_+^2 = \overline{D}_+^2 = 0$, the chirality conditions $\overline{D}_+\Phi^i = 0$, $\overline{D}_+\Lambda^a = E^a(\Phi)$ and the superspace equations of motion as written above. All differential corollaries of the relations should also be included as relations of course.

It is not too hard to find a set of independent generators $\mathcal{G}$, so that all the relations will be taken into account and we will have simply $\widehat{\mathcal{F}}_0 \simeq \mathbb{C}[\mathcal{G}]$, a polynomial algebra generated by those generators.

We will now find this $\mathcal{G}$. First of all, due to the chirality conditions, Φ^i can only appear with the D_+ derivative (moreover, with at most one, because $D_+^2 = 0$), and

$\overline{\Phi}^i$ – with $\overline{D}_+$. The chirality condition for Λ^a allows to replace $\overline{D}_+\Lambda^a$ by $E^a(\Phi)$, while the equation of motion $D_+\Lambda^a = -2\overline{J}_a(\overline{\Phi})$ allows to replace $D_+\Lambda^a$ by an expression without derivatives. Therefore it is enough to consider only bosonic derivatives acting on Λ^a. However the simple relation:

$$2i\partial_{++}\Lambda^a = \{D_+, \overline{D}_+\}\Lambda^a = D_+\overline{D}_+\Lambda^a + \overline{D}_+D_+\Lambda^a = D_+E^a(\Phi) + 2\overline{D}_+\overline{J}_a(\overline{\Phi}) \quad (3.52)$$

shows that ∂_{++} derivatives acting on Fermi superfields can also be removed. Therefore, in the generating set $\mathcal{G}$, it is enough to include only $\partial^n_{--}\Lambda^a$ and $\partial^n_{--}\overline{\Lambda}^a$, with $n \geq 0$, and the appropriate derivatives of bosonic chiral superfields. By appropriate derivatives of bosonic chiral superfields we mean the following. First, we need to include $\partial^n_{--}\Phi^i$ and $\partial^n_{--}\overline{\Phi}^i$ with $n \geq 0$. $D_+\Phi^i$ and $\overline{D}_+\overline{\Phi}^i$ should also be included, but there is no need to include expressions like $\overline{D}_+\partial^n_{--}\Phi^i$, because, as equations of motion for Φ^i show, $D_+\partial_{--}\Phi^i$ and $\overline{D}_+\partial_{--}\overline{\Phi}^i$ can be replaced by expressions without derivatives. Expressions like $\partial^n_{++}\Phi^i$, $D_+\partial^n_{++}\Phi^i$ and their complex conjugates have to be included, they cannot be reduced to expressions without derivatives. Finally, there is no need to include both ∂_{++} and ∂_{--} derivatives because of:

$$2i\partial_{++}\partial_{--}\overline{\Phi}^i = D_+\overline{D}_+\partial_{--}\overline{\Phi}^i = D_+\left(i\overline{\Lambda}^a\frac{\partial E^a}{\partial\Phi^i} - 2i\Lambda^a\frac{\partial J_a}{\partial\Phi^i}\right). \quad (3.53)$$

So, to summarize, we write the generating set explicitly:

$$\mathcal{G} = \left\{\partial^n_{--}\Phi^i, \partial^n_{--}\overline{\Phi}^i, \partial^n_{++}\Phi^i, D_+\partial^n_{++}\Phi^i, \partial^n_{++}\overline{\Phi}^i, \overline{D}_+\partial^n_{++}\overline{\Phi}^i, \partial^n_{--}\Lambda^a, \partial^n_{--}\overline{\Lambda}^a, n \geq 0\right\}. \quad (3.54)$$

To emphasize once again, we claim that:

$$\widehat{\mathcal{F}}_0 \simeq \mathbb{C}[\mathcal{G}]. \quad (3.55)$$

Using the relations satisfied by the fields, it is not hard to describe the action of $\overline{D}_+$ in terms of the generators in $\mathcal{G}$. We have:

$$\begin{aligned}
&\overline{D}_+(\partial_{--}^n \Phi^i) = 0, \\
&\overline{D}_+(\partial_{--}^n \overline{\Phi}^i) = \partial_{--}^{n-1}\left(i\overline{\Lambda}^a \frac{\partial E^a}{\partial \Phi^i} - 2i\Lambda^a \frac{\partial J_a}{\partial \Phi^i}\right), \\
&\overline{D}_+(\partial_{++}^n \Phi^i) = 0, \quad \overline{D}_+(D_+\partial_{++}^n \Phi^i) = 2i\partial_{++}^{n+1}\Phi^i, \\
&\overline{D}_+(\partial_{++}^n \overline{\Phi}^i) = \overline{D}_+\partial_{++}^n \overline{\Phi}^i, \quad \overline{D}_+(\overline{D}_+\partial_{++}^n \overline{\Phi}^i) = 0, \\
&\overline{D}_+(\partial_{--}^n \Lambda^a) = \partial_{--}^n E^a(\Phi), \\
&\overline{D}_+(\partial_{--}^n \overline{\Lambda}^a) = -2\partial_{--}^n J_a(\Phi).
\end{aligned} \tag{3.56}$$

From these formulas we can guess that polynomials of $\partial_{--}^n \Phi^i$ should be in the cohomology. However, we need some extra assumptions about E^a and J_a in order to move further.

3.2.1 Quasihomogeneous case

As we have already learned, it is interesting to consider the case when the theory has an R-symmetry. In such a case, we expect to have an explicit stress-energy tensor in the cohomology. It is not hard to check that the following transformation:

$$\begin{aligned}
&\theta^+ \to e^{-i\epsilon}\theta^+, \\
&\Phi^i \to e^{-i\epsilon\alpha_i}\Phi^i, \\
&\Lambda^a \to e^{-i\epsilon\widetilde{\alpha}_a}\Lambda^a
\end{aligned} \tag{3.57}$$

is a symmetry of the classical action if and only if the following quasihomogeneity conditions are satisfied:

$$\begin{aligned} \widetilde{\alpha}_a J_a + \sum_i \alpha_i \Phi^i \frac{\partial J_a}{\partial \Phi^i} &= J_a, \\ -\widetilde{\alpha}_a E^a + \sum_i \alpha_i \Phi^i \frac{\partial E^a}{\partial \Phi^i} &= E^a, \end{aligned} \tag{3.58}$$

where α_i and $\widetilde{\alpha}_a$ are real numbers. It is a matter of a standard calculation to find the real conserved current $j_{\pm\pm}$ for this R-symmetry. It is then straightforward to write a superfield which has it as the lowest component. The answer is:

$$\begin{aligned} \mathcal{R}_{++} &= -\frac{i}{2}\sum_i \alpha_i \left(\Phi^i \partial_{++}\overline{\Phi}^i - \overline{\Phi}^i \partial_{++}\Phi^i\right) + \frac{1}{2}\sum_i (1-\alpha_i) D_+\Phi^i \overline{D}_+\overline{\Phi}^i \\ \mathcal{R}_{--} &= -\frac{i}{2}\sum_i \alpha_i \left(\Phi^i \partial_{--}\overline{\Phi}^i - \overline{\Phi}^i \partial_{--}\Phi^i\right) - \sum_a \widetilde{\alpha}_a \Lambda^a \overline{\Lambda}^a. \end{aligned} \tag{3.59}$$

It is not a coincidence that we called it $\mathcal{R}$. In fact, one can check that the equations of motion imply $\partial_{--}\mathcal{R}_{++} + \partial_{++}\mathcal{R}_{--} = 0$. Therefore, higher components of $\mathcal{R}$ are also conserved currents. This is the supercurrent multiplet discussed before provided we can find another real superfield $\mathcal{Y}_{----}$ (which has possibly improved stress-energy tensor T_{----} as its lowest component) satisfying the required constraints. As one can check from (3.14), the condition[4] on $\mathcal{Y}_{----}$ is:

$$\overline{D}_+ \mathcal{Y}_{----} = \frac{i}{2} \overline{D}_+ \partial_{--} \mathcal{R}_{--}. \tag{3.60}$$

Note that $D_+\mathcal{Y}_{----} = -\frac{i}{2} D_+ \partial_{--}\mathcal{R}_{--}$ is then satisfied automatically. This defines $\mathcal{Y}_{----}$ uniquely up to an arbitrary function of x^{--}, as $\mathcal{Y}_{----} \to \mathcal{Y}_{----} + f(x^{--})$ preserves the above constraints. A simple computation allows to find a real superfield

[4]The fact that this $\mathcal{Y}_{----}$ together with $\mathcal{Y}_{++++} = \frac{i}{4}[D_+, \overline{D}_+]\mathcal{J}_{++}$ and $\mathcal{Y}_{++--} = \frac{i}{4}[D_+, \overline{D}_+]\mathcal{J}_{--}$ form a conserved superspace current then follows automatically.

such that it satisfies these constraints:

$$\mathcal{Y}_{----} = \sum_i \left[\partial_{--}\Phi^i\partial_{--}\overline{\Phi}^i - \frac{\alpha_i}{4}\partial_{--}^2(\overline{\Phi}^i\Phi^i)\right] + \sum_a \left[\frac{i}{2}\Lambda^a\partial_{--}\overline{\Lambda}^a - \frac{i}{2}\partial_{--}\Lambda^a\overline{\Lambda}^a\right]. \tag{3.61}$$

Therefore, we actually have a supercurrent multiplet described by $\mathcal{R}_{++}$, $\mathcal{R}_{--}$ and $\mathcal{Y}_{----}$, which is, moreover, an R-multiplet in the terminology of [80], as reviewed in Section 3.1.2. The $\overline{Q}_+$-cohomology class represented by the twisted stress-energy tensor $\widetilde{T}_{----}$ from (3.27) promotes to the $\overline{D}_+$-cohomology class represented by the superfield:

$$\begin{aligned}\widetilde{\mathcal{Y}} &= \mathcal{Y}_{----} - \frac{i}{2}\partial_{--}\mathcal{J}_{--} \\ &= \sum_i \left[\partial_{--}\Phi^i\partial_{--}\overline{\Phi}^i - \frac{\alpha_i}{2}\partial_{--}(\Phi^i\partial_{--}\overline{\Phi}^i)\right] \\ &+ \sum_a \left[\frac{i}{2}\Lambda^a\partial_{--}\overline{\Lambda}^a - \frac{i}{2}\partial_{--}\Lambda^a\overline{\Lambda}^a + \frac{i\widetilde{\alpha}_a}{2}\partial_{--}(\Lambda^a\overline{\Lambda}^a)\right].\end{aligned} \tag{3.62}$$

This is precisely the stress-energy tensor in the cohomology as found in [79]. At first sight, one could think that this is the end of the story. However, there are some subtleties here, which we will now discuss.

First of all, how is this R-multiplet related to the more general supercurrent multiplet which we found in (3.51)? The answer is simple. If we also define

$$\mathcal{V}_- = \frac{i}{2}\overline{D}_+\mathcal{R}_{--}, \tag{3.63}$$

then $\mathcal{R}_{++}$, $\mathcal{V}_{-}$ and $\mathcal{Y}_{----}$ form a supercurrent multiplet related to $\mathcal{S}_{++}$, $\mathcal{W}_{-}$ and $\mathcal{T}_{----}$ by the improvement transformation:

$$\begin{aligned}
\mathcal{R}_{++} &= \mathcal{S}_{++} + [D_+, \overline{D}_+]U, \\
\mathcal{V}_{-} &= \mathcal{W}_{-} + \partial_{--}\overline{D}_+ U, \\
\mathcal{Y}_{----} &= \mathcal{T}_{----} + \partial_{--}^2 U, \\
U &= -\sum_i \frac{\alpha_i}{4}\Phi^i \overline{\Phi}^i.
\end{aligned} \tag{3.64}$$

Note that the superfield U cannot be represented as a real part of some *local* chiral superfield. Therefore this is an example of the improvement transformation allowed for the general supercurrent multiplet but not allowed for the R-multiplet. As we will see momentarily, there might exist several R-multiplets which are not equivalent to each other as R-mulitplets (cannot be related to each other by the R-multiplet improvements), but they all are related to the same supercurrent multiplet $\mathcal{S}_{++}, \mathcal{W}_{-}, \mathcal{T}_{----}$ by the more general improvement described above.

So now we will discuss the possibility of having several inequivalent R-multiplets. Note that the quasihomogeneity conditions (3.58) might have more than one solution. This corresponds to having an extra flavor $U(1)$ symmetry, which can then mix with the R-symmetry to give another solution of (3.58) (in terms of current, this means to replace the R-symmetry current j_R by $j_R + j_F$, where j_F is a Flavor symmetry current).

The flavor symmetry does not rotate the thetas, so it acts just as:

$$\begin{aligned}
\Phi^i &\to e^{-i\epsilon q_i}\Phi^i, \\
\Lambda^a &\to e^{-i\epsilon \widetilde{q}_a}\Lambda^a.
\end{aligned} \tag{3.65}$$

The condition that this is a symmetry of the classical action is:

$$\begin{aligned}\widetilde{q}_a J_a + \sum_i q_i \Phi^i \frac{\partial J_a}{\partial \Phi^i} &= 0, \\ -\widetilde{q}_a E^a + \sum_i q_i \Phi^i \frac{\partial E^a}{\partial \Phi^i} &= 0.\end{aligned} \tag{3.66}$$

We can see now that if $\{\alpha_i, \widetilde{\alpha}_a\}$ is some solution of (3.58) and $\{q_i, \widetilde{q}_a\}$ is some solution of (3.66), then $\{\alpha_i + q_i, \widetilde{\alpha}_a + \widetilde{q}_a\}$ is another solution of (3.58). This is actually the ambiguity of the supercurrent multiplet which we were discussing before. In case we have extra superspace currents, the basic supercurrent mutiplet $\mathcal{S}_{++}, \mathcal{W}_-, \mathcal{T}_{----}$ can be shifted. Let us belabor this point somewhat further.

One can compute the current corresponding to the flavor symmetry (3.65) and find the real superfield which contains it as the lowest component:

$$\begin{aligned}\mathcal{I}_{--} &= -\sum_a \widetilde{q}_a \Lambda^a \overline{\Lambda}^a - \frac{i}{2} \sum_i q_i \left(\Phi^i \partial_{--} \overline{\Phi}^i - \overline{\Phi}^i \partial_{--} \Phi^i \right), \\ \mathcal{I}_{++} &= -\frac{1}{2} \sum_i q_i D_+ \Phi^i \overline{D}_+ \overline{\Phi}^i + \frac{i}{2} \sum_i q_i \left(\overline{\Phi}^i \partial_{++} \Phi^i - \Phi^i \partial_{++} \overline{\Phi}^i \right).\end{aligned} \tag{3.67}$$

On shell these are conserved at the level of superfields:

$$\partial_{++} \mathcal{I}_{--} + \partial_{--} \mathcal{I}_{++} = 0. \tag{3.68}$$

One can do a small computation to check that the following superfield:

$$\mathcal{F}_{--} = -\sum_a \widetilde{q}_a \Lambda^a \overline{\Lambda}^a - i \sum_i q_i \Phi^i \partial_{--} \overline{\Phi}^i \tag{3.69}$$

is chiral on-shell, i.e., it satisfies $\overline{D}_+ \mathcal{F}_{--} = 0$ provided the equations of motion hold. In particular, it means that this $\mathcal{F}_{--}$ gives rise to the left-moving $U(1)$ current in the cohomology. But it is also true that:

$$\operatorname{Re}\mathcal{F}_{--} = \mathcal{I}_{--}. \tag{3.70}$$

Therefore, there exists a local expression for the superfield $\widetilde{\mathcal{I}}_{--}$:

$$\widetilde{\mathcal{I}}_{--} = \operatorname{Im}\mathcal{F}_{--} = -\frac{1}{2}\sum_i q_i \partial_{--}(\Phi^i \overline{\Phi}^i). \tag{3.71}$$

So, according to the general discussion from the Section 3.1.2, we can shift the R-multiplet using this $\mathcal{I}_{\pm\pm}$. Recall that the shift is:

$$\begin{aligned}
\mathcal{R}_{++} &\to \mathcal{R}_{++} + \mathcal{I}_{++}, \\
\mathcal{R}_{--} &\to \mathcal{R}_{--} + \mathcal{I}_{--}, \\
\mathcal{Y}_{----} &\to \mathcal{Y}_{----} + \frac{1}{2}\partial_{--}\widetilde{\mathcal{I}}_{--}.
\end{aligned} \tag{3.72}$$

For the $\overline{D}_+$-closed element $\widetilde{\mathcal{Y}}_{----} = \mathcal{Y}_{----} - \frac{i}{2}\mathcal{J}_{--}$, we have:

$$\widetilde{\mathcal{Y}}_{----} \to \widetilde{\mathcal{Y}}_{----} - \frac{i}{2}\partial_{--}(\mathcal{I}_{--} + i\widetilde{\mathcal{I}}_{--}) = \widetilde{\mathcal{Y}}_{----} - \frac{i}{2}\partial_{--}\mathcal{F}_{--}. \tag{3.73}$$

So the cohomology class $[\widetilde{\mathcal{Y}}_{----}]$ gets shifted by $-\frac{i}{2}[\partial_{--}\mathcal{F}_{--}]$.

Let us summarize. We have the family of R-current multiplets generated by shifts using the superspace current $\mathcal{I}_{\pm\pm}$. In the cohomology this corresponds to having an extra left-moving $U(1)$ current $[\mathcal{F}_{--}]$ generating an ambiguity of the stress-energy tensor in the cohomology, as we can do shifts of the cohomology class $[\widetilde{\mathcal{Y}}_{----}]$ by $[\partial_{--}\mathcal{F}_{--}]$.

But the conformal theory to which our LG model flows in the IR supposedly should have a unique stress-energy tensor, which thus gives a preferred stress-energy tensor in the $\overline{Q}_+$-cohomology. One can ask a natural question: which of the R-current multiplets above corresponds to the true stress-energy tensor of the theory in the IR? The answer is simple: the correct stress-energy tensor is the one, for which the $U(1)$ current $[\mathcal{F}_{--}]$ is a primary operator in the cohomology, at least when it is possible to make it primary (we will discuss this point later). It is clear that this corresponds to extremizing the central charge of the corresponding Virasiro algebra (see the next subsection). To turn this statement into a criteria for picking the unique solution $(\alpha_i, \widetilde{\alpha}_a)$ of (3.58), we need to understand first how to compute the operator product expansions (OPE) in the cohomology.

The OPE in the cohomology

The component action of the model that we study is:

$$S = S_D + S_F, \tag{3.74}$$

where the D-term action is:

$$\begin{aligned} S_D = \frac{1}{\pi}\int \mathrm{d}^2x\Big(&- \partial_{--}\overline{\phi}^i\partial_{++}\phi^i - \frac{i}{2}\overline{\psi}^i_+\partial_{--}\psi^i_+ - i\overline{\lambda}^a\partial_{++}\lambda^a - \frac{1}{2}G^a\overline{G}^a \\ &+\frac{i}{2}\partial_i E^a(\phi)\overline{\lambda}^a\psi^i_+ - \frac{i}{2}\partial_i\overline{E}^a(\overline{\phi})\overline{\psi}^i_+\lambda^a + \frac{1}{2}E^a(\phi)\overline{E}^a(\overline{\phi})\Big), \end{aligned} \tag{3.75}$$

and the F-term is:

$$S_F = \frac{1}{\pi}\int \mathrm{d}^2x \left(G^a J_a(\phi) + \overline{G}^a\overline{J}_a(\overline{\phi}) - i\lambda^a\psi^i_+\partial_i J_a(\phi) - i\overline{\lambda}^a\overline{\psi}^i_+\partial_i\overline{J}_a(\overline{\phi})\right). \tag{3.76}$$

All couplings come from the E and J-type superpotentials. Note that ϕ is dimensionless in 2d, (and fermions are of dimension 1/2), therefore both E^a and J_a should have dimension 1. We will include an explicit coupling μ of dimension 1 in the theory, and replace $E^a \to \mu E^a$ and $J_a \to \mu J_a$ in the above action, thinking of $E^a(\phi)$ and $J_a(\phi)$ as dimensionless functions of dimensionless fields ϕ^i now.

In Section 3.1.2 we saw that in order to compute the OPE of the cohomology classes we can turn off all dimensionful couplings in the theory. In particular, we can tune μ to zero. This will remove all interactions from the above action. Thus to compute the OPE of the cohomology classes, it is enough to consider the free theory:

$$S_0 = \frac{1}{\pi}\int \mathrm{d}^2x\Big(- \partial_{--}\overline{\phi}^i\partial_{++}\phi^i - \frac{i}{2}\overline{\psi}^i_+\partial_{--}\psi^i_+ - i\overline{\lambda}^a\partial_{++}\lambda^a - \frac{1}{2}G^a\overline{G}^a\Big). \tag{3.77}$$

Its correlators can be conveniently combined into superfield correlators:

$$\begin{aligned}\Big\langle\overline{\Phi}^i(x,\theta')\Phi^j(y,\theta)\Big\rangle &= \delta^{ij}\log\left(r^{--}r^{++}\right),\\ \big\langle\overline{\Lambda}^a(x,\theta')\Lambda^b(y,\theta)\big\rangle &= \delta^{ab}\frac{i}{r^{--}},\end{aligned} \tag{3.78}$$

where

$$r^{--} = x^{--} - y^{--}, \quad r^{++} = x^{++} - y^{++} + i\theta^+\overline{\theta}^+ + i\theta'^+\overline{\theta}'^+ + 2i\overline{\theta}'^+\theta^+. \tag{3.79}$$

Now we want to compute the OPE of $\widetilde{\mathcal{Y}}$ from (3.62) with itself. $\widetilde{\mathcal{Y}}$ represented a candidate stress-energy tensor in the cohomology and was given by:

$$\begin{aligned}\widetilde{\mathcal{Y}} = &\sum_i\left[\Big(1-\frac{\alpha_i}{2}\Big)\partial_{--}\Phi^i\partial_{--}\overline{\Phi}^i - \frac{\alpha_i}{2}\Phi^i\partial^2_{--}\overline{\Phi}^i\right]\\ &+\sum_a\left[\frac{i}{2}(1+\widetilde{\alpha}_a)\Lambda^a\partial_{--}\overline{\Lambda}^a - \frac{i}{2}(1-\widetilde{\alpha}_a)\partial_{--}\Lambda^a\overline{\Lambda}^a\right].\end{aligned} \tag{3.80}$$

Using the OPE above, we find that:

$$\widetilde{\mathcal{Y}}(x)\widetilde{\mathcal{Y}}(y) \sim \frac{c/2}{(x^{--}-y^{--})^4} + \frac{2\widetilde{\mathcal{Y}}(y)}{(x^{--}-y^{--})^2} + \frac{\partial_{--}\widetilde{\mathcal{Y}}(y)}{x^{--}-y^{--}} + \{\overline{Q}_+,\dots\}, \tag{3.81}$$

where the notation $\{\overline{Q}_+, \dots\}$ for the unimportant term is slightly inaccurate: what we actually mean is that the term that we drop becomes $\overline{Q}_+$-exact after we put $\theta^+ = \overline{\theta}^+ = 0$, but as a shorthand we will denote it as $\{\overline{Q}_+, \dots\}$. The central term is:

$$c = \sum_i (2 - 6\alpha_i + 3\alpha_i^2) + \sum_a (1 - 3\widetilde{\alpha}_a^2). \tag{3.82}$$

This matches the result of [79] and shows that we indeed have the stress-energy tensor in the cohomology.

Before we found that in case there is a $U(1)$ flavor symmetry, there is another $\overline{D}_+$-closed superfield $\mathcal{F}_{--}$, which gives rise to the left-moving $U(1)$ current in the cohomology. Recall that:

$$\mathcal{F}_{--} = -\sum_a \widetilde{q}_a \Lambda^a \overline{\Lambda}^a - i \sum_i q_i \Phi^i \partial_{--} \overline{\Phi}^i. \tag{3.83}$$

We can similarly compute its OPE:

$$\mathcal{F}_{--}(x)\mathcal{F}_{--}(y) \sim \frac{\sum_i q_i^2 - \sum_a \widetilde{q}_a^2}{(x^{--} - y^{--})^2} + \{\overline{Q}_+, \dots\}. \tag{3.84}$$

This current creates ambiguity, as we explained before: we can replace $\widetilde{\mathcal{Y}}$ by $\widetilde{\mathcal{Y}} + \lambda \partial_{--}\mathcal{F}_{--}$ for any $\lambda \in \mathbb{R}$ and get another stress-energy tensor in the cohomology. The unique one is picked by requiring that the $[\mathcal{F}_{--}]$ cohomology class be primary with respect to the correct stress-energy tensor, whenever it is possible to impose such a condition. Equivalently, since shifting by the current shifts the central charge, one can ask that the value of the central charge (3.82) be extremal with respect to the shifts $(\alpha_i, \widetilde{\alpha}_a) \to (\alpha_i + \lambda q_i, \widetilde{\alpha}_a + \lambda \widetilde{q}_a)$. Any of these two criteria of course give the same equation:

$$\sum_i q_i(1 - \alpha_i) + \sum_a \widetilde{q}_a \widetilde{\alpha}_a = 0. \tag{3.85}$$

In a generic situation, this equation allows to pick a unique solution $(\alpha_i, \widetilde{\alpha}_a)$ and write a correct stress-energy tensor. If the action admits f independent $U(1)$ flavor symmetries described by charges $(q_i^n, \widetilde{q}_a^n), n = 1 \dots f$, we should write the equation (3.85) for each of them. Again, generically, one can expect this to give a condition to pick the unique stress-energy tensor in the cohomology.

However, non-generic situations are possible, when this equation might either not fix the stress-energy tensor completely, or might have no solutions at all. As we will see later, this can happen for the $\mathcal{N} = (2,2)$ theories. In such a case, indices a and i take the same set of values, we have $\widetilde{\alpha}_a = \alpha_{i=a}$, and flavor symmetries (which should differ from the $\mathcal{N} = (2,2)$ R-symmetries) have $\widetilde{q}_a = q_{i=a}$. Therefore the equation (3.85) reduces to just $\sum_i q_i = 0$, which either holds identically and therefore imposes no constraints on α_i, or does not hold at all. In a former situation, the ambiguity of choosing the unique stress-energy tensor is not removed and is just present in the IR. In a latter situation, there is no solution to (3.85), which means that $[\mathcal{F}_{--}]$ cannot be made primary by choosing the proper stress-energy tensor. There is an unwanted central term in the $\widetilde{\mathcal{Y}}\mathcal{F}_{--}$ OPE, which cannot be removed and signals that there is an obstruction for the IR CFT to exist. We will see in examples that there is a flat direction in the potential and the RG flow simply never ends at any fixed point.

One can also note that if we decide to study the gauge theory obtained by gauging the flavor symmetry with charges $(q_i, \widetilde{q}_a)$, then the above equations become related to anomalies. Namely, the central term in the $\mathcal{F}_{--}\mathcal{F}_{--}$ OPE becomes just the gauge anomaly (so it is the t'Hooft anomaly in the LG model context): we need $\sum_i q_i^2 - \sum_a \widetilde{q}_a^2 = 0$ for the gauge theory to exist [73]. Then equation (3.85) becomes the condition for the R-symmetry defined by charges $(\alpha_i, \widetilde{\alpha}_a)$ to be non-anomalous [79].

Classical and quantum chiral algebra

When we were discussing the OPE in the cohomology, we argued that, as a consequence of conformal invariance, there should be no dimensionful couplings present in the OPE. We can generalize that further to say that the chiral algebra should not depend on any dimensionful couplings at all. Any algebraic relations that involve dimensionful coupling constants would violate the combination of scale and Lorentz invariance.

One of the basic facts about theories we study is that they are free in the UV. In fact, this provides an alternative argument for why the singular part of the OPE is independent of couplings. Short-distance singularities of operators are simply governed by the free theory, even before passing to the $\overline{Q}_+$ cohomology (however, we find the argument based on Lorentz and scale invariance in the cohomology to be more transparent in our case).

Independence of chiral algebras on dimensionful couplings implies a useful property, which can be thought of as a sort of non-renormalization theorem. The exact quantum chiral algebra in our theories is "almost determined" by the classical chiral algebra. All we need to do to find the quantum counterpart is renormalize composite operators. Composite operators can be thought of as several fundamental fields brought into one point, and in the process we should subtract short-distance singularities. It might well happen (and will happen in concrete examples later) that even though the classical operator is in $\overline{Q}_+$ cohomology, the infinite piece you have to subtract is not annihilated by $\overline{Q}_+$. In this way, renormalization of composite operators representing classical cohomology classes can remove part of the classical cohomology. The claim is that what you obtain using this procedure is the exact answer.

To understand why this is true, we will think of an exact quantum theory as a set of local operators, which satisfy OPE relations and operator equations of motion. As we said, short-distance singularities are governed by the free theory, so singular part of the OPE does not care about interactions and operator equations of motion. Non-singular part of the OPE can be thought of as a definition of composite operators, and this is the point where we should be careful, as already noted before. The remaining thing we need to care about are operator equations of motion.

If we stare at classical equations (3.47), we can understand that they do not have any short-distance singularities and can be made into operator equations. The question one might ask is whether they receive any corrections at the quantum level. If there were such corrections, they would be a result of interactions and would depend on the dimensionful coupling[5] μ. If this could change the answer for chiral algebra, it would mean that the algebra depends on a dimensionful constant μ. We know that this is impossible on general grounds, so we expect that quantum corrections to operator equations of motion are not important for the chiral algebra computation.

In fact, thinking slightly more general, the situation might be even simpler. Suppose we have some renormalizable field theory, and we define it in the path integral approach. This means that we choose our favorite regularization to make path integral finite-dimensional, define the action and the measure in this regularization and add counterterms, if needed. Or, alternatively, think in terms of bare fields and couplings, without any counterterms. The standard way to derive equations of motion which hold under correlators, i.e., operator equations of motion, is through integration by parts. For renormalizable field theories defined in this way, these equations of motion hold exactly when written in terms bare fields. If we write them in terms of physical fields and counterterms, then counterterms of course contribute to equations of motion, but their role is to renormalize composite operators that appear in equations of motion. This becomes very clear in the example of the $\lambda\phi^4$ theory. The equation of

[5]In fact, the right hand side of (3.47) is already proportional to μ, so additional terms would be multiplied by higher powers of μ

motion of the $\lambda\phi^4$ with counterterms is:

$$(\Box + m^2)\phi = \lambda\phi^3 + \delta_m\phi + \delta_\phi\Box\phi + \delta_\lambda\phi^3, \tag{3.86}$$

and by some simple manipulations with diagrams, one can see that these three terms on the right are precisely what one needs to define a composite operator $\lambda\phi^3$. The mass counterterm $\delta_m\phi$ removes singularity coming from the self-contraction in ϕ^3, while the other two remove singularities coming from contractions between ϕ^3 and one insertion of the interaction vertex $\lambda\phi^4/4$. It is quite obvious that this continues to higher orders of perturbation theory, simply because the theory is renormalizable and has only these three counterterms.

It is not completely clear how general this argument is and whether it holds for gauge theories, but it definitely works for our LG models. Moreover, it is possible to show that our models do not need any counterterms at all.

So our conclusion is that equations (3.47) hold exactly once we properly define composite operators appearing there. This supports our claim that to compute quantum chiral algebra, we need to find the classical one and then check which part of it survives after the renormalization of composite operators.

All these statements are true in perturbation theory. They might not hold if non-perturbative corrections become relevant. For example, instantons might lift cohomology classes [84], and this has to be studied separately. In our case we assume that the worldsheet and the target are topologically trivial, so non-perturbative corrections are not expected.

Non-abelian global symmetries

In addition to $U(1)$ global symmetries, the action may also have non-abelian linearly realized global symmetries that commute with SUSY. They generally are of the form

$\Phi^i \to A^i_j \Phi^j, \Lambda^a \to B^a_b \Lambda^b$. The kinetic part of the action implies that $A \in U(N_\Phi)$ and $B \in U(N_\Lambda)$, where N_Φ is the number of chiral superfields $\Phi^i, i = 1 \dots N_\Phi$, and N_Λ is the number of Fermi superfields $\Lambda^a, a = 1 \dots N_\Lambda$.

It is clear that by a unitary transformation $\Phi^i \to U^i_j \Phi^j$, $\Lambda^a \to V_b^a \Lambda^b$ one can always bring A and B into the diagonal form, and in such a basis they will describe just the $U(1)$ global symmetry. Therefore, in order to have something new compared to the previous discussion, we assume that the action has some $U(1)$ global symmetries and, on top of that, also has some non-abelian symmetries. Altogether, they close to a subgroup $G \subset U(N_\Phi) \times U(N_\Lambda)$. The free theory has the full $U(N_\Phi) \times U(N_\Lambda)$ symmetry, which is then broken to the subgroup G by the E and J superpotentials.

Embedding $G \subset U(N_\Phi) \times U(N_\Lambda)$ defines an $(N_\Phi + N_\Lambda)$-dimensional representation of G on superfields of our model. This representation is reducible and can be decomposed as a direct sum of an N_Φ-dimensional representation R_Φ on chiral superfields and an N_Λ-dimensional representation R_Λ on Fermi superfields. Let the Hermitian generators of this subgroup in the representation R_Φ be called t_α, $\alpha = 1 \dots |G|$, and in the representation R_Λ – τ_α, $\alpha = 1 \dots |G|$. The infinitesimal transformation is:

$$\begin{aligned} \Phi^i &\to \Phi^i + i\epsilon^\alpha (t_\alpha)^i_j \Phi^j, \\ \Lambda^a &\to \Lambda^a + i\epsilon^\alpha (\tau_\alpha)^a_b \Lambda^b. \end{aligned} \tag{3.87}$$

The condition on J and E for this to be a symmetry is:

$$\begin{aligned} (t_\alpha)^i_j \Phi^j \partial_i J_a(\Phi) + (\tau_\alpha)^b_a J_b(\Phi) = 0, \\ (t_\alpha)^i_j \Phi^j \partial_i E^a(\Phi) - (\tau_\alpha)^a_b E^b(\Phi) = 0. \end{aligned} \tag{3.88}$$

It is straightforward to repeat what we had done for abelian symmetries and to find the corresponding element in the $\overline{D}_+$-cohomology:

$$\mathcal{J}_\alpha = (\tau_\alpha)^a_b \Lambda^b \overline{\Lambda}^a + i(t_\alpha)^i_j \Phi^j \partial_{--} \overline{\Phi}^i. \tag{3.89}$$

If we write $[t_\alpha, t_\beta] = i f^\gamma_{\alpha\beta} t_\gamma$, then the OPE of these currents is given by:

$$\mathcal{J}_\alpha(x)\mathcal{J}_\beta(y) \sim \frac{\mathrm{tr}(t_\alpha t_\beta) - \mathrm{tr}(\tau_\alpha \tau_\beta)}{(x^{--} - y^{--})^2} + \frac{f^\gamma_{\alpha\beta}\mathcal{J}_\gamma(y)}{x^{--} - y^{--}} + \{\overline{Q}_+, \dots\}. \tag{3.90}$$

We have $\mathrm{tr}(t_\alpha t_\beta) = 2x_\Phi \delta_{\alpha\beta}$ and $\mathrm{tr}(\tau_\alpha \tau_\beta) = 2x_\Lambda \delta_{\alpha\beta}$, where x_Φ and x_Λ are Dynkin indices of the representations R_Φ and R_Λ respectively. Therefore, in the cohomology we find a current algebra of G at the level $r = 2(x_\Phi - x_\Lambda)$.

3.3 $\mathcal{N} = (2,2)$ models

If in a general $\mathcal{N} = (0,2)$ LG model as described before we put $E^a = 0$, take a to be the same sort of index as i, i.e., just put $N_\Lambda = N_\Phi$ (recall that everything is topologically trivial in our discussion) and take $J_a(\Phi) = \frac{\partial W(\Phi)}{\partial \Phi^{i=a}}$ for some holomorphic superpotential $W(\Phi)$, we get a general $\mathcal{N} = (2,2)$ LG model. In such a case $(0,2)$ superfields are promoted to $(2,2)$ chiral superfields:

$$\mathbf{\Phi}^i = \Phi^i + i\sqrt{2}\theta^- \Lambda^i - i\theta^- \overline{\theta}^- \partial_{--} \Phi^i. \tag{3.91}$$

With $\mathcal{N} = (2,2)$ supersymmetry, we can go further in the discussion of general properties of the chiral algebra in the $\overline{Q}_+$-cohomology. First of all, let us get rid of the trivially reducible case. Suppose that we can organize superfields $\mathbf{\Phi}^i$ into two nonempty sets: $\{\mathbf{\Phi}^1, \mathbf{\Phi}^2, \dots, \mathbf{\Phi}^s\}$, $\{\mathbf{\Phi}^{s+1}, \mathbf{\Phi}^{s+2}, \dots, \mathbf{\Phi}^{N_\Phi}\}$, so that the superpotential can be written as a sum:

$$W(\mathbf{\Phi}) = W^{(1)}(\mathbf{\Phi}^1, \mathbf{\Phi}^2, \dots, \mathbf{\Phi}^s) + W^{(2)}(\mathbf{\Phi}^{s+1}, \mathbf{\Phi}^{s+2}, \dots, \mathbf{\Phi}^{N_\Phi}), \tag{3.92}$$

This superpotential just describes 2 separate LG models which do not interact with each other. The space of observables in such a model is just the graded-symmetric tensor product of the spaces for each of the two models, and the supercharge is the sum $\overline{Q}_+ = \overline{Q}_+^{(1)} + \overline{Q}_+^{(2)}$, where each term in the sum acts on the corresponding factor in the graded-symmetric tensor product. It is a simple algebraic exercise to prove that the cohomology of such a $\overline{Q}_+$ is just the graded-symmetric tensor product of the cohomologies of $\overline{Q}_+^{(1)}$ and $\overline{Q}_+^{(2)}$.

Therefore, without any loss of generality, it is enough to study superpotentials which cannot be decomposed as in (3.92), and can never be brought into such a decomposable form by a holomorphic change of coordinates on the target. We will assume this from now on. Note that it was shown in [76] that with such an assumption, no accidents happen in the IR, which also simplifies life a lot.

Since we put $E^a = 0$, quasihomogeneity conditions (3.58) now always have at least one solution, $\widetilde{\alpha}_a = 1, \forall a$, $\alpha_i = 0, \forall i$. Therefore, according to our previous discussion, there is always a stress-energy tensor in the cohomology. It is interesting, however, to study the case when $W(\mathbf{\Phi})$ is quasi-homogeneous itself:

$$\sum_i \beta_i \Phi^i \frac{\partial W}{\partial \Phi^i} = W(\Phi). \tag{3.93}$$

After all, as was noted in [103], this is the case most relevant for studying the IR fixed point of the LG model. With this property, if we take $\alpha_i = \beta_i$ and $\widetilde{\alpha}_a = \beta_{i=a}$, we get another solution of (3.58). In other words, there exists a $U(1)$ flavor symmetry corresponding to the solution $q_i = \beta_i$, $\widetilde{q}_a = \beta_{i=a} - 1$ of (3.66).

If there is only one such flavor symmetry, we can see that the equation (3.85) picks $\alpha_i = \beta_i$ and $\widetilde{\alpha}_a = \beta_{i=a}$ as defining the correct stress-energy tensor. Indeed, these values satisfy (3.85), while another solution, $\widetilde{\alpha}_a = 1$, $\alpha_i = 0$, inserted in (3.85), gives $\sum_i q_i + \sum_a \widetilde{q}_a = \sum_i (2\beta_i - 1)$, which is generically non-zero. The last sum being zero corresponds to various degenerate cases, for example if superpotential is just a quadratic polynomial (which means that all fields are massive, the IR theory is trivial and the chiral algebra should be trivial too). We will not concentrate on such cases.

On the other hand, there can be more flavor symmetries in the model:

$$\mathbf{\Phi}^i \to e^{-i\gamma_i \epsilon} \mathbf{\Phi}^i, \tag{3.94}$$

if one can find such a system of charges γ_i that:

$$\sum_i \gamma_i \Phi^i \frac{\partial W}{\partial \Phi^i} = 0. \tag{3.95}$$

This gives a solution $q_i = \gamma_i$, $\widetilde{q}_a = \gamma_{i=a}$ of (3.66). Note that both the solution $q_i = \beta_i$, $\widetilde{q}_a = \beta_{i=a} - 1$ and the solution $q_i = \gamma_i$, $\widetilde{q}_a = \gamma_{i=a}$ describe flavor symmetries from the $\mathcal{N} = (0,2)$ point of view, since they just satisfy (3.66). However, from the $\mathcal{N} = (2,2)$ point of view, only the latter one is a flavor symmetry, while the former one becomes the left-handed R-symmetry of the $\mathcal{N} = (2,2)$ SUSY, which is seen from the fact that Φ's and Λ's charges differ by one.

The action of the LG model in the $(2,2)$ superspace is:

$$S = \frac{1}{4\pi} \int \mathrm{d}^2 \mathrm{d}^4\theta \, \overline{\mathbf{\Phi}}^i \mathbf{\Phi}^i + \frac{1}{4\pi} \int \mathrm{d}^2 x \mathrm{d}^2\theta \, W(\mathbf{\Phi}) + \frac{1}{4\pi} \int \mathrm{d}^2 x \mathrm{d}^2\overline{\theta} \, \overline{W}(\overline{\mathbf{\Phi}}) \tag{3.96}$$

The superspace equations of motion are simply:

$$\overline{D}_+ \overline{D}_- \overline{\mathbf{\Phi}}^i = \frac{\partial W}{\partial \mathbf{\Phi}^i}. \tag{3.97}$$

As was first noted in [78], we can find an element in the $\overline{D}_+$-cohomology represented by the $(2,2)$ superfield:

$$\mathcal{J} = \sum_i \left(\frac{1-\beta_i}{2} D_- \mathbf{\Phi}^i \overline{D}_- \overline{\mathbf{\Phi}}^i - i\beta_i \mathbf{\Phi}^i \partial_{--} \overline{\mathbf{\Phi}}^i \right), \tag{3.98}$$

which can then be expanded in components with respect to θ^- and $\overline{\theta}^-$: the lowest component is the left-handed R-current, the top component is the stress-energy tensor (which, using our earlier $\mathcal{N}=(0,2)$ terminology, corresponds to the solution $\alpha_i = \beta_i$, $\widetilde{\alpha}_a = \beta_{i=a}$ of (3.58)), and the fermionic components are the two left-handed supersymmetries. Therefore, this $\mathcal{J}$ generates a left-moving $\mathcal{N}=2$ superconformal algebra in the $\overline{D}_+$-cohomology.

If there exist additional $U(1)$ flavor symmetries characterized by weights γ_i satisfying (3.95), then there is another $\overline{D}_+$-cohomology class represented by:

$$\Psi = \frac{1}{2} \sum_i \gamma_i \mathbf{\Phi}^i \overline{D}_- \overline{\mathbf{\Phi}}^i, \tag{3.99}$$

so the derivative:

$$D_- \Psi = \sum_i \gamma_i \left(\frac{1}{2} D_- \mathbf{\Phi}^i \overline{D}_- \overline{\mathbf{\Phi}}^i + i \mathbf{\Phi}^i \partial_{--} \overline{\mathbf{\Phi}}^i \right) \tag{3.100}$$

generates ambiguity, because we can replace $\mathcal{J} \to \mathcal{J} + \lambda D_- \Psi$, $\forall \lambda \in \mathbb{R}$. Of course, this is still the same ambiguity of the $\mathcal{N}=(0,2)$ stress-tensor multiplet related to $U(1)$ flavor symmetries that we were discussing before. The only difference is that by now we have dealt with the $U(1)$ global symmetry which is the left-handed R-symmetry from the $\mathcal{N}=(2,2)$ point of view (it was described by the charges $q_i = \beta_i$, $\widetilde{q}_a = \beta_{i=a} - 1$), and what we are left with in (3.99) corresponds to the actual $\mathcal{N}=(2,2)$ flavor symmetry. Similar to what we had for a more general $\mathcal{N}=(0,2)$ case, we could have analyzed this ambiguity using the $\mathcal{N}=(2,2)$ supersurrent multiplet, especially since its structure is described in details in the Appendix C of [80]. However, we chose not to do this, as it would not give us anything essentially new compared to what we have already understood.

Previous discussion of the OPE in the cohomology being determined by the free propagators of course still holds. The free propagator of chiral superfields is:

$$\left\langle \overline{\mathbf{\Phi}}^i(x_1,\theta_1)\mathbf{\Phi}^j(x_2,\theta_2)\right\rangle = \delta^{ij}\log\left(R_{12}^{--}R_{12}^{++}\right), \tag{3.101}$$

where

$$\begin{aligned} R_{12}^{--} &= x_1^{--} - x_2^{--} + i\theta_1^-\overline{\theta}_1^- + i\theta_2^-\overline{\theta}_2^- + 2i\overline{\theta}_1^-\theta_2^-, \\ R_{12}^{++} &= x_1^{++} - x_2^{++} + i\theta_1^+\overline{\theta}_1^+ + i\theta_2^+\overline{\theta}_2^+ + 2i\overline{\theta}_1^+\theta_2^+. \end{aligned} \tag{3.102}$$

We can compute the OPEs:

$$\begin{aligned} \mathcal{J}(x_1,\theta_1)\mathcal{J}(x_2,\theta_2) \sim -\frac{c}{3(\mathbf{r}_{12})^2} - \frac{2\theta_{12}^-\overline{\theta}_{12}^-}{(\mathbf{r}_{12})^2}\mathcal{J}(x_2,\theta_2) - \frac{i\theta_{12}^-}{\mathbf{r}_{12}}D_-\mathcal{J}(x_2,\theta_2) \\ -\frac{i\overline{\theta}_{12}^-}{\mathbf{r}_{12}}\overline{D}_-\mathcal{J}(x_2,\theta_2) - \frac{2\theta_{12}^-\overline{\theta}_{12}^-}{\mathbf{r}_{12}}\partial_{--}\mathcal{J}(x_2,\theta_2) + \{\overline{Q}_+,\dots\}, \end{aligned} \tag{3.103}$$

where

$$\begin{aligned} \theta_{12}^- &= \theta_1^- - \theta_2^-, \quad \overline{\theta}_{12}^- = \overline{\theta}_1^- - \overline{\theta}_2^-, \\ \mathbf{r}_{12} &= x_1^{--} - x_2^{--} + i\overline{\theta}_1^-\theta_2^- - i\overline{\theta}_2^-\theta_1^-, \end{aligned} \tag{3.104}$$

and the central charge is:

$$c = 3\sum_i (1-2\beta_i). \tag{3.105}$$

Equation (3.103) encodes the $\mathcal{N}=2$ superconformal algebra with the central charge c (this equation, but in slightly different conventions, was present in [78]). Of course we could have obtained the same value of the central charge using the more general equation (3.82), which holds for more general $\mathcal{N}=(0,2)$ LG models. One would have to put $\alpha_i=\beta_i$, $\widetilde{\alpha}_a=\beta_{i=a}$ there.

Notice that the central charge (3.105) is linear in β_i. This means that if we have $U(1)$ flavor symmetries such that (3.95) holds, we can no longer get rid of the ambiguity $\beta_i\to\beta_i+\lambda\gamma_i$ by simply asking the central charge to take the extremal value. This is related to the fact that the OPE of the cohomology class represented by (3.99) with itself is regular:

$$\Psi(x_1,\theta_1)\Psi(x_2,\theta_2)\sim\{\overline{Q}_+,\dots\}. \tag{3.106}$$

So that the OPE of $[\Psi]$ with $[\mathcal{J}]$ is the same as with $[\mathcal{J}+\lambda D_-\Psi]$, $\forall\lambda\in\mathbb{R}$. The $\mathcal{J}\Psi$ OPE is:

$$\begin{aligned}\mathcal{J}(x_1,\theta_1)\Psi(x_2,\theta_2)\sim\kappa\frac{\theta^-_{12}}{(\mathbf{r}_{12})^2}-\frac{\theta^-_{12}\overline{\theta}^-_{12}}{(\mathbf{r}_{12})^2}\Psi(x_2,\theta_2)-\frac{i\theta^-_{12}}{\mathbf{r}_{12}}D_-\Psi(x_2,\theta_2)\\-\frac{2\theta^-_{12}\overline{\theta}^-_{12}}{\mathbf{r}_{12}}\partial_{--}\Psi(x_2,\theta_2)-\frac{i}{\mathbf{r}_{12}}\Psi(x_2,\theta_2)+\{\overline{Q}_+,\dots\},\end{aligned} \tag{3.107}$$

where $\kappa=\sum_i\gamma_i$. Compare this with what one expects for the OPE of $\mathcal{J}$ with some superconformal primary superfield $\mathcal{P}$:

$$\begin{aligned}\mathcal{J}(x_1,\theta_1)\mathcal{P}(x_2,\theta_2)\sim-\frac{2\theta^-_{12}\overline{\theta}^-_{12}}{(\mathbf{r}_{12})^2}\Delta\mathcal{P}(x_2,\theta_2)-\frac{i\theta^-_{12}}{\mathbf{r}_{12}}D_-\mathcal{P}(x_2,\theta_2)\\-\frac{i\overline{\theta}^-_{12}}{\mathbf{r}_{12}}\overline{D}_-\mathcal{P}(x_2,\theta_2)-\frac{2\theta^-_{12}\overline{\theta}^-_{12}}{\mathbf{r}_{12}}\partial_{--}\mathcal{P}(x_2,\theta_2)-\frac{i}{\mathbf{r}_{12}}q\mathcal{P}(x_2,\theta_2)+\{\overline{Q}_+,\dots\},\end{aligned} \tag{3.108}$$

where Δ is the conformal dimension of $\mathcal{P}$ and q is its R-charge. What we see is that for non-zero values of κ, Ψ is non-primary, and moreover it is not a descendant of any primary, as can be seen from unitarity and global superconformal invariance of the vacuum of the IR theory.[6] Therefore, the non-zero κ becomes an obstruction for the IR CFT to exist. This happens for example in a model with two superfields X and Y and superpotential $W = XY^2$.

Notice that since the $\Psi\Psi$ OPE is regular, so is the $D_-\Psi D_-\Psi$ OPE. The absence of central term in it means, as we have mentioned in Section 3.2.1, that the corresponding flavor symmetry can be gauged without encountering gauge anomalies. Possible non-zero value of κ then becomes the anomaly for the right-handed R-symmetry. This would be relevant if we were studying gauge theories.

The theory can also have non-abelian flavor symmetries, which lead, as we have argued before, to the current algebra in the cohomology. In our discussion of general $(0,2)$ theories, the level of this current algebra was given by the difference of Dynkin indices: $r = 2x_\Phi - 2x_\Lambda$. The first term here corresponded to the way flavor symmetry acted on Φ's, and the second – on Λ's. In the $(2,2)$-supersymmetric case, the flavor symmetry acts in the same way on Φ's and Λ's, as they are just components of the $(2,2)$ chiral superfields $\mathbf{\Phi}^i$. So $x_\Phi = x_\Lambda$. We conclude that the current algebra in the cohomology corresponding to some flavor symmetry of the $\mathcal{N} = (2,2)$ supersymmetric LG model always has level zero.

[6] The argument is as follows. Presence of the central term in (3.107) implies through the operator-state correspondence that there is a state $|\psi\rangle$ in the IR CFT such that $G^-_{+1/2}|\psi\rangle = \kappa|0\rangle + \overline{Q}_+|\phi\rangle$, where $G^-_{+1/2}$ is one of the superconformal generators, $|0\rangle$ is the vacuum state and $|\phi\rangle$ is some state. Taking the dimension-zero component of this equality, we can assume that ψ has dimensions $(1/2, 0)$, so that $G^-_{+1/2}|\psi\rangle$ has dimension zero. Since in a unitary theory there are no operators of negative dimension, $\overline{Q}_+|\phi\rangle$ should not be there: $G^-_{+1/2}|\psi\rangle = \kappa|0\rangle$. Invariance of the vacuum implies $G^+_{-1/2}G^-_{+1/2}|\psi\rangle = 0$. Since in a unitary theory $(G^-_{+1/2})^\dagger = G^+_{-1/2}$, by multiplying with $\langle\psi|$, the last equality implies $G^-_{+1/2}|\psi\rangle = 0$, which gives a contradiction unless $\kappa = 0$.

3.4 Examples

In this section we will consider a few examples of applications of our machinery to the $\mathcal{N} = (2, 2)$ LG models, where we can say something about the chiral algebra and therefore draw some conclusions about the theory to which the model flows in the IR.

3.4.1 Degenerate examples

Consider the theory of two chiral superfields X and Y with superpotential

$$W = XY^2. \tag{3.109}$$

This theory has a non-trivial flavor symmetry. A possible charge assignment is: $\gamma_X = 2$, $\gamma_Y = -1$, so that

$$\gamma_X X \frac{\partial W}{\partial X} + \gamma_Y Y \frac{\partial W}{\partial Y} = 0. \tag{3.110}$$

As we know from the equation (3.99), there is an extra operator Ψ in the cohomology as a result of this flavor symmetry. Since $\gamma_X + \gamma_Y = 1 \neq 0$, the OPE (3.107) tells us that this operator is not primary. Moreover, as we explained in the footnote 6, an operator satisfying (3.107) cannot be made primary in a unitary CFT with invariant vacuum. Therefore, its existence indicates that the RG flow does not end at any CFT. In fact, the superpotential has a flat direction $Y = 0$, and we can conclude that the low-energy theory describing propagation along this flat direction, as well as some other low-energy modes, cannot be conformal. One can get a conformal fixed point if we add a perturbation ϵX^{2n+1} to the superpotential. This will actually correspond to having the D series of minimal models at the IR fixed point, with the exact choice of the model depending on n, even for small ϵ. By sending $\epsilon \to 0$, the IR fixed point will most likely go to infinity, so that the degenerate theory $W = XY^2$ will not have an endpoint for its RG flow.

This flavor symmetry could be gauged, however, as we have noted before, this would make right-handed R-symmetry anomalous because of $\gamma_X + \gamma_Y \neq 0$.

By considering a slightly different superpotential, namely:

$$W = X^2 Y^2, \tag{3.111}$$

we get again a theory with a flavor symmetry, but the charges now can be chosen to be $\gamma_X = 1$, $\gamma_Y = -1$, so that $\gamma_X + \gamma_Y = 0$. Therefore, the bad central term does not appear in (3.107), and the theory in the IR has a chance to be conformal, even though the superpotential has flat directions. Also, if we gauge this flavor symmetry, we still get a theory with the right-handed R-symmetry. We are not going to study this example any further.

3.4.2 $\mathcal{N} = 2$ minimal models

A series of $\mathcal{N} = (2, 2)$ LG models are known to flow in the IR to the $\mathcal{N} = (2, 2)$ minimal models. These superconformal theories are relatively simple. The central charge is given by [104–108]:

$$c = \frac{3k}{k+2},\ k \geq 1, \tag{3.112}$$

and there is a known spectrum of possible superconformal primaries. The A-D-E classification of modular-invariant theories is known [109–112], and the corresponding LG superpotentials have been identified before. So, we can try to compute the chiral algebra both for the LG model and for the minimal model which is supposed to arise in the IR, therefore providing more evidence for this relation, as well as demonstrating the power of chiral algebras.

The A_{k+1} series

For a given k, the diagonal A_{k+1} minimal model is the simplest one. Its set of primaries has a subset of $k+1$ fields which are chiral primary with respect to $\widetilde{\text{SVir}}$. Let us call them $\mathcal{O}_s$, $s = 0, \ldots k$, where $\mathcal{O}_0 = 1$ is the identity operator and $\mathcal{O}_s$ has left-right conformal dimensions $(h, \overline{h}) = (\frac{s}{2(k+2)}, \frac{s}{2(k+2)})$ and left-right $U(1)$ charges $(q, \overline{q}) = (\frac{s}{k+2}, \frac{s}{k+2})$. As we see, they all are chiral primaries with respect to both SVir and $\widetilde{\text{SVir}}$. Therefore, together with the $\mathcal{N} = 2$ currents, they generate the chiral algebra of the theory, as well as the anti-chiral algebra abtained analogously by taking the cohomology of $\overline{Q}_-$.

We expect to get the same result from the LG model description. It is obtained by considering only one chiral superfield $\mathbf{\Phi}$ with the superpotential:

$$W(\mathbf{\Phi}) = \frac{\mathbf{\Phi}^{k+2}}{k+2}. \tag{3.113}$$

The equations of motion are

$$\begin{aligned} &\overline{D}_+ \overline{D}_- \overline{\mathbf{\Phi}} - \mathbf{\Phi}^{k+1} = 0, \\ &D_- D_+ \mathbf{\Phi} - \overline{\mathbf{\Phi}}^{k+1} = 0. \end{aligned} \tag{3.114}$$

Differentiating these equations and multiplying them by arbitrary polynomials of $\mathbf{\Phi}$, $\overline{\mathbf{\Phi}}$ and their derivatives, we get a differential ideal $\mathcal{I}$. The algebra $\widehat{\mathcal{F}}_0$ consists of arbitrary polynomials of variables $\partial_+^n \partial_-^m D_+^k D_-^p \mathbf{\Phi}$ and $\partial_+^n \partial_-^m \overline{D}_+^k \overline{D}_-^p \overline{\mathbf{\Phi}}$ for non-negative integers n, m, k, p, modulo the ideal $\mathcal{I}$:

$$\widehat{\mathcal{F}}_0 = \mathbb{C}[\ldots, \partial_+^n \partial_-^m D_+^k D_-^p \mathbf{\Phi}, \partial_+^n \partial_-^m \overline{D}_+^k \overline{D}_-^p \overline{\mathbf{\Phi}}, \ldots]/\mathcal{I}. \tag{3.115}$$

It is not hard to find another set of generators, which will generate $\widehat{\mathcal{F}}_0$ as a super-commutative polynomial algebra itself. We already explained it in the context of general $\mathcal{N}=(0,2)$ LG models. Namely, we can take:

$$\mathcal{G}=\{\partial_{--}^n\mathbf{\Phi},\partial_{++}^n\mathbf{\Phi},D_-\partial_{--}^n\mathbf{\Phi},D_+\partial_{++}^n\mathbf{\Phi},\partial_{--}^n\overline{\mathbf{\Phi}},\partial_{++}^n\overline{\mathbf{\Phi}},\overline{D}_-\partial_{--}^n\overline{\mathbf{\Phi}},\overline{D}_+\partial_{++}^n\overline{\mathbf{\Phi}},n\geq 0\}. \tag{3.116}$$

All other derivatives of elementary superfields $\mathbf{\Phi}$ and $\overline{\mathbf{\Phi}}$ can be expressed, using equations of motion, as polynomials of these generators, and moreover, there are no further algebraic relations between these generators. So we have:

$$\widehat{\mathcal{F}}_0\simeq\mathbb{C}[\mathcal{G}]. \tag{3.117}$$

We will first compute the classical cohomology of $\overline{D}_+$ acting on this space. After that we will check which part of it survives at the quantum level, when we take care to subtract singular parts from composite operators. It is clear that the cohomology classes can only be destroyed by this subtraction. Indeed, suppose we define:

$$:AB:(z)=\lim_{\epsilon\to 0}\left(A(z+\epsilon)B(z)-(\text{singular in}\,\epsilon)\right). \tag{3.118}$$

If AB was classically in the cohomology but the singular part is not $\overline{D}_+$-closed, the operator $:AB:$ is no longer in the cohomology. If AB was not in the cohomology even classically, then neither is $:AB:$, which is quite obvious. Finally, if AB was classically $\overline{D}_+$-exact, then there is no need to consider $:AB:$. Even if the singular part represented some non-trivial quantum cohomology class, we would find it by starting with some other classical cohomology class anyways. So, we will look for the classical cohomology first, and then check which part of it survives subtraction of singularities.[7]

[7] In fact, computation of the classical cohomology is a hard combinatorial problem, while we are really only interested in quantum cohomology. So we will not determine the classical cohomology completely, only partly. As we will see, there is an $\mathcal{N}=2$ super-Virasoro algebra in cohomology, so our approach will be to look for those classical cohomology classes which have a chance to be superconformal primaries at the quantum level.

To find how $\overline{D}_+$ acts on $\widehat{\mathcal{F}}_0$ in terms of the generators, we act with $\overline{D}_+$ on the generators from the set $\mathcal{G}$ and, using the equations of motion, express the result in terms of these generators again. To explicitly describe $\overline{D}_+$, it is convenient to write it as a sum:

$$\overline{D}_+ = d_0 + d_1, \tag{3.119}$$

where d_0 acts as follows:

$$\begin{aligned} d_0 :& \partial_{--}^n \mathbf{\Phi} \mapsto 0, \quad \partial_{++}^n \mathbf{\Phi} \mapsto 0, \quad D_- \partial_{--}^n \mathbf{\Phi} \mapsto 0, \quad \partial_{--}^{n+1} \overline{\mathbf{\Phi}} \mapsto 0, \quad \overline{D}_+ \partial_{++}^n \overline{\mathbf{\Phi}} \mapsto 0, \\ & \overline{D}_- \partial_{--}^n \overline{\mathbf{\Phi}} \mapsto 0, \quad D_+ \partial_{++}^n \mathbf{\Phi} \mapsto 2i \partial_{++}^{n+1} \mathbf{\Phi}, \quad \partial_{++}^n \overline{\mathbf{\Phi}} \mapsto \overline{D}_+ \partial_{++}^n \overline{\mathbf{\Phi}}, \end{aligned} \tag{3.120}$$

and d_1 acts as:

$$\begin{aligned} d_1 :& \partial_{--}^n \mathbf{\Phi} \mapsto 0, \quad \partial_{++}^n \mathbf{\Phi} \mapsto 0, \quad D_- \partial_{--}^n \mathbf{\Phi} \mapsto 0, \quad D_+ \partial_{++}^n \mathbf{\Phi} \mapsto 0, \quad \partial_{++}^n \overline{\mathbf{\Phi}} \mapsto 0, \\ & \overline{D}_+ \partial_{++}^n \overline{\mathbf{\Phi}} \mapsto 0, \quad \overline{D}_- \partial_{--}^n \overline{\mathbf{\Phi}} \mapsto \partial_{--}^n (\mathbf{\Phi}^{k+1}), \quad \partial_{--}^{n+1} \overline{\mathbf{\Phi}} \mapsto \frac{i}{2} D_- \partial_{--}^n (\mathbf{\Phi}^{k+1}). \end{aligned} \tag{3.121}$$

This explicitly describes how $\overline{D}_+$ acts on the generators, and then extends to the full algebra $\widehat{\mathcal{F}}_0$ by linearity and Leibniz rule. Notice that d_0 is just the $\overline{D}_+$ in the theory with zero superpotential, while d_1 includes corrections due to the superpotential. This splitting of $\overline{D}_+$ is motivated by a perturbative computation of the $\overline{D}_+$-cohommology, i.e., the spectral sequence, which we are about to perform.

Let us introduce a filtration degree on $\widehat{\mathcal{F}}_0$ by saying that for generators:

$$\forall x \in \mathcal{G}, \mathrm{fdeg}(x) = 1, \tag{3.122}$$

which then extends multiplicatively on the whole $\widehat{\mathcal{F}}_0$. We then define:

$$\widehat{\mathcal{F}}_0^{(p)} = \{S \in \widehat{\mathcal{F}}_0 : \mathrm{fdeg}(S) \geq p\}, \tag{3.123}$$

which gives a filtration:

$$\widehat{\mathcal{F}}_0 \cong \widehat{\mathcal{F}}_0^{(0)} \supset \widehat{\mathcal{F}}_0^{(1)} \supset \widehat{\mathcal{F}}_0^{(2)} \supset \dots \tag{3.124}$$

Our differential $\overline{D}_+$ obviously preserves this filtration. In particular, d_0 does not change the filtration degree, while d_1 increases it by k, if $k > 0$. This allows us to apply spectral sequences to compute the cohomology of $\overline{D}_+$. But before that we will mention a trivial technical lemma we will need later.

Lemma 5.1: Let V be a $\mathbb{Z}_2$-graded vector space and $S(V) = \oplus_{k\geq 0} S^k(V)$ be the graded-symmetric algebra of V. If there is a degree-1 differential $d : V \to V$, i.e., $d^2 = 0$, then by the Leibniz rule it extends to a differential acting on the graded-symmetric algebra $d : S(V) \to S(V)$, and moreover, its cohomology is:

$$H(S(V), d) = S\left(H(V, d)\right). \tag{3.125}$$

Now, having this Lemma, we will proceed to compute the cohomology of $\overline{D}_+$.

First let us consider the trivial case $k = 0$. Then both d_0 and d_1 do not change the filtration degree. We can define a vector space spanned by the elements of $\mathcal{G}$: $V = \mathrm{Span}(\mathcal{G})$. Since $\overline{D}_+ = d_0 + d_1$ does not change the filtration degree, it acts as a linear operator on this V. Next we notice that $\widehat{\mathcal{F}}_0 \simeq S(V)$, so by the Lemma $H(\widehat{\mathcal{F}}_0, \overline{D}_+) = S\big(H(V, \overline{D}_+)\big)$. To compute the cohomology of $\overline{D}_+$ acting as a linear operator on V, we notice that all elements of $\mathcal{G}$ are either not $\overline{D}_+$-closed or are $\overline{D}_+$-exact as a consequence of the equation of motion $\overline{D}_+\overline{D}_-\overline{\mathbf{\Phi}} = \mathbf{\Phi}$. So the cohomology is trivial for $k = 0$ (stress-energy supercurrent $\mathcal{J}$ becomes $\overline{D}_+$-exact for $k = 0$ ad well). This could be expected because the $k = 0$ model is massive, and therefore the IR theory it flows to is empty.

Now, suppose $k > 0$. Then at the zeroth order of spectral sequence we have:

$$E_0^p = \widehat{\mathcal{F}}_0^{(p)}/\widehat{\mathcal{F}}_0^{(p+1)}, \quad E_0 \equiv \mathrm{Gr}(\widehat{\mathcal{F}}_0) \cong \bigoplus_{p\geq 0} E_0^p, \tag{3.126}$$

where $\mathrm{Gr}(\widehat{\mathcal{F}}_0)$ is the graded space associated with the filtered space $\widehat{\mathcal{F}}_0$, and the differential acting on it is just d_0, which preserves grading. We note that $E_0^1 \simeq \mathrm{Span}(\mathcal{G})$, the vector space spanned by the generators from $\mathcal{G}$ (which all have degree 1). Since d_0 preserves grading and, as one can easily see, $E_0 \simeq S(E_0^1)$, we just apply Lemma and get $H(E_0, d_0) = S\left(H(E_0^1, d_0)\right)$. By inspecting equations (3.120), we easily find the cohomology of d_0 acting on E_0^1. The answer is $H(E_0^1, d_0) = \mathrm{Span}(S_0)$, where the set S_0 is:

$$S_0 = \{\partial_{--}^n \mathbf{\Phi},\, D_- \partial_{--}^n \mathbf{\Phi},\, \partial_{--}^{n+1} \overline{\mathbf{\Phi}},\, \overline{D}_- \partial_{--}^n \overline{\mathbf{\Phi}},\, n \geq 0\}. \tag{3.127}$$

Therefore, we find the first term of the spectral sequence:

$$E_1 = H(E_0, d_0) \simeq \mathbb{C}[S_0]. \tag{3.128}$$

Now, if $k = 1$, then for the first step of spectral sequence, d_1 becomes the differential acting on E_1. If $k > 1$, then the differential acting on E_1 is just zero, and $E_2 = H(E_1, 0) \simeq E_1$. Next, if $k > 2$, we find that $E_3 \simeq E_1$, and so on. This procedure goes on until we get to the k-th term of spectral sequence: $E_k \simeq E_1$. As we know from spectral sequences, the differential acting on E_k should be the degree-k part of $\overline{D}_+$, i.e., d_1. So for the next term we have:

$$E_{k+1} \simeq H(E_1, d_1). \tag{3.129}$$

Since there are no components of $\overline{D}_+$ of degree higher than k, the spectral sequence collapses here and we conclude that:

$$H(\widehat{\mathcal{F}}_0, \overline{D}_+) \simeq H(E_1, d_1) \simeq H(\mathbb{C}[S_0], d_1). \tag{3.130}$$

So all we need to do now is compute the cohomology of d_1 acting on $\mathbb{C}[S_0]$. The way d_1 acts on the elements of S_0 is:

$$\begin{aligned} d_1: \quad & \partial^n_{--}\mathbf{\Phi} \mapsto 0, \quad D_-\partial^n_{--}\mathbf{\Phi} \mapsto 0, \\ & \partial^{n+1}_{--}\overline{\mathbf{\Phi}} \mapsto \frac{i}{2} D_-\partial^n_{--}(\mathbf{\Phi}^{k+1}), \quad \overline{D}_-\partial^n_{--}\overline{\mathbf{\Phi}} \mapsto \partial^n_{--}(\mathbf{\Phi}^{k+1}). \end{aligned} \tag{3.131}$$

Even though we have considerably simplified the original problem, the direct computation of the d_1 cohomology is still too nasty. We can simplify it more by recalling that we already have a stress-energy supercurrent in the cohomology, and therefore it is enough to look for its superconformal primaries only. Our superpotential is of a quasi-homogeneous class, with $\beta = \frac{1}{k+2}$, so the stress-energy supercurrent is:

$$\mathcal{J} = \frac{k+1}{2(k+2)} D_-\mathbf{\Phi}\overline{D}_-\overline{\mathbf{\Phi}} - \frac{i}{k+2}\mathbf{\Phi}\partial_{--}\overline{\mathbf{\Phi}} \tag{3.132}$$

and the corresponding central charge is $c = \frac{3k}{k+2}$. Now suppose we found some polynomial $P \in \mathbb{C}[S_0]$ which represents a $\overline{D}_+$-cohomology class. We have the following technical Lemma:

Lemma 5.2: Every d_1-cohomology class $[P]$ which is a superconformal primary with respect to $\mathcal{J}$, can be represented as a polynomial of $\mathbf{\Phi}$, $D_-\mathbf{\Phi}$, $\overline{D}_-\overline{\mathbf{\Phi}}$ and $\partial_{--}\overline{\mathbf{\Phi}}$, that is $P \in \mathbb{C}[\mathbf{\Phi}, D_-\mathbf{\Phi}, \overline{D}_-\overline{\mathbf{\Phi}}, \partial_{--}\overline{\mathbf{\Phi}}]$.

The idea is that having higher derivatives of $\mathbf{\Phi}$ and $\overline{\mathbf{\Phi}}$ in the expression for P will result in higher poles in the $\mathcal{J}(x,\theta)P(0,0)$ OPE, which should not be there if

$[P]$ is primary. Elegant proof of this statement is not available at the moment, but calculations seem to show that it is true, so we leave it as a conjecture.

The operators $\mathbf{\Phi}$, $\mathbf{\Phi}^2, \ldots$ $\mathbf{\Phi}^k$ are all in the cohomology and are primaries – we will write their OPE's with $\mathcal{J}$ later. $\mathbf{\Phi}^{k+1}$ is exact and so is not in the cohomology, so any polynomial of $\mathbf{\Phi}$ is just a linear combination of $1, \mathbf{\Phi}, \mathbf{\Phi}^2, \ldots, \mathbf{\Phi}^k$ in the cohomology. Since $D_-(P(\mathbf{\Phi})) = P'(\mathbf{\Phi})D_-\mathbf{\Phi}$ and $(D_-\mathbf{\Phi})^2 = 0$, any polynomial of $\mathbf{\Phi}$ and $D_-\mathbf{\Phi}$ is $A(\mathbf{\Phi}) + D_-B(\mathbf{\Phi})$, where the second term is a descendant. Let us figure out now if there are any other primaries in the cohomology. We try to construct d_1-closed (or equivalently, $\overline{D}_+$-closed) polynomials from $\mathbf{\Phi}, D_-\mathbf{\Phi}, \overline{D}_-\overline{\mathbf{\Phi}}$ and $\partial_{--}\overline{\mathbf{\Phi}}$, which are not just polynomials of $\mathbf{\Phi}$ and $D_-\mathbf{\Phi}$. A simple computation shows that the most general such combination with even statistics is:

$$\mathcal{E} = \sum_{n=0}^{\infty} P_n(\mathbf{\Phi})(\partial_{--}\overline{\mathbf{\Phi}})^n \left[\frac{(n+1)(k+1)}{2} D_-\mathbf{\Phi}\overline{D}_-\overline{\mathbf{\Phi}} - i\mathbf{\Phi}\partial_{--}\overline{\mathbf{\Phi}}\right], \tag{3.133}$$

where P_n are arbitrary polynomials, while the most general odd closed element is:

$$\mathcal{O} = \sum_{n=1}^{\infty} C_n(\mathbf{\Phi})(\partial_{--}\overline{\mathbf{\Phi}})^n D_-\mathbf{\Phi}, \tag{3.134}$$

where again C_n are arbitrary polynomials.

To slightly simplify computations, we notice that since the operator d_1 increases the introduced above filtration degree fdeg by k, one can grade the cohomology by this degree, and it is enough to assume that $\mathcal{E}$ has a given fixed degree (i.e., it is a homogeneous polynomial). Next, we notice that we could introduce another grading – by the number of derivatives in the expression. If we assign the bosonic derivative ∂_{--} a "derivative degree" 1 and the fermionic derivatives D_- and $\overline{D}_-$ a "derivative degree" 1/2, we can see that the operator d_1 actually lowers the "derivative degree" by 1/2. Therefore, again, we can grade the cohomology by this degree, and it is enough to study the cohomology within the sector with a given "derivative degree". Fixing values of these two degrees – the filtration degree and the "derivative degree" – we see that it is enough, without loss of generality, to consider:

$$\mathcal{E}^{s,n} = \mathbf{\Phi}^s(\partial_{--}\overline{\mathbf{\Phi}})^n \left[\frac{(n+1)(k+1)}{2} D_-\mathbf{\Phi}\overline{D}_-\overline{\mathbf{\Phi}} - i\mathbf{\Phi}\partial_{--}\overline{\mathbf{\Phi}}\right],$$

$$\mathcal{O}^{s,n} = \mathbf{\Phi}^s(\partial_{--}\overline{\mathbf{\Phi}})^n D_-\mathbf{\Phi}. \tag{3.135}$$

where s and n are non-negative integers. A simple calculation gives:

$$D_-\mathcal{E}^{s,n} = -i\left[s+1+(n+1)(k+1)\right]\mathcal{O}^{s,n+1}. \tag{3.136}$$

This suggests that any odd element of the above form $\mathcal{O}^{s,n+1}$ that we could have possibly found in the cohomology would always by a descendant of some even element. This is also true for $\mathcal{O}^{s,0} = \mathbf{\Phi}^s D_-\mathbf{\Phi} = \frac{1}{s+1}D_-\mathbf{\Phi}^{s+1}$. Therefore, it is enough to study the expression $\mathcal{E}^{s,n}$ given above. Can it represent a nontrivial cohomology class, and can it be a superconformal primary?

Observables $\mathcal{E}^{s,n}$ and their lifting

Notice that for $s \geq k$:

$$d_1\left[\mathbf{\Phi}^{s-k}(\partial_{--}\overline{\mathbf{\Phi}})^{n+1}\overline{D}_-\overline{\mathbf{\Phi}}\right] = i\mathcal{E}^{s,n}, \tag{3.137}$$

so $\mathcal{E}^{s,n}$ is exact for $s \geq k$. On the other hand, for $s < k$, $\mathcal{E}^{s,n}$ is obviously not exact, because, as we can see from the equation (3.121), the image of d_1 always contains the field $\mathbf{\Phi}$ at least $k+1$ times, while $\mathcal{E}^{s,n}$ contains it $s+1$ times. So we conclude that $\mathcal{E}^{s,n}$ for $s < k$ indeed represents a non-trivial classical cohomology class.

Classical observables $\mathcal{E}^{s,n}$ satisfy the following multiplication rule:

$$\mathcal{E}^{s,n}\mathcal{E}^{t,m} = -i\mathcal{E}^{s+t+1,n+m+1}. \tag{3.138}$$

They can be combined with the observables $\mathbf{\Phi}^s$, for which we have:

$$\mathbf{\Phi}^s \mathcal{E}^{t,n} = \mathcal{E}^{s+t,n}. \tag{3.139}$$

We see that $\mathbf{\Phi}$ and $\mathcal{E}^{s,n}$ generate a closed sector in the classical cohomology. As we will find soon, these are not all classical cohomology classes, there exist more. But all observables that have a chance of being superconformal primaries in the quantum cohomology are within this sector.

The stress-energy supercurrent $\mathcal{J}$ that we identified before is of course among these observables:

$$\mathcal{J} = \frac{1}{k+2}\mathcal{E}^{0,0}. \tag{3.140}$$

In particular:

$$\mathcal{J}\mathcal{E}^{s,n} = -\frac{i}{k+2}\mathcal{E}^{s+1,n+1}. \tag{3.141}$$

This equation implies that the only observables which have a chance of being superconformal primaries at the quantum level are $\mathcal{E}^{s,0}$ and $\mathcal{E}^{0,n}$. But because of:

$$\mathcal{E}^{s,0} = (k+2)\mathcal{J}\mathbf{\Phi}^s, \tag{3.142}$$

the former are simply descendants of $\mathbf{\Phi}^s$. So we only have $\mathcal{E}^{0,n}$ left.

Can $\mathcal{E}^{0,n}$ represent cohomology classes in quantum theory? It turns out that only for $n = 0$. The reason is that for $n > 0$, the infinite piece that one has to subtract in order to define the composite operator $\mathcal{E}^{0,n}$ is not $\overline{Q}_+$-closed.

Consider the simplest operator $\mathcal{E}^{0,1}$. We call its lowest component e_1:

$$e_1 = (k+1)\partial_{--}\overline{\phi}\psi_-\overline{\psi}_- - i\phi(\partial_{--}\overline{\phi})^2. \tag{3.143}$$

This is a composite operator whose precise definition requires subtraction of singularities:

$$e_1(x) = \lim_{\epsilon\to 0}\left((k+1)\partial_{--}\overline{\phi}(x)\psi_-(x)\overline{\psi}_-(x-\epsilon) - i(\partial_{--}\overline{\phi}(x))^2\phi(x-\epsilon) - \frac{2ki}{\epsilon^{--}}\partial_{--}\overline{\phi}(x)\right). \tag{3.144}$$

We see that the piece that we subtract is not $\overline{Q}_+$-closed, which already suggests that $e_1(x)$ is probably not in the cohomology. Careful computation of $[\overline{Q}_+, e_1]$, followed by taking the $\epsilon \to 0$ limit, shows that:

$$[\overline{Q}_+, e_1] = -(k+1)\left[(k+1)\phi^k\partial_{--}\psi_- - \frac{1}{2}\psi_-\partial_{--}\phi^k\right] - i(k+2)[\overline{Q}_+, \partial^2_{--}\overline{\phi}]. \tag{3.145}$$

So indeed, e_1 is not in quantum cohomology. We know that classical observables should be lifted from cohomology in pairs. Therefore, the combination we got on the right, $r_1 = (k+1)\phi^k\partial_{--}\psi_- - \frac{1}{2}\psi_-\partial_{--}\phi^k$, should be some classical cohomology class which disappears together with e_1. And indeed, it is in the classical cohomology, as it is easy to check. Before, we found classical cohomology classes which had a chance of being superconformal primaries, and this r_1 was not among them, which suggests that it should be a descendant. Another computation shows that it is indeed a descendant. The lowest component of $\mathcal{J}$ is:

$$j = \mathcal{J}| = \frac{k+1}{2(k+2)}\psi_-\overline{\psi}_- - \frac{i}{k+2}\phi\partial_{--}\overline{\phi}, \tag{3.146}$$

it is a $U(1)$ current in the $\mathcal{N}=2$ super-Virasoro. A computation shows that:

$$j_{-1}(\phi^k\psi_-) =: j\phi^k\psi_- := \frac{i}{k+2}\left((k+1)\phi^k\partial_{--}\psi_- - \psi_-\partial_{--}\phi^k\right) + [\overline{Q}_+, \dots]. \tag{3.147}$$

So this new operator, $r_1 = (k+1)\phi^k\partial_{--}\psi_- - \frac{1}{2}\psi_-\partial_{--}\phi^k$, is actually a superconformal descendant of $\phi^k\psi_-$. One can ask a similar question: what is this $\phi^k\psi_-$? Clearly, it is in the classical cohomology. But in fact, $\phi^k\psi_- = \frac{1}{k+1}[Q_-, \phi^{k+1}]$, and recall that we have a relation $\phi^{k+1} = 0$ in the classical cohomology. Therefore $\phi^k\psi_-$ also vanishes in the classical cohomology. So we have discovered the following: classically, we have cohomology classes e_1 and r_1, but quantum-mechanically, we have $[\overline{Q}_+, e_1] = r_1$. And this r_1 is a superconformal descendant of $\phi^k\psi_-$, which is actually zero in the classical cohomology.

This might look confusing – how is it possible that superconformal descendant of zero is not zero? The resolution of this apparent paradox is that, actually, super-Virasoro algebra does not act in the classical cohomology. It only acts in the quantum cohomology by the OPE with the stress-energy supercurrent $\mathcal{J}$, while there is no notion of OPE in the classical cohomology. Therefore, there is no contradiction between the facts that $\phi^k\psi_-$ vanishes in the classical cohomology, while its superconformal descendant r_1 does not vanish classically. The fact that latter is a descendant of the former is borrowed from the chiral algebra in the quantum cohomology. And in the quantum cohomology, because of this relation, both of them indeed have to vanish. This is quite satisfactory, because it also explains why r_1 should be lifted from the classical cohomology – because it vanishes in quantum chiral algebra!

In fact, by taking all possible superconformal descendants of the relation $\phi^k = 0$, we will get a lot of (probably, infinitely many) operators which vanish in the quantum cohomology but represent non-vanishing classical cohomology classes. They all should be lifted from the cohomology through the mechanism which we have just described.

Also, it is not hard to convince oneself that not only $\mathcal{E}^{0,1}$, but all operators $\mathcal{E}^{0,n}$, $n > 0$ get lifted from the cohomology at quantum level for the same reasons. Clearly, there is some interesting (or at least non-trivial) mathematical structure in how classical cohomology classes get paired and lifted from the cohomology. It is quite possible that our observables $\mathcal{E}^{s,n}$ and superconformal descendants of ϕ^{k+1} are not the only classical cohomology classes involved in this. However, we are not going to study this question here.

We are only interested in the quantum cohomology here, so the conclusion we need now is that the only primary operators in the cohomology are $1, \mathbf{\Phi}, \mathbf{\Phi}^2, \ldots, \mathbf{\Phi}^k$. They, together with the stress-energy supercurrent $\mathcal{J}$, generate the full chiral algebra in the $\overline{Q}_+$-cohomology. One can find that:

$$\mathcal{J}(x_1,\theta_1)\mathbf{\Phi}^s(x_2,\theta_2) \sim -\left(\frac{2\theta_{12}^-\overline{\theta}_{12}^-}{(\mathbf{r}_{12})^2}h_s + \frac{i\theta_{12}^-}{\mathbf{r}_{12}}D_- + \frac{2\theta_{12}^-\overline{\theta}_{12}^-}{\mathbf{r}_{12}}\partial_{--} + \frac{i}{\mathbf{r}_{12}}q_s\right)\mathbf{\Phi}^s, \quad (3.148)$$

where $h_s = \frac{q_s}{2} = \frac{s}{2(k+2)}$. We see that dimensions and charges match exactly our expectations for the A_{k+1} minimal model.

D and E series of minimal models

We will not go into much details about the chiral algebras of D and E series of minimal models. Instead we will just look at some of their features, leaving a more detailed study for the future.

The LG models which are expected to flow to D_{2n+2} minimal models in the IR are described by the superpotential:

$$W = XY^2 + \frac{X^{2n+1}}{2n+1}. \quad (3.149)$$

Consider the $n = 1$ theory. It has $W = XY^2 + \frac{X^3}{3}$. If we make a change of variables

$$V = \frac{X+Y}{\sqrt{2}}, \quad U = \frac{X-Y}{\sqrt{2}}, \quad (3.150)$$

We will get an LG model with $W = \frac{\sqrt{2}}{3}V^3 + \frac{\sqrt{2}}{3}U^3$. This is just a pair of non-interacting A_2 models. Thus the theory in the IR is expected to be just $A_2 \otimes A_2$, with the chiral algebra being a tensor product as well. Recall from the previous subsection that, for the $W \propto V^3$, the chiral algebra has only two primaries: the identity 1 and V, and there is also a stress-energy supercurrent $\mathcal{J}_V$. Similarly for the second one: we have 1 and U as primaries, and we have $\mathcal{J}_U$. By taking the tensor product of these two, we can identify primaries in the chiral algebra of $A_2 \otimes A_2$ as:

$$1, V, U, VU, \mathcal{J}_V - \mathcal{J}_U. \tag{3.151}$$

Going back to X amd Y, the first four are simply:

$$1, X, Y, X^2 - Y^2. \tag{3.152}$$

Moreover, since in the cohomology $V^2 = U^2 = 0$, these X and Y satisfy relations in the chiral algebra:

$$\begin{aligned} X^2 + Y^2 &= 0, \\ XY &= 0, \end{aligned} \tag{3.153}$$

which are just the relations of the chiral ring, so we get the familiar result (explained in Section 3.1.3) that operators from the chiral ring of the $\mathcal{N} = (2, 2)$ theory are primaries of the chiral algebra. However, we have an extra primary operator of dimension 1:

$$\mathcal{P} = \mathcal{J}_V - \mathcal{J}_U, \tag{3.154}$$

which is not part of the chiral ring. The existence of this extra primary current in the cohomology was already noticed in [113], where the author also conjectured that every D_{2n+2} model has, in addition to the generators of the chiral ring, a single dimension-n primary in the cohomology.

We are not going to study $n > 1$ cases here. The only thing we want to mention is that the spectral sequence approach we used for the A_{k+1} models can be clearly generalized to the D_{2n+2} case. For $n > 1$, the operator $\overline{D}_+$ will split as a sum of three terms:

$$\overline{D}_+ = d_0 + d_1 + d_2, \tag{3.155}$$

where d_0 corresponds to the zero superpotential, d_1 takes into account the effect of XY^2 term in the superpotential, and d_2 encodes the effect of X^{2n+1} interaction. It should be possible, though more technical than in the A_{k+1} case, to compute the cohomology using this splitting and check the conjecture made in [113].

Finally, a small remark about the E series. The models E_6 and E_8 correspond to superpotentials X^3+Y^4 and X^3+Y^5. Therefore, their chiral algebras are immediately identified as those of $A_2 \otimes A_3$ and $A_2 \otimes A_4$ respectively. Therefore, they will also contain extra primary operators, in addition to the chiral ring elements. The E_7 model has:

$$W = X^3 + XY^3, \tag{3.156}$$

therefore it has to be studied separately. In this case again we will have:

$$\overline{D}_+ = d_0 + d_1 + d_2, \tag{3.157}$$

where d_0 is a $\overline{D}_+$ operator in the theory of two free chiral superfields without any superpotential, d_1 takes into account the X^3 term and d_2 takes care of XY^3. It is clear that at the second step of the spectral sequence computation, when we consider the cohomology of d_1, we will essentially get the cohomology of the A_2 model multiplied by the free theory described by the chiral superfield Y. Computing the cohomology of d_2 at the next step then becomes much simpler, since we already know the answer for A_2.

3.5 Some further remarks

We have only scratched the surface of the subject, demonstrating some general properties of chiral algebras of $\mathcal{N} = (0, 2)$ theories and giving several simple examples. The most general property was the RG invariance of the answer, which makes chiral algebras interesting objects to study in the context of dualities.

One obvious extension of this work would be to get a better description of chiral algebras of $\mathcal{N} = (2, 2)$ LG models with quasi-homogeneous (or even general) polynomial superpotentials. Our treatment allowed us to find answers in some cases, but it would be much nicer to have a more general result, which would associate chiral algebra to any polynomial superpotential. It would also be useful to find some classes of $\mathcal{N} = (0, 2)$ models in which the chiral algebra could be described completely.

But the most interesting and immediate extension is, of course, the application of chiral algebras to gauge theories. If the LG model has some flavor symmetry, one can gauge it by coupling to gauge multiplets. One can argue that perturbatively, the way this gauging is implemented in the chiral algebra is as follows. If G is the gauge group, one should first take the G-invariant subalgebra of the ungauged chiral algebra, then tensor multiply it by the "small" bc-system of dimension $(1, 0)$ (where "small" means that zero mode of c is excluded from the algebra). The ungauged chiral algebra has a current in it which corresponds to the flavor symmetry we want to gauge. Using this current and the bc-ghosts, one can construct a BRST operator. The condition of its nilpotency is precisely the condition that there is no gauge anomaly, i.e., that the symmetry we want to gauge really can be gauged. Then we have to compute the cohomology of this BRST operator. The answer is the gauged chiral algebra.

This procedure seems to hold in perturbation theory. One way to argue it is by writing equations of motion of the gauge theory and, similar to what we did in this paper, computing the cohomology of $\overline{D}_+$ using perturbation theory (or spectral sequence) in gauge coupling. This approach is somewhat ugly, but it allows to argue that the answer is as we claimed above. Another, more conceptual proof would be to define the gauge theory using the BRST formalism and the holomorphic gauge $v_{++} = 0$. This would give the action:

$$S = S_0 + \{Q_B, \Psi\} = S_0 + \int \mathrm{d}^2x\, l^A v^A_{++} + \int \mathrm{d}^2x\, b^A \mathcal{D}_{++} c^A, \qquad (3.158)$$

where l^A is the auxiliary field implementing gauge $v^A_{++} = 0$, and we added Faddeev-Popov ghosts. One can extend supersymmetry to act trivially on ghosts. Then the supercharge $\overline{Q}_+$ and the BRST charge Q_B anticommute: $\{\overline{Q}_+, Q_B\} = 0$, and we really have two commuting complexes. The theory is defined as the cohomology of Q_B, and within that cohomology we want to find the chiral algebra in the cohomology of $\overline{Q}_+$. Since the complexes commute, we could first find the cohomology of $\overline{Q}_+$, and then compute the cohomology of Q_B. It is quite nice to discover that the gauging procedure we explained above arises in this way. However, some techical details still have to be clarified, and this is a part of an ongoing research.

A question of utmost importance is to understand how the gauging procedure should be modified to account for non-perturbative effects, such as instantons.

Another extension, which is also important for gauge theories, is to study models without R-symmetry. We can easily find gauge theories with an anomalous R-symmetry. In case they are constructed by gauging some LG models that have (right-handed) R-symmetry, it becomes natural to ask what special happens to their chiral algebra during gauging.

Appendix A

Supergravity Conventions

Here we describe in detail our supergravity conventions in dimensions 11, 5, and 4, and also the dimensional reduction relating them. Our conventions are generally those of [87].

A.1 Gamma-Matrices and Spinors

Euclidean gamma-matrices will always satisfy a Clifford algebra with a plus sign, e.g., $\{\Gamma_I, \Gamma_J\} = 2\delta_{IJ}$. For a fermion ψ, sometimes we write

$$\overline{\psi} \equiv \psi^T C, \tag{A.1}$$

where C is usually called the charge-conjugation matrix.

A.1.1 Four Dimensions With Euclidean Signature

The flat space 4d gamma-matrices are denoted γ_a, while the curved-space matrices are $\gamma_\mu = e^a_\mu \gamma_a$, where e^a_μ is the 4d vielbein. Negative chirality (or left-handed) spinor indices are denoted $A, B, C, \ldots$, while positive chirality (or right-handed) ones are denoted $\dot{A}, \dot{B}, \dot{C}, \ldots$.

Indices $A, B, C, \ldots$ and $\dot{A}, \dot{B}, \dot{C}, \ldots$ are lowered or raised by antisymmetric tensors ε_{AB} and $\varepsilon_{\dot{A}\dot{B}}$, where we choose as usual $\varepsilon_{12} = \varepsilon^{12} = 1$. In lowering/raising indices, we adhere to the so-called NW-SE ("Northwest-Southeast") convention, when indices are always summed in the NW-SE direction: $\psi_A = \psi^B \varepsilon_{BA}$, $\psi^A = \varepsilon^{AB}\psi_B$.

We choose the following representation for the 4d Euclidean gamma-matrices:

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \; i = 1, \ldots, 3, \quad \gamma_4 = \begin{pmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{pmatrix}, \tag{A.2}$$

where σ_i are the usual Pauli matrices. The chirality matrix is:

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{A.3}$$

With this choice, the "upper" or "lower" components of a 5d spinor ψ^α are 4d spinors ψ^A and $\psi^{\dot{A}}$ of negative or positive chirality, respectively. Moreover, gamma-matrices with both spinor indices lowered behave under complex conjugation as follows:

$$(\gamma^\mu_{A\dot{A}})^* = -\varepsilon^{AB}\varepsilon^{\dot{A}\dot{B}}\gamma^\mu_{B\dot{B}}. \tag{A.4}$$

As usual in even dimensions, there are two possible charge conjugation matrices, which we will denote as C and $\widetilde{C} = -C\gamma_5$, satisfying $\gamma_\mu^T = C\gamma_\mu C^{-1}$ and $\gamma_\mu^T = -\widetilde{C}\gamma_\mu\widetilde{C}^{-1}$ (note that $\gamma_\mu^T = \gamma_\mu^*$ in Euclidean signature):

$$C = \gamma_2\gamma_4 = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \quad \widetilde{C} = -C\gamma_5 = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.5}$$

By saying that we lower/raise both left and right 4d spinor indices by ε, we have automatically picked $\widetilde{C}$ in $d = 4$.

In Lorentz signature, fermions always carry a real structure. This is typically not the case in Euclidean signature (for example, if the Standard Model of particle physics is formulated in Euclidean signature, the fermions carry no real structure). For our purposes in this paper, spinors in 4d Euclidean space can be considered to come by dimensional reduction from 5d Minkowski spacetime, and therefore they carry a real structure. Since the spinor representation of Spin(4) (or of Spin(4, 1)) is pseudo-real rather than real, to define a reality condition, one has to add an extra index $i = 1, 2$ (which can also be lowered/raised by an antisymmetric tensor ε_{ij}). Then the reality conditions for left-handed and right-handed spinors ψ^{Ai} and $\psi^{\dot{A}i}$ respectively are:

$$
\begin{aligned}
(\psi^{Ai})^* &= \psi_{Ai}, \\
(\psi^{\dot{A}i})^* &= \psi_{\dot{A}i}.
\end{aligned}
\tag{A.6}
$$

A.1.2 5d Gamma-Matrices and Spinors

We denote 5d gamma-matrices as $\Gamma_{\mathfrak{a}}$ with flat index $\mathfrak{a}$ (or Γ_M with the curved index M). In Lorentz signature, we choose the following relation between 5d and 4d gamma-matrices:

$$
\begin{aligned}
\Gamma_{\mathfrak{a}=a} &= \gamma_a, a = 1 \dots 4, \\
\Gamma_0 &= i\gamma_5.
\end{aligned}
\tag{A.7}
$$

In 5d Euclidean signature, we take the fifth gamma-matrix to be $\Gamma_5 = \gamma_5$.

We denote 5d spinor indices by $\alpha, \beta, \gamma, \dots$. They are lowered/raised by the matrix $C_{\alpha\beta}$ that was defined in eqn. (A.5) (in $d = 5$, Lorentz invariance leaves no choice in this matrix) and again a NW-SE rule is applied. We sometimes write a 5d spinor Ψ^α in terms of the 4d chiral basis and think of it as a pair of Weyl spinors Ψ^A and $\Psi^{\dot{A}}$, but with indices raised or lowered by the 5d matrix $C_{AB} = \varepsilon_{AB}, C_{\dot{A}\dot{B}} = -\varepsilon_{\dot{A}\dot{B}}$. In particular, that is how we usually treat the 5d supersymmetry algebra, writing it in terms of the chiral components Q_{Ai} and $Q_{\dot{A}i}$. Of course, such a splitting explicitly breaks part of the Spin$(4,1)$ symmetry, but this part is broken by the Kaluza-Klein reduction anyway.

To define the reality condition satisfied by 5d spinors in Lorentz signature, we first introduce

$$B = -i\Gamma_0 C = \begin{pmatrix} -\varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}, \tag{A.8}$$

and define

$$\Psi^c = B^{-1}\Psi^*. \tag{A.9}$$

To satisfy a reality condition, a spinor also needs an additional index $i = 1, 2$, since the spinor representation of Spin$(4,1)$ is pseudoreal. Finally, the reality condition on Ψ^i is $\Psi^i = \varepsilon^{ij}(\Psi^j)^c$. In terms of the chiral components Ψ^{Ai} and $\Psi^{\dot{A}i}$, this condition is:

$$\begin{aligned} (\Psi^{Ai})^* &= \Psi^{Bj}\varepsilon_{BA}\varepsilon_{ji} \equiv \Psi_{Ai}, \\ (\Psi^{\dot{A}i})^* &= \Psi^{\dot{B}j}\varepsilon_{\dot{B}\dot{A}}\varepsilon_{ji} \equiv -\Psi_{\dot{A}i}. \end{aligned} \tag{A.10}$$

The 5d spinor $\Psi^{i\alpha}$ in (4+1)d reduces to a pair of 4d spinors ψ^{Ai} and $\psi^{\dot{A}i}$. We make the identification with the indices raised: $\psi^{Ai} = \Psi^{Ai}$, $\psi^{\dot{A}i} = \Psi^{\dot{A}i}$. It is necessary to specify this because we have introduced a slightly different convention in raising and lowering 4d spinor indices.

A.1.3 6d and 11d Gamma-Matrices

We denote the 6d gamma-matrices along the Calabi-Yau manifold Y as $\widetilde{\gamma}_n, n = 6 \dots 11$ (here we do not specify whether the index is "curved" or "flat"). The 6d chirality matrix is $\widetilde{\gamma}_* = i\widetilde{\gamma}_6 \cdots \widetilde{\gamma}_{11}$. We think about spinors on Y as $(0,p)$-forms for $p = 0 \dots 3$. If z^i are local coordinates on Y and $\boldsymbol{Q}_{i\bar{j}}$ is a metric on Y, the gamma-matrices act as:

$$\begin{aligned} \widetilde{\gamma}_{z^i} &= \sqrt{2}\boldsymbol{Q}_{i\bar{j}} \mathrm{d}\bar{z}^{\bar{j}} \wedge \\ \widetilde{\gamma}_{\bar{z}^{\bar{i}}} &= \sqrt{2}\iota_{\frac{\partial}{\partial \bar{z}^{\bar{i}}}}. \end{aligned} \tag{A.11}$$

We choose chirality in such a way that a covariantly constant spinor λ_- of negative chirality corresponds to an antiholomorphic $(0,3)$-form $\overline{\Omega}$, while a covariantly constant spinor λ_+ of positive chirality corresponds to a constant function 1. We choose the 6d charge conjugation matrix C_6 satisfying

$$\widetilde{\gamma}_n^T = -C_6 \widetilde{\gamma}_n C_6^{-1}. \tag{A.12}$$

The choice of C_6 lets us define a bilinear pairing $(\; , \;)$ on fermions, and we require that $(\lambda_+, \lambda_-) = (\lambda_-, \lambda_+) = 1$.

Let us use calligraphic letters for the 11d indices and denote 11d gamma-matrices by slanted capital gamma. So we write $\mathit{\Gamma}_{\mathcal{A}}$ for 11d gamma-matrices referred to a flat basis and $\mathit{\Gamma}_{\mathcal{M}}$ for the ones referred to a curved basis. We choose the 11d gamma-matrices to be related as follows to the 5d and 6d gamma-matrices:

$$\begin{aligned} \mathit{\Gamma}_{\mathcal{A}=a} &= \Gamma_a \otimes \widetilde{\gamma}_*, \; a = 1 \dots 5 \\ \mathit{\Gamma}_{\mathcal{A}=n} &= \mathbb{1}_4 \otimes \widetilde{\gamma}_n, \; n = 6 \dots 11, \end{aligned} \tag{A.13}$$

where $\mathbb{1}_4$ is the unit 4×4 matrix. In Lorentz signature (where $\varGamma_5$ is replaced by $\varGamma_0$), we require:

$$\varGamma_0 \varGamma_1 \ldots \varGamma_4 \varGamma_6 \varGamma_7 \ldots \varGamma_{11} = 1. \tag{A.14}$$

We will use large lower-case Greek letters to denote 11d spinorial indices: $\alpha, \beta, \ldots$ With the above choice of the 6d charge conjugation matrix, the 11d charge conjugation matrix C_{11} is related to the 5d and 6d matrices in an obvious way:

$$C_{11} = C_5 \otimes C_6. \tag{A.15}$$

In Lorentz signature, the supersymmetry generators are an 11d Majorana fermion η. In compactification on Y, the unbroken supersymmetries are those for which η is the tensor product of λ_+ or λ_- with a 5d spinor ϵ^1 or ϵ^2:

$$\eta = \epsilon^2 \otimes \lambda_+ + \epsilon^1 \otimes \lambda_-. \tag{A.16}$$

A.2 5d SUSY Algebra

From (A.4), (A.7) and (A.10), one can find, in 5d Minkowski space, the SUSY algebra compatible with the 5d reality conditions:

$$\{Q_{\alpha i}, Q_{\beta j}\} = -i\varepsilon_{ij}\Gamma^M_{\alpha\beta}P_M + \varepsilon_{ij}C_{\alpha\beta}\zeta, \tag{A.17}$$

where ζ is a real central charge. In a chiral basis, the algebra is

$$\begin{aligned} \{Q_{Ai}, Q_{Bj}\} &= \varepsilon_{AB}\varepsilon_{ij}(H+\zeta) \\ \{Q_{Ai}, Q_{\dot{B}j}\} &= -i\varepsilon_{ij}\Gamma^{\mu}{}_{A\dot{B}}P_\mu \\ \{Q_{\dot{A}i}, Q_{\dot{B}j}\} &= \varepsilon_{\dot{A}\dot{B}}\varepsilon_{ij}(H-\zeta), \end{aligned} \tag{A.18}$$

where $H = P^0$ is the 5d Hamiltonian.

A.3 11d Supergravity

Though not explicitly used in the main part of the paper, the following form of the 11d supergravity action is implicitly assumed:

$$\mathcal{L} = \frac{1}{2\kappa_{11}^2}\Bigg(ER - \frac{E}{48}G^2 + \frac{1}{12^4}\epsilon^{\mathcal{MNLP}_1\ldots\mathcal{P}_4\mathcal{Q}_1\ldots\mathcal{Q}_4}C_{\mathcal{MNL}}G_{\mathcal{P}_1\ldots\mathcal{P}_4}G_{\mathcal{Q}_1\ldots\mathcal{Q}_4} \\ -E\overline{\psi}_{\mathcal{M}}\Gamma^{\mathcal{MNP}}D_{\mathcal{N}}\left[\frac{1}{2}(\omega+\widehat{\omega})\right]\psi_{\mathcal{P}} \\ -\frac{E}{192}(\overline{\psi}_{\mathcal{Q}}\Gamma^{\mathcal{MNLPQR}}\psi_{\mathcal{R}} + 12\overline{\psi}^{\mathcal{M}}\Gamma^{\mathcal{NL}}\psi^{\mathcal{P}})(G+\widehat{G})_{\mathcal{MNLP}}\Bigg), \tag{A.19}$$

where E is the determinant of the 11d vielbein, G is a curvature of the C-field, $\psi_{\mathcal{M}}$ is a gravitino field (a Majorana vector-spinor), and hatted quantities include some extra corrections quadratic in fermions (the exact expressions are not important to us). The supersymmetry transformations are:

$$\begin{aligned} \delta E^{\mathcal{A}}_{\mathcal{M}} &= \frac{1}{2}\overline{\eta}\Gamma^{\mathcal{A}}\psi_{\mathcal{M}}, \\ \delta\psi_{\mathcal{M}} &= D_{\mathcal{M}}(\widehat{\omega})\eta + T_{\mathcal{M}}^{\ \mathcal{NPQR}}G_{\mathcal{NPQR}}\eta, \\ \delta C_{\mathcal{MNP}} &= -\frac{3}{2}\overline{\eta}\Gamma_{[\mathcal{MN}}\psi_{\mathcal{P}]}, \end{aligned} \tag{A.20}$$

where

$$T_{\mathcal{M}}^{\ \mathcal{NPQR}} = \frac{1}{288}\left(\Gamma_{\mathcal{M}}^{\ \mathcal{NPQR}} - 8\delta_{\mathcal{M}}^{[\mathcal{N}}\Gamma^{\mathcal{PQR}]}\right). \tag{A.21}$$

In the action above, κ_{11} is the 11-dimensional gravitational constant. It is actually related to the M2-brane tension T_2 (see for example [90]), the relation being:

$$2\kappa_{11}^2(T_2)^3 = (2\pi)^2. \tag{A.22}$$

We will work in units with $T_2 = 1$ and thus:

$$2\kappa_{11}^2 = (2\pi)^2. \tag{A.23}$$

A.4 Reduction from 11d to 5d

We review the dimensional reduction of 11d supergravity on a Calabi-Yau Y (an original reference is [114]). We denote the 5-dimensional metric as G (this will hopefully not be confused with the field strength G of the 11d C-field). We denote the Calabi-Yau Ricci metric of Y as $\boldsymbol{Q}$, and the compatible complex structure as I. The Kahler form of Y is $\omega = \boldsymbol{Q}(I\cdot, \cdot)$. The volume form is:

$$\mathrm{Vol} = \frac{1}{6}\omega \wedge \omega \wedge \omega. \tag{A.24}$$

For arbitrary $(1,1)$-forms α and β, we have identities:

$$\begin{aligned} \beta \wedge \omega \wedge \omega &= (\beta, \omega)\mathrm{Vol}, \\ \alpha \wedge \beta \wedge \omega &= \frac{1}{4}\left[(\alpha,\omega)(\beta,\omega) - 2(\alpha,\beta)\right]\mathrm{Vol}, \\ *\alpha &= -2\alpha \wedge \omega + \frac{1}{2}(\alpha,\omega)\omega \wedge \omega. \end{aligned} \tag{A.25}$$

Here the inner product on 2-forms is defined by $(\alpha, \beta) = \alpha_{nr}\beta_{ms}\boldsymbol{Q}^{nm}\boldsymbol{Q}^{rs}$.

Because Y is Ricci-flat, its Kahler form is harmonic and thus can be expanded in a basis of harmonic $(1,1)$-forms (ω_I):

$$\omega = \sum_I v^I \omega_I, \tag{A.26}$$

where v^I are Kahler moduli. Define also:

$$\mathcal{C}_{IJK} = \frac{1}{6}\int_Y \omega_I \wedge \omega_J \wedge \omega_K. \tag{A.27}$$

Now let us reduce the bosonic part of the 11d action. We will be interested in the Kahler moduli of only. (The complex structure moduli of Y give rise to hypermultiplets, which decouple at low energies from the vector multiplet couplings that are described by the GV formula.)

One can find the following formula for the 11d Ricci scalar in terms of the 5d Ricci scalar and the Calabi-Yau metric:

$$\sqrt{\boldsymbol{Q}}R^{(11)} = \sqrt{\boldsymbol{Q}}R^{(5)} - \nabla_M(\sqrt{\boldsymbol{Q}}\boldsymbol{Q}^{mn}\partial^M \boldsymbol{Q}_{mn}) - \sqrt{\boldsymbol{Q}}\left(\frac{1}{4}(\partial_M \boldsymbol{Q}, \partial^M \boldsymbol{Q}) - \frac{1}{4}(\boldsymbol{Q}, \partial_M \boldsymbol{Q})(\boldsymbol{Q}, \partial^M \boldsymbol{Q})\right). \tag{A.28}$$

Here M is a 5d index, and covariant derivatives are with respect to the 5d metric. The total derivative part clearly drops out of the action.

Denote the volume of Y by V. Introduce also $v = V^{1/3}$ and

$$h^I = \frac{v^I}{v}. \tag{A.29}$$

The volume is part of a hypermultiplet, so we are not interested in the action for it right now. Using (A.25), we can find:

$$\begin{aligned}\int_Y R^{(11)}\mathrm{Vol} &= V(x)R^{(5)} + \int_Y \partial_M \omega \wedge \partial^M \omega \wedge \omega + \frac{1}{4}\int_Y \partial_M \omega \wedge *(\partial^M \omega) \\ &= V(x)\left(R^{(5)} + 3C_{IJK}h^I \partial_M h^J \partial^M h^K + \text{hypermultiplet part}\right).\end{aligned} \tag{A.30}$$

Now take a look at the 3-form field. At low energies, we expand

$$C = \sum_I V^I \wedge \omega_I,$$
$$G = \sum_I \mathrm{d}V^I \wedge \omega_I, \tag{A.31}$$

where the V^I are abelian gauge fields in five dimensions. Then

$$(G, G) = 6(\mathrm{d}V)^I_{MN}(\mathrm{d}V)^J_{PQ}\mathrm{G}^{MP}\mathrm{G}^{NQ}(\omega_I, \omega_J). \tag{A.32}$$

The kinetic term for C in 11 dimensions reduces in $d = 5$ to

$$-\frac{v}{4}a_{IJ}(\mathrm{d}V^I \cdot \mathrm{d}V^J), \tag{A.33}$$

where, using (A.25):

$$a_{IJ} = \frac{1}{4}\int_Y \omega_I \wedge *\omega_J = -3\mathcal{C}_{IJK}h^K + \frac{9}{2}(\mathcal{C}hh)_I(\mathcal{C}hh)_J. \tag{A.34}$$

The 11d Chern-Simons term reduces to

$$-\frac{1}{2}\mathcal{C}_{IJK}V^I \wedge \mathrm{d}V^J \wedge \mathrm{d}V^K. \tag{A.35}$$

So, ignoring hypermultiplets, the bosonic part of the action is (remember that $\kappa_{11}^2 = 2\pi^2$):

$$2\pi^2\mathcal{L}_5\mathrm{Vol} = \left[V(x)\left(\frac{1}{2}R^{(5)} + \frac{3}{2}\mathcal{C}_{IJK}h^I\partial_M h^J\partial^M h^K\right) - \frac{v}{4}a_{IJ}(\mathrm{d}V^I \cdot \mathrm{d}V^J)\right]\mathrm{Vol}$$
$$-\frac{1}{2}\mathcal{C}_{IJK}V^I \wedge \mathrm{d}V^J \wedge \mathrm{d}V^K. \tag{A.36}$$

Now make a Weyl rescaling of the 5d metric $\mathrm{G}_{MN} \to \frac{1}{v^2}\mathrm{G}_{MN}$, to bring the 5d action to the Einstein frame:

$$2\pi^2 \mathcal{L}_5 \mathrm{Vol} = \Big[\frac{1}{2}R^{(5)} + \frac{3}{2}\mathcal{C}_{IJK}h^I\partial_M h^J\partial^M h^K - \frac{1}{4}a_{IJ}(dV^I \cdot dV^J)\Big]\mathrm{Vol} \\ -\frac{1}{2}\mathcal{C}_{IJK}V^I \wedge dV^J \wedge dV^K + \text{hypermultiplets.} \qquad \text{(A.37)}$$

Our conventions in this action are slightly different from those often found in the literature. To get the action in the conventions of [96], one has to rescale by $h^I \to \sqrt{\frac{3}{2}}h^I$ and $\mathcal{C}_{IJK} \to \frac{2\sqrt{2}}{3\sqrt{3}}\mathcal{C}_{IJK}$ (and also do appropriate rescalings to get rid of the factor $2\pi^2$ coming from the gravitational constant). However, the action normalized as in (A.37) is more convenient for us.

Some quantities that appeared in section 2.1 are

$$\begin{aligned} h_I &= \mathcal{C}_{IJK}h^J h^K, \\ a_{IJ} &= -3\mathcal{C}_{IJK}h^K + \frac{9}{2}h_I h_J, \\ h_I &= \frac{2}{3}a_{IJ}h^J. \end{aligned} \qquad \text{(A.38)}$$

The constraint $\mathcal{C}_{IJK}h^I h^J h^K = 1$ (eqn. (2.34), which implies that $h_I h^I = 1$, was used in the last line.

The scalar kinetic energy in $2\pi^2\mathcal{L}_5$ can be rewritten as:

$$-\frac{1}{2}a_{IJ}\partial_M h^I \partial^M h^J. \qquad \text{(A.39)}$$

A.5 Reduction from 5d to 4d

We reduce the $N = 1, d = 5$ supergravity on a circle and make the field redefinitions required to relate it to the standard $N = 2, d = 4$ supergravity in the Einstein frame metric (a similar procedure was performed in [97]).

Assume that the fifth direction is a circle parametrized by an angular variable y ($0 \leq y \leq 2\pi$). After integrating over y, the overall factor of $1/(2\pi^2)$ in front of the 5d action (A.37) will be replaced by an overall $1/\pi$ in front of the 4d action. This factor is sometimes removed by rescalings, but we will find it more convenient not to do so.

Take the following ansatz for the funfbein e_M^a:

$$e_M^a = \begin{pmatrix} e^{-\sigma/2} e_\mu^a & e^\sigma B_\mu \\ 0 & e^\sigma \end{pmatrix}, \quad e_a^M = \begin{pmatrix} e^{\sigma/2} e_a^\mu & -e^{\sigma/2} e_a^\mu B_\mu \\ 0 & e^{-\sigma} \end{pmatrix}. \tag{A.40}$$

The 5-dimensional Ricci scalar $R^{(5)}$ takes the following form in terms of the 4-dimensional Ricci scalar $R^{(4)}$ and other fields present in the funfbein:

$$R^{(5)} = e^\sigma R^{(4)} + e^\sigma \Box \sigma - \frac{3}{2} e^\sigma (\partial\sigma)^2 - \frac{1}{4} e^{4\sigma} (dB)^2. \tag{A.41}$$

Set $\alpha^I = V_y^I$. Define the 4-dimensional gauge fields as $A^\Lambda, \Lambda = 0 \ldots b_2(Y)$, where $\Lambda = I = 1, \ldots, b_2(Y)$ come from reduction of the 5-dimensional vectors, while $\Lambda = 0$ corresponds to the Kaluza-Klein (KK) vector:

$$\begin{aligned} A_\mu^I &= V_\mu^I - \alpha^I B_\mu \\ A_\mu^0 &= -B_\mu, \end{aligned} \tag{A.42}$$

The scalar kinetic term in 4d originates from the curvature term in 5d, the scalar kinetic term in 5d and the vector kinetic term in 5d. It takes the form:

$$-\frac{1}{\pi} \left(\frac{1}{2} e^{-2\sigma} a_{IJ} \partial_\mu (e^\sigma h^I) \partial^\mu (e^\sigma h^J) + \frac{1}{2} e^{-2\sigma} a_{IJ} \partial_\mu \alpha^I \partial^\mu \alpha^J \right). \tag{A.43}$$

If we define a complex scalar

$$Z^I = \alpha^I + ie^{\sigma}h^I, \tag{A.44}$$

then the kinetic term becomes

$$-\frac{1}{\pi}g_{I\overline{J}}\partial_\mu Z^I \partial^\mu \overline{Z}^J, \tag{A.45}$$

where

$$g_{L\overline{M}} = \frac{1}{2}e^{-2\sigma}a_{LM} = \frac{\partial}{\partial Z^L}\frac{\partial}{\partial \overline{Z^M}}\log\left[\mathcal{C}_{IJK}(Z^I - \overline{Z^I})(Z^J - \overline{Z^J})(Z^K - \overline{Z^K})\right]. \tag{A.46}$$

The vector kinetic term takes the standard form:

$$-\frac{i}{4\pi}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda+}_{\mu\nu}F^{\Sigma+\mu\nu} + \text{c.c.} \tag{A.47}$$

with coefficients

$$\begin{aligned}
\mathcal{N}_{IJ} &= -i(e^{\sigma}a_{IJ} - 3i\mathcal{C}_{IJK}\alpha^K), \\
\mathcal{N}_{I0} &= i(e^{\sigma}a_{IJ}\alpha^J - \frac{3i}{2}\mathcal{C}_{IJK}\alpha^J\alpha^K), \\
\mathcal{N}_{00} &= -i(e^{\sigma}a_{IJ}\alpha^I\alpha^J - i\mathcal{C}_{IJK}\alpha^I\alpha^J\alpha^K + \frac{1}{2}e^{3\sigma}).
\end{aligned} \tag{A.48}$$

One can check (see (2.50)) that this corresponds to the prepotential:

$$\mathcal{F}_0^{\text{cl}} = -\frac{1}{2}\frac{\mathcal{C}_{IJK}X^IX^JX^K}{X^0}. \tag{A.49}$$

Another useful relation in KK reduction from $d = 5$ to $d = 4$ is the expression for the 5d Dirac operator

$$\not{D} = \Gamma^M(\partial_M + \frac{1}{4}\omega_M^{ab}\Gamma_{ab} - iq_I V_M^I) \tag{A.50}$$

in terms of the 4d fields:

$$\not{D} = e^{\sigma/2}\gamma^\mu(\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - iq_I V_\mu^I - B_\mu\partial_y + iq_I\alpha^I B_\mu) + e^{-\sigma}\gamma_5\partial_y - iq_I\alpha^I e^{-\sigma}\gamma_5 + \frac{1}{8}e^{2\sigma}(\mathrm{d}B)_{\mu\nu}\gamma^{\mu\nu}\gamma_5 - \frac{1}{4}e^{\sigma/2}\not{\partial}\sigma. \tag{A.51}$$

Taking σ to be constant, taking $V_\mu^I = h^I V_\mu = \frac{h^I}{4}e^{-\sigma/2}U_\mu$, and acting on a field with the KK mode number $-n$, this reduces to:

$$\begin{aligned} \not{D} &= e^{\sigma/2}\gamma^\mu D_\mu - ie^{-\sigma}(n + q_I\alpha^I)\gamma_5 - \frac{i}{32}e^{\sigma/2}\mathsf{W}^-_{\mu\nu}\gamma^{\mu\nu}\gamma_5, \\ D_\mu &= \partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} - i\frac{\overline{\mathbb{Z}}}{4}U_\mu. \end{aligned} \tag{A.52}$$

One can see that the first term in the expression for $\not{D}$ is just a 4d Dirac operator, the second term corrects the 5d mass term (replacing M by $\mathbb{Z}$ or $\overline{\mathbb{Z}}$ depending on the 4d chirality), and the third term shifts the 5d magnetic moment coupling. This expression is important in the dimensional reduction of the 5d hypermultiplet action performed in section 2.3.2.

The fields that we have described can be organized in 4d supermultiplets $\mathcal{X}^\Lambda$ and $\mathcal{W}_{AB}$ as described in the main text. General references on these superfields are [49–51].

Appendix B

M2-brane on a Holomorphic Curve

Here we derive the 5d BPS superparticle action describing an M2-brane wrapped on a smooth isolated holomorphic curve Σ in a Calabi-Yau manifold Y.

B.1 Membrane in Superspace

An M-theory membrane can be described as a submanifold Ω of dimension $3|0$ living in a superspace $\mathfrak{M}$ of dimension $11|32$. If the background fields are purely bosonic (as we can assume for our purposes), then $\mathfrak{M}$ is split, with reduced space some 11-dimensional spin-manifold $\mathfrak{M}_{\mathrm{red}}$ and odd directions that parametrize the spin bundle $S(\mathfrak{M}_{\mathrm{red}})$. We consider Ω as an abstract three-manifold with an embedding in $\mathfrak{M}$:

$$\widehat{X} : \Omega \to \mathfrak{M}. \tag{B.1}$$

Since $\mathfrak{M}$ projects to its reduced space $\mathfrak{M}_{\mathrm{red}}$, $\widehat{X}$ can be projected to an embedding $X : \Omega \to \mathfrak{M}_{\mathrm{red}}$. The additional information in $\widehat{X}$ is a fermionic section of the pull-back of the spinor bundle:

$$\Theta \in \Pi\Gamma\left(\Omega, X^* S(\mathfrak{M}_{\mathrm{red}})\right). \tag{B.2}$$

Here $\Gamma(\Omega, \cdot)$ represents the space of sections, and the symbol Π tells us that Θ has odd statistics. X and Θ are the fields that are governed by the membrane world-volume theory. The κ-symmetric action for these fields was constructed in [115, 116]. Our main reference for expanding the component action is [117]. Their conventions for 11d supergravity are slightly different from ours and can be translated by $\omega_\mu^{ab} \to -\omega_{\mathcal{M}}^{\mathcal{AB}}$, $R \to -R$, $\psi_\mu \to \frac{1}{2}\psi_{\mathcal{M}}$, and $\eta \to \frac{1}{2}\eta$ (here η is the 11d supersymmetry generator), while also reversing the orientations of $\mathfrak{M}_{\text{red}}$ and Ω and multiplying the action by an overall constant.

We parametrize Ω by local coordinates $\zeta^0, \zeta^1, \zeta^2$. We denote the fields on the membrane as $Z^{\widehat{\mathcal{M}}}(\zeta) = (X^{\mathcal{M}}(\zeta), \Theta^{\alpha}(\zeta))$ (an index like $\widehat{\mathcal{M}}$ with a hat denotes a superspace index). Let $E_{\widehat{\mathcal{M}}}^{\widehat{\mathcal{A}}}$ be the supervielbein, where $\widehat{\mathcal{M}}$ is a curved and $\widehat{\mathcal{A}}$ is a flat superspace index. Let $B_{\widehat{\mathcal{M}}\widehat{\mathcal{N}}\widehat{\mathcal{P}}}$ be the superspace three-form gauge superfield. $E_{\widehat{\mathcal{M}}}^{\widehat{\mathcal{A}}}$ and $B_{\widehat{\mathcal{M}}\widehat{\mathcal{N}}\widehat{\mathcal{P}}}$ encode the target space geometry. The pull-back of the supervielbein to the membrane is $\Pi_i^{\widehat{\mathcal{A}}} = E_{\widehat{\mathcal{M}}}^{\widehat{\mathcal{A}}} \partial Z^{\widehat{\mathcal{M}}}/\partial\zeta^i$. The induced metric is $g_{ij} = \Pi_i^{\mathcal{A}}\Pi_j^{\mathcal{B}}\eta_{\mathcal{AB}}$, where $\eta_{\mathcal{AB}}$ is the 11-dimensional Minkowski metric. Here $\mathcal{A}, \mathcal{B}$ are ordinary flat 11-dimensional indices. Then the membrane action is:

$$S = \int d^3\zeta \left[-\sqrt{-g} + \frac{1}{6}\varepsilon^{ijk}\Pi_i^{\widehat{\mathcal{A}}}\Pi_j^{\widehat{\mathcal{B}}}\Pi_k^{\widehat{\mathcal{C}}} B_{\widehat{\mathcal{C}}\widehat{\mathcal{B}}\widehat{\mathcal{A}}} \right]. \tag{B.3}$$

Define the matrix:

$$\varGamma = -\frac{\varepsilon^{ijk}}{6\sqrt{-g}}\Pi_i^{\mathcal{A}}\Pi_j^{\mathcal{B}}\Pi_k^{\mathcal{C}}\varGamma_{\mathcal{ABC}}. \tag{B.4}$$

It satisfies $\varGamma^2 = 1$ and enters in defining the κ-symmetry of the membrane action:

$$\delta Z^{\widehat{\mathcal{M}}} E_{\widehat{\mathcal{M}}}^{\mathcal{A}} = 0, \quad \delta Z^{\widehat{\mathcal{M}}} E_{\widehat{\mathcal{M}}}^{\alpha} = (1 - \varGamma)^{\alpha}{}_{\beta}\kappa^{\beta}, \tag{B.5}$$

where $\kappa(\zeta)$ is a local fermionic parameter. The κ-symmetry allows one to gauge away half of the fermionic degrees of freedom on the membrane. (Instead of saying that the membrane has a worldvolume of dimension 3|0 and is governed by a κ-symmetric action, an equivalent point of view that has some advantages is to say that the membrane worldvolume has dimension 3|8. The 3|8-dimensional membrane worldvolume in the second point of view is obtained by applying all possible κ transformations to the 3|0-dimensional membrane worldvolume in the first point of view. This refinement will not be important for us.)

B.2 Wrapped BPS Membrane

We focus on the case $\mathfrak{M}_{\text{red}} = M \times Y$, where Y is a Calabi-Yau manifold and M is a five-manifold with a large radius of curvature. In our application, M will eventually be either Minkowski spacetime or the supersymmetric Gödel universe (also called the graviphoton background in this paper). Let $\Sigma \subset Y$ be a 2-cycle inside of Y. Consider an M2-brane wrapping Σ. It propagates as a 5d particle on M, given that the radius of curvature of M is large enough. More precisely, a propagating M2-brane wrapped on Σ generates a whole infinite set of 5d particles corresponding to its different internal excitations. These excitations may or may not preserve some supersymmetry, and correspondingly the particles propagating on M form short or long multiplets of SUSY. We are interested in those particles that preserve as much of the 5d supersymmetry as possible, namely half of it. These arise from the supersymmetric ground states of the internal motion. So those are the states that we must understand.

A supersymmetry of the ambient superpace $\mathfrak{M}$ remains unbroken in the presence of an M2-brane if in the M2-brane theory the supersymmetry transformation can be compensated by a κ-transformation [67]. For this to be possible, Σ must be[1] a holomorphic curve in Y [67]. In this appendix, we will consider only the case that Σ is isolated; in other words, we assume that it has no deformations (even infinitesimal

[1]In [67], this is shown for a string worldsheet in a superstring theory compactified on Y. Our case can be reduced to this by considering an M2-brane wrapping Σ and winding the M-theory circle once.

ones) as a holomorphic curve in Y. Otherwise, the moduli of Σ, along with fermionic zero-modes that will be related to them by supersymmetry, must be quantized in order to determine the supersymmetric states of the M2-brane.

So now we consider a membrane with worldvolume $\Sigma \times \gamma$, where $\gamma \subset M$ is a 5d worldline. We parametrize γ by a coordinate t and Σ by a local holomorphic coordinate z. The membrane worldvolume is parametrized as:

$$\begin{aligned} X^M(t,z,\overline{z}) &= x^M(t),\ M = 0\dots 4, \\ X^m(t,z,\overline{z}) &= X^m(z,\overline{z}),\ m = 6\dots 11, \end{aligned} \tag{B.6}$$

where $x^M(t)$ parametrizes $\gamma \subset M$, and $X^m(z,\overline{z})$ parametrizes $\Sigma \subset Y$. If M is taken to have Euclidean signature, we replace here X^0 by $X^5 = iX^0$. We pick local holomorphic coordinates (z, w^1, w^2) on Y so that Σ is locally defined by $w^1 = w^2 = 0$.

To describe the fermionic fields of the M2-brane, we first note that $S(M \times Y) = S(M) \otimes S(Y)$. Then we recall that on a Calabi-Yau manifold, one has an isomorphism $S(Y) \cong \Omega^{0,\bullet}Y$, where $\Omega^{0,\bullet}$ is the space of $(0,q)$-forms, $q = 0,\dots,3$. In this isomorphism, the Dirac operator on Y is simply $\sqrt{2}(\overline{\partial} + \overline{\partial}^*)$ (where $\overline{\partial}^*$ is the adjoint of $\overline{\partial}$ with respect to the natural L^2-scalar product). Thus the field Θ^α has a 5d spinor index α and takes values in the $(0,p)$-forms on Y, restricted to Σ. The fact that Y is Calabi-Yau implies various isomorphisms between bundles. Let Ω be a holomorphic 3-form on Y, normalized so that the volume form of Y is $i\Omega \wedge \overline{\Omega}$. Let $\mathcal{G}_{z\overline{z}}$ be the restriction to Σ of the Kahler metric of Y and let $G^N_{w^i\overline{w}^j}$ be the induced metric on the normal bundle $N\Sigma$ to Σ in Y. We write for various components of Θ^α:

$$\begin{aligned}
\theta^\alpha_{\overline{w}^i} &= G^N_{\overline{w}^i w^j}\chi^{\alpha w^j} \\
\theta^\alpha_{\overline{z}\,\overline{w}^i} &= \overline{\Omega}_{\overline{z}\,\overline{w}^i\overline{w}^j}\widetilde{\chi}^{\alpha\overline{w}^j} \\
\theta^\alpha_{\overline{w}^1\overline{w}^2} &= \overline{\Omega}_{\overline{w}^1\overline{w}^2\overline{z}}\mathcal{G}^{z\overline{z}}\theta^\alpha_z \\
\theta^\alpha_{\overline{z}\,\overline{w}^1\overline{w}^2} &= \overline{\Omega}_{\overline{z}\,\overline{w}^1\overline{w}^2}\widetilde{\theta}^\alpha.
\end{aligned} \tag{B.7}$$

The fields χ and $\widetilde{\chi}$ are sections of the normal bundle $N\Sigma$ (tensored with the 5d spin bundle). They are related by supersymmetry to the normal deformations of Σ inside of Y. Since we are considering the case of an isolated holomorphic curve, neither normal deformations nor fermions χ, $\widetilde{\chi}$ have any zero modes. Thus we can discard them. Θ^α then reduces to

$$\Theta^\alpha = \theta^\alpha + \widetilde{\theta}^\alpha_{\overline{z}}\mathrm{d}\overline{z} + \overline{\Omega}_{\overline{w}^1\overline{w}^2\overline{z}}\mathcal{G}^{z\overline{z}}\theta^\alpha_z\mathrm{d}\overline{w}^1\wedge\mathrm{d}\overline{w}^2 + \overline{\Omega}_{\overline{z}\overline{w}^1\overline{w}^2}\widetilde{\theta}^\alpha\mathrm{d}\overline{z}\wedge\mathrm{d}\overline{w}^1\wedge\mathrm{d}\overline{w}^2. \tag{B.8}$$

Because of our assumption that Σ is rigid, in the quantization of an M2-brane wrapping Σ, the only bosonic zero-modes are the ones associated to the center of mass motion along the five-manifold M. Hence those are the only bosonic modes in the effective action that describes such a superparticle; they parametrize the particle orbit $\gamma\subset M$. The fermionic modes in this action arise as the zero-modes of the internal motion, that is, the zero-modes of the fermionic variables on Σ. We find these modes by studying the part of the M2-brane action that is of order Θ^2, using formulas in [117]. (Terms in the action of higher order in Θ give only irrelevant contributions.)

First of all, with bosonic fields taken as in (B.6), one finds

$$\mathit{\Gamma} = -\frac{i}{\sqrt{-\dot{x}^2}}\mathcal{G}^{z\overline{z}}\dot{x}^M\mathit{\Gamma}_{Mz\overline{z}} + \mathcal{O}(\Theta^2). \tag{B.9}$$

This implies that the linearized κ-symmetry is:

$$\begin{aligned}\delta\Theta &= (1-\mathit{\Gamma})\kappa + \mathcal{O}(\Theta^2)\\ \delta X^{\mathcal{M}} &= \overline{(1-\mathit{\Gamma})\kappa}\mathit{\Gamma}^{\mathcal{M}}\Theta + \mathcal{O}(\Theta^2).\end{aligned} \tag{B.10}$$

This can be used to gauge-away the $\mathit{\Gamma} = -1$ part of Θ (up to higher orders in Θ). So we may assume that $\mathit{\Gamma} = 1 + \mathcal{O}(\Theta^2)$ when acting on Θ, that is $(\mathit{\Gamma} - 1)\Theta = \mathcal{O}(\Theta^3)$. This implies:

$$(\dot{x}^M \mathit{\Gamma}_M - i\mathcal{G}^{z\overline{z}}\mathit{\Gamma}_{z\overline{z}}\sqrt{-\dot{x}^2})\Theta = \mathcal{O}(\Theta^3). \tag{B.11}$$

Using this and taking the ansatz (A.31) for the C-field, we find the action (using results of [117]):

$$\begin{aligned}S = \int \mathrm{d}t\, \mathrm{d}^2 z\Big[&- 2\mathcal{G}_{z\overline{z}}\sqrt{-\dot{x}^2} - 2i\dot{x}^M V^I_M \omega_{Iz\overline{z}} + 4i\overline{\Theta}\mathit{\Gamma}_{z\overline{z}}\nabla_t\Theta - 4\sqrt{-\dot{x}^2}\overline{\Theta}(\mathit{\Gamma}_z\nabla_{\overline{z}} + \mathit{\Gamma}_{\overline{z}}\nabla_z)\Theta\\ &- \frac{1}{2}\sqrt{-\dot{x}^2}\mathcal{G}^{z\overline{z}}\overline{\Theta}\mathit{\Gamma}_{z\overline{z}}\mathit{\Gamma}^{MN}\Theta(\mathrm{d}V^I)_{MN}\omega_{Iz\overline{z}}\\ &- \frac{1}{2}\sqrt{-\dot{x}^2}\mathcal{G}_{z\overline{z}}\overline{\Theta}\mathit{\Gamma}^{w^i\overline{w}^j}\mathit{\Gamma}^{MN}\Theta(\mathrm{d}V^I)_{MN}\omega_{Iw^i\overline{w}^j} + \mathcal{O}(\Theta^4)\Big],\end{aligned} \tag{B.12}$$

where $\mathrm{d}^2 z = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}$. Here the covariant derivative ∇_t is defined using the pullback to the membrane worldvolume of the Levi-Civita connection of M.

B.3 Fermionic Zero-Modes

Now we can find the fermionic zero-modes. Expanding around a membrane that wraps Σ and is at rest in $M = \mathbb{R}^5$, so that $\dot{x}^2 = -1$, the fermionic part of the action becomes

$$\int \mathrm{d}t\mathrm{d}^2z\Big[4i\overline{\Theta}\mathit{\Gamma}_{z\overline{z}}\nabla_t\Theta - 4\overline{\Theta}(\mathit{\Gamma}_z\nabla_{\overline{z}} + \mathit{\Gamma}_{\overline{z}}\nabla_z)\Theta - \frac{1}{2}\mathcal{G}^{z\overline{z}}\overline{\Theta}\mathit{\Gamma}_{z\overline{z}}\mathit{\Gamma}^{MN}\Theta(\mathrm{d}V^I)_{MN}\omega_{Iz\overline{z}} - \frac{1}{2}\mathcal{G}_{z\overline{z}}\overline{\Theta}\mathit{\Gamma}^{w^i\overline{w}^j}\mathit{\Gamma}^{MN}\Theta(\mathrm{d}V^I)_{MN}\omega_{Iw^i\overline{w}^j}\Big]. \tag{B.13}$$

If the $U(1)$ background fields vanish, i.e., at $V^I = 0$, then only the first two terms in the action survive, the Hamiltonian becomes simply $H = 4\overline{\Theta}(\mathit{\Gamma}_z\nabla_{\overline{z}} + \mathit{\Gamma}_{\overline{z}}\nabla_z)\Theta$, and thus the fermion zero-modes are characterized by

$$(\mathit{\Gamma}_z\nabla_{\overline{z}} + \mathit{\Gamma}_{\overline{z}}\nabla_z)\Theta = 0. \tag{B.14}$$

Once we find the solutions in this idealized case, we can turn on the curvature of M and a graviphoton background as small perturbations.

To solve eqn. (B.14), we first note that

$$(\mathit{\Gamma}_z\nabla_{\overline{z}} + \mathit{\Gamma}_{\overline{z}}\nabla_z) = \mathbb{1}_4 \otimes \mathcal{G}_{z\overline{z}}\mathcal{D}, \quad \mathcal{D} = \mathcal{G}^{z\overline{z}}(\widetilde{\gamma}_z\nabla_{\overline{z}} + \widetilde{\gamma}_{\overline{z}}\nabla_z), \tag{B.15}$$

where $\mathbb{1}_4$ is the identity operator acting on $S(\mathbb{R}^{4,1})$ and $\mathcal{D}$ is simply the natural Dirac operator on Σ acting on spinors with values in the pullback to Σ of $S(Y)$, the spinors on Y. If we expand Θ as in (B.8), then the components all obey the most obvious equations:

$$\begin{aligned} \overline{\partial}\theta &= 0 \\ \partial(\widetilde{\theta}_{\overline{z}}\mathrm{d}\overline{z}) &= 0 \\ \overline{\partial}(\theta_z\mathrm{d}z) &= 0 \\ \partial\widetilde{\theta} &= 0. \end{aligned} \tag{B.16}$$

Because Σ is compact, these equations imply that θ and $\widetilde{\theta}$ are constant along Σ, while $\theta_z dz$ and $\widetilde{\theta}_{\overline{z}} d\overline{z}$ are holomorphic $(1,0)$ and antiholomorphic $(0,1)$-forms on Σ respectively. The κ symmetry gauge condition (B.11) implies that all these modes have left-handed 4d chirality, that is, they transform as $(1/2,0)$ under the 4d rotation group $SU(2)_\ell \times SU(2)_r$. Thus θ and $\widetilde{\theta}$ have 2 zero-modes each, and if Σ has genus g, then θ_z and $\widetilde{\theta}_{\overline{z}}$ each have $2g$ zero-modes.

To match the notation that we used in section 2.2, we write the constant $(1/2,0)$ modes of θ and $\widetilde{\theta}$ as $\theta^A = \frac{1}{2}\psi_1^A$ and $\widetilde{\theta}^A = \frac{1}{2}\psi_2^A$, respectively, where $A = 1,2$ is a left-handed spinor index. The fields ψ_1^A and ψ_2^A together make up the field that in section 2.2 was called ψ_i^A, $i, A = 1,2$. Introduce a basis of holomorphic $(1,0)$-forms λ_σ, $\sigma = 1 \ldots \mathfrak{g}$ and a complex conjugate basis of antiholomorphic $(0,1)$-forms $\overline{\lambda}_\sigma$, $\sigma = 1 \ldots \mathfrak{g}$, such that:

$$i \int_\Sigma \lambda_\sigma \wedge \overline{\lambda}_\kappa = \delta_{\sigma\kappa}. \tag{B.17}$$

We expand the $(1/2,0)$ parts of θ_z and $\widetilde{\theta}_{\overline{z}}$ in this basis:

$$\begin{aligned} \theta_z^A \mathrm{d}z &= \frac{1}{2}\sum_{\sigma=1}^{\mathfrak{g}} \rho_\sigma^A \lambda_\sigma, \\ \widetilde{\theta}_{\overline{z}}^A \mathrm{d}\overline{z} &= \frac{1}{2}\sum_{\sigma=1}^{\mathfrak{g}} \widetilde{\rho}_\sigma^A \overline{\lambda}_\sigma. \end{aligned} \tag{B.18}$$

The fields ρ_σ^A and $\widetilde{\rho}_\sigma^A$ were introduced in section 2.2.4.

If Σ were not isolated and χ, $\widetilde{\chi}$ had some zero modes, then the κ gauge-fixing condition (B.11) would force them to be of positive chirality in the 4d sense; that is, they would satisfy $-i\Gamma_0\chi = +\chi$ and would transform as $(0,1/2)$ under the 4d rotation group. The possible role of such modes was discussed in section 2.2.4.

B.4 Superparticle Action

We now give a slow t-dependence to the fermionic zero-modes ψ_i^A and ρ_σ^A, $\widetilde{\rho}_\sigma^A$ and turn on the background gauge fields $V_M^I = h^I V_M$. The mass and charges of the wrapped M2-brane

$$
\begin{aligned}
M &= \int_\Sigma \omega = \int_\Sigma \mathrm{d}^2 z 2\mathcal{G}_{z\bar{z}} \\
q_I &= \int_\Sigma \omega_I
\end{aligned}
\tag{B.19}
$$

are related by the usual formula[2] $M = q_I h^I$.

Starting from eqn. (B.12), it is not hard to write the action for an arbitrary spacetime with small and slowly varying curvature and for an arbitrary worldline γ that has everywhere a large radus of curvature. For an arbitrary worldline, the κ gauge-fixing conditions look as follows:

$$
\begin{aligned}
&\frac{\dot{x}^M \Gamma_M}{i\sqrt{-\dot{x}^2}} \psi_i = -\psi_i, \\
&\frac{\dot{x}^M \Gamma_M}{i\sqrt{-\dot{x}^2}} \rho_\sigma = -\rho_\sigma, \\
&\frac{\dot{x}^M \Gamma_M}{i\sqrt{-\dot{x}^2}} \widetilde{\rho}_\sigma = -\widetilde{\rho}_\sigma.
\end{aligned}
\tag{B.20}
$$

These conditions state that the fermions ψ_i, ρ_σ, and $\widetilde{\rho}_\sigma$ all transform as $(1/2, 0)$ under rotations of the normal plane to the worldline.

[2] One might recall from eqn. (2.31) that in general $\omega = v^I \omega_I$, and so the mass of the BPS particle in M-theory units is $\sum_I q_I v^I$. The 5d Einstein frame metric is related to the 11d metric by rescaling by a certain power of the Calabi-Yau volume, and the BPS mass in 5d Einstein frame is instead $M = q_I h^I$. For simplicity, in this appendix, one can just assume that the volume of Calabi-Yau is 1 from the beginning, so the rescaling is unnecessary. Then $\omega = h^I \omega_I$, where $\mathcal{C}_{IJK} h^I h^J h^K = 1$.

The action takes the form:

$$S = \int \mathrm{d}t \Big[-M\sqrt{-\dot{x}^2} + q_I V_M^I \dot{x}^M + \frac{i}{2} M \varepsilon^{AB} \varepsilon^{ij} \psi_{Ai} \nabla_t \psi_{Bj} - \frac{i}{16} M \sqrt{-\dot{x}^2} \mathsf{T}^-_{AB} \varepsilon^{ij} \psi_i^A \psi_j^B + \sum_{\sigma=1}^{\mathrm{g}} \Big(i\varepsilon^{AB} \widetilde{\rho}_{A\sigma} \nabla_t \rho_{B\sigma} + \frac{3i}{8} \sqrt{-\dot{x}^2} \mathsf{T}^-_{AB} \widetilde{\rho}^A_\sigma \rho^B_\sigma \Big) \Big]. \tag{B.21}$$

Here ∇_t is the pull-back to the particle world-line γ of the Levi-Civita connection of M, projected onto the plane normal to the worldline. And as usual, $\mathsf{T}^-_{AB} = \mathsf{T}^-_{\mu\nu} \gamma^{\mu\nu}_{AB}$ is the anti-selfdual part of T in the normal plane or equivalently in the local rest frame of the particle. One interesting thing about this action is that the kappa-symmetry gauge (B.20) ensures that only the projections of T^- and of the Levi-Civita connection ω_M^{ab} to the plane normal to γ enter this action, while the components along γ drop out automatically. If we were writing corresponding equations of motion, we would have to impose this by hand.

To get the particle action used in section 3, one has to specialize this action to the graviphoton background and assume that the particle is almost at rest, i.e., do a non-relativistic expansion. In the graviphoton background, the spin-connection contribution cancels the magnetic moment coupling of ψ_i and modifies it for ρ and $\widetilde{\rho}$. In the end, we get just the following familiar result (where we did not perform the non-relativistic expansion for the bosonic kinetic energy):

$$S = \int \mathrm{d}t \Big[-M\sqrt{-\dot{x}^2} + q_I V_M^I \dot{x}^M + \frac{i}{2} M \varepsilon^{AB} \varepsilon^{ij} \psi_{Ai} \frac{\mathrm{d}}{\mathrm{d}t} \psi_{Bj} + \sum_{\sigma=1}^{\mathrm{g}} \Big(i\varepsilon^{AB} \widetilde{\rho}_{A\sigma} \frac{\mathrm{d}}{\mathrm{d}t} \rho_{B\sigma} + \frac{i}{2} \mathsf{T}^-_{AB} \widetilde{\rho}^A_\sigma \rho^B_\sigma \Big) \Big]. \tag{B.22}$$

All fermions transform as $(1/2, 0)$ under $SU(2)_\ell \times SU(2)_r$.

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