Aspects of the Renormalization Group in Three-Dimensional Quantum Field Theory

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Abstract

The concept of renormalization group (RG) flow is one of the most novel and broad-reaching aspects of quantum field theory (QFT). The RG flow is implemented by constructing effective descriptions of a QFT at decreasing energy scales. One reason that RG flow is useful is that often one is interested in low-energy properties of theories with complicated short-distance structures. RG flows are subject to C theorems in relativistic QFT. The C theorems order the space of Lorentz-invariant QFTs. RG flows generically begin at scale-invariant fixed points known as conformal field theories (CFTs) and end in trivial massive theories. With tuning, the RG flows may end at non-trivial CFTs. Each CFT has an associated dimensionless C value. The C theorem states that under RG flow from a UV to an IR fixed point, the C value decreases.

In this Dissertation I present the F-theorem, which is a C theorem in three spacetime dimensions. I show that the correct quantity to consider is the Euclidean free energy of the CFT conformally mapped to the three-sphere, known as the F value. After motivating the F-theorem, I develop tools for calculating the F value in a variety of CFTs, with and without supersymmetry, including free field theories and gauge theories with large numbers of flavors. I also show that the F value is itself a useful quantity for probing the gauge/gravity duality and understanding other aspects of CFT, such as the scaling dimensions of monopole operators. The F theorem is closely related to quantum entanglement entropy. At conformal fixed points, the F value is equal to minus the renormalized entanglement entropy (REE) in flat Minkowski space across a circle. Away from the fixed points, the REE is a monotonically decreasing function along the RG flow. I compute the REE in a variety of holographic and non-holographic theories. I conclude the Dissertation by discussing a somewhat surprising result: the REE is not stationary at conformal fixed points.

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Mariangela introduced me to the Dark Side. Over the past year she has taken me from a curious student, interested in learning more about dark matter, to an active phenomenologist. We have already written four papers together, with more forthcoming. I'm confident our collaboration will continue for many years. It saddens me that I cannot include our work in this Dissertation.

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To my family.

Contents

Chapter 1

Introduction

Degrees of freedom decrease with energy.

Consider a crystal of table salt. The effective number of degrees of freedom of this system increases with the energy of my probe. At low energies, I can excite vibrational modes within the lattice, but I cannot resolve the individual lattice sites. If I probe the system at higher energies, the complexity increases. I can irradiate the material and knock out inner-shell electrons within the constituent atoms. At even higher energies, I can resolve the atomic nuclei, for example, by exciting nuclear transitions. As I keep increasing the energy of my probe, I don't even see individual nuclei. Eventually, I don't even see protons and neutrons. Instead, I see the quarks and gluons buried inside.

It is intuitive that the number of degrees of freedom should decrease with energy; if I have less energy, I have fewer options. This is something we all experience in our lives. But sometimes physics qualitatively changes between different energy scales. In these cases, it may not be straightforward to even have a common definition of 'degrees of freedom' that is valid at high and low energies. After all, the physics of quarks and gluons sure looks different from crystals of salt. It even looks quite different from the physics of neutrons and protons. What do we mean by degrees of freedom? Is there really a universal quantity that always decreases with energy, even when the systems undergo drastic transitions as we lower the energy? These are, at a high-level, the motivations for the work in this Dissertation.

I address these questions within the framework of relativistic, unitary quantum field theory (QFT). Many interesting, real world systems are described by such theories. Within this context, quantities C that may be calculated uniquely in all QFTs and that decrease with energy in all circumstances are said to obey a C-theorem. It is instructive to think of the C values as roughly measuring the degrees of freedom, with the analogy above in mind. However, in other ways – as I mention below – the C values are notably different from the normal notions of degrees of freedom.

C-theorems constrain how systems evolve with energy.

In four (spacetime) dimensional relativistic QFT, a proposal for a C theorem was given by Cardy in the 1980's [1]. Only recently, in 2011, was his proposal – called the a-theorem – finally proven [2]. Over the past 20 years, the a-theorem has been a useful tool for understanding aspects of four-dimensional QFT. QFT is four spacetime dimensions is interesting because it has direct application to particle physics.

Cardy's a-theorem is based on Zamolodchikov's c-theorem [3]. The c-theorem gives an appropriate measure of degrees of freedom in two spacetime-dimensional QFTs. The c-theorem is not directly useful for particle physics, since real spacetime is four dimensional. However, it still had and continues to have many interesting applications. This is for two reasons; (i) many condensed matter and statistical systems, under the right conditions, are described by two-dimensional QFTs, and (ii) the world-sheet of string theory is described by a two-dimensional QFT. QFT is a commonly used tool in multiple subfields of physics, and so understanding the basic structure of QFT tends to have impacts across the field.

Odd dimensions are different.

Historically, the c-theorem was proposed the year I was born – 1986 – and Cardy's a-theorem paper came 2 years later. With that said, for nearly twenty five years there was no known measure for the degrees of freedom in three-dimensional QFT. That's not to say that people didn't try, but no conjecture withstood further scrutiny. And that's also not to say that people didn't care. Many real-world systems are described, under certain conditions, by three-dimensional QFTs. For instance, the critical point of the water-vapor phase diagram is described by one of the simplest (to formulate) three-dimensional QFTs – the 3-D Ising model. In field theory language, this is the critical point obtained by perturbing a free scalar field theory by a ϕ^4 deformation, while keeping the mass of the field tuned to zero. This field theory is extremely hard to solve in practice, and one often must resort to numerical tools in order to make predictions. Other physical systems described by non-trivial three-dimensional QFTs include critical points found in insulating antiferromagnets and d-wave superconductors and between quantum Hall states, among many other examples.

Why are odd-dimensional field theories different from even dimensional field theories? A crucial difference – the one that is important here – is that odd-dimensional field theories do not, for the most part, have quantum anomalies, while even-dimensional QFTs do. A quantum anomaly occurs when a symmetry of the classical theory is not respected quantum mechanically. In even dimensions there exists a conformal anomaly; theories that classically are scale invariant acquire a scale dependence at the quantum level. The quantum breaking of scale invariance is associated with a coefficient, and it is this coefficient (a in 4-D, c in 2-D) that satisfies a C-theorem in two and four spacetime dimensions. In three dimensions, this term simply does not exists, since the conformal symmetry is not anomalous. We must try something else.

C-Theorems Order the Landscape of RG Flows

It is useful to be more precise about the idea of 'degrees of freedom decreasing with energy.' In QFT, decreasing the energy-scale of a system is called renormalization group (RG) flow. The word 'flow' is meant to imply that the RG only goes one direction. RG flows generally begin and end at conformal fixed points. At conformal fixed points, the QFTs are called conformal field theories (CFTs). These theories – at least classically – have no dependence on scale. In particular, this means that they are invariant under the RG flow, since the RG is associated with a change of scale. The high-energy fixed point is called the UV fixed point, while the low-energy one is called the IR fixed point.

C-theorems provide an ordering on the space of QFTs. These concepts are nicely visualized with the help of the following analogy. We can imagine the landscape of QFTs as the physical landscape of a mountainside. RG flow is the action of going down the mountain, just like a river. And, like a river, we are not allowed to go back up the mountain.

Suppose I point to two points on this mountain and ask "is it possible for a river to flow between these two points?" If the first point is higher than the second, then the answer is "possibly yes." But if the first point is lower, then the answer is "definitely no."

The C-theorems give us a concept of height. In particular, they give us a quantity C that always decreases under RG flow. That is, $C_{\mathrm{UV}} > C_{\mathrm{IR}}$, just like the quantity called 'height' in our analogy. In this analogy, the conformal fixed points may be visualized as basins, or lakes. These are places where the RG flow ends.

Just like rivers, RG flows only go down to lower energies. The flow down a mountainside takes place along a unique path. Contrast this to going up the mountain. If I am standing at a lake – a conformal fixed point – I may go back up the mountain in many different directions. Going down is more unique than going up. The concepts are illustrated in Fig. 1.1.

The analogy here is meant to be intuitive, but there are important differences as well between RG flow and the flow of water down a mountainside. The latter phenomena obeys a gradient flow; that is, rivers always take the path of steepest descent. This is not necessarily true with RG flows.

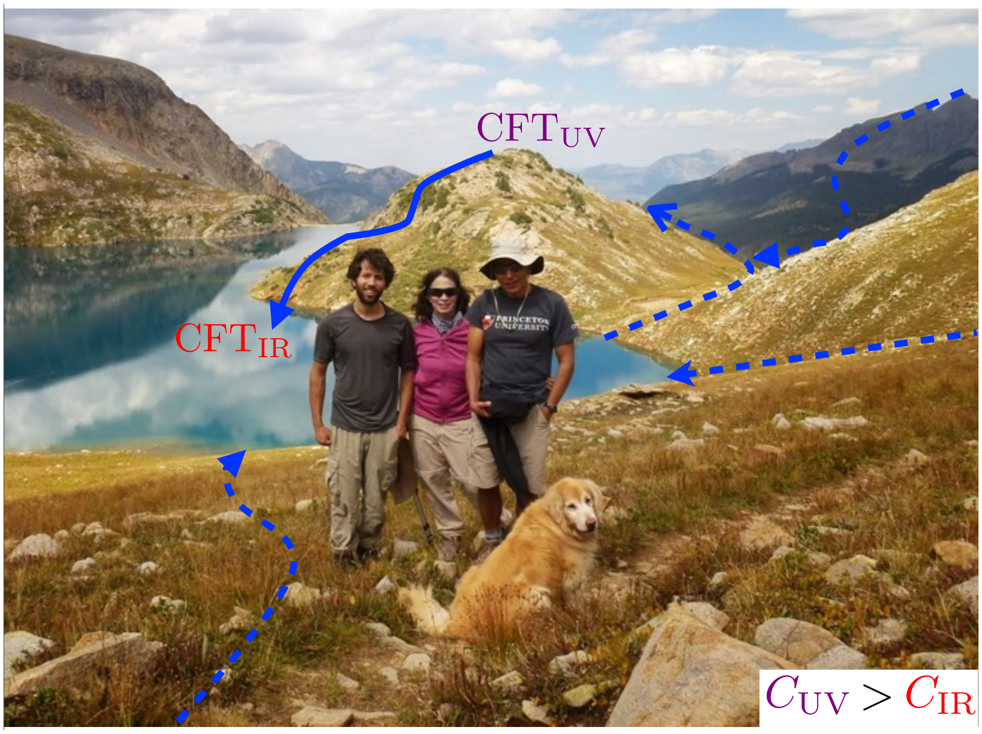


Figure 1.1: RG flows may be visualized as the paths of rivers down mountains. These flows begin and end at places where there is no gradient (CFTs). The C-theorem tells us that $C_{\text{UV}} > C_{\text{IR}}$; that is, C decreases under RG flow. This is analogous to the decrease in height, or gravitational potential energy, along the flow of the river.

I introduced the concept of RG flow in the context of decreasing the energy of my probe. Since energy and length are inversely proportional, I may also think of RG flow as increasing the minimum length scale in my system. For example, suppose I have a lattice with lattice-spacing ϵ. At each lattice site, I have a fundamental degree of freedom, perhaps an atom. If I clump together the behavior of nearby atoms, I can rewrite my theory in terms of an effective theory with lattice spacing, say, 2ϵ. By repeating this exercise, I flow between the fundamental UV theory with lattice spacing ϵ to the low-energy (IR) effective theory, with some large effective lattice spacing. If the low-energy theory is a conformal fixed point, then, in fact, the theory does not depend on the lattice spacing at all.

What about thermal entropy?

It is important to distinguish 'C' from the normal counting of free-field degrees of freedom. In four dimensional field theory, we normally say that the real scalar field has one degree of freedom, while the photon has two degrees of freedom, corresponding to the two physical polarization states. On the contrary, the a coefficient of the photon is 62 times that of the real scalar. The C-coefficients do not necessarily match our intuition. These coefficients should simply be thought of as ordering coefficients on the space of QFTs.

The standard counting of free-field degrees of freedom may be recovered from the thermal entropy S_{therm} of a system. More precisely, in a relativistic theory $\mathcal{F}_{\text{therm}} \propto c_{\text{therm}} T^d$, where T is the temperature, $\mathcal{F}_{\text{therm}}$ is the Helmholtz free energy, and d is the spacetime dimension. The quantity c_{therm} is a dimensionless quantity that, in four spacetime dimensions, is twice as big for the photon as it is for the real scalar field. It is a characteristic of the underlying theory. But does it satisfy a C-theorem? That is, does c_{therm} always decrease under RG flow? It turns out that in two spacetime dimensions, c_{therm} does satisfy a C-theorem, but in higher dimensions it does not. It is non-trivial that thermal entropy does not order QFTs in three and four spacetime dimensions [4–7]. In practice, though, it is often the case that c_{therm} does decrease under RG flow, even though it does not do so in all cases [4, 5].

Thermal entropy $\rightarrow$ quantum entanglement entropy in QFT

Just a few years ago, C-theorems were mysterious. Over the past few years, a much clearer picture has emerged. It is this picture that I will describe in this Dissertation.

In 2011 we proposed the F-theorem [8, 9]. The F-theorem is the conjecture that the ground-state Euclidean free energy on the three-sphere satisfies a C-theorem in three dimensions. Around the same time that we proposed the F-theorem, an independent group of researchers proposed another seemingly independent C-theorem in three dimensions [10, 11]. They proposed that the entanglement entropy of the ground state decreases under RG flow.

The entanglement entropy is a uniquely quantum entropy that may be computed even at zero temperature. There is an important qualitative difference between theories at zero temperature and finite temperature. When $T > 0$, one has contributions from all of the excited states of the theory, while at zero temperature we are only sensitive to the ground state.

Entanglement entropy measures the quantum entanglement across a spatial surface. The entanglement C-theorem proposal took that surface to be a circle. The RG flow is easily implemented by simply increasing the radius of the circle.

Shortly after our independent proposals, it was realized that the F-theorem and the entanglement C-theorem were actually the same. The reason is that at conformal fixed points, the Euclidean free energy of the CFT on the three-sphere is equal to the ground-state entanglement entropy in flat spacetime across a circle [11].

Casini and Huerta proved the F-theorem in 2012 [12]. They did so by using the relation to entanglement entropy and explicitly constructing a function that is monotonically decreasing as the radius of the circle increases. Their proof holds for all unitary relativistic QFTs in three spacetime dimensions.

These ideas also led to a unified understanding of C-theorems in two, three, and four dimensions. In D dimensions, the appropriate quantity to consider at the conformal fixed points is the Euclidean free energy of the theory on the D-sphere. Equivalently, this is equal to the ground-state entanglement entropy of the theory in flat spacetime across the $(D-2)$-sphere. In even dimensions, the appropriately regularized versions of these quantities are dominated by a conformal anomaly term. The coefficient of this term gives us exactly c in $D = 2$ and a in $D = 4$. In odd dimensions, there is no conformal anomaly term. However, there is still a finite contribution to the free energy, which in $D = 3$ we call F. Just like a and c decrease monotonically under RG flow, F also satisfies a C-theorem. However, only in $D = 2$ is the zero-temperature quantum entanglement entropy directly related to the finite temperature thermal entropy. We have conjectured that entanglement entropy provides a C-theorem in all dimensions, but so far this has only been proven for $D = 2, 3$, and 4.

This Dissertation

In this Dissertation I introduce the F-theorem. Chapter 2 is a slightly modified version of the paper [9], which is where we first explained the F-theorem in generality. This chapter gives simple examples of RG flows where we may compute F in the UV and the IR and see explicitly see that it decreases. This chapter also begins to develop tools for computing F on the three-sphere.

Chapters 3 and 4 are modified versions of the papers [13] and [14], respectively. I include these chapters because they develop tools for calculating F in more non-trivial field theories. Chapter 3 discusses three-dimensional gauge theories with large numbers of flavors. Chapter 4 is closely related; here I consider higher-spin gauge theories instead of just spin-1 gauge theory. These chapters also show that F is useful as a general probe of field theory. For example, I show that we may use F to provide non-trivial tests the gauge-gravity duality. These tests arise from computing F on both sides of the duality.

Chapters 5 and 6, which are based on the papers [15] and [16], respectively, focus on the RG flow instead of the conformal fixed points. Away from the conformal fixed points, one should calculate the entanglement entropy instead of the three-sphere free energy. Chapter 5 shows how to do this holographically, while in chapter 6 I show how to numerically calculate the free-field massive entanglement entropy using lattice techniques. This dissertation ends in chapter 6 with an open question concerning the stationarity of entanglement entropy.

I have chosen to base this Dissertation on the papers [9,13–16] because they tell a cohesive story. However, due to space constraints I must leave out many other works of mine that are also related to this story, namely [8, 17–21].

The remainder of this Introduction reviews some of the basic results and concepts needed to understand the later chapters.

1.1 C-Theorems in QFT

In this subsection we review the $D = 2$ c-theorem and the $D = 4$ a-theorem. We then introduce the F-theorem and its connection to entanglement entropy.

C-theorems in $D = 2$ and 4

The first example of a C-theorem in QFT was given in two-dimensions by Zamolodchikov [3], who used the two-point functions of the stress-energy tensor to define the Zamolodchikov c-function that had the desired properties. The Zamolodchikov c-function has the additional property that at the RG fixed points it coincides with the Weyl anomaly coefficient c, which is given by the expectation value of the trace of the stress-energy tensor on a curved (Euclidean signature) manifold:

$$\langle T^a{}_a \rangle = -\frac{c}{12} R \,. \tag{1.1}$$

Here R is the curvature scalar, and we normalize c so that it equals unity for a real conformal scalar field.

In four dimensions there are two Weyl anomaly coefficients, a and c, such that

$$\langle T^a{}_a \rangle = \frac{c}{16\pi^2} W_{abcd} W^{abcd} - 2a E_4 - \frac{a'}{16\pi^2} \nabla^2 R \,, \tag{1.2}$$

where W_{abcd} is the Weyl tensor and E_4 is the Euler density, which has the normalization $\int_{S^4} d^4x \sqrt{g} E_4 = 2$. Cardy has conjectured [1] that it should be the a-coefficient that decreases under RG flow. He was led to this conjecture by the observation that, since in two-dimensions we can isolate c by considering $\int_{S^2} d^2x \sqrt{g} \langle T^a{}_a \rangle$, we can naturally single out a in four dimensions by considering the analogous integral on S^4. This follows because the Weyl tensor vanishes on the four-sphere. Cardy's conjecture that the quantity a obeys a C-theorem is called the a-theorem.

Recently a general proof of the a-theorem was constructed in [2], where the authors explicitly constructed a function that is monotonically decreasing along RG-flow, stationary at conformal fixed points, and equal to the a-anomaly coefficient at those fixed points. This work was preceded by more than 20 years of evidence towards the a-theorem. Considerable evidence came from studying 4-D supersymmetric field theories, where a is determined by the $U(1)_R$ charges [22]. The prescription for determining the superconformal R-charges is called a-maximization [23], which states that at superconformal fixed points the correct R-symmetry locally maximizes a. This has passed many consistency checks that rely both on field theoretic methods and on the AdS/CFT correspondence [24–26]. For large N superconformal gauge theories dual to type IIB string theory on $AdS_5 \times Y_5$, Y_5 being a Sasaki-Einstein space, a-maximization is equivalent to the statement that the Sasaki-Einstein metric on Y_5 is a volume minimizer within the set of all Sasakian metrics on this space [27]. This equivalence was proved in [28, 29].

The F-theorem

As already mentioned, a long-standing problem in QFT is to find a three-dimensional C-theorem. This is of particular interest since there are an abundance of fixed points in three dimensions with relevance to real world systems. However, since there is no conformal anomaly in 3-D, the trace of the stress-energy tensor simply vanishes at conformal fixed points. Over the years there have been a number of attempts at constructing a C-theorem in 3-D. One such proposal was to consider the free energy at finite temperature T [4, 5]:

$$F_T = -\frac{\Gamma(D/2)\zeta(D)}{\pi^{D/2}} c_{\text{Therm}} V_{D-1} T^D \,, \tag{1.3}$$

where D is the dimension of space-time, V_{D-1} is the spatial volume, and c_{Therm} is a dimensionless number normalized so that a massless scalar field gives $c_{\text{Therm}} = 1$. However, it was recognized right away that c_{Therm} may increase under RG flow if the UV is not asymptotically free [4,5]. For example, c_{Therm} increases under RG flow from the critical $D = 3$ $O(N)$ model to the Goldstone phase described by $N-1$ free fields [6,7]. This rules out the possibility of a c_{Therm}-theorem. The quantity c_{Therm} also violates another requirement for a good c-function: it varies along lines of fixed points. This may be seen, for example, in the four-dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [30].

The F-theorem is a C-theorem in $D = 3$ [8,9]. A general proof of the theorem was proposed in [12]. In fact, the F-theorem naturally extends to all spacetime dimensions D. At the conformal fixed points, one is instructed to conformally map the CFT to the D-sphere and compute the Euclidean free-energy

$$F = -\log |Z_{S^D}| \,, \tag{1.4}$$

where Z_{S^D} is the partition function. Since CFTs have no scale-dependence by definition, the regularized free energy should have no dependence on the radius R of the S^D. However, in even dimensions D there is a conformal anomaly, and scale-invariance is not respected quantum mechanically. We may explicitly compute the dependence of F on R, and we find

$$\frac{\partial F}{\partial \log(R)} = -D \int d^D x \sqrt{g} \langle T^a{}_a \rangle \,. \tag{1.5}$$

In even dimensions, we may integrate this equation so that $F \sim a_D \log R$, where a_D is the a-type Weyl anomaly coefficient (c in $D = 2$, and a in $D = 4$). In even-dimensions, Cardy has conjectured that the a-type anomaly satisfies a C-theorem [9].

In odd-dimensions D, the right-hand side of (1.5) vanished identically. F has no dependence on R since there is no conformally anomaly. Integrating (1.5) we find that $F \sim F_0$, where F_0 is a dimensionless number, called the F-value, that characterizes the CFT. The F-theorem proposes that in odd-dimensions, F_0 satisfies a C-theorem.

Jafferis [31] conjectured that the 3-D analogue of a-maximization is that the R-symmetry of $\mathcal{N} = 2$ superconformal theories in three dimensions extremizes F, and in [8] this was sharpened into the principle of F-maximization, which proposes that the F-value of the IR CFT is locally maximized by the trial R-charge. The principle of F-maximization passed many theoretical tests [31–41] before being proven in [42]. Also, for large N theories with $AdS_4 \times Y_7$ dual descriptions in M-theory, F-maximization is correctly mapped to the minimization of the volume of the Sasaki-Einstein spaces Y_7 [8, 43].

The principle of F-maximization was part of our original motivation for conjecturing the F-theorem [8]. In that paper, we considered various RG flows between CFTs with $\mathcal{N} \geq 2$ supersymmetry, and in all examples we found that $F_{\rm UV} > F_{\rm IR}$. Moreover, we found that F remained constant for exactly marginal deformations. F-maximization naturally leads to the F-theorem in the context of $\mathcal{N} \geq 2$ RG flows induced by superpotential deformations of the UV theory. At the level of the localized partition function, the superpotential deformations have the effect of constraining the R-symmetry. In the IR, we are instructed to maximize F over the appropriately constrained R-symmetry, while in the UV the same functional is maximized but without the constraints. Naturally, this then implies that $F_{\rm UV} \geq F_{\rm IR}$ for these RG flows.

Connections to entanglement entropy

The D-sphere free energy F at conformal fixed points is related to entanglement entropy (EE). We will review EE and its connection to F in detail in the following subsections. Here, we summarize the relationship.

In D-dimensions, the EE $S(R)$ in flat Minkowski spacetime across a spatial $(D-2)$-sphere of radius R is dominated by the area-law term: $S(R) \propto (R/\epsilon)^{D-2}$, where ϵ is the short-distance cut-off. At conformal fixed points, we should remove the power-like divergent terms in ϵ to construct a renormalized EE. In D odd, the renormalized EE has no dependence on R, while in D even there is a logarithmic dependence on R because of the conformal anomaly. A direct calculation, which we review below, shows that in both D even and odd,

$S(R) = -F$ at conformal fixed points [10,11]. That is, the renormalized EE is exactly equal to minus the renormalized D-sphere free energy.

Casini and Huerta used the EE to construct an entropic proof of the c-theorem in $D = 2$ [44]. Their proof of the F-theorem in $D = 3$ is similar [12]. Away from the conformal fixed points, one is instructed to compute the renormalized EE across the circle of radius R [45]

$$\mathcal{F}(R) = -S(R) + RS'(R)\,. \tag{1.6}$$

This is a finite function for theories that are conformal in the UV. For a CFT this function takes the constant value F_0. An important property of $\mathcal{F}(R)$ is that in the limit of large R it approaches the IR F-value $F_{\rm IR}$ [45]. Furthermore, $\mathcal{F}'(R) = RS''(R)$. It was shown in [12] that for any Lorentz invariant field theory $S''(R) \leq 0$. This demonstrates that $\mathcal{F}(R)$ is a non-increasing function and therefore proves the F-theorem.

1.2 Entanglement entropy in quantum mechanics

In this subsection I briefly reviewing the concepts of quantum entanglement, entanglement entropy, and Rényi entropy. Suppose the Hilbert space $\mathcal{H}$ of a quantum system has a basis of orthonormal states $\{|\psi_s\rangle\}$. A general normalized state $|\psi\rangle$ in the Hilbert space may be written as a superposition of the basis vectors:

$$|\psi\rangle = \sum_s c_s |\psi_s\rangle\,, \qquad \sum_s |c_s|^2 = 1\,. \tag{1.7}$$

The state $|\psi\rangle$ is called a pure state. The expectation value of an observable operator O in a pure state $|\psi\rangle$ is given by $\langle O\rangle = \langle\psi|O|\psi\rangle$. We may also define the density matrix $\rho_\psi = |\psi\rangle\langle\psi|$ and write $\langle\psi|O|\psi\rangle = \mathrm{tr}(\rho_\psi O)$, where the trace is over the Hilbert space $\mathcal{H}$.

Quantum mechanical systems may also be in mixed states. Mixed states are defined as quantum states where the density matrix cannot be written in the form $\rho = |\psi\rangle\langle\psi|$ for some pure state $|\psi\rangle$. However, the density matrix ρ may still be used to compute the expectation value of an observable operator O in a mixed state: $\langle O\rangle = \mathrm{tr}(\rho O)$. The most general density matrix is given by

$$\rho = \sum_s p_s |\psi_s\rangle\langle\psi_s| \,, \qquad \sum_s p_s = 1\,. \tag{1.8}$$

This says that the probability the mixed state is found to be in the pure state $|\psi_s\rangle$ is p_s.

The von Neumann entropy S of some quantum mechanical state with density matrix ρ is defined by

$$S \equiv -\,\mathrm{tr}(\rho \log \rho)\,. \tag{1.9}$$

Using the expansion for $|\psi\rangle$ in eq. (1.8) we may write

$$S = -\sum_s p_s \log(p_s)\,. \tag{1.10}$$

From this equation it clear that only when ρ describes a pure state does $S = 0$. If ρ describes a mixed state then $S > 0$.

A natural generalization of the von Neumann entropy is the Rényi entropy S_q [46, 47]:

$$S_q \equiv \frac{1}{1-q}\log(\mathrm{tr}\,\rho^q) = \frac{1}{1-q}\log\left(\sum_s p_s^q\right)\,, \qquad q \geq 0\,, q \neq 1\,. \tag{1.11}$$

It is clear that the Rényi entropies are also strictly positive for mixed states and zero for pure states. Note that $\lim_{q\to 1} S_q = S$, so a computation of the Rényi entropies also gives the von Neumann entropy.

Now suppose the Hilbert space $\mathcal{H}$ may be written as a direct product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and let the system be in a pure state with density matrix ρ. We define the reduced density matrix of ρ in $\mathcal{H}_A$ to be

$$\rho_A = \mathrm{tr}_B\, \rho\,, \tag{1.12}$$

where tr_B is defined to be a trace over $\mathcal{H}_B$. The entanglement entropy is then the Von Neumann entropy of the reduced density matrix ρ_A [48]:

$$S_A \equiv -\,\mathrm{tr}(\rho_A \log \rho_A) = -\,\mathrm{tr}(\rho_B \log \rho_B)\,. \tag{1.13}$$

The state $|\psi\rangle$ is said to be entangled between $\mathcal{H}_A$ and $\mathcal{H}_B$ if $S_A > 0$.

1.2.1 Example: two-spin system

We illustrate the above points with a simple example: two spin-$\frac{1}{2}$ degrees of freedom, with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We take the basis of $\mathcal{H}_A$ to be $\{|\uparrow\rangle_A\,, |\downarrow\rangle_A\}$ and the basis of $\mathcal{H}_B$ to be $\{|\uparrow\rangle_B\,, |\downarrow\rangle_B\}$. Consider the pure state in $\mathcal{H}$

$$|\psi\rangle = \cos\theta |\uparrow\rangle_A |\downarrow\rangle_B + \sin\theta |\downarrow\rangle_A |\uparrow\rangle_B\,, \qquad 0 \le \theta \le \frac{\pi}{2}\,. \tag{1.14}$$

Before calculating the entanglement entropy, let us think intuitively about the entanglement of this state. Suppose we have two observers Adam and Bob who measure the spin states of A and B, respectively. Suppose Adam performs his measurement first. When $\theta = 0$, Adam will always measure spin-up, and Bob will always measure spin-down. The opposite happens when $\theta = \pi/2$. These states are not entangled because Adam's measurement does not effect Bob's subsequent measurement. On the contrary, when $\theta \neq 0, \pi/2$ sometimes Adam will measure spin-up and sometimes he will measure spin-down. When he measures spin-up Bob will measure spin-down and vice versa. These states are entangled because Adam's measurement determines Bob's. When $\theta = \pi/4$ by symmetry we know that half of the time Adam will measure spin-up and half of the time he will measure spin-down. This case seems to the one of maximal entanglement. Now let us confirm our intuition with a calculation of the entanglement entropy.

The density matrix for this state is simply $\rho = |\psi\rangle\langle\psi|$. The reduced density matrix in $\mathcal{H}_A$ is easily calculated to be

$$\rho_A = \cos^2\theta |\uparrow\rangle_A\langle\uparrow|_A + \sin^2\theta |\downarrow\rangle_A\langle\downarrow|_A\,, \tag{1.15}$$

which gives the entanglement entropy

$$S_A = -(\cos^2\theta \log\cos^2\theta + \sin^2\theta \log\sin^2\theta)\,. \tag{1.16}$$

In fig. 1.2 we plot the entanglement entropy S_A as a function of θ. The entanglement entropy

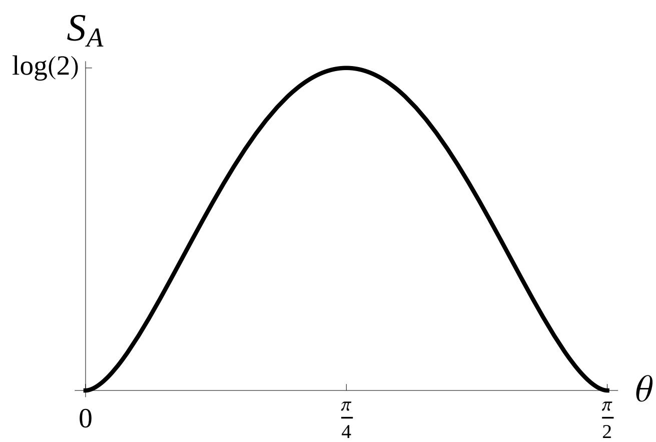


Figure 1.2: The entanglement entropy S_A of the state $|\psi\rangle$ between the sub-spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Notice that the maximum is at $\theta = \pi/4$.

is maximal at $\theta = \pi/4$, where $S_A = \log 2$. As expected, the entanglement entropy vanishes at $\theta = 0, \pi/2$.

1.3 Entanglement entropy in QFT

We would like to generalize the ideas of Rényi and entanglement entropy to quantum field theory. Consider a field theory in $D = d + 1$ dimensions on flat Minkowski space-time $\mathbb{R}^{d,1}$. Let the vacuum $|0\rangle$ of the theory have density matrix $\rho = |0\rangle\langle 0|$. We introduce a space-like 'entangling surface' Σ, which separates $\mathbb{R}^d$ into two spaces A and B such that $A \cup B = \mathbb{R}^d$ and $A \cap B = 0$. We will be especially interested in the case where Σ is a $(d-1)$-sphere in the spatial dimensions of radius R. We call this entangling surfaces $\Sigma_{S^{d-1}}$, and we let A be the space inside of the sphere and B be the space outside. The reduced density matrix $\rho_\Sigma = \mathrm{tr}_B\,|0\rangle\langle 0|$ is given by integrating out the degrees of freedom outside of the sphere. We can then calculate the entanglement entropy and the Rényi entropies using eqs. (1.13) and (1.11), respectively, for ρ_Σ.

1.3.1 Example: System of weakly interacting spins

Before embarking in earnest on a calculation in QFT, it is useful to consider a toy example that will demonstrate some of the main structure of entanglement entropy in field theory. We consider a simple lattice model in 2 spatial dimensions of very weakly interacting spins. Each spin is paired up with one of its nearest neighbors, so that if there are N spins then there are $N/2$ pairs. A pair of spins does not interact with any other pair. The spins are arranged on a square lattice of length ϵ. We illustrate this setup in figure 1.3. The form of the interaction is to put the ground-state of each pair in the maximally entangled state $\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle)$. This example may seem contrived, but it is designed to illustrate an important point about the nature of entanglement entropy.

We want to calculate the entanglement entropy of the ground-state across a circular entangling surface S^1 of radius R. Label the region inside of the circle by A and region outside by B. The only pairs which contribute to the entanglement entropy are those which have one spin in A and one spin in B. Each such pair has entanglement entropy equal

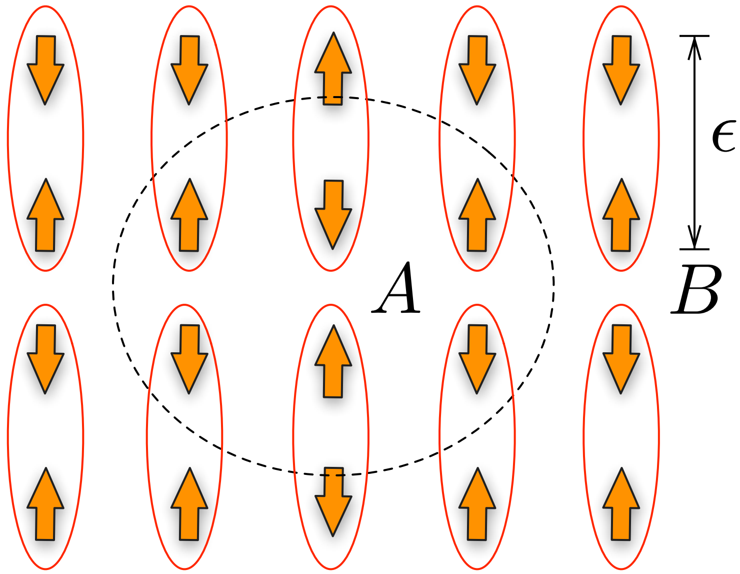


Figure 1.3: A simple dimer state, where the red circles denote the pairs of spins, which are arranged on a square lattice of spacing ϵ. The spins will always be measured to be in opposite spin states. The pairs of spins do not interact with each other. The dotted black circle is the entangling surface of radius R, the region A is the interior of the circle, and B is the region outside of the circle. We really want to consider the limit $R/\epsilon \gg 1$.

to $\log 2$. For large R, the number of such pairs scales like the circumference of the circle divided by the lattice spacing ϵ. Thus we find that the entanglement entropy of this system is approximately

$$S_A \sim \alpha \frac{2\pi R}{\epsilon} \log 2\,, \tag{1.17}$$

where α is some un-important, order one numerical constant that counts more precisely how many spin pairs are intersected by the circle.

Our simple example illustrates what is know as the area law for the entanglement entropy in quantum field theory (see, for example, ref. [49]):

$$S_A = g_{d-1}\,\mathrm{Vol}(\Sigma)\epsilon^{d-1} + \cdots + g_1\,\mathrm{Vol}(\Sigma)^{1/(d-1)}\epsilon^{-1} + g_0 \log\left(\frac{\epsilon^{d-1}}{\mathrm{Vol}(\Sigma)}\right) + S_0\,, \tag{1.18}$$

where ϵ is the short distance cut-off of the theory. The constants g_{d-1} through g_1 depend on the UV behavior of the theory. The constant g_0 is expected to be universal. If we rescale ϵ, the coefficient g_0 remains unchanged, though the ϵ-independent term S_0 will receive a shift. Thus, when $g_0 \neq 0$ we conclude that S_0 is also not physical. However, when g_0 equals zero then it appears that S_0 has a well defined meaning independent of the UV cut-off. At conformal fixed points we will see that when d is odd g_0 is non-vanishing and when d is even g_0 vanishes. This is because g_0 is proportional to the anomaly coefficient that multiplies the Euler density, and there are no anomalies when $D = d + 1$ is odd. Also note that while it must be true that $S_A \geq 0$, there are no obvious positivity requirements for g_0 and S_0.

Heuristically, the ϵ-dependent terms in eq. (1.18) come from quantum fluctuations across Σ. These fluctuations take place at the UV length scale ϵ. The 'number' of fluctuations that occur across Σ is then schematically given by $\mathrm{Vol}(\Sigma)\epsilon^{d-1}$, which gives the leading term in eq. (1.18). In fact, our quantum mechanical dimer model can be thought of as a toy model for vacuum fluctuations, where each pair of spins is thought of as a particle anti-particle pair.

We will be especially interested in the entanglement and Rényi entropies of QFTs at conformal fixed points. A $(d + 1)$-dimensional relativistic CFT on $\mathbb{R}^{d,1}$ is a field theory which is invariant under conformal transformations (see, for example, ref. [50]). The group of conformal transformations includes, in addition to the usual Poincaré symmetries, scale transformations and special conformal transformations. Relativistic theories that are invariant under scale transformations are expected to be invariant under the full conformal group.

One reason we are interested in CFTs is the following. Consider a D-dimensional many-body quantum system with Hamiltonian $H(g)$, where g is some parameter of the theory. Suppose there is a special value of g, which we call g_c, where the system undergoes a continuous phase transition. When $g = g_c$ we say that g is at a quantum critical point (QCP). For g near g_c the correlation length ζ scales as $\zeta \sim |g - g_c|^{-\nu} \gg \epsilon$, where $\nu > 0$ is a critical exponent and ϵ is the lattice spacing. This implies that it is not necessary to know the details of the UV theory in order to describe the properties of the low-energy degrees of freedom near a QCP. If the low-energy degrees of freedom have linear dispersion relations, then the theory is well approximated by a relativistic CFT.

1.4 Methods for calculating entanglement entropy

We would like to calculate the EE across the entangling surface $\Sigma_{S^{d-1}}$ introduced in the previous section. This section summarizes the methods introduced in [10, 11] and references therein. Let us use a Cartesian chart, with metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + \sum_{i=1}^{d} (dx^i)^2 \,. \tag{1.19}$$

At $x^0 = 0$, the surface S^{d-1} of radius R is described by the equation $r \leq R$, where $r \equiv \sqrt{(x^1)^2 + \cdots + (x^d)^2}$ and R is the radius of the sphere. The causal development of this region, which we call $\mathcal{D}$, is the region in space-time such that any time-like or null curve which passes through $\mathcal{D}$ must necessarily intersect the S^{d-1} at $x^0 = 0$. This implies that $\mathcal{D}$ is given by the intersection of two cones in space-time:

$$\mathcal{D} = \{r + x^0 \leq R\} \cap \{r - x^0 \leq R\} \,, \qquad -R \leq x^0 \leq R \,. \tag{1.20}$$

In general there is no simple expression for the reduced density matrix on $\mathcal{D}$. However, when we have a CFT it is possible to conformally map the region $\mathcal{D}$ to the Rindler wedge $\mathcal{R}$ of Minkowski space. In the Rindler wedge the density matrix describes a thermal state with respect to boosts about the origin. This is the Unruh effect [51], but we are getting ahead of ourselves. First let us illustrate the conformal map between $\mathcal{D}$ and $\mathcal{R}$ explicitly.

Consider Minkowski space with coordinates X^μ and metric $ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu$. The Rindler wedge of Minkowski space is the causal development of the $X^0 = 0$ slice corresponding to the right-half plane $X^1 \geq 0$ for all X^i, $i \geq 2$:

$$\mathcal{R} = \{X^1 + X^0 \geq 0\} \cap \{X^1 - X^0 \geq 0\}\,, \qquad -\infty < X^i,\, X^0 < \infty\ , i \geq 2\,. \tag{1.21}$$

An explicit conformal mapping of $\mathcal{R}$ to $\mathcal{D}$ is given by

$$x^\mu = \frac{X^\mu - (X \cdot X) C^\mu}{1 - 2(X \cdot C) + (X \cdot X)(C \cdot C)} + 2R^2 C^\mu\,, \tag{1.22}$$

where we take $C^\mu = (0, 1/(2R), 0, \dots, 0)$. This is a combination of a special conformal transformation plus a translation, so the metric can only change by an overall conformal factor. Indeed, it is straightforward to verify that this is the case. Since we are studying CFTs, this overall conformal factor is irrelevant. It is also easy to see that that the coordinates x^μ given in (1.22) exactly cover $\mathcal{D}$.

The vacuum $|0\rangle$ of the entire Minkowski space is a pure state with density matrix $\rho = |0\rangle\langle 0|$. We want to obtain the non-trivial reduced density matrix inside of $\mathcal{D}$. If our theory is conformally invariant, we may equivalently consider the reduced density matrix inside the Rindler wedge $\mathcal{R}$. The Unruh effect states that if one takes the vacuum of a QFT in Minkowski space and reduces to the Rindler wedge, the resulting state is thermal with respect to translations along the directions of boosts. Let us make this more precise. We define new coordinates t and z such that

$$X^1 \pm X^0 = z e^{\pm t/R}\,, \qquad z > 0\,, \quad \forall\, t \in \mathbb{R}\,, \tag{1.23}$$

so that the metric is the Rindler metric:

$$ds^2 = -\frac{z^2}{R^2} dt^2 + dz^2 + \sum_{i=2}^{d} (dX^i)^2\,. \tag{1.24}$$

Notice that the Rindler coordinates naturally cover only the Rindler wedge of Minkowski space. The parameter R is arbitrary and is put in so that t has dimension of length. The precise statement of the Unruh effect is that the vacuum of Minkowski space restricted to the Rindler wedge is described by a thermal state at inverse temperature $\beta = 2\pi R$, with the Hamiltonian $\hat{H}_t$ being the operator that generates translations in the Rindler time t. From the point of view of the Cartesian chart, $\hat{H}_t$ generates boosts around the origin. The density matrix of a thermal state at inverse temperature β is given by

$$\rho = \frac{e^{-\beta \hat{H}_t}}{Z}\,, \qquad Z = \mathrm{tr}(e^{-\beta \hat{H}_t})\,. \tag{1.25}$$

Note that Z is simply the thermal partition function. With this definition we can calculate thermal expectation values:

$$\langle O \rangle = \mathrm{tr}(\rho O) = \frac{1}{Z}\sum_n e^{-\beta E_n}\langle n|O|n\rangle\,, \tag{1.26}$$

where the states $\{|n\rangle\}$ are the eigenstates of $\hat{H}_t$ with eigenvalues $\{E_n\}$.

The thermal partition function of a QFT in Lorentz signature is equivalent to the Euclidean path integral, where we analytically continue to imaginary time by defining $\tau = it$ and compactify τ on a circle of radius equal to the inverse temperature β (see, for example, [50]). Now let us calculate the Rényi entropies S_q given in eq. (1.11). We see that the first step is to compute the quantity $\mathrm{tr}\,\rho^q$, which in turn requires us to compute the thermal partition function

$$Z_q = \mathrm{tr}\exp(-2\pi q R\hat{H}_t) = e^{-\mathcal{F}_q}\,, \qquad F_q = \frac{1}{2\pi R q}\mathcal{F}_q\,, \tag{1.27}$$

where the inverse temperature is $\beta = (2\pi q R)$. The quantity Z_q is equal to the Euclidean partition function of the QFT on the space

$$ds^2 = \frac{z^2}{R^2} d\tau^2 + dz^2 + \sum_{i=2}^{d} dX_i^2 \,, \tag{1.28}$$

where now $0 \leq \tau < 2\pi q R$. If we define the angle $\theta = \tau/R$, then it is clear that the metric in eq. (1.28) describes flat space where, for $q \neq 1$, there is a line of conical singularities at $z = 0$. When $q = 1$ there is no conical singularity. Before moving on, notice that $\mathcal{F}_q = -\log |Z_q|$ is the Euclidean free energy. The thermal free energy F_q is by convention taken to be this quantity times the temperature.

We are now in a position to given an expression for the Rényi entropies across $\Sigma_{S^{d-1}}$ in terms of Euclidean path integrals:

$$S_q = \frac{1}{1-q} \log \frac{\mathrm{tr}\, e^{-2\pi R q \mathcal{H}_t}}{(\mathrm{tr}\, e^{-2\pi R \mathcal{H}_t})^q} = \frac{q\mathcal{F}_1 - \mathcal{F}_q}{1-q} = \frac{2\pi R q \,(F_1 - F_q)}{1-q} \,. \tag{1.29}$$

Let us think about how to calculate the entanglement entropy S_1. Taking the limit $q \to 1$ in (1.29), one obtains

$$S_1 = \left. \frac{dF(T)}{dT} \right|_{T=1/(2\pi R)} = -S_{\mathrm{therm}}(1/(2\pi R)) \,, \tag{1.30}$$

where $S_{\mathrm{therm}}(T)$ is the thermal entropy at temperature T. In general, the definition of the Helmholtz free energy is

$$F(T) = E(T) - T S_{\mathrm{therm}}(T) \,, \tag{1.31}$$

where $E(T) = \mathrm{tr}(\rho_q \hat{H}_t)$ is the total energy. It follows immediately from (1.30) and (1.31) that $S_1 = -2\pi R[F_1 - E(1/2\pi R)] = -\mathcal{F}_1 + \mathcal{F}_1'$.

In D odd, $E(1/(2\pi R)) = 0$ since the trace of the stress-energy tensor vanishes. In D even, $E(1/(2\pi R))$ does not vanish, and it is proportional to the a-type anomaly coefficient. However, the energy only gives a finite shift to S_1, and $S_1 \sim \log R$. Thus, in both D even and odd we may write $S_1 = -\mathcal{F}_1$, where it is understood that both sides of the relation refer to the properly renormalized, universal quantities.

1.4.1 Mapping to $R \times \mathbb{H}^d$

Notice that the metric in eq. (1.28) may be written as

$$ds^2 = \Omega^2 \left(d\tau^2 + \frac{R^2}{z^2} \left[dz^2 + \sum_{i=2}^{d} dX_i^2 \right] \right) , \tag{1.32}$$

with $\Omega = z/R$. This implies that by applying a conformal transformation we can map the theory from the Rindler wedge to the space $S^1 \times \mathbb{H}^d$, where $\mathbb{H}^d$ is the hyperbolic space of radius R. We subsequently refer to this space simply as $H_q^{(d+1)}$. The metric on $H_q^{(d+1)}$ may be written in slightly more convenient coordinates:

$$ds_H^2 = R^2 (d\tau^2 + d\eta^2 + \sinh^2 \eta \, d\Omega_{d-1}^2) \, , \tag{1.33}$$

where $0 \leq \tau < 2\pi q$, $0 \leq \eta < \infty$, and $d\Omega_{d-1}^2$ is the volume element on the unit S^{d-2}.

The spaces $\mathbb{H}^d$ are non-compact, which implies their volumes are infinite and require regularization. The proper regularization of these volumes uses a hard cutoff at some value $\eta = \eta_0$ [11]:

$$\mathrm{Vol}(\mathbb{H}^d) = \frac{d\pi^{d/2} R^d}{\Gamma\left(\frac{d}{2} + 1\right)} \int_0^{\eta_0} d\eta \, \sinh^{d-1} \eta \, . \tag{1.34}$$

The case of most interest is when $d = 2$, which gives

$$\text{Vol}(\mathbb{H}^2) = 2\pi R^2 \left[\frac{e^{\eta_0}}{2} - 1 + \frac{e^{-\eta_0}}{2}\right] . \tag{1.35}$$

Taking the finite part of this expression as η_0 goes to infinity gives us the regularized volume $\text{Vol}(\mathbb{H}^2) = -2\pi R^2$. When d is odd the regularized volume picks up a logarithmic divergence. For example, in $d = 3$ we find the regularized volume $\text{Vol}(\mathbb{H}^3) = -2\pi R^3 \log(R/\epsilon)$, where we have redefined the UV regulator to be the short distance regulator ϵ.

1.4.2 Mapping to the q-fold branched covering of S^d

The metric in eq. (1.33) may be conformally mapped to the metric on the q-fold branched covering of S^d:

$$ds_C^2 = R^2 \left[\cos^2\theta d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-1}^2\right] , \tag{1.36}$$

where the ranges of the coordinates are $0 \le \tau < 2\pi q$, $0 \le \phi < 2\pi$, and $0 \le \theta < \pi/2$. This space is referred to as $C_q^{(d+1)}$ [52]. The conformal equivalence of the metrics on $H_q^{(d+1)}$ and $C_q^{(d+1)}$ is made explicit by identifying $\sinh\eta = \cos\theta$, which implies $ds_C^2 = \cos^2\theta ds_H^2$. Note that when $q = 1$ the space $C_q^{(d+1)}$ is simply the round sphere $S^{(d+1)}$. Thus we have shown that the entanglement entropy of a CFT across the entangling surface $\Sigma_{S^{d-1}}$ is equivalent to minus the free energy of the CFT conformally coupled to curvature on the sphere S^{d+1}.

Chapter 2

The F-Theorem

This chapter is a lightly-modified version of the paper [9].

2.1 Introduction

In this chapter I will introduce the F theorem. In particular, I will subject the F-theorem to tests that do not rely on supersymmetry. Supersymmetric tests of the F theorem were first performed in [8]. Our approach here is similar to Cardy's [1] who, in the absence of a proof of the a-theorem, presented some evidence for it in the context of a CFT on S^4 perturbed by weakly relevant operators. (His work generalizes similar calculations in $d = 2$ [3, 53].) In sections 2.2 and 2.3 we use the perturbed conformal field theory on S^3 to present evidence for the F-theorem. We also discuss other odd-dimensional Euclidean theories on S^d where similar perturbative calculations provide evidence that $(-1)^{(d-1)/2} \log |Z|$ decreases along RG flow. In the particular case $d = 1$ these calculations were carried out in [54, 55] providing evidence for the g-theorem.[1] In section 2.4 we review the calculations of F for theories involving free massless boson, fermion, and vector fields. We show that these values are consistent with the F-theorem for some RG flows. In section 2.5 we consider another class

[1]The $d = 1$ dynamics is often assumed to take place on the boundary of a $d = 2$ conformal field theory. A proof of the g-theorem in this context was given in [56].

of examples which involve RG flows in large N field theories perturbed by relevant double-trace operators. In these cases, the theories flow to IR fixed points, and $F_{\rm IR} - F_{\rm UV}$ can be calculated even when the double-trace operator is not weakly relevant [57–59]. The results are consistent with the F-theorem. An explicit example of this kind is the critical $O(N)$ model. In particular, we show that the flow from the critical $O(N)$ model to the Goldstone phase, which was earlier found to violate the $c_{\rm Therm}$-theorem [6, 7], does not violate the F-theorem.

2.2 Perturbed Conformal Field Theory

In this section we discuss Euclidean conformal field theories perturbed by a slightly relevant scalar operator of dimension $\Delta = d - \epsilon$, where $0 < \epsilon \ll 1$. Our approach follows closely that in [1, 3, 53–55]. To keep the discussion fairly general, we will work in an arbitrary odd dimension d throughout most of the following calculation, though the case of most interest to us is $d = 3$. Our calculations generalize those carried out for $d = 1$ to provide evidence for the g-theorem [54, 55, 60]. We take the action of the perturbed field theory to be

$$S = S_0 + \lambda_0 \int d^d x \sqrt{G} O(x) \,, \tag{2.1}$$

where S_0 is the action of the field theory at the UV fixed point, λ_0 is the UV bare coupling defined at some UV scale μ_0, G is the determinant of the background metric, and $O(x)$ the bare operator of dimension Δ.

2.2.1 Beta function and the running coupling

For the purposes of finding the beta function it is sufficient to work in the flat $\mathbb{R}^d$. For the CFT on $\mathbb{R}^d$, conformal invariance fixes the functional form of the connected two-point and three-point functions [61], and we choose the normalization of O to be such that

$$\begin{aligned} \langle O(x)O(y)\rangle_0 &= \frac{1}{|x-y|^{2(d-\epsilon)}} \,, \\ \langle O(x)O(y)O(z)\rangle_0 &= \frac{C}{|x-y|^{d-\epsilon}\,|y-z|^{d-\epsilon}\,|z-x|^{d-\epsilon}} \,, \end{aligned} \tag{2.2}$$

for some constant C. These correlators correspond to the OPE

$$O(x)O(y) = \frac{1}{|x-y|^{2(d-\epsilon)}} + \frac{CO(x)}{|x-y|^{d-\epsilon}} + \ldots \qquad \text{as } x \to y \,. \tag{2.3}$$

In the perturbed theory, the coupling runs. The beta function is [1, 50][2]

$$\beta(g) = \mu\frac{dg}{d\mu} = -\epsilon g + \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} C g^2 + \mathcal{O}(g^3) \,, \tag{2.4}$$

where μ is the renormalization scale, and $g = \lambda\mu^{-\epsilon}$ is the dimensionless renormalized coupling. Integrating this equation with the boundary condition $g(\mu_0) = g_0 \ll 1$, where μ_0 is a UV cutoff, we obtain the running coupling

$$g(\mu) = g_0\left(\frac{\mu_0}{\mu}\right)^{\epsilon} - \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\frac{Cg_0^2}{2\epsilon}\left[\left(\frac{\mu_0}{\mu}\right)^{2\epsilon} - \left(\frac{\mu_0}{\mu}\right)^{\epsilon}\right] + \mathcal{O}(g_0^3) \,. \tag{2.5}$$

One can understand the two equations above from the following RG argument. Correlation functions in the interacting theory differ from the ones in the free theory by an extra insertion of

$$e^{-\lambda_0 \int d^dx\, O(x)} = 1 - \lambda_0 \int d^dx\, O(x) + \frac{\lambda_0^2}{2}\int d^dx \int_{|x-y|>\frac{1}{\mu_0}} d^dy\, O(x)O(y) + \ldots \,, \tag{2.6}$$

where the condition $|x-y| > \frac{1}{\mu_0}$ comes from imposing the UV cutoff μ_0. In obtaining an effective action at some scale μ, one simply isolates the contribution from modes between

[2] This equation differs from eq. (9) in [1] by the sign of the second term because our coupling g differs from the one in [1] by a minus sign.

energy scales μ_0 and μ: for example, we write the last integral in (2.6) as

$$\int_{|x-y|>\frac{1}{\mu_0}} d^d y\, O(x)O(y) = \int_{|x-y|>\frac{1}{\mu}} d^d y\, O(x)O(y) + \int_{\frac{1}{\mu_0}<|x-y|<\frac{1}{\mu}} \frac{d^d y\, C O(x)}{|x-y|^{d-\epsilon}} + \dots \tag{2.7}$$

where in the region $\frac{1}{\mu_0} < |x-y| < \frac{1}{\mu}$ we only exhibited the contribution from the second term in the OPE (2.3). The first term in eq. (2.7) should be thought of as arising from the effective action at scale μ, while the second term should be interpreted as a renormalization of the coupling. Combining (2.7) with (2.6), one can deduce that the effective coupling $\lambda(\mu)$ is:

$$\lambda(\mu) = \lambda_0 - \frac{C\lambda_0^2}{2} \int_{\frac{1}{\mu_0}<|x-y|<\frac{1}{\mu}} \frac{d^d y}{|x-y|^{d-\epsilon}} + \dots = \lambda_0 - \frac{C\lambda_0^2}{2\epsilon} \mathrm{Vol}(S^{d-1}) \left[\frac{1}{\mu^\epsilon} - \frac{1}{\mu_0^\epsilon}\right] . \tag{2.8}$$

Using $\mathrm{Vol}(S^{d-1}) = 2\pi^{d/2}/\Gamma(d/2)$, one can further check that this expression agrees with (2.5) provided $g_0 = \lambda_0 \mu_0^{-\epsilon}$ and $g(\mu) = \lambda(\mu)\mu^{-\epsilon}$.

If $C < 0$ then both terms in the beta function (2.4) are positive; thus, g grows along the flow, and the fate of the IR theory depends on the coefficients of the g^3 and higher order terms. However, if $C > 0$ then there exists a robust IR fixed point at

$$g^* = \frac{\Gamma\left(\frac{d}{2}\right)\epsilon}{\pi^{d/2} C} + \mathcal{O}(\epsilon^2) , \tag{2.9}$$

whose position depends on the coefficient of the g^3 term only through the terms of order ϵ^2.

2.2.2 Free energy on S^d

From now on, let us consider the field theory on a d-dimensional sphere S^d of radius a. Putting the field theory on S^d effectively sets the RG scale μ to be of order $1/a$. For convenience we will set $\mu = 1/(2a)$ and express our answers in terms of the renormalized coupling g at this scale. In the following computations we will also send the UV cutoff $\mu_0 \to \infty$ after appropriately subtracting any UV divergences.

The metric on S^d is most conveniently described through stereographic projection to $\mathbb{R}^d$ because in these coordinates the metric is manifestly conformally flat:

$$ds^2 = \frac{4a^2}{\left(1+|x|^2\right)^2} \sum_{i=1}^{d} (dx_i)^2 \,, \qquad |x|^2 \equiv \sum_{i=1}^{d} (x_i)^2 \,. \tag{2.10}$$

In the unperturbed theory, the connected correlation functions of O on S^d can be obtained from those in flat space given in eq. (2.2) by conformal transformation:

$$\begin{aligned} \langle O(x)O(y)\rangle_0 &= \frac{1}{s(x,y)^{2(d-\epsilon)}} \,, \\ \langle O(x)O(y)O(z)\rangle_0 &= \frac{C}{s(x,y)^{d-\epsilon} s(y,z)^{d-\epsilon} s(z,x)^{d-\epsilon}} \,, \end{aligned} \tag{2.11}$$

where

$$s(x,y) = 2a \frac{|x-y|}{\left(1+|x|^2\right)^{1/2} \left(1+|y|^2\right)^{1/2}} \tag{2.12}$$

is the "chordal distance" between points x and y.

The path integral on S^3 has UV divergences that should be subtracted away. After this regularization, which we will perform through analytic continuation, one can essentially remove the UV cutoff μ_0 by sending it to infinity. The resulting regularized path integral $Z_0(\lambda_0)$ depends on the bare coupling λ_0. As is standard in perturbative field theory, one can write down the following series expansion for $\log|Z(\lambda_0)|$ in terms of the connected correlators of the unperturbed theory:

$$\log\left|\frac{Z(\lambda_0)}{Z(0)}\right| = \sum_{n=1}^{\infty} \frac{(-\lambda_0)^n}{n!} \int d^d x_1 \sqrt{G} \cdots \int d^d x_n \sqrt{G} \langle O(x_1) \cdots O(x_n)\rangle_0 \,. \tag{2.13}$$

We have $\langle O(x)\rangle_0 = 0$ because the unperturbed theory is a CFT. Using the definition $F \equiv -\log Z$, we can write the first few terms in the above expression as

$$\delta F(\lambda_0) \equiv F(\lambda_0) - F(0) = -\frac{\lambda_0^2}{2} I_2 + \frac{\lambda_0^3}{6} I_3 + \mathcal{O}(\lambda_0^4)\,, \tag{2.14}$$

where

$$\begin{aligned} I_2 &= \int d^d x \sqrt{G} \int d^d y \sqrt{G}\, \langle O(x) O(y)\rangle_0 = \frac{(2a)^{2\epsilon}\pi^{d+1/2}}{2^{d-1}} \frac{\Gamma\left(-\frac{d}{2}+\epsilon\right)}{\Gamma\left(\frac{d+1}{2}\right)\Gamma(\epsilon)}\,, \\ I_3 &= \int d^d x \sqrt{G} \int d^d y \sqrt{G} \int d^d z \sqrt{G}\, \langle O(x) O(y) O(z)\rangle_0 = \frac{8\pi^{3(d+1)/2} a^{3\epsilon}}{\Gamma(d)} \frac{\Gamma\left(-\frac{d}{2}+\frac{3\epsilon}{2}\right)}{\Gamma\left(\frac{1+\epsilon}{2}\right)^3} C\,. \end{aligned} \tag{2.15}$$

These integrals were evaluated through analytic continuation in ϵ from a region where they are absolutely convergent [1].

One can simplify equation (2.14) by expressing it in terms of the renormalized coupling g instead of the bare coupling λ_0 and performing a series expansion in ϵ. Solving eq. (2.8) for λ_0 (with $\mu_0 \to \infty$ and $\mu = 1/(2a)$), one obtains

$$\lambda_0 (2a)^\epsilon = g + \frac{C\pi^{d/2}}{\epsilon\Gamma\left(\frac{d}{2}\right)} g^2 + \mathcal{O}(g^3)\,. \tag{2.16}$$

Substituting this expression together with the expressions for I_2 and I_3 from eq. (2.15) into eq. (2.14) gives

$$\delta F(g) = (-1)^{\frac{d+1}{2}} \frac{2\pi^{d+1}}{d!} \left[-\frac{1}{2}\epsilon g^2 + \frac{1}{3} \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} C g^3 + \mathcal{O}(g^4) \right]\,, \tag{2.17}$$

where we expanded each coefficient of g^n to the first nonvanishing order in ϵ. By comparing this formula with the beta function (2.4), we observe that, to the third order in g, the derivative of the free energy is proportional to the beta function:

$$\frac{dF}{dg} = (-1)^{\frac{d+1}{2}} \frac{2\pi^{d+1}}{d!} \beta(g) + \mathcal{O}(g^2) \,. \tag{2.18}$$

The proportionality between dF/dg and $\beta(g)$ to this order in perturbation theory is not unexpected: one can show that $dF(g)/dg$ equals the integrated one-point function of the renormalized operator O,

$$\frac{dF}{dg} = \mu^{\epsilon} \int d^d x \sqrt{G} \langle O_{\text{ren}}(x) \rangle_{\lambda} \,. \tag{2.19}$$

This one point function is required by conformal invariance to vanish at the RG fixed points. To the order in g we have been working at, both the beta function and the one point function of O are quadratic functions, so the fact that conformal invariance forces them to have the same zeroes implies that they must be proportional. To higher orders in perturbation theory, we expect that dF/dg will equal $\beta(g)$ times a nonvanishing function of g.

One can also note that for both signs of C the beta function $\beta(g)$ is negative to second order in perturbation theory. Eq. (2.18) then tells us that the quantity $\tilde{F} = (-1)^{\frac{d+1}{2}} F$ is a monotonically decreasing function of the radius of the sphere in all odd dimensions. We interpret this behavior as a monotonic decrease in $\tilde{F}$ along RG flow between the UV and IR fixed points. $\tilde{F}$ is stationary at conformal fixed points, supporting the F-theorem in three dimensions and the g-theorem in one dimension.

Recall that when $C > 0$ there is a perturbative fixed point at the value of the coupling g^* given in (2.9). Eq. (2.17) tells us that the difference between the free energy at this perturbative fixed point and that at the UV fixed point $g = 0$ is

$$\delta\tilde{F}(g^*) = -\frac{(d-2)!!}{2^{d-1} d (d-1)!!} \frac{\pi^2 \epsilon^3}{3C^2} \,. \tag{2.20}$$

The case of most interest is $d = 3$, where

$$\delta F(g^*)|_{d=3} = -\frac{\pi^2 \epsilon^3}{72C^2} \,. \tag{2.21}$$

We will be able to reproduce this expression in a specific example in section 2.5.1.

The arguments above relied heavily on $O(x)$ being a scalar operator. If instead $O(x)$ is a pseudo-scalar, then the relation

$$\langle O(-x_1)O(-x_2)\ldots O(-x_n)\rangle_0 = (-1)^n \langle O(x_1)O(x_2)\ldots O(x_n)\rangle_0 \,. \tag{2.22}$$

implies that the integrated n-point functions of $O(x)$ vanish if n is odd. In particular $I_3 = 0$ in equation (2.15), and so the first non-linear correction to the beta function is of order g^3; it comes from integrating the four-point function of $O(x)$ as opposed to the three-point function as was the case for a scalar operator. Because the form of the four-point function is not fixed by conformal invariance but rather depends on the details of the theory, it is hard to say anything general in this case. A specific example of a slightly relevant pseudo-scalar deformation is discussed in section 2.5.3, with the deformation coming from a fermionic double trace operator.

2.3 Towards a more general proof of the F-theorem

Let us consider a CFT on S^d perturbed by multiple operators,

$$S = S_0 + \lambda_0^i \int d^d x \sqrt{G} O_i(x) \,, \tag{2.23}$$

where the bare operators O_i have dimensions $\Delta_i = d - \epsilon_i$ with $\epsilon_i > 0$, and S_0 is a conformally-invariant action. In section 2.2 we studied the special case where there was only one such perturbing operator.

In terms of the dimensionless running couplings g^i, which we will denote collectively by $\mathbf{g}$, a simple application of the chain rule gives

$$\frac{dF}{d\log\mu} = \beta^i(\mathbf{g})\frac{\partial F}{\partial g^i}\,, \qquad \beta^i(\mathbf{g}) \equiv \frac{dg^i}{d\log\mu}\,, \tag{2.24}$$

where we introduced the beta functions $\beta^i(\mathbf{g})$. Differentiating the partition function with respect to g^i, one can see that the gradients $\partial F/\partial g^i$ are given by the general relation:

$$\begin{aligned}\frac{\partial F}{\partial g^i} &= \mu^{\epsilon_i}\int d^dx\sqrt{G}\,\langle O_{\text{ren}i}(x)\rangle_\lambda \\ &= (-1)^{\frac{d+1}{2}}\frac{2\pi^{d+1}}{d!}h_{ij}(\mathbf{g})\beta^j\,.\end{aligned} \tag{2.25}$$

In the last line of this eq. (2.25) we have defined the matrix $h_{ij}(\mathbf{g})$, which can be thought of as a metric on the space of coupling constants. Consequently, introducing $\tilde{F} = (-1)^{\frac{d+1}{2}}F$ as in the previous section, we have[3]

$$\frac{d\tilde{F}}{d\log\mu} = \beta^i h_{ij}\beta^j\,. \tag{2.26}$$

In principle, the entries of the matrix $h_{ij}(\mathbf{g})$ could be singular for certain values of the coupling. A sufficient condition for the F-theorem to hold is that $h_{ij}(\mathbf{g})$ is strictly positive definite for all $\mathbf{g}$. We will see that this is the case at least perturbatively in small $\mathbf{g}$.

The perturbative construction of $\beta^i(\mathbf{g})$ and $h_{ij}(\mathbf{g})$ generalizes the computation in section 2.2. We can choose our operators $O_i(x)$ so that in flat space the two and three-point functions at the UV fixed point are

$$\begin{aligned}\langle O_i(x)O_j(y)\rangle_0 &= \frac{\delta_{ij}}{|x-y|^{2\Delta_i}}\,, \\ \langle O_i(x)O_j(y)O_k(z)\rangle_0 &= \frac{C_{ijk}}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}\,|y-z|^{\Delta_j+\Delta_k-\Delta_i}\,|z-x|^{\Delta_i+\Delta_k-\Delta_j}}\,,\end{aligned} \tag{2.27}$$

[3]This equation is analogous to that derived for the c-function in two dimensional field theory [3], where it contains a metric on the space of coupling constants well-known as the Zamolodchikov metric.

for some structure constants C_{ijk}. The corresponding OPE is

$$O_i(x)O_j(y) = \frac{\delta_{ij}}{|x-y|^{2\Delta_i}} + \frac{C_{ij}^k O_k(x)}{|x-y|^{\Delta_i+\Delta_j-\Delta_k}} + \ldots \qquad \text{as } x \to y\,, \tag{2.28}$$

where $C_{ij}^k = \delta^{kl} C_{lij}$. These correlators yield the beta functions

$$\beta^i(\mathbf{g}) = \mu \frac{dg^i}{d\mu} = -\epsilon_i g^i + \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \sum_{j,k} C^i_{jk} g^j g^k + \mathcal{O}(\mathbf{g}^3)\,, \tag{2.29}$$

and the free energy

$$\delta F = (-1)^{\frac{d+1}{2}} \frac{2\pi^{d+1}}{d!} \left[-\frac{1}{2} \sum_i \epsilon_i (g^i)^2 + \frac{\pi^{d/2}}{3\Gamma\left(\frac{d}{2}\right)} \sum_{i,j,k} C_{ijk} g^i g^j g^k + \mathcal{O}(\mathbf{g}^4) \right]\,. \tag{2.30}$$

We see that (2.25) is satisfied with

$$h_{ij}(\mathbf{g}) = \delta_{ij} + \mathcal{O}(\mathbf{g})\,, \tag{2.31}$$

so the matrix $h_{ij}(\mathbf{g})$ is positive definite to first nonvanishing order in $\mathbf{g}$. Of course, as long as the perturbative expansion converges, $h_{ij}(\mathbf{g})$ will continue to be positive definite at the very least in a small neighborhood of $\mathbf{g} = 0$. A potential route towards proving the F-theorem is to construct the metric $h_{ij}(\mathbf{g})$ non-perturbatively and demonstrate that it is positive definite. Such an approach was undertaken in [56] for one-dimensional field theories that can be realized as boundaries of two-dimensional field theories.

2.4 F-values for free conformal fields

2.4.1 Free conformal scalar field

In this section we calculate the free energy of a free scalar field conformally coupled to the round S^d. Similar results have appeared in [62, 63].

In d-dimensions the action of a free scalar field conformally coupled to S^d is given by

$$S_S = \frac{1}{2} \int d^d x \sqrt{G} \left[(\nabla \phi)^2 + \frac{d-2}{4(d-1)} R \phi^2 \right] . \tag{2.32}$$

We take the radius of the round S^d to be a, so that the Ricci scalar is $R = d(d-1)/a^2$. Up to a constant additive term,

$$F_S = -\log |Z_S| = \frac{1}{2} \log \det \left[\mu_0^{-2} \mathcal{O}_S \right] , \qquad \mathcal{O}_S \equiv -\nabla^2 + \frac{d-2}{4(d-1)} R . \tag{2.33}$$

where μ_0 is the UV cutoff needed to properly define the path integral. At the end of the day F_S will not depend on μ_0 or a in odd dimensions. When $d \geq 2$, the eigenvalues of $\mathcal{O}_S$ are

$$\lambda_n = \frac{1}{a^2} \left(n + \frac{d}{2} \right) \left(n - 1 + \frac{d}{2} \right) , \qquad n \geq 0 , \tag{2.34}$$

and each has multiplicity

$$m_n = \frac{(2n + d - 1)(n + d - 2)!}{(d-1)! n!} . \tag{2.35}$$

The free energy is therefore

$$F_S = \frac{1}{2} \sum_{n=0}^{\infty} m_n \left[-2 \log(\mu_0 a) + \log \left(n + \frac{d}{2} \right) + \log \left(n - 1 + \frac{d}{2} \right) \right] . \tag{2.36}$$

This sum clearly diverges at large n, but it can be regulated using zeta-function regularization. By explicit computation, one can see that, unlike in even dimensions, in odd dimensions we have

$$\sum_{n=0}^{\infty} m_n = 0 , \tag{2.37}$$

d	F_S
3	$\frac{1}{2^4}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx 0.0638$
5	$\frac{-1}{2^8}\left(2\log 2 + \frac{2\zeta(3)}{\pi^2} - \frac{15\zeta(5)}{\pi^4}\right) \approx -5.74 \times 10^{-3}$
7	$\frac{1}{2^{12}}\left(4\log 2 + \frac{82\zeta(3)}{15\pi^2} - \frac{10\zeta(5)}{\pi^4} - \frac{63\zeta(7)}{\pi^6}\right) \approx 7.97 \times 10^{-4}$
9	$\frac{-1}{2^{16}}\left(10\log 2 + \frac{1588\zeta(3)}{105\pi^2} - \frac{2\zeta(5)}{\pi^4} - \frac{126\zeta(7)}{\pi^6} - \frac{255\zeta(9)}{\pi^8}\right) \approx -1.31 \times 10^{-4}$
11	$\frac{1}{2^{20}}\left(28\log 2 + \frac{7794\zeta(3)}{175\pi^2} + \frac{1940\zeta(5)}{63\pi^4} - \frac{1218\zeta(7)}{5\pi^6} - \frac{850\zeta(9)}{\pi^8} - \frac{1023\zeta(11)}{\pi^{10}}\right) \approx 2.37 \times 10^{-5}$

Table 2.1: The F-value for a free conformal scalar field on S^d.

so there is no logarithmic dependence on $\mu_0 a$. This is in agreement with the fact that there is no conformal anomaly in this case. The remaining contribution to this sum can be computed from the function

$$-\frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{m_n}{\left(n+\frac{d}{2}\right)^s} + \frac{m_n}{\left(n-1+\frac{d}{2}\right)^s}\right] = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{m_n + m_{n-1}}{\left(n-1+\frac{d}{2}\right)^s}\,, \tag{2.38}$$

whose derivative at $s = 0$ formally gives (2.36). One can check that $m_n + m_{n-1}$ is a polynomial of degree $d-1$ in $n-1+\frac{d}{2}$, so the sum in (2.38) converges absolutely for $s > d$ and can be evaluated in terms of $\zeta(s-k, \frac{d}{2}-1)$, with k ranging over all even integers between 0 and $d-1$. In $d = 3$, for example, we have $m_n = (n+1)^2$ and $m_n + m_{n-1} = 2\left(n+\frac{1}{2}\right)^2 + \frac{1}{2}$, so

$$F_S = -\frac{1}{2}\frac{d}{ds}\left[2\zeta\left(s-2, \frac{1}{2}\right) + \frac{1}{2}\zeta\left(s, \frac{1}{2}\right)\right]\Bigg|_{s=0} = \frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx 0.0638\,. \tag{2.39}$$

For other values of d, see table 2.1. These results agree with earlier work [62–64]. For all odd d we note that the the free energy on S^d equals minus the entanglement entropy across S^{d-2} calculated in [65].

2.4.2 Free massless fermion field

In this section we calculate the free energy of a free massless complex Dirac fermion on the round S^d. We begin with the free fermion action

$$S_D = \int d^d x \sqrt{G} \psi^\dagger (i \not{D}) \psi \,. \tag{2.40}$$

Unlike in the case of the free conformal scalar action, the conformal fermion action does not contain a coupling between the fermion fields and curvature. The free energy is given by

$$F_D = -\log |Z_D| = -\log \det \left[\mu_0^{-1} \mathcal{O}_D \right] \,, \qquad \mathcal{O}_D \equiv i \not{D} \,. \tag{2.41}$$

The eigenvalues of $\mathcal{O}_D$ are

$$\pm \frac{1}{a} \left(n + \frac{d}{2} \right) \,, \qquad n \geq 0 \,, \tag{2.42}$$

each with multiplicity

$$\hat{m}_n = \dim \gamma \begin{pmatrix} n + d - 1 \\ n \end{pmatrix} \,. \tag{2.43}$$

Here, $\dim \gamma$ is the dimension of the gamma matrices in d dimensions. For odd d, $\dim \gamma = 2^{\frac{d-1}{2}}$ in the fundamental representation.

One can then write (2.41) as

$$F_D = -2 \sum_{n=0}^{\infty} \hat{m}_n \left[-\log(\mu_0 a) + \log \left(n + \frac{d}{2} \right) \right] \,, \tag{2.44}$$

and again one can check that $\sum_{n=0}^{\infty} \hat{m}_n = 0$ in odd dimensions using zeta-function regularization, so there is no logarithmic dependence on $\mu_0 a$. To compute F_D, one can write it

d	$F_D/\dim\gamma$
3	$\frac{1}{2^4}\left(2\log 2+\frac{3\zeta(3)}{\pi^2}\right)\approx 0.110$
5	$\frac{-1}{2^8}\left(6\log 2+\frac{10\zeta(3)}{\pi^2}+\frac{15\zeta(5)}{\pi^4}\right)\approx -2.16\times 10^{-2}$
7	$\frac{1}{2^{12}}\left(20\log 2+\frac{518\zeta(3)}{15\pi^2}+\frac{70\zeta(5)}{\pi^4}+\frac{63\zeta(7)}{\pi^6}\right)\approx 4.61\times 10^{-3}$
9	$\frac{-1}{2^{16}}\left(70\log 2+\frac{12916\zeta(3)}{105\pi^2}+\frac{282\zeta(5)}{\pi^4}+\frac{378\zeta(7)}{\pi^6}+\frac{255\zeta(9)}{\pi^8}\right)\approx -1.02\times 10^{-3}$
11	$\frac{1}{2^{20}}\left(252\log 2+\frac{234938\zeta(3)}{525\pi^2}+\frac{69124\zeta(5)}{63\pi^4}+\frac{8778\zeta(7)}{5\pi^6}+\frac{1870\zeta(9)}{\pi^8}+\frac{1023\zeta(11)}{\pi^{10}}\right)\approx 2.32\times 10^{-4}$

Table 2.2: The F-value for a free massless Dirac fermion field on S^d. Here, $\dim\gamma$ is the dimension of the gamma matrices on S^d and is equal to $2^{\frac{d-1}{2}}$ in odd dimensions d.

formally as the derivative at $s=0$ of the function

$$2\sum_{n=0}^{\infty}\frac{\hat{m}_n}{\left(n+\frac{d}{2}\right)^s}\,. \tag{2.45}$$

One can check that $\hat{m}_n$ is a polynomial of degree $d-1$ in $n+\frac{d}{2}$, so the sum in (2.45) converges absolutely for $s>d$ and can be expressed in terms of $\zeta(s-k,\frac{d}{2})$, with k ranging over the even integers between 0 and $d-1$.

For $d=3$, $\hat{m}_n=(n+2)(n+1)$, and the F-value of a massless Dirac fermion is

$$F_D=2\zeta'(-2,3/2)-\frac{1}{2}\zeta'(0,3/2)=\frac{\log 2}{4}+\frac{3\zeta(3)}{8\pi^2}\approx 0.219\,. \tag{2.46}$$

This result agrees with earlier work [64]. [4] For other values of d, see table 2.2. The F-value of a Majorana fermion is one half the result in table 2.2.

[4] We note that F_D/F_S is not a rational number and is quite large, ≈ 3.43. For comparison, we note that the contribution of a $d=3$ massless Dirac fermion to the thermal free energy is $3/2$ times that of a massless real scalar. In $d=4$ the a-coefficient of a massless Dirac fermion is 11 times that of a conformal scalar, while its contribution to the thermal free energy is $7/2$ times that of a massless scalar. Only in $d=2$ does the c-coefficient of a massless Dirac fermion equal that of a massless scalar.

2.4.3 Chern-Simons Theory

In three dimensions $U(N)$ Yang-Mills theory does not have a UV fixed point. Instead, we will consider $U(N)$ Chern-Simons gauge theory with level k. The F-value for $N = 1$ is $\frac{1}{2}\log k$, while for $N > 1$ it was found to be [66]

$$F_{\text{CS}}(k, N) = \frac{N}{2}\log(k+N) - \sum_{j=1}^{N-1}(N-j)\log\left(2\sin\frac{\pi j}{k+N}\right) . \tag{2.47}$$

In the weak coupling limit $k \gg N$, and for sufficiently large N, this expression may be approximated by $\frac{1}{2}N^2\left(\log\frac{k}{2\pi N} + \frac{3}{2}\right)$. Thus, somewhat surprisingly, the CS theory has a large F-value, even though it has no propagating degrees of freedom.

In four dimensions, one of the first tests of the a-theorem was provided by the $SU(N)$ gauge theory coupled to N_f massless Dirac fermions in the fundamental representation [1]. This theory is asymptotically free for $N_f < 11N/2$. If this is the case, then in the UV the a-coefficient receives contributions from the $N_c^2 - 1$ gauge bosons and the $N_f N$ free fermions. In the IR, it is believed that chiral symmetry breaking produces $N_f^2 - 1$ Goldstone bosons, which are the only degrees of freedom that contribute to a_{IR}. The asymptotic freedom condition $N_f < 11N/2$ imposes an upper bound on the IR value of a that is restrictive enough to not violate the a-theorem [1].

In three dimensions we cannot construct similar tests involving $U(N)$ Yang-Mills theory coupled to fundamental fermions because the UV theory is not conformal. Instead we consider Chern-Simons gauge theories. As a first example take the $U(1)$ Chern-Simons gauge theory coupled to N_f massless Dirac fermions of charge 1. For $k \gg 1$ this theory is weakly coupled, so the F-value is

$$F_{\text{UV}} \approx \frac{1}{2}\log k + N_f\left(\frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2}\right) + O(N_f/k) . \tag{2.48}$$

Now, let us add a mass for the fermion. The IR fixed point is then described by the $U(1)$ Chern-Simons gauge theory with CS level $k \pm N_f/2$ generated through the parity anomaly [67, 68], where the sign is determined by the sign of the fermion mass. Therefore, the IR free energy is $\frac{1}{2}\log(k \pm N_f/2)$. It is not hard to check that this is smaller than (5.43).

Now we consider $U(N)_k$ Chern-Simons gauge theory coupled to N_f massless fundamental Dirac fermions. For $k \gg N$ this is a weakly coupled conformal field theory whose F-value is

$$F_{\text{UV}} \approx \frac{1}{2}N^2\left(\log\frac{k}{2\pi N} + \frac{3}{2}\right) + NN_f\left(\frac{\log 2}{4} + \frac{3\zeta(3)}{8\pi^2}\right) + O(N_f N^2/k)\,. \tag{2.49}$$

Now, let us add a $U(N_f)$ symmetric mass for the fermions. The IR fixed point is then described by the $U(N)$ Chern-Simons gauge theory with CS level $k \pm N_f/2$. Therefore, $F_{\text{IR}} = F_{\text{CS}}(k \pm N_f/2, N)$. It is not hard to check that $F_{\text{UV}} > F_{\text{IR}}$ for any N_f if $k \gg N$. The comparison is the simplest if, in addition, we assume $k \gg N_f$. Then

$$F_{\text{IR}} \approx \frac{1}{2}N^2\left(\log\frac{k}{2\pi N} + \frac{3}{2}\right) \pm \frac{N_f N^2}{4k} + \dots\ , \tag{2.50}$$

making it obvious that $F_{\text{UV}} > F_{\text{IR}}$.

2.5 Double trace deformations

In this section we study the change in free energy under a relevant double trace deformation in a d-dimensional large N field theory, starting from a UV fixed point and flowing to an IR fixed point. Some of this section is a review of the earlier work [57–59].

2.5.1 Bosonic double trace deformation

Consider a bosonic single trace operator Φ within the UV conformal field theory. Let the dimension of this operator, Δ, lie inside the range $(d/2-1, d/2)$. The lower limit on the dimension is the unitarity bound. We impose the upper limit on the dimension because we will be adding the operator Φ^2 to the lagrangian and we want this to be a relevant operator. There are general arguments [57, 69] that this deformation will cause an RG flow to an IR fixed point where Φ has dimension $d - \Delta$.

We begin with the partition function

$$Z = \int D\phi \exp\left(-S_0 - \frac{\lambda_0}{2} \int d^d x \sqrt{G} \Phi^2\right) = Z_0 \left\langle \exp\left(-\frac{\lambda_0}{2} \int d^d x \sqrt{G} \Phi^2\right) \right\rangle_0 , \qquad (2.51)$$

where, as in section 2.2, λ_0 is the bare coupling defined at the UV scale μ_0, Φ is the bare operator, and expectation values $\langle \cdots \rangle_0$ are taken with respect to the conformal action S_0. The measure $D\phi$ is schematic for integration over all degrees of freedom in the theory. We are interested in calculating the difference δF_Δ between the free energies of the IR and UV fixed points,

$$\delta F_\Delta = -\log \left| \frac{Z}{Z_0} \right| . \qquad (2.52)$$

We explicitly write δF_Δ as a function of the UV scaling dimension Δ to emphasize the dependence of the IR free energy on the UV scaling dimension of the single trace operator Φ.

As in [57] we proceed through a Hubbard-Stratonovich transformation. That is, we introduce an auxiliary field σ so that

$$\frac{Z}{Z_0} = \frac{1}{\int D\sigma \exp(\frac{1}{2\lambda_0} \int d^d x \sqrt{G} \sigma^2)} \int D\sigma \left\langle \exp\left[\int d^d x \sqrt{G} \left(\frac{1}{2\lambda_0}\sigma^2 + \sigma\Phi\right)\right]\right\rangle_0 . \qquad (2.53)$$

In this context, large N implies that the higher point functions of Φ are suppressed relative to the two-point function by factors of $1/N$, where we take N large. This allows us to write

$$\left\langle \exp\left(\int d^d x \sqrt{G} \sigma(x) \Phi(x)\right)\right\rangle_0 = \exp\left[\frac{1}{2}\left\langle \left(\int d^d x \sqrt{G} \sigma(x) \Phi(x)\right)^2 \right\rangle_0 + O(1/N)\right] . \qquad (2.54)$$

The integral in equation (2.53) is then simply a gaussian integral, which integrates to give

$$\delta F_\Delta = \frac{1}{2}\,\mathrm{tr}\log(K)\,, \tag{2.55}$$

where

$$K(x,y) = \frac{1}{\sqrt{G(x)}}\delta(x-y) + \lambda_0 a^d \left\langle \Phi(x)\Phi(y)\right\rangle_0 \,. \tag{2.56}$$

We choose to normalize the operator Φ so that the perturbing operator Φ^2 has the same normalization as the operator O in section 2.2. Specifically we take the two-point function of Φ on the round S^d to be given by

$$\left\langle \Phi(x)\Phi(y)\right\rangle_0 = \frac{1}{\sqrt{2}}\frac{1}{s(x,y)^{2\Delta}}\,. \tag{2.57}$$

We then proceed by expanding the right hand side of equation (2.57) in S^d spherical harmonics using

$$\frac{1}{s(x,y)^{2\Delta}} = \frac{1}{a^{2\Delta}}\sum_{n,m} g_n Y^*_{nm}(x) Y_{nm}(y)\,, \tag{2.58}$$

where we normalize the $Y_{nm}(x)$ to be orthonormal with respect to the standard inner product on the unit S^d. The g_n coefficients in equation (2.58) can be found in [57], where they are shown to be

$$g_n = \pi^{d/2} 2^{d-\Delta}\frac{\Gamma(\frac{d}{2}-\Delta)}{\Gamma(\Delta)}\frac{\Gamma(n+\Delta)}{\Gamma(d+n-\Delta)}\,, \qquad n \geq 0\,. \tag{2.59}$$

The expression for δF_Δ (2.55) was evaluated using dimensional regularization in [58] . Here we briefly review their argument. The eigenvalues of the operator K only depend on the angular momentum n through the g_n coefficients of equation (2.59). States on the sphere

S^d with angular momentum n have the degeneracy m_n given in equation (2.35). One can therefore write the change in free energy as

$$\delta F_\Delta = \frac{1}{2}\sum_{n=0}^{\infty} m_n \log\left[1+\lambda_0 a^{d-2\Delta} g_n\right] . \tag{2.60}$$

Because $d-2\Delta > 0$, in the IR limit $a^{d-2\Delta}$ goes to infinity. Continuing to dimension $d<0$, the sum in equation (2.60) converges and δF_Δ becomes

$$\delta F_\Delta = \frac{1}{2}\sum_{n=0}^{\infty} m_n \log\left(\frac{\Gamma(n+\Delta)}{\Gamma(d+n-\Delta)}\right) , \tag{2.61}$$

where in simplifying equation (2.60) one uses $\sum_n m_n = 0$ as in eq. (2.37).

The sum in equation (2.61) is evaluated exactly in [58]. In odd dimensions they find

$$\frac{d\left(\delta F_\Delta\right)}{d\Delta} = \frac{(-1)^{(d+1)/2}\pi^2(d-2\Delta)}{2\Gamma(1+d)}\frac{\sec\left[\pi\left(\Delta-\frac{d}{2}\right)\right]\tan\left[\pi\left(\Delta-\frac{d}{2}\right)\right]}{\Gamma(1-\Delta)\Gamma(1-d+\Delta)} . \tag{2.62}$$

The result agrees exactly with the dual calculation in AdS_{d+1} [58]. In the case of most interest, where $d=3$, it reduces to the following simple expression:

$$\frac{d\left(\delta F_\Delta\right)}{d\Delta} = -\frac{\pi}{6}(\Delta-1)(\Delta-\frac{3}{2})(\Delta-2)\cot(\pi\Delta) . \tag{2.63}$$

As a last step, we integrate eq. (2.63) with respect to Δ to get the final expression for δF_Δ,

$$\delta F_\Delta = -\frac{\pi}{6}\int_{\Delta}^{3/2} dx(x-1)(x-\frac{3}{2})(x-2)\cot(\pi x) . \tag{2.64}$$

The upper limit of integration in eq. (2.64) is chosen to be $3/2$ because we know that $\delta F_{\Delta=3/2} = 0$, as can be seen directly in eq. (2.61), where each term in the sum vanishes when $\Delta = d/2$. The reason why $\delta F_{\Delta=3/2} = 0$ is that when $\Delta = d/2$ the operator Φ^2 is marginal.

In figure 2.1 we plot δF_Δ over the allowed range of Δ when $d=3$ (solid curve). There

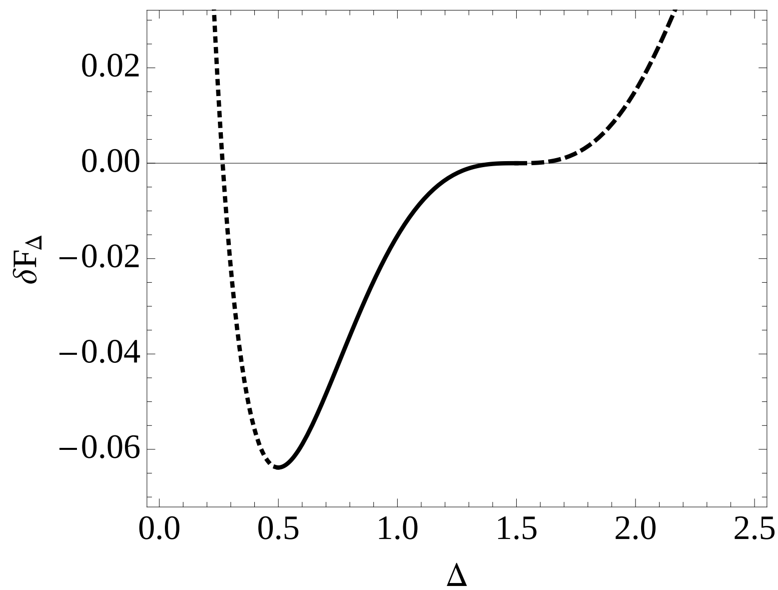


Figure 2.1: The change in free energy δF between the UV theory and the IR theory when the UV theory is perturbed by a double trace operator O^2, where O has dimension Δ. The allowed range of Δ corresponds to the solid section of the curve. The UV theory is non unitary when $\Delta < 1/2$ (dotted), and the double-trace deformation is irrelevant when $\Delta > 3/2$ (dashed).

are two cases of special interest. The first is when $\Delta = 1$, which corresponds to the $O(N)$ models we discuss in section 2.5.2. The numerical value for the difference in the free energy between the IR and UV fixed points in this case is

$$\delta F_{\Delta=1} = -\frac{\zeta(3)}{8\pi^2} \approx -0.0152\,. \tag{2.65}$$

The second case of interest is when $\Delta = 1/2$, so that the operator Φ^2 corresponds to adding a mass term for the free scalar field Φ. In this case δF evaluates to

$$\delta F_{\Delta=1/2} = -\frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx -0.0638\,. \tag{2.66}$$

The change in free energy in equation (2.66) is simply minus the free energy of a massive real scalar field in equation (2.39). This makes sense since in this case we simply integrated out the real free scalar field Φ. This result can be thought of as a check of our procedure.

There is one further consistency check we can easily perform. If we take $\Delta = (3-\epsilon)/2$, then the IR fixed point is the perturbative fixed point of section 2.2. The coefficient of the three point function C is easily calculated to be $C = 4/\sqrt{2}$ in this case. Equation (2.21) predicts that the difference in free energy between the IR and UV fixed points is

$$\delta F_{\Delta=(3-\epsilon)/2} = -\frac{\pi^2\epsilon^3}{576} + o(\epsilon^3)\,. \tag{2.67}$$

Indeed, expanding the integral in equation (2.64) for $\Delta = (3-\epsilon)/2$ with ϵ small reproduces exactly equation (2.67). This provides another consistency check between the double trace calculation and the perturbative calculation.

In Fig. 2.1 we also show δF_Δ for $\Delta > 3/2$ (dashed) and $\Delta < 1/2$ (dotted). In the former case, the UV double-trace deformation is irrelevant, and so it is not surprising that δF_Δ is positive; we are exchanging the UV and IR fixed points. When $\Delta < 1/2$, the UV fixed point is non-unitary and the F-theorem is not required to hold. Indeed, for Δ less than ≈ 0.2658 there is a region where δF_Δ is positive.[5]

2.5.2 RG flows in $O(N)$ vector models

In this section we discuss RG flows in the $O(N)$ vector models and compare the free energies of the various fixed points. We begin with the classical $O(N)$ model action in flat 3-dimensional Euclidean space,

$$S[\vec{\Phi}] = \frac{1}{2}\int d^3x \left[\partial\vec{\Phi}\cdot\partial\vec{\Phi} + m_0^2\vec{\Phi}^2 + \frac{\lambda_0}{2N}\left(\vec{\Phi}\cdot\vec{\Phi}\right)^2\right], \tag{2.68}$$

where $\vec{\Phi}$ is an N-component vector of real scalar fields. The F-value of the UV fixed point of this theory is of course just that of N massless free scalar fields: $F_{\text{UV}}^{\text{bos}} = \frac{N}{16}\left(2\log 2 - 3\frac{\zeta(3)}{\pi^2}\right)$. If we take $m^2 > 0$, then all N scalar fields become massive in the IR and we end up with the

[5]Similarly, the Zamolodchikov c-theorem [3] is not applicable to non-unitarity theories. For explicit violations of the c-theorem in non-unitary theories see, for example, [70, 71].

trivial empty theory whose F-value vanishes. The critical model comes from maintaining the vanishing renormalized mass. This theory has a non-trivial IR fixed point. The difference between the IR and UV F-values is given in equation (2.65). Therefore for large N the F-value of the IR fixed point in the critical $O(N)$ model is

$$F_{\rm crit}^{\rm bos} = \frac{N}{16}\left(2\log 2 - 3\frac{\zeta(3)}{\pi^2}\right) - \frac{\zeta(3)}{8\pi^2} + O(1/N). \tag{2.69}$$

The free and critical $O(N)$ vector models have been conjectured [72] to be dual to the minimal Vasiliev higher-spin gauge theory in AdS_4 [73–75], with different boundary conditions. Recently, this conjecture has been subjected to some non-trivial tests [76–78], and new ideas have appeared on how to prove it [79, 80]. It would be very interesting to match our field theory results for $F_{\rm UV}^{\rm bos}$ and $F_{\rm crit}^{\rm bos}$ using the higher spin theory in Euclidean AdS_4.

Now consider perturbing the critical $O(N)$ model by the scalar mass term with $m_0^2 < 0$. As the theory flows to the IR the potential breaks the symmetry from $O(N)$ to $O(N-1)$, and so by Goldstone's theorem we pick up $N-1$ flat directions in field space. In the far IR these Goldstone modes simply become $N-1$ free massless scalar fields, with F-value

$$F_{\rm Goldstone} = \frac{N-1}{16}\left(2\log 2 - 3\frac{\zeta(3)}{\pi^2}\right) . \tag{2.70}$$

Thus,

$$F_{\rm Goldstone} - F_{\rm crit}^{\rm bos} = -\frac{1}{16}\left(2\log 2 - 5\frac{\zeta(3)}{\pi^2}\right) \approx -0.0486 \tag{2.71}$$

in agreement with the conjectured F-theorem.

This conclusion should be contrasted with the evolution of the thermal free energy coefficient $c_{\rm Therm}$. In the critical $O(N)$ model $c_{\rm Therm} = 4N/5 + O(1)$ [6, 7], while in the Goldstone phase $c_{\rm Therm} = N - 1$. Thus, for large enough N the flow from the critical $O(N)$ model to the Goldstone phase rules out the possibility of a $c_{\rm Therm}$ theorem. On the other hand, the coefficient of the stress-energy tensor 2-point function c_T decreases when the $O(N)$ model flows from the critical to the Goldstone phase [81]. Thus, such a flow does not rule out the possibility of a c_T theorem.

Another interesting $O(N)$ model to consider is the $d = 3$ Gross-Neveu model with N massless Majorana fermions ψ^i and the interaction term $(\bar{\psi}^i\psi^i)^2$. This model has an interacting UV fixed point where the pseudoscalar operator $\bar{\psi}^i\psi^i$ has dimension $1+O(1/N)$. The IR fixed point is described simply by N free fermions. Thus, we find that

$$\begin{aligned} F_{\rm UV}^{\rm ferm} &= \frac{N}{16}\left(2\log 2 + 3\frac{\zeta(3)}{\pi^2}\right) + \frac{\zeta(3)}{8\pi^2} + O(1/N) \ , \\ F_{\rm IR}^{\rm ferm} &= \frac{N}{16}\left(2\log 2 + 3\frac{\zeta(3)}{\pi^2}\right) \ . \end{aligned} \tag{2.72}$$

The higher-spin duals of these theories in AdS_4 were conjectured in [82, 83], and recently these conjectures were subjected to non-trivial tests [76,77]. It would be interesting to derive the results (2.72) using the higher-spin gauge theory in Euclidean AdS_4.

2.5.3 Fermionic double trace deformation

In this section we study the change in free energy under a fermionic double trace deformation in a large N field theory on S^d. The calculation proceeds analogously to that in section 2.5.1, where we deformed the UV fixed point by a bosonic double trace deformation. The difference is that we replace the bosonic operator $\Phi(x)$ by a fermionic, single-trace operator $\chi(x)$. In this section we will assume that χ is a complex Grassmann-valued spinor field. In order to obtain the difference in free energy between the IR and UV fixed points when χ is Majorana, all one has to do is divide the final result by two.

Let the dimension of the operator χ be Δ, with Δ inside the range $[(d-1)/2, d/2]$. The lower limit on the dimension is the unitarity bound on spinor operators. The upper limit on the dimension comes from requiring the operator $\bar{\chi}\chi$ to be relevant. Just as in the case of the bosonic double trace deformation, one can argue that the double-trace deformation will induce an RG flow that takes the theory to an IR fixed point where χ has dimension $d - \Delta$ [59].

We want to compute the F-value of the IR fixed point, so we need to calculate the free energy of the theory on the round S^d. The partition function on S^d is given by

$$Z = Z_0 \left\langle \exp\left(-\lambda_0 \int d^dx \sqrt{G} \bar{\chi}\chi\right)\right\rangle_0 , \tag{2.73}$$

where λ_0 is the coupling of dimension $d - 2\Delta$. The calculation of the expectation value in equation (2.73) was presented in [59], and here we summarize their derivation. First, we introduce a complex auxiliary spinor field η and write

$$\frac{Z}{Z_0} = \frac{1}{\int D\eta D\bar{\eta} \exp(\int d^dx \sqrt{G}\bar{\eta}\eta)} \int D\eta D\bar{\eta} \left\langle \exp\left[\int d^dx \sqrt{G}\left(\bar{\eta}\eta + \sqrt{\lambda_0}(\bar{\eta}\chi + \bar{\chi}\eta)\right)\right]\right\rangle_0 . \tag{2.74}$$

Just as in the bosonic case, the assumption of large N comes into play by taking the expectation value inside of the exponential, giving

$$\left\langle \exp\left[\int d^dx \sqrt{G}\sqrt{\lambda_0}(\bar{\eta}\chi + \bar{\chi}\eta)\right]\right\rangle_0 = \exp\left[\lambda_0 \int d^dx \sqrt{G} \int d^dy \sqrt{G} \bar{\eta}(x) \left\langle \chi(x)\bar{\chi}(y)\right\rangle_0 \eta(y) + o(1/N^0)\right] . \tag{2.75}$$

We assume that n-point functions, with $n > 2$, are suppressed by inverse power of N. The integral in equation (2.74) is then Gaussian. Exponentiating the result to give the change in free energy δF_Δ we find

$$\delta F_\Delta = -\operatorname{tr}\log(\hat{K}) , \tag{2.76}$$

where

$$\hat{K}(x,y) = \frac{1}{\sqrt{G(x)}}\delta(x-y) + \lambda_0 a^d \left\langle \chi(x)\bar{\chi}(y)\right\rangle_0 \,. \tag{2.77}$$

In flat space we choose the fermion two-point function to have the normalization

$$\hat{G}(x,y) = \left\langle \chi(x)\bar{\chi}(y)\right\rangle_0 = \frac{\gamma\cdot(x-y)}{|x-y|^{2\Delta+1}}\,. \tag{2.78}$$

We need to find the eigenvalues and degeneracies of the operator $\hat{G}$ on the sphere. This problem is solved in [59] and here we simply quote the result.[6] The eigenvalues

$$\hat{g}_n \propto \pm i\frac{\Gamma(n+\Delta+1/2)}{\Gamma(n+d-\Delta+1/2)}\,, \qquad n\geq 0 \tag{2.79}$$

come in conjugate pairs and are indexed by the integer n that runs from zero to infinity. In equation (2.79) we leave off any n independent proportionality factors, because as we will see below these factors do not contribute to the free energy in the IR limit. At each level n there is a degeneracy $\hat{m}_n$ given in equation (2.43).

Analytically continuing to the region of the complex plane where $\mathrm{Re}(d) < 1$, the trace in equation (2.76) converges and in the IR limit we can write

$$\delta F_\Delta = -2\sum_{n=0}^{\infty}\hat{m}_n \log\frac{\Gamma(n+\Delta+1/2)}{\Gamma(n+d-\Delta+1/2)}\,. \tag{2.80}$$

The sum in equation (2.80) is easily evaluated using the methods in [58]. After a simple calculation one finds the result (specifying to three-dimensions)

$$\delta F_\Delta = -\frac{2\pi}{3}\int_\Delta^{3/2} dx\left(x-\frac{1}{2}\right)\left(x-\frac{3}{2}\right)\left(x-\frac{5}{2}\right)\tan(\pi x)\,, \qquad \Delta\in\left(1,\frac{3}{2}\right)\,. \tag{2.81}$$

[6] See section 5.3 of that paper.

In figure 2.2 we plot the change in free energy δF over this range of Δ.

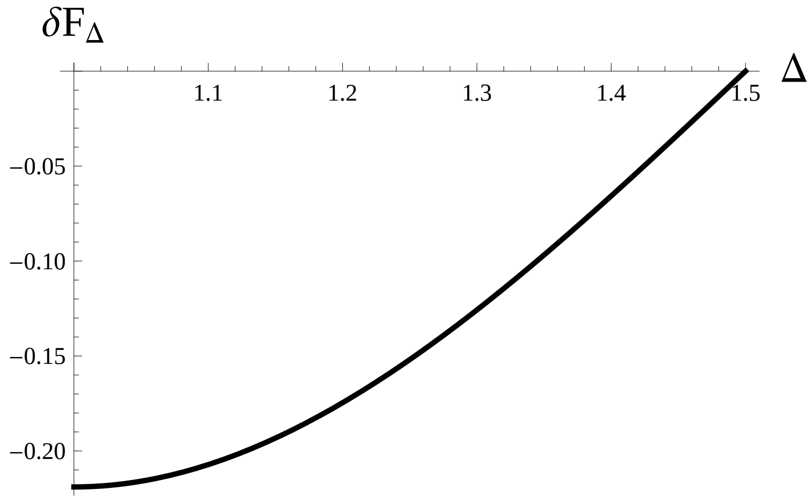


Figure 2.2: The change in free energy δF between the IR theory and the UV theory when the UV theory is perturbed by a a double trace operator $\bar{\chi}\chi$, where χ is a fermionic operator of dimension Δ.

Just as in the bosonic case, we can check this procedure by evaluating δF when χ has the dimensionality $\Delta = 1$ of a free spinor field. The integral in equation (2.81) evaluates to

$$\delta F_{\Delta=1} = -\frac{1}{8}\left(2\log 2 + 3\frac{\zeta(3)}{\pi^2}\right) \approx -.219\,. \tag{2.82}$$

Comparing to equation (2.46), we see that this is the F-value of a massless complex spinor field. Intuitively this makes sense, because in this case all we have done is to integrate out a massive free complex spinor.

We note that, as in section 2.5.1, δF vanishes for $\Delta = 3/2$ where the double-trace operator is marginal. One might be tempted to expand equation (2.81) for Δ near 3/2 and attempt to compare with the perturbative result in equation (2.21). One would quickly find that this does not work. Indeed, letting $\Delta = (3-\epsilon)/2$ we see that

$$\delta F_{\Delta=(3-\epsilon)/2} = -\frac{\epsilon}{3} + O(\epsilon^3)\,. \tag{2.83}$$

The leading change in the free energy is order ϵ while in the perturbative calculation of section 2.2 the change in free energy is order ϵ^3.

The resolution is that the perturbing operator $O = \bar{\chi}\chi$ is a pseudo-scalar, as can be seen, for example, from the fact that the correlation functions of an odd number of $O(x_i)$ change sign under $x_i \to -x_i$. From eq. (2.78), one can compute explicitly the connected correlation functions in flat space:

$$\begin{aligned}
\langle O(x)O(y)\rangle &= \frac{\dim(\gamma)}{|x-y|^{4\Delta}}\,, \\
\langle O(x)O(y)O(z)\rangle_0 &= 2i\dim(\gamma)\frac{\epsilon_{ijk}(x-y)_i(y-z)_j(z-x)_k}{(|x-y||y-z||z-x|)^{2\Delta+1}}\,, \\
\langle O(x)O(y)O(z)O(w)\rangle_0 &= -\dim(\gamma)\Big(2X_{xzyw}X_{xwyz} + 2X_{xyzw}X_{xzyw} + 2X_{xyzw}X_{xwyz} \\
&\qquad + X_{xyyz}X_{xwzw} + X_{xyxw}X_{yzzw}\Big)\,,
\end{aligned} \tag{2.84}$$

where we have defined

$$X_{abcd} \equiv \frac{(a-b)\cdot(c-d)}{|a-b|^{2\Delta+1}\,|c-d|^{2\Delta+1}}\,. \tag{2.85}$$

That the n-point functions change sign under reflection implies that only odd powers of the coupling appear in the beta function, and also that only even powers of the coupling appear in an expansion of the free energy as in eq. (2.14). We believe that all coefficients in the expansions of the beta function and δF as power series in the coupling constant are $\mathcal{O}(\epsilon)$, as can be checked explicitly for the case of the four-point function in (2.84). It follows that the IR fixed point does not occur when the coupling g is small. One would therefore need to calculate all the terms in eq. (2.14) in order to find the change in the F-value along the RG flow.

Chapter 3

Entanglement Entropy of 3-d Conformal Gauge Theories with Many Flavors

This chapter is a lightly-modified version of the paper [13].

3.1 Introduction

Many interesting quantum critical points of condensed matter systems in two spatial dimensions [84–97] are described by conformal field theories (CFT) in three spacetime dimensions where massless fermionic and/or bosonic matter interacts with deconfined gauge fields. These include critical points found in insulating antiferromagnets and d-wave superconductors and between quantum Hall states. Such CFTs can be naturally analyzed by an expansion in $1/N_F$, where N_F is the number of 'flavors' of matter. This large N_F limit is taken at fixed N_c, where N_c is a measure of the size of the gauge group *e.g.* the non-abelian gauge group $U(N_c)$. Classic examples of such CFTs include three-dimensional $U(1)$ gauge theory coupled to a large number of massless charged scalars [98] or Dirac fermions [99,100]. These theories are conformal to all orders in the $1/N_F$ expansion, and they are widely believed to be conformal for $N_F > N_{\text{crit}}$, where N_{crit} is a conjectured critical number of flavors dependent on the choice of the gauge group [99,100]. The 3-dimensional CFTs may also contain Chern-Simons terms whose coefficients k may be taken to be large.

In any 3-dimensional field theory with $\mathcal{N} \geq 2$ supersymmetry, the S^3 free energy F may be calculated using the method of localization [31, 101–103]. It has also been calculated in some simple non-supersymmetric CFTs, such as free field theories [9, 65, 104, 105] and the Wilson-Fisher fixed point of the $O(N)$ model for large N [9], which has been conjectured [72] to be dual to Vasiliev's higher-spin gauge theory in AdS_4 [74]. In this chapter we present the calculation of F in certain 3-d gauge theories coupled to a large number of massless flavors, to the first subleading order in $1/N_F$. We will find that this subleading term is of order $\log N_F$.

The CFTs we study have the following general structure. The matter sector has Dirac fermions ψ_α, $\alpha = 1 \dots N_f$, and complex scalars, z_a, $a = 1 \dots N_b$. We will always take the large N_f limit with N_b/N_f fixed, and use the symbol N_F to refer generically to either N_f or N_b. These matter fields are coupled to each other and a gauge field A_μ by a Lagrangian of the form

$$\mathcal{L}_m = \sum_{\alpha=1}^{N_f} \overline{\psi}_\alpha \gamma^\mu D_\mu \psi_\alpha + \sum_{a=1}^{N_b} \left(|D_\mu z_a|^2 + s|z_a|^2 + \frac{u}{2} \left(|z_a|^2\right)^2 \right) + \dots , \tag{3.1}$$

where $D_\mu = \partial_\mu - iA_\mu$ is the gauge covariant derivative, and the ellipses represent additional possible contact-couplings between the fermions and bosons, such as a Yukawa coupling. The scalar "mass" s generally has to be tuned to reach the quantum critical point at the renormalization group (RG) fixed point, which is described by a three dimensional CFT; however, this is the only relevant perturbation at the CFT fixed point, and so only a single parameter has to be tuned to access the fixed point. In some cases, such scalar mass terms are forbidden, and then the CFT describes a quantum critical phase. All other couplings, such as u and the Yukawa coupling, reach values associated with the RG fixed point, and so their values are immaterial for the universal properties of interest in the present chapter.

The gauge sector of the CFT has a traditional Maxwell term, along with a possible Chern-Simons term

$$\mathcal{L}_A = \frac{1}{2e^2} \operatorname{Tr} F^2 + \frac{ik}{2\pi} \operatorname{Tr} \left(F \wedge A - \frac{1}{3} A \wedge A \wedge A \right) . \tag{3.2}$$

The gauge coupling e^2 has dimension of mass in three spacetime dimensions. It flows to an RG fixed point value, and so its value is also immaterial; indeed, we can safely take the limit $e^2 \to \infty$ at the outset. However, our results will depend upon the value of the Chern-Simons coupling k, which is RG invariant. We will typically take the large N_F limit with k/N_F fixed at fixed N_c, and in most of this chapter we set $N_c = 1$ for simplicity. (This is to be contrasted with the 't Hooft type limit of large N_c where k/N_c is held fixed; see, for example, recent work [36, 106, 107].) One of our principal results, established in section 3.3, is that for the $U(1)$ gauge theory with Chern-Simons level k, coupled to N_f massless Dirac fermions and N_b massless complex scalars of charge 1 as in (3.1) with $s = u = 0$,

$$F = \frac{\log 2}{4} (N_f + N_b) + \frac{3\zeta(3)}{8\pi^2} (N_f - N_b) + \frac{1}{2} \log \left[\pi \sqrt{ \left(\frac{N_f + N_b}{8} \right)^2 + \left(\frac{k}{\pi} \right)^2 } \right] + \ldots . \tag{3.3}$$

This formula shows that the entanglement entropy is not simply the sum of the topological contribution $-\frac{1}{2} \log k$ and the contribution of the gapless bulk modes, unlike in the models of [108]. For CFTs with interacting scalars relevant for condensed matter applications, we have to consider the $u \to \infty$ limit, and this yields a correction of order unity, with $F \to F - \zeta(3)/(8\pi^2)$ at this order [9]. All higher order corrections to (3.3) are expected to be suppressed by integer powers of $1/N_F$, whose coefficients do not contain any factors of $\log N_F$.

In section 3.4 we will examine similar $\mathcal{N} = 2$ supersymmetric CFTs on S^3 using the localization approach. We consider theories with chiral and non-chiral flavorings. The partition function Z on S^3 is given by a finite-dimensional integral, which has to be locally minimized with respect to the scaling dimensions of the matter fields [31]. For a theory with N charged superfields we develop $1/N$ expansions for the scaling dimensions and for the entanglement entropy. As for the non-supersymmetric case, the subleading term in F is of order $\log N$. The coefficient of this term computed via localization agrees with the direct perturbative calculation (3.3).

In the supersymmetric case it is possible to develop the $1/N$ expansions to a rather high order, and we compare them with precise numerical results. This comparison yields an unprecedented test of the validity and accuracy of the $1/N$ expansion. At least for supersymmetric CFTs, we find the $1/N$ expansion is accurate down to rather small values of N. We also note a recent numerical study [109], which found reasonable accuracy in the $1/N_b$ expansion for a non-supersymmetric CFT.

3.2 Mapping to S^3 and large N_F expansion

Let us start by examining the case of a $U(1)$ gauge field. After sending $e^2 \to \infty$, the combined Lagrangian $\mathcal{L}_m + \mathcal{L}_A$ obtained from (3.1) and (3.2) contains two relevant couplings s and u, and we should first understand to what values we need to tune them in order to describe an RG fixed point. Let's ignore for the moment the fermions and the gauge field and focus on the complex scalar fields. The path integral on a space with arbitrary metric is

$$Z = \int Dz_a \exp\left[-\int d^3r\,\sqrt{g}\left(|\partial_\mu z_a|^2 + s|z_a|^2 + \frac{u}{2}\left(|z_a|^2\right)^2\right)\right] . \qquad (3.4)$$

With the help of an extra field λ, this path integral can be equivalently written as

$$Z = \mathcal{C} \int Dz_a D\lambda \exp\left[- \int d^3r \sqrt{g} \left(|\partial_\mu z_a|^2 + s|z_a|^2 - i\lambda |z_a|^2 + \frac{1}{2u}\lambda^2 \right) \right] , \tag{3.5}$$

where the normalization factor $\mathcal{C}$ defined through $\mathcal{C} \int D\lambda \exp\left[- \int d^3r \sqrt{g} \left(\frac{1}{2u}\lambda^2 \right) \right] = 1$ was introduced so that the value of the path integral stays unchanged.

In flat three-dimensional space, we can tune $s = u = 0$ and describe a non-interacting CFT of N_b complex scalars. If instead we tune $s = 0$ and send $u \to \infty$, the path integrals (3.4) and (3.5) describe the interacting fixed point that we will primarily be interested in in this chapter. We can also send both s and u to infinity, in which case the path integrals above describe the empty field theory. Using conformal symmetry, we can map each of these fixed points to S^3 by simply mapping all the correlators in the theory. Indeed, since the metric on S^3 is equal to that on $\mathbb{R}^3$ up to a conformal transformation,

$$ds^2_{S^3} = \frac{4}{\left(1 + |\vec{r}|^2\right)^2} ds^2_{\mathbb{R}^3} , \tag{3.6}$$

the mapping of correlators to S^3 is achieved by replacing

$$\vec{r} - \vec{r}' \to \frac{2(\vec{r} - \vec{r}')}{\left(1 + |\vec{r}|^2\right)^{1/2} \left(1 + |\vec{r}'|^2\right)^{1/2}} \tag{3.7}$$

in all the flat-space expressions.[1] While the theory on S^3 defined this way certainly has the correlators of a CFT, it may be a priori not clear which action, and in particular which values of s and u, one should choose in order to reproduce these correlators.

In order to study the free theory on S^3 one should tune $s = 3/4$ and $u = 0$. This result holds to all orders in N_b and one can understand it as follows. The two-point connected

[1] This replacement certainly works for correlators of scalar operators. In the case of vector operators it is still true that one can use (3.7) provided that the S^3 correlators are expressed in a frame basis, as in the following section.

correlator of z_a on $\mathbb{R}^3$ is

$$\langle \bar{z}_a(r) z_b(r') \rangle_{\text{free}}^{\mathbb{R}^3} = \frac{\delta_{ab}}{4\pi \left| \vec{r} - \vec{r'} \right|} \,, \tag{3.8}$$

because it is the unique solution to the equation of motion following from (3.4) with a delta-function source, $-\nabla^2_{\mathbb{R}^3} \langle \bar{z}_a(r) z_b(r') \rangle_{\text{free}}^{\mathbb{R}^3} = \delta_{ab} \delta^{(3)}(\vec{r} - \vec{r'})$. Using the mapping (3.7) we infer that the corresponding two-point correlator on S^3 should be

$$\langle \bar{z}_a(r) z_b(r') \rangle_{\text{free}}^{S^3} = \frac{\delta_{ab} \left(1 + |\vec{r}|^2\right)^{1/2} \left(1 + |\vec{r'}|^2\right)^{1/2}}{8\pi \left| \vec{r} - \vec{r'} \right|} \,. \tag{3.9}$$

An explicit computation shows that $\left(-\nabla^2_{S^3} + 3/4\right) \langle z_a(r) z_b(r') \rangle_{\text{free}}^{S^3} = \delta_{ab} \delta^{(3)}(\vec{r} - \vec{r'}) / \sqrt{g(r)}$, which is indeed the equation of motion that would follow from (3.4) with $s = 3/4$ and $u = 0$. This result was of course to be expected because a mass squared given by $s = 3/4$ corresponds to a conformally coupled scalar.

A more subtle issue is how to map to S^3 the interacting fixed point, which in flat space had $s = 0$ and $u = \infty$. As explained for example in [57], the generating functional of connected correlation functions of the singlet operator $|z_a|^2$ in the theory with $u = \infty$ equals the Legendre transform of the corresponding generating functional in the theory with $u = 0$, to leading order in a large N_b expansion. This result holds on any manifold, and in particular on both $\mathbb{R}^3$ and S^3, and it assumes the other couplings in the theory are held fixed. If we set $s = 0$ on $\mathbb{R}^3$ and $s = 3/4$ on S^3, the Legendre transform assures us, for example, that to leading order in N_b the two-point correlators in the theory with $u = \infty$ are

$$\begin{aligned} \langle |z_a(r)|^2 |z_b(r')|^2 \rangle_{\text{critical}}^{\mathbb{R}^3} &= \frac{c N_b}{|\vec{r} - \vec{r'}|^4} \,, \\ \langle |z_a(r)|^2 |z_b(r')|^2 \rangle_{\text{critical}}^{S^3} &= \frac{c N_b \left(1 + |\vec{r}|^2\right)^2 \left(1 + |\vec{r'}|^2\right)^2}{16 \, |\vec{r} - \vec{r'}|^4} \,, \end{aligned} \tag{3.10}$$

with the same normalization constant c, which is consistent with the conformal mapping of correlators realized through eq. (3.7). While in the free theory z_a is a free field and the operator $|z_a|^2$ therefore has dimension 1, in the interacting theory $|z_a|^2$ is a dimension 2 operator. To study the interacting fixed point on S^3 we therefore should set $s = 3/4 + O(1/N_b)$ and take $u \to \infty$ in (3.5).

Reintroducing the fermionic and gauge fields, we can write down the action as

$$S = \int d^3r \sqrt{g} \left[\overline{\psi}_\alpha \gamma^\mu D_\mu \psi_\alpha + |D_\mu z_a|^2 + (s - i\lambda) |z_a|^2 + \frac{1}{2u}\lambda^2 \right] + \frac{ik}{4\pi} \int A \wedge dA \, . \tag{3.11}$$

This action is of course invariant under gauge transformations, and therefore a correct definition of the path integral requires gauge fixing:

$$Z = \frac{1}{\text{Vol}(G)} \int DA \, DX \, e^{-S[A,X]} \, , \tag{3.12}$$

where $\text{Vol}(G)$ is the volume of the group of gauge transformations, and X denotes generically all fields besides the gauge field. One justification for this normalization of the path integral is that for a pure Chern-Simons gauge theory on S^3 it yields the expected answer [66] $Z = 1/\sqrt{k}$, as will emerge from our computations below. Because the first cohomology of S^3 is trivial, we can write uniquely any gauge field configuration A as $A = B + d\phi$, where $d{*}B = 0$ and ϕ is defined only up to constant shifts. One should think of B as the gauge-fixed version of A and of $d\phi$ as the possible gauge transformations of A. Since the action $S[A, X]$ is gauge-invariant, it is independent of ϕ and only depends on B: $S[A, X] = S[B, X]$.

We claim that

$$DA = DB \, D(d\phi) = DB \, D'\phi \sqrt{\det{}'(-\nabla^2)} \, , \tag{3.13}$$

where $D'\phi$ means that we're not integrating over configurations with $\phi = \text{constant}$, and $\det'$ denotes the determinant with the zero modes removed from the spectrum. To understand this relation, first note that the space $\Omega^p(S^3)$ of p-forms on S^3 is a metric space with the distance function $\mathcal{D}(\omega, \omega+\delta\omega) = \left(\int |\delta\omega|^2\right)^{1/2}$. Then $\mathcal{D}(d\phi, d\phi + d\delta\phi) = \left(\int |d\delta\phi|^2\right)^{1/2} = \left(\int \delta\phi\,(-\nabla^2)\,\delta\phi\right)^{1/2}$ after integration by parts, and also $\mathcal{D}(\phi, \phi+\delta\phi) = \left(\int |\delta\phi|^2\right)^{1/2}$. In other words, for each component of ϕ in a basis of eigenfunctions of the Laplacian, the distance between $d\phi$ and $d\phi + d\delta\phi$ is larger than the distance between ϕ and $\phi+\delta\phi$ by a factor of the square root of the eigenvalue with respect to $-\nabla^2$. Eq. (3.13) follows as a straightforward change of variables.

The gauge transformations are maps from S^3 into the Lie algebra of the gauge group. The volume of the group of gauge transformations $\mathrm{Vol}(G)$ can be expressed as

$$\mathrm{Vol}(G) = \mathrm{Vol}(H) \int D'\phi\,, \tag{3.14}$$

where H is the group of constant gauge transformations, and $\int D'\phi$ is an integral over the non-constant gauge transformations with the measure given by the metric function $\mathcal{D}$ introduced in the previous paragraph. In the case of a compact $U(1)$ with $\mathrm{Vol}(U(1)) = 2\pi$, a constant gauge transformation $\phi = c$ has $c \in [0, 2\pi)$. Therefore

$$\mathrm{Vol}(H) = \int_0^{2\pi} dc \frac{\mathcal{D}(c, c+\delta c)}{\delta c} = \int_0^{2\pi} dc \sqrt{\int 1} = 2\pi \sqrt{\mathrm{Vol}(S^3)}\,. \tag{3.15}$$

Combining (3.12)–(3.15) we obtain

$$Z = \frac{\mathcal{C}\sqrt{\det{}'(-\nabla^2)}}{2\pi\sqrt{\mathrm{Vol}(S^3)}} \int D\psi_\alpha\, Dz_a\, DB\, D\lambda\, e^{-S[\psi_\alpha, z_a, B, \lambda]}\,. \tag{3.16}$$

In this chapter we will use the partition function in eq. (3.16) to compute $F = -\log|Z|$ in the limit where N_f, N_b, and k are taken to be large and of the same order.

To leading order in the number of flavors we can ignore the gauge field and the Lagrange multiplier field λ. Setting $s = 3/4$ as discussed above, we can write down the resulting path integral as

$$Z_0 = \int D\psi_\alpha \, Dz_a \, \exp\left[-\int d^3r \, \sqrt{g} \left(\overline{\psi}_\alpha \gamma^\mu \nabla_\mu \psi_\alpha + |\partial_\mu z_a|^2 + \frac{3}{4}|z_a|^2 \right) \right] . \tag{3.17}$$

In this approximation we have a theory of free N_f Dirac fermions and N_b complex scalars with the free energy [9]

$$F_0 = \frac{\log 2}{4} \, (N_f + N_b) + \frac{3\zeta(3)}{8\pi^2} (N_f - N_b) \, . \tag{3.18}$$

To find the corrections to F_0 we write (3.16) approximately as

$$Z \approx e^{-F_0} \frac{\mathcal{C}\sqrt{\det{}' \, (-\nabla^2)}}{2\pi\sqrt{\mathrm{Vol}(S^3)}} \int DB \, D\lambda \, e^{-S_{\mathrm{eff}}[\lambda] - S_{\mathrm{eff}}^{\mathrm{vec}}[B]} \, , \tag{3.19}$$

with

$$\begin{aligned} S_{\mathrm{eff}}[\lambda] &= \int d^3r \, \sqrt{g(r)} \frac{1}{2u} \lambda(r)^2 - \frac{1}{2} \int d^3r \, \sqrt{g(r)} \int d^3r' \, \sqrt{g(r')} \lambda(r) \lambda(r') \left\langle |z_a(r)|^2 |z_b(r')|^2 \right\rangle_{\mathrm{free}}^{S^3} \\ S_{\mathrm{eff}}^{\mathrm{vec}}[B] &= \frac{ik}{4\pi} \int B \wedge dB - \frac{1}{2} \int d^3r \, \sqrt{g(r)} \int d^3r' \, \sqrt{g(r')} B_\mu(r) B_\nu(r') \left\langle J^\mu(r) J^\nu(r') \right\rangle_{\mathrm{free}}^{S^3} \, , \end{aligned} \tag{3.20}$$

where

$$J^\mu(r) = \bar{\psi}_\alpha(r) \gamma^\mu \psi_\alpha(r) + i\bar{z}_a(r) \partial^\mu z_a(r) - i z_a(r) \partial^\mu \bar{z}_a(r) \, . \tag{3.21}$$

In writing the effective action (3.20) we used $\langle |z_a(r)|^2 \rangle_{\mathrm{free}}^{S^3} = 0$, which follows from the fact that the free theory (3.17) is a CFT.

Defining

$$\delta F_\lambda = -\log\left|\mathcal{C}\int D\lambda\, e^{-S_{\text{eff}}[\lambda]}\right| ,$$
$$\delta F_A = -\log\left|\frac{\sqrt{\det{}'(-\Delta)}}{2\pi\sqrt{\text{Vol}(S^3)}}\int DB\, e^{-S^{\text{vec}}_{\text{eff}}[B]}\right| , \tag{3.22}$$

we can then write $F = F_0 + \delta F_\lambda + \delta F_A + o(N^0)$. The quantity δF_λ was computed in [9]:

$$\delta F_\lambda = -\frac{\zeta(3)}{8\pi^2} . \tag{3.23}$$

We devote the next section of this chapter to calculating δF_A.

3.3 Gauge field contribution to the free energy

3.3.1 Performing the Gaussian integrals

Let's denote by $K^{\mu\nu}$ the integration kernel appearing in S^A_{eff}, namely

$$K^{\mu\nu}(r,r') = -\langle J^\mu(r)J^\nu(r')\rangle^{S^3}_{\text{free}} + \frac{ik}{2\pi}\frac{\delta^3(r-r')}{\sqrt{g(r)}}\frac{1}{\sqrt{g(r')}}\epsilon^{\mu\nu\rho}\partial'_\rho . \tag{3.24}$$

As discussed above, when one writes $A = d\phi + B$ the action should be independent of ϕ, so pure gauge configurations $A_\nu(r') = \partial'_\nu\phi(r')$ are exact zero modes of the kernel $K^{\mu\nu}(r,r')$. Since we should integrate over the gauge-fixed field B only, the Gaussian integration of the effective theory $S^{\text{vec}}_{\text{eff}}[B]$ yields $1/\sqrt{\det{}'(K^{\mu\nu}/(2\pi))}$. Again, the prime means that we remove the zero modes from the spectrum when we calculate the determinant.

Reinstating the radius R of the sphere measured in units of some fixed UV cutoff, the discussion in the previous two paragraphs can be summarized as

$$\delta F_A = \frac{1}{2}\,\text{tr}'\log\left[\frac{K^{\mu\nu}}{2\pi R}\right] - \frac{1}{2}\,\text{tr}'\log\left[-\frac{\nabla^2}{R^2}\right] + \log\left(2\pi\sqrt{R^3\,\text{Vol}(S^3)}\right) , \tag{3.25}$$

where all the operators are defined on an S^3 of unit radius. Out of the first two terms in this expression, the second one is the easier one to calculate (see also [64]). The spectrum of the Laplacian on a unit-radius S^3 consists of eigenvalues $n(n+2)$ with multiplicity $(n+1)^2$ for any $n \geq 0$. One first rearranges the terms in the sum as

$$\frac{1}{2}\,\mathrm{tr}'\log\left(-\frac{\nabla^2}{R^2}\right) = \frac{1}{2}\sum_{n=1}^{\infty}(n+1)^2\log\frac{n(n+2)}{R^2} = \sum_{n=1}^{\infty}(n^2+1)\log\frac{n}{R} - \frac{\log(2/R)}{2}\,. \tag{3.26}$$

Then, using zeta-function regularization one writes

$$\frac{1}{2}\,\mathrm{tr}'\log\left(-\frac{\nabla^2}{R^2}\right) = -\frac{\log(2/R)}{2} - \frac{d}{ds}\sum_{n=1}^{\infty}\frac{n^2+1}{(n/R)^s}\bigg|_{s=0} = \frac{\zeta(3)}{4\pi^2} + \frac{\log(\pi R^2)}{2}\,. \tag{3.27}$$

Combining this expression with eq. (3.25) and using $\mathrm{Vol}(S^3) = 2\pi^2$, we obtain

$$\delta F_A = \frac{1}{2}\,\mathrm{tr}'\log\left[\frac{K^{\mu\nu}}{2\pi R}\right] - \frac{\zeta(3)}{4\pi^2} + \frac{\log\left(8\pi^3 R\right)}{2}\,. \tag{3.28}$$

The only remaining task is the computation of the first term in this equation that we perform in the next subsection by explicit diagonalization of $K^{\mu\nu}$.

3.3.2 Diagonalizing the kernel $K^{\mu\nu}(r, r')$

Ultimately we would like to diagonalize the kernel $K^{\mu\nu}$ on S^3. However, as a warm up it is instructive to consider the same diagonalization problem in flat space first.

Warm-up: Diagonalization on $\mathbb{R}^3$

The first step is to calculate the two-point function of current $\langle J^\mu(r)J^\nu(0)\rangle^{\mathbb{R}^3}_{\text{free}}$, where we use the superscript $\mathbb{R}^3$ to emphasize that we are in flat space. If we normalize the two-point functions of z_a and ψ_α to be

$$\begin{aligned} \langle \bar{z}_a(r) z_b(0) \rangle^{\mathbb{R}^3}_{\text{free}} &= \int \frac{d^3p}{(2\pi)^3} \frac{\delta_{ab}}{|p|^2} e^{ip\cdot r} = \frac{\delta_{ab}}{4\pi\,|r|} \\ \langle \psi_\alpha(r) \bar{\psi}_\beta(0) \rangle^{\mathbb{R}^3}_{\text{free}} &= \int \frac{d^3p}{(2\pi)^3} \frac{\delta_{\alpha\beta}\gamma^\mu p_\mu}{|p|^2} e^{ip\cdot r} = \frac{i}{4\pi} \frac{\delta_{\alpha\beta}\gamma^\mu r_\mu}{|r|^3}\,, \end{aligned} \tag{3.29}$$

then the two-point function of the current may be straightforwardly calculated to be

$$\langle J^\mu(r) J^\nu(0) \rangle^{\mathbb{R}^3}_{\text{free}} = \frac{N_f + N_b}{8\pi^2} \frac{|r|^2\,\delta^{\mu\nu} - 2r^\mu r^\nu}{|r|^6}\,. \tag{3.30}$$

It is simple to check that eq. (3.30) is of the right form. This correlator is fixed up to an overall constant by the requirements that it should be homogeneous of degree -4 in r (J^μ is a dimension 2 operator) and that away from $r = 0$ it should satisfy the conservation equation $\partial_\mu \langle J^\mu(r) J^\nu(0) \rangle_{\mathbb{R}^3} = 0$. Using

$$\int d^3r\,\frac{e^{ip\cdot r}}{|r|^4} = -\pi^2\,|p|\,, \qquad \int d^3r\,\frac{e^{ip\cdot r}}{|r|^6} = \frac{\pi^2}{12}\,|p|^3\,, \tag{3.31}$$

and introducing the Fourier space representation of the kernel $K^{\mu\nu}$ via

$$K_{\mu\nu}(r, r') = \int \frac{d^3p}{(2\pi)^3} K_{\mu\nu}(p) e^{ip\cdot(r-r')}\,, \tag{3.32}$$

one obtains [96]

$$K_{\mu\nu}(p) = \frac{N_f + N_b}{16}\,|p| \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{|p|^2} \right) + \frac{k}{2\pi} \epsilon_{\mu\nu}{}^\rho p_\rho\,. \tag{3.33}$$

For fixed p, the eigenvalues of this matrix are

$$0\,, \qquad \frac{|p|}{2} \left(\frac{N_f + N_b}{8} \pm i\frac{k}{\pi} \right)\,. \tag{3.34}$$

The eigenvector associated with the zero eigenvalue is as expected $ip_\nu e^{ip\cdot r'}$, corresponding to a gauge configuration $A_\nu = \partial_\nu \phi$. We will now see that on S^3, while the diagonalization of $K^{\mu\nu}$ is significantly more complicated, the answer is equally simple: the magnitude of the momentum p appearing in (3.34) should be replaced by a positive integer label n.

Diagonalization on S^3

When we work with vector fields on S^3 it is convenient to introduce the dreibein

$$e^i(r) = \frac{2}{1+|r|^2} dr^i \tag{3.35}$$

and work only with frame indices. For example,

$$\langle J^i(r) J^j(r')\rangle^{S^3}_{\text{free}} = e^i_\mu(r) e^j_\nu(r') \langle J^\mu(r) J^\nu(r')\rangle^{S^3}_{\text{free}} . \tag{3.36}$$

The frame indices i and j are raised and lowered with the flat metric, so there is no distinction between lower and upper frame indices in Euclidean signature.

Using the transformation of correlators under Weyl rescalings in eq. (3.7), one can immediately write down the current two-point function on S^3:

$$\langle J^i(r) J^j(0)\rangle^{S^3}_{\text{free}} = \frac{N_f + N_b}{8\pi^2} \frac{\left(1+|r|^2\right)^2}{2} \frac{|r|^2 \delta^{ij} - 2r^i r^j}{|r|^6} . \tag{3.37}$$

As in flat space, the form of this correlator is fixed by the requirement that away from $r = 0$ we must have $\nabla_i \langle J^i(r) J^j(0)\rangle = 0$.

To understand the diagonalization of K^{ij} on S^3 we need to know that the space of square-integrable one-forms on S^3, being a vector space acted on by the $SO(4) \cong SU(2)_L \times SU(2)_R$ isometry group, decomposes into irreducible representations of $SO(4)$ as follows. Any one-form ω can be written as a sum of a closed one-form and a co-closed one-form. The closed one-forms on S^3 are of course cohomologous to zero, so they're also exact. A basis for them therefore consists of the gradients of the usual spherical harmonics. Like the spherical harmonics, they transform in irreps with $j_L = j_R$. On the other hand, the co-closed one-forms transform in irreps with $j_L = j_R \pm 1$. So an arbitrary one-form can be written as

$$\omega_i(r) = \sum_{n,\ell,m} \left[\omega^S_{n\ell m}\mathbb{S}^{n\ell m}_i(r) + \omega^L_{n\ell m}\mathbb{V}^{n\ell m}_{L,i}(r) + \omega^R_{n\ell m}\mathbb{V}^{n\ell m}_{R,i}(r)\right] , \tag{3.38}$$

where we denoted by $\mathbb{S}^{n\ell m}_i$ the closed component transforming in the irrep with $j_L = j_R = (n-1)/2$ and by $\mathbb{V}^{n\ell m}_{L,i}$ and $\mathbb{V}^{n\ell m}_{R,i}$ the co-closed components transforming in the irreps with $j_R = j_L + 1 = n/2$ and $j_L = j_R + 1 = n/2$, respectively. All the harmonics appearing in (3.38) have $n \geq 2$. For $\mathbb{S}^{n\ell m}_i$ there are n^2 states in each irrep indexed by the integers ℓ and m satisfying $0 \leq \ell < n$ and $-\ell \leq m \leq \ell$. For the other two classes of vector harmonics, we have the same bounds on m but now $0 < \ell < n$, giving a total dimension of $n^2 - 1$ for each irrep.

The $SO(4)$ generators commute with the kernel K_{ij}, so the eigenvectors of this kernel can be taken to be $\mathbb{S}^{n\ell m}_i$, $\mathbb{V}^{n\ell m}_{L,i}$, and $\mathbb{V}^{n\ell m}_{R,i}$. The spectral decomposition of K_{ij} is therefore

$$K_{ij}(r,r') = \sum_{n,\ell,m} \left[s_n\mathbb{S}^{n\ell m}_i(r)\mathbb{S}^{n\ell m}_j(r')^* + v^L_n\mathbb{V}^{n\ell m}_{L,i}(r)\mathbb{V}^{n\ell m}_{L,j}(r')^* + v^R_n\mathbb{V}^{n\ell m}_{R,i}(r)\mathbb{V}^{n\ell m}_{R,j}(r')^*\right] , \tag{3.39}$$

where s_n, v^L_n, and v^R_n are the corresponding eigenvalues. These eigenvalues are independent of ℓ and m because for fixed n one can change ℓ and m by acting with the $SO(4)$ generators, which commute with K_{ij}. The degeneracy of s_n is n^2 and that of v^L_n and v^R_n is $n^2 - 1$, with $n \geq 2$.

Given K_{ij} one can find its eigenvalues by taking inner products with the eigenvectors. Using rotational invariance, one can actually set $r' = 0$ after summing over ℓ and m. For example,

$$s_n = \frac{\mathrm{Vol}(S^3)}{n^2}\sum_{\ell=0}^{n-1}\sum_{m=-\ell}^{\ell}\int_{S^3} d^3r\, \mathbb{S}^{n\ell m}_i(r)^* K_{ij}(r,0)\mathbb{S}^{n\ell m}_j(0) , \tag{3.40}$$

where the n^2 in the denominator is the dimension of the $SO(4)$ irrep to which $\mathbb{S}_i^{n\ell m}$ belong. We notice that only the harmonics with $\ell = 1$ contribute, so

$$s_n = \frac{\mathrm{Vol}(S^3)}{n^2} \sum_{m=-1}^{1} \int_{S^3} d^3r \, \mathbb{S}_i^{n1m}(r)^* K_{ij}(r,0) \mathbb{S}_j^{n1m}(0) \,, \tag{3.41}$$

with similar expressions for v_n^L and v_n^R, the only difference being that n^2 should be replaced by $n^2 - 1$.

Using explicit formulae for the harmonics (see Appendix A of [13]), one obtains

$$s_n = \frac{N_f + N_b}{64\pi n(n^2-1)} \int_0^\pi d\chi \, \csc^6 \frac{\chi}{2} \sin\chi \left[-2n\cos(n\chi)\sin\chi + \big(1 - n^2 + \cos\chi(n^2+1)\big)\sin(n\chi) \right] \tag{3.42}$$

and

$$v_n^{L,R} = \frac{N_f + N_b}{64\pi n(n^2-1)} \int_0^\pi d\chi \, \csc^6 \frac{\chi}{2} \sin\chi \left[n\sin\chi\cos(n\chi) + \big(n^2 - n^2\cos\chi - 1\big)\sin n\chi \right] \pm \frac{ikn}{2\pi} \,. \tag{3.43}$$

The integration variable χ appearing here is related to r through $|r| = \cot(\chi/2)$.

We expect $s_n = 0$ because of gauge invariance. Both (3.43) and (3.42) are divergent at $\chi = 0$, and need to be regulated. A way of regulating them that doesn't preserve gauge invariance is to replace $\csc^6 \frac{\chi}{2}$ by $\csc^\alpha \frac{\chi}{2}$, compute these integrals for values of α for which they are convergent, and then formally set $\alpha = 6$. Another way would be to assume $s_2 = 0$, and calculate $s_n - s_2$ and $v_n - s_2$, which are now convergent integrals. Both of these ways of regulating (3.42) and (3.43) give

$$s_n = 0 \,, \qquad v_n^L = \frac{n(N_f + N_b)}{16} + \frac{ikn}{2\pi} \,, \qquad v_n^R = \frac{n(N_f + N_b)}{16} - \frac{ikn}{2\pi} \,. \tag{3.44}$$

Note the similarity between these expressions and the corresponding flat-space ones in eq. (3.34).

3.3.3 Contribution to the free energy

We can now evaluate the first term in (3.28):

$$\begin{aligned}\frac{1}{2}\,\mathrm{tr}'\log\left(\frac{K^{\mu\nu}}{2\pi R}\right) &= \sum_{n=2}^{\infty}(n^2-1)\log\left|\frac{v_n^L}{2\pi R}\right| \\ &= \frac{1}{2}\log\left[\frac{1}{8\pi^2}\sqrt{\left(\frac{N_f+N_b}{8}\right)^2+\left(\frac{k}{\pi}\right)^2}\,\right]+\frac{\zeta(3)}{4\pi^2}-\frac{\log R}{2}\,,\end{aligned} \tag{3.45}$$

where the second line was obtained with the help of zeta-function regularization. Combining this expression with (3.28) yields

$$\delta F_A = \frac{1}{2}\log\left[\pi\sqrt{\left(\frac{N_f+N_b}{8}\right)^2+\left(\frac{k}{\pi}\right)^2}\,\right]\,. \tag{3.46}$$

Note that all of the $\log R$ terms cancel in the final answer, as they should since we are describing a conformal fixed point, for which the path integral should be independent of R. Another check of this result is that when $N_f = N_b = 0$ we recover the standard result for the free energy of $U(1)$ CS theory on S^3 [66], $\delta F_A = \frac{1}{2}\log k$.

As an aside, we note that if we included the Maxwell term in (3.2), eq. (3.45) would be modified to

$$\frac{1}{2}\,\mathrm{tr}'\log\left(\frac{K^{\mu\nu}}{2\pi R}\right) = \sum_{n=2}^{\infty}(n^2-1)\log\left|\frac{1}{2\pi R}\left(v_n^L+\frac{n^2}{e^2 R}\right)\right|\,, \tag{3.47}$$

with v_n^L still defined as in (3.44). Of course, e^2 flows to infinity in the IR, so as long as we have a non-zero CS level or non-zero numbers of flavors one can safely ignore the contribution from the Maxwell term in (3.47). If however one studies pure Maxwell theory with $k = N_f = N_b = 0$ so that $v_n^L = 0$ in (3.47), the S^3 free energy becomes[2]

$$F_{\text{Maxwell}} = -\frac{\log(e^2 R)}{2} + \frac{\zeta(3)}{4\pi^2}\,. \tag{3.48}$$

The logarithmic dependence on R is consistent with the fact that the free Maxwell theory is not conformal in three spacetime dimensions. F_{Maxwell} decreases monotonically from the UV (small R) to the IR (large R).

Of course, since the Maxwell theory is not a conformal theory in three spacetime dimensions, it is not immediately obvious whether the entanglement entropy should have the same dependence on $\log(e^2 R)$ at the three-sphere free energy. However, recently it has been shown that the renormalized entanglement entropy across the circle of radius R of the Maxwell theory with compact $U(1)$ gauge group behaves like $-\log(e^2 R)/2$ in the UV ($e^2 R \ll 1$) and approaches the F-value of a real conformal scalar field in the IR ($e^2 R \gg 1$) [110]. This work used the duality between the gauge theory and the free compact scalar field. In the UV, the theory is well-described by the non-compact Maxwell theory, while in the IR the theory approaches that of the non-compact real scalar field. The entanglement entropy calculation was performed directly using the replica trick.

3.3.4 Generalization to $U(N_c)$ theory

Eq. (3.28) generalizes straightforwardly to the case of $U(N_c)$ gauge theory with N_f Dirac fermions and N_b complex bosons transforming in the fundamental representation of the gauge group. At large k the contribution of the $\operatorname{Tr} A^3$ term in the Chern-Simons Lagrangian (3.2) to the S^3 partition function is suppressed by $1/\sqrt{k}$, and the quadratic term proportional to $\operatorname{Tr} A \wedge dA$ is the same as that of N_c^2 $U(1)$ gauge fields with Chern-Simons coupling k. There are N_f Dirac fermions and N_b complex bosons charged under each of these $U(1)$ gauge fields.

[2] We thank D. Jafferis and Z. Komargodski for very useful discussions of the free Maxwell field on S^3.

One then just has to multiply the $U(1)$ answer (3.46) by a factor of N_c^2:

$$\delta F_A = \frac{N_c^2}{2} \log \left[\pi \sqrt{\left(\frac{N_f + N_b}{8}\right)^2 + \left(\frac{k}{\pi}\right)^2} \right] + \log \frac{\mathrm{Vol}(U(N_c))}{\mathrm{Vol}(U(1))^{N_c^2}} \,. \tag{3.49}$$

The second term in this expression comes from the different gauge fixing of the $U(N_c)$ gauge theory compared to a theory of N_c^2 $U(1)$ gauge fields. As explained in section 3.2, the gauge fixing procedure involves dividing the partition function by the volume of the gauge group, so the partition function for the $U(N_c)$ theory has a prefactor of $1/\operatorname{Vol}(U(N_c))$ while the $U(1)^{N_c^2}$ theory obtained by multiplying (3.46) by N_c^2 would have a prefactor of $1/\operatorname{Vol}(U(1))^{N_c^2}$. We have (see for example [64])

$$\frac{\mathrm{Vol}(U(N_c))}{\mathrm{Vol}(U(1))^{N_c^2}} = \frac{(2\pi)^{-N_c(N_c-1)/2}}{1! \cdot 2! \cdots (N_c - 1)!} \,. \tag{3.50}$$

Thus, for $U(N_c)$ gauge theory with N_f fundamental fermions and N_b fundamental bosons we have

$$\begin{aligned} F = \frac{N_c \log 2}{4} (N_f + N_b) + \frac{3\zeta(3)}{8\pi^2} N_c (N_f - N_b) + \frac{N_c^2}{2} \log \left[\pi \sqrt{\left(\frac{N_f + N_b}{8}\right)^2 + \left(\frac{k}{\pi}\right)^2} \right] \\ - \frac{1}{2} N_c (N_c - 1) \log(2\pi) - \log\left(1! \cdot 2! \cdots (N_c - 1)!\right) + \dots \,, \end{aligned} \tag{3.51}$$

with corrections expected to vanish in the limit of large N_F. In writing (3.51) we kept N_c of order one while scaling N_f, N_b, and k to infinity with their ratios fixed. Generalizing (3.51) to different gauge groups proceeds in a similar way.

3.4 SUSY gauge theory with flavors

In this section we compute the free energy of $U(1)$ Chern-Simons matter theories with $\mathcal{N} \geq 2$ supersymmetry coupled to a large number of flavors. These computations allow us to check the first sub-leading correction to the non-SUSY result in the equation (3.46) in a different way. The computations in this section have as starting point the results of refs. [31, 101], which used the technique of supersymmetric localization to rewrite the S^3 partition function of theories with $\mathcal{N} \geq 2$ SUSY as finite-dimensional integrals. Our computations also involve finding the scaling dimensions of the gauge-invariant operators.

3.4.1 $\mathcal{N} = 4$ theory

As a warmup to the $\mathcal{N} = 2$ calculations, consider the $\mathcal{N} = 4$ parity-preserving supersymmetric $U(1)$ theory consisting of N $\mathcal{N} = 4$ hypermultiplets coupled to an $\mathcal{N} = 4$ vector multiplet. In $\mathcal{N} = 2$ notation, the $\mathcal{N} = 4$ vector multiplet consists of an $\mathcal{N} = 2$ vector and a neutral chiral superfield Φ of dimension 1. The $\mathcal{N} = 4$ supersymmetry does not allow a Chern-Simons term. The hypermultiplets can be rewritten as N pairs of oppositely charged chiral-multiplets Q_a of $U(1)$ charge $+1$ and $\tilde{Q}_a$ with $U(1)$ charge -1. The $\mathcal{N} = 4$ SUSY requires a superpotential interaction $W \sim \tilde{Q}_a \Phi Q_a$. The superpotential has R-charge equal to 2. Then the $SU(2)$ subgroup of the $SO(4)_R$ R-symmetry, under which $\tilde{Q}_a$ and $\bar{Q}_a$ transform as a doublet, fixes the R-charge of the matter chiral multiplets to have the canonical free-field value: $\Delta_Q = \Delta_{\tilde{Q}} = 1/2$. The partition function is then given by [101]

$$Z = \frac{1}{2^N} \int_{-\infty}^{\infty} \frac{d\lambda}{\cosh^N(\pi\lambda)} = \frac{2^{-N}\Gamma\left(\frac{N}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{N+1}{2}\right)} \,. \tag{3.52}$$

Expanding this at large N we find

$$F = -\log Z = N \log 2 + \frac{1}{2} \log\left(\frac{N\pi}{2}\right) - \frac{1}{4N} + \frac{1}{24N^3} + \dots \,. \tag{3.53}$$

This large N expansion is asymptotic, but it provides a very good approximation of the exact answer (3.52) down to $N = 1$—see Figure 3.1. Including more terms in (3.53) makes the approximation worse at $N = 1$.

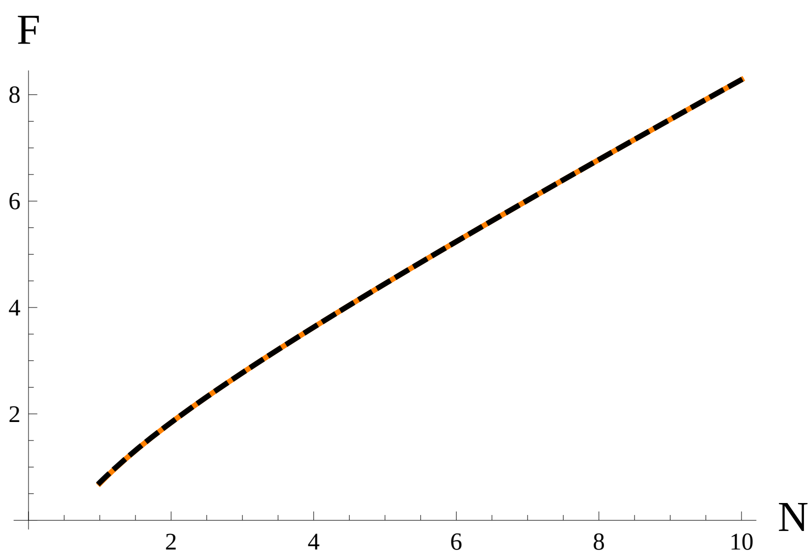


Figure 3.1: The exact free energy of the $\mathcal{N} = 4$ theory obtained from eq. (3.52) (solid orange) and the analytical approximation (3.53) (dashed black).

With N pairs of hyper-multiplets we have a total of $2N$ physical complex bosons and $2N$ Dirac fermions. We then see perfect agreement of the first two terms in (3.53) with eqs. (3.18) and (3.46).

3.4.2 $\mathcal{N} = 3$ theory

Let us add the Chern-Simons term for the $\mathcal{N} = 2$ abelian vector multiplet; it breaks $\mathcal{N} = 4$ down to $\mathcal{N} = 3$ supersymmetry. The field content is the same as that of an $\mathcal{N} = 4$ vector multiplet and N hypermultiplets, namely an $\mathcal{N} = 2$ vector, a neutral chiral Φ, and N pairs of chiral multiplets Q_a and $\tilde{Q}_a$ charged under the $\mathcal{N} = 2$ vector. The superpotential required by $\mathcal{N} = 3$ SUSY is

$$W = -\frac{k}{4\pi}\Phi^2 + \tilde{Q}_a \Phi Q_a \,. \tag{3.54}$$

After integrating out the massive field Φ, the superpotential can be rewritten as [111]

$$W = \frac{2\pi}{k}(\tilde{Q}_a Q_a)^2 \,. \tag{3.55}$$

The conformal dimensions of Q_a and $\tilde{Q}_a$ are still equal to 1/2, as is required by the marginality of W and by the $\mathbb{Z}_2$ symmetry under which Q_a and $\tilde{Q}_a$ are interchanged and all the fields in the vector multiplet change sign. The partition function is [101]

$$Z = \frac{1}{2^N}\int d\lambda \frac{e^{i\pi k\lambda^2}}{\cosh^N(\pi\lambda)} \,. \tag{3.56}$$

While this expression cannot be evaluated analytically, one can evaluate it using a saddle point approximation in the limit where both k and N are taken to be large. Let us define $\kappa = 2k/(N\pi)$ and take N to infinity while keeping κ fixed. The saddle point is at $\lambda = 0$, and in order to obtain a systematic expansion, one should write

$$Z = \frac{1}{2^N}\int d\lambda\, e^{-N\pi^2\lambda^2(1-i\kappa)/2}\left[1 + \frac{N\pi^4\lambda^4}{12} - \frac{N\pi^6\lambda^6}{45} + \frac{N(68+35N)\pi^8\lambda^8}{10080} + \dots\right] , \tag{3.57}$$

where the parenthesis contains the small λ expansion of the function $e^{N\pi^2\lambda^2/2}\cosh^{-N}(\pi\lambda)$. Order by order in this expansion one can perform the integrals in (3.57) analytically. The result is

$$Z = \frac{1}{2^N}\sqrt{\frac{2}{N\pi(1-i\kappa)}}\left[1 + \frac{1}{4N(1-i\kappa)^2} - \frac{1}{3N^2(1-i\kappa)^3} + \frac{68+35N}{96N^3(1-i\kappa)^4} + \dots\right] . \tag{3.58}$$

Calculating $F = -\log|Z|$ and expanding in N, we obtain

$$F = N\log 2 + \frac{1}{2}\log\left(\frac{N\pi}{2}\sqrt{1+\kappa^2}\right) + \frac{\kappa^2-1}{4(\kappa^2+1)^2 N} - \frac{4\kappa^2(\kappa^2-1)}{3(1+\kappa^2)^4 N^2} + O(N^{-3}) \,. \tag{3.59}$$

We note that in order to calculate the $O(N^{-\alpha})$ term in F we need the expansion in (3.57) to be up to order $O(\lambda^{4\alpha})$. The expression (3.59) is also in agreement with eqs. (3.18) and (3.46) given that the $\mathcal{N}=3$ theory has $N_b = N_f = 2N$.

Let us extend our discussion to the non-abelian theory with gauge group $U(N_c)$. The field content now consists of an $\mathcal{N}=4$ vector multiplet in the adjoint representation of the gauge group and N pairs of chiral multiplets Q_a and $\tilde{Q}_a$, in the fundamental and anti-fundamental representations of the gauge group, respectively. After localization the partition function for this theory is given by [101]

$$Z = \frac{2^{N_c(N_c-1)}}{2^{NN_c}N_c!} \int \left(\prod_{i=1}^{N_c} d\lambda_i \right) \left(\prod_{i<j}^{N_c} \sinh^2[\pi(\lambda_i - \lambda_j)] \right) \exp\left(i\pi k \sum_{i=1}^{N_c} \lambda_i^2 \right) \prod_{i=1}^{N_c} \cosh^{-N}(\pi\lambda_i)\,. \tag{3.60}$$

In the limit where $N_c/N \ll 1$, the integral has a saddle point at $\lambda_i = O\left[(N_c/N)^{1/2}\right]$. Through next to leading order the partition function of the non-abelian theory reduces to

$$Z = \frac{(2\pi)^{N_c(N_c-1)}}{2^{NN_c}N_c!N^{\frac{N_c^2}{2}}} \left(\int \prod_{i=1}^{N_c} d\tilde{\lambda}_i \right) \left(\prod_{i<j}^{N_c} (\tilde{\lambda}_i - \tilde{\lambda}_j)^2 \right) \exp\left(-\frac{\pi^2\,(1-i\kappa)}{2} \sum_{i=1}^{N_c} \tilde{\lambda}_i^2 \right) + \dots\,, \tag{3.61}$$

where κ is defined as in the abelian theory and $\tilde{\lambda}_i = \sqrt{N}\lambda_i$. We have rescaled the integration variables so that the remaining integrals in eq. (3.61) produce numbers independent of N. Taking the log of eq. (3.61) we then see immediately that

$$F = N_c N \log 2 + \frac{N_c^2}{2} \log(N) + O(N^0)\,. \tag{3.62}$$

Given that the non-abelian theory has $N_b = N_f = 2N$, the equation above is in agreement with eq. (3.51).

3.4.3 Non-chiral $\mathcal{N} = 2$ theory

Moving up one notch in complexity, we now consider the $\mathcal{N} = 2$ Chern-Simons theory coupled to the chiral fields Q_a and $\tilde{Q}_a$ introduced above, this time without the superpotential (3.55). The absence of the superpotential leaves the R-charges of Q_a and $\tilde{Q}_a$ a priori unrestricted. It was proposed in [31] that one way of finding the correct IR R-charges in an $\mathcal{N} = 2$ theory is by calculating the partition function on S^3 for any choice of trial R-charges consistent with the marginality of the superpotential and then extremizing over all such R-charge assignments. The R-charges of Q_a and $\tilde{Q}_a$ can be taken to be equal to some common value Δ because of the following symmetries: the action is invariant under two $U(N)$ symmetries under which the Q_a and $\tilde{Q}_a$ transform as fundamental vectors, as well as under a charge conjugation symmetry that flips the sign of all the fields in the vector multiplet and at the same time interchanges Q_a and $\tilde{Q}_a$.

As a function of Δ, the partition function is [31]

$$Z = \int_{-\infty}^{\infty} d\lambda \, e^{i\pi k\lambda^2} e^{N\left(\ell(1-\Delta+i\lambda)+\ell(1-\Delta-i\lambda)\right)} \,, \tag{3.63}$$

where the function $\ell(z)$ is given by

$$\ell(z) = -z\log\left(1-e^{2\pi iz}\right) + \frac{i}{2}\left(\pi z^2 + \frac{1}{\pi}\mathrm{Li}_2\left(e^{2\pi iz}\right)\right) - \frac{i\pi}{12} \,. \tag{3.64}$$

This function can be found by solving the differential equation $\partial_z \ell(z) = -\pi z\cot(\pi z)$ with the boundary condition $\ell(0) = 0$. It is a real function when z is real.

We again take N to infinity while keeping $\kappa = 2k/(N\pi)$ fixed. In this limit one can use the saddle point approximation to calculate the partition function (3.63) as in the previous section. The exponent in (3.63) is an even function of λ, so there is a saddle point at $\lambda = 0$, and we will assume this is the only relevant saddle. To leading order in N we therefore have

$$F(\Delta) = -2N\ell(1-\Delta) + O(\log N)\,. \tag{3.65}$$

This function is maximized when $dF/d\Delta = 2\pi(\Delta - 1)\cot(\pi\Delta) = 0$, which implies $\Delta = 1/2 + O(N^{-1})$. We will find that this anomalous dimension affects F only at order $1/N$, i.e. the first two leading orders in the large N expansion of F are the same for the $\mathcal{N} = 2$ theory and the $\mathcal{N} = 3$ theory studied in the previous section.

One can develop a systematic expansion to study $1/N$ corrections in a similar way to what was done at the end of the previous section for the $\mathcal{N} = 3$ theory. The fact that now Δ depends on N introduces an extra complication. We expand Δ as

$$\Delta = \frac{1}{2} + \frac{\Delta_1}{N} + \frac{\Delta_2}{N^2} + \dots\,, \tag{3.66}$$

and we rescale $\lambda = \tilde\lambda/\sqrt{N}$. One can then write

$$Z = \frac{1}{2^N\sqrt{N}} \int_{-\infty}^{\infty} d\tilde\lambda\, e^{-\pi^2\tilde\lambda^2(1-i\kappa)/2} \left[1 + \frac{6\pi^2\Delta_1^2 + 24\Delta_1\tilde\lambda^2 + \tilde\lambda^4}{12N} + \dots\right], \tag{3.67}$$

where the expansion in parenthesis is in powers of $1/N$ while holding $\tilde\lambda$ fixed. Term by term in this expansion, these integrals can be evaluated analytically. The free energy is

$$F(\Delta) = N\log 2 + \frac{1}{2}\log\left(\frac{N\pi}{2}\sqrt{1+\kappa^2}\right) - \left(\frac{\pi^2\Delta_1^2}{2} + \frac{2\Delta_1}{1+\kappa^2} + \frac{1-\kappa^2}{4(1+\kappa^2)^2}\right)\frac{1}{N} + \dots\,. \tag{3.68}$$

Maximizing this expression with respect to Δ_1 we obtain

$$\Delta_1 = -\frac{2}{\pi^2(1+\kappa^2)}\,. \tag{3.69}$$

For $k \gg N \gg 1$ this result agrees with section 6.3 of [31]. Repeating this procedure two more orders in F we find

$$\Delta = \frac{1}{2} - \frac{2}{\pi^2(1+\kappa^2)}\frac{1}{N} - \frac{2\left[\pi^2 - 12 + \kappa^2(4-2\pi^2) + \pi^2\kappa^4\right]}{\pi^4(1+\kappa^2)^3}\frac{1}{N^2} + O(N^{-3})\,. \tag{3.70}$$

This series appears to be perfectly convergent. In fig. 3.2 we plot $\Delta(N)$ for a few values of κ using both the precise numerical result and the approximation (3.70).

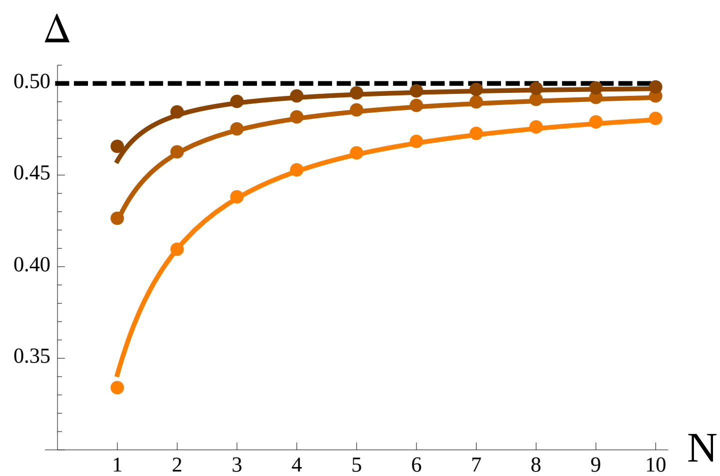


Figure 3.2: The R-charge Δ plotted as a function of N for $\kappa = 0, 4/\pi, 8/\pi$, with darker plots corresponding to larger κ. The solid lines are calculated using the approximation in eq. (3.70). The circles are computed by numerically maximizing the free energy with respect to Δ. Note that the two computations match well even for small N.

Using eqs. (3.68) and (3.69) we find that the free energy is

$$F = N\log 2 + \frac{1}{2}\log\left(\frac{N\pi}{2}\sqrt{1+\kappa^2}\right) + \left(\frac{\kappa^2-1}{4(1+\kappa^2)^2} + \frac{2}{\pi^2(1+\kappa^2)^2}\right)\frac{1}{N} + O(N^{-2})\,. \tag{3.71}$$

Using $N_b = N_f = 2N$, we see that this expression agrees with eqs. (3.18) and (3.46) that were derived directly from a large N expansion without the use of supersymmetric localization.

Let us perturb the $\mathcal{N} = 2$ theory discussed above by the quartic superpotential

$$W = g(Q_a\tilde{Q}_a)^2\,. \tag{3.72}$$

Since, as can be seen from (3.69), the dimension of Q_a and $\tilde{Q}_a$ is slightly smaller than 1/2, the perturbation (3.72) is a slightly relevant perturbation of the UV $\mathcal{N} = 2$ theory. This theory should flow to an IR fixed point where the superpotential is exactly marginal, i.e. the IR R-charges of Q_a and $\tilde{Q}_a$ are 1/2. The calculation of $F_{\rm IR}$ is thus exactly the same as for the $\mathcal{N} = 3$ superconformal $U(1)$ theory discussed in section 4.2. The infrared theory is conformal for any g, and for the special value $g = 2\pi/k$ it is the $\mathcal{N} = 3$ theory in eq. (3.55). Eqs. (3.59) and (3.71) imply that the change in free energy between the UV and IR fixed points is

$$F_{\rm UV} - F_{\rm IR} = \frac{2}{\pi^2(1+\kappa^2)^2 N} + O(N^{-2})\,, \tag{3.73}$$

which can be explicitly seen to be positive, in agreement with the conjectured F-theorem [8].

Since the superpotential deformation (3.72) is only slightly relevant, one may wonder how the result (3.73) compares with the perturbative computation performed in [9]. In [9] it was shown that if the Lagrangian is perturbed by a slightly relevant scalar operator of dimension $3 - \epsilon$, then there is a perturbative IR fixed point and $F_{\rm UV} - F_{\rm IR} \propto \epsilon^3$. If however the Lagrangian is perturbed by a pseudoscalar operator of dimension $3 - \epsilon$, then there is no perturbative fixed point; it was seen in an example that if a fixed point exists then one might expect $F_{\rm UV} - F_{\rm IR} \propto \epsilon$. In our case, the superpotential deformation (3.72) translates into perturbations of the Lagrangian by both a scalar operator $\mathcal{O}_1$ and a pseudoscalar operator $\mathcal{O}_2$. Indeed, denoting

$$Q_a = \phi_a + \sqrt{2}\theta\psi_a + \theta^2 F_a\,, \qquad \tilde{Q}_a = \tilde{\phi}_a + \sqrt{2}\theta\tilde{\psi}_a + \theta^2\tilde{F}_a\,, \tag{3.74}$$

we have

$$\delta\mathcal{L} = g^2\mathcal{O}_1 + g\mathcal{O}_2\,,$$
$$\mathcal{O}_1 = -8\left|\phi_a\tilde{\phi}_a\phi_b\right|^2 - 8\left|\phi_a\tilde{\phi}_a\tilde{\phi}_b\right|^2\,, \tag{3.75}$$
$$\mathcal{O}_2 = -2\psi_a\tilde{\psi}_a\phi_b\tilde{\phi}_b - \psi_a\psi_b\tilde{\phi}_a\tilde{\phi}_b - \tilde{\psi}_a\tilde{\psi}_b\phi_a\phi_b - 2\psi_a\tilde{\psi}_b\tilde{\phi}_a\phi_b + \text{c.c}\,.$$

The scaling dimensions of these operators are

$$\Delta(\mathcal{O}_1) = 3 + 6\frac{\Delta_1}{N} + O(N^{-2})\,, \qquad \Delta(\mathcal{O}_2) = 3 + 4\frac{\Delta_1}{N} + O(N^{-2})\,, \tag{3.76}$$

so the pseudoscalar operator $\mathcal{O}_2$ is the more relevant one. One might expect the IR fixed point should be non-perturbative and that $F_{\text{UV}} - F_{\text{IR}} \propto -\Delta_1/N$ times a function of order one. That the IR fixed point is non-perturbative can be seen after writing $g = \hat{g}/N$ so that $\hat{g}$ stays of order 1 as we take N to infinity. The IR coupling $g_{\text{IR}} = 2\pi/k$ corresponds to $\hat{g}_{\text{IR}} = 4/\kappa$, which is of order one in the large N limit, meaning that the IR fixed point is non-perturbative. That $F_{\text{UV}} - F_{\text{IR}} \propto -\Delta_1/N$ times a function of order one can be immediately seen from eqs. (3.73) and (3.69).

3.4.4 Chiral $\mathcal{N} = 2$ theory

We now consider a natural generalization of the non-chiral $\mathcal{N} = 2$ theory discussed in the previous section—the chiral $\mathcal{N} = 2$ theory. This theory is given by $\mathcal{N} = 2$ Chern-Simons theory coupled to N chiral fields Q_a and $\tilde{N}$ anti-chiral fields $\tilde{Q}_a$ with no superpotential. When $N = \tilde{N}$ this theory reduces to the non-chiral theory discussed in the previous section. Without loss of generality, we now assume that $N > \tilde{N}$. Instead of dealing with N and $\tilde{N}$ it is convenient to define the following quantities,

$$\bar{N} \equiv \frac{N + \tilde{N}}{2}\,, \qquad \mu \equiv \frac{N - \tilde{N}}{N + \tilde{N}}\,, \qquad 0 < \mu \leq 1\,. \tag{3.77}$$

The R-charges of Q_a and $\tilde{Q}_a$, which we denote by Δ and $\tilde{\Delta}$, are not gauge-invariant observables. Gauge invariant operators may be constructed from combinations of Q_a, $\tilde{Q}_a$, and the monopole operators T_m, which create m units of magnetic flux through 2-spheres surrounding their insertion points. The R-charge of T_m is given by [112, 113]

$$R[T_m] = \gamma_{|m|} + m\,\delta\,, \tag{3.78}$$

where $\gamma_{|m|}$ is determined in terms of Δ and $\tilde{\Delta}$, while δ is so far arbitrary. In the F-maximization procedure one finds that in the space of δ, Δ, and $\tilde{\Delta}$ there is exactly one flat direction: F remains unchanged if we send simultaneously $\Delta \to \Delta + r$, $\tilde{\Delta} \to \tilde{\Delta} - r$, and $\delta \to \delta + kr$ for any r [8]. The R-charges of the gauge-invariant operators are of course independent of r. As long as $k \neq 0$, we can set $\delta = 0$ as a gauge choice and work only with Δ and $\tilde{\Delta}$, which are not necessarily equal when $\mu \neq 0$.

As a function of the R-charges Δ and $\tilde{\Delta}$, the partition function we need to consider is then

$$Z = \int_{-\infty}^{\infty} d\lambda\, e^{i\pi k\lambda^2} e^{N\ell(1-\Delta+i\lambda)+\tilde{N}\ell(1-\tilde{\Delta}-i\lambda)}\,. \tag{3.79}$$

We want to calculate the partition function in the limit where $\bar{N}$ goes to infinity and $\kappa = 2k/(\bar{N}\pi)$ and μ are held fixed. In the large $\bar{N}$ limit we again find a saddle point at $\lambda = 0$. The saddle point equation requires $\Delta = 1/2 + O(1/\bar{N})$ and $\tilde{\Delta} = 1/2 + O(1/\bar{N})$. In order to study $1/\bar{N}$ corrections, we expand the R-charges as

$$\Delta = \frac{1}{2} + \frac{\Delta_1}{\bar{N}} + \frac{\Delta_2}{\bar{N}^2} + \dots\,, \qquad \tilde{\Delta} = \frac{1}{2} + \frac{\tilde{\Delta}_1}{\bar{N}} + \frac{\tilde{\Delta}_2}{\bar{N}^2} + \dots\,. \tag{3.80}$$

Using the methods developed in the previous sections, we can calculate the free energy perturbatively in the $1/\bar{N}$ expansion and maximize the resulting expression term by term with respect to the Δ_i and $\tilde{\Delta}_i$. Going through the procedure we find the following results for the free energy and the R-charges:

$$\Delta = \frac{1}{2} - \frac{2(1+\mu)}{\pi^2(1+\kappa^2)\bar{N}} + O(\bar{N}^{-2}), \qquad \tilde{\Delta} = \frac{1}{2} - \frac{2(1-\mu)}{\pi^2(1+\kappa^2)\bar{N}} + O(\bar{N}^{-2}),$$
$$F = \bar{N}\log 2 + \frac{1}{2}\log\left(\frac{\bar{N}\pi}{2}\sqrt{1+\kappa^2}\right) + \left[\frac{\kappa^2-1}{4(1+\kappa^2)^2} + \frac{2}{\pi^2(1+\kappa^2)^2} - \frac{4\mu^2}{\pi^2}\left(\frac{1}{(1+\kappa^2)^2} - \frac{4}{3(1+\kappa^2)^3}\right)\right]\frac{1}{\bar{N}} + O(\bar{N}^{-2}). \tag{3.81}$$

Using $N_b = N_f = 2\bar{N}$, we see that the expression for F agrees with eqs. (3.18) and (3.46) that were derived without the use of supersymmetric localization. The combination $\Delta + \tilde{\Delta}$, which gives the R-charge of the the gauge invariant meson operators $Q_a\tilde{Q}_b$, is in agreement with the pertrubative calculations in [37, 114].

We can perturb this theory by adding in the superpotential[3]

$$W \sim \sum_{a,b}(Q_a\tilde{Q}_b)^2\,. \tag{3.82}$$

Since $\Delta + \tilde{\Delta} < 1$ in the UV $\mathcal{N} = 2$ CFT, this superpotential deformation is relevant and causes an RG flow to the fixed point where the superpotential is exactly marginal. At the IR $\mathcal{N} = 2$ fixed point we have the constraint $\Delta + \tilde{\Delta} = 1$. To determine the free energy at the IR fixed point we simply have to repeat the F-maximization procedure above subject to this constraint. That in the UV one has to maximize F without any constraints while in the IR one has to maximize F under the constraint $\Delta + \tilde{\Delta} = 1$ means that the free energy of the IR fixed point is necessarily at most equal to the free energy of the UV fixed point. Indeed, we find that

$$F_{\rm UV} - F_{\rm IR} = \frac{2(1-\mu^2)}{\pi^2(1+\kappa^2)^2}\frac{1}{\bar{N}} + O(\bar{N}^{-2}), \tag{3.83}$$

[3]Changing the relative coefficients of the terms in (3.82) is an exactly marginal deformation [115] and does not change F.

which is manifestly positive when $\mu^2 < 1$. When $\mu = 1$ there are no $\tilde{Q}_a$ fields, and so we are not allowed to add in the superpotential deformation. The R-charges at the IR fixed point are given by

$$\Delta = \frac{1}{2} - \frac{4\mu}{\pi^2(1+\kappa^2)}\frac{1}{\bar{N}} + O(\bar{N}^{-2}) \,, \qquad \tilde{\Delta} = 1 - \Delta \,. \tag{3.84}$$

3.5 Discussion

In this chapter we studied certain 3-dimensional gauge theories coupled to a large number N_F of massless charged fields. Such theories are conformal for a sufficiently large N_F, and a good tool for studying them is the $1/N_F$ expansion. In this chapter we used such an expansion to study the disk entanglement entropy, which is related to the free energy F on the 3-sphere.

For the $U(N_c)$ gauge theory coupled to N_f massless Dirac fermions and N_b massless scalars we found the first subleading term in the expansion, (3.51). We have also studied the $\mathcal{N} = 2$ supersymmetric abelian gauge theory coupled to N positively charged chiral superfields Q and N negatively charged chiral superfields $\tilde{Q}$. In this case, F can be calculated numerically for any N using the methods of localization. We compared these numerical results with their $1/N$ expansion and found excellent agreement down to small N.

An important question concerning such CFTs is whether there is a breakdown of conformal invariance for sufficiently small N_F. In the $\mathcal{N} = 2$ supersymmetric $U(1)$ gauge theory, even for a single non-chiral flavor the theory is conformal and unitary. This is indicated by the mirror symmetry arguments [116] and confirmed by explicit calculation of the localized path integral in [31], which indicates that the dimension of Q and $\tilde{Q}$ is exactly $1/3$. However, in the non-supersymmetric $U(1)$ theories there typically is a lower bound for the conformal window. For example, in the extreme limit $N_F = 0$ we find the free Maxwell theory, which is not conformally invariant. We studied it on the S^3 of radius R in section 3 and found that F_{Maxwell} varies logarithmically with R, eq. (3.48), indicating the lack of conformal invariance.

One possible phenomenon for small N_F is the chiral symmetry breaking in 3-dimensional QED coupled to massless fermions [99,100]. The numerical studies of lattice antiferromagnets [117] suggest the QED theory with $N_f = 8$ Dirac fermions is a stable CFT, while the $N_f = 4$ theory is unstable to symmetry breaking towards a non-conformal ground state [93]. More generally, one of the signs of crossing the lower edge of the conformal window could be that the assumption of conformality leads to certain gauge invariant operators having scaling dimensions that violate the 3-dimensional unitarity bound $\Delta > 1/2$.

In [8, 9] it was conjectured that F must be positive in a unitary CFT. Since as N_F decreases so does F, it is possible that F may become negative for sufficiently small N_F. This could serve as another criterion for theories outside the conformal window. It would be interesting to explore the different criteria above and to see if they are related.

Chapter 4

AdS Description of Induced Higher-Spin Gauge Theory

This chapter is a lightly-modified version of the paper [14].

4.1 Introduction and summary

A Conformal Field Theory (CFT) in d dimensions is dual to a gravitational theory in AdS_{d+1} endowed with a particular choice of boundary conditions [24–26]. For example, a local scalar operator $\mathcal{O}(x^\mu)$ with dimension Δ is dual to a scalar field $\Phi(z, x^\mu)$ that behaves as z^Δ near the AdS boundary. The possible values of Δ are determined by the mass of the scalar field in the bulk:

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m^2} \, , \tag{4.1}$$

where the AdS radius has been set to 1. The dimension Δ_- is allowed only in the range $-(d/2)^2 < m^2 < -(d/2)^2 + 1$ [118,119]; using it for greater values of m^2 results in an operator dimension that violates the unitarity bound. An RG flow from a large N CFT where the operator $\mathcal{O}$ has dimension Δ_- to another CFT where it has dimension Δ_+ takes place when the double-trace operator $\mathcal{O}^2$ is added to the action [57, 69]. The effect of this flow on the partition function of the Euclidean CFT on the d-dimensional sphere has been studied in a number of papers [9, 57, 58, 120, 121].

These results have interesting applications to AdS_4/CFT_3 dualities involving Vasiliev's interacting higher-spin gauge theories in AdS_4 [73–75, 122]. These theories have been conjectured to be dual to 3-d CFTs such as the critical $O(N)$ model [72], or the Gross-Neveu model [82, 83], or various large N Chern-Simons theories coupled to conformal matter in the fundamental representation of the gauge group [106, 107]. Such AdS/CFT dualities are often called "vectorial" because the dynamical fields in the CFT are N-vectors rather than $N \times N$ matrices. In particular, the scalar $O(N)$ model has been conjectured [72] to be dual to the minimal type-A Vasiliev theory containing gauge fields of all even spin in AdS_4, while the Gross-Neveu model has been conjectured [82, 83] to be dual to the minimal type-B Vasiliev theory.[1] Considerable evidence has been accumulated in favor of the vectorial AdS_4/CFT_3 dualities [76–78, 123–128], and we will make further use of them in this chapter.

The possibility of two different conformally invariant AdS boundary conditions extends in an interesting way to fields of spin $s > 0$. For example, to a spin 1 conserved $U(1)$ current J_μ in a 3-dimensional CFT there corresponds a massless gauge field A_μ in AdS_4 with the boundary condition that the magnetic field F_{ij} vanishes at the AdS boundary $z = 0$. If instead the electric field F_{iz} is required to vanish at the boundary, then the $U(1)$ symmetry of the CFT becomes gauged [129]. These facts have applications to the versions of Vasiliev theory that contain gauge fields of all integer spin in AdS_4. The type A such model is dual to the $U(N)$ symmetric 3-d CFT of N complex scalar fields [72], while the type B model is dual to the theory of N Dirac fermions [82, 83]. The ability to change the boundary conditions for the spin 1 field makes it plausible [130] that the type A or B Vasiliev theory in AdS_4 with the electric boundary condition on the spin 1 field is dual to 3-dimensional CFTs where the $U(1)$ gauge field is coupled to a large number N of conformally invariant

[1] An important distinction between the type A and B parity invariant Vasiliev theories is that in the former the scalar field has positive parity, while in the latter it has negative parity [73, 83].

complex scalar or fermion fields, i.e. the 3-dimensional “induced” QED [98] restricted to the $SU(N)$ singlet sector. A more general, mixed boundary condition on the $U(1)$ gauge field in AdS_4 results in addition of the Chern-Simons term for the dynamical $U(1)$ gauge field in QED_3 [129]. There is an $SL(2,\mathbb{Z})$ action on the resulting set of 3-d CFTs [129]. The possibility of imposing modified boundary conditions on spins $s \leq 1$ in Vasiliev's theory was also used in [131] in constructing higher-spin duals of various supersymmetric Chern-Simons matter theories. Besides considering the $U(1)$ symmetries Ref. [131] also considered gauging non-abelian symmetries. Non-abelian gauge fields can appear in supergravity as well as in Vasiliev theory; with standard boundary conditions they correspond to non-abelian global symmetries in the dual field theory. Changing the boundary conditions in AdS_{d+1} is expected to lead to a non-abelian induced gauge theory in d dimensions.

Another very interesting special case is $s = 2$. Modifying the boundary condition for the graviton in AdS_4 makes the metric fluctuating also in the dual boundary theory [132, 133]. The resulting 3-d theory then describes a Weyl invariant gravity induced by coupling to conformal matter. The effective action for this theory was explored at the quadratic order for gravitons in [132]. A further study of the modified boundary conditions in AdS_4 indicated that the correspondence with 3-d induced gravity works at the full non-linear level [133]. Furthermore, the conformal graviton spectrum around flat space was found in [133] to be free of ghost-like modes for all odd d, suggesting that these induced theories are unitary at least in perturbation theory (on the other hand, in even d there are ghosts, as familiar in the case of $d = 4$ Weyl gravity [134]). Using these ideas, we will conjecture, for example, that modifying the graviton boundary conditions in Vasiliev's minimal type A theory makes it dual to the $O(N)$ singlet sector of the Weyl invariant 3-d gravity coupled to N conformal scalar fields ϕ^i, $i = 1, \ldots, N$. The path integral for this theory is

$$Z_{\text{3-d gravity}} = \int \frac{[Dg_{\mu\nu}][D\phi^i]}{\text{Vol(Diff)Vol(Weyl)}} e^{-S} , \tag{4.2}$$

$$S = \int d^3x \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^i + \frac{1}{8} R (\phi^i)^2 \right) . \tag{4.3}$$

Similarly, it is plausible that the minimal type B Vasiliev theory with modified graviton boundary conditions is dual to the $O(N)$ singlet sector of the Weyl invariant 3-d gravity coupled to N massless fermions. As for the $s = 1$ case, for $s = 2$ there is a possibility of mixed parity-violating boundary conditions in AdS_4 [132, 133, 135, 136], which correspond to adding to the 3-d action the gravitational Chern-Simons term $i\kappa \int \mathrm{tr}(\omega \wedge d\omega + \frac{2}{3}\omega^3)$ [137, 138]. Similarly, the $\mathcal{N} = 8$ superconformal gravity coupled to the BLG/ABJM theory was studied in [139–141]. The crucial role of alternate boundary conditions in AdS_4 was noted there as well.

In analogy with the above discussions, it is possible to modify the AdS_4 boundary conditions for higher-spin fields with $s > 2$. This modification results in gauging the corresponding higher-spin symmetries in the 3-d boundary theory,[2] as was proposed some time ago at the level of the linearized approximation [132] (see also [143]) and studied more recently in the context of the fully non-linear Vasiliev higher-spin theory [142]. The non-linearities have the important effect that, when an $s > 2$ current is gauged, one may need to gauge all remaining currents too.[3] In that case, the 3-d dual of a minimal Vasiliev theory in AdS_4 is expected to be a Weyl invariant theory of gauge fields of all even spins induced by the coupling to N conformal scalar or fermion fields. On the other hand, the gauged $s = 1$ and $s = 2$ examples discussed above do not require gauging higher-spin symmetries, because the non-linear gauge transformations for spin $s \leq 2$ form a closed subalgebra of the higher-spin algebra. The 3-d theory where currents of all spin are gauged is clearly more complicated than either 3-dimensional QED or the induced gravity theory in (4.2). Such an induced higher-spin gauge theory was studied in [144], and some progress has been recently made

[2] One motivation for studying the theories where some of the currents are gauged, which was stressed in [142], is that they do not obey the theorem of [123]. This theorem requires theories with exactly conserved higher spin currents to be free. However, when some of the currents are gauged the remaining ones are not conserved; therefore, the theorem of [123] does not apply. For example, the 3-d QED coupled to N flavors is obviously not a free theory, even when N is large. The theory obtained by gauging the whole set of HS currents also does not obey the theorem of [123], being a higher spin gauge theory (in particular including gravity), while [123] assumes a CFT with global HS symmetries and corresponding exactly conserved currents.

[3] We thank M. Vasiliev for stressing this to us.

using twistor space techniques in the unfolded formulation [142]. It is also interesting to ask if a truncation of this 3-d theory to a finite number of higher-spins is possible.

In this chapter we will subject these Anti-de Sitter/Induced Gauge Theory (AdS/IGT) correspondences to some new tests in the regime where N is very large; in this limit the Vasiliev theories in AdS_4 become weakly coupled while the path integrals in the 3-d theory can be studied semi-classically. We will calculate the change in the 3-sphere free energy $F = -\log |Z_{S^3}|$ produced by the gauging of a symmetry with $s \geq 1$. We will then show that this change agrees with the corresponding calculation in Euclidean AdS_4, which uses modified boundary conditions for a spin s gauge field. In fact, in QED_3 coupled to N conformal scalar or fermion fields the 3-sphere free energy was studied in the previous chapter with the result $F_{\rm QED} - F_{\rm free} = \frac{1}{2}\log N + O(N^0)$. We will show that for the gauging of spin s current this expression generalizes to

$$F^{(s)}_{\rm gauged} - F^{(s)}_{\rm free} = \frac{(4s^2-1)s}{6}\log N + O(N^0)\ . \tag{4.4}$$

As we will discuss in section 4.4, the coefficient of $\frac{1}{2}\log N$ is the number of spin $s-1$ conformal Killing tensors (equivalently, these are the conformal higher-spin currents which were found in [145] following [146]). Each such tensor corresponds to a missing gauge invariance (a zero mode of the operator $\mathcal{O}_g$ defined in (4.47) that takes a rank $s-1$ traceless symmetric tensor to a pure gauge mode of a spin s gauge field) in the 3-dimensional theory of the spin s gauge field. These tensors transform in the $[s-1, s-1]$ irreducible representation of the conformal group $SO(4,1)$ (its Young tableaux has two rows of length $s-1$) [147,148]. The AdS/CFT correspondence relates a conformal Killing tensor in d dimensions to a traceless Killing tensor in AdS_{d+1} [149]. In section 4.7 we will study this relation in detail with special emphasis on the AdS boundary behavior of the Killing tensors.

In addition to studying the gauging of conserved higher-spin currents, we will study the closely related problem of deforming a 3-d CFT by a double-trace operator $J_{\mu_1\mu_2\ldots\mu_s}J^{\mu_1\mu_2\ldots\mu_s}$, where the spin s single-trace operator $J_{\mu_1\mu_2\ldots\mu_s}$ has dimension Δ. If $\Delta > 3/2$, then the double-trace operator is irrelevant; such irrelevant deformations were discussed for $s \geq 1$ in [132]. For large N it is possible to show that the deformed theory possesses a UV fixed point where the spin s operator has dimension $\Delta_- = 3 - \Delta + O(1/N)$. In this case, we will find using both the 3-d field theory and AdS$_4$ calculations that

$$\delta F^{(s)}_{\Delta} \equiv F^{(s)}_{\rm UV} - F^{(s)}_{\rm IR} = \frac{(2\,s+1)\pi}{6} \int_{3/2}^{\Delta} \big(x - \frac{3}{2}\big)(x+s-1)(x-s-2)\cot(\pi x)\,. \tag{4.5}$$

For spin $s \geq 1$, Δ_- cannot satisfy the unitarity bound $\Delta^{(s)} \geq s+1$. The only cases where unitarity appears to be restored is when the spin s current is conserved and has $\Delta = s+1$; then Δ_- is the dimension of the dual spin s gauge field, which is not a gauge invariant operator, so there is no obvious issue with unitarity.

While in odd dimensions d the parity invariant conformal higher-spin gauge theories have induced non-local actions, in even d there are theories that are *local* and Weyl invariant for any spin s (these local actions are the coefficients of the induced logarithmically divergent terms [133, 150–152]). For example, in $d = 4$ they are the free Maxwell theory ($s = 1$), the conformal gravity ($s = 2$) [153], and their Fradkin-Tseytlin higher-spin generalizations [134]. These conformal higher-spin theories have actions involving more than two derivatives in contrast with the two-derivative quadratic Fronsdal actions [154]. This is evident already for the $s = 2$ conformal theory whose action is the square of the Weyl tensor. The role of the Weyl-squared gravity in the AdS/CFT correspondence has been explored for some time [133, 150]. A relation between conformal $d = 4$ higher-spin theories and massless higher-spin theories in AdS$_5$ was proposed in [151, 152]. Our approach of using alternate boundary conditions for massless spin s gauge fields in Euclidean AdS$_{d+1}$ indeed relates them to conformal spin s gauge fields on S^d. As an application of these ideas, in section 4.9 we will demonstrate that the massless spin s fields in AdS$_{d+1}$ endowed with alternate boundary conditions provide an efficient way for calculating the Weyl anomaly a-coefficients of conformal spin s theories in even d. In particular, we will reproduce the Weyl anomaly a-coefficient of the $d = 4$ conformal gravity [134, 153] and conjecture a formula generalizing it to all conformal 4-d gauge theories of integer spin $s > 0$:

$$a_s = \frac{s^2}{180}(1+s)^2[3 + 14s(1+s)]\,. \tag{4.6}$$

Similarly, we may consider higher-spin theories in AdS_3 [155–157] whose dual $d = 2$ CFTs have W symmetries [158–163]. Changing the boundary conditions in the bulk corresponds to gauging these symmetries. From the one-loop determinants of graviton and higher-spin gauge fields with alternate boundary conditions in AdS_3, we reproduce the well-known central charge $c = -26$ of the bc ghosts in 2-d gravity [164], as well as its higher-spin generalization [165]: $c_s = -2(1 + 6s(s-1))$.

4.2 Double-trace deformations with higher-spin operators

We start by analyzing the double-trace deformations with $s \geq 1$ in the case where the single-trace spin s operator has dimension $\Delta \neq s + 1$. As remarked in the introduction, these deformations are somewhat less desirable than those with $\Delta = s + 1$ due to the appearance of operators that violate the unitarity bound. Nevertheless, the theories with $\Delta \neq s + 1$ are still interesting conceptually, and they are somewhat simpler computationally because we do not have to worry about gauge invariance. As a consequence of this fact—and we will show this in detail in the following sections—the difference in free energies $\delta F^{(s)}_{\Delta}$ is order N^0 when $\Delta \neq s + 1$, while it is order $\log N$ when $\Delta = s + 1$, as advertised in (4.5) and (4.4). In this section we begin with the cases $\Delta \neq s + 1$ and use field theoretic arguments to demonstrate (4.5) for small values of s. In section 4.4 we then discuss the implications of gauge invariance when $\Delta = s + 1$.

Before turning to the calculation, however, we mention two interesting features of the result in (4.5). The first observation is that $\delta F_\Delta^{(s)}$ is positive for $3/2 < \Delta < 2$ for all s. When $s = 0$ this is required by the F-theorem [8–12]—in fact, in that case δF must be positive when $3/2 < \Delta < 5/2$, as discussed previously. But when $s \geq 1$ one of the fixed

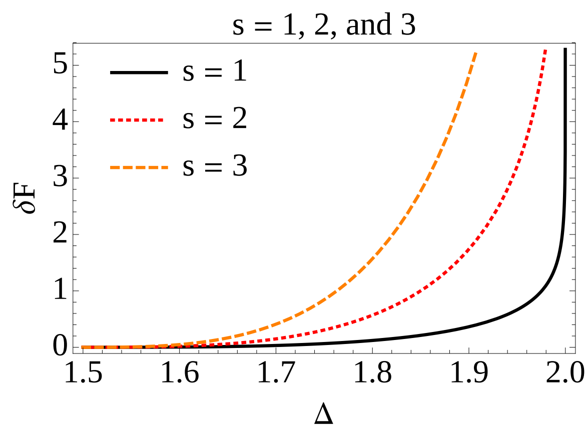


Figure 4.1: $\delta F_\Delta^{(s)}$ plotted as a function of Δ for $s = 1$, 2, and 3. When $s = 0$ this quantity is plotted in Fig. 2.1. The F-theorem does not apply to the $s \geq 1$ theories since one or both of the fixed points is non-unitary. The exception is when $\Delta = s + 1$, since in this case the naive unitarity arguments are not valid.

points is always non-unitary, and so the F-theorem does not require δF to be positive. It is therefore interesting that $\delta F_\Delta^{(s)}$ is always positive for $3/2 < \Delta < 2$, but the significance of this observation is unclear.

The second observation, which is also illustrated in figure 4.1, is that $\delta F_\Delta^{(s)}$ diverges logarithmically as $\Delta \to 2$ when $s \geq 1$. Furthermore, if we take $\Delta = s + 1 - \epsilon$, where $\epsilon \ll 1$, and concentrate on the contribution of the upper integration limit in (4.5), then we find

$$\delta F_\Delta^{(s)} = -\frac{(4s^2 - 1)s}{6} \log \epsilon + O(\epsilon^0) \ . \tag{4.7}$$

This result shows, in some sense, how the result in (4.4), which is valid strictly when $\Delta = 1+s$, emerges from the case of more general double-trace deformation. The conclusion is that gauging a symmetry in a large N CFT makes δF logarithmically large.

4.2.1 General strategy

The RG flow we are considering may be constructed explicitly as follows. Let S_0 be the action of a large N CFT defined on a conformally flat background with metric $g_{\mu\nu}$. We perturb S_0 by the irrelevant deformation proportional to the double-trace operator J^2 to obtain the action

$$S = S_0 + \frac{\lambda_0}{2} \int d^3x \sqrt{g} J_{\mu_1\mu_2\ldots\mu_s}(x) J^{\mu_1\mu_2\ldots\mu_s}(x)\,, \tag{4.8}$$

where $J_{\mu_1\mu_2\ldots\mu_s}$ is a symmetric traceless tensor. This theory has a UV fixed point where $J_{\mu_1\mu_2\ldots\mu_s}$ has dimension $\Delta_- = 3 - \Delta + O(1/N)$. To demonstrate this, we use the Hubbard-Stratonovich transformation to write the action with the help of a spin s auxiliary field $h_{\mu_1\mu_2\ldots\mu_s}$:

$$S = S_0 - \int d^3x \sqrt{g(x)} \left[h_{\mu_1\ldots\mu_s}(x) J^{\mu_1\ldots\mu_s}(x) + \frac{1}{2\lambda_0} h_{\mu_1\ldots\mu_s} h^{\mu_1\ldots\mu_s} \right] . \tag{4.9}$$

A study of the induced action for $h_{\mu_1\mu_2\ldots\mu_s}$ shows that the last term is negligible at the UV fixed point [132]. When the current $J_{\mu_1\mu_2\ldots\mu_s}$ is conserved, the auxiliary field $h_{\mu_1\mu_2\ldots\mu_s}$ assumes the role of a spin s gauge field.

One can evaluate the ratio Z/Z_0 of the partition functions corresponding to S and S_0 perturbatively in $1/N$ as follows. Integrating out the fields that appear in the undeformed action S_0, one can write the partition function of the deformed theory (4.9) as

$$Z/Z_0 = \int Dh_{\mu_1\ldots\mu_s} \left\langle \exp\left(\int d^3x \sqrt{g(x)} h_{\mu_1\ldots\mu_s}(x) J^{\mu_1\ldots\mu_s}(x) \right) \right\rangle_0 , \tag{4.10}$$

where on the right-hand side the expectation value is computed with the measure $\exp[-S_0]$. Expanding the exponential and using the fact that $\langle J_{\mu_1\ldots\mu_s}(x)\rangle_0 = 0$, as appropriate for a

CFT on a conformally flat space, one obtains

$$Z = Z_0 \int Dh_{\mu_1 \ldots \mu_s} e^{-S_{\text{eff}}[h_{\mu_1 \ldots \mu_s}]} \,, \tag{4.11}$$

where the effective action for the auxiliary field is to quadratic order given by

$$S_{\text{eff}} = -\frac{1}{2} \int d^3x \, d^3y \, \sqrt{g(x)} \sqrt{g(y)} \, h_{\mu_1 \ldots \mu_s}(x) h_{\nu_1 \ldots \nu_s}(y) \langle J^{\mu_1 \ldots \mu_s}(x) J^{\nu_1 \ldots \nu_s}(y) \rangle_0^{\text{conn}} + \ldots \,. \tag{4.12}$$

The expansion in (4.12) is given in terms of connected correlators of the spin s operator, which are all assumed to be $O(N)$. At large N the typical fluctuations of $h_{\mu_1 \ldots \mu_s}$ are $O(N^{-1/2})$, and therefore the contributions to the partition function of the higher order terms in $h_{\mu_1 \ldots \mu_s}$, that were not exhibited in (4.12), become negligible. The functional integral (4.11) can then be evaluated in the saddle-point approximation:

$$Z \approx Z_0 (\det K)^{-1/2} \,, \tag{4.13}$$

where the operator K given as an integration kernel can be expressed as

$$K_{\mu_1 \ldots \mu_s ; \nu_1 \ldots \nu_s}(x, y) = -\langle J_{\mu_1 \ldots \mu_s}(x) J_{\nu_1 \ldots \nu_s}(y) \rangle_0^{\text{conn}} \,. \tag{4.14}$$

The expression (4.13) is valid on any conformally flat space.

Specializing to the case where the background metric is that of the unit S^3, (4.13) implies

$$\delta F_\Delta^{(s)} = -\log \left| \frac{Z}{Z_0} \right| = \frac{1}{2} \operatorname{tr} \log K + O(1/N) \,. \tag{4.15}$$

To calculate $\delta F_\Delta^{(s)}$ one would therefore need to sum the logarithms of the eigenvalues of the kernel K on S^3 weighted by their multiplicities.

An explicit formula for K can be written down most easily if we parameterize S^3 through the stereographic projection from $\mathbb{R}^3$. In other words, let us introduce the metric

$$ds^2_{S^3} = \frac{4}{\left(1+|x|^2\right)^2}\left[(dx^1)^2+(dx^2)^2+(dx^3)^2\right] \,, \tag{4.16}$$

as well as the frame

$$e^i = \frac{2}{1+|x|^2}dx^i \,. \tag{4.17}$$

In this frame, the kernel (4.14) is constrained by conformal invariance to be[4]

$$K_{i_1\ldots i_s}{}^{j_1\ldots j_s}(x,y) = N\,C\left(\frac{\left(1+|x|^2\right)\left(1+|y|^2\right)}{4\,|x-y|^2}\right)^{\Delta} I_{(i_1}{}^{(j_1}I_{i_2}{}^{j_2}\cdots I_{i_s)}{}^{j_s)} \,, \tag{4.18}$$

where C is an N-independent normalization constant, and

$$I^{ij} \equiv \delta^{ij} - 2\frac{(x^i-y^i)(x^j-y^j)}{|x-y|^2} \,. \tag{4.19}$$

In (4.18), the symmetrizations are performed with total weight one and include the removal of all the traces. Importantly, the kernel K is linear in N.

4.3 Explicit field theory calculations

4.3.1 Symmetric traceless tensor harmonics on S^3

The eigenvalues of K can be found with the help of rotational symmetry on S^3; the eigenfunctions of K must be symmetric traceless tensor harmonics on S^3. For spin 0, these harmonics are the usual spherical harmonics on S^3 which transform as the $(\mathbf{n},\mathbf{n})$ irreps[5] of

[4]Frame indices are raised and lowered with the flat metric.
[5]We write the spin j representation of $SU(2)$ as $\mathbf{2j+1}$.

the isometry group $SU(2)_L \times SU(2)_R$—they are just traceless symmetric polynomials in the standard embedding coordinates of S^3 into $\mathbb{R}^4$. The space of normalizable functions on S^3 therefore decomposes under $SO(4)$ as

$$\bigoplus_{n=1}^{\infty} (\mathbf{n}, \mathbf{n}) \, . \tag{4.20}$$

For every positive integer n, there are n^2 scalar harmonics, which we denote by $Y_{n\ell m}(x)$, with $0 \leq \ell < n$ and $|m| \leq \ell$. Explicit expressions for these scalar harmonics are given in [14].

For spin s, the space of rank s symmetric traceless tensors on S^3 decomposes under $SO(4)$ as

$$\bigoplus_{n=s+1}^{\infty} \bigoplus_{s'=-s}^{s} (\mathbf{n} + \mathbf{s}', \mathbf{n} - \mathbf{s}') \, . \tag{4.21}$$

In other words, there are $2s+1$ towers of modes indexed by s', where there are $n^2 - s'^2$ modes in each tower, with $n > s$. We denote these harmonics by $\mathbb{H}^{s', n\ell m}_{\mu_1 \cdots \mu_s}(x)$, with $s' \leq \ell < n$ and $-\ell \leq m \leq \ell$. Explicit expressions for $s \leq 3$ are given in [14].

The reason for the decomposition (4.21) is easy to state. Starting with the three $SU(2)_L$ Killing vectors (or the corresponding one-forms obtained by lowering indices with the metric), one can construct rank-s traceless symmetric tensors by taking traceless symmetric tensor products of these Killing vectors. Angular momentum addition guarantees that these tensors transform as $(\mathbf{2s+1}, \mathbf{1})$ under $SU(2)_L \times SU(2)_R$. The most general rank-s traceless symmetric tensor on S^3 is a linear combination of these $(\mathbf{2s+1}, \mathbf{1})$ tensors with coefficients that depend on position. These coefficients are functions on S^3, so they can be expanded in the basis of scalar spherical harmonics, which as mentioned above transform as $(\mathbf{n}, \mathbf{n})$ under $SO(4)$. The traceless symmetric tensors therefore transform as the tensor sum of products $(\mathbf{n}, \mathbf{n}) \otimes (\mathbf{2s+1}, \mathbf{1})$ over all $n \geq 1$. This description yields (4.21) after a shift in n.

All the harmonics in a given irreducible representation of $SO(4)$ are eigenfunctions of K corresponding to the same eigenvalue. Let $k_{n,s'}$ be the eigenvalue corresponding to each term in (4.21):

$$\int d^3y\, \sqrt{g(y)} K_{\mu_1 \ldots \mu_s}{}^{\nu_1 \ldots \nu_s}(x, y)\, \mathbb{H}^{s', n\ell m}_{\nu_1 \ldots \nu_s}(y) = k_{n,s'} \mathbb{H}^{s', n\ell m}_{\mu_1 \ldots \mu_s}(x) \,. \tag{4.22}$$

Then

$$\delta F^{(s)}_{\Delta} = \frac{1}{2} \sum_{n=s+1}^{\infty} \sum_{s'=-s}^{s} (n^2 - s'^2) \log k_{n,s'} \,. \tag{4.23}$$

Because the kernel (4.18) is invariant under the $\mathbb{Z}_2$ reflection symmetry that exchanges $SU(2)_L$ with $SU(2)_R$, we must have $k_{n,s'} = k_{n,-s'}$. Since the eigenvalue $k_{n,s'}$ doesn't depend on the quantum numbers ℓ and m, we can write

$$k_{n,s'} = \frac{1}{n^2 - s'^2} \sum_{\ell, m} \int d^3x\, d^3y\, \sqrt{g(x)} \sqrt{g(y)}\, \mathbb{H}^{s', n\ell m}_{\mu_1 \ldots \mu_s}(x)^*\, K^{\mu_1 \ldots \mu_s; \nu_1 \ldots \nu_s}(x, y)\, \mathbb{H}^{s', n\ell m}_{\nu_1 \ldots \nu_s}(y) \,. \tag{4.24}$$

The average over all the states in a given irreducible representation of $SO(4)$ makes the product $\mathbb{H}(x)^* K(x, y) \mathbb{H}(y)$ depend only on the relative angle between x and y. One can then perform five of the six integrals in (4.24), which gives

$$k_{n,s'} = \frac{64\pi^3}{n^2 - s'^2} \int dr\, \frac{r^2}{(1 + r^2)^3}\, \mathbb{Z}^{s', n}_{\mu_1 \ldots \mu_s; \nu_1 \ldots \nu_s}(r\hat{v})\, K^{\mu_1 \ldots \mu_s; \nu_1 \ldots \nu_s}(r\hat{v}, 0) \,, \tag{4.25}$$

where $\hat{v}$ is an arbitrary unit vector, say $\hat{v} = (0, 0, 1)$, and $\mathbb{Z}$ is a tensor "zonal" harmonic defined as

$$\mathbb{Z}^{s', n}_{\mu_1 \ldots \mu_s; \nu_1 \ldots \nu_s}(x) \equiv \sum_{\ell, m} \mathbb{H}^{s', n\ell m}_{\mu_1 \ldots \mu_s}(x)^* \mathbb{H}^{s', n\ell m}_{\nu_1 \ldots \nu_s}(0) \,. \tag{4.26}$$

We can thus find $k_{n,s'}$ by performing only a one-dimensional integral. All that remains to do is to find explicit expressions for the tensor zonal harmonics $\mathbb{Z}^{s', n}$ and the kernel K. We will do so in specific examples.

Before discussing the order N^0 corrections to $\delta F_\Delta^{(s)}$, however, we are already in position to show that the $\log N$ correction vanishes when $\Delta \neq s+1$. From (4.25) and (4.18), we see that each $k_{n,s'}$ is proportional to N and the normalization factor C. The $\log N$ correction to $\delta F_\Delta^{(s)}$ is then found by evaluating the divergent sum

$$\begin{aligned}\delta F_\Delta^{(s)} &= \left(\frac{1}{2}\sum_{n=s+1}^{\infty}\sum_{s'=-s}^{s}(n^2 - s'^2)\right)\log N + O(N^0)\\ &= \left(s+\frac{1}{2}\right)\left[\zeta(-2,s+1) - \frac{s(s+1)}{3}\zeta(0,s+1)\right]\log N + O(N^0)\\ &= O(N^0)\end{aligned} \tag{4.27}$$

through zeta function regularization. In simplifying the second line above we have used a standard identity for the Hurwitz zeta-function. We may use the same computation to show that (i) the $O(N^0)$ term does not depend on the normalization factor C, and (ii) if we reinstate the radius R of the S^3, the potential $\log R$ term vanishes. This latter point is important; since there is no anomaly in 3-d, the quantity $\delta F_\Delta^{(s)}$ must not have any dependence on the radius R through terms that cannot be removed by the addition of local counter-terms. A $\log R$ term is an example of such a term that cannot be removed.

4.3.2 Particular cases

We now calculate the order N^0 term in $\delta F_\Delta^{(s)}$ explicitly for $s = 0$, 1, and 2, and we show that the results are consistent with (4.5). The $s = 0$ calculation was performed in Sec. 2.5. As a warmup we begin by reviewing that computation in the current notation. We have also performed the $s = 3$ calculation explicitly. Some of the details may be found in the original paper [14].

Spin 0

For $s = 0$ we have only one type of eigenvalue, $k_{n,0}$. Using (4.18) we see that the kernel is given simply by

$$K(r\hat{v}, 0) = \frac{N\,C}{(2\sin(\chi/2))^{2\Delta}}\,, \tag{4.28}$$

where we have defined

$$r \equiv \tan\frac{\chi}{2}\,. \tag{4.29}$$

To compute the zonal harmonics we use the definition in (4.26) along with the explicit expressions for the spherical harmonics, given in [14], and we find

$$\begin{aligned} \mathbb{Z}^{0,n}(r\hat{v}) &= \sum_{\ell,m} Y_{n\ell m}(\chi,\theta,\phi) Y_{n\ell m}(\chi = 0) \\ &= Y_{n00}(\chi,\theta,\phi) Y_{n00}(\chi = 0) = \frac{n\csc\chi\sin(n\chi)}{2\pi^2}\,. \end{aligned} \tag{4.30}$$

The integral in (4.25) may then be performed explicitly:

$$\begin{aligned} k_{n,0} &= \frac{N\,C\,2^{2(1-\Delta)}\,\pi}{n} \int_0^\pi d\chi\, \frac{\sin\chi\sin n\chi}{\left(\sin\frac{\chi}{2}\right)^{2\Delta}} \\ &= 4\pi\,N\,C\sin(\pi\Delta)\frac{\Gamma(2-2\Delta)\Gamma(n-1+\Delta)}{\Gamma(2+n-\Delta)}\,. \end{aligned} \tag{4.31}$$

The change in the free energy may be evaluated using (4.23), which leads to the expression

$$\delta F_\Delta^{(0)} = \frac{1}{2}\sum_{n=1}^\infty n^2 \log\frac{\Gamma(n-1+\Delta)}{\Gamma(2+n-\Delta)}\,. \tag{4.32}$$

When $\Delta = 3/2$ the operator J^2 is marginal, and so in that case we expect $\delta F_{3/2}^{(0)} = 0$. Indeed, taking $\Delta = 3/2$ in (4.32), we see that each of the terms in the sum vanishes independently.

The sum in (4.32) was evaluated explicitly for general Δ in [58], and their regularized result is a particular case of (4.5). Below we give a more simple, though perhaps slightly less rigorous, derivation that will be useful when going on to the more complicated, higher-spin theories. First we take a derivative of (4.32) with respect to Δ, and then we insert a factor of $\exp[-\epsilon\, n]$, $\epsilon > 0$, into the sum to make it convergent:

$$\begin{aligned}\partial_\Delta \delta F^{(0)}_\Delta &= \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\left[\sum_{n=1}^{\infty}\big[\psi(2+n-\Delta)+\psi(n-1+\Delta)\big]e^{-\epsilon\, n}\right] \\ &= \frac{3-2\gamma-2\log\epsilon}{\epsilon^3} - \frac{13+6\Delta(\Delta-3)}{12\,\epsilon} \\ &+ \frac{\pi}{6}(\Delta-1)\left(\Delta-\frac{3}{2}\right)(\Delta-2)\cot(\pi\Delta) + O(\epsilon)\,. \end{aligned} \tag{4.33}$$

Subtracting the divergent terms from (4.33) and using the relation $\delta F^{(0)}_\Delta = \int_{3/2}^{\Delta} dx\,(\partial_x \delta F^{(0)}_x)$, which follows from the fact that $\delta F^{(0)}_{3/2} = 0$, we arrive at the result in (4.5) with $s = 0$.

Spin 1

When $s = 1$, a similar computation—using the Appendices of [14]—gives

$$\begin{aligned} k_{n,0} &= N\,C\frac{4\pi(2-\Delta)\Gamma(2-2\Delta)\sin(\pi\Delta)}{\Delta}\frac{\Gamma(n-1+\Delta)}{\Gamma(n+2-\Delta)}\,, \\ k_{n,\pm1} &= \frac{\Delta-1}{2-\Delta}k_{n,0}\,. \end{aligned} \tag{4.34}$$

This allows us to write $\delta F^{(1)}_\Delta$ as the sum

$$\delta F^{(1)}_\Delta = \frac{1}{2}\sum_{n=2}^{\infty}\left[n^2\log k_{n,0} + 2(n^2-1)\log k_{n,1}\right]\,, \tag{4.35}$$

which simplifies to

$$\delta F^{(1)}_\Delta = \frac{1}{2}\log\left|\frac{\Delta-1}{2-\Delta}\right| + \sum_{n=2}^{\infty}\left(\frac{3}{2}n^2-1\right)\log\left|\frac{\Gamma(n-1+\Delta)}{\Gamma(n+2-\Delta)}\right|\,. \tag{4.36}$$

As a first check of (4.36), we should verify that this expression vanishes when $\Delta = 3/2$. Indeed, in this case each of the terms in the sum vanishes independently. To evaluate (4.36) for more general Δ, it is again convenient to take a derivative with respect to Δ and to insert a factor of $e^{-\epsilon n}$ into the sum to make it convergent. The identity

$$\begin{aligned}&\lim_{\epsilon\to 0^+}\left[\frac{3}{2}\partial_\epsilon^2 - 1\right]\sum_{n=2}^{\infty}\left[\psi(n+2-\Delta)+\psi(n-1+\Delta)\right]e^{-\epsilon n}\\ &= \frac{1}{2}\left(\frac{1}{2-3\Delta+\Delta^2}\right)+\frac{\pi}{2}\Delta(\Delta-3)\left(\Delta-\frac{3}{2}\right)\cot(\pi\Delta)\end{aligned} \tag{4.37}$$

then allows us to conclude that

$$\partial_\Delta \delta F_\Delta^{(1)} = \frac{\pi}{2}\Delta(\Delta-3)\left(\Delta-\frac{3}{2}\right)\cot(\pi\Delta)\,, \tag{4.38}$$

which is consistent with (4.5).

Spin 2

The calculation of the eigenvalues is again straightforward when $s = 2$, and it leads to

$$\begin{aligned}k_{n,0} &= c(\Delta)\frac{\Gamma(n-1+\Delta)}{\Gamma(n+2-\Delta)}\,, \qquad k_{n,1} = \frac{\Delta-1}{2-\Delta}k_{n,0}\,,\\ k_{n,2} &= \frac{\Delta(\Delta-1)}{(\Delta-2)(\Delta-3)}k_{n,0}\,,\end{aligned} \tag{4.39}$$

where the common factor

$$c(\Delta) = N\,C\,\frac{8\pi(\Delta-3)(\Delta-2)(2\Delta-1)\Gamma(-2\Delta)\sin(\pi\Delta)}{\Delta+1} \tag{4.40}$$

is independent of n. We then find that $\delta F_\Delta^{(2)}$ may be written as the sum

$$\delta F_\Delta^{(2)} = \sum_{n=3}^{\infty}\left(\frac{5}{2}n^2-5\right)\log\left|\frac{\Gamma(n-1+\Delta)}{\Gamma(n+2-\Delta)}\right| + \frac{5}{2}\log\left|\frac{2\Delta^2(\Delta-1)}{(\Delta-2)(\Delta-3)^2}\right|\,. \tag{4.41}$$

When $\Delta = 3/2$, each of the terms in the sum vanishes identically, leading to the expected result $\delta F^{(2)}_{3/2} = 0$. To evaluate this sum for more general Δ, we follow the by now familiar procedure of taking a derivative with respect to Δ and inserting a factor $e^{-\epsilon n}$ into the sum to make it convergent. Using an identity analogous to (4.37), we find the result

$$\partial_\Delta \delta F^{(2)}_\Delta = \frac{5\pi}{6}(\Delta - 4)\left(\Delta - \frac{3}{2}\right)(\Delta + 1)\cot(\pi\Delta)\,, \tag{4.42}$$

which is consistent with (4.5).

4.3.3 A conjecture for arbitrary spin

The spin 3 calculation is worked out explicitly in [14]. From these examples with $s \leq 3$ we conjecture that at arbitrary integer spin s the eigenvalues are related to each other by

$$k_{n,0} = c_s(\Delta)\frac{\Gamma(n-1+\Delta)}{\Gamma(n+2-\Delta)}\,, \qquad k_{n,i} = \frac{\Gamma(2-\Delta)}{\Gamma(\Delta-1)}\frac{\Gamma(-1+i+\Delta)}{\Gamma(2+i-\Delta)}k_{n,0}\,. \tag{4.43}$$

Importantly, the common factor $c_s(\Delta)$ is n-independent. The calculation in (4.27) that showed that $\delta F^{(s)}_\Delta$ does not depend on the radius R and N then also shows that $\delta F^{(s)}_\Delta$ is independent of $c_s(\Delta)$. Moreover, when $\Delta = 3/2$ we find that $k_{n,i} = k_{n,0}$, which immediately implies that $\delta F^{(s)}_{3/2} = 0$. To test the eigenvalue conjecture for more general Δ, we may calculate $\partial_\Delta \delta F^{(s)}_\Delta$ using the identity

$$\begin{aligned}
&\lim_{\epsilon\to 0^+}\left[(1+2s)\partial_\epsilon^2 - \frac{s(1+s)(1+2s)}{3}\right]\sum_{n=1+s}^{\infty}\left[\psi(n+2-\Delta)+\psi(n-1+\Delta)\right]e^{-\epsilon n} \\
&+ \frac{(1+2s)}{3}\sum_{i=1}^{s}\left(s(s+1)-3\,i^2\right)\left[\psi(2-\Delta)+\psi(\Delta-1)-\psi(2+i-\Delta)-\psi(-1+i+\Delta)\right] \\
&= \frac{(2\,s+1)\pi}{3}\left(\Delta-\frac{3}{2}\right)(\Delta+s-1)(\Delta-s-2)\cot(\pi\Delta)\,,
\end{aligned} \tag{4.44}$$

and we find the desired formula (4.5) at arbitrary integer spin. In section 4.6.2 we prove (4.5) for arbitrary spin s from a much simpler calculation in the bulk.

4.4 Conserved Currents and Gauge Symmetries

Let us now return to the case where $\Delta = s+1$ and the operator $J_{\mu_1\mu_2\ldots\mu_s}$ is a conserved current of spin s. Specifically, we will consider the theories of N free conformal complex scalars or Dirac fermion fields, which possess such currents of all $s > 0$. The conformal theory in this case is the gauge theory for the spin s gauge field $h_{\mu_1\mu_2\ldots\mu_s}$ with quadratic and higher-order terms induced by the one-loop diagram with conformal matter propagating around the loop. We will derive the result advertised in (4.4), and we will also show explicitly that δF is independent of the radius R of the three-sphere. This independence of R is crucial for the interpretation of the induced theory as a conformal theory.

For more generality, we work in d dimensions, with d odd. The restriction to odd dimensions is put in to avoid the Weyl anomaly, which occurs when d is even. We return to the even dimensional case in later sections. Note that in all d the scaling dimension of the spin s gauge field is $\Delta_- = 2-s$.

The expression (4.4), as well as its generalization to arbitrary odd d, follows from a careful treatment of the gauge symmetry in the path integral. At the linearized level, the induced conformal higher-spin theory has the following local symmetries[6]

$$\delta h_{\mu_1\ldots\mu_s} = \nabla_{(\mu_1} v_{\mu_2\ldots\mu_s)} + g_{(\mu_1\mu_2}\lambda_{\mu_3\ldots\mu_s)}\,, \tag{4.45}$$

where the rank $s-1$ symmetric traceless gauge parameter v_{s-1} is the generalization of the familiar diffeomorphisms for spin 2, and the rank $s-2$ parameter λ_{s-2} generalizes the local Weyl invariance of conformal gravity [134]. We may use this symmetry to gauge away completely the trace of $h_{\mu_1\ldots\mu_s}$, and the remaining gauge symmetry is then obtained by

[6]We symmetrize with total weight one. In other words $v_{(\mu_1\mu_2\ldots\mu_s)} = \frac{1}{s!}\sum_{\sigma\in S_s} v_{\sigma_{\mu_1}\ldots\sigma_{\mu_s}}$.

restricting to the traceless part of (4.45)

$$\delta h_{\mu_1\mu_2\ldots\mu_s} = \left(\mathcal{O}_g v\right)_{\mu_1\mu_2\ldots\mu_s}, \tag{4.46}$$

where the operator $\mathcal{O}_g$ takes the rank $s-1$ traceless symmetric tensor $v_{\mu_1\ldots\mu_{s-1}}$ to a rank s traceless symmetric tensor, namely:

$$\left(\mathcal{O}_g v\right)_{\mu_1\mu_2\ldots\mu_s} = \nabla_{(\mu_1} v_{\mu_2\mu_3\ldots\mu_s)} - \frac{s-1}{d+2(s-2)} g_{(\mu_1\mu_2}\nabla^{\nu} v_{\mu_3\mu_4\ldots\mu_s)\nu}\,. \tag{4.47}$$

One can then decompose the gauge field $h_{\mu_1\ldots\mu_s}$ as

$$h_{\mu_1\ldots\mu_s} = t_{\mu_1\ldots\mu_s} + \left(\mathcal{O}_g v\right)_{\mu_1\mu_2\ldots\mu_s}, \qquad \nabla^{\mu_1} t_{\mu_1\ldots\mu_s} = 0\,. \tag{4.48}$$

The first term in (4.48) represents the physical modes, while the second term represents the pure gauge modes. The requirement $\nabla^{\mu_1} t_{\mu_1\ldots\mu_s} = 0$ on the physical modes is a gauge fixing condition.

After integrating out the conformally invariant matter fields, the partition function at the conformal fixed point takes the form

$$Z = \frac{1}{\mathrm{Vol}(G)} \int Dh e^{-S_{\mathrm{eff}}[h]}\,, \tag{4.49}$$

where G is the group of gauge transformations, and the effective action for the spin s gauge field h is given explicitly in the quadratic approximation by

$$S_{\mathrm{eff}}[h] = \frac{1}{2}\int d^d x\sqrt{g(x)}\int d^d y\sqrt{g(y)}\, h^{i_1\ldots i_s}(x) K_{i_1\ldots i_s}{}^{j_1\ldots j_s}(x,y) h_{j_1\ldots j_s}(y)\,, \tag{4.50}$$

for some kernel K as in (4.18) for $d=3$. It is important that $K \propto N$, where N is the number of conformally coupled matter fields; when N is large, the quadratic approximation (4.50) to the effective action becomes arbitrarily accurate. The action $S_{\rm eff}[h]$ is of course independent of the pure-gauge modes, so $S_{\rm eff}[h] = S_{\rm eff}[t]$. Performing the split (4.48) and writing the volume of the group of gauge transformations as an integral over gauge parameters, we have

$$Z \approx \frac{\int D(\mathcal{O}_g v)}{\int Dv} \int Dt\, e^{-S_{\rm eff}[t]} \,. \tag{4.51}$$

We are interested in studying the dependence on the S^d radius R and on the number N of conformally coupled matter fields. While only the last factor in (4.51) depends on N, the R-dependence of each of the two factors in (4.51) is more subtle. The absence of a Weyl anomaly guarantees, however, that Z is independent of R, as we now explain.

On general grounds, the absence of a Weyl anomaly in odd dimensions means that the integration measure in the path integral is invariant under constant rescalings of the integration variables. For instance, for a rank s traceless symmetric tensor $h_{\mu_1\ldots\mu_s}$, this means that $Dh = D(\lambda h)$ for any constant λ. We checked this fact explicitly in (4.27) in $d = 3$: the Jacobian $D(\lambda h)/Dh$ equals λ raised to the sum of the degeneracies of all symmetric traceless tensor modes, and we checked that this sum vanishes in zeta-function regularization in $d = 3$. Similar checks are straightforward to perform for other odd d.

The action in (4.49) remains unchanged if we send $g_{\mu\nu} \to \tilde\lambda^2 g_{\mu\nu}$ and $h_{i_1\ldots i_s} \to \tilde\lambda^{s-2+d/2} h_{i_1\ldots i_s}$ (where $i_1, i_2, \ldots$ are frame indices). Since the integration measure also remains unchanged (because all the modes are rescaled by the same factor), it follows that the partition function does not change either. One then concludes that the partition function on S^d is independent of R, because we can compute Z for a sphere of unit radius, and then reinstate R by performing a scale transformation.

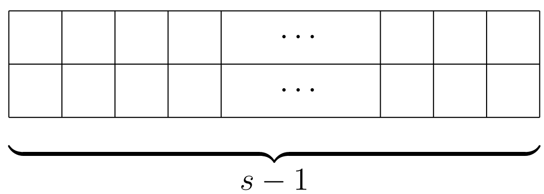
In order to understand the dependence of (4.51) on N, we should first examine the zero modes of the operator $\mathcal{O}_g$. These zero modes are important because in the numerator of the first factor in (4.51) we should not integrate over these modes, while in the denominator we should. The zero modes of $\mathcal{O}_g$ are solutions to the conformal Killing tensor equation

$$\nabla_{(\mu_1} v_{\mu_2\mu_3\ldots\mu_s)} = \frac{s-1}{d+2(s-2)} g_{(\mu_1\mu_2} \nabla^\nu v_{\mu_3\mu_4\ldots\mu_s)\nu} \,. \tag{4.52}$$

As shown in [148], see also [145, 146], the symmetric traceless conformal Killing tensors of rank $s-1$ form an irreducible representation of $SO(d+1,1)$ of dimension

$$n_{s-1} = \frac{(d+2s-4)(d+2s-3)(d+2s-2)(d+s-4)!(d+s-3)!}{s!(s-1)!d!(d-2)!} \,. \tag{4.53}$$

This is the representation of corresponding to the Young diagram



, (4.54)

which has two rows of length $s-1$.[7] The representation may be labelled by the set of integers with $m_1 = m_2 = s-1$ and $m_3 = \ldots = 0$ corresponding to the length of each row, and we conventionally denote it as $[s-1, s-1]$.

Note that when $s = 2$, (4.52) reduces to the more familiar conformal Killing vector equation

$$\nabla_\mu v_\nu + \nabla_\nu v_\mu = \frac{2g_{\mu\nu}}{3} \nabla \cdot v \,, \tag{4.55}$$

and it is well-known that there are $n_1 = (d+1)(d+2)/2$ linearly independent conformal Killing vectors; they transform in the adjoint (antisymmetric two-index tensor) representation of $SO(d+1,1)$. An equivalent counting of conformal Killing tensors is in terms of representations of $SO(d+1)$, where the solutions of (4.52) transform as irreps whose Young

[7] The same rectangular two-row representation appears naturally in the frame-like description of higher-spin gauge fields in AdS_{d+1} [147].

diagrams have two rows: $s-1$ boxes in the first row and any number of boxes in the second row.

We can now have a more detailed understanding of how each factor in (4.51) depends on R. Let us start with the denominator of the first factor, $\mathrm{Vol}(G)=\int Dv$. This quantity by itself is R-independent, as guaranteed by the absence of a Weyl anomaly and by the fact that we are integrating over all the modes of a rank $s-1$ traceless symmetric tensor. We can split, however, the integral over all gauge parameters into an integral over the kernel of $\mathcal{O}_g$, which is the stabilizer of the gauge orbits, and an integral over the transverse space:

$$\mathrm{Vol}(G)=\mathrm{Vol}(H)\int D'v\,, \qquad \mathrm{Vol}(H)=\int_{\mathrm{Ker}\,\mathcal{O}_g} Dv\,. \tag{4.56}$$

The discussion above implies that $g_{\mu\nu}\propto R^2$, $t_{i_1\ldots i_s}\propto R^{s-2+d/2}$, and $v_{i_1\ldots i_{s-1}}\propto R^{s-1+d/2}$. Since $\mathrm{Vol}(H)$ contains n_{s-1} integrals and each integral contributes a factor of $R^{2-1+d/2}$, we have

$$\mathrm{Vol}(H)\propto R^{n_{s-1}(s-1+d/2)}\,, \qquad \int D'v\propto R^{-n_{s-1}(s-1+d/2)}\,, \tag{4.57}$$

where the R-dependence of $\int D'v$ is such that $\mathrm{Vol}(G)$ is R-independent.[8] The number of integration variables in $\int D'v$ is therefore equal to $-n_{s-1}$ in zeta-function regularization. The R-dependence of the two other ingredients of (4.51) is

$$\int D(\mathcal{O}_g v)\propto R^{-n_{s-1}(s-2+d/2)}\,, \qquad \int Dt\, e^{-S_{\mathrm{eff}}[t]}\propto R^{n_{s-1}(s-2+d/2)}\,. \tag{4.58}$$

The first expression follows because that the number of integration variables equals $-n_{s-1}$ in zeta function regularization—for they're the same integration variables as in the $\int D'v$ integral—and because by dimensional analysis each integral contributes one fewer power

[8] The factor $\mathrm{Vol}(H)$ is also proportional to the volume of the gauge group. While for $s=1$ the gauge group is compact, an extra complication that arises when $s>1$ is that the gauge group is now non-compact and its volume is formally infinite.

of R than each of the $\int D'v$ integrals. The second expression in (4.58) is such that the R-dependence cancels when integrating over all rank-s traceless symmetric tensor modes. The number of integration variables equals $+n_{s-1}$ in zeta-function regularization, and each integral contributes a factor of $R^{s-2+d/2}$.

The dependence on N in (4.51) comes entirely from the integrand of the second factor where $K \propto N$. As a consequence of there not being a Weyl anomaly, we can write

$$Z \approx \frac{\int D(\sqrt{N}\mathcal{O}_g v)}{\int D(\sqrt{N}v)} \int D(\sqrt{N}t)\, e^{-S_{\text{eff}}[\sqrt{N}t]}\,. \tag{4.59}$$

The second factor is now N-independent, while the first factor is proportional to $\left(1/\sqrt{N}\right)^{n_{s-1}}$, simply because the denominator contains n_{s-1} more integrals than the numerator. Therefore

$$\delta F = \frac{n_{s-1}}{2} \log N + O(N^0)\,, \tag{4.60}$$

In $d = 3$, this expression reduces to (4.4). This result was obtained in the leading large N approximation where only the terms quadratic in the spin s gauge field needed to be included in the induced action. In this approximation we could simultaneously gauge the currents with spins $s_1, s_2, \ldots, s_k$. In such a theory,

$$\delta F = \frac{1}{2} \log N \sum_{i=1}^{k} n_{s_i - 1} + O(N^0)\,. \tag{4.61}$$

When non-linear effects are included in the induced gauge theory for higher-spin gauge fields, or equivalently in the dual Vasiliev theory in AdS_{d+1} space, it may be necessary to gauge all the higher-spin symmetries simultaneously [142].

4.5 The calculation in AdS: general setup

Let us consider a free massive spin s field propagating in Euclidean AdS_{d+1}, i.e. the hyperbolic space $\mathbb{H}^{d+1}$. This can be described by a totally symmetric tensor[9] $h_{\mu_1\cdots\mu_s}$ satisfying the Fierz-Pauli equations

$$\begin{aligned}&\left(\nabla^2-\kappa^2\right)h_{\mu_1\cdots\mu_s}=0\,,\\&\kappa^2=m^2-2+(s-2)(s+d-3)\,,\\&\nabla^\mu h_{\mu\mu_2\cdots\mu_s}=0\,,\qquad g^{\mu\nu}h_{\mu\nu\mu_3\cdots\mu_s}=0\,.\end{aligned}\tag{4.62}$$

The mass term in the wave equation above is defined so that m^2 correspond to the physical mass of the field,[10] while the extra spin-dependent shift arises from the coupling to the curvature of AdS (here and throughout we will set the AdS radius to one). These equations of motion and constraints may be derived from a Lagrangian, but we will not need the details of the general construction here. As a simple example, the $s=1$ case can be described by the Proca action

$$S=\int d^{d+1}x\sqrt{g}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{m^2}{2}A_\mu A^\mu\right)\,.\tag{4.63}$$

The equations of motion coming from this action, $\nabla^\mu F_{\mu\nu}=m^2A_\nu$, can be shown to be equivalent to (4.62) as long as $m^2\neq 0$. For massive fields, the equations (4.62) describe the propagation of $g(s)=\frac{(2s+d-2)(s+d-3)!}{(d-2)!s!}$ on-shell degrees of freedom.

In the massless case $m^2=0$, the spin $s\geq 1$ fields become gauge fields, with linearized gauge invariance

$$\delta h_{\mu_1\cdots\mu_s}=\nabla_{(\mu_1}\epsilon_{\mu_2\cdots\mu_s)},\tag{4.64}$$

[9] For $d=3$ a totally symmetric traceless tensor is the only possibility for a spin s field. In higher dimensions, more general mixed symmetry fields are possible, but we will not consider them in this chapter.

[10] Except for $s=0$, where in this normalization $m=0$ gives a scalar with mass-squared equal to $4-2d$. For $d=3$, this is a conformally coupled scalar field.

where the gauge parameter is a rank $s-1$ symmetric traceless tensor. The gauge invariant equations of motion and action are known [154], but we will not need their explicit form. The simple equations (4.62) may be still used to describe the propagation of on-shell degrees of freedom. In this case, however, the second line of (4.62) does not follow from the equations of motion but can be imposed as a consistent on-shell gauge condition (see e.g. [149]). Due to the usual counting of gauge symmetries, the number of propagating degrees of freedom in this case is

$$g(s) - g(s-1) = \frac{(2s+d-3)(s+d-4)!}{(d-3)!s!} \,. \tag{4.65}$$

In $d+1=4$, this number gives 2 degrees of freedom for all non-zero spins, corresponding to helicities $\pm s$. In $d+1=3$ dimensions, there are no propagating degrees of freedom for $s>1$, and one for $s=1$.

The conformal dimension of the spin s field theory operator dual to $h_{\mu_1\cdots\mu_s}$ can be obtained by studying the near-boundary behavior of a solution to the equations of motion. To be concrete, if we use Poincaré coordinates for AdS_{d+1}

$$ds^2 = \frac{dz^2 + \sum_{i=1}^{d} dx_i^2}{z^2} \,, \tag{4.66}$$

a solution to (4.62) behaves as $z \to 0$ as (see e.g. [76]) $h_{i_1\cdots i_s} \sim z^{\Delta - s}$, where Δ is a root of the equation $(\Delta + s - 2)(\Delta + 2 - d - s) = m^2$. The solutions to this equation are

$$\Delta_\pm = \frac{d}{2} \pm \nu \,, \qquad \nu = \sqrt{m^2 + \left(\frac{d}{2} + s - 2\right)^2} \,. \tag{4.67}$$

The same bulk theory describes two different CFTs depending on the boundary conditions for the field $h_{\mu_1\cdots\mu_s}$, and these CFTs are exactly the endpoints of the RG flow obtained from the action in (4.8). The boundary condition $h_{(s)} \sim z^{\Delta_- - s}$ corresponds to the UV CFT, with J_s having dimension Δ_-, and the boundary condition $h_{(s)} \sim z^{\Delta_+ - s}$ describes the IR fixed point, with J_s of dimension $\Delta \equiv \Delta_+$. In the massless case, $\Delta_+ = s + d - 2$ is the dimension of the spin s conserved current in the free theory, while $\Delta_- = 2 - s$ is the dimension of the spin s auxiliary field that becomes a dynamical gauge field in the induced theory.

The contribution of h to the free energy is given by evaluating the one-loop determinant

$$F^{(s)}_{\Delta_\pm} = -\log \int Dh\, e^{-s_h} \Big|_{\Delta_\pm} , \tag{4.68}$$

where the symbol $|_{\Delta_\pm}$ indicates which boundary conditions we are to impose at small z. Thus, the change in free energy between the UV and IR fixed points is given by

$$\delta F^{(s)}_{\Delta} = F^{(s)}_{\Delta_-} - F^{(s)}_{\Delta_+} = \frac{1}{2}\left[\mathrm{tr}^{(s)}_- \log(-\nabla^2 + \kappa^2) - \mathrm{tr}^{(s)}_+ \log(-\nabla^2 + \kappa^2)\right] , \tag{4.69}$$

where the operator $\nabla^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu$ acts on symmetric transverse-traceless (STT) tensors of rank s. Using the approach of [58, 120] and taking a derivative with respect to Δ gives the more convenient expression

$$\partial_\Delta \delta F^{(s)}_{\Delta} = (2\Delta - d)\frac{\partial \delta F^{(s)}_{\Delta}}{\partial m^2} = \frac{2\Delta - d}{2}\int \mathrm{vol}_{\mathbb{H}^{d+1}} \left(\mathrm{Tr}\, G^{(s)}_{\Delta_-}(x,x) - \mathrm{Tr}\, G^{(s)}_{\Delta_+}(x,x)\right) \tag{4.70}$$

in terms of the Green's functions $G^{(s)}_{\Delta_\pm}(x,y)$ for the spin s field with the respective boundary conditions. Here $\mathrm{Tr}\, G^{(s)}(x,x)$ denotes the Green's function at coincident points traced over the space-time indices, namely $\mathrm{Tr}\, G^{(s)}(x,x) = \lim_{y\to x} g^{\mu_1\nu_1}\cdots g^{\mu_s\nu_s} G_{\mu_1\ldots\mu_s\nu_1\ldots\nu_s}(x,y)$. Of course, the Green's function at coincident points is divergent, but the divergence is just the usual short-distance singularity of flat space propagators, which cancels when taking the difference between the two boundary conditions in (4.70) [120].

4.6 Massive spin s fields in AdS

4.6.1 Some lower spin examples

As a warm-up we begin by considering a scalar field in $d = 3$. In this case the Green's functions may be written down simply in terms of the chordal distance u.[11] Using the Poincaré coordinates (4.66), let us denote two points on AdS_4 by $x^\mu = (z, x^i)$ and $y^\mu = (w, y^i)$. Then the chordal distance is given by

$$u(x,y) \equiv \frac{(z-w)^2 + (x^i - y^i)(x^i - y^i)}{2zw} . \tag{4.71}$$

We then use the standard result for the Green's function of the massive scalar field on AdS_{d+1} (see, for example, [166]),

$$\begin{aligned} G_\Delta(x,y) = G_\Delta(u) &= \tilde{C}_\Delta (2u^{-1})^\Delta F\big(\Delta, \Delta - \frac{d}{2} + \frac{1}{2}; 2\Delta - d + 1; -2u^{-1}\big) , \\ \tilde{C}_\Delta &= \frac{\Gamma(\Delta)\Gamma(\Delta - \frac{d}{2} + \frac{1}{2})}{(4\pi)^{(d+1)/2}\Gamma(2\Delta - d + 1)} . \end{aligned} \tag{4.72}$$

Taking $d = 3$, in the short-distance limit $u \to 0$ we find

$$G_\Delta(u) = \frac{1}{8\pi^2 u} + O(\log u) , \tag{4.73}$$

and

$$G_{3-\Delta}(u) - G_\Delta(u) = \frac{1}{8\pi}(\Delta - 1)(\Delta - 2)\cot(\pi\Delta) + O(u) . \tag{4.74}$$

The only other ingredient needed to complete the computation is the regularized volume of $\mathbb{H}^4$, which is $4\pi^2/3$ (see (4.82)). Combining this fact with (4.74) and (4.70) then allows us to reproduce (4.5) with $s = 0$.

[11] The chordal distance u is related to the geodesic distance r by $u = \cosh r - 1$.

The spin 1 calculation may be carried out in an analogous fashion to the spin 0 calculation presented above. The massive bulk-to-bulk vector field propagator was worked out explicitly in [167]:[12]

$$(G^{(1)}_{\mu\nu})_\Delta(u) = -\big[G_\Delta(u) + L_\Delta(u)\big]T_{\mu\nu} - L'_\Delta(u)S_{\mu\nu}\,, \tag{4.75}$$

where $G_\Delta(u)$ is the scalar propagator defined in (4.72) and

$$\begin{aligned} L_\Delta(u) &= -\frac{1}{(\Delta-1)(\Delta-2)}\big[2G_\Delta(u) + (1+u)G'_\Delta(u)\big]\,, \\ T_{\mu\nu} &= \partial_\mu\partial_\nu u\,, \qquad S_{\mu\nu} = \partial_\mu u\partial_\nu u\,. \end{aligned} \tag{4.76}$$

Using the explicit definition of u in (4.71), we may work out that in the limit $u \to 0$ the trace $T^\mu_\mu \to -4$ while $S^\mu_\mu \to 0$. A straightforward calculation using the results above then leads to equation (4.38). One may perform an analogous computation using the massive spin 2 propagator derived in [167]. Following the same steps as above, one can evaluate the trace of the Green's function at coincident points. Taking the difference of the two boundary conditions readily allows one to reproduce the CFT result (4.42).

4.6.2 Arbitrary spin

In principle one may proceed to arbitrary spin by generalizing the method presented above for the spin 0 and 1 cases to general spin s. Thankfully, however, there is a shortcut which saves us from having to solve for the massive bulk-to-bulk propagator at arbitrary spin. Moreover, we may keep arbitrary the boundary spacetime dimension $d \geq 2$ in the following calculation without adding much complexity. We begin by considering the integer spin cases, and we comment on the generalization to half-integer spin in Section 4.8.

[12] The propagator we use, (4.75), differs by an overall minus sign compared to the one in [167]. In these conventions the propagator reduces in the flat space limit to the Fourier transform of $(g_{\mu\nu} - k_\mu k_\nu/m^2)/(k^2 + m^2)$.

Let us start by recalling the familiar definition of the heat kernel for the operator $-\nabla^2+\kappa^2$ acting on transverse symmetric traceless spin s tensors. The heat kernel $K_{\mu_1\cdots\mu_s}{}^{\nu_1\cdots\nu_s}(x,x',t)$ on $\mathbb{H}^{d+1}$ is a solution to the equations

$$\begin{aligned}\left(\frac{\partial}{\partial t}-\nabla^2+\kappa^2\right)K_{\mu_1\cdots\mu_s}{}^{\nu_1\cdots\nu_s}(x,y,t)=0\,,\\ K_{\mu_1\cdots\mu_s}{}^{\nu_1\cdots\nu_s}(x,y,0)=\delta_{(\mu_1\cdots\mu_s)}{}^{(\nu_1\cdots\nu_s)}(x,y)\,,\end{aligned} \tag{4.77}$$

where $\delta_{(\mu_1\cdots\mu_s)}{}^{(\nu_1\cdots\nu_s)}(x,x')$ is the STT δ-function on $\mathbb{H}^{d+1}$. An explicit expression for the heat kernel may be written down in terms of the STT eigenfunctions $\hat{h}^{\lambda,u}_{\mu_1\cdots\mu_s}$, which are taken to be orthonormal with respect to the standard inner product on $\mathbb{H}^{d+1}$ and which satisfy the equation

$$-\nabla^2\,\hat{h}^{\lambda,u}_{\mu_1\cdots\mu_s}(x)=\left(\lambda^2+\frac{d^2}{4}+s\right)\hat{h}^{\lambda,u}_{\mu_1\cdots\mu_s}(x) \tag{4.78}$$

as well as transversality and tracelessness. Here u is a multi-index labeling different eigenfunctions with the same eigenvalue under $-\nabla^2$, and it corresponds to the set of integers which specify the spherical harmonics on the S^d boundary. Additionally, the eigenvalue in (4.78) has been shifted in such a way that $\lambda\geq 0$. In terms of these eigenfunctions, the heat kernel may be written formally as

$$\begin{aligned}K_{\mu_1\cdots\mu_s}{}^{\nu_1\cdots\nu_s}(x,y,t)=\sum_u\int_0^\infty d\lambda\,\hat{h}^{\lambda,u}_{\mu_1\cdots\mu_s}(x)\hat{h}^{\lambda,u\;\nu_1\cdots\nu_s}(x')^*\\ \exp\left[-\left(\lambda^2+\frac{d^2}{4}+s+\kappa^2\right)t\right]\,.\end{aligned} \tag{4.79}$$

Note that using (4.62) and (4.67) we can write

$$\lambda^2+\frac{d^2}{4}+s+\kappa^2=\lambda^2+\left(\Delta-\frac{d}{2}\right)^2\,, \tag{4.80}$$

where Δ is the dimension of the dual operator. The spectral zeta function $\zeta^H(z;x)$ is defined by evaluating the trace of the heat kernel at coincident points $x=y$, inserting a factor of t^{z-1}, and integrating over t:

$$\begin{aligned}\zeta^H(z;x) &\equiv \frac{1}{\Gamma(z)}\int_0^\infty dt\, t^{z-1} K_{\mu_1\cdots\mu_s}{}^{\mu_1\cdots\mu_s}(x,x,t) \\ &= \sum_u \int_0^\infty d\lambda\, \frac{\hat{h}^{\lambda,u}_{\mu_1\cdots\mu_s}(x)\hat{h}^{\lambda,u\;\nu_1\cdots\nu_s}(x)^*}{(\lambda^2+(\Delta-d/2)^2)^z}\,.\end{aligned} \tag{4.81}$$

Since the space $\mathbb{H}^{d+1}$ is homogeneous, the zeta function does not depend on the position x. We may define the integrated zeta function $\zeta^H(z)$ to be the integral of $\zeta^H(z,x)$ over the whole space, but for the reason just given this only has the effect of multiplying the expression in (4.81) by a factor of the regularized volume of $\mathbb{H}^{d+1}$. This regularized volume may be found by writing the metric as $d\rho^2+\sinh^2\rho\; d\Omega^2_{S^d}$ and imposing a cut-off on ρ at a large value ρ_c. In even and odd dimensions this then gives [11, 58, 104]

$$\int \mathrm{vol}_{\mathbb{H}^{d+1}} = \begin{cases} \pi^{d/2}\Gamma\left(-\frac{d}{2}\right)\,, & d \text{ odd}\,, \\ \frac{2(-\pi)^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)}\log R\,, & d \text{ even}\,, \end{cases} \tag{4.82}$$

where R is the radius of S^d located at $\rho=\rho_c$.[13] Since the integral over proper time t of the heat kernel gives the Green's function, it is clear from the definition (4.81) that the spectral zeta function is related to the trace of the Green's function at coincident points by

$$\zeta^H(z=1) = \int \mathrm{vol}_{\mathbb{H}^{d+1}}\,\mathrm{Tr}\, G^{(s)}_\Delta(x,x)\,. \tag{4.83}$$

The boundary conditions for the Green's function are determined by the boundary conditions we take for the eigenfunctions $h^{\lambda,u}_{\mu_1\cdots\mu_s}(x)$. The authors of [168, 169] calculated $\zeta^H(z)$ for arbitrary spin and in arbitrary dimension d, assuming certain regularity conditions on the eigenfunctions that correspond to imposing the Δ_+ boundary condition on the Green's

[13] Only the logarithmic divergence was retained in the even d case. One may, for example, work in dimensional regularization with $d\to d-\varepsilon$, and identify the $1/\varepsilon$ pole with the $\log R$ divergence.

function. To obtain the result for the Δ_- boundary condition, we will analytically continue their final result to arbitrary Δ, as explained below.

Assuming for the moment $\Delta = \Delta_+$, the zeta function (4.81) may be written in terms of the integral over λ

$$\zeta^H(z) = \left(\frac{\int \mathrm{vol}_{\mathbb{H}^{d+1}}}{\int \mathrm{vol}_{S^d}}\right) \frac{2^{d-1}}{\pi} g(s) \int_0^\infty d\lambda \, \frac{\mu(\lambda)}{\left[\lambda^2 + \left(\Delta_+ - \frac{d}{2}\right)^2\right]^z} \,, \tag{4.84}$$

with $g(s)$ the spin factor, which in $d = 2$ is given by $g(0) = 1$ and $g(s) = 2$ for $s \geq 1$, and in $d > 2$ by

$$g(s) = \frac{(2s + d - 2)(s + d - 3)!}{(d-2)!s!} \,, \qquad d \geq 3 \,. \tag{4.85}$$

This spin factor is the number of propagating degrees of freedom of a massive spin s field in $d+1$ dimensions. In $3+1$ dimensions, these are the familiar $2s+1$ degrees of freedom of a massive spin s field.

The function $\mu(\lambda)$ is known as the spectral function, and it is obtained from (4.81) by summing over all discrete indices of the eigenfunctions. The result of [169] gives

$$\mu(\lambda) = \frac{\pi \left[\lambda^2 + \left(s + \frac{d-2}{2}\right)^2\right]}{\left(2^{d-1}\Gamma\left(\frac{d+1}{2}\right)\right)^2} \left|\frac{\Gamma\left(i\lambda + \frac{d-2}{2}\right)}{\Gamma(i\lambda)}\right|^2 \,. \tag{4.86}$$

We now turn to the evaluation of the integral in (4.84), beginning with the case of most interest, $d = 3$. The spectral function in $d = 3$ may be simplified to

$$\mu(\lambda) = \frac{\pi\lambda}{16}\left[\lambda^2 + \left(s + \frac{1}{2}\right)^2\right] \tanh \pi\lambda \,, \tag{4.87}$$

and from this we see that to evaluate $\zeta^H(z)$ we need to compute the integral

$$I_3(z) = \int_0^\infty d\lambda \, \lambda \left[\lambda^2 + \left(s + \frac{1}{2}\right)^2\right] \frac{\tanh \pi\lambda}{[\lambda^2 + \nu^2]^z} \,, \qquad \nu \equiv \Delta_+ - \frac{d}{2} \,. \tag{4.88}$$

The integral only converges for $\mathrm{Re}(z) > 2$, and so we proceed by assuming $\mathrm{Re}(z) > 2$, evaluating $I_3(z)$ explicitly, and then analytically continuing to the other values of z. One way to evaluate $I_3(z)$ is to use the identity $\tanh(\pi\lambda) = 1 - 2(1 + e^{2\pi\lambda})^{-1}$ to write

$$\begin{aligned} I_3(z) =& \frac{\nu^{2(1-z)}}{2(2-z)(1-z)} \left[\nu^2 + (z-2)\left(s + \frac{1}{2}\right)^2 \right] \\ &- 2\int_0^\infty d\lambda\, \lambda \left[\lambda^2 + \left(s + \frac{1}{2}\right)^2 \right] \frac{1}{(1 + e^{2\pi\lambda})\left[\lambda^2 + \nu^2\right]^z} . \end{aligned} \tag{4.89}$$

The integral appearing above is now perfectly convergent for all z, and it may be evaluated explicitly for specific z using, for example, the identities in [170]. The analytic continuation necessary to extract the result for $\Delta = \Delta_-$ can be done as follows. We first compute the integral (4.89) assuming $\Delta = \Delta_+$, so that $\nu \geq 0$. We then interpret the final result as an analytic function of ν (for instance, by replacing $|\nu| \to \nu$) and obtain the $\Delta_- = d - \Delta_+$ boundary condition by sending $\nu \to -\nu$.

An example of particular interest is $z = 1$, and in this case we find

$$\begin{aligned} I_3(z \approx 1) = & \left[\left(s + \frac{1}{2}\right)^2 - \nu^2 \right] \frac{1}{2(z-1)} + \left[\nu^2 - \left(s + \frac{1}{2}\right)^2 \right] \psi\left(\nu + \frac{1}{2}\right) \\ & - \frac{1}{24} - \frac{\nu^2}{2} + O(z-1) . \end{aligned} \tag{4.90}$$

Substituting the result above into (4.84), we obtain an expression for $\zeta^H(z \approx 1)$ with the Δ_+ boundary condition. The pole at $z = 1$ is just the expected short-distance singularity of the propagator, which will cancel when we compute the difference of the two boundary conditions $\zeta^H(z \approx 1) - \zeta^H_-(z \approx 1)$, where the minus subscript refers to the Δ_- boundary condition. As explained above, we find that a shortcut to obtaining $\zeta^H_-(z \approx 1)$ is to analytically continue the result in (4.90) letting $\nu \to -\nu$.[14] Then, making use of the identity

$$\psi\left(\frac{1}{2}+\nu\right)-\psi\left(\frac{1}{2}-\nu\right)=\pi\tan\nu\pi\,, \tag{4.91}$$

we obtain

$$\zeta^H(z)-\zeta^H_-(z)\big|_{z=1}=-\frac{\pi}{3}\left(s+\frac{1}{2}\right)(\Delta_+-s-2)(\Delta_++s-1)\cot\pi\Delta_+\,, \tag{4.92}$$

which, together with (4.70), immediately confirms the result for $\delta F^{(s)}_\Delta$ in (4.5).

The method used to derive (4.92) becomes more cumbersome when generalizing to arbitrary space-time dimensions. There is however a slightly more formal shortcut to evaluating (4.84) based on extending the region of integration in λ to $(-\infty,+\infty)$ and closing the contour of integration in the complex plane. One may then argue that

$$\zeta^H(z)-\zeta^H_-(z)\big|_{z=1}=2^d\left(\frac{\int \mathrm{vol}_{\mathbb{H}^{d+1}}}{\int \mathrm{vol}_{S^d}}\right)g(s)\frac{\mu\left[i\left(\Delta_+-\frac{d}{2}\right)\right]}{2\Delta_+-d}\,. \tag{4.93}$$

When d is odd we then find (even d will be discussed in section 4.9)

$$\begin{aligned}\partial_\Delta\delta F^{(s)}_\Delta=(-1)^{(d-1)/2}g(s)\frac{\Gamma\left(-\frac{d}{2}\right)}{2^d\sqrt{\pi}\Gamma\left(\frac{d+1}{2}\right)}\left(\Delta-\frac{d}{2}\right)(\Delta+s-1)(\Delta-s-d+1)\\ \Gamma(\Delta-1)\Gamma(d-1-\Delta)\cos(\pi\Delta)\,.\end{aligned} \tag{4.94}$$

Note that when $s=0$ this agrees with the result in [9, 58]. In $d=3$, it leads to the result quoted in eq. (4.5).

Moreover, we conjecture the identity

$$\zeta^H(z)-\zeta^H_-(z)\big|_{z=0}=2^d\left(\frac{\int \mathrm{vol}_{\mathbb{H}^{d+1}}}{\int \mathrm{vol}_{S^d}}\right)g(s)\,i\left(\operatorname*{Res}_{\lambda=i(\Delta_+-d/2)}\mu(\lambda)\right)\,, \tag{4.95}$$

[14]This analytic continuation becomes more subtle when $\Delta_+=s+1$, and so we treat this case separately later.

which is useful when $\Delta_+ = d+s-2$, corresponding to a conserved current at the boundary. Note that this expression vanishes in even d for all Δ_+ and vanishes in odd d when $\Delta_+ \neq d+s-2$. However, when $\Delta_+ = d+s-2$ and d is odd, we find

$$\zeta^H(z) - \zeta^H_-(z)\big|_{z=0} = -n_{s-1}\,, \tag{4.96}$$

with n_{s-1} defined in (4.53). We will explain the significance of these results in the next section.

4.7 Massless higher-spin fields in AdS and gauge symmetries

In this section we discuss directly the case of massless higher-spin fields, the corresponding gauge fixing and the bulk interpretation of the coefficient of $\log N$ associated to the Δ_- boundary conditions. As usual, in computing the one-loop partition function for a higher-spin gauge field, we must properly gauge fix the local symmetry (4.64). Using a covariant gauge fixing procedure and introducing the corresponding ghosts,[15] the end result is that the one-loop partition function in AdS_{d+1} may be written as the ratio of determinants (see for example [171–174] for the spin 2 case, and [160, 175, 176] for the generalization to arbitrary spin)

$$Z_{(s)} = \frac{\left[\det^{STT}_{s-1}\left(-\nabla^2 + (s-1)(d+s-2)\right)\right]^{\frac{1}{2}}}{\left[\det^{STT}_{s}\left(-\nabla^2 + (s-2)(d+s-3) - 2\right)\right]^{\frac{1}{2}}}\,, \tag{4.97}$$

where each determinant is computed on the space of symmetric traceless transverse tensors. The numerator corresponds essentially to the spin $s-1$ ghost contribution. The structure of the associated kinetic operator may be obtained basically by "squaring" the gauge

[15] Alternatively, one may use a procedure similar to the one discussed in Section 4 by explicitly decomposing the higher-spin gauge field into its transverse, trace and pure gauge parts.

transformation

$$
\begin{aligned}
&\int d^{d+1}x\sqrt{g}\nabla_{(\mu_1}\xi_{\mu_2\dots\mu_s)}\nabla^{(\mu_1}\xi^{\mu_2\dots\mu_s)} \\
&= \int d^{d+1}x\sqrt{g}\xi^{\mu_1\dots\mu_{s-1}}\left(-\nabla^2+(s-1)(d+s-2)\right)\xi_{\mu_1\dots\mu_{s-1}}\,,
\end{aligned}
\tag{4.98}
$$

where we have integrated by parts, restricted to transverse ξ_{s-1}, and related commutators of covariant derivatives to the curvature of AdS (we set the AdS radius to one).

Recall that we are interested in computing the ratio of the partition functions with $\Delta_+ = d+s-2$ and $\Delta_- = 2-s$ boundary conditions imposed on the physical spin s gauge fields. However, when computing the ghost determinant in (4.97), we also have in principle two choices of boundary behavior for the Green's function associated to the kinetic operator $-\nabla^2+(s-1)(d+s-2)$. Working in Poincare coordinates and using (4.67), one finds that the two boundary conditions on the spin $s-1$ transverse field with such kinetic operator are

$$
\xi_{i_1\dots i_{s-1}}(z,x_i) \sim z^{\delta_\pm}c_{i_1\dots i_{s-1}}(x_i), \qquad \delta_+ = d, \qquad \delta_- = 2-2s\,, \tag{4.99}
$$

where $i_1,\dots,i_{s-1}$ are indices along the flat d-dimensional boundary. As we now explain, the choice of $\delta_\pm$ ghost behavior is correlated with the choice $\Delta_\pm$ on the physical gauge field. To see this, we can look at the structure of the allowed gauge transformations on the spin s gauge field

$$
\delta h_{\mu_1\dots\mu_s} = \nabla_{(\mu_1}\xi_{\mu_2\dots\mu_s)}\,. \tag{4.100}
$$

The boundary behavior of the gauge field is

$$
h_{i_1\dots i_s}(z,x_i) \sim z^{\Delta_\pm - s}\alpha_{i_1\dots i_s}(x_i)\,, \qquad \Delta_+ = s+d-2\,, \qquad \Delta_- = 2-s\,. \tag{4.101}
$$

In the case of the ordinary Δ_+ boundary condition, we see that in order for the gauge transformation to preserve the boundary behavior of the spin s gauge field, we must choose in (4.99) the $\xi_{s-1} \sim z^d$ behavior for the ghost. The bulk gauge transformations then fall off fast enough at the boundary so that the bulk spin s field is dual to a gauge invariant conserved current. On the other hand, with the alternate Δ_- boundary condition, $h_{(s)}$ is dual to a gauge field at the boundary. In this case, we expect that the bulk gauge transformations should reproduce in the $z \to 0$ limit the gauge transformations in the boundary theory. From (4.99), we see that the $\delta_- = 2 - 2s$ behavior for the ghost is precisely what we need for this to happen, since in this case the spin s gauge field (4.101) and the ghost have the same scaling in the boundary limit.

In section 4.4 we explained that the coefficient of $\log N$ in the free energy can be understood as counting the numbers of missing gauge transformations, or equivalently ghost zero modes. We thus expect that an analogous interpretation should hold in the bulk. Indeed, the quadratic action for the bulk spin s fields has the schematic form

$$S \sim N \int d^{d+1}x \sqrt{g} h_{(s)} \mathcal{D}^{(s)} h_{(s)} \,, \tag{4.102}$$

where N plays the role of the (inverse of the) coupling constant. The ghost action does not carry N dependence. However, by general arguments (see e.g. [177] for a related discussion), the Gaussian path integral on the spin s field gives a coupling dependence in the partition function

$$\left(\frac{1}{\sqrt{N}}\right)^{d_s - (d_{s-1} - n_{s-1})} \,, \tag{4.103}$$

where d_s is the dimension of the space of unconstrained spin s fields, d_{s-1} the dimension of the spin $(s-1)$ gauge parameter space, and n_{s-1} the number of gauge transformations that act trivially on the gauge field. Using a regularization such that $d_s = d_{s-1} = 0$ (such as the ζ-function regularization we used in the boundary), the N dependence of the one-loop free energy will then be $F = \frac{1}{2} n_{s-1} \log N$. To prove agreement with the boundary calculation, we just have to show that we have the same number n_{s-1} of trivial gauge transformations (or ghost zero modes) in the bulk as we do in the boundary, and also, importantly, that such zero modes of the gauge transformation are only present with the Δ_- boundary condition.

The trivial bulk gauge transformations that we should count are the solutions to

$$\nabla_{(\mu_1}\xi_{\mu_2\ldots\mu_s)} = 0, \qquad \xi^{\mu}{}_{\mu\mu_3\ldots\mu_{s-1}} = 0\,; \tag{4.104}$$

namely, they are the traceless spin $s-1$ Killing tensors of the AdS background. Note that due to (4.98) these are also zero modes of the ghost kinetic operator. The traceless Killing tensors of AdS_{d+1} are expected to be in one-to-one correspondence with the conformal Killing tensors in the boundary CFT [149]. So we anticipate that solutions to (4.104) should fall into the $[s-1, s-1]$ representation of $SO(d+1,1)$, and hence we should have the same number of zero modes in the bulk and in the boundary. However, since the boundary behavior of these modes is crucial in our analysis, it is important to analyze explicitly the solutions to (4.104).

Let us first look at the simplest $s = 1$ case. Here we are just counting solutions to

$$\nabla_\mu \xi = 0\,. \tag{4.105}$$

Clearly the only solution is $\xi =$ constant over the whole AdS. If the gauge field is quantized with the Δ_+ boundary condition, then, as we have argued above, the analysis of allowed gauge transformations requires $\xi \sim z^d$ near the boundary. Therefore, as expected, this constant mode should not be counted as a trivial gauge transformation in the Δ_+ theory. On the other hand, with the Δ_- boundary condition the scalar ghost should have precisely the behavior $\xi \sim z^0$ at small z (see (4.99)), and so the constant mode solving (4.105) should indeed be intepreted as a trivial gauge transformation of the Δ_- theory. Of course, the projection of this mode to the boundary (trivially) coincides with the single constant gauge transformation on S^3, leading to $\delta F_{s=1} = 1/2 \log N + O(N^0)$.

For $s = 2$, we should look for solutions to

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0\,. \tag{4.106}$$

These are just the Killing vectors generating the isometries of AdS_{d+1}, and the solution is well known. There are $(d+1)(d+2)/2$ Killing vectors transforming in the adjoint representation of $SO(d+1,1)$. We may describe (Euclidean) AdS_{d+1} as the hyperboloid in $\mathbb{R}^{d+1,1}$

$$\eta_{AB} X^A X^B = -1 \qquad A, B = 0, 1, \ldots d+1\,, \tag{4.107}$$

where $\eta_{AB} = (-1, +1, \ldots, +1)$. Choosing an explicit parameterization $X^A(x^\mu)$, where x^μ are coordinates on AdS_{d+1}, the Killing vectors are given by

$$\xi^{AB}_\mu = X^A \partial_\mu X^B - X^B \partial_\mu X^A\,. \tag{4.108}$$

For instance, in the Poincare coordinates

$$X^A = \left(\frac{z}{2}\big[1 + \frac{1}{z^2}(1 + z^2 + x^i x^i)\big], \frac{x^i}{z}, \frac{z}{2}\big[1 + \frac{1}{z^2}(1 - z^2 - x^i x^i)\big] \right), \qquad i = 1, \ldots, d\,. \tag{4.109}$$

A simple calculation shows that the Killing vectors behave at small z as

$$\xi^{AB}_i = z^{-2} v^{AB}_i(x_i) + O(z^0), \qquad \xi^{AB}_z = z^{-1} f^{AB}(x_i)\,. \tag{4.110}$$

From (4.99) and the discussion thereafter, we conclude that these are truly zero modes of the bulk gauge transformations only when the graviton is quantized with the alternate Δ_- boundary condition. Therefore, we reproduce the result $F^{(2)}_{\Delta_-} - F^{(2)}_{\Delta_+} = 5 \log N$ in $d = 3$. As a remark, note that the boundary limit of the AdS Killing vectors yields as expected the conformal Killing vectors on the boundary, as one can explicitly check[16]

$$\lim_{z\to 0} z^2 \xi^{AB}_i = v^{AB}_i(x_i), \qquad \nabla_i v^{AB}_j + \nabla_j v^{AB}_i - \frac{2}{d} g_{ij} \nabla^k v^{AB}_k = 0\,. \tag{4.111}$$

[16] We have used Poincaré coordinates for simplicity in discussing the boundary behavior. However this result is general. For instance, using the metric $d\rho^2 + \sinh^2\rho \, d\Omega^2_{S^d}$ one can reproduce the conformal Killing vectors on S^d from the $\rho \to \infty$ limit of the AdS Killing vectors.

To proceed with the higher-spin cases, we can use the result [178] that in spaces of constant curvature (such as AdS) all Killing tensors of rank greater than or equal to two are reducible; i.e. they can be constructed from symmetrized tensor products of the Killing vectors. It is clear that when we take the tensor product of $s-1$ Killing vectors, which transform in the $[1,1]$ representation, we get a sum of irreducible representations including in particular $[s-1,s-1]$. In fact, after imposing that the resulting Killing tensor is traceless in the spacetime indices, all representations except $[s-1,s-1]$ are projected out. Let us see this more explicitly. At rank $s-1$, we construct the symmetric tensor

$$\xi_{\mu_1\dots\mu_{s-1}} = C_{A_1B_1,A_2B_2,\dots,A_{s-1}B_{s-1}} \left[\xi_{\mu_1}^{A_1B_1}\xi_{\mu_2}^{A_2B_2}\cdots\xi_{\mu_s}^{A_{s-1}B_{s-1}} + \dots\right] , \tag{4.112}$$

where the term in the square brackets is completely symmetrized in the spacetime indices, and $C_{A_1B_1,\dots,A_{s-1}B_{s-1}}$ is a constant tensor, which, by construction, is antisymmetric in each pair of indices and symmetric under exchange of any pair. It is easy to see that this solves the Killing tensor equation, and the theorem guarantees that there are no additional non-trivial solutions in AdS. To impose the tracelessness condition, we note that the Killing vectors satisfy an indentity of the form

$$g^{\mu\nu}\xi_\mu^{AB}\xi_\nu^{CD} = \frac{1}{d}\left[\eta^{AC}\xi_\mu^{EB}\xi_E^{\mu\ D} \pm 3 \text{ terms} - \frac{1}{d-1}(\eta^{AC}\eta^{BD} - \eta^{AD}\eta^{BC})\xi_\mu^{EB}\xi_{EB}^{\mu}\right] . \tag{4.113}$$

Therefore, as long as all traces are removed from the coefficient tensor $C_{A_1B_1,\dots,A_{s-1}B_{s-1}}$, we obtain a traceless Killing tensor. Finally, we note that if $C_{A_1B_1,\dots,A_{s-1}B_{s-1}}$ were totally antisymmetric in 3 or more indices, (4.112) would vanish identically. To summarize, $C_{A_1B_1,\dots,A_{s-1}B_{s-1}}$ is constrained to be antisymmetric in each pair of indices, completely traceless, and such that the antisymmetrization over any 3 indices gives zero. Indeed, this can be seen to be a realization of the $[s-1,s-1]$ representation of $SO(d+1,1)$. As a familiar example, at $s=3$ we see that $C_{A_1B_1,A_2B_2}$ is constrained to have the symmetries of the Weyl tensor (in $d+2$ dimensions), which correspond to the $[2,2]$ representation. From the explicit tensor product construction, it is clear that the boundary behavior of these traceless Killing tensors is $\xi_{i_1\ldots i_{s-1}} \sim z^{2-2s}$. From (4.99), we see that this is precisely the behavior we should impose on the ghosts when the spin s field is quantized with the Δ_- boundary condition. Therefore, we find the expected $n_{s-1} = \dim([s-1, s-1])$ "missing" gauge transformation in the Δ_- theory and reproduce from the bulk the result

$$F^{(s)}_{\Delta_-} - F^{(s)}_{\Delta_+} = \frac{1}{2} n_{s-1} \log N + O(N^0)\,. \tag{4.114}$$

To conclude this section, let us observe that it appears to be possible to reproduce the correct coefficient of $\log N$ also by some formal manipulations on the spectral ζ-function, as discussed in Section 4.6.2. Because the overall coupling in front of the bulk higher-spin action is proportional to N, the coefficient of $\frac{1}{2}\log N$ can be understood (see (4.103)) as counting the dimension of the space of the physical spin s field. Therefore, we may try to formally compute

$$\delta F^{(s)}_{\Delta} = \frac{\log N}{2}\left(\mathrm{tr}^{(s)}_{-} - \mathrm{tr}^{(s)}_{+}\right) = -\,\frac{\log N}{2}\left[\zeta^H(z) - \zeta^H_-(z)\right]\Big|_{z=0}\,. \tag{4.115}$$

From the discussion of the previous section we see that this expression vanishes unless $\Delta = d+s-2$ and d is odd. In that case, we may use the result in (4.96) to calculate $\delta F^{(s)}_{d+s-2}$, and one can see that this indeed leads to the expected result.

4.8 Comments on half-integer spins

So far our discussion has been restricted to the case where J_s is a bosonic single-trace operator of integer spin s. Of course, it is also possible to consider cases where J_s is a fermionic single-trace operator of half-odd-integer spin; the double-trace operator is still bosonic and can be added to the action. The simplest case of $s = 1/2$ in $d = 3$ has already been studied in the literature [9, 59]. In this section we briefly consider generalizations of this result to higher half-integer spin. As we have seen, the dual AdS_4 calculations tend to be simpler than the field theory calculations on S^3. In this section we list some results obtained in the bulk, leaving comparisons with the explicit field theory calculations for future work.

Following [179] we see that in the half-integer spin case the spectral function is modified to

$$\mu(\lambda) = \frac{\pi\lambda}{16}\left[\lambda^2 + \left(s+\frac{1}{2}\right)^2\right]\coth\pi\lambda\,. \tag{4.116}$$

With the operator J_s a real fermion, the change $\delta F^{(s)}_{\Delta}$ acquires an additional minus sign compared to (4.69) because of the closed fermion loop. We then find that for half-integer spin

$$\delta F^{(s)}_{\Delta} = \frac{(2\,s+1)\pi}{6}\int_{3/2}^{\Delta}\big(x-\frac{3}{2}\big)(x+s-1)(x-s-2)\tan(\pi x)\,, \tag{4.117}$$

so that for arbitrary integer or half-integer spin we have the general formula

$$\delta F^{(s)}_{\Delta} = \frac{(2\,s+1)\pi}{6}(-1)^{2s}\int_{3/2}^{\Delta}\big(x-\frac{3}{2}\big)(x+s-1)(x-s-2)\cot\big(\pi(x+s)\big)\,. \tag{4.118}$$

Note that for spin 1/2 this agrees with the result in [9, 59].

We note that for $\Delta = s+1-\epsilon$ we find a logarithmic divergence of the form

$$\delta F^{(s)} = -\frac{s(4s^2-1)}{6}\log\epsilon\,. \tag{4.119}$$

This again suggests that for $\epsilon = 0$, $\delta F^{(s)} = \frac{1}{2}n^{d=3}_{s-1}\log N$, where for $d=3$

$$n^{d=3}_{s-1} = \frac{s(4s^2-1)}{3} = \frac{(2s+1)!}{3!(2s-2)!}\,. \tag{4.120}$$

This formula is the restriction to $d = 3$ of (4.53).[17] As we have discussed, the logarithmic divergence in $\delta F^{(s)}$ appears for $s \geq 1$ and is associated with gauge transformations that act trivially on the spin s gauge field. For example, for $s = 3/2$ such gauge transformations are simply the 4 Killing spinors in AdS_4. More generally, for half-integer s, the Killing tensors transform in the $m_1 = m_2 = s - 1$ spinor representation of $SO(4,1)$. The counting of degeneracies of such representations is particularly simple because they are symmetric tensors of rank $2s - 2$ with spinor indices. Indeed, the formula (4.120) is simply the number of such tensors where each index takes 4 values. We note that this applies to integer s as well. Note also that (4.120) precisely vanishes at $s = 0$ and $s = 1/2$, which correspond to the only cases in which we do not have gauge symmetries.

4.9 Calculation of Weyl anomalies in even d

In this section we discuss an interesting application of alternate boundary conditions in AdS_{d+1}: we will show that they provide an efficient method for finding the Weyl anomaly coefficients of conformal higher-spin field theories in even dimensions d. In the $d = 4$ case such theories were introduced in [134]; an interacting conformal higher-spin theory including each spin once was proposed in [152].

For all d the alternate boundary conditions in AdS_{d+1} correspond to a theory where the dynamics of the spin s gauge field is "induced" by its coupling to the conserved current $J_{\mu_1\mu_2\ldots\mu_s}$. However, some properties of the theory depend significantly on whether d is even or odd. In odd d the induced conformally invariant action is necessarily non-local as, for example, in 3-dimensional QED. In even d we instead find a local conformally invariant term multiplied by $\log(q^2/\Lambda^2)$. Well-known examples of this in $d = 4$ include $F_{\mu\nu}F^{\mu\nu}$ for

[17] For $d > 3$ the formula (4.53) does not apply to half-integer s because that formula was calculated with $m_3 = \ldots = 0$, which does not make sense for spinors. It is plausible that we should instead consider the representations $m_1 = m_2 = s - 1$ and $m_3 = \ldots = 1/2$. For example, for $d = 5$ the dimension of the representation with $m_1 = m_2 = s - 1$ and $m_3 = 1/2$ is $\frac{(2s+3)(2s+2)(2s+1)(s-\frac{1}{2})(s+\frac{1}{2})(s+\frac{3}{2})(s+\frac{5}{2})}{3\times 5!}$. It would be interesting to check by a direct calculation that this gives the correct number of fermionic Killing tensors.

$s = 1$ and the Weyl tensor squared, $C_{\mu\nu\kappa\sigma}C^{\mu\nu\kappa\sigma}$, for $s = 2$. Their appearance is due to the structure of 2-point functions; for example,

$$\langle J_\mu(q) J_\nu(-q)\rangle \sim (q_\mu q_\nu - \delta_{\mu\nu} q^2)\log(q^2/\Lambda^2)\,. \tag{4.121}$$

The logarithmic term is due to the fact that in QED_4, the quantum effects of the charged fields lead to a logarithmic flow of the charge. Far in the IR the dynamics reduces to that of the free Maxwell field decoupled from the charged field. This is a conformal field theory, and we will show how considering a massless gauge field in AdS_5 with alternate boundary conditions gives the familiar anomaly coefficient $a_1 = 31/45$.[18] Similarly, for $s = 2$ we will obtain $a_2 = 87/5$ in agreement with the direct calculation [134, 153] in the conformal Weyl-squared gravity.[19]

First, let us calculate the change in the Weyl anomaly coefficient produced by the double-trace flows with operators $J_{\mu_1\mu_2\ldots\mu_s}J^{\mu_1\mu_2\ldots\mu_s}$, where $J_{\mu_1\mu_2\ldots\mu_s}$ is a spin s single-trace operator of dimension Δ, extending the earlier work of [57, 58, 120]. When d is even, the $\log R$ term in the free energy on S^d is identified with the anomaly a-coefficient. Using (4.93) we then find

$$\delta a^{(s)}_\Delta = -\frac{2g(s)}{\pi\, d!}\int_{\frac{d}{2}}^{\Delta} dx\left(x - \frac{d}{2}\right)(x+s-1)(x-s-d+1)\Gamma(x-1)\Gamma(d-1-x)\sin(\pi x)\,. \tag{4.122}$$

For $s = 0$ this expression agrees with the results in [57, 58, 120]. With $s = 0, \Delta = \frac{d}{2} + 1$ this formula agrees with the coefficient of the logarithmic divergence in the S^d free energy for a conformally coupled scalar field [104]. For instance, $\delta a^{(0)} = -\frac{1}{3}, \frac{1}{90}, -\frac{1}{756}, \frac{23}{113400}$ in

[18] We recall that a conventionally denotes the coefficient of the Euler density term in the Weyl anomaly. By our methods we do not have access to the c coefficient, which is the one associated with the square of the Weyl tensor.

[19] The relation to the notation for anomaly coefficients used in [153] is $a = 2\beta_2 - 4\beta_1$; see also [134].

$d = 2, 4, 6, 8$ respectively. This is because in this case the Hubbard-Stratonovich field has the dimension of a free conformal scalar.

An interesting special case is $d = 4$. Integrating (4.122) over Δ we obtain the change in the a-anomaly coefficient:

$$\delta a^{(s)} = a_{\rm UV}^{(s)} - a_{\rm IR}^{(s)} = \frac{(s+1)^2}{180}(\Delta-2)^3\left[5(1+s)^2 - 3(\Delta-2)^2\right], \tag{4.123}$$

where a is normalized such that $a = 1/90$ for a real conformal scalar field.

The higher-spin conformal gauge theories are obtained by taking $\Delta = 2 + s$ with $s \geq 1$, but in this case we must be careful to also include the contribution of the spin $s-1$ ghosts with alternate boundary conditions. Since the ghost determinant appears in the numerator of (4.97), the contribution of the ghosts to the anomaly a-coefficient of the induced theory may be computed from (4.123) with $\Delta = 3 + s$ (recall that for the spin $s-1$ ghosts we have $\Delta_\pm = \delta_\pm + s - 1$, where $\delta_\pm$ is given in (4.99)). More explicitly, defining $a_s = a_s^{\rm gauged} - a_s^{\rm ungauged}$ so that a_s is the anomaly a-coefficient for the conformal spin s field, we have

$$a_s = a_s^{\rm phys} - a_{s-1}^{\rm ghost}, \tag{4.124}$$

with $a_s^{\rm phys}$ the contribution from the physical modes and $a_{s-1}^{\rm ghost}$ that from the ghosts. We find

$$\begin{aligned} a_s^{\rm phys} &= \frac{s^3}{180}(1+s)^2\left[5 + 2\,s\,(5+s)\right], \\ a_{s-1}^{\rm ghost} &= -\frac{s^2}{180}(1+s)^3\left[3 + 2\,s\,(3-s)\right], \end{aligned} \tag{4.125}$$

which leads to the result quoted in (4.6). Using this result, we can calculate the Weyl anomaly of the 4-d conformal gauge theory including the fields of each positive integer spin once. One way to try constructing such an induced gauge theory is to start with N conformal charged scalars or fermions in $d = 4$ and gauge all the currents with $s \geq 1$. Using (4.6) and the zeta-function regularization, we find that the sum of all Weyl anomaly coefficients

$$\sum_{s=1}^{\infty} a_s = \frac{1}{90}\big[10\zeta(-3) + 21\zeta(-5)\big] = 0 \,, \tag{4.126}$$

where we have used the fact that $\zeta(-2n) = 0$ for $n > 1$. Thus, the theory with such a field content has no a-type Weyl anomaly. This provides partial evidence for the consistency of such a conformal higher-spin theory, but the c anomaly coefficient remains to be determined.

Since the a-type Weyl anomaly cancels in the conformal higher-spin theory, the leading term in the S^4 free energy of the induced theory is the $\log N$ type term that comes from (4.114). When the a-type anomaly does not cancel in an even dimensional induced gauge theory, this term is subdominant compared to the $\log R$ term. The sum over all of the $\log N$ contributions in zeta-function regularization gives

$$F = \frac{1}{2}\sum_{s=1}^{\infty} n_{s-1}\log N = \frac{\log N}{24}\big(\zeta(-2) + 4\zeta(-3) + 5\zeta(-4) + 2\zeta(-5)\big) = \frac{\log N}{945}\,, \tag{4.127}$$

where we have used (4.53) to calculate n_{s-1} in $d = 4$.

A similar calculation may be carried out in other even dimensions; for example, in $d = 2$ we find that for generic Δ the change in central charge is given by

$$c_{\text{UV}} - c_{\text{IR}} = g(s)(\Delta - 1)\big[(\Delta - 1)^2 - 3s^2\big] \tag{4.128}$$

in units where $c = 1$ for a real scalar field. When the dimension Δ equals the spin so that we are dealing with a spin s gauge theory, we may include the contribution of the ghosts to calculate $c_s = c_s^{\text{gauged}} - c_s^{\text{ungauged}}$. We find that

$$c_1 = -1\,, \qquad c_s = -2\big[1 + 6\,s\,(s-1)\big] \quad (s \geq 2)\,. \tag{4.129}$$

The central charges c_s with $s \geq 2$ agree with those in the W-gravity theories [165]; they are the central charges of the higher-spin bc ghost system with weights $(s, 1-s)$. In particular, for $s = 2$ we find the well-known result $c_2 = -26$ for the central charge of the ghost system in the 2-d gravity [164]. Thus, we have found a dual AdS_3 approach to the critical dimension of the bosonic string. We note that the result for $s = 2$ does not include the contribution of the conformal factor, the Liouville mode. This mode is frozen because in the dual AdS_3 calculation the trace of the graviton at the boundary is kept fixed to zero. Similarly, in the calculation of the Weyl anomaly for 4-d conformal gravity the conformal factor is frozen. The result $a_2 = 87/5$ of [134,153] is obtained in a "quantum Weyl gauge," where the trace of the graviton is set to zero off-shell, and so a_2 receives contributions only from the traceless gravitons and ghosts.[20]

As noted in [165], in zeta function regularization

$$\sum_{s=2}^{\infty} c_s = 2\big[1 + 6\zeta(-1) - \zeta(0)\big] = 2\,. \tag{4.130}$$

Thus, a conformal 2-d theory with $s \geq 2$ fields does not have a vanishing Weyl anomaly. However, as observed in [165], it is possible to cancel the total anomaly by adding a suitable matter sector with $c_{\text{mat}} = -2$. A well-known example is the "topological" $\eta\xi$ theory with weights $(1, 0)$; it is the $s = 1$ case of the bc ghost systems with weights $(s, 1-s)$.

[20] Of course, in the presence of a net non-zero anomaly, the conformal factor does not really decouple and becomes dynamical, as in the quantum Liouville theory [164]. But the result $a_2 = 87/5$ does not include the contribution of this trace mode.

Chapter 5

On Shape Dependence and RG Flow of Entanglement Entropy

This chapter is a lightly-modified version of the paper [15].

5.1 Introduction

The ground state entanglement entropies have emerged as a useful set of quantities for probing quantum entanglement and the degrees of freedom of many-body ground states (see [180–185] for reviews and references to earlier work). If we consider the entanglement entropy (EE) of a d-dimensional spatial region and its complement, then the leading term is typically proportional to the area of the $(d-1)$-dimensional boundary in units of the lattice spacing ϵ. The useful information is then encoded in the sub-leading terms which depend on the shape of the boundary. For example, in $(3+1)$-dimensional CFT, it has been found [186] that the expansion of the entanglement entropy (EE) for a smooth closed entangling surface Σ has the simple geometrical structure,

$$S = \alpha \frac{A_\Sigma}{\epsilon^2} + \log \epsilon \left(\frac{a}{720\pi} \int_\Sigma R_\Sigma + \frac{c}{240\pi} \int_\Sigma (k_a^{\mu\nu} k_{\nu\mu}^a - \frac{1}{2} k_a^{\mu\mu} k_{\nu\nu}^a) \right) , \tag{5.1}$$

where $k^a_{\mu\nu} = -\gamma^\rho_\mu \gamma^\sigma_\nu \nabla_\rho n^a_\sigma$ is the second fundamental form, $\gamma_{\mu\nu} = g_{\mu\nu} - n^a_\mu n^a_\nu$ is the induced metric (first fundamental form) on Σ, and R_Σ is the Ricci scalar of Σ, which equals twice its Gaussian curvature. The Weyl anomaly coefficients a and c are normalized above in such a way that $(a, c) = (n_0 + 11n_{1/2}, n_0 + 6n_{1/2})$, where n_0 and $n_{1/2}$ are the numbers of real scalar and Dirac fields, respectively.

In $(2 + 1)$-dimensional CFT the structure of the entanglement entropy for a smooth contour Σ in a plane is

$$S = \alpha \frac{\ell_\Sigma}{\epsilon} - F \ , \tag{5.2}$$

where ℓ_Σ is the length of the contour. There is no known expression for F in terms of the curvature of the boundary. However, if the boundary contains a cusp of length $r_{\max}$ and opening angle Ω, then S contains an additional singular term $-f_{\rm cusp}(\Omega) \log(r_{\max}/\epsilon)$ [187–189]. In both field theoretic [187] and holographic [188, 189] calculations, $f_{\rm cusp}(\Omega)$ turns out to be a smooth convex function that interpolates monotonically between $\sim 1/\Omega$ behavior at small angles and zero at $\Omega = \pi$. However, the details of the function are not universal—the holographic, free scalar, and free fermion calculations produce different functions $f_{\rm cusp}(\Omega)$.

In this chapter we present new results on the shape dependence of entanglement entropy in two and three spatial dimensions, studying both smooth and singular boundary geometries. Many of our calculations rely on the geometrical approach to the calculation of entanglement entropy [190–193] based on the gauge/gravity duality [24–26], but we also present some purely field theoretic arguments. In three spatial dimensions we will consider EE for a conical entangling surface with opening angle Ω and show that the calculation in AdS_5 produces a term $\sim \frac{\cos^2\Omega}{\sin\Omega} \log^2(r_{\max}/\epsilon)$. For a wedge of length L and opening angle Ω, we will show that EE contains a divergent term $\sim f_{\rm wedge}(\Omega)L/\epsilon$. Surprisingly, we find $f_{\rm wedge}(\Omega) = f_{\rm cusp}(\Omega)$ both in the free scalar field theory and in the holographic calculations.

We also present new calculations of the renormalized entanglement entropy. In [45, 194], some holographic calculations of the renormalized entanglement entropy $\mathcal{F}(R)$ were presented. Similarly, we will calculate $\mathcal{F}(R)$ for the smooth Cvetič-Gibbons-Lu-Pope (CGLP) solution [195] of 11-dimensional supergravity, which is a warped product of $\mathbb{R}^{2,1}$ and 8-dimensional Stenzel space [196], $\sum_{i=1}^{5} z_i^2 = \epsilon^2$. This smooth warped throat is similar to the KS background of type IIB string theory [197]. The CGLP background describes RG flow from the CFT$_3$ dual to $AdS_4 \times V_{5,2}$ in the UV (for its different field theoretic descriptions, see [113, 198]), to a gapped theory in the IR. The masses of some of the bound states in the CGLP theory were calculated in [199]. Using the holographic approach to the entanglement entropy, we will confirm that $\mathcal{F}(R)$ for the CGLP background is a monotonic function that approaches zero as $\sim 1/R$ for large R. This function exhibits an interesting second-order phase transition at a special value of R, where the bulk surface reaches the bottom of the throat and its topology changes. Transitions of this type have been observed in earlier holographic calculations [45, 200] (see also [192, 193, 201–203]).

Generally, for theories with a mass gap of order m, the large R expansion of the disk entanglement entropy is expected to have the form [183, 204–206]

$$S(R) = \alpha \frac{2\pi R}{\epsilon} + \beta\, m\, (2\pi R) - \gamma + 2\pi \sum_{n=0}^{\infty} \frac{c_{-1-2n}}{(mR)^{2n+1}}\,, \tag{5.3}$$

where γ is the topological entangelement entropy [207, 208] (in the simple cases we will consider, γ vanishes). Following [183, 204, 206] we will show that the terms $\sim (mR)^{-2n-1}$ are related to the anomaly terms in $2n+3$ spatial dimensions.

In order to gain better insight into the structure of entanglement entropy for gapped theories, we will generalize from a circle of radius R to an arbitrary smooth contour Σ. In this case, the general structure of the EE in gapped $(2+1)$-dimensional theories is

$$S_\Sigma = \alpha \frac{\ell_\Sigma}{\epsilon} + \beta\, m\, \ell_\Sigma - \gamma + \sum_{n=0}^{\infty} \frac{\tilde{c}^{\Sigma}_{-1-2n}}{m^{2n+1}}\,, \tag{5.4}$$

where the coefficients $\tilde{c}^{\Sigma}_{-1-2n}$ are integrals of functions of the extrinsic curvature and its derivatives [205]. The expansion of S_Σ has only odd power of $1/m$ because on dimensional grounds these terms are multiplied by even powers of the extrinsic curvature and its derivatives. Since in any pure state, and in particular in the vacuum, the EE of a region is equal to that of its complement, we have the symmetry $\kappa \to -\kappa$ [205]. Generalizing the arguments of [183, 204, 206], we give a prescription for calculating the $\tilde{c}^{\Sigma}_{-1-2n}$ for massive free scalar and Dirac fields. Using a holographic description of large N theories with a mass gap, we calculate the coefficients $\tilde{c}^{\Sigma}_{-1}$ and $\tilde{c}^{\Sigma}_{-3}$ explicitly. We check the infrared expansion for the specific case of CGLP background.

5.2 The $(2+1)$-dimensional entanglement entropy in free massive theories

In this section we show how to calculate the $1/m$ expansion of the entanglement entropy for massive free scalar and fermion fields in $(2+1)$-dimensions. We will take the entangling surface to be a smooth, closed curve Σ_1 of length ℓ_{Σ_1} and extrinsic curvature κ in the $t = 0$ slice of flat $\mathbb{R}^{2,1}$.

More generally, one could consider the case where the $(2+1)$-dimensional spacetime is described by a general manifold $\mathcal{M}$. Using the replica trick one is then able to show that the entanglement entropy has the large mass expansion of the form given in (5.4) with $\beta = -(n_0 + n_{1/2})/12$ (see, for example, [184, 209, 210]). The integers n_0 and $n_{1/2}$ denote the numbers of real scalar and Dirac fields, respectively, in $(2+1)$-dimensions. The coefficients $\tilde{c}^{\Sigma_1}_{-1-2n}$ are known explicitly in the case where Σ_1 has vanishing extrinsic curvature [184, 209, 210]. We will henceforth take $\mathcal{M} = \mathbb{R}^{2,1}$ and allow the surface Σ_1, which is taken to lie in the $t = 0$ plane, to have a non-trivial extrinsic curvature. We want to determine the coefficients $\tilde{c}^{\Sigma_1}_{-1-2n}$ in terms of integrals of functions of the extrinsic curvature and its derivatives. Our approach to the computation follows that of Casini and Huerta [183, 204, 206], who showed how to compute the coefficients $\tilde{c}^{\Sigma_1}_{-1} = c_{-1}/R$ in the special case where Σ_1 is a circle of radius R.

The calculation proceeds by considering a higher, even dimensional QFT consisting of free fields in $\mathbb{R}^{2,1} \times T^k$, with $k \geq 1$ odd and T^k the symmetric k-torus of a large volume

$\mathrm{Vol}(T^k) = L^k$. In the following argument one can replace T^k by an arbitrary scalable k-dimensional smooth manifold. We give the free fields a small mass M, which will act as an infrared regulator for the conformal anomaly. We want to calculate the entanglement entropy in this theory across the $(1+k)$-dimensional surface $\Sigma_{1+k} = \Sigma_1 \times T^k$, which fills the k-torus and is described by the smooth curve Σ_1 in the $t = 0$ plane of $\mathbb{R}^{2,1}$. We may Fourier decompose the field modes in the compact directions to obtain an infinite tower of massive $(2+1)$-dimensional fields, with masses

$$m^2_{n_1,\cdots,n_k} = M^2 + \left(\frac{2\pi}{L}\right)^2 \sum_{i=1}^{k} n_i^2 \,, \qquad n_i \in \mathbb{Z} \,. \tag{5.5}$$

The entanglement entropy in the $(2+k+1)$-dimensional theory then becomes equal to the sum over $(2+1)$-dimensional entanglement entropies for massive fields across the curve Σ_1. Taking the large L limit, the spectrum of masses becomes continuous and we find

$$S^{(2+k+1)}_{\Sigma_{1+k}}(M) = \frac{k\,\mathrm{Vol}(T^k)}{2^k \pi^{k/2} \Gamma(\frac{k}{2}+1)} \int_0^{1/\epsilon} dp\, p^{k-1} S^{(2+1)}_{\Sigma_1}(\sqrt{M^2+p^2}) \,, \tag{5.6}$$

where ϵ is the UV cut-off. We now substitute the expansion of $S^{(2+1)}_{\Sigma_1}(m)$ given in (5.4) into (5.6). We see that the term in the expansion of $S^{(2+1)}_{\Sigma_1}(m)$ which goes as $1/m^k$ determines the logarithmic conformal anomaly term in $S^{(2+k+1)}_{\Sigma_{1+k}}(M)$. Turning this argument around, suppose the entropy of the $(2n+4)$-dimensional theory has the anomaly term

$$S^{(2n+4)}_{\Sigma_{2n+2}}(M)\Big|_{\log} = s^{(2n+4)}_{\Sigma_{2n+2}} \log(M\epsilon) \,, \tag{5.7}$$

then we can immediately read off the coefficient $\tilde{c}^{\Sigma_1}_{-1-2n}$:

$$\tilde{c}^{\Sigma_1}_{-1-2n} = -\frac{\pi(2\pi)^n (2n-1)!!}{\mathrm{Vol}(T^{2n+1})} s^{(2n+4)}_{\Sigma_{2n+2}} \,. \tag{5.8}$$

The above formula is slightly modified for fermions. Dirac fermions in $(2n+4)$ dimensions are in a 2^{n+2}-dimensional representation, which after dimensional reduction reduces to 2^{n+1} $(2+1)$-dimensional Dirac fermions. Thus, the right hand side of (5.8) should be divided by 2^{n+1} for Dirac fermions. As a corollary to this argument, we see that the absence of the $\log\epsilon$ terms in odd dimensional CFTs implies that the IR expansion of $S^{(2+1)}_{\Sigma_1}(m)$ contains only odd powers of $1/m$, in agreement with the arguments in [205].

Let's see how this works explicitly when $n=0$. The expression for $s^{(3+1)}_{\Sigma_2}$ is given in (5.1). The Euler number $\chi(\Sigma_2)$ vanishes for $\Sigma_2=\Sigma_1\times S^1$. The two normal vectors to Σ_2 are within $\mathbb{R}^{2,1}$, which we write with coordinates

$$ds^2_{(2+1)} = -dt^2 + dr^2 + r^2 d\theta^2\,, \tag{5.9}$$

where θ has period 2π. One of the normal vectors to Σ_2 is timelike, $n^1_\mu=(1,0,0,0)$, where the fourth component is in the direction of the S^1 of length L, and its second fundamental form vanishes. Suppose Σ_1 is defined by a curve $r=R(\theta)$. Then, the other normal vector is spacelike, $n^2_\mu=(0,r,-rR'(\theta),0)/\sqrt{r^2+R'^2(\theta)}$, and this gives a second fundamental form with non-vanishing component

$$k^{2\,\theta}_{\theta} = \frac{R^2(\theta)+2R'^2(\theta)-R(\theta)R''(\theta)}{(R^2(\theta)+(R'(\theta))^2)^{3/2}} \equiv \kappa(\theta)\ , \tag{5.10}$$

where $\kappa(\theta)$ is the extrinsic curvature of the surface Σ_1 in the $\mathbb{R}^2$ plane. It follows that both $k^a_{\mu\nu}k^{\mu\nu}_a$ and $k^{\mu\mu}_a k^a_{\nu\nu}$ in (5.1) become $\kappa^2(\theta)$. This leads to

$$\tilde{c}^{\Sigma_1}_{-1} = -\frac{1}{480}(n_0+3n_{1/2})\oint ds\,\kappa^2 \tag{5.11}$$

and

$$S^{(2+1)}_{\Sigma_1}(m) = \alpha\frac{\ell_{\Sigma_1}}{\epsilon} - \frac{m(n_0+n_{1/2})\ell_{\Sigma 1}}{12} - \frac{n_0+3n_{1/2}}{480\,m}\oint ds\,\kappa^2 + O(1/m^3)\,, \tag{5.12}$$

where we stress that $n_{1/2}$ the number of $(2+1)$-dimensional Dirac fermions.

In principle the calculation of the higher order corrections to the entanglement entropy in powers of $1/m$ would proceed analogously. For example, to calculate the coefficient $\tilde{c}_{-3}^{\Sigma_1}$, which gives the order $1/m^3$ correction to the entropy, we would need to first calculate $s_{\Sigma_4}^{(5+1)}$, with $\Sigma_4 = \Sigma_1 \times T^3$. We then expect

$$s_{\Sigma_4}^{(5+1)} = \mathrm{Vol}(T^3)\left[(A_0\, n_0 + A_{1/2}\, n_{1/2}^{(6)}) \oint ds\, \kappa^4 + (B_0\, n_0 + B_{1/2}\, n_{1/2}^{(6)}) \oint ds \left(\frac{d\kappa}{ds}\right)^2\right] \,, \quad (5.13)$$

for some coefficients $(A_0, A_{1/2})$ and $(B_0, B_{1/2})$, which should be functions of the 6-dimensional anomaly coefficients. We use the notation $n_{1/2}^{(6)}$ to stress that this counts the number of $(5+1)$-dimensional Dirac fermions. This then leads to

$$\tilde{c}_{-3}^{\Sigma_1} = -2\pi^2\left[\left(A_0\, n_0 + \frac{A_{1/2}}{4}\, n_{1/2}\right) \oint ds\, \kappa^4 + \left(B_0\, n_0 + \frac{B_{1/2}}{4}\, n_{1/2}\right) \oint ds \left(\frac{d\kappa}{ds}\right)^2\right] \,. \quad (5.14)$$

This formula is consistent with the general arguments in [205].

5.3 Holographic computation of the $(2+1)$-dimensional entanglement entropy in gapped backgrounds

The (renormalized) entanglement entropy may be calculated holographically by following the usual procedure for holographic entanglement entropy [190–193]. Consider a $(d+1)$-dimensional large N field theory with a D-dimensional gravitational dual. While we will ultimately be interested in $(2+1)$-dimensional QFT, for now we keep the dimension d general. As in [193], let the gravitational background have the Einstein-frame metric

$$ds_D^2 = \alpha(u)[du^2 + \beta(u)dx^\mu dx_\mu] + g_{ij}dy^i dy^j \,, \qquad dx^\mu dx_\mu = -dt^2 + dr^2 + r^2 d\Omega_{d-1}^2 \,, \quad (5.15)$$

with $\alpha(u) > 0$ and $\beta(u) > 0$ and $i, j = d+3, \cdots, D$. The compact, internal $(D-d-2)$-dimensional manifold is taken to have a volume

$$V(u) \equiv \int \prod_{i=d+3}^{D} dy^i \sqrt{\det g}\,, \tag{5.16}$$

which is a function of the holographic radial coordinate u. We assume that u has a minimal value u_0 where a p-sphere in the internal manifold shrinks to zero size, resulting in $V(u_0) = 0$. At u_0 we assume that all supergravity fields are regular, which implies $\alpha(u_0)$ and $\beta(u_0)$ are finite. The coordinate u ranges from infinity in the far UV to u_0 in the far IR. Such geometries typically describe confining gauge theories.

We further assume that the gravitational theory approaches a conformal fixed point in the UV ($u = \infty$), and we work in coordinates where

$$\lim_{u\to\infty} \alpha(u) = \alpha_{\mathrm{UV}}\,, \qquad \lim_{u\to\infty} V(u) = V_{\mathrm{UV}}\,, \qquad \beta(u) = \exp\left(\frac{2u\sqrt{\alpha_{\mathrm{UV}}}}{L_{\mathrm{UV}}}\right) + \ldots\,, \tag{5.17}$$

where α_{UV} and V_{UV} are constants, and L_{UV} is the radius of AdS_{d+2}.

We want to calculate the entanglement entropy in the QFT across a codimension two spacelike surface Σ_{d-1}. The entanglement entropy [190–193] is calculated holographically by finding the $(D-2)$-dimensional surface Σ_{D-2}, which approaches Σ_{d-1} at the boundary of the bulk manifold, is extended in the rest of the spatial dimensions, and minimizes the area functional

$$S_\Sigma = \frac{1}{4G_N^{(D)}} \int_{\Sigma_{D-2}} d^{D-2}\sigma \sqrt{G_{\mathrm{ind}}^{(D-2)}}\,, \tag{5.18}$$

where $G_{\mathrm{ind}}^{(D-2)}$ is the induced metric on Σ_{D-2}. The entanglement entropy is then given by the functional S_Σ evaluated at the extremum.

A case of particular interest is when the region Σ_{d-1} is the $(d-1)$-sphere of radius R. Writing the radial coordinate r as a function of the holographic coordinate u, the induced metric on Σ_{D-2} is

$$ds_\Sigma^2 = \alpha(u)[(1+\beta(u)(\partial_u r)^2)du^2 + \beta(u)r^2(u)d\Omega_{d-1}^2] + g_{ij}dy^i dy^j \,, \tag{5.19}$$

which gives the following expression for the area functional in terms of the unknown function $r(u)$:

$$\begin{aligned} S(R) &= \frac{\mathrm{Vol}(S^{d-1})}{4G_N^{(D)}} \int_{u_0}^{\infty} du\, r^{d-1}(u) g(u) \sqrt{1+\beta(u)(\partial_u r)^2} \,, \\ g(u) &= \alpha^{d/2}(u)\beta^{(d-1)/2}(u) V(u) \,. \end{aligned} \tag{5.20}$$

In general we need to first solve the Euler-Lagrange equation,

$$(d-1)r^{d-2}(u)g(u)\sqrt{1+\beta(u)(\partial_u r)^2} = \frac{d}{du}\left[\frac{r^{d-1}(u)g(u)\beta(u)(\partial_u r)}{\sqrt{1+\beta(u)(\partial_u r)^2}}\right] \,, \tag{5.21}$$

for the function $r(u)$, then evaluate the area functional in (5.20) on the solution with a UV cut-off $u < u_{\mathrm{UV}}$, then use (1.6) to construct the finite renormalized entanglement entropy. For non-trivial backgrounds this must be done numerically. To solve the equation of motion (5.21), we also need to specify the boundary conditions. There are two types of solutions with different topologies.

One of them, which we will call the cylinder-type solution, terminates at $u = u_0$ where the volume of the internal space becomes zero: $V(u_0) = 0$. One can find the form of the solutions $r(u)$ for u near u_0 by expanding (5.21) around $u = u_0$:

$$r(u) = r_0 + \frac{d-1}{4r_0\beta(u_0)}(u-u_0)^2 + O((u-u_0)^3) \,, \qquad r_0 > 0 \,. \tag{5.22}$$

The other type of solution, which we call the disk-type solution, has a tip at $u = u_{\min} > u_0$, where the radius of the sphere becomes zero: $r(u_{\min}) = 0$. For u near $u_{\min}$, the solutions behave like

$$r(u) = 2\sqrt{\frac{dg(u_{\min})}{2\beta(u_{\min})g'(u_{\min}) + g(u_{\min})\beta'(u_{\min})}}(u-u_{\min})^{1/2} + O((u-u_{\min})^{3/2})\,. \tag{5.23}$$

5.3.1 IR behavior of the EE for a circle

We may obtain the IR asymptotic behavior of the entanglement entropy for $\Sigma_1 = S^1$ through an analytic procedure, and in doing so we show that the renormalized entanglement entropy approaches zero in the IR from above like $1/R$, where R is the radius of the S^1. Note that in this section we restrict to the physical dimension $d = 2$. In the following section we generalize the computation by allowing for a general entangling surface Σ_1.

For now, we take $\Sigma_1 = S^1$ of radius R. We assume that at large R the solutions to the Euler-Lagrange equations will be of the form $r(u) = R + \delta(u)/R$, with $\delta(u)$ independent of R. Expanding the Euler-Lagrange equation in powers of $1/R$, we find the equation

$$\frac{d}{du}[g(u)\beta(u)\delta'(u)] = g(u)\,, \tag{5.24}$$

which may be integrated to obtain

$$\delta(u) = -\int_u^\infty du' \frac{1}{g(u')\beta(u')}\int_{u_0}^{u'} du''\, g(u'')\,. \tag{5.25}$$

Expanding the area functional in (5.20) and using the equation of motion in (5.24) gives

$$\begin{aligned} S(R) = \frac{2\pi}{4G_N^{(D)}}\Bigg[&R\frac{V_{\mathrm{UV}}L_{\mathrm{UV}}}{\epsilon} + R\left(\int_{u_0}^{u_\infty} du g(u) - \frac{V_{\mathrm{UV}}L_{\mathrm{UV}}}{\epsilon}\right) \\ &-\frac{1}{2R}\int_{u_0}^\infty du\, g(u)\beta(u)[\delta'(u)]^2 + O(R^{-3})\Bigg]\,, \end{aligned} \tag{5.26}$$

where we used the boundary conditions $\delta(u_\infty) = 0$ and $\delta'(u_0) = 0$ for the solution, which make the surface term vanish. The UV cut-off ϵ defined by

$$\frac{1}{\epsilon^2} = \alpha_{\rm UV} \exp\left(\frac{2u_\infty\sqrt{\alpha_{\rm UV}}}{L_{\rm UV}}\right) . \tag{5.27}$$

To compare with (5.4), we set the mass m to unity and use the dimensionless radius R for convenience. We then see that we can make the identifications

$$\begin{aligned} \alpha &= \frac{V_{\rm UV}L_{\rm UV}}{4G_N^{(D)}} , \qquad \beta = \frac{1}{4G_N^{(D)}}\left(\int_{u_0}^{u_\infty} du g(u) - \frac{V_{\rm UV}L_{\rm UV}}{\epsilon}\right) , \\ \tilde{c}_{-1}^{\Sigma} &= \frac{-1}{8G_N^{(D)}}\int_{u_0}^{\infty}\frac{du}{g(u)\beta(u)}\left(\int_{u_0}^{\infty} du' g(u')\right)^2 \oint ds\, \kappa^2 . \end{aligned} \tag{5.28}$$

Notice that the coefficient β is finite and independent of the UV cut-off ϵ. To calculate the coefficients $\tilde{c}_{-3}^{\Sigma}$ we must consider a more general entangling surface. This is because $d\kappa/ds = 0$ for the circle. In the following section we generalize the above calculation to allow for a general, smooth entangling surface, and in doing so we calculate $\tilde{c}_{-3}^{\Sigma}$.

5.3.2 IR behavior of the EE for a general entangling surface

We would like to repeat the calculation in the previous section allowing for a general spacelike entangling surface Σ_1. While we believe that the computation can be carried out in full generality, it is enough to restrict ourselves to a closed curve Σ_1 that is a boundary of a star-shaped domain.[1] Such a curve can be parameterized using polar coordinates by a function $R_\Lambda(\theta)$. We write the entangling surface as $R_\Lambda(\theta) = \Lambda R(\theta)$, with $R(\theta)$ a smooth function and $\Lambda \geq 1$. The IR limit corresponds to Λ large enough such that the extrinsic curvature is small, $\kappa_\Lambda(\theta) = \Lambda^{-1}\kappa(\theta) \ll 1$, along the entire curve.

[1] A star-shaped domain is a set $S \subset \mathbb{R}^n$ with the property that there exists a point $x_0 \in S$ such that the line segments joining x_0 to all other points in S are contained in S.

The induced metric on the bulk surface Σ_{D-2} is now

$$ds^2_\Sigma = \alpha(u)\left[(1+\beta(u)(\partial_u r)^2)du^2 + \beta(u)(r^2(u,\theta) + \left(\partial_\theta r\right)^2\right)d\theta^2 \\ + 2\beta(u)(\partial_\theta r)(\partial_u r)d\theta\, du] + g_{ij}dy^i dy^j \,, \tag{5.29}$$

where the radial coordinate $r(u,\theta)$ is taken to be a function of the holographic coordinate u and the angular coordinate θ. We require that $\lim_{u\to\infty} r(u,\theta) = R_\Lambda(\theta)$. The area functional for the entanglement entropy may be written as

$$S_\Sigma = \frac{1}{4G_N^{(D)}}\int_0^{2\pi} d\theta \int_{u_0}^{\infty} du\, g(u)\sqrt{(r^2(u,\theta)+(\partial_\theta r)^2)\left(1+\beta(u)(\partial_u r)^2\right) - \beta(u)(\partial_\theta r)^2(\partial_u r)^2}\,, \tag{5.30}$$

with $g(u)$ and $\beta(u)$ defined as before.

We assume that in the IR the solutions to the Euler-Lagrange equation give

$$r(u,\theta) = \Lambda R(\theta) + \frac{\delta(u,\theta)}{\Lambda} + O(1/\Lambda^3)\,, \tag{5.31}$$

with $\delta(u,\theta)$ order Λ^0. We Substitute the ansatz in (5.31) into the area functional in (5.30) and expand in inverse powers of Λ up to and including terms of order $1/\Lambda$:

$$S_\Sigma = \frac{1}{4G_N^{(D)}}\int_0^{2\pi} d\theta \int_{u_0}^{\infty} du\, \Big[\Lambda g(u)\sqrt{R(\theta)^2 + R'(\theta)^2} \\ + \frac{g(u)R(\theta)}{\Lambda}\left(\frac{R(\theta)\beta(u)}{\sqrt{R^2(\theta)+R'(\theta)^2}}(\partial_u\delta)^2 + 2\kappa(\theta)\delta(u,\theta)\right)\Big] + O(1/\Lambda^3)\,, \tag{5.32}$$

where the extrinsic curvature of the entangling surface, $\kappa(\theta)$, is given explicitly in (5.10). Applying the variational principle to find the Euler-Lagrange equation for $\delta(u,\theta)$ gives

$$\frac{d}{du}[g(u)\beta(u)\partial_u\delta(u,\theta)] = \frac{\kappa(\theta)\sqrt{R(\theta)^2+R'(\theta)^2}}{R(\theta)}g(u)\,, \tag{5.33}$$

which may be integrated to give

$$\delta(u,\theta) = -\frac{\kappa(\theta)\sqrt{R(\theta)^2 + R'(\theta)^2}}{R(\theta)} \int_u^\infty du' \frac{1}{g(u')\beta(u')} \int_{u_0}^{u'} du''\, g(u'')\,. \tag{5.34}$$

We want to calculate the terms in the expansion of S_Σ of order $1/\Lambda^3$. These terms are completely determined by the expansion of $r(u,\theta)$ in (5.31) through order $1/\Lambda$. Expanding the area function in (5.30) through order $1/\Lambda^3$ and evaluating on the solution for $\delta(u,\theta)$ given in (5.34) allows us to determine the $\tilde{c}^\Sigma_{-3}$ coefficients in (5.4):

$$\tilde{c}^\Sigma_{-3} = a^{(1)}_{-3}\left(\frac{1}{2}\oint ds\,\kappa^4 - \oint ds\left(\frac{d\kappa}{ds}\right)^2\right) + a^{(2)}_{-3}\oint ds\,\kappa^4\,, \tag{5.35}$$

with

$$\begin{aligned} a^{(1)}_{-3} &= -\frac{1}{4G_N^{(D)}}\int_{u_0}^\infty \frac{du}{g(u)\beta(u)}\left[\int_{u_0}^u du' g(u')\right]^2 \int_u^\infty \frac{du'}{g(u')\beta(u')}\int_{u_0}^{u'} du''\, g(u'')\,, \\ a^{(2)}_{-3} &= -\frac{1}{32G_N^{(D)}}\left(\oint ds\,\kappa^4\right)\int_{u_0}^\infty \frac{du}{g(u)^3\beta(u)^2}\left(\int_{u_0}^u du'\, g(u')\right)^4 . \end{aligned} \tag{5.36}$$

5.4 An example: CGLP background of M-theory

The CGLP background [195] of M-theory is the gravitational dual of a gapped (2 + 1)-dimensional field theory, which nicely illustrates the general features discussed in the previous section. The supergravity background is a warped product of $\mathbb{R}^{2,1}$ and an eight-dimensional Stenzel space [196]

$$\sum_{i=1}^5 z_i^2 = \varepsilon^2\,, \tag{5.37}$$

where ε is a real deformation parameter. When $\varepsilon = 0$ this equation describes an eight-dimensional cone whose base is the Stiefel manifold $V_{5,2}$.

As explained in [195, 211], the Stenzel space (5.37) can be parameterized by a radial coordinate τ ranging from 0 to ∞ and the seven angles in $V_{5,2}$. At $\tau = 0$ a 3-sphere shrinks to zero size, and the $\tau = 0$ section is a round S^4.

The 11-dimensional metric is of the form of the metric in (5.15) if we identify the holographic coordinate u with τ, where $\tau_0 = 0$, and[2]

$$\begin{aligned} \alpha(\tau) &= \frac{H^{1/3}(\tau)c^2(\tau)}{4}\,, \qquad \beta(\tau) = \frac{4}{c^2(\tau)H(\tau)}\,, \\ V(\tau) &= \frac{9}{2}3^{1/8}\pi^4 H^{7/6}(\tau)(2+\cosh\tau)^{3/8}\sinh^{3/2}\left(\frac{\tau}{2}\right)\sinh^{3/2}(\tau)\,. \end{aligned} \tag{5.38}$$

The functions $H(\tau)$ and $c(\tau)$ are defined by

$$H(\tau) = \frac{(2\pi\ell_P)^6 N}{81\pi^4}2^{3/2}3^{11/4}\int_{(2+\cosh\tau)^{1/4}}^{\infty}\frac{dt}{(t^4-1)^{5/2}}\,, \qquad c^2(\tau) = \frac{3^{7/4}}{2}\frac{\cosh^3\frac{\tau}{2}}{(2+\cosh\tau)^{3/4}}\,, \tag{5.39}$$

where N is the number of units of asymptotic G_4 flux. In particular, notice that $V(\tau = 0) = 0$, which is a result of the vanishing 3-sphere. For more details on the CGLP background see, for example, [195, 211].

We begin by studying the entanglement entropy and renormalized entanglement entropy in the simpler case where the entangling region Σ_1 is taken to be a circle of radius R. In this case the Euler-Lagrange equation for the function $r(\tau)$ in (5.21) may be solved numerically with the boundary condition $r(\tau = \infty) = R$. In practice, we cut the space off at some large $\tau_{\rm UV}$. For each $R > 0$ there exists a value $\tau_{\rm min}(R)$, which is the smallest value of τ for which the function $r(\tau)$ is defined. There exists a critical value $R_{\rm crit} \approx .73$ for which $r(\tau_{\rm min} = 0) = 0$. For $R < R_{\rm crit}$ the solutions to the equation of motion describe surfaces of disk type that behave as in (5.23) for τ near $\tau_{\rm min}$. The topology of these surfaces is that of a disk times $V_{5,2}$. The solutions for $R > R_{\rm crit}$ are surfaces of cylindrical type that stretch to

[2]We follow the conventions of [211] and work in units where $\varepsilon = 1$.

the bottom of the Stenzel space and behave as in (5.22) for τ near $\tau_0 = 0$. The topology of these surfaces is that of a circle times the Stenzel space.

In Figure 5.1 (a) we plot the numerical solutions to the equation of motion for a range

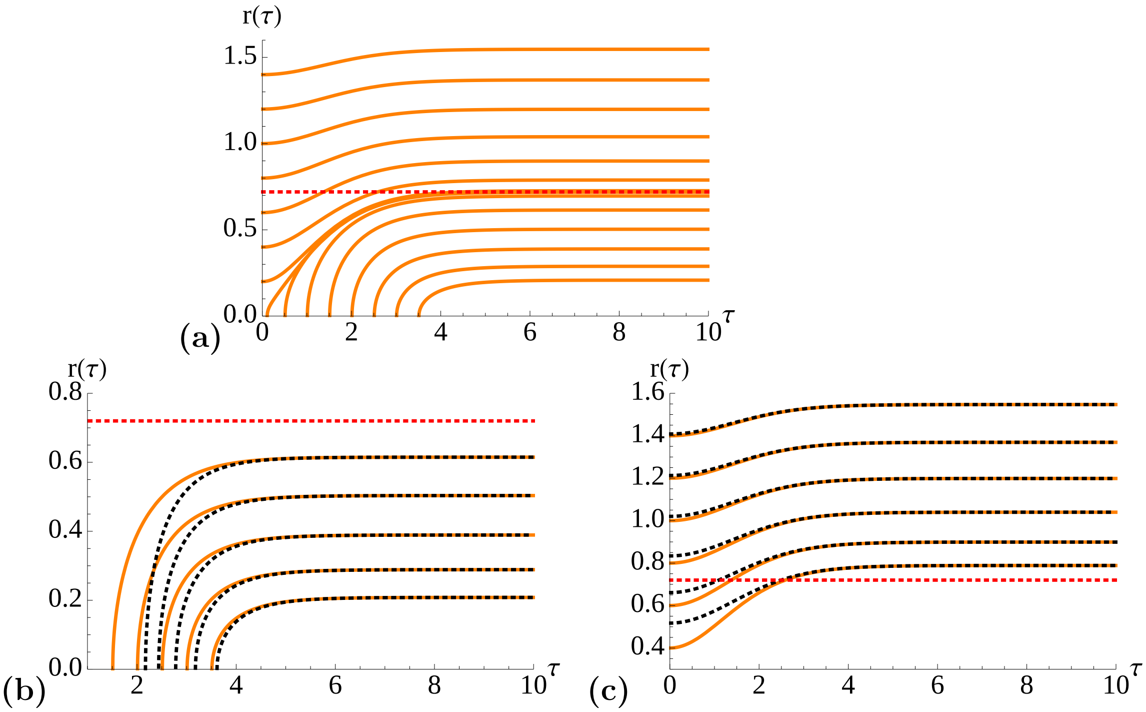


Figure 5.1: **(a)** Numerical solutions to the equation of motion for the holographic entangling surface, given by $r(\tau)$, in the CGLP theory. The dotted red line indicates the critical value R_{crit}, where the solutions change from disk-type to cylinder-type. **(b)** A zoomed-in plot of the UV region, with disk-type solutions, where we plot the AdS approximation in (5.40) in dotted black. **(c)** A zoomed-in plot of the IR region, with cylinder-type solutions, with the analytic approximation given by $\delta(\tau)$ in (5.25) plotted in dotted black.

of $R < R_{\text{crit}}$ and $R > R_{\text{crit}}$. In the far UV the solution for $r(\tau)$ should approach the AdS solution

$$r(\tau) = \sqrt{R^2 - 2^{5/2}3^{1/2}e^{-3\tau/2}}\,. \tag{5.40}$$

In Figure 5.1 (b) we zoom in on some of the disk-type solutions in the far UV and plot the solutions along with the asymptotic in (5.40). In the far IR region the cylinder-type solutions should be well approximated by the function $\delta(\tau)$ in (5.25). In Figure 5.1 (c) we plot some of the cylinder-type solutions along with the analytic approximation.

As was discussed in section 5.3, to calculate the renormalized entanglement entropy it is sufficient to evaluate the entanglement entropy with a strict UV cut-off. We cut off the space at a large τ value τ_{UV}. We then numerically integrate the area functional and differentiate it to construct $\mathcal{F}$. A plot of the renormalized entanglement entropy along the RG flow is given in Figure 5.2.

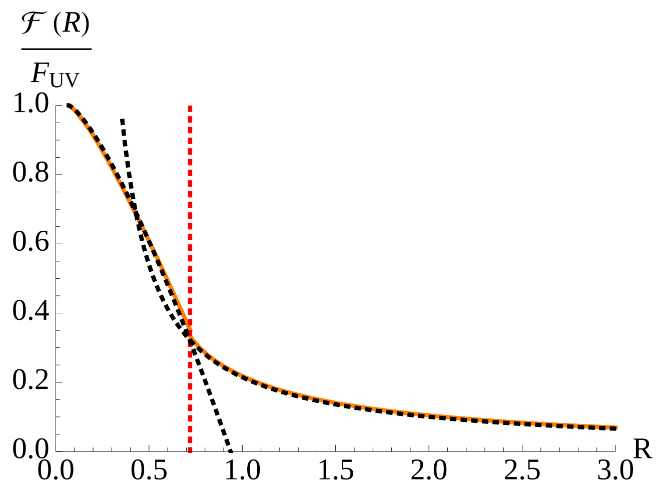


Figure 5.2: The renormalized entanglement entropy $\mathcal{F}(R)$ along the RG flow in the CGLP theory plotted in orange. The left dotted black curve is the asymptotic UV approximation to $\mathcal{F}(R)$ given in (5.43). The right dotted black curve is the IR approximation to $\mathcal{F}(R)$ given in (5.44). The dotted red line marks the value $R = R_{\text{crit}}$.

5.4.1 The renormalized entanglement entropy in the UV and the IR

In the far UV we can treat the CGLP M-theory background as a perturbation of the $AdS_4 \times V_{5,2}$ background. From (6.23), we know that the UV fixed point has a renormalized entanglement entropy

$$F_{\text{UV}} = \frac{16\pi N^{3/2}}{27} + O(N^{1/2}) \,, \tag{5.41}$$

where we have used $\mathrm{Vol}(V_{5,2}) = 27\pi^4/128$. To describe the RG flow in the vicinity of the UV fixed point, it is convenient to use the effective 4-dimensional metric in the form of (6.17). A straightforward calculation shows that at small y the function $f(y)$ has the expansion

$$f(y) = 1 + 2^{1/3}3^{2/3}y^{4/3} + \cdots , \tag{5.42}$$

which, using (6.28), implies the RG flow is driven by an operator in the UV field theory of dimension $\Delta = 7/3$, which is consistent with [211]. Using (6.30), we then see that at small R

$$\mathcal{F}(R) = \frac{16\pi N^{3/2}}{27}\left(1 - \frac{3}{7}2^{1/3}3^{2/3}R^{4/3} + \cdots\right) . \tag{5.43}$$

This function is plotted together with the numerical solution in Figure 5.2. Note that it is a very good approximation to the actual renormalized entanglement entropy for $R < R_{\mathrm{crit}}$. We also see explicitly that $\partial_R\mathcal{F} = 0$ at $R = 0$.

In the IR we may use (5.28) and (5.35) to get an asymptotic expression for the renormalized entanglement entropy, which gives

$$\mathcal{F} \approx \frac{16\pi N^{3/2}}{27}\left(\frac{0.1959}{R} + \frac{1.845\times 10^{-2}}{R^3} + O(1/R^5)\right) . \tag{5.44}$$

This function is plotted in Figure 5.2, which shows that it is a good approximation to the actual renormalized entanglement entropy at large R.

5.4.2 Tests of the shape dependence of the entanglement entropy

In this section we will consider a more general spacelike entangling surface Σ_1, specified by the function $R(\theta)$ in polar coordinates. We want to check (5.34), which gives an approximation to the cylinder-type solutions in the far IR. In particular, this equation claims that the variation of the bulk entangling surface Σ_2 away from the straight cylinder is proportional to the combination $\kappa(\theta)(\sqrt{R(\theta)^2 + R'(\theta)^2}/R(\theta))$.

As an example, we consider the entangling surface Σ_1 plotted in Figure 5.3, which has a small extrinsic curvature along the entire curve. A good way of measuring the accuracy of

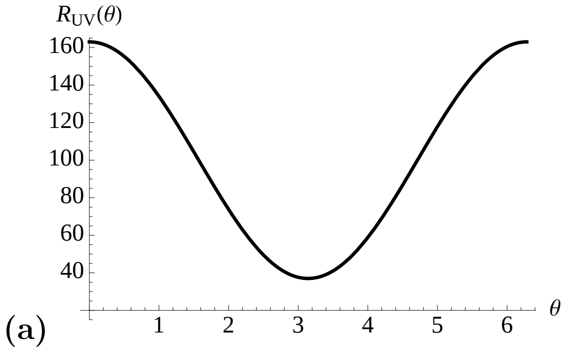


Figure 5.3: **(a)** A plot of an example entangling surface Σ_1, described by the function $R_{\mathrm{UV}}(\theta)$ in polar coordinates, in the CGLP theory. **(b)** The extrinsic curvature $\kappa_{\mathrm{UV}}(\theta)$ for the entangling surface $R_{\mathrm{UV}}(\theta)$. The extrinsic curvature is small over the whole curve.

the analytic approximation in (5.34) is to compare the function $R_{\mathrm{UV}}(\theta) - R_{\mathrm{IR}}(\theta)$ as computed

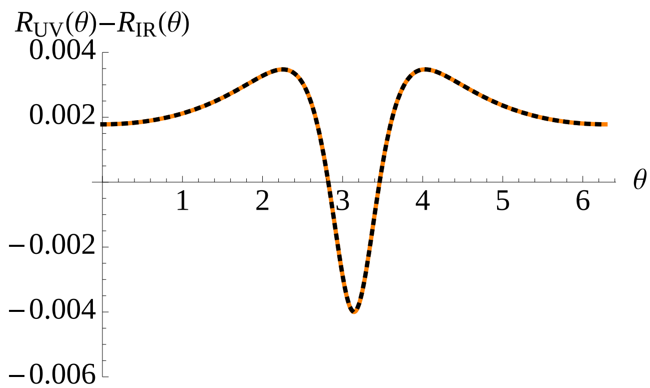


Figure 5.4: The function $R_{\mathrm{UV}}(\theta) - R_{\mathrm{IR}}(\theta)$ in the CGLP theory, with $R_{\mathrm{UV}}(\theta)$ plotted in fig. 5.3. The solid orange curve is computed by numerically solving the equation of motion for the holographic entangling surface Σ_2. The dotted black curve is an analytic approximation, which is equal to $-\delta(\tau = 0, \theta)$, with $\delta(0, \theta)$ given in (5.34).

both numerically and using (5.34), which is done in Figure 5.4. Here the function $R_{\mathrm{IR}}(\theta)$ is the profile of the cylinder when it reaches $\tau = 0$: $R_{\mathrm{IR}}(\theta) = r(\tau = 0, \theta)$. The analytic approximation simply gives $R_{\rm UV}(\theta) - R_{\rm IR}(\theta) \approx -\delta(\tau = 0, \theta)$, with $\delta(0, \theta)$ given in (5.34). The two curves match extremely well.

5.5 Shape dependence of the entanglement entropy in $(3+1)$-dimensional CFT

The entanglement entropy for a smooth entangling surface Σ in a $(3+1)$-dimensional CFT is given in (5.1). However, if the surface is not smooth, for example if it has conical or wedge singularities, then there may be additional contributions to (5.1). In this section we consider the entanglement entropy for a wedge and a cone in a $(3+1)$-dimensional CFT through both field theoretical and holographic computations. We find that wedge entanglement entropy acquires a $1/\epsilon$ divergence not present in (5.1). The cone entanglement entropy has a $\log^2 \epsilon$ divergence as predicted by (5.1), but its correct coefficient is twice smaller than for a regulated version of (5.1).

5.5.1 The entanglement entropy for a wedge

The wedge is the surface in $(3+1)$-dimensions given by $(x^1, x^2, x^3) = (r \sin\phi, r\cos\phi, z)$, where $0 \le r < \infty$, $\phi = 0$ and Ω, and $z \sim z + L$. We have compactified the z direction on a large circle of length L to avoid unwanted infrared divergences. We begin by using the replica trick to calculate the entanglement entropy with this geometry for a free scalar field, and this is followed by a holographic computation.

The free scalar field

Using the replica trick one can show that the entanglement entropy for the massive scalar field is given by [212]

$$S = \frac{1}{1-\alpha}\sum_{k=0}^{\alpha-1} \log Z_k\Big|_{\alpha\to 1} \;, \tag{5.45}$$

where Z_k is the partition function of a scalar field ϕ_k on 4-dimensional Euclidean space with boundary conditions

$$\phi_k(\vec{x}, t=0^+) = e^{2\pi i \frac{k}{\alpha}} \phi_k(\vec{x}, t=0^-) \;, \qquad \vec{x} \in A \;, \tag{5.46}$$

where A is the region bounded by the wedge. Since the theory we consider is free, the partition function Z_k is obtained from the integral of the Green's function:

$$\frac{\partial}{\partial m^2} \log Z_k = -\frac{1}{2}\int d^{d+1}X \, G_k(\vec{X}, \vec{X}) \;. \tag{5.47}$$

The Green's function is the two-point correlation function of the free massive scalar and is subject to the following conditions:

$$\begin{aligned} (-\Delta_{\vec{X}} + m^2)\, G_k(\vec{X}, \vec{X}') &= \delta(\vec{X} - \vec{X}') \;, \\ \lim_{\epsilon\to 0^+} G_k((\vec{x},\epsilon), \vec{X}') &= e^{2\pi i \frac{k}{\alpha}} \lim_{\epsilon \to 0^-} G_k((\vec{X},\epsilon), \vec{X}') \;, \qquad \vec{x} \in A \;. \end{aligned} \tag{5.48}$$

We expand the Green's function in Fourier modes along the z-direction. The problem of finding the 4-dimensional Green's function then reduces to that of finding the 3-dimensional Green's functions for a cusp entangling surface for a tower of massive fields, with masses

$$M_n^2 = m^2 + \left(\frac{2\pi n}{L}\right)^2 \;, \qquad n \in \mathbb{Z} \,. \tag{5.49}$$

Using the result in [187] for the Green's function with a cusp entangling surface in 3-dimensions, we find

$$G_k(\vec{X}, \vec{X}') = \frac{2}{L} \sum_{\nu} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\lambda \frac{\lambda}{\lambda^2 + M_n^2} g_n(\vec{x}) g_n^*(\vec{x}) \,, \tag{5.50}$$

where we use $\vec{x}$ to denote the 3-dimensional coordinates (t, x^1, x^2). The g_n are the eigenfunctions of the 3-dimensional Laplace operator $(-\Delta_{\vec{x}} + M_n^2)$, whose eigenvalues we denote by $(\lambda^2 + M_n^2)$. Using spherical coordinates with $(t, x_1, x_2) = (\rho\cos\theta, \rho\sin\theta\sin\phi, \rho\sin\theta\cos\phi)$, the eigenfunctions are given explicitly by

$$g_n(\vec{x}) = \psi_\nu(\theta, \phi) \frac{J_{\frac{1}{2}+\nu}(\lambda\rho)}{\sqrt{\rho}} \,, \tag{5.51}$$

where J is the Bessel function of the first kind. The functions ψ_ν are the eigenfunctions of the angular laplacian Δ_Ω on the two-sphere,

$$\Delta_\Omega \psi_\nu(\theta, \phi) = -\nu(\nu+1)\psi_\nu(\theta, \phi) \,, \qquad \int d\theta\, d\phi\, \sin(\theta) |\psi_\nu(\theta, \phi)|^2 = 1 \,, \tag{5.52}$$

subject to the boundary condition

$$\lim_{\epsilon \to 0^+} \psi_\nu\left(\frac{\pi}{2} + \epsilon, \phi\right) = e^{2\pi i \frac{k}{\alpha}} \lim_{\epsilon \to 0^+} \psi_\nu\left(\frac{\pi}{2} - \epsilon, \phi\right) \,, \qquad \phi \in [0, \Omega] \,. \tag{5.53}$$

Preforming the integral over λ in (5.50) and taking a derivative of the partition function in (5.47) with respect to m^2 gives [213]

$$\frac{\partial}{\partial m^2} \log Z_k = -\frac{L}{4m} \coth\left(\frac{mL}{2}\right) \sum_{\nu} \left(\nu + \frac{1}{2}\right) \,. \tag{5.54}$$

The sum over the eigenvalues ν is divergent and needs regularization. The computation of the sum was carried out in detail in [187], and one finds that the regularized sum only depends on k and the angle of the cusp Ω. After integrating (5.54) with respect to m^2, we find that the entanglement entropy for a wedge has the angle dependent UV divergence

$$S_{\text{wedge}} = \int^{1/\epsilon^2} dm^2 \frac{1}{\alpha - 1} \sum_{k=0}^{\alpha-1} \left(\frac{\partial}{\partial m^2} \log Z_k \right) \Big|_{\alpha\to 1} = f^{(\text{scalar})}_{\text{cusp}}(\Omega) \frac{L}{\epsilon} + O(\epsilon^0) \ , \tag{5.55}$$

where the function $f^{(\text{scalar})}_{\text{cusp}}(\Omega)$ is the same function as for the cusp in $(2+1)$-dimensional CFT [187]. It behaves as $f^{(\text{scalar})}_{\text{cusp}}(\Omega) \sim 1/\Omega$ when the angle is very small, while it becomes zero at $\Omega = \pi$ where there is no cusp in the entangling surface. The function $f_{\text{cusp}}(\Omega)$ is not universal and depends on the type of matter.

A holographic computation

Next we compute the holographic entanglement entropy for the wedge. To this end, we use the following AdS_5 metric,

$$ds^2 = \frac{dy^2 - dt^2 + dr^2 + r^2 d\phi^2 + dz^2}{y^2} \ . \tag{5.56}$$

For simplicity, we have set the AdS radius to 1. The central charges a and c of the dual CFT_4, normalized as in (5.1), are then determined by the 5-dimensional Newton constant [214].

$$a = 3\,c = \frac{45\pi}{G_N^{(5)}} \ . \tag{5.57}$$

The wedge is defined by $\Sigma = \{0 \le r < r_{\max},\ \phi = \pm\frac{\Omega}{2},\ z \sim z + L\}$ at the AdS boundary $y = 0$. The large radius cut-off $r_{\max}$ and the length L are introduced to regularize the volume of the wedge. As usual, to introduce a UV cut-off we will restrict $y \ge \epsilon$. The entanglement entropy functional is given by

$$S = \frac{L}{4G_N^{(5)}} \int dr \int d\phi \, \frac{1}{y^3(r,\phi)} \sqrt{r^2 + r^2(\partial_r y)^2 + (\partial_\phi y)^2} \ , \tag{5.58}$$

where we take the holographic coordinate y to be a function of (r, ϕ).

We must find the function $y(r, \phi)$ which minimizes the entanglement entropy functional and approaches the wedge at the boundary of AdS_5. The scaling symmetry of the spacetime and the wedge implies the following ansatz for the minimal surface [188, 189]:

$$y(r, \phi) = \frac{r}{g(\phi)} \; . \tag{5.59}$$

With this ansatz the initial value problem becomes first order[3]

$$\frac{dg}{d\phi} = g\sqrt{(1+g^2)\left(\frac{g^2(1+g^2)^2}{g_0^2(1+g_0^2)^2} - 1\right)} \, , \qquad g_0 = g(0) \, , \quad g'(0) = 0 \, . \tag{5.60}$$

It follows that the angle of the wedge determines g_0 as

$$\frac{\Omega}{2} = \int_{g_0}^{\infty} dg \frac{1}{g\sqrt{(1+g^2)\left(\frac{g^2(1+g^2)^2}{g_0^2(1+g_0^2)^2} - 1\right)}} \; . \tag{5.61}$$

Integrating this equation we find that, as in the (2+1)-dimensional cusp calculation [188,189], the limiting value where $g_0 = 0$ is $\Omega = \pi$.

The entanglement entropy is then found by evaluating the regularized functional in (5.58) on the solution to the equation of motion:

$$\begin{aligned} S &= \frac{2L}{4G_N^{(5)}} \int_{g_0 \epsilon}^{r_{\max}} \frac{dr}{r^2} \int_{g_0}^{r/\epsilon} dg \, h(g, g_0) \\ &= \frac{2L}{4G_N^{(5)} \epsilon} \int_{g_0}^{r_{\max}/\epsilon} \frac{dr}{r^2} \left[\frac{r^2 - g_0^2}{2} + \int_{g_0}^{r} dg \left(h(g, g_0) - g \right) \right] \\ &= \frac{1}{4G_N^{(5)}} \left[\frac{A_\Sigma}{2\epsilon^2} - f_{\text{wedge}}^{(\text{hol})}(\Omega) \frac{L}{\epsilon} + O(\epsilon^0) \right] \; . \end{aligned} \tag{5.62}$$

[3]Note that one must first find the equation of motion for $y(r, \phi)$ and then subsequently substitute the ansatz in (5.59).

The functions $h(g, g_0)$ and $f^{(\text{hol})}_{\text{wedge}}(\Omega)$ are defined by

$$
\begin{aligned}
h(g, g_0) &= \frac{g^2(1+g^2)}{\sqrt{g^2(1+g^2)^2 - g_0^2(1+g_0^2)^2}} \ , \\
f^{(\text{hol})}_{\text{wedge}}(\Omega) &= g_0 - \int_{g_0}^{\infty} \frac{dr}{r^2} \int_{g_0}^{r} dg \left(h(g, g_0) - g \right) \ .
\end{aligned}
\tag{5.63}
$$

Just like the free scalar field calculation for the wedge, (5.55), the holographic result has an L/ϵ divergence that was absent for smooth entangling surfaces.[4] A numerical plot of the function $f^{(\text{hol})}_{\text{wedge}}(\Omega)$ is shown in Figure 5.5, where it can be seen that it goes to zero as Ω approaches π. When Ω is small $f^{(\text{hol})}_{\text{wedge}}(\Omega)$ diverges as

$$
f^{(\text{hol})}_{\text{wedge}}(\Omega) \sim \frac{0.646}{\Omega} \ . \tag{5.64}
$$

A pole at $\Omega = 0$ also appeared in the field theory computation (5.55). With that said, the function $f^{(\text{hol})}_{\text{wedge}}(\Omega)$ is different from that of the scalar field theory, $f^{(\text{scalar})}_{\text{cusp}}(\Omega)$, which appeared in (5.55).

A surprising result, however, is that after an overall rescaling the function $f^{(\text{hol})}_{\text{wedge}}(\Omega)$ agrees with function $f^{(\text{hol})}_{\text{cusp}}(\Omega)$ in [189] describing the holographic cusp anomaly in $(2+1)$-dimensions. The normalization of the function $f_{\text{wedge}}(\Omega)$ depends on the choice of the UV cut-off scale ϵ. We introduce the normalized function $\tilde{f}^{(\text{hol})}_{\text{wedge}}(\Omega) = a f^{(\text{hol})}_{\text{wedge}}(\Omega)$ by tuning the constant a such that $\tilde{f}^{(\text{hol})}_{\text{wedge}}(\Omega)$ agrees with $f^{(\text{hol})}_{\text{cusp}}(\Omega)$ in the limit of $\Omega \to 0$. We find $a \sim 1.11$ numerically, and in Figure 5.5 we plot both $\tilde{f}^{(\text{hol})}_{\text{wedge}}(\Omega)$ and $f^{(\text{hol})}_{\text{cusp}}(\Omega)$. The plot shows that in fact the normalized function $\tilde{f}^{(\text{hol})}_{\text{wedge}}(\Omega)$ is the same (within the numerical accuracy) as $f^{(\text{hol})}_{\text{cusp}}(\Omega)$, although the definitions (5.61) and (5.63) appear quite different from those for the cusp in [189]. For a free scalar field the function $f^{(\text{scalar})}_{\text{wedge}}(\Omega)$ for the wedge also turned out to be the same as $f^{(\text{scalar})}_{\text{cusp}}(\Omega)$ for the cusp (see (5.55)). It is very interesting that the

[4] Since on the sides of the wedge the extrinsic curvature vanishes, it is reasonable to think of this term in the entanglement entropy as due to the wedge singularity.

appropriately normalized functions $f(\Omega)$ agree for the cusp and wedge geometries both in the free and in the strongly coupled theories that we have studied.

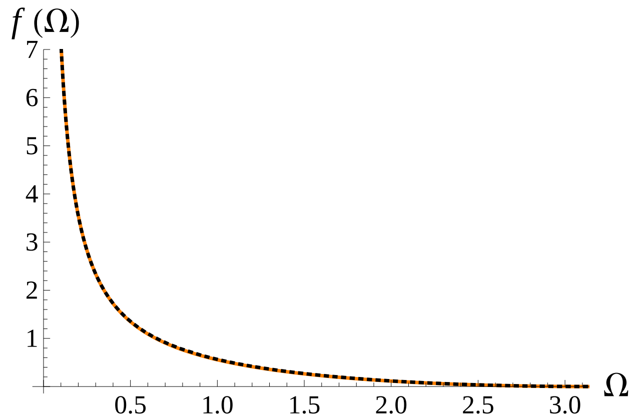


Figure 5.5: The entanglement entropy for the wedge has a $1/\epsilon$ divergent term, whose coefficient depends on the angle of the wedge. This coefficient $f^{(\text{hol})}_{\text{wedge}}(\Omega)$ is given explicitly in (5.61) and (5.63), and its normalization depends on the UV cut-off ϵ. The black dotted line is the normalized function $\tilde{f}^{(\text{hol})}_{\text{wedge}}(\Omega) = a f^{(\text{hol})}_{\text{wedge}}(\Omega)$ for the wedge with $a \sim 1.11$ and the orange line is the function $f^{(\text{hol})}_{\text{cusp}}(\Omega)$ for the cusp in $(2+1)$-dimensions.

5.5.2 The entanglement entropy for a cone

In this section we show that when the entangling surface in $(3+1)$-dimensional CFT has a conical singularity, the entanglement entropy acquires a $\log^2(r_{\text{max}}/\epsilon)$ divergence. We take the entangling surface to be the cone defined by $(r, \theta, \phi) = (r, \Omega, \phi)$, where $0 \leq r < r_{\text{max}}$, $\phi \sim \phi + 2\pi$ is the azimuthal angle, and Ω is the opening angle of the cone. The large distance cut-off r_{max} regulates the area of the cone.

To begin, we will evaluate (5.1) for this surface. Even though this equation is only valid for smooth entangling surfaces, it does provide a quick way of seeing how the $\log^2 \epsilon$ divergence appears. The cone has two normal vectors in $\mathbb{R}^{1,3}$, given by $n^1_\mu = (1,0,0,0)$ and $n^2_\mu = (0,0,r,0)$ in (t, r, θ, ϕ) coordinates. Only the second fundamental form associated with n^2_μ is non-vanishing, with non-zero component $k^2_{\phi\phi} = \frac{1}{2} r \sin 2\Omega$. The c-anomaly term in (5.1) then contributes

$$
\begin{aligned}
&\frac{c}{480\pi}\log\epsilon\int_{r_0}^{r_{\max}}dr\int_0^{2\pi}d\phi\, r\sin\theta(k^2_{\phi\phi})^2(g^{\phi\phi})^2\\
&=-\frac{c}{240}\frac{\cos^2\Omega}{\sin\Omega}\log^2\epsilon+\cdots\ .
\end{aligned}
\tag{5.65}
$$

In going from the first to the second line in (5.65), we assume that the tip of the cone is cut-off at some short distance $r_0 \propto \epsilon$. Note that the a-anomaly does not give an additional contribution to the singularity because $\int R_\Sigma = 0$. The new UV divergent term (5.65) vanishes at $\Omega = \frac{\pi}{2}$, where there is no conical singularity, while its coefficient diverges as Ω goes to zero.

A more heuristic way to obtain the $\log^2\epsilon$ term, similar to an argument for the cusp geometry in [187] is as follows. When Ω is small, the cone may be approximately decomposed into a union of cylinders with radius $R = L\sin\Omega$ and the length ΔL, where L is the length from the apex of the cone to one of the cylinders and ΔL is supposed to be small. The logarithmic term of the entanglement entropy of the cylinder comes from the c-anomaly given in (5.1)

$$
S_{\text{cylinder}} = \frac{c}{240}\frac{\Delta L}{R}\log(R/\epsilon)\ . \tag{5.66}
$$

It follows that the entanglement entropy of the cone has the square of the logarithmic divergence

$$
S_{\text{cone}} \approx \int_\epsilon^{r_{\max}} dL\,\frac{c}{240}\frac{1}{L\sin\Omega}\log(L\sin\Omega/\epsilon) = -\frac{c}{480}\frac{1}{\sin\Omega}\log^2(r_{\max}/\epsilon)+\cdots\ , \tag{5.67}
$$

which reproduces the leading behavior of (5.65) in the small Ω limit, except it is smaller by a factor of 2. We will see below that the factor in (5.67) is correct.

We now present a more precise holographic derivation of the $\log^2 \epsilon$ divergence, which correctly takes into account the conical singularity. We use the AdS_5 metric

$$ds^2 = \frac{dy^2 - dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi)}{y^2}\,, \tag{5.68}$$

so that the entangling surface is given by the cone $\Sigma = \{0 \leq r < \infty,\, \theta = \Omega,\, 0 \leq \phi < 2\pi\}$ at the AdS boundary $y = 0$. Taking the holographic coordinate y to be a function of (r, θ), we use the conformal and rotational symmetries of AdS spacetime and the entangling surface to restrict the ansatz to

$$y(r,\theta) = \frac{r}{\tilde{g}(\theta)}\ . \tag{5.69}$$

The entanglement entropy functional is then given by

$$\begin{aligned} S &= \frac{2\pi}{4G_N^{(5)}} \int \frac{dr}{r} \int d\theta \sin\theta\, \tilde{g}\sqrt{\tilde{g}^4 + \tilde{g}^2 + (\tilde{g}')^2} \\ &= -\frac{\pi}{2G_N^{(5)}} \int \frac{dr}{r} \int ds\, g\sqrt{g^4 + g^2 + (1-s^2)(g')^2} \\ &= -\frac{\pi}{2G_N^{(5)}} \int \frac{dr}{r} \int dg\, g\sqrt{(g^4+g^2)s'(g)^2 + 1 - s(g)^2}\ , \end{aligned} \tag{5.70}$$

where we introduced the new variable $s = \cos\theta$, which runs from $s_0 \equiv \cos\Omega$ to unity, and redefined $g(s) = \tilde{g}(\theta(s))$. In the last equality, we changed the integration variable from s to g.

The entanglement entropy is given by evaluating the EE functional in the last line of (5.70) on the function $s(g)$ which solves the Euler-Lagrange equation,

$$\frac{g\, s(g)}{\sqrt{(g^4+g^2)s'(g)^2 + 1 - s(g)^2}} + \frac{d}{dg}\left[\frac{g(g^4+g^2)s'(g)}{\sqrt{(g^4+g^2)s'(g)^2 + 1 - s(g)^2}}\right] = 0\,, \tag{5.71}$$

subject to the boundary condition $s(g = r/\epsilon) = s_0$, where the UV cutoff is put at $y = \epsilon$ in the AdS spacetime. However, to find the leading divergences in the entanglement entropy, it only is necessary to know the function $s(g)$ near the boundary at $g = r/\epsilon$, which we may assume is a large number. Taking the large g limit of (5.71), one may verify that the asymptotic expansion for $s(g)$ near the boundary is

$$s(g) = s_0 + \frac{s_0}{4g^2} + O\left(\frac{\log g}{g^4}\right) . \tag{5.72}$$

While the solution above only satisfies the boundary conditions up to a term of order $(\epsilon/r)^2$, the difference does not affect the leading two singular terms in the EE. In evaluating the entanglement entropy functional in (5.70) on the solution $s(g)$ in (5.72), we must evaluate the integral

$$\int_{r/\epsilon}^{g_0} dg \left[\sqrt{1-s_0^2}\, g - \frac{s_0^2}{8\sqrt{1-s_0^2}}\frac{1}{g} + O\left(\frac{\log g}{g^3}\right)\right] = -\frac{\sin\Omega}{2}\frac{r^2}{\epsilon^2} + \frac{\cos^2\Omega}{8\sin\Omega}\log(r/\epsilon) + O(\epsilon^0) , \tag{5.73}$$

where g_0 is the minimum value of $g(s)$. Then, preforming the r integral from $r = g_0\epsilon$ to $r_{\max}$, we obtain the entanglement entropy for the cone:

$$S_{\text{cone}} = \frac{1}{4G_N^{(5)}}\left[\frac{A_\Sigma}{2\epsilon^2} - \frac{\pi\cos^2\Omega}{8\sin\Omega}\log^2(r_{\max}/\epsilon) + \dots\right] . \tag{5.74}$$

In order to compare this result with the naive calculation in (5.65), we use (5.57). One then sees that (5.74) is smaller than (5.65) by a factor of 2. As stressed above, the approach of (5.1) is not precise for singular entangling surfaces. It is nice, therefore, that it is only a factor of 2 off from the precise holographic result (5.74).

Chapter 6

Is Renormalized Entanglement Entropy Stationary at RG Fixed Points?

This chapter is a lightly-modified version of the paper [16].

6.1 Introduction

Zamolodchikov used the two-point functions of the stress-energy tensor to define a monotonic c-function in 2 spacetime dimensions [3]. At RG fixed points the Zamolodchikov c-function equals the Weyl anomaly coefficient c. An important property of the Zamolodchikov c-function is that it is stationary at the fixed points. For example, if we consider perturbing a CFT by a slightly relevant operator $\mathcal{O}$ of dimension $2-\delta$ then

$$c(g) = c_{\mathrm{UV}} - g^2\delta + O(g^3) \ , \tag{6.1}$$

where g is the renormalized dimensionless coupling. More generally, in a theory with more than one coupling constant,

$$\frac{\partial c}{\partial g^i} = G_{ij}\beta^j \ , \tag{6.2}$$

where G_{ij} is the Zamolodchikov metric, and $\beta^j = \mu\frac{dg^j}{d\mu}$ is the beta-function for the coupling g^j. The fact that the metric G_{ij} is non-singular guarantees the stationarity of the Zamolodchikov c-function at any fixed point in two-dimensions.

It is of great interest to find out if these results extend to field theory in dimension $d > 2$. In four-dimensional conformal field theory there are two Weyl anomaly coefficients, a and c. Long ago Cardy conjectured [1] that it should be the a-coefficient that decreases under RG flow. This coefficient can be calculated from the expectation of value of the trace of the stress-energy tensor in the Euclidean theory on S^4. Using conformal perturbation theory, it is possible to establish the analogue of (6.1) in four-dimensions [1]:

$$c(g) = a_{\rm UV} - g^2\delta + O(g^3) \ , \tag{6.3}$$

where the perturbing operator $\mathcal{O}$ has dimension $4-\delta$.

In this chapter we address a question about the proposed C-function $\mathcal{F}(R)$ in three-dimensions that has not yet been elucidated: namely, is it stationary for arbitrary perturbations around a CFT?

In a general field theory which does not have a gravity dual, the calculation of REE is a difficult problem even if we resort to numerical methods. In this chapter we consider a particularly simple example of RG flow provided by a free massive scalar field on $\mathbb{R}^{2,1}$ with the action

$$I = -\frac{1}{2}\int d^3x \left[(\partial_\mu\phi)^2 + m^2\phi^2\right] \ . \tag{6.4}$$

In this case there are efficient numerical algorithms for calculating $\mathcal{F}$ [206, 215]. On dimensional grounds, $\mathcal{F}$ must be a function of $g = (mR)^2$, which is the dimensionless coupling associated with the relevant perturbation $m^2\phi^2/2$. The UV fixed point of this theory is that of a massless free scalar, for which the value of $\mathcal{F}$ is known analytically [9, 65]; therefore,

$$\mathcal{F}(g=0) = \mathcal{F}_{\rm UV} = \frac{\ln 2}{8} - \frac{3\zeta(3)}{16\pi^2} \approx 0.0638 \ . \tag{6.5}$$

The stationarity of the c-function at the UV fixed point would require $\mathcal{F}'(0) = 0$. However, our numerical calculation instead indicates that $\mathcal{F}'(g)$ is negative at small g. In fact, it may diverge in the limit $g \to 0$, but the limitations of our numerical work do not allow us to make a precise statement about the small g behavior of $\mathcal{F}'(g)$.

Our numerical results suggest that REE does not in general define a c-function in the Zamolodchikov sense, because it is not always stationary at conformal fixed points.[1] It is not clear how to compare this statement with the proof [12] of the monotonicity of $\mathcal{F}$. If g can have either sign, then $\mathcal{F}'(0) \neq 0$ would violate monotonicity. In the example we have considered, however, insisting on a stable UV fixed point requires that g is positive. So, there is no immediate conflict with the work of [44]. We note, however, that the absence of stationarity means that an equation like (6.2) cannot in general apply to $\mathcal{F}$.

Examination of other examples where the stationarity of $\mathcal{F}$ can be tested is highly desirable. In studies of holographic entanglement entropy [190, 191], $\mathcal{F}$ initially appeared to be stationary [15, 45]. However, a closer analysis showed that in fact the holographic EE is not stationary [216]. The holographic calculations are presented in Sec. 6.4.

6.2 Strip entanglement entropy at small mass

Although the renormalized entanglement entropy across a circle is of our main interest, as a warm-up we start with a slightly different but simpler quantity, the entanglement entropy of a strip.

[1] In $(1+1)$-dimensions the monotonic c-function derived from the entanglement entropy of a segment [44] is not stationary with respect to m^2 either. In $(1+1)$-dimensions this is likely due to the fact that ϕ^2 is not a conformal primary field in the CFT of a massless scalar.

The entanglement entropy between the strip of width R and its complement in $(2+1)$-dimensional free field theory is simply related to the entanglement entropy between the interval of length R and its complement in the $(1+1)$-dimensional theory. More precisely, for the $(2+1)$-dimensional massive scalar field of mass m we have the relation [204]

$$S^{(2+1)}_{\text{strip}}(m,R) = \frac{L}{\pi}\int_0^\infty dp\, S^{(1+1)}_{\text{interval}}\big(\sqrt{p^2+m^2},R\big)\,, \tag{6.6}$$

where $\sqrt{p^2+m^2}$ is the mass of the $(1+1)$-dimensional scalar field, and L is the length of the strip. To derive this equation one first compactifies the direction parallel to the strip to a large circle of length L. Decomposing the $(2+1)$-dimensional scalar field into angular momentum modes along this circle and then taking $L\to\infty$ leads to (6.6).

It is useful to define the entropic c-function for the strip through

$$C_{\text{strip}} \equiv R^2\,\partial_R\,\hat{S}^{(2+1)}_{\text{strip}}\,, \tag{6.7}$$

where $\hat{S}^{(2+1)}_{\text{strip}} \equiv S^{(2+1)}_{\text{strip}}/L$ is the entanglement entropy per unit length. This function is manifestly finite and cut-off independent [191–193, 203]. For the massive scalar field, C_{strip} is also simply related to the $(1+1)$-dimensional entropic c-function, defined [44] by $c(t = mR) \equiv R\,\partial_R\, S^{(1+1)}_{\text{interval}}$, through (6.6):

$$C_{\text{strip}}(mR) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\; c\big(\sqrt{x^2+(mR)^2}\big)\,. \tag{6.8}$$

We should note, however, that in general C_{strip} is not expected to be a good c-function; for example, it is not constant along lines of fixed points.

The entropic c-function for the $(1+1)$-dimensional massive scalar field is a well studied quantity (see [183] for a review). The function interpolates between $1/3$ in the UV ($t=0$) and 0 in the IR ($t\to\infty$). At small and large values of t one can calculate the leading behavior of $c(t)$ analytically [204],

$$\begin{aligned} c(t) &= \frac{1}{3} + \frac{1}{2\log t} + \cdots\,, \qquad t \ll 1\,, \\ c(t) &= \frac{t}{4} K_1(2t) + \cdots\,, \qquad t \gg 1\,. \end{aligned} \tag{6.9}$$

Unfortunately there is no known closed form expression for $c(t)$ along the entire RG flow. The function may be constructed numerically by solving an infinite sequence of nonlinear differential equations [204]. Following this procedure leads to the curve in figure 6.1.

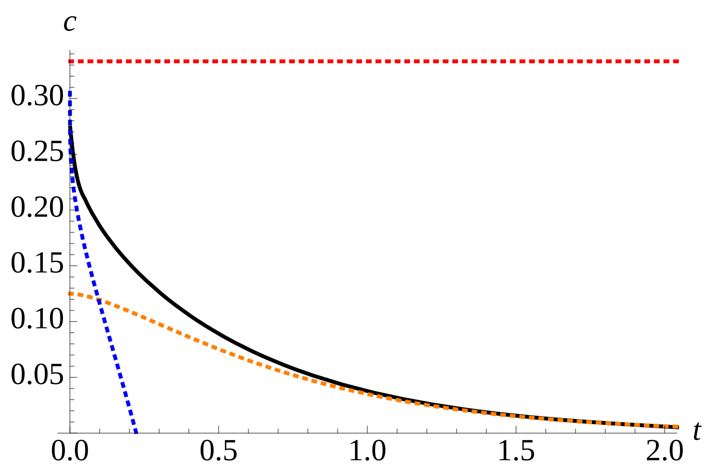


Figure 6.1: The entropic c-function $c \equiv R\,\partial_R\, S^{(1+1)}_{\text{interval}}$ for the $(1+1)$-dimensional massive scalar field as a function of $t \equiv mR$, where m is the mass and R is the length of the interval. The black curve comes from a numerical calculation using the prescription in [204]. The blue and orange curves are the analytic approximations in (6.9) at small and large values of t, respectively. The dotted red line marks the conformal value $c(0) = 1/3$.

The function $C_{\text{strip}}(mR)$ is a monotonically decreasing function that approaches zero exponentially fast in the IR. Using (6.8) one may determine numerically its value at $mR = 0$ [204],

$$C_{\text{strip}}(0) = \frac{1}{\pi} \int_0^\infty dt\, c(t) \approx 3.97 \times 10^{-2}\,. \tag{6.10}$$

At large mR we may use the approximation for $c(t)$ in the second line of (6.9) to write

$$C_{\text{strip}}(mR) \approx \frac{1}{16} e^{-2mR} \left(mR + \frac{1}{2} \right) , \qquad mR \gg 1 . \tag{6.11}$$

In figure 6.2 we numerically plot C_{strip} along the RG flow together with the IR analytic approximation.

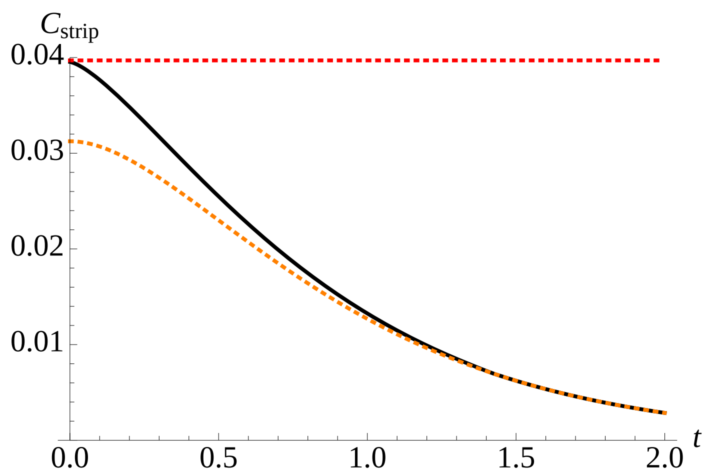


Figure 6.2: The function $C_{\text{strip}} \equiv R^2 \, \partial_R \, \hat{S}^{(2+1)}_{\text{strip}}$ for the free massive scalar field in black, where $\hat{S}^{(2+1)}_{\text{strip}} \equiv S^{(2+1)}_{\text{strip}}/L$ is the entanglement entropy per unit length across the strip of width R. The orange curve is the IR approximation in (6.11). The dotted red line is the initial value at $t = mR = 0$ given in (6.10).

It is interesting to ask whether C_{strip} is stationary at $mR = 0$. The first derivative of the entropic c-function (6.8) with respect to $t = mR$ gives

$$\begin{aligned} C'_{\text{strip}}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{t}{\sqrt{x^2+t^2}} \, c'(\sqrt{x^2+t^2}) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \frac{1}{\sqrt{x^2+1}} \left[-\frac{1}{2\sqrt{x^2+1}\log^2(t\sqrt{x^2+1})} + O\left(\log^{-3} t\right) \right] \\ &\xrightarrow[t\to 0]{} -\frac{1}{4\log^2 t} , \end{aligned} \tag{6.12}$$

where we used the asymptotic form of the $(1+1)$-dimensional entropic c-function (6.9) and rescaled $x \to tx$ in the second line. While this implies that $C'_{\text{strip}}(0) = 0$, stationarity additionally requires that $C'_{\rm strip}(t)/t$ vanishes at $t=0$. This is clearly not the case since this quantity diverges as $-1/(4t\log^2 t)$ as $t\to 0$. As can be seen in figure 6.3, we confirm this behavior numerically by plotting $C'_{\rm strip}(t)/t$ along with the analytic prediction.

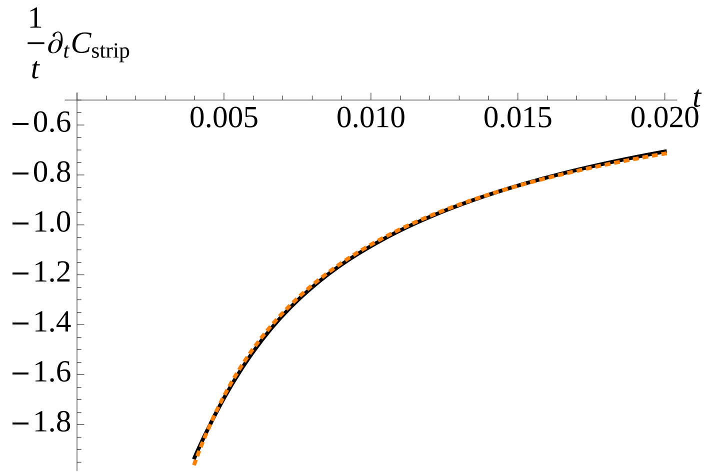


Figure 6.3: The first derivative $C'_{\rm strip}(t)/t$ as $t=mR\to 0$. The black curve comes from the numerical calculation, and the dotted orange curve comes from fitting the numerics to the function $C'_{\rm strip}(t)/t \approx a/(t\log^2 t)$ as $t\to 0$. We find $a\approx -0.25$, which agrees with the analytic result in (6.12). This means that $C_{\rm strip}$ is not stationary for the massive scalar field.

To summarize the results of this section, we have found that the small mass expansion of the UV finite function (6.7) for a strip of width R is

$$C_{\rm strip}(mR) \approx 0.0397 - \frac{|mR|}{4\log^2(mR)} + \cdots \ . \tag{6.13}$$

In the next section we will present evidence that the REE across a circle of radius R has a similar structure, with $d\mathcal{F}/dg$ looking clearly negative at small $g=(mR)^2$.

6.3 Disk entanglement entropy at small mass

The entanglement entropy for the massive scalar field across the circle of radius R may be calculated numerically following the prescription in [183, 206, 215, 217]. This numerical method has passed many non-trivial checks. For example, in [206] the terms proportional to (mR) and $1/(mR)$ in the large (mR) expansion of the disk entanglement entropy were matched numerically to the analytic predictions to high accuracy. A similar analysis in [18] matched the $1/(mR)^3$ term in the large mass expansion of the EE to the analytic prediction, and in [45] the value of the REE at $mR = 0$ was shown to agree with (6.5).

The numerical procedure [183,206,215,217] expands the scalar field into modes of integer angular momentum n. The radial direction is discretized into a lattice of N units. For each n the discrete Hamiltonian takes the form $H_n = \frac{1}{2}\sum_i \pi_i^2 + \frac{1}{2}\sum_{ij} \phi_i K_n^{ij} \phi_j$, where π_i is the conjugate momentum to ϕ_i and $i = 1, \ldots, N$. The nonzero entries of the $N \times N$ matrix K_n are

$$K_n^{11} = \frac{3}{2} + n^2 + m^2\,, \qquad K_n^{ii} = 2 + \frac{n^2}{i^2} + m^2\,, \qquad K_n^{i,i+1} = K_n^{i+1,i} = -\frac{i+1/2}{\sqrt{i(i+1)}}\,. \tag{6.14}$$

To compute the entanglement entropy we need to know the two-point correlators $X \equiv \langle \phi_i \phi_j \rangle = \frac{1}{2}(K^{-1/2})_{ij}$ and $P \equiv \langle \pi_i \pi_j \rangle = \frac{1}{2}(K^{+1/2})_{ij}$. If the radius of the entangling circle R is a half-integer in units of the lattice spacing, then we must reduce the matrices X_{ij} and P_{ij} to the $r \times r$ matrices X_{ij}^r and P_{ij}^r, which are defined by taking $1 \le i, j \le r$ with $r = R - \frac{1}{2}$. The entanglement entropy is then given by

$$S(R) = S_0 + 2\sum_{n=1}^{\infty} S_n\,, \tag{6.15}$$

with

$$S_n = \mathrm{tr}\left[\left(\sqrt{X_n^r P_n^r} + \frac{1}{2}\right)\log\left(\sqrt{X_n^r P_n^r} + \frac{1}{2}\right) - \left(\sqrt{X_n^r P_n^r} - \frac{1}{2}\right)\log\left(\sqrt{X_n^r P_n^r} - \frac{1}{2}\right)\right]. \tag{6.16}$$

Further details of the numerical calculation can be found in [206].

In order to achieve the continuum limit we need to take $N \gg r \gg 1$ and $m \ll 1$ (restoring the lattice spacing, the latter condition is $m\epsilon \ll 1$). Then mR can be either small or large, and we can explore the REE as a function of this parameter.

In our calculations we use a radial lattice consisting of $N = 200$ points. We want to calculate the entanglement entropy for $.06 < mR < 2$, but to minimize lattice effects we restrict $30 < r < 50$ in lattice units. To accomplish this we calculate the entanglement entropy for $m = .002 \cdot i$ in inverse lattice units, with $i = 1, \ldots, 20$. To take into account finite lattice effects, we follow [45] and for the lowest 10 angular momentum modes we repeat the calculation on lattices of sizes $N = 200 + 10 \cdot j$, with $j = 0, \ldots, 50$, and then extrapolate to $N = \infty$. We perform the numerical calculation for the first 3000 angular momentum modes. We take into account higher angular momentum modes by using the asymptotic behavior of S_n derived in the original paper [16].

From the entanglement entropy we construct the renormalized entanglement entropy $\mathcal{F}$ along the RG flow. A plot of this function versus $(mR)^2$ is given in figure 6.4. From this

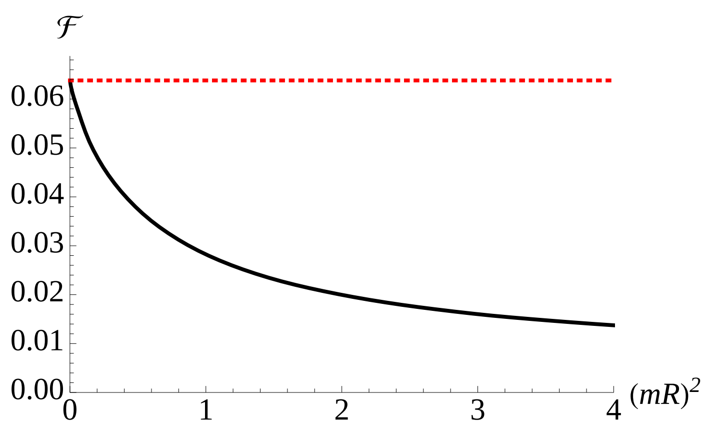


Figure 6.4: The renormalized entanglement entropy $\mathcal{F}$ across the circle of radius R for the massive real free scalar plotted in black as a function of $(mR)^2$. In this plot it can clearly be seen that $\partial_{(mR)^2}\mathcal{F}$ is negative and nonzero at $(mR)^2 = 0$, which implies that the REE $\mathcal{F}$ is not stationary at the UV fixed point of a free massless scalar field. The dotted red line is the zero mass value $\mathcal{F}_{\text{UV}} = \frac{\ln 2}{8} - \frac{3\zeta(3)}{16\pi^2}$.

plot we can see that $\partial_{(mR)^2}\mathcal{F}$ is clearly negative and nonzero at $(mR)^2 = 0$. Recent, more precise numerical calculations suggest that $\mathcal{F} \approx F_{\rm UV} - 0.133(mR)^2$ near $mR = 0$ [216].

6.4 Non-stationarity from holography

In this section we discuss the entanglement entropy of $(2+1)$-dimensional CFTs which have gravitational duals, perturbed by relevant operators. This section is adapted from [15], but importantly it is updated to include the recent stationarity observations found in [216]. We will see that non-stationarity is also present in the holographic context.

We work with an effective $(3+1)$-dimensional gravitational theory, with metric

$$ds_4^2 = \frac{L_{\rm UV}^2}{y^2}\left(\frac{dy^2}{f(y)} - dt^2 + dr^2 + r^2 d\theta^2\right), \qquad y \geq 0\,, \tag{6.17}$$

and we assume that in the UV (small y) the metric asymptotes to AdS_4 with radius $L_{\rm UV}$, i.e. $f(y) = 1 + O(y^\alpha)$, $\alpha > 0$. The entanglement entropy across a circle of radius R in the boundary QFT is then given by the area functional

$$S(R) = \frac{\pi L_{\rm UV}^2}{2G_N^{(4)}}\int_\epsilon^{y_{\rm IR}} dy\, \frac{r(y)}{y^2\sqrt{f(y)}}\sqrt{1 + f(y)(\partial_y r)^2}\,, \tag{6.18}$$

where the function $r(y)$ satisfies the Euler-Lagrange equation, ϵ is the short-distance cut-off, and $y_{\rm IR}$ is the maximal value of y for the solution.

If the effective $(3+1)$-dimensional gravitational theory comes from an exact D-dimensional string or M-theory background, with metric as in (5.15), then we may identify

$$f(y) = \beta(u)\left(\frac{\partial y}{\partial u}\right)^2, \qquad \frac{L_{\rm UV}^2}{y^2} = \frac{V(u)}{V_{\rm UV}}\alpha(u)\beta(u)\,. \tag{6.19}$$

Let us begin with the conformal limit, where we may take $f(y) = 1$. We also define the $(3+1)$-dimensional Newton's constant $G_N^{(4)} = G_N^{(D)}/V_{\rm UV}$, where, in 10- and 11-dimensions, the Newton's constant takes the values $G_N^{(10)} = 8\pi^6\alpha'^4 g_s^2$ and $G_N^{(11)} = (32\pi^2)^{-1}(2\pi\ell_p)^9$, respectively. The solution to the equation of motion is then $r(y) = \sqrt{R^2 - y^2}$. Evaluating the area functional on this solution and expanding in ϵ gives

$$S(R) = \frac{\pi L_{\rm UV}^2}{2G_N^{(4)}} R \int_\epsilon^R \frac{dy}{y^2} = \frac{\pi R L_{\rm UV}^2}{2G_N^{(4)}\epsilon} - \frac{\pi L_{\rm UV}^2}{2G_N^{(4)}} \,. \tag{6.20}$$

Suppose that the 3-dimensional CFT comes from the near horizon limit of N M2-branes at the tip of the cone $\mathcal{C}_Y = \mathbb{R} \times Y$, where Y is some 7-dimensional internal Sasaki-Einstein space. In the large N limit the theory is well described by the supergravity background

$$ds_{11}^2 = ds_{AdS_4}^2 + 4L_{\rm UV}^2 ds_Y^2 \,, \qquad F_4 = \frac{3}{L_{\rm UV}}\,{\rm vol}_{AdS_4} \,, \qquad F_7 = *_{11}F_4 = 384 L_{\rm UV}^6\,{\rm vol}_Y \,, \tag{6.21}$$

where the radius $L_{\rm UV}$ of AdS_4 is quantized in plank units:

$$N = \frac{1}{(2\pi\ell_p)^6}\int_Y F_7 = \frac{6\,{\rm Vol}(Y)}{\pi^6}\frac{L_{\rm UV}^6}{\ell_p^6} \,. \tag{6.22}$$

Substituting the relation between N and $L_{\rm UV}$ in (6.22) into (6.20) gives the renormalized entanglement entropy

$$F_{\rm UV} = \frac{\pi L_{\rm UV}^2}{2G_N^{(4)}} = N^{3/2}\sqrt{\frac{2\pi^6}{27\,{\rm Vol}(Y)}} + O(N^{1/2}) \,. \tag{6.23}$$

As a consistency check, we may verify that this is equal to the finite part of the Euclidean free energy of the theory on the 3-sphere [8,218], in agreement with the general result of [11].

Now suppose that there is an RG flow in the boundary QFT caused by perturbing the UV CFT by a relevant scalar operator $\mathcal{O}$ of dimension $1/2 \leq \Delta < 3$. The operator $\mathcal{O}$ is dual to a massive scalar field in the bulk, with $\Delta(\Delta - 3) = m^2$. The bulk action is then given by

$$I_4 = \frac{1}{16\pi G_N^{(4)}}\int d^4x\,\sqrt{-g}\left[R + \frac{6}{L_{\rm UV}^2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 + \cdots\right] \,, \tag{6.24}$$

where the dots stand for higher order terms in ϕ and the contributions of other fields, which won't be relevant for this discussion. When the field ϕ vanishes the equation of motion for the metric simply gives AdS_4.

For each bulk mass m, there are two possible boundary dimensions $\Delta_\pm$ [119]:

$$\Delta_\pm = \frac{3}{2} \pm \sqrt{m^2 + \frac{9}{4}}\,. \tag{6.25}$$

Both boundary dimensions are allowed if $\frac{1}{2} \leq \Delta_- \leq \frac{3}{2}$, while only Δ_+ is allowed if $\Delta_+ > 5/2$. The equations of motion for ϕ in the AdS_4 background give the following asymptotic solution at small y,

$$\phi(y,x) = y^{3-\Delta_+}[\phi_0^+(\vec{x}) + O(y^2)] + y^{3-\Delta_-}[\phi_0^-(\vec{x}) + O(y^2)]\,. \tag{6.26}$$

For $3/2 \leq \Delta_+ < 3$, taking the Δ_+ boundary condition corresponds to perturbing the action of the UV boundary CFT to

$$I_3 = I_{\rm UV} + \int d^3x \phi_0^+(\vec{x})\mathcal{O}(\vec{x})\,, \tag{6.27}$$

where $\mathcal{O}$ has dimension $\Delta = \Delta_+$. For $1/2 \leq \Delta_- < 3/2$, the Δ_- boundary condition has this interpretation (with ϕ_0^+ replaced by ϕ_0^- in (6.27), and $\mathcal{O}$ an operator of dimension $\Delta = \Delta_-$).

The field ϕ has a back-reaction on the metric. Taking the Δ_+ boundary condition and setting $\phi_0^+(\vec{x}) = \phi_0$ constant, we find that at small y the Einstein equation gives us

$$f(y) = 1 + \frac{3-\Delta_+}{4}\phi_0^2 y^{2(3-\Delta_+)} + \cdots\,, \qquad (\Delta_+ \text{ b.c.}). \tag{6.28}$$

Taking the Δ_- boundary condition with $\phi_0^-(\vec{x}) = \phi_0$ gives [216]

$$f(y) = 1 + \frac{\Delta_-}{4}\phi_0^{2\Delta_-/(3-\Delta_-)} y^{2\Delta_-} + \cdots\,, \qquad (\Delta_- \text{ b.c.}). \tag{6.29}$$

To find the first correction to the renormalized entanglement entropy as a result of the relevant deformation of the UV CFT, with Δ_+ boundary condition, it is sufficient to evaluate the action in (6.18) with $f(y)$ given in (6.28) on the UV solution $r(y) = \sqrt{R^2 - y^2}$. A straightforward calculation then gives

$$\mathcal{F}(R) = F_{\mathrm{UV}} \left(1 - \frac{(3-\Delta_+)}{8\left(\frac{7}{2} - \Delta_+\right)} \phi_0^2 \, R^{2(3-\Delta_+)} + \cdots \right) . \tag{6.30}$$

The factor $3 - \Delta_+$ plays an important role; it ensures that the renormalized entanglement entropy is not changed by marginal perturbations. When instead we take the Δ_- boundary condition, we find that [216]

$$\mathcal{F}(R) = F_{\mathrm{UV}} \left(1 - \frac{\Delta_-}{8\left(\frac{1}{2} + \Delta_-\right)} \phi_0^{2\Delta_-/(3-\Delta_-)} R^{2\Delta_-} + \cdots \right) . \tag{6.31}$$

Note that $\partial_{\phi_0} \mathcal{F}$ only vanishes as $\phi_0 R^{3-\Delta} \to 0$ if $\Delta > 1$. If the dimension Δ of the perturbing operator is $\frac{1}{2} \leq \Delta \leq 1$, then the REE is not stationary at the UV fixed point [216].

In particular, the mass perturbation in the free scalar field theory corresponds to $\Delta = 1$, and we expect $\partial_{(mR)^2} \mathcal{F}|_{mR=0} = \text{const}$. Of course, we do not expect the overall coefficient of $(mR)^2$ to match the holographic calculation, since the free scalar field theory does not have a simple gravitational dual.

6.5 A toy model: the massive scalar on $\mathbb{H}^2 \times S^1$

For a CFT in $(2+1)$-dimensions the calculation of the Rényi q-entropy of a disk may be mapped to a calculation on $\mathbb{H}^2 \times S^1$ [11, 104], where q is the radius of the circle. For a free scalar field of mass m the free energy on this space is [17]

$$\mathcal{F}_q(m^2) = \int_0^\infty d\lambda \mathcal{D}(\lambda) \left[\log \left(1 - e^{-2\pi q \sqrt{\lambda + (mR)^2}} \right) + \pi q \sqrt{\lambda + (mR)^2} \right] , \tag{6.32}$$

where the density of states is given by

$$\mathcal{D}(\lambda)d\lambda = \frac{\mathrm{Vol}(\mathbb{H}^2)}{4\pi}\tanh(\pi\sqrt{\lambda})d\lambda \ , \tag{6.33}$$

and the regularized volume of the hyperbolic space is $\mathrm{Vol}(\mathbb{H}^2) = -2\pi$. The formula for the finite part of entanglement entropy is

$$S_1 = \left.\frac{\partial \mathcal{F}_q}{\partial q}\right|_{q=1} - \mathcal{F}_1 \ . \tag{6.34}$$

Applying this relation at $m = 0$, which corresponds to a massless scalar on a disk, one obtains $S_1 = -\frac{\ln 2}{8} + \frac{3\zeta(3)}{16\pi^2}$, in agreement with other methods [17].

The status of the calculation on $\mathbb{H}^2 \times S^1$ for $m^2 > 0$ is less clear since this does not correspond to turning on mass on the original disk geometry. It is still interesting to inquire whether S_1 defined above is stationary with respect to turning on m^2. An explicit calculation yields

$$\left.\frac{\partial S_1}{\partial m^2}\right|_{m^2=0} = \frac{\pi^2}{16}R^2 \,. \tag{6.35}$$

Similarly, the calculation of the Rényi entropy can be mapped to that on the q-fold covering of S^3 [17]. For a massive free scalar field of mass m on this space, the derivative of S_1 with respect to the mass of the scalar field is $\frac{\partial S_1}{\partial m^2}|_{m^2=0} = -\frac{\pi^2}{16}R^2$, which agrees in absolute value with (4.4) but has a different sign. Either way, S_1 is not stationary in these toy models. This lack of stationarity of S_1, which is easily established analytically, is reminiscent of the numerical result for the disk entanglement found in the previous section.

Chapter 7

Conclusion

In this Dissertation, I presented the F-theorem and some of its applications. The F-theorem is useful because it orders the space of three-dimensional QFTs. If the unitary CFT A has F-value F_A, while the unitary CFT B has $F = F_B > F_A$, then there is no RG flow that begins at A and ends at B. The CFT F-values may be computed in two equivalent ways; F equals the renormalized free energy of the Euclidean CFT conformally mapped to the three-sphere, and it also equals the renormalized entanglement entropy across a circle in flat Minkowski space. Along the RG flow, the renormalized entanglement entropy is a monotonically decreasing function. I demonstrated this explicitly in both holographic and non-holographic examples.

The CFT F-values themselves are – apart from the F-theorem – useful quantities for studying various aspects of QFT. For example, I demonstrated that they may be used as a probe of the gauge-gravity correspondence, and I showed how they help determine the scaling dimensions of monopole operators in $\mathcal{N} = 2$ supersymmetric theories, through the principle of F-maximization.

There remain many open and interesting problems related to the F-theorem. Foremost among them, in my opinion, is physical application. Can F be measured in a lab? F is an inherently non-local quantity, so doing this directly may be difficult. Perhaps it is possible to approximate F through more-easily measurable quantities. Entanglement entropy is certainly measurable on the lattice. Numerical lattice simulations of physical systems may be able to compute F in order to help identity the conformal fixed points.

At the conformal fixed points themselves, we have developed a large array of tools for calculating F. While there is certainly more work that can be done here, it is also important to understand the REE away from the fixed points. Currently we have two methods for computing the REE in these cases: (i) numerical lattice techniques, and (ii) holographic methods. These methods may, realistically, only be applied to a very small subset of physically interesting RG flows. Better tools are needed for calculating the REE. In fact, we do not even know how to compute the REE in the vicinity of a conformal fixed point directly in field theory.

Developing tools for calculating the REE perturbatively near conformal fixed points is an outstanding problem. For example, I showed in chapter 6 that lattice calculations of the real massive scalar field indicate that $\partial_{m^2}\mathcal{F}$ is non-vanishing at $m^2 = 0$. This indicates that the REE is non-stationary. Ultimately, however, we need analytic methods for calculating $\mathcal{F}$ perturbatively in m^2 near $m^2 = 0$ in order to understand the meaning of the non-stationarity. More generally, if we deform the action at a conformal fixed point by $\int d^3x \lambda \mathcal{O}$, where $\mathcal{O}$ is a relevant operator in the CFT of dimension $\Delta < 3$, how do we calculate $\mathcal{F}$ perturbatively in λ? I answered this question in the holographic context in Sec. 6.4, following [216]. However, a purely field theoretic understanding of this behavior would be instructive.

The tools for holographically calculating EE and F are often, as indicated above, much more developed than their field theory counterparts. With that said, there remain many interesting open problems on the gravity side. One such problem is the calculation of the leading term in F, $F_0 = NF_S$, in the theory of N free fields. I showed in Sec. 4 how we may calculate the first $1/N$ correction to F in the bulk Vasiliev theory. The bulk 1-loop calculations have been further developed recently in [20, 219]. Still, it is fascinating, and somewhat troubling, that we are unable to calculate the leading contribution to F in the bulk. Such a calculation would probably require knowing the action for Vasiliev's higher-spin theory.

Over the past few years, the role of the EE across the circle has emerged as a useful probe of QFT. In Sec. 5 I briefly described EE for more general entangling surfaces in the holographic framework. However, in general it is much less straightforward to calculate the EE in QFT for non-spherical entangling surfaces, even at the conformal fixed points. At four-dimensional conformal fixed points, the universal contribution to the EE, for general entangling surfaces, has been worked out by Solodukhin [220]. I found a similar formula, valid for any entangling geometry, in six spacetime dimensions [18]. However, both of our calculations are indirect and rely heavily on holography. It would be extremely useful to find an efficient field theory method for calculating the EE across general entangling surfaces.

The Rényi entropies are also much less explored than the EE. When the entangling surface is spherical, these entropies are related to the thermal free energies on the hyperbolic space. A holographic prescription for the spherical Rényi entropies also exists [221]. However, there is no proposal for how to calculate the Rényi entropies holographically when the entangling surface is not spherical. Recently it has been shown that derivatives of the spherical Rényi entropies with respect to the Rényi parameter are proportional to integrated correlation functions of stress tensors on the hyperbolic space [222]. With that said, the role of the non-spherical Rényi entropies remains mysterious. Understanding what information may be extracted from these entropies remains an open question.

I have shown that quantum entropy is a powerful probe of QFT. Yet I have the feeling that we are only just beginning to understand its role in nature.

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