Cosmological Perturbations in Inflation and in de Sitter space

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Abstract

This thesis focuses on various aspects of inflationary fluctuations. First, we study gravitational wave fluctuations in de Sitter space. The isometries of the spacetime constrain to a few parameters the Wheeler-DeWitt wavefunctional of the universe, to cubic order in fluctuations. At cubic order, there are three independent terms in the wavefunctional. From the point of view of the bulk action, one term corresponds to Einstein gravity, and a new term comes from a cubic term in the curvature tensor. The third term is a pure phase and does not give rise to a new shape for expectation values of graviton fluctuations. These results can be seen as the leading order non-gaussian contributions in a slow-roll expansion for inflationary observables. We also use the wavefunctional approach to explain a universal consistency condition of n-point expectation values in single field inflation. This consistency condition relates a soft limit of an n-point expectation value to $n-1$-point expectation values. We show how these conditions can be easily derived from the wavefunctional point of view. Namely, they follow from the momentum constraint of general relativity, which is equivalent to the constraint of spatial diffeomorphism invariance.

We also study expectation values beyond tree level. We show that subhorizon fluctuations in loop diagrams do not generate a mass term for superhorizon fluctuations. Such a mass term could spoil the predictivity of inflation, which is based on the existence of properly defined field variables that become constant once their wavelength is bigger than the size of the horizon. Such a mass term would be seen in the two point expectation value as a contribution that grows linearly with time at late times. The absence of this mass term is closely related to the soft limits studied in previous chapters. It is analogous to the absence of a mass term for the photon in quantum electrodynamics, due to gauge symmetry.

Finally, we use the tools of holography and entanglement entropy to study superhorizon correlations in quantum field theories in de Sitter space. The entropy has interesting terms that have no equivalent in flat space field theories. These new terms are due to particle creation in an expanding universe. The entropy is calculated directly for free massive scalar theories. For theories with holographic duals, it is determined by the area of some extremal surface in the bulk geometry. We calculate the entropy for different classes of holographic duals. For one of these classes, the holographic dual geometry is an asymptotically Anti-de Sitter space that decays into a crunching cosmology, an open Friedmann-Robertson-Walker universe. The extremal surface used in the calculation of the entropy lies almost entirely on the slice of maximal scale factor of the crunching cosmology.

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[1] Brazil 2, Argentina 0. World Cup Final prediction!

[2] Hopefully Sasha, Grisha Tarnopolskiy and myself will be able to write a paper about these ideas soon!

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"Não chores, meu filho;

não chores, que a vida

é luta renhida:

viver é lutar.

A vida é combate

Que os fracos abate,

Que os fortes, os bravos

Só pode exaltar."

-Gonçalves Dias

To all the great teachers I had in my life.

Contents

Chapter 1

Introduction

Our current understanding of the universe at its most fundamental level is based on Quantum Mechanics and General Relativity. The former is responsible for the physics of particles and its interactions, and dominates our understanding of short distance physics, when materials cease to be homogeneous and atomic structure becomes relevant. The latter explains the motions of planets, stars, galaxies and the universe as a whole, and dominates our understanding of physics at very large distances.

The early universe is a remarkable example of a situation that requires both General Relativity and Quantum Mechanics to be used simultaneously. This happens for the following reason. At large distance scales, the universe looks very homogeneous. This raised a puzzle, given that the universe is not old enough for regions far from each other to have ever been in causal contact. To solve this problem, the inflationary paradigm was proposed [1, 2, 3]. It is the proposal that the universe undergoes exponential expansion in its very early stages, thus making our visible universe arise from a small region. An alternative scenario which is described by similar equations is the following. Our universe was in a false vacuum, namely, a local minimum of a potential, and then tunnels to the true vacuum. A bubble nucleation follows in the region that was previously in a false vacuum. Inside of this bubble we have an expanding universe. Both of these scenarios solve the puzzle as the visible universe arises from a region small enough to be homogeneous and in causal contact.

One beautiful outcome of the inflationary idea is that it does more than solving the problem for which it was designed. As we are dealing with a small region that expands, quantum mechanical effects become important. We can visualize the local rate of expansion of the universe by the value of the potential for a certain quantum field. We can track how much a region of spacetime expands by using this fluctuating field in the resulting geometry as a clock. The small quantum fluctuations of this field dictate how long inflation lasts in each region of the universe. In the end we have an inhomogeneous "reheating" surface, which indicates the end of inflation. This means that different regions inflated more or less than an average value, due to the quantum fluctuations of the clock. So, at the end of the inflationary period we are left with a large universe with small inhomogeneities [4, 5, 6, 7, 8]. These inhomogeneities eventually source anisotropies in the Cosmic Microwave Background [9], and dictate the Large Scale Structure of the universe [10, 11]. The formation of visible structure in the universe is due to quantum fluctuations of the inflationary epoch.

1.1 Inflation and small fluctuations

To study the inflationary paradigm in more detail, we need a toy model. There are two simple models that capture most of the features we see in the data.

The first model is de Sitter space (dS). dS is a solution of Einstein's equations with positive cosmological constant. The dS background captures the essential idea of inflation, as it can be noted by writing its line element in flat coordinates:

$$ds^2 = -dt^2 + e^{2Ht} dx_i dx^i, \quad i = 1, 2, 3. \tag{1.1}$$

The curvature of dS is set by the parameter H, which is its intrinsic Hubble constant. It enjoys as many symmetries as Minkowski space, thus allowing one to make exact statements about its quantum fluctuations. In dS inflation goes on forever, at least semiclassically. In that sense, there is no transition to a radiation or matter dominated universe in dS. However most of the basic features of inflationary perturbations can be understood in this simplified setting.

When we study the dynamics of quantum fluctuations on this background, we have to distinguish between distance scales bigger or smaller than the dS "radius", $R_{dS} = H^{-1}$. Consider a fluctuation with a given "comoving" wavelength. This corresponds to a Fourier mode k^i conjugate to the coordinates x^i in (1.1). Its physical wavelength grows with time, as $k_{phys}^{-1} = e^{Ht} k^{-1}$. For distances shorter than R_{dS} there is little difference between the time evolution of a field in flat space and a field in dS, as it can not feel the effects of curvature. For distances larger than R_{dS}, the fluctuations feel the influence of the curved spacetime. Now consider a free massless scalar field in this background. By solving its equations of motion, one finds out that fluctuations of this field freeze after their wavelength becomes bigger than R_{dS}. When treating the case with dynamical gravity, it will turn out that the fluctuations can be written as massless fields in a quasi-dS background.

The overall picture is the following. If we keep track of a given comoving wavelength mode, it fluctuates in the same way as we expect in flat space, with some oscillatory behavior. As the mode's wavelength becomes bigger than R_{dS}, these fluctuations freeze. Technically, this means that the power spectrum for the fluctuations becomes time independent. This intuitive picture of the evolution is important, so that we refer to R_{dS} as the "horizon", and we distinguish the dynamics of a mode depending on whether it has "crossed the horizon" or it is "inside the horizon"[1].

From the isometries of the background, it turns out that the two point function of massless fields is fixed up to the choice of vacuum state. There is a well motivated vacuum state, called Bunch-Davies vacuum, which reproduces the expected behavior of fluctuations at subhorizon distances[12, 13, 14]. It turns out that the quantum fluctuations at superhorizon distances enjoy two remarkable properties. First, their amplitude is proportional to Hubble's constant. Second, their spectrum is scale invariant, i.e. $P_k \propto k^{-3}$.

These are predictions of any inflationary scenario. As inflation has to end, some mechanism has to kick in to stop it. This will produce symmetry breaking effects, thus inflationary perturbations are near scale invariant[4, 5, 6, 7, 8].

To model the stoppage of inflation, we consider a scalar field rolling down a potential $V(\phi)$, with the initial condition that ϕ has a non-zero expectation value in the vacuum[2, 3]. This model is called Single Field Inflation, and contains the basic features shared by all inflationary models.

The breaking of dS symmetry is parametrized by the slope of the inflationary potential - if it is a flat potential, the model is equivalent to a massless scalar in dS - and we assume that this scale breaking is very small. The intuitive picture is easy to follow if we consider surfaces of constant ϕ. The value of ϕ decreases as it rolls down the potential. Because of quantum fluctuations, in some regions the field moves down in the potential more than in others. The net effect is that inflation lasts longer in regions where the roll is slower. Inhomogeneities are generated due to quantum fluctuations of this "inflaton" field. Surfaces of constant ϕ set constant time surfaces in a certain gauge. A fluctuation in ϕ is a time delay or advance. This fluctuation can be written as a local change in the scale factor; thus, it is a fluctuation in the local curvature. This curvature variable behaves like a massless field in this background, and its fluctuations are quasi scale invariant, as the dS symmetry is slightly broken.

In both scenarios, the gravitational field itself is a quantum field, and thus gravitational fluctuations are produced, with an almost scale invariant spectrum in single field inflation, and exact scale invariance in dS [15, 16, 17]. Any inflationary model will generate quantum fluctuations of the gravitational field itself, with a scale invariant spectrum. Once we consider fluctuations in the gravitational field, the metric becomes a dynamical variable, and its gauge symmetry needs to be treated properly. Gauge invariant observables can

[1] The technical reason why R_{dS} is referred as the horizon size is related to the description of dS from a free falling observer, who sees a cosmological horizon at distance R_{dS} from its position.

be defined order by order in perturbation theory[2] [18, 19, 20].

The gravitational wave fluctuations behave as a massless tensor field, or as two free massless scalar fields in the background geometry, one for each polarization mode of the graviton. The inflaton perturbations, written as curvature perturbations, behave as a massless scalar field. Because the scalar mode is tied to the dS symmetry breaking, its action is weighted by this factor. Thus, its fluctuations are enhanced by the dS-breaking factor, with respect to the tensor fluctuations. This is the origin of a small tensor-to-scalar ratio, which complicates the detection of primordial gravitational waves in the features of the Cosmic Microwave Background (CMB) radiation.

In summary, the inflationary paradigm - which consists of a phase of quasi-de Sitter expansion - produces quasi scale invariant scalar and tensor fluctuations, due to the inflaton and graviton fields. These fluctuations freeze at the horizon scale, and remain frozen until their physical wavelength becomes comparable to the size of the horizon, later in the history of the universe.

1.2 Probing details of the inflationary scenario

In the previous section we highlighted universal features of inflationary theories. To understand the inflationary period better, we need to look for features that are model dependent. The two point function was determined in a free field approximation. Properly defined field variables behave roughly as massless fields in a quasi-dS background. However, the non-linearity of gravity automatically introduces interactions. In a free theory, the spectrum of perturbations is Gaussian. In other words, it is fixed by the two point expectation value of fluctuations. For example, the three point expectation value is zero, and the four point expectation value is fixed by Wick's theorem.

The simplest observable that carries new information about inflation beyond the free field approximation is the three point function of inflationary perturbations. It gives rise to the so-called bispectrum in the CMB[21, 22, 23, 24]. This is a function of two momentum variables and thus can be affected in different ways depending on the shape of the potential. Current experiments, like the PLANCK satellite, are looking for a non-gaussian signature in the CMB data[25]. If there exists a non-zero three point function - as the non-linearity of gravity predicts - we will probe details of the inflationary scenario.

Defining gauge-invariant observables is a hard problem in gravitational theories. One candidate for that are the late-time expectation values of fields. As we have to fix a gauge, in principle we compute some gauge-invariant object. However, they are not fully gauge-invariant, as some large diffeomorphisms

[2] Local observables can not be gauge invariant in gravity. So, when dealing with gauge invariance in cosmological perturbation theory, one should either fix the gauge *a priori* or talk about gauge invariance to a certain order in the generator of diffeomorphisms.

are still allowed that maintain the metric unchanged. At the level of the three point function, this effect translates to some "consistency conditions" that fixes a piece of the expectation values in terms of lower-point expectation values. They arise in the same fashion as Ward identities appear in gauge theories, fixing correlation functions that involve the longitudinal part of the gauge field. It is important to understand these consistency conditions in detail, as they carry information about the frame dependence of our observable.

A more technical issue, not directly related to observations of the CMB, is the following. Most inflationary calculations are done at tree level. Due to the non-linearity of gravity, an infinite tower of interaction vertices appear in the Feynman diagrams. As a non-renormalizable theory, one should interpret computations done in this setting as effective field theory calculations, with the cutoff set by the Planck scale. One could then wonder whether loop diagrams can spoil the tree level prediction for inflationary observables. In other words, we stated previously that, in a certain gauge, the inflaton field is massless. In principle, a mass term can be generated dynamically, by loop diagrams. It is important to check whether this is possible in inflation, and if not, what protects the effective action from having such a term. In the cause of quantum electrodynamics, gauge symmetry forbids the appearance of a mass term for the photon. If the inflaton remains massless beyond the tree approximation, some symmetry must protect it.

The purpose of this thesis is to explore some of these questions in specific models in detail, with the hope to understand general features of inflationary theories and of dS space.

1.3 Overview of the thesis

The thesis is based on already published work by myself with collaborators. The goal is to understand quantum fluctuations in the inflationary setting, from various points of view. The focus is in two and three point expectation values, at tree level and one loop. In the last chapter we use a different probe of these superhorizon correlations using entanglement entropy.

Chapter 2, based on [26], was written with Juan Maldacena. We study non-gaussianities in a dS background. The isometries of dS constrain the possible shapes for the three point function. As discussed in the previous section, the inflationary scalar mode is associated to the breaking of dS symmetry, so we focus on gravitational fluctuations, which persist in the limit of exact dS. In a slow-roll expansion, we write the most general three point function to order $\mathcal{O}(\epsilon^0)$ for three graviton fluctuations. We also analyze the general constraints of dS symmetry on more general correlation functions.

Chapter 3 is based on [27]. I argue that the non-gaussian consistency conditions, which relate soft limits of n-point functions to $n-1$-point functions, are a general consequence of diffeomorphism invariance of the Wheeler-De Witt wavefunctional of the universe. In this language the consistency conditions are very easy to derive, to arbitrary order in the soft leg expansion. The focus is on three point functions of single field inflation and Einstein gravity in dS, but the construction is easily extensible to other settings.

Chapter 4, based on [28], was written with Leonardo Senatore and Matias Zaldarriaga. We study loop effects in the two point function of scalar fluctuations in dS. By examining in detail all the Schwinger-Keldysh diagrams, we show that there is no backreaction of modes running in the loop at late times, inducing some late time dependence on the power spectrum. In other words, we rule out that the one loop contribution from short wavelength modes can produce a mass term in the effective action for a long wavelength mode. This is a direct consequence of diffeomorphism invariance of the action, and is closely related to the soft limits discussed in chapter 3. It is crucial that we can trust the tree level calculation to believe in the inflationary paradigm, as we rely on the fact that fluctuations remain frozen after they exit the horizon. Such an effect would jeopardize such an important property of the inflationary perturbations.

Chapter 5, based on [29], was written with Juan Maldacena. We study superhorizon correlations by calculating the entanglement entropy of spherical regions. These spheres are of size much bigger than the horizon scale. It turns out that this entropy has an interesting contribution, that grows with the number of inflationary e-folds. We calculate this contribution for free massive scalars and for theories with holographic duals. A curious observation is that for theories dual to crunching cosmologies, the entropy computation is related to a specific property of the crunching cosmology. This maybe a hint to an FRW/CFT type of duality.

Chapter 2

On graviton non-gaussianities during inflation

2.1 Introduction

Recently there has been some effort in understanding the non-gaussian corrections to primordial fluctuations generated during inflation. The simplest correction is a contribution to the three point functions of scalar and tensor fluctuations. For scalar fluctuations there is a classification of the possible shapes for the three point function that appear to the leading orders in the derivative expansion for the scalar field [30, 31, 32, 33].

In this chapter we consider tensor fluctuations. We work in the de Sitter approximation and we argue that there are only three possible shapes for the three point function *to all orders* in the derivative expansion. Thus the de Sitter approximation allows us to consider arbitrarily high order corrections in the derivative expansion. The idea is simply that the three point function is constrained by the de Sitter isometries. At late times, the interesting part of the wavefunction becomes time independent and the de Sitter isometries act as the conformal group on the spatial boundary. We are familiar with the exact scale invariance, but, in addition, we also have conformal invariance. The conformal invariance fixes the three point functions almost uniquely. By "almost", we simply mean that there are three possible shapes allowed, two that preserve parity and one that violates parity. We compute explicitly these shapes and we show that they are the only ones consistent with the conformal symmetry. In particular, we analyze in detail the constraints from conformal invariance. In order to compute these three shapes it is enough to compute them for a simple Lagrangian that is general enough to produce them. The Einstein gravity Lagrangian produces one of these three shapes [34, 22]. The other parity conserving shape can be obtained by adding a $\int W^3$ term to the action, where W

is the Weyl tensor. Finally, the parity violating shape can be obtained by adding $\int W^2\widetilde{W}$, where $\widetilde{W}$ is the Weyl tensor with two indices contracted with an ϵ tensor. The fact that the gravitational wave expectation value is determined by the symmetries is intimately connected with the following fact: in four dimensional flat space there are also three possible three point graviton scattering amplitudes [35] [1]. Though the parity violating shape is contained in the wavefunction of the universe (or in related AdS partition functions), it does not arise for expectation values [36, 37]. Thus for gravitational wave correlators in dS we only have two possible shapes, both parity conserving.

We show that, under general principles, the higher derivative corrections can be as large as the term that comes from the Einstein term, though still very small compared to the two point function. In fact, we expect that the ratio of $\langle\gamma\gamma\gamma\rangle/\langle\gamma\gamma\rangle^2$ is of order one for the ordinary gravity case, and can be as big as one for the other shape. When it becomes one for the other shape it means that the derivative expansion is breaking down. This happens when the scale controlling the higher derivative corrections becomes close to the Hubble scale. For example, the string scale can get close to the Hubble scale. Even though ordinary Einstein gravity is breaking down, we can still compute this three point function from symmetry considerations, indicating the power of the symmetry based approach for the three point function. This gravity three point function appears to be outside the reach of the experiments occurring in the near future. We find it interesting that by measuring the gravitational wave three point function we can directly assess the size of the higher derivative corrections in the gravity sector of the theory. Of course, there are models of inflation where higher derivatives are important in the scalar sector [38, 39] and in that case too, the non-gaussian corrections are a direct way to test those models [40, 41, 32]. The simplicity of the results we find here is no longer present when we go from de Sitter to an inflationary background. However, the results we find are still the leading approximation in the slow roll expansion. Once we are away from the de Sitter approximation, one can still study the higher derivative corrections in a systematic fashion as explained in [30, 33, 42].

Our results have also a "dual" use. The computation of the three point function for gravitational waves is mathematically equivalent to the computation of the three point function of the stress tensor in a three dimensional conformal field theory. This is most clear when we consider the wavefunction of the universe as a function of the metric, expanded around de Sitter space at late times [22]. This is a simple consequence of the symmetries, we are not invoking any duality here, but making a simple statement [2]. From this point of view it is clear why conformal symmetry restricts the answer. If one were dealing with scalar operators, there would be only one possible three point function. For the stress tensor, we have three possibilities, two

[1] In flat space, one has to complexify the momenta to have nonvanishing three point amplitudes. In de Sitter, they are the natural observables.

[2] Of course, this statement is consistent with the idea that such a wavefunction can be computed in terms of a dual field theory. Here we are not making any assumption about the existence of a dual field theory. Discussions of a possible dual theory in the de Sitter context can be found in [43, 44].

parity conserving and one parity violating. The parity conserving three point functions were computed in [45]. Here we present these three point functions in momentum space. Momentum space is convenient to take into account the conservation laws, since one can easily focus on the transverse components of the stress tensor. However, the constraints from special conformal symmetry are a little cumbersome, but manageable. We derived the explicit form of the special conformal generators in momentum space and we checked that the correlators we computed are the only solutions. In fact, we found it convenient to introduce a spinor helicity formalism, which is similar to the one used in flat four dimensional space. This formalism simplifies the algebra involving the spin indices and it is a convenient way to describe gravitational waves in de Sitter, or stress tensor correlators in a three dimensional conformal field theory. In Fourier space the stress tensor has a three momentum $\vec{k}$, whose square is non-zero. The longitudinal components are determined by the Ward identities. So the non-trivial information is in the transverse, traceless components. The transverse space is two dimensional and we can classify the transverse indices in terms of their helicity. Thus we have two operators with definite helicity, $T^{\pm}(\vec{k})$. In terms of gravitational waves, we are considering gravitational waves that have circular polarization. These can be described in a convenient way by defining two spinors λ and $\bar{\lambda}$, such that $\lambda^a \bar{\lambda}^{\dot{b}} = (\vec{k}, |\vec{k}|)^{a\dot{b}}$. In other words, we form a null four vector, and we proceed as in the four dimensional case. We only have $SO(3)$ symmetry, rather than $SO(1,3)$, which allows us to mix dotted and undotted indices. We can then write the polarization vectors as $\xi^i \propto \sigma^i_{a\dot{b}} \lambda^a \lambda^{\dot{b}}$ (no bar), etc. This leads to simpler expressions for the three point correlation functions of the stress tensor in momentum space. We have expressed the special conformal generator in terms of these variables. One interesting aspect is that this formalism makes the three point function completely algebraic (up to the delta function for momentum conservation). As such, it might be a useful starting point for computing higher point functions in a recursive fashion, both in dS and AdS. This Fourier representation might also help in the construction of conformal blocks. The connection between bulk symmetries and the conformal symmetry on the boundary was discussed in the inflationary context in [46, 47, 48, 49, 50].

The idea of using conformal symmetry to constrain cosmological correlators was also discussed in [51]. Though the point of view is similar, some of the details differ. In that paper, scalar fluctuations were considered. However, scalar fluctuations, and their three point function, crucially depend on departures from conformal symmetry. It is likely that a systematic treatment of such a breaking could lead also to constraints, specially at leading order in slow roll. On the other hand, the gravitational wave case, which is discussed here, directly gives us the leading term in the slow roll expansion.

The chapter is organized as follows. In section 2.2 we perform the computation of the most general three point function from a bulk perspective. We also discuss the possible size of the higher derivative corrections. In section 2.3 we review the spinor helicity formalism in 4D flat spacetime, and propose a similar formalism that is useful for describing correlators of CFTs and expectation values in dS and AdS. We then write the previously computed three point functions using these variables. In section 2.4 we review the idea of viewing the wavefunction of the universe in terms of objects that have the same symmetries as correlators of stress tensors in CFT. We also emphasize how conformal symmetry constrains the possible shapes of the three point function. In section 2.5 we explicitly compute the three point function for the stress tensor for free field theories in 3D, and show that, up to contact terms, they have the same shapes as the ones that do not violate parity, computed from the bulk perspective. The appendices contain various technical points and side comments.

2.2 Direct computation of general three point functions

In this section we compute the three point function for gravitational waves in de Sitter space. We do the computation in a fairly straightforward fashion. In the next section we will discuss in more detail the symmetries of the problem and the constraints on the three point function.

2.2.1 Setup and review of the computation of the gravitational wave spectrum

The gravitational wave spectrum in single-field slow roll inflation was derived in [16]. Here we will compute the non-gaussian corrections to that result. As we discussed above, we will do all our computations in the de Sitter approximation. Namely, we assume that we have a cosmological constant term so that the background spacetime is de Sitter. There is no inflaton or scalar perturbation in this context. This approximation correctly gives the leading terms in the slow roll expansion. We leave a more complete analysis to the future.

It is convenient to write the metric in the ADM form

$$g_{00} = -N^2 + g_{ij}N^iN^j, \quad g_{0i} = g_{ij}N^j, \quad g_{ij} = e^{2Ht}\exp(\gamma)_{ij} \tag{2.1}$$

Where H is Hubble's constant. N and N_i are Lagrange multipliers (their equations of motion will not be dynamical), and γ_{ij} parametrizes gravitational degrees of freedom. The action can be expressed as

$$S = \frac{M_{Pl}^2}{2}\int \sqrt{-g}(R - 6H^2) = \frac{M_{Pl}^2}{2}\int \sqrt{g_3}\left(NR^{(3)} - 6NH^2 + N^{-1}(E_{ij}E^{ij} - (E^i{}_i)^2)\right) \tag{2.2}$$

Where $E_{ij} = \frac{1}{2}(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i)$ and we define $M_{Pl}^{-2} \equiv 8\pi G_N$. We fix the gauge by imposing that gravity fluctuations are transverse traceless, $\gamma_{ii} = 0$ and $\partial_i\gamma_{ij} = 0$. Up to third order in the action, we only need to compute the first order values of the Lagrange multipliers N and N_i [22]. By our gauge choice, these are $N = 1$ and $N_i = 0$ as there cannot be a first order dependence on the gravity fluctuations. Expanding the action up to second order in perturbations we find

$$S_2 = \frac{M_{Pl}^2}{8} \int \left(e^{3Ht} \dot{\gamma}_{ij} \dot{\gamma}_{ij} - e^{Ht} \partial_l \gamma_{ij} \partial_l \gamma_{ij} \right) \tag{2.3}$$

We can expand the gravitational waves in terms of polarization tensors and a suitable choice of solutions of the classical equations of motion. If we write

$$\gamma_{ij}(\mathbf{x}, t) = \int \frac{d^3k}{(2\pi)^3} \sum_{s=+,-} \epsilon_{ij}^s e^{i\mathbf{k}.\mathbf{x}} \gamma_{cl}(t) a_{s,\vec{k}}^\dagger + h.c. \tag{2.4}$$

the gauge fixing conditions imply that the polarization tensors are traceless, $\epsilon_{ii} = 0$, and transverse $k_i \epsilon_{ij} = 0$. The helicities can be normalized by $\epsilon_{ij}^A \epsilon_{ij}^{*B} = 4\delta^{AB}$. The equations of motion are then given by

$$0 = \gamma_{cl}''(\eta) - \frac{2}{\eta} \gamma_{cl}'(\eta) + k^2 \gamma_{cl}(\eta) \;, \qquad\qquad \eta = -\frac{e^{-Ht}}{H} \tag{2.5}$$

where we introduced conformal time, η. We take the classical solutions to be those that correspond to the Bunch-Davies vacuum [13], so $\gamma_{cl}(\eta) = \frac{H}{\sqrt{2k^3}} e^{ik\eta}(1 - ik\eta)$. Here we have denoted by $k = |\vec{k}|$ the absolute value of the 3-momentum of the wave. We are interested in the late time contribution to the two-point function, so we take the limit where $\eta \to 0$. After Fourier transforming the late-time dependence of the two-point function and contracting it with polarization tensors of same helicities we find:

$$\langle \gamma_k^{s_1} \gamma_{k'}^{s_2} \rangle = (2\pi)^3 \delta^3(k + k') \frac{1}{2k^3} \left(\frac{H}{M_{Pl}} \right)^2 4\delta_{s_1 s_2} \tag{2.6}$$

In the inflationary context (with a scalar field), higher derivative terms could give rise to a parity breaking contribution to the two point function. This arises from terms in the effective action of the form $\int f(\phi) W \widetilde{W}$ [52, 53]. This parity breaking term leads to a different amplitude for positive and negative helicity gravitational waves, leading to a net circular polarization for gravitational waves. If f is constant this term is a total derivative and it does not contribute. Thus, in de Sitter there is no contribution from this term. In other words, the parity breaking contribution is proportional to the time derivative of f. Some authors have claimed that one can get such parity breaking terms even in pure de Sitter [54]. However, such a contribution would break CPT. Naively, we would expect that CPT is spontaneously broken because of the expansion of the universe. However, in de Sitter we can go to the static patch coordinates where the metric is static. For such an observer we expect CPT to be a symmetry. A different value of the left versus right circular polarization for gravitational waves would then violate CPT.

In the AdS case, or in a general CFT, there can be parity violating contact terms in the two point function[3]. This is discussed in more detail in appendix 2.9.

2.2.2 Three point amplitudes in flat space

In order to motivate the form of the four dimensional action that we will consider, let us discuss some aspects of the scattering of three gravitational waves in flat space. This is relevant for our problem since at short distances the spacetime becomes close to flat space.

In flat space we can consider the on shell scattering amplitude between three gravitational waves. Due to the momentum conservation condition we cannot form any non-zero Mandelstam invariant from the three momenta. Thus, all the possible forms for the amplitude are exhausted by listing all the possible ways of contracting the polarization tensors of the gravitational waves and their momenta, [35][4]. There are only two possible ways of doing this, in a parity conserving manner. One corresponds to the amplitude that comes from the Einstein action. The other corresponds to the amplitude we would get from a term in the action that has the form W^3, where W is the Weyl tensor. In addition, we can write down a parity violating amplitude that comes from a term of the form $\widetilde{W}W^2$, where $\widetilde{W}_{abcd} = \epsilon_{abef}W^{ef}{}_{cd}$. These terms involving the Weyl tensor are expected to arise from higher derivative corrections in a generic gravity theory. By using field redefinitions, any other higher derivative interaction can be written in such a way that it does not contribute to the three point amplitude.

By analogy, in our de Sitter computation we will consider only the following terms in the gravity action

$$\begin{aligned} S_{eff} = \int d^4x \Big[\sqrt{-g} \left(\frac{M_{Pl}^2}{2} \left(-6H^2 + R\right) + \Lambda^{-2} \left(a\ W^{ab}{}_{cd} W^{cd}{}_{mn} W^{mn}{}_{ab} \right) \right) + \\ + \Lambda^{-2} \left(b\ \epsilon^{abef} W_{efcd} W^{cd}{}_{mn} W^{mn}{}_{ab} \right) \Big] \end{aligned} \tag{2.7}$$

Here Λ is a scale that sets the value of the higher derivative corrections. We will discuss its possible values later. This form of the action is enough for generating the most general gravity three point function that is consistent with de Sitter invariance. This will be shown in more detail in section 2.4, by using the action of the special conformal generators. For the time being we can accept it in analogy to the flat space result. Instead of the Weyl tensor in (2.7) we could have used the Riemann tensor. The disadvantage would be that the R^3 term would not have vanished in a pure de Sitter background and it would also have contributed to the two point function. However, these extra contributions are trivial and can be removed by field redefinitions. So it is convenient to consider just the W^3 term.

[3]A contact term is a contribution proportional to a delta function of the operator positions.
[4]In flat space, the three point amplitude is non-trivial only after analytically continuing to complex values of the momentum.

2.2.3 Three-point function calculations

In this subsection we compute the three point functions that emerge from the action in (2.7). First we compute the three point function coming from the Einstein term, and then the one from the W^3 term.

2.2.4 Three point function from the Einstein term

This was done in [34, 22][5]. For completeness, we review the calculation and give some further details. To cubic order we can set $N = 1$, $N_i = 0$ in (2.1). Then the only cubic contribution from (2.2) comes from the term involving the curvature of the three dimensional slices, $R^{(3)}$. Let us see more explicitly why this is the case. On the three dimensional slices we define $g = e^{2Ht}\hat{g}$, with $\hat{g}_{ij} = (e^{\gamma})_{ij}$. All indices will be raised and lowered with $\hat{g}$. The action has the form

$$S_R^{(3)} = \frac{1}{2}\int dtd^3x \left[e^{Ht}\hat{R}^{(3)} + e^{-Ht}(\hat{E}_{ij}\hat{E}^{ij} - (\hat{E}^i{}_i)^2)\right] \tag{2.8}$$

Now we prove that the second term does not contribute any third order term to the action. To second order in γ we have $\hat{E}^i_j = (\dot{\gamma} + \frac{1}{2}[\dot{\gamma}, \gamma])_{ij}$. Then we find

$$\hat{E}_{ij}\hat{E}^{ij} - (\hat{E}^i{}_i)^2 \quad = \quad \dot{\gamma}_{ij}\dot{\gamma}_{ij} + o(\gamma^4) \tag{2.9}$$

which does not have any third order term. Thus, the third order action is proportional to the curvature of the three-metric. This is then integrated over time, with the appropriate prefactor in (2.8). Of course, if we were doing the computation of the flat space three point amplitude, we could also use a similar argument. The only difference would be the absence of the e^{Ht} factor in the action (2.8). Thus, the algebra involving the contraction of the polarization tensors and the momenta is the same as the one we would do in flat space (in a gauge where the polarization tensors are zero in the time direction). Thus the de Sitter answer is proportional to the flat space result, multiplied by a function of $|\vec{k}_i|$ only, which comes from the fact that the time dependent part of the wavefunctions is different.

We want to calculate the tree level three-point function that arises from this third-order action. In order to do that, we use the in-in formalism. The general prescription is that any correlator is given by the time evolution from the "in" vacuum up to the operator insertion and then time evolved backwards, to the "in" vacuum again, $\langle O(t)\rangle = \left\langle \mathrm{in}\left|\bar{T}e^{-i\int H_{int}(t')dt'}O(t)Te^{i\int H_{int}(t')dt'}\right|\mathrm{in}\right\rangle$. We are only interested in the late-time

[5] In [34] the AdS case was considered.

limit of the expectation value. We find

$$\langle\gamma^{s_1}(x_1,t)\gamma^{s_2}(x_2,t)\gamma^{s_3}(x_3,t)\rangle_{t\to+\infty}=-i\int_{-\infty}^{+\infty}dt'\ [H_{int}(t'),\ \gamma^{s_1}(x_1,+\infty)\gamma^{s_2}(x_2,+\infty)\gamma^{s_3}(x_3,+\infty)] \quad (2.10)$$

We write the gravitational waves in terms of oscillators as in (2.4). We calculate correlators for gravitons of specific helicities and 3-momenta. Note that, because there are no time derivatives in the interaction Lagrangian, then it follows that $H^3_{int}=-L^3_{int}$. Once we put in the wavefunctions, the time integral that we need to compute is of the form $Im[\int_{-\infty}^0 d\eta\frac{1}{\eta^2}(1-ik_1\eta)(1-ik_2\eta)(1-ik_3\eta)e^{i(k_1+k_2+k_3)\eta}]$ (in conformal time). Two aspects of the calculation are emphasized here. One is that we need to rotate the contour to damp the exponential factor at early times, which physically corresponds to finding the vacuum of the interacting theory [22], as is done in the analogous flat space computation. Another aspect is that, around zero, the primitive is of the form $-\frac{e^{i(k_1+k_2+k_3)\epsilon}}{\epsilon}=-\frac{1}{\epsilon}-i(k_1+k_2+k_3)+O(\epsilon)$, where ϵ is our late-time cutoff. This divergent contribution is real and it drops out from the imaginary part. We get

$$\begin{aligned}\langle\gamma^{s_1}_{k_1}\gamma^{s_2}_{k_2}\gamma^{s_3}_{k_3}\rangle_R=(2\pi)^3\delta^3\left(\mathbf{k_1}+\mathbf{k_2}+\mathbf{k_3}\right)&\left(\frac{H}{M_{Pl}}\right)^4\frac{4}{(2k_1k_2k_3)^3}\times\\ &\left[(k_i^2k_j^2\epsilon^1_{ij})\epsilon^2_{kl}\epsilon^3_{kl}-2\epsilon^1_{ij}(k_l^3\epsilon^2_{li})(k_m^2\epsilon^3_{mj})+\text{cyclic}\right]\times\\ &\left(k_1+k_2+k_3-\frac{k_1k_2+k_1k_3+k_2k_3}{k_1+k_2+k_3}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right)\end{aligned} \quad (2.11)$$

The second line is the one that is the same as in the flat space amplitude. The third line comes from the details of the time integral. Below we will see how this form for the expectation value is determined by the de Sitter isometries, or the conformal symmetry.

2.2.5 Three point amplitude from W^3 in flat space

Let us calculate the following term in flat space, to which we will refer as W^3: $W^{\alpha\beta}{}_{\gamma\delta}W^{\gamma\delta}{}_{\sigma\rho}W^{\sigma\rho}{}_{\alpha\beta}$. We can write the following first order expressions for the components of the Weyl tensor

$$\begin{aligned}W^{0i}{}_{0j}&=\frac{1}{2}\ddot{\gamma}_{ij}\\ W^{ij}{}_{0k}&=\frac{1}{2}\left(\dot{\gamma}_{ki,j}-\dot{\gamma}_{kj,i}\right)\\ W^{0i}{}_{jk}&=\frac{1}{2}\left(\dot{\gamma}_{ik,j}-\dot{\gamma}_{ij,k}\right)\\ W^{ij}{}_{kl}&=\frac{1}{2}\left(-\delta_{ik}\ddot{\gamma}_{jl}+\delta_{il}\ddot{\gamma}_{jk}+\delta_{jk}\ddot{\gamma}_{il}-\delta_{jl}\ddot{\gamma}_{ik}\right)\end{aligned} \quad (2.12)$$

where we used that γ is an on shell gravitational wave. i.e. γ obeys the flat space equations of motion. We also used that $\gamma_{ii} = \partial_i \gamma_{ij} = 0$, $N_i = 0$, $N = 1$. We can then write

$$\begin{aligned} W^{\alpha\beta}{}_{\gamma\delta} W^{\gamma\delta}{}_{\sigma\rho} W^{\sigma\rho}{}_{\alpha\beta} = W^{ij}{}_{kl} W^{kl}{}_{mn} W^{mn}{}_{ij} + 6 W^{0i}{}_{jk} W^{jk}{}_{lm} W^{lm}{}_{0i} + \\ + 12 W^{0i}{}_{0j} W^{0j}{}_{kl} W^{kl}{}_{0i} + 8 W^{0i}{}_{0j} W^{0j}{}_{0k} W^{0k}{}_{0i} \end{aligned} \tag{2.13}$$

Evaluating these terms leads us to

$$S^{(3)} = \int \Lambda^{-2} \left[2\ddot{\gamma}_{ij}\ddot{\gamma}_{jk}\ddot{\gamma}_{ki} + 3\ddot{\gamma}_{ij}\dot{\gamma}_{kl,i}\dot{\gamma}_{kl,j} + 3\ddot{\gamma}_{ij}\dot{\gamma}_{ik,l}\dot{\gamma}_{jl,k} - 6\ddot{\gamma}_{ij}\dot{\gamma}_{ik,l}\dot{\gamma}_{kl,j} \right] \tag{2.14}$$

Plugging $\gamma_{ij} = \epsilon^1_{ij} e^{\mathrm{i}k_1 \cdot x} + \epsilon^2_{ij} e^{\mathrm{i}k_2 \cdot x} + \epsilon^3_{ij} e^{\mathrm{i}k_3 \cdot x}$ where $k \cdot x = k^i x_i - kt$, we get the following expression for the vertex due to the W^3 term:

$$\begin{aligned} V_{W^3,flat} = 6\ k_1 k_2 k_3 \Big[k_1\ k_2^i k_2^j \epsilon^1_{ij} \epsilon^2_{kl} \epsilon^3_{kl} + \text{cyclic} - \\ (k_1 + k_2 + k_3)(\epsilon^1_{ij} k_3^k \epsilon^2_{ki} k_2^l \epsilon^3_{lj} + \text{cyclic}) - 2\ k_1 k_2 k_3 \epsilon^1_{ij} \epsilon^2_{jk} \epsilon^3_{ki} \Big] \end{aligned} \tag{2.15}$$

By choosing a suitable basis for the polarization tensors, one can show that this agrees with the gauge invariant covariant expression $V_{W^3,flat} = 6 k_1^\mu k_1^\nu \epsilon^1_{\rho\sigma} k_2^\rho k_2^\sigma \epsilon^2_{\eta\tau} k_3^\eta k_3^\tau \epsilon^3_{\mu\nu}$.

2.2.6 Three point function from W^3 in dS

The straightforward way of performing the computation would be to insert now the expressions for the wavefunctions in the W^3 term in de Sitter space, etc. There is a simple observation that allows us to perform the de Sitter computation. First we observe that the Weyl tensor is designed so that it transforms in a simple way under overall Weyl rescaling of the metric. Thus the Weyl tensor for the metric in conformal time is simply given by $W_{\mu\nu\delta\sigma}(g) = \frac{1}{H^2\eta^2} W_{\mu\nu\delta\sigma}(\hat{g} = e^\gamma)$. Note also that, for this reason, the Weyl tensor vanishes in the pure de Sitter background. Thus, we only need to evaluate the Weyl tensor at linearized order[6]. For on shell wavefunctions $\gamma = (1 - ik\eta) e^{ik\eta + i\vec{k}.\vec{x}}$ we can show that

$$W_{\mu\nu\delta\sigma}(\gamma) = -i|\vec{k}|\eta W^{flat}_{\mu\nu\delta\sigma}(e^{ik\eta + i\vec{k}.\vec{x}}) \tag{2.16}$$

where W^{flat} is the expression for the flat space Weyl tensor that we computed in the previous section, computed at linearized order for a plane wave around flat space. Thus, when we insert these expressions in

[6] Note that the W^3 term does not contribute to the two point function.

the action we have

$$S = \int W^3 = \int_{-\infty}^{0} d\eta d^3x (k_1 k_2 k_3 \eta^3)(H^2\eta^2)(W^{flat})^3 \tag{2.17}$$

The whole algebra involving polarization tensors and momenta is exactly the same as in flat space. The only difference is the time integral, which now involves a factor of the form $\int d\eta \eta^5 e^{iE\eta} \propto 1/E^6$, where we have defined $E = k_1 + k_2 + k_3$, and we rotated the contour appropriately. Putting all this together, we get the following result for the three-point function due to the W^3 term

$$\begin{aligned} \langle \gamma_{k_1}^{s_1} \gamma_{k_2}^{s_2} \gamma_{k_3}^{s_3} \rangle_{W^3} &= (2\pi)^3 \delta^3 \left(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} \right) \times \\ & \left(\frac{H}{M_{Pl}} \right)^6 \left(\frac{H}{\Lambda} \right)^2 a \frac{(-30)}{(k_1+k_2+k_3)^6 (k_1 k_2 k_3)^2} V_{W^3,flat} \end{aligned} \tag{2.18}$$

where $V_{W^3,flat}$ was introduced in (2.15). There are also factors of $1/k_i^3$ that were included to get this result. The parity violating piece will be discussed after we introduce spinor variables, because they will make the calculation much simpler.

2.2.7 Estimating the size of the corrections

Let us write the effective action in the schematic form

$$S = \frac{M_{Pl}^2}{2} \left[\int [\sqrt{g} R - 6H^2 \sqrt{g}] + L^4 \int W^3 \right] + \cdots \tag{2.19}$$

where the dots denote other terms that do not contribute to the three point function. Here L is a constant of dimensions of length. We have pulled out an overall power of M_{Pl}^2 for convenience. The gravitational wave expectation values coming from this Lagrangian have the following orders of magnitude

$$\langle \gamma\gamma \rangle \sim \frac{H^2}{M_{Pl}^2} \ , \qquad \langle \gamma\gamma\gamma \rangle_R = \frac{H^4}{M_{Pl}^4} \qquad \langle \gamma\gamma\gamma \rangle_{W^3} = \frac{H^4}{M_{Pl}^4} (LH)^4 \tag{2.20}$$

Thus the ratio between the two types of non-gaussian corrections is

$$\frac{\langle \gamma\gamma\gamma \rangle_{W^3}}{\langle \gamma\gamma\gamma \rangle_R} \sim L^4 H^4 \tag{2.21}$$

We know that H^2/M_{Pl}^2 is small. This parameter controls the size of the fluctuations. In the AdS context, we know that when the right hand side in (2.21) becomes of order one we have causality problems [55, 56, 57, 58]. We expect that the same is true in dS, but we have not computed the precise value of the numerical coefficient where such causality violation would occur. So we expect that

$$HL \lesssim 1 \tag{2.22}$$

In a string theory context we expect L to be of the order of the string scale, or the Kaluza Klein scale. Thus the four dimensional gravity description is appropriate when $HL \ll 1$. In fact, in string theory we expect important corrections when $H\ell_s \sim 1$. In that case, the string length is comparable to the Hubble scale and we expect to have important stringy corrections to the gravity expansion. Note that in the string theory context we can still have $H^2/M_{Pl}^2 \sim g_s^2$ being quite small. So we see that there are scenarios where the higher derivative corrections are as important as the Einstein contribution, while we still have a small two point function, or small expansion parameter H^2/M_{Pl}^2. In general, in such a situation we would not have any good argument for neglecting higher curvature corrections, beyond the W^3 term. However, in the particular case of the three point function, we can just consider these two terms and that is enough, since these two terms (the Einstein term and the W^3 term) are enough to parametrize all the possible three point functions consistent with de Sitter invariance. If we define an $f_{NL-gravity} = \langle\gamma\gamma\gamma\rangle/\langle\gamma\gamma\rangle^2$, then we find that the Einstein gravity contribution of $f_{NL-gravity}$ is of order one. This is in contrast to the f_{NL} for scalar fluctuations which, for the simplest models, is suppressed by an extra slow roll factor[7].

In an inflationary situation we know that the fact that the fluctuations are small is an indication that the theory was weakly coupled when the fluctuations were generated. However, it could also be that the stringy corrections, or higher derivative corrections were sizable. In that case, we see that the gravitational wave three point function (or bispectrum) gives a direct measure of the size of higher derivative corrections. Other ways of trying to see these corrections, discussed in [59], involves a full reconstruction of the potential, etc. In an inflationary context terms involving the scalar field and its time variation could give rise to new shapes for the three point function since conformal symmetry would then be broken. However, one expects such terms to be suppressed by slow roll factors relative to the ones we have considered here. However a model specific analysis is necessary to see whether terms that contain slow roll factors, but less powers of LH dominate over the ones we discussed. For example, a term of the form $M_{Pl}^2 L^2 f(\phi) W^2$ is generically present in the effective action[33]. Such a term could give a correction of the order $\langle\gamma\gamma\gamma\rangle_{fW^2}/\langle\gamma\gamma\gamma\rangle_R \sim \epsilon_f (HL)^2$. Here ϵ_f is a small quantity of the order of a slow roll parameter, involving the time derivatives of f. Whether this dominates or not relative to (2.21) depends on the details of the inflationary scenario. In most cases, one indeed expects it to dominate. It would be very interesting if (2.21) dominates because it is a direct signature of higher derivative corrections in the *gravitational* sector during inflation.

[7] Note that we have divided by the *gravity* two point function to define $f_{NL-gravity}$. If we had divided by scalar correlators, we would have obtained a factor of ϵ^2.

Notice that the upper bound (2.22) is actually smaller than the naive expectation from the point of view of the validity of the effective theory. From that point of view we would simply demand that the correction due to W^3 at the de Sitter scale H should be smaller than one. This requires the weaker bound $H^4 L^4 < \frac{M_{Pl}}{H}$. This condition is certainly too lax in the AdS context, where one can argue for the more restrictive condition (2.22).

In summary, we can make the higher derivative contribution to the gravity three point function of the same order as the Einstein Gravity contribution. Any of these two terms are, of course, fairly small to begin with.

2.3 Spinor helicity variables for de Sitter computations

In this section we introduce a technical tool that simplifies the description of gravitons in de Sitter. The same technique works for anti-de Sitter and it can also be applied for conformal field theories, as we will explain later.

The spinor helicity formalism is a convenient way to describe scattering amplitudes of massless particles with spin in four dimensions. We review the basic ideas here. For a more detailed description, see [60, 35, 61, 62]. In four dimensions the Lorentz group is $SO(1,3) \sim SL(2) \times SL(2)$. A vector such as k_μ can be viewed as having two $SL(2)$ indices, $k^{a\dot{b}}$. The new indices run over two values. A 4-momentum that obeys the mass shell condition, $k^2 = 0$ can be represented as a product of two (bosonic) spinors $k^{a\dot{b}} = \lambda^a \bar{\lambda}^{\dot{b}}$. Note that if we rescale $\lambda \to w\lambda$ and $\bar{\lambda} \to \frac{1}{w}\bar{\lambda}$ we get the same four vector. We shall call this the "helicity" transformation. Similarly, the polarization vector of a spin one particle ξ_μ with negative helicity can be represented as

$$\xi^{-a\dot{b}} = \frac{\lambda^a \bar{\mu}^{\dot{b}}}{\langle \bar{\lambda}, \bar{\mu} \rangle} \tag{2.23}$$

where we used the $SL(2)$ invariant contraction of indices $\langle \lambda, \mu \rangle \equiv \epsilon_{ab} \lambda^a \mu^b$, where ϵ_{ab} is the SL(2) invariant epsilon tensor. We have a similar tensor $\epsilon_{\dot{a}\dot{b}}$ to contract the dotted indices. We cannot contract an undotted index with a dotted index. Note that this polarization vector (2.23) is not invariant under the helicity transformation. In fact, we can assign it a definite helicity weight, which we call minus one. This polarization tensor (2.23) is independent of the choice of $\bar{\mu}$. More precisely, different choices of $\bar{\mu}$ correspond to gauge transformations on the external particles. For negative helicity we exchange $\lambda, \bar{\eta} \leftrightarrow \bar{\lambda}, \eta$ in (2.23). For the graviton we can write the polarization tensor as a "square" of that of the vector

$$\xi^{+ab\dot{a}\dot{b}} = \frac{\mu^a \mu^b \bar{\lambda}^{\dot{a}} \bar{\lambda}^{\dot{b}}}{\langle \mu, \lambda \rangle^2}, \qquad \xi^{-ab\dot{a}\dot{b}} = \frac{\lambda^a \lambda^b \bar{\mu}^{\dot{a}} \bar{\mu}^{\dot{b}}}{\langle \bar{\lambda}, \bar{\mu} \rangle^2} \tag{2.24}$$

The product of two four vectors can be written as $k.k' = -2\langle\lambda,\lambda'\rangle\langle\bar\lambda,\bar\lambda'\rangle$.

Now let us turn to our problem. We are interested in computing properties of gravitational waves at late time. We still have the three momentum $\vec{k}$. This is not null. However, we can just define a null four momentum $(|\vec{k}|,\vec{k})$. This is just a definition. We can now introduce λ and $\bar\lambda$ as we have done above for the flat space case. In other words, given a three momentum $\vec{k}$ we define λ, $\bar\lambda$ via

$$(|\vec{k}|,\vec{k})^{a\dot b} = (|k|\,\sigma^{0\,a\dot b} + \vec{k}.\vec{\sigma}^{a\dot b}) = \lambda^a\bar\lambda^{\dot b} \tag{2.25}$$

In the de Sitter problem we do not have full $SL(2)\times SL(2)$ symmetry. We only have one $SL(2)$ symmetry which corresponds to the $SO(3)$ rotation group in three dimensions. This group is diagonally embedded into the $SL(2)\times SL(2)$ group we discussed above. In other words, as we perform a spatial rotation we change both the a and $\dot a$ indices in the same way. This means that we now have one more invariant tensor, $\epsilon_{\dot b a}$ which allows us to contract the dotted with the undotted indices. For example, out of λ^a and $\bar\lambda^{\dot b}$ we can construct $\langle\lambda,\bar\lambda\rangle$ by contracting with $\epsilon_{\dot b a}$. This is proportional to $|\vec{k}|$. Thus, this contraction is equivalent to picking out the zero component of the null vector. When we construct the polarization tensors of gravitational waves, or of vectors, it is convenient to choose them so that their zero component vanishes. But, we have already seen that extracting the zero component involves contracting dotted and undotted indices. We can now then choose a special $\bar\mu$ in (2.23) which makes sure that the zero component vanishes. Namely, we choose $\bar\mu^{\dot b} = \lambda^b$. This would not be allowed under the four dimensional rules, but it is perfectly fine in our context. In other words, we choose polarization vectors of the form

$$\xi^{+a\dot b} = \frac{\bar\lambda^a\bar\lambda^{\dot b}}{\langle\bar\lambda,\lambda\rangle}\ , \qquad\qquad \xi^{-a\dot b} = \frac{\lambda^a\lambda^{\dot b}}{\langle\lambda,\bar\lambda\rangle} \tag{2.26}$$

Notice that the denominator is just what we were calling $k = |\vec{k}|$. Note also that the zero component of ξ is zero, since this involves contracting the a and $\dot b$ indices. This gives a vanishing result due to the antisymmetry of the inner product. In our case we have a delta function for momentum conservation due to translation invariance, but we do not have one for energy conservation. The delta function for momentum conservation can be written by contracting $\sum_I \lambda_I^a\bar\lambda_I^{\dot b}$ with $\sigma^{i\,a\dot b}$ in order to get the spatial momentum. Alternatively we can say that $\sum_I \lambda_I^a\bar\lambda_I^{\dot b} \propto \epsilon^{a\dot b}$. This is just saying that the fourvector has only a time component.

For the graviton, we likewise take $\mu = \bar\lambda$ and $\bar\mu = \lambda$ in (2.24). With these choices we make sure that the polarization vector has zero time components and that it is transverse to the momentum.

Everything we said here also applies for correlation function of the stress tensor in three dimensional field theories. If we have the stress tensor operator $T_{ij}(k)$ in Fourier space, we can then contract it with a polarization vector transverse to k constructed from λ and $\bar{\lambda}$. In other words, we construct operators of the form $T^+ = \xi_i^+ \xi_j^+ T_{ij}$ with ξ^+ as in (2.26). This formalism applies for any case where we have a four dimensional bulk and a three dimensional boundary, de Sitter, Anti-de Sitter, Hyperbolic space, Euclidean boundary, Lorentzian boundary, etc. The only difference between various cases are the reality conditions. For example, in the de Sitter case that we are discussing now, the reality condition is $(\bar{\lambda}^{\dot{a}})^* = \epsilon_{\dot{a}b} \lambda^b$.

In summary, we can use the spinor helicity formalism tyo describe gravitational waves in de Sitter, or any inflationary background. It is a convenient way to take into account the rotational symmetry of the problem. One can rewrite the expressions we had above in terms of these variables.

2.3.1 Gravitational wave correlators in the spinor helicity variables

Let us first note the form of the two point function. The only non-vanishing two point functions are the $++$ and $--$ two point functions. This is dictated simply by angular momentum conservation along the direction of the momentum. Since the momenta of the two insertions are opposite to each other, their spins are also opposite and sum to zero as they should. The two point functions are then

$$\langle \gamma^+ \gamma^+ \rangle = \delta^3(k+k') \frac{\langle \lambda, \lambda' \rangle^2}{\langle \lambda, \bar{\lambda} \rangle^5} = \delta^3(k+k') \frac{1}{\langle \lambda, \bar{\lambda} \rangle^3} \tag{2.27}$$

where in the last formula we have used a particular expression for λ' in terms of $\bar{\lambda}$. More precisely, if the momentum of one wave if $\vec{k}$, with its associated λ and $\bar{\lambda}$, then for $\vec{k}' = -\vec{k}$ we can choose $\lambda' = \bar{\lambda}$ and $\bar{\lambda}' = -\lambda$. Here we have used that the matrices $\sigma^i_{\ a\dot{b}}$ are symmetric. In the first expression we can clearly see the helicity weights of the expression. For the $--$ one we get a similar expression.

We can now consider the three point functions. The simplest to describe are the ones coming from the W^3 interaction. In fact, these contribute only to the $+++$ and $---$ correlators, but not to the $++-$ correlators. This is a feature which is also present in the flat space case. These non vanishing correlators can be rewritten as

$$\begin{aligned}
\langle \gamma^+_{k_1} \gamma^+_{k_2} \gamma^+_{k_3} \rangle_{W^3} &= \mathcal{M} \frac{(-2^8 \times 3^2 \times 5)}{(k_1+k_2+k_3)^6 (k_1 k_2 k_3)^2} [\langle \bar{1}, \bar{2} \rangle \langle \bar{2}, \bar{3} \rangle \langle \bar{3}, \bar{1} \rangle]^2 \\
\langle \gamma^-_{k_1} \gamma^-_{k_2} \gamma^-_{k_3} \rangle_{W^3} &= \mathcal{M} \frac{(-2^8 \times 3^2 \times 5)}{(k_1+k_2+k_3)^6 (k_1 k_2 k_3)^2} [\langle 1, 2 \rangle \langle 2, 3 \rangle \langle 3, 1 \rangle]^2 \\
\mathcal{M} &= (2\pi)^3 \delta^3 \left(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} \right) \left(\frac{H}{M_{Pl}} \right)^6 \left(\frac{H}{\Lambda} \right)^2
\end{aligned} \tag{2.28}$$

where the k_n in the denominators can also be written in terms of brackets such as $k_n = -\langle n, \bar{n} \rangle$, if so desired. Note that, when rewritten in terms of the λ_n and $\bar{\lambda}_n$, the above expressions are just rational functions of the spinor helicity variables (up to the overall momentum conservation delta function). One can check that indeed the $++-$ and $--+$ vertices vanish for the W^3 term, which is straightforward by using the expressions in appendix 2.8. Note that this is not trivial because we do not have four-momentum conservation, only the three-momenta are conserved. The parity violating interaction $W^2\widetilde{W}$ does not contribute to the de Sitter expectation values [36, 37].

The Einstein term contributes to all polarization components

$$\langle\gamma^+_{k_1}\gamma^+_{k_2}\gamma^+_{k_3}\rangle_R = (2\pi)^3\delta^3\left(\sum_i k_i\right)\left(\frac{H}{M_{Pl}}\right)^4\frac{2}{(k_1k_2k_3)^5}\left[(k_1+k_2+k_3)^3-\right.$$
$$\left.-(k_1k_2+k_1k_3+k_2k_3)(k_1+k_2+k_3)-k_1k_2k_3\right]\left[\langle\bar{1},\bar{2}\rangle\langle\bar{2},\bar{3}\rangle\langle\bar{3},\bar{1}\rangle\right]^2 \tag{2.29}$$
$$\langle\gamma^+_{k_1}\gamma^+_{k_2}\gamma^-_{k_3}\rangle_R = (2\pi)^3\delta^3\left(\sum_i k_i\right)\left(\frac{H}{M_{Pl}}\right)^4\frac{1}{8(k_1k_2k_3)^5}\left[(k_1+k_2-k_3)(k_1-k_2+k_3)(k_2+k_3-k_1)\right]^2$$
$$\left(k_1+k_2+k_3-\frac{k_1k_2+k_1k_3+k_2k_3}{k_1+k_2+k_3}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right)\left[\frac{\langle\bar{1},\bar{2}\rangle^3}{\langle\bar{1},\bar{3}\rangle\langle\bar{3},\bar{2}\rangle}\right]^2 \tag{2.30}$$

and similar expressions for $--+$ and $---$. Note that the Einstein gravity contribution to $+++$ or $---$ is non-vanishing. This is in contradistinction to what happens in flat space, where it does not contribute to the $+++$ or $---$ cases. This might seem surprising, given that we had said before that the polarization tensor contribution to the time integrand is the same as the flat space one. After doing the time integral, in flat space we get energy conservation, which we do not have here. This explains why we got a non-vanishing answer. In fact, the flat space amplitude is recovered from the above expressions by focusing on the coefficients of the double poles in $E = k_1+k_2+k_3$. The fact that (2.29) does not have a double pole ensures that the flat space answer is zero for those polarizations[8]. Similarly, the flat space answers for W^3 are obtained by looking at the coefficient of the 6^{th} order pole in E in (2.28).

The expressions (2.29) can also be written in a form that shows explicitly the effect of changing the helicity of one particle:

$$\langle\gamma^+_{k_1}\gamma^+_{k_2}\gamma^+_{k_3}\rangle_R = \mathcal{N}(k_1+k_2+k_3)^2(\langle\bar{1},\bar{2}\rangle\langle\bar{2},\bar{3}\rangle\langle\bar{3},\bar{1}\rangle)^2 \tag{2.31}$$
$$\langle\gamma^+_{k_1}\gamma^+_{k_2}\gamma^-_{k_3}\rangle_R = \mathcal{N}(k_1+k_2-k_3)^2(\langle\bar{1},\bar{2}\rangle\langle\bar{2},3\rangle\langle3,\bar{1}\rangle)^2 \tag{2.32}$$
$$\mathcal{N} = (2\pi)^3\delta^3\left(\sum_i k_i\right)\left(\frac{H}{M_{Pl}}\right)^4\frac{2}{(k_1k_2k_3)^5}\times$$
$$\times\left(k_1+k_2+k_3-\frac{k_1k_2+k_1k_3+k_2k_3}{k_1+k_2+k_3}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right) \tag{2.33}$$

In the next section we will show that the forms of these results follow from demanding conformal sym-

[8] In comparing to the flat space result, there are also factors of $(k_1k_2k_3)$ that come from the normalization of the wavefunction.

metry.

2.4 Gravitational wave correlation function and conformal symmetry

In this section we will show how the three point functions we discussed above are constrained by conformal symmetry.

2.4.1 Wavefunction of the universe point of view

In order to express the constraints of conformal symmetry it is convenient to take the following point of view on the computation of the gravity expectation values. Instead of computing expectation values for the gravitational waves, we can compute the probability to observe a certain gravitational wave, or almost equivalently, the wavefunction $\Psi(\gamma)$. The expectation values are given by simply taking $|\Psi(\gamma)|^2$ and integrating over γ. This point of view is totally equivalent to the usual one, where one computes expectation values of γ. It is useful because it makes the connection to AdS very transparent[9]. It also makes the action of the symmetries more similar to the action of the symmetries in a conformal field theory. This is explained in more detail in [22] (see also [63]).

One writes the wavefunction in the form:

$$\begin{aligned}\Psi = \exp\Bigg(\frac{1}{2}\int d^3x d^3y \langle T^s(x) T^{s'}(y)\rangle \gamma^s(x)\gamma^{s'}(y) + \\ + \frac{1}{6}\int d^3x d^3y d^3z \langle T^s(x) T^{s'}(y) T^{s''}(z)\rangle \gamma^s(x)\gamma^{s'}(y)\gamma^{s''}(z) + \ \cdots\Bigg)\end{aligned} \tag{2.34}$$

The first term expresses the simple fact that the wavefunction is gaussian. From this point of view, the quantity $\langle T^s(x)T^{s'}(y)\rangle$ is just setting the variance of the gaussian. Namely, this is just a convenient name that we give to this variance. Similarly for the cubic term, which is responsible for the first non-gaussian correction, etc. Here we have ignored local terms that are purely imaginary and which drop out when we take the absolute value of the wavefunction. From this expression for the wavefunction one can derive the

[9] In fact, the perturbative de Sitter computation is simply an analytic continuation of the perturbative Anti-de Sitter computation [22].

following forms for the two and three point functions [22], to leading order in the loop expansion,

$$\langle \gamma^{s_1}_{k_1} \gamma^{s_2}_{k_2} \rangle = - \frac{1}{2 \langle T^{s_1}_{k_1} T^{s_2}_{k_2} \rangle} \tag{2.35}$$

$$\langle \gamma^{s_1}_{k_1} \gamma^{s_2}_{k_2} \gamma^{s_3}_{k_3} \rangle = - \frac{\langle T^{s_1}_{k_1} T^{s_2}_{k_2} T^{s_3}_{k_3} \rangle + \langle T^{s_1}_{-k_1} T^{s_2}_{-k_2} T^{s_3}_{-k_3} \rangle^*}{\Pi_i (2 \langle T^{s_i}_{k_i} T^{s_i}_{-k_i} \rangle)} \tag{2.36}$$

So we see that it is easy to go from the description in terms of a wavefunction to the description in terms of expectation values of the metric. The complex conjugate arises from doing $|\Psi|^2$ and we used that $\gamma^s(-\vec{k})^* = \gamma^s(\vec{k})$. However, if the wavefunction contains terms that are pure phases, we can loose this information when we consider expectation values of the metric. Precisely this happens when we have the parity violating interaction $\int W^2 \widetilde{W}$. It contributes to a term that is a pure phase.

Here Ψ is the usual Wheeler de Witt wavefunction of the universe, evaluated in perturbation theory. It is expressed in a particular gauge, because we have imposed the $N = 1$, $N_i = 0$ conditions. The usual reparametrization constraints and Hamiltonian constraints boil down to some identities on the functions appearing in (2.34). These identities are precisely the Ward identities obeyed by the stress tensor in a three dimensional conformal field theory[10]. In the AdS case, this is of course familiar from the AdS/CFT point of view. In the de Sitter case, it is also true since this wavefunction is a simple analytic continuation of the AdS one. It is an analytic continuation where the radius is changed by i times the radius. In any case, one can just derive directly these Ward identities from the constraints of General Relativity. These identities express the fact that the wavefunction is reparametrization invariant. For the case that we have scalar operators (and corresponding scalar fields in dS) we get an identity of the form $\partial_i \langle T_{ij}(x) \prod_k O(x_k) \rangle = - \sum_l \delta^3(x - x_l) \partial_{x^j_l} \langle \prod_k O(x_k) \rangle$. These are derived by starting with the reparametrization constraint, taking multiple derivatives with respect to the arguments of the wavefunction, and setting all fluctuations to zero after taking the derivatives. There is also another identity coming from the Hamiltonian constraint. This involves the trace of T and it takes into account the dimension of the operator. Namely we have $\langle T_{ii}(x) \prod_k O(x_k) \rangle = - \sum_l \delta^3(x - x_l) \Delta_l \langle \prod_k O(x_k) \rangle$. From these two identities, we can derive equations for the correlation functions if we have a conformal Killing vector [11]. Alternatively, we can derive these equations simply by noticing that a conformal reparametrization does not change the metric on the boundary, up to a rescaling (or a shift of time in the bulk). Thus, this leaves the wavefunction explicitly invariant, without even changing the metric, which is why we get equations on correlation functions for each isometry of the background space. From the

[10] Though the Ward identities are the same, some of the positivity constraints of ordinary CFT's are not obeyed. For example, the $\langle TT \rangle$ two point function is negative. Thus, if there is a dual CFT, it should have this unusual property.

[11] If v^j is the conformal Killing vector, then we can multiply the $\partial_i T_{ij} \cdots$ equation by v^j, integrate over x, integrate by parts, use the conformal Killing vector equation $\partial_{(i} v_{j)} = \frac{1}{3} \eta_{ij} (\partial . v)$, use the T_{ii} equation and obtain the equations $\sum_s [v^i(x_s) \partial_{x^i_s} + \frac{\partial . v(x_s)}{3} \Delta_s] \langle \prod_k O(x_k) \rangle = 0$. These encode all the equations obeyed by correlators that are a consequence of the de Sitter isometries at late times. For the metric, or stress tensors, the equations contain more indices and we write them in detail below.

general relativity point of view, this is just the statement that each isometry of the background leads to a constraint on the wavefunction. In our case, the operators are other insertions of the stress tensor. Thus, we can think of the coefficients $\langle TT \rangle$ and $\langle TTT \rangle$ appearing in (2.34) as correlation functions of "stress tensors". We are not assuming the existence of a dual CFT, we are simply saying that these quantities obey the same Ward identities as the ones for the stress tensor in a CFT. A more precise discussion of these identities can be found in appendix 2.11 and in section 2.4.2.

The isometries of de Sitter translate into symmetries of the wavefunction. Some of these are simple, like translation invariance. A less trivial one is dilatation invariance, or scale invariance. This simply determines the overall scaling of the three point function in terms of the momentum. If we think of γ as a dimensionless variable, then its Fourier components have dimension minus three. Thus the total dimension of any n point function is $-3n$. The delta function of momentum conservation takes into account a -3, and the remainder is the overall degree of homogeneity in the momentum. For the two point function it is -3 and for the three point function it is -6. It is a simple matter to count powers of momenta in the expressions we have given in order to check that this is indeed the case.

If instead we look at correlators of the stress tensor, then in position space, we have that the operator has dimension three, while in momentum space it has dimension zero.

The constraints from special conformal transformation are harder to implement and we discuss them in the next section. The results of the following section are also valid in any three dimensional conformal field theory. The general form for the three point function in position space was given in [45]. Here we study the same problem in momentum space. The expressions we find seem a bit simpler to us than the ones in [45], but the reader can judge by him or herself.

2.4.2 Constraints from special conformal transformations

de Sitter space is invariant under a full $SO(1,4)$ symmetry group. The metric $ds^2 = \frac{-d\eta^2 + dx^2}{\eta^2}$ makes some of these isometries manifest. In particular the scaling symmetry changes $x_i \to \alpha x_i$ and $\eta \to \alpha \eta$. There are also three more isometries that are given in infinitesimal form by

$$x^i \to x^i + b^i(-\eta^2 + \vec{x}.\vec{x}) - 2x^i(\vec{b}.\vec{x}) \ , \qquad \eta \to \eta - 2\eta(\vec{b}.\vec{x}) \tag{2.37}$$

where $\vec{b}$ is infinitesimal. When $\eta \to 0$, which is the future boundary of the space, the time rescaling acts in a simple way on the wavefunction. In addition we can drop the η^2 in the space part. The transformation then becomes what is called a "special conformal" transformation on the boundary parametrized by $\vec{x}$. In this section we study the action of these transformations in detail.

Now we work with coordinates on the three dimensional boundary. A special conformal transformation is given by

$$\delta x^i = x^2 b^i - 2x^i(x.b) \tag{2.38}$$

$$\Sigma^i{}_j \equiv \frac{\partial \delta x^i}{\partial x^j} = 2(x^j b^i - x^i b^j) - 2\delta^i_j(x.b) \equiv 2\widehat{M}^j{}_i - 2\delta^i_j(x.b) \tag{2.39}$$

$$J = \det(1+\Sigma)^{1/3} \sim 1 + \frac{1}{3}\sum_{\nu=1}^{3}\Sigma^\nu{}_\nu = 1 - 2(x.b) \tag{2.40}$$

where b^i is an infinitesimal parameter. The transformation law for a tensor is

$$T'_{\nu'_1\cdots\nu'_n}(x') = \frac{1}{J^{\Delta-n}}\left(\frac{\partial x^{\sigma_1}}{\partial x'^{\nu'_1}}\cdots\frac{\partial x^{\sigma_n}}{\partial x'^{\nu'_n}}\right)T_{\sigma_1\cdots\sigma_n}(x) \tag{2.41}$$

where Δ is the conformal dimension. For a current or the stress tensor we have $\Delta = 2, 3$ respectively. These transformation laws describe the (infinitesimal) action of the de Sitter isometries on the comoving coordinates at late time.

Now, in order to compute the variation of a correlator, we are interested in its change as a function. This means that the transformed correlator, as a function of the new variables, x' should be the same as the old correlator as a function of x. Thus we can evaluate $T'(x)$ (and not x'). Then we write $x = x' - \delta x$. In that way we find that the change is

$$\begin{aligned}\delta T_{\sigma_1\cdots\sigma_n} =& \Delta 2(x.b)T_{\sigma_1\cdots\sigma_n} - 2\sum_{l=1}^{n}\widehat{M}^{\nu_l}{}_{\sigma_l}T_{\sigma_1\cdots\nu_l\cdots\sigma_n} - DT_{\sigma_1\cdots\sigma_n}\\ D \equiv& \, x^2(b.\partial) - 2(b.x)(x.\partial)\end{aligned} \tag{2.42}$$

The matrix $\widehat{M}$ was defined in (2.39). We now Fourier transform (2.42) . The terms that contain a single power of x are easy to transform. They are given simply by inserting factors of $x \to -i\partial_k$. For the term involving a D, it is important that we first replace the $x \to -i\partial_k$ and *then* we change the derivatives by factors of k, $\partial_x \to -ik$. Thus a term like

$$x^2\partial_i \to i\vec{\partial}_k^2 k_i = i(k_i\vec{\partial}_k^2 + 2\partial_{k^i}) \tag{2.43}$$

$$x^i(x.\partial_x) \to i(\partial_{k^i}(\partial_{k^j}k_j)) = i[4\partial_{k^i} + k_j\partial_{k_j}\partial_{k^i}] \tag{2.44}$$

We then find that the special conformal generator (up to an overall i), now has the form

$$\begin{aligned}\delta T_{i_1\cdots i_n}(k) =& -(\Delta-3)2(b.\partial_k)T_{i_1\cdots i_n}(k) + 2\sum_{l=1}^{n}\widetilde{M}^{j_l}{}_{i_l}T_{i_1\cdots j_l\cdots i_n}(k) - \widetilde{D}T_{i_1\cdots i_n}(k)\\ \widetilde{M}^i{}_j \equiv& (b^i\partial_{k^j} - b^j\partial_{k^i})\\ \widetilde{D} \equiv& (b.k)\vec{\partial}_k^2 - 2k_j\partial_{k_j}(b.\partial_k)\end{aligned} \tag{2.45}$$

Where the (-3) in $\Delta-3$ comes from the commutators we had in (2.43) .

In momentum space any generator has an overall momentum conserving delta function $\delta(\sum_I \vec{k}_i)$. It is possible to pull the momentum space operator through the delta function. One can show that all terms involving derivatives of the delta function vanish. This is argued in detail in appendix 2.10. The final result is that we can simply act with the operator (2.45) on the coefficient of the delta function.

We would now like to express the action of the special conformal generator in terms of the spinor helicity variables. This problem is very similar to the one analyzed for amplitudes in [60]. There it was shown that the special conformal generator is given by

$$b.\widehat{\mathcal{O}} \equiv b^i\sigma^{ia\dot{a}}\frac{\partial^2}{\partial\lambda^a\partial\bar{\lambda}^{\dot{a}}} \tag{2.46}$$

The closure of the algebra implies that this simple form can only be consistent when it is applied to objects of scaling dimension minus one (in Fourier space). In our case, we will see that (2.46) differs from the special conformal generator only by terms proportional to the Ward identity for the corresponding tensor (the current or the stress tensor). This will be discussed in more detail below.

2.4.3 Constraints of special conformal invariance on scalar operators

The correlation function of three scalar operators is very simple in position space and it is given by a well known formula. In momentum space, it is hard to find an explicit expression because it is hard to do the Fourier transform. For the case of the three point function, the answer is a function of the $|\vec{k}_I|$. In that case we can rewrite the special conformal generator as

$$\vec{b}.\vec{k}\left[-2(\Delta-2)\frac{1}{|k|}\partial_{|k|} + \frac{\partial^2}{\partial|k|^2}\right] \tag{2.47}$$

We see that the case of $\Delta = 2$ is particularly simple[12]. So we consider a situation with three scalar

[12]Note that in this case the Fourier transform has dimension minus one, and then the special conformal generator is given by the simple expression in (2.46).

operators of dimension $\Delta = 2$. Invariance under special conformal transformations then implies

$$k_1^i \partial_{k_1}^2 f + k_2^i \partial_{k_2}^2 f + k_3^i \partial_{k_3}^2 f = 0 \tag{2.48}$$

where f is the Fourier transform of the correlator. Using momentum conservation we can conclude that all second derivatives should be equal, for any $m \neq n$:

$$(\partial_{k_m}^2 - \partial_{k_n}^2) f(k_1, k_2, k_3) = 0 \tag{2.49}$$

For each pair of variables this looks like a two dimensional wave equation. Thus the general solution is given by $f(k_1, k_2, k_3) = g(k_1 + k_2 + k_3) + h(k_1 - k_2 - k_3) + l(k_2 - k_3 - k_1) + m(k_3 - k_1 - k_2)$. To fix the form of these functions we look at the dilatation constraint:

$$(k_1 \partial_{k_1} + k_2 \partial_{k_2} + k_3 \partial_{k_3}) f(k_1, k_2, k_3) = c \tag{2.50}$$

In principle, $c = 0$. We would be tempted to conclude that this implies that each of the functions in f should be scaling invariant. This would leave only a constant solution. One can see that a logarithm is also allowed. The variation of a logarithm is a constant, and in position space, this is just a contact term. In other words, we can allow $c \neq 0$ in the right hand side of (2.50). Another way to see this is to consider the Fourier transform of the dilatation constraint. When we substitute $x \to i\partial_k$ we are implicitly integrating by parts. In general we neglect the surface terms because they are not singular. In the case we are considering, one can see that these terms are non-zero, hence $c \neq 0$.

In order to fix the combination of logarithms we can impose permutation symmetry as well as a good OPE expansion. The OPE expansion in position space says that $\langle OOO \rangle \sim \frac{1}{x_{23}^2} \frac{1}{x_{12}^4}$ as $x_{23} \to 0$. This translates into $\langle OOO \rangle \sim \frac{|\vec{k}_1|}{|\vec{k}_3|}$ as $\vec{k}_1 \to 0$ [13]. We then find that only the following solution is allowed

$$f(k_1, k_2, k_3) \sim \log(k_1 + k_2 + k_3) \tag{2.51}$$

It is possible to check that this is also the Fourier transform of the usual position space expression, $\frac{1}{x_{12}^2 x_{13}^2 x_{23}^2}$. It is also possible to show that one has simple solutions when operators of $\Delta = 2, 1$ are involved. This is done as follows. After we obtain (2.51) we can express the Fourier transform of the three point function as

[13] This OPE requirement is imposed up to contact terms. Thus, for example, a term of the form $\log k_2$, in this limit gives us a $\delta^2(x_{12})$ (since it is independent of k_1) and is consistent with the OPE requirement, which is only imposed at separated points. In other words, when we expand (2.51) for small $\vec{k}_1$ we get $2 \log k_2 + \frac{k_1}{2k_2} + \cdots$, we drop the first term and the second leads to the correct OPE.

$f = \prod_{I=1}^{3} k_I^{\Delta_I - 2} g$. Then g has scaling dimension zero, and the special conformal generator on each particle acquires the form

$$\left(\prod_{I=1}^{3} k_I^{\Delta_I - 2}\right) \vec{b}.\vec{k} \left[\frac{-(\Delta-2)(\Delta-1)}{|k|^2} + \frac{\partial^2}{\partial |k|^2}\right] g \tag{2.52}$$

We then see that for $\Delta = 1, 2$ the computation is the same as what we have done above. If all operators have $\Delta = 1$, then the answer is $g = 1$ or $f_{\Delta=1} = \frac{1}{k_1 k_2 k_3}$. When some operators have $\Delta = 1$ and some $\Delta = 2$ we cannot use permutation symmetry to select the solution, but it should be simple to find it.

2.4.4 Constraints of special conformal invariance on conserved currents

The constraints for special conformal invariance in momentum space are given by (2.45). Here we would like to express the constraint of special conformal invariance in the spinor helicity variables. We would like to express the special conformal generator in terms of a simple operator such as (2.46). The operator we want to consider is the current in Fourier space, multiplied by a polarization vector proportional to $\xi^{-a\dot{b}} = \frac{\lambda^a \lambda^{\dot{b}}}{\langle \lambda, \bar{\lambda} \rangle}$. In fact, just multiplying by this vector has a nice property, it leads to an operator of dimension minus one, since the Fourier transform of a conserved current has dimension minus one, and this choice of polarization vector does not modify the scaling dimension. If J is the conserved current, we take $\xi^-.J$ and we act with the operator (2.46). The lambda derivatives can act on ξ and also on J, when they act on J, we can express them in terms of $\vec{k}$ derivatives. After a somewhat lengthy calculation, one can rearrange all terms so that we get the action of (2.45) on the current, plus a term proportional to the divergence of J, or $\vec{k}.\vec{J}$. More explicitly, we find[14]

$$b^i \sigma^{i\beta\dot{\alpha}} \frac{\partial}{\partial \lambda^\beta} \frac{\partial}{\partial \bar{\lambda}^{\dot{\alpha}}} (\xi^-.J) = \xi_i^- (\delta^i_j 2b.\vec{\partial} + 2\widetilde{M}^i{}_j - \delta^i_j \widetilde{D}) J^j - (b.\xi^-) \frac{k^i}{|\vec{k}|^2} J^i \tag{2.53}$$

The first term in the right hand side vanishes due to the special conformal generator. The second term in the right hand side involves a longitudinal component of the current. One is tempted to set that to zero. However, we should recall that, inside a correlation function, we get contact terms at the positions of other charged operators. These terms are simply given by (the Fourier transform of) the Ward identity

$$k_1^i \langle J^i(k_1) O_2(k_2) \cdots O_n(k_n) \rangle = -\sum_{l=2}^{n} Q_l \langle O_2(k_2) \cdots O_l(k_l + k_1) \cdots O_n(k_n) \rangle \tag{2.54}$$

Where Q_l is the charge of the operator O_l. These are lower point functions. The conclusion is that there is a simple equation we can write down, by acting with the special conformal generator in spinor helicity variables, (2.46).

[14] $\widetilde{M}^i{}_j - \delta^i_j$ and $\widetilde{D}$ are defined in (2.45).

Note that in position space, we normally impose the special conformal transformation at separated points. In other words, we do not consider local terms. A local term will contribute to the three point function in position space as

$$\langle O(x)O(y)O(z)\rangle_{Local} \sim \mathcal{D}[\delta^3(x-y)]f(x-z) + \text{cyclic} \tag{2.55}$$

$\mathcal{D}$ is an arbitrary differential operator, which, when integrated by parts, will just yield an analytic function of the ks (like k_1^i, $k_1^4 k_3^i$, ...). Upon Fourier transforming the first term, we see that its form will be [analytic piece] $\times$ $F(k_3)$ + cyclic, where F is the Fourier transform of f. So, any piece in the three point function that is analytic in two of its variables, like k_1, $k_1 k_2^2$, k_3^4/k_1, corresponds to a local term. Something like $k_1 k_2$ is analytic in k_3 but not in k_1 and k_2, so it is non-local.

Let us see how this works more explicitly. We can start with the $--$ two point function

$$\langle \xi^-.J(k_1)\xi^-.J(k_2)\rangle = \delta^3(k_1+k_2)\frac{\langle 1,2\rangle^2}{\langle 1,\bar{1}\rangle} \tag{2.56}$$

Here the right hand side of the Ward identity vanishes and indeed, this function is annihilated by (2.46).

Now we consider the three point functions of currents J^a which are associated to a non-abelian symmetry. In the bulk they arise from a non-abelian gauge theory. The Ward identity is given by

$$k_1^i\langle J_i^a(k_1)J_j^b(k_2)J_l^c(k_3)\rangle = f^{abc}[\langle J_j(k_1+k_2)J_l(k_3)\rangle - \langle J_j(k_2)J_l(k_3+k_1)\rangle] \tag{2.57}$$

Where the color factor was stripped off the two point correlators [15].

Just as a check, let us compute the three point function for a gauge theory with a Yang Mills action in the bulk. Since the gauge field is conformally coupled, we can do the computation in flat space. We compute this correlator between three gauge fields in flat space, all set at $t=0$ and Fourier transformed in the spatial directions. We have the usual f^{abc} non-abelian coupling in the bulk. In Feynman gauge, the final answer is

$$\langle A_{\mu_1}^{a_1}(k_1)A_{\mu_2}^{a_2}(k_2)A_{\mu_3}^{a_3}(k_3)\rangle \propto \delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{f^{a_1a_2a_3}}{|k_1||k_2||k_3|}\frac{1}{E}[\delta^{\mu_1\mu_2}(k_1^{\mu_3}-k_2^{\mu_3}) + \text{cyclic}] \tag{2.58}$$

where $E=\sum_I |k_I|$.[16]

[15] One unpleasant feature of this equation, (2.57), is that the currents in the right hand side are evaluated at a shifted momentum, so it is not trivial to express this in terms of the λ and $\bar{\lambda}$ variables. In this particular case, this is not a problem since the two point functions can be explicitly computed, but this might lead to a more complicated story in the case of higher point functions.

[16] This is done as follows. The three point function in the bulk is the term in square brackets. We attach the propagators. We write the energy conservation condition as $\int dt' e^{it'\sum k_n^0}$. Then we integrate over k_n^0 to localize things at $t=0$. We can deform the integration contour and we only pick the residues of each of the poles of the propagators which give rise to the factor in the denominator. We close the contour up or down depending on the sign of t'. In each case, integrating over t' gives us the factor of $1/E$.

We now multiply by ξ^- for each particle to compute the $---$ correlator and we get

$$\langle \xi_1^- .A^{a_1}(k_1)\xi_2^- .A^{a_2}(k_2)\xi_3^- .A^{a_3}(k_3)\rangle \propto \delta^3(\vec{k}_1 + k_2 + k_3) f^{a_1 a_2 a_3} \frac{\langle 1,2\rangle\langle 2,3\rangle\langle 1,3\rangle}{[\langle 1,\bar{1}\rangle\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle]^2} \tag{2.59}$$

Note that the expectation value we started from, (2.58), is not gauge invariant. On the other hand, once we put in the polarization vectors and we compute the transverse part, as in (2.59), we get a gauge invariant result (under linearized gauge transformations). Note now that this expectation value is related to the currents, via a formula similar to (2.36), which introduces extra factors of the two point function, which here are simply factors of $|\vec{k}|$. Thus we find

$$\langle \xi_1 .J^{a_1}(k_1)\xi_2 .J^{a_2}(k_2)\xi_3 .J^{a_3}(k_3)\rangle \propto k_1 k_2 k_3 \langle \xi_1 .A^{a_1}(k_1)\xi_2 .A^{a_2}(k_2)\xi_3 .A^{a_3}(k_3)\rangle \tag{2.60}$$

Now let us check that this expectation value obeys the conformal Ward identity, with the operator (2.46). The action of (2.46) on the first current is

$$\begin{aligned} b^i\sigma^{ia\dot{a}} \frac{\partial^2}{\partial\lambda_1^a \partial\bar{\lambda}_1^{\dot{a}}} \left[\frac{\langle 1,2\rangle\langle 2,3\rangle\langle 1,3\rangle}{\langle 1,\bar{1}\rangle\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle}\right] =& b^i\sigma^{ia\dot{a}} \left[\frac{\lambda_{2a}\lambda_{1\dot{a}}\langle 2,3\rangle\langle 3,1\rangle}{\langle 1,\bar{1}\rangle^2\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle} - \frac{\lambda_{3a}\lambda_{1\dot{a}}\langle 1,2\rangle\langle 2,3\rangle}{\langle 1,\bar{1}\rangle^2\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle} - \right. \\ & \left. - \frac{2\bar{\lambda}_{1a}\lambda_{1\dot{a}}\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle}{\langle 1,\bar{1}\rangle^3\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle}\right] \end{aligned} \tag{2.61}$$

We now use the Schouten identity - a consequence of the fact that the spinors live in a 2D space - to simplify this further. Expressing λ_{2a} in terms of λ_{1a} and $\bar{\lambda}_{1a}$ we have $\langle 1,\bar{1}\rangle\lambda_{2a} = -\langle \bar{1},2\rangle\lambda_{1a} + \langle 1,2\rangle\bar{\lambda}_{1a}$ and thus:

$$\frac{\lambda_{2a}\lambda_{1\dot{a}}\langle 2,3\rangle\langle 3,1\rangle}{\langle 1,\bar{1}\rangle^2\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle} = \frac{\bar{\lambda}_{1a}\lambda_{1\dot{a}}\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle}{\langle 1,\bar{1}\rangle^3\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle} - \frac{\lambda_{1a}\lambda_{1\dot{a}}\langle \bar{1},2\rangle\langle 2,3\rangle\langle 3,1\rangle}{\langle 1,\bar{1}\rangle^3\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle} \tag{2.62}$$

We can do the same for λ_{3a} and then all that remains is a term proportional to $\lambda_{1\dot{a}}\lambda_{1a}$, given by

$$b^i\sigma^{ia\dot{a}} \frac{\partial^2}{\partial\lambda_1^a \partial\bar{\lambda}_1^{\dot{a}}} \left[\frac{\langle 1,2\rangle\langle 2,3\rangle\langle 1,3\rangle}{\langle 1,\bar{1}\rangle\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle}\right] = - \frac{b.\xi_1^-}{\langle 1,\bar{1}\rangle^2\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle} [\langle \bar{1},2\rangle\langle 2,3\rangle\langle 3,1\rangle - \langle \bar{1},3\rangle\langle 1,2\rangle\langle 2,3\rangle] \tag{2.63}$$

Using the momentum conservation condition we can express "cross-products" of the form $\langle m,\bar{n}\rangle$ for $m \neq n$ in terms of other brackets (the details are worked out in an appendix) through $(k_1+k_2+k_3)\langle m,\bar{n}\rangle = -2\langle m,o\rangle\langle \bar{o},\bar{n}\rangle$, where $m \neq n \neq o$ and then the term in (2.63) is

$$\langle \bar{1},2\rangle\langle 2,3\rangle\langle 3,1\rangle - \langle \bar{1},3\rangle\langle 1,2\rangle\langle 2,3\rangle = -\langle 2,3\rangle^2(k_2 - k_3) \tag{2.64}$$

Putting the pieces together, this is the expected contribution from the Ward identity

$$
\begin{aligned}
&\sum_{I=1}^{3} b^i \widehat{\mathcal{O}}_I^i \left[\langle \xi_1 .J^a(k_1)\xi_2.J^b(k_2)\xi_3.J^c(k_3)\rangle\right] = \\
&\qquad = f^{abc}\left\{\frac{b.\xi_1}{\langle 1,\bar{1}\rangle^2}\frac{\langle 2,3\rangle^2}{\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle}[\langle 2,\bar{2}\rangle - \langle 3,\bar{3}\rangle] + cyclic\right\} \\
&\qquad = f^{abc}\left\{\frac{b.\xi_1}{k_1^2}\xi_2.\xi_3[k_2-k_3] + cyclic\right\}
\end{aligned}
\tag{2.65}
$$

The expectation values that we have computed for gauge fields in de Sitter can also be computed in flat space, since gauge fields are conformal invariant[17]. So, we are simply computing correlation function of gauge invariant field strengths in R^4 but on a particular spatial slice. We are putting all operators at $t = 0$. We can think in momentum space and consider the operators $F_{ab}(t = 0, \vec{k})$, $F_{\dot{a}\dot{b}}(t = 0, \vec{k})$ where we Fourier transformed in the spatial coordinates but not in the time coordinate. Given $\vec{k}$ for each operator, we can define λ and $\bar{\lambda}$ via (2.25). We then can write the operators we considered above as:

$$
2\xi^-.A = -\frac{1}{k}\lambda^a\lambda^b(F^+_{ab} + F^-_{\dot{a}\dot{b}}) = -i\xi^-_i F_{jl}\epsilon^{ijl} \ , \qquad 2\xi^+.A = \frac{1}{k}\bar{\lambda}^{\dot{a}}\bar{\lambda}^{\dot{b}}(F^+_{ab} + F^-_{\dot{a}\dot{b}}) = i\xi^+_i F_{jl}\epsilon^{ijl}
$$

In both of these expressions, when we write F_{ab}, or $F_{\dot{a}\dot{b}}$ we mean the self dual and anti-self dual parts, but the indices are summed over with the indices of the indicated λ's. These expressions involve contractions that are not natural in flat space, but are reasonable once we break the full Lorentz symmetry to the rotation group. These operators are set at $t = 0$, and in Fourier space in the spatial section, with momentum $\vec{k}$.[18]

One can write higher derivative operators that give rise to three point functions that are annihilated by the special conformal generator (2.46). These operators would be $aTr[F^3]$ and $bTr[\widetilde{F}F^2]$. Now it is important to put in the dS metric. The wavefunctions for A are still the same as those in flat space, if we compute the three point functions perturbatively. These give the following three point functions[19]

$$
\langle J^{a,+}(1)J^{b,+}(2)J^{c,+}(3)\rangle \propto (2\pi)^3\delta^3(\sum k_i)(a + i\ b)f^{abc}\frac{\langle \bar{1},\bar{2}\rangle\langle \bar{2},\bar{3}\rangle\langle \bar{3},\bar{1}\rangle}{(\langle 1,\bar{1}\rangle + \langle 2,\bar{2}\rangle + \langle 3,\bar{3}\rangle)^3} \tag{2.66}
$$

$$
\langle J^{a,-}(1)J^{b,-}(2)J^{c,-}(3)\rangle \propto (2\pi)^3\delta^3(\sum k_i)(a - i\ b)f^{abc}\frac{\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle}{(\langle 1,\bar{1}\rangle + \langle 2,\bar{2}\rangle + \langle 3,\bar{3}\rangle)^3} \tag{2.67}
$$

The result (2.59), which comes from the usual Yang Mills term can be converted into a correlator of

[17] This is true for the tree correlators we are discussing but it is not true if loop corrections are taken into account.

[18] Note that both the self dual and anti-self dual parts of F contribute to each of the helicities. The reason is that we have defined a four momentum $(|k|, \vec{k})$ in order to define λ, $\bar{\lambda}$. (We could also have reversed the sign of the zeroth component to $|k| \to -|k|$, which exchanges $\bar{\lambda} \leftrightarrow \lambda$). With this definition, the time component of this four momentum is not necessarily equal to the total four momentum of on shell waves coming in or out of the F insertions (it can differ by a sign).

[19] These couplings do not require a non-abelian theory. They are antisymmetric in the Lorentz indices, thus they require an antisymmetric tensor. Thus, we could have three abelian gauge fields F^I and then the cubic couplings $\epsilon_{IJL}tr[F^IF^JF^L]$, where the trace is over the Lorentz indices.

curents of the form

$$
\begin{aligned}
\langle J^{a,+}(1)J^{b,+}(2)J^{c,+}(3)\rangle &= \left.\frac{\delta^3\Psi[A]}{\delta A^{a,+}(1)\delta A^{b,+}(2)\delta A^{c,+}(3)}\right|_{A=0} \\
&\propto \delta^3(\vec{k}_1+\vec{k}_2+\vec{k}_3) f^{a_1a_2a_3}\frac{\langle 1,2\rangle\langle 2,3\rangle\langle 1,3\rangle}{\langle 1,\bar{1}\rangle\langle 2,\bar{2}\rangle\langle 3,\bar{3}\rangle}
\end{aligned} \tag{2.68}
$$

is completely analytic in momentum space, and could be viewed as arising from a factor in the wavefunction of the form $\Psi \sim e^{Tr[A\wedge A\wedge A]}$. However, we would need a term in the wavefunction with the opposite sign to remove the $---$ correlator. Thus, although it looks like a local term, it does not seem possible to remove both the $+++$ and the $---$ correlator with the same factor. On the other hand, (2.66) is definitely non analytic in momentum, due to the $1/E^3$ singularity.

The current correlators are derivatives of the wavefunction. The expectation values of A can be obtained from them. In that case the parity violating b term in (2.66) drops out.

2.4.5 Constraints of special conformal invariance on the stress tensor

In this section we consider the constraints of conformal invariance in momentum space for the stress tensor.

We multiply the stress tensor by a convenient polarization tensor $\epsilon^-_{ij}T_{ij}(k)$, with $\epsilon^-_{ij} = \xi^-_i\xi^-_j$. In order to study the action of the special conformal generator it is convenient to define an operator containing an extra power of k as

$$
\widehat{T}^- = \frac{\epsilon^-_{ij}T_{ij}(k)}{k}\ , \qquad \epsilon^-_{ij} \equiv \xi^-_i\xi^-_j = \frac{\lambda^a\lambda^b\lambda^{\dot{a}}\lambda^{\dot{b}}}{\langle\lambda,\bar{\lambda}\rangle^2} \tag{2.69}
$$

The power of k was chosen so that the special conformal generator has the simple expression given by (2.46). The expression for ϵ_{ij} is that one that would give a naturally normalized tensor, when we take the reality conditions into account. Again, this operator does not quite annihilate the correlator, but it produces a term involving the Ward identity in the right hand side. Although more laborious in terms of manipulations, the general ideas are the same as in the current case so we will be more brief in the details of the conformal symmetry check.

We find that (2.46) acts on the stress tensor as

$$
b.\widehat{\mathcal{O}}\widehat{T}^- = b.\widehat{\mathcal{O}}\left[\frac{\epsilon^-_{ij}T_{ij}}{k}\right] = \frac{\epsilon^-_{ij}}{k}[4\widetilde{M}^j{}_l - \delta^j{}_l\widetilde{D}]T_{il} - 3\frac{1}{k^3}b_i\epsilon^-_{ij}k_l(T_{lj}+T_{jl}) \tag{2.70}
$$

The first term is what we expect from (2.45) for the stress tensor, and it vanishes. The second term can be computed by using the Ward identity. Again, such terms are analytic in some of the momenta. So if we disregard terms that are analytic in the momenta, we can drop also the term involving the Ward identity.

Let us first ignore this Ward identity terms and compute homogeneous solutions of the equation. Let us consider first a general $---$ three point function of the stress tensor. Such a general three point function is given by

$$\langle \hat{T}^- \hat{T}^- \hat{T}^- \rangle = \delta^3(\sum \vec{k}_I)[\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle]^2 f(k_1,k_2,k_3) \tag{2.71}$$

With f a symmetric function of dimension minus six. After some algebra, using Schouten identities, we get

$$\begin{aligned}\sum_n (\sigma^i)^{a\dot{a}} \frac{\partial^2}{\partial\lambda_n^a \partial\bar{\lambda}_n^{\dot{a}}} \left[(\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle)^2 f(k_1,k_2,k_3)\right] &= -2\frac{\xi_1^i}{k_1}\langle 1,2\rangle\langle 2,3\rangle^3\langle 3,1\rangle[k_3-k_2]\partial_{k_1} f + \\ &+ [\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle]^2 \left[\frac{4}{k_1}\partial_{k_1} f + \partial_{k_1}^2 f\right] k_1^i + \text{cyclic}\end{aligned} \tag{2.72}$$

Although the ξs are linearly independent, the ks are not. A convenient way to rewrite (2.72) is to choose special conformal transformation parameters b^i to project out a few components. Let us take $b^i \sim (\lambda_2^a \lambda_3^{\dot{b}} + \lambda_3^a \lambda_2^{\dot{b}})^i$, for example. This combination was chosen so that the time component of b is zero. We find

$$\begin{aligned}(\lambda_2\lambda_3).\widehat{\mathcal{O}}[(---)f] &= \langle 1,2\rangle^2\langle 2,3\rangle^3\langle 3,1\rangle^2 \left\{4(\partial_{k_2}-\partial_{k_3})f + k_3(\partial_{k_1}^2 - \partial_{k_3}^2)f - k_2(\partial_{k_1}^2-\partial_{k_2}^2)f\right\} \\ \rightarrow \qquad 0 &= 4(\partial_{k_2}-\partial_{k_3})f + k_3(\partial_{k_1}^2-\partial_{k_3}^2)f - k_2(\partial_{k_1}^2-\partial_{k_2}^2)f\end{aligned} \tag{2.73}$$

It is straightforward to check that the gravity result we obtained from a W^3 in interaction contribution gets annihilated by this operator. Such a contribution is simply $f = (k_1+k_2+k_3)^{-6}$. The Einstein contribution is not annihilated. It gives a nonzero answer that matches the expected answer from the Ward identity. One can solve the equation (2.73) and its two other cyclic cousins by brute force. One finds the expected solution, mentioned above, plus a new solution that has the form

$$f = \frac{1}{[(k_1+k_2-k_3)(k_2+k_3-k_1)(k_3+k_1-k_2)]^2} \tag{2.74}$$

This new solution does not have the right limit when $\vec{k}_1 \to 0$. Namely, if we start with $T_{ij}(\vec{k}_1)$, when the momentum goes to zero we do not expect any singular term when $\vec{k} \to 0$. In fact, an insertion of $T_{ij}(\vec{k}=0)$ corresponds to a constant metric or a change of coordinates. So, in fact this limit has a precise form. On the other hand if we look at this limit in (2.74), we find that $f \sim 1/k_1^4$, which is too singular compared to the expected behavior.

Let us now turn our attention to the $--+$ correlator. We first write an ansatz of the form

$$\langle \hat{T}^- \hat{T}^- \hat{T}^+ \rangle = \delta^3(\sum \vec{k}_n)[\langle 1,2\rangle\langle 2,\bar{3}\rangle\langle 1,\bar{3}\rangle]^2 g(k_1,k_2,k_3) \tag{2.75}$$

where g is a homogeneous function of degree six [20]. We can now use a trick to get the homogeneous solutions of the special conformal generator. We note that if we exchange $\lambda \leftrightarrow \bar{\lambda}$, then the sign of $|k|$ is changed, but $\vec{k}$ does not change. Now, the ansatz for (2.75) differs from (2.71) precisely by such a change in the third particle. Thus, the two solutions for g in (2.75) are simply given by the two solutions for f but with $k_3 \to -k_3$. More explicitly, the two solutions are

$$g = \frac{1}{(k_1 + k_2 - k_3)^6} \tag{2.76}$$

$$g = \frac{1}{[(k_1 + k_2 + k_3)(k_2 + k_3 - k_1)(k_3 + k_1 - k_2)]^2} \tag{2.77}$$

However, now both solutions are inconsistent with the small $\vec{k}_i$ limit. The first solution has a problem when $\vec{k}_1 \to 0$ and the second when $\vec{k}_3 \to 0$. Thus, both are discarded since these limits are too singular. Even though this trick of exchanging $\lambda \leftrightarrow \bar{\lambda}$ was useful for generating solutions of the homogeneous equation, the full results for the correlators are *not* given by such a simple exchange.

In conclusion, we have shown that there are no other solution of the conformal Ward identities beyond the ones we have already considered. It remains to be shown that the Einstein gravity answer obeys the conformal Ward identity. One can check that the Einstein gravity answer is not annihilated by the operator (2.46). It gives a nonzero term. This term is indeed what is expected from the Ward identity for the stress tensor, which is the second term in (2.70). Of course, this is expected since Einstein gravity has these symmetries. The relevant expressions are left to appendix 2.11.

2.5 Remarks on field theory correlators

In this section we compute the free field theory three point correlation function for scalars and fermions. This is very similar to what was done in [45] in position space. Here we work in momentum space. We will compare these expressions to the gravity ones computed above. The idea is that by considering a theory of a free scalar and a theory with a free fermion we obtain two independent shapes for the three point function of the stress tensor. Since we have computed the most general shapes above, this will serve as a check of our previous arguments. In addition, the momentum space expressions for the correlators might be useful for further studies. The two correlation functions that we obtain for scalars or fermions are parity conserving. Our results indicate that there can be field theories that give rise to the parity breaking contribution. Such field theories are not free, and it would be interesting to find the field theories that produce such correlators

[20] In analogy to flat space, one is tempted to write this in terms of $[\frac{\langle 1,2\rangle^3}{\langle 2,3\rangle\langle 1,3\rangle}]^2$. This is easy to do using the identities in appendix 2.8, but we found simpler expressions in terms of (2.75).

[21]. We will concentrate here on the free theory case[22].

The computation is in principle straightforward, one simple has to compute a one loop diagram with three stress tensor insertions. The only minor complication is the proliferation of indices. To compute the diagram itself one needs to use the standard Feynman parametrization of the loop integral. In particular, it is convenient to use the following Feynman parametrization:

$$\frac{1}{ABC} = \int_0^\infty \frac{2\, d\alpha\, d\beta}{(A + \alpha B + \beta C)^3} \tag{2.78}$$

The final expression will have several contractions of polarization tensors with 3-momenta, e.g. $\epsilon^1_{ij}\epsilon^2_{jk}\epsilon^3_{ki}$, $k^2_i k^2_j \epsilon^1_{ij} k^3_l \epsilon^2_{lm} k^2_n \epsilon^3_{nm}$, etc. Then one needs to use the expressions presented in the appendix to convert these to spinor brackets. The final answers have a simpler form than the ones with the polarization tensors.

We treat first the scalar case and then the fermion case.

2.5.1 Three point correlators for a free scalar

The stress-energy tensor for a real, canonically normalized scalar field is

$$T_{ij}(x) = \frac{3}{4}\partial_i\phi(x)\partial_j\phi(x) - \frac{1}{4}\phi\partial_i\partial_j\phi(x) - \frac{1}{8}\delta_{ij}\partial^2\phi^2(x) \tag{2.79}$$

The two point function is, up to the delta function:

$$\langle T^+ T^+\rangle_\phi = \frac{k^3}{256} \tag{2.80}$$

The three point functions are

$$\begin{aligned}\langle T^+(k_1)T^+(k_2)T^+(k_3)\rangle_\phi &= \Bigg[-\frac{k_1^3+k_2^3+k_3^3}{64} + \frac{(k_1k_2k_3)^3}{2(k_1+k_2+k_3)^6} - \\ &-\frac{(k_1+k_2+k_3)^2}{128}\left((k_1+k_2+k_3) - \frac{(k_1k_2+k_1k_3+k_2k_3)}{(k_1+k_2+k_3)} - \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right)\Bigg]\frac{(\langle\bar{1},\bar{2}\rangle\langle\bar{2},\bar{3}\rangle\langle\bar{3},\bar{1}\rangle)^2}{k_1^2k_2^2k_3^2}\end{aligned} \tag{2.81}$$

$$\begin{aligned}\langle T^+(k_1)T^+(k_2)T^-(k_3)\rangle_\phi &= \Bigg[-\frac{k_1^3+k_2^3+k_3^3}{64} - \\ &-\frac{(k_1+k_2-k_3)^2}{128}\left((k_1+k_2+k_3) - \frac{(k_1k_2+k_1k_3+k_2k_3)}{(k_1+k_2+k_3)} - \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right)\Bigg]\frac{(\langle\bar{1},\bar{2}\rangle\langle\bar{2},3\rangle\langle\bar{1},3\rangle)^2}{k_1^2k_2^2k_3^2}\end{aligned} \tag{2.82}$$

[21] There are bounds similar to the ones derived in [58] for the parity breaking and parity conserving coefficients that appear in a three point function. The parity conserving bounds were considered in [64]. These bounds can be derived by considering a thought experiment where we insert a stress tensor at the origin with some energy and then we look at the angular dependence of the energy one point function, as measured by calorimeters placed at infinity. The stress tensor at the origin has spin ± 2 under the SO(2) rotation group of the spatial plane. The energy one point function as a function of the angle is $\langle\mathcal{E}(\theta)\rangle \sim e^{i4\theta}+1+e^{-i4\theta}$, with coefficients that depend on the parts of the stress tensor three point functions that we have characterized as coming from $(W^+)^3$, R, $(W^-)^3$.

[22] Similar calculations involving the trace of the stress tensor were considered in [50].

2.5.2 Three point correlators for a free spinor

The stress tensor for a complex Dirac spinor is given by

$$T_{ij}(x) = \frac{1}{4}(\bar{\Psi}\gamma_i\partial_j\Psi - \partial_j\bar{\Psi}\gamma_i\Psi) + (i \leftrightarrow j) \tag{2.83}$$

The two point function is

$$\langle T^+T^+\rangle_\psi = \frac{k^3}{128} \tag{2.84}$$

The three point function is given by:

$$\begin{aligned}\langle T^+(k_1)T^+(k_2)T^+(k_3)\rangle_\psi &= \Big[-\frac{k_1^3+k_2^3+k_3^3}{64} - \frac{(k_1k_2k_3)^3}{(k_1+k_2+k_3)^6} - \\ &-\frac{(k_1+k_2+k_3)^2}{64}\left((k_1+k_2+k_3) - \frac{(k_1k_2+k_1k_3+k_2k_3)}{(k_1+k_2+k_3)} - \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right)\Big]\frac{(\langle\bar{1},\bar{2}\rangle\langle\bar{2},\bar{3}\rangle\langle\bar{3},\bar{1}\rangle)^2}{k_1^2k_2^2k_3^2}\end{aligned} \tag{2.85}$$

$$\begin{aligned}\langle T^+(k_1)T^+(k_2)T^-(k_3)\rangle_\psi &= \Big[-\frac{k_1^3+k_2^3+k_3^3}{64} + \\ &-\frac{(k_1+k_2-k_3)^2}{64}\left((k_1+k_2+k_3) - \frac{(k_1k_2+k_1k_3+k_2k_3)}{(k_1+k_2+k_3)} - \frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right)\Big]\frac{(\langle\bar{1},\bar{2}\rangle\langle\bar{2},3\rangle\langle\bar{1},3\rangle)^2}{k_1^2k_2^2k_3^2}\end{aligned} \tag{2.86}$$

2.5.3 Comparison with the gravity computation

We see that these results contain the general shapes discussed in gravity, but they also have an extra term proportional to $\sum k_i^3$. This is a contact term. Namely, it is non-zero only when some operators are on top of each other. In position space we get a delta function of the relative displacement between two of the insertions. These terms are easily recognized in momentum space because they are analytic in two of the momenta. These contact terms represent an ambiguity in the definition of the stress tensor. There is no ambiguity in taking the first derivative with respect to the metric. However, these contact terms involve a second derivative with respect to the metric. So if we define the metric as $g = e^\gamma$ and we take derivatives with respect to γ we are going to get one answer. If we took $g = 1 + \gamma'$ and took derivatives with respect to γ' we would get a different answer. In fact, we have the same ambiguity in the gravity results if we define $\gamma'_{ij} = \gamma_{ij} + \frac{1}{2}\gamma_{il}\gamma_{lj}$. In that case the two results will differ precisely by such a term. It is interesting to note that the non-gaussian consistency condition discussed in [22] does depend on this precise definition of the metric, since a constant γ gives rise to different coordinate transformations depending on how we defined γ. The one derived in [22] holds when the metric is defined in terms of $g = e^\gamma$.

We can note that if we have n_ϕ scalars and n_ψ dirac fermions, then we have to sum the two contributions to the three point functions that we have written above . If $n_\phi = 2n_\psi$, then we see that the term going like $1/E^6$ cancels. This is the contribution that comes from a W^3 term in the bulk. This combination is also the one that appears in a supersymmetric theory. In fact, in a supersymmetric theory the three point function of the stress tensor does not have any free parameters[23]. It is the same as the one given by the pure Einstein theory in the bulk, which does not contain any $1/E^6$ terms. This is related to the fact that in four flat dimensions supersymmetry forces the $+++$ and $---$ amplitudes to vanish [65].

2.6 Discussion

In this chapter we have computed the possible shapes for non-gaussianity for gravitational waves in the de Sitter approximation. Though three possible shapes are allowed by the isometries, only two arise in de-Sitter expectation values. The parity violating shape contributes with a pure phase to the wavefunction and it drops out from expectation values. The two parity conserving shapes were given in equations (2.11), (2.18). One of these shapes is given by the Einstein theory. The other shape arise from higher derivative terms. Under general principles the other contribution can be as big as the Einstein term contribution. Of course, in such a case the derivative expansion is breaking down. However, the symmetries allow us to compute the three point function despite this breakdown. This is expected for an inflationary scenario where the string scale is close to the Hubble scale. This requires a weak string coupling, so that we get a small value of $H^2/M_{Pl}^2 \sim g_s^2$. One of these shapes is parity breaking. These three point functions of gravitational waves are expected to be small, having an $f_{NL-gravity} = \frac{\langle\gamma\gamma\gamma\rangle}{\langle\gamma\gamma\rangle^2}$ of order one. In a more realistic inflationary scenario, which includes a slow rolling scalar field, then we expect that these results give the answer to leading order in the slow roll expansion. It would be interesting to classify the general leading corrections to the graviton three point function in a general inflationary scenario. This can probably be done using the methods of [30, 31, 32, 33]. Here by assuming exact de Sitter symmetry we have managed to compute the correction to all orders in the derivative expansion.

We have presented the result in terms of the three point function for circularly polarized gravitational waves. We used a convenient spinor helicity description of the kinematics. These spinor helicity formulas are somewhat similar to the ones describing flat space amplitudes. It would be interesting to see if this formalism helps in computing higher order amplitudes in de Sitter space. The problem of computing gravitational wave correlators in de Sitter is intimately related with the corresponding problem in Anti-de Sitter. (The two are formally related by taking $R_{dS}^2 \to -R_{AdS}^2$, where R are the corresponding radii of curvature). Thus, all that we have discussed here also applies to the AdS situation. The dS wavefunction is related to the AdS partition function. In this case all three shapes can arise. The parity violating shape arises from the $\int W^2 \widetilde{W}$ term in the action. These three point correlators, for Einstein gravity in AdS, were computed in [34]. It would be

[23]Of course, to have supersymmetry we need to consider an AdS, rather than a dS bulk.

interesting to see if the spinor helicity formalism is useful for computing higher point tree level correlation functions. It is likely to be useful if one uses an on shell method like the one proposed in [66, 67] [24]. In the spinor helicity formalism that we have introduced, we have defined the "time" component of the momentum to be $|\vec{k}|$. The choice of sign here was somewhat arbitrary. When we are in four dimensional flat space, there is a simple physical interpretation for the results we get by analytically continuing to $-|\vec{k}|$, as exchanging an incoming into an outgoing particle. It would be interesting to understand better the interpretation for this analytic continuation of the correlators we have been discussing. This continuation was important in [66, 67].

These computations of three point functions in de Sitter or anti-de Sitter are intimately related to the computation of stress tensor correlators in a three dimensional field theory. In fact, the symmetries are the same in both cases. Therefore the constraints of conformal symmetry are the same. The physical requirements are also very similar. The only minor difference is whether we require the two point function to be positive or not, etc. But in terms of possible shapes that are allowed the discussion is identical. Thus, our results can also be viewed as giving the three point correlation functions for a three dimensional field theory. The position space version of these three point functions was discussed in [34, 68, 69, 70]. For some three point functions the position space version is much simpler. On the other hand, for the stress tensor, the position space correlator has many terms due to the different ways of contracting the indices [45]. The momentum space versions we have written here are definitely shorter than the position space ones. They are a bit convention dependent due to the contact terms. Thus, they depend on precisely how we are defining the metric to non-linear orders. Here we have made a definite choice. An elegant and simple way to write correlation functions is to go to the embedding space formalism [71, 72, 73]. It is likely that one can obtain relatively simple expressions for the three point correlators using that formalism. On the other hand, the momentum space formalism might be useful for constructing conformal blocks, since, in momentum space, there is only one state propagating in the intermediate channel.

2.7 Appendix A: Expression for the three point function in terms of an explicit choice for polarization tensors

As we are studying three point functions, there is a way to define polarization tensors that are similar to the usual "$\times$" and "$+$" of General relativity. We call them X and P here, so as not to cause confusion with the helicity labels $+$ and $-$. The choice of helicity states is based on the little group of an Euclidean 3D CFT.

[24] Raju's proposal [66, 67] is only for $D > 4$ dimensions, but it might be possible that something similar exists in $D = 4$.

Basically, one takes two possible polarizations, P and X, as functions of a vector orthogonal to the plane of the triangle and of a vector orthogonal to one of the momenta we are looking at. So, using the notation defined in figure 2.1, we have that

$$\epsilon_{ij}^{P,m} = 2(z_i z_j - u_i^m u_j^m) \tag{2.87}$$

$$\epsilon_{ij}^{X,m} = 2(u_i^m z_j + z_i u_j^m) \tag{2.88}$$

And the previously discussed $+$ and $-$ polarizations will be given by $\pm = P \pm iX$. P and X are the polarizations known as $+$ and $\times$ in general relativity, but we choose to use different labels so that the former is not interpreted as positive helicity by mistake.

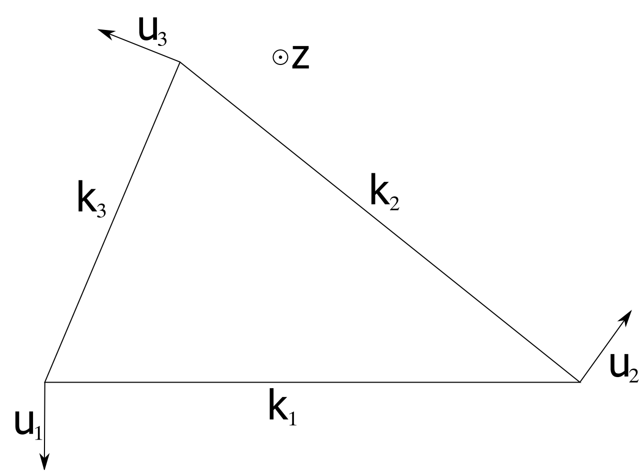

Figure 2.1: The 3-momenta and the auxiliary vectors used to define the polarizations.

We list here the results for the non-gaussianities due to the Einstein term and the Weyl term. The relevant pieces are labeled by the polarization choices PPP and XXP. We always take particle three to have polarization P. The other structures are obtained by cyclic permutation. There are no PPX and XXX structures because they break parity, since z flips under parity so that X is odd and P is even. We use here the notation $J(k_1, k_2, k_3) \equiv 2(k_1^2 k_2^2 + k_1^2 k_3^2 + k_2^2 k_3^2) - (k_1^4 + k_2^4 + k_3^4)$.

$$\langle\gamma^P_{k_1}\gamma^P_{k_2}\gamma^P_{k_3}\rangle_R = (2\pi)^3\delta^3\left(\sum_i k_i\right)\left(\frac{H}{M_{Pl}}\right)^4\frac{-1}{4(k_1k_2k_3)^5}\left[J(k_1,k_2,k_3)\left(\sum_{i=1}^3 k_i^4+6\sum_{i<j}k_i^2k_j^2\right)\right]$$
$$\left(k_1+k_2+k_3-\frac{k_1k_2+k_1k_3+k_2k_3}{k_1+k_2+k_3}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right) \tag{2.89}$$

$$\langle\gamma^X_{k_1}\gamma^X_{k_2}\gamma^P_{k_3}\rangle_R = (2\pi)^3\delta^3\left(\sum_i k_i\right)\left(\frac{H}{M_{Pl}}\right)^4\frac{1}{(k_1k_2k_3)^4}\left[J(k_1,k_2,k_3)\frac{k_1^2+k_2^2+3k_3^2}{k_3}\right]$$
$$\left(k_1+k_2+k_3-\frac{k_1k_2+k_1k_3+k_2k_3}{k_1+k_2+k_3}-\frac{k_1k_2k_3}{(k_1+k_2+k_3)^2}\right) \tag{2.90}$$

$$\langle\gamma^P_{k_1}\gamma^P_{k_2}\gamma^P_{k_3}\rangle_{W^3} = (2\pi)^3\delta^3\left(\sum_i k_i\right)\left(\frac{H}{M_{Pl}}\right)^6\left(\frac{H}{\Lambda}\right)^2 a\frac{2160}{(k_1+k_2+k_3)^4(k_1k_2k_3)^2}J(k_1,k_2,k_3)=$$
$$=(2\pi)^3\delta^3\left(\sum_i k_i\right)\left(\frac{H}{M_{Pl}}\right)^6\left(\frac{H}{\Lambda}\right)^2 a\frac{270(k_1+k_2-k_3)(k_2+k_3-k_1)(k_3+k_1-k_2)}{(k_1+k_2+k_3)^3(k_1k_2k_3)^2} \tag{2.91}$$

$$\langle\gamma^X_{k_1}\gamma^X_{k_2}\gamma^P_{k_3}\rangle_{W^3} = -\langle\gamma^P_{k_1}\gamma^P_{k_2}\gamma^P_{k_3}\rangle_{W^3} \tag{2.92}$$

2.8 Appendix B: Details on the spinor helicity formalism

Here we summarize some conventions that we have used.

- Metric: $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$; $\epsilon_{\dot{a}a} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; $\epsilon^{a\dot{a}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
- Sigma matrices: $\sigma^{\mu a}{}_b = (-\delta^a{}_b, \sigma^{ia}{}_b)$;
- Scalar product: $p.q = -2\langle\lambda_p,\lambda_q\rangle\langle\bar\lambda_p,\bar\lambda_q\rangle$; Energies: $p^0 \equiv p = -\langle\lambda_p,\bar\lambda_p\rangle = -\epsilon_{ab}\lambda^a_p\bar\lambda^b_p$
- Polarization vectors used for the expressions in the appendix (normalization is not the same as the one used in the previous sections for the stress tensor): $\xi^{-a\dot{a}} = -\frac{\lambda^a\lambda^{\dot{a}}}{k}$ and $\xi^{+a\dot{a}} = \frac{\bar\lambda^a\bar\lambda^{\dot{a}}}{k}$

Starting from a three momentum $\vec{k}$, we define a four momentum $k^\mu = (|k|,\vec{k})$. This obeys $k^\mu k_\mu = 0$. This defintion can be done for dS, AdS, or a three dimensional CFT. Note that $k^i = \hat\sigma^i{}_{\dot{a}b}\lambda^b\bar\lambda^{\dot{a}}$, where $\hat\sigma^i$ are the Pauli matrices with an index lowered by $\epsilon_{\dot{a}c}$. These matrices are symmetric. Thus we conclude that if we exchange $\lambda^a \leftrightarrow \bar\lambda^a$ we keep the value of $\vec{k}$, but we change the sign of the energy $k^0 \equiv k$. This change is not consistent with the reality conditions which are $(\lambda^*)^a = \epsilon_{\dot{b}a}\bar\lambda^{\dot{b}}$, $(\bar\lambda^*)^{\dot{b}} = -\epsilon_{\dot{b}a}\lambda^a$. However, if we just forget about the reality condition, then the exchange of $\lambda \leftrightarrow \bar{\lambda}$ is allowed. Note also that the following symmetry is consistent with the reality conditions and it reverses the sign of $\vec{k}$ but does not change the sign of k. Namely, the exchange $\lambda' = \bar{\lambda}$, $\bar{\lambda}' = -\lambda$. This is useful for the two point function, where the momentum conservation condition forces the spatial momenta to be opposite. Note that in some formulas we write k_i, which means $|\vec{k}|$. This can also be written in terms of $k_i = -\langle \lambda, \bar{\lambda} \rangle$.

For a given 3-momentum $\vec{k} = (k_1, k_2, k_3)$ if we define $|\vec{k}| \equiv k_0$ then one can for example take the following explicit choice of spinors, assuming that the reality condition is satisfied:

$$\lambda^a = \left(\sqrt{\frac{k_0 + k_3}{2}}, \frac{-k_1 + ik_2}{\sqrt{2(k_0 + k_3)}} \right)^T \quad ; \quad \bar{\lambda}^{\dot{a}} = \left(\frac{-k_1 - ik_2}{\sqrt{2(k_0 + k_3)}}, -\sqrt{\frac{k_0 + k_3}{2}} \right) \tag{2.93}$$

Let us now summarize a few identities that are useful for the treatment of the three point function. For the case of the three point function the 3-momentum conservation condition reads

$$\lambda_1^a \bar{\lambda}_1^{\dot{a}} + \lambda_2^a \bar{\lambda}_2^{\dot{a}} + \lambda_3^a \bar{\lambda}_3^{\dot{a}} = -\frac{E}{2} \epsilon^{a\dot{a}} \ , \qquad E \equiv k_1 + k_2 + k_3 = -\sum_{n=1}^{3} \langle \lambda_n, \bar{\lambda}_n \rangle \tag{2.94}$$

Where the coefficient two is determined from contracting the a and $\dot{a}$. We can contract this expression with, say, λ_1 and $\bar{\lambda}_2$. The purpose of that is to derive an expression for an object of the form $(m, \bar{n})$, which has no interpretation as an energy, if $m \neq n$. By doing that we find that $2\langle 1, 3 \rangle \langle \bar{3}, \bar{2} \rangle = E(1, \bar{2})$ and hence, we can write the general expression:

$$\langle m, \bar{n} \rangle = -\sum_{o \neq m,n} \frac{2\langle m, o \rangle \langle \bar{o}, \bar{n} \rangle}{E} \tag{2.95}$$

Where $m \neq n \neq o$. Also, note that the 4-momentum product of two distinct 4-momenta, in terms of the three energies and the total energy, is given by:

$$-2\langle m, n \rangle \langle \bar{m}, \bar{n} \rangle = k^m_{\ \mu} k^{n\mu} = -k^m k^n + \mathbf{k^m}.\mathbf{k^n} = \frac{1}{2}(k_o - k_m - k_n)E \tag{2.96}$$

And we also use the Schouten identity, which is useful to write a given spinor λ in terms of two reference spinors μ and ξ

$$\langle \xi, \mu \rangle \lambda^a = \langle \xi, \lambda \rangle \mu^a - \langle \mu, \lambda \rangle \xi^a \tag{2.97}$$

Whenever we have an expression in terms of angle brackets we can write it completely in terms of $\langle m, n\rangle$ brackets and the $k_m = -\langle m, \bar{m}\rangle$ brackets. Useful identities to do so are (for $m \neq n$)

$$\langle m, \bar{n}\rangle = \frac{\langle m, o\rangle}{\langle n, o\rangle}\frac{(k_o + k_n - k_m)}{2} \ , \qquad \langle \bar{m}, \bar{n}\rangle = -\frac{(k_o - k_m - k_n)E}{4\langle m, n\rangle} \tag{2.98}$$

2.8.1 Some expressions involving polarization vectors

Now, let us calculate all the possible contractions of polarization vectors and momenta of different helicities:

$$\xi^+_m.\xi^+_n = -2\frac{\langle \bar{m}, \bar{n}\rangle^2}{k_m k_n} \tag{2.99}$$

$$\xi^-_m.\xi^-_n = -2\frac{\langle m, n\rangle^2}{k_m k_n} \tag{2.100}$$

$$\xi^+_m.\xi^-_n = 2\frac{\langle \bar{m}, n\rangle^2}{k_m k_n} \tag{2.101}$$

As for contractions of momenta with polarization vectors, we have [25]

$$k_m.\xi^+_n = -\frac{2\langle m, \bar{n}\rangle\langle \bar{m}, \bar{n}\rangle}{k_n} \tag{2.102}$$

$$k_m.\xi^-_n = \frac{2\langle m, n\rangle\langle \bar{m}, n\rangle}{k_n} \tag{2.103}$$

These can also be used to convert the gravity expressions into expressions in the spinor helicity variables since the gravity polarization tensor is $\epsilon_{ij} \sim \xi_i \xi_j$.

2.9 Appendix C: Comments on the parity breaking piece of the two point function

As pointed out in [52] , the gravitational wave two point correlation function (or gravitational wave spectrum) can be different for the two circularly polarized waves without breaking rotation symmetry. In fact, a bulk coupling of the form $\int f(\phi) W\widetilde{W}$ is enough to produce this. This mechanism requires an inflaton. One can ask whether a parity breaking two point function is possible in de Sitter space, as some authors have suggested [54]. Here we make some comments on the parity breaking pieces of the two point function of the stress tensor.

The summary is that parity breaking terms are allowed in the gaussian part of the wavefunction of the

[25] As there is no time component of the polarization vector, $k^{a\dot{a}}\epsilon_{\dot{a}a} \sim k_\mu \epsilon^\mu = k_i \epsilon_i$ so we are only taking the space components into account.

universe, or in the two point function of CFT's. However, such terms are local, and contribute with a phase to the wavefunction. Thus they do not lead to different amplitudes for left and right circular polarizations.

Let us start by discussing this from the wavefunction of the universe point of view. From that point of view the question is whether there can be a parity breaking two point function for the stress tensor. One is tempted to say that the answer is no. The argument is the following. The stress tensor is in a single representation of the conformal group, thus its two point function should be uniquely fixed. In fact, this is correct if we consider the two point function at different spatial points. However, there can be a parity breaking contact term. In order to understand this, let us discuss first the case of a current, or a gauge field in the bulk, and then discuss the case of the graviton or the stress tensor.

2.9.1 Parity breaking terms in the two point functions for currents

These were discussed in the AdS context in [74]. We just summarize the discussion here. The Fourier transform of the conserved current two point function is (in a 3D CFT)

$$\langle J_i(k_1)J_j(k_2)\rangle = \delta^3(k_1+k_2)\left[(\delta_{ij}k^2 - k_ik_j)|k|^{-1} + \theta\epsilon_{ijl}k_{1,l}\right] \tag{2.104}$$

This is consistent with conformal symmetry. It is annihilated by (2.45). The θ term breaks parity. Since this term is analytic in the momentum, it gives rise to a contact term in position space, a term proportional to $i\epsilon_{ijl}\partial_{x_l}\delta^3(x-y)$. If we couple the current to an external source A_μ and compute $\Psi[A] = Z[A.J]$, then we are just adding a local term to the wavefunction of the Chern-Simons form $\psi_\theta(A) = e^{i\theta\int d^3xAdA}\psi_{\theta=0}(A)$ [26]. This is what we would get in a dS situation if we have an ordinary θ term in the bulk. In other words, if we have a gauge field in the bulk with an interaction $\theta\int Tr[F\wedge F]$, then the wavefunction contains a term proportional to the Chern-Simons action on the spatial slice. In a unitary (and gauge invariant) bulk theory this term has a real value of θ, so that it contributes as a phase in the wavefunction. Thus, when we compute the square of the wavefunction, this terms drops out. More explicitly, we can now compute the wavefunction in momentum space

$$\psi(A) = \exp\{-A_i(k)A_j(-k)(\delta_{ij}|k| - \frac{k_ik_j}{|k|} + \theta\epsilon_{ijs}k_s)\} \tag{2.105}$$

We see that the θ term is imaginary if we take θ to be real. (We use that $A(k)^* = A(-k)$). Then, if we compute $|\psi(A)|^2$ we find that the θ term drops out if θ is real.

Now we can ask, could it be that some unknown unitary Hamiltonian produces a wavefunction that

[26] For the non abelian case, we can complete this quadratic action into the full cubic one.

contains a Chern-Simons part with a purely imaginary θ? The arguments leading to the Chern-Simons term in the wavefunction were based purely on demanding conformal symmetry and did not rely on any assumptions about the bulk Hamiltonian, or even its existence. All we are assuming is that we have a wavefunction that is conformal invariant. In particular, purely from conformal symmetry, the θ term could be imaginary. An imaginary θ leads to different amplitudes for the two circular polarization states of the gauge field[27].

One problem with this is that the resulting probability amplitude is now not invariant under large gauge transformations. This is due to the fact that the Chern-Simons action shifts by a certain real factor under a large gauge transformation. Thus the wavefunctions produced in this way are not gauge invariant [28]. This argument is most clear in a non-abelian situation.

However, if we ignore this problem, then we should also point out another issue. A different amplitude for left and right circularly polarized waves violates CPT invariance[29]. This is most clear if we think about the observer in static patch coordinates. This observers sees de Sitter as static. For this observer CPT is a symmetry, it is not spontaneously broken by the background. However, CPT transforms + circular polarization into $-$. Thus these amplitudes cannot be different. Note that the wavefunctions that we discussed are the late times ones, the wavefunctions for fluctuations outside the horizon. On the other hand, the static patch observer probes the wavefunction inside the horizon. So, here we have assumed, by continuity, that if we get a parity breaking effect outside the horizon, then we should also see some effect inside the horizon.

2.9.2 Parity breaking two point functions for the stress tensor

Similar arguments can be used for the two point function of the stress tensor in momentum space. The only term we can write that breaks parity, by power counting, is

$$\langle T_{ij}(k)T_{mn}(-k)\rangle_{odd} \sim [(\epsilon_{iml}k_l\delta_{jn}k^2 + (i \leftrightarrow j)) + (m \leftrightarrow n)] \tag{2.106}$$

This is a function of k_i and k^2, hence, it is analytic and corresponds to a local term in position space.

In gravity, an analog of the θ term is the topological invariant

$$\int Tr[R\wedge R] = \int \epsilon^{\mu\nu\rho\sigma} R^{ab}_{\mu\nu} R^{ab}_{\rho\sigma} = \int \epsilon^{\mu\nu\rho\sigma} R^{\gamma\delta}{}_{\mu\nu} R^{\gamma\delta}{}_{\rho\sigma} = \int W\widetilde{W} \tag{2.107}$$

[27]We are not assuming that we have an ordinary θ term in the bulk, but simply that some unknown dynamics gives rise to a θ term in the wavefunction as in (2.105).

[28]This argument was suggested to us by E. Witten.

[29]It is not known whether CPT invariance holds in quantum gravity. In theories that have a dual CFT description, like the ones in AdS, CPT is a good symmetry, so it seems reasonable to assume that CPT will still be a symmetry of dS.

The last equality follows from the symmetries of the Riemann curvature. Adding this term as $\theta \int W\widetilde{W}$ to the bulk action we get a contribution to the wavefunction of the form

$$e^{i\theta S_{CS}(\omega)} \tag{2.108}$$

where ω is the spin connection. It can also be written in terms of the Christoffel connection [30]. Let us check that this term indeed produces (2.106). We expand the Chern-Simons term to quadratic order and obtain

$$S_{CS} \sim \int \epsilon^{ijk} \Gamma^r_{is} \partial_k \Gamma^s_{jr} \tag{2.109}$$

Using the first order expressions of the connection we find

$$S_{CS} \propto \epsilon_{ijl} (\partial_r \gamma_{si} - \partial_s \gamma_{ri}) \partial_l \partial_r \gamma_{sj} \qquad \to \qquad \epsilon_{ijl} k^2 \gamma(k)_{si} k_l \gamma(-k)_{si} \tag{2.110}$$

where we used that γ is transverse, $k_s \gamma_{si} = 0$. This indeed reproduces (2.106).

As in the gauge field case, if θ is real, this term disappears from $|\Psi|^2$. On the other hand, if θ is imaginary, we do get an extra contribution to $|\Psi|^2$ which leads to a different amplitude for left and right circularly polarized gravitational waves, as pointed out in [54].

Note that S_{CS} in (2.108) is not invariant under large gauge transformations of the local Lorentz indices of the spin connection.

Again CPT invariance forbids a different amplitude for left and right circular polarization.

Note that all the remarks in this section apply to the case of pure de Sitter. In the case that we have an inflationary background, time reversal symmetry is broken by the inflaton and we can certainly have a parity violating two point function [52].

2.10 Appendix D: Commuting through the delta function

In this appendix we show that the action of the special conformal generator (2.45) on a correlator or expectation value of the form $\delta(P)\mathcal{M}$ is equal to the action of the operator on $\mathcal{M}$. In other words, (2.45) commutes with the momentum conserving delta function. The point is to understand how to get all the momentum derivatives through the momentum conserving δ function. From now on all derivatives will be k derivatives.

[30] In fact, we can use the relation between the two connections that comes from demanding that $D_\mu \epsilon^a_\nu = 0$, which is $\omega^{ab}_\mu = e^a_\alpha \Gamma^\alpha_{\mu\nu} E^{\nu b} - \partial_\mu e^a_\nu E^{\nu b}$ In this form it has the form of a $GL(N)$ gauge transformation, $\omega_\mu = g \Gamma_\mu g^{-1} - \partial_\mu g g^{-1}$ with $g = e^a_\alpha$. Of course, gauge transformations are a symmetry of the CS action. Thus we get the same action in terms of both connections. The first (upper) and last indices of Γ are viewed as the "internal" GL(N) indices of the connection.

The delta function depends only on the sum of all momenta, let us call that P. Then the sum over all particles of D_a where a runs over particle number has the form

$$\left(\sum_a \widetilde{D}_a \delta^3(P)\right)\mathcal{M} = 6[(b.\partial_{\vec{P}})\delta^3]\mathcal{M} \tag{2.111}$$

In order to derive this we have done the following. In each term the derivatives are with respect to k_a, which end up ∂_P when acting on the δ function. Then the k_a in $\widetilde{D}$ all sum up to P. Thus we have a term of the form $P(\partial_P\partial_P\delta)\mathcal{M}$. We then integrate by parts the derivatives to act on $\mathcal{M}$ in such a way that we get terms of the form in (2.111) and also terms of the form $P\delta(P)\partial_P\partial_P\mathcal{M}$. Such terms vanish. Thus (2.111) is the total contribution from terms with two derivatives on the delta function.

We can now consider terms which have only one derivative on the delta functions. There are terms coming from $\widetilde{D}$. Let us consider those first. The first term in $\widetilde{D}$ contributes with

$$2[\partial_{P_j}\delta](2b^i.k_a^i)\partial_{k_a^j} \tag{2.112}$$

The second term gives

$$2[\partial_{P_j}\delta]\left\{(-2k_a^j)(b.\partial_{k_a}) - 2b^j(k.\partial_k)\right\} \tag{2.113}$$

The (3.46) and the first term in (2.113) give the action of the rotation generators on the term multiplying the δ function. These would make the correlator vanish if it was rotational invariant. On the other hand, we have indices, thus the correlator is rotational covariant. The action of the rotation generators has to results in some action on the indices. These must arise from the action of $\widetilde{M}$ on the δ function, producing the spin generators. Finally we have the last term, which adds up to the dilatation generator. Such terms combine with a derivative of a δ function in (2.111) and also a derivative that comes from the first term in (2.45) . They altogether sum up to

$$-(b.\partial_P\delta)2\left\{\sum_a(\Delta_a - 3) \quad + 3 - \sum_a k_a.\partial_{k_a}\right\} \tag{2.114}$$

The last term is computing the overall dimension of the term that multiplies the δ function, the $+3$ is taking into account the dimensions of the δ function. And the first term is the total dimension of the (Fourier transform) of the external states. Thus, if the answer is dilatation covariant these terms will vanish. The total dilatation eigenvalue $k\partial_k$ of the coefficient of the delta function is then $3 + \sum_a(\Delta_a - 3)$.

2.11 Appendix E: Checking the Ward identities for the stress tensor

The Ward identities come from the statement that the wavefunction of the universe is reparametrization invariant $\Psi[g_{ij}] = \Psi[g_{ij} + \nabla_{(i}v_{j)}]$. This then implies that

$$\nabla^i \left[\frac{1}{\sqrt{g}} \frac{\delta \Psi}{\delta g^{ij}} \right] = 0 \tag{2.115}$$

The stress tensor correlators are defined by taking multiple derivatives of the wavefunction and then setting g to the flat metric. By taking multiple derivatives of (2.115) we get the Ward identity which looks like

$$\partial_i \langle T_{ij}(x) T_{l_1,m_1}(y_1) \cdots T_{l_n,m_n}(y_r) \rangle = \sum_{s=1}^{r} D_{l_s m_s}^{l'_s m'_s} \delta^3(x - y_s) \langle T_{l_1,m_1}(y_1) \cdots T_{l'_s,m'_s}(y_s) \cdots T_{l_n,m_n}(y_r) \rangle \tag{2.116}$$

where D is a first order derivative (acting on x or y_s) and the indices are contracted in some way. These terms come from acting with the metric derivatives on the explicit metric dependence in the covariant derivative, etc. Notice that if all the points in the left hand side are different from each other, then the right hand side is zero. In this section we will assume that all the points $y_1, \cdots, y_r$ are different from each other. The precise form of the contact terms in the right hand side depends on the precise definition of the "derivative with respect to the metric". If we define the stress tensor as derivatives of the from $\frac{\delta}{\delta g^{ij}}$, then the Ward identities can be found in [45]. However, in our case, we defined the stress tensor correlators as derivatives of the wavefunction with respect to γ_{ij}, where we write the metric as $g = e^{\gamma}$, see (2.34). This leads to slightly different expressions. The difference is only present as extra contact terms that arise when we use the chain rule $T_{ij}^{\text{ours}} = \frac{\delta}{\delta \gamma^{ij}} = \frac{\delta g^{lm}}{\delta \gamma^{ij}} \frac{\delta}{\delta g^{lm}}$.

One can keep track of these extra terms and write the precise version of the Ward identity.

For the case of the three point function, after going to Fourier space, we get the simple expression [31]

$$\langle [k_1 \xi^1 T(k_1)][\xi_2 \xi_2 T(k_2)][\xi_3 \xi_3 T(k_3)] \rangle = \xi_2.\xi_3 \left\{ \xi_1.k_3 \xi_2.\xi_3 + \xi_2.k_1 \xi_1.\xi_3 + \xi_3.k_2 \xi_1.\xi_2 \right\} \left[2k_2^3 - 2k_3^3 \right] \tag{2.117}$$

Which, for $---$ is given by:

$$\langle [k_1 \xi^1 T(k_1)][\xi_2 \xi_2 T(k_2)][\xi_3 \xi_3 T(k_3)] \rangle = 8 \langle 1, 2 \rangle \langle 2, 3 \rangle^3 \langle 3, 1 \rangle \frac{(k_1 + k_2 + k_3)(k_2^3 - k_3^3)}{(k_1 k_2 k_3)^2} \tag{2.118}$$

From the point of view of the operator (2.46), this expression (2.118) should be multiplying a term that

[31] This gets simplified thanks to the fact that $\langle T_{ij}(-k)[\xi \xi T(k)] \rangle = 2\xi^i \xi^j k^3$.

goes like $(k_1^{-3})\xi^1(...)$. The coefficient of the three point function is then fixed by comparing this with the result of acting on the three point functions of Einstein gravity with (2.46).

For a general $---$ three point function, given by $(\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle)^2 f(k_1,k_2,k_3)$, the action of (2.46) is [32]

$$(\sigma^i)^{a\dot a}\frac{\partial^2}{\partial\lambda_1^a\partial\bar\lambda_1^{\dot a}}\left[(\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle)^2 f(k_1,k_2,k_3)\right] =$$
$$= [\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle]^2\left[(6+2k_1^m\partial_{k_1^m})\partial_{k_1^i}f - k_1^i\partial_{k_1^l}\partial_{k_1^l}f\right]+$$
$$+\, i\epsilon^{ijk}\frac{\xi_1^k k_1^j}{k_1}[2\langle 1,2\rangle\langle 2,3\rangle^2\langle 3,1\rangle[\langle 2,\bar 1\rangle\langle 3,1\rangle-\langle 3,\bar 1\rangle\langle 1,2\rangle]]\partial_{k_1}f \tag{2.119}$$

In order to derive this, we used Schouten to express λ_a^2,λ_a^3 each as a function of λ_a^1 and $\bar\lambda_a^1$. As ξ^1 is a complex vector, one can show that $i\epsilon^{ijk}\xi_1^k\frac{k_1^j}{k_1}=\xi_1^i$. Then the third line is proportional to ξ^1. The next step is to write the first piece as a function of k_1 and not of its components. The final result is:

$$\sum_n(\sigma^i)^{a\dot a}\frac{\partial^2}{\partial\lambda_n^a\partial\bar\lambda_n^{\dot a}}\left[(\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle)^2 f(k_1,k_2,k_3)\right] = -2\xi_1^i(1,2)(2,3)^3(3,1)[k_3-k_2]\partial_{k_1}f+$$
$$+\,[\langle 1,2\rangle\langle 2,3\rangle\langle 3,1\rangle]^2\left[\frac{4}{k_1}\partial_{k_1}f+\partial_{k_1}^2 f\right]k_1^i+\text{cyclic} \tag{2.120}$$

Although the ξs are linearly independent, the ks are not. A convenient way to rewrite (2.120) is to choose special conformal transformation parameters b^i to project out a few components. Let us take $b^i\sim(\lambda_2\lambda_3+\lambda_3\lambda_2)^i$ for example. The constraint is

$$(\lambda_2\lambda_3).\widehat{\mathcal{O}}[(---)f] = \langle 1,2\rangle^2\langle 2,3\rangle^3\langle 3,1\rangle^2\left\{4(\partial_{k_2}-\partial_{k_3})f+k_3(\partial_{k_1}^2-\partial_{k_3}^2)f-k_2(\partial_{k_1}^2-\partial_{k_2}^2)f\right\} \tag{2.121}$$

For the $--+$ correlator, the derivation is similar.

[32]When there is no index in the derivative, it is understood that we are taking the derivative with respect to the energy, or $|k|$. We hope this does not cause confusion in the expressions here.

Chapter 3

Inflationary Consistency Conditions from a Wavefunctional Perspective

3.1 Introduction

Recently, there has been much interest in calculating non-gaussian deviations for the statistics of primordial perturbations generated by inflation. Signatures of primordial non-gaussianity could falsify various models of the early universe. One is in general interested in computing three point expectation values of fields, evaluated at late times, when all modes have exited the horizon.

Maldacena pointed out [22] that there is a nice consistency check for the three point function of (single field) inflation. Namely, when one considers a “squeezed” triangle shape, where one of the momenta is much smaller than the others (their sum needs to be zero due to translational invariance), the three point function can be written in terms of the tilt of the spectrum of the two point function.

The intuition behind this consistency condition is as follows. In the squeezed regime, the long wavelength mode has exited the horizon earlier than the other modes, so its effect is to rescale the coordinates at which one computes the power spectrum for the other, shorter wavelength fluctuations. This intuition turns out to be correct for all models with a single field setting the natural “clock” for the inflationary period [75] , and is thus a way to falsify single field inflation.

The consistency condition has been checked in many different models, and was derived in various different ways. An incomplete list of references is [75, 32, 31]. The original consistency condition concerns only the leading term on the momentum of the long wavelength mode. Considerable progress has been made since, by several groups, on studying subleading corrections to the squeezed limit. Also, consistency conditions coming from soft internal momenta were found, relevant to squeezed limits of expectation values with four or more legs [76, 77, 78, 79, 80, 81, 82]. The derivations attack the problem from various perspectives. They either explore the broken symmetries of the (quasi-de Sitter) background, or some residual diffeomorphism invariance of the metric that was not completely fixed. There are also approaches that take the long mode as a classical background perturbation over which the shorter modes evolve.

In this chapter, a different derivation of these results is provided. The object of primary interest will be the wavefunction of the universe, $\Psi[h, \phi]$, which has information on the probability for spacetime to have a spacelike slice with a given 3-metric and additional field profiles (for single field inflation, we also specify the profile of the inflaton on the slice). In this formalism, the wavefunction is specified by the so-called Wheeler-DeWitt equation [83]. We will show that coordinate reparametrization invariance of the three slice, also known as the momentum constraint, has all the information on squeezed limits of inflationary expectation values. In other words, all known consistency conditions follow from a symmetry of the wavefunction of the universe, or a constraint on its form. The wavefunctional perspective was also used to derive consistency conditions in [82].

Our situation here is analogous to the following in a gauge theory. We can compute Feynman diagrams and find correlation functions for the gauge fields A_μ. These correlation functions satisfy some transversality condition, which basically removes the unphysical longitudinal modes from physical observables, like scattering amplitudes, and preserve unitarity etc. These are the Ward identities satisfied by the correlation functions. In gauge theory, we know what the good, gauge invariant observables should be (for example, correlations of field strengths $F_{\mu\nu}$, or Wilson loops). In gravity, a good observable should be diffeomorphism invariant.

When we compute the expectation values, there are still "longitudinal modes", or unphysical information, in these functions. The consistency condition basically tells us that the leading and next to leading order terms in the squeezed limit are fixed by this pure gauge information. From the point of view of a "metaobserver" that sees our universe from outside, these would be unphysical modes. Because we have to pick a frame to make observations in cosmology, we would measure a squeezed non-gaussianity. The point is that it is fixed basically by the field content of the inflationary theory, and not from the details of the field interactions etc. This effect was recently computed and discussed in [84], in the context of translating the inflationary expectation values to CMB fluctuations in the sky.

In [85], it was pointed out that, in fact there is an infinite number of such consistency conditions, and, at each order in the long mode Taylor series, there should be terms fixed by diffeomorphism invariance. With the wavefunctional approach, that becomes very clear, as the consistency conditions can be extracted from a power series expansion of an expression of the schematic form:

$$k_L^i \frac{\delta^n \Psi}{\delta h_{ij}(k_L)\delta h(k_1)\delta h(k_2)...} = -k_1^j \frac{\delta^{n-1}\Psi}{\delta h(k_1+k_L)\delta h(k_2)...} - k_2^j \frac{\delta^{n-1}\Psi}{\delta h(k_1)\delta h(k_2+k_L)...} - \cdots \tag{3.1}$$

Where we omit indices of the other metric insertions for simplicity. These functional derivatives can be mapped to tree level expectation values of the fluctuating fields (metric, inflaton etc.). We can expand (3.1) around $k_L = 0$ and, at each order, it will provide a consistency condition. In fact, (3.1) totally determines the form of the derivative of the wavefunction to leading and subleading orders. From quadratic order on, we cannot fully constrain this functional derivative, and that is when the truly physical contributions to the squeezed limit appear [84, 86].

One nice feature of this wavefunctional formalism is that it is easily extended to theories with more fluctuating fields. Also, tracking the consequences of other symmetries of the wavefunction, like some flavor symmetry of the scalar sector, seems to be straightforward in this language.

The chapter is structured as follows. In section 2, we review the Wheeler-DeWitt formalism, in particular for Einstein gravity in de Sitter space, and for single field inflation. In section 3, we briefly treat the Hamiltonian constraints and comment on their implications. In section 4, we write the consistency condition from the wavefunctional perspective. In section 5, we derive the consequences for expectation values of fields, focusing on three point functions. In section 6, we make a few observations on gauge/gravity duality. In section 7 we discuss our results, followed by a few appendices on technical details. In particular, appendix B shows a somewhat simple but still interesting extension of the consistency condition to a single field inflation model with an additional massless scalar.

3.2 Gravity in the Schrodinger picture: the Wheeler-DeWitt equations

In the Wheeler-DeWitt approach to perturbative quantum gravity, the object of interest is the wavefunction of the universe. It gives the probability that the spacetime has a spatial slice with given 3-metric and field profile. The equations express the time and space reparametrization invariance of the wavefunction. The spatial reparametrization invariance implies the so-called momentum constraint on the wavefunction, and is an expression analogous to Gauss' law in electromagnetism. The time reparametrization invariance encodes the dynamics of the theory. These are properly described in the $3+1$ decomposition of the metric, or the ADM formalism.

In this section we review how to obtain the Wheeler-DeWitt equations from the ADM decomposition of the metric. We analyze two particular cases of interest, namely, Einstein gravity in de Sitter space and single field inflation.

3.2.1 Einstein Gravity with positive Cosmological constant

Start from the action:

$$S = \kappa \int \sqrt{-g}({}^{(4)}R - 2\Lambda) \tag{3.2}$$

With $\kappa \equiv (16\pi G_N)^{-1} = \frac{M_{Pl}^2}{2}$. Then, in the ADM decomposition, $ds^2 = -N^2dt^2 + h_{ij}(N^i dt + dx^i)(N^j dt + dx^j)$, the action reads:

$$S = \kappa \int N\sqrt{h}\left[K_{ij}K^{ij} - K^2 + R - 2\Lambda\right], \quad K_{ij} \equiv \frac{1}{2N}(\dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i) \tag{3.3}$$

Indices in (3.3) and from now on are raised with h^{ij} instead of the 4D metric. We are omitting some boundary contributions which were subtracted by the Gibbons-Hawking-York term. The conjugate momenta to the metric are:

$$\begin{aligned} &\pi \equiv \frac{\delta L}{\delta \dot{N}} = 0, \qquad \pi^i \equiv \frac{\delta L}{\delta \dot{N}_i} = 0 \\ &\pi^{ij} \equiv \frac{\delta L}{\delta \dot{h}_{ij}} = \kappa\sqrt{h}(K^{ij} - h^{ij}K) \end{aligned} \tag{3.4}$$

So the Hamiltonian will be of the form:

$$\begin{aligned} &H = \int \left\{ N\left[\frac{1}{2\kappa\sqrt{h}}G_{ij,kl}\pi^{ij}\pi^{kl} - \kappa\sqrt{h}(R - 2\Lambda)\right] + 2\nabla_i N_j \pi^{ij}\right\} \\ &G_{ij,kl} \equiv (h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}) \end{aligned} \tag{3.5}$$

$G_{ij,kl}$ is called the DeWitt metric[1]. Quantization on a basis that is diagonal in the three metric h_{ij} corresponds to promoting $\pi^{ij} \to -i\hbar\frac{\delta}{\delta h_{ij}}$. Then, variation with respect to the lapse and shift yields the equations:

$$\begin{aligned} &\left[\frac{\hbar^2}{2\kappa\sqrt{h}}G_{ij,kl}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{kl}} + \kappa\sqrt{h}(R - 2\Lambda)\right]\Psi = 0 \\ &-2i\hbar\nabla_i\left[\frac{1}{\sqrt{h}}\frac{\delta\Psi}{\delta h_{ij}}\right] = 0 \end{aligned} \tag{3.6}$$

[1] In the literature, the factor of $1/\sqrt{h}$ is usually absorbed in the definition of G, so $G_{here} = \sqrt{h}G_{DeWitt}$

3.2.2 Gravity plus a Scalar field

Now write the action as follows:

$$S = \kappa \int (R - (\nabla\phi)^2 - 2V(\phi)) \tag{3.7}$$

Again, using the ADM decomposition, the action reads:

$$S = \kappa \int \sqrt{h} \left[N(K_{ij}K^{ij} - K^2 + R) + \frac{1}{N}(\dot{\phi} - N^i \partial_i \phi)^2 - N h^{ij} \partial_i \phi \partial_j \phi - 2NV(\phi) \right] \tag{3.8}$$

The conjugate momentum for the metric is the same, and the gravitational Hamiltonian is the same, but for the cosmological constant. The conjugate momentum for the scalar field is:

$$\pi_\phi = \frac{2\kappa\sqrt{h}}{N}(\dot{\phi} - N^i \partial_i \phi) \tag{3.9}$$

The total Hamiltonian will be:

$$\begin{aligned} H = \int \Big\{ & N \left[\frac{1}{2\kappa\sqrt{h}} G_{ij,kl} \pi^{ij} \pi^{kl} - \kappa\sqrt{h} R + \frac{1}{4\kappa\sqrt{h}} \pi_\phi^2 + \kappa\sqrt{h} \left(h^{ij} \partial_i \phi \partial_j \phi + 2V(\phi) \right) \right] + \\ & + 2\nabla_i N_j \pi^{ij} + h^{ij} N_j \partial_i \phi \pi_\phi \Big\} \end{aligned} \tag{3.10}$$

Now, the wavefunction $\Psi[h, \phi]$ is subject to the constraints:

$$\begin{aligned} & \left[\frac{\hbar^2}{2\kappa\sqrt{h}} G_{ij,kl} \frac{\delta}{\delta h_{ij}} \frac{\delta}{\delta h_{kl}} + \kappa\sqrt{h} R + \frac{\hbar^2}{4\kappa\sqrt{h}} \frac{\delta^2}{\delta\phi^2} - \kappa\sqrt{h} \left(h^{ij} \partial_i \phi \partial_j \phi + 2V(\phi) \right) \right] \Psi = 0 \\ & - 2i\hbar \nabla_i \left[\frac{1}{\sqrt{h}} \frac{\delta \Psi}{\delta h_{ij}} \right] + i\hbar \frac{1}{\sqrt{h}} h^{ij} \partial_i \phi \frac{\delta \Psi}{\delta \phi} = 0 \end{aligned} \tag{3.11}$$

3.2.3 Tree level Wheeler-DeWitt equation

Write $\Psi = \text{Exp}\{iW/\hbar\}$ and expand the equations (3.6), (3.11) to zeroth order in $\hbar$. We get Hamilton-Jacobi equations for W, of the form:

$$\begin{aligned} & \left[-\frac{1}{2\kappa\sqrt{h}} G_{ij,kl} \frac{\delta W}{\delta h_{ij}} \frac{\delta W}{\delta h_{kl}} + \kappa\sqrt{h}(R - 2\Lambda) \right] = 0 \\ & 2\nabla_i \left[\frac{1}{\sqrt{h}} \frac{\delta W}{\delta h_{ij}} \right] = 0 \end{aligned} \tag{3.12}$$

And, for single field inflation, of the form:

$$\left[-\frac{1}{2\kappa\sqrt{h}}G_{ij,kl}\frac{\delta W}{\delta h_{ij}}\frac{\delta W}{\delta h_{kl}}+\kappa\sqrt{h}R-\frac{1}{4\kappa\sqrt{h}}\left(\frac{\delta W}{\delta\phi}\right)^2-\kappa\sqrt{h}\left(h^{ij}\partial_i\phi\partial_j\phi+2V(\phi)\right)\right]=0$$
$$2\nabla_i\left[\frac{1}{\sqrt{h}}\frac{\delta W}{\delta h_{ij}}\right]-\frac{1}{\sqrt{h}}h^{ij}\partial_i\phi\frac{\delta W}{\delta\phi}=0 \tag{3.13}$$

3.3 Structure of the Wavefunction at large "volume" and time independence

We are interested in computing the wavefunction at late times. Time is absent in the Wheeler-DeWitt approach to quantum gravity, so, in the context of inflation, we will be looking at the wavefunction for a spatial slice with large "volume". In other words, take the spatial metric and redefine it as $h_{ij}=a^2\hat{h}_{ij}$. We can then consider the asymptotics of (3.12) and (3.13) as $a\to\infty$.

The time reparametrization constraint of general relativity is encoded in the Hamiltonian constraint. In principle, it will fix the wavefunction, given suitable boundary conditions. Here we just want to point out that there is a "time-independent" piece of the wavefunction, which is nonlocal and encodes the superhorizon fluctuations in inflation.

3.3.1 Pure gravity

Begin by substituting $h_{ij}=a^2\hat{h}_{ij}$ to (3.12). The Hamiltonian constraint becomes:

$$\left[-\frac{1}{2\kappa a^3\sqrt{\hat{h}}}\hat{G}_{ij,kl}\frac{\delta W}{\delta\hat{h}_{ij}}\frac{\delta W}{\delta\hat{h}_{kl}}+\kappa\sqrt{\hat{h}}(a\hat{R}-a^3(2\Lambda))\right]=0 \tag{3.14}$$

Now write $W=a^3\alpha\int\sqrt{\hat{h}}+a\beta\int\sqrt{\hat{h}}\hat{R}+W_0+\mathcal{O}(1/a)$. Solving (3.14) order by order in a, we get:

$$\alpha=4\kappa\sqrt{\frac{\Lambda}{3}}$$
$$\beta=-\kappa\sqrt{\frac{3}{\Lambda}} \tag{3.15}$$
$$\hat{h}_{ij}\frac{\delta W_0}{\delta\hat{h}_{ij}}=0$$

So (3.15) tells us that W_0 is scale invariant, or a independent. Note that, as we consider the $a\to\infty$ limit and compute expectation values, only W_0 is important, as the local terms are imaginary, so they will not appear in $|\Psi[h]|^2=\mathrm{Exp}[2\mathrm{Re}\left(\frac{i}{\hbar}W_0\right)]$.

3.3.2 Single Field Inflation

Here the method is essentially the same, though the structure of the wavefunction is more complicated. A similar construction was carried for an arbitrary number of scalars, for a 5-D spacetime, in [63]. See also [47, 48] for a detailed analysis of the Hamilton-Jacobi equation for single field inflation. The Hamiltonian constraint is:

$$\left[-\frac{1}{2\kappa a^3\sqrt{h}}\hat{G}_{ij,kl}\frac{\delta W}{\delta \hat{h}_{ij}}\frac{\delta W}{\delta \hat{h}_{kl}}+\kappa\sqrt{\hat{h}}aR-\frac{1}{4\kappa a^3\sqrt{\hat{h}}}\left(\frac{\delta W}{\delta\phi}\right)^2-\kappa\sqrt{\hat{h}}\left(a\hat{h}^{ij}\partial_i\phi\partial_j\phi+2a^3V(\phi)\right)\right]=0 \quad (3.16)$$

Now write $W=a^3\int\sqrt{\hat{h}}U(\phi)+a\int\sqrt{\hat{h}}\left[\Phi(\phi)\hat{R}+\Theta(\phi)(h^{ij}\partial_i\phi\partial_j\phi)\right]+W_0+\mathcal{O}(1/a)$. Solving (3.16) order by order in a, we get:

$$\begin{aligned}
&V=\frac{1}{8\kappa^2}\left[\frac{3U^2}{2}-U'^2\right]\\
&\frac{U\Phi}{2}-U'\Phi'=-2\kappa^2\\
&\frac{U'\Theta'}{2}-U\left(\Phi''-\frac{\Theta}{4}\right)=\kappa^2\\
&\frac{U'}{U}=\frac{\Phi'}{\Theta}\\
&\hat{h}_{ij}\frac{\delta W_0}{\delta\hat{h}_{ij}}=\frac{U'}{U}\frac{\delta W_0}{\delta\phi}
\end{aligned} \quad (3.17)$$

The auxiliary potential U is related to the potential V via (3.17). The variation of W_0 with respect to the trace of the metric is related to a variation of W_0 with respect to the scalar field. This relates two different gauge choices, one in which the trace of the metric is a fluctuating degree of freedom (ζ gauge) and the other where the scalar field fluctuates ($\delta\phi$ gauge). The factor that relates one to the other is related to the slow-roll parameter of single field inflation[46, 22, 48].

The existence of a "time-independent" piece of the wavefunction, for large volume (late times), is equivalent to the statement that there are fluctuations of the metric that freeze at late times [87] . Those are the fluctuating fields whose correlations are calculated using the in-in formalism in inflation.

With Hartle-Hawking boundary conditions, W_0 can be computed explicitly. One evaluates the classical action with a solution for the equations of motion that obeys these boundary conditions. At very early times, the modes are in their flat space vacuum, as their physical wavelength is too small to probe any curvature effects of spacetime. W_0 has an imaginary piece that gives the tree level contribution to inflationary expectation values [22, 14].

3.4 Consistency condition as a Ward Identity for derivatives of the Wavefunction

In the previous section, we showed that the wavefunction has a piece that is late time independent. Now we want to show that the consistency condition for the cosmological correlators arises from reparametrization invariance of the wavefunction of the universe, in particular, of W_0. We write $h_{ij} = a^2(\delta_{ij} + p_{ij})$ and consider the limit of $a \to \infty$, which would correspond to a late time slice in the semiclassical approximation.

Now we impose that the wavefunction is invariant under spatial diffeomorphisms. This means that $\Psi[h_{ij} + \nabla_{(i}v_{j)}] = \Psi[h_{ij}]$. This implies that:

$$\delta\Psi[h_{ij}] = 2\int d^3x \nabla_a v_b(x) \frac{\delta\Psi[h_{ij}]}{\delta h_{ab}(x)} = 0 \Rightarrow \nabla_i \left[\frac{1}{\sqrt{h(x)}} \frac{\delta\Psi}{\delta h_{ij}(x)} = 0\right] \tag{3.18}$$

Equation (3.18) is the Ward identity for the wavefunction of the universe. It is a statement on its reparametrization invariance. Of course, this is the same as equation (3.6). Specializing to W_0, the scale invariant piece of the wavefunction, as in (3.15), we see that it also satisfies (3.18) with $\Psi \to W_0$, as the other terms that survive in the $a \to \infty$ limit automatically satisfy (3.18), as they are invariant under spatial reparametrizations.

Let us now perform a perturbative expansion in W_0. We take the perturbations to be around the flat metric, as $\delta_{ij} + p_{ij}$. Of course, this is due to what we know about the universe being approximately flat after inflation. The consistency conditions can be easily generalized to expansions around different backgrounds, as the WdW equations are invariant under the choice of background metric.

As we are interested in the non-local piece of the wavefunction, we start quadratic in the metric[2]:

$$\begin{aligned} W_0[\hat{h}] = &\frac{1}{2!}\int d^3x d^3y \left(\frac{\delta^2 W_0[\delta]}{\delta\hat{h}_{ab}(x)\delta\hat{h}_{cd}(y)}\right) p_{ab}(x)p_{cd}(y) + \\ &+ \frac{1}{3!}\int d^3x d^3y d^3z \left(\frac{\delta^3 W_0[\delta]}{\delta\hat{h}_{ab}(x)\delta\hat{h}_{cd}(y)\hat{h}_{ef}(z)}\right) p_{ab}(x)p_{cd}(y)p_{ef}(z) + \\ &+ \frac{1}{4!}\int d^3x d^3y d^3z d^3w \frac{\delta^4 W_0[\delta]}{\delta\hat{h}_{ab}(x)\delta\hat{h}_{cd}(y)\delta\hat{h}_{ef}(z)\delta\hat{h}_{ij}(w)} p_{ab}(x)p_{cd}(y)p_{ef}(z)p_{ij}(w) + \cdots \end{aligned} \tag{3.19}$$

We are Taylor expanding around the flat metric. Square brackets in the derivatives mean that we calculate the derivative at the background metric. As we will be mostly dealing with W_0 from here on, we will omit the hat on $\hat{h}_{ij}$.

[2] In principle there can be a scale invariant, local contribution to the wavefunction, proportional to the gravitational Chern-Simons term. It can be argued that this term is a pure phase [26], and thus will not contribute to the sort of correlators we consider here. In any case, even if it were real, it would contribute a local term to expectation values, and we are interested in consistency conditions for the nonlocal terms.

We now work out the consequences of (3.18) to the coefficients in the perturbative expansion (3.19). The idea is to commute an insertion of $\delta/\delta\hat{h}_{ij}$ into (3.18) and then evaluate the resultant expression for $h_{ij} = \delta_{ij}$. We rewrite (3.18) as:

$$\nabla_i \left[\frac{\delta W_0}{\delta h_{ij}(x)} \right] = \partial_i \left(\frac{\delta W_0}{\delta h_{ij}(x)} \right) + \Gamma^j_{ik}(x) \frac{\delta W_0}{\delta h_{ik}(x)} = 0 \tag{3.20}$$

The only issue here is how to commute through the Christoffel symbol. Writing $\Gamma^a_{bc} = h^{ad}\Gamma_{dbc}$ the following expressions are useful:

$$\begin{aligned}
&\frac{\delta^n \Gamma^a_{bc}(x)}{\delta h_{i_1 j_1}(y_1) \cdots \delta h_{i_n j_n}(y_n)} = \frac{\delta^n h^{ad}(x)}{\delta h_{i_1 j_1}(y_1) \cdots \delta h_{i_n j_n}(y_n)} \Gamma_{dbc}(x) + \cdots \\
&\cdots + \sum_{k=1}^{n} \frac{\delta^{n-1} h^{ad}(x)}{\delta h_{i_1 j_1}(y_1) \cdots \delta h_{i_n j_n}(y_n)} \frac{\delta \Gamma_{dbc}(x)}{\delta h_{i_k j_k}(y_k)} \\
&\frac{\delta h^{ab}(x)}{\delta h_{cd}(y)} = -h^{am}(x) h^{bn}(x) \delta^{cd}_{mn} \delta(x-y), \quad \delta^{cd}_{mn} \equiv \frac{1}{2}(\delta^c_m \delta^d_n + \delta^c_n \delta^d_m) \\
&\frac{\delta \Gamma_{dbc}(x)}{\delta h_{ij}(y)} = \frac{1}{2} \left(\delta^{ij}_{bd}\ \partial^x_c \delta(x-y) + \delta^{ij}_{cd}\ \partial^x_b \delta(x-y) - \delta^{ij}_{bc}\ \partial^x_d \delta(x-y) \right)
\end{aligned} \tag{3.21}$$

Second derivative

First let us just commute one insertion of $\delta/\delta h_{ij}$ through (3.18). We get:

$$\frac{\partial}{\partial x^i} \left(\frac{\delta^2 W_0[\delta]}{\delta h_{ij}(x) \delta h_{kl}(y)} \right) = 0 \tag{3.22}$$

Third derivative

We now commute two insertions of $\delta/\delta h_{ij}$ through (3.18) . We get:

$$\begin{aligned}
&\frac{\partial}{\partial x^i} \left(\frac{\delta^3 W_0[\delta]}{\delta h_{ij}(x) \delta h_{kl}(y) \delta h_{mn}(z)} \right) = -\frac{1}{2} \left[\left(\delta^{jk} \frac{\delta^2 W_0[\delta]}{\delta h_{il}(x) \delta h_{mn}(z)} \frac{\partial}{\partial x^i} \delta(x-y) + \{k \leftrightarrow l\} \right) + \right. \\
&+ \left(\delta^{jm} \frac{\delta^2 W_0[\delta]}{\delta h_{kl}(y) \delta h_{in}(x)} \frac{\partial}{\partial x^i} \delta(x-z) + \{m \leftrightarrow n\} \right) - \left(\frac{\delta^2 W_0[\delta]}{\delta h_{kl}(x) \delta h_{mn}(z)} \frac{\partial}{\partial x_j} \delta(x-y) + \right. \\
&\left. \left. + \frac{\delta^2 W_0[\delta]}{\delta h_{kl}(y) \delta h_{mn}(x)} \frac{\partial}{\partial x_j} \delta(x-z) \right) \right]
\end{aligned} \tag{3.23}$$

3.5 Consequences for expectation values

The expectation values of insertions of the metric is given by:

$$\langle h_{ij}(x) h_{kl}(y) \cdots \rangle = \int dh |\Psi[h]|^2 h_{ij}(x) h_{kl}(y) \cdots \tag{3.24}$$

It is convenient to do the following before moving on. We want to write the expectation values of operators in momentum space. We also use a basis of polarization tensors that is traceless and transverse with respect to the flat metric:

$$\epsilon^s_{ij}\epsilon^{s'}_{ij} = 2\delta^{ss'}, \qquad k_i\epsilon^s_{ij} = 0 \tag{3.25}$$

Indices are contracted with δ^{ij}. We call the helicity modes $+$ and $-$. Angular momentum conservation and momentum conservation tells us that the only two point functions allowed are $\langle p^+p^+\rangle$ and $\langle p^-p^-\rangle$, with the perturbation being $p_{ij} \equiv h_{ij} - \delta_{ij}$. Now write the wavefunction as follows:

$$\Psi = \Psi_{local} \times \text{Exp}\left\{\sum_n \int dk_1 \cdots dk_n \frac{1}{n!}\frac{\delta^n W_0[\delta]}{\delta h^{s_1}(k_1)\cdots\delta h^{s_n}(k_n)} p^{s_1}(k_1)\cdots p^{s_n}(k_n)\right\} \tag{3.26}$$

In terms of (3.26), the two point expectation value for gravitational wave perturbations is given by:

$$\langle p^{s_1}(k_1)p^{s_2}(k_2)\rangle = -\frac{1}{2\text{Re}\frac{\delta^2 W_0[\delta]}{\delta h^{s_1}(k_1)\delta h^{s_2}(k_2)}} \tag{3.27}$$

Three Point Function

In terms of (3.26), the three point expectation value for gravitational wave perturbations is given by:

$$\langle p^{s_1}(k_1)p^{s_2}(k_2)p^{s_3}(k_3)\rangle = -\frac{2\text{Re}\left(\frac{\delta^3 W_0[\delta]}{\delta h^{s_1}(k_1)\delta h^{s_2}(k_2)\delta h^{s_3}(k_3)}\right)}{\Pi_i \text{Re}\left(2\frac{\delta^2 W_0[\delta]}{\delta h^{s_i}(k_i)\delta h^{s_i}(-k_i)}\right)} \tag{3.28}$$

Let us now understand how (3.23) constrains the shape of the three point function in the squeezed limit. Start from looking at (3.22) and (3.23) in momentum space. As the background is translation invariant, the derivatives of the wavefunction should only depend on distances between points. In momentum space, this means that there is always a subtended momentum conserving delta function in front of an expectation value[3]. Thus, an n-point expectation value will explicitly depend on $n-1$ momenta, the last momentum dependence removed by translation invariance.

Before giving the final forms for (3.22) and (3.23), we need to do one more thing. The variable we actually use in the bulk computations is γ_{ij}, such that $h_{ij} = \text{Exp}(\gamma)_{ij}$. To cubic order, $h_{ij} = \delta_{ij} + \gamma_{ij} + \frac{1}{2}\gamma_{ik}\gamma_{kj}$. Translating the relation for the three point derivative (3.23) will induce new contact terms in the Ward identity, due to the use of the chain rule. In momentum space, (3.22) and (3.23) will read:

$$k_{1,i}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{ij}(k_1)\delta\gamma_{kl}(k_2)} = 0 \tag{3.29}$$

[3]The notation used in [22] and in many papers in the literature is to use a $'$ in front of the expectation value, e. g. $\langle h(k_1)h(k_2)\rangle = \delta(k_1+k_2)\langle h(k_1)h(-k_1)\rangle'$. We will always assume that the delta function is taken care of, and use it to eliminate one momentum variable in the various expectation values.

$$
\begin{aligned}
&k_{1,a}\frac{\delta^3 W_0[\delta]}{\delta\gamma_{aj}(k_1)\delta\gamma_{kl}(k_2)\delta\gamma_{mn}(k_3)}=\\
&=\frac{1}{2}\left[\delta^{jk}k_{2,a}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{al}(-k_3)\delta\gamma_{mn}(k_3)}+\delta^{jl}k_{2,a}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{ak}(-k_3)\delta\gamma_{mn}(k_3)}+\right.\\
&+\delta^{jm}k_{3,a}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{kl}(k_2)\delta\gamma_{an}(-k_2)}+\delta^{jn}k_{3,a}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{kl}(k_2)\delta\gamma_{ma}(-k_2)}-\\
&\left.-k_{2,j}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{kl}(-k_3)\delta\gamma_{mn}(k_3)}-k_{3,j}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{kl}(k_2)\delta\gamma_{mn}(-k_2)}\right]+\\
&+k_{1,a}\left[\delta^{kl}_{bd}\delta^{aj}_{dc}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{bc}(-k_3)\delta\gamma_{mn}(k_3)}+\delta^{mn}_{bd}\delta^{ij}_{dc}\frac{\delta^2 W_0[\delta]}{\delta\gamma_{kl}(k_2)\delta\gamma_{bc}(-k_2)}\right]
\end{aligned}
\tag{3.30}
$$

The last line of (3.30) comes from the change of variables $p \to \gamma$. The derivatives are evaluated around the flat background, meaning that we set $\gamma = 0$ after taking the derivative. Dummy indices are $a, \cdots, d$ in (3.30). (3.30) encodes all consistency conditions for the three point function of inflationary perturbations.

3.5.1 Extracting the consistency condition

To get the consistency conditions of inflation, we need to consider the squeezed limit, or $k_1 \to 0$. We want to show that (3.30) implies an infinite number of such consistency conditions, as recently discussed by [85] . The leading correction to the consistency condition, which is also completely fixed by the longitudinal modes, was first discussed in [78] . Now, all one needs to do is to Taylor expand (3.30) around $k_1 = 0$. Let us introduce the simplified notation:

$$
D^{ij}_k W_0 \equiv \frac{\delta W_0[\delta]}{\delta\gamma_{ij}(k)}
\tag{3.31}
$$

Then, the three point function for gravitational waves is given by:

$$
\langle\gamma^{s_1}_{k_1}\gamma^{s_2}_{k_2}\gamma^{s_3}_{k_3}\rangle = -\frac{2\mathrm{Re}\left[D^{s_1}_{k_1}D^{s_2}_{k_2}D^{s_3}_{k_3}W_0\right]}{2\mathrm{Re}\left[D^{s_1}_{k_1}D^{s_1}_{-k_1}W_0\right]2\mathrm{Re}\left[D^{s_2}_{k_2}D^{s_2}_{-k_2}W_0\right]2\mathrm{Re}\left[D^{s_3}_{k_3}D^{s_3}_{-k_3}W_0\right]}
\tag{3.32}
$$

And the consistency condition is encoded in the identity:

$$
\begin{aligned}
&k^a_1 D^{aj}_{k_1}D^{kl}_{k_2}D^{mn}_{k_3}W_0=\\
&=\frac{1}{2}\left[\delta^{jk}k^a_2 D^{al}_{-k_3}D^{mn}_{k_3}W_0+\delta^{jl}k^a_2 D^{ak}_{-k_3}D^{mn}_{k_3}W_0+\delta^{jm}k^a_3 D^{kl}_{k_2}D^{an}_{-k_2}W_0+\delta^{jn}k^a_3 D^{kl}_{k_2}D^{ma}_{-k_2}W_0-\right.\\
&\left.-k^j_2 D^{kl}_{-k_3}D^{mn}_{k_3}W_0-k^j_3 D^{kl}_{k_2}D^{mn}_{-k_2}W_0\right]+k^a_1\left[\delta^{kl}_{bd}\delta^{aj}_{dc}D^{bc}_{-k_3}D^{mn}_{k_3}W_0+\delta^{mn}_{bd}\delta^{aj}_{dc}D^{kl}_{k_2}D^{bc}_{-k_2}W_0\right]
\end{aligned}
\tag{3.33}
$$

Now we expand (3.32) and (3.33) around $k_1 = 0$. Assuming that $\lim_{k_1\to 0} k^a_1 D^{aj}_{k_1}D^{kl}_{k_2}D^{mn}_{k_3}W_0 = 0$, which is true if there are no terms of the form $\log k_1$ in the three point function of cosmological perturbations, we get:

$$\begin{aligned}
&\lim_{k_1\to 0} D^{ij}_{k_1} D^{kl}_{k_2} D^{mn}_{-k_1-k_2} W_0 = \\
&= -\frac{1}{2}\left[\delta^{jk} D^{il}_{-k_2} D^{mn}_{k_2} W_0 + \delta^{jl} D^{ik}_{-k_2} D^{mn}_{k_2} W_0 + \delta^{jm} D^{kl}_{-k_2} D^{in}_{k_2} W_0 + \delta^{jn} D^{kl}_{-k_2} D^{mi}_{k_2} W_0 + \right. \\
&\left. + k_{2,j}\frac{\partial}{\partial k_{2,i}} D^{kl}_{-k_2} D^{mn}_{k_2} W_0 - \delta^{ij} D^{kl}_{-k_2} D^{mn}_{k_2} W_0\right] + \left[\delta^{kl}_{bd}\delta^{ij}_{cd} D^{bc}_{-k_2} D^{mn}_{k_2} W_0 + \delta^{mn}_{bd}\delta^{ij}_{cd} D^{kl}_{-k_2} D^{bc}_{k_2} W_0\right]
\end{aligned} \tag{3.34}$$

Now, we contract (3.34) with polarization tensors for the fluctuations. We obtain:

$$\lim_{k_1\to 0} D^{s_1}_{k_1} D^{s_2}_{k_2} D^{s_2}_{-k_1-k_2} W_0 = -\frac{1}{2}\epsilon^{ij}_1 k^i_2 \frac{\partial}{\partial k^j_2} D^{s_2}_{-k_2} D^{s_2}_{k_2} W_0 = -\epsilon^{ij}_1 k^i_2 k^j_2 \frac{\partial}{\partial k^2_2} D^{s_2}_{-k_2} D^{s_2}_{k_2} W_0 \tag{3.35}$$

We now see that the leading order contribution to the squeezed limit of the three point function of gravitational waves is:

$$\begin{aligned}
&\lim_{k_1\to 0}\langle \gamma^{s_1}_{k_1}\gamma^{s_2}_{k_2}\gamma^{s_3}_{k_3}\rangle = -\frac{2\mathrm{Re}\lim_{k_1\to 0}\left[D^{s_1}_{k_1} D^{s_2}_{k_2} D^{s_3}_{k_3} W_0\right]\delta^{s_2 s_3}}{2\mathrm{Re}\left[D^{s_1}_{k_1} D^{s_1}_{-k_1} W_0\right]\left(2\mathrm{Re}\left[D^{s_2}_{k_2} D^{s_2}_{-k_2} W_0\right]\right)^2} = \\
&= -\left[-\frac{1}{2\mathrm{Re}[D^{s_1}_{k_1} D^{s_1}_{-k_1} W_0]}\right]\left[-\epsilon^{ij}_1 k^i_2 k^j_2 \frac{\partial}{\partial k^2_2}\left(\frac{1}{2\mathrm{Re}[D^{s_2}_{k_2} D^{s_2}_{-k_2} W_0]}\right)\right]\delta^{s_2 s_3} = \\
&= -\langle \gamma^{s_1}_{k_1}\gamma^{s_1}_{-k_1}\rangle \epsilon^{ij}_1 k^i_2 k^j_2 \frac{\partial}{\partial k^2_2}\langle \gamma^{s_2}_{k_2}\gamma^{s_2}_{-k_2}\rangle \delta^{s_2 s_3}
\end{aligned} \tag{3.36}$$

Which is the standard consistency condition for the gravitational wave three point function [22].

3.5.2 Scalar Fluctuations

Although there is no scalar mode in pure gravity in de Sitter space, (as is illustrated by equation (3.15)) we can still make use of (3.33) to extract the consistency condition for the inflaton, ζ. The reason is that we evaluate the wavefunction at a surface of constant background field, $W_0[h,\phi]$ with $\partial_i\phi = 0$. Then, the momentum constraint (3.11) reduces to the one in pure gravity, and thus (3.33) applies. As the ζ mode is also taken as the exponential of the metric, but of its trace, instead of its traceless transverse component, all we need to do is to contract (3.33) with δ^{ij}, etc. At the level of the three point function, we take our metric to be $h_{ij} = \delta_{ij}(1 + 2\zeta + 2\zeta^2)$. That corresponds to the substitution $\gamma_{ij} \to 2\zeta\delta_{ij}$. In (3.33) that corresponds to $2D^{ij}_k\delta^{ij} \to D_k$, where $D_k \equiv \delta/\delta\zeta_k$. We obtain:

$$\lim_{k_1\to 0} D_{k_1} D_{k_2} D_{-k_1-k_2} W_0 = \left[\left(3 - k^i_2\frac{\partial}{\partial k^i_2}\right) D_{-k_2} D_{k_2} W_0\right] = \left[\left(3 - k_2\frac{\partial}{\partial k_2}\right) D_{-k_2} D_{k_2} W_0\right] \tag{3.37}$$

Thus, for the three point function, we obtain:

$$\begin{aligned}
&\lim_{k_1\to 0}\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = -\frac{2\mathrm{Re}\lim_{k_1\to 0}\left[D_{k_1}D_{k_2}D_{k_3}W_0\right]}{2\mathrm{Re}\left[D_{k_1}D_{-k_1}W_0\right]\left(2\mathrm{Re}\left[D_{k_2}D_{-k_2}W_0\right]\right)^2} = \\
&= -\left[-\frac{1}{2\mathrm{Re}[D_{k_1}D_{-k_1}W_0]}\right]\left[-\frac{1}{2\mathrm{Re}[D_{k_2}D_{-k_2}W_0]}\right]\left[-\frac{d\log\left(-2\mathrm{Re}\left[D_{k_2}D_{-k_2}W_0k_2^{-3}\right]\right)}{d\log k}\right] = \\
&= -\langle\zeta_{k_1}\zeta_{-k_1}\rangle\langle\zeta_{k_2}\zeta_{-k_2}\rangle\frac{\partial\log\left[k_2^3\langle\zeta_{k_2}\zeta_{-k_2}\rangle\right]}{\partial\log k_2}
\end{aligned} \tag{3.38}$$

3.5.3 Mixed three point functions

For three point functions of one long scalar and two short tensor fluctuations, and vice-versa, the derivation is essentially the same. One just needs to contract (3.33) with the proper polarization tensors etc. We just quote the final results. For a long scalar mode and short tensor modes we have:

$$\begin{aligned}
&\lim_{k_1\to 0}\langle\zeta_{k_1}\gamma_{k_2}^{s_2}\gamma_{k_3}^{s_3}\rangle = -\frac{2\mathrm{Re}\lim_{k_1\to 0}\left[D_{k_1}D_{k_2}^{s_2}D_{k_3}^{s_3}W_0\right]\delta^{s_2s_3}}{2\mathrm{Re}\left[D_{k_1}D_{-k_1}W_0\right]\left(2\mathrm{Re}\left[D_{k_2}^{s_2}D_{-k_2}^{s_2}W_0\right]\right)^2} = \\
&= -\left[-\frac{1}{2\mathrm{Re}[D_{k_1}D_{-k_1}W_0]}\right]\left[-\frac{\delta^{s_2s_3}}{2\mathrm{Re}[D_{k_2}^{s_2}D_{-k_2}^{s_2}W_0]}\right]\left[-\frac{d\log\left(-2\mathrm{Re}\left[D_{k_2}^{s_2}D_{-k_2}^{s_2}W_0k_2^{-3}\right]\right)}{d\log k}\right] = \\
&= -\langle\zeta_{k_1}\zeta_{-k_1}\rangle\langle\gamma_{k_2}^{s_2}\gamma_{-k_2}^{s_2}\rangle\delta^{s_2s_3}\frac{\partial\log\left[k_2^3\langle\gamma_{k_2}^{s_2}\gamma_{-k_2}^{s_2}\rangle\right]}{\partial\log k_2}
\end{aligned} \tag{3.39}$$

While, for a long tensor mode and two short scalar modes, we get:

$$\begin{aligned}
&\lim_{k_1\to 0}\langle\gamma_{k_1}^{s_1}\zeta_{k_2}\zeta_{k_3}\rangle = -\frac{2\mathrm{Re}\lim_{k_1\to 0}\left[D_{k_1}^{s_1}D_{k_2}D_{k_3}W_0\right]}{2\mathrm{Re}\left[D_{k_1}^{s_1}D_{-k_1}^{s_1}W_0\right]\left(2\mathrm{Re}\left[D_{k_2}D_{-k_2}W_0\right]\right)^2} = \\
&= -\left[-\frac{1}{2\mathrm{Re}[D_{k_1}^{s_1}D_{-k_1}^{s_1}W_0]}\right]\left[-\epsilon_1^{ij}k_2^ik_2^j\frac{\partial}{\partial k_2^2}\left(\frac{1}{2\mathrm{Re}[D_{k_2}D_{-k_2}W_0]}\right)\right] = \\
&= -\langle\gamma_{k_1}^{s_1}\gamma_{-k_1}^{s_1}\rangle\epsilon_1^{ij}k_2^ik_2^j\frac{\partial}{\partial k_2^2}\langle\zeta_{k_2}\zeta_{-k_2}\rangle
\end{aligned} \tag{3.40}$$

3.5.4 Higher order consistency conditions

Let us expand the three point function of fluctuations in a Taylor series around $k_1 = 0$. For simplicity, we consider scalar fluctuations:

$$\langle\zeta_{k_1}\zeta_{k_2}\zeta_{-k_1-k_2}\rangle = \langle\zeta_{k_1}\zeta_{-k_1}\rangle\left[Z(k_2) + k_1^aF^a(k_2) + \frac{1}{2}k_1^ak_1^bS^{ab}(k_2) + \cdots\right] \tag{3.41}$$

It was already pointed out in [22] that the leading order term $Z(k_2)$ is fixed by the two point function, which is what we call the inflationary consistency condition. In [78] it was argued that the first order term $F^a(k_2)$ is also completely fixed by some residual conformal symmetry of the background. Reference [85] studied constraints to the higher order terms in (3.41), and found general Ward identities that should be obeyed by some combinations of gravitational wave and inflaton expectation values.

All of these consistency conditions follow from a Taylor expansion of the longitudinal mode Ward identities (3.33). So the inflationary consistency conditions can be explained by the reparametrization invariance, or momentum constraint, of the wavefunction of the universe. The terms that have physical content, and are probing the primordial non-gaussianity of inflationary perturbations, start quadratic in k_1 in (3.41). In [84] it was pointed out that the squeezed three point function of single field inflation gives rise to no effect in a physical observable. This is of course consistent with the picture that the squeezed limit is totally fixed by diffeomorphism invariance, as physical observables are diff-invariant. In other words, there is residual gauge symmetry in the squeezed limit of expectation values of inflationary fluctuations, and these can be tracked down from the original symmetry.

Here we derive the consistency condition discussed in [78], which completely fixes the linear term in k_1 in (3.41). We also discuss the generalized consistency conditions of [85], pointing out why from quadratic order on, the longitudinal modes do not fix completely the three point function. Note that our derivation makes no use of conformal symmetry; we rely purely on reparametrization invariance of the wavefunction.

First, contract (3.33) with $4\delta^{kl}\delta^{mn}$. We get[4]:

$$k_1^a D_{k_1}^{aj} D_{k_2} D_{k_3} W_0 = \frac{1}{2}\left[-k_2^j D_{k_3} D_{-k_3} W_0 - k_3^j D_{k_2} D_{-k_2} W_0\right] \tag{3.42}$$

Now, take the first order correction to the three point function in the squeezed limit. We Taylor expand (3.32), for scalar fluctuations, to first order in k_1. Comparing with the formula (3.41) we get:

$$\begin{aligned} Z(k_2) &= \frac{1}{2}\frac{\mathrm{Re}\lim_{k_1\to 0} D_{k_1} D_{k_2} D_{k_3} W_0}{(\mathrm{Re}\left[D_{k_2} D_{-k_2} W_0\right])^2} = -\langle\zeta_{k_2}\zeta_{-k_2}\rangle \frac{\partial \log\left[k_2^3\langle\zeta_{k_2}\zeta_{-k_2}\rangle\right]}{\partial \log k_2} \\ F^a(k_2) &= \frac{1}{2}\left(\frac{\lim_{k_1\to 0}\partial_{k_1^a}\mathrm{Re} D_{k_1} D_{k_2} D_{k_3} W_0}{(\mathrm{Re}\left[D_{k_2} D_{-k_2} W_0\right])^2} - \frac{\mathrm{Re}\lim_{k_1\to 0} D_{k_1} D_{k_2} D_{k_3} W_0}{(\mathrm{Re}\left[D_{k_2} D_{-k_2} W_0\right])^3}\partial_{k_2^a}\mathrm{Re}\left[D_{k_2} D_{-k_2} W_0\right]\right) \end{aligned} \tag{3.43}$$

Now we take two derivatives of (3.33) with respect to k_1 and take $k_1 \to 0$. That will give:

$$\frac{\partial}{\partial k_1^l} D_{k_1}^{ij} D_{k_2} D_{k_3} W_0 + \frac{\partial}{\partial k_1^i} D_{k_1}^{lj} D_{k_2} D_{k_3} W_0 = -\frac{1}{2} k_2^j \frac{\partial^2}{\partial k_2^l \partial k_2^i} D_{k_2} D_{-k_2} W_0 \tag{3.44}$$

Note that the index j in (3.44) is singled out, and the left hand side is symmetric in i, l. We contract (3.44)

[4] In general, the Ward identity will look like $k_1^i D_{k_1}^{ij} D_{k_2}\cdots D_{k_n} W_0 = -k_2^j D_{k_1+k_2}\cdots D_{k_n} W_0 - \cdots - k_n^j D_{k_2}\cdots D_{k_n+k_1} W_0$, i.e., it relates the n-derivative to the $n-1$-derivative of the wavefunction, evaluated at shifted momenta [26].

with $2\delta^{ij}$ and δ^{il} and subtract the equations we obtain, getting:

$$\begin{aligned}\lim_{k_1\to 0}\frac{\partial}{\partial k_1^i}D_{k_1}D_{k_2}D_{k_3}W_0 &= -k_2^a\frac{\partial^2}{\partial k_2^a\partial k_2^i}D_{k_2}D_{-k_2}W_0+\frac{1}{2}k_2^i\frac{\partial^2}{\partial k_2^a\partial k_2^a}D_{k_2}D_{-k_2}W_0 = \\ &= -k_2^i\left(\frac{1}{k_2}\frac{\partial}{\partial k_2}-\frac{1}{2}\frac{\partial^2}{\partial k_2^2}\right)D_{k_2}D_{-k_2}W_0\end{aligned} \tag{3.45}$$

Where we used that the second derivative of the wavefunction depends only on the absolute value of k_2. Then, plugging this back in (3.43) we get:

$$F^a(k_2) = -\frac{1}{2}\partial_{k_2^a}Z(k_2) = \frac{1}{2}\partial_{k_2^a}\left[\langle\zeta_{k_2}\zeta_{-k_2}\rangle\frac{\partial\log\left[k_2^3\langle\zeta_{k_2}\zeta_{-k_2}\rangle\right]}{\partial\log k_2}\right] \tag{3.46}$$

It was observed in [77] that under the substitution $k_1\to k_L$, $k_2\to k_S-k_L/2$, the linear term in (3.41), $F^a(k_S)$ is absent. One can check that changing variables from k_2 to k_S such is the case, so (3.46) is compatible with the claims made in [77, 78].

We can also study the case of one long tensor mode and two short scalar modes, as in [78][5]. There is a small point to be made, which is the following. We obtain from our method the object $\partial_{k^a}D_k^{bc}DDW_0$. Then, we need to contract this with a polarization tensor. But the expectation value we consider is already contracted with the polarization tensor, so we could be neglecting a term where the derivative acts on the polarization tensor, and the resulting tensor is contracted with the expectation value. In other words, we do not calculate the contribution coming from $(\partial_{k^a}\epsilon^{bc})D_k^{bc}DDW_0$. We show in appendix C that this contribution is zero, and so we capture the entire linear term in the long momentum. The consistency condition to linear order will be:

$$\lim_{k_1\to 0}\langle\gamma^s_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = \langle\gamma^s_{k_1}\gamma^s_{k_1}\rangle\left\{Z_\gamma(k_2)+\frac{k_1.k_2}{4k_2^2}\frac{k_2.\epsilon_1.k_2}{k_2^2}\left[k_2\partial_{k_2}-k_2^2\partial_{k_2}^2\right]\langle\zeta_{k_2}\zeta_{k_2}\rangle\right\} \tag{3.47}$$

With $Z_\gamma(k_2)$ can be read out from (3.40). This result agrees with the prescription given in [78], and, in particular, with the case of single field inflation[22].

To obtain the higher order consistency conditions described in [85] , note the following. It is clear that, taking multiple derivatives with respect to the momentum being squeezed, the best we can do is obtain an expression for the symmetrized derivative $\partial^{(i_1}\partial^{i_2}\cdots\partial^{i_{n-1}}D_{k_1}^{i_n)\ j}D_{k_2}D_{k_3}W_0$. We can project out some components of this symmetrized derivative, and relate it to linear combinations of three point functions, as is done in [85]. Of course, one would not expect to be able to obtain all derivatives of the wavefunction from the Ward identity, as we are just finding the parts of the expectation value fixed by gauge invariance.

[5] In particular, we want to check equations (66) and (67) of [78].

Let us study in more detail the case of the second derivative. We obtain:

$$\lim_{k_1\to 0}\left[\frac{\partial^2}{\partial k_1^a \partial k_1^b}D_{k_1}^{cj}+\frac{\partial^2}{\partial k_1^c \partial k_1^a}D_{k_1}^{bj}+\frac{\partial^2}{\partial k_1^b \partial k_1^c}D_{k_1}^{aj}\right]D_{k_2}D_{k_3}W_0=-\frac{1}{2}k_2^j\frac{\partial^3}{\partial k_2^a\partial k_2^b\partial k_2^c}D_{k_2}D_{-k_2}W_0 \qquad (3.48)$$

There are two types of indices in some sense here, the index that is not symmetrized and the symmetrized ones. Just as we did for the first derivative, we can either contract symmetrized indices or one symmetrized index with the separate one. Take δ^{ab} and δ^{ac}, and after some manipulation, the best one can obtain is (we write explicitly D^{cc}, so there is no confusion with derivatives with respect to the scalar, $D \equiv 2D^{cc}$):

$$\begin{aligned}\lim_{k_1\to 0}&\left[\frac{\partial^2}{\partial k_1^a \partial k_1^b}D_{k_1}^{cc}-\frac{\partial^2}{\partial k_1^c \partial k_1^c}D_{k_1}^{ab}\right]D_{k_2}D_{k_3}W_0=\\ &=-\frac{1}{2}\left[k_2^c\frac{\partial^3}{\partial k_2^c\partial k_2^a\partial k_2^b}-k_2^a\frac{\partial^3}{\partial k_2^b\partial k_2^c\partial k_2^c}\right]D_{k_2}D_{-k_2}W_0\end{aligned} \qquad (3.49)$$

Which is not good enough to isolate what we would like, $\lim_{k_1\to 0}\frac{\partial^2}{\partial k_1^a\partial k_1^b}D_{k_1}D_{k_2}D_{k_3}W_0$.

3.6 Comment on gauge/gravity duality

In gauge/gravity duality, dS/CFT [44, 43, 22] is the proposal that an asymptotically de Sitter space can be described by a dual field theory. In the approach of [22], the proposal is that the wavefunction of the universe with a certain 3-metric profile is equal to the partition function of a CFT, where this 3-metric is a parameter of the partition function.

Then, the stress tensor of the dual field theory, T^{ij}, is given by functional derivatives of the partition function with respect to the metric. So, when we take functional derivatives of Ψ with respect to the flat background, in the dual picture we are computing correlation functions of the stress tensor in the vacuum of the field theory. So, from the field theory perspective, the functional derivatives of the metric we considered throughout the chapter, $D^{ij}D^{kl}\cdots\Psi$ are equal to correlation functions of the stress tensor, $\langle T^{ij}T^{kl}\cdots\rangle$.

From this point of view, the consistency condition has a simple interpretation. (3.18) expresses the conservation of the stress tensor of the dual theory, $\nabla_i\langle T^{ij}\rangle=0$. So, (3.18) is equivalent to Ward identities obeyed by the stress tensor, which can be found in [45]. Note, though, that we do NOT need anything like dS/CFT or gauge/gravity duality to use the Ward identities that the T^{ij} satisfy. These have a pure bulk interpretation from diffeomorphism invariance.

Note also that the final equations in (3.15) and (3.17) can be interpreted as identities obeyed by the trace of the stress tensor. (3.15) states that an insertion of the trace of the stress tensor should render any correlation function to be zero. This means that $\langle T^{ii}\cdots\rangle=0$. This is expected, as de Sitter has isometries at late times that are isomorphic to the conformal group [26] . For the single field case, there is no conformal symmetry, due to the presence of the inflaton. It corresponds to the insertion of an operator that deforms the CFT [22, 46, 47, 48]. This operator breaks the conformal symmetry, and induces a trace to the stress tensor. The relation between the operator and the trace is given by the last equation of (3.17).

3.7 Discussion

In this chapter, we gave a different perspective on how to derive inflationary consistency conditions. The objectives of this approach were two-fold. First, to show that the origin of these conditions stems from diffeomorphism invariance of the wavefunction. Second, this approach seems to be generalizable to other inflationary theories, and thus could be exploited in more generality, in the same fashion that Ward identities are derived from symmetries of the path integral.

It is important to notice that we are always dealing with the "mathematical" squeezed limit, in the sense of taking the long mode wave number to zero. There are several models where the consistency condition is violated, in the sense of the ratio of the sides of the triangle being small, but not zero. This physical squeezed limit can probe different scales in the theory, and is usually associated to the long modes not freezing at this scale. It would be interesting to use the methods in [86] to see if one can say something in general about the leading order term in theories that violate the consistency condition. Let us also observe that, throughout the chapter, we used a technical assumption, namely, that $\lim_{k\to 0} \frac{d}{d\log k} D_k D \cdots D\Psi = 0$ [6].

The consistency condition can be also stated in terms of modes that are still inside the horizon. In our language we are always dealing with the superhorizon wavefunction. In a semiclassical setup, where cosmology is treated as an effective field theory, we can evaluate the semiclassical wave function at a given time slice, $\Psi[h, \phi, \eta]$. It is not clear that this object makes sense beyond effective field theory, but for the purposes of studying inflation as an effective theory, it is well defined and one can follow the same steps of the previous sections. The long mode would still correspond to taking $k \to 0$, but the other modes in the expectation value will be inside the horizon and the same consistency conditions would follow. The novelty here is that expectation values with derivatives in the subhorizon modes are non-zero, and thus one can derive new consistency conditions for those. Still, they should follow from the same diffeomorphism invariance constraint.

[6] This assumption seems to have no drawbacks for the following reason: a term that violates it would have to be an analytic function of one of the soft momenta. Thus, it is a contact term in position space and is not the piece fixed by these consistency conditions. Also, note that if we have a field theory that produces an almost scale invariant spectrum, either the logarithm is the indicator of an anomalous dimension coming from loop effects, or it should be discarded as it comes from a contact term in the expectation value. Note that a term of the form $\log(k_1 + k_2 + k_3)$ is allowed on a three point function, as it satisfies the assumption we made. An argument based on analyticity was made in [88] that such terms would violate the attractor structure of single field models.

The leading terms of these inflationary expectation values are thus fixed by gauge invariance. Of course, it would be nice if we could compute observables free of these pure gauge pieces. In gauge theory we know the answer to this question. In gravitational theories the answer is not so clear, unless there is a dual description in terms of a field theory. From the wavefunctional point of view, this would be equivalent to regarding its derivatives as the fundamental observables. In gauge gravity duality, these would translate to expectation values of the stress tensor of the theory. There, the consistency condition has the interpretation that, at zero momentum, there is an ambiguity related to the definition of the stress tensor [89].

The analysis carried in this note involved tree level expectation values. But the general structure of the Ward Identities coming from coordinate reparametrization invariance are valid to any loop order. The Hamiltonian constraint would necessarily involve some UV regularization, which leads to renormalization of observables etc. (this has been already carried out for gauge theories in [90]). But for the one-loop cosmology and beyond, the momentum constraint remain unchanged. So there might be some generalized consistency condition to n-loops. Maybe it can be generated recursively, just like in the recent proofs of conservation of the inflaton ζ outside of the horizon, which rely on the consistency condition [91, 92, 28, 93].

3.8 Appendix A: Four point functions and beyond

The procedure in the chapter can be generalized to higher order expectation values. Here we outline the general features of this procedure. The main difference is that there are two different squeezed limits. Namely, the limit when an external leg has zero momentum, or when an internal leg has zero momentum - a collinear limit.

Let us illustrate that point by considering a four point expectation value of scalar fluctuations in single field inflation. Its form, in terms of derivatives of the wavefunction, is given by[7]:

$$\begin{aligned}\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\zeta_{k_4}\rangle = &\frac{1}{\Pi_i 2\mathrm{Re}\left[D_{k_i}D_{-k_i}W_0\right]}\Bigg\{2\mathrm{Re}[D_{k_1}D_{k_2}D_{k_3}D_{k_4}W_0]+\\ &+\sum_a \frac{2\mathrm{Re}[D_{k_1}D_{k_2}D^a_{-k_1-k_2}W_0]2\mathrm{Re}[D^a_{-k_3-k_4}D_{k_3}D_{k_4}W_0]}{2\mathrm{Re}\left[D^a_{k_1+k_2}D^a_{-k_1-k_2}W_0\right]} + \text{permutations}\Bigg\}\end{aligned} \tag{3.50}$$

Where the $\sum_a$ represents the sum over all degrees of freedom (two graviton polarizations and one scalar). The permutations account for the different exchange channels for the internal leg, like the s, t and u channels in four particle scattering.

[7] We consider only the connected part of the four point expectation value. There is an additional contribution coming from disconnected diagrams, which is present in the free theory, which is thus Gaussian.

The external squeezed limit is completely analogous to the one in the main text. One needs to commute three functional derivatives through the momentum constraint and take the limit of an external momentum to zero etc. We analyze the internal momentum squeezed limit in detail, as it has no analogue for the three point expectation value.

Consider the limit $k_1 \to -k_2$. Of course, due to translation invariance, $k_4 \to -k_3$. We see that the overall denominator in (3.50) does not diverge. The only singular piece comes from the exchange diagrams that involve two vertices, as its denominator involves a second derivative of W_0 evaluated at $k_1 + k_2$ momentum. The contributions from the four-derivative term and other exchange terms are thus subleading. For the leading term, its numerator is the square of the squeezed limit of a third derivative, so we use (3.34)to relate these to two point expectation values. Thus, to leading order in $k_1 + k_2$, we obtain:

$$
\begin{aligned}
&\lim_{k_2 \to -k_1} \langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \zeta_{k_4} \rangle = \\
&= \langle \zeta_{k_1+k_2} \zeta_{k_1+k_2} \rangle \left(\langle \zeta_{k_1} \zeta_{-k_1} \rangle \frac{\partial \log \left[k_1^3 \langle \zeta_{k_1} \zeta_{-k_1} \rangle \right]}{\partial \log k_1} \right) \left(\langle \zeta_{k_3} \zeta_{-k_3} \rangle \frac{\partial \log \left[k_3^3 \langle \zeta_{k_3} \zeta_{-k_3} \rangle \right]}{\partial \log k_3} \right) + \\
&+ \sum_s \langle \gamma^s_{k_1+k_2} \gamma^s_{k_1+k_2} \rangle \left(\epsilon^{s,\ ij}_{k_1+k_2} k_1^i k_1^j \frac{\partial}{\partial k_1^2} \langle \zeta_{k_1} \zeta_{-k_1} \rangle \right) \left(\epsilon^{s,\ ij}_{k_1+k_2} k_3^i k_3^j \frac{\partial}{\partial k_3^2} \langle \zeta_{k_3} \zeta_{-k_3} \rangle \right)
\end{aligned}
\tag{3.51}
$$

Which is consistent with the results described in [76, 79, 80].

Note that this procedure can be extended to an n-point expectation value, but the amount of diagrams contributing beyond leading order makes the general expressions become cumbersome. This problem is treated in detail in [85] . There, a prescription to calculate the contribution from diagrams with exchanged particles, like the one we consider for the collinear limit, is given in detail.

3.9 Appendix B: Massless scalar spectator field

Let us analyze an example of an inflationary theory with an inflaton plus a massless scalar field, similar to the example discussed in [94]. We have the metric, h_{ij}, the inflaton ϕ and the spectator field σ. The condition of reparametrization invariance of the wavefunction is a simple extension of (3.13):

$$
2\nabla_i \left[\frac{1}{\sqrt{h}} \frac{\delta W_0}{\delta h_{ij}} \right] - \frac{1}{\sqrt{h}} h^{ij} \partial_i \phi \frac{\delta W_0}{\delta \phi} - \frac{1}{\sqrt{h}} h^{ij} \partial_i \sigma \frac{\delta W_0}{\delta \sigma} = 0 \tag{3.52}
$$

As in the single field case, we are interested on the wavefunction calculated on a slice of constant inflaton field. So we are effectively treating a momentum constraint of the form:

$$2\nabla_i\left[\frac{1}{\sqrt{h}}\frac{\delta W[h(x),\phi,\sigma(x)]}{\delta h_{ij}}\right]-\frac{1}{\sqrt{h}}h^{ij}\partial_i\sigma\frac{\delta W[h(x),\phi,\sigma(x)]}{\delta\sigma}=0 \tag{3.53}$$

Now, we take two functional derivatives with respect to the massless field, say, $\delta^2/\delta\sigma(y)\delta\sigma(z)$. They commute through the covariant derivative, and each one may hit the $\partial_i\sigma(x)$ term in (3.53). Because we are dealing with a scalar operator, it should be no surprise that we obtain a Ward identity identical to (3.42). After going to momentum space, and using the D notation for the functional derivatives, we finally obtain:

$$k_1^a D^{aj}_{k_1}D^\sigma_{k_2}D^\sigma_{k_3}W_0=\frac{1}{2}\left[-k_2^j D^\sigma_{k_3}D^\sigma_{-k_3}W_0-k_3^j D^\sigma_{k_2}D^\sigma_{-k_2}W_0\right] \tag{3.54}$$

So we have the same consistency condition as the one for scalar operators, namely:

$$\lim_{k_1\to 0}\langle\zeta_{k_1}\sigma_{k_2}\sigma_{k_3}\rangle=\langle\zeta_{k_1}\zeta_{-k_1}\rangle\frac{\partial\log\left[k_2^3\langle\sigma_{k_2}\sigma_{-k_2}\rangle\right]}{\partial\log k_2}=-n_\sigma\langle\zeta_{k_1}\zeta_{-k_1}\rangle\langle\sigma_{k_2}\sigma_{-k_2}\rangle \tag{3.55}$$

The field σ behaves as a free field in (quasi) de Sitter space, so its spectral index is the same as that of a gravitational wave, given by $n_\sigma=-2\epsilon$ [22], where the slow roll factor is related to the variation of Hubble's constant, $\epsilon H^2=-\dot{H}$. In fact, the computation of the three point function and the check for the squeezed limit is quite similar to the case of two gravitons and one scalar studied in [22].

To check (3.55) we need to compute the three point expectation value using the in-in formalism. We just state the main equations here. The quadratic actions, with corresponding (late time) two point functions are given by:

$$\begin{aligned}
S_{\zeta\zeta}&=\frac{1}{2}\int dtd^3x(2\epsilon)\left[a^3\dot{\zeta}^2-a(\partial\zeta)^2\right], &\qquad \langle\zeta_{k_1}\zeta_{k_2}\rangle&=(2\pi)^3\delta^3(k_1+k_2)\frac{H^2}{4\epsilon k^3}\\
S_{\sigma\sigma}&=\frac{1}{2}\int dtd^3x\left[a^3\dot{\sigma}^2-a(\partial\sigma)^2\right], &\qquad \langle\sigma_{k_1}\sigma_{k_2}\rangle&=(2\pi)^3\delta^3(k_1+k_2)\frac{H^2}{2k^3}
\end{aligned} \tag{3.56}$$

The cubic action is given by equation (27) of [94] . We can integrate it by parts, following the strategy in [22], to see that the interaction is of order ϵ, the slow roll parameter:

$$\begin{aligned}
S_{\zeta\sigma\sigma}=\int\Big[&-\frac{a}{2}\zeta(\partial\sigma)^2-\frac{a}{2H}\dot{\zeta}(\partial\sigma)^2+a\partial_i\left(\frac{\zeta}{H}-\epsilon a^2\partial^{-2}\dot{\zeta}\right)\dot{\sigma}\partial_i\sigma-\\
&-\frac{a^3}{2H}\dot{\zeta}\dot{\sigma}^2+\frac{3a^3}{2}\zeta\dot{\sigma}^2\Big]=\int\left\{\epsilon\left[-\zeta L_{\sigma\sigma}-a^3\partial_i\partial^{-2}\dot{\zeta}\dot{\sigma}\partial_i\sigma\right]+\frac{\zeta\dot{\sigma}}{H}\frac{\delta L_{\sigma\sigma}}{\delta\sigma}\right\}
\end{aligned} \tag{3.57}$$

The last term is proportional to the quadratic equations of motion, and can be removed from the action by a proper σ field redefinition[22]. This redefinition does not alter the final three point expectation value though, as it vanishes outside the horizon. What is left are the terms in square brackets. It is clear that, when the ζ mode becomes very long in wavelength, it is simply rescaling the two point Lagrangian for the σ field. This is the standard bulk intuition to justify the consistency relation. In fact, one can check that the second term in square brackets is subleading in the squeezed limit, thus making this intuition rigorous. To leading order in slow roll, the squeezed three point expectation value is given by:

$$\lim_{k_1 \to 0} \langle \zeta_{k_1} \sigma_{k_2} \sigma_{k_3} \rangle = \frac{H^4}{4 k_1^3 k_2^3} = -(-2\epsilon) \frac{H^2}{4\epsilon k_1^3} \frac{H^2}{2k_2^3} = -n_\sigma \langle \zeta_{k_1} \zeta_{-k_1} \rangle \langle \sigma_{k_2} \sigma_{-k_2} \rangle \tag{3.58}$$

An important observation is the following. Because the massless σ field is free to fluctuate, it can convert itself into ζ fluctuations during reheating and other phases beyond inflation. Also, we assumed that there is a quasi-de Sitter background over which the fluctuations evolve. Thus, it is not necessarily true that the three point expectation values computed here are kept frozen and will induce temperature correlations in the CMB.

3.10 Appendix C: Derivative of the polarization tensor

Take a three point expectation value that involves a long tensor mode. If we expand it to linear order in the long mode, we obtain:

$$\langle \gamma^s_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle = \langle \gamma^s_0 \zeta_{k_2} \zeta_{-k_2} \rangle + k_1^a \left[\langle \gamma^{bc}_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \frac{\partial}{\partial k_1^a} \epsilon^1_{bc} + \epsilon^1_{bc} \frac{\partial}{\partial k_1^a} \langle \gamma^{bc}_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \right] + \cdots \tag{3.59}$$

We want to show that the first term does not contribute in brackets does not contribute to (3.59). In order to do that, we take derivatives of the defining expressions for the polarization tensor (3.25) and obtain:

$$\left(\frac{\partial}{\partial k^a} \epsilon_{bc} \right) \epsilon_{bc} = 0, \quad \epsilon_{ab} + k_c \frac{\partial}{\partial k^a} \epsilon_{bc} = 0 \Rightarrow \frac{\partial}{\partial k^a} \epsilon_{bc} = - \frac{k_b \epsilon_{ac} + k_c \epsilon_{ab}}{k^2} \tag{3.60}$$

Which, when contracted with k_1^a, will give zero. This is why we are free to contract the polarization tensor directly with $\frac{\partial}{\partial k_1^a} \langle \gamma^{bc}_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle$ to derive the linear consistency condition.

Chapter 4

Time Independence of ζ in Single Field Inflation

4.1 Introduction

4.1.1 Motivation

The purpose of this chapter is to prove that in single clock inflation, where there is only one relevant degree of freedom during inflation, the correlation function of the curvature perturbation ζ for separations outside the horizon is time independent at one loop level. We believe this to be a very important result to prove for several reasons. As it becomes more and more likely that Inflation was part of the early history of our Universe it becomes more and more important to understand how the theory behaves at quantum level, even if the expected corrections are small. We could make an analogy with the 1950s when QED was studied to all orders in perturbation theory. Similarly to what happened in that case, it is not so obvious that quantum corrections are as small as one might expect. While a simple parametric analysis tells that the corrections to the curvature perturbation should be of order

$$\langle\zeta^2\rangle_{1-\text{loop}} \sim \langle\zeta^2\rangle^2_{tree} \sim 10^{-9}\langle\zeta^2\rangle_{tree} \ , \tag{4.1}$$

no symmetry forbids the presence of potentially large infrared factors, such as

$$\langle\zeta_k^2\rangle_{1-\text{loop}} \sim k^3\langle\zeta_k^2\rangle^2_{tree}\log(kL) \ , \tag{4.2}$$

where L is the comoving size of the inflationary space, or of the form

$$\langle\zeta_k^2\rangle_{1-\text{loop}} \sim k^3\langle\zeta_k^2\rangle_{tree}^2 Ht \ , \tag{4.3}$$

where H is the Hubble constant during inflation and t is time. All these terms have appeared in partial calculations of the one-loop corrections to the power spectrum [94].

Log(H/μ) **effects:** Additionally, infrared effects of the form

$$\langle\zeta_k^2\rangle_{1-\text{loop}} \sim k^3\langle\zeta_k^2\rangle_{tree}^2 \log(k/\mu) \ , \tag{4.4}$$

with μ being the renormalization scale of the theory, have been found in several papers (see references in [91]). Strictly speaking, a correction of the form $\log(k/\mu)$ is not allowed by symmetries, representing a breaking of zero-mode gauge invariance $x \to \lambda x, a \to a/\lambda$, where a is the scale factor of the FRW metric. Its presence was due to a mistake in the implementation of a diff. invariant regularization, and this issue is addressed in [91], where it was shown that the logarithmic running takes the form

$$\langle\zeta_k^2\rangle_{1-\text{loop}} \sim k^3\langle\zeta_k^2\rangle^2 \log(H/\mu) \ . \tag{4.5}$$

Notice that if a result of the form $\log(k/\mu)$ were to be correct, then the effect could have been potentially very large when $k \to 0$.

Contrary to the case of $\log(k/\mu)$, logarithmic corrections of the form $\log(kL)$ or $\log(a(t)) \sim Ht$ are allowed by symmetries.

Log(kL) **effects.** The factor of $\log(kL)$ can be potentially very large, as $\log(kL)$ is of order $N_{\text{beginning}}$, the number of e-foldings of Inflation that have occurred *before* the mode k has crossed the horizon. Even for the standard inflation that we might have in our past, $N_{\text{beginning}}$ can be a large enhancement factor. Furthermore in situations where $N_{\text{beginning}}$ might be large, $\langle\zeta^2\rangle$ for modes exiting the horizon at the beginning of inflation might also be significantly larger as one could be near an eternal inflation regime. The infrared factor $\log(kL)$ does appear in the one-loop correction to the power spectrum [95, 96], and in [92] it was shown that it is simply a projection effect that is completely removed when one computes observable quantities and that does not affect our ability to extract predictions from inflation.

Log$(a(t))$ **effects and the predictivity of Inflation.** In this chapter we try to address the question of whether the one-loop correction to the power spectrum is time dependent, or in other words if at loop level ζ_k is constant after the mode k has crossed the horizon. We notice that for our current inflationary patch, since we observe around 50 e-foldings of inflation and $\zeta \sim 3 \times 10^{-5}$, such a correction factor, even if present, would represent a correction at most of order $50 \times 10^{-9} \sim 5 \times 10^{-8}$. From an observational perspective this is a very small correction. Regardless of this fact, as a matter of principle if such a time-dependent factor were to be present the consequences for the inflationary theory would be profound. Such a result would imply that short scale fluctuations, say of the size of the horizon, can change the amplitude of a mode after it has crossed the horizon. In standard inflation the amplitude of the short perturbations is very small and the duration of inflation is relatively short so the resulting evolution of the long modes is negligible. However, fluctuations might not be small during other epochs of the evolution of the universe, such as reheating and baryogenesis or if the dynamics of inflation changes dramatically at some point. We know little about these epochs, but if perturbations were to be large on Hubble scales during those times, the time-dependence induced on long, observable, modes could change their amplitude significantly. We would lose the predictions of Inflation unless we know the details of the physics governing reheating or baryogenesis, which we hardly do.

The potential for a time dependence of the power spectrum at loop level was pointed out by Weinberg in [94]. He noticed that many diagrams naively induce a time-dependence of ζ [1]. The question of weather a time dependence persists after we sum all the diagrams has remained open. [91] addressed this issue in certain simplified examples involving spectator fields running in the loops. Although the physics identified in that paper will basically apply unchanged in this study, the fact of the matter is that no proper calculation in the context of single clock inflation has been presented. Ref. [99] claims to have done this and to have found a time dependence. In reality they only presented results for a severely truncated and simplified Lagrangian and of course they did not recover the cancellations we identify in this chapter and thus claimed a spurious time dependence.

Slow Roll Eternal Inflation. From a more theoretical point of view, a time-dependence of ζ would have important consequences for slow-roll eternal inflation. In recent years [100, 101, 102, 104], there has been remarkable progress in understanding slow roll eternal inflation at a quantitative level. The study of eternal inflation (usually of the false vacuum type) has been largely motivated by the fact that the universe is currently accelerating and by the apparent existence of a landscape of vacua in String Theory which put together suggest that the current acceleration can be understood as resulting from an anthropic selection of the vacuum energy made possible by an epoch of eternal inflation in our past. Another piece of motivation to study eternal inflation relies on the perhaps mysterious connections between gravity and quantum mechanics in the presence of a horizon. De Sitter space, with its supposedly finite entropy, represents a mystery, and slow roll (eternal) inflation represents a natural regularization of de Sitter space. In [100] it was shown that

[1]Such a result would not be in contradiction with the many proofs available in the literature on the conservation of ζ outside of the horizon (see for example [22, 97, 98]). The fact that the constant solution is the attractor one, and not simply one of the two solutions, was proven in [31]. All these proofs work in the limit in which all modes are longer than the horizon, so that gradients of all fluctuations can be neglected.

slow roll inflation undergoes a phase transition when a parameter

$$\Omega = \frac{2\pi^2}{3}\frac{\dot{\phi}^2}{H^4} \ , \tag{4.6}$$

becomes less than one. At that point, the probability to develop an infinite volume goes from being strictly zero to non-zero. This is the phase transition to eternal inflation. Subsequently, in [101], it was found that there is a sharp upper bound to how large a finite volume can be created: the probability to produce a finite volume larger than e^{6N_c}, with N_c representing the classical number of e-foldings, is non-perturbatively small from the point of view of quantum gravity:

$$P\left(V_{\text{finite}} > e^{6N_c}\right) < e^{-M_{\rm Pl}^2/H^2} \ . \tag{4.7}$$

By connecting the classical number of e-foldings to the the entropy of de Sitter space S_{dS} at the end of inflation, this bound can be recast as

$$P\left(V_{\text{finite}} > e^{S_{dS}/2}\right) < e^{-M_{\rm Pl}^2/H^2} \ . \tag{4.8}$$

This bound is a generalization to the quantum and eternal regime of the bound found in [103], that was much stronger than the one in (4.8). Further, in [102], it was shown that this bound is actually universal: it holds for any number of spacetime dimensions and for any number of inflating fields. Moreover it holds unchanged also when considering higher-order corrections to the theory of gravity and of the inflaton, and it does so to all orders in perturbation theory. In [104], it is shown that it holds also when including slow-roll corrections. All of these results strongly suggest that the bound in (4.8) is a true fact of nature connected to the holographic interpretation of de Sitter space.

All these new results on Eternal Inflation assumed that the ζ two-point function at coincidence takes the form[2]

$$\langle\zeta(x)^2\rangle \sim H^3 t \ , \tag{4.9}$$

which is a direct consequence of its scale invariance and time-independence in Fourier space

$$\langle\zeta_k^2\rangle \sim \frac{H^2}{k^3} \ . \tag{4.10}$$

[2]Studies of the phase transition to slow-roll eternal inflation have only been done at lowest order in slow-roll, where there is basically no distinction between ζ and $\delta\phi$.

If the two point function of the inflaton in Fourier space were to go as

$$\langle\zeta_k^2\rangle \sim \frac{H^2}{k^3}\log(kL) \ , \qquad \text{or} \qquad \langle\zeta_k^2\rangle \sim \frac{H^2}{k^3}Ht \tag{4.11}$$

then in real space it would go as

$$\langle\zeta(x)^2\rangle \sim H^4 t^2 \ , \tag{4.12}$$

and all the above-mentioned new results on slow roll eternal inflation would fail [3]. Depending on the sign of the loop correction, we would be lead to conclude that all inflationary models are either eternal or never-eternal. This motivates us to study the possible time-dependence of ζ at loop level.

4.1.2 Simple Arguments

There are several simple intuitive arguments that suggest that short scale fluctuations cannot induce a time dependence on a long wavelength ζ mode that is much longer than the horizon. The simplest and most intuitive argument relies on the fact that at long wavelengths a ζ mode is indistinguishable in practice from a rescaling of the scale factor $a \to a\, e^{\zeta}$. This means that a time dependent ζ is more or less equivalent to a change in the local value of the expansion rate H: $\dot{\zeta} \sim \delta H$. In order for short-scale fluctuations to create a time-dependent long wavelength ζ, the short scale fluctuations should create a modulation of the Hubble parameter that is coherent over a very large scale, the scale of the long wavelength ζ mode.

One could imagine two mechanisms through which this could happen. The random small scale fluctuations could lead by chance to a large scale fluctuation, but simple 'square root of N' type of arguments show that this is not the case. Another option is that the short modes are sensitive to the long wavelength fluctuations through tidal-type effects and thus their expectation values, their energy density say, varies over the long scales and leads to a modulation in the expansion rate. This last possibility also sounds quite unreasonable. Because of the attractor nature of the inflationary background, a long wavelength ζ fluctuation is locally almost indistinguishable from a rescaling of the background, with corrections that rapidly redshift to zero. This means that short wavelength fluctuations should behave in very much the same way in the presence of a long ζ mode as they do in its absence (apart for a trivial rescaling of the coordinates). This is what the so-called Maldacena consistency condition of curvature fluctuations actually states [22, 75, 31], and it has been shown to work at tree-level in several calculations.

Perhaps a better way to illustrate the point we are trying to make is the following. Assume that short wavelength modes running in the loop lead to a time dependence of the two point function of a long wave-

[3] We acknowledge David Gross for pointing this out to us.

length mode. This one loop calculation is just giving the change of the long modes produced by the short modes when averaged over the short ones. If the short modes can be observed directly the effect of the short modes on the long should lead to an observable correlation between short and long modes. In other words, it should lead for example to a non-zero three point function in the squeezed limit. However, since the work of Maldacena [22] we know that there is no such effect in the squeezed three point function. It is hard to imagine that one would not be able to detect a correlation between short and long modes when both short and long modes are measured, but that on average the short modes do lead to an evolution of the long modes.

All of this suggests that it would be quite surprising if short modes were to induce time-dependence in a long wavelength ζ fluctuation [4]. We note that the essence of these arguments were already given by some of us in [91].

4.1.3 Summary of the Strategy

Let us make the simple arguments above a bit more precise highlighting our strategy to prove the time-independence of ζ. Since we are interested in a late time-dependence of ζ, we can restrict ourselves to the case in which we let only short wavelength modes run in the loops. The constancy of ζ when all modes are outside the horizon was already proved in [22]. In the present case, computing one-loop effects can be thought as solving the non-linear evolution equations for a long wavelength ζ operator, ζ_L, up to cubic order in the fluctuations. This will take the form

$$\hat{O}\left[\zeta_L\right] = S\left[\zeta_S, \zeta_S, \zeta_L\right] \ , \tag{4.13}$$

where S represent a generic sum of operators that are quadratic in the short wavelength ζ, ζ_S, and that can also eventually depend on ζ_L both explicitly and implicitly through a dependence of ζ_S on ζ_L. Each monomial in S can contain derivatives acting on the various ζ's. The solution is schematically of the form

$$\zeta_L = \hat{O}^{-1}\left[S\left[\zeta_S, \zeta_S, \zeta_L\right]\right] \ . \tag{4.14}$$

It should be noted the $\langle S\left[\zeta_S, \zeta_S, \zeta_L\right]\rangle$ is in general not zero. There are tadpole contributions for ζ because at loop level we are expanding around the incorrect background history. We will add tadpole counterterms to the action to ensure that the background solution we started with satisfies the equations of motion. These

[4]There is one subtlety which has to do with the renormalization of the background. Short wavelength fluctuations do renormalize the background, so that $H(t)$ is different from its value at tree level when the short fluctuations are neglected. It is important to take this fact into account properly in order for ζ not to have a time-dependence.

counterterms lead to additional diagrams that will cancel many of the one loop diagrams in our power spectrum calculation.

The one loop power spectrum will be given by

$$\langle \zeta_L \zeta_L \rangle \sim \langle \hat{O}^{-1} \left[S \left[\zeta_S, \zeta_S, \zeta_L \right] \right] \zeta_L \rangle + \langle \hat{O}^{-1} \left[S \left[\zeta_S, \zeta_S, \zeta_L = 0 \right] \right] \hat{O}^{-1} \left[S \left[\zeta_S, \zeta_S, \zeta_L = 0 \right] \right] \rangle \; . \tag{4.15}$$

We call the first contribution on the right the cut-in-the-side (CIS) diagrams, while the second contribution on the right cut-in-the-middle (CIM) diagrams.

The CIM diagrams represent the effect of the short scale modes in their unperturbed state directly on the power spectrum of the long wavelength modes. These diagrams will not lead to any time-dependence of the long modes simply because it is very hard for short mode fluctuations to be coherent over long scales.

Many of the CIS diagrams cancel with diagrams coming from the tadpole counterterms. The remaining CIS diagrams represent instead the evolution of ζ_L due to the effect that ζ_L itself has on the expectation value of quadratic operators made of short modes. These diagrams involve the correlation between this short-mode expectation value and the long wavelength mode itself [5]. This short-mode long-mode correlation sources ζ_L.

The Maldacena consistency condition implies that this short-mode long-mode correlation actually vanishes,

$$\langle \hat{O}^{-1} \left[S \left[\zeta_S, \zeta_S, \zeta_L \right] \right] \zeta_L \rangle = 0. \tag{4.16}$$

This is so because the consistency condition means that in the limit in which the long mode has a wavelength much longer than the horizon, it simply acts as a rescaling of the coordinates. So the correlation function between short and long modes can be understood in terms of the power spectrum of the short modes computed in a rescaled background. Since in the loop the short-mode expectation value is integrated over all the short-mode momenta the rescaling is irrelevant and as a result there is no correlation between the short scale power and the long mode.

Even though the former arguments are quite compelling, the calculation is very complex, and many subtleties are hidden in the above equations. They include the identification of the Lagrangian of the ζ zero-mode, that will turn out to be delicate and to affect the definition of the tadpole counterterms. Because of diff. invariance, these counterterms will play a role even for the finite momentum correlation functions. Additionally, it will be non-trivial to see how the Maldacena consistency condition works when dealing with operators involving derivatives.

In summary, since the interactions are dominated by the gravitational ones, our one-loop computation

[5] It will become clear later that this remaining CIS diagrams depend both on the cubic and quartic Hamiltonians.

amounts to doing a one loop calculation in gravity in an accelerating universe. This is quite a hard task, at least for us! In particular, there are *many many many* diagrams involved, and *many many* of these naively induce a time-dependence on ζ. The time independence will result from cancellations among diagrams. We will now try to move step by step to make our arguments explicit and precise, finally proving that ζ is constant outside of the horizon also at one-loop level.

4.2 An Intuitive Organization of the Diagrams

It is possible to organize the one-loop diagrams in a way that is particularly close to our intuition. This approach was originally developed in [105] for a restricted set of theories, and it was noted in [106] that the derivation was not consistent with the $i\,\epsilon$ prescription for choosing the interacting vacuum in the past. This approach has been generalized in [91] to more generic theories and a correct $i\,\epsilon$ prescription has been implemented. Here we will see that the implementation of the $i\,\epsilon$ prescription can be performed in a very simple way.

For concreteness let us specialize to the ζ two-point function. We have to compute

$$\langle\Omega|\zeta^2(t)|\Omega\rangle = \langle 0|U_{int}(t,-\infty_+)^\dagger \zeta_I^2(t) U_{int}(t,-\infty_+)|0\rangle \ , \tag{4.17}$$

where $|\Omega\rangle$ is the vacuum of the interacting theory, $|0\rangle$ is the one of the free theory,

$$U_{int}(t,-\infty_+) = Te^{-i\int_{-\infty_+}^{t} dt'\ H_{int}(t')} \ , \tag{4.18}$$

and the subscript $_I$ stays for interaction picture. Finally, the symbol $-\infty_+$ represents the fact that the time-integration contour has been rotated so as to project the free vacuum on the interacting vacuum. In practice, this amounts to choosing the contour that suppresses the oscillatory terms in the infinite past.

We start by taking expression (4.17) and inserting the unit operator

$$1 = U_{int}(t,-\infty)U_{int}^\dagger(t,-\infty) \ , \tag{4.19}$$

between the two ζ's, to obtain

$$\langle\zeta^2(t)\rangle = \langle\Big(U_{int}^\dagger(t,-\infty_-)\zeta_I(t)U_{int}(t,-\infty)\Big)\ \Big(U_{int}^\dagger(t,-\infty)\zeta_I(t)U_{int}(t,-\infty_+)\Big)\rangle \ , \tag{4.20}$$

where we have ignored to specify the state upon which we compute the correlation function, either $|\Omega\rangle$ or

$|0\rangle$, as it is clear from the context. Ignoring for a moment the issue of the $i\,\epsilon$ prescription, we have written the expectation of the operator $\zeta(t)^2$ as the product of two interaction picture $\zeta_I(t)$'s, each evolved with the interaction picture time evolution operator U_{int}. In other words, the $\zeta(t)^2$ correlation function is simply given by the correlation function of the evolved $\zeta(t)$'s. We can Taylor expand in H_{int} to obtain

$$\begin{aligned}
&\langle\zeta^2(t)\rangle = \\
&= \langle \left(\sum_{N=0}^{\infty} i^N \int_{-\infty}^{t} dt_N \int_{-\infty}^{t_N} dt_{N-1} \ldots \int_{-\infty}^{t_2} dt_1 \,[H_{int}(t_1),[H_{int}(t_2),\ldots[H_{int}(t_N),\zeta_I(t)]\ldots]]\right) \\
&\times \left(\sum_{N=0}^{\infty} i^N \int_{-\infty}^{t} dt'_N \int_{-\infty}^{t_N} dt'_{N-1} \ldots \int_{-\infty}^{t_2} dt'_1 \,[H_{int}(t'_1),[H_{int}(t'_2),\ldots[H_{int}(t'_N),\zeta_I(t)]\ldots]]\right)^{\dagger} \rangle \;.
\end{aligned} \tag{4.21}$$

Expanding (4.21) up to second order in H_{int}, we obtain

$$\langle\zeta^2(t)\rangle = \langle\zeta^2(t)\rangle_{CIS} + \langle\zeta^2(t)\rangle_{CIM} \;, \tag{4.22}$$

where we have defined

$$\begin{aligned}
\langle\zeta^2(t)\rangle_{CIS} &= -2\,\mathrm{Re}\left[\left(\int_{-\infty}^{t} dt_2 \int_{-\infty}^{t_2} dt_1 \langle[H_3(t_1),[H_3(t_2),\zeta_I(t)]]\right)\zeta_I(t)\rangle\right. \\
&\qquad \left. -i\left(\int_{-\infty}^{t} dt_1 \langle[H_4(t_1),\zeta_I(t)]\right)\zeta_I(t)\rangle\right] \;, \\
\langle\zeta^2(t)\rangle_{CIM} &= -\left(\int_{-\infty}^{t} dt_1 \langle[H_3(t_1),\zeta_I(t)]\right)\left(\int_{-\infty}^{t} dt'_1\,[H_3(t'_1),\zeta_I(t)]\rangle\right) \;.
\end{aligned} \tag{4.23}$$

The subscript $_{CIS}$ denotes what we call cut-in-the-side diagrams, while $_{CIM}$ denotes cut-in-the-middle diagrams. Here by H_3, $H_4,\ldots$ we mean the cubic, quartic, $\ldots$ interaction Hamiltonians. We see that the CIM diagrams are made up by evolving each of the two ζ's to first order in the cubic interactions. The CIS diagrams corresponds to evolving only one of the two ζ's, either twice with cubic interactions or once with a quartic interaction.

This form of organizing the diagrams is particularly intuitive. If we remind ourselves that the ζ retarded Green's function is given by

$$G^R_\zeta(x,x') = i\theta(t-t')\,[\zeta_I(x),\zeta_I(x')] \;, \tag{4.24}$$

we have that

$$[H^{(3)}_{int},\zeta] \sim G^R\frac{\delta\mathcal{L}_3}{\delta\zeta} \;. \tag{4.25}$$

Then the CIM diagrams approximately correspond to considering the sourcing of ζ from the vacuum correlation function of $\delta\mathcal{L}_3/\delta\zeta$. This is very similar to the case when we try to solve some equations of motion perturbatively. We can define the solution of order n in the perturbation as $\zeta^{(n)}$. If we have schematically:

$$\mathcal{D}\zeta^{(2)} = \zeta^{(1)2} \qquad \Rightarrow \qquad \zeta^{(2)} = \int dt' G_\zeta(t,t')\zeta^{(1)}(t')^2 \tag{4.26}$$

where $\mathcal{D}$ is the differential operator of the free equations of motion, of which the Green's function is the inverse, then the CIM diagram is represented by the following

$$CIM = \langle \zeta^{(2)}\zeta^{(2)} \rangle \ . \tag{4.27}$$

The CIM diagram is diagrammatically represented in Fig. 4.1. Intuitively, it can be thought of as taking into account of the backreaction on ζ from the quantum variance of the operator $\delta\mathcal{L}_3/\delta\zeta$.

On the other hand, the CIS diagrams correspond to two sort of diagrams. The ones involving the quartic interactions, CIS_4, correspond to considering the effect of the expectation value of the vacuum fluctuations of two fluctuations on the external ζ. Schematically, it is given by

$$\mathcal{D}\zeta = \zeta^{(1)3} \qquad \Rightarrow \qquad \zeta^{(3)} = \int dt' G_\zeta(t,t')\zeta^{(1)}(t')^3 \qquad \Rightarrow \qquad CIS_4 = \langle \zeta^{(3)}\zeta^{(1)} \rangle \ , \tag{4.28}$$

and it is represented in Fig. 4.2.

The CIS diagrams that involve two cubic interactions can in turn be divided in two subclasses. The first are of the non-1PI form, $CIS_{non-1PI}$, and describe the effect of the expectation value of two fluctuations on the ζ zero mode, ζ_0, and how then the zero mode affects the ζ propagation. Schematically, this is given by

$$\begin{aligned} &\mathcal{D}\zeta_0 = \zeta^{(1)2} \qquad \Rightarrow \qquad \langle\zeta_0^{(2)}\rangle = \int dt' G_\zeta(t,t')\langle\zeta^{(1)}(t')^2\rangle \\ &\mathcal{D}\zeta^{(3)} = \zeta^{(1)}\langle\zeta_0^{(2)}\rangle \qquad \Rightarrow \qquad \zeta^{(3)} = \int dt' G_\zeta(t,t')\zeta^{(1)}(t')\langle\zeta_0^{(2)}\rangle(t') \qquad \Rightarrow \qquad CIS_{non-1PI} = \langle \zeta^{(3)}\zeta^{(1)} \rangle \ , \end{aligned} \tag{4.29}$$

and it is represented in Fig. 4.3. This diagram intuitively represents how a perturbation to the background (the zero mode) affects the evolution of the finite-k modes.

The second kind of CIS diagram, CIS_{1PI} is 1PI and corresponds to considering the sourcing on ζ from two fluctuations, one of which has been perturbed by an initial ζ fluctuation.

$$\begin{aligned} &\mathcal{D}\zeta^{(2)} = \zeta^{(1)2} \qquad \Rightarrow \qquad \zeta^{(2)} = \int dt' G_\zeta(t,t')\zeta^{(1)}(t')^2 \\ &\mathcal{D}\zeta^{(3)} = \zeta^{(1)}\zeta^{(2)} \qquad \Rightarrow \qquad \zeta^{(3)} = \int dt' G_\zeta(t,t')\zeta^{(1)}(t')\zeta^{(2)}(t') \qquad \Rightarrow \qquad CIS_{1PI} = \langle \zeta^{(3)}\zeta^{(1)} \rangle \ , \end{aligned} \tag{4.30}$$

and it is represented in Fig. 4.4. Physically, this represents how a fluctuation is affected by two fluctuations,

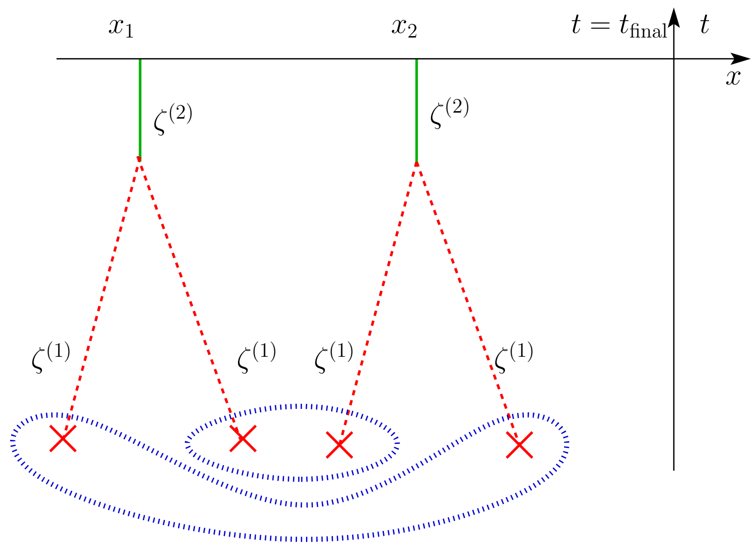

Figure 4.1: Cut-in-the-middle (CIM) diagrams. Green continuous lines represent Green's functions, red dashed lines represent free fields, and red crosses circled by a blue dotted line represent correlations of free fields. Two crosses have to be contracted together in order for the diagram not to be zero. This diagram represents how vacuum correlation functions of quadratic operators $\zeta^{(1)2}$, $\langle\zeta^{(1)2}\zeta^{(1)2}\rangle$ source perturbed correlation functions for $\zeta^{(2)}$: $\langle\zeta^{(2)}\zeta^{(2)}\rangle$.

one of which has been already perturbed. If we imagine for a moment that only short fluctuations run in the loop, this diagram would represent how a long mode affects through tidal effects the dynamics of the short modes, and how these backreact on the long mode.

Let us finally comment on how to implement the $i\,\epsilon$ prescription. When we insert the unit operator in (4.20), we should keep in mind that the integration contours of the time evolutors on the sides of the expectation value are rotated, while the ones in the middle are not. This means that when we Taylor expand in H_{int}, the various terms do not really regroup and form commutators, because they are evaluated on different contours. A solution to this problem was provided in [91] where the rotation was performed only at very early times and the commutator form applied only at late time. Here we implement the correct $i\,\epsilon$ rotation in a different way. We perform no contour rotation, but we multiply our expression by $e^{i\epsilon(\sum k_i)t}$, where the sum runs over all the momenta involved in the loops and $\epsilon > 0$, so that the time integrals are convergent in the far past, and then take the limit $\epsilon \to 0^+$. While the multiplication by $e^{i\epsilon(\sum k_i)t}$ is not a rotation of the contour of integration, it converges to one in the limit $\epsilon \to 0^+$. It can be easily checked that this procedure agrees with the rotation of the contour.

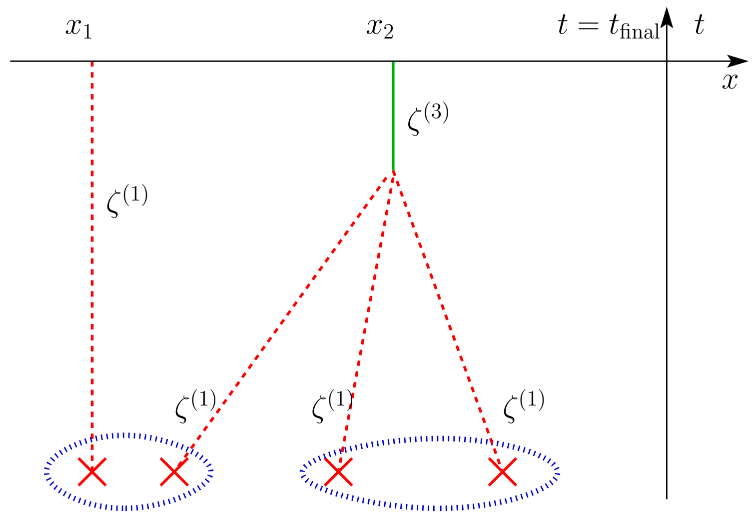

Figure 4.2: Cut-in-the-side quartic (CIS_4) diagrams. These diagrams represent how vacuum expectation values of quadratic operators $\langle \zeta^{(1)2} \rangle$ affect the propagation of a mode $\zeta^{(3)}$, and therefore the ζ correlation function: $\langle \zeta^{(3)} \zeta^{(1)} \rangle$

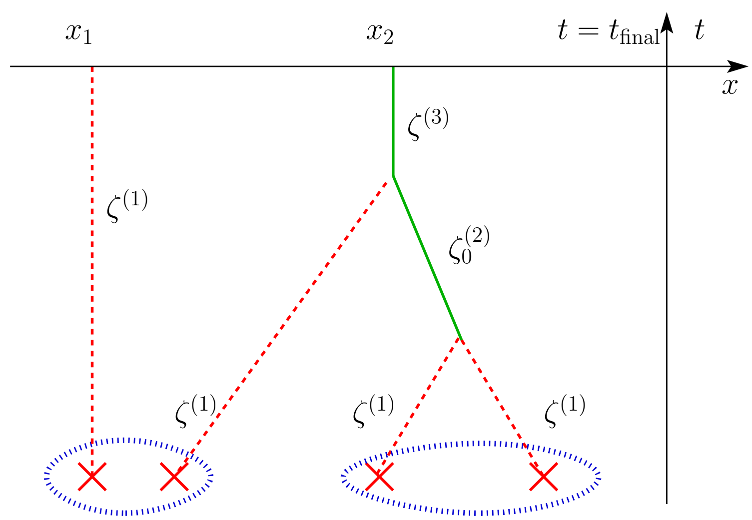

Figure 4.3: Non-1PI cut-in-the-side quartic ($CIS_{non-1PI}$) diagrams. These diagrams represent how vacuum expectation values of quadratic operators $\langle \zeta^{(1)2} \rangle$ affect the propagation of the zero mode $\zeta_0^{(2)}$, and therefore the evolution of a mode by a non linear coupling $\zeta^{(3)} \sim \zeta^{(1)} \zeta_0^{(2)}$. This sources a correlation function of the form: $\langle \zeta^{(3)} \zeta^{(1)} \rangle$

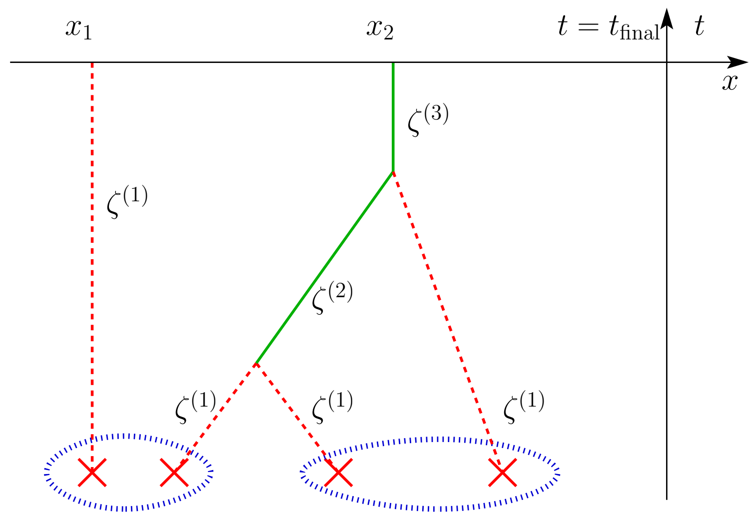

Figure 4.4: 1PI cut-in-the-side quartic (CIS_{1PI}) diagrams. These diagrams represent how the propagation of a mode is perturbed at two different times by two fluctuations that are correlated among themselves. This sources a correlation function of the form: $\langle\zeta^{(1)}\zeta^{(3)}\rangle$

4.3 Loops as the integral of the three-point function

Let us consider a cubic Lagrangian of the form

$$\mathcal{L}_3 = \sum_n \mathcal{L}_3^{(n)} \tag{4.31}$$

where the sum over n runs over all possible monomials constituting $\mathcal{L}_3$. We will schematically write

$$\mathcal{L}_3^{(n)} \propto \mathcal{D}_1^{(n)}\zeta\mathcal{D}_2^{(n)}\zeta\mathcal{D}_3^{(n)}\zeta, \tag{4.32}$$

where $\mathcal{D}_a^{(n)}$, $a = 1, 2, 3$ are the differential operator acting on $\zeta(x)$ in position a. It includes both time and spatial derivatives, as well as the identity operator.

There are certain quartic diagrams which we call $Quartic_{3,\partial_t}$. They are the quartic diagrams with the quartic vertices that arise because the cubic Lagrangian contains $\dot{\zeta}$, $H_4 \supset H_{4,3^2} = \delta\dot{\zeta}/\delta P \times (\delta\mathcal{L}_3/\delta\dot{\zeta})^2/2$. We want to prove that we can write the sum of $CIS_{1PI} + CIM + Quartic_{3,\partial_t}$ diagrams as:

$$\begin{aligned}\langle\zeta_k\zeta_k\rangle_{CIS_{1PI}+CIM+Quartic_{3,\partial_t}} &= \lim_{\epsilon\to 0}\int_{-\infty}^{t} dt_1\; a(t_1)^{3+\delta} \\ \sum_{a,n}\mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)2\mathrm{Re}\langle&\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k U_{int}^\dagger(t_1,-\infty)\zeta_{k,I}(t)U_{int}(t_1,-\infty)\rangle\; e^{\epsilon k t_1}\;.\end{aligned} \tag{4.33}$$

In this formula

$$\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k \tag{4.34}$$

represents the k-Fourier component of what is left of the the cubic Lagrangian term $\mathcal{L}_3^{(n)}$ after the removal of $a(t_1)^{3+\delta}\,\mathcal{D}_a^{(n)}\zeta_a(t_1)$. ζ_I is again the interaction picture field.

Eq. (4.33) is a remarkably simple formula given that it sums up a very large number of diagrams. It shows that the sum of all these diagrams can be written as a sum of integrals of three-point functions. Since we are interested in the case in which the fluctuations running in the loop are much shorter-wavelength than the one in the external fields, the three-point functions are computed in the squeezed limit, a fact that simplifies largely their behavior and makes them describable using the consistency condition of three-point functions. This will turn out to be very useful.

4.3.1 Quasi 3-point function

In order to prove the master eq. (4.33), let us start by considering the 3-point function appearing there:

$$\begin{aligned}
&2\mathrm{Re}\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k U_{int}^\dagger(t_1,-\infty)\zeta_{I,k}(t)U_{int}(t_1,-\infty)\rangle = \\
&2\mathrm{Re}\left\{\langle U_{int}(t_1,-\infty)^\dagger\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_{I,k} U_{int}(t_1,-\infty)U_{int}^\dagger(t_1,-\infty)\zeta_{I,k}(t)U_{int}(t_1,-\infty)\rangle+\right.\\
&\quad\sum_b\frac{i}{2}\langle\left[\mathcal{D}_b^{(n,out)}\left[H_3(t_1),\zeta(t_1)\right]\left(\frac{1}{a(t_1)^{3+\delta}}\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\delta\dot\zeta_b(t_1)}\right)+\right.\\
&\quad\left.\left.\left(\frac{1}{a(t_1)^{3+\delta}}\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\delta\dot\zeta_b(t_1)}\right)\mathcal{D}_b^{(n,out)}\left[H_3(t_1),\zeta(t_1)\right]\right]_k\zeta_k(t)\rangle\right\}\ ,
\end{aligned}\tag{4.35}$$

where

$$\left(\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\delta\dot\zeta_b(t_1)}\right) \tag{4.36}$$

represents the removal of $\dot\zeta(t_1)$ in position b from the quadratic term $\left(\delta\mathcal{L}_3^{(n)}(t_1)/\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\right)$. Finally $\mathcal{D}_b^{(n,out)}$ is the derivative operator acting on ζ_b outstripped of the time derivative. For example if $\mathcal{D}_b\zeta_b=\partial\dot\zeta_b$, then $\mathcal{D}_b^{(out)}=\partial$. The last contact terms are due to the fact that $\dot\zeta$ is not the momentum conjugate to ζ. The simplest way to obtain its time evolution is using $\partial_t(U_{int}^\dagger(t)\zeta_I(t)U_{int}(t))$. When the time derivative acts on the U_{int}s it results in contact terms. We have also symmetrized its expression because $\mathcal{L}_3$ is hermitian.

Straightforward manipulations lead to

$$2\text{Re}\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k U^\dagger_{int}(t_1,-\infty)\zeta_{I,k}(t)U_{int}(t_1,-\infty)\rangle = \tag{4.37}$$

$$2\text{Re}\left\{\langle\left[i\int_{-\infty}^{t_1}dt_2\ H_3(t_2),\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\zeta_k(t)\rangle\right. \tag{4.38}$$

$$+\sum_{m,b}\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\int_{-\infty}^{t_1}dt_2\ \mathcal{D}_b^{(m)}G_{\zeta_k}(t,t_2)\left[\frac{\delta\mathcal{L}_3^{(m)}(t_2)}{\delta\tilde{\mathcal{D}}_b^{(m)}\zeta_b(t_2)}\right]_k\rangle \tag{4.39}$$

$$-\frac{1}{2}\sum_b\langle\left[\mathcal{D}_b^{(n,out)}\left(\frac{\delta\tilde{\mathcal{L}}_3(t_1)}{\delta P(t_1)}\right)\left(\frac{1}{a(t_1)^{3+\delta}}\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\delta\dot{\zeta}_b(t_1)}\right)\right. \tag{4.40}$$

$$\left.\left.+\left(\frac{1}{a(t_1)^{3+\delta}}\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\delta\dot{\zeta}_b(t_1)}\right)\mathcal{D}_b^{(n,out)}\left(\frac{\delta\tilde{\mathcal{L}}_3(t_1)}{\delta P(t_1)}\right)\right]_k\zeta_k(t)\rangle\right\}\ ,$$

where

$$G_\zeta(t,t_1)=i\theta(t-t_1)[\zeta(t),\zeta(t_1)] \tag{4.41}$$

is the ζ Green's function from t_1 to t. The second term is obtained upon noticing that

$$[\int_{-\infty}^{t}dt_2\ H_3(t_2),\zeta_k(t)]=-i\int_{-\infty}^{t}dt_2\ \sum_{m,b}\mathcal{D}_b^{(m)}G_{\zeta_k}(t,t_2)\left[\frac{\delta\mathcal{L}_3^{(m)}(t_2)}{\delta\mathcal{D}_b^{(m)}\zeta_b(t_2)}\right]_k\ , \tag{4.42}$$

and the third term through the following

$$[H_3(t_1),\zeta(t_1)]=-i\frac{\delta\tilde{H}_3}{\delta P}=i\frac{\delta\tilde{\mathcal{L}}_3}{\delta P}\ , \tag{4.43}$$

where P is the momentum conjugate to ζ in the interaction picture: $P=\delta\mathcal{L}_2/\delta\dot{\zeta}$, and we introduced $\tilde{H}_3$ because any additional (spatial) derivatives acting on P have been integrated by parts and now act on H_3. Let us label the term in line (4.38) by $\mathcal{I}_1^{(n)}$, the one in line (4.39) by $\mathcal{I}_2^{(n)}$ and the one in line (4.40) by $\mathcal{I}_3^{(n)}$:

$$\begin{aligned}\mathcal{I}_1^{(n,a)}(t_1) &= 2\text{Re}\left\{\langle\left[i\int_{-\infty}^{t_1}dt_2\ H_3(t_2),\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\zeta_k(t)\rangle\right\}\ , \\ \mathcal{I}_2^{(n,a)}(t_1) &= 2\sum_{m,b}\text{Re}\left\{\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\int_{-\infty}^{t_1}dt_2\ \mathcal{D}_b^{(m)}G_{\zeta_k}(t,t_2)\left[\frac{\delta\mathcal{L}_3^{(m)}(t_2)}{\delta\tilde{\mathcal{D}}_b^{(m)}\zeta_b(t_2)}\right]_k\rangle\right\}\ , \\ \mathcal{I}_3^{(n,a)}(t_1) &= -\sum_b\text{Re}\left\{\langle\left[\mathcal{D}_b^{(n,out)}\left(\frac{\delta\tilde{\mathcal{L}}_3(t_1)}{\delta P(t_1)}\right)\left(\frac{1}{a(t_1)^{3+\delta}}\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\delta\dot{\zeta}_b(t_1)}\right)\right.\right. \\ &\quad \left.\left.+\left(\frac{1}{a(t_1)^{3+\delta}}\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)\delta\dot{\zeta}_b(t_1)}\right)\mathcal{D}_b^{(n,out)}\left(\frac{\delta\tilde{\mathcal{L}}_3(t_1)}{\delta P(t_1)}\right)\right]_k\zeta_k(t)\rangle\right\}\ ,\end{aligned} \tag{4.44}$$

so that

$$2\text{Re}\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\zeta_k(t)\rangle=\mathcal{I}_1^{(n,a)}(t_1)+\mathcal{I}_2^{(n,a)}(t_1)+\mathcal{I}_3^{(n,a)}(t_1)\ . \tag{4.45}$$

We are now going to see that the CIS diagrams reduce to the sum over a and n of the integral of the Green's function times $\mathcal{I}_1^{(n,a)}$, the CIM diagrams reduce to the integral of Green's function or of its derivatives times $\mathcal{I}_2^{(n,a)}$, and finally the quartic diagrams using the quartic vertices associated to the cubic Lagrangian reduce to the integral of Green's function times the sum over a and n of $\mathcal{I}_3^{(n,a)}$.

4.3.2 CIS_{1PI} diagrams

The CIS_{1PI} diagrams read

$$\begin{aligned} CIS_{1PI} \quad = \quad & -2\ \text{Re}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\langle[H_3(t_2),[\ H_3(t_1),\zeta_k(t)]]\,\zeta_k(t)\rangle= \\ & 2\ \text{Re}\sum_{n,a}i\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\langle\left[H_3(t_2),\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)\right]_k\zeta_k(t)\rangle= \\ & 2\ \text{Re}\sum_{n,a}\int_{-\infty}^{t}dt_1\mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)\langle\left[i\int_{-\infty}^{t_1}dt_2H_3(t_2),\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\zeta_k(t)\rangle \\ & =\sum_{n,a}\int_{-\infty}^{t}dt_1\,a(t_1)^{3+\delta}\,\mathcal{D}_a^{(n)}G_{\zeta_k}^{(n)}(t,t_1)\,\mathcal{I}_1^{(n,a)}(t_1)\ . \end{aligned} \tag{4.46}$$

So the CIS diagrams are the integral of the Green's function times the sum over a and n of $\mathcal{I}_1^{(n,a)}$.

4.3.3 CIM diagrams

The CIM diagrams read

$$\begin{aligned} CIM \quad = \quad & -\langle\left[\int_{-\infty}^{t}dt_1\ H_3(t_1),\zeta_k(t)\right]\left[\int_{-\infty}^{t}dt_2\ H_3(t_2),\zeta_k(t)\right]\rangle \\ & =\sum_{n,m,a,b}\int_{-\infty}^{t}dt_1\ \mathcal{D}_a^{(n)}G_{\zeta_k}^{(n)}(t,t_1)\langle\left[\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\int_{-\infty}^{t}dt_2\mathcal{D}_b^{(m)}G_{\zeta_k}(t,t_2)\left[\frac{\delta\mathcal{L}_3^{(m)}(t_2)}{\delta\mathcal{D}_b^{(m)}\zeta_b(t_2)}\right]_k\rangle \\ & =2\sum_{n,m,a,b}\int_{-\infty}^{t}dt_1\ \mathcal{D}_a^{(n)}G_{\zeta_k}^{(n)}(t,t_1)\langle\left[\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\int_{-\infty}^{t_1}dt_2\mathcal{D}_b^{(m)}G_{\zeta_k}(t,t_2)\left[\frac{\delta\mathcal{L}_3^{(m)}(t_2)}{\delta\mathcal{D}_b^{(m)}\zeta_b(t_2)}\right]_k\rangle \\ & =\sum_{n,a}\int_{-\infty}^{t}dt_1\,a(t_1)^{3+\delta}\ \mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)\,\mathcal{I}_2^{(n,a)}(t_1) \end{aligned} \tag{4.47}$$

so the CIM diagrams are the integral of the Green's function times the sum over a and n of $\mathcal{I}_2^{(n,a)}$.

4.3.4 Quartic Diagrams from Cubic Lagrangian

The fact that the cubic Lagrangian depends on $\dot{\zeta}$ means that the interaction picture quartic Hamiltonian receives a contribution that we call $H_{4,3^2}$, equal to

$$H_{4,3^2} = \frac{1}{2}\frac{\delta P}{\delta\dot{\zeta}}\left(\frac{\delta\tilde{\mathcal{L}}_3}{\delta P}\right)^2 = \frac{1}{2}\frac{\delta P}{\delta\dot{\zeta}}\sum_{b,n}\mathcal{D}_b^{(n,out)}\left(\frac{\delta\tilde{\mathcal{L}}_3}{\delta P}\right)\left(\frac{\delta\mathcal{L}_3^{(n)}}{\delta P_b}\right) \ , \tag{4.48}$$

where in the second term we have explicitly stressed the sum over b and we have integrated by parts any possible residual derivative (notice that the sign is re-absorbed in the definition of $\tilde{\mathcal{L}}_3$). The resulting quartic diagram is

$$\begin{aligned}
&Quartic_{3,\partial_t} = \\
&2\,\mathrm{Re}\left\{\langle\left[i\int_{-\infty}^{t}dt_1 H_{4,3^2}(t_1),\zeta(t)\right]_k\zeta_k(t)\rangle\right\} = \mathrm{Re}\left\{\langle\left[i\int_{\infty}^{t}dt_1\frac{\delta P}{\delta\dot{\zeta}}\left(\frac{\delta\tilde{\mathcal{L}}_3}{\delta P}\right)^2,\zeta(t)\right]_k\zeta_k(t)\rangle\right\} \\
&= -\mathrm{Re}\sum_{n,a}\left\{\int_{-\infty}^{t}dt_1\ \mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)\ \times\right. \\
&\left.\langle\left[\left(\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a\delta\dot{\zeta}_b}\right)\mathcal{D}_b^{(n,out)}\left(\frac{\delta\tilde{\mathcal{L}}_3(t_1)}{\delta P}\right)+\mathcal{D}_b^{(n,out)}\left(\frac{\delta\tilde{\mathcal{L}}_3(t_1)}{\delta P}\right)\left(\frac{\delta^2\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a\delta\dot{\zeta}_b}\right)\right]_k\zeta_k(t)\rangle\right\} \\
&= \sum_{n,a}\int_{-\infty}^{t}dt_1\,a(t_1)^{3+\delta}\ \mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)\ \mathcal{I}_3^{(n,a)}(t_1)\ .
\end{aligned} \tag{4.49}$$

So the $Quartic_{3,\partial_t}$ diagrams are the integral of the Green's function times the sum over a and n of $\mathcal{I}_3^{(n,a)}$. By summing the final expressions from the CIS_{1PI}, CIM and $Quartic_{3,\partial_t}$, we obtain the remarkably simple formula in eq. (4.33), as we wanted to show.

4.4 Time-(in)dependence of ζ from cubic diagrams

We can now ask ourselves if the contribution from the diagrams considered in the former section can lead to a time dependence on the ζ_k correlation function after the comoving mode k has crossed the horizon.

4.4.1 Quartic$_{\partial_i}$ diagrams

To understand wether the diagrams considered so far can lead to a time dependence, it will turn out to be useful to first add the quartic diagrams that are associated to the rescaling of the spatial derivatives in the cubic vertices. We call these Quartic$_{\partial_i}$. They take the form

$$Quartic_{\partial_i} = -\sum_n \partial_i\partial_t^n\zeta \int d\zeta \frac{\partial\mathcal{L}_3}{\partial(\partial_i\partial_t^n\zeta)} \ , \tag{4.50}$$

The symbol $\int d\zeta$ represents the fact that we multiply $\partial\mathcal{L}_3/\partial(\partial_i\partial_t^n\zeta)$ by ζ if there is no ζ without any derivative acting on it in $\partial\mathcal{L}_3/\partial(\partial_i\partial_t^n\zeta)$, we multiply by $\zeta/2$ if there is one ζ without any derivative acting on it [6]. The reason why we wish to include these quartic diagrams with the former is due to the fact that whenever an operator contains a spatial derivative, we expect that in the presence of long ζ mode the coordinates are effectively rescaled in a form $\partial_i \rightarrow e^{-\zeta}\partial_i$. As we will explain more in detail, the former interactions do not take into account of this rescaling, which is instead implemented by the quartic terms we are singling out. More formally, we can understand the presence of these terms in the following way. In the ADM parametrization

$$ds^2 = -N^2dt^2 + h_{ij}(dx^i + N^idt)(dx^j + N^jdt) \ , \tag{4.51}$$

ζ gauge and the ζ perturbation are defined by fixing the spatial diff.s by imposing the spatial metric to take the form

$$h_{ij} = a(t)^2 e^{2\zeta(\vec{x},t)}\delta_{ij} \ , \tag{4.52}$$

and the time diff.s are fixed by imposing the inflaton perturbations to be zero. This gauge choice leaves some zero-mode spatial diff.s unfixed. For example those that are associated to a time dependent rescaling and translation of the spatial coordinates:

$$x^i \quad \rightarrow \quad x^i = e^{\beta(t)}\tilde{x}^i + C^i(t) \ , \tag{4.53}$$

with $\beta(t)$, $C_i(t)$ generic functions of time. Under this rescaling, ζ and N^i transform as

$$\begin{aligned} \zeta \quad &\rightarrow \quad \tilde{\zeta} = \zeta + \beta(t) \ , \\ N^i \quad &\rightarrow \quad \tilde{N}^i = N^i e^{-\beta} + \dot{\beta}(t)\tilde{x}^i + e^{-\beta}\dot{C}^i(t). \end{aligned} \tag{4.54}$$

Thus the ζ zero mode has not been gauge fixed. For our purposes, we therefore learn that the ζ action must be diff. invariant under this restricted group of diff.s. Therefore, any combination of ∂_i must actually take the form $e^{-\zeta}\partial_i$ to be diff. invariant. By Taylor expanding this exponential, we clearly see that there is a

[6] The last remaining option, two ζ's in $\partial\mathcal{L}_3/\partial(\partial_i\partial_t^n\zeta)$ without any derivatives acting on them, is forbidden by rotational invariance.

connection between linear and quadratic terms, or from cubic and quartic terms.

To be even more explicit, let us give some examples. Given a vertex in the cubic Lagrangian, we identify the necessary vertex to be considered from the quartic Lagrangian in the following way

$$\begin{aligned}
\mathcal{L}_3 \quad &\supset \quad \zeta(\partial_i\zeta)^2 \qquad \rightarrow \qquad \mathcal{L}_4 \quad \supset \quad -\zeta^2(\partial_i\zeta)^2 \ , \\
\mathcal{L}_3 \quad &\supset \quad \dot\zeta(\partial_i\zeta)^2 \qquad \rightarrow \qquad \mathcal{L}_4 \quad \supset \quad -2\zeta\dot\zeta(\partial_i\zeta)^2 \ .
\end{aligned} \tag{4.55}$$

4.4.2 Time independence and the consistency condition

It is useful to split formula (4.33) into the sum of two terms. Let us introduce a time $t_{k_{out}}$ quite after the mode k has crossed the horizon $k/a(t_{k_{out}}) = \epsilon_{out} H(t_{k_{out}})$, with $\epsilon_{out} \ll 1$. Eq. (4.33) can be written as

$$\begin{aligned}
\langle\zeta_k\zeta_k\rangle_{CIS_{1PI}+CIM+Quartic_{3,\partial_t}} = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{t_{k_{out}}} dt_1 + \int_{t_{k_{out}}}^{t} dt_1\right) \\
\left[a(t_1)^{3+\delta}\sum_{a,n}\mathcal{D}_a^{(n)} G_{\zeta_k}(t,t_1) 2\mathrm{Re}\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k U^\dagger_{int}(t_1,-\infty)\zeta_{k,I}(t)U_{int}(t_1,-\infty)\rangle\, e^{\epsilon k t_1}\right] \ .
\end{aligned} \tag{4.56}$$

The contribution from the first term represents the case where the three-point function is evaluated at a time before the time $t_{k_{out}}$, while the second integral represents the contribution from evaluating the contribution of the three-point function from time $t_{k_{out}}$ up to the present time t.

Clearly, the first term is time-independent. The only dependence on t appears in the last term $\zeta_I(t)$, the interaction picture field that is constant at $t \gg t_{k_{out}}$. Let us therefore concentrate on the second term. Since we are considering times when the mode k is very outside of the horizon, we can expand the Green's function at late times $k/a(t_1) \ll H$. In conformal time, we have

$$G_{\zeta_k}(\eta,\eta_1) \simeq \frac{H^2}{3}\left(\eta^3 - \eta_1^3\right)\theta(\eta-\eta_1) \ , \tag{4.57}$$

obtaining

$$\begin{aligned}
\langle\zeta_k\zeta_k\rangle_{CIS_{int}+CIM+Quartic_{3,\partial_t},t} \simeq \lim_{\epsilon\to 0}\int_{\eta_{k_{out}}}^{\eta} d\eta_1 \left(-\frac{1}{H\eta_1}\right)^{4+\delta}\sum_{a,n}\mathcal{D}_a^{(n,out)}\frac{H^2}{3}\left(\eta^3-\eta_1^3\right)\theta(\eta-\eta_1) \\
2\,\mathrm{Re}\langle\left[\frac{1}{a(\eta_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(\eta_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(\eta_1)}\right]_k U^\dagger(\eta_1,-\infty)\zeta_{k,I}(\eta)U(\eta_1,-\infty)\rangle\, e^{\epsilon\, k\log(a(\eta_1))/H} \ ,
\end{aligned} \tag{4.58}$$

where the subscript $_t$ in $\langle\zeta_k\zeta_k\rangle_{CIS_{1PI}+CIM+Quartic_{3,\partial_t},t}$ refers to the fact that we are concentrating only on the time dependent part, and where the appearance of $\mathcal{D}_a^{(n,out)}$ is due to the fact that the commutators of $[\zeta_k, \dot{\zeta}_k]$ and $[\zeta_k, \zeta_k]$ scale in the same way at late times. Neglecting any possible time dependence from the terms in the second line, we see that naively the time integral diverges as

$$\int^{\eta} d\eta_1 \frac{1}{\eta_1^{1+\delta}} \sim \frac{1}{\eta_1^{\delta}} \quad \to \quad \log(-\eta) \sim Ht \quad \text{as} \quad \delta \to 0 \ . \tag{4.59}$$

We see the potential risk of linear infrared divergencies in cosmic time t (logarithmic in conformal time η) in the case the three-point function's contribution, that we have neglected in this formula, does not decay in time. Contributions from terms with $\mathcal{D}_a^{(n,out)}$ being non-unity are clearly more convergent by powers of η_1.

Let us therefore concentrate on the three point function, which can be schematically written as a convolution:

$$\sim \int d^{3+\delta}q \ \langle \mathcal{D}_1 \zeta_{\vec{q}}(t_1) \mathcal{D}_2 \zeta_{\vec{k}-\vec{q}}(t_1) \ U(t_1, -\infty)^{\dagger} \zeta_{k,I}(t) U(t_1, -\infty) \rangle \tag{4.60}$$

where $\mathcal{D}_{1,2}$ represent generic differential operators (including the identity operator) that could be present. The integral in q runs from very small wavenumbers (much smaller than k) up to infinity because we are working in dimensional regularization.

The contribution from momenta smaller than k/ϵ_{out} cannot give a time dependence. This is so because as these modes are longer than k/ϵ_{out}, the three-point function is evaluated when all the Fourier modes are very outside of the horizon. A remarkable property of the cubic interaction Lagrangian of ζ, which can be traced back to the original diff. invariance of the Lagrangian, is the fact that it can be written in a form where there are *no* operators with either no derivative or just a time derivative [22]. This means that if we decide to consider the contribution from terms where $\mathcal{D}_a^{(n,out)}$ is absent, so that they are potentially IR divergent, we are forced to consider an operator $\left[\delta\mathcal{L}_3^{(n)}(\eta_1)/\delta(\mathcal{D}_a^{(n)}\zeta_a(\eta_1))\right]_k \sim \zeta_{\vec{q}}(t_1)\zeta_{\vec{k}-\vec{q}}(t_1)$ with at least a derivative acting on one of the two operators. This therefore leads to a time-convergent integral [7].

We are finally lead to consider the remaining part of the integral where we include modes $q \gtrsim k/\epsilon_{out}$. These modes are at horizon crossing or well inside the horizon when the three-point function is evaluated, and so, contrary to what happens in the former regime $q \lesssim k/\epsilon_{out}$, there is no suppression for derivatives acting on these modes. However, in this regime we can use a remarkable property of the three-point function in the regime $k \ll q$, the so called 'consistency condition' of the three-point function, which states that at

[7] Similar conclusion applies also to the case where we consider the operator

$$\frac{\partial_i}{\partial^2}\dot{\zeta}\partial_i\zeta\dot{\zeta} \ . \tag{4.61}$$

as even if we remove first non-local term by inserting it in the Green's function, we are left with $\partial_i\zeta\dot{\zeta}$ that has enough derivatives to compensate for the non local term.

leading order in $k/q \ll 1$, $k/(aH) \ll 1$, the three-point function has the following form

$$\langle \left[\frac{1}{a(\eta_1)^{3+\delta}} \frac{\delta \mathcal{L}_3^{(n)}(\eta_1)}{\delta \mathcal{D}_a^{(n)} \zeta_a(\eta_1)} \right]_{k,\ (q \gg k)} \zeta_{k,I}(\eta_1) \rangle \simeq \tag{4.62}$$

$$\simeq \frac{1}{q^{3+\delta}} \frac{\partial \langle \left[q^{3+\delta} \frac{1}{a(\eta_1)^{3+\delta}} \frac{\delta \mathcal{L}_3^{(n)}(\eta_1)}{\delta \mathcal{D}_a^{(n)} \zeta_a(\eta_1)} \right]_q \rangle}{\partial \log q} \langle \zeta_k(\eta_1)^2 \rangle + \mathcal{O}\left(\mathrm{Max}\left[\left(\frac{k}{a(\eta_1)H(\eta_1)} \right)^2, \left(\frac{k}{q} \right)^2 \right] \right) .$$

The last term represents the subleading correction to the squeezed limit. Let us understand the Max $\left[\frac{k}{a(\eta_1)H(\eta_1)}, \frac{k}{q} \right]$ term. If we expand in gradients in the long wavelength fluctuation, the natural quantity to consider is clearly the physical wavenumber $k/(aH)$. So, this is the natural size of the correction in the squeezed limit. The calculation of the three-point function in this limit involves a time integral in a variable that we can call η_2. Subleading corrections in the squeezed limit are contained in the integrand are proportional to $k/(a(\eta_2)H(\eta_2))$. If the short modes q are longer than the horizon at the time η_1, then the time integral is peaked at the time η_2 when the modes q crossed the horizon. This gives $q/(a(\eta_2)H(\eta_2)) \sim 1$, which gives a correction of the form k/q. If the modes q are instead still inside the horizon at η_1, the integral is peaked at $\eta_2 \sim \eta_1$, giving a correction of the form $k/(a(\eta_1)H(\eta_1))$.

There are two subtleties to discuss about the above formula (4.62). The first regards the case in which the operator $\left[\delta \mathcal{L}_3^{(n)}(\eta_1)/\delta(\mathcal{D}_a^{(n)} \zeta_a(\eta_1)) \right]_k$ contains spatial derivatives of the short modes, for example if it is of the form $(\partial_i \zeta)^2$. In this case the consistency condition does not hold. The consistency condition implies that in the squeezed limit the 3-point function follows directly from the fact that in this limit the long mode acts as a rescaling of the spatial coordinates $\vec{x} \to e^{-\zeta} \vec{x}$. However, when we compute the 3-point function with the usual formulas $\sim [\int dt\, H_{int}, \zeta^3]$, we are evolving in the interaction picture the operators ζ, not the spatial coordinates themselves. This means that evaluation of the 3-point function amounts to effectively rescaling the *argument* of the operators $\zeta(\vec{x}) \to \zeta(e^{-\zeta}\vec{x})$. The computation does not implement the rescaling of the spatial derivatives, simply because they 'go along with the ride', unaffected by the interacting Hamiltonian. Formula (4.62) does not hold. Although this seems to challenge the very intuitive result that a long wavelength ζ acts as a rescaling of the coordinates, diff. invariance provides a solution. The quartic vertex Quartic$_{\partial_i}$in eq. (4.50) provides precisely the contact term necessary to rescale the coordinates in the spatial derivative. So, eq. (4.62) holds after we add to all the diagrams considered so far also the Quartic$_{\partial_i}$. In App. 4.8, we discuss examples of three-point functions in the squeezed limit in which one of the modes has much longer wavelength than the others, involving short modes that are still inside the horizon and that are acted upon by space and time derivatives. There we show that the consistency condition holds after the addition of the relevant contact operators.

The second subtlety in using (4.62) is that in the three-point function we are computing the last term should be

$$U_{int}(\eta_1,-\infty)^\dagger \zeta_{k,I}(\eta) U_{int}(\eta_1,-\infty)\ ,$$

which is different from $\zeta_{k,I}(\eta_1)$. This is equivalent to the situation where we were to arbitrarily shut down H_{int} at t_1 and the theory become free after that. Even though this is not the case in the actual physical system, it can be straightforwardly realized that this difference does not matter, because at the time t_1 the k-mode is already well outside the horizon. We therefore are free to use (4.62) at leading order in $k/(aH)$.

By substituting (4.62) into (4.58), we obtain:

$$\begin{aligned}\langle\zeta_k\zeta_k\rangle_{CIS_{1PI}+CIM+Quartic_{3,\partial_t}+Quartic_{\partial_i},t} &\simeq \lim_{\epsilon\to 0}\int_{\eta_{k_{out}}}^{\eta} d\eta_1\ \left(-\frac{1}{H\eta_1}\right)^{4+\delta} \quad (4.63)\\ \sum_{a,n}\mathcal{D}_a^{(n,out)}\frac{H^2}{3}\left(\eta^3-\eta_1^3\right)\theta(\eta-\eta_1)2\,\mathrm{Re}\int_{k/\epsilon_{out}}^{+\infty} d^{3+\delta}q\ \frac{1}{q^{3+\delta}} &\frac{\partial\langle\left[q^{3+\delta}\frac{1}{a(\eta_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(\eta_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(\eta_1)}\right]_q\rangle}{\partial\log q}\langle\zeta_k(t)^2\rangle\ .\end{aligned}$$

The rotational integral is trivially performed, and the remaining momentum q-integral is a total derivative. This leads to

$$\begin{aligned}\langle\zeta_k\zeta_k\rangle_{CIS+CIM+Quartic_{3,\partial_t}+Quartic_{\partial_i},t} \quad\simeq\quad & \lim_{\epsilon\to 0}\int_{\eta_{k_{out}}}^{\eta} d\eta_1\ \left(-\frac{1}{H\eta_1}\right)^{4+\delta}\sum_{a,n}\mathcal{D}_a^{(n,out)}\frac{H^2}{3}\left(\eta^3-\eta_1^3\right)\theta(\eta-\eta_1)\\ & 8\pi\ \langle\left[\frac{1}{a(\eta_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(\eta_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(\eta_1)}\right]_q\rangle\Bigg|_{q=k/\epsilon_{out}}\langle\zeta_k(t)^2\rangle\ , \qquad (4.64)\end{aligned}$$

where the contribution from $q=\infty$ is zero as the integral is made convergent in dim-reg. As we evaluate the term

$$\langle\left[q^{3+\delta}\frac{1}{a(\eta_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(\eta_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(\eta_1)}\right]_q\rangle\Bigg|_{q=k/\epsilon_{out}} \qquad (4.65)$$

and we take the limit $\eta_1\to 0$ as $\eta\to 0$, we notice the property of the cubic ζ-Lagrangian that we mentioned before: there is *no* operator $\left[\delta\mathcal{L}_3^{(n)}(\eta_1)/\delta(\mathcal{D}_a^{(n)}\zeta_a(\eta_1))\right]_k$ that does not vanish as some power of $k/(a(\eta_1)H(\eta_1))\sim k\eta_1\to 0$. This is so because in order for this term to have any chance to contribute at late times we had to restrict ourselves to choosing an operator that had at least one derivative acting on one of the two ζ operators. Since this terms is evaluated when momenta are outside the horizon, it vanishes as $\eta_1\to 0$. This means that the resulting time integral is convergent.

We stress that there is no time dependence because, as a result of the consistency condition, the integrand in the internal momenta q becomes a total derivative. If this had not been the case, it would have been less trivial to show that the result of the integration leads to a time independent answer [8].

This result can probably be stated more intuitively by simply noticing that the consistency condition implies that in the extreme squeezed limit $k \ll q$ the effect of the long mode on the dynamics is to do *nothing*: its effect is simply a trivial rescaling of the comoving momenta. Since we compute the integrals over the whole high momentum modes, this rescaling has no effect apart for changing the boundary of integration for the most infrared modes of order k/ϵ_{out}. But the integral has no support in that region. This is a simple explanation of the reason why the loop integral becomes a total derivative in the squeezed limit.

This is enough to make the subleading corrections time convergent. We have at this point gone through the whole phase space in $CIM + CIS + Quartic_{3,\partial_t} + Quartic_{\partial_i}$ diagrams, finding that their sum leads to no time dependence.

A note on the counterterms: It is important to realize that (4.64) is the result of the full loops integrals in the squeezed limit $k \ll q,\ k/(aH) \ll 1$. The integral is therefore UV finite, even in the limit in which we send the number of spatial dimensions to three, or the regulator to infinity. This is a very important consistency check. If the integral in this regime were to be UV divergent we would have had a divergent time dependence piece and we would have needed a counterterm that cancelled the divergent time-dependence of ζ. But there are no counterterms in the action that induce a time-dependence for ζ because that is equivalent to inducing at quadratic level a mass for ζ which does not happen for the terms allowed by the symmetries. As we will see in the next section, the only quadratic counterterms that induce a mass for ζ are the ones associated to the tadpole terms, that induce *also* a linear tadpole for ζ. We will verify they will exactly cancel the time-dependence from the diagrams built with the quartic vertices.

4.4.3 Example

It is instructive to find a simple example where this can be seen explicitly. Thanks to the Effective Field Theory of Inflation [30, 42], it is possible to find a consistent inflationary Lagrangian which has the properties

[8] Let us comment on the contribution of the subleading corrections in eq. (4.62), which do not take the form of a total derivative. Those contributions are not scale invariant in the external wavenumber k, having one additional factor of k in the numerator with respect to the leading, scale invariant contribution. This means that the resulting contribution goes to zero at late times as $(k\eta_1)^2$ and so they lead to a time-convergent contribution as $\eta \to 0$. The fact that the contribution to the subleading corrections in eq. (4.62) is not scale invariant comes from the following. If we consider the contribution from any fixed momentum shell in q between $q \sim k/\epsilon_{out}$ to $q \sim \gamma k$ with $\gamma \gg 1/\epsilon_{out}$, with γ a time independent number small enough so that q is outside the horizon, the contribution from that shell of momenta goes to zero as some power of $k\eta_1$. This is so because the operator $\left[\delta\mathcal{L}_3^{(n)}(\eta_1)/\delta(\mathcal{D}_a^{(n)}\zeta_a(\eta_1))\right]_k$ contains some derivatives of the fields. This means that the contributions in the integrand coming from momenta outside of the horizon is peaked at those momenta at horizon crossing $q \sim aH$, which meanS that the subleading corrections are of the form $k/q \sim k/(aH)$ and so the integrand goes to zero as $\eta_1 \to 0$. Finally, the contribution from momenta q that are inside the horizon is explicitly down by powers of $k/(aH)$ and so they are as well convergent.

we discussed [9]. By parametrizing the fluctuations in terms of the Goldstone boson π and going to the decoupling limit, the algebra becomes very simple. Let us take for example the following Lagrangian in the decoupling limit

$$S=\int d^4x\,\sqrt{-g}\,\left[-\dot{H}M_{\rm Pl}^2\left(\dot{\pi}^2-\frac{1}{a^2}(\partial_i\pi)\right)+M^4(t+\pi)\left(\dot{\pi}^2+\ldots\right)\right] \tag{4.66}$$

where ... represent cubic or quartic terms in π that have one derivative acting on each fluctuation. Those terms do not lead to any diagram with an explicit time dependence, and we neglect them here. For illustrative purposes, let us suppose now that the function $M^4(t)$ depends linearly on time. By Taylor expanding in π, we notice that we have the cubic interaction

$$\mathcal{L}_3=\partial_t\left(M^4(t)\right)\pi\dot{\pi}^2\ . \tag{4.67}$$

This interaction in very dangerous. If we imagine forming a loop with two of these vertices and using a π in the first vertex to contract with the external leg, the resulting diagram will become time-dependent. This means that time-independence can come only from a quartic interaction. Indeed, this is exactly the kind of cubic Lagrangians that leads to a non-trivial $\mathcal{H}_{4,3}$.

Bu concentrating only on the effects proportional to $(\partial_t M^4)^2$, the action can be recast as

$$\mathcal{S}=\int d^4x\,\sqrt{-g}\,\left[\frac{-\dot{H}M_{\rm Pl}^2}{c_s^2}\left(\dot{\pi}^2-\frac{c_s^2}{a^2}(\partial_i\pi)^2\right)+\left(\partial_t\left(M^4(t)\right)\pi\dot{\pi}^2\right)\right]\ . \tag{4.68}$$

The speed of sound is $c_s^2=-\dot{H}M_{\rm Pl}^2/(-\dot{H}M_{\rm Pl}^2+M^4(t))$. The momentum conjugate to π, P, is given by

$$P=\frac{\delta\mathcal{L}}{\delta\dot{\pi}}=2a^3\left(\frac{-\dot{H}M_{\rm Pl}^2}{c_s^2}+\partial_t\left(M^4(t)\right)\pi\right)\dot{\pi}\ , \tag{4.69}$$

and the Hamiltonian is therefore

$$\mathcal{H}=P\,\dot{\pi}(\pi,P)-\mathcal{L}(\pi,\dot{\pi}(\pi,P))=\frac{P^2}{4a^3\left(\frac{-\dot{H}M_{\rm Pl}^2}{c_s^2}+\partial_t\left(M^4(t)\right)\pi\right)}+a^3\left(-\dot{H}M_{\rm Pl}^2\right)\frac{1}{a^2}(\partial_i\pi)^2\ . \tag{4.70}$$

We can identify the quartic Hamiltonian of order $(\partial_t M^4)^2$ to be

$$\mathcal{H}_{4,3}=\frac{\left[\partial_t\left(M^4(t)\right)\right]^2}{4a^3\left(\frac{-\dot{H}M_{\rm Pl}^2}{c_s^2}\right)^3}P^2\pi^2=a^3\,\frac{\left[\partial_t\left(M^4(t)\right)\right]^2}{\frac{-\dot{H}M_{\rm Pl}^2}{c_s^2}}\dot{\pi}_I^2\pi_I^2\ . \tag{4.71}$$

[9] The Effective Field Theory of Inflation is a quite powerful new formalism to describe the theory of inflation in very general terms. A sample of recent works that have been developing it is given by [30, 42, 31, 91, 107]

where in the second passage we have written the expression in terms of the interaction picture fields. It can be easily checked that this agrees with (4.48). Quartic diagrams built with $\mathcal{H}_{4,3}$ lead also to time dependence, a time-dependence that indeed cancels the one from the cubic diagrams built with $(\partial_t(M^4(t))\pi\dot{\pi}^2$. This example is discussed in detail in appendix 4.8.2.

4.5 Time-(in)dependence of ζ from quartic diagrams

In order to complete the study of the possible infrared effects we need to look at the contribution from the remaining quartic interactions $H_4 \supset H_{4,4} = -\mathcal{L}_4 - \mathcal{L}_{4,\partial_i}$, where $\mathcal{L}_{4,\partial_i}$ represents the terms that were borrowed in the former section to give the $Quartic_{\partial_i}$ diagrams. These remaining diagrams contribute to the two point function as

$$\langle\zeta_k\zeta_k\rangle_{Quartic_4} = \tag{4.72}$$
$$-\lim_{\epsilon\to 0}\int_{-\infty}^{t} dt_1\ a(t)^{3+\delta}\sum_{a,n}\mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)2\mathrm{Re}\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_4^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]_k\zeta_k(t_1)\rangle\ e^{\epsilon k t_1}\ .$$

Like in the former section, it is straightforward to see that the factor before the four-point function on the left of the above formula leads to a time dependence proportional to Ht if the four-point function does not have a suppression at late time. Contrary to what happened in the former section with the three-point function after it was integrated over comoving monenta, there is no such a cancellation from diagrams within H_4. So there is a subset of diagrams that naively lead to a time-dependence. We are now going to show that there is a cancellation that leads to absence of a time dependence of ζ at late times after adding a new set of diagrams. These new diagrams come from effectively quartic vertices that arise when we insert the couterterms for the tadpoles.

Let us see this in detail. At one loop order the first diagrams we should consider are the tadpole diagrams, that can be written as

$$\langle\zeta_k\rangle_{Tad} = \lim_{\epsilon\to 0}\int_{-\infty}^{t} dt_1\ a(t)^{3+\delta}\sum_{a,n}\mathcal{D}_a^{(n)}G_{\zeta_k}(t,t_1)\langle\left[\frac{1}{a(t_1)^{3+\delta}}\frac{\delta\mathcal{L}_3^{(n)}(t_1)}{\delta\mathcal{D}_a^{(n)}\zeta_a(t_1)}\right]\rangle\ e^{\epsilon k t_1}\ . \tag{4.73}$$

Very simple counting arguments shows that these diagrams can lead to a time dependence of the zero mode $\zeta_{k=0}$. If these diagrams are not zero it is because we are expanding around the wrong unperturbed history. Indeed, by translation invariance, only the $k=0$ mode is directly affected, and the zero mode can be totally reabsorbed in the definition of the unperturbed history. However this does not mean that these diagrams affect only the zero mode: they can be attached with a cubic vertex to a propagator to affect the two point function of modes at finite k in a non-1PI diagram (Fig. 4.3), and possibly induce a time dependence even there. The fact that this diagrams is not zero is clearly a nuisance.

Fortunately, these diagrams can be set to zero by inserting proper counterterms. In order to cancel tadpole diagrams, they must start linear in the fluctuations. In principle, there are many possible operators of this form, but luckily we can use a theorem proved in the context of the Effective Field Theory of inflation [30, 42]. It states that all the possible tadpole counterterms can be reduced to just two operators [10]. In unitary gauge, these are

$$\mathcal{S}_{tad,counter} = \int d^4x\sqrt{-g}\ \left[g^{00}\delta M^4(t) + \delta\Lambda(t)\right]\ . \tag{4.74}$$

Up to one loop level, the terms starting linear in the fluctuations take the form

$$\mathcal{S}_{tad} = \int d^4x\sqrt{-g}\ \left[g^{00}\left(M_{\rm Pl}^2\dot{H} + \delta M^4\right) - M_{\rm Pl}^2\left(\left(3H^2 + \dot{H}\right) + \delta\Lambda\right)\right]\ , \tag{4.75}$$

The coefficients $\dot{H}M_{\rm Pl}^2$ and $-M_{\rm Pl}^2(3H^2 + \dot{H})$ are uniquely fixed by the background, as proven in [30, 42], while the terms δM^4 and $\delta\Lambda$ represent the one-loop counterterms that are chosen to cancel the tadpole diagrams. The most important point that we need to realize is that these operators that start linear in the fluctuations necessarily contain higher order terms. This is so because of the non-linear realization of time diffs. In particular this means that there will be quadratic terms that can contribute to the two-point function effectively as one-loop terms. In this section we are going to prove that they exactly cancel the quartic diagrams constructed with H_4 that would lead to a time dependence.

4.5.1 Example:

Since the algebra quickly becomes very complicated, we use the Effective Field Theory of Inflation [30, 42] to find a consistent inflationary model where this cancellation can be studied in the simplest context. Let us consider the following Lagrangian in unitary gauge

$$\mathcal{S} = \int d^4x\sqrt{-g}\ \left[g^{00}\left(M_{\rm Pl}^2\dot{H} + \delta M^4\right) - M_{\rm Pl}^2\left(\left(3H^2 + \dot{H}\right) + \delta\Lambda\right) + M_3^4\left(\delta g^{00}\right)^3\right] \tag{4.76}$$

and let us imagine that M_3^4 depends rapidly linearly in time. This means that we can concentrate on that interaction and study it in the decoupling limit. Upon reinserting the Goldstone boson π by performing a

[10] We stress that this is one of the advantages of using the Effective Field Theory of Inflation: by concentrating directly on the fluctuations, it allows immediately to identify the operators with the correct number of fluctuating fields to be tadpole counterterms.

time-diff $t \to t + \pi$, the Lagrangian reduces to

$$\mathcal{S} = \int d^4x\sqrt{-g}\ \Big[(-1-\dot\pi+(\partial\pi)^2)\left(M_{\rm Pl}^2\dot H+\delta M^4(t+\pi)\right)-M_{\rm Pl}^2\left(\left(3H^2+\dot H\right)+\delta\Lambda\right)+ \\ M_3^4(t+\pi)\dot\pi^3\Big]\ , \tag{4.77}$$

where we have stopped at quartic level and we have kept only the interactions proportional to $M_3(t)$. By Taylor expanding the last term, we have a vertex of the form $\dot{M_3^4}\pi\dot\pi^3$ which, if we contract $\dot\pi$ as the final leg in the Green's function, leads to a quartic diagram that naively induces a time-dependence. Let us see how it cancels with the operators induced by the tadpole counterterms. By the non-linear realization of time diffs., this same operator starts cubic, and it therefore induces a tadpole. All diagrams with only one vertex can be most simply studied directly in the Lagrangian by taking the expectation value of the quadratic operators contracted in the loop, and studying the resulting quadratic Lagrangian. This is equivalent to resuming all the non-1PI diagrams obtained by multiple insertion of the same loop. So we notice that the last term induces a tadpole term of the form

$$\delta\mathcal{S}_{3\to1} = \int d^4x\sqrt{-g}\left[3M_3^4(t)\delta g^{00}\langle(\delta g^{00})^2\rangle\right]\ . \tag{4.78}$$

This means that in order to cancel this diagram we have to choose δM^4 as

$$\delta M(t)^4 = -3M_3^4(t)\langle(\delta g^{00})^2\rangle\ . \tag{4.79}$$

This is shown diagrammatically in Fig. 4.5 where we call the variables directly ζ. The cancellation of the tadpole terms automatically guarantees the cancellation of the non-1PI diagrams, that otherwise should be included (see Fig. 4.6).

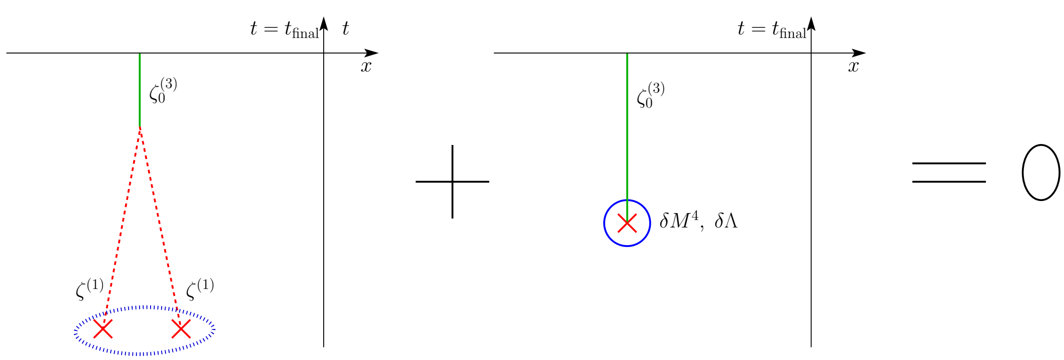

Figure 4.5: Cancellation between the tadpole diagram and the tadpole counterterm.

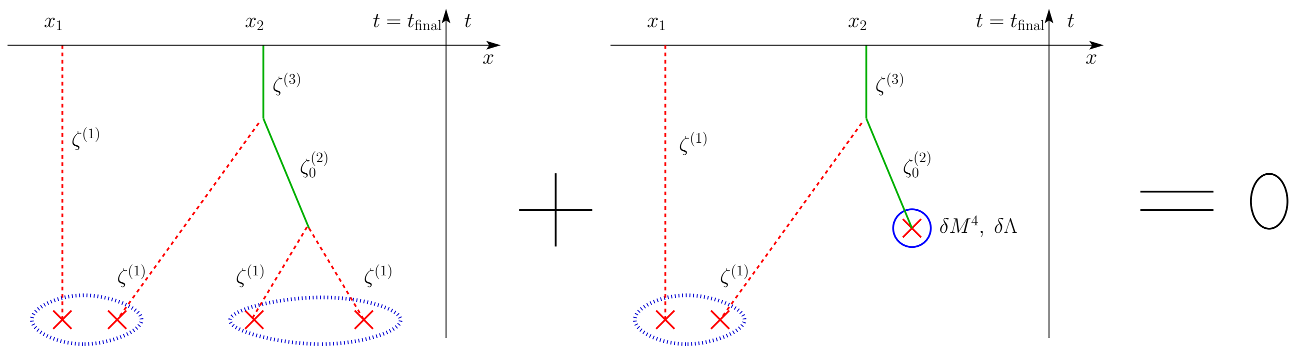

Figure 4.6: Cancellation of the $CIS_{non-1PI}$ diagrams with the $CIS_{non-1PI}$ diagrams constructed with the tadpole counterterms

In unitary gauge, the resulting tadpole operator in δg^{00} is of the form

$$\mathcal{S}_{Tad,counter} = \int d^4x\sqrt{-g}\ \left[-\delta g^{00} 3M_3^4(t)\langle(\delta g^{00})^2\rangle\right]\ , \tag{4.80}$$

But since this has the same form as the induced tadpole operator that we have from $\left(\delta g^{00}\right)^3$, then the resulting quadratic (and higher order) terms that we obtain by expanding $\sqrt{-g}M_3^4(t)$ will also cancel. This removes the contribution from the quartic operators that would induce a time dependence.

This can also be checked directly at the level of π. The dangerous term $\dot{M}_3^4\pi\dot{\pi}^3$ effectively gives a contribution that in the action can be represented as

$$\delta\mathcal{S}_{4\to 2} = \int d^4x\sqrt{-g}\ \left[3\dot{M}_3^4\pi\dot{\pi}\langle\dot{\pi}^2\rangle\right]\ , \tag{4.81}$$

which is exactly cancelled by the tadpole term at second order

$$\mathcal{S}^{(2)}_{Tad,counter} = \int d^4x\sqrt{-g}\ \left[-\dot{\pi}3M_3^4(t+\pi)\langle\dot{\pi}^2\rangle\right] \supset \int d^4x\sqrt{-g}\ \left[-3\dot{M}_3^4(t)\pi\dot{\pi}\langle\dot{\pi}^2\rangle\right]\ . \tag{4.82}$$

This is represented in Fig. 4.7. Other quadratic terms induced by this tadpole operator are of the form $\dot{\pi}^2$ and $(\partial_i\pi)^2$ and so do not induce time-dependent effects.

This cancellation can be intuitively summarized by noticing that the ζ action at tree-level cannot have any mass term once expressed around the correct background. This is so because ζ constant must be a solution of the equations of motion when the mode is outside the horizon. The counterterms for tadpole diagrams ensure that we are around the correct history, and so the quartic diagrams must cancel with the induced-quadratic diagrams from the tadpoles counterterms.

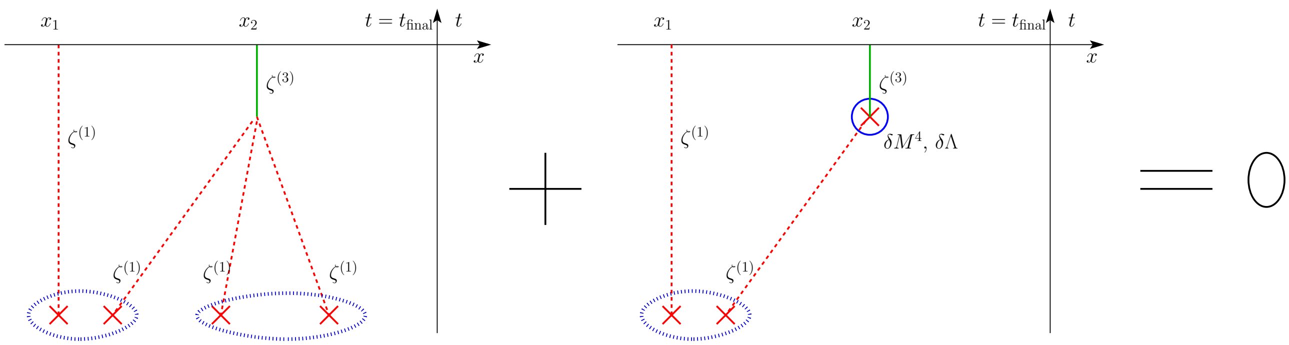

Figure 4.7: Cancellation of some quartic diagrams with the tree diagrams with an insertion of a counterterm-induced quadratic vertex.

4.6 Quartic diagrams: Verification for purely gravitational interactions

Let us now move on and consider the most generic example for H_4 where we take generic coefficients and we do not neglect interactions mediated by gravity. Because of the complexity of this kind of interactions, the discussion becomes quite complicated even though all the essential points have already been highlighted using the Effective Field Theory of Inflation in the former section. We will therefore perform the study in several steps.

The first step will be to study the induced time dependence on the ζ zero mode, ζ_0. As we discussed in eq. (4.53) and (4.54), the zero mode is not gauge fixed in the ordinary ζ gauge. We can fix the two functions in eq. (4.54) in the following way: first we impose periodic boundary conditions. We imagine that the system is in a very large periodic box of comoving size L. In this way we forbid any dependence proportional to x^i. This fixes $\beta(t)$. Second, we can fix $C^i(t)$ by imposing that the zero mode component of N^i vanishes: $N^i_{\vec{k}=0}(t) = 0$.

4.6.1 On the gauge choice for the zero mode

Before proceeding, it is very interesting to notice the following. At finite k, N^i is determined by being the solution of a constraint equation. At linear level, for example, the equation reads:

$$\partial_i N^i \sim \dot{\zeta} \tag{4.83}$$

which can be solved at finite k to give

$$N^i \sim \frac{k^i}{k^2}\dot{\zeta} \ . \tag{4.84}$$

In real space this term is often reported in a non-local fashion as $N^i \sim \frac{\partial_i}{\partial^2}\dot{\zeta}$. The zero momentum limit of that expression gives something that in real space reads as

$$N^i(t) \sim \dot{\zeta} x^i \ . \tag{4.85}$$

By using our freedom in choosing the function β, we decided to set this term to zero. Therefore our solution for N^i is *not* the $k \to 0$ limit of the solution for N^i at finite k. We choose to work in a gauge where the limit is discontinuous. Of course any gauge choice should be as good as any other one.

Working within the gauge where the limit $k \to 0$ of N^i is continuous, that we can call 'continuous gauge', raises several complications that we prefer to avoid. First of all, the continous gauge looks very unfamiliar when there is only a zero mode present. In this case the spacetime is described by an FRW metric but the gauge choice makes us use unusual coordinates where $g_{0i} \neq 0$. But the situation becomes even more complicated. For example if in the continuous gauge we naively Taylor expand the action at linear level, we find that there is a tadpole term for the zero mode. The action starts linear, proportional to

$$\begin{aligned} S = \frac{M_{\rm Pl}^2}{2} \int d^4x \sqrt{-g} \ \left[R + \dot{H}\delta g^{00} + 3H^2 + \dot{H} + \ldots \right] \quad \supset \\ \sim M_{\rm Pl}^2 \int d^4x \, a^3 \, H \partial_i N^i \sim M_{\rm Pl}^2 \int d^4x \ a^3 \ \frac{\dot{H}}{H} \dot{\zeta} \ . \end{aligned} \tag{4.86}$$

where $\ldots$ stands for terms that start explicitly quadratic in the fluctuations. This is of course a wrong result, as the action for the fluctuations should start at quadratic order if we expand around a solution to the classical equations of motion, as we are doing. The reason for the mistake is that in this case the action has a boundary term that does not decouple in the limit in which we send the boundary to infinity. This is due to the behavior of $N^i \propto x^i$. Indeed the boundary term is the Gibbons-Hawking-York one:

$$S_{GHY} = M_{\rm Pl}^2 \int_{\partial V^{(4)}} d^3\tilde{x} \ \sqrt{-h} \, K \ , \tag{4.87}$$

where h is the induced metric on the boundary described by coordinates $\tilde{x}$ and K the trace of the extrinsic curvature. It is easy to check that this boundary term cancels the tadpole for the zero mode that we obtain from the bulk action.

The situation is instead much simpler in the 'discontinuous gauge' where the limit $k \to 0$ of N_i is discontinuous. In this case, for a fixed comoving box, the boundary terms become irrelevant as we send the boundary to infinity, and indeed the bulk action starts quadratic in the fluctuations. Furthermore, zero mode fluctuations appear to be directly in a standard FRW slicing. We will therefore work with this discontinuous gauge.

4.6.2 Time-independence for the zero-mode

We are now going to prove that the zero-mode is time-independent at one-loop. In order to do this, we need to expand the action to quadratic order in the zero-mode and independently up to quadratic order in the non-zero-modes. We count them as independent parameters. Since we expand only up to second order in each of the parameters, we need to solve the constraint solutions in the zero and in the short modes only at linear level in each of those. We work in Fourier space directly, and write

$$\begin{aligned} N &= 1 + \delta N_k(t) + \delta N_0(t) \ , \\ N^i_k &= \partial_i \psi_k(t) \ . \end{aligned} \tag{4.88}$$

We start from the action

$$\begin{aligned} S = &\int d^3x\, dt\, \sqrt{h} \\ &\left\{ \frac{1}{2} M_{\rm Pl}^2 \left(\frac{E_{ij}E^{ij} - {E^i}_i{}^2}{N} + NR \right) - \frac{M_{\rm Pl}^2 \dot{H}}{N} - N M_{\rm Pl}^2 \left(3H^2 + \dot{H} \right) - N\delta\Lambda(t) - \frac{\delta M^4(t)}{N} \right\} \ , \end{aligned} \tag{4.89}$$

where the δM^4 and $\delta\Lambda$ terms represent the only two tadpole counterterms allowed by symmetries (all other possible choices are equivalent to those [30, 42]), and should be intended as objects that are of order ζ_k^2. The constraint equations read

$$\begin{aligned} &\frac{M_{\rm Pl}^2}{2} \left[R - \frac{1}{N^2} \left({E^i}_j {E^j}_i - {E^l}_l \right)^2 \right] + \frac{1}{N^2} \left(M_{\rm Pl}^2 \dot{H} + \delta M^4 \right) - \left[M_{\rm Pl}^2 \left(3H^2 + \dot{H} \right) + \delta\Lambda \right] = 0 \ , \\ &\hat{\nabla}_i \left[\frac{1}{N} \left({E^i}_j - {\delta^i}_j {E^l}_l \right) \right] = 0 \ , \end{aligned} \tag{4.90}$$

and are solved by

$$\begin{aligned} \delta N_0(t) &= \frac{3H}{\dot{H} + 3H^2} \dot{\zeta}_0 \ , \\ \delta N_k &= \frac{(1 + \delta N_0)}{H + \dot{\zeta}_0} \dot{\zeta}_k \ , \\ \psi_k &= \frac{e^{-2(\zeta_0 + \rho(t))}}{k^2 \left(H + \dot{\zeta}_0 \right)^2} \left(\dot{H} \dot{\zeta}_k e^{2(\zeta_0 + \rho(t))} - (H + \dot{\zeta}_0) k^2 \zeta_k(t) (1 + \delta N_0)^2 \right) \ . \end{aligned} \tag{4.91}$$

We plug back the above solutions into the action. At linear order the action is a total derivative, as it should be. At quadratic order, the zero-mode action reads

$$S_{\zeta_0^2} = \int d^3x\, dt\, e^{3\rho(t)} \left(-\frac{3M_{\rm Pl}^2 \dot{H}}{3H^2 + \dot{H}} \right) \dot{\zeta}_0(t)^2 \ , \tag{4.92}$$

where we are writing $a(t) = e^{\rho(t)}$. It is interesting to notice that the quadratic action for the zero-mode is not the $k \to 0$ limit of the finite k ζ action, the prefactor of $\dot{\zeta}_k^2$ being different. This is indeed

$$S_{\zeta_k^2} = \int d^3k\, dt\, e^{3\rho(t)} \left(-\frac{M_{\rm Pl}^2 \dot{H}}{H^2} \right) \left(\dot{\zeta}_{\vec{k}}(t)\dot{\zeta}_{-\vec{k}}(t) - e^{-2\rho(t)} k^2 \zeta_{-\vec{k}}\zeta_{\vec{k}} \right) \ . \tag{4.93}$$

Tadpole Counterterms' Coefficients

At this point we need to find the expressions for the tadpole counterterms $\delta\Lambda$ and δM^4 that ensure the cancellation of the tadpoles for ζ_0. This is done by finding the cubic action at order $\zeta_0\zeta_k^2$, taking the expectation value on the short modes and canceling the resulting tadpole coefficients [11]. Leaving out the simple algebra, the solution for the tadpole counterterms reads

$$\begin{aligned}
\delta M^4 \quad &= \quad \frac{M_{\rm Pl}^2 e^{-2\rho(t)}}{3H^4} \Big(H \Big(-2\langle \partial_i\zeta\partial_i\dot{\zeta}\rangle \dot{H} + H(\langle \partial_i\dot{\zeta}\partial_i\dot{\zeta}\rangle + \langle \partial_i\zeta\partial_i\ddot{\zeta}\rangle) - H^2\langle \partial_i\zeta\partial_i\dot{\zeta}\rangle + H^3(-\langle \partial_i\zeta\partial_i\zeta\rangle) \Big) \\
&\quad -e^{2\rho(t)} \Big(6H^2\langle \dot{\zeta}\dot{\zeta}\rangle \dot{H} - 3\langle \dot{\zeta}\dot{\zeta}\rangle \dot{H}^2 - 9H^3\langle \zeta\dot{\zeta}\rangle \dot{H} + H\Big(\langle \dot{\zeta}\dot{\zeta}\rangle \ddot{H} + 2\langle \dot{\zeta}\ddot{\zeta}\rangle \dot{H} \Big) + \\
&\quad 6H^4(\langle \dot{\zeta}\dot{\zeta}\rangle + \langle \zeta\ddot{\zeta}\rangle) \Big) \Big) \ , \\
\delta\Lambda \quad &= \quad \frac{M_{\rm Pl}^2 e^{-2\rho(t)}}{3H^4} \Big(H \Big(H(H(2H\langle \partial_i\zeta\partial_i\zeta\rangle + 5\langle \partial_i\zeta\partial_i\dot{\zeta}\rangle) + \langle \partial_i\dot{\zeta}\partial_i\dot{\zeta}\rangle + \langle \partial_i\zeta\partial_i\ddot{\zeta}\rangle) - 2\langle \partial_i\zeta\partial_i\dot{\zeta}\rangle \dot{H} \Big) \\
&\quad -e^{2\rho(t)} \Big(3H^2\langle \dot{\zeta}\dot{\zeta}\rangle \dot{H} - 3\langle \dot{\zeta}\dot{\zeta}\rangle \dot{H}^2 + 9H^3\langle \zeta\dot{\zeta}\rangle \dot{H} + H\Big(\langle \dot{\zeta}\dot{\zeta}\rangle \ddot{H} + 2\langle \dot{\zeta}\ddot{\zeta}\rangle \dot{H} \Big) \\
&\quad +6H^4(\langle \dot{\zeta}\dot{\zeta}\rangle + \langle \zeta\ddot{\zeta}\rangle) + 36H^5\langle \zeta\dot{\zeta}\rangle \Big) \Big) \ .
\end{aligned} \tag{4.94}$$

In these expressions, a term such as $\langle \partial_i\zeta\partial_i\zeta\rangle$ stands for $\langle \partial_i\zeta(\vec{x},t)\partial_i\zeta(\vec{x},t)\rangle$. A term like $\langle \zeta\dot{\zeta}\rangle$ stays for $\langle \zeta\dot{\zeta} + \dot{\zeta}\zeta\rangle/2$. No slow roll approximation has been performed nor it has ever been performed in this chapter. There are three subtleties to stress here. The first is that the cubic action of order $\zeta_0\zeta_k^2$ is not the cubic action ζ_k^3 with one of the momenta taken to zero. As before, the limit is discontinuous and the action is different. We do not report it here because it is very long and comes from trivial substitution of the solutions of the constraint equations into the action. Second, in taking expectation values $\langle \zeta^2\rangle$, one might worry about the contribution of the zero-mode, which has a different action than $\zeta_0\zeta_k^2$. This is irrelevant because the zero mode has measure zero when we perform the expectation value. The difference in the action is important

[11] Notice that since we are working in the gauge $N_0^i = 0$ and we choose a fixed comoving box in this gauge, there is no need to introduce boundary counterterms.

for the tadpole terms and for the non 1-PI diagrams because the ζ_0 propagator is the only one singled out by translation invariance. Finally, the third subtlety is about the expectation values involving two derivatives of ζ: $\langle\zeta\ddot{\zeta}\rangle$. Here one can use the linear equation of motion for the short modes as derived from (4.93) to relate it to expectation values of the form $\langle\zeta\partial^2\zeta\rangle$ or $\langle\zeta\dot{\zeta}\rangle$.

Cancellation between quartic diagrams and diff.-enhanced tadpole counterterms

At this point we are able to address the time (in)dependence of the zero mode two-point function. In the former section we have discussed the contribution of the diagrams involving two cubic terms. We saw that upon the addition of some quartic diagrams, they induced no time dependence on ζ. We have now to deal with the remaining quartic diagrams, that in this case come from the action of the form $\zeta_0^2\zeta_k^2$.

The simplest way to evaluate the contribution of these diagrams to the ζ_0 two-point function is to derive the quartic action and substitute directly the quadratic pieces in the short modes with their expectation value. For example

$$\int d^3k\,dt\,e^{3\rho(t)}\;\zeta_0(t)^2\zeta_{\vec{k}}\zeta_{-\vec{k}}\quad\rightarrow\quad\int dt\,e^{3\rho(t)}\;\zeta_0(t)^2\langle\zeta^2\rangle\ , \tag{4.95}$$

and then derive the resulting linear equation of motion for ζ_0. In this way we can incorporate the effect of this quartic diagrams by simply studying the corrections to the quadratic action. The symmetries of the problem imply that the quadratic action will have a kinetic term $\dot{\zeta}_0^2$ and a mass terms ζ_0^2. There is also a term proportional to $\dot{\zeta}_0\zeta_0$ that can be reduced to a mass term upon integration by parts. Clearly a time dependence on $\langle\zeta_0^2\rangle$ can come only from a non vanishing mass term. These terms read

$$\begin{aligned}
&S^{(4)}_{\zeta_0^2,\,\zeta_0\dot{\zeta}_0} = \\
&\int dt\left[\frac{\zeta_0^2}{2H^2}\left(M_{\rm Pl}^2He^{\rho(t)}(H\langle\partial_i\zeta\partial_i\zeta\rangle+2\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle)-9e^{3\rho(t)}\left(M_{\rm Pl}^2\dot{H}\left(3H^2\langle\zeta\zeta\rangle+\langle\dot{\zeta}\dot{\zeta}\rangle\right)\right.\right.\right.\\
&\qquad\left.\left.+H^2\left(3M_{\rm Pl}^2H\left(3H\langle\zeta\zeta\rangle+2\langle\zeta\dot{\zeta}\rangle\right)+2(\delta\Lambda+\delta M^4)\right)\right)\right)\\
&\quad+\frac{\zeta_0\dot{\zeta}_0}{H^3\left(\dot{H}+3H^2\right)}\left(M_{\rm Pl}^2He^{\rho(t)}\left(3H^3\langle\partial_i\zeta\partial_i\zeta\rangle-2\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle\dot{H}\right)\right.\\
&\qquad-3e^{3\rho(t)}\left(3M_{\rm Pl}^2H^2\dot{H}\left(2H\langle\zeta\dot{\zeta}\rangle+3H^2\langle\zeta\zeta\rangle-3\langle\dot{\zeta}\dot{\zeta}\rangle\right)\right.\\
&\qquad\left.\left.\left.-2M_{\rm Pl}^2\langle\dot{\zeta}\dot{\zeta}\rangle\dot{H}^2+3H^4\left(9M_{\rm Pl}^2H^2\langle\zeta\zeta\rangle+2(\delta\Lambda-\delta M^4)\right)\right)\right)\right]\ .
\end{aligned} \tag{4.96}$$

After we substitute in the counterterm solutions from (4.94), and we integrate by parts the term $\zeta_0\dot{\zeta}_0$, the above expression simplifies to

$$S^{(4)}_{\zeta_0^2} = \int dt \frac{M_{\rm Pl}^2 e^{-\rho(t)}}{H^2\left(\dot{H}+3H^2\right)^2}\ \zeta_0^2 \tag{4.97}$$
$$\Big(2\langle\partial_i\partial_j\zeta\partial_i\partial_j\zeta\rangle\dot{H}\left(\dot{H}+3H^2\right)+e^{2\rho(t)}\Big(2\dot{H}\Big(\dot{H}(H(H\langle\partial_i\zeta\partial_i\zeta\rangle+7\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle)-\langle\partial_i\dot{\zeta}\partial_i\dot{\zeta}\rangle)$$
$$-3H^2(H(2H\langle\partial_i\zeta\partial_i\zeta\rangle-\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle)+\langle\partial_i\dot{\zeta}\partial_i\dot{\zeta}\rangle)\Big)+\ddot{H}\left(2\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle\dot{H}-3H^3\langle\partial_i\zeta\partial_i\zeta\rangle\right)\Big)\Big)\ .$$

Clearly, a mass term seems to have survived after we have taken into account of the quadratic terms generated by the counterterm solutions. Unless these remaining terms are exactly those quartic terms of eq. (4.50), the terms associated with a rescaling of the spatial coordinates in cubic vertices, we would have a time-dependence for the ζ_0 two point function. Luckily [12], this is exactly what happens. It is indeed indicative that all the surviving terms have spatial derivatives acting on the ζ's inside the expectation values, suggesting that they are indeed associated to a rescaling of the spatial coordinates. Let us therefore discover what are those terms in (4.50) by first finding the cubic Lagrangian of order $\zeta_0\zeta_k^2$ and then taking the expectation value of the finite-k modes. With the usual procedure, we obtain

$$S^{(3)}_{\zeta_0\zeta_k^2} = \int d^3x\ dt\ \left(-\frac{e^{\rho(t)}}{H^3\left(\dot{H}+3H^2\right)}\right) \tag{4.98}$$
$$\Big(H^3\dot{H}\Big(M_{\rm Pl}^2\Big(\zeta_0\Big(-\left(\langle\partial_i\zeta\partial_i\zeta\rangle-9e^{2\rho(t)}\langle\zeta\zeta\rangle\dot{H}\right)+12e^{2\rho(t)}\dot{H}+9\langle\dot{\zeta}\dot{\zeta}\rangle e^{2\rho(t)}\Big)+6\langle\zeta\dot{\zeta}\rangle e^{2\rho(t)}\dot{\zeta}_0\Big)$$
$$+6\delta\Lambda(t)\zeta_0 e^{2\rho(t)}+6\delta M^4\zeta_0 e^{2\rho(t)}\Big)-3H^4\Big(\dot{\zeta}_0\Big(M_{\rm Pl}^2\Big(\left(\langle\partial_i\zeta\partial_i\zeta\rangle-3e^{2\rho(t)}\langle\zeta\zeta\rangle\dot{H}\right)-4e^{2\rho(t)}\dot{H}\Big)$$
$$-2\delta\Lambda e^{2\rho(t)}+2\delta M^4 e^{2\rho(t)}\Big)+2M_{\rm Pl}^2\zeta_0\left(\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle-3e^{2\rho(t)}\langle\zeta\dot{\zeta}\rangle\dot{H}\right)\Big)$$
$$+3H^5\zeta_0\Big(M_{\rm Pl}^2\Big(-\Big(\left(\langle\partial_i\zeta\partial_i\zeta\rangle-18e^{2\rho(t)}\langle\zeta\zeta\rangle\dot{H}\right)-24e^{2\rho(t)}\dot{H}\Big)\Big)+6\delta\Lambda e^{2\rho(t)}+6\delta M^4 e^{2\rho(t)}\Big)$$
$$-M_{\rm Pl}^2H^2\dot{H}\left(2\zeta_0\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle+9\langle\dot{\zeta}\dot{\zeta}\rangle e^{2\rho(t)}\dot{\zeta}_0\right)+M_{\rm Pl}^2H\dot{H}\left(3\zeta_0\langle\dot{\zeta}\dot{\zeta}\rangle e^{2\rho(t)}\dot{H}+2k^2\langle\zeta\dot{\zeta}\rangle\dot{\zeta}_0\right)$$
$$-2M_{\rm Pl}^2\langle\dot{\zeta}\dot{\zeta}\rangle e^{2\rho(t)}\dot{H}^2\dot{\zeta}_0+9M_{\rm Pl}^2H^6e^{2\rho(t)}\left(3\langle\zeta\zeta\rangle\dot{\zeta}_0+6\zeta_0\langle\zeta\dot{\zeta}\rangle\right)+81M_{\rm Pl}^2H^7\zeta_0\langle\zeta\zeta\rangle e^{2\rho(t)}\Big)\ .$$

According to the results of sec. 4.4, loops formed with cubic operators that contain spatial derivatives would induce time dependence unless we combine them with quartic loops constructed with the operators derived

[12] Or obviously, depending on the point of view.

from formula (4.50). Applying it to the cubic action above, we obtain

$$S_{Quartic,\partial_i} = \int d^3x\, dt\, \left(-\frac{M_{\rm Pl}^2 e^{\rho(t)}}{H^2\left(\dot{H}+3H^2\right)}\zeta_0 \right) \tag{4.99}$$

$$\Big(-4\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle\dot{H}\dot{\zeta}_0 + 2H\zeta_0\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle\dot{H} + H^2\zeta_0\langle\partial_i\zeta\partial_i\zeta\rangle\dot{H} + 6H^3\langle\partial_i\zeta\partial_i\zeta\rangle\dot{\zeta}_0$$
$$+6H^3\zeta_0\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle + 3H^4\zeta_0\langle\partial_i\zeta\partial_i\zeta\rangle\Big)\ .$$

Upon integration by parts, and after using the equation of motions in terms of the form $\langle\ddot{\zeta}\zeta\rangle$, it is easy to see that these terms are exactly the ones left out in (4.97). Notice that we do not even need to compute explicitly the value of $\langle\partial\zeta\partial\zeta\rangle$: it cancels with the corresponding terms. This shows that one can combine the terms in (4.97) with the diagrams built with cubic interactions to see that all those diagrams do not give a time dependence to ζ_0. The remaining quartic diagrams cancel with the quadratic terms induced by the tadpole terms.

This concludes all the diagrams that appear at one loop. We see that both the 1-PI and non 1-PI diagrams are important to cancel each other so that, even though naively many diagrams are dangerous and can potentially give a time dependence to the ζ_0 correlation function, the time-dependence cancels in the sum, and we conclude that the ζ_0 two-point function is time independent.

4.6.3 Time-independence for the non-zero-modes

We are now ready to begin the study of the case in which the external momentum is finite. This task is very challenging [13], as the interactions are even more complicated than for the case of the zero mode. Luckily we will be able to do it by employing a trick. As we discussed, the time-dependence we are interested in ruling out is the one that appears when the wavelength of the mode is much longer than the horizon, and the loop effect is due to short wavelength modes running in the loop (modes longer than our mode clearly cannot induce a time dependence). For this reason, we can simplify the action by taking the leading term in the smallness of the derivatives of the external mode.

In ζ-gauge, this simplification is not trivial at all. After substituting the solutions to the constraint equations, N^i becomes larger and larger as we move to finite but smaller and smaller k's. This is due to the fact that at finite k, N^i has the non-local-looking expression $N^i \sim k^i\dot{\zeta}/k^2$ [14]. Armed with the experience of the zero-mode, we realize that it would probably be much better if we could find a gauge where N^i does

[13] At least for our standards.

[14] We stress that since we are trying to investigate if ζ_k becomes time-dependent, we cannot assume that $\dot{\zeta} \sim k^2\zeta/a^2$ out of the horizon, as it happens in the free theory. Indeed time derivatives do not count as a suppression when the mode is part of a commutator in a Green's function.

not have this bad behavior at low momenta. Since at finite k all gauge freedoms are completely fixed by the ζ-gauge conditions, this is globally impossible. However, we can do this locally. Indeed, we can find a frame valid in a region of space very small compared to the wavelength of the mode, where the universe looks like an anisotropic flat universe. Corrections to the results obtained in this frame will be down by powers of $k/(aH)$ and so will lead to a contribution that is convergent with time. Since we are dealing with a time-dependent finite-k Fourier mode, the local frame is not a local FRW universe as it was for the zero mode, but it is an anisotropic universe. For simplicity, we can choose to work directly with a single Fourier mode

$$\zeta_k(\vec{x},t) = \mathrm{Re}\left[\tilde{\zeta}_0(t)\, e^{i\vec{k}\cdot\vec{x}}\right] \ , \qquad \mathrm{Re}\left[\tilde{\zeta}_0\right] = \zeta_0 \ . \tag{4.100}$$

Using rotational invariance, we can take the momentum $\vec{k}$ to be along the $\hat{z}$ direction without loss of generality. The resulting spatial metric in the ADM parametrization is given by the following:

$$\begin{aligned}
&\hat{h}_{11} = \hat{h}_{22} = e^{2\rho(t)+2\zeta_0(t)+2\lambda_0(t)} e^{2\zeta(\vec{x},t)} \ , \\
&\hat{h}_{33} = e^{2\rho(t)+2\zeta_0(t)-4\lambda_0(t)} e^{2\zeta(\vec{x},t)} \ , \\
&N_i = \partial_i\psi(\vec{x},t) + \tilde{N}_i(\vec{x},t) \ , \qquad \partial^i\tilde{N}_i(\vec{x},t) = 0 \\
&N = 1 + \delta N_0(t) + \delta N(\vec{x},t) \ .
\end{aligned} \tag{4.101}$$

Here the fields with the argument $\vec{x}$ represent short wavelength fields that will be integrated over in the loops. We see that there is no $N_{i,0}(t)$ component. This is so because we can make N_0^i and $\partial_i N_0^j$ vanish. The field λ_0 is the (traceless) anisotropic component of the metric, related to ζ_0 by

$$\lambda_0(t) = -\frac{1}{3}\int^t dt' \ \frac{\dot{H}}{H^2}\dot{\zeta}_0 \ , \tag{4.102}$$

up to an irrelevant constant that can be set to zero using a constant rescaling of the spatial coordinates. The details of this change of coordinates are given in App. 4.9.

Apart from the terms proportional to λ_0, the treatment is very parallel to the one of the former subsection. First we find the solution to the tadpole counterterms δM^4 and $\delta\Lambda$. As expected, there is no tadpole for the terms in λ_0 because of rotational invariance: the free vacuum expectation value of product of fields must be rotational invariant and cannot source any anisotropy. This is indeed the case, and the solutions for δM^4 and $\delta\Lambda$ are exactly the same as before eq. (4.94) [15].

[15] There is only one subtlety here that distinguishes this case from the former one. In the former section we were studying the effect of loops on the zero mode, and therefore loop integrals whose range is over momenta that are shorter than the external one, were basically running over all momenta. Here instead, since we are Taylor expanding in derivatives of the long external mode, loops should formally include only modes that are shorter than the external one. This is hardly a problem however

At this point we proceed to find the action for the short modes in this background. We start with the solution to the constraint equations, that read:

$$
\begin{aligned}
\delta N_0(t) &= \frac{3H}{3H^2+\dot H}\dot\zeta_0 \ , \\
\delta N_k &= \frac{\dot\zeta_k}{H^2}\left(H\delta N_0 + H - \dot\zeta_0\right) + \frac{k_{\rm ani}^2}{2k^2H^2}\left(\dot\zeta_k - 3H\zeta_k\right)\dot\lambda_0 \ , \\
\psi_k &= \frac{e^{2\rho(t)}}{2k^4H^3}\Big[2\dot H\dot\zeta_k\left(H\left(2k_{\rm ani}^2\lambda_0 + 2k^2\zeta_0 + k^2\right) - 2k^2\dot\zeta_0\right) + \\
&\quad \dot\lambda_0\left(-3H\zeta_k\left(3H^2+\dot H\right) + \left(3H^2+2\dot H\right)\dot\zeta_k\right) + \\
&\quad k^2H\zeta_k\left(-k_{\rm ani}^2\dot\lambda_0 - 2k^2\left(2H\delta N_0 + H - \dot\zeta_0\right)\right)\Big] \ , \\
\tilde N_i &= \frac{k_i e^{2\rho(t)}}{k^4 H} 2\left(k^2 - k_{\rm ani}^2\right)\dot\lambda_0\left(3H\zeta_k - \dot\zeta_k\right) \ , \quad i = 1,2 \ , \\
\tilde N_3 &= \frac{k_3 e^{2\rho(t)}}{k^4 H} 2\left(2k^2 - k_{\rm ani}^2\right)\dot\lambda_0\left(3H\zeta_k - \dot\zeta_k\right) \ ,
\end{aligned}
\tag{4.103}
$$

where $k^2 = k_x^2 + k_y^2 + k_z^2$ and $k_{\rm ani}^2 = k_x^2 + k_y^2 - 2k_z^2$. $k_{\rm ani}$ has the nice property that $\int d^2\hat k\ k_{\rm ani}^2 = 0$. After substitution of the above solutions in the action, we obtain the quartic action at order $\zeta_0^2\zeta_k^2$. As before we evaluate the expectation value on the ζ_k-modes and isolate the terms in ζ_0 (and λ_0) that could lead to a time-dependence for ζ_0. Clearly, we need to keep track only of the terms that contain at least one λ_0, the terms quadratic in ζ_0 will cancel exactly as in the former section. Furthermore, because of rotational invariance, terms proportional to $\lambda_0\zeta_0$ are absent. We are left with

$$
\begin{aligned}
S^{(4)}_{\lambda_0^2} &= \int d^3x\, dt\ e^{\rho(t)}\lambda_0^2\frac{4M_{\rm Pl}^2}{H}\left(H\langle\partial_i\zeta\partial_i\zeta\rangle + 2\langle\partial_i\zeta\partial_i\dot\zeta\rangle\right) \ , \\
S^{(4)}_{\lambda_0\dot\lambda_0} &= \int d^3x\, dt\ \left(-2e^{\rho(t)}\frac{M_{\rm Pl}^2}{H^3}\right)\left(-3H\left(\langle\frac{\partial_{\rm ani}^2}{\partial^2}\zeta\frac{\partial_{\rm ani}^2}{\partial^2}\dot\zeta\rangle - 2\langle\zeta\dot\zeta\rangle\right)e^{2\rho(t)}\dot H - 6H^2\langle\partial_i\zeta\partial_i\zeta\rangle\right. \\
&\quad \left. + \left(\langle\frac{\partial_{\rm ani}^2}{\partial^2}\dot\zeta\frac{\partial_{\rm ani}^2}{\partial^2}\dot\zeta\rangle - 2\langle\dot\zeta\dot\zeta\rangle\right)e^{2\rho(t)}\dot H + 2H\langle\partial_i\zeta\partial_i\dot\zeta\rangle\right) \ .
\end{aligned}
\tag{4.104}
$$

Here $\partial^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ while $\partial_{\rm ani}^2 = \partial_x^2 + \partial_y^2 - 2\partial_z^2$. The second expression above can be further simplified by noticing that by rotational invariance

$$
\langle\frac{\partial_{\rm ani}^2}{\partial^2}\dot\zeta\frac{\partial_{\rm ani}^2}{\partial^2}\dot\zeta\rangle = \frac{4}{5}\langle\dot\zeta\dot\zeta\rangle \ , \tag{4.105}
$$

and similar for similar terms. After integrating by parts the term in $\lambda_0\dot\lambda_0$ and summing with the term in

because in order to prove that there is no induced time-dependence, we are interested in the case where the external mode k is outside of the horizon. The contribution from modes longer than the horizon is equivalent to the contribution of modes that are all out of the horizon. At this point, a nice property of the ζ action tells that there are no vertices without at least a derivative acting on one ζ fluctuation [22]. This guarantees that when all the modes are outside of the horizon each vertex is suppress by powers of $k/(aH)$. So those contributions would give rise to a time-convergent contribution and can be safely ignored.

λ_0^2, we obtain the final expression

$$S^{(4)}_{\lambda_0^2} = \int d^3x\, dt\, e^{-\rho(t)}\, \lambda_0^2 \frac{2M_{\rm Pl}^2}{5H^4\dot{H}} \tag{4.106}$$
$$\Big(-20H^3 e^{2\rho(t)}\dot{H}\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle - 5H^4\langle\partial_i\zeta\partial_i\zeta\rangle e^{2\rho(t)}\dot{H} - 3e^{4\rho(t)}\dot{H}^3\langle\dot{\zeta}\dot{\zeta}\rangle$$
$$+H^2\left(-5e^{2\rho(t)}\ddot{H}\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle + e^{2\rho(t)}\dot{H}\left(18e^{2\rho(t)}\dot{H}\langle\dot{\zeta}\dot{\zeta}\rangle + 5\langle\partial_i\dot{\zeta}\partial_i\dot{\zeta}\rangle\right)\right.$$
$$\left.-\dot{H}\left(5\langle\partial_j\partial_i\zeta\partial_j\partial_i\zeta\rangle - 6e^{2\rho(t)}\dot{H}\langle\partial_i\zeta\partial_i\zeta\rangle\right)\right) + 3He^{2\rho(t)}\dot{H}\left(e^{2\rho(t)}\ddot{H}\langle\dot{\zeta}\dot{\zeta}\rangle + 2\langle\partial_i\zeta\partial_i\dot{\zeta}\rangle\dot{H}\right)\Big)\ .$$

As in the former section, if these terms were not to be exactly the ones in Quartic$_{\partial_i}$ then we will have a time dependence for the ζ correlation function. To check for this, we move to the cubic action. Again, we need simply to investigate terms proportional to $\lambda_0\zeta_k^2$. We have

$$S^{(3)}_{\lambda_0\zeta_k^2} = \int d^3x\, dt\, 2\frac{M_{\rm Pl}^2 e^{\rho(t)}}{H^3} \tag{4.107}$$
$$\Big(e^{2\rho(t)}\dot{H}\dot{\lambda}_0\left(3H\frac{\partial^2_{\rm ani}}{\partial^2}\zeta\dot{\zeta} - \frac{\partial^2_{\rm ani}}{\partial^2}\dot{\zeta}\dot{\zeta}\right) - H\Big(\dot{\lambda}_0\left(\partial^2_{\rm ani}\zeta\dot{\zeta} - 3H\partial^2_{\rm ani}\zeta\zeta\right)$$
$$-2H\lambda_0\left(H\partial^2_{\rm ani}\zeta\zeta + 2\partial^2_{\rm ani}\zeta\dot{\zeta}\right)\Big)\Big)\ .$$

The identification of the quartic vertices starting from the cubic vertices is slightly more complicated due to the anisotropy. In practice, everytime in the cubic Lagrangian there are two derivatives that are contracted, they should be thought of as originating from being contracted with the spatial metric $\hat{h}_{ij}$, and we take the resulting relevant quartic operator. Let us give a few examples:

$$\begin{aligned}
\mathcal{L}_3 &\supset \zeta_0(\partial_i\zeta)^2 &&\to &\mathcal{L}_4 &\supset -\zeta_0^2(\partial_i\zeta)^2 - 2\lambda_0\zeta_0(\partial_{\rm ani}\zeta)^2\ ,\\
\mathcal{L}_3 &\supset \dot{\zeta}_0(\partial_i\zeta)^2 &&\to &\mathcal{L}_4 &\supset -2\zeta_0\dot{\zeta}_0(\partial_i\zeta)^2 - 2\lambda_0\dot{\zeta}_0(\partial_{\rm ani}\zeta)^2\ ,\\
\mathcal{L}_3 &\supset \lambda_0(\partial_i\zeta)^2 &&\to &\mathcal{L}_4 &\supset -2\zeta_0\lambda_0(\partial_i\zeta)^2 - \lambda_0^2(\partial_{\rm ani}\zeta)^2\ ,\\
\mathcal{L}_3 &\supset \dot{\lambda}_0(\partial_i\zeta)^2 &&\to &\mathcal{L}_4 &\supset -2\zeta_0\dot{\lambda}_0(\partial_i\zeta)^2 - 2\lambda_0\dot{\lambda}_0(\partial_{\rm ani}\zeta)^2,\\
\mathcal{L}_3 &\supset \zeta_0(\partial_{\rm ani}\zeta)^2 &&\to &\mathcal{L}_4 &\supset (-\zeta_0+2\lambda_0)\zeta_0(\partial_{\rm ani}\zeta)^2 - 4\lambda_0\zeta_0(\partial_i\zeta)^2\ ,\\
\mathcal{L}_3 &\supset \dot{\zeta}_0(\partial_{\rm ani}\zeta)^2 &&\to &\mathcal{L}_4 &\supset (-2\zeta_0+2\lambda_0)\dot{\zeta}_0(\partial_{\rm ani}\zeta)^2 - 4\lambda_0\dot{\zeta}_0(\partial_i\zeta)^2\ ,\\
\mathcal{L}_3 &\supset \lambda_0(\partial_{\rm ani}\zeta)^2 &&\to &\mathcal{L}_4 &\supset (-2\zeta_0+\lambda_0)\lambda_0(\partial_{\rm ani}\zeta)^2 - 2\lambda_0^2(\partial_i\zeta)^2\ ,\\
\mathcal{L}_3 &\supset \dot{\lambda}_0(\partial_{\rm ani}\zeta)^2 &&\to &\mathcal{L}_4 &\supset (-2\zeta_0+2\lambda_0)\dot{\lambda}_0(\partial_{\rm ani}\zeta)^2 - 4\lambda_0\dot{\lambda}_0(\partial_i\zeta)^2\ ,
\end{aligned} \tag{4.108}$$

where $\vec{\partial}_{\rm ani} = (\partial_x, \partial_y, i\sqrt{2}\partial_z)$. Upon implementing the promotion of the spatial derivative to include the ζ_0

and λ_0 factors, we have

$$S_{Quartic,\partial_i} = \int d^3x\, dt\, e^{\rho(t)}\, \lambda_0\, \frac{4M_{\rm Pl}^2}{5H(t)^3} \tag{4.109}$$
$$\Big(5H\left(\dot\lambda_0\left(3H\langle\partial_i\zeta\partial_i\zeta\rangle - \langle\partial_i\zeta\partial_i\dot\zeta\rangle\right) + H\lambda_0\left(H\langle\partial_i\zeta\partial_i\zeta\rangle + 2\langle\partial_i\zeta\partial_i\dot\zeta\rangle\right)\right) -$$
$$3e^{2\rho(t)}\dot H\dot\lambda_0\left(3H\langle\zeta\dot\zeta\rangle - \langle\dot\zeta\dot\zeta\rangle\right)\Big) =$$
$$= \int d^3x\, dt\, e^{-\rho(t)}\, \lambda_0^2 \frac{M_{\rm Pl}^2}{5H^4\dot H}$$
$$\Big(-20H^3e^{2\rho(t)}\dot H\langle\partial_i\zeta\partial_i\dot\zeta\rangle - 5H^4\langle\partial_i\zeta\partial_i\zeta\rangle e^{2\rho(t)}\dot H - 3e^{4\rho(t)}\dot H^3\langle\dot\zeta\dot\zeta\rangle$$
$$+H^2\Big(-5e^{2\rho(t)}\ddot H\langle\partial_i\zeta\partial_i\dot\zeta\rangle + e^{2\rho(t)}\dot H\left(18e^{2\rho(t)}\dot H\langle\dot\zeta\dot\zeta\rangle + 5\langle\partial_i\dot\zeta\partial_i\dot\zeta\rangle\right)$$
$$-\dot H\left(5\langle\partial_j\partial_i\zeta\partial_j\partial_i\zeta\rangle - 6e^{2\rho(t)}\dot H\langle\partial_i\zeta\partial_i\zeta\rangle\right)\Big) + 3He^{2\rho(t)}\dot H\left(e^{2\rho(t)}\ddot H\langle\zeta\dot\zeta\rangle + 2\langle\partial_i\zeta\partial_i\dot\zeta\rangle\dot H\right)\Big)\ .$$

where in the first passage we have used that by rotational invariance terms involving $\partial^2_{\rm ani}$ are zero and those involving $\partial^4_{\rm ani}$ are equal to the same expression with $(\partial^2_{\rm ani})^2 \to 4(\partial^2)^2/5$, and in the second passage we have performed an integration by parts. We see that this $Quartic_{\partial_i}$ term is exactly the one being left out from the loops with the quartic diagrams, and so its time-dependent contribution will cancel the one coming from the $CIS_{1PI} + CIM + Quartic_{\partial_t}$ diagrams. This completes the exploration of all the diagrams entering at one-loop, proving that the ζ_k correlator does not have a time-dependence even at finite momentum k.

A note on tensor modes: Since in this section we have dealt with gravitational interactions, it is logical to wonder on the contribution of the tensor modes. Indeed, for standard slow roll inflation, at one-loop the contribution from tensor modes is parametrically the same as the one from the ζ short modes. One might wonder why we could neglect them, or alternatively why time-dependent effects from loops of ζ modes cancel independently of the ones from loops of tensor modes. It is easy to realize that the contribution from tensor modes must cancel independently. Let us analyze the various diagrams. It is pretty clear that the diagrams built with cubic vertices will cancel independently in the same way as they independently did for the ζ modes. This cancellation in fact relies on the consistency condition, that holds for tensor modes as well as for ζ modes. A bit less obvious is to understand why the graviton and ζ contribution from quartic and tadpole terms cancel independently. The fact that the contribution of tensor modes and scalar modes is parametrically the same is an accident of standard slow roll inflation. It is possible to engineer inflationary models where the contribution is parametrically different. If for example we add to the Effective Field Theory of Inflation an operator of the form $(\delta g^{00})^2$, we change the speed of sound of the ζ fluctuations, without changing the ones of the tensor modes. Since the tadpoles and the quartic loops are evaluated on the linear solutions, this shows that those loops are parametrically different, and they have to cancel independently.

We have explicitly verified that this is the case for the effect on the ζ zero-mode.

4.7 Conclusions

Understanding the behavior of the theory of inflationary fluctuations at one-loop order, with particular attention to the possible infrared factors, is a very important task. We have stressed how this is important for the predictivity of inflation as well as for slow roll eternal inflation and its universal volume bound. In general, it is also important to understand how the theory we believe to be the strongest contender for describing the first instants in the history of our universe behaves at quantum level.

In this chapter we have proven that the ζ_k correlation function does not receive corrections that grow with time $\sim Ht$ after the mode has crossed the horizon. This result is achieved by proving that there is a cancellation among the various diagrams that would naively induce a time-dependence, if taken alone. While this cancellation happens in an intricate way, its physical origin can be stated in a very simple form. First, since there is a vacuum contribution to the stress tensor due to the fluctuations, it is important to define the ζ fluctuations around the correct one-loop spacetime background. This can be achieved either by automatically including $non-1PI$ diagrams in the calculation, or, as we do here, by inserting diff. invariant counterterms that cancel the tadpole correction. Because of diff. invariance, these tadpole counterterms contain terms quadratic in the fluctuations that modify the ζ propagator and account for a cancellation of the time-dependence induced by many of the diagrams built from quartic vertex. Some of these quartic diagrams indeed look very much like coming from a renormalization of the background, as they involve vacuum expectation values of quadratic operators on the unperturbed background. It is not so surprising that they cancel with the tadpole counterterms.

The remaining quartic vertices, that we have called $Quartic_{\partial_t}$ and $Quartic_{\partial_i}$, induce a time dependence that cancels with the one from the cubic diagrams that we call $CIS_{1PI} + CIM$. The sum of all these diagrams describes how the vacuum expectation value of the short-wavelength modes is affected by the presence of a long-wavelength mode, and how the perturbation in this expectation value in turn backreacts on the long-wavelength mode. Because of the attractor nature of the inflationary solution, a long wavelength ζ fluctuation is equivalent to a trivial rescaling of the coordinates in the unperturbed background. So the vacuum expectation value of the short-wavelength modes should not be affected at all by the presence of a long wavelength mode making this effect disappear.

Since the ζ fluctuations are not derivatively coupled, a feature shared also by the graviton, showing this is not easy. In order to do it we wrote the sum of these diagrams as the three-point function between two short-wavelength modes and one long-wavelength mode, integrated over the short-wavelength Fourier components. In this way, after adding the terms from $Quartic_{\partial_t}$ and $Quartic_{\partial_i}$, we could use the consistency condition to show that the presence of a long-wavelength ζ does not change the expectation value of short modes in a way that correlates with the long mode and therefore that these diagrams do not give any time dependence.

By accounting for all the diagrams at one loop order we proved that ζ is a constant at this order.

There are many possible generalizations to our results. In the introduction we gave arguments that could be easily generalized to arbitrary loops. Furthermore it would be nice to include in the treatment gravitons both inside the loops as well as in the external legs. All of this seems doable. The physical principles responsible for the cancellations we found should hold unchanged also for these more general cases.

4.8 Appendix A: Consistency Condition inside the Horizon

In this Appendix we discuss the three-point function in the squeezed limit in which one of the modes is much longer than the other two. While so far the literature has always concentrated in the limit in which the two short modes are outside of the horizon, as this is the relevant limit for observed modes in tree-level correlation functions, at loop level we are also interested in the case in which the two short modes are inside the horizon. We will verify that the consistency condition also holds in this regime. We will do this at leading order in slow roll parameters.

For the case in which the short modes are still inside the horizon, the proof at leading order in slow roll parameters is very easy. In fact, contrary to what happens when we are interested in computing the correlation function of modes at a time when they are outside the horizon, in this case the leading interaction is of zeroth order in the slow roll parameters. Indeed, it is not true that the ζ cubic action starts at first order in slow roll parameters (relative to the quadratic action). This is so only up to terms that can be removed by a field redefinition and that can therefore be evaluated at the final time. For modes that are outside of the horizon at the time of evaluation, these vanish. For modes that are not yet outside of the horizon, they do not, and they therefore represent the leading contribution in the slow roll expansion.

Following [22], the term we are discussing comes from the field redefinition:

$$\zeta = \zeta_n + \frac{\zeta\dot{\zeta}}{H} + \dots \ , \tag{4.110}$$

where $\dots$ represent terms suppressed by slow roll parameters. The variable ζ_n has a cubic action that is suppressed by slow roll parameters, and so negligible. At this point computing the three-point function is very straightforward. In the limit in which the long mode k_3 is much longer than the horizon $k_3/a(\eta) \ll H$

and $k_3 \ll k_2 \simeq k_1$, we have

$$
\begin{aligned}
\langle \zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)\rangle &\simeq (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{1}{H}\langle \dot{\zeta}_{k_1}\zeta_{k_1}+\zeta_{k_1}\dot{\zeta}_{k_1}\rangle' \,\langle \zeta^2_{k_3}\rangle' \\
&= (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{1}{H}\partial_t\langle \zeta^2_{k_1}\rangle' \,\langle \zeta^2_{k_3}\rangle' \ , \qquad k_1 \ll k_3 \ ,
\end{aligned}
\tag{4.111}
$$

where the $\langle\rangle'$ symbol stays for the fact that we have removed the delta function from the expectation value. Using the wavefunction of the modes at leading order in slow roll parameters

$$
\zeta^{cl}_k(\eta) = \frac{H}{2\sqrt{\epsilon}M_{\rm Pl}}\frac{1}{k^{3/2}}\left(1-ik\eta\right)e^{ik\eta} \ , \tag{4.112}
$$

where ϵ is the slow roll parameter $\epsilon = -\dot{H}/H^2$, we obtain

$$
\langle \zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)\rangle \simeq -\frac{H^4}{8M_{\rm Pl}^4\epsilon^2}\cdot\frac{\eta^2}{k_1k_3^3} \ . \tag{4.113}
$$

In order to satisfy the consistency condition, the above result should be equal to

$$
\langle \zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)\rangle \simeq -(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{\partial\left[k_1^3\langle\zeta^2_{k_1}\rangle'\right]}{\partial\log k_1}\langle\zeta^2_{k_1}\rangle'\langle\zeta^2_{k_3}\rangle' \ . \tag{4.114}
$$

Notice that since the short modes are still inside the horizon, their power spectrum is not yet scale invariant, so $\partial\left[k_1^3\langle\zeta^2_{k_1}\rangle'\right]/\partial\log k_1$ is not slow-roll suppressed. Upon substitution of (4.112), this is indeed equal to (4.113), verifying the consistency condition for modes inside the horizon.

4.8.1 Consistency condition for operators with spatial derivatives

Let us now consider the three-point function in the same regime of momenta as above for a derivative operator of the form

$$
\left\langle \frac{1}{a(\eta)^2}\partial_i\zeta_{k_1}(\eta)\partial_i\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)\right\rangle \ . \tag{4.115}
$$

Since when we compute the three-point function we simply evolve the operators and not their spatial derivatives, the result can be trivially obtained from the one above in eq. (4.113) to be

$$
\begin{aligned}
\left\langle \frac{1}{a(\eta)^2}\left(\partial_i\zeta\right)_{k_1}(\eta)\left(\partial_i\zeta\right)_{k_2}(\eta)\zeta_{k_3}(\eta)\right\rangle &\simeq (2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\ \frac{k_1^2}{a(\eta)^2}\frac{1}{H}\partial_t\langle\zeta^2_{k_1}\rangle'\,\langle\zeta^2_{k_3}\rangle' \\
&= -\frac{H^6}{8M_{\rm Pl}^4\epsilon^2}\cdot\frac{\eta^4\,k_1}{k_3^3} \ , \qquad k_1 \ll k_3 \ .
\end{aligned}
\tag{4.116}
$$

This operator does not satisfy the consistency condition, that reads

$$\left\langle \frac{1}{a(\eta)^2} \left(\partial_i\zeta\right)_{k_1}(\eta)\left(\partial_i\zeta\right)_{k_2}(\eta)\zeta_{k_3}(\eta)\right\rangle \simeq -(2\pi)^3\delta^{(3)}(\vec{k}_1+\vec{k}_2+\vec{k}_3)\frac{1}{a(\eta)^2}\frac{\partial\left[k_1^5\langle\zeta_{k_1}^2\rangle'\right]}{\partial\log k_1}\langle\zeta_{k_1}^2\rangle'\langle\zeta_{k_3}^2\rangle'$$
$$= -\frac{H^6}{8M_{\rm Pl}^4\epsilon^2}\frac{\eta^2\left(1+\eta^2k_1^2\right)}{k_1k_3^3}\ . \tag{4.117}$$

The reason for this mismatch is that in the consistency condition we are rescaling all the momenta, including the ones representing the external derivatives.

An operator that instead satisfies the consistency condition (4.117) is one in which the derivatives go together with factors of $e^{-\zeta_{k_3}}$. In the squeezed limit we have

$$\left\langle \frac{1}{a(\eta)^2e^{2\zeta_{k_3}(\vec{x},\eta)}} \left(\partial_i\zeta\right)_{k_1}(\eta)\left(\partial_i\zeta\right)_{k_2}(\eta)\zeta_{k_3}(\eta)\right\rangle = \tag{4.118}$$
$$\left\langle \frac{1}{a(\eta)^2} \left(\partial_i\zeta\right)_{k_1}(\eta)(\partial_i\zeta)_{-k_1}(\eta)\zeta_{k_3\simeq0}(\eta)\right\rangle - 2\left\langle \frac{1}{a(\eta)^2} \left(\partial_i\zeta\right)_{k_1}(\eta)\left(\partial_i\zeta\right)_{-k_1}(\eta)\zeta_{k_3\simeq0}(\eta)\zeta_{-k_3\simeq0}(\eta)\right\rangle\ ,$$

as it can be readily verified.

We see that the consistency condition is satisfied by considering the sum of the operator we considered initially $\left(\partial_i\zeta\right)_{k_1}\left(\partial_i\zeta\right)_{k_2}\zeta_{k_3}$ plus a contact quartic operator of the form $-2\left(\partial_i\zeta\right)_{k_1}\left(\partial_i\zeta\right)_{k_2}{\zeta_{k_3}}^2$. As we argued in the main text, this additional contact operator comes automatically in the quartic Lagrangian, its presence being indeed guaranteed by the residual diff. invariance that we have in ζ gauge. The factor of 2 apparent mismatch in the contact operator we insert in (4.118) and the one we identify in the quartic Lagrangian in (4.55) takes into account the combinatorial factor that we have when we contract the operator with the external wavefunctions. This is the kind of combination of operators we consider in sec. 4.4 when we use the consistency condition to show that some combination of diagrams do not lead to time dependence in the ζ correlators.

4.8.2 Consistency condition for operators with time derivatives

Here we want to show that time derivatives of operators, even when inside the horizon, will obey the consistency condition, when correlated with a long wavelength mode. In particular, we want to study a correlation function of the form $\langle\dot\zeta_{k_1}(\eta)\dot\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)\rangle$ in the regime $k_3\ll k_1\approx k_2$, and the long mode has exited the horizon.

As discussed in Sec. 4.4.3, there is a contribution from contact terms that is essential for the consistency condition to be satisfied. For operators that involved spatial derivatives, we had to borrow terms from the quartic Hamiltonian. Here, we have a very similar situation. For operators with time derivatives, these operators came naturally from $H_{4,3}$, i.e., the quartic Hamiltonian induced by the cubic Lagrangian. A more rigorous parallel between these cases is made at the end of this subsection.

We will study an example in the Effective Field Theory of inflation where the speed of sound deviates from the speed of light, and the other background quantities, like H and $\dot{H}$, are assumed to be constant, for effects of computing the tilt of the spectrum. The action was written in (4.66) but let us write it before taking the decoupling limit:

$$S = \int d^4x\sqrt{-g}\left[\dot{H}M_{\rm Pl}^2 g^{00} + M^4(t)(\delta g^{00})^2\right] \ . \tag{4.119}$$

Let us further assume that $M^4(t)$ has a linear dependence in time and we will concentrate only on the effects that are proportional to $\partial_t(M^4)$. That is, we imagine that $M^4(t)$ varies on time scales that are slow with respect to H^{-1}, but fast with respect to ϵH^{-1}. Using the Stueckelberg procedure to recover gauge invariance, we perform a diffeomorphism $t \to t+\pi$ and consider the limit where the longitudinal mode decouples from the graviton. The action, up to cubic order, reads:

$$S = \int dtd^3xa^3\left\{\frac{-\dot{H}M_{\rm Pl}^2}{c_s^2}\left[\dot{\pi}^2 - c_s^2\left(\frac{\partial_i\pi}{a}\right)^2\right] + 4\partial_t(M^4(t))\pi\dot{\pi}^2 + 4M^4(t)\dot{\pi}^3 - 4M^4(t)\dot{\pi}\left(\frac{\partial_i\pi}{a}\right)^2\right\} \ . \tag{4.120}$$

The speed of sound breaks the equivalent footing of time and space derivatives in the quadratic term, and is given by

$$c_s^2 = \frac{-\dot{H}M_{\rm Pl}^2}{4M^4(t) - \dot{H}M_{\rm Pl}^2} \ . \tag{4.121}$$

To write $\langle\dot{\zeta}_{k_1}\dot{\zeta}_{k_2}\zeta_{k_3}\rangle$ we first compute $\langle\dot{\pi}_{k_1}\dot{\pi}_{k_2}\pi_{k_3}\rangle$. In order to do it, we need the following operator equation, in Heisemberg picture:

$$\begin{aligned}\dot{\pi}(t) &= \partial_t(U^\dagger_{int}(t,-\infty_+\pi_I(t)U_{int}(t,-\infty_+)) = \\ &\quad iU^\dagger_{int}(t,-\infty_+)[H_{int}(t),\pi_I(t)]U_{int}(t,-\infty_+) + U^\dagger_{int}(t,-\infty_+)\dot{\pi}_I(t)U_{int}(t,-\infty_+) \ .\end{aligned} \tag{4.122}$$

The quantum field π can be written as $\pi_k(\eta) = a_{-k}\pi^{cl}(k,\eta) + a^\dagger_k\pi^{cl}(k,\eta)^*$, with the classical wavefunction given by:

$$\pi^{cl}(k,\eta) = -\frac{i}{2\sqrt{\epsilon c_s k^3}M_{\rm Pl}}(1-ic_sk\eta)e^{ic_sk\eta} \ . \tag{4.123}$$

Then the three point function $\langle \dot\pi_{k_1}(\eta)\dot\pi_{k_2}(\eta)\pi_{k_3}(\eta)\rangle$ is given by:

$$\begin{aligned}\langle \dot\pi_{k_1}(\eta)\dot\pi_{k_2}(\eta)\pi_{k_3}(\eta)\rangle = i\int_{-\infty_+}^{\eta} d\tau \langle [H_3(\tau), \dot\pi_{k_1}(\eta)\dot\pi_{k_2}(\eta)\pi_{k_3}(\eta)]\rangle + \\ + i\langle [H_3(\eta), \pi_{k_1}(\eta)]\dot\pi_{k_2}(\eta)\pi_{k_3}(\eta)\rangle + i\langle \dot\pi_{k_1}(\eta)[H_3(\eta), \pi_{k_2}(\eta)]\pi_{k_3}(\eta)\rangle \ .\end{aligned} \tag{4.124}$$

The first term in the right hand side is the usual in-in expression, and the terms in the second line are the extra contact terms that come from using (4.122). A straightforward computation of the three terms yields, in the squeezed limit:

$$\begin{aligned}& i\int_{-\infty_+}^{\eta} d\tau \langle [H_3(\tau), \dot\pi_{k_1}(\eta)\dot\pi_{k_2}(\eta)\pi_{k_3}(\eta)]\rangle = (2\pi)^3\delta^{(3)}\left(\sum k_i\right)\frac{1}{8}\frac{c_s^4\partial_t(M^4(t))}{M_{\rm Pl}^6\epsilon^3}\frac{(k_1\eta)^4}{k_1^3k_3^3} \ , \\ & i\langle [H_3(\eta), \pi_{k_1}(\eta)]\dot\pi_{k_2}(\eta)\pi_{k_3}(\eta)\rangle = \\ & \qquad = i\langle \dot\pi_{k_1}(\eta)[H_3(\eta), \pi_{k_2}(\eta)]\pi_{k_3}(\eta)\rangle = -(2\pi)^3\delta^{(3)}\left(\sum k_i\right)\frac{1}{4}\frac{c_s^4\partial_t(M^4(t))}{M_{\rm Pl}^6\epsilon^3}\frac{(k_1\eta)^4}{k_1^3k_3^3} \ .\end{aligned} \tag{4.125}$$

So adding these terms will give us $\langle \dot\pi_{k_1}(\eta)\dot\pi_{k_2}(\eta)\pi_{k_3}(\eta)\rangle$. Now, we are interested in $\langle \dot\zeta_{k_1}(\eta)\dot\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)\rangle$. But the ζ and π fields are related through $\zeta = -H\pi + H\dot\pi\pi$ [31][16], so we can write our desired correlator:

$$\begin{aligned}& \langle \dot\zeta_{k_1}(\eta)\dot\zeta_{k_2}(\eta)\zeta_{k_3}(\eta)\rangle = \\ & \quad (2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\left[\frac{3}{8}\frac{c_s^4\partial_t(M^4(t))H^3}{M_{\rm Pl}^6\epsilon^3}\frac{(k_1\eta)^4}{k_1^3k_3^3} + \frac{1}{H}\frac{\partial\langle\dot\zeta_{k_1}^2\rangle'}{\partial t}\langle\zeta_{k_3}^2\rangle'\right] \\ & \quad = -(2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\frac{c_s^3H^4}{4\epsilon M_{\rm Pl}^2}4\frac{(k_1\eta)^4}{k_1^3}\langle\zeta_{k_3}^2\rangle' = \\ & \quad = -(2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\frac{1}{k_1^3}\frac{d}{d\log k_1}\left(k_1^3\langle\dot\zeta_{k_1}^2\rangle'\right)\langle\zeta_{k_3}^2\rangle' \ , \quad k_3 \ll k_1 \approx k_2 \ ,\end{aligned} \tag{4.126}$$

where we have used that

$$\begin{aligned}\langle \zeta_{k_1}(\eta)\zeta_{k_2}(\eta)\rangle &= (2\pi)^3\delta^{(3)}(k_1+k_2)\frac{H^2(1+c_s^2k_1^2\eta^2)}{4c_s\epsilon M_{\rm Pl}^2k_1^3} \ , \\ \langle \dot\zeta_{k_1}(\eta)\dot\zeta_{k_2}(\eta)\rangle &= (2\pi)^3\delta^{(3)}(k_1+k_2)\frac{c_s^3H^4(k_1\eta)^4}{4\epsilon M_{\rm Pl}^2\,k_1^3} \ .\end{aligned} \tag{4.127}$$

Notice that the effect of the field redefinition is to remove the time derivatives associated to terms that do not depend explicitly on $k\eta$, such as c_s, so that the consistency condition works. This concludes our check of the consistency condition for modes inside the horizon, with time derivative operators.

As a last remark, we now discuss the relation between the contact terms that contributed to $\langle\dot\pi\dot\pi\pi\rangle$, and

[16]There are additional quadratic corrections to this expression, but they will give corrections to the three point function that are subleading when at least one of the modes is outside of the horizon or that are slow roll suppressed.

the contact terms arising from the quartic Hamiltonian $H_{4,3}$, which is discussed in the main text. They are playing the exact same role: accounting for the action of the time derivative on U_{int}. The results of this subsection can be cast in a form that makes this connection more manifest. We use here the notation “S, L” for short and long modes.

In the main text, we are computing a three point function of the following schematic form:

$$\sum_a \int^\eta d\eta' \left\langle \left(\frac{\delta L_3}{\delta \mathcal{D}^a \zeta^a}\right)_S (\eta')\zeta_L(\eta) \right\rangle \mathcal{D}^a G_\zeta(\eta',\eta) \sim \int^\eta d\eta' \left\langle i[H_{4,3}^{S,S,L,L}(\eta'),\zeta_L(\eta)]\zeta_L(\eta) \right\rangle + \dots \ , \tag{4.128}$$

where $\dots$ are contributions to the one-loop two point function coming from other diagrams. Now, we can recast the three point function $\langle \dot\pi\dot\pi\pi\rangle$ as:

$$\langle \dot\pi_S(\eta)\dot\pi_S(\eta)\pi_L(\eta)\rangle = \left\langle \left(\frac{\delta L_3^{\pi\dot\pi^2}}{\delta\dot\pi}\right)_S (\eta)\ \pi_L(\eta) \right\rangle \ , \tag{4.129}$$

and the contact term as:

$$i\langle [H_3(\eta),\pi_S(\eta)]\dot\pi_S(\eta)\pi_L(\eta)\rangle \sim -\left\langle \left(\frac{\delta L_3}{\delta P}\right)_{S,L}(\eta) \left(\frac{\delta L_3^{\pi\dot\pi^2}}{\delta\dot\pi}\right)_{S,L}(\eta) \right\rangle \ . \tag{4.130}$$

So we see that if the three point function involved the full Lagrangian, the contact term would be proportional to the squeezed quartic Hamiltonian, $\langle H_{4,3}^{S,S,L,L}\rangle$. As the one loop diagram involves a commutator instead of a tree level four point function, we need to insert the Green’s function on the left hand side of (4.128), thus seeing how both three point functions are affected by contact terms coming from $H_{4,3}$.

4.9 Appendix B: Local Anisotropic Universe

We aim here to provide the change of coordinates that locally takes us from the metric written in standard ζ gauge to a form that is locally of the form of (4.101). We need to work only at linear order in the long wavelength fluctuations ζ_L, because in the loop we integrate over the short wavelength fluctuations ζ_S. We start from the metric in ADM parametrization

$$ds^2 = -N^2dt^2 + \sum_{ij}\delta_{ij}a(t)^2e^{2\zeta}\left(dx^i+N^idt\right)\left(dx^j+N^jdt\right) \ , \tag{4.131}$$

where in this appendix we suspend the convention of summing over repeated indices. We can perform the following change of coordinates

$$x^i = e^{\beta_{ij}(t)}\tilde{x}^j + C^i(t) \ , \tag{4.132}$$

without introducing perturbations in the field that is driving inflation. Since we can work at linear order in the long modes, we can use rotational invariance to consider a long mode with wavenumber only along the $\hat{z}$ direction,

$$\zeta_L(\vec{x},t) = \mathrm{Re}\left[\tilde{\zeta}_0(t)\, e^{ikz}\right] \ , \qquad \mathrm{Re}\left[\tilde{\zeta}_0\right] = \zeta_0 \ . \tag{4.133}$$

It will be enough to take $\beta_{ij} = \beta(t)\delta_{i3}\delta_{j3}$. The only subtle point in this change of variables is that at linear order in the long modes, we have

$$\vec{N}_L = \left\{0, 0, \mathrm{Re}\left[i\frac{\dot{H}}{H^2}\frac{1}{k}\dot{\tilde{\zeta}}_0\, e^{ik\, e^{\beta}\tilde{z}}\right]\right\} + \mathcal{O}(k^2\tilde{\zeta}_0) \ , \tag{4.134}$$

which does not have a nice behavior for $k \to 0$. We need therefore to enforce that our change of coordinates not only fixes to zero N^i at one point, say the origin, $N^i_0 = 0$, but also it must set to zero $\partial_i N^j$ at the origin, $(\partial_i N^j)_0 = 0$. This will guarantee that neglected terms are suppressed in the limit $k \to 0$.

Simple algebra shows that the solution is

$$\vec{C} = \int dt \left\{0, 0, \frac{\dot{H}}{H^2}\frac{1}{k}\mathrm{Im}\left[\dot{\tilde{\zeta}}_0\right]\right\} \ , \tag{4.135}$$

$$\beta = \int dt\, \frac{\dot{H}}{H^2}\dot{\zeta}_0 \ . \tag{4.136}$$

The metric then takes the form of (4.101), with, in the new coordinates

$$\tilde{N}^i_0 = 0 \ , \qquad \left(\partial_j \tilde{N}^i\right)_0 = 0 \ , \qquad \tilde{\zeta}(\vec{\tilde{x}},t) = \zeta\left(\vec{x}(\vec{\tilde{x}},t), t\right) + \frac{2}{3}\int^t dt\, \frac{\dot{H}}{H^2}\dot{\zeta}_0(t) \ . \tag{4.137}$$

Notice that the short mode fluctuations ζ_S transform as a scalar under this change of coordinates

$$\tilde{\zeta}_S(\vec{\tilde{x}},t) = \zeta_S\left(\vec{x}(\vec{\tilde{x}},t), t\right) \ . \tag{4.138}$$

The same procedure can be clearly performed at non-linear level in ζ_L using a generic matrix β_{ij}, but this is not necessary for a one-loop calculation.

Chapter 5

Entanglement entropy in de Sitter space

5.1 Introduction

Entanglement entropy is a useful tool to characterize states with long range quantum order in condensed matter physics (see [108, 109] and references therein). It is also useful in quantum field theory to characterize the nature of the long range correlations that we have in the vacuum (see e.g. [110, 111, 112] and references therein).

We study the entanglement entropy for quantum field theories in de Sitter space. We choose the standard vacuum state [13, 12, 14] (the Euclidean, Hartle-Hawking/Bunch-Davies/Chernikov-Tagirov vacuum). We do not include dynamical gravity. In particular, the entropy we compute should not be confused with the gravitational de Sitter entropy.

Our motivation is to quantify the degree of superhorizon correlations that are generated by the cosmological expansion.

We consider a spherical surface that divides the spatial slice into the interior and exterior. We compute the entanglement entropy by tracing over the exterior. We take the size of this sphere, R, to be much bigger than the de Sitter radius, $R \gg R_{dS} = H^{-1}$, where H is Hubble's constant. Of course, for $R \ll R_{dS}$ we expect the same result as in flat space. If $R = R_{dS}$, then we would have the usual thermal density matrix in the static patch and its associated entropy[1]. As usual, the entanglement entropy has a UV divergent contribution which we ignore, since it comes from local physics. For very large spheres, and in four dimensions, the finite

[1]This can be regarded as a (UV divergent) $\mathcal{O}(G_N^0)$ correction to the gravitational entropy of de Sitter space [113].

piece has a term that goes like the area of the sphere and one that goes like the logarithm of the area. We focus on the coefficient of the logarithmic piece. In odd spacetime dimensions there are finite terms that go like positive powers of the area and a constant term. We then focus on the constant term.

We first calculate the entanglement entropy for a free massive scalar field. To determine it, one needs to find the density matrix from tracing out the degrees of freedom outside of the surface. When the spherical surface is taken all the way to the boundary of de Sitter space the problem develops an $SO(1,3)$ symmetry. This symmetry is very helpful for computing the density matrix and the associated entropy. Since we have the density matrix, it is also easy to compute the Rényi entropies.

We then study the entanglement entropy of field theories with a gravity dual. When the dual is known, we use the proposal of [114, 115] to calculate the entropy. It boils down to an extremal area problem. The answer for the entanglement entropy depends drastically on the properties of the gravity dual. In particular, if the gravity dual has a hyperbolic Friedman-Robertson-Walker spacetime inside, then there is a non-zero contribution at order N^2 for the "interesting" piece of the entanglement entropy. Otherwise, the order N^2 contribution vanishes.

This provides some further hints that the FRW region is indeed somehow contained in the field theory in de Sitter space [116]. More precisely, it should be contained in the superhorizon correlations of colored fields[2].

The chapter is organized as follows. In section 2, we discuss general features of entanglement entropy in de Sitter. In section 3, we consider a free scalar field and compute its entanglement entropy. In section 4, we write holographic duals of field theories in de Sitter, and compute the entropy of spherical surfaces in these theories. We end with a discussion. Some more technical details are presented in the appendices.

5.2 General features of entanglement entropy in de Sitter

Entanglement entropy is defined as follows [113]. At some given time slice, we consider a closed surface Σ which separates the slice into a region inside the surface and a region outside. In a local quantum field theory we expect to have an approximate decomposition of the Hilbert space into $H = H_{in} \times H_{out}$ where H_{in} contains modes localized inside the surface and H_{out} modes localized outside. One can then define a density matrix $\rho_{in} = \mathrm{Tr}_{H_{out}} |\psi\rangle\langle\psi|$ obtained by tracing over the outside Hilbert space. The entanglement

[2] A holographic calculation of the entanglement entropy associated to a quantum quench is presented in [117]. A quantum quench is the sudden perturbation of a pure state. The subsequent relaxation back to equilibrium can be understood in terms of the entanglement entropy of the quenched region. There, one has a contribution to the (time dependent) entropy coming from the region behind the horizon of the holographic dual.

entropy is the von Neumann entropy obtained from this density matrix:

$$S = -\mathrm{Tr}\rho_{in} \log \rho_{in} \tag{5.1}$$

5.2.1 Four dimensions

We consider de Sitter space in the flat slicing

$$ds^2 = \frac{1}{(H\eta)^2}(-d\eta^2 + dx_1^2 + dx_2^2 + dx_3^2) \tag{5.2}$$

where H is the Hubble scale and η is conformal time. We consider surfaces that sit at constant η slices. We consider a free, minimally coupled, scalar field of mass m in the usual vacuum state [13, 12, 14].

As in any quantum field theory, the entanglement entropy is UV divergent

$$S = S_{\mathrm{UV-divergent}} + S_{\mathrm{UV-finite}} \tag{5.3}$$

The UV divergencies are due to local effects and have the the form

$$S_{\mathrm{UV-divergent}} = c_1 \frac{A}{\epsilon^2} + \log(\epsilon H)(c_2 + c_3 A m^2 + c_4 A H^2) \tag{5.4}$$

where ϵ is the UV cutoff. The first term is the well known area contribution to the entropy [110, 111], coming from entanglement of particles close to the surface considered. The logarithmic terms involving c_2 and c_3 also arise in flat space. Finally, the last term involves the curvature of the bulk space[3]. All these UV divergent terms arise from local effects and their coefficients are the same as what we would have obtained in flat space. We have included H as a scale inside the logarithm. This is just an arbitrary definition, we could also have used m [118], when m is non-zero.

Our focus is on the UV finite terms that contain information about the long range correlations of the quantum state in de Sitter space. The entropy is invariant under the isometries of dS. This is true for both pieces in (5.3). In addition, we expect that the long distance part of the state becomes time independent. More precisely, the long range entanglement was established when these distances were subhorizon size. Once

[3]In de Sitter there is only one curvature scale, but in general we could write terms as

$$S_{\log \epsilon H} = \int_\Sigma \left(a R_{\mu\nu\rho\sigma} n_i^\mu n_i^\rho n_j^\nu n_j^\sigma + b R_{\mu\nu} n_i^\mu n_i^\nu + cR + d K_i^{\mu\nu} K_{i\mu\nu} + e K_{i\mu}^\mu K_{i\nu}^\nu + \cdots \right) \tag{5.5}$$

where K are the extrinsic curvatures and i, j label the two normal directions and $\mu, \nu, \cdots$ are spacetime indices The extrinsic curvatures also contribute to c_2 in (5.4). One could also write a term that depends on the intrinsic curvature of the surface, R_Σ, but the Gauss-Codazzi relations can be used to relate it to the other terms in (5.5).

they moved outside the horizon we do not expect to be able to modify this entanglement by subsequent evolution. Thus, we expect that the long range part of the entanglement entropy should be constant as we go to late times. So, if we fix a surface in comoving x coordinates in (5.2), and we keep this surface fixed as we move to late times, $\eta \to 0$, then we naively expect that the entanglement should be constant. This expectation is not quite right because new modes are coming in at late times. However, all these modes only give rise to entanglement at short distances in comoving coordinates. The effects of this entanglement could be written in a local fashion.

In conclusion, we expect that the UV-finite piece of the entropy is given by

$$S_{\mathrm{UV-Finite}} = c_5 AH^2 + \frac{c_6}{2}\log(AH^2) + \text{finite} = c_5\frac{A_c}{\eta^2} + c_6\log\eta + \text{finite} \tag{5.6}$$

where A is the proper area of the surface and A_c is the area in comoving coordinates ($A = \frac{A_c}{H^2\eta^2}$). The finite piece is a bit ambiguous due to the presence of the logarithmic term.

The coefficient of the logarithmic term, c_6, contains information about the long range entanglement of the state. This term looks similar to the UV divergent logarithmic term in (5.4), but they should not be confused with each other. If we had a conformal field theory in de Sitter they would be equal. However, in a non-conformal theory they are not equal ($c_6 \neq c_2$). For general surfaces, the coefficient of the logarithm will depend on two combinations of the extrinsic curvature of the surface in comoving coordinates. For simplicity we consider a sphere here[4]. This general form of the entropy, (5.6), will be confirmed by our explicit computations below.

We define the "interesting" part of the entropy to be the coefficient of the logarithm, $S_{\mathrm{intr}} \equiv c_6$. The *UV*-finite area term, with coefficient c_5, though physically interesting, is not easily calculable with our method. It receives contributions from the entanglement at distances of a few Hubble radii from the entangling surface. It would be nice to find a way to isolate this contribution and compute c_5 exactly. We could only do that in the case where the theory has a gravity dual.

5.2.2 Three dimensions

For three dimensional de Sitter space we can have a similar discussion.

$$\begin{aligned} S &= d_1\frac{A}{\epsilon} + S_{\mathrm{UV-finite}} \\ S_{\mathrm{UV-finite}} &= d_2 AH + d_3 = d_2\frac{A_c}{\eta} + d_3 \end{aligned} \tag{5.7}$$

[4]It is enough to do the computation for another surface, say a cylinder, to determine the second coefficient and have a result that is valid for general surfaces [119]. In other words, for a general surface we have $c_6 = f_1\int K_{ab}K_{ab} + f_2\int (K_{aa})^2$ where $f_1,\ f_2$ are some constants and K_{ab} is the extrinsic curvature of the surface within the spatial slice.

Here there is no logarithmic term. The interesting term is d_3 which is the finite piece. So we define $S_{\text{intr}} \equiv d_3$.

A similar discussion exists in all other dimensions. For even spacetime dimensions the interesting term is the logarithmic one and for odd dimensions it is the constant. One can isolate these interesting terms by taking appropriate derivatives with respect to the physical area, as done in [120] in a similar context[5].

Note that we are considering quantum fields in a fixed spacetime. We have no gravity. And we are making no contact with the gravitational de Sitter entropy which is the area of the horizon in Planck units.

5.3 Entanglement entropy for a free massive scalar field in de Sitter

Here we compute the entropy of a free massive scalar field for a spherical entangling surface.

5.3.1 Setup of the problem

Consider, in flat coordinates, a spherical surface S^2 defined by $x_1^2 + x_2^2 + x_3^2 = R_c^2$. We consider $R_c \gg \eta$. This means that the surface is much bigger than the horizon.

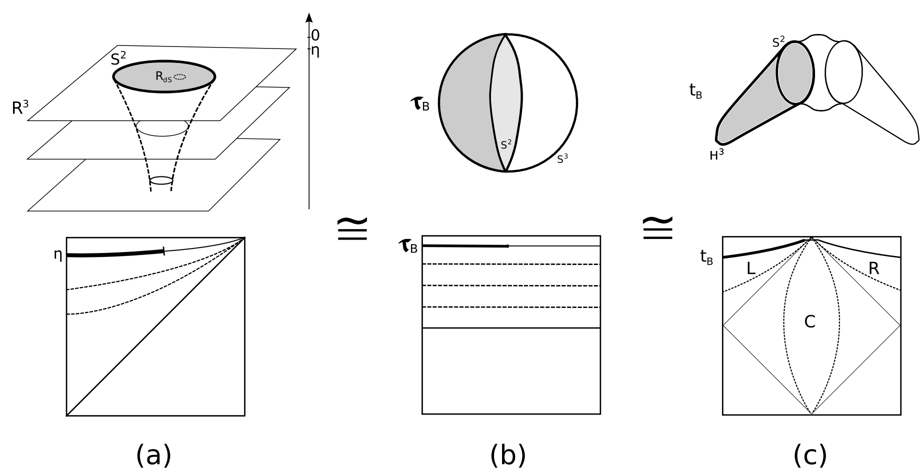

Figure 5.1: Setup of the problem: (a) We consider a sphere with radius much greater than the horizon size, at late conformal time η, in flat slices. (b) This problem can be mapped to half of a 3-sphere S^3, also with boundary S^2, but now the equator, at late global time τ_B. (c) We can also describe this problem using hyperbolic slices. The interior of the sphere maps to the "left" (L) hyperbolic slice. The Penrose diagrams for all situations are depicted below the geometric sketches.

If we could neglect the η dependent terms, we can take the limit $\eta \to 0$, keeping R_c fixed. This then becomes a surface on the boundary. This surface is left invariant by an $SO(1,3)$ subgroup of the $SO(1,4)$ de Sitter isometry group. We expect that the coefficient of the logarithmic term that we discussed above is

[5] See formula (1.1) of [120].

also invariant under this group. It is therefore convenient to choose a coordinate system where $SO(1,3)$ is realized more manifestly. This is done in two steps. First we can consider de Sitter in global coordinates, where the equal time slices are three-spheres. Then we can choose the entangling surface to be the two-sphere equator of the three-sphere. In fact, at $\eta = 0$, we can certainly map any two sphere on the boundary of de Sitter to the equator of S^3 by a de Sitter isometry. Finally, to regularize this problem we can then move back the two sphere to a very late fixed global time surface.

We can then choose a coordinate system where the SO(1,3) symmetry is realized geometrically in a simple way. Namely, this SO(1,3) is the symmetry group acting on hyperbolic slices in some coordinate system that we describe below.

5.3.2 Wavefunctions of free fields in hyperbolic slices and the Euclidean vacuum

The hyperbolic/open slicing of de Sitter space was studied in detail in [121, 122]. It can be obtained by analytic continuation of the sphere S^4 metric, sliced by S^3s. The S^4 is described in embedding coordinates by $X_1^2 + ... + X_5^2 = H^{-2}$. The coordinates are parametrized by angles in the following way:

$$X_5 = H^{-1} \cos\tau_E \cos\rho_E, \qquad X_4 = H^{-1} \sin\tau_E, \qquad X_{1,2,3} = H^{-1} \cos\tau_E \sin\rho_E n_{1,2,3} \tag{5.8}$$

where n_i are the components of a unit vector in R^3. The metric in Euclidean signature is given by:

$$ds_E^2 = H^{-2}(d\tau_E^2 + \cos^2\tau_E(d\rho_E^2 + \sin^2\rho_E \; d\Omega_2^2)) \tag{5.9}$$

We analytically continue $X_5 \to iX_0$. Then the Lorentzian manifold is divided in three parts, related to the Euclidean coordinates by:

$$\begin{aligned}
R: &\quad \begin{cases} \tau_E = \frac{\pi}{2} - it_R & t_R \geq 0 \\ \rho_E = -ir_R & r_R \geq 0 \end{cases} \\
C: &\quad \begin{cases} \tau_E = \tau_C & -\pi/2 \leq t_C \leq \pi/2 \\ \rho_E = \frac{\pi}{2} - ir_C & -\infty < r_C < \infty \end{cases} \\
L: &\quad \begin{cases} \tau_E = -\frac{\pi}{2} + it_L & t_L \geq 0 \\ \rho_E = -ir_L & r_L \geq 0 \end{cases}
\end{aligned} \tag{5.10}$$

The metric in each region is given by:

$$
\begin{aligned}
ds_R^2 &= H^{-2}(-dt_R^2 + \sinh^2 t_R(dr_R^2 + \sinh^2 r_R d\Omega_2^2)) \\
ds_C^2 &= H^{-2}(dt_C^2 + \cos^2 t_C(-dr_C^2 + \cosh^2 r_C d\Omega_2^2)) \\
ds_L^2 &= H^{-2}(-dt_L^2 + \sinh^2 t_L(dr_L^2 + \sinh^2 r_L d\Omega_2^2))
\end{aligned}
\tag{5.11}
$$

We now consider a minimally coupled[6] massive scalar field in dS_4, with action given by $S = \frac{1}{2}\int \sqrt{-g}(-(\nabla\phi)^2 - m^2\phi^2)$. The equations of motion for the mode functions in the R or L regions are

$$
\left[\frac{1}{\sinh^3 t}\frac{\partial}{\partial t}\sinh^3 t\frac{\partial}{\partial t} - \frac{1}{\sinh^2 t}\mathbf{L^2_{H^3}} + \frac{9}{4} - \nu^2\right] u(t, r, \Omega) = 0 \tag{5.12}
$$

Where $\mathbf{L^2_{H^3}}$ is the Laplacian in the unit hyperboloid, and the parameter ν is

$$
\nu = \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \tag{5.13}
$$

When $\nu = \frac{1}{2}$ (or $\frac{m^2}{H^2} = 2$) we have a conformally coupled massless scalar. In this case we should recover the flat space answer for the entanglement entropy, since de Sitter is conformally flat. We will consider first situations where $\frac{m^2}{H^2} \geq 2$, so that $0 \leq \nu \leq 1/2$ or ν imaginary. The minimally coupled massless case corresponds to $\nu = 3/2$. We will later comment on the low mass region, $\frac{m^2}{H^2} < 2$ or $1/2 < \nu \leq 3/2$.

The wavefunctions are labeled by quantum numbers corresponding to the Casimir on H^3 and angular momentum on S^2:

$$
u_{plm} \sim \frac{H}{\sinh t}\chi_p(t) Y_{plm}(r, \Omega_2) \ , \qquad -\mathbf{L_{H^3}} Y_{plm} = (1 + p^2) Y_{plm} \tag{5.14}
$$

The Y_{plm} are eigenfunctions on the hyperboloid, analogous to the standard spherical harmonics. Their expressions can be found in [122].

The time dependence (other than the $1/\sinh t$ factor) is contained in the functions $\chi_p(t)$. The equation of motion (5.12) is a Legendre equation and the solutions are given in terms of Legendre functions $P_a^b(x)$. In order to pick the "positive frequency" wavefunctions corresponding to the Euclidean vacuum we need to demand that they are analytic when they are continued to the lower hemisphere. These wavefunctions have

[6]If we had a coupling to the scalar curvature $\xi R\phi^2$, we can simply shift the mass $m_{eff}^2 = m^2 + 6\xi H^2$ and consider the minimally coupled one.

support on both the Left and Right regions. This gives [122]

$$\chi_{p,\sigma} = \begin{cases} \dfrac{1}{2\sinh \pi p}\left(\dfrac{e^{\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu + ip + 1/2)} P^{ip}_{\nu-1/2}(\cosh t_R) - \dfrac{e^{-\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu - ip + 1/2)} P^{-ip}_{\nu-1/2}(\cosh t_R)\right) \\ \dfrac{\sigma}{2\sinh \pi p}\left(\dfrac{e^{\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu + ip + 1/2)} P^{ip}_{\nu-1/2}(\cosh t_L) - \dfrac{e^{-\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu - ip + 1/2)} P^{-ip}_{\nu-1/2}(\cosh t_L)\right) \end{cases} \tag{5.15}$$

The index σ can take the values ± 1. For each σ the top line gives the function on the R hyperboloid and the bottom line gives the value of the function on the L hyperboloid. There are two solutions (two values of σ) because we started from two hyperboloids.

The field operator is written in terms of these mode functions as

$$\hat{\phi}(x) = \int dp \sum_{\sigma,l,m} (a_{\sigma plm} u_{\sigma plm}(x) + a^{\dagger}_{\sigma plm} \overline{u}_{\sigma plm}(x)) \tag{5.16}$$

To trace out the degrees of freedom in, say, the R space, we change basis to functions that have support on either the R or L regions. It does not matter which functions we choose to describe the Hilbert space. The crucial simplification of this coordinate system is that the entangling surface, when taken to the de Sitter boundary, preserves all the isometries of the H^3 slices. This implies that the entanglement is diagonal in the p, l, m indices since these are all eigenvalues of some symmetry generator. Thus, to compute this entanglement we only need to look at the analytic properties of (5.15) for each value of p.

Let us first consider the case that ν is real. For the R region we take basis functions equal to the Legendre functions $P^{ip}_{\nu-1/2}(\cosh t_R)$ and $P^{-ip}_{\nu-1/2}(\cosh t_R)$, and zero in the L region. These are the positive and negative frequency wavefunctions in the R region. We do the same in the L region. These should be properly normalized with respect to the Klein-Gordon norm, which would yield a normalization factor N_p. We can write the original mode functions, (5.15), in terms of these new ones in matricial form:

$$\begin{gathered} \begin{cases} \chi^\sigma = N_p^{-1} \sum_{q=R,L} (\alpha^\sigma_q P^q + \beta^\sigma_q \overline{P}^q) \\ \overline{\chi}^\sigma = N_p^{-1} \sum_{q=R,L} (\overline{\beta}^\sigma_q P^q + \overline{\alpha}^\sigma_q \bar{P}^q) \end{cases} \Rightarrow \chi^I = M^I_J P^J N_p^{-1} \\ \sigma = \pm 1, \quad P^{R,L} \equiv P^{ip}_{\nu-1/2}(\cosh t_{R,L}), \quad \chi^I \equiv \begin{pmatrix} \chi^\sigma \\ \overline{\chi}^\sigma \end{pmatrix} \end{gathered} \tag{5.17}$$

The capital indices (I, J) run from 1 to 4, as we are grouping both the χ_σ and $\overline{\chi}_\sigma$. The coefficients α and β are simply the terms multiplying the corresponding P functions in (5.15), see appendix A for their explicit values. As the field operator should be the same under this change of basis, then it follows that:

$$
\begin{aligned}
&\phi = a_I \chi^I = b_J P^J N_p^{-1} \Rightarrow a_J = b_I (M^{-1})^I_J \\
&M = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} \gamma & \delta \\ \overline{\delta} & \overline{\gamma} \end{pmatrix} \Rightarrow \quad a_\sigma = \sum_{q=R,L} \gamma_{q\sigma} b_q + \overline{\delta}_{q\sigma} b_q^\dagger
\end{aligned}
\tag{5.18}
$$

Here $a^I = (a_\sigma, a_\sigma^\dagger)$, $b^J = (b_{L,R}, b_{L,R}^\dagger)$, and $P^J = (P_{L,R}, \bar{P}_{L,R})$. M is a 2×2 matrix whose elements are 2×2 matrices. The expression for M^{-1} is the definition of δ, γ, etc. The vacuum is defined so that $a_\sigma |\Psi\rangle = 0$. We want to write $|\Psi\rangle$ in terms of the $b_{R,L}$ oscillators and the vacua associated to each of these oscillators, $b_R|R\rangle = 0$ and $b_L|L\rangle = 0$. As we are dealing with free fields, their Gaussian structure suggests the ansatz

$$
|\Psi\rangle = e^{\frac{1}{2}\sum_{i,j=R,L} m_{ij} b_i^\dagger b_j^\dagger} |R\rangle |L\rangle \tag{5.19}
$$

and one can solve for m_{ij} demanding that $a_\sigma|\Psi\rangle = 0$. This gives

$$
m_{ij}\gamma_{j\sigma} + \overline{\delta}_{i\sigma} = 0 \Rightarrow m_{ij} = -\overline{\delta}_{i\sigma}(\gamma^{-1})_{\sigma j} \tag{5.20}
$$

Using the expressions in (5.15) (see appendix A) we find for m:

$$
m_{ij} = e^{i\theta} \frac{\sqrt{2}e^{-p\pi}}{\sqrt{\cosh 2\pi p + \cos 2\pi\nu}} \begin{pmatrix} \cos\pi\nu & i\sinh p\pi \\ i\sinh p\pi & \cos\pi\nu \end{pmatrix} \tag{5.21}
$$

Where θ is an unimportant phase factor, which can be absorbed in the definition of the $b^\dagger$ oscillators. In m_{ij} the normalization factors N_p drop out, so they never need to be computed.

The expression (5.19), with (5.21), needs to be simplified more before we can easily trace out the R degrees of freedom. We would like to introduce new oscillators c_L and c_R (and their adjoints) so that the original state Ψ has the form

$$
|\Psi\rangle = e^{\gamma c_R^\dagger c_L^\dagger} |R\rangle' |L\rangle' \tag{5.22}
$$

where $|R\rangle'|L\rangle'$ are annihilated by c_R, c_L. The details on the transformation are in appendix A. Here we state the result. The b's and c's are related by:

$$
\begin{aligned}
&c_R = u b_R + v b_R^\dagger \\
&c_L = \bar{u} b_L + \bar{v} b_L^\dagger, \quad |u|^2 - |v|^2 = 1
\end{aligned}
\tag{5.23}
$$

Requiring that $c_R|\Psi\rangle = \gamma c_L^\dagger|\Psi\rangle$ and $c_L|\Psi\rangle = \gamma c_R^\dagger|\Psi\rangle$ imposes constraints on u and v. The system of equations has a solution with γ given by

$$\gamma = i\frac{\sqrt{2}}{\sqrt{\cosh 2\pi p + \cos 2\pi\nu} + \sqrt{\cosh 2\pi p + \cos 2\pi\nu + 2}} \tag{5.24}$$

We have considered the case of $0 \le \nu \le 1/2$. For ν imaginary, (5.24) is analytic under the substitution $\nu \to i\nu$, which corresponds to substituting $\cos 2\pi\nu \to \cosh 2\pi i\nu$, so (5.24) is also valid for this range of masses. One can check directly, by redoing all the steps in the above derivation, that the same final answer is obtained if we had assumed that ν was purely imaginary.

5.3.3 The density matrix

The full vacuum state is the product of the vacuum state for each oscillator. Each oscillator is labelled by p, l, m. For each oscillator we can write the vacuum state as in (5.22). Expanding (5.22) and tracing over the right Hilbert space we get

$$\rho_{p,l,m} = Tr_{H_R}(|\Psi\rangle\langle\Psi|) \propto \sum_{n=0}^{\infty} |\gamma_p|^2 |n; p, l, m\rangle\langle n; p, l, m| \tag{5.25}$$

So, for given quantum numbers, the density matrix is diagonal. It takes the form $\rho_L(p) = (1-|\gamma_p|^2)\mathrm{diag}(1, |\gamma_p|^2, |\gamma_p|^4, \cdots)$, normalized to $\mathrm{Tr}\rho_L = 1$. The full density matrix is simply the product of the density matrix for each value of p, l, m. This reflects the fact that there is no entanglement among states with different $SO(1,3)$ quantum numbers. The density matrix for the conformally coupled case was computed before in [123].

Here, one can write the resulting density matrix as $\rho_L = e^{-\beta\mathcal{H}_{ent}}$ with $\mathcal{H}_{ent}$ called the entanglement hamiltonian. Here it seems natural to choose $\beta = 2\pi$ as the inverse temperature of dS. Because the density matrix is diagonal, the entanglement Hamiltonian should be that of a gas of free particles, with the energy of each excitation a function of the H^3 Casimir and the mass of the scalar field. This does not appear to be related to any ordinary dynamical Hamiltonian in de Sitter. In other words, take $\rho_L \propto \mathrm{diag}(1, |\gamma_p|^2, |\gamma_p|^4, ...)$ then the entanglement Hamiltonian for each particles is $H_p = E_p c_p^\dagger c_p$, with $E_p = -\frac{1}{2\pi}\log|\gamma_p|^2$. For the conformally coupled scalar then $E_p = p$ and we have the entropy of a free gas in H^3. In other words, in the conformal case the entanglement Hamiltonian coincides with the Hamiltonian of the field theory on $R \times H^3$ [124, 125].

5.3.4 Computing the Entropy

With the density matrix (5.25) we can calculate the entropy associated to each particular set of $SO(1,3)$ quantum numbers

$$S(p,\nu) = -\text{Tr}\rho_L(p)\log\rho_L(p) = -\log(1-|\gamma_p|^2) - \frac{|\gamma_p|^2}{1-|\gamma_p|^2}\log|\gamma_p|^2 \tag{5.26}$$

The final entropy is then computed by summing (5.26) over all the states. This sum translates into an integral over p and a volume integral over the hyperboloid. In other words, we use the density of states on the hyperboloid:

$$S(\nu) = V_{H^3}\int dp\mathcal{D}(p)S(p,\nu) \tag{5.27}$$

The density of states for radial functions on the hyperboloid is known for any dimensions [126]. For example, for H^3, $\mathcal{D}(p) = \frac{p^2}{2\pi^2}$. Here V_{H^3} is the volume of the hyperboloid. This is of course infinite. This infinity is arising because we are taking the entangling surface all the way to $\eta = 0$. We can regularize the volume with a large radial cutoff in H^3. This should roughly correspond to putting the entangling surface at a finite time. Since we are only interested in the coefficient of the logarithm, the precise way we do the cutoff at large volumes should not matter. The volume of a unit size H^3 for radius less that r_c is given by

$$V_{H^3} = V_{S^2}\int_0^{r_c} dr\sinh^2 r \sim 4\pi\left(\frac{e^{2r_c}}{8} - \frac{r_c}{2}\right) \tag{5.28}$$

The first term goes like the area of the entangling surface. The second one involves the logarithm of this area. We can also identify $r_c \to -\log\eta$. This can be understood more precisely as follows. If we fix a large t_L and we go to large r_L, then we see from (5.8)(5.10) that the corresponding surface would be at an $\eta \propto e^{-r_L}$, for large r_L. Thus, we can confidently extract the coefficient c_6 in (5.6). For such purposes we can define $V_{H^3,\text{reg}} = 2\pi$. The leading area term , proportional to e^{2r_c} depends on the details of the matching of this IR cutoff to the proper UV cutoff. These details can change its coefficient. In appendix B, we compute in detail the regularized volume of the hyperboloid in any dimension.

Thus, the final answer for the logarithmic term of the entanglement entropy is

$$\begin{gathered} S = c_6\log\eta + \text{other terms} \\ S_{\text{intr}} \equiv c_6 = \frac{1}{\pi}\int_0^\infty dp\, p^2 S(p,\nu) \end{gathered} \tag{5.29}$$

with $S(p,\nu)$ given in (5.26), (5.24). This is plotted in figure 5.2.

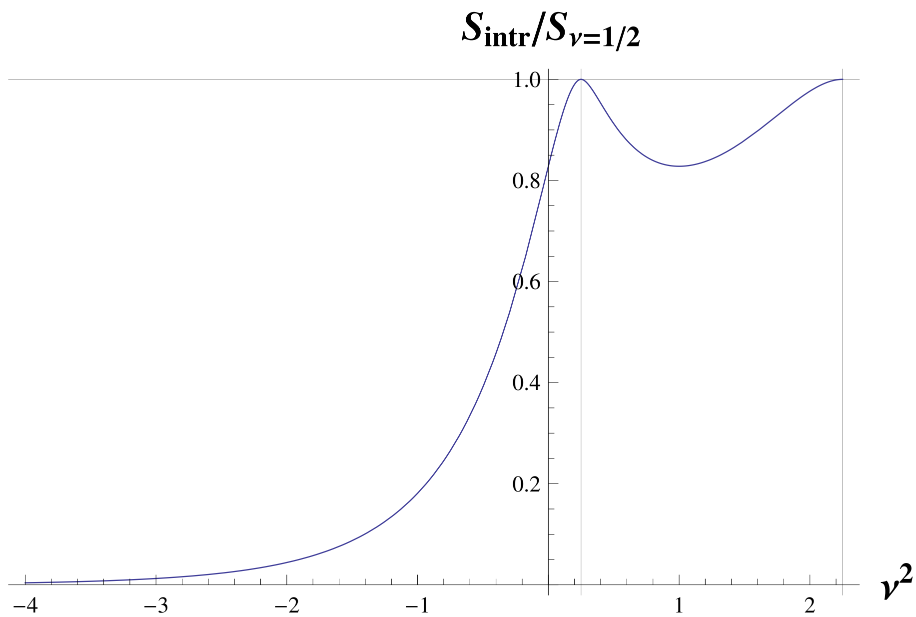

Figure 5.2: Plot of the entropy $S_{intr}/S_{intr,\nu=1/2}$ of the free scalar field, normalized to the conformally coupled scalar, versus its mass parameter squared. The minimally coupled massless case corresponds to $\nu^2 = 9/4$, the conformally coupled scalar to $\nu^2 = 1/4$ and for large mass (negative ν^2) the entropy has a decaying exponential behavior.

5.3.5 Extension to general dimensions

These results can be easily extended to a real massive scalar field in any number of dimensions D. Again we have hyperbolic H^{D-1} slices and the decomposition of the time dependent part of the wavefunctions is identical, provided that we replace ν by the corresponding expression in D dimensions

$$\nu^2 = \frac{(D-1)^2}{4} - \frac{m^2}{H^2} \tag{5.30}$$

Then the whole computation is identical and we get exactly the same function $S(p,\nu)$ for each mode. The final result involves integrating with the right density of states for hyperboloids in $D-1$ dimensions which is [126]

$$\begin{aligned}
&\mathcal{D}_2(p) = \frac{p}{2\pi}\,\mathrm{th}\,\pi p, \quad \mathcal{D}_3(p) = \frac{p^2}{2\pi^2} \\
&\mathcal{D}_{D-1}(p) = \frac{p^2 + \left(\frac{D-4}{2}\right)^2}{2\pi(D-3)}\mathcal{D}_{D-3}(p) = \frac{2}{(4\pi)^{\frac{D-1}{2}}\Gamma(\frac{D-1}{2})}\frac{|\Gamma(ip+\frac{D}{2}-1)|^2}{|\Gamma(ip)|^2}, \\
&-\mathbf{L}_{\mathbf{H}^{\mathbf{D-1}}}Y_p = \left(p^2 + \left(\frac{D-2}{2}\right)^2\right)Y_p
\end{aligned} \tag{5.31}$$

We also need to define the regularized volumes of hyperbolic space in $D-1$ dimensions. They are related to the volume of spheres

$$V_{H^{D-1},\mathrm{reg}} = \begin{cases} \dfrac{(-1)^{\frac{D}{2}} V_{S^{D-1}}}{\pi} & \mathrm{D}even \\ \dfrac{(-1)^{\frac{D-1}{2}} V_{S^{D-1}}}{2} & \mathrm{D}odd \end{cases} , \qquad V_{S^{D-1}} = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})} \tag{5.32}$$

When D is even, we defined this regularized volume as minus the coefficient of $\log \eta$. When D is odd, we defined it to be the finite part after we extract the divergent terms. A derivation of these volume formulas is given in appendix B. Then the final expression for any dimension is

$$S_{\mathrm{intr}} = V_{H_{D-1},\mathrm{reg}} \int_0^\infty dp \mathcal{D}_{D-1}(p) S(p,\nu) \tag{5.33}$$

with the expressions in (5.32), (5.31), (5.26), (5.24), (5.30). We have defined S_{reg} as

$$\begin{aligned} S =& S_{\mathrm{intr}} \log \eta + \cdots \qquad \text{for} \quad D \quad \text{even} \\ S =& S_{\mathrm{intr}} + \cdots \qquad\qquad \text{for} \quad D \quad \text{odd} \end{aligned} \tag{5.34}$$

where the dots denote terms that are UV divergent or that go like powers of η for small η.

5.3.6 Rényi Entropies

We can also use the density matrix to compute the Rényi entropies, defined as:

$$S_q = \frac{1}{1-q} \log \mathrm{Tr} \rho^q, \quad q > 0 \tag{5.35}$$

We first calculate the Rényi entropy associated to each $SO(1,3)$ quantum number. It is given by:

$$S_q(p,\nu) = \frac{q}{1-q} \log(1 - |\gamma_p|^2) - \frac{1}{1-q} \log(1 - |\gamma_p|^{2q}) \tag{5.36}$$

Then, just like we did for the entanglement entropy (which corresponds to $q \to 1$), one integrates (5.36) with the density of states for $D-1$ hyperboloids:

$$S_{q,\mathrm{intr}} = V_{H^{D-1},\mathrm{reg}} \int_0^\infty dp \mathcal{D}_{D-1}(p) S_q(p,\nu) \tag{5.37}$$

With $S_{q,\mathrm{intr}}$ being the finite term in the entropy, for odd dimensions, and the term that multiplies $\log \eta$, for even dimensions.

5.3.7 Negativity

A measure of the *quantum* entanglement between two regions is the so called negativity of a system[127, 128, 129]. It is defined as follows. Take the density matrix $\rho = |\Psi\rangle\langle\Psi|$ of a system (in our case, $R \cup L$). Note that no tracing is made to write ρ, so it is the density matrix of a pure state for the case we consider.

If we write a basis of states for the R and L spaces and denote them as R^i and L^i, we define the transposed matrix ρ^{T_L} as follows:

$$\langle R^i L^j|\rho^{T_L}|R^k L^m\rangle = \langle R^i L^m|\rho|R^k L^j\rangle \tag{5.38}$$

Then, the negativity is defined as $\mathcal{E} \equiv \ln \mathrm{Tr}|\rho^{T_L}|$, where $\mathrm{Tr}|\rho^{T_L}|$ is the sum of the absolute value of the eigenvalues of ρ^{T_L}. For a pure state [127] , the negativity is equal to the $q = 1/2$ Rényi entropy from tracing out the L states. So it follows that:

$$\mathcal{E}_{\mathrm{intr}} = V_{H^{D-1},\mathrm{reg}} \int_0^\infty dp \mathcal{D}_{D-1}(p) S_{1/2}(p,\nu) \tag{5.39}$$

With $\mathcal{E}_{\mathrm{intr}}$ being the finite term in the negativity, for odd dimensions, and the term that multiplies $\log\eta$, for even dimensions. $S_{1/2}(p,\nu)$ is given by (5.36) with $q = 1/2$.

5.3.8 Consistency checks: conformally coupled scalar and large mass limit

As a consistency check of (5.27), we analyze the cases of the conformally coupled scalar, and of masses much bigger than the Hubble scale.

Conformally coupled scalar

For the conformally coupled scalar in any dimensions we need to set the mass parameter to $\nu = 1/2$. The entropy should be the same as that of flat space. For a spherical entangling surface, the universal term is $g_e \log \epsilon_{UV}/R$ for even dimensions, and is a finite number, g_o, for odd dimensions [124, 125]. The only difference here is that we are following a surface of constant comoving area, so its radius is given by $R = R_c/(H\eta)$. So, one sees that the term that goes like $\log\eta$, in even dimensions, has the exact same origin as the UV divergent one; in particular, we expect $c_6 = g_e$ for the four dimensional case, and g_o is the finite piece in the three dimensional case.

Four dimensions:

The entropy is given by (5.29)

$$S_{\mathrm{intr}} = \frac{1}{\pi}\int_0^\infty dp \frac{p^2}{2\pi^2} S\left(p, \frac{1}{2}\right) = \frac{1}{90} \tag{5.40}$$

This indeed coincides with the coefficient of the logarithm in the flat space result [124].

Three dimensions:

The entropy is given by:

$$\begin{aligned} S_{\text{intr}} =& V_{H^2,reg} \int \frac{d^2p}{(2\pi)^2} \operatorname{th} \pi p S\left(p, \frac{1}{2}\right) = -\int_0^\infty p dp \operatorname{th} \pi p S\left(p, \frac{1}{2}\right) = \\ =& \frac{3\zeta(3)}{16\pi^2} - \frac{\log(2)}{8} \end{aligned} \tag{5.41}$$

This corresponds to half the value computed in [130] , because there a complex scalar is considered, and also matches to half the value of the Barnes functions in [125].

Conformally coupled scalar in other dimensions

For even dimensions, S_{intr} has been reported for dimensions up to $d = 14$ in [124], and for odd dimensions, numerical values were reported up to $d = 11$ [125]. Using (5.33) we checked that the entropies agree for all the results in [124, 125].

Large mass limit

Here we show the behavior of the entanglement entropy for very large mass, in three and four dimensions. The eigenvalues of the density matrix as a function of the SO(1,3) Casimir are given in terms of (5.24). For large mass, there are basically two regimes, $0 < p < |\nu|$ and $p > |\nu|$

$$|\gamma_p|^2 = \begin{cases} e^{-2\pi|\nu|} & 0 < p < |\nu| \\ e^{-2\pi p} & |\nu| < p \end{cases} \tag{5.42}$$

In this regime we can approximate $|\gamma| \ll 1$ everywhere and the entropy per mode is

$$S(p) \sim -|\gamma_p|^2 \log |\gamma_p|^2 \tag{5.43}$$

Most of the contribution will come from the region $p < |\nu|$, up to $1/\nu$ corrections. This gives

$$\frac{S_{\text{intr}}}{V_{H^{D-1},\text{reg}}} \sim \int_0^\nu dp \mathcal{D}(p) S(p) \sim (2\pi\nu e^{-2\pi\nu}) \int_0^\nu dp \mathcal{D}(p) = \begin{cases} \frac{\nu^3}{2} e^{-2\pi\nu} & \text{d=3} \\ \frac{\nu^4}{3\pi} e^{-2\pi\nu} & \text{d=4} \end{cases} \tag{5.44}$$

which is accurate up to multiplicative factors of order $(1 + \mathcal{O}(1/\nu))$.

5.3.9 Low mass range: $1/2 < \nu \leq 3/2$

In this low mass range the expansion of the field involves an extra mode besides the ones we discussed so far [122]. This is a mode with a special value of p. Namely $p = i(\nu - \frac{1}{2})$. This mode is necessary because all the other modes, which have real p, have wavefunctions whose leading asymptotics vanish on the S^2 equator of the S^3 future boundary. This mode has a different value for the Casimir (a different value of p) than all other modes, so it cannot be entangled with them. So we think that this mode does not contribute to the long range entanglement. It would be nice to verify this more explicitly.

Note that we can analytically continue the answer we obtained for $\nu \leq 1/2$ to larger values. We obtain an answer which has no obvious problems, so we suspect that this is the right answer for the entanglement entropy, even in this low mass range. The full result is plotted in Figure 5.2, and we find that for $\nu = 3/2$, which is the massless scalar, we get exactly the same result as for a conformally coupled scalar.

5.4 Entanglement entropy from gravity duals.

After studying free field theories in the previous section, we now consider strongly coupled field theories in de Sitter. We consider theories that have a gravity dual. Gauge gravity duality in de Sitter was studied in [131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145], and references there in. When a field theory has a gravity dual, it was proposed in [114] that the entanglement entropy is proportional to the area of a minimal surface that ends on the entangling surface at the AdS boundary. This formula has passed many consistency checks. It is certainly valid in simple cases such as spherical entangling surfaces [146]. Here we are considering a time dependent situation. It is then natural to use extremal surfaces but now in the full time dependent geometry [115]. This extremality condition tells us how the surface moves in the time direction as it goes into the bulk.

First, we study a CFT in de Sitter. This is a trivial case since de Sitter is conformally flat, so we can go to a conformal frame that is not time dependent and obtain the answer [114, 147]. Nevertheless we will describe it in some detail because it is useful as a stepping stone for the non-conformal case. We then consider non-conformal field theories in some generality. We relegate to appendix C the discussion of a special case corresponding to a non-conformal field theory in four dimensions that comes from compactifying a five dimensional conformal field theory on a circle.

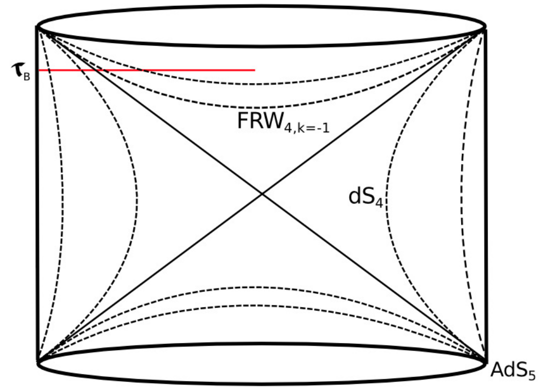

Figure 5.3: The gravity dual of a CFT living on dS_4. We slice AdS_5 with dS_4 slices. Inside the horizon we have an FRW universe with H^4 slices. The minimal surface is an H^3 that lies on a constant global time surface. The red line represents the radial direction of this H^3, and the S^2 shrinks smoothly at the tip.

5.4.1 Conformal field theories in de Sitter

As the field theory is defined in dS_4, it is convenient to choose a dS_4 slicing of AdS_5. These slices cover only part of the spacetime, see Figure 5.3. They cover the region outside the lightcone of a point in the bulk. The interior region of this lightcone can be viewed as an FRW cosmology with hyperbolic spatial slices.

We then introduce the following coordinate systems:

1. Embedding coordinates:

$$\begin{aligned} &-Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_4^2 = -1 \\ &ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2 + ... + dY_4^2 \end{aligned} \tag{5.45}$$

2. dS_4 and FRW coordinates:

   (a) dS slices:

$$\begin{aligned} &Y_{-1} = \cosh\rho, \quad Y_0 = \sinh\rho\sinh\tau, \quad Y_i = \sinh\rho\cosh\tau n_i \\ &ds^2 = d\rho^2 + \sinh^2\rho(-d\tau^2 + \cosh^2\tau(d\alpha^2 + \cos^2\alpha d\Omega_2)) \end{aligned} \tag{5.46}$$

   (b) FRW slices. We substitute $\rho = i\sigma$ and $\tau = -i\frac{\pi}{2} + \chi$ in (5.46).

$$\begin{aligned} &Y_{-1} = \cos\sigma, \quad Y_0 = \sin\sigma\cosh\chi, \quad Y_i = \sin\sigma\sinh\chi n_i \\ &ds^2 = -d\sigma^2 + \sin^2\sigma(d\chi^2 + \sinh^2\chi(d\alpha^2 + \cos^2\alpha d\Omega_2)) \end{aligned} \tag{5.47}$$

3. Global coordinates

$$\begin{aligned} &Y_{-1} = \cosh \rho_g \cos \tau_g, \quad Y_0 = \cosh \rho_g \sin \tau_g, \quad Y_i = \sinh \rho_g n_i \\ &ds^2 = d\rho_g^2 - \cosh^2 \rho_g d\tau_g^2 + \sinh^2 \rho_g (d\alpha^2 + \cos^2 \alpha d\Omega_2) \end{aligned} \tag{5.48}$$

As the entangling surface we choose the S^2 at $\alpha = 0$, at a large time τ_B and at $\rho = \infty$. In terms of global coordinates the surface lies at a constant τ_g, or at

$$\frac{Y_0}{Y_{-1}} = \sinh \tau_B = \tan \tau_{gB}, \qquad Y_4 = 0 \tag{5.49}$$

Its area is

$$A = 4\pi \int_0^{\rho_{gc}} \sinh^2 \rho_g d\rho_g \sim 4\pi \left(\frac{e^{2\rho_{gc}}}{8} - \frac{\rho_{gc}}{2} \right) \tag{5.50}$$

where ρ_{gc} is the cutoff in the global coordinates. It is convenient to express this in terms of the radial coordinate in the dS slicing using $\sinh \rho_g = \sinh \rho \cosh \tau$. In the large ρ_{gc}, ρ_c, τ_B limit we find $\rho_{gc} \approx \rho_c + \tau_B - \log 2$. Then (5.50) becomes

$$A \sim 4\pi \left(\frac{e^{2\rho_c + 2\tau_B}}{16} - \frac{1}{2}(\rho_c + \tau_B) \right) \sim 4\pi \left(\frac{1}{16(\eta \ \epsilon_{UV})^2} + \frac{1}{2}(\log \epsilon_{UV} + \log \eta \ ) \right) \tag{5.51}$$

We see that the coefficients of the two logarithmic terms are the same, as is expected in any CFT. Here $\epsilon_{UV} = e^{-\rho_c}$ is the cutoff in the de Sitter frame and $\eta \sim e^{-\tau_B}$ is de Sitter conformal time.

5.4.2 Non-conformal theories

A simple way to get a non-conformal theory is to add a relevant perturbation to a conformal field theory. Let us first discuss the possible Euclidean geometries. Thus we consider theories on a sphere. In the interior we obtain a spherically symmetric metric and profile for the scalar field of the form

$$ds^2 = d\rho^2 + a^2(\rho) d\Omega_D^2 \ , \qquad \qquad \phi = \phi(\rho) \tag{5.52}$$

Some examples were discussed in [148, 133, 149] [7]. If the mass scale of the relevant perturbation is small compared to the inverse size of the sphere, the dual geometry will be a small deformation of Euclidean AdS_{D+1}. Then we find that, at the origin, $a = \rho + \mathcal{O}(\rho^3)$, and the sphere shrinks smoothly. In this case we will say that we have the "ungapped" phase. For very large ρ we expect that $\log a \propto \rho$, if we have a CFT as

[7] We are interpreting the solutions of [149] as explained in appendix A of [116]. This geometry also appears in decays of AdS space [150, 149].

the UV fixed point description.

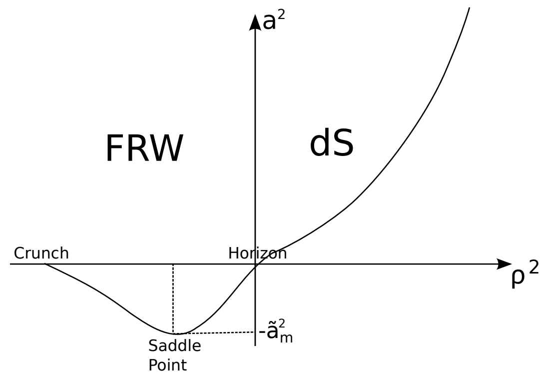

Figure 5.4: The typical shape for the scale factor for the gravity dual of a CFT perturbed by a relevant operator in the "ungapped" phase. The region with negative ρ^2 corresponds to the FRW region. In that region, we see that $\tilde{a}^2 = -a^2$ reaches a maximum value, $\tilde{a}_m$, and then contracts again into a big crunch.

On the other hand, if the mass scale of the relevant perturbation is large compared to the inverse size of the sphere then the boundary sphere does not have to shrink when we go to the interior. For example, the space can end before we get to $a = 0$. This can happen in multiple ways. We could have an end of the world brane at a non-zero value of a. Or some extra dimension could shrink to zero at this position. This typically happens for the holographic duals of theories with a mass gap, especially if the mass gap is much bigger than H. We call this the "gapped" phase. See [135, 137, 138, 140, 145, 151, 152] for some examples. In principle, the same field theory could display both phases as we vary the mass parameter of the relevant perturbation. Then, there is a large N phase transition between the two regimes[8].

As we go to lorentzian signature, the ungapped case leads to a horizon, located at $\rho = 0$. The metric is smooth if $a = \rho + \mathcal{O}(\rho^3)$. The region behind this horizon is obtained by setting $\rho = i\sigma$ in (5.52) and $d\Omega_D^2 \to -ds_{H_D}^2$. This region looks like a Friedman-Robertson-Walker cosmology with hyperbolic spatial sections.

$$ds^2 = -d\sigma^2 + (\tilde{a}(\sigma))^2 ds_{H_D}^2 \ , \qquad \tilde{a}(\sigma) \equiv -ia(i\rho) \tag{5.53}$$

If the scalar field is non-zero at $\rho = 0$ we typically find that a singularity develops at a non-zero value of σ, with the scale factor growing from zero at $\sigma = 0$ and then decreasing again at the big crunch singularity. The scale factor then achieves a maximum somewhere in between, say at σ_m. See Figure 5.4.

[8]Since we are at finite volume we might not have a true phase transition. In de Sitter, thermal effects will mix the two phases. We will nevertheless restrict our attention to one of these phases at a time.

We can choose global coordinates for dS_D

$$ds_{dS_D} = -d\tau^2 + \cosh^2\tau(\cos^2\alpha d\Omega_{D-2} + d\alpha^2) \tag{5.54}$$

We pick the entangling surface to be the S^{D-2} at $\alpha = 0$ and some late time τ_B. We assume that the surface stays at $\alpha = 0$ as it goes into the bulk. In that case we simply need to find how τ varies as a function of ρ as we go into the interior. We need to minimize the following action

$$S = \frac{V_{S^{D-2}}}{4G_N}\int (a\cosh\tau)^{D-2}\sqrt{d\rho^2 - a^2 d\tau^2} \tag{5.55}$$

The equations of motion simplify if we assume τ is very large and we can approximate $\cosh\tau \sim \frac{1}{2}e^{\tau}$. In that case the equations of motion give a first order equation for $y \equiv \frac{d\tau}{d\rho}$.

5.4.3 Non-conformal theories - gapped phase

In the gapped phase, we can solve the equation for y. Inserting that back into the action will give an answer that will go like $e^{(D-2)\tau_B}$ times some function which depends on the details of the solution. Thus, this produces just an area term. We can expand the action in powers of $e^{-2\tau}$ and obtain corrections to this answer. However, if the solution is such that the range of variation of τ is finite in the the interior, then we do not expect that any of these corrections gives a logarithmic term (for even D) or a finite term (for odd D). Thus, in the gapped phase we get that

$$S_{\text{intr}} = 0 \tag{5.56}$$

to leading order. The discussion is similar to the one in [153, 154] for a large entangling surface.

5.4.4 Non conformal theories - ungapped phase

In the ungapped phase, something more interesting occurs. The surface goes all the way to the horizon at $\rho = 0$. Up to that point the previous argument still applies and we expect no contributions to the interesting piece of the entropy from the region $\rho > 0$.

When the surface goes into the FRW region note that the S_{D-2} can shrink to zero at the origin of the hyperbolic slices. If we call $\rho = i\sigma$ and $\tau = \chi - i\pi/2$, then we see that the metric of the full space has the form

$$ds^2 = -d\sigma^2 + (\tilde{a}(\sigma))^2[d\chi^2 + \sinh^2\chi(\cos^2\alpha d\Omega_{D-2} + d\alpha^2)] \tag{5.57}$$

where $\tilde{a}(\sigma) = -ia(i\sigma)$ is the analytic continuation of $a(\rho)$. We expect that the surface extends up to $\chi = 0$

where the S^{D-1} shrinks smoothly. Clearly this is what was happening in the conformal case discussed in the previous subsection. Thus, by continuity we expect that this also happens in this case.

More explicitly, in this region we can write the action (5.55),

$$S = \frac{V_{S^{D-2}}}{4G_N} \int (\tilde{a} \sinh \chi)^{D-2} \sqrt{-\left(\frac{d\sigma}{d\chi}\right)^2 + \tilde{a}^2} \tag{5.58}$$

If we first set $\frac{d\sigma}{d\chi} = 0$, we can extremize the area by sitting at σ_m where $\tilde{a} = \tilde{a}_m$, which is the maximum value

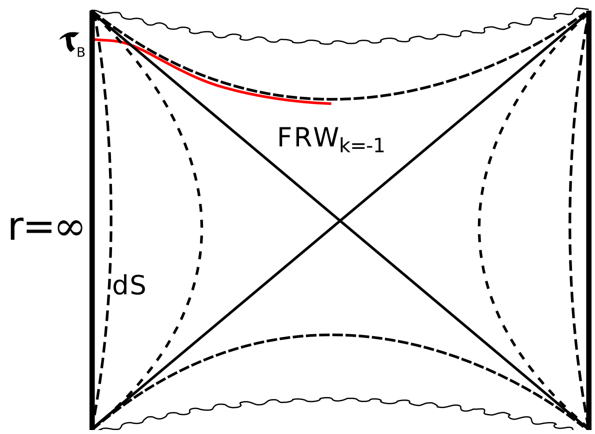

Figure 5.5: The typical shape for the scale factor for the gravity dual of a CFT perturbed by a relevant operator in the "ungapped" phase. The region with negative ρ^2 corresponds to the FRW region. In that region, we see that $\tilde{a}^2 = -a^2$ reaches a maximum value, $\tilde{a}_m$, and then contracts again into a big crunch.

for $\tilde{a}$. We can then include small variations around this point. We find that we get exponentially increasing or decreasing solutions as we go away from σ_m. Since the solution needs to join into a solution with a very large value of τ_B, we expect that it will start with a value of σ at $\chi = 0$ which is exponentially close to σ_m. Then the solution stays close to σ_m up to $\chi \sim \tau_B$ and then it moves away and approaches $\sigma \sim 0$. Namely, we expect that for $\sigma \sim 0$ the solution will behave as $\chi = \tau_B - \log \sigma$+rest, where the rest has an expansion in powers of $e^{-2\tau_B}$. This then joins with the solution of the form $\tau = \tau_B - \log \rho$+rest in the $\rho > 0$ region. The part of the solution which we denote as "rest", has a simple expansion in powers of $e^{-2\tau_B}$, with the leading term being independent of τ_B. All those terms are not expected to contribute to the interesting part of the entanglement entropy. The qualitative form of the solution can be found in figure 5.5.

The interesting part of the entanglement entropy comes from the region of the surface that sits near σ_m. In this region the entropy behaves as

$$S = \frac{\tilde{a}_m^{D-1}}{4G_N} V_{H^{D-1}} \quad \longrightarrow \quad S_{\text{intr}} = \frac{\tilde{a}_m^{D-1}}{4G_N} V_{H^{D-1}\text{reg}} \tag{5.59}$$

Here we got an answer which basically goes like the volume of the hyperbolic slice H^{D-1}. This should be cutoff at some value $\chi \sim \tau_B$. We have extracted the log term or the finite term, defined as the regularized volume.

Thus (5.59) gives the final expression for the entanglement entropy computed using the gravity dual. We see that the final expression is very simple. It depends only on the maximum value, $\tilde{a}_m$, of the scale factor in the FRW region.

Using this holographic method, and finding the precise solution for the extremal surface one can also compute the coefficient c_5 in (5.6) (or analogous terms in general dimensions). But we will not do that here.

In appendix C we discuss a particular example in more detail. The results agree with the general discussion we had here.

5.5 Discussion

In this chapter we have computed the entanglement entropy of some quantum field theories in de Sitter space. There are interesting features that are not present in the flat space case. In flat space, a massive theory does not lead to any long range entanglement. On the other hand, in de Sitter space particle creation gives rise to a long range contribution to the entanglement. This contribution is specific to de Sitter space and does not have a flat space counterpart. We isolated this interesting part by considering a very large surface and focusing on the terms that were either logarithmic (for even dimensions) or constant (for odd dimensions) as we took the large area limit.

In the large area limit the computation can be done with relative ease thanks to a special SO(1,D-1) symmetry that arises as we take the entangling surface to the boundary of dS_D. For a free field, this symmetry allowed us to separate the field modes so that the entanglement involves only two harmonic oscillator degrees of freedom at a time. So the density matrix factorizes into a product of density matrices for each pair of harmonic oscillators. The final expression for the entanglement entropy for a free field was given in (5.33). We checked that it reproduces the known answer for the case of a conformally coupled scalar. We also saw that in the large mass limit the entanglement goes as $e^{-m/H}$ which is due to the pair creation of massive particles. Since these pairs are rare, they do not produce much entanglement.

We have also studied the entanglement entropy in theories that have gravity duals. The interesting contribution to the entropy only arises when the bulk dual has a horizon. Behind the horizon there is an FRW region with hyperbolic cross sections. The scale factor of these hyperbolic cross sections grows, has a maximum, and then decreases again. The entanglement entropy comes from a surface that sits within the hyperbolic slice at the time of maximum expansion. This gives a simple formula for the holographic entanglement entropy (5.59). From the field theory point of view, it is an N^2 term. Thus, it comes from the long range entanglement of colored fields. It is particularly interesting that the long range entanglement comes from the FRW cosmological region behind the horizon. This suggests that this FRW cosmology is indeed somehow contained in the field theory on de Sitter space [116, 155] . More precisely, it is contained in colored modes that are correlated over superhorizon distances.

In the gapped phase the order N^2 contribution to the long range entanglement entropy vanishes. We expect to have an order one contribution that comes from bulk excitations which can be viewed as color singlet massive excitations in the boundary theory. From such contributions we expect an order one answer which is qualitatively similar to what we found for free massive scalar fields above.

5.6 Appendix A: Bogoliubov coefficients

Here we give the explicit form of the coefficients in (5.17).

$$\begin{aligned} \alpha_R^\sigma =& \frac{e^{\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu + ip + 1/2)} \ , \qquad & \alpha_L^\sigma = \sigma \frac{e^{\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu + ip + 1/2)} \\ \beta_R^\sigma =& -\frac{e^{-\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu - ip + 1/2)} \ , \qquad & \beta_L^\sigma = -\sigma \frac{e^{-\pi p} - i\sigma e^{-i\pi\nu}}{\Gamma(\nu - ip + 1/2)} \end{aligned} \tag{5.60}$$

We also find

$$\begin{aligned} \gamma_{j\sigma} =& \frac{\Gamma(\nu + ip + \frac{1}{2}) i e^{\pi p + i\pi\nu}}{4\sinh \pi p} \begin{pmatrix} \frac{1}{ie^{\pi p + i\pi\nu}+1} & \frac{1}{ie^{\pi p + i\pi\nu}-1} \\ \frac{1}{ie^{\pi p + i\pi\nu}+1} & -\frac{1}{ie^{\pi p + i\pi\nu}-1} \end{pmatrix}_{j\sigma} \\ \bar{\delta}_{j\sigma} =& \frac{\Gamma(\nu - ip + \frac{1}{2}) i e^{\pi p + i\pi\nu}}{4\sinh \pi p} \begin{pmatrix} \frac{1}{ie^{\pi p + i\pi\nu}+e^{2\pi p}} & \frac{1}{ie^{\pi p + i\pi\nu}-e^{2\pi p}} \\ \frac{1}{ie^{\pi p + i\pi\nu}+e^{2\pi p}} & -\frac{1}{ie^{\pi p + i\pi\nu}-e^{2\pi p}} \end{pmatrix}_{j\sigma} \end{aligned} \tag{5.61}$$

these were used to obtain (5.21).

We define c_R and c_L via (5.23) and the state in (5.22). We demand that $c_R|\Psi\rangle = \gamma c_L^\dagger|\Psi\rangle, c_L|\Psi\rangle = \gamma c_R^\dagger|\Psi\rangle$. Using (5.23) and denoting $m_{RR} = m_{LL} = \rho$, $m_{RL} = \zeta$ these two conditions become

$$\begin{aligned} &(u\rho + v - \gamma v\zeta)b_R^\dagger + (u\zeta - \gamma v\rho - \gamma u)b_L^\dagger = 0 \\ &(\bar{u}\zeta - \gamma\bar{u} - \gamma\bar{v}\rho)b_R^\dagger + (\bar{u}\rho + \bar{v} - \gamma\bar{v}\zeta)b_L^\dagger = 0 \end{aligned} \tag{5.62}$$

which imply that each of the coefficients is zero.

From the structure of (5.62), one sees that under the substitution $u \to \bar{u}$, $v \to \bar{v}$ we have the same set of equations. If one tries to solve them together then $\frac{u}{v} = \frac{\bar{u}}{\bar{v}}$; hence this ratio must be real. One can show that this is indeed the case and γ is given by (5.24).

5.7 Appendix B: Regularized volume of the Hyperboloid

Here we calculate the regularized volume of a hyperboloid in $D-1$ dimensions. We have to consider the cases of D even and D odd separately. First, note that the volume is given by the integral:

$$V_{H^{D-1}} = V_{S^{D-2}} \int_0^{\rho_c} d\rho (\sinh\rho)^{D-2} \tag{5.63}$$

Now we expand the integrand:

$$\frac{V_{H^{D-1}}}{V_{S^{D-2}}} = \frac{1}{2^{D-2}} \int_0^{\rho_c} d\rho \sum_{n=0}^{D-2} \binom{D-2}{n} (-1)^n e^{(D-2-2n)\rho} \tag{5.64}$$

But the integral of any exponential is given by:

$$\int_0^{\rho_c} d\rho \; e^{a\rho} = -\frac{1}{a} + \begin{cases} 0, & \text{a¡0} \\ & \\ \text{divergent}, & \text{a¿0} \end{cases} \tag{5.65}$$

Now we treat even or odd dimensions separately.

Even D: Here, the integrand of (5.64) contains a term independent of ρ in the summation, which gives rise to the logarithm (a term linear in ρ_c). The term we are interested in corresponds to setting $n = D/2 - 1$:

$$V_{H^{D-1},\text{reg}} = \frac{(-1)^{\frac{D}{2}}}{\frac{D-2}{2}} \binom{D-2}{\frac{D-2}{2}} V_{S^{D-2}} = \frac{(-1)^{\frac{D}{2}} \pi^{\frac{D-2}{2}}}{\frac{D-2}{2}} \frac{(D-2)!}{\left(\frac{D-2}{2}\right)!^3} = \frac{(-1)^{\frac{D}{2}}}{\pi} V_{S^{D-1}} \tag{5.66}$$

Odd D: Now, there is no constant term in the integrand of (5.64). Performing the summation in (5.64), and using (5.65), we get:

$$V_{H^{D-1},\text{reg}} = -\frac{1}{2^{D-2}} \sum_{n=0}^{D-2} \binom{D-2}{n} \frac{(-1)^n}{D-2-2n} V_{S^{D-2}} = \pi^{\frac{D-2}{2}} \Gamma\left[-\frac{D-2}{2}\right] = \frac{(-1)^{\frac{D-1}{2}}}{2} V_{S^{D-1}} \tag{5.67}$$

A more direct way to relate the regularized volumes of hyperbolic space to volume of the corresponding spheres is by a shift of the integration contour. We change $\rho_c \to \rho_c + i\pi$. This does not change the constant term, but we get an $i\pi$ from the log term. We then shift the contour to run from $\rho = 0$ along $\rho = i\theta$, with $0 \le \theta \le \pi$ and then from $i\pi$ to $i\pi + \rho_c$. The θ integral gives the volume of a sphere and the new integral with $Im(\rho) = \pi$ gives an answer which is either the same or minus the original integral. The fact that these regularized volumes are given by volume of spheres is related to the analytic continuation between AdS and dS wavefunctions [22, 156].

5.8 Appendix C: Entanglement entropy for conformal field theories on $dS_4 \times S^1$

Let us first discuss the gravity dual in the Euclidean case. The boundary is $S^4 \times S^1$. This boundary also arises when we consider a thermal configuration for the field theory on S^4. We will consider antiperiodic boundary conditions for the fermions along the S^1. There are two solutions. One is AdS with time compactified on a circle. The other is the Schwarzschild AdS black hole. Depending on the size of the circle one or the other solution is favored [157, 158]. Here we want to continue $S^4 \to dS^4$. An incomplete list of references where these geometries were explored is [135, 137, 138, 140, 145, 151, 152] .

As a theory on dS^4 we have a scale set by the radius of the extra spatial circle. At large N we have a sharp phase transition. At finite N we can have tunneling back and forth between these phases. Here we restrict attention to one of the phases, ignoring this tunneling. The Schwarzschild AdS solution looks basically like the gapped solutions we discussed in general above. Here, the S^4, or the dS^4, never shrinks to zero. It can be viewed as a bubble of nothing. On the other hand, the periodically identified AdS_6 solution gives the ungapped case, with the S^4 or dS^4 shrinking, which leads to a hyperbolic FRW region behind the horizon.

Gapped phase - Cigar geometry

We now consider the cigar geometry. In the UV we expect to see the divergence structure to be that of a 5D CFT, but in the IR it should behave like a gapped 4D non-conformal theory. The metric is given by

$$ds^2 = f d\phi^2 + r^2 ds^2_{dS^4} + \frac{dr^2}{f}, \qquad f = 1 + r^2 - \frac{m}{r^3} \tag{5.68}$$

The period β of the ϕ circle is given in terms of r_h, the largest root of $f(r_h) = 0$, by

$$\beta = \frac{4\pi}{f'(r_h)} = \frac{4\pi r_h}{2 + 4r_h^2} \tag{5.69}$$

Note that $\beta_{max} = \pi/\sqrt{2}$. This solution only exists for $\beta \leq \beta_{max}$. This geometry is shown in Figure 5.6. We consider an entangling surface which is an S^2 at a large value of τ_B.

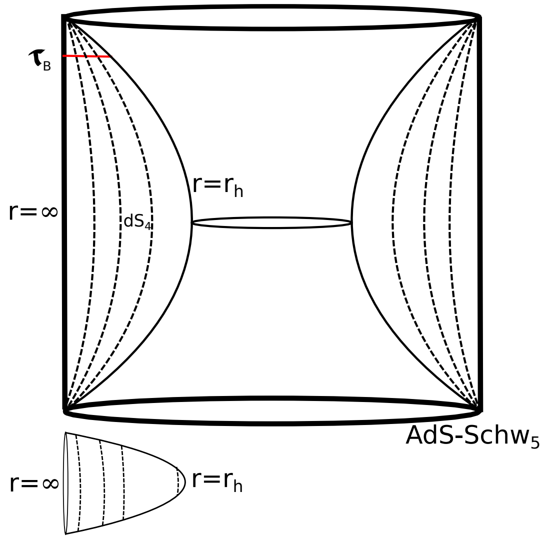

Figure 5.6: The gravity dual of a 5D CFT on $dS_4 \times S^1$. The spacetime ends at $r = r_h$, where the circle shrinks in a smooth fashion. We display an extremal surface going from τ_B to the interior.

We need to consider the action

$$A \propto \int dr r^2 \cosh^2 \tau \sqrt{1 - f r^2 \tau'^2} \tag{5.70}$$

This problem was also discussed in [115]. Since we are interested in large τ_B we can approximate this by

$$A_{approx} = \int dr r^2 e^{2\tau} \sqrt{1 - f r^2 (\tau')^2} \tag{5.71}$$

If the large tau approximation is valid throughout the solution then we see that the dependence on τ_B drops out from the equation and it only appears normalizing the action. In that case the full result is proportional to the area, $e^{2\tau_B}$, with no logarithmic term. In the approximation (5.71), the equation of motion involves only τ' and τ''. So we can define a new variable $y \equiv \tau'$ and the equation becomes first order. One can expand the equation for y and get that y has an expansion of the form $y = [-2/(3r^3) + 10/(27r^5) - 4m/(14r^6) + \cdots] + a(1/r^6 + \cdots)$ where a is an arbitrary coefficient representing the fact that we have one integration constant.

This undetermined coefficient should be set by requiring that the solution is smooth at $r = r_h$. If one expands the equation around $r = r_h$, assuming the solution has a power series expansion around r_h, then we get that y should have a certain fixed value at r_h and then all its powers are fixed around that point. Notice that if $y = y_h + y'_h(r - r_h) + \cdots$, implies that τ is regular around that point, since $(r - r_h) \propto x^2$ where x is the proper distance from the tip.

The full solution can be written as

$$\tau = \tau_B - \int_{r_h}^{\infty} y(r) dr \tag{5.72}$$

where y is independent of τ_B.

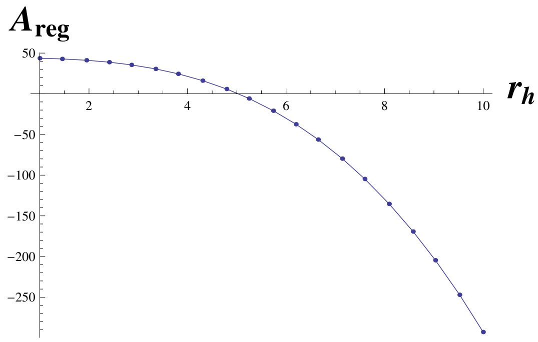

Figure 5.7: The regulated area A_{reg} is defined by $A_{\text{total}} = e^{2\tau_B} A_{\text{reg}} + A_{\text{div}}$.

At large r we get $\tau - \tau_B = \frac{1}{3r^2} + \mathcal{O}(1/r^4)$ and the action (5.71) evaluates to

$$A_{approx} \propto e^{2\tau_B} \int dr \left[r^2 + \frac{4}{9} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] \sim e^{2\tau_B} \left(\frac{r_c^3}{3} + \frac{4}{9} r_c + \text{finite}\right) = e^{2\tau_B} A_{reg} + A_{div} \tag{5.73}$$

We see that we get the kind of UV divergencies we expect in a five dimensional theory, as expected.

These can be subtracted and we can compute the finite terms. These are plotted in Figure 5.7 as a function of r_h.

So far, we have computed the finite term that grows like the area. By expanding (5.70) to the next order in the $e^{-2\tau}$ expansion we can get the next term. The next term will give a constant value, independent of the area. In particular, it will not produce a logarithmic contribution. In other words, there will not be a contribution proportional to τ_B.

In conclusion, in this phase, there is no logarithmic contribution to the entanglement entropy, at order $1/G_N$ or N^2.

Ungapped phase - Crunching geometry

Now the geometry is simply AdS_6 with an identification. This construction is described in detail in [151, 152]. The resulting geometry has a big crunch singularity where the radius of the spatial circle shrinks to zero. This geometry is sometimes called “ topological black hole”, as a higher dimensional generalization of the BTZ solution in $3D$ gravity.

It is more convenient to use a similar coordinate system as the one used to describe the cigar geometry in the previous case. The metric is given by (5.68), with $m = 0$. Those coordinates only cover the region outside of the lightcone at the origin, $r = 0$. To continue into the FRW region, one needs to use $r = i\sigma$ and $\tau = -i\pi/2 + \chi$ in (5.68).

The equation in the $r > 0$ region is such that we can make the large τ approximation and it thus reduces to a first order equation for $y = \tau' = \frac{d\tau}{dr}$. For small r, an analysis of the differential equation tells us that

$$y \sim -\frac{1}{r} - 2r^3 + \cdots + b\left(r + \frac{10-7b}{2}r^3 + \cdots\right) \tag{5.74}$$

where there is only one undetermined coefficient (or integration constant) which is b (it is really non-linear in b). This leads to a τ which is

$$\tau \sim -\log r - \frac{r^4}{2} + \cdots + c + b\left(\frac{r^2}{2} + \frac{10-7b}{8}r^4 + \cdots\right) \tag{5.75}$$

where c is a new integration constant. We expect that the evolution from this near horizon region to infinity only gives a constant shift. In other words, we expect that $c = \tau_B$+constant. This constant appears to depend on the value of b that is yet to be determined. We find that b should be positive in order to get a solution that goes to infinity and is non-singular.

We are now supposed to analytically continue into the FRW region. For that purpose we set $r = i\sigma$ and $\tau = -i\pi/2 + \chi$. Thus the equation (5.75) goes into

$$\chi \sim -\log\sigma - \frac{\sigma^4}{2} + \cdots + c + b\left(-\frac{\sigma^2}{2} + \frac{10-7b}{8}\sigma^4 + \cdots\right) \tag{5.76}$$

Then we are supposed to evolve the equation. It is convenient to change variables and write the Lagrangian in terms of $\sigma(\chi)$ as:

$$A \sim \int d\chi \sigma^2 \sinh^2\chi\sqrt{-(\sigma')^2 + \sigma^2(1-\sigma^2)} \tag{5.77}$$

In this case, at $\chi = 0$ we can set any starting point value for $\sigma(\chi = 0)$ and we have to impose that $\sigma' = 0$. Then we get only one integration constant which is $\sigma(0)$, as the second derivative is fixed by regularity of the solution to be $\sigma''(0) = -\sigma(0)(3 - 4\sigma(0)^2)$. We see that its sign depends on the starting value of $\sigma(0)$.

Since we want our critical surface to have a "large" constant value when we get to $\sigma \to 0$ as $\chi \to \infty$, we need to tune the value of $\sigma(0)$ so that it gives rise to this large constant. This can be obtained by tuning the coefficient in front of $\sigma(0)$. This critical value of $\sigma(0)$ is easy to understand. It is a solution of the equations of motion with $\sigma'(\rho) = 0$ (for a constant σ), it is a saddle point for the solution, located at $\sigma = \sqrt{3}/2$. If σ is slightly bigger than the critical value, the minimal surface will collapse into the singularity, so we tune this value to be slightly less than the critical point.

So, in conclusion, we see that the surface stays for a while at $\sigma \approx \sqrt{3}/2$ which is the critical point stated above. The value of the action (5.77) in this region is then

$$\frac{3\sqrt{3}}{16}\int_0^{\chi_c} d\chi \sinh^2\chi = \frac{3\sqrt{3}}{16}\left[\frac{e^{2\chi_c}}{8} - \frac{\chi_c}{2} + \cdots\right] \tag{5.78}$$

Here $\chi_c \sim \tau_B \sim \log\eta$ is the value of χ at the transition region. Thus, we find that the interesting contribution to the entanglement entropy is coming from the FRW region.

The transition region and the solution all the way to the AdS boundary is expected to be universal and its action is not expected to contribute further logarithmic terms.

In conclusion, the logarithmic term gives

$$S_{\text{intr}} = \frac{R^4_{AdS_6}}{4G_N}\beta\frac{4\pi 3\sqrt{3}}{32} \tag{5.79}$$

Here we repeat this computation in another coordinate system which is non-singular at the horizon. We use Kruskal-like coordinates [151, 152, 137] . It also makes the numerical analysis much simpler. In terms of embedding coordinates for AdS_6:

$$\begin{aligned} &-Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_5^2 = -1 \\ &ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2 + ... + dY_5^2 \end{aligned} \tag{5.80}$$

The Kruskal coordinates are given by:

$$\begin{gathered} Y_{-1} = \frac{1+y^2}{1-y^2}\cosh\phi, \quad Y_5 = \frac{1+y^2}{1-y^2}\sinh\phi, \quad Y_{0,\dots,4} = \frac{2y_{0,\dots4}}{1-y^2}, \quad y^2 \equiv -y_0^2 + y_1^2 + ... + y_4^2 \\ ds^2 = \frac{4}{(1-y^2)^2}(-dy_0^2 + ... + dy_4^2) + \left(\frac{1+y^2}{1-y^2}\right)^2 d\phi^2 \end{gathered} \tag{5.81}$$

In these coordinates, the dS region corresponds to $0 < y^2 < 1$ and the FRW region to $-1 < y^2 < 0$, with the singularity located at $y^2 = -1$. The AdS boundary is at $y^2 = 1$. We can relate the pair (r, τ) and (χ, σ), connected by the analytic continuation $(r = i\sigma,\ \tau = -i\pi/2 + \chi)$ to (y^2, y_0) by the formulas:

$$\begin{aligned} r &= \frac{2\sqrt{y^2}}{1-y^2}, \quad \sinh\tau = \frac{y_0}{\sqrt{y^2}} \\ \sigma &= \frac{2\sqrt{-y^2}}{1-y^2}, \quad \cosh\chi = \frac{y_0}{\sqrt{-y^2}} \end{aligned} \tag{5.82}$$

The area functional gets simplified to:

$$A = \int \sqrt{\frac{16(1+y^2)^2}{(1-y^2)^8}(y_0^2+y^2)\left[(d(y^2))^2 + 4y_0 dy_0 d(y^2) - 4y^2(dy_0)^2\right]} \tag{5.83}$$

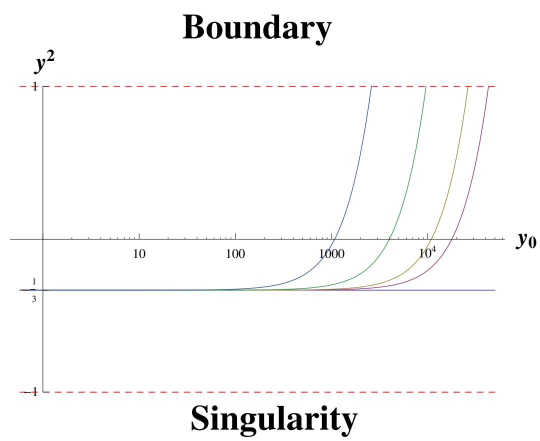

Figure 5.8: We plot here the value of y^2 versus y_0. For small y_0 the solution starts very close to the surface of maximum expansion at $y^2 = -1/3$, stays there for a while and then they go into the AdS boundary at $y^2 = 1$. The closer y_0 is to the saddle point $\tilde{y}_m^2 = -1/3$, the longer the solution will stay on this slice, giving a contribution that goes like the volume of an H^3. Then, at a time y_0 of the order of the time the surface reaches the boundary, it exits the FRW region. The interesting (logarithmic) contribution to the entropy is coming from the volume of the H^3 surface along the FRW slice at $y^2 = -1/3$.

If one looks for the saddle point described in the FRW coordinates, then one obtains $y^2 = -1/3$. The same situation can be described in simple fashion in terms of these coordinates. So $y^2 \sim -1/3$ for a large range of y_0, and it crosses to the dS-sliced region at a time of order τ_B, in dS coordinates, so part of the surface just gives the volume of an H^3, as in (5.78):

$$A \sim \frac{9}{16}\int_0^{\frac{\cosh\chi_c}{\sqrt{3}}} \sqrt{3y_0^2 - 1}\; dy_0 = \frac{3\sqrt{3}}{16}\left[\frac{e^{2\chi_c}}{8} - \frac{\chi_c}{2} + \cdots\right] \tag{5.84}$$

Some plots for the minimal surfaces are shown in Figure 5.8.

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