

EXPLORATIONS IN HOLOGRAPHIC
ENTANGLEMENT ENTROPY

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Abstract

In this Dissertation, different aspects of holographic entanglement entropy are explored. In quantum information theory, entanglement is a useful resource needed to perform quantum operations. Entanglement entropy quantifies this resource and even if computable in quantum field theories, it is quite hard to calculate explicitly. However, in the context of gauge/gravity duality, (holographic) entanglement entropy was proposed by Ryu and Takayanagi (RT) to be captured by the area of a minimal surface in Anti DeSitter (AdS) space, which is extremely simple to compute.

We begin by using the holographic dictionary to derive the RT formula. This is done by first considering smooth bulk geometries which are dual to n copies of the boundary geometry glued in a particular way. The action of these geometries computes the Renyi entropies S_n for any integer n . We give a prescription to analytically continue the action to non-integer n and argue that, in the $n \rightarrow 1$ limit, S_1 , the entanglement entropy, is given by the area of a minimal surface, reproducing the RT conjecture. That is, we reduced the RT proposal to the equality between bulk and boundary partition functions, a standard entry in the dictionary.

We then generalize the previous formalism to account for corrections due to bulk quantum fields ($1/N$ corrections in the boundary). This allows us to derive a new formula: boundary entanglement entropy is given by the area of the minimal surface plus bulk entanglement entropy. This extends RT beyond the planar limit and we present several predictions of the proposal.

Our exploration of holographic entanglement entropy continues by considering different states: by comparing the quantum corrected entropy for nearby states we obtain the modular hamiltonian operator in bulk perturbation theory. From this expression of the modular hamiltonian, we derive that relative entropy is bulk relative entropy and that the modular flow is the bulk modular flow.

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Chapter 1

Introduction

1.1 Holography: A crossroad of fields

High energy theory has historically been concerned with the study of very small physics like fundamental particles and interactions or the theory of everything and unification. In trying to understand strong interactions, string theory was discovered. However, string theory was found out to be far more interesting, among other things, it was shown to be a theory of quantum gravity: the unification between the gravitational forces, which governs the evolution of the Universe, and quantum mechanics, which dictates the probabilistic evolution of electrons and subatomic particles. The realm of quantum gravity doesn't necessarily concern the very small. Quantum effects are important in cosmology: in the context of inflation, the quantum fluctuations of the inflaton (primordial fluctuations) seed the large scale structure of the universe. Quantum effects are also important in macroscopic black holes, they are responsible for its evaporation, as was argued in [1] . Black hole evaporation is necessarily a quantum mechanical effect, however it is still not clear how this whole process is completely compatible with quantum mechanics (information loss [2]). So, even if gravity has been technically quantized with the discovery of string theory, it is still

far from clear how several paradoxes are practically solved in quantum gravity. In fact, some people argue that an observer who falls into a black hole will hit a firewall at the black hole horizon [3], and even if the general feeling is that this shouldn't happen, the community is confused about how to counter the firewall argument.

However, there is yet another approach into quantum gravity (motivated from string theory) which might make the problem more tractable. In 1997 [4] proposed the striking *AdS/CFT* duality: string theory in a “box” (Anti De Sitter space) is dual to field theory in the boundary of the “box”. This is a weak/strong duality, when the gravitational theory is weakly coupled, the field theory is strongly coupled. This duality relates two very different theories and soon after the proposal an extensive dictionary between quantities in gravity and in the field theory was developed.

On a different note, some strongly coupled systems studied by experimentalists can be modeled by complicated field theories. It is hard to extract predictions from these models, because long and elaborated computer simulations are often needed. The dynamics of these systems are generally so complicated that it is impossible to make progress analytically. However, *AdS/CFT* provides an insight into strongly coupled field theory physics in terms of simple gravitational calculations. Even if the duality is a property of a specific theory, it has been useful to understand the properties of certain classes of strongly coupled systems.

More concretely, the dictionary has made it possible to understand the physics of collisions of heavy ions in accelerators or exotic phases of matter like superconductivity, by just doing simple calculations. A considerable amount of the initial efforts in holography was devoted to understand how one could study a variety of (boundary) field theory phenomena in terms of the simpler calculations in gravity.

In this way, most of the developments in holography have been focused on using *AdS* to understand the *CFT*. However, these two theories are still dual in the presence of corrections to the classical and weakly coupled gravitational bulk theory. Thus,

even if the boundary theory is complicated, it provides a proper definition of quantum gravity. In this way, the *AdS/CFT* dictionary should be thought as a useful tool in both directions. This powerful dictionary which was derived in the high energy theory context is thus useful for a variety of other fields: condensed matter, high energy phenomenology, cosmology, quantum gravity (quantum information theory),...

1.2 Holography and quantum gravity

Even if the *AdS/CFT* conjecture provided an explicit definition of quantum gravity in terms of a holographic field theory, the idea that gravity is holographic is much older.

In local quantum systems, the total Hilbert space factorizes into the product of the Hilbert spaces at every point and the observables in this Hilbert space are local operators. Gauge theories are also local, only gauge invariant observables can be measured and they necessarily have support in more than one point. For example, in electromagnetism one can measure the electric or magnetic field, but operators like the vector potential are not physically observable. Furthermore, conserved charges, which are normally defined in a time slice, are better understood as boundary observables. For example, Gauss' law implies that the electric charge contained in a spatial region is equal to the charge measured in the associated flux surface. Gravity is also a gauge theory: nothing in gravity can depend on the choice of coordinates and thus any physical observable that one considers should be coordinate independent. In this case, the associated conserved charge is the Hamiltonian, the operator which governs time evolution. In analogy with the electric charge in the electromagnetic case, the energy in some space region can be obtained by measuring the Hamiltonian in the boundary of the region. Given that any quantum state has a unitary time evolution and that the Hamiltonian is a boundary term, it seems suggestive to think that

maybe one should associate a Hilbert space to the boundary where the Hamiltonian has support. This would imply that the degrees of freedom in gravity are holographic: they live in one less dimension.

Black hole entropy

A more concrete realization of the holographic principle is the fact that black holes are thermodynamic systems: they have a temperature and an entropy given by the Bekenstein-Hawking formula

$$S_{BH} = \frac{A}{4G_N} \tag{1.1}$$

with A the area of the black hole horizon. From this we learn that even if usual thermal systems have an entropy that scales like the volume of space, thermal systems in quantum gravity seem to scale like the area. The holographic principle [5] suggests that one should give serious consideration to this boundary theory, and think about this entropy as the thermal entropy of a theory which lives in one less dimension. Even if the BH formula was originally proposed from the fact that it satisfies a first law, it was later derived [6] by thinking about the classical action of the black hole background as the saddle point approximation to the euclidean quantum gravity path integral, which equates the classical action with the free energy. However, there is no general first principle to count the gravitational degrees of freedom and obtain this entropy from some microscopic counting.

Black hole entropy as entanglement entropy

After this thermodynamic interpretation, it was suggested [7] that another natural interpretation would be to think of this entropy as the contribution from all fields (including gravitational) to the entanglement entropy of the black hole exterior. Even if this seems natural, it is not clear how to do this calculation in gravity. Note that these two ways of thinking about it: the black hole being a thermal system or thinking about the exterior in terms of tracing out interior degrees of freedom are equivalent.

In the previous section, I described how *AdS/CFT* is useful to understand strongly coupled field theories from simple classical relativity calculations. However, given that we know so little about (even perturbative) quantum gravity, we can use the field theoretical definition of the Hilbert space to understand better how to do certain things in quantum gravity.

One could think of this as the guiding principle of this present work: by understanding how the dictionary works for questions concerning splitting (boundary) Hilbert spaces, one might be able to probe complicated properties of quantum gravity. There are lots of questions in quantum gravity that we can explore in this way: how can one split a given spacetime into subregions? Field theories are local and thus the Hilbert space factorizes, but it is far from clear to what extent one could do something like this in gravity. If there is some notion of quantum gravity for subregions, can one compare different states in the same subregion, ...? In this work, we are going to pursue this line of thought by doing concrete calculations in *AdS/CFT*. Even if these questions are interesting enough in the *AdS/CFT* context, we also expect that they will provide with good intuition for how quantum gravity works more generally.

1.3 Entanglement in quantum mechanics

A characteristic property of quantum systems is that they can be entangled. On a first sight, entanglement can appear very similar to classical correlations. Consider a magician who has a white and a black ball. He gives one of them to Alice and the other to Bob, and the magician assigns the balls randomly, so that, if the experiment is repeated several times, they will get each color the same number of times. When Alice looks at her ball, she will know the color of Bob's ball. So, the outcome of Alice measurement will be correlated with that of Bob. This correlation is classical: before

observing the ball its color is already determined and the magician could know which color each person got.

One can try to repeat a similar experiment using quantum mechanics [8]. The magician now has a pair of *entangled* spins and he gives one to Alice and the other to Bob, the spins are prepared so that if Alice measures the spin in some direction it will point up half of the times and down half of the times. Her measurement will be perfectly anticorrelated with that of Bob, once Alice measures that the spin is up, she will know that, if Bob's measures the spin in the same direction, it will be pointing down. Naively, the outcomes of the measurement look the same as in the classical case. However, in the quantum case, spin doesn't have a definite direction before being measured, there are no hidden variables which could explain their outcomes. For example, we can consider Alice and Bob measuring their respective correlated particle in the x, y axes, so that a measurement in the x axes can give an outcome $\pm x$. The simplest way to see that there is no hidden variables theory is by considering an experiment where Alice and Bob measure their spins in orthogonal directions. If Bob measures his spin in x direction and gets $+x$ and Alice measures her spin in the y axes with outcome $-y$, a hidden variables theory would predict that Alice's spin is in the $-x$ direction, because it is anticorrelated with Bob's. However, after measuring her spin the y direction and obtaining $-y$, a measurement of her spin in the x direction will give $+x$ with probability $\frac{1}{2}$. A way of summing up why entanglement is different is that it induces correlations between non commuting observables, which is not possible classically.

It is often useful to describe the quantum state restricted to a subsystem. If the system is in a pure state, it will be described by a wave function $|\Psi\rangle$ and it will have an associated density matrix $\rho = |\Psi\rangle\langle\Psi|$. The reduced density matrix of a subsystem

R is defined by tracing over \bar{R} , the degrees of freedom which are not in R .

$$\rho_R = \text{tr}_{\bar{R}} \rho \tag{1.2}$$

This reduced density matrix can be diagonalized and the number of non-zero elements quantifies how entangled this subsystem is with the environment. The Von Neumann entropy of ρ_R provides with a measure of quantum entanglement, the entanglement entropy:

$$S_{EE} = -\text{tr} \rho_R \log \rho_R \tag{1.3}$$

This quantity can be understood as quantifying of the number of Bell pairs that are shared between R and \bar{R} .

1.4 Entanglement in QFT

Quantum field theories have a very entangled vacuum and thus, entanglement entropy of subregions is divergent. In algebraic approaches to quantum field theory one never considers entanglement entropy but instead there are other entanglement measures that can be formally defined, like the relative entropy.

Despite of these divergences, the entanglement entropy of a subregion R can be regularized by introducing a cutoff near the entangling surface ∂R . This regularization is far from universal and thus there is some ambiguity in how one does it. However, one can often extract some unambiguous information from the entanglement entropy. Up to coefficients, the expression for the entanglement entropy in four dimensions looks like:

$$S_{EE}(R) = \frac{A(\partial R)}{\epsilon^2} + \dots + \# \log \epsilon + \dots \tag{1.4}$$

Here $A(\partial R)$ is the area of the entangling surface, so this first term is often called "area law" . In this case the coefficient of the logarithmic term is universal. One

can also get rid of these ambiguities by comparing the entropies of different states or considering the mutual information.

Even after considering the previous points, the calculation of entanglement entropy is hard. One can compute the density matrices explicitly for free fields and compute its von Neumann entropy, and this is often calculated by putting the theory on a lattice. More generally, only for specially symmetric situations one can make more progress: for example, if one considers the vacuum state of an arbitrary Lorentz invariant theory, the density matrix of half a region is thermal, this is the Unruh effect, which is formalized in the Bisognano-Wichmann theorem.

In general, it is hard to get the density matrix functional explicitly for quantum field theories. In order to avoid taking the log of a complicated matrix, one can equivalently define the entanglement entropy as the analytic continuation of the partition function of our theory in a different background. This is called the replica trick and consists in instead evaluating $\text{tr}\rho_R^n$. This object corresponds to the partition function in a geometry where one inserts a conical excess around the entangling surface ∂R . The entanglement entropy can be obtained by analytically continuing this conical excess partition functions:

$$S_{EE} = \lim_{n \rightarrow 1} \frac{1}{1-n} (\text{tr}\rho_R^n - n \text{tr}\rho_R) \quad (1.5)$$

Setting up the computation in this way makes it better defined: it provides a path integral description of the entanglement entropy. However, it is still hard to complete the calculation explicitly, unless there is some symmetry or one considers free fields.

Because of the intrinsic complications of computing entanglement entropy in quantum field theories it was rather surprising when Ryu and Takayanagi (RT) [9] proposed that, in holography, entanglement entropy has a simple expression. They suggested that one should compute the area of the minimal bulk surface that ends in the bound-

ary entangling surface:

$$S_{EE}(R) = \min \frac{A(S)}{4G_N}, \partial S = \partial R \tag{1.6}$$

This proposal passed several checks and in the next chapter we will provide a derivation.

While the RT formula is very interesting by itself and it makes it easy to compute entanglement entropy in strongly interacting field theories, it also provides a new window to understand entanglement in quantum gravity. We will explore these ideas in the next subsection.

1.5 Entanglement and spacetime

The beauty of RT's proposal is that entanglement becomes a geometrical quantity. The previous formula is not the first instance where this was observed though. In [10], Maldacena described the thermofield double state, which can roughly be thought as a particular quantum superposition of many different and disconnected geometries, in terms of a unique, connected geometry. In this case, it looks like the massive amount of entanglement between the two disconnected geometries is responsible for gluing them. This fact (often dubbed $EPR \rightarrow ER$) was further explored in [11] and it is probably one of the most mysterious properties of quantum gravity. Mark Van Raamsdonk suggested [12, 13] that maybe one should think about entanglement as the "glue of spacetime".

Even if these ideas are very appealing, it is complicated to quantify under what conditions geometries "emerge from entanglement", because this necessarily involves the dynamics of non-perturbative quantum gravity.

Another interesting direction is to understand to what extent the bulk subregion between the RT surface and the boundary subregion R , the entanglement wedge, is

dual to R . This is sometimes called subregion-subregion duality. From the boundary point of view, there is a well defined Hilbert space of the subregion R , but since the study of entanglement in quantum gravity has been less explored, it is less clear how one traces out in the bulk. Because the bulk theory is gravitational, the only way that we understand the bulk degrees of freedom is in the semiclassical approximation, in terms of quantum fields in a curved background. This provides with the bulk with a relatively simple Hilbert space, but once gravity is incorporated there are lots of subtleties in how to define the Hilbert space of this bulk subregion (treating the gravitons as perturbative fields) because in gauge theories there is no Hilbert space factorization. However, the Hilbert space of the boundary subregion is well defined, so it looks like one should be able to understand very non-trivial questions in (perturbative) quantum gravity using *AdS/CFT*.

1.6 Main results

In this Dissertation, various aspects of entanglement in holography are studied. The different chapters correspond to a selection of the author's published work [14, 15, 16]. The papers [17, 18, 19] that the author wrote during these years haven't been included in this thesis.

In the first chapter, we consider the dual geometries of boundaries which have a conical excess $2\pi(n - 1)$ around the entangling surface ∂R . That is, we use the established *AdS/CFT* dictionary to understand entanglement entropy, which allows for a derivation of RT from the usual equality between boundary partition function and bulk action. An important point is that these bulk geometries are smooth: even if the boundary is singular, the bulk is regular and it satisfies the gravitational equations of motion. This means that in order to compute the partition function in the replicated geometry (which in gravity is given by the action of the gravitational

solution), one has to solve some complicated non-linear equations. In general, it is hard to get the partition function for any finite $n - 1$. However, in this chapter we discuss how one should think about the $n - 1 \rightarrow 0$ limit and how one can solve the equations of motion perturbatively in $n - 1$. We observe the entanglement entropy is given by the area of a special surface, and the equations of motion close to $n - 1$ determine that this surface has to be minimal. This provides a derivation for the RT formula.

Since the previous construction is fairly general, one easily go beyond Einstein gravity. In the next chapter, we consider the effect of quantum corrections in the bulk. That is, we consider the $1/N$ (or \hbar) corrections to the RT formula. A striking property of RT is the fact that it is given by the integral of a local bulk quantity. The quantum corrections have a contribution which is also a local integral in the RT surface. However, there is also an important non-local contribution: one has to consider the entanglement entropy of bulk fields in the region delimited between the RT surface and the boundary. So, boundary entanglement has a geometric contribution but there is also a non-local bulk entanglement piece.

Then we proceed to use the previous ideas to extract more general information about our boundary region R . In particular, we study how one can think about the boundary density matrix as an operator in bulk perturbation theory. We observe that it can be given a very simple expression: the bulk entanglement entropy gives the bulk density matrix but the area contribution becomes just a linear operator (within the small subspace of bulk perturbations). This simple expression is obtained by considering small perturbations of the density matrix. Having access to the density matrix, one can consider other interesting quantities. For example, we discuss how the relative entropy (a measure of entanglement which encodes how close are two states) becomes the bulk relative entropy and how the modular evolution ¹ is generated by

¹A generalization of Rindler evolution for less symmetric systems.

the bulk modular hamiltonian. These ideas also provide a framework to think about subregion-subregion duality and we speculate how one could try to quantify it using our observations.

To conclude, in this Dissertation we have studied in detail what the dual of entanglement entropy in *AdS/CFT* is. We have shown that it is given by the area of the minimal surface plus the bulk entanglement entropy and that the modular hamiltonian of a boundary subregion is the minimal area operator plus the bulk modular hamiltonian. All this suggests that entanglement is dual to entanglement (in the absence of strong back-reaction). In this way, one might expect that the RT formula could be understood in terms of gravitational bulk entanglement, but it is not completely clear how to compute entanglement entropy in (perturbative) quantum gravity. These properties also identify a bulk subregion which captures the entanglement properties of the boundary subregion. Because the entanglement structure of the two subregions is roughly the same, this presents a strong case for an extension of *AdS/CFT* to subregions. This is interesting by itself and this might also be very helpful to understand how one should think of subregions, reduced density matrices and entanglement in perturbative quantum gravity.

Chapter 2

Generalized gravitational entropy

2.1 Introduction

Originally the concept of entropy arose from equilibrium thermodynamics. However, we now think of entropy as a measure of information. In particular, we can assign an entropy to a general density matrix via

$$S = -Tr[\rho \log \rho] \tag{2.1}$$

By thinking about the thermodynamics of black holes the area formula for gravitational entropy was discovered [20, 21, 1]. Gibbons and Hawking introduced a thermodynamic interpretation of euclidean gravity solutions with a $U(1)$ isometry [6]. The idea is that one considers Euclidean solutions with prescribed boundary conditions. The boundary conditions, as well as the solutions, are invariant under a $U(1)$ symmetry¹. These solutions can be viewed as describing the computation of the partition function of a quantum theory in the classical approximation. In other words, one thinks of the Euclidean gravitational action as $\log Z(\beta) = -S_{E,grav}$. Then

¹Here we assume that there is a single $U(1)$ symmetry, otherwise we need to add the corresponding chemical potentials, etc.

the entropy, obtained as $S = -(\beta\partial_\beta - 1) \log Z$, is equal to the area of the codimension two surface which is a fixed point for the $U(1)$ symmetry in the bulk. Classically, the boundary can be chosen to be any surface where we put boundary conditions. It can also be an asymptotic boundary such as the AdS boundary.

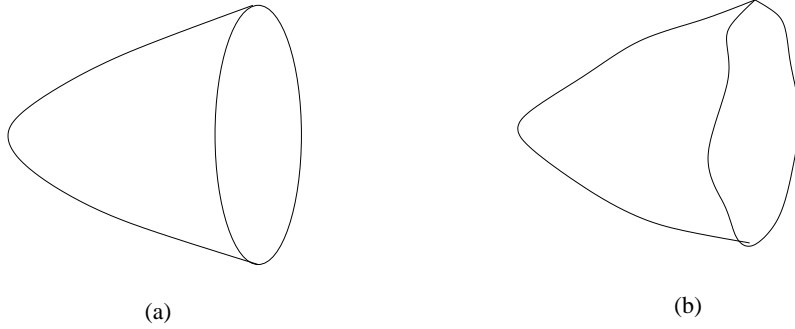


Figure 2.1: (a) A euclidean solution with a $U(1)$ symmetry is interpreted as computing the equilibrium thermodynamic partition function of the gravity theory. (b) We consider a euclidean solution with a circle but without a $U(1)$ symmetry. This is interpreted as computing $Tr[\rho]$ for an un-normalized density matrix in the gravity theory. This is the density matrix produced by euclidean evolution.

Interestingly, one can extend the notion of gravitational entropy to situations without a $U(1)$ symmetry as follows.

Let us first consider a general quantum system. Its Euclidean evolution generates an un-normalized density matrix

$$\rho = P e^{-\int_{\tau_0}^{\tau_f} d\tau H(\tau)} \quad (2.2)$$

where we considered a general time dependent Euclidean Hamiltonian. We can compute the entropy of this density matrix by the “replica trick”. Namely, first notice that $Tr[\rho]$ can be computed by considering euclidean evolution on a circle, identifying $\tau_f = \tau_0 + 2\pi$ ². Similarly, we can compute $Tr[\rho^n]$ by considering time evolution over

²Throughout this chapter we set the coordinate length of the initial circle to 2π . Of course, its physical length depends on the metric.

a circle of n times the length of the original one, where the couplings in the theory are strictly periodic under shifts of the original circle, $H(\tau + 2\pi) = H(\tau)$.

We then can compute the entropy as

$$S = -n\partial_n [\log Z(n) - n \log Z(1)]|_{n=1} = -Tr[\hat{\rho} \log \hat{\rho}] ,$$

$$Z(n) \equiv Tr[\rho^n] , \quad \hat{\rho} \equiv \frac{\rho}{Tr[\rho]} \quad (2.3)$$

here now $\hat{\rho}$ is a properly normalized density matrix. This involves computing $Z(n)$ and then performing an analytic continuation in n .

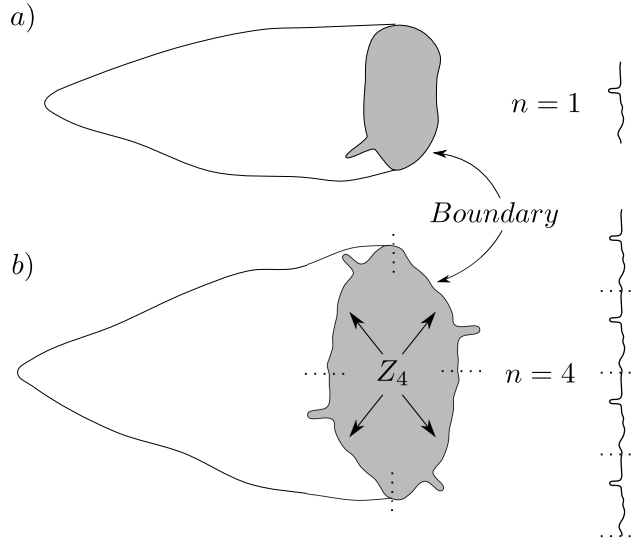


Figure 2.2: Computing the entropy using the replica trick. (a) Euclidean solution for $n = 1$. (b) Solution for $n = 4$. At the boundary we go around the original circle n times before making the identification. We then find a smooth gravity solution with these boundary conditions. The curves in the right hand side are schematically giving the boundary conditions at infinity. We see that in (b), we simply repeat n times the boundary conditions we had in (a).

Going back to the gravitational context, we can consider metrics which end on a boundary. We assume that the boundary has a direction with the topology of the circle. The boundary data can depend on the position along this circle but it respects

the periodicity of the circle. We define the coordinate $\tau \sim \tau + 2\pi$ on the circle. We can then consider a spacetime in the interior which is smooth. Its Euclidean action is defined to be $\log Z(1)$. See 2.1 (b). We can also consider other spacetimes where we take the same boundary data but consider a new circle with period $\tau \sim \tau + 2\pi n$. Their Euclidean action is defined to be $\log Z(n)$. These computations can be viewed as computing $Tr[\rho^n]$ for the density matrix produced by the Euclidean evolution. See 2.1 . If we are sufficiently diligent, we can find these actions, analytically continue in n the corresponding answers and compute S as in (2.3) . This has been explicitly done in [22, 23] for some examples in three dimensional gravity.

Note that we are implicitly assuming that gravity is holographic. We are imagining that setting boundary conditions on some boundary defines the theory and that the interior geometry is an approximation to the full computation. We do not know how (and whether) holography works for general boundaries. Here we only need it to be approximately valid so that this classical computation has the interpretation of computing an approximate density matrix in some approximate theory. In cases where the boundary is a true asymptotic boundary (such as a locally asymptotically *AdS* boundary) the situation is well understood. This corresponds to computing the entropy of a perfectly well defined density matrix in the dual field theory .

Interestingly, there is a simple conjecture for the final answer. The entropy is also given by the area of a special codimension two surface in the bulk of the original ($n = 1$) solution. At this surface the circle shrinks smoothly to zero size. The surface obeys a minimal area condition.

$$S \equiv - n \partial_n [\log Z(n) - n \log Z(1)]|_{n=1} = \frac{A_{\text{minimal}}}{4G_N} \quad (2.4)$$

From now on, $\log Z(n)$ denotes the classical gravity action $\log Z(n) = -S_{\text{Grav}}$ of the n^{th} solution. This formula was first conjectured by Ryu and Takayanagi in the context

of the computation of entanglement entropy of conformal field theories with gravity duals [9] (see [24] for a review³). Proving their formula amounts to proving the above conjecture, as we explain below. Notice that (2.4) can be viewed as a statement about classical general relativity. It is a relation between the actions for classical solutions that are produced by the replica trick and the area of the minimal area solution with $n = 1$. Of course, for solutions with a $U(1)$ symmetry, (2.4) reduces to the standard Gibbons-Hawking computation. In that case, the $U(1)$ symmetry also ensures that the horizon is a minimal surface, with zero extrinsic curvature.

In this chapter we will give an argument for (2.4) based on reasonable assumptions regarding the analytic continuation of the solutions away from integer values of n .

We will also explain why proving (2.4) is equivalent to proving the Ryu Takayanagi conjecture. The Ryu-Takayanagi conjecture for the case of asymptotically AdS_3 pure gravity was proven in [22, 23]. Previous arguments include [27], whose assumptions were criticized in [28].⁴

This chapter is organized as follows. In section 2 we perform some explicit computations in a simple example. In section 3 we review the derivation of the entropy formula for the case with a $U(1)$ symmetry. In section 4 we present the arguments for the main formula (2.4). There we explain how the solution looks for n close to one. We also derive the minimal area condition for the surface. In section 5 we discuss the connection to entanglement entropy in field theories with gravity dual. In section 6 we present the conclusions. In the appendices we present some further explicit examples and more details on the computations.

³ See [25, 26, 27] for related work.

⁴ Fursaev [27] took the solution for $n = 1$ and set $\tau \sim \tau + 2\pi n$ everywhere in the bulk. This introduces a conical singularity in the bulk. As noted by Headrick [28], for integer n , one should instead consider solutions which are non-singular in the bulk.

2.2 A simple example without a $U(1)$ symmetry

Since our discussion has been a bit abstract, let us discuss a very simple concrete example. This example will also motivate some assumptions that we will make later.

Let us start with the BTZ geometry

$$ds^2 = \left[\frac{dr^2}{(1+r^2)} + r^2 d\tau^2 + (1+r^2) dx^2 \right] \quad (2.5)$$

This metric has a $U(1)$ isometry along the circle labeled by τ , $\tau \sim \tau + 2\pi$. All functions will be invariant under translations in x . This direction will not play any role in this discussion and we take it to be compact of size L_x . Computing the entropy for this solution gives the standard area formula, S_0 , for this solution.

We now add a complex, minimally coupled, massless scalar field ϕ . We set boundary conditions that are not $U(1)$ invariant

$$\phi = \eta e^{i\tau}, \quad \text{at } r = \infty \quad (2.6)$$

We now compute the gravitational action to second order in η for the family of solutions described above. The metric is changed at order η^2 , but since the original background obeys Einstein's equations, there is no contribution from the gravitational term to order η^2 . So, to this order, the whole contribution comes from the scalar field term in the action.

Namely, for the n^{th} case, we need to consider a spacetime with the same boundary conditions as in (2.6) but where $\tau \sim \tau + 2\pi n$. This implies that the spacetime in the interior is

$$ds^2 = \left[\frac{dr^2}{(n^{-2} + r^2)} + r^2 d\tau^2 + (n^{-2} + r^2) dx^2 \right] \quad (2.7)$$

And we need to consider a scalar field in this spacetime. We can write the wave equation. The solution of the wave equation that is regular at the origin and obeys

(2.6) at infinity is

$$\phi = \eta e^{i\tau} f_n(r), \quad f_n(r) = (nr)^n \frac{\Gamma(\frac{n}{2} + 1)^2}{\Gamma(n + 1)} {}_2F_1\left(\frac{n}{2}, \frac{n}{2} + 1; n + 1; -(nr)^2\right) \quad (2.8)$$

Note that $f_n \rightarrow 1$ as $r \rightarrow \infty$.

We now evaluate the gravitational action for every n . We evaluate it to second order in η , so we consider the quadratic action for the field ϕ . Using standard formulas we can write

$$\begin{aligned} \log Z(n)|_{\eta^2} &= - \int_{AdS_3} |\nabla\phi|^2 = -(2\pi n)L_x [r^3\phi^*\partial_r\phi]_{r=\infty} = \\ &= (2\pi L_x) [1 - n \log n + n\psi(n/2) + (\text{linear in } n)] \end{aligned} \quad (2.9)$$

where L_x is the length of the x direction and ψ is the Euler ψ function. The terms linear in n include divergent terms that should be subtracted. However, they do not contribute to the entropy (2.3).

We analytically continue in n and compute the entropy via (2.3) to find

$$S = S_0 + \eta^2 \pi L_x \left(4 - \frac{\pi^2}{2}\right) \quad (2.10)$$

We can now compare this with the answer we expect from the area formula. This non-zero configuration for the scalar field changes the geometry to second order in η . Thus it produces a second order change in the area of the horizon. This change can be computed from Einstein's equation. We obtain the same answer (2.10). This is done in detail in appendix A, where we also consider a scalar field with an arbitrary mass.

So, we have explicitly checked the conjecture for this special case. Now, let us make some remarks.

We considered a complex scalar field, but the computation can be done also for a real scalar field with boundary conditions $\phi = \eta \cos \tau$ at infinity. The result is essentially the same. See appendix A.

Notice that the solution for the n^{th} case has a Z_n symmetry. This is a replica symmetry of the boundary conditions which extends to the bulk solution. So, in this case we are not breaking the replica symmetry. Notice that $r = 0$ is a fixed point of the action of the Z_n replica symmetry for all $n > 1$. In this case, the metric has a $U(1)$ symmetry. However, the full scalar field configuration is only symmetric under the Z_n ⁵.

Here we have computed $\log Z(n)$ and then analytically continued the answer. The geometry (2.7) is well defined also for non-integer n and we can trivially continue it to non-integer values of n , and it remains smooth. We could ask whether we can also analytically continue the whole field configuration to non integer values of n . Notice that as we vary n , the τ dependence at the boundary is kept fixed. Thus, even for non-integer n , we will keep the same boundary condition. This boundary condition is not compatible with a non-integer period for τ . We will ignore this. In other words, we will integrate τ between $[0, 2\pi]$ and multiply the result by n . However, as we go to $r = 0$, we find that the scalar field behaves as $\phi \sim r^n e^{i\tau}$, which leads to a singularity for the scalar field at $r = 0$. The scalar field, or its stress tensor do not diverge if $n > 1$. In other words, this appears to be a relatively harmless integrable singularity. This singularity seems physically questionable. But we are not trying to give a physical interpretation to the solution with non-integer n . We are only trying to define it mathematically, as an intermediate step in computing the replica trick answer. One could worry that if we allow singularities, then the solution will not be uniquely defined. However, we are allowing a very specific behavior which determines a unique solution for given boundary conditions. More explicitly, note that when we

⁵ In this case there is a $U(1)$ symmetry which is shift in τ combined with a phase rotation of the complex field. But for a similar computation with a real scalar field we only have the Z_n symmetry.

solve the wave equation near $r \sim 0$, we get two solutions $r^n e^{i\tau}$ and $r^{-n} e^{i\tau}$. We set to zero the coefficient of the second solution at the origin. This prescription uniquely selects a solution, both for integer and non-integer n .

This gravity theory in AdS_3 with a massless scalar field can arise from a Kaluza Klein reduction of a higher dimensional theory. For example, it can come from a ten dimensional solution of the form $AdS_3 \times S^3 \times T^4$. Then the massless field can be an off-diagonal component of the metric on of the four torus [29, 30]. More explicitly, we can deform the metric of the four torus as

$$ds_{T^4}^2 = e^{2\phi_1} dy_1^2 + e^{2\phi_2} dy_2^2 + e^{-2\phi_1} dy_3^2 + e^{-2\phi_2} dy_4^2 \quad (2.11)$$

where $\phi = \phi_1 + i\phi_2$. We see that the singularity of the field ϕ at the origin translates into a singularity for some of the Riemann tensor components. Let us consider $n = 1 + \epsilon$. Then since $\phi \propto r^{1+\epsilon} e^{i\tau}$ this leads to a singularity in some of the Riemann tensor components $R_{\alpha i \beta i} \sim \frac{\epsilon}{r}$ (no sum over i), where i are the directions on the four torus and α denotes the directions along the two transverse components (labeled also by r and τ) Despite these singularities the action is finite, as we saw when we computed it explicitly.

An alternative way to view the solution labeled by n is the following. We consider the τ circle to have period 2π but introduce a conical singularity at the origin with opening angle $2\pi/n$. This is not the same as the gravity solution with $n = 1$ since the field configuration has to adjust to the presence of the conical singularity. Then, when we evaluate the gravitational action, we integrate τ over $[0, 2\pi]$ but multiply the resulting answer by a factor of n . This factor of n arises because the real period of τ is $2\pi n$ instead of 2π . It is important that we evaluate the gravitational action *without introducing any contributions from the tip of the conical singularity*, since the

full space (with the right period for τ) is non-singular ^{6 7}. This picture makes sense both for integer or non-integer n .

2.3 Computation of gravitational entropy when there is a $U(1)$ symmetry

In this section we will describe the computation of the entropy using Euclidean methods in a way that it emphasizes the fact that the contribution comes from the horizon. This has been discussed by various authors in a similar form [21, 31, 32, 33, 34, 35]. Here we say it two ways that we particularly liked.

2.3.1 Entropy from rounded off cones

Setting the period of the circle to be $\tau \sim \tau + 2\pi n$, then we find that the formula for the entropy can be written as

$$S = -n\partial_n [\log Z(n) - n \log Z(1)]_{n=1} \quad (2.12)$$

Let us consider this expression for n close to one. We interpret the first term in the square brackets as the correct, smooth solution when n is not one. We interpret the second term as the solution for $n = 1$ but with a τ which has period $\tau \sim \tau + 2\pi n$. This solution has a conical singularity at the origin. However, we *do not* include any contribution from the conical singularity. We simply integrate the gravitational action density away from the tip.

We now evaluate the difference in the square brackets in (2.13) by adding and subtracting a smooth geometry which is the same as that of the cone far away from the origin, but it is a regularized cone near the origin, see 2.3.1. This smooth

⁶This looks superficially similar to what was discussed in [27], but it is different in detail.

⁷ This point of view was also suggested to us by T. Faulkner.

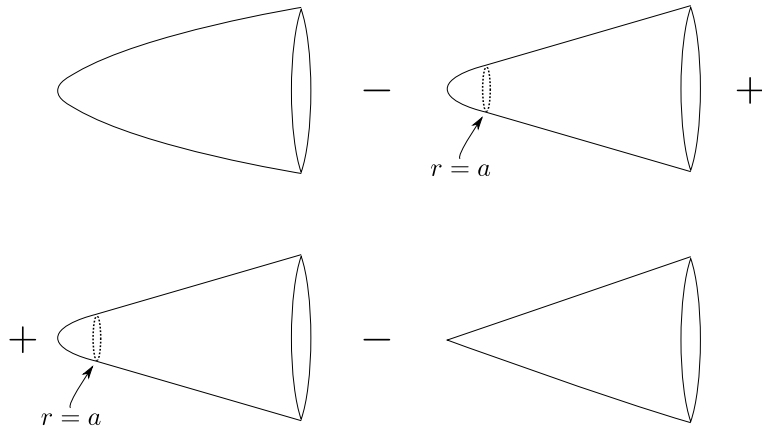


Figure 2.3: A particular combination of geometries that is useful for computing the entropy. The first geometry is the correct solution with period $2\pi n$. The last geometry contains a conical singularity. It is the solution with $n = 1$ but with the circle identified after $\tau \rightarrow \tau + 2\pi n$. For $n = 1$ the deficit angle of the cone is very small and it has been greatly exaggerated here for artistic reasons. The two middle ones are identical and correspond to a regularized version of the last solution. They only differ for $r < a$, where a is small regulator. This is not a solution, it is an off-shell configuration. All of the configurations obey the same boundary conditions at infinity.

geometry is not a solution of the equations of motion of the theory, it is an off-shell configuration. We are simply introducing it to help us perform the computation. It is possible to choose this off-shell configuration in such a way that the metric differs only by an amount of order $n - 1$ from the true solution.

Thus we get

$$S = -n\partial_n [(\log Z(n) - \log Z^{\text{off}}(n)) + (\log Z^{\text{off}}(n) - n \log Z(1))]_{n=1} \quad (2.13)$$

Each of the terms in the brackets is the action for one of the configurations in 2.3.1. Since the off shell configuration that corresponds to a regularized cone differs by a first order term in $n - 1$ from a solution of the equations, we see that we can interpret the first parenthesis as the result of doing a first order variation away from a solution

(the solution with period n). This first order variation vanishes due to the equations of motion for the solution with period n . Notice that both metrics obey the same boundary conditions at the boundary, so that there are no boundary terms⁸.

So all that remains is the second parenthesis. The second parenthesis contains the difference between a smooth cone and a regularized cone. This receives a contribution only from the region near the tip of the cone. This contribution is extensive in the area of the horizon, namely the area of the surface transverse to the tip of the cone. The region near the rounded tip of the cone contains an integral of $\int d^2x \sqrt{g} R$ along the cone directions which gives

$$\int_{\text{Reg Cone}} d^2x \sqrt{g} R \sim 4\pi(1 - n) \quad (2.14)$$

Thus, the final answer has the form

$$S = \frac{1}{16\pi G_N} (\text{Area}) \left(-n \partial_n \int_{\text{Reg Cone}} d^2x \sqrt{g} R \right) = \frac{\text{Area}}{4G_N} \quad (2.15)$$

One can consider a metric that explicitly regularizes the cone, such as [34]

$$ds^2 = dr^2 g^2(r) + r^2 d\tau^2 \quad (2.16)$$

where $g = n + o(r^2)$ at $r \sim 0$ and $g = 1$ for $r > a$, where a is a small distance which sets the size of the regularization. Inserting this metric into the gravitational action we get (2.14). One can choose a completely explicit function such as $g = 1 + (n - 1)e^{-r^2/a^2}$, for example. In this case we can see explicitly that the metric perturbation is of order $(n - 1)$.

⁸The absence of boundary terms is clearest if we write the action in a non-manifestly covariant form using only first derivatives of the metric. Then the fact that the two configurations obey the same boundary conditions for the metric implies that there are no boundary terms.

2.3.2 Entropy from apparent conical singularities

Another way to think about this problem is as follows. First we note that, since the solutions are invariant under time translation, the evaluation of $\log Z(n)$ is the same as

$$\log Z(n) = n[\log Z(n)]_{2\pi} \quad (2.17)$$

where $[\log Z(n)]_{2\pi}$ is the gravitational action density for the solution labeled by n but integrated over τ from $[0, 2\pi]$ (instead of $[0, 2\pi n]$). We can now write the entropy as

$$S = -n^2 \partial_n [\log Z(n)]_{2\pi} \quad (2.18)$$

Note that the solution labeled by n is a smooth geometry if the τ circle has period $2\pi n$. On the other hand, imagine we wanted to view it as a configuration where the τ period continues to be 2π . In that case, it is a geometry with a conical singularity whose opening angle is $2\pi/n$. Thus we can view

$$[\log Z(n)]_{2\pi} \quad (2.19)$$

as the gravitational action of a configuration with $\tau = \tau + 2\pi$ but with a conical singularity with opening angle $2\pi/n$, *without including any curvature contribution from the conical singularity*. Then we see that the expression of the entropy (2.18) involves taking a derivative with respect to n . When we change n we are changing the opening angle of the singularity. In addition, we are changing the metric and other fields everywhere since they have to adjust to this new strength of the singularity. However, since the original solution (the solution with $n = 1$) is a solution of the equations, we would naively expect that a first order variation of the metric and other fields should vanish due to their equations of motion. This naive expectation is essentially right, except for the fact that we are changing the boundary conditions

at the origin, since the strength of the conical singularity is being changed. Thus, the only change in the action comes from a boundary term. In other words, when we change n the action changes as

$$\begin{aligned}
-\partial_n[\log Z(n)]_{2\pi}|_{n=1} = & \int E_g \partial_n g + E_\phi \partial_n \phi + \\
& + \frac{1}{8G_N} \int_{r \sim 0} dy^{D-2} \sqrt{g} (\nabla^\mu \partial_n g_{\mu r} - g^{\mu\nu} \nabla_r \partial_n g_{\mu\nu}) = \frac{A}{4G_N} (2.20)
\end{aligned}$$

where E_g and E_ϕ are the equations of motion for the metric and other fields, which vanish. Here y are the coordinates along the $r = 0$ surface. The boundary term vanishes at the large r boundary since we are choosing boundary conditions in such a way that the variation of the action gives the equations of motion without extra boundary terms. On the other hand, at the horizon (at $r = 0$), we do get a contribution from the boundary term. This boundary term produces the area contribution. Note that the n derivatives of the metric are evaluated at the horizon. For example, in the parametrization $ds^2 = n^2 dr^2 + r^2 d\tau^2$ near the origin, we get, as the only non-vanishing component, $\partial_n g_{rr}|_{n=1} = 2$. With these expressions we can evaluate the parenthesis in (2.20) and obtain $2/r$.

This derivation easily generalizes to theories with higher derivative actions, giving the Wald entropy [36, 37, 38].

Note that in both cases we used explicitly the locality of the action along the τ direction. It would be interesting to find the corresponding formula in weakly coupled string theory exactly in α' .

2.4 Argument for the entropy formula (2.4)

2.4.1 Properties of the metric for n integer

For $n = 1$, the boundary contains a circle which we label by the coordinate τ . Recall that the boundary is the surface where we are putting boundary conditions. This circle is non-contractible on the boundary, but it can be contractible in the interior of the geometry. Here by boundary, we mean the boundary where we set boundary conditions for the gravitational action. It need not be an asymptotic boundary.

The metric and all fields are periodic on this circle. Let us collectively denote these fields as $\psi(\tau)$, with

$$\psi(\tau) \sim \psi(\tau + 2\pi) \quad (2.21)$$

Of course, the fields depend on other coordinates, but here we are highlighting their τ dependence. We impose boundary conditions

$$\psi(\tau)|_{\text{Boundary}} = \hat{\psi}_B(\tau), \quad \hat{\psi}_B(\tau) = \hat{\psi}_B(\tau + 2\pi) \quad (2.22)$$

where we specify the functions $\hat{\psi}_B(\tau)$, which are periodic.

The solution with $n > 1$, has exactly the same boundary conditions (2.22), but we require the periodicity $\tau = \tau + 2\pi n$ on the τ circle. This implies that the boundary conditions have a Z_n symmetry. We assume that the bulk solution continues to have this Z_n symmetry.⁹

9

In principle, the replica Z_n symmetry can be broken. Our discussion assumes that it is not broken. The simplest gravity solutions can also develop other instabilities. For example, if one considers gravity in AdS_{d+1} with a boundary $H_{d-1} \times S^1$. If the radius of S^1 is equal to the radius of H_{d-1} then the full solution is AdS , viewed as a black brane with a hyperbolic spatial section. If we make the S^1 n times larger, then for large n , we approach an extremal black hole with an $AdS_2 \times H_{d-1}$ near horizon geometry. This can lead to bad tachyons for $m^2 R_{AdS_{d+1}}^2 < -d/4$. Thus if the original AdS_{d+1} has tachyons in the allowed range $-d^2/4 \leq m^2 R_{AdS}^2 < -d/4$, then we will have an instability. This is similar to the discussion of [39], where an extremal Reissner-Nordstrom black brane in AdS_4 , with a near horizon geometry $AdS_2 \times R^2$ was considered. Good tachyons in AdS_4 can be bad tachyons in the AdS_2 region if $-\frac{9}{4} < m^2 R_{AdS}^2 < -\frac{3}{2}$. See [40] for further discussion. Here we

Each of the solutions for $n > 1$ has a special codimension two surface which is left invariant by the action of Z_n . We will focus on this surface. We can choose a coordinate r which is a radial coordinate away from this surface and an angle τ . The true angle around the surface is really $\alpha = \tau/n$, we have chosen τ to have the same period as the one we have at infinity ($\alpha \sim \alpha + 2\pi$). The metric in the two directions transverse to this surface has the form

$$ds^2 = n^2 dr^2 + r^2 d\tau^2 + \dots \tag{2.23}$$

where the factor of n comes from demanding that there is no singularity at $r = 0$. In addition, all fields are required to have an $e^{ik\tau}$ dependence, with integer k . This comes from the period of τ and the Z_n symmetry. Thus, a scalar field would behave as $r^n e^{i\tau} \sim r^n e^{in\alpha}$ near the origin, as results from demanding that it is non-singular.

As a side remark, notice that if the bulk space has no fixed points under the Z_n action, then this means that we can choose the coordinate τ in the interior so that this circle never shrinks. An example is a space with topology $R^{d-1} \times S^1$, but with a metric that depends on the coordinate along the S^1 . In these cases the entropy is zero. The reason is very simple, the solution for the n^{th} replica is the same as the solution with $n = 1$ but with a longer circle so that $\log Z(n) = n \log Z(1)$. Here, of course, we used the locality of the classical action.

2.4.2 Metric for n non-integer

Here we make some assumptions on the form of the metric when n is not an integer. We will continue to impose exactly the same boundary condition (2.22), which is periodic with period 2π . This is *not* compatible with $\tau \rightarrow \tau + 2\pi n$. Now, in the

assume that we have no dangerous tachyons that can lead to these instabilities. Similar instabilities were observed computing the Renyi entropies of circular regions in the three dimensional interacting $O(N)$ model, we thank Igor Klebanov for this observation .

region where the τ circle has positive size, we can ignore this problem and think of τ as being non-compact. When we evaluate the action, we can integrate the τ direction from 0 to 2π and then multiply by n .

However, we expect that there is still a surface where the τ circle shrinks to zero. For the two dimensions transverse to this surface we impose that the metric continues to behave as in (2.23) , even though n is not an integer. The rest of the fields, including other components of the metric, are chosen so that they are periodic in $\tau \rightarrow \tau + 2\pi$ as in (2.21). This implies that the field configuration is singular at $r = 0$. However, we expect that this singularity is as harmless as the one we had for the scalar field in section 2 .

This is seems a reasonable assumption. As evidence for its validity we can point to the explicit example mentioned in section 2 .

An equivalent way to specify the solutions is to compactify the τ circle to $\tau + 2\pi$ in all cases (all values of n) and demand that there is a conical defect angle with opening angle $2\pi/n$ in the interior. We do not introduce any contribution to the action from the tip of the cone. In addition, we multiply the gravitational action by a factor of n . This is mathematically equivalent to what was discussed above and the reader can choose the preferred interpretation.

Note that this is similar to introducing a cosmic string (or cosmic $D - 3$ brane) with opening angle $2\pi/n$ in the original solution, with the metric backreacting as necessary to account for its presence.

As $n \rightarrow 1$ the solution goes over to the solution with $n = 1$. Thus, this analytically continued solution is close to the $n = 1$ solution and we can expand it in powers of $n - 1$.

2.4.3 Derivation of the minimal surface condition

We emphasized that for $n > 1$ we have a special surface where the circle shrinks, and is fixed under the Z_n action. But for $n = 1$ there is no obvious special surface, since there is no unique way to choose the coordinate τ in the interior once it is not associated to a $U(1)$ isometry. So, when we expand the solution in $n - 1$ we need to select a surface. Motivated by the Ryu-Takayanagi conjecture we want to select a minimal area surface. In this subsection we will explain the origin of this minimal area condition. The final conclusion is that the condition comes from demanding that the solution obeys the Einstein equations to leading order in $n - 1$. This derivation is essentially the same as the derivation of the equations of motion for a cosmic string (or $D - 3$ brane) from the behavior of the metric near the conical singularity. This problem was analyzed previously in [41, 42].

Two dimensional dilation gravity

It is good to start with a simple situation first. For that purpose we will consider a two dimensional dilaton gravity where the action is

$$-S_{\text{Grav}} = \frac{1}{16\pi} \int d^2x \sqrt{g} e^{-2\varphi} [R + 4(\nabla\varphi)^2 + \dots] \quad (2.24)$$

where the dots indicate other fields, or a potential for φ , etc. Notice that if we have a solution with a horizon, then $e^{-2\varphi}$ at the horizon plays the role of the area in Planck units of the higher dimensional gravity solutions. In this case the codimension two surface is just a point. The minimum area condition is that $e^{-2\varphi}$ is a minimum (or really an extremum) at this point. We will derive this condition from demanding that the configuration for small $\epsilon \equiv n - 1$ obeys the linearized field equations near $r = 0$. In other words, expanding the fields around the $n = 1$ solution, and *assuming the periodicity condition for the fields*, (2.21), we will see that we can only obey the equations if $\partial_i\varphi = 0$.

Let us say that as $n \rightarrow 1$ the special surface goes over to some point of the $n = 1$ manifold. Let us pick this point to be the origin in some coordinate system x^1, x^2 . Then the metric of the $n = 1$ solution around this point is $ds^2 = dx_1^2 + dx_2^2 + o(x^2)$. The field φ is regular at this point. Now, for $n - 1 = \epsilon$ we expect a metric of the form $ds^2 = e^{2\rho}(dx_1^2 + dx_2^2)$, with $e^{2\rho} = r^{2(\frac{1}{n}-1)}$, as $r \rightarrow 0$. Then to first order in ϵ we have $\delta\rho \sim -\epsilon \log r$ to be the first order solution. We consider the two following equations for two dimensional dilaton gravity

$$\begin{aligned} 0 &= e^{-2\varphi}(4\partial_z\varphi\partial_z\rho + 2\partial_z^2\varphi) + T_{zz}^{\text{matter}} \\ 0 &= e^{-2\varphi}(4\partial_{\bar{z}}\varphi\partial_{\bar{z}}\rho + 2\partial_{\bar{z}}^2\varphi) + T_{\bar{z}\bar{z}}^{\text{matter}} \end{aligned} \quad (2.25)$$

where $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. Here T^{matter} denotes the stress tensor for the rest of the fields of the theory, coming from the dots in (2.24). Expanding the first equation to first order we find

$$-2\partial_z\varphi(0)\frac{\epsilon}{z} + 2\partial_z^2\delta\varphi + \delta T_{zz}^{\text{matter}} = 0 \quad (2.26)$$

and a similar equation by expanding the second. Here $\partial_z\varphi(0)$ is the derivative of the field for the $n = 1$ solution at the origin. It is just a z independent constant. Since the matter stress tensor is not expected to be singular at order $1/r$, we find that

$$\partial_z^2\delta\varphi \propto \frac{\epsilon\partial_z\varphi(0)}{z}, \quad \partial_{\bar{z}}^2\delta\varphi \propto \frac{\epsilon\partial_{\bar{z}}\varphi(0)}{\bar{z}} \quad (2.27)$$

up to terms that are less singular as $r \rightarrow 0$. Now we assume that the solution for $\delta\varphi$ has a fourier expansion with integer powers of $e^{i\tau}$. The first equation in (2.27) suggests that we try a solution proportional to $\delta\varphi \propto z \log z$. However, the periodicity condition under shifts of τ suggests that we should consider $\delta\varphi \propto z \log(z\bar{z})$. However,

this is not a solution of the second equation. Thus this implies that the gradients of the field should vanish at the origin.

More formally, we can argue as follows. The periodicity condition implies that if we take the τ derivative of any field and integrate over τ between zero and 2π , we should get zero. This is true both for $\delta\varphi$ and its derivatives. In particular, note that the following combination of derivatives gives

$$\partial_\tau [(r\partial_r - 1)\partial_z\delta\varphi] \propto (z\partial_z - \bar{z}\partial_{\bar{z}})(z\partial_z + \bar{z}\partial_{\bar{z}} - 1)\partial_z\delta\varphi \propto \epsilon\partial_z\varphi(0) \quad (2.28)$$

where we used both equations in (2.27). Now the integral over τ of (2.28) should be zero, according to our assumption about the periodicity of $\delta\varphi$. This then implies that $\partial_z\varphi(0) = 0$.

In summary, in this case we found that the condition comes from the zz and $\bar{z}\bar{z}$ components of the Einstein equations. In higher dimensions we expect that this will come from Einstein's equations in the directions normal to the surface.

Note that if we changed the coefficient of the dilaton kinetic term in (2.24) from $4(\nabla\varphi)^2$ to $(4 + \sigma)(\nabla\varphi)^2$, then we would be adding terms of the form $\sigma\partial_z\varphi\partial_z\varphi$ to the equations in (2.25). Expanding around the background solution such terms lead to contributions that are subleading, in the expansion around the origin, compared to the terms already taken into account in (2.27). Thus, if we had a two dimensional action with a different coefficient for the dilaton kinetic term, we would have reached the same conclusion¹⁰.

Einstein gravity in D dimensions

We now go back to the case of Einstein gravity. In general, we can expand the metric

¹⁰This is to be expected since this coefficient can be changed by a field redefinition of the metric.

of the $n = 1$ solution around the special surface as

$$\begin{aligned}
ds^2 &= dr^2 + R^2 d\tau^2 + b_i d\tau dy^i + g_{ij} dy^i dy^j , \\
g_{ij} &= h_{ij} + r \cos \tau K_{ij}^1 + r \sin \tau K_{ij}^2 + o(r^2) \\
R &= r + o(r^3) , \quad b_i = o(r^2)
\end{aligned} \tag{2.29}$$

where r is coordinate normal to the surface and y_i are coordinates along the surface. Here K_{ij}^α are the two extrinsic curvature tensors. h_{ij} depends only on y_i but not on r or τ . When we deform away from $n = 1$ we assume that we cannot change the period of the cosines above.

When $n = 1 + \epsilon$ some of the metric components generically go like $r^{1+\epsilon}$. This can give rise to terms in the equations of motion going like $1/r$. These terms can only come from situations where we have two derivatives along the transverse directions (the r and τ directions). Such terms in the equations of motion are the same as the ones we would obtain by performing a dimensional reduction from D dimensions to the two transverse directions. This brings us back to the previous case. More explicitly, we write the full D dimensional metric as

$$ds^2 = e^{2\rho} (dx_1^2 + dx_2^2) + e^{-\frac{4\varphi}{D-2}} \hat{g}_{ij} dy^i dy^j + o(r^2) , \quad \det(\hat{g}_{ij}) = 1 \tag{2.30}$$

where \hat{g}_{ij} is the transverse metric appearing in (2.29) but rescaled so that its determinant is one. The off diagonal terms in (2.29) do not contribute to terms of order $1/r$ in the equations of motion. Both \hat{g}_{ij} and φ depend on all the coordinates, the y_i and the x_i . Here we have pulled out the overall volume factor of the transverse space and parametrized it by φ . Dimensionally reducing to the first two dimensions gives us (2.24), but with a different coefficient for the dilaton kinetic term. Thus, we obtain the same conditions that $\partial_{x^\alpha} \varphi = 0$ for $\alpha = 1, 2$. Now if we translate between

φ and the original metric (2.29) we find that

$$-4\varphi = \log(\det(h_{ij})) + x^1 K^1 + x^2 K^2 + o(r^2) , \quad K^\alpha = h^{ij} K_{ij}^\alpha \quad (2.31)$$

where K^α are the traces of the extrinsic curvature tensors. We then see that the condition $\partial_\alpha \varphi = 0$ implies that

$$K^1 = K^2 = 0 \quad (2.32)$$

Namely, the traces of the extrinsic curvatures should vanish. There are two directions that are transverse to the surface so we have two relevant extrinsic curvatures. These coincide with the equations of motion for a minimal area surface. There are two transverse directions to the surface and thus two equations. In appendix B we derive (2.32) directly in D dimensions, without doing the dimensional reduction.

Note that the non-trace part of the extrinsic curvatures are *not* constrained to vanish. In fact, already in our simple example of section 2 we have non-vanishing extrinsic curvature if we interpret the scalar field as coming from a component of a higher dimensional metric as in (2.11) .¹¹

2.4.4 Computation of the entropy using the cone method

Once we have established the form of the solution, we can compute the entropy using the cone method as explained in section 3. The arguments are similar, but one has to check that the mild singularities we discussed above cause no problems.

Let us discuss this first for the case of AdS plus a scalar field discussed in section 2. There, the singularity is only present in the scalar field which behaves as $\phi \sim r^\epsilon r e^{i\tau}$. With this mild singularity, if we integrate by parts in order to use the equation of mo-

¹¹ More explicitly, in the notation of that section, if the field at the origin goes as $\phi = r e^{i\tau}$ this leads to $\phi_1 = r \cos \tau = x^1$ and there is an extrinsic curvature component $K_{y^1 y^1}^1 = -K_{y^3 y^3}^1$ using the coordinates (2.11) and (2.5) for the 3-d part.

tion for ϕ , it is clear that we do not run into any problem at $r = 0$. The most dangerous term seems to come from the variation of $\delta \int dr r (\partial_r \phi)^2 = 2 \int dr r \partial_r \phi \partial_r \delta \phi \rightarrow r \partial_r \phi \delta \phi|_0$. However, $\delta \phi$ would also vanish at the origin if we are considering the variations that come from varying n in the solution. In other words, when we compare the correct configuration with $n - 1 > 0$ and a regularized cone, we can consider a regularized cone where ϕ has the same type of singularity at the origin. This shows that the first parenthesis in (2.13) vanishes.

The second parenthesis only gives us something interesting if we consider terms that have two derivatives acting on the metric, otherwise their contribution is going to be small as we remove the regulator. Thus, only the metric in the two directions transverse to the minimal surface are relevant. And in those dimensions the computation reduces to the usual one, with the contribution coming only from the curvature term in the action.

In conclusion, evaluating the differences in (2.13) we find that the answer is equal to the area, as we wanted to prove to argue (2.4) .

The discussion so far was completely local in the directions transverse to the “horizon”. Here by horizon we mean the point in the two transverse directions where the circle is shrinking to zero size. In some cases this “horizon” can have multiple disconnected regions. Then, we should sum over the areas of each of the horizons. Even when we have multiple horizons, the period of the τ circle is the same in the whole solution.

2.4.5 A comments on other $U(1)$ symmetries

Throughout this discussion we have focused on the particular geometric circle that we used to define the density matrix and the replica trick. We considered cases where we have no translation symmetry along the circle. However, we can have other $U(1)$ symmetries. As a simple example, we can have a $U(1)$ gauge field in the bulk. Then

as in the ordinary case, [6], the integral of the gauge field along the τ circle should vanish at the origin $\int_0^{2\pi n} d\tau A_\tau \Big|_{r \rightarrow 0} = 0$ (up to global gauge transformations), where the τ circle shrinks. This should hold for all n , both integer and non-integer. At the boundary we can fix the holonomy of A along the τ circle as we please. In order to compute the entropy of the density matrix with a chemical potential we should fix the integral $\hat{\mu} \equiv \int_0^{2\pi} A_\tau$ at the large r boundary. If we keep everything else fixed at the boundary but we vary $\hat{\mu}$, this has the interpretation of changing the density matrix $\rho \rightarrow e^{i\hat{\mu}Q}\rho$ where Q is the charge associated to the $U(1)$ symmetry. We can compute the entropy of this density matrix by treating this boundary condition as we treated all other boundary conditions. Namely, $A_\tau(\tau)$ is kept fixed. Therefore its circulation over the τ circle of length $2\pi n$ is $n\hat{\mu}$. Of course, if the holonomy in the τ circle is different at the origin ($r = 0$) than at the boundary, then we will have a non-zero field strength in the bulk. The computation of the entropy is identical to what we discussed in general.

This other $U(1)$ symmetry can also be an ordinary geometric isometry, and its treatment is similar.

2.5 Connection to the Ryu-Takayanagi formula

We presented the computation of the entropy of the gravitational density matrix in a form that is very general. The objective was to emphasize that (2.4) is really a statement about an analytic continuation of classical solutions. In this section we explain why the conjecture (2.4) for the entropy is related to the Ryu-Takayanagi formula for entanglement entropy.

The Ryu-Takayanagi formula is a conjecture in the AdS/CFT context [9]. In the quantum field theory one is interested in computing the entanglement entropy of a spatial region A on the boundary of the field theory. This spatial region has a

boundary ∂A . The conjecture is that this entanglement entropy is given by the area (in Planck units) of a codimension two minimal surface in the bulk whose boundary ends on ∂A .

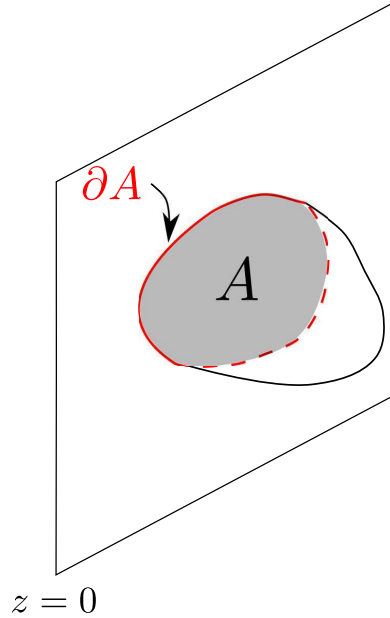


Figure 2.4: The Ryu-Takayanagi conjecture. The entanglement entropy of a region A in a conformal field theory is given by the area of a minimal surface in the bulk of AdS that ends on ∂A (the boundary of region A) at the boundary of AdS .

In principle, we can compute the entanglement entropy of the region A by using the replica trick [43, 44]. This is a general method for computing entanglement entropy in quantum field theories. The idea is to take n copies of the field theory and match them together so that by moving in a circle around ∂A we go from one copy to the next. Going n times around this circle we come back to the original copy. Thus at ∂A there is a conical defect with a $2\pi n$ opening angle. This appears to be a singular metric. However, one can choose a conformal factor that diverges at ∂A in such a way that the size of the circle around ∂A is finite.

This is most easily understood for simple regions [45, 46]. Imagine we have a conformal field theory $R^{1,d-1}$ at the boundary. Then we can choose a region A defined

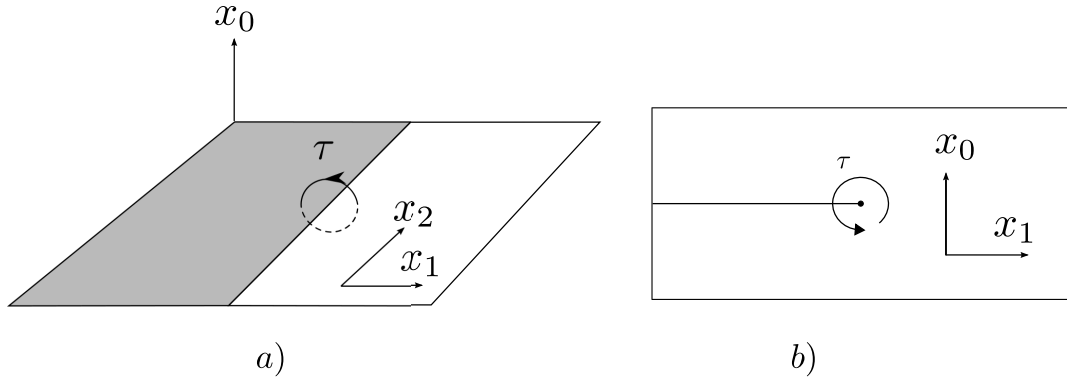


Figure 2.5: Geometries that we are considering to compute the entanglement entropy in the field theory. a) Semiplane. Region A is half of the plane and its boundary, ∂A , is at $x_0 = x_1 = 0$. b) x_0, x_1 view of the semiplane and the coordinate τ .

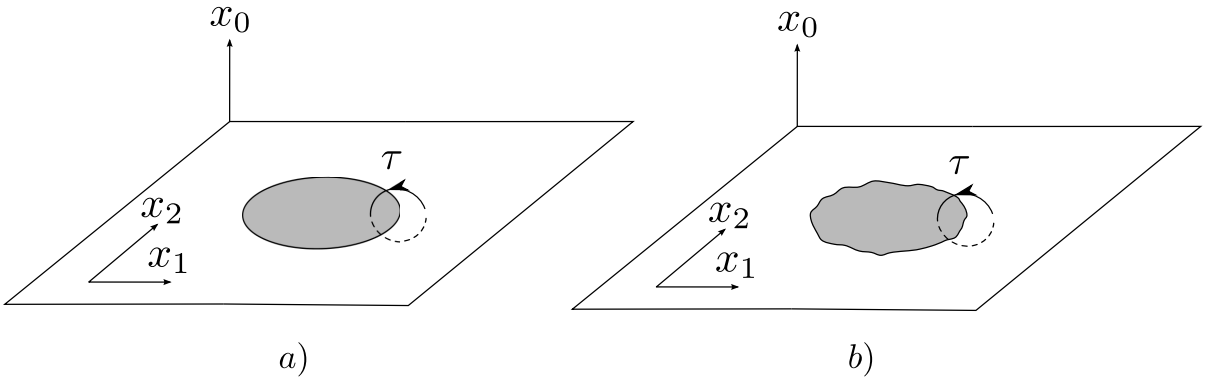


Figure 2.6: Other geometries in three dimensions. a) Disk configuration. b) Slightly deformed disk. The τ coordinate goes around ∂A .

by $x_1 > 0$, see figure 2.5 . The boundary of the region is the surface $x_0 = 0, x_1 = 0$. Going to Euclidean space, R^d , we can combine the directions x_0 and x_1 into two directions labeled in polar coordinates by r and τ . The metric is

$$ds^2 = r^2 d\tau^2 + dr^2 + d\vec{x}^2 \quad \rightarrow \quad d\tau^2 + \frac{dr^2 + d\vec{x}^2}{r^2} \quad (2.33)$$

where \vec{x} are the rest of the spatial coordinates. In the right hand side of (2.33) we have multiplied by an overall conformal factor $1/r^2$ to put the metric in the form of $S^1 \times H_{d-1}$. We can now easily perform the replica trick, it corresponds to changing the length of S^1 from 2π to $2\pi n$. Clearly this metric, $S_n^1 \times H_{d-1}$ is a perfectly legal metric and we can consider its gravity dual. It is a certain black brane. In this case, we have a $U(1)$ isometry in the rescaled coordinates and then the entropy computed using the replica trick or using the ordinary Gibbons-Hawking formula is exactly the same. Note that at the AdS boundary the circle S^1 has a nonzero size everywhere. In the interior of AdS it shrinks to zero at a “horizon”. Notice, in particular, that the half space region we discussed above can be conformally mapped to a spherical region $\sum x_i^2 \leq 1$. In this case, the circle S^1 appearing in (2.33) corresponds to a coordinates that goes around ∂A as in figure 2.5 .

Now, this was a very simple region. If we consider more complicated regions, then it is not possible to choose a system of coordinates and a conformal rescaling such that the metric is independent of the angular direction τ . In all cases we will have an angular direction, τ , since it is the direction we used to perform the replica trick construction. The choice of this coordinate is completely arbitrary, as long as it goes around the boundary of region A . As we go near ∂A we have a problem which locally looks like (2.33) , and we can choose a conformal factor which makes the metric non-singular as in (2.33) for all the replicas. The difference with (2.33) is that, as we increase r , we will have extra terms in the metric that can have some τ dependence. This dependence always involves powers of $e^{\pm i\tau}$ since this is just the statement that the τ direction is parametrizing circles in the original boundary geometry. All the statements in this paragraph involve the boundary geometry, the geometry where the field theory lives. These replica trick boundary geometries simply amount to letting

the circle have size $\tau \sim \tau + 2\pi n$, *without changing any of the functions that appear in the boundary geometry*. All such functions are periodic under $\tau \rightarrow \tau + 2\pi$ ¹².

Thus, the field theory replica trick, can be translated, via the standard AdS/CFT dictionary [47, 48], to a problem in gravity which is identical to the problem that we discussed in section 4. Here no conjecture is involved other than the original AdS/CFT relation. The replica trick then defines the entropy as in (2.3) . In order to do that, we need to analytically continue in n to $n \sim 1$.

The Ryu-Takayanagi conjecture boils down to a statement in classical geometry. It is the statement we discussed in section 4. Computing $\log Z(n)$, using smooth geometries, analytically continuing in n , and computing the entropy defined in (2.3) gives the area formula in (2.4) ¹³

Notice that in a setup where A is a spatial region contained at $x^0 = 0$ on the boundary, then there is a time reflection symmetry $x^0 \rightarrow -x^0$, which translates into $\tau \rightarrow -\tau$ for the circle in the Euclidean solution. This implies that we can go to Lorentzian signature, as usual with $x^0 \rightarrow ix^0$. This translates into $\tau \rightarrow it$. Now the region where the τ circle is shrinking to zero corresponds to a horizon in the bulk. It is a horizon for an observer sitting at fixed small r .

There is a generalization of the Ryu-Takayanagi conjecture for situations that are time dependent [49] . It again involves an extremal surface ending on ∂A , but in the full Lorentzian spacetime. In those cases there is no obvious Euclidean continuation to perform the replica trick. This suggests that there should be a way to think about the problem which does not go through the Euclidean solutions and the replica trick.

We should remark that in some cases we can perform a replica trick in the Euclidean

¹² If we want to explicitly parameterize the metric in this way, we might need to choose different coordinate patches, as usual. When the coordinate patches are chosen in a τ dependent fashion, then the $\tau \rightarrow \tau + 2\pi n$ identification can produce spaces with an n dependent topology. This happens, for example, in the case that we have two separate intervals in a two dimensional CFT. (We thank Xi Dong for a discussion on this.)

¹³Of course, if there is more than one bulk solution we should in principle sum over all of them and the one with the minimal action dominates. If we have two sequences of solutions, labeled by n , we can take the $n \rightarrow 1$ limit for each sequence and select the one with the minimal action.

geometry for regions that depend on the Euclidean time and then one can analytically continue to the Lorentzian signature solutions. Some examples were discussed in [50]

2.5.1 General entanglement interpretation

In the introduction, we presented the computation of the generalized gravitational entropy as a property of the density matrix constructed by integrating over a circle in Euclidean time. It is natural to ask whether there is a general Lorentzian interpretation that involves entanglement. This is indeed the case in the Ryu-Takayanagi discussion of entanglement of a subregion of the boundary.

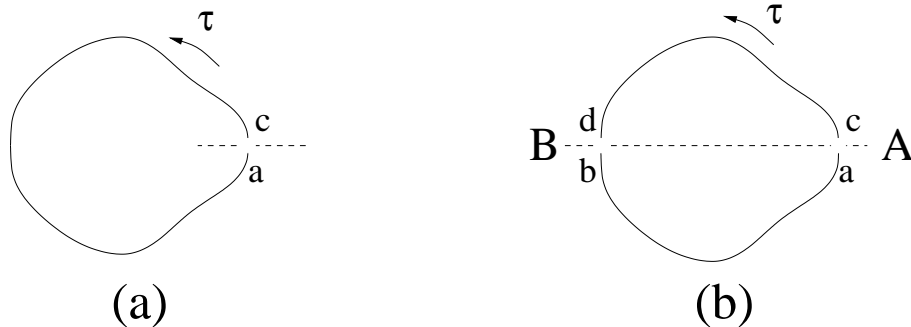


Figure 2.7: We consider periodic boundary conditions with a reflection symmetry $\tau \rightarrow -\tau$. In (a) we see that by cutting at $\tau = 0$ we get a density matrix ρ_{ac} , where a and c label the states on the two sides of the cut. In (b) we note that we can cut along the moment of time reflection symmetry $\tau = 0, \pi$. Then we get a pure state in two separate Hilbert spaces labeled by A and B . The bottom half of the picture can be viewed as a state ψ_{ab} and the top part as ψ_{cd}^\dagger . Tracing out over the B Hilbert space, we recover $\rho_{ac} = \sum_b \psi_{ab} \psi_{cb}^\dagger$. At this moment of time reflection symmetry we can also continue to Lorentzian signature.

Here we would like to point out that in very general situations we can also have an entanglement interpretation. Suppose that the boundary conditions have a moment of time reflection symmetry. Say that this acts as $\tau \rightarrow -\tau$. Then by cutting the boundary conditions at $\tau = 0, \pi$ we can interpret the lower part of the evolution as

specifying a pure state $|\Psi\rangle$ in the product of two theories, which we call A and B . See 2.5.1 . Similarly, the upper part can be viewed as specifying the state $\langle\Psi|$. The density matrix can then arise by tracing over one or the other subsystem. And the entropy can be interpreted as entanglement entropy for system A with B . This is the same as in the eternal black hole discussion [51, 10].

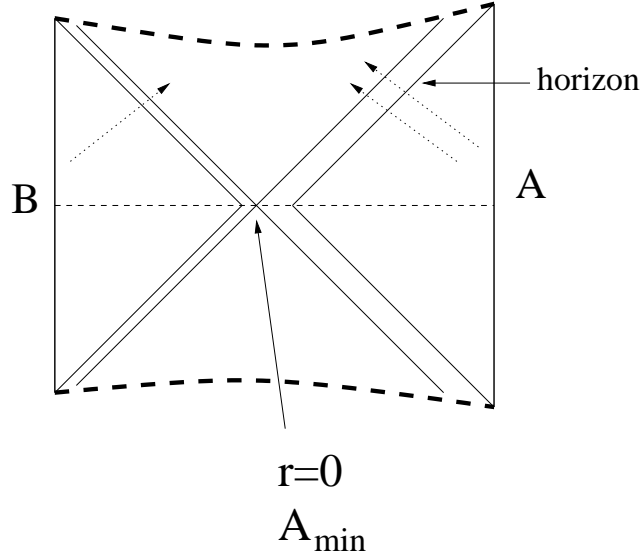


Figure 2.8: Here we consider a situation with asymptotically AdS boundary conditions. The boundary conditions contain a small time dependent deformation which vanishes at infinity. So in the far future we settle down into a stationary black hole on both sides. The entropy of these black holes is bigger than the entropy of the initial entanglement since, the time dependent boundary conditions have sent in energy and have increased the entropy of the system. In other words, there was a non-zero flux of energy through the horizon which increased its area. The dotted lines indicate the matter falling through the horizon.

The bulk solution is also expected to have a time reflection symmetry in this case. Under $\tau \rightarrow i t$ we get a Lorentzian solution. The vicinity of $r = 0$ looks locally like Rindler space. This procedure generically produces a time dependent solution and we might get singularities or horizons in the boundary conditions. We can consider a situation where the Lorentzian time evolution can be performed out to infinite time without ever connecting again the two boundary regions or encountering singularities

on the boundary. An example is the following. We start from a Euclidean black hole but with a small perturbation of the boundary conditions which is smooth in Euclidean time and goes to zero at large Lorentzian time. More concretely, we can consider the model of section 2 and set the boundary conditions $\phi_B = \frac{\eta(1+\cos\tau)}{2+\cos 2\tau}$. When we go to Lorentzian time this becomes $\phi_B = \frac{\eta(1\pm\cosh\tau)}{2+\cosh 2t}$ where the \pm corresponds to the A and B sides respectively. Note that these go to zero at large times. We expect that solution should be qualitatively like 2.5.1 . A very explicit solution with these characteristics was studied in [52, 53] ¹⁴.

In cases that arise from entanglement of subregions via AdS/CFT, the fact that the causal horizon is closer to the boundary than the minimal surface that computes the entanglement entropy was noted in [49](see also [54, 55, 56, 57]).

2.6 Conclusions and discussion

In this chapter we have noted that we can generalize the concept of Euclidean gravitational entropy to more general situations than the ones associated to thermal equilibrium. In particular, we have considered euclidean solutions that contain a circle $\tau \rightarrow \tau + 2\pi$. We have introduced a boundary, setting boundary conditions which are τ dependent but periodic under $\tau \rightarrow \tau + 2\pi$. Thinking of gravity as a holographic theory, we view these boundary conditions as defining the system. Euclidean evolution on the circle produces an un-normalized density matrix. The Euclidean solution gives us the trace of this density matrix. By performing the gravity version of the replica trick we have defined traces of n^{th} powers of the density matrix. These are geometries with exactly the same boundary condition as functions of τ , but where the τ variable is taken to have period $\tau \rightarrow \tau + 2\pi n$. For integer n the bulk geometries

¹⁴ The solutions in [52, 53] are based on Janus solutions. Their boundary in Euclidean space has the form $S^1 \times \Sigma$ where Σ is a quotient of hyperbolic space. The S^1 is divided in two equal parts and the dilaton has a different value on each part. The Lorentzian continuation is obtained by continuing across the moment with a time reflection symmetry. The two boundaries different values for the dilaton. These values are constant in time. The bulk smoothly interpolates between the two.

are smooth and free of any conical defects. These geometries are computing the trace of the n^{th} power of the density matrix. By analytically continuing in n and taking a derivative near $n = 1$ we can compute a quantity that is interpreted as the Von-Neumann entropy of the underlying density matrix. Note that all computations are classical. The density matrix we are talking about is a hypothetical density matrix in some underlying theory of quantum gravity. In AdS/CFT situations we can give a precise definition for this density matrix.

A version of the Ryu-Takayanagi conjecture is that this generalized gravitational entropy, computed in this fashion, is given by the area of a minimal area surface in the original geometry (the solution with $n = 1$).

We have given some arguments for the correctness of the Ryu-Takayanagi conjecture. The arguments involved the assumption that we can analytically continue the geometries away from integer values of n . We further made the assumption that these analytically continued geometries, for small $\epsilon \equiv n - 1$, are smooth in the two directions transverse to the minimal area surface but can have mild singularities which are not important for evaluating the action. We do not view these metrics as physically meaningful, we view them just as a tool for deriving the Ryu-Takayanagi formula. Our assumptions were motivated by considering a simple example, described in section 2. But we have no further justification other than the fact that they hold in this example and seem reasonable assumptions. We have derived the minimal area condition by demanding the existence of a small deviation away from the $n = 1$ solution that is consistent with our assumptions on the type of singularities that are allowed. One simple way to state the type of allowed singularities is to do a dimensional reduction of the whole configuration to the two dimensions transverse to the minimal surface. Then we have a two dimensional metric, a dilaton field that multiplies the two dimensional curvature in the action and a set of other fields. Then the metric should be smooth and the gradient of the dilaton at the minimal surface should be

zero, which is the minimal area condition. All other fields can have mild singularities of the form $\phi \sim z|z|^{2\epsilon}$ at the origin. When n is not an integer we evaluate the gravitational action by integrating τ between $[0, 2\pi]$ and then multiplying by n . We have also argued that this method gives rise to the area formula for the entropy, essentially for the same reasons as for the case with the $U(1)$ symmetry. One way to understand this is that all non $U(1)$ -invariant fields are going to zero at the origin. Then the methods described in section 3 give the usual formula.

An alternative way to view the solutions is to imagine that we keep the original period of the circle, $\tau \sim \tau + 2\pi$ but we introduce a cosmic string (or cosmic $D - 3$ brane) with a $2\pi/n$ opening angle. In addition, we multiply the resulting action by a factor of n . For n close to one we have a very light cosmic string that deforms the geometry very slightly. We can then view the entropy formula as arising from the Nambu action for this cosmic string. Also the minimal area condition comes from minimizing this Nambu action. The long and detailed discussion that we presented tried to justify these statements in detail.

Although we have mainly discussed surfaces anchored in the boundary of AdS , where we have an underlying density matrix, these methods could be extended to more general cases. For our point of view it is necessary that we consider a surface at some distance from the entangling surface and perform the replica trick holding this surface fixed. The method selects a minimal surface. For example, in section 5.1, we saw that the RT formula encodes the entanglement between two regions of spacetime that are prepared in the Hartle-Hawking state, here we get the boundary reduced density matrix by tracing over the degrees of freedom of one of the subsystems.

Bianchi and Myers, [58], proposed that the area of arbitrary surfaces measures the entanglement of quantum gravity degrees of freedom between the inside and outside of this surface. While this is an interesting possibility, it cannot be derived from our method since we get *minimal* surfaces. Furthermore, drawing a small tubular

neighborhood around an arbitrary surface and performing our replica trick would lead to entanglement which is not computed by the original surface but by a minimal one, if there is one within the tubular region.

One interesting open question is whether one can generalize the derivation to the time dependent case considered in [49] , where, generically, there is no obvious Euclidean continuation.

Another interesting direction is to generalize the discussion to gravity with higher derivatives. The most naive conjecture is that the entropy is given by the Wald formula. However, this conjecture was argued to be wrong in [59] , where a modified conjecture was made for the case of Lovelock gravity. A more informed conjecture is to say that we get the Wald-Iyer formula proposed in section 7 of [37] . In fact, this reduces to the proposal in [59] for Lovelock gravity. It would be interesting to see whether this is correct and what the equations for the surface are.

2.7 Appendix A: Example of a scalar field in AdS_3

In this appendix, we consider a massive scalar field in AdS_3 and show explicitly that the entropy that we compute using the replica trick is equal to the modification of the area due to the presence of a non-zero scalar field background.

2.7.1 Massive scalar field

For a massive scalar field we have equations which are very similar to the ones in the text. We consider a complex scalar field of mass m . Setting the radius of AdS_3 to one we need to impose the boundary condition

$$\phi|_{r_c} = \eta e^{i\tau} r_c^{\Delta-2} , \quad \Delta = 1 + \sqrt{m^2 + 1} \quad (2.34)$$

where Δ is the scaling dimension of the corresponding operator and r_c is a large value of r which represents the cutoff surface. The relevant solution of the wave equation on the metric (2.7) is

$$\phi = \eta e^{i\tau} \frac{f(nr)}{f(nr_c)} r_c^{\Delta-2}, \quad f(r) = r^n {}_2F_1\left(\frac{n}{2} - \frac{\Delta}{2} + 1, \frac{n}{2} + \frac{\Delta}{2}; n+1; -r^2\right) \quad (2.35)$$

We then evaluate

$$\begin{aligned} \log Z(n) &= - \int d^3x \sqrt{g} [|\nabla\phi|^2 + m^2|\phi|^2] = -(2\pi n) L_x \phi_{r_c}^* r_c^3 \partial_r \phi|_{r_c} = \\ &= (2\pi L_x) |\eta|^2 [B(n, \Delta) + \text{linear in } n] \end{aligned} \quad (2.36)$$

where the terms linear in n also include all divergent terms. It is important that these counterterms do not give rise to any non-trivial n dependence. This is due to the fact that we keep the τ -dependence of the boundary conditions fixed as we vary n . We also defined

$$B(n, \Delta) = - \frac{2n^{3-2\Delta} \Gamma(2-\Delta) \Gamma\left(\frac{n+\Delta}{2}\right)^2}{\Gamma(\Delta-1) \Gamma\left(\frac{n-\Delta}{2} + 1\right)^2} \quad (2.37)$$

We can then compute the entropy to order η^2 from (2.3), which gives

$$\begin{aligned} S|_{\eta^2} &= - n \partial_n [\log Z(n) - n \log Z(1)]|_{n=1} = \\ &= - \eta^2 \left\{ \frac{4\pi [2(\Delta-2)\Delta + (1-\Delta)\pi \tan(\pi\Delta/2)] \Gamma(2-\Delta) \Gamma\left(\frac{\Delta+1}{2}\right)^2}{\Gamma\left(\frac{3-\Delta}{2}\right)^2 \Gamma(\Delta)} \right\} \end{aligned} \quad (2.38)$$

2.7.2 Change in the metric from Einstein's equations

Now we will study the backreaction of the scalar in the metric. The action is

$$-S = \int_{AdS_3} [R - 2\Lambda - |\nabla\phi|^2 + m^2|\phi|^2] \quad (2.39)$$

with $\Lambda = -1$. The equations of motion are

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}(R - 2) = T_{(\mu\nu)} \quad (2.40)$$

where $T_{\mu\nu} = \partial_\mu\phi^*\partial_\nu\phi - \frac{g_{\mu\nu}}{2}(|\nabla\phi|^2 + m^2|\phi|^2)$. The ansatz for the metric is

$$ds^2 = \frac{1}{r^2 + g(r) + 1}dr^2 + (r^2 + 1)(1 + v(r))dx^2 + r^2dt^2 \quad (2.41)$$

where $g(r), v(r)$ will be $O(\eta^2)$. If we expand Einstein equations to first order we obtain three equations for the diagonal components. There are only two independent equations since the last one will give us the scalar wave equation when the first two are satisfied

$$\begin{aligned} g'(r) &= T_{xx} \frac{2r}{(r^2 + 1)} \\ v'(r) &= 2rT_{rr} - \frac{2rg(r)}{(r^2 + 1)^2} \end{aligned} \quad (2.42)$$

Since we consider a configuration with $\partial_x\phi = 0$, we can relate the components of the stress energy tensor: $T_{rr} = (\partial_r\phi)^2 + (1 + r^2)^{-2}T_{xx}$. We then find

$$v'(r) = 2r(\partial_r\phi)^2 + \frac{\partial}{\partial r} \left(\frac{g(r)}{r^2 + 1} \right) \quad (2.43)$$

And

$$v(0) = -2 \int_0^\infty dr r |\partial_r\phi|^2 \quad \rightarrow \quad S|_{\eta^2} = 4\pi\delta A = -\eta^2(4\pi L_x) \int_0^\infty dr r |\partial_r\phi(r)|^2 \quad (2.44)$$

where we use that the second term in (2.43) is a total derivative and that $g(0) = 0$ due to the regularity condition for the metric at the origin. In addition $g/r^2 \rightarrow 0$ at

infinity. In our units ($16\pi G_N = 1$), the black hole formula is $S = 4\pi A = 4\pi A_0(1 + \frac{v(0)}{2}) = 4\pi(A_0 + \delta A)$. Substituting the solution for $\phi(r)$ for $n = 1$ (2.35), and integrating, we get the same as in (2.38). We checked this only numerically, but below we will show it without performing the explicit calculation.

2.7.3 The two quantities are the same

In the above computation we actually did not need to solve all the equations to the end in order to show that the two results are the same.

We will rearrange the entropy formula for the scalar so that we get an expression that is simpler to compare with the area contribution. The lagrangian $\mathcal{L}(g_{\mu\nu}, \phi, \nabla_\mu \phi)$ is a function of τ . When we evaluate the gravitational action, we integrate over all coordinates except τ . Then we first integrate over τ from zero to 2π and then multiply by n . We can do this both for integer or non-integer n . We denote the τ integral as $[\log Z(n)]_{2\pi}$. Then we have

$$\log Z(n) = n[\log Z(n)]_{2\pi} \quad (2.45)$$

Then the entropy formula (2.3) simplifies and we get

$$S = -n\partial_n \{n[\log Z(n)]_{2\pi} - n \log Z(1)\}_{n=1} = -\partial_n [\log Z(n)]_{2\pi}|_{n=1} \quad (2.46)$$

And the later expression can be straightforwardly evaluated, using¹⁵ $\sqrt{g}T_{\mu\nu} = \frac{\partial\sqrt{g}\mathcal{L}}{\partial g^{\mu\nu}}$,

$$\begin{aligned} -\partial_n[\log Z^{\text{matter}}(n)]_{2\pi} &= \int_0^{2\pi} d\tau \int dx dr \sqrt{g} \left(T_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} + \frac{\partial \mathcal{L}}{\partial \phi} \partial_n \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_n \partial^\mu \phi \right) = \\ &= \int_0^{2\pi} d\tau \int dx dr \sqrt{g} T_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} \end{aligned} \quad (2.47)$$

In the last line we used the equations of motion (of course $\frac{\delta S}{\delta \phi} = 0$). One can check that the expression with the stress energy tensor gives us $\sqrt{g}T_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} \Big|_{n=1} = -2\eta^2 r f'^2$, so

$$S - S_0 = -\eta^2 4\pi L_x \int dr r f'^2 \quad (2.48)$$

In writing (2.47) we have only included the action of the scalar field in the computation.

We can now show that we get the area, without using explicit expressions. This can be done as follows. First note that in the second line of (2.47) we can use Einstein's equation to write $T_{\mu\nu}$ in terms of the Einstein tensor, which is related to the variation of the gravitational action. We end up with an expression of the form

$$\int d\tau dx dr \sqrt{g} G_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} \Big|_{n=1} \quad (2.49)$$

This is closely related to the derivative of the gravitational part of the action. As we explained above we know that the gravitational part of the action has no term of order η^2 . Thus we know that the ∂_n derivative of the gravitational part vanishes at order η^2 . This derivative is the same as (2.49) up to a total derivative term

$$\partial_n[\log Z^{\text{Grav}}(n)]_{2\pi} \Big|_{\eta^2} = 0 = 2\pi \left[\int dx dr \sqrt{g} G_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} - \int dx \sqrt{g} \nabla_\mu \partial_n g^{\mu r} \Big|_{r=0} \right]_{\eta^2} \quad (2.50)$$

¹⁵We define it like this because the action is $\log Z(n) = \int_{AdS_3} (\mathcal{L}_{\text{Grav}} - \mathcal{L}_{\text{matter}})$ so the field equations read $G_{\mu\nu} = T_{\mu\nu}$.

The last term gives the area of the horizon, or more precisely the area of the horizon at order η^2 .

These are the same manipulations that one can do in general, but we have done all steps explicitly above to check that everything indeed works in situations with no U(1) symmetry.

2.7.4 Real scalar

We now consider the case of a real scalar field $\phi = f(r) \cos \tau$, $f(r)$ is the same as before but now the stress energy tensor no longer has the $U(1)$ symmetry

$$T_{\mu\nu} = T_{\mu\nu}^0 + T_{\mu\nu}^1 e^{i2\tau} + T_{\mu\nu}^{-1} e^{-i2\tau} \quad (2.51)$$

And $T_1^* = T_{-1}$. The metric has the same fourier decomposition, so v and g in (2.41) also have three fourier components. The entropy coming from the change in the area is

$$S|_{\eta^2} = \int_0^{2\pi} d\tau v(0) = 2\pi v^0(0) \quad (2.52)$$

where v^0 is the constant component of v . It is easy to check that $v^0(0) = -\int dr r f'^2$.

The scalar action contributes as follows

$$S|_{\eta^2} = \int d\tau dr (-2r f'^2 \cos^2(k\tau)) = -2\pi \int dr r f'^2 \quad (2.53)$$

So we find agreement once more, and the result is precisely half of the complex scalar.

2.8 Appendix B: Derivation of minimal area condition for the general case from a explicit calculation

In this appendix, we obtain the minimal area condition of section 4 without using dimensional reduction. As in section 4, we derive this condition from requiring that the analytically continued solution satisfies the linearized equations of motion near $r = 0$.

The metric of the $n = 1$ solution, which satisfies (locally) the equations of motion is

$$\begin{aligned}
 ds^2 &= dx_1^2 + dx_2^2 + g_{ij}(dy^i + b_\alpha^i dx^\alpha)(dy^j + b_\alpha^j dx^\alpha) + o(r^2) , \\
 g_{ij} &= h_{ij} + x_1 K_{ij}^1 + x_2 K_{ij}^2 , \quad b_\alpha^i \sim o(r)
 \end{aligned} \tag{2.54}$$

Here, y_i are the directions along the surface. Now, we do the replica trick, that is, we change the periodicity of the τ circle from 2π to $2\pi n$ and analytically continue n to $1 + \epsilon$. In this way, the metric will be modified to linear order in ϵ

$$ds^2 = e^{2\rho}(dr^2 + r^2 d\tau^2) + g_{ij}(dy^i + b_\alpha^i dx^\alpha)(dy^j + b_\alpha^j dx^\alpha) + \delta g \tag{2.55}$$

Where we decomposed the perturbation in a part that makes the metric smooth $\rho = \delta\rho = -\epsilon \log r$ and a perturbation δg that has components δg_{ab} valued in all directions. For simplicity we work with z, \bar{z} coordinates: $x_1 = \frac{z+\bar{z}}{2}, x_2 = \frac{z-\bar{z}}{2i}$. As a gauge condition, we set $\delta g_{zz} = \delta g_{\bar{z}\bar{z}} = 0$. We also set $\delta g_{z\bar{z}} = 0$, since this variation is included in ρ . We require the perturbation, δg_{ab} to be periodic: $\delta g_{ab}(\tau) \sim \delta g_{ab}(\tau + 2\pi)$.

We want to compute the linearized equation of motion $\delta G_{zz} = \delta T_{zz}$. In particular, we want to focus on the terms that can be divergent, going like $1/r$ near the origin.

We find

$$\begin{aligned}\delta R_{zz} &= \frac{-\epsilon}{z} K_z + \frac{1}{2} (2\delta g^p_{z;zp} - \delta g_{;zz} - \nabla^2 \delta g_{zz}) + (\text{regular as } r \rightarrow 0) \\ &= \frac{-\epsilon}{z} K_z - \frac{1}{2} \partial_z^2 \delta \gamma + \dots\end{aligned}\tag{2.56}$$

where $\delta \gamma \equiv g^{ij} \delta g_{ij}$ and $K_z = \frac{K^1 - iK^2}{2}$. In (2.56) we neglected the terms that have y_i derivatives because we expect them to be regular, only terms with two x^α derivatives can contribute to this order.

Now, since the stress energy tensor is not expected to be singular, the equations of motion imply that the two potentially divergent terms should cancel

$$\begin{aligned}\frac{1}{2} \partial_z^2 \delta \gamma &= \frac{-\epsilon}{z} K_z \\ \frac{1}{2} \partial_{\bar{z}}^2 \delta \gamma &= \frac{-\epsilon}{\bar{z}} K_{\bar{z}}\end{aligned}\tag{2.57}$$

These are the same equations as before (2.27), which are only satisfied for a periodic function, $\delta \gamma(\tau) \sim \delta \gamma(\tau + 2\pi)$, if $K_z = K_{\bar{z}} = 0$. Note that although the equations of motion are well behaved for $K_z = 0$, the Riemann tensor diverges, as we discussed in section 2. This discussion is similar to the analysis in [41] for the motion of a cosmic string.

2.9 Appendix C : Computation of the entropy for a disk

Here we consider a very simple example of gravitational entropy. We go through it to explain how one can put boundary conditions at fixed distance.

Consider the metric $ds^2 = dr^2 + r^2 d\tau^2$. In addition, we can have other dimensions, but let us assume we can ignore them. In this case, we can say that we pick an $r = r_c$

and we set up the boundary conditions there. We demand that the metric in the angular direction is

$$ds_{bdy}^2 = r_c^2 d\tau^2 \tag{2.58}$$

at the boundary $r = r_c$. We now consider the situation with $\tau \sim \tau + 2\pi n$. We should consider now metrics with the same boundary condition (2.58), but compatible with the new period. These metrics are

$$ds^2 = n^2 dr^2 + r^2 d\tau^2 \tag{2.59}$$

We can evaluate the gravitational action for these spaces and obtain

$$\log Z(n) = \frac{1}{16\pi G_N} \left[\int \sqrt{g} R + 2 \int_{bdy} K \right] = \frac{A}{4G_N} \tag{2.60}$$

which is independent of n . Here A is the area of the transverse directions which were not explicitly mentioned above. Using the usual formula, we get the expected area formula for the entropy.

We have included this trivial computation to explicitly show how gravity regularizes the divergent contribution that one normally gets in field theory. In fact, there is no divergence because there was no conical space in this computation!. Of course, this begs the question of whether the finite part of the one loop correction computed by performing a one loop computation around the above geometries is indeed the same as the finite part of the one loop corrections computed using the conical spaces that appear in the field theory discussion of the replica trick.

Chapter 3

Quantum corrections to holographic entanglement entropy

3.1 Holographic entanglement entropy

In quantum field theories, it is interesting to compute the entanglement entropy among various subregions. For example, we can consider a region A and compute the entanglement entropy between region A and the rest of the system, see figure 4.1. In theories with a gravity dual there is a very simple prescription for computing this entropy [9, 24]. We first find a minimal area surface that ends on the boundary of region A , at the boundary of the bulk, see figure 4.1. Then the entropy is given by the area of this surface,

$$S_{cl}(A) = \frac{(\text{Area})_{\min}}{4G_N} \quad (3.1)$$

In situations where we can apply the replica trick, this formula was proven for AdS_3 in [22, 23] and more generally in [14]. This is the correct result to leading order in the G_N expansion. If the boundary theory is a large N gauge theory, then (3.1) is of order N^2 . The leading term (3.1) comes from classical physics in the bulk. Here

we consider the quantum corrections to this formula. Namely, corrections that come from quantum mechanical effects in the bulk. These are of order G_N^0 (or N^0).

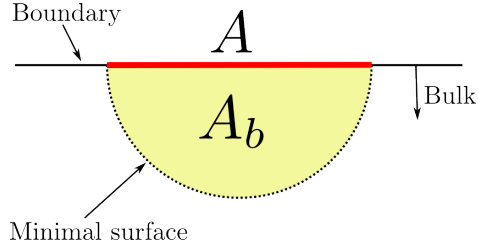


Figure 3.1: The red segment indicates a spatial region, A , of the boundary theory. The leading contribution to the entanglement entropy is computed by the area of a minimal surface that ends at the boundary of region A . This surface divides the bulk into two, region A_b and its complement. Region A_b lives in the bulk and has one more dimension than region A . The leading correction to the boundary entanglement entropy is given by the bulk entanglement entropy between region A_b and the rest of the bulk.

We find that the quantum corrections are essentially given by the bulk entanglement entropy. More precisely, the minimal surface that appears in (3.1) divides the bulk into two regions. We denote by A_b the bulk region that is connected to the boundary region A , see figure 4.1. Then the bulk quantum correction is essentially given by the bulk entanglement entropy between region A_b and the rest of the bulk. Namely, at this order, we can think of the bulk as an effective field theory living on a fixed background geometry and compute the entanglement entropy of region A_b as we would normally do in any quantum field theory¹. This is a computation in the bulk effective field theory, it depends on the details of the bulk fields. We can then

¹Caution: do not confuse the bulk entanglement entropy (3.3) with the one computed by the area formula (3.1). Both are computed in the bulk and are entanglement entropies, so unfortunately we have a clash of terminology. Hopefully, this will not cause confusion. Note also that [58] discussed a proposal of entanglement entropy in gravitational theories which does not require the surfaces to be minimal.

write the quantum correction as

$$S(A) = S_{cl}(A) + S_q(A) + \mathcal{O}(G_N) , \tag{3.2}$$

$$S_q(A) = S_{\text{bulk-ent}}(A_b) + \dots \tag{3.3}$$

The dots in (3.3) denote some extra one loop terms that can be expressed (like the classical term (3.1)) as an integral of local quantities. We will give a more detailed discussion of these terms below. They include terms that cancel the UV divergencies of the bulk entanglement entropy, so that S_q is a finite quantity. In the case of black holes, this expression for the quantum correction has been discussed in [7, 60, 43, 33, 61, 62, 63, 64], with increasing degrees of precision.

We first present a sketch of an argument for this formula. We then consider various simple checks.

3.2 An argument

In static situations one can use the replica trick to compute the entropy. This can be done to any order in the G_N expansion. In particular, it can be used to compute the quantum corrections. The procedure is the following. First we find the smooth bulk solutions for each integer n . The full partition functions around these geometries, including the classical action and all quantum corrections, gives the n^{th} Renyi entropies. One then computes the analytic continuation in n . At order G_N^0 this involves computing the one loop determinants around each of the classical solutions. There are many difficulties with this method, including constructing the smooth bulk solutions and then continuing the replica index to non-integer n . Despite these difficulties, in [65] this method was used to compute the quantum correction in a few cases using the classical bulk solutions constructed in [22]. On the other hand, the formula (3.3) is a shortcut, or an alternative expression, for the final answer in the same way that

(3.1) is a shortcut for the classical version of the replica method. The final answer (3.3) is physically clearer and easier to compute.

3.2.1 Review of the classical argument

Let us begin by reviewing the derivation of (3.1) in the classical case [14]. First consider the boundary field theory. The replica method is based on going to euclidean time and then considering an angular direction with origin at the boundary of region A . We label this by τ , with $\tau = \tau + 2\pi$, see figure 3.2 for an illustration.

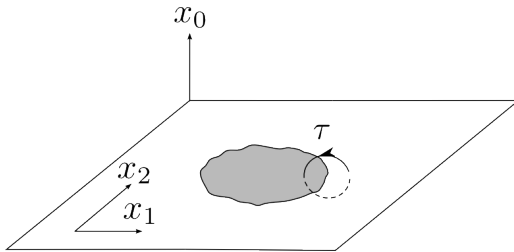


Figure 3.2: Slightly deformed disk and angular direction around the boundary.

We then consider the quantum field theory in a series of spaces given by the same metric but with $\tau = \tau + 2\pi n$, with integer n . With the naive boundary metric this τ circle shrinks at the boundary of the region A . However, we can rescale the metric by choosing a Weyl factor so that the circle does not shrink according to the boundary metric². We need to compute the partition function of the quantum field theory on this sequence of spaces and then analytically continue in n to compute

$$S = -\partial_n(\log Z_n - n \log Z_1)|_{n=1} = -Tr[\rho \log \rho] \quad (3.4)$$

where $\rho = \rho_A$ is the density matrix of region A in the boundary theory.

² If the theory is conformal this rescaling does not change the interesting physics. If it is not conformal we can still do it, but we will have spatially varying dimensionful couplings in the new space.

In theories with gravity duals, the partition functions can be computed by considering bulk solutions, g_n , which end at the boundary on the geometries we have defined above. Then one computes the gravitational action and partition functions for these solutions. This can be done to any order in the G_N expansion. The leading order answer comes from evaluating the classical action. We discuss this first.

These bulk geometries, g_n , are typically such that the circle τ shrinks smoothly in the interior. These geometries have a Z_n symmetry generated by $\tau \rightarrow \tau + 2\pi$, since the metric and all other couplings are periodic under this shift. See figure 3.3. It is convenient to introduce the geometries $\hat{g}_n = g_n/Z_n$. These are bulk geometries with exactly the same boundary conditions as the original geometry, g_1 , with $\tau = \tau + 2\pi$. However, these geometries typically contain a conical defect, or cosmic “string” (a codimension two surface) with opening angle $2\pi/n$. These sit at the points where the Z_n symmetry had fixed points, the points where the circle shrinks. Then the classical action obeys the condition $I[g_n] = nI[\hat{g}_n]$. This just follows from the fact that the classical action is the τ integral of a local lagrangian density. In evaluating $I[\hat{g}_n]$ we *do not* include any contributions from the singularity, not even a Gibbons-Hawking boundary term near the singularity³. We simply integrate the usual bulk lagrangian away from the singularity. We can now analytically continue the geometries \hat{g}_n to non-integer n . They have the same boundary as the $n = 1$ solution, but in the interior they contain cosmic “string” singularity of opening angle $2\pi/n$. When $n \rightarrow 1$ we have a very light cosmic string. The minimal area condition comes from the equations of motion of this cosmic string and the area formula (3.1) follows essentially from its action, see [14] for more details.

³ We still include the Gibbons-Hawking boundary term at the AdS boundary, as usual.

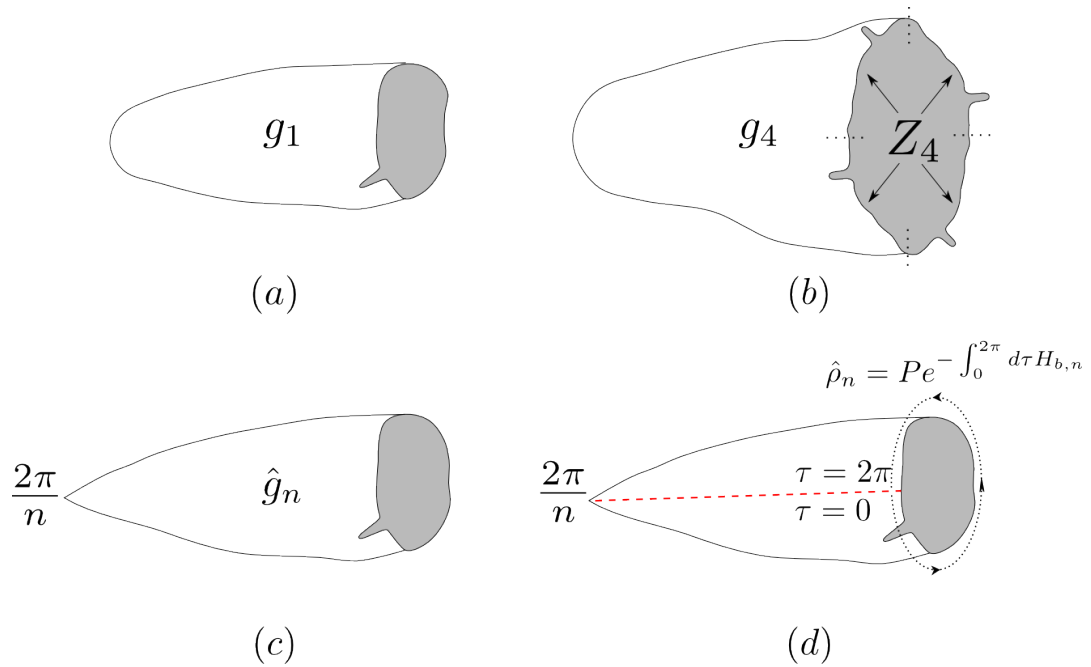


Figure 3.3: Computation of the entropy using the replica trick. *a*) Original geometry with no $U(1)$ symmetry. *b*) Replicated smooth geometry g_4 . *c*) After a Z_n quotient of the g_n geometry of *b*) we get the geometry $\hat{g}_n = g_n/Z_n$. It has a conical singularity with opening angle $2\pi/n$. This geometry has the same asymptotic boundary conditions as the original one in *a*). We can analytically continue this geometry to non-integer values of n . *d*) We use the geometries in *c*) to construct the density matrix $\hat{\rho}_n$. $\hat{\rho}_n$ is defined as a path integral on this geometry with arbitrary boundary conditions at $\tau = 0, 2\pi$. It can be computed using the bulk Hamiltonian for τ evolution.

3.2.2 Quantum argument

This is a generalization of the black hole discussion in [63, 64] to situations without the $U(1)$ symmetry.

At the quantum level, the replica trick instructs us to compute the partition function of all the bulk quantum fields around the black hole geometry. This involves computing the functional determinants for the quadratic fluctuations around the ge-

ometries g_n ⁴. In performing this computation we can view τ as a time evolution, so that the quantum partition function can be written as

$$\begin{aligned} Z_{q,n} &= \text{Tr}[Pe^{-\int_0^{2\pi n} d\tau H_{b,n}(\tau)}] = \text{Tr}[\hat{\rho}_n^n], \\ \hat{\rho}_n &\equiv Pe^{-\int_0^{2\pi} H_{b,n}(\tau)} \end{aligned} \tag{3.5}$$

Here $H_{b,n}(\tau)$ is the bulk time dependent hamiltonian that evolves the system along the τ direction⁵. It depends on n because the equal τ slices of the geometry g_n do depend on n . In the second equality we have used the fact that $H_b(\tau) = H_b(\tau + 2\pi)$. We have also defined a bulk (non-normalized) density matrix $\hat{\rho}_n$. The n subscript reminds us that the definition depends on n because the bulk geometry depends on n . In fact, we can assign $\hat{\rho}_n$ also to the bulk geometry $\hat{g}_n = g_n/Z_n$. Up to now the discussion was for integer n .

Now we analytically continue to non-integer n as follows. We consider the bulk geometry \hat{g}_n that we defined for the classical computation. We define again $\hat{\rho}_n$ as given by the same expression as in (3.5). Now $H_{n,b}$ is a Hamiltonian defined on equal τ slices of the geometry \hat{g}_n , with non-integer n . In summary, we define the partition function for non-integer n via

$$Z_{q,n} = \text{Tr}[\hat{\rho}_n^n], \quad \hat{\rho}_n \equiv Pe^{-\int_0^{2\pi} H_{b,n}(\tau)} \tag{3.6}$$

Here we are ignoring UV divergencies. More precisely, we can consider a UV regulator that is local and general covariant so that the discussion is valid for the regulated theory.

⁴Part of the bulk fields could be strongly coupled. For example, we can have a non-trivial CFT in the bulk. In that case, the bulk computation is more complicated, but the principle is the same (at this order in the G_N expansion): computing the partition function in the geometry g_n .

⁵ $H_{b,n}(\tau)$ is a local integral over a constant τ spatial slice. This should not be confused with the so called ‘‘modular hamiltonian’’, K , defined through $e^{-K} = Pe^{-\int_0^{2\pi} H_{b,n}(\tau)}$ which is a non-local operator.

We can now write the expression for the quantum correction as

$$\begin{aligned}
S_q &= -\partial_n (\log Z_{q,n} - n \log Z_{q,1})_{n=1} = -\partial_n (\log \text{Tr}[\hat{\rho}_n^n] - n \log \text{Tr}[\hat{\rho}_1])_{n=1} = \\
&= S_{\text{bulk-ent}} + S_{\dots} \\
S_{\text{bulk-ent}} &= -\partial_n (\log \text{Tr}[\rho_1^n] - n \log \text{Tr}[\rho_1])_{n=1} , \quad S_{\dots} \equiv -\frac{\text{Tr}[\partial_n \hat{\rho}_n]_{n=1}}{\text{Tr}[\rho_1]} \quad (3.7)
\end{aligned}$$

Here $S_{\text{bulk-ent}}$ involves only $\rho_1 \equiv \hat{\rho}_1$, which is the density matrix in the original ($n = 1$) geometry. This term computes the bulk entanglement entropy. The second term, S_{\dots} , arises due to the n dependence of the bulk solution and gives rise to the dots in (3.3). Let us find a more explicit expression for this term. For simplicity, we assume that the solution is such that only the metric is non-zero in the classical background and the rest of the fields are zero. This can be easily generalized. To evaluate S_{\dots} we go again to the Lagrangian formalism. The Lagrangian, $\mathcal{L}(\hat{g}_n, h, \varphi)$, depends on the background metric and the small fluctuations of all the fields: the metric fluctuations, h , as well as all the other fields denoted by φ . We can then write

$$S_{\dots} = \langle \int d\tau \partial_n \mathcal{L} \rangle = \int d\tau \langle E_{\mu\nu}(\hat{g} + h, \varphi) \partial_n \hat{g}^{\mu\nu} + d\Theta(\hat{g}, h, \varphi; \partial_n \hat{g}) \rangle - \int d\tau d\Theta(\hat{g}, \partial_n g) \quad (3.8)$$

where the brackets indicate quantum expectation values. In other words, we integrate over the fields h and φ . Here $E_{\mu\nu}$ represent equations of motion for the metric. These do not vanish because the quantum fluctuations are off shell. And Θ is related to all the partial integrations involved in going from a variation of the lagrangian to the equations of motion. We are using a notation similar to [37], where the reader can find explicit expressions. Θ is linear in $\partial_n \hat{g}$. The Θ term is the same as the one that gives rise to the Wald-like entropy formula [37]. We say Wald-like because we are considering a situation without a $U(1)$ symmetry. For the usual two derivative action, it gives rise to the area formula. Here we are evaluating it for a generic off shell configuration (since we have general variations h, φ) and computing the expectation

value. We have also subtracted the classical result. The simplest example where this term is nonzero is the following. Consider a theory with a scalar field with a coupling $\zeta\phi^2R$. If the scalar field is zero in the classical solution, this does not contribute to the classical black hole entropy. However, if we consider the small fluctuations of ϕ , we will get a term proportional to $\zeta\langle\phi^2\rangle(\text{Area})$. Such a term arises from the Θ term in (3.8). In general, we denote such terms as $\langle\Delta S_{\text{W-like}}\rangle$ ⁶. This is the expectation value of the formal expression for the Wald-like entropy⁷. We expect that the graviton gives rise to possible contributions to this term.

Now let us focus on the first term in (3.8). The equations of motion are non-zero because we are considering quantum fluctuations. We can formally write this term as

$$\int d\tau\langle E_{\mu\nu}\rangle\partial_n\hat{g}^{\mu\nu} = -\frac{1}{2}\int d\tau\langle T_{\mu\nu}\rangle\partial_n\hat{g}^{\mu\nu} \quad (3.9)$$

Here we have viewed the quantum expectation value of the equations of motion as a quantum generated expectation value for the stress tensor. This expectation value of the equations of motion will force us to change in the classical background. Indeed, to avoid ‘‘tadpoles’’ we will need to change the classical background $\hat{g} \rightarrow \hat{g} + \bar{h}$, where \bar{h} is small classical correction of order G_N in such a way that

$$E_{\mu\nu}(\hat{g} + \bar{h}) = -\langle E_{\mu\nu}\rangle = \frac{1}{2}\langle T_{\mu\nu}\rangle \quad (3.10)$$

where we are expanding the left hand side only to first order in \bar{h} . We can then reexpress (3.9) as

$$\int d\tau E(\hat{g} + \bar{h})\partial_n\hat{g} = \partial_n I_n(\hat{g}_n + \bar{h})|_{n=1} - \int d\tau d\Theta(\hat{g} + \bar{h}, \partial_n\hat{g}) \quad (3.11)$$

⁶ Note that for solutions where $R = 0$, we do not expect any ζ dependence on S_q or $S_{\text{bulk-ent}}$. $S_{\text{bulk-ent}}$ does not depend on ζ and one can easily show that the ζ dependence on the finite part of S_{\dots} cancels between δA and $\langle\Delta S_{\text{W-like}}\rangle$ terms.

⁷ In situations without a $U(1)$ symmetry, the general Wald-like expression for a general higher derivative theory is not known. For the purposes of this discussion we simply assume that such an expression exists. In the case of an action with R^2 terms the expression was found in [59, 66, 67, 68].

To first order in \bar{h} , $I_n(\hat{g}_n + \bar{h}) = I_n(\hat{g}_n)$ due to the equations of motion for \hat{g} . In (3.11) we are considering n very close to one. Here \bar{h} is the solution for $n = 1$, and we have kept it fixed as we vary n away from one. We have also ignored higher order terms in \bar{h} . The right hand side of (3.11) can be then rewritten as the change in the area due to the shift of the classical solution, $\frac{\delta A}{4G_N}$. Since the change in the background is of order G_N , this term is of order one. In a general higher derivative theory this will presumably become $\delta S_{\text{W-like}}$. A diagrammatic interpretation of this contribution is given in figure 3.4.

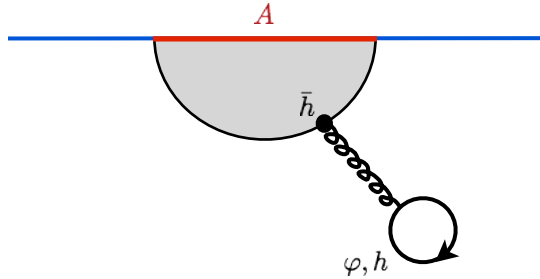


Figure 3.4: The contribution to S_{\dots} from the change in the area of the minimal surface, δA , due to the quantum corrections of the background. We can interpret this diagram as solving (3.10) for \bar{h} in terms of 1-loop stress tensor. We need to solve for \bar{h} along the minimal surface and integrate the stress tensor over all space.

In addition, we should add terms arising from the counterterms that render the bulk quantum theory finite. Such counterterms are given by local expressions in terms of the metric and the curvature, etc. Thus they look like the classical action itself. They contribute to the entropy via local terms of the same form as the ones we get for a general higher derivative local action. For example a counterterm of the form $\frac{1}{\epsilon^{D-2}} \int R$ gives a contribution $\frac{(\text{Area})}{\epsilon^{D-2}}$. There are similar contributions from higher derivative terms. We just apply the Wald-like formula for the counterterms⁸.

⁸As we mentioned this formula is unknown for general non-U(1) invariant situations. However see [59, 66, 67, 68] for R^2 corrections. Here we simply assume that such a formula exists.

In conclusion, the full expression for the quantum correction to the entropy is given by

$$S_q = S_{bulk-ent} + \frac{\delta A}{4G_N} + \langle \Delta S_{W-like} \rangle + S_{counterterms} \quad (3.12)$$

The first term is the bulk entanglement. The second is the change in the area due to the shift in the classical background due to quantum corrections. The third is the quantum expectation value of the formal expression of the Wald-like entropy. The final term arises because we need to introduce counterterms in order to render the computation finite⁹. The last three terms in (3.12) fill in the dots in (3.3). Some articles, e.g. [63], compute the entanglement entropy by smoothing the tip of the cone and, when fields are coupled to curvature, they obtain an extra contact term, this is precisely our Wald-like term, $S_{reg-cone} = S_{bulk-ent} + \langle \Delta S_{W-like} \rangle$.

Let us finish with some comments. The expression (3.12) for the case of black holes was discussed in [63, 64]¹⁰. Notice that, in the black hole case, we can compute the entropy using the Gibbons-Hawking method, which is to change the period of τ (called β), considering always the smooth solution. In this case, we get the full quantum result from the determinants, computed on the n - (or β -)dependent geometry. In other words, at this order, there is no need to shift the classical background due to quantum corrections, or to evaluate quantum expectation values of the formal expression for the Wald entropy.¹¹ However, if we evaluate the quantum correction using bulk entanglement (as opposed to the Gibbons-Hawking method) we need to take them into account to get the right answer. Similarly, if we compute the quantum correction using the replica trick, we can just compute the determinants, and analytically continue them without worrying about the changes in the classical background due to the quantum corrections, as was done for AdS_3 in [65].

⁹Some aspects of these counterterms have been discussed recently in [69].

¹⁰ In the black hole case, where one has a $U(1)$ symmetry, it is easier to define the quantum computation for non-integer n . Here we had to define it as (3.6).

¹¹ For example, this has been carried out explicitly to find the logarithmic corrections to black hole entropy, see [70] and references within.

The last three terms in (3.12) are given by local integrals on the original minimal surface. Thus, they contribute terms which are qualitatively similar to the classical contribution. The classical Ryu-Takayanagi formula was shown to obey various nontrivial inequalities also obeyed by entanglement entropy [71]. One of these is the strong subadditivity condition. In fact, this inequality follows from the fact that we are minimizing a quantity in the bulk [71]. Thus if we add the last three terms in (3.12) to the Ryu-Takayanagi formula, we still get a result that can be viewed as the minimization of a local expression. To order G_N^0 , the corrections in (3.12) do not change the shape of the surface because they are small corrections. Moreover, the bulk entanglement contribution, the first term in (3.12), obeys the entropy strong subadditivity condition on its own, since it can be viewed as a field theory computation in the bulk. Thus, we have argued that the classical plus first quantum contribution should also obey the strong subadditivity condition.

3.3 Applications

Here we discuss some applications of the above formula. We will concentrate on cases where the quantum correction gives a qualitatively new effect.

3.3.1 Almost gapped large N theory

Consider the Klebanov-Strassler theory in the large N limit, where it is described by the gravity dual found in [72]. The shape of the corresponding geometry is such that most of the bulk fields give rise to massive excitations from the four dimensional point of view. The only massless excitations are associated to the spontaneous breaking of the $U(1)$ baryon symmetry [73, 74]. Since it is a supersymmetric theory, the usual Goldstone boson is part of a massless chiral superfield.

Now consider a region A of a size which is larger than the inverse mass of the lightest massive modes. The classical contribution for such a region was computed in [75]. This arises from a minimal area surface which comes down from the boundary into the bottom of the throat with a topology as indicated in figure 3.5 . The result is that it goes as

$$S_d \propto c_0 R^2 + \text{constant} + \dots \quad (3.13)$$

for large R , where R is the size of the region. Here c_0 has both UV divergent and finite contributions. c_0 is proportional to N^2 ¹²

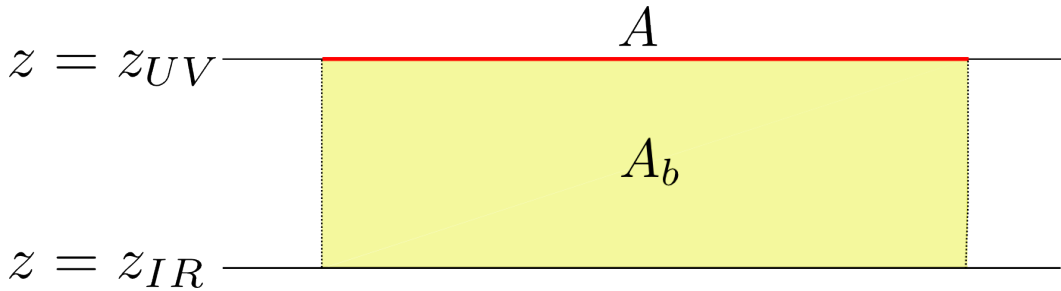


Figure 3.5: Shape of the minimal area surface in the Klebanov-Strassler theory. The yellow region is the interior. The quantum correction is given by the entanglement between the interior and the exterior.

The quantum correction is given by the entanglement in the bulk between the interior and the exterior of region bounded by the Ryu-Takayanagi minimal area surface in the bulk, see figure 3.5. For a large region, we can approximately compute the bulk contributions by doing a Kaluza-Klein decomposition of all the bulk fields, and then doing the entanglement computation in four dimensions. To the order we are working, all the bulk fields are free. All the massive bulk modes contribute only with terms that give rise to contributions similar to (3.13). However, the massless modes (two bosons and two fermions) give rise to a qualitatively new logarithmic

¹² Here by N we mean the value of N in the last step of the cascade [72]. The UV divergent contribution has a larger effective value of N .

term of the form

$$S_{q-\log} = -\alpha \log R\Lambda \tag{3.14}$$

where Λ is the scale setting the mass of the massive modes. Here α is a numerical constant that depends on the shape of the region [76]. For a spherical region $\alpha = 4a$ where a is the conformal anomaly coefficient for a chiral superfield, $a = -\frac{1}{48}$.

A similar correction to the Ryu-Takayanagi formula was argued for in [77]. In Section 3 of [77] they consider an AdS soliton geometry which is dual to a 3d confining gauge theory. A Chern-Simons term was added to the boundary theory resulting in a topological theory in the IR. The expected topological term in the entanglement entropy is reproduced by the entanglement of bulk fields. This provides a further check of (3.3).

3.3.2 Thermal systems in the bulk

We can consider a confining theory whose geometry can be modelled by an AdS space with an infrared end of the world brane. In this case, let us consider a theory with no massless modes. Then the entanglement entropy of a large region of size R will behave as in (3.13). This will be the case as long as we consider the theory in the vacuum. However, if we consider the theory in a thermal bulk state, with a gas of particles in the bulk, we get a contribution to the entropy from this gas. We are considering the phase with no black brane. Then we get a contribution proportional to the volume, $S(A) \propto V_A$, in addition to (3.13). This contribution is of order G_N^0 (or N^0). We obtain this contribution from the bulk entanglement entropy of region A_b , see figure (3.6).

Another case which is qualitatively similar arises when we consider a fermi surface in the bulk¹³. Since we end up computing the bulk entanglement entropy, we reproduce the logarithmic terms that are expected in that context [78, 79]. This

¹³We thank S. Hartnol for pointing out this application.

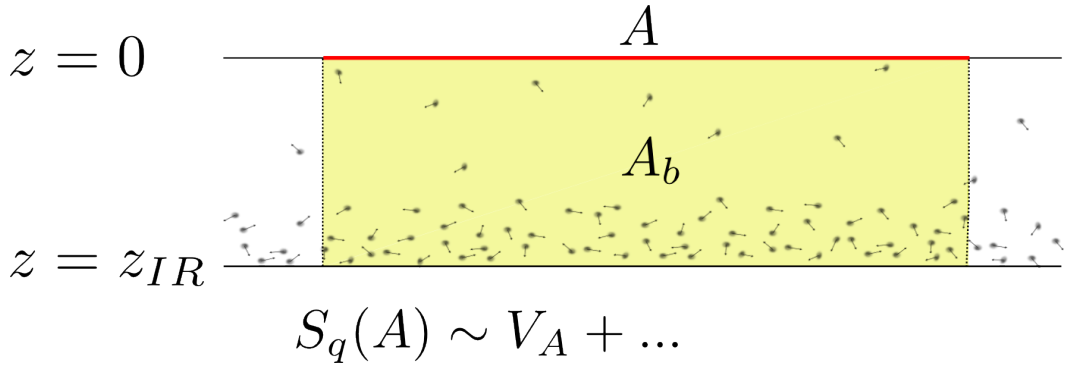


Figure 3.6: Confining theory and thermal gas in the bulk. Here V_A is the volume of region A in the boundary

is important for applications of AdS/CFT to non-Fermi liquids. See for example [80, 81, 82], where such logarithmic violations are expected due to the appearance of bulk fermi surfaces. This should be contrasted with [83] where the logarithmic violations to the entanglement entropy were found from the leading geometric term.

3.3.3 Non-contractible circle

If the τ circle that appeared in our discussion in section 3.2 is not contractible in the bulk, then the classical contribution to the entropy vanishes. In this case, the whole contribution to the generalized gravitational entropy comes from the quantum correction. It involves the propagation of the quantum particles around the bulk circle. It is a finite contribution. In the case that the system has a $U(1)$ symmetry, this is just the thermal entropy of a gas of particles in the bulk. In general, this setup leads to a bulk mixed state under analytical continuation to Lorentzian signature and we just get the entropy of this bulk mixed state.

3.3.4 Mutual information, generalities

For two disjoint regions, A and B , we define the mutual information

$$I(A, B) = S(A) + S(B) - S(A \cup B) \quad (3.15)$$

A feature of the Ryu-Takayanagi formula is that, for well separated disjoint regions, the mutual information is zero [28]. See figure 3.7. In other words, the classical bulk answer is zero. This is due to the fact that the surface for $S(A \cup B)$ is the union of the surfaces that we use to compute $S(A)$ and $S(B)$. We will see that the quantum correction gives us something different from zero. Note that all the local contributions (coming from the second, third and fourth terms in (3.12)) also cancel for the same reason as in the classical case. Thus mutual information comes purely from the bulk entanglement term (the first term in (3.12)). Thus the quantum contribution to the mutual information is simply equal to the bulk mutual information for the two bulk regions:

$$I(A, B) = I_{bulk,ent}(A_b, B_b) \quad (3.16)$$

Here A and B are two regions in the boundary CFT. A_b and B_b are the two corresponding regions in the bulk, see figure 3.7. As explained in [12, 13], a non zero answer is necessary for having non-vanishing correlators. The argument is based on the general bound for correlators [84]

$$I(A, B) \geq \frac{(\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle)^2}{2|\mathcal{O}_A|^2 |\mathcal{O}_B|^2} \quad (3.17)$$

where $|\mathcal{O}_A|$ is the absolute value of the maximum eigenvalue¹⁴. Thus, the non-zero one loop correction will enable us to obey this bound. We will discuss this in more detail below.

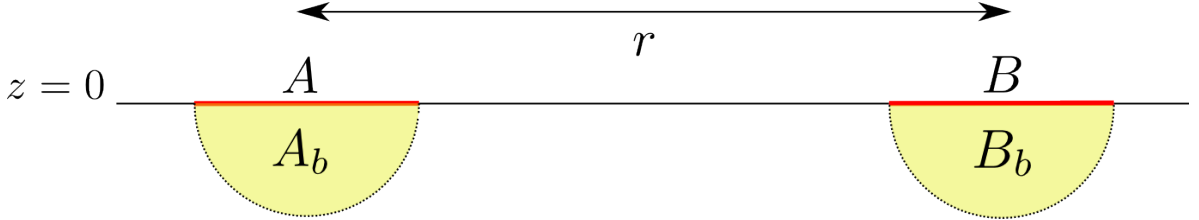


Figure 3.7: We consider two regions A and B on the boundary which are separated by a long distance, $r \gg r_A, r_B$, where $r_{A,B}$ are their sizes. The minimal area surfaces have the shape indicated. In the bulk, they define regions A_b and B_b , which are shown in yellow. The surface for $S(A \cup B)$ is simply the sum of the two surfaces.

Long distance expansion for the mutual information in quantum field theory

Here we consider two disjoint regions, A and B that are separated by a large distance in the boundary theory. In this situation, one can do a kind of operator product expansion for the mutual information. As discussed in [85, 28, 86, 87], the expected leading contribution comes from the exchange of a pair of operators each with dimension Δ ¹⁵. In other words, we have [85, 28, 86, 87]

$$I(A, B) \sim \sum C_\Delta \frac{1}{r^{4\Delta}} + \dots \quad (3.18)$$

¹⁴Of course, we should choose \mathcal{O}_A to be a suitably smeared function of a local operator so that the maximum eigenvalue is finite. For example, $\mathcal{O}_A \sim e^{i \int O(x)g(x)}$, where $g(x)$ is a localized smooth function.

¹⁵An idea for an OPE expansion of mutual information was discussed in [88]. However, we think that it is not correct because it includes the exchange of single particle states, as opposed to two particle states.

where C_Δ comes from squares of OPE coefficients. These OPE coefficients $C_{\mathcal{O}}^A$ arise by replacing region A of the replica space by a sum $\sum C_{\mathcal{O}}^A \mathcal{O}$ over local operators in the n copies of the original CFT. Such operators take the form of products of operators of the original CFT living on the different replicas. Once we have these OPE coefficients we can find:

$$C_\Delta = \partial_n \left[\sum C_{\mathcal{O}}^A C_{\mathcal{O}}^B \right]_{n=1} \quad (3.19)$$

where the sum is over all operators contributing at the same order as (3.18). This involves sums over operators in different replicas and the analytic continuation in n appears non-trivial.

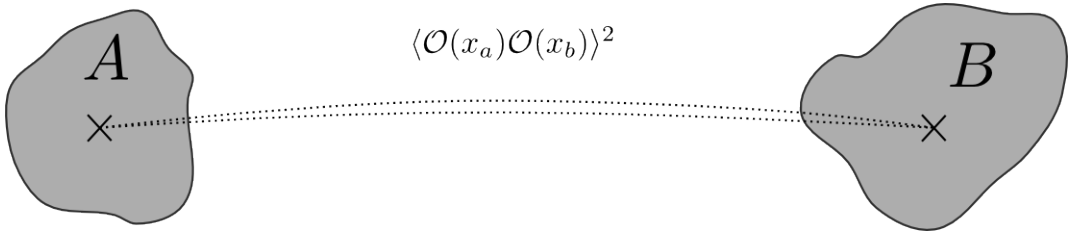


Figure 3.8: OPE-like expansion for mutual information.

For a single operator living on a single replica the OPE coefficient $C_{\mathcal{O}}^A$, in principle, could be calculated. However, it vanishes as $(n - 1)$ since the one point functions of the un-replicated space vanishes. Therefore, the square of the OPE coefficient in (3.19) vanishes at $n = 1$. The two operator case in (3.18) gives the first non-zero answer. We expect that the leading contribution comes from pairs of operators with lowest anomalous dimension.

At integer n we are doing a standard OPE expansion in terms of operators of the replicated theory. However, the final result at $n = 1$ cannot be interpreted as an ordinary OPE expansion in the original theory. For example, the leading behavior in (3.18) might not be reproduced by operators of the original theory. For example, the

theory, at $n = 1$, might not have an operator with dimension $\Delta' = 2\Delta$ to reproduce (3.18)¹⁶. In general, the individual OPE coefficients cannot be continued to $n = 1$. However the sums of squares of all the OPE coefficients contributing at the same order in (3.18) can be continued to $n = 1$ [86]. Here we will not compute the OPE coefficients, we simply focus on the r dependence.

Notice that this behavior of the mutual information, (3.18), is consistent with the bound (3.17). In addition, this implies that the C_Δ coefficient for the lightest operator cannot vanish.

In large N theories, the standard large N counting rules imply that the OPE coefficients $C_{\mathcal{O}}^A$ for the leading contribution are of order one, since they come from a connected two point function in the replicated geometry. This is in the normalization where the two point function of single trace operators is normalized to one. Thus, the leading contribution to C_Δ vanishes at order N^2 and is non-vanishing at order one. Similar large N counting for more general operators leads us to expect that the mutual information vanishes exactly at order N^2 in large N theories, for well separated regions, as is the case in large N theories with gravity duals. For this argument, the crucial feature is that the contribution from the exchange of a single operator vanishes¹⁷.

We can similarly consider mutual information in non-conformal theories. For example we can consider a massive theory. In this case the long distance expansion can be done in terms of the excitations of the massive theory, in terms of the lightest massive excitation. As before, these excitations will propagate along the n separate copies of the replicated theory. And the leading contribution comes from pairs of the

¹⁶ For example, in the Ising model, the leading term comes from the spin operator of dimension $\Delta = 1/4$. However there is no (spin zero) operator in the theory with dimension $1/2$ that can reproduce (3.18).

¹⁷This is no longer true for the mutual Renyi entropies [28].

lightest particle¹⁸. Again the bound (3.17) implies that the corresponding coefficient cannot vanish. So far we have discussed theories in flat space. We can similarly consider theories in curved spaces. Again, for well separated regions, we have a long distance expansion of the mutual information that involves the propagation of the lightest excitations, but now in curved spacetimes. Thus the mutual information behaves as

$$I(A, B) \sim CG(x_A, x_B)^2 + \dots \quad (3.20)$$

where G is the propagator for the lightest excitation of the theory in the curved manifold. More precisely, the one whose $G(x_A, x_B)$ propagator is the largest.

Long distance expansion for mutual information using gravity duals

Now we consider a theory with a gravity dual. For well separated regions, as argued around (3.16), the leading order term comes from the bulk entanglement between regions A_b and B_b , see figure 3.7. In this approximation, we have a quantum field theory in a fixed background geometry. Then the long distance expansion of the mutual information reduces to the expression in (3.20), where we should consider the lightest bulk particle. If the theory reduces to pure gravity in the bulk, then this is the graviton. Again, the coefficient is non-zero due to the bulk version of (3.17).

But at long distances $G(x_{A_b}, x_{B_b}) \sim \frac{1}{|x_A - x_B|^{2\Delta}}$ due to the standard AdS/CFT dictionary [47, 48]. Here x_{A_b} is some point in the bulk region A_b and x_A is some point in the boundary region A . Inserting this into (3.20) we reproduced the expected field theory result (3.18).

¹⁸ In general, the contribution from the exchange of a single particle should vanish when $n \rightarrow 1$. In free theories, the single particle contribution vanishes for all n due to a Z_2 symmetry that multiplies the field by a minus sign.

3.3.5 Corrections to the Entanglement Plateaux

Another situation where we expect quantum corrections to be the dominant answer comes from considering entanglement entropy for subsystems in thermal states. They satisfy the Araki-Lieb inequality [89]:

$$\Delta S = S(\rho) - |S_{A^c} - S_A| \geq 0 \quad (3.21)$$

where ρ is the density matrix describing the state of the full system. Here A^c is the complement of region A in the boundary theory ($A \cup A^c$ gives the full system). For a thermal state $S(\rho)$ is just the thermal entropy of the full system.

In holographic theories this inequality can be saturated when A is small enough (or equivalently A^c is small.) This was discussed extensively in [90] where this saturation was named the Entanglement Plateaux (see also [91, 9, 71].) That is, for region A small enough the minimal surface for region A^c is the disconnected sum of the minimal surface for region A and the horizon of a black hole in the bulk, see figure 3.9. The thermal entropy is computed by the black hole horizon. Thus the classical answer gives a vanishing contribution to (3.21). In the bulk, the first non-zero contribution to (3.21) comes from the bulk entanglement contribution to the quantum correction (3.12). This reduces to

$$\Delta S = S_H - S_{A_b^c} + S_{A_b} = S_H + S_{A_b} - S_{H \cup A_b} = I(H, A_b) > 0 \quad (3.22)$$

where region H is the region behind the horizon. We are imagining we have the eternal black hole and region H is the second bulk space joined to the first by the Einstein-Rosen bridge. We see that ΔS is the same as the bulk mutual information of regions H and A_b . This is positive by the subadditivity condition applied to the

bulk field theory. The inequality in (3.22) is strict because of (3.17) applied to the bulk theory.

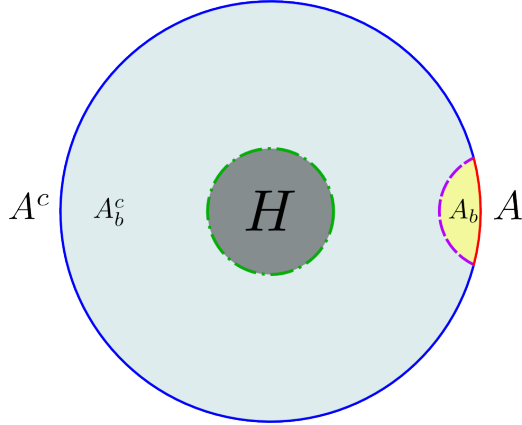


Figure 3.9: We consider a small region A and its complement A^c in a finite temperature state. The bulk contains a black hole. The region A_b is the region outside the black hole horizon. The minimal surface that gives the leading answer to $S(A)$ is the one indicated by a purple dashed line surrounding region A_b . The surface associated to $S(A^c)$ is the one associated to $S(A)$ plus the black hole horizon. The thermal entropy is computed by the surface at the black hole horizon. The region H is the interior of the black hole.

3.3.6 EPR pair in the bulk

Imagine two well separated regions A and B in such a way that their mutual information vanishes according to the classical RT formula. In the vacuum, the mutual information decays at long distances. Here we add two spins that are EPR correlated as indicated in figure 3.10 . We can imagine these as arising from the spin of two (fermionic) glueballs in the boundary theory which corresponds to two particles in the bulk.

In this case the bulk entanglement entropy contains a non-zero piece which is independent of the separation, for large separations. This is just simply the usual mutual information of two spins, $I = 2 \log 2$. Of course we can consider a more

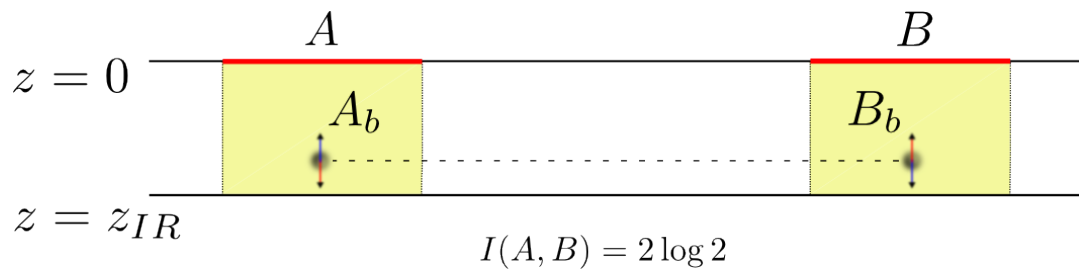


Figure 3.10: We consider two regions and their mutual information. In each bulk region we have a quantum spin. The two spins are in an EPR configuration.

complex system with the same type of result. This contribution is given by the bulk entanglement term in (3.12).

Chapter 4

Relative entropy is bulk relative entropy

4.1 Introduction and summary of results

Recently there has been a great deal of effort in elucidating patterns of entanglement for theories that have gravity duals. The simplest quantity that can characterize such patterns is the von Neumann entropy of subregions, sometimes called the “entanglement entropy”. This quantity is divergent in local quantum field theories, but the divergences are well understood and one can extract finite quantities. Moreover, one can construct strictly finite quantities that are well-defined and have no ambiguities. A particularly interesting quantity is the so called “relative entropy” [92, 93]. This is a measure of distinguishability between two states, a reference “vacuum state” σ and an arbitrary state ρ

$$S(\rho|\sigma) = Tr[\rho \log \rho - \rho \log \sigma] \tag{4.1}$$

If we define a modular Hamiltonian $K = -\log \sigma$, then this can be viewed as the free energy difference between the state ρ and the “vacuum” σ at temperature $\beta = 1$, $S(\rho|\sigma) = \Delta K - \Delta S$.

Relative entropy has nice positivity and monotonicity properties. It has also played an important role in formulating a precise version of the Bekenstein bound [94] and arguments for the second law of black hole thermodynamics [95, 96].

In some cases the modular hamiltonian has a simple local expression. The simplest case is the one associated to Rindler space, where the modular Hamiltonian is simply given by the boost generator.

In this article we consider quantum field theories that have a gravity dual. We consider an arbitrary subregion on the boundary theory R , and a reference state σ , described by a smooth gravity solution. σ can be the vacuum state, but is also allowed to be any state described by the bulk gravity theory. We then claim that the modular Hamiltonian corresponding to this state has a simple bulk expression. It is given by

$$K_{\text{bdy}} = \frac{\text{Area}_{\text{ext}}}{4G_N} + K_{\text{bulk}} + \dots + o(G_N) \quad (4.2)$$

The first term is the area of the Ryu Takayangi surface \mathcal{S} (see figure 4.1), viewed as an operator in the semiclassically quantized bulk theory. This term was previously discussed in [97]. The $o(G_N^0)$ term K_{bulk} is the modular Hamiltonian of the bulk region enclosed by the Ryu-Takayanagi surface, R_b , when we view the bulk as an ordinary quantum field theory, with suitable care exercised to treat the quadratic action for the gravitons. Finally, the dots represent local operators on \mathcal{S} , which we will later specify. We see that the boundary modular Hamiltonian has a simple expression in the bulk. In particular, to leading order in the $1/G_N$ expansion it is just the area term, which is a very simple local expression in the bulk. Furthermore, this simple expression is precisely what appears in the entropy. This modular Hamiltonian makes

sense when we compute its action on bulk field theory states ρ which are related to σ by bulk perturbation theory. Roughly speaking, we consider a ρ which is obtained from σ by adding or subtracting particles without generating a large backreaction.

Due to the form of the modular Hamiltonian (4.2), we obtain a simple result for the relative entropy

$$S_{\text{bdy}}(\rho|\sigma) = S_{\text{bulk}}(\rho|\sigma) \quad (4.3)$$

where the left hand side is the expression for the relative entropy on the boundary. In the right hand side we have the relative entropy of the bulk quantum field theory, with ρ and σ in the right hand side, being the bulk states associated to the boundary states ρ, σ appearing in the left hand side. Note that the area term cancels.

Another consequence of (4.2) is that the action of K_{bdy} coincides with the action of K_{bulk} in the interior of the entanglement wedge¹,

$$[K_{\text{bdy}}, \phi] = [K_{\text{bulk}}, \phi] \quad (4.4)$$

for ϕ a local operator in R_b . This follows from causality in bulk perturbation theory: terms in K_{bdy} localized on \mathcal{S} do not contribute to its action in the interior of the entanglement wedge, \mathcal{S} being space-like to the interior. Note K_{bulk} is the bulk modular Hamiltonian associated to a very specific subregion, that bounded by the extremal surface \mathcal{S} . Implications of (4.4) for entanglement wedge reconstruction are described in section 4.5.2.

The bulk dual of relative entropy for subregions with a Killing symmetry was considered before in [91, 98, 99, 100, 101, 102]. In particular, in [102], the authors related it to the classical canonical energy. In fact, we argue below that the bulk modular hamiltonian is equal to the canonical energy in this case. This result extends that discussion to the quantum case. Note (4.2) and (4.3) are valid for arbitrary

¹The entanglement wedge is the domain of dependence of the region R_b .

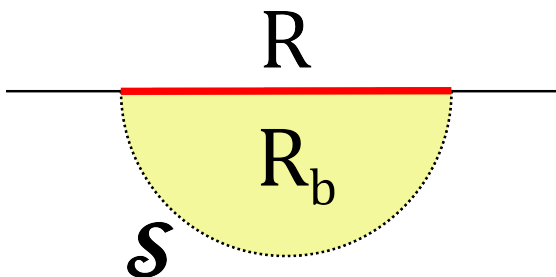


Figure 4.1: The red segment indicates a spatial region, R , of the boundary theory. The leading contribution to the entanglement entropy is computed by the area of an extremal surface \mathcal{S} that ends at the boundary of region R . This surface divides the bulk into two, region R_b and its complement. Region R_b lives in the bulk and has one more dimension than region R . The leading correction to the boundary entanglement entropy is given by the bulk entanglement entropy between region R_b and the rest of the bulk.

regions, with or without a Killing symmetry. In addition, we are not restricting σ to be the vacuum state. Recently a different extension of [102] has been explored in [103], which extends it to situations where one has a very large deformation relative to the vacuum state. That discussion does not obviously overlap with ours.

This chapter is organized as follows. In section two, we recall definitions and properties of entanglement entropy, the modular Hamiltonian, and relative entropy. In section three, we present an argument for the gravity dual of the modular hamiltonian and the bulk expression for relative entropy. In section four, we discuss the case with a $U(1)$ symmetry, relating to previous work. In section five, we discuss the flow generated by the boundary modular hamiltonian in the bulk. We close in section six with some discussion and open questions.

4.2 Entanglement entropy, the modular hamiltonian, and relative entropy

We consider a system that is specified by a density matrix ρ . This can arise in quantum field theory by taking a global state and reducing it to a subregion R . We can compute the von Neuman entropy $S = -Tr[\rho \log \rho]$. Due to UV divergences this is infinite in quantum field theory. However, these divergences are typically independent of the particular state we consider, and when they depend on the state, they do so via the expectation value of an operator. See [104, 105].

4.2.1 Modular Hamiltonian

It is often useful to define the modular hamiltonian $K_\rho \equiv -\log \rho$. From its definition, it is not particularly clear why this is useful – it is in general a very non-local complicated operator. However, for certain symmetric situations it is nice and simple.

The simplest case is a thermal state where $K = H/T$, with H the Hamiltonian of the system. Another case is when the subregion is the Rindler wedge and the state is the vacuum of Minkowski space, when K is the boost generator. This is a simple integral of a local operator, the stress tensor. For a spherical region in a conformal field theory, we have a similarly simple expression, which is obtained from the previous case by a conformal transformation [45]. In free field theory one can also obtain a relatively simple expression that is bilocal in the fields [106] for a general subregion of the vacuum state.

In this chapter we consider another case in which simplification occurs. We consider a quantum system with a gravity dual and a state that can be described by a gravity solution. We will argue that the modular Hamiltonian is given by the area of the Ryu-Takayanagi minimal surface plus the bulk modular Hamiltonian of the bulk region enclosed by the Ryu-Takayanagi surface.

4.2.2 Relative entropy

Modular Hamiltonians also appear in the relative entropy

$$S_{rel}(\rho|\sigma) = \text{tr}\rho(\log \rho - \log \sigma) = \Delta\langle K_\sigma \rangle - \Delta S \quad (4.5)$$

where $K_\sigma = -\log \sigma$ is the modular Hamiltonian associated to the state σ . If σ was a thermal state, the relative entropy would be the free energy difference relative to the thermal state. As such it should always be positive.

Relative entropies have a number of interesting properties such as positivity and monotonicity [92]. Moreover, while the entanglement entropy is not well defined for QFT's, relative entropies have a precise mathematical definition [93].

If $\rho = \sigma + \delta\rho$, then, because of positivity, the relative entropy is zero to first order in $\delta\rho$. This is called the first law of entanglement:

$$\delta S = \delta\langle K_\sigma \rangle \quad (4.6)$$

When we consider a gauge theory, the definition of entanglement entropy is ambiguous. If we use the lattice definition, there are different operator algebras that can be naturally associated with a region R [104]. Different choices give different entropies. These algebras differ in the elements that are kept when splitting space into two, so that ambiguities are localized on the boundary of the region, ∂R . One natural way of defining the entanglement entropy is by fixing a set of boundary conditions and summing over all possibilities, since there is no physical boundary. This was carried out for gauge fields in [107, 108, 109] and gives the same result as the euclidean prescription of [110]. However, the details involved in the definition of the subalgebra are localized on the boundary. Because of the monotonicity of relative

entropy, these do not contribute to the relative entropy (see section 6 of [104] for more details).

In the case of gravitons we expect that similar results should hold. We expect that we similarly need to fix some boundary conditions and then sum over these choices. For example, we could choose to fix the metric fluctuations on the Ryu-Takayanagi surface, viewing it as a classical variable, and then integrate over it. As argued in [104], we expect that the detailed choice should not matter when we compute the relative entropy. See appendix A for more details.

As we mentioned above, it often occurs that two different possible definitions of the entropy give results that differ by the expectation value of a local operator, $S(\rho) = \text{tr}(\rho\mathcal{O}) + \tilde{S}(\rho)$. A trivial example is the divergent area term which is just a number. In these cases the two possible modular Hamiltonians are related by

$$S(\rho) = \text{tr}(\rho\mathcal{O}) + \tilde{S}(\rho) \longrightarrow K = \mathcal{O} + \tilde{K} \quad (4.7)$$

This implies that relative entropies are unambiguous, $S(\rho|\sigma) = \tilde{S}(\rho|\sigma)$. For the equality of relative entropies, it is not necessary for \mathcal{O} to be a state independent operator. It is only necessary that \mathcal{O} is the same operator for the states ρ and σ .²

4.3 Gravity dual of the modular hamiltonian

A leading order holographic prescription for computing entanglement entropy was proposed in [9, 49] and it was extended to the next order in G_N in [15] (see also [65]). The entanglement entropy of a region R is the area of the extremal codimension-two surface \mathcal{S} that asymptotes to the boundary of the region ∂R , plus the bulk von

² In other words, if we consider a family of states, with ρ and σ in that family, then \mathcal{O} should be a state independent operator within that family.

Neuman entropy of the region enclosed by \mathcal{S} , denoted by R_b . See figure 4.1.

$$S_{\text{bdy}}(R) = \frac{A_{\text{ext}}(\mathcal{S})}{4G_N} + S_{\text{bulk}}(R_b) + S_{\text{Wald-like}} \quad (4.8)$$

$S_{\text{Wald-like}}$ indicates terms which can be written as expectation values of local operators on \mathcal{S} . They arise when we compute quantum corrections [15], we discuss examples below.

We can extract a modular Hamiltonian from this expression. We consider states that can be described by quantum field theory in the bulk. We consider a reference state σ , which could be the vacuum or any other state that has a semiclassical bulk description. We consider other states ρ which likewise can be viewed as semiclassical states built around the bulk state for σ . To be concrete we consider the situation where the classical or quantum fields of ρ are a small perturbation on σ so that the area is only changed by a small amount. Now the basic and simple observation is that both the area term and the $S_{\text{Wald-like}}$ are expectation values of operators in the bulk effective theory. Therefore, for states that have a bulk effective theory, we can use (4.7) to conclude that

$$K_{\text{bdy}} = \frac{\hat{A}_{\text{ext}}}{4G_N} + \hat{S}_{\text{Wald-like}} + K_{\text{bulk}} \quad (4.9)$$

This includes the contribution from the gravitons, as we will explain in detail below. The area term was first discussed in [97]. We view the area of the extremal surface as an operator in the bulk effective theory. This contains both the classical area as well as any changes in the area that result from the backreaction of quantum effects. Since we are specifying the surface using the extremality condition, this area is a gauge invariant observable in the gravity theory.³ Note that the area changes as

³ If we merely define a surface by its coordinate location in the background solution, then a pure gauge fluctuation of the metric can change the area. If the original surface is not extremal this already happens to first order.

we change the state, but we can choose a gauge where the position of the extremal surface is fixed. Finally $\hat{S}_{\text{Wald-like}}$ are the operators whose expectation values give us $S_{\text{Wald-like}}$.

Interestingly, all terms that can be written as local operators drop out when we consider the relative entropy. The relative entropy has a very simple expression

$$S_{\text{bdy}}(\rho|\sigma) = S_{\text{bulk}}(\rho|\sigma) \tag{4.10}$$

Note that the term going like $1/G_N$ cancels out and we are only left with terms of order G_N^0 . There could be further corrections proportional to G_N which we do not discuss in this article. It is tempting to speculate that perhaps (4.10) might be true to all orders in the G_N expansion (i.e. to all orders within bulk perturbation theory).

Of course, using the equation for the entropy (4.8) and (4.9) we can check that the first law (4.6) is obeyed. In the next section we discuss this in more detail for a spherical subregion in the vacuum.

4.4 Regions with a local boundary modular Hamiltonian

For thermal states, Rindler space, or spherical regions of conformal field theories we have an explicit expression for the boundary modular Hamiltonian. In all these cases there is a continuation to Euclidean space with a compact euclidean time and a $U(1)$ translation symmetry along Euclidean time. We also have a corresponding symmetry in Lorentzian signature generated by a Killing or (conformal Killing) vector ξ . The modular Hamiltonian is then given in terms of the stress tensor as $K_{\text{bdy}} = \mathcal{E}_R \equiv \int *(\xi.T_{\text{bdy}})$, where the integral is over a boundary space-like slice. When the theory

has a gravity dual, the bulk state σ is also invariant under a bulk Killing vector ξ . In this subsection we will discuss (4.9) for states constructed around σ .

For this discussion it is useful to recall Wald's treatment of the first law [21, 36, 37]

$$\delta\mathcal{E}_R = \frac{A_{\text{lin}}(\delta g)}{4G_N} + \int_{\Sigma} *(\xi \cdot E_g(\delta g)) \quad (4.11)$$

where $E_g(\delta g)$ is simply the linearized Einstein tensor with the proper cosmological constant. It is just the variation of the gravitational part of the action and does not include the matter contribution. Here A_{lin} is the first order variation in the area due to a metric fluctuation δg . And Σ is any Cauchy slice in the entanglement wedge R_b . Equation (4.11) is a tautology, it arises by integrating by parts the linearized Einstein tensor. It is linear in δg and we can write it as an operator equation by sending $\delta g \rightarrow \delta\hat{g}$, where $\delta\hat{g}$ is the operator describing small fluctuations in the metric in the semiclassically quantized theory.

4.4.1 Linear order in the metric

For clarity we will first ignore dynamical gravitons, and include them later (we would have nothing extra to include if we were in three bulk dimensions). We consider matter fields with an $o(G_N^0)$ stress tensor in the bulk, assuming the matter stress tensor was zero on the σ background.⁴ Such matter fields produce a small change in the metric that can be obtained by linearizing the Einstein equations around the vacuum. These equations say $E_g(\delta g)_{\mu\nu} = T_{\mu\nu}^{\text{mat}}$, where $T_{\mu\nu}^{\text{mat}}$ is the stress tensor of matter. Inserting this in (4.11) we find that [21, 36, 37]

$$\delta\mathcal{E}_R = \frac{A_{\text{lin}}(\delta g)}{4G_N} + \int_{\Sigma} *(\xi \cdot T) = \frac{A_{\text{lin}}(\delta g)}{4G_N} + K_{\text{bulk}} \quad (4.12)$$

⁴This discussion can be simply extended when there is a non-zero but $U(1)$ -symmetric background matter stress tensor, such as in a charged black hole. In that case we need to subtract the background stress tensor to obtain the bulk modular Hamiltonian.

where we used that the bulk modular Hamiltonian also has a simple local expression in terms of the stress tensor due to the presence of a Killing vector with the right properties at the entangling surface \mathcal{S} . Notice that we can disregard additive constants in both the area and \mathcal{E} , which are the values for the state σ . We only care about deviations from these values. This is basically the inverse of the argument in [111]. This shows how (4.9) works in this symmetric case. The term $\hat{S}_{\text{Wald-like}}$ in (4.9) arises in some cases as we discuss below.

Let us now discuss the $\hat{S}_{\text{Wald-like}}$ term. There can be different sources for this term. A simple source is the following. The bulk entanglement entropy has a series of divergences which include an area term, but also terms with higher powers of the curvature. Depending on how we extract the divergences we can get certain terms with finite coefficients. Such terms are included in $S_{\text{Wald-like}}$. A different case is that of a scalar field with a coupling $\alpha\phi^2(R - R_0)$ where R is the Ricci scalar in the bulk, and R_0 the Ricci scalar on the unperturbed background, the one associated to the state σ . Then there exists an additional term in the entropy of the form $\hat{S}_{\text{Wald-like}} = 2\pi\alpha \int_{\mathcal{S}} \phi^2$. If we compute the entropy as the continuum limit of the one on the lattice, then it will be independent of α . Under these conditions the bulk modular Hamiltonian is also independent of α and is given by the canonical stress tensor, involving only first derivatives of the field. However, the combination of $K_{\text{bulk}} + \hat{S}_{\text{Wald-like}} = \int_{\Sigma} *(\xi \cdot T_{\mu\nu}^{\text{grav}}(\phi))$, where $T_{\mu\nu}^{\text{grav}}(\phi)$ is the standard stress tensor that would appear in the right hand side of Einstein's equations. $T_{\mu\nu}^{\text{grav}}(\phi)$ does depend on α . The α dependent contribution is a total derivative which evaluates to $2\pi\alpha\phi^2$ at the extremal surface. A related discussion in the field theory context appeared in [105, 18].

4.4.2 The graviton contribution

We expect that we can view the propagating gravitons as one more field that lives on the original background, given by the metric g_σ . In fact, we can expand Einstein's equations in terms of $g = g_\sigma + \delta g_2 + h$. Here h , which is of order $\sqrt{G_N}$, represents the dynamical graviton field and obeys linearized field equations. δg_2 takes into account the effects of back-reaction and obeys the equation

$$E(\delta g_2)_{\mu\nu} = T_{\mu\nu}^{\text{grav}}(h) + T_{\mu\nu}^{\text{matter}} \quad (4.13)$$

where $T_{\mu\nu}^{\text{grav}}(h)$ comes simply from expanding the Einstein tensor (plus the cosmological constant) to second order and moving the quadratic term in h to the right hand side. h obeys the homogeneous linearized equation of motion, so the term linear in h in the equation above vanishes. We can now use equations (44-46) in [112], which imply that

$$K_{\text{bdy},1+2} = \mathcal{E}_{1+2} = \frac{\hat{A}_{\text{lin}}(h + \delta g_2) + \hat{A}_{\text{quad}}(h)}{4G_N} + E_{\text{can}} \quad (4.14)$$

where $K_{\text{bdy},1+2}$ is the boundary modular Hamiltonian (or energy conjugate to τ translations) expanded to quadratic order in fluctuations. Similarly, the area is expanded to linear and quadratic order. Finally, E_{can} is the bulk canonical energy⁵ defined by $E_{\text{can}} = \int \omega(h, \mathcal{L}_\xi h) + \text{matter contribution}$, where ω is the symplectic form defined in [112]. From this expression we conclude that the modular Hamiltonian is the canonical energy

$$K_{\text{bulk}} = E_{\text{can}} \quad (4.15)$$

We can make contact with the previous expression (4.12) as follows. If we include the gravitons by replacing $T_{\mu\nu}^{\text{mat}} \rightarrow T_{\mu\nu}^{\text{mat}} + T_{\mu\nu}^{\text{grav}}(h)$ in (4.12), then we notice that we get $A_{\text{lin}}(\delta g_2)$, without the term $A_{\text{quad}}(h)$. However, one can argue that (see eqn. (84)

⁵This differs from the integral of the gravitational stress tensor by boundary terms.

of [112])

$$\int_{\Sigma} *(\xi.T^{\text{grav}}(h)) = E_{\text{can}}(h) + \frac{A_{\text{quad}}(h)}{4G_N} \quad (4.16)$$

thus recovering (4.14).

In appendix A we discuss in more detail the boundary conditions that are necessary for quantizing the graviton field.

4.4.3 Quadratic order for coherent states

The problem of the gravity dual of relative entropy was considered in [102] in the classical regime for quadratic fluctuations around a background with a local modular Hamiltonian. They argued that the gravity dual is equal to the canonical energy. Here we rederive their result from (4.10).

We simply view a classical background as a coherent state in the quantum theory. $e^{i\lambda \int \Pi \hat{\phi} + \hat{\phi} \hat{\Pi}} |\psi_{\sigma}\rangle$, where $|\psi_{\sigma}\rangle$ is the state associated to σ ⁶. We see that in free field theory we can view coherent states as arising from the action of a product of unitary operators, one acting inside the region and one outside. For this reason finite coherent excitations do not change the bulk von Neuman entropy of subregions, or $\Delta S_{\text{bulk}} = 0$. Thus, the contribution to the bulk relative entropy comes purely from the bulk Hamiltonian, which we have argued is equal to canonical energy (4.15). Therefore, in this situation we recover the result in [102]

$$S_{\text{bdy}}(\rho|\sigma) = S_{\text{bulk}}(\rho|\sigma) = \Delta K_{\text{bulk}} - \Delta S_{\text{bulk}} = \Delta K_{\text{bulk}} = E_{\text{canonical}} \quad (4.17)$$

4.5 Modular flow

The modular hamiltonian generates an automorphism on the operator algebra, the modular flow. Consider the unitary transformation $U(s) = e^{iKs}$. Even if the modular

⁶ Here λ could be $O(1/\sqrt{G_N})$ as long as the backreaction is small.

hamiltonian is not technically an operator in the algebra, the modular flow of an operator, $O(s) \equiv U(s)OU(-s)$, stays within the algebra. For a generic region, the modular flow might be complicated, see [113] for some discussion about modular flows for fermions in $1 + 1$ dimensions. However, in our holographic context it can help us understand subregion-subregion duality. In particular, it can help answer the question of whether the boundary region R describes the entanglement wedge or only the causal wedge [114, 56, 57, 115]. The entanglement wedge is the causal domain of the spatial region bounded by the interior of \mathcal{S} .

From (4.2), we have that

$$[K_{\text{bdy}}, \phi] = [K_{\text{bulk}}, \phi] \tag{4.18}$$

where ϕ is any operator with support only in the interior of the entanglement wedge, and where on the right-hand side we have suppressed terms subleading in G_N . On the left-hand side terms in K_{bdy} localized on \mathcal{S} have dropped out, similarly as in (4.10). Thus the boundary modular flow is equal to the bulk modular flow of the entanglement wedge, the causal wedge does not play any role.

One may also consider the flow generated by the total modular operator, $K_{\text{bdy,Total}} = K_{\text{bdy},R} - K_{\text{bdy},\bar{R}}$, which should be a smooth operator without any ambiguities. From our full formula for the bulk dual of the modular Hamiltonian we see that $K_{\text{bdy,Total}} = K_{\text{bulk,Total}} + o(G_N)$. If the global state is pure, then K_{Total} annihilates it.

4.5.1 Smoothness of the full modular Hamiltonian in the bulk

For problems that have a $U(1)$ symmetry, such as thermal states and Rindler or spherical subregions of CFTs, we know the full boundary modular Hamiltonian \mathcal{E} .

We can define a time coordinate τ which is translated by the action of \mathcal{E} in the boundary theory. In these situations the bulk state also has an associated symmetry generated by the Killing vector ξ . We can choose coordinates so that we extend τ in the bulk and ξ simply translates τ in the bulk. Then the bulk modular Hamiltonian is the bulk operator that performs a translation of the bulk fields along the bulk τ direction.

Let us now consider an eternal black hole and the thermofield double state [10]. This state is invariant under the action of $H_R - H_L$. Let us now consider the action of only the right side boundary Hamiltonian H_R ⁷. It was argued in [11] that this corresponds to the same gravity solution but where the origin of the time direction on the right side is changed. This implies that the Wheeler de Witt patch associated to $t_L = t_R = 0$ looks as in figure 4.2(b), after the action of e^{-itH_R} . On the other hand, if we consider the bulk quantum field theory and we act with only the right side bulk modular Hamiltonian $K_{\text{bulk,R}}$ we would produce a state that is singular at the horizon. By the way, it is precisely for this reason that algebraic quantum field theorists like to consider the total modular Hamiltonian instead. It turns out that the change in the bulk state is the same as the one would obtain if we were quantizing the bulk field theory along a slice which had a kink as shown in figure 4.2(b). Interestingly the area term in the full modular Hamiltonian (4.9) has the effect of producing such a kink. In other words, the area term produces a shift in the τ coordinate, or a relative boost between the left and right sides [31]. The action of only the area term or only $K_{\text{Bulk,R}}$ would lead to a state that is singular at the horizon, but the combined action of the two produces a smooth state, which is simply the same bulk geometry but with a relative shift in the identification of the boundary time coordinates⁸.

Let us go back to a general non- $U(1)$ invariant case. Since the bulk modular Hamiltonian reduces to the one in the $U(1)$ -symmetric case very near the bulk entan-

⁷Here left and right denote the two copies in the thermofield double state.

⁸We thank D. Marolf for discussions about this point.

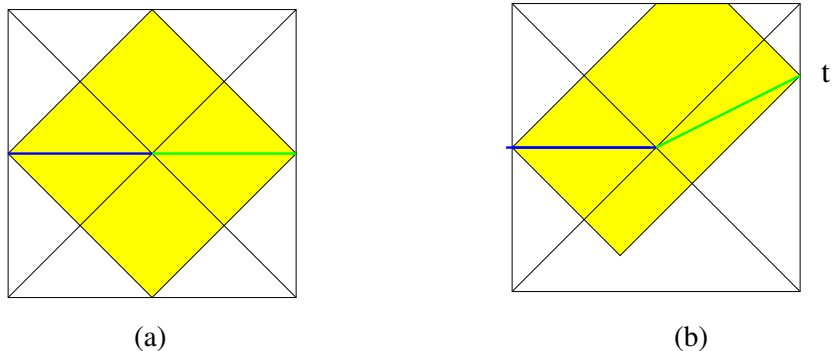


Figure 4.2: In this figure we are considering the thermofield double state. (a) Acting with the bulk modular Hamiltonian $e^{-itK_{\text{bulk,R}}}$ we get a new state on the horizontal line that has a singularity at the horizon. (b) The area term introduces a kink, or a relative boost between the left and right sides. Then the state produced by the full right side Hamiltonian is non-singular, and locally equal to the vacuum state.

gling surface \mathcal{S} , we expect that the action of the full boundary modular Hamiltonian, including the area term, will not be locally singular in the bulk – though it can be singular from the boundary point of view due to boundary UV divergences.

4.5.2 Implications for entanglement wedge reconstruction

One is often interested in defining local bulk operators as smeared operators in the boundary. This operator should be defined order by order in G_N over a fixed background and should be local to the extent allowed by gauge constraints. If we consider a $t = 0$ slice in the vacuum state, then we can think of a local bulk operator $\phi(X)$ as a smeared integral of boundary operators [116]

$$\phi(X) = \int_{\text{bdy}} dx^{d-1} dt G(X|x, t) O(x, t) + o(G_N) \quad (4.19)$$

One would like to understand to what extent this ϕ operator can be localized to a subregion in the boundary.

Given a region in the boundary R , we have been associating a corresponding region in the bulk, the so-called entanglement wedge which is the domain of dependence of R_b , $D[R_b]$. There is another bulk region one can associate to R , the causal wedge (with space-like slice R_C) which is the set of all bulk points in causal contact with $D[R]$, [55]. R_C is generically smaller than R_b [117, 115].

In situations with a $U(1)$ symmetry, such as a thermal state or a Rindler or spherical subregion of a CFT, we have time-translation symmetry and a local modular Hamiltonian that generates translations in the time τ . We can express bulk local operators in the entanglement wedge (which coincides with the causal wedge) in terms of boundary operators localized in $D[R]$ [116, 118]⁹

$$\phi(X) = \int_R dy^{d-1} \int d\tau G'(X|y, \tau) O(y, \tau) + o(G_N), \quad X \in R_b \quad (4.20)$$

A natural proposal for describing operators in that case is that we can replace τ in (4.20) by the modular parameter s . In other words, we consider modular flows of local operators on the boundary, defined as $O_R(x, s) \equiv U(s)O_R(x, 0)U^{-1}(s)$

A simple case in which R_b is larger than R_C is the case of two intervals in a 1+1 CFT such that their total size is larger than half the size of the whole system, see figure 4.3. Here, it is less clear how to think about the operators in the entanglement wedge. We would like to use the previous fact that the modular flow is bulk modular flow to try to get some insight into this issue.

The modular flow in the entanglement wedge will be non-local, but highly constrained: the bulk modular hamiltonian is bilocal in the fields [106]. If we have an operator near the boundary of the causal wedge and modular evolve it, it will quickly develop a non zero commutator with a nearby operator which does not lie in the causal wedge. Alternatively, an operator close to the boundary of the entanglement wedge will have an approximately local modular flow. It will follow the light rays em-

⁹It is sometimes necessary to go to Fourier space to make this formula precise [118, 119].

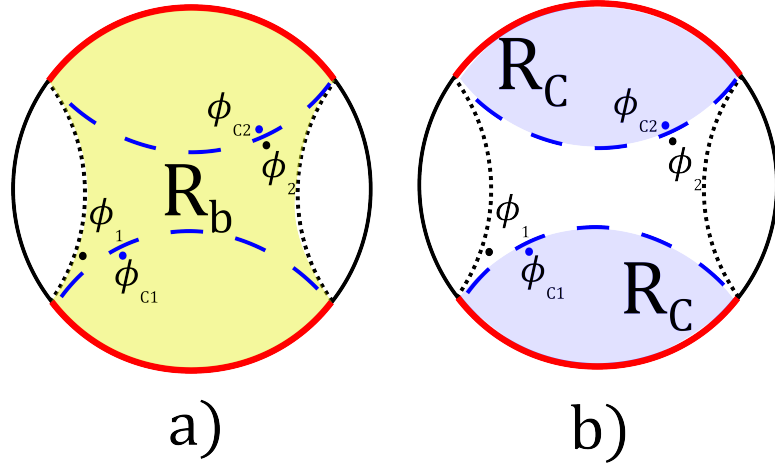


Figure 4.3: In both figures the region R is the union of the two red intervals and the Ryu-Takayanagi surface is the dotted black line, while the boundary of R_C is the blue dashed line (color online). In $a)$, the shaded region denotes the defining spatial slice R_b of the entanglement wedge. In $b)$, the shaded region is the defining spatial slice R_C of the causal wedge. The modular flow of an operator close to the Ryu-Takayanagi surface will be approximately local, so that $\phi_1(s)$ will be almost local and, after some s , it will be in causal contact with ϕ_{C1} . This flow takes the operator out of this slice to its past or to its future. Alternatively, if we consider an operator near the boundary of the causal wedge ϕ_{C2} , it is clear that, under modular flow, $[\phi_{C2}(s), \phi_2] \neq 0$.

anating from the extremal surface and it can be on causal contact with the operators in the causal wedge. See figure 4.3.

So we see that to reconstruct the operator in the interior of the entanglement wedge, one necessarily needs to understand better the modular flow. It seems natural to conjecture that one can generalize (4.20) to two intervals (or general regions) by considering the modular parameter instead of Rindler time, ie the simplest generalization of the AdS/Rindler formula which accounts for the non-locality of the modular hamiltonian would be

$$\phi(X) = \int_R dx \int ds G''(X|x, s) O(x, s), \quad X \in R_b \quad (4.21)$$

Here G'' is a function that should be worked out. It will depend on the bilocal kernel that describes the modular Hamiltonian for free fields [106].

So we see that to reconstruct the operator in the interior of the entanglement wedge, it is necessary to understand better modular flows in the quantum field theory of the bulk. To make these comments more precise, a more detailed analysis would be required, which should include a discussion about gravitational dressing and the constraints. We leave this to future work.

Here we have discussed how the operators in the entanglement wedge can be thought of from the boundary perspective. However, note that from (4.9) (and consequently the formula for the relative entropy), it is clear that one should think of the entanglement wedge as the only meaningful candidate for the “dual of R ”, see also [114]. If we add some particles to the vacuum in the entanglement wedge R_b (which do not need to be entangled with \bar{R}_b), the bulk relative entropy will change. According to (4.3), the boundary relative entropy also changes and, therefore, state is distinguishable from the vacuum, even if we have only access to R .

4.6 Comments and discussion

4.6.1 The relative entropy for coherent states

If we consider coherent states, since their bulk entanglement entropy is not changed, the relative entropy will just come from the difference in the bulk modular hamiltonian. Since our formulation is completely general, one could in principle compute it for any reference region or state and small perturbations over it.

A particularly simple case would be the relative entropy for an arbitrary subregion between the vacuum and a coherent state of matter. To second order in the perturbation, one only needs to work out how the modular hamiltonian for the free

fields [106] looks like for that subregion of AdS, and then evaluate it in the coherent state background.

4.6.2 Positivity of relative entropy and energy constraints

Our formula (4.10) implies that the energy constraints obtained from the positivity of the relative entropy can be understood as arising from the fact that the relative entropy has to be positive in the bulk.

4.6.3 Higher derivative gravity

Even though we focused on Einstein gravity, our discussion is likely to apply to other theories of gravity. The modular hamiltonian will likely be that of an operator localized on the entangling surface plus the bulk modular Hamiltonian in the corresponding entanglement wedge. Thus the relative entropy will be that of the bulk. There could be subtleties that we have not thought about.

4.6.4 Beyond extremal surfaces

A. Wall proved the second law by using the monotonicity of relative entropy [95, 96]. If we consider two Cauchy slices $\Sigma_0, \Sigma_{t>0}$ outside a black hole, then $S_{rel,t} < S_{rel,0}$ is enough to prove the generalized second law. Interestingly, section 3 of [112] shows the “decrease of canonical energy”: $E_{can}(t) < E_{can}(0)$. The setup (Cauchy slices) that they both consider is the same. Due to the connection between relative entropy and canonical energy, [102], we expect a relation between these two statements. This does not obviously follow from what we said due to the following reason.

Here we limited our discussion to the entanglement wedge. In other words, we are always considering the surface \mathcal{S} to be extremal. We expect that the discussion should generalize to situations where the surface \mathcal{S} is along a causal horizon. The question

is: what is the precise boundary dual of the region exterior to such a horizon? Even though we can think about the bulk computation, we are not sure what boundary computation it corresponds to. A proposal was made in [120], and perhaps one can understand it in that context.

Being able to define relative entropies for regions which are not bounded by minimal surfaces is also crucial to the interesting proposal in [121] to derive Einstein’s equations from (a suitable extension to non-extremal surfaces of) the Ryu-Takayanagi formula for entanglement.

4.6.5 Distillable entanglement

In the recent papers [122, 123] it was argued that for gauge fields, only the purely quantum part of the entanglement entropy corresponds to distillable entanglement. The “classical” piece that cannot be used as a resource corresponds to the Shannon entropy of the center variables of [104]. Our terms local in \mathcal{S} are the gravitational analog of this classical piece and one might expect that a bulk observer with access only to the low-energy effective field theory can only extract Bell pairs from the bulk entanglement. This seems relevant for the AMPS paradox [3, 124, 125].

4.7 Appendix A: Subregions of gauge theories

4.7.1 U(1) gauge theory

The problem of defining the operator algebra of a subregion of a gauge theory was considered in [104]. It was shown that for a lattice gauge theory there are several possible definitions of the subalgebra. It was further found that the subalgebra can have a center, namely some operators that commute with all the other elements of the subalgebra. In this case we can view the center as classical variables. Calling the classical variables x_i , then for each value of x_i we have a classical probability p_i and a

density matrix ρ_i for each irreducible block. The relative entropy between two states is then

$$S(\rho|\sigma) = H(p|q) + \sum_i p_i S(\rho_i|\sigma_i) \quad (4.22)$$

where p_i, q_i are the probabilities of variables x_i in the state ρ and σ respectively. H is the classical (Shanon) relative entropies of two probability distributions, $H = \sum_i p_i \log(p_i/q_i)$.

In the continuum we expect that the relative entropy is finite and independent of the microscopic details regarding the precise definition of the algebra [93].

These microscopic details have a continuum counterpart. When we consider a region R we would like to be able to define a consistent quantum theory within the subregion. In particular, imagine that we consider all classical solutions restricted to the subregion. Then we define a presymplectic product between two such solutions, which we will use to quantize the gauge orbits. This presymplectic product should be gauge invariant so that it does not depend on the particular representative. Let us consider a free Maxwell field. The presymplectic product is given by integrating

$$\Omega(A^1, A^2) = \int_{\Sigma} \omega(A^1, A^2) = \int_{\Sigma} (A^1 \wedge *F^2 - A^2 \wedge *F^1) \quad (4.23)$$

where $A^1 = A^1_{\mu} dx^{\mu}$ is a gauge field configuration. Here we imagine that both A^1 and A^2 are solutions to the equations of motion. Σ is any spacelike surface.

Demanding gauge invariance amounts to the statement

$$0 = \Omega(A, d\epsilon) = \int_{\partial\Sigma} \epsilon \wedge F \quad (4.24)$$

where $\partial\Sigma$ is the boundary of the spacelike surface. We have used the equations of motion for F and integrated by parts. In order to make this vanish we need some boundary conditions. In particular, let us concentrate on the boundary conditions

required at the boundary of Σ corresponding to the boundary of a region $\mathcal{S} = \partial\Sigma$. One possible boundary condition is to set $A_i = A_i^{\text{cl}}$ for components along the surface, where A_i^{cl} is a classical gauge field on the surface. In this case, it is natural to set $\epsilon = 0$ on the surface. We can quantize the problem for each fixed A_i^{cl} and then integrate over all A_i^{cl} . These values of A_i^{cl} are the “center” variables x_i in the above discussion. This is called the “magnetic” center, since the gauge field A_i^{cl} defines a magnetic field $F = dA^{\text{cl}}$ on the surface.

There are other possibilities, such as fixing the electric field, or “electric center”, where the perpendicular electric field is fixed.

These would correspond to specific choices on the lattice. Since we expect that relative entropy is a finite and smooth function of the shape of the region, [104] has shown that the detailed boundary condition does not matter, as long as we choose something that makes physical sense. Recently, [107, 108] carried out explicitly the field theory calculation, being careful with the center variables.

4.7.2 Gravity

Here we consider the problem of defining a subregion in a theory of Einstein gravity. We consider only the problem at the quadratic level where we need to consider free gravitons moving around a fixed background (which obeys Einstein’s equations). These gravitons can be viewed as a particular example of a gauge theory. We can also compute the symplectic form, as given in [37], and then impose that the symplectic inner product between a pure gauge mode and another solution to the linearized equations vanishes. Here the gauge transformations are reparametrizations, generated by a vector field ζ . Note that ζ is *not* a killing vector, it is a general vector field and it should not be confused with ξ discussed in section 4.4. Writing the metric as $g + \delta g$, where g is the background metric and δg is a small fluctuation. Then the gauge transformation acts as $\delta g \rightarrow \delta g + \mathcal{L}_\zeta g$, where \mathcal{L}_ζ is the Lie derivative. Then, as

shown in [112], there is a simple expression for the symplectic product with a such a pure gauge mode

$$\int_{\Sigma} \omega(\delta g, \mathcal{L}_{\zeta} g) = \int_{\partial\Sigma} \delta Q_{\zeta} - \zeta \cdot \Theta(g, \delta g) \quad (4.25)$$

with Q_{ζ} and $\Theta(g, \delta g)$ given in eqns (32) and (17) of [112].

We would like to choose boundary conditions on the surface which make the right hand side zero. We choose boundary conditions similar to the “magnetic” ones above. Namely, we fix the metric along the entangling surface \mathcal{S} to $\delta g_{ij} = \gamma_{ij}$. We treat γ_{ij} as classical and then integrate over it. This is enough to make all terms in (4.25) vanish. Let us be more explicit. By a change of coordinates we can always set the metric to have the following form near the entangling surface. For simplicity we write it in Euclidean space, but the same is true in Lorentzian signature

$$ds^2 = d\rho^2 + [\rho^2 + o(\rho^4)](d\tau + a_i dy^i)^2 + h_{ij} dy^i dy^j \quad (4.26)$$

here a_i and h_{ij} can be functions of τ and ρ , with a regular expansion around $\rho = 0$. In these coordinates the extremal surface \mathcal{S} is always at $\rho = 0$, both for the original metric and the perturbed metric. Extremality implies that the trace of the extrinsic curvature is zero, or $K^A = h^{ij} \partial_{X^A} h_{ij} = 0$, where $X^A = (X^1, X^2) = (\rho \cos \tau, \rho \sin \tau)$. This is true for the background and the fluctuations

$$K^A = 0, \quad \delta K^A = 0 \quad (4.27)$$

which ensures that even on the perturbed solution we are considering the minimal surface. These conditions ensure that the splitting between the two regions is defined in a gauge invariant way.

We demand that all fluctuations are given in the gauge (4.26). Thus, near $\rho = 0$, δg leads to δa_i and δh_{ij} . We now further set a boundary condition that $\delta h_{ij} = \gamma_{ij}$

where γ_{ij} is a classical function which we will later integrate over. For defining the quantum problem we will view it as being classical. We will quantize the fields in the subregion for fixed values of γ_{ij} and then integrate over the classical values of γ_{ij} .

With these boundary conditions we see that all terms in (4.25) vanish. In fact, (4.25), has three terms¹⁰

$$\begin{aligned} \int_{\Sigma} \omega(\delta h, \mathcal{L}_{\xi} g) &= \int_{\partial\Sigma} \delta_{\delta h} Q(\zeta) - i_{\zeta} \Theta(g, \delta h) \\ &= \int_{\partial\Sigma} \left[\delta a_i \zeta^i + \zeta^{\tau} \delta h_i^i + (-h^{ij} \partial_A \delta h_{ij} + \frac{1}{2} \delta h^{ij} \partial_A h_{ij}) \zeta_B \epsilon^{AB} \right] \end{aligned} \quad (4.28)$$

Since the fluctuation of the metric is zero at the entangling surface, $\delta h_{ij} = 0$, we see that many terms vanish. In addition, since we are setting $\delta h_{ij} = 0$, it is also natural to restrict the vector fields so that $\zeta^i = 0$ on the surface. This ensures that the first term in (4.28) vanishes. Note that the middle term is related to the fact that the area generates a shift in the coordinate τ . After all the area is the Noether charge associated to such shifts [36, 37].

The extremality condition makes sure that we are choosing a (generically) unique surface for each geometry. We then treat the induced geometry on the surface as a classical variable, quantize the metric in the subregion, and then sum over this classical variable. In this region, we seem to have a gauge invariant symplectic product.

We have not explicitly computed the entanglement entropy for gravitons with these choices, but we expect that it should lead to a well defined problem and that relative entropies will be finite.

¹⁰We did not keep track of the numerical coefficients in front of each of the three terms

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