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Consensus and Clustering of Networked Agent Opinions: A Belief Theoretic View

Ranga Dabarera
University of Miami, dabareranr@gmail.com

UNIVERSITY OF MIAMI

CONSENSUS AND CLUSTERING OF NETWORKED AGENT OPINIONS:
A BELIEF THEORETIC VIEW

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Ranga Dabarera

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Ranga Dabarera

Approved:

Kamal Premaratne, Ph.D.
Professor of Electrical and Computer Engineering

Manohar Murthi, Ph.D.
Associate Professor of Electrical and Computer Engineering

Miroslav Kubat, Ph.D.
Associate Professor of Electrical and Computer Engineering

Jie Xu, Ph.D.
Assistant Professor of Electrical and Computer Engineering

Thanuka Wickramarathne, Ph.D.
Assistant Professor of Electrical and Computer Engineering
University of Massachusetts Lowell

Guillermo Prado, Ph.D.
Dean of the Graduate School

DABARERA, RANGA (Ph.D., Electrical and Computer
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Opinion dynamics is the study of the exchange of opinions among agents embedded and communicating over a network. In particular, issues related to the formation of consensus of opinions, opinion clustering, and opinion influencing have drawn the attention of researchers in areas as diverse as sociophysics, economics, finance, computer science, and engineering, due to their potential application in social networks, marketing, cooperative control of autonomous agents, etc.

In this dissertation, we utilize the Dempster-Shafer (DS) belief theoretic framework to capture agent opinions. The DS theoretic formulation allows us to account for the types of uncertainties that are inherent in social opinions in a more convenient and intuitive manner. Opinions that are modeled as probability mass functions (p.m.f.s) can also be captured as a special case of this belief theoretic formulation. The opinion exchange among neighboring agents are modeled using the Conditional Update Equation (CUE) and the opinion exchange models adhere to notions in Social Judgement Theory (SJT) which examines the basic psychological processes underlying the expression of attitudes and their modifiability through communication.

To study consensus and opinion clustering with this belief theoretic viewpoint, we take two different approaches. In the first approach, using matrix theoretic analysis,

we introduce the notion of *opinion dynamic chains* to account for opinions that are modeled as p.m.f.s. In particular, we explore how the presence of opinion leaders affects consensus and opinion cluster formation. This analysis however assumes synchronous (i.e., delay-less) communication between agents. In the second approach, we utilize notions from paracontractions theory to account for ad-hoc and dynamic networks with possibly asynchronous message passing. With agents embedded within such an ad-hoc, dynamic, and asynchronous network structure, and agent opinions captured via the more general DS belief theoretic models, conditions under which consensus and opinion clustering occur are explored, giving special attention to the presence of opinion leaders.

Another aspect of the work undertaken in this dissertation is how to generate a network in order for the agents to reach a consensus. In particular, given the agent opinion distribution and the bound of confidence, we determine the edge formation probability among agents in an Erdös-Rènyi random network. The ultimate objective is to build the network topology for consensus-based distributed decision making.

Dedicated to my brother...

Acknowledgements

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Acronyms

BBA Basic Belief Assignment. 12, 18

BoE Body of Evidence. 7, 12, 143

CCT Conditional Core Theorem. 105

CUE Conditional Update Equation. iii, 20

DS Dempster-Shafer. iii, 7

DST Dempster-Shafer Theoretic. 6, 138, 143

DW Deffuant-Weisbuch. 32

EDTI Event-based Discrete Time Index. 90, 93, 95, 96

ER Erdős-Rényi. 133

FH Fagin-Halpern. 17

FJ Friedkin-Johnsen. 32

FoD Frame of Discernment. 12

HK Hegselmann-Krause. 33

NP Nonparametric. 129

Glossary

belief The total support of all proper subsets of a proposition. 14, 21

cautious The leader like agents. They act like opinion leaders. 7, 22

confidence level A threshold of the difference of agent opinions
    in opinion updating. 33

consensus A general agreement among agents. iii, 2, 7, 9, 30, 90

ignorance Lack of evidence. 13

opinion clustering Subsets of agents agreeing to local common agreements. iii, 9,
    90

opinion influencing Spreading a particular idea among a network of agents. iii,
    145

plausibility The total belief of a proposition that does not contradict with itself.
    15, 21

receptive Agents who are more open to opinions of other agents. 7, 22

uncertainty The interval between belief and plausibility of a proposition. 15

Nomenclature

$m(\cdot)$ DS masses

$\mathrm{Bl}(\cdot)$ DS Belief

$\mathrm{Pl}(\cdot)$ DS Plausibility

Θ Frame of Discernment (FoD)

$\mathcal{F}$ Core, i.e. the set of focal elements

$\widehat{\mathcal{F}}$ Set of propositions with nonzero beliefs

$\mathcal{E}$ Body of Evidence (BoE)

$\mathscr{E}$ Set of BoEs

$m(\cdot|A)$ Conditional mass function given A

$\mathrm{Bl}(\cdot|A)$ Conditional belief function given A

$\mathrm{Pl}(\cdot|A)$ Conditional plausibility function given A

$|\Psi|$ Number of elements in any finite set Ψ (cardinality)

$\tau(\cdot)$ Contraction coefficient

$\tau_B(\cdot)$ Birkhoff contraction coefficient

$\mathbb{N}$ The set of integers, or the set of all natural numbers

$\mathbb{N}_0$ The set of all natural numbers (including zero)

$\mathbb{R}$ The set of all real numbers

$\mathbb{R}_{\geq 0}$ The set of non-negative real numbers

$\mathbb{R}_{>0}$ The set of positive real numbers

$(\boldsymbol{\cdot})_{\geq 0}$ Non-negative counterpart

$\mathbf{1}_N$ All ones N-element vector

ε Bound of confidence

$H_{\triangleleft}$ CUE operator

α_i CUE inertia parameter of $\mathcal{E}_i$

β_{ij} CUE parameter which controls the updating strategy

$\|\mathcal{E}_i - \mathcal{E}_j\|_J$ Jousselme distance between $\mathcal{E}_i$ and $\mathcal{E}_j$

$\mathscr{G}$ Set of graphs

$\mathcal{G}$ Graph

$\mathcal{G}_i \circ \mathcal{G}_j$ Graph composition of $\mathcal{G}_i$ and $\mathcal{G}_j$

A Adjacency matrix

V Vertex set of a graph

E Edge set of a graph

$\boldsymbol{\pi}(B)$ Opinion profile of proposition B

$\mathcal{N}_{i,k}$ Neighborhood of i-th agent at time k

$\mathcal{N}_{i,k}(\varepsilon_i)$ $\mathcal{N}_{i,k}$ after confidence bound notions

$\mathcal{G}_k^{\dagger}(\boldsymbol{\varepsilon})$ Graph after confidence bound notions

$E_k^{\dagger}(\boldsymbol{\varepsilon})$ Edges after confidence bound notions

W_k Confidence matrix with p.m.f. opinions at time k

$\breve{W}_k$ Confidence matrix with Dirichlet BoE opinions at time k

ξ Fixed point

xv

ζ Common fixed point

$\mathcal{H}$ Pool of operators

$\mathcal{X}_0$ Initial conditions

$\mathcal{I}$ Index sequence of updating operators

$\mathcal{S}$ Sequence of delays

Part I

Introduction

Chapter 1

Introduction

1.1 Overview

Collective behavior of networked multi-agent systems, where the agents behavior is determined by local interactions between neighboring agents, finds application in many areas of interest. In such a multi-agent system, study of how an agent's opinions can be modeled, how an agent exchanges its opinion with its neighbors, and how opinions change over time, and the study of other issues related to opinion modeling and opinion dynamics, have attracted researchers from areas as diverse as sociology, economics, psychology, politics, finance, physics, computer science, and engineering, for over six decades. Over these years, different models have been proposed to capture agent opinions and opinion exchange mechanisms of networked agents as they give valuable insights in many real-world applications. In distributed control problems, reaching a consensus state is often desired for purposes of achieving a control objective [1]; in estimation problems, the agents collectively attempt to estimate an underlying statistic of a signal [2]; within the context of fusion, the agents attempt to pool their evidence to arrive at a consensus decision [3]. In politics, information regarding

consensus or clustering of voter opinions can be of enormous benefit to candidates in planning their campaigns [4, 5]; in social networks, these models can be utilized to study consensus [6–8] as well as to identify influential nodes [9]; in viral marketing, knowledge on how opinion propagation occurs via intermediaries helps promotional activities [10, 11]. In general, such mathematical models can describe how evidence is exchanged among agents and the formation of consensus and opinion clusters.

Mutual exchange of agent opinions can be seen as an iterative process, where each agent updates its opinion based on its prior opinion, as well as the opinions of its neighboring agents. In such an iterative process, one is then led to assess whether the process converges so that each agent reaches a 'stable' opinion, or a fixed point. In general, agents interactions tend to make the opinion of agents more 'similar' [12]. Repeated interactions lead to higher degrees of homogeneity, that can be either partial or complete depending on the initial opinion distribution and the iterative updating process. When a group of agents share the same fixed point, that same group of agents is said to belong to the same *opinion cluster* [13]. Advancing further, when all the agents in a multi-agent system congregate under one common fixed point, they are said to reach a *consensus.* See Figures 1.1a and 1.1b. In a network where the agents may represent soft sources, consensus usually refers to a common agreement about an opinion of interest. To reach a consensus regarding a variable or some phenomenon of interest, the agents typically start with their own initial states and then iteratively exchange their states regarding their beliefs about the variable. Convergence analysis involves the study of such iterated belief revision processes among agents embedded in a networked environment.

It should be noted that, modeling of the agent opinion and the opinion exchange and update mechanisms is an extremely difficult task when it comes to human agents. A particular agent is neither simply share nor completely disregard the opinion of

(A) Opinion clusters. (B) Consensus of opinions.
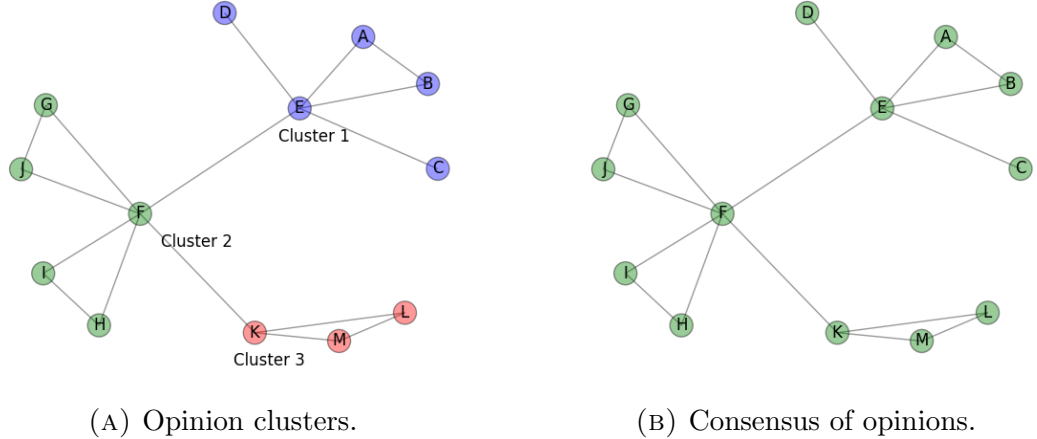
FIGURE 1.1: Opinion clustering and consensus. The different colors depict the 'stable' opinion that each agent reaches.

another agent, but rather take portions from opinions of other agents in forming its own opinion [14]. The detailed behavior of human agents is an outcome of complex physiological and phycological process, still largely unknown [12]. However, in many situations certain properties of large scale phenomena do not depend on the microscopic details of the process. In this dissertation we discuss opinion dynamic models with endogenous interpersonal communications, i.e., one agent's attitude affects another's [15]. All other affects on attitude, i.e., exogenous factors, are not addressed in the opinion dynamic models that we consider in this dissertation. Modeling dynamics of opinion exchange by trying to include the tractable important properties as much as possible, and yield crucial macroscopic information is vital for decision making.

1.2 Motivation

In multi-agent systems like social networks or a group of mobile robots completing a task, agent behavior is determined by local interactions between neighboring agents. In distributed control problems, achieving a consensus state is often necessary for the agents to achieve a control objective [1, 16]. Within the context of fusion, the agents

attempt to pool their evidence to arrive at a consensus assessment or decision [3]. To reach a consensus regarding an opinion, agents in a social network typically exchange opinions in some manner with their neighbors, and update their own opinions, possibly resulting in a consensus. Understanding the conditions under which a group of agents will achieve a consensus in a network with time-varying links is a challenging problem that has recently garnered much interest.

The presence of opinion leaders uniquely influence the consensus of a multi-agent system. Often time, leaders are the driving force which guides the opinion of the followers in a particular opinion direction [17]. Hence the analysis of consensus in agent systems which includes opinion leaders deserve rigorous analysis. In this thesis we have provided theorems and corollaries weighing on the leader-followers scenario. The study of leader-follower scenario can be utilized to understand the emergence of extremism, minority opinion spreading/survival, emergence of political parties, etc. [18].

With the advances in man-machine interaction (MMI) recent multi-agent systems often consists of heterogeneous sources. These sources can include a combination of *soft* sensors (i.e., human-based sources such as, expert opinions, subjective evidence, etc.) and *hard* sensors (i.e., conventional physics-based sensors) [19]. Hence opinion representation of such a diverse group consisting nuanced-opinions can be achieved using belief theoretic framework. This makes the agent system an complex fusion environment. The analysis of convergence of such a complex system demands additional mathematical avenues beyond the traditional matrix theoretic analysis. The study of these heterogenous systems expands the application of multi-agent system analysis to more wider areas of economics, political science, sociology and many other fields in addition to the traditional signal processing realm.

One crucial application of consensus is to make decisions in distributed agent environments. For instance, in a scenario of autonomous mobile robot group navigation, it is necessary for them to reach a consensus in order to navigate as one unit. Hence designing the communication infrastructure with the most optimum topology is vital for the group to reach a consensus and navigate together. Equipped with the knowledge on the conditions for consensus formation, generation of the underlying network topology with fault torrent capabilities and outlier removal mechanisms can boost the deployment of autonomous multi-agent systems in many real world applications.

1.3 Contributions and Their Significance

1.3.1 Accounting for Social Judgement Theoretic Notions

Social Judgement Theory (SJT) examines the underlying psychological processes of attitude expressions [20]. In particular, we capture three main concepts stemming from SJT in our opinion models [21].

Bounded confidence: is the phenomenon of agents' willingness to update their opinions with the neighboring agents' opinions only if those opinions are within their acceptable range. Our opinion dynamic model accounts for bound of confidence using a distance measure and a threshold value, as to represent the acceptable range of each agent, in updating opinions from neighbors.

Global affinity: considers multivariate opinions to capture closeness among agents. For instance, had we used only single valued opinions, we would not have captured the true opinion proximity of agents, because even if two agents are distanced apart from one scalar opinion, they could still be close by having other very similar opinions. In our models, we use Dempster-Shafer Theoretic (DST)

Body of Evidence (BoE) to represent agents, where opinions are quantified using Dempster-Shafer (DS) masses. This allows us to capture uncertainty involved in agent's opinion space. In certain analysis, the opinions are modeled with probability mass function, which is a special case of DS mass assignment.

Nature of persuasion: recognizes the agent's *ego* involvement in opinion updating. For instance, individuals with smaller ego are easy to persuade. In our model, we handle this by assigning relevant opinion updating strategies to agent's namely *receptive* and *cautious.*

1.3.2   Opinion Dynamics in the Presence of Opinion Leaders

The problem of analyzing the performance of network agents exchanging evidence in a dynamic network has recently grown in importance. This problem has relevance in signal and data fusion network applications, and in studying opinion and consensus dynamics in social networks. Most of the analysis that have been carried out on opinion dynamics have been focused on group of homogeneous receptive agents, without giving much attention to the leaders. Our analysis in [21] interpret leaders with cautiously updating agents. In that we have shown conditions when we get a consensus among a group of agents under the presence of leaders. Furthermore, we analyze conditions for which clustering occurs under the presence of multiple opinion leaders.

In Chapter 3 of this dissertation, we examine consensus formation in asynchronous dynamic/ad-hoc networks within the DST framework, focusing on the specific special cases of agent opinions represented with p.m.f.s and Dirichlet BoEs. Similar to [21] (the analytical part in [21] can be found in AppendixA), we utilize bounded confidence notions to assume that agents only exchange evidence and update their states with

their neighbors whose states are close as measured by a suitable norm. In addition, we again assume that the network connectivity graph is dynamic, and therefore that links can appear, or disappear, meaning that the agents may not have the same neighbors at each discrete-time instant. In contrast to [21], our work in Chapter 3 requires less restrictive analytical assumptions. For example, we do not assume the *strictly order preserved* arrangement of agent opinions that was required in [21]. Instead, we utilize notions from matrix and graph theory and networks to analytically examine the convergence of agent states to consensus. We assume that each agent may assume either *cautious* or *receptive* opinion updating strategies, which allows us to study the issue of opinion leaders and opinion followers. We examine several cases of interest in both opinion dynamics and fusion. For example, we study the *leader-follower problem* in which a subset of the networked agents achieve an 'opinion cluster' that is followed and adopted by the rest of the network [17]. In addition, we study the case of multiple opinion leaders where we show how the states may converge to opinion clusters in which subsets of network nodes converge to distinct states. Simulation results are provided to demonstrate the validity of the analytical results. More general analysis of convergence with agents opinions represented as DS BoEs is presented in Chapter 4.

1.3.3 Network Generation for Consensus

One of the prominent aspects of the study of opinion dynamics is to utilize consensus as a way of distributed decision making in a collaborative environment. In Chapter 5 a method to generate a network in order to facilitate agent communications to reach a consensus has been presented. We have utilized analytical results on Erdős-Rényi random graph and accounted for notions of Social Judgement Theory in our proposed network generation mechanism. A case example has been given to explain how the model can be used to generate a network to tolerate faults. The random nature of

the generation process makes it difficult to fragment the group through preplanned attacks.

1.4 Organization of the Dissertation

The dissertation has been divided into three main parts.

Part I - Introduction and Background Theory.

Part II - Theoretical Analysis of Opinion Dynamics.

Part III - Network Generation for Consensus.

1.4.1 Introduction and Background Theory

Chapter 1 and Chapter 2 give an overview of the dissertation and background theories used in the rest of the chapters in the dissertation. Among all the background theories provided, the Dempster-Shafer Theory Preliminaries in Section 2.1 play a pivotal role in modeling of opinion dynamics in Part II and network generation in Part III of the dissertation. It should be noted that, certain times, when presenting background theory, the notations used in cited sources have been altered in order to have a consistent notation throughout this dissertation. A summary of important acronyms, glossary terms and nomenclature can be found right after the table of contents.

1.4.2 Theoretical Analysis of Opinion Dynamics

Our analytical studies on opinion dynamics, with special attention to consensus and opinion clustering, are given in Part II of dissertation. Chapter 3 explains how we

model the opinions in accordance with the notions in Social Judgement Theory (SJT). Furthermore, in Chapter 3 conditions, under which consensus and opinion clustering formed, are analyzed with opinions restricted to probability mass functions and Dirichlet BoEs, specially under the presence of opinion leaders. In Chapter 4 the assumptions made on opinion modeling are relaxed and the formation of consensus and opinion clustering are analyzed in view of nonlinear paracontracting operators.

1.4.3 Network Generation for Consensus

Chapter 5 presents a network generation mechanism which can be utilized for consensus-based distributed decision making. We have utilized analytical results on Erdős-Rényi random graph and accounted for notions of Social Judgement Theory in our proposed network generation mechanism. The random nature of the generation process makes it difficult to fragment the group through preplanned attacks. This network generation has applications in social networks, autonomous mobile robots, distributed sensor systems, viral marketing etc.

Chapter 2

Preliminaries

2.1 Dempster-Shafer Belief Theory

Generally, probability models the extent an event is likely to occur. Often, lack of information and ignorance make it difficult to give precise single values as a measure of the likelihood of an event. Dempster-Shafer (DS) theory captures the effect of ignorance and imprecise information in a mathematically rigorous yet intuitive manner. Following is a brief insight into the basics of DS theory (DST) and associated ideas that are used throughout this dissertation.

2.1.1 Basic Notions

We use $\mathbb{N}$ and $\mathbb{R}$ to denote the integers and reals, respectively. Subscript $(\cdot)_{\geq 0}$ attached to these are their non-negative counterparts; $\mathbb{R}_{[0,1]}$ denotes the reals taking values in $[0, 1]$.

Frame of Discernment (FoD)

In DST, and within the context of the work described in this dissertation, the Frame of Discernment (FoD) refers to the finite discrete set $\Theta = \{\theta_1, \cdots, \theta_M\}$ of mutually exclusive and exhaustive propositions [22]. The cardinality $|\Theta| = M$ of Θ is the number of independent singleton propositions. A singleton proposition $\theta_i \in \Theta$ represents the lowest level of discernible information. The power set of the FoD $2^\Theta = \{A : A \subseteq \Theta\}$ denotes all the possible subsets of Θ. For $A \subseteq \Theta$, $\overline{A}$ denotes all singletons in Θ that are not in A.

Basic Belief Assignment (BBA)

In DST, the Basic Belief Assignment (BBA) constitutes the counterpart to the probability measure in probability theory.

Definition 2.1 (Basic Belief Assignment (BBA)). A *basic belief assignment (BBA)* or *mass assignment* is a mapping $m(\cdot) : 2^\Theta \mapsto [0,1]$ such that the following conditions are satisfied.

1. $\sum_{A \subseteq \Theta} m(A) = 1$.

2. $m(\emptyset) = 0$. ∎

The BBA measures the "support" assigned to proposition $A \subseteq \Theta$. Propositions that receive non-zero masses are referred to as *focal elements*. The set of focal elements is the *core* $\mathcal{F}$. The triplet $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$ is referred to as the Body of Evidence (BoE). The *mass vector* corresponding to the BoE $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$ is

$$\mathbf{m} = [m(\emptyset), m(\theta_1), \ldots, m(\theta_M), m(\theta_1\theta_2), \ldots, m(\theta_1\theta_M),$$
$$m(\theta_1\theta_2\theta_3), \ldots, m(\theta_1\theta_2\theta_M), \ldots, m(\Theta)]^T \in \mathbb{R}^{2^M}_{[0,1]}. \quad (2.1)$$

Notion of Ignorance

In DST, focal elements can be any singleton or composite (i.e., non-singleton) proposition. DST captures the notion of *ignorance* by allocating masses to composite propositions. For instance, the composite proposition $\{\theta_i\theta_j\}$, $\theta_i, \theta_j \in \Theta$, is a doubleton and the mass assignment $m(\theta_i\theta_j) > 0$ represents ignorance or lack of evidence to differentiate between the two constituent singletons. The state of complete ignorance can be easily captured via the *vacuous BBA* which has Θ as its only focal element, i.e., the mass assignment structure of the vacuous BBA is $m(A) = 1$ for $A = \Theta$ (and $m(A) = 0$ for $A \subset \Theta$).

A BBA is called *Bayesian* if each focal element is a singleton. For a Bayesian BBA, the BBA, belief, and plausibility (which will be explained in next section), all reduce to a probability assignment.

Belief and Plausibility

The mass assignment for a particular proposition A, i.e., $m(A)$, measures the support assigned to proposition A itself only. On the other hand, the belief of proposition A accounts for the support of all proper subsets of A.

Definition 2.2 (Belief). Given a BoE $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$, the *belief* assigned to a proposition $A \in \Theta$ is Bl $: 2^\Theta \mapsto [0, 1]$ where

$$\mathrm{Bl}(A) = \sum_{B \subseteq A} m(B). \qquad \blacksquare$$

$\mathrm{Bl}(A)$ represents the total belief that is committed to A without also being committed to its complement $\overline{A}$. The propositions that possess nonzero beliefs are denoted by

$$\widehat{\mathcal{F}} = \{A \subseteq \Theta : \mathrm{Bl}(A) \neq 0\}. \tag{2.2}$$

Note that, $\mathcal{F} \subseteq \widehat{\mathcal{F}}$, i.e., $\mathrm{Bl}(A) > 0$, $\forall A \in \mathcal{F}$.

We can characterize beliefs without any reference to the underlying BBA.

Theorem 2.3. *For a given FoD Θ, the function $\mathrm{Bl} : 2^\Theta \mapsto [0,1]$ constitutes a belief function iff the following conditions are satisfied:*

1. *$\mathrm{Bl}(\emptyset) = 0$.*

2. *$\mathrm{Bl}(\Theta) = 1$.*

3. *$\forall \{A_i\}_{i=\overline{1,n}}$ st. $A_i \subseteq \Theta$,*

$$\mathrm{Bl}\left(\bigcup_{i=\overline{1,n}} A_i\right) \geq \sum_{\substack{I \subseteq \{1,\ldots,n\} \\ I \neq 0}} (-1)^{|I|+1} \mathrm{Bl}\left(\bigcap_{i \in I} A_i\right). \qquad \blacksquare$$

Given the beliefs associated with a particular FoD, we may compute the corresponding BBA as follows:

Theorem 2.4. *For a given FoD Θ, suppose $\mathrm{Bl} : 2^\Theta \mapsto [0,1]$ constitutes a valid belief function in the sense of Theorem 2.3. Then the function $m : 2^\Theta \mapsto [0,1]$ defined as*

$$m(A) = \sum_{B \subseteq A} (-1)^{|A-B|} \mathrm{Bl}(B),$$

constitutes a valid bba as in the sense of Definition 2.1. $\qquad \blacksquare$

Definition 2.5 (Plausibility)**.** Given a BoE, $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$, the *plausibility* assigned to a proposition $A \subseteq \Theta$ is $\mathrm{Pl} : 2^\Theta \mapsto [0, 1]$ where

$$\mathrm{Pl}(A) = 1 - \mathrm{Bl}(\overline{A}). \qquad \blacksquare$$

Plausibility corresponds to the total belief that does not contradict A. We may calculate plausibility for any $A \subseteq \Theta$ via

$$\mathrm{Pl}(A) = \sum_{B \cap A \neq \emptyset} m(B). \qquad (2.3)$$

Clearly, $\mathrm{Pl}(A) \geq \mathrm{Bl}(A)$.

The *uncertainty* of A is the interval $[\mathrm{Bl}(A), \mathrm{Pl}(A)]$.

Belief and Probability Functions

Probability measures can be taken as a special case of belief functions.

Definition 2.6. A belief function $\mathrm{Bl} : 2^\Theta \mapsto [0, 1]$ associated with the BoE $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$ is said to be a *Bayesian belief function* if it constitutes a probability measure on 2^Θ. $\blacksquare$

Theorem 2.7 gives equivalent conditions for Bayesian belief functions.

Theorem 2.7. *Given a BoE* $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$ *and a belief function* $\mathrm{Bl} : 2^\Theta \mapsto [0, 1]$, *the following statements are equivalent:*

- $\mathrm{Bl}(\cdot)$ *is a Bayesian belief function.*

- $\mathrm{Bl}(\cdot) = \mathrm{Pl}(\cdot)$.

- *All elements of $\mathcal{F}$ are singletons, i.e., $m(A) = 0, \forall A$ s.t. $|A| > 1$.*

- $\text{Bl}(A) + \text{Bl}(\overline{A}) = 1, \forall A \subseteq \Theta.$ ∎

The following additional statements hold true for Bayesian belief functions:

- $m(\cdot) = \text{Bl}(\cdot) = \text{Pl}(\cdot) \equiv \text{P}(\cdot).$

- $\text{Bl}(A \cup B) = \text{Bl}(A) + \text{Bl}(B)$ whenever $A, B \subseteq \Theta$ and $A \cap B = \emptyset.$

Dirichlet BoE

A special DST model which retains the ability to capture complete ignorance with only a slight increase in computational complexity compared to Bayesian BBA is the *Dirichlet BoE* (so named because of its close relationship with Dirichlet probability distributions [23, 24]). The singletons and Θ constitute the only focal elements of a Dirichlet BoE.

2.1.2 DS Theoretic Conditionals

Several notions of DST conditionals exist in the literature. For example,

- Dempster's rule of conditioning (including the non-normalized version),

- Yager-Kohlas' rule of conditioning,

- Resemblance based rule of conditioning,

- Fagin-Halpern' rule of conditioning.

Of these different notions, the Fagin-Halpern (FH) conditional offers a unique probabilistic interpretation and hence a natural transition to the Bayesian conditional notion [25, 26]. The extensive study in [27] identifies several attractive properties of the FH conditionals including its equivalence to other popular notions of DST conditionals.

Definition 2.8 (Fagin-Halpern (FH) Conditionals)**.** For the BoE $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$ and $A \subseteq \Theta$ s.t. $A \in \widehat{\mathcal{F}}$, the *conditional belief* $\mathrm{Bl}(B|A) : 2^{\Theta} \mapsto [0,1]$ and *conditional plausibility* $\mathrm{Pl}(B|A) : 2^{\Theta} \mapsto [0,1]$ of B given A are

$$\mathrm{Bl}(B|A) = \frac{\mathrm{Bl}(A \cap B)}{\mathrm{Bl}(A \cap B) + \mathrm{Pl}(A \cap \overline{B})};$$
$$\mathrm{Pl}(B|A) = \frac{\mathrm{Pl}(A \cap B)}{\mathrm{Pl}(A \cap B) + \mathrm{Bl}(A \cap \overline{B})}. \qquad \blacksquare$$

The Conditional Core Theorem [28] can be utilized to directly identify the conditional focal elements to improve computational performance when applying FH conditionals.

2.1.3 Evidence Fusion

To facilitate our discussion, we wish to differentiate between the following two notions, both related to evidence fusion, i.e., how a knowledge base incorporates new evidence that it receives.

- **Evidence combination:** This corresponds to merging multiple knowledge bases to obtain a combined knowledge base that accommodates the viewpoints of its constituents. For instance, in the Bayesian framework, this is done via the application of the Bayes' rule.

- **Evidence updating:** This refers to the refinement of one's knowledge base to accommodate a new piece of evidence that is received. For example, in Bayesian framework, updating is done via the conditional notion.

The evidence being received could have been generated BoEs with different underlying FoDs. The following notions characterize the nature of these FoDs:

- **Non-Exhaustive or Exhaustive:** If all the FoDs are defined in the same 'context' so that the 'complete' FoD can be thought of as the union of all the constituent FoDs, then a member FoD that is a strict subset of the complete FoD is said to be *non-exhaustive.* In contrast, if the member FoDs possess all the elements of the complete FoD, then that member FoD is said to be *exhaustive.*

- **Heterogeneous or Homogeneous:** FoDs whose singletons are defined within different 'contexts' are said to be *heterogeneous.* The 'complete' FoD is then constructed via the cross-product of these heterogeneous member FoDs. In contrast, if the singletons of all FoDs are defined in the same context then those FoDs are said to be *homogeneous.*

2.1.4 Evidence Combination

Of all DST evidence combination strategies, *Dempster's evidence combination function (DECF)* is perhaps the most widely used evidence combination strategy that yields a combined BBA which accounts for evidences of the constituent FoDs.

Dempster's Evidence Combination Function (DECF)

Consider two BoEs $\mathcal{E}_1 = \{\Theta, \mathcal{F}_1, m_1\}$ and $\mathcal{E}_2 = \{\Theta, \mathcal{F}_2, m_2\}$ that span across the same FoD. Before defining the DECF let us first define the notion of *conflict.*

Definition 2.9. The conflict between the evidence present in BoEs $\mathcal{E}_1$ and $\mathcal{E}_2$ is

$$K_{12} = \sum_{B \in \mathcal{F}_1, C \in \mathcal{F}_2 \; C \cap B = \emptyset} m_1(B) m_2(C). \qquad \blacksquare$$

The BoEs $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be

- **incompatible,** if $K_{12} = 1$;

- **compatible,** if $K_{12} \in [0, 1)$; and

- **completely compatible,** if $K_{12} = 0$.

DECF applies to compatible BoEs only, and it is defined as

Definition 2.10 (Dempster's Evidence Combination Function (DECF))**.** Consider two compatible BoEs $\mathcal{E}_1$ and $\mathcal{E}_2$ with *conflict* K_{12}. Then the DECF generates the BBA

$$m(A) \equiv (m_1 \oplus m_2)(A) = \sum_{B \in \mathcal{F}_1, C \in \mathcal{F}_2 \; C \cap B = A} \frac{m_1(B)\, m_2(C)}{1 - K_{12}}, \; \forall A \subseteq \Theta. \qquad \blacksquare$$

The fusion operation of the DECF is denoted as $m(\cdot) = (m_1 \oplus m_2)(\cdot)$. The $\oplus$ operator is both associative and commutative thus allowing for convenient combination of multiple BoEs. A variation of the DECF which accounts for evidence reliability is $m(\cdot) = (\hat{m}_1 \oplus \hat{m}_2)(\cdot)$, where

$$\hat{m}_k(A) = \begin{cases} b_k m(A), & \text{for } A \subset \Theta \\ (1 - b_k) + b_k m_k(\Theta), & \text{for } A = \Theta. \end{cases} \qquad (2.4)$$

Here, $b_k \in [0, 1]$ is referred to as the *discounting factor* [22].

2.1.5 Evidence Updating

As stated before, in evidence updating one's knowledge base accommodates a new piece of evidence that is received. The *conditional approach* does exactly this.

Conditional Update Equation (CUE)

The Conditional Update Equation (CUE) [29, 30] offers a strategy to update evidence from different BoEs $\mathcal{E}_j = \{\Theta_j, \mathcal{F}_j, m_j\}$, $j = 1, 2, \ldots, N$, to arrive at a new updated BoE $\mathcal{E} = \{\Theta, \mathcal{F}, m\}$. Without loss of generality, let us consider updating the BoE $\mathcal{E}_i$ with the evidence in $\mathcal{E}_j$, $j = \{1, 2, \ldots, N\} \setminus i$, to generate the BoE $\mathcal{E}$. In the CUE, the updated belief of an arbitrary proposition B in the BoE $\mathcal{E}$ is given by

$$\mathrm{Bl}_i(B)_{k+1} = \alpha_i \mathrm{Bl}_i(B)_k + \sum_{\substack{j=1;\\ j\neq i}}^{N} \sum_{A\in\mathcal{F}_j} \beta_{ij}(A) \, \mathrm{Bl}_j(B|A)_k. \tag{2.5}$$

Here, α_i and β_{ij} are non-negative parameters that satisfy $\alpha_i + \sum_{j\neq i} \sum_{A\in\mathcal{F}_j} \beta_{ij}(A) = 1$. The integer subscripts k and $k+1$ denote the BoE before and after the updating operation.

The CUE-based fusion operator can be defined as follows.

Definition 2.11 ($H_\lhd$ operator)**.** [19] Let $\mathscr{E}_\Theta \equiv \{\mathcal{E} | \mathcal{E} = \{\Theta, \mathcal{F}, m(\cdot)\}\}$ denote the set of all possible BoEs defined on Θ. Then the set of N BoEs corresponding to the N agents can be represented as $\mathcal{E}_i = \{\Theta, \mathcal{F}_i, m_i(\cdot)\} \in \mathscr{E}_\Theta$, $i = 1, 2, \ldots, N$. Then the CUE operator $H_\lhd^i : \mathscr{E}_\Theta^N \mapsto \mathscr{E}_\Theta$ that updates $\mathcal{E}_i$ with all $\mathcal{E}_j$, $j \in \{1, 2, \ldots, m\} \setminus \{i\}$ can be written as,

$$H_\lhd^i\big(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N\big) \equiv \mathcal{E}_i \lhd \big(\mathcal{E}_1 \bowtie \mathcal{E}_2 \bowtie \cdots \bowtie \mathcal{E}_{i-1}$$
$$\bowtie \mathcal{E}_{i+1} \bowtie \cdots \bowtie \mathcal{E}_N\big). \tag{2.6}$$

■

Selection of the CUE Parameters

The work in [30, 31] provides different strategies for the selection of the CUE parameters. The parameters α_i determines the flexibility of accepting updates from neighbors. The lower the value of α_i, the higher the flexibility of the CUE update towards changes. Two approaches for selecting α_i, $i = 1, 2, \ldots, N$, are as follows:

1. **Inertia of available evidence:** The weight α_i can be taken as a measure of the inertia of a particular agent toward updating with the incoming evidence. Some inertial based strategies of selecting α_i are as follows:

   (a) **Infinite inertia based updating:** $\alpha_i = 1$.

   (b) **Zero inertia based updating:** $\alpha_i = 0$.

   (c) **Proportional inertia based updating:** $\alpha_i = N/(N + 1)$, where N denotes the number of 'pieces' of evidence that has been received so far.

2. **Integrity of available evidence:** The selection of α_i is done such that the integrity of the originally assigned belief and plausibility values are maintained in the following sense:

$$\mathrm{Bl}_i(B)_{k+1} \leq \mathrm{Pl}_i(B)_k \text{ and } \mathrm{Pl}_i(B)_{k+1} \geq \mathrm{Bl}_i(B)_k. \tag{2.7}$$

   Then, the range of values for α_i can be given as

$$\alpha_i \in \begin{cases} \left[\dfrac{1 - \mathrm{Pl}_i(B)_k}{1 - \mathrm{Bl}_i(B)_k}, 1\right], & \text{for } \mathrm{Bl}_i(B)_k < 1; \\ [0, 1], & \text{for } \mathrm{Bl}_i(B)_k = 1. \end{cases} \tag{2.8}$$

The parameters $\beta_{ij}(A)$ allows one to weigh the incoming evidence. These parameters can be selected to be proportional to the support available for the corresponding focal element from $\mathcal{E}_i$ or $\mathcal{E}_j$, under the constraint that they satisfy $\alpha_i + \sum_{j \neq i} \sum_{A \in \mathcal{F}_j} \beta_{ij}(A) = 1$. This generates two main strategies which are referred to as *cautious* and *receptive* strategies [30]:

1. **Receptive Updating:** This strategy weighs the incoming evidence according to the support of each focal element receives from $\mathcal{E}_j$, $j = \{1, 2, \ldots, N\} \setminus i$, by selecting $\beta_{ij}(A)$ as

$$\beta_{ij}(A) = K_{ij} m_{\Theta_j}(A), \tag{2.9}$$

where K_{ij} is a constant such that $\alpha_i + \sum_{j \neq i} \sum_{A \in \mathcal{F}_j} K_{ij} m_{\Theta_j}(A) = 1$.

2. **Cautious Updating:** This strategy weighs the incoming evidence according to the support of each focal element receives from $\mathcal{E}_i$ by selecting $\beta_{ij}(A)$ as

$$\beta_{ij}(A) = K_{ij} m_{\Theta_i}(A), \tag{2.10}$$

where K_{ij} is a constant such that $\alpha_i + \sum_{j \neq i} \sum_{A \in \mathcal{F}_j} K_{ij} m_{\Theta_i}(A) = 1$.

One can extend the notions of *cautious* and *receptive* to model moderate agents, with *partially cautious agents*.

3. **Partially cautious updating:** This strategy weighs the incoming evidence according to the support of each focal element receives from $\mathcal{E}_i$ by selecting $\beta_{ij}(A)$ as

$$\beta_{i,j}(A) = K_{i,j} \big(\rho_i m_i(A) + (1 - \rho_i) m_j(A) \big), \tag{2.11}$$

where $K_{i,j}$ and ρ_i, $0 \leq \rho_i \leq 1$ are constants, such that

$$\alpha_i + \sum_{j \neq i} \sum_{A \in \mathcal{F}_j} K_{i,j}\big(\rho_i m_i(A) + (1 - \rho_i)m_j(A)\big) = 1. \qquad (2.12)$$

2.1.6   Distance Measure

In the work to follow in this dissertation, we will need a distance measure which captures the closeness between DST BoEs. Among possible alternatives that have appeared in the literature, for our work, we use the DST distance measure in [32, 33].

Definition 2.12 (Distance Between BoEs). [32] The distance between the two agent BoEs $\mathcal{E}_i = \{\Theta, \mathcal{F}_i, m_i\}$ and $\mathcal{E}_j = \{\Theta, \mathcal{F}_j, m_j\}$ is

$$\|\mathcal{E}_i - \mathcal{E}_j\|_J = \left[0.5\,(\mathbf{m}_i - \mathbf{m}_j)^T \mathcal{D}\,(\mathbf{m}_i - \mathbf{m}_j)\right]^{1/2} \in \mathbb{R}_{[0,1]},$$

where $\mathbf{m}_i, \mathbf{m}_j \in \mathbb{R}_{\geq 0}^{2^M}$ are the mass vectors associated with the BoEs $\mathcal{E}_i$ and $\mathcal{E}_j$, respectively; $\mathcal{D} = \{d_{mn}\} \in \mathbb{R}_{\geq 0}^{2^M \times 2^M}$, with

$$d_{mn} = \begin{cases} 0, & \text{if } A_m = A_n = \emptyset; \\ \dfrac{|A_m \cap A_n|}{|A_m \cup A_n|}, & \text{otherwise.} \end{cases} \qquad \blacksquare$$

2.2   Relevant Notions from Graph Theory

2.2.1   Basic Notions

A *network* or *graph* is a relational structure constituted of a finite set of objects and the relationships between pairs of objects [34]. Vertices V (also termed as 'nodes', 'points', 'sites' or 'actors') denote the finite set of objects. Edges E (also termed

as 'lines', 'arcs', 'links', 'bonds' or 'ties') denote the relationship between a part of vertices. The adjacency matrix A is an $(N \times N)$-sized matrix $A = \{A_{ij}\}$ s.t.

$$
A_{ij} = \begin{cases} 1, & \text{if there is an edge between vertices } i \text{ and } j; \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}
$$

Here $N = |V|$, the cardinality of the vertex set V.

Throughout the dissertation, we represent agents with vertices and interaction between agents with edges. A graph can be explained with set of vertices and edges $\mathcal{G}(V, E)$, or with its associated adjacency matrix A.

Networks can be either *directed* or *undirected*. In directed networks the edge set E contains ordered vertex pairs. In contrast, the undirected network contains unordered vertex pairs.

We use $\mathcal{G}_k = (V, E_k)$ to denote a time varying directed graph at discrete-time instant $k \in \mathbb{N}_{\geq 0}$. Here, $e_{ij} \in E_k$ represents an unidirectional edge from node $V_j \in V$ to node $V_i \in V$. We use A_k to identify the $(N \times N)$ adjacency matrix associated with E_k.

Consider the directed graph $\mathcal{G}_k = (V, E_k)$. The *out-component of vertex $V_i \in V$* is the set of vertices (including vertex V_i itself) reachable via directed paths from vertex V_i. The *in-component of vertex $V_i \in V$* is the set of vertices (including vertex V_i itself) from which vertex V_i is reachable via directed paths.

2.2.2   Random Graphs

Random graphs can be characterized by the random process which generates them. In this dissertation, we use the Erdös-Rènyi (ER) random graph model to generate

random graphs, mainly due to its analytical tractability. The information on the expected structure and attributes of the ER network can be extracted easily [34].

Erdös-Rènyi Random Graph Model

Erdös-Rènyi random graph model is typically denoted as $\mathcal{G}(N, p)$ where N is the number of vertices and p is the probability of independently placing an edge between each distinct vertex pair. The mean number of edges of $\mathcal{G}(N, p)$ is $^{N}C_2 p$ whereas the mean vertex degree is given by $c \equiv (N-1)p$. Cleary, for a given number of vertices, increasing p yields higher values for the mean number of edges and the mean vertex degree.

2.2.3 Centrality Measures

Different centrality measures have been proposed in order to quantify the most central or influential vertices [34]. Centrality measures can be categorized according to the property they attempt to capture:

- **Degree:** connectedness of a vertex.

- **Neighbor characteristics:** presence of influential neighbors.

- **Closeness:** ease reaching other vertices.

- **Betweenness:** role in connecting other vertices.

2.2.4 Graph Composition

The concept of Graph Composition has been used in the study of reaching a consensus in a dynamically changing environment [35]. Let $\mathscr{G}$ be the set of all directed graphs

with vertex set $V = \{V_1, V_2, \ldots, V_N\}$ and $\mathcal{G}_i \in \mathcal{G}$, where $\mathcal{G}_i = (V, E_i)$, $i = 1, 2, \ldots$, with E_i being an edge set.

Composition: The composition $\mathcal{G}_1 \circ \mathcal{G}_2$ of graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ is the directed graph with vertex set V and the edge list $E_{1 \circ 2}$ in such a way, that $e_{ik} \in E_{1 \circ 2}$ is an edge of the composition graph $\mathcal{G}_1 \circ \mathcal{G}_2$ if there are edges $e_{ij} \in E_1$ and $e_{jk} \in E_2$.

Rooted graph: A vertex V_i is a *root* of a directed graph $\mathcal{G}$ if there is a directed path from V_i to every other vertex $V_j, j \in \{1, 2, \ldots, N\} \setminus i$ of $\mathcal{G}$. A *rooted graph* is a graph $\mathcal{G} \in \mathcal{G}$ such that the graph $\mathcal{G}$ contains at least one root.

Jointly rooted sequence of graphs: Consider a finite sequence of directed graphs $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k \in \mathcal{G}$. We say that sequence of graphs is *jointly rooted* if the graph composition $\mathcal{G}_1 \circ \mathcal{G}_2 \circ \cdots \circ \mathcal{G}_k$ is a rooted graph.

Repeatedly jointly rooted sequence of graphs: An infinite sequence of graphs $\mathcal{G}_1, \mathcal{G}_2, \ldots$ in $\mathcal{G}$ is *repeatedly jointly rooted* with period ℓ if there is a positive integer ℓ for which each finite sequence $\mathcal{G}_{\ell(k-1)+1}, \mathcal{G}_{\ell(k-1)+2}, \ldots, \mathcal{G}_{\ell k}, k \geq 1$ is jointly rooted.

2.3   Relevant Notions from Matrix and Stochastic Matrix Theory

2.3.1   Basic Notions

In this thesis we use the superscripts $(.)^N$ and $(.)^{M \times N}$ to denote N-sized vector and $(M \times N)$-sized matrix counterparts. For $X = \{X_{ij}\} \in \mathbb{R}^{M \times N}$, $\|X\|$ denotes its ∞-norm, i.e., $\|X\| = \max_{i \in \overline{1,M}} \sum_{j=1}^{N} |X_{ij}|$. We use $X > 0$ and $X \geq 0$ to denote a matrix/vector with positive and non-negative entries, respectively. A matrix whose

entries are all 0s (and is of compatible size) is denoted by $\mathbf{0}$; an N-element vector whose elements are all 1s is denoted by $\mathbf{1}_N$.

Stochastic/sub-stochastic matrices and the limiting behavior of their products play a critical role in our work.

Definition 2.13 (Stochastic matrix). If $X \in \mathbb{R}_{[0,1]}^{N \times N}$ is stochastic, then $\sum_{j=1}^{N} X_{ij} = 1$, $\forall i \in \overline{1,N}$, and $\|X\| = 1$. Here, $\|X\| = \max_{i=1,\ldots,N} \sum_{j=1}^{N} |X_{ij}|$, i.e., $\|X\|$ denotes the ∞-norm. $\blacksquare$

Definition 2.14 (Sub-stochastic matrix). The stochastic matrix $X \in \mathbb{R}_{[0,1]}^{N \times N}$ is said to be sub-stochastic if $\exists i \in \overline{1,N}$ s.t. $\sum_{j=1}^{N} X_{ij} < 1$. $\blacksquare$

2.3.2 Stochastic Chains

Definition 2.15 (Properties of Stochastic Chains). [36] Consider the stochastic chain $\{W_k = \{w_{ij,k}\}, \ k \in \mathbb{N}_{\geq 0}$, where $w_{ij,k}$ represents the weight of the edge from vertex j to vertex j in the directed graph $\mathcal{G}_k = (V, E_k)$, for $k \in \mathbb{N}_{\geq 0}$. For a given pair of $S, \overline{S} \subseteq V$ s.t. $S \cup \overline{S} = V$ and $S \cap \overline{S} = \emptyset$, let $W_{S\overline{S},k} = \sum_{i \in S, j \in \overline{S}} w_{ij,k}$.

(a) *Balanced chains:* $\{W_k\}$ is *balanced* if there exists a scalar $\sigma > 0$ s.t. $W_{S\overline{S},k} \geq \sigma W_{\overline{S}S,k}$ for any non-trivial $S \subseteq V$ and $k \in \mathbb{N}_{\geq 0}$. The scalar σ is referred to as a *balancedness coefficient.*

(b) *Strongly aperiodic chains:* $\{W_k\}$ is *strongly aperiodic* if $w_{ii,k} \geq \varsigma$ for some $\varsigma > 0$, and for all $i \in \overline{1,N}$ and $k \in \mathbb{N}_{\geq 0}$.

(c) *Infinite flow graph:* The *infinite flow graph* associated with $\{W_k\}$ is the directed graph $\mathcal{G}_\infty = (V, E_\infty)$ where

$$E_\infty = \left\{ (i,j) : \sum_{k=0}^{\infty} (w_{ij,k} + w_{ji,k}) = \infty, \ i \neq j \in \overline{1,N} \right\}. \quad \blacksquare$$

Graph theoretically, $S \times \overline{S}$ corresponds to the 'flow' entering the subset S; hence $W_{S\overline{S}}$ is the sum of weights of 'flows' entering S, i.e., the 'flow' leaving $\overline{S}$. The balancedness property requires that the 'flows' leaving and entering each subset of vertices do not vanish over time [36].

2.3.3   Contraction Coefficients

Let Σ be a matrix set and $\Lambda = \Sigma \cup \Sigma^2 \cup \cdots$. A non-negative function $\tau : \Lambda \to \mathbb{R}$ is called a *contraction coefficient* for Σ if

$$\tau(AB) \leq \tau(A)\,\tau(B), \ \forall A, B \in \Lambda. \tag{2.14}$$

Contraction coefficients can be used to show that a sequence of vectors or a sequence of matrices converges in some sense [37]. *Birkhoff contraction coefficient* is one such contraction coefficient. Before we proceed to discuss on Birkhoff contraction coefficient in Section 2.3.3, let us look at some preliminary notions that will help us understand Birkhoff contraction coefficient better.

Projective Metrics

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ where $\mathbf{x} = (x_1, \cdots, x_n)^T$ and $\mathbf{y} = (y_1, \cdots, y_n)^T$. If $x_i \geq 0 \ \forall i \in \{1, \cdots, n\}$, then $\mathbf{x}$ is said to be non-negative; if $x_i > 0 \ \forall i \in \{1, \cdots, n\}$, then $\mathbf{x}$ is said to be positive. If $x_i \geq y_i$ (or, $x_i > y_i$), $\forall i \in \{1, \cdots, n\}$, then we write $\mathbf{x} \geq \mathbf{y}$ (or, $\mathbf{x} > \mathbf{y}$).

Definition 2.16 (Projective Metric)**.** Let $\mathbf{x}, \mathbf{y} > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then the *projective metric* p is given by

$$p(\mathbf{x}, \mathbf{y}) = \ln \frac{\max_i \frac{x_i}{y_i}}{\min_j \frac{x_j}{y_j}}. \qquad \blacksquare$$

The positive orthant denoted by $(\mathbb{R}^n)^+$, is the set of all positive vectors in $\mathbb{R}^n$. The *projective metric* p given in Definition 2.16 defines a scaled distance between any two vectors in $(\mathbb{R}^n)^+$. If $\mathbf{x}$ and $\mathbf{y}$ are positive vectors in $\mathbb{R}^n$, then $p(\mathbf{x}, \mathbf{y}) = p(\alpha \mathbf{x}, \beta \mathbf{y})$, for any positive constants α and β.

Definition 2.17. An $m \times n$ matrix $A \geq 0$ is said to be *row-allowable* if it has at least one positive entry in each of its rows; it is said to be *column-allowable* if A^T is row-allowable. The matrix A is said to be allowable if it is both row and column allowable. $\blacksquare$

Lemma 2.18. *Let A be an $m \times n$ row allowable matrix and $\mathbf{x}$ and $\mathbf{y}$ are positive vectors. Then*

$$p(A\mathbf{x}, A\mathbf{y}) \leq p(\mathbf{x}, \mathbf{y}). \qquad \square$$

Proof. See [37], Section 2.2.1. $\blacksquare$

Birkhoff Contraction Coefficient

Definition 2.19 (Birkhoff Contraction Coefficient)**.** The *Birkhoff Contraction Coefficient* of an $n \times n$ row allowable matrix A is

$$\tau_B(A) = \sup \frac{p(A\mathbf{x}, A\mathbf{y})}{p(\mathbf{x}, \mathbf{y})},$$

where the supremum is taken over all positive vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. $\blacksquare$

It can be seen that $p(A\mathbf{x}, A\mathbf{y}) \leq \tau_B(A)\, p(\mathbf{x}, \mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} > 0$. Hence from Lemma 2.18,

$$\tau_B(A) \leq 1. \tag{2.15}$$

Theorem 2.20. *Let $A \geq 0$ be a row-allowable matrix. Then $\tau_B(A) = 0$ iff A is of rank 1.* $\square$

Proof. See [38], Section 3.1. ∎

Theorem 2.21. *Let A and B be $n \times n$ row-allowable, non-negative matrices. Then*

$$\tau_B(AB) \leq \tau_B(A)\tau_B(B). \qquad \square$$

Proof. See [37], Section 2.2. ∎

2.4   Opinion Models and Opinion Dynamics

2.4.1   Basic Notions

Previous work within signal processing, control, and data fusion geared towards analyzing agent consensus formation has mostly modeled an agent state as a real-valued vector [1–3, 16] (see also [19] and references therein). While such an assumption is useful in the context of agents achieving consensus on a control vector, or sensing network agents achieving a consensus signal estimate, the assumption of a real-valued vector may not necessarily be always suitable within higher-level fusion, or the analysis of opinion dynamics in social networks. In higher-level fusion, agents attempt to achieve a consensus assessment regarding a situation, while in opinion dynamics agents attempt to achieve a consensus opinion. The inherent uncertainty within such

applications may necessitate a more structured agent state vector such as a probability mass function (p.m.f.) or a Dempster-Shafer (DS) theoretic mass function. However, much of the work on opinion dynamics in social networks has focused on modeling the agent states as either scalar real numbers, or a vector of real numbers [18], e.g., see the *Hegselmann-Krause (HK) model* [14, 39] and the *Deffuant-Weisbuch (DW) model* [40, 41].

2.4.2 Opinion Dynamics Models

Suppose there are N number of agents in the group and the opinion updating occurs at discrete time steps, $T = \{0, 1, 2, 3, \cdots\}$. In order to accommodate generalized opinion representation in Chapter 3 we use an extended notation that has been used in [14].

For an agent $i \in \{1, \cdots, N\}$, let its opinion on a particular issue or proposition θ at discrete time instance k, be denoted by $m_i(\theta)_k$. To introduce and explain the opinion models currently being used in the literature, for the remainder of this chapter, we assume that $m_i(\theta)_k \in \mathbb{R}$. Later, in Chapter 3, we will introduce our DST opinion model.

The vector $\mathbf{m}(\theta)_k = [m_1(\theta)_k, m_2(\theta)_k, \cdots, m_N(\theta)_k]^T \in \mathbb{R}_{\geq 0}^N$ is the opinion profile of θ at instance k (more formal definition is given in Definition 3.1). Fixing an agent i, the weight given to any other agent, say j, is denoted by w_{ij}, such that $w_{i1} + w_{i2} + \cdots + w_{iN} = 1$ and $w_{ij} \geq 0$ for all i, j. Then the opinion formation of agent i can be given as in

$$m_i(\theta)_{k+1} = w_{i1}m_1(\theta)_k + w_{i2}m_2(\theta)_k + \cdots + w_{iN}m_N(\theta)_k. \tag{2.16}$$

The weights may change with time or with the opinion profile, i.e., $w_{ij} = w_{ij}(k, \mathbf{m}(\theta)_k)$. Let $W(k) = \{w_{ij}(k)\}$. Here, whenever $w_{ij}(k) > 0$, agent j communicates its state to agent i, with a weight of w_{ij} at instance k.

Classical Model

When $W(k)$ is a fixed stochastic matrix we can denote it without the time dependence, i.e., as W. The classical model with fixed weights W as in

$$\mathbf{m}(\theta)_{k+1} = W\mathbf{m}(\theta)_k, \forall k \in T, \tag{2.17}$$

has been used in [42] and [6].

Friedkin-Johnsen (FJ) Model

The Friedkin-Johnsen (FJ) model [15] accounts for the susceptibility of agents in opinion updating. Following the same notation as in (2.17), this FJ model is given as

$$\mathbf{m}(\theta)_{k+1} = AW\mathbf{m}(\theta)_k + (I - A)\mathbf{m}(\theta)_0, \tag{2.18}$$

where A is a sub-stochastic diagonal matrix. Each diagonal element in A accounts for the susceptibility of corresponding agent.

Deffuant-Weisbuch (DW) Model

In [40] Deffuant *et al.* propose a model considering the notion of *bounded confidence* (See Section 2.5). Consider a population of N agents be embedded in a graph, where agents may discuss with each other if the corresponding nodes are connected. Each

agent's initial opinion on θ, i.e., $m_i(\theta)_0$, $i \in \{1, 2, \ldots, N\}$ is randomly chosen in the interval $[0, 1]$. The opinion updating is carried out via pairwise interactions, where at each time step, a randomly selected pair of agents are chosen [41, 43]. Let p and q be the pair of interacting agents at time k, with opinions $m_p(\theta)_k$ and $m_q(\theta)_k$, respectively. DW dynamics is as follows:

$$
m_r(\theta)_{k+1} = \begin{cases} m_r(\theta)_k + \mu[m_r(\theta)_k - m_q(\theta)_k], & \text{if } r = p \text{ and } |m_p(\theta)_k m_q(\theta)_k| < \varepsilon; \\ m_r(\theta)_k + \mu[m_r(\theta)_k - m_p(\theta)_k], & \text{if } r = q \text{ and } |m_p(\theta)_k - m_q(\theta)_k| < \varepsilon; \\ m_r(\theta)_k, & \text{otherwise}, \end{cases}
$$

$$(2.19)$$

where μ is the parameter named *convergence parameter* with a value lying in the interval $[0, 1/2]$, and ε is a threshold value which determines the maximum social proximity in order to exchange opinions.

Hegselmann-Krause (HK) Model

Hegselmann-Krause (HK) model [14] accounts for the similarity of opinions among agents in opinion updating. An agent i takes only those agents j into account whose opinions differ from his/her own by not more than a certain threshold or *confidence level ε_i*. Assuming all agents can communicate with each other, for an agent i, the 'opinion-wise' closer agents set (for opinion θ) is given by

$$
\Psi(i, \mathbf{m}(\theta)) = \left\{ 1 \leq j \leq N : |m_i(\theta) - m_j(\theta)| \leq \varepsilon_i \right\}.
$$

$$(2.20)$$

Assume that agent i puts an equal emphasis on all $j \in \Psi(i, \mathbf{m}(\theta))$. Then the weights are given by

$$w_{ij} = \begin{cases} 0, & \text{for } j \notin \Psi(i, \mathbf{m}(\theta)); \\ |\Psi(i, \mathbf{m}(\theta))|^{-1}, & \text{for } j \in \Psi(i, \mathbf{m}(\theta)). \end{cases} \tag{2.21}$$

Here $|\Psi|$ is the cardinality of the finite set Ψ. Then, the model of *bounded confidence* is given by

$$m_i(\theta)_{k+1} = |\Psi(i, \mathbf{m}(\theta)_k)|^{-1} \sum_{j \in \Psi(i, \mathbf{m}(\theta)_k)} m_j(\theta)_k, \text{ for } k \in T. \tag{2.22}$$

Even though the bound of confidence notions of HK model is similar to that of DW model, the major difference holds in the nature of agent interactions. In DW model agent interactions are always pairwise whereas in HW model an agent can be considered to update from multiple agents at a particular time index.

2.5 Social Judgement Theory

2.5.1 Basic Notions

Social Judgement Theory (SJT) discusses the basic psychological processes underlying the expression of attitudes and their modifiability through communication [20].

Boundedness

When a group of agents communicate between each other, a particular agent may adjust its opinion based mainly on the opinions of neighboring agents with similar opinions. In other words, an agent may be willing to update its opinion with the neighboring agent's opinion only if the 'distance' to that opinion is less than a certain

value ε. *Bounded confidence* refers to this phenomenon. The rationale for bounded confidence stems from the concept of *latitude of acceptance* in SJT.

In the Sociophysics community, the *Hegselmann-Krause (HK) model* [14, 39] and the *Deffuant-Weisbuch (DW) model* [40, 41, 43] have attracted considerable attention for modeling real-valued opinions under bounded confidence [18]. Most of the work on HK and DW models have been carried out on a single opinion which is usually taken to be bounded in the range $[0, 1]$, where 0 and 1 represent the two strong extreme opinions whereas values in $(0, 1)$ represent weaker/stronger opinions towards the extremes.

Global Affinity

Fortunato, et al. [44] have considered vector opinions under bounded confidence. In such a scenario, the affinity of a single opinion may not capture the global affinity of the agents. For example, consider a situation where two agents A and B initially have different views on a particular political party, say T. Thus, if only this single opinion is considered, under a bounded confidence model, A and B may not exchange opinions. But, if A and B agree from a global point-of-view (e.g., they may have similar opinions about the other parties), under a bounded confidence model, they may exchange opinions (including opinions about T).

Nature of Persuasion

SJT further mentions that a receiving agent's *ego* involvement should also be taken into account when assessing opinion change [20]. Individuals with smaller ego involvement are easy to persuade. While such individuals tend to have a higher latitude of acceptance, nature of persuasion is a different notion than bounded confidence (which

is accounted for via ε). For instance, one may find it difficult to persuade an agent possessing high ego in spite of it having neighboring agents with similar opinions. Hence, different opinion updating strategies may have to be implemented to account for nature of persuasion of each agent. In our models, we use two main opinion updating strategies referred to *cautious* and *receptive* update strategies.

2.5.2 Accounting for SJT Notions in Opinion Dynamics

In our work [21], we have used various notions stemming from SJT. The work in [21] can also account for the concept of *forceful agents* as put forth in [45]. There are two ways to view forceful agents: *stubborn agents* and *community leaders/news media.* Stubbornness can be considered a form of egocentricity of agents. Community leaders can be modeled as cautious updating agents with a higher number of neighboring agents. They influence other agents in the community and change the opinion of receptive agents in a particular direction.

Chapter 3 gives an extensive analysis on the consensus and opinion cluster formation under the presence of multiple opinion leaders while accounting for notions from SJT. Appendix Section A.1 also gives an alternative analysis on consensus formation under bounded confidence. The analysis in Appendix Section A.1 is focused on opinion models with singleton propositions only, i.e. using p.m.f.s only. However, Chapter 3 extends the analysis to agent opinions represented with Dirichlet BoEs. The analysis in Chapter 4 gives a paracontractive theoretic analysis on consensus and cluster formation for agent opinions represented with general DS masses.

Part II

Opinion Dynamics

Chapter 3

Consensus in the Presence of Multiple Opinion Leaders: Effect of Bounded Confidence

In this chapter, we use the framework of DS theory to capture the opinion of an agent and examine the opinion dynamics in networks where an agent can utilize either a *cautious* or *receptive* updating strategy. In particular, we examine the case of *bounded confidence updating* where an agent exchanges its evidence only with neighboring nodes possessing 'similar' evidence as measured by a suitable norm. In a fusion network, this captures the case in which nodes only update their state based on evidence consistent with the node's own evidence. In opinion dynamics, this captures the notions of SJT in which agents update their opinions only with other agents possessing opinions closer to their own. Focusing on the case where an agent state is modeled as a p.m.f. which emerges as a special case of the DST opinion model, in this chapter we utilize results from matrix theory, graph theory, and networks to prove the existence of consensus agent states in several time-varying network cases of interest. For example, we show the existence of a consensus in which a subset of network nodes achieves a consensus that is adopted by follower network nodes. Of particular interest is the case of multiple opinion leaders, where we show that the

agents do not reach a consensus in general, but rather converge to 'opinion clusters'. Simulation results are provided to illustrate the main results.

In [19], the DST framework has been utilized to address consensus formation in asynchronous dynamic/ad-hoc networks with applications to high-level fusion networks, and opinion dynamics in social networks. By utilizing DST, the methods by which an agent models an opinion and updates its opinion are now equipped with a powerful mechanism for grappling with the uncertainty inherent in the problem, whether in the form of vague agent opinions, or imprecise data in a fusion network (e.g., vague witness statements). The foundations of analyzing opinion dynamics and revision of agent state beliefs in asynchronous dynamic/ad-hoc, demonstrating conditions under which consensus is achieved can be found in [19].

Our work, which has appeared in [21], utilizes ideas from psychology, namely SJT [20], which examines the basic psychological processes underlying the expression of attitudes and their modifiability through communication. In particular, we use the notion of *bounded confidence* which stems from the concept of *latitude of acceptance* in SJT. In a social network, agents may only communicate and exchange opinions with their neighbors that have similar opinions on a particular topic. In other words, an agent may be willing to update its opinion with the neighboring agent's opinion only if the 'distance' to that opinion is less than a certain bound of confidence. The opinion exchange models in the HK and DW models account for these bounded confidence notions. The work in [46] addresses statistical estimation of the bound of confidence. In [47] agents are treated as Bayesian decision-makers and Bayes' risk error has been used to estimate the bounds. The bounded confidence assumption is also useful within high-level fusion networks to capture the cases in which agents may only exchange evidence with agents that have states similar to their own, and hence consistent, and perhaps avoiding the use of faulty outlier sensors/agents.

3.1 DST Modeling of Opinion Dynamics

In the work that follows, we consider N agents embedded in the directed graph $\mathcal{G}_k = (V, E_k)$, where the node $V_i \in V$, $i \in \overline{1, N}$, represents the i-th agent. Unless otherwise mentioned, the opinion of the i-th agent at time instant $k \in \mathbb{N}_{\geq 0}$ is taken to be captured via the BoE $\mathcal{E}_{i,k} = \{\Theta, \mathcal{F}_{i,k}, m_i(\cdot)_k\}$, $i \in \overline{1, N}$. Note that we assume that the FoDs associated with the agent opinion BoEs are identical and equal to Θ. To proceed, we will need

Definition 3.1 (Opinion Profile). Consider the agent BoEs $\mathcal{E}_{i,k} = \{\Theta, \mathcal{F}_{i,k}, m_i(\cdot)_k\}$, $i \in \overline{1, N}$, $k \in \mathbb{R}_{\geq 0}$. The *opinion profile* of $B \subseteq \Theta$ at $k \in \mathbb{R}_{\geq 0}$ is

$$\boldsymbol{\pi}(B)_k = [m_1(B)_k, \ldots, m_N(B)]^T \in \mathbb{R}^N_{[0,1]},$$

with $\boldsymbol{\pi}(B)_0 \in \mathbb{R}^N_{[0,1]}$ denoting its *initial state.* ∎

As an example, consider $N = 5$ agents with identical FoD $\Theta = \{\theta_1, \theta_2\}$. Then the possible set of propositions is $\{\theta_1, \theta_2, \theta_1\theta_2\}$. Now the *opinion profiles of the propositions* at the time instance $k \in \mathbb{R}_{\geq 0}$ can be given as

$$\boldsymbol{\pi}(\theta_1)_k = \begin{bmatrix} m_1(\theta_1)_k \\ m_2(\theta_1)_k \\ m_3(\theta_1)_k \\ m_4(\theta_1)_k \\ m_5(\theta_1)_k \end{bmatrix} \; ; \; \boldsymbol{\pi}(\theta_2)_k = \begin{bmatrix} m_1(\theta_2)_k \\ m_2(\theta_2)_k \\ m_3(\theta_2)_k \\ m_4(\theta_2)_k \\ m_5(\theta_2)_k \end{bmatrix} \; ; \; \boldsymbol{\pi}(\theta_1\theta_2)_k = \begin{bmatrix} m_1(\theta_1\theta_2)_k \\ m_2(\theta_1\theta_2)_k \\ m_3(\theta_1\theta_2)_k \\ m_4(\theta_1\theta_2)_k \\ m_5(\theta_1\theta_2)_k \end{bmatrix} . \qquad (3.1)$$

On the other hand, the *opinions of the agents* in this scenario are represented via the BoEs $\mathcal{E}_{1,k}$, $\mathcal{E}_{2,k}$, $\mathcal{E}_{3,k}$, $\mathcal{E}_{4,k}$ and $\mathcal{E}_{5,k}$ for the five agents $\forall k \in \mathbb{R}_{\geq 0}$.

3.1.1 Bounded Confidence

For each agent, define the following sets of neighborhood agents at time instant k:

$$\mathcal{N}_{i,k} = \left\{ V_j \in V : j \in \overline{1,N}, \text{ and } e_{ij} \in E_k \right\};$$

$$\mathcal{N}_{i,k}(\varepsilon_i) = \left\{ V_j \in \mathcal{N}_{i,k} : \|\mathcal{E}_{i,k} - \mathcal{E}_{j,k}\|_J \le \varepsilon_i \right\}, \tag{3.2}$$

where $\| \cdot \|_J$ denotes the distance measure in Definition 2.12 (while any valid norm applicable for DST BoEs could be used); $\varepsilon_i \ge 0$ is the *latitude of acceptance* or *bound of confidence* associated with the i-th agent and $\boldsymbol{\varepsilon} = [\varepsilon_1, \ldots, \varepsilon_N]^T$. So, $\mathcal{N}_{i,k}(\varepsilon_i)$ denotes the neighbors of the i-th agent at time k left after 'pruning' the links subjected to the bound of confidence requirement. Also, let

$$\mathcal{G}_k^\dagger(\boldsymbol{\varepsilon}) = (V, E_k^\dagger(\boldsymbol{\varepsilon})), \tag{3.3}$$

where $E_k^\dagger(\boldsymbol{\varepsilon}) = \{e_{ij} \in E_k : \|\mathcal{E}_{i,k} - \mathcal{E}_{j,k}\|_J \le \varepsilon_i\}$.

The bounded confidence process of updating an agent's opinion is as follows [18, 44]: the i-th agent updates its BoE $\mathcal{E}_i$ in response to the opinion BoE $\mathcal{E}_j$ of its neighbor, the j-th agent, only if $j \in \mathcal{N}_i(\varepsilon_i)$. In [40], the threshold ε_i is referred to as an *openness character.* Another interpretation views ε_i as an *uncertainty,* i.e., if the i-th agent possesses an opinion with some degree of uncertainty ε_i, then it ignores the views of those neighboring agents who fall outside its uncertainty range.

3.1.2 Opinion Updating and Consensus Formation

In what follows, we will use $\mathcal{S}$ to identify the indices corresponding to a subset $V(\mathcal{S})$ of the agents in V, i.e., $V(\mathcal{S}) = \{V_i \in V \mid v_i \in \mathcal{S} \subseteq \{a, 2, \ldots, N\}\}$. To proceed, we adopt the following

Definition 3.2 (Opinion Clusters, Consensus). Let $\mathcal{E}_{i,k}$, $i \in \overline{1, N}$, $k \in \mathbb{N}_{\geq 0}$, denote the opinions of N agents embedded within the network $\mathcal{G}_k = (V, E_k)$, where each agent repeatedly updates its state by iterative opinion exchange. Let $V(\mathcal{S})$ identify a subset of the agents in V.

(a) Suppose $\lim_{k \to \infty} \|\mathcal{E}_{i,k} - \mathcal{E}_{j,k}\|_J = 0$, $\forall i, j \in \mathcal{S}$ (or equivalently, $\lim_{k \to \infty} \mathcal{E}_{i,k} \equiv \mathcal{E}_*$, $\forall i \in \mathcal{S}$) and suppose $\lim_{k \to \infty} \mathcal{E}_{i,k} \neq \mathcal{E}_*$, $\forall i \in \overline{1, N} \setminus \mathcal{S}$. Then, the agents in $\mathcal{S}$ are said to form an *opinion cluster*.

(b) The agents are said to reach a *consensus* if $\mathcal{S} = V$, i.e., all the agents in V form a single opinion cluster. ∎

Henceforth, our results will only be stated with the formation of a consensus in mind (e.g., see Lemma 1). Due to (b) above, these results can easily be reformulated so that they pertain to the formation of opinion clusters.

In our work, we assume that each agent updates its opinion in accordance with the CUE [29, 30]. Depending on the number of cautious agents present, we consider three cases:

1. *No Opinion Leaders:* This is the most common scenario that appears in the literature [14, 18, 40]. Here, all agents are receptive and no opinion leaders are present.

2. *Single Opinion Leader:* Here, all agents are receptive except one cautious agent or opinion leader. This is the scenario considered in typical leader-follower models [2], and the recent work in [19]).

3. *Multiple Opinion Leaders:* Here, there are multiple cautiously updating opinion leaders, generally with different initial opinions. To our knowledge, this case has not been addressed prior to this present work of ours.

Our work, which is based on the work in [13, 39], provides sufficient conditions for consensus/cluster formation under certain strong assumptions (e.g., the agent opinions can be ordered as what is referred to as an ε-chain). We relax such assumptions in this work by utilizing properties of products of stochastic/sub-stochastic matrices.

Suppose the i-th agent updates its opinion $\mathcal{E}_i$ by taking into account its neighboring agents $j \in \mathcal{N}_{i,k}(\varepsilon_i)$, where $\varepsilon_i > 0$ is the i-th agent's bound of confidence. The CUE being the update strategy being employed, the i-th agent's updated opinion can be expressed as [30]

$$m_i(B)_{k+1} = \alpha_{i,k}m_i(B)_k + \sum_{j \neq i}\sum_{A \in \mathcal{F}_{j,k}} \beta_{ij}(A)_k m_j(B|A)_k. \tag{3.4}$$

Here, $j \in \mathbb{I}_{i,k}$ where $\mathbb{I}_{i,k}$ denotes the index set of agents in $\mathcal{N}_{i,k}(\varepsilon_i)$; the CUE parameters $\alpha_{i,k}, \beta_{ij}(\cdot)_k \in \mathbb{R}_{[0,1]}$ satisfy

$$\alpha_{i,k} + \sum_{j \neq i}\sum_{A \in \mathcal{F}_{j,k}} \beta_{ij}(A)_k = 1. \tag{3.5}$$

The work in [30] identifies two update strategies which determine the CUE parameter set $\{\beta_{i,j}(\cdot)_k\}$ to be employed:

(a) *Receptive updating:* Select $\beta_{ij}(A)_k \propto m_j(A)_k$. Receptively updating agents are referred to as *opinion followers.*

(b) *Cautious updating:* Select $\beta_{ij}(A)_k \propto m_i(A)_k$. Cautiously updating agents are referred to as *opinion leaders.*

3.2 Probabilistic Agent Opinions

When the DST BoEs $\mathcal{E}_{i,k}$, $i \in \overline{1, N}$, $k \in \mathbb{N}_{\geq 0}$, possess only singleton focal elements, we essentially have the case of probabilistic agent opinions. In this case, the CUE-based opinion update in (3.4) reduces to the discrete-time dynamic system

$$\boldsymbol{\pi}(\theta_p)_{k+1} = W_k \, \boldsymbol{\pi}(\theta_p)_k, \ p \in \overline{1, M}, \tag{3.6}$$

where $\boldsymbol{\pi}(\textbf{.})_k \in \mathbb{R}_{[0,1]}^N$, $W_k = \{w_{ij,k}\} \in \mathbb{R}_{[0,1]}^{N \times N}$ is row-stochastic [37], and its elements are given by

$$w_{ij,k} = \begin{cases} \alpha_{i,k}, & \text{for } i = j; \\ (1 - \alpha_{i,k})/|\mathcal{N}_{i,k}(\varepsilon_i)|, & \text{for } j \in \mathcal{N}_{i,k}(\varepsilon_i); \\ 0, & \text{otherwise.} \end{cases} \tag{3.7}$$

As in [48], we refer to W_k as the *confidence matrix* because $w_{ij,k}$ represents the weight the i-th agent attaches to the opinion of the j-th agent at time step k. Note that, W_k constitutes the weighted adjacency matrix of $\mathcal{G}_k^\dagger(\boldsymbol{\varepsilon})$ in (3.3).

The work in [19] studies in detail how cautious agents possessing DST opinions behave under a CUE-based update strategy. For our purposes, we would only need

Proposition 1. When agents possessing probabilistic agent opinions employ a CUE-based update strategy, the opinion of a cautiously updating agent is invariant. $\square$

Proof. Without loss of generality, suppose the i-th agent is a cautious agent. Its CUE-based update is given by (3.4), where

$$\beta_{ij}(B)_k = \mu_{ij,k} m_i(A)_k; \ \ \alpha_{i,k} + \sum_{j \neq i} \sum_{A \in \mathcal{F}_{j,k}} \mu_{ij,k} = 1,$$

for singletons $A, B \in \Theta$. But, we know that [26], for singleton propositions $A, B \in \Theta$, $m(B|A) = 1$ only if $B = A$, and $m(B|A) = 0$ otherwise. With this substitution, (3.4) reduces to $m_1(B)_{k+1} = m_1(B)_k, \forall k \in \mathbb{N}_{\geq 0}$. ■

To proceed, we will need

Definition 3.3 (Left (or Backward) Products). [49]

(a) *Left Product:* The *left product* of the sequence of matrices $\{W_k\}$, $W_k \in \mathbb{R}^{N \times N}$, is

$$W_{k:\ell} = \begin{cases} I, & \text{for } k < \ell; \\ W_\ell, & \text{for } k = \ell; \\ W_k W_{k-1} \cdots W_\ell, & \text{for } k > \ell. \end{cases}$$

(b) *Left-Converging product:* The left product $W_{k:0}$ is said to be *left-converging* if $\lim_{k \to \infty} W_{k:0}$ exists, in which case we write $W_\infty = \lim_{k \to \infty} W_{k:0}$. ■

Note that, with Definition 3.3, the dynamic system in (3.6) can be expressed as

$$\boldsymbol{\pi}(\theta_p)_{k+1} = W_{k:0}\, \boldsymbol{\pi}(\theta_p)_0. \tag{3.8}$$

Thus, whenever W_∞ exists,

$$\lim_{k \to \infty} \boldsymbol{\pi}(\theta_p)_{k+1} = W_\infty\, \boldsymbol{\pi}(\theta_p)_0. \tag{3.9}$$

Clearly, the convergence of agent opinions depends on the existence and the nature of W_∞. When W_∞ exists, let us denote the converged opinion profile for θ_p as $\boldsymbol{\pi}_*(\theta_p) \in \mathbb{R}^N_{[0,1]}$. Consensus (in the sense of Definition 3.2) is a special case of a converged opinion profile.

Lemma 1. The agents form a consensus iff $\exists \boldsymbol{\eta} = \{\eta_p\} \in \mathbb{R}^M_{[0,1]}$ s.t.

$$\boldsymbol{\pi}_*(\theta_p) = \eta_p \mathbf{1}_N, \forall p \in \overline{1, M}, \forall i \in \overline{1, N}. \qquad \square$$

Proof. Suppose $\boldsymbol{\pi}_*(\theta_p) = \eta_p \mathbf{1}_N, \forall p \in \overline{1, M}, \forall i \in \overline{1, N}$, for some $\boldsymbol{\eta} = \{\eta_p\} \in \mathbb{R}^M_{[0,1]}$. This clearly means that

$$\lim_{k \to \infty} \mathcal{E}_{i,k} \equiv \mathcal{E}_*, \forall i \in \overline{1, N}.$$

Thus, the agents form a consensus.

Conversely, if

$$\lim_{k \to \infty} \mathcal{E}_{i,k} \equiv \mathcal{E}_*, \forall i \in \overline{1, N},$$

we must have $\boldsymbol{\pi}_*(\theta_p) = \eta_p \mathbf{1}_N, \forall p \in \overline{1, M}, \forall i \in \overline{1, N}$. ∎

The limiting behavior of the stochastic matrix product $\{W_k\}$ plays a crucial role in consensus analysis when DST BoEs possess only singleton focal elements.

Lemma 2. Consider the stochastic chain $\{W_k\}$, $k \in \mathbb{N}_{\geq 0}$, s.t. $W_\infty = \mathbf{1}\,\mathbf{v}^T$ for some stochastic vector $\mathbf{v} \in \mathbb{R}^N_{[0,1]}$. Then, the agents reach the consensus $\boldsymbol{\pi}_*(\theta_p) = (\mathbf{v}^T \boldsymbol{\pi}(\theta_p)_0)\,\mathbf{1}$, where $\boldsymbol{\pi}(\theta_p)_0 \in \mathbb{R}^N_{[0,1]}$ denotes the initial opinion profile. $\square$

Proof. Use (3.9):

$$\lim_{k \to \infty} \boldsymbol{\pi}(\theta_p)_{k+1} = \mathbf{1}\,\mathbf{v}^T \boldsymbol{\pi}(\theta_p)_0 = \eta_p \mathbf{1},\ \eta_p = \mathbf{v}^T \boldsymbol{\pi}(\theta_p)_0.$$

So, from Lemma 1, we achieve a consensus. ∎

3.2.1 No Opinion Leaders

This is the most widely studied scenario and many consensus-related results can be found in the literature [2, 18, 48, 50, 51]. The work in [48] studies convergence when each element in the stochastic chain $\{W_k\}$ has positive diagonals. From a graph theoretic viewpoint, this implies that there is a path from each vertex to itself; from an opinion dynamics perspective, this is referred to as having the *self-communicating* property [48]. This self-communicating property in [48] is closely related to the strong aperiodicity property in [51]: given a balanced and strongly aperiodic stochastic chain $\{W_k\}$, $W_{\infty:k_0}$ is rank one for all $k_0 \in \mathbb{N}_{\geq 0}$ iff the infinite flow graph of $\{W_k\}$ is connected.

3.2.2 Single Opinion Leader

We first introduce

Definition 3.4 (Opinion Dynamics Chain Driven By One Group (1-ODC)). The directed dynamic network $\mathcal{G}_k^\dagger(\varepsilon) = (V, E_k^\dagger(\varepsilon))$ in (3.3) is said to be an *opinion dynamics chain driven by one group (1-ODC)* if its corresponding confidence matrix W_k in (3.6) can be expressed as the lower block triangular matrix

$$W_k = \begin{bmatrix} A_k & \mathbf{0} \\ C_k & D_k \end{bmatrix},$$

where $A_k \in \mathbb{R}_{[0,1]}^{N_C \times N_C}$ and $D_k \in \mathbb{R}_{[0,1]}^{N_{out} \times N_{out}}$, and the other matrices have compatible sizes. ∎
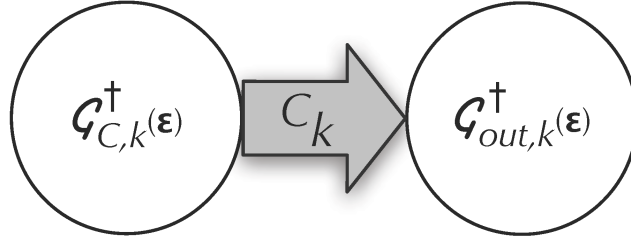
FIGURE 3.1: An opinion dynamics chain driven by one group (1-ODC).

One may view an 1-ODC as consisting of a 'central' component

$$\mathcal{G}_{C,k}^{\dagger}(\boldsymbol{\varepsilon}) = (V_C, E_{C,k}^{\dagger}(\boldsymbol{\varepsilon})) \text{ (with } N_C \text{ agents)}, \tag{3.10}$$

and another component

$$\mathcal{G}_{out,k}^{\dagger}(\boldsymbol{\varepsilon}) = (V_{out}, E_{out,k}^{\dagger}(\boldsymbol{\varepsilon})) \text{ (with } N_{out} \text{ agents)}, \tag{3.11}$$

s.t. no agent in $\mathcal{G}_{out,k}^{\dagger}(\boldsymbol{\varepsilon})$ belongs to the in-component of $\mathcal{G}_{C,k}^{\dagger}(\boldsymbol{\varepsilon})$, for any $k \in \mathbb{N}_{\geq 0}$. See Fig. 3.1. Note that, A_k and D_k correspond to the confidence matrices of agents in $\mathcal{G}_{C,k}^{\dagger}(\boldsymbol{\varepsilon})$ and $\mathcal{G}_{out,k}^{\dagger}(\boldsymbol{\varepsilon})$, respectively. In a social setting the agents embedded in the central $\mathcal{G}_{C,k}^{\dagger}(\boldsymbol{\varepsilon})$ can be viewed as an elite group which influences the opinions of the followers. For instance, the elite group can be a media organization or set of high profile leaders.

Theorem 1. Consider agents embedded in a 1-ODC employing a CUE-based update strategy. Furthermore, suppose that

- $\lim_{n \to \infty} A_{n:0} = \mathbf{1}_{N_C} \mathbf{v}_A^T$, where $\mathbf{v}_A \in \mathbb{R}_{[0,1]}^{N_C}$ is a stochastic vector so that the agents in V_C reach their own consensus, and

- $\|D_k\| \leq \rho < 1, \forall k \in \mathbb{N}_{\geq 0}$.

Then, the agents in V (i.e., agents in V_C and V_{out}) will reach a consensus at the consensus reached by the agents in V_C. □

Proof. The CUE-based opinion update strategy yields the dynamic system

$$\begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_{k+1} \\ \boldsymbol{\pi}_D(\theta_p)_{k+1} \end{bmatrix} = W_k \begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_k \\ \boldsymbol{\pi}_D(\theta_p)_k \end{bmatrix}, \tag{3.12}$$

where $k \in \mathbb{N}_{\geq 0}$, $\theta_p \in \Theta$, and

$$W_k = \begin{bmatrix} A_k & \mathbf{0} \\ C_k & D_k \end{bmatrix} \implies W_{n:0} = \begin{bmatrix} A_{n:0} & \mathbf{0} \\ P_n & D_{n:0} \end{bmatrix}. \tag{3.13}$$

Here the sub matrices have sizes compatible with W_k. Use $W_{n+1:0} = W_{n+1}W_{n:0}$ to get

$$P_{n+1} = C_{n+1}A_{n:0} + D_{n+1}P_n, \ P_0 = C_0, \tag{3.14}$$

for $n \in \mathbb{N}_{\geq 0}$. Due to the row-stochasticity of W_k, we have

$$\mathbf{1}_{N_{out}} = C_k \mathbf{1}_{N_C} + D_k \mathbf{1}_{N_{out}}, \ \forall k \in \mathbb{N}_{\geq 0}. \tag{3.15}$$

Subtract $\mathbf{1}_{N_{out}}\mathbf{v}_A^T$ from both sides of (3.14):

$$\begin{aligned} P_{n+1} - \mathbf{1}_{N_{out}}\mathbf{v}_A^T &= C_{n+1}A_{n:0} + D_{n+1}P_n - \mathbf{1}_{N_{out}}\mathbf{v}_A^T \\ &= C_{n+1}A_{n:0} + D_{n+1}P_n - [C_{n+1}\mathbf{1}_{N_C} + D_{n+1}\mathbf{1}_{N_{out}}]\mathbf{v}_A^T \\ &= C_{n+1}[A_{n:0} - \mathbf{1}_{N_C}\mathbf{v}_A^T] + D_{n+1}[P_n - \mathbf{1}_{N_{out}}\mathbf{v}_A^T], \end{aligned}$$

for $n \in \mathbb{N}_{\geq 0}$. Here, we have used (3.15). Employing the notation $\Delta P_n = P_n - \mathbf{1}_{N_{out}} \mathbf{v}_A^T$, $n \in \mathbb{N}_{\geq 0}$, we express this as

$$\Delta P_{n+1} = C_{n+1}[A_{n:0} - \mathbf{1}_{N_C} \mathbf{v}_A^T] + D_{n+1} \Delta P_n, \ n \in \mathbb{N}_{\geq 0}.$$

Then, we may bound $\|\Delta P_{n+1} - D_{n+1} \Delta P_n\|$ as

$$|\|\Delta P_{n+1}\| - \|D_{n+1} \Delta P_n\|| \leq \|\Delta P_{n+1} - D_{n+1} \Delta P_n\|$$
$$= \|C_{n+1}[A_{n:0} - \mathbf{1}_{N_C} \mathbf{v}_A^T]\|$$
$$\leq \|A_{n:0} - \mathbf{1}_{N_C} \mathbf{v}_A^T\|. \tag{3.16}$$

We proceed by noting that $\lim_{n \to \infty} A_{n:0} = \mathbf{1}_{N_C} \mathbf{v}_A^T$, where N_C-sized stochastic vector, implies that the agents in V_C converges to a consensus (see Lemma 2). Hence, given an arbitrary $\epsilon_A > 0$, $\exists N_A \in \mathbb{N}_{\geq 0}$ s.t.

$$\|A_{n:0} - \mathbf{1}_{N_C} \mathbf{v}_A^T\| < \epsilon_A, \ \forall n \geq N_A. \tag{3.17}$$

From (3.16) and (3.17), we can obtain the following: given an arbitrary $\epsilon_A > 0$, $\exists N_A \in \mathbb{N}_{\geq 0}$ s.t.
$$|\|\Delta P_{n+1}\| - \|D_{n+1} \Delta P_n\|| < \epsilon_A, \ \forall n \geq N_A.$$

So, for $n \geq N_A$, we have

$$\|\Delta P_{n+1}\| < \|D_{n+1} \Delta P_n\| + \epsilon_A < \rho \|\Delta P_n\| + \epsilon_A, \tag{3.18}$$

where we use the fact that $\|D_{n+1}\| \leq \rho < 1$, $\forall n \geq N_A$. This upper bound for $\|\Delta P_{n+1}\|$ yields

$$\|\Delta P_{N_A+L}\| < \epsilon_A \sum_{\ell=0}^{L-1} \rho^\ell + \rho^L \|\Delta P_{N_A}\|, \ L \geq 0. \tag{3.19}$$

We note that

$$\lim_{L \to \infty} \left(\epsilon_A \sum_{\ell=0}^{L-1} \rho^\ell + \rho^L \|\Delta P_{N_A}\| \right) = \frac{\epsilon_A}{1 - \rho},$$

i.e., given an arbitrary $\epsilon_P > 0$, $\exists L_P \in \mathbb{N}_{\geq 0}$ s.t.

$$\epsilon_A \sum_{\ell=0}^{L-1} \rho^\ell + \rho^L \|\Delta P_{N_A}\| - \frac{\epsilon_A}{1 - \rho} < \epsilon_P, \ \forall L \geq L_P.$$

Use this in (3.19):

$$\|\Delta P_{N_A + L}\| < \epsilon_P + \frac{\epsilon_A}{1 - \rho}, \ \forall L \geq L_P;$$

in other words,

$$\lim_{n \to \infty} \|\Delta P_n\| = \lim_{n \to \infty} \|P_n - \mathbf{1}_{N_{out}} \mathbf{v}_A^T\| = 0.$$

Using (3.13) in (3.12), we get

$$\begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_{n+1} \\ \boldsymbol{\pi}_D(\theta_p)_{n+1} \end{bmatrix} = \begin{bmatrix} A_{n:0} & \mathbf{0}_{N_C \times N_{out}} \\ P_n & D_{n:0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_0 \\ \boldsymbol{\pi}_D(\theta_p)_0 \end{bmatrix}. \tag{3.20}$$

Let us take the consensus among agents in V_C as

$$\boldsymbol{\pi}_A^*(\theta_p) = \left(\mathbf{1}_{N_C} \mathbf{v}_A^T \right) \boldsymbol{\pi}_A(\theta_p)_0.$$

Then, by (3.20), we can write

$$\begin{bmatrix} \boldsymbol{\pi}_A^*(\theta_p) \\ \boldsymbol{\pi}_D^*(\theta_p) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{1}_{N_C} \mathbf{v}_A^T & \mathbf{0}_{N_C \times N_{out}} \\ \mathbf{1}_{N_C} \mathbf{v}_A^T & \mathbf{0}_{N_{out} \times N_{out}} \end{bmatrix}}_{\mathbf{1}_{N_C + N_{out}} \mathbf{v}^{*T}} \begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_0 \\ \boldsymbol{\pi}_D(\theta_p)_0 \end{bmatrix}, \tag{3.21}$$

where $\mathbf{v}^*$ is a $(N_C + N_{out})$-sized stochastic vector created by concatenating the vectors $\mathbf{v}_A$ and $\mathbf{0}_{N_{out} \times 1}$. But (3.21) satisfies the conditions of Lemma 2. So, all agents in V

achieve a consensus. Moreover, as (3.21) shows, this consensus is the same consensus achieved by the agents in V_C. ∎

An immediate consequence of Theorem 1 is

Corollary 1. Consider the network $\mathcal{G}_k^\dagger(\varepsilon) = (V, E_k^\dagger(\varepsilon))$ in (3.3) populated with receptively updating opinion followers and a single cautiously updating opinion leader. Suppose D_k corresponds to the confidence matrix of the receptively updating opinion followers with $\|D_k\| \le \rho < 1$, $\forall k \in \mathbb{N}_{\ge 0}$. Then, with a CUE-based update strategy, all the agents reach a consensus opinion, and this consensus opinion is equal to the opinion of the opinion leader. □

Proof. Construct a 1-ODC as in Definition 3.4 with $\mathcal{G}_{C,k}^\dagger(\varepsilon)$ containing the opinion leader only and $\mathcal{G}_{out,k}^\dagger(\varepsilon)$ populated with the opinion followers. Proposition 1 implies that $V_C^\dagger$ (which consists of only the cautiously updating opinion leader) generates a consensus. Therefore, according to Theorem 1, all the agents in V reach a consensus, and this consensus is the opinion leader's opinion. ∎

3.2.3   Two Opinion Leaders
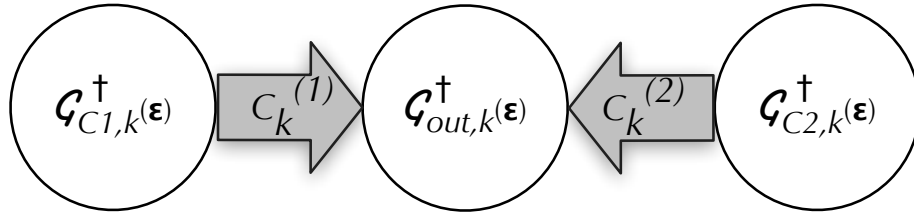
FIGURE 3.2: An opinion dynamics chain driven by two groups (2-ODC).

The situation turns out to be significantly more complicated when there are multiple opinion leaders. To look at the two opinion leader case, let us introduce

Definition 3.5 (Opinion Dynamics Chain Driven By Two Groups (2-ODC))**.** The directed dynamic graph $\mathcal{G}_k^\dagger(\varepsilon) = (V, E_k^\dagger(\varepsilon))$ in (3.3) is said to be an *opinion dynamics*

chain driven by two groups (2-ODC) if its corresponding confidence matrix W_k in (3.6) can be expressed as the lower block triangular matrix

$$W_k = \begin{bmatrix} A_k^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_k^{(2)} & \mathbf{0} \\ C_k^{(1)} & C_k^{(2)} & D_k \end{bmatrix}.$$

where $A_k^{(1)} \in \mathbb{R}_{[0,1]}^{N_{C1} \times N_{C1}}$, $A_k^{(2)} \in \mathbb{R}_{[0,1]}^{N_{C2} \times N_{C2}}$, and $D_k \in \mathbb{R}_{[0,1]}^{N_{out} \times N_{out}}$, and the other matrices have compatible sizes. ∎

One may view a 2-ODC as consisting of two 'central' components

$$\mathcal{G}_{C1,k}^{\dagger}(\varepsilon) = (V_{C1}, E_{C1,k}^{\dagger}(\varepsilon)) \text{ (with } N_{C1} \text{ agents), and;} \tag{3.22}$$

$$\mathcal{G}_{C2,k}^{\dagger}(\varepsilon) = (V_{C2}, E_{C2,k}^{\dagger}(\varepsilon)) \text{ (with } N_{C2} \text{ agents),} \tag{3.23}$$

plus a third component

$$\mathcal{G}_{out,k}^{\dagger}(\varepsilon) = (V_{out}, E_{out,k}^{\dagger}(\varepsilon)) \text{ (with } N_{out} \text{ agents),} \tag{3.24}$$

s.t. no agent in $\mathcal{G}_{out,k}^{\dagger}$ belongs to the in-components of either $\mathcal{G}_{C1,k}^{\dagger}(\varepsilon)$ or $\mathcal{G}_{C1,k}^{\dagger}(\varepsilon)$. See Fig. 3.2. Note that, $A_k^{(1)}$, $A_k^{(2)}$, and D_k correspond to the confidence matrices of agents in $\mathcal{G}_{C1,k}^{\dagger}(\varepsilon)$, $\mathcal{G}_{C2,k}^{\dagger}(\varepsilon)$, and $\mathcal{G}_{out,k}^{\dagger}(\varepsilon)$, respectively.

In a social setting the two 'central' components can be viewed as two elite groups. We assume that the two elite groups do not exchange opinions directly among each other. We now demonstrate that the agents in even a 2-ODC cannot yield a consensus in general, even though the agents in V_{C1} and V_{C2} achieve their own consensus opinions.

Theorem 2. Consider agents embedded in a 2-ODC employing a CUE-based update strategy. Furthermore, suppose that

- $\lim_{n \to \infty} A_{n:0}^{(1)} = \mathbf{1}_{N_{C1}} \mathbf{v}_1^T$ and $\lim_{n \to \infty} A_{n:0}^{(2)} = \mathbf{1}_{N_{C2}} \mathbf{v}_2^T$, where $\mathbf{v}_1 \in \mathbb{R}_{[0,1]}^{N_{C1}}$ and $\mathbf{v}_2 \in \mathbb{R}_{[0,1]}^{N_{C2}}$ are stochastic vectors so that the agents in V_{C1} and V_{C2} achieve their own consensus opinions, and

- $\|D_k\| \leq \rho < 1$, $\forall k \in \mathbb{N}_{\geq 0}$.

Then, the following are true:

(a) The agents in V (i.e., agents in V_{C1}, V_{C2}, and V_{out}) reach a consensus iff the consensus opinions of the agents in V_{C1} and V_{C2} are equal.

(b) When the consensus opinions of the agents in V_{C1} and V_{C2} are not equal, a consensus among the agents in V_{out} occurs if $\exists \lambda^{(1)}, \lambda^{(2)} \in (0, 1)$ s.t. $\lambda^{(1)} + \lambda^{(2)} = 1$ and $\lambda^{(1)} C_k^{(1)} \mathbf{1}_{N_{C1}} = \lambda^{(2)} C_k^{(2)} \mathbf{1}_{N_{C2}}$, $\forall k \in \mathbb{N}_{\geq 0}$. $\qquad\square$

Proof. Note that we may write

$$
W_{n:0} = \begin{bmatrix} A_{n:0}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{n:0}^{(2)} & \mathbf{0} \\ P_n^{(1)} & P_n^{(2)} & D_{n:0} \end{bmatrix},
$$

where, for $n \in \mathbb{N}_{\geq 0}$,

$$
P_{n+1}^{(1)} = C_{n+1}^{(1)} A_{n:0}^{(1)} + D_{n+1} P_n^{(1)}, \;\; P_0^{(1)} = C_0^{(1)};
$$
$$
P_{n+1}^{(2)} = C_{n+1}^{(2)} A_{n:0}^{(2)} + D_{n+1} P_n^{(2)}, \;\; P_0^{(2)} = C_0^{(2)}. \tag{3.25}
$$

(a) Suppose the two consensus opinions of the agents in V_{C1} and V_{C2} are equal. One may then recast the 2-ODC as a 1-DOC with the confidence matrices corresponding to the central component and the other component taken as $\begin{bmatrix} A_k^{(1)} & \mathbf{0} \\ \mathbf{0} & A_k^{(2)} \end{bmatrix}$ and D_k, respectively. Noting that the central component reaches a

common consensus opinion, apply Theorem 1 to show that all the agents must reach a consensus which is identical to the common consensus opinion formed within the central component. Conversely, if the two consensus opinions of the agents in V_{C1} and V_{C2} are not equal, no consensus is possible among the agents in V_{C1} and V_{C2} because these two sets of agents do not update from each other.

(b) Suppose the two consensus opinions of the agents in V_{C1} and V_{C2} are not equal. Due to the row stochasticity of W_k, we have

$$
\begin{aligned}
\mathbf{1}_{N_{out}} &= C_k^{(1)}\mathbf{1}_{N_{C1}} + C_k^{(2)}\mathbf{1}_{N_{C2}} + D_k\mathbf{1}_{N_{out}}; \\
\lambda^{(2)}\mathbf{1}_{N_{out}} &= C_k^{(1)}\mathbf{1}_{N_{C1}} + \lambda^{(2)}D_k\mathbf{1}_{N_{out}}, \ k \in \mathbb{N}_{\geq 0},
\end{aligned}
\tag{3.26}
$$

where we used

$$
\lambda^{(1)}C_k^{(1)}\mathbf{1}_{N_{C1}} = \lambda^{(2)}C_k^{(2)}\mathbf{1}_{N_{C2}}.
$$

Now, proceeding as we did in the proof of Theorem 1, subtract $\lambda^{(2)}\mathbf{1}_{N_{out}}\mathbf{v}_1^T$ from both sides of (3.25) and substitute for $\mathbf{1}_{N_{out}}$ from (3.26) to get

$$
P_{n+1}^{(1)} - \lambda^{(2)}\mathbf{1}_{N_{out}}\mathbf{v}_1^T = C_{n+1}^{(1)}[A_{n:0}^{(1)} - \mathbf{1}_{N_{C1}}\mathbf{v}_1^T] + D_{n+1}[P_n^{(1)} - \lambda^{(2)}\mathbf{1}_{N_{out}}\mathbf{v}_1^T],
$$

for $n \in \mathbb{N}_{\geq 0}$. As before, we use the notation $\Delta P_n^{(1)} = P_n^{(1)} - \lambda^{(2)}\mathbf{1}_{N_{out}}\mathbf{v}_1^T$ to express this as

$$
\Delta P_{n+1}^{(1)} = C_{n+1}^{(1)}[A_{n:0}^{(1)} - \mathbf{1}_{N_{C1}}\mathbf{v}_1^T] + D_{n+1}\Delta P_n^{(1)},
$$

for $n \in \mathbb{N}_{\geq 0}$. Now, bound $\|\Delta P_{n+1}^{(1)} - D_{n+1}\Delta P_n^{(1)}\|$ as

$$\|\|\Delta P_{n+1}^{(1)}\| - \|D_{n+1}\Delta P_n^{(1)}\|\| \leq \|\Delta P_{n+1}^{(1)} - D_{n+1}\Delta P_n^{(1)}\|$$
$$= \|C_{n+1}^{(1)}[A_{n:0}^{(1)} - \mathbf{1}_{N_{C1}}\mathbf{v}_1^T]\|$$
$$\leq \|A_{n:0}^{(1)} - \mathbf{1}_{N_{C1}}\mathbf{v}_1^T\|,$$

where we used the sub-stochasticity of $C_n^{(1)}$.

Now, as in the proof of Theorem 1, use the fact that the agents in V_{C1} and V_{C2} each achieve a consensus to show that

$$\lim_{n\to\infty} \|P_n^{(1)} - \lambda^{(2)}\mathbf{1}_{N_{out}}\mathbf{v}_1^T\| = \lim_{n\to\infty} \|P_n^{(2)} - \lambda^{(1)}\mathbf{1}_{N_{out}}\mathbf{v}_2^T\| = 0.$$

Denote the consensus among the agents in V_{C1} and V_{C2} as

$$\boldsymbol{\pi}_{A^{(1)}}^*(\theta_p) = (\mathbf{1}_{N_{C1}}\mathbf{v}_1^T)\,\boldsymbol{\pi}_{A^{(1)}}(\theta_p)_0;$$
$$\boldsymbol{\pi}_{A^{(2)}}^*(\theta_p) = (\mathbf{1}_{N_{C2}}\mathbf{v}_2^T)\,\boldsymbol{\pi}_{A^{(2)}}(\theta_p)_0,$$

respectively. Then we have

$$\begin{bmatrix} \boldsymbol{\pi}_{A^{(1)}}^*(\theta_p) \\ \boldsymbol{\pi}_{A^{(2)}}^*(\theta_p) \\ \boldsymbol{\pi}_D^*(\theta_p) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{N_{C1}}\mathbf{v}_1^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{N_{C2}}\mathbf{v}_2^T & \mathbf{0} \\ \lambda^{(2)}\mathbf{1}_{N_{C1}}\mathbf{v}_1^T & \lambda^{(1)}\mathbf{1}_{N_{C2}}\mathbf{v}_2^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_{A^{(1)}}(\theta_p)_0 \\ \boldsymbol{\pi}_{A^{(2)}}(\theta_p)_0 \\ \boldsymbol{\pi}_D(\theta_p)_0 \end{bmatrix},$$

where $\boldsymbol{\pi}_D^*(\theta_p)$ is given as

$$\boldsymbol{\pi}_D^*(\theta_p) = \mathbf{1}_{N_{out}} \begin{bmatrix} \lambda^{(2)}\mathbf{v}_1^T & \lambda^{(1)}\mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_{A^{(1)}}(\theta_p)_0 \\ \boldsymbol{\pi}_{A^{(2)}}(\theta_p)_0 \end{bmatrix}.$$

Hence, from Lemma 2, we conclude that the agents in V_{out} reach a consensus if $\lambda^{(1)} + \lambda^{(2)} = 1$, $\forall k \in \mathbb{N}_{\geq 0}$. $\blacksquare$

One may interpret this result in the following manner: the matrices $C_k^{(1)}$ and $C_k^{(2)}$ signify the 'weights' or 'bias' that agents in V_{out} give to the agents in V_{C1} and V_{C2}, respectively. To reach a consensus, each agent in V_{out} must give the same proportion of weights to the agents in V_{C1} and V_{C2} for all $k \in \mathbb{N}_{\geq 0}$: $\lambda^{(2)} > \lambda^{(1)}$ implies that a higher weight is given to the opinions of the agents in V_{C1} than to the opinions of the agents in V_{C2}. This might be due to stronger interconnections between the agents in V_{C1} and V_{out} or it could simply be due to a bias towards the opinions of the agents in V_{C1}.

As an immediate consequence of Theorem 2 we get

Corollary 2. Consider the network $\mathcal{G}_k^{\dagger}(\varepsilon) = (V, E_k^{\dagger}(\varepsilon))$ in (3.3) populated with receptively updating opinion followers and two cautiously updating opinion leaders. Suppose D_k corresponds to the confidence matrix of the receptively updating opinion followers with $\|D_k\| \leq \rho < 1$, $\forall k \in \mathbb{N}_{\geq 0}$. Then, with a CUE-based update strategy, the following are true:

(a) All the agents reach a consensus iff the opinions of the two opinion leaders are equal.

(b) When the opinion leaders do not possess the same opinion, the opinion followers form an opinion cluster if the proportion of weights that each receptive agent gives to the two opinion leaders is identical and no opinion leader is given zero weight. $\blacksquare$

3.3 Dirichlet Agent Opinions

Next, let us suppose that agents opinions are modeled via Dirichlet BoEs $\mathcal{E}_{i,k}$, $i \in \overline{1,N}$, $k \in \mathbb{N}_{\geq 0}$. Then, using the properties $Bl(\theta_i|\theta_i) = 1$, $Bl(\theta_i|\theta_j) = 0$, $i \neq j$, and $Bl(B|\Theta) = B$, $\forall B \subseteq \Theta$, one can easily show that the CUE-based update mechanism in (3.4) retains the Dirichlet property of the updated BoEs at each step [30]. For this Dirichlet BoE case, the CUE-based opinion update reduces to the following DT dynamic system:

$$\boldsymbol{\pi}(\theta_p)_{k+1} = \breve{W}_k \, \boldsymbol{\pi}(\theta_p)_k, \ p \in \overline{1, M}. \tag{3.27}$$

Here $\boldsymbol{\pi}(\cdot)_k$ is as in (3.6) and $\breve{W}_k = \{\breve{w}_{ij,k}\} \in \mathbb{R}^{N \times N}$ where $\breve{w}_{ij,k}$, $i, j \in \overline{1, N}$, $k \in \mathbb{N}_{\geq 0}$. When the i-th agent is receptively updating,

$$\breve{w}_{ij,k} = \begin{cases} \alpha_{i,k}, & \text{for } i = j; \\ \dfrac{(1 - \alpha_{i,k})(1 + m_j(\Theta)_k)}{|\mathcal{N}_{i,k}(\varepsilon_i)|}, & \text{for } j \in \mathcal{N}_{i,k}(\varepsilon_i); \\ 0, & \text{otherwise}; \end{cases} \tag{3.28}$$

when the i-th agent is cautiously updating,

$$\breve{w}_{ij,k} = \begin{cases} 1, & \text{for } i = j; \\ \dfrac{(1 - \alpha_{i,k}) m_i(\Theta)_k}{|\mathcal{N}_{i,k}(\varepsilon_i)|}, & \text{for } j \in \mathcal{N}_{i,k}(\varepsilon_i); \\ 0, & \text{otherwise}. \end{cases} \tag{3.29}$$

While we may still refer to $\breve{W}_k$ as the corresponding *confidence matrix*, unlike W_k, $\breve{W}_k$ is not necessarily stochastic.

To proceed we take inspiration from [30], where it is shown that under mild conditions, the masses for complete ambiguity Θ vanish when two Dirichlet agents mutually update each other. The same result turns out to hold true for multiple BoEs.

Lemma 3. Consider the CUE-based updating of the Dirichlet BoEs as in (3.27). If

$$\alpha_{i,k} + \sum_{j \neq i} \beta_{ij}(\Theta)_k \leq \rho < 1, \ \forall i, j \in \overline{1, N}, \ \forall k \in \mathbb{N}_{\geq 0},$$

then $\lim_{n \to \infty} m_i(\Theta)_n = 0, \ \forall i \in \overline{1, N}$. $\qquad\qquad\square$

Proof. Observe that the update of $\boldsymbol{\pi}(\Theta)$ can be written as

$$\boldsymbol{\pi}(\Theta)_{k+1} = \boldsymbol{\Gamma}_k \boldsymbol{\pi}(\Theta)_k,$$

where

$$\boldsymbol{\Gamma}_k = \begin{bmatrix} \alpha_{1,k} & \beta_{12}(\Theta)_k & \beta_{13}(\Theta)_k & \cdots & \beta_{1N}(\Theta)_k \\ \beta_{21}(\Theta)_k & \alpha_{2,k} & \beta_{23}(\Theta)_k & \cdots & \beta_{2N}(\Theta)_k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \beta_{N1}(\Theta)_k & \beta_{N2}(\Theta)_k & \beta_{N3}(\Theta)_k & \cdots & \alpha_{N,k} \end{bmatrix}.$$

The condition in the statement implies that $\|\boldsymbol{\Gamma}_k\|_\infty \leq \rho < 1, \ \forall k \in \mathbb{N}_{\geq 0}$, which guarantees the claim. $\qquad\qquad\blacksquare$

Note that each BoE being updated possessing at least one singleton focal element ensures that the condition in Lemma 3 is satisfied, which in turn ensures that the mass for each completely ambiguous proposition vanishes in the limit. Recall that Corollaries 1 and 2 apply to probabilistic agents. In the following sections, we present the counterparts to these results which apply to Dirichlet agents.

3.3.1 Single Opinion Leader

Let us consider a 1-ODC as in Definition 3.4 and the corresponding confidence matrix $\breve{W}_k$ as

$$\breve{W}_k = \begin{bmatrix} A_k & \mathbf{0} \\ C_k & D_k \end{bmatrix},$$ (3.30)

where $A_k \in \mathbb{R}_{[0,1]}^{N_C \times N_C}$ and $D_k \in \mathbb{R}_{[0,1]}^{N_{out} \times N_{out}}$, and $\mathbf{0}$ and C_k have compatible sizes. As explained in Section 3.2.2, in this 1-ODC consists of a 'central' component $\mathcal{G}_{C,k}^{\dagger}(\varepsilon) = (V_C, E_{C,k}^{\dagger}(\varepsilon))$ with N_C agents (see Fig. 3.1) and a 'follower' component $\mathcal{G}_{out,k}^{\dagger}(\varepsilon) = (V_{out}, E_{out,k}^{\dagger}(\varepsilon))$ with N_{out} agents. Note that in the 'follower' component all the agents are receptively updating. A_k and D_k correspond to the confidence matrices of agents in $\mathcal{G}_{C,k}^{\dagger}(\varepsilon)$ and $\mathcal{G}_{out,k}^{\dagger}(\varepsilon)$, respectively.

Theorem 3. Consider agents possessing Dirichlet opinions embedded in a 1-ODC employing a CUE-based update strategy. Furthermore, suppose that

- condition in Lemma 3 is satisfied,

- $\lim_{n \to \infty} A_{n:0} = \mathbf{1}_{N_C} \mathbf{v}_A^T$, where $\mathbf{v}_A \in \mathbb{R}_{[0,1]}^{N_C}$ is a stochastic vector so that the agents in V_C reach their own consensus, and

- $\exists N_D$ s.t., $\|D_k\| \leq \rho < 1$, $\forall k > N_D$ and $k, N_D \in \mathbb{N}_{\geq 0}$.

Then the agents in V (i.e., agents in V_C and V_{out}) will reach a consensus at the consensus reached by the agents in V_C. $\qquad \square$

Proof. The CUE-based update strategy yields the dynamic system

$$\begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_{k+1} \\ \boldsymbol{\pi}_D(\theta_p)_{k+1} \end{bmatrix} = \breve{W}_k \begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_k \\ \boldsymbol{\pi}_D(\theta_p)_k \end{bmatrix},$$ (3.31)

where $k \in \mathbb{N}_{\geq 0}$, $\theta_p \in \Theta$, and

$$\breve{W}_k = \begin{bmatrix} A_k & \mathbf{0} \\ C_k & D_k \end{bmatrix} \implies \breve{W}_{n:0} = \begin{bmatrix} A_{n:0} & \mathbf{0} \\ P_n & D_{n:0} \end{bmatrix}. \tag{3.32}$$

The sub matrices have compatible sizes with $\breve{W}_k$. From $\breve{W}_{n+1:0} = \breve{W}_{n+1}\breve{W}_{n:0}$, we can get

$$P_{n+1} = C_{n+1}A_{n:0} + D_{n+1}P_n, \ P_0 = C_0,$$

for $n \in \mathbb{N}_{\geq 0}$. Unlike with agents possessing probabilistic opinions, in this scenario with Dirichlet agent opinions, $\breve{W}_k$ is not necessarily stochastic. However, since the conditions in Lemma 3 are satisfied, $\forall \epsilon > 0$, $\exists k_D \in \mathbb{N}_{\geq 0}$ such that $|\breve{W}_k \mathbf{v} - \mathbf{1}_N| < \epsilon$, $\forall k > k_D$ and with $\mathbf{v}$ being any stochastic vector. Then $\forall \epsilon > 0, \exists k_D \in \mathbb{N}_{\geq 0}$ s.t.

$$\|C_k \mathbf{1}_{N_C} + D_k \mathbf{1}_{N_{out}} - \mathbf{1}_{N_{out}}\| < \epsilon, \ \forall k > k_D.$$

Using the notation $\Delta P_n = P_n - \mathbf{1}_{N_{out}} \mathbf{v}_A^T$, $n \in \mathbb{N}_{\geq 0}$, and following similar arguments as in the proof of Theorem 1, we get, $\forall \epsilon_D > 0$, $\exists k_D \in \mathbb{N}_0$ s.t.

$$|\|\Delta P_{n+1}\| - \|D_{n+1}\Delta P_n\|| \leq \|A_{n:0} - \mathbf{1}_{N_C} \mathbf{v}_A^T\| + \epsilon_D, \ \forall n > k_D. \tag{3.33}$$

From $\lim_{n \to \infty} A_{n:0} = \mathbf{1}_{N_C} \mathbf{v}_A^T$, given an arbitrary $\epsilon_A > 0$, $\exists N_A \in \mathbb{N}_{\geq 0}$ s.t.

$$\|A_{n:0} - \mathbf{1}_{N_C} \mathbf{v}_A^T\| < \epsilon_A, \ \forall n \geq N_A. \tag{3.34}$$

Then from (3.33) and (3.34), we can obtain the following:

$$|\|\Delta P_{n+1}\| - \|D_{n+1}\Delta P_n\|| < \epsilon_A + \epsilon_D, \ \forall n \geq \max(k_D, N_A).$$

Let $\max\{k_D, N_A, N_D\} = N_B$. So, $\forall n \geq N_B$, we have

$$\|\Delta P_{n+1}\| < \|D_{n+1}\Delta P_n\| + \epsilon_A + \epsilon_D < \rho\|\Delta P_n\| + \epsilon_A + \epsilon_D,$$

where we use the fact that $\|D_{n+1}\| \leq \rho < 1$, $\forall n \geq N_D$. This upper bound for $\|\Delta P_{n+1}\|$ yields

$$\|\Delta P_{N_B+L}\| < (\epsilon_A + \epsilon_D)\sum_{\ell=0}^{L-1}\rho^\ell + \rho^L\|\Delta P_{N_B}\|, \ L \geq 0. \tag{3.35}$$

We note that

$$\lim_{L\to\infty}\left((\epsilon_A + \epsilon_D)\sum_{\ell=0}^{L-1}\rho^\ell + \rho^L\|\Delta P_{N_B}\|\right) = \frac{\epsilon_A + \epsilon_D}{1 - \rho},$$

i.e., given an arbitrary $\epsilon_P > 0$, $\exists L_P \in \mathbb{N}_{\geq 0}$ s.t.

$$(\epsilon_A + \epsilon_D)\sum_{\ell=0}^{L-1}\rho^\ell + \rho^L\|\Delta P_{N_B}\| - \frac{\epsilon_A + \epsilon_D}{1 - \rho} < \epsilon_P, \ \forall L \geq L_P.$$

Use this in (3.35):

$$\|\Delta P_{N_B+L}\| < \epsilon_P + \frac{\epsilon_A + \epsilon_D}{1 - \rho}, \ \forall L \geq L_P.$$

In other words,

$$\lim_{n\to\infty}\|\Delta P_n\| = \lim_{n\to\infty}\|P_n - \mathbf{1}_{N_{out}}\mathbf{v}_A^T\| = 0.$$

Using (3.32) in (3.31), we get

$$\begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_{n+1} \\ \boldsymbol{\pi}_D(\theta_p)_{n+1} \end{bmatrix} = \begin{bmatrix} A_{n:0} & \mathbf{0}_{N_C \times N_{out}} \\ P_n & D_{n:0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_0 \\ \boldsymbol{\pi}_D(\theta_p)_0 \end{bmatrix}.$$

Let us take the consensus among agents in V_C as

$$\boldsymbol{\pi}_A^*(\theta_p) = \left(\mathbf{1}_{N_C}\mathbf{v}_A^T\right)\boldsymbol{\pi}_A(\theta_p)_0.$$

Then, by (3.20), we can write

$$\begin{bmatrix} \boldsymbol{\pi}_A^*(\theta_p) \\ \boldsymbol{\pi}_D^*(\theta_p) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{1}_{N_C}\mathbf{v}_A^T & \mathbf{0}_{N_C \times N_{out}} \\ \mathbf{1}_{N_C}\mathbf{v}_A^T & \mathbf{0}_{N_{out} \times N_{out}} \end{bmatrix}}_{\mathbf{1}_{N_C+N_{out}}\mathbf{v}^{*T}} \begin{bmatrix} \boldsymbol{\pi}_A(\theta_p)_0 \\ \boldsymbol{\pi}_D(\theta_p)_0 \end{bmatrix}, \tag{3.36}$$

where $\mathbf{v}^*$ is a $(N_C + N_{out})$-sized stochastic vector created by concatenating the vectors $\mathbf{v}_A$ and $\mathbf{0}_{N_{out} \times 1}$. But (3.36) satisfies the conditions of Lemma 2. So, all agents in V achieve a consensus. Moreover, as (3.36) shows, this consensus is the same consensus achieved by the agents in V_C. ∎

An immediate consequence of Theorem 3 is

Corollary 3. Consider the network $\mathcal{G}_k^\dagger(\boldsymbol{\varepsilon}) = (V, E_k^\dagger(\boldsymbol{\varepsilon}))$ in (3.3) populated with receptively updating Dirichlet opinion followers and a single cautiously updating Dirichlet opinion leader. Suppose that the condition in Lemma 3 is satisfied. Let D_k denote the confidence matrix of the receptively updating opinion followers with $\|D_k\| \leq \rho < 1, \forall k \geq N_D$, for some $N_D \in \mathbb{N}_{\geq 0}$. Then all the agents reach a consensus opinion. □

Proof. Construct a 1-ODC as in Definition 3.4 with $\mathcal{G}_{C,k}^\dagger(\boldsymbol{\varepsilon})$ containing the opinion leader only and $\mathcal{G}_{out,k}^\dagger(\boldsymbol{\varepsilon})$ populated with the opinion followers. Proposition 1 implies that $V_C^\dagger$ (which consists of only the cautiously updating opinion leader) generates a consensus. Therefore, according to Theorem 3, all the agents in V reach a consensus opinion. ∎

3.3.2 Two Opinion Leaders

Let us consider a 2-ODC as in Definition 3.5 and the corresponding confidence matrix $\breve{W}_k$ as

$$\breve{W}_k = \begin{bmatrix} A_k^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_k^{(2)} & \mathbf{0} \\ C_k^{(1)} & C_k^{(2)} & D_k \end{bmatrix}. \tag{3.37}$$

where $A_k^{(1)} \in \mathbb{R}_{[0,1]}^{N_{C1} \times N_{C1}}$, $A_k^{(2)} \in \mathbb{R}_{[0,1]}^{N_{C2} \times N_{C2}}$, and $D_k \in \mathbb{R}_{[0,1]}^{N_{out} \times N_{out}}$, and the other matrices have compatible sizes. As explained in Section 3.2.3, this 2-ODC consists of two 'central' components $\mathcal{G}_{C1,k}^{\dagger}(\varepsilon) = (V_{C1}, E_{C1,k}^{\dagger}(\varepsilon))$ (with N_{C1} agents) and $\mathcal{G}_{C2,k}^{\dagger}(\varepsilon) = (V_{C2}, E_{C2,k}^{\dagger}(\varepsilon))$ (with N_{C2} agents), plus a third component $\mathcal{G}_{out,k}^{\dagger}(\varepsilon) = (V_{out}, E_{out,k}^{\dagger}(\varepsilon))$ (with N_{out} agents) s.t. no agent in $\mathcal{G}_{out,k}^{\dagger}$ belongs to the in-components of either $\mathcal{G}_{C1,k}^{\dagger}(\varepsilon)$ or $\mathcal{G}_{C1,k}^{\dagger}(\varepsilon)$. See Fig. 3.2. Note that, $A_k^{(1)}$, $A_k^{(2)}$, and D_k correspond to the confidence matrices of agents in $\mathcal{G}_{C1,k}^{\dagger}(\varepsilon)$, $\mathcal{G}_{C2,k}^{\dagger}(\varepsilon)$, and $\mathcal{G}_{out,k}^{\dagger}(\varepsilon)$, respectively.

Theorem 4 below shows that a 2-ODC cannot yield a consensus in general. Furthermore, it provides conditions for consensus among the N_{out} agents embedded in $\mathcal{G}_{out,k}^{\dagger}(\varepsilon)$.

Theorem 4. Consider agents embedded in a 2-ODC employing a CUE-based update strategy. Furthermore, suppose that

- condition in Lemma 3 is satisfied,

- $\lim_{n \to \infty} A_{n:0}^{(1)} = \mathbf{1}_{N_{C1}} \mathbf{v}_1^T$ and $\lim_{n \to \infty} A_{n:0}^{(2)} = \mathbf{1}_{N_{C2}} \mathbf{v}_2^T$, where $\mathbf{v}_1 \in \mathbb{R}_{[0,1]}^{N_{C1}}$ and $\mathbf{v}_2 \in \mathbb{R}_{[0,1]}^{N_{C2}}$ are stochastic vectors so that the agents in V_{C1} and V_{C2} achieve their own consensus opinions, and

- $\|D_k\| \leq \rho < 1$, $\forall k \geq N_D$, for some $N_D \in \mathbb{N}_{\geq 0}$.

Then, the following are true:

(a) The agents in V (i.e., agents in V_{C1}, V_{C2}, and V_{out}) reach a consensus iff the consensus opinions of the agents in V_{C1} and V_{C2} are equal.

(b) When the consensus opinions of the agents in V_{C1} and V_{C2} are not equal, a consensus among the agents in V_{out} occurs if $\exists \lambda^{(1)}, \lambda^{(2)} \in (0,1)$ s.t. $\lambda^{(1)} + \lambda^{(2)} = 1$ and $\lambda^{(1)} C_k^{(1)} \mathbf{1}_{N_{C1}} = \lambda^{(2)} C_k^{(2)} \mathbf{1}_{N_{C2}}$, $\forall k \in \mathbb{N}_{\geq 0}$. $\qquad \square$

Proof. From (3.37) we can get

$$\breve{W}_{n:0} = \begin{bmatrix} A_{n:0}^{(1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_{n:0}^{(2)} & \mathbf{0} \\ P_n^{(1)} & P_n^{(2)} & D_{n:0} \end{bmatrix},$$

where, for $n \in \mathbb{N}_{\geq 0}$,

$$P_{n+1}^{(1)} = C_{n+1}^{(1)} A_{n:0}^{(1)} + D_{n+1} P_n^{(1)}, \ P_0^{(1)} = C_0^{(1)};$$
$$P_{n+1}^{(2)} = C_{n+1}^{(2)} A_{n:0}^{(2)} + D_{n+1} P_n^{(2)}, \ P_0^{(2)} = C_0^{(2)}.$$

(a) Suppose the two converged consensus opinions of the agents in V_{C1} and V_{C2} are equal. Then by following a similar argument as in the proof of Theorem 2(a) and using the result of Theorem 3 we can show that all agents must reach a consensus. Conversely, if the two converged consensus opinions of the agents in V_{C1} and V_{C2} are not equal, no consensus is possible among agents.

(b) Suppose the two consensus opinions of the agents in V_{C1} and V_{C2} are not equal. Unlike with agents possessing probabilistic opinions, in this scenario with Dirichlet agent opinions $\breve{W}_k$ is not necessarily stochastic. However, since the condition

in Lemma 3 is satisfied, $\forall \epsilon > 0, \exists k_D \in \mathbb{N}_{\geq 0}$ such that, $|\breve{W}_k \mathbf{v} - \mathbf{1}_N| < \epsilon, \forall k > k_D$ with $\mathbf{v}$ being any stochastic vector. Then $\forall \epsilon > 0, \exists k_D \in \mathbb{N}_{\geq 0}$ s.t.

$$\|C_k^{(1)}\mathbf{1}_{N_{C1}} + C_k^{(2)}\mathbf{1}_{N_{C2}} + D_k\mathbf{1}_{N_{out}} - \mathbf{1}_{N_{out}}\| < \epsilon;$$

$$\|C_k^{(1)}\mathbf{1}_{N_{C1}} + \lambda^{(2)}D_k\mathbf{1}_{N_{out}} - \lambda^{(2)}\mathbf{1}_{N_{out}}\| \quad < \epsilon, \ \forall k > k_D,$$

where we used

$$\lambda^{(1)}C_k^{(1)}\mathbf{1}_{N_{C1}} = \lambda^{(2)}C_k^{(2)}\mathbf{1}_{N_{C2}}.$$

Now, proceeding as we did in the proof of Theorem 3, and with the notation $\Delta P_n^{(1)} = P_n^{(1)} - \lambda^{(2)}\mathbf{1}_{N_{out}}\mathbf{v}_1^T$ we get, $\forall \epsilon_D > 0, \exists k_d \in \mathbb{N}_{\geq 0}$ s.t.

$$\|\|\Delta P_{n+1}^{(1)}\| - \|D_{n+1}\Delta P_n^{(1)}\|\| \leq \|\Delta P_{n+1}^{(1)} - D_{n+1}\Delta P_n^{(1)}\|$$

$$= \|C_{n+1}^{(1)}[A_{n:0}^{(1)} - \mathbf{1}_{N_{C1}}\mathbf{v}_1^T] + \epsilon_D\|$$

$$\leq \|A_{n:0}^{(1)} - \mathbf{1}_{N_{C1}}\mathbf{v}_1^T\| + \epsilon_D,$$

where we used the sub-stochasticity of $C_n^{(1)}$ for $\forall n > k_D$.

Now, as in the proof of Theorem 3, use the fact that the agents in V_{C1} and V_{C2} each achieve a consensus to show that

$$\lim_{n \to \infty} \|P_n^{(1)} - \lambda^{(2)}\mathbf{1}_{N_{out}}\mathbf{v}_1^T\| = \lim_{n \to \infty} \|P_n^{(2)} - \lambda^{(1)}\mathbf{1}_{N_{out}}\mathbf{v}_2^T\| = 0.$$

Denote the consensus among the agents in V_{C1} and V_{C2} as

$$\boldsymbol{\pi}_{A^{(1)}}^*(\theta_p) = (\mathbf{1}_{N_{C1}}\mathbf{v}_1^T)\boldsymbol{\pi}_{A^{(1)}}(\theta_p)_0;$$

$$\boldsymbol{\pi}_{A^{(2)}}^*(\theta_p) = (\mathbf{1}_{N_{C2}}\mathbf{v}_2^T)\boldsymbol{\pi}_{A^{(2)}}(\theta_p)_0,$$

respectively. Then we have

$$
\begin{bmatrix} \boldsymbol{\pi}^*_{A^{(1)}}(\theta_p) \\ \boldsymbol{\pi}^*_{A^{(2)}}(\theta_p) \\ \boldsymbol{\pi}^*_D(\theta_p) \end{bmatrix} = \begin{bmatrix} \mathbf{1}_{N_{C1}}\mathbf{v}_1^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_{N_{C2}}\mathbf{v}_2^T & \mathbf{0} \\ \lambda^{(2)}\mathbf{1}_{N_{C1}}\mathbf{v}_1^T & \lambda^{(1)}\mathbf{1}_{N_{C2}}\mathbf{v}_2^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_{A^{(1)}}(\theta_p)_0 \\ \boldsymbol{\pi}_{A^{(2)}}(\theta_p)_0 \\ \boldsymbol{\pi}_D(\theta_p)_0 \end{bmatrix},
$$

where $\boldsymbol{\pi}^*_D(\theta_p)$ is given as

$$
\boldsymbol{\pi}^*_D(\theta_p) = \mathbf{1}_{N_{out}} \begin{bmatrix} \lambda^{(2)}\mathbf{v}_1^T & \lambda^{(1)}\mathbf{v}_2^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\pi}_{A^{(1)}}(\theta_p)_0 \\ \boldsymbol{\pi}_{A^{(2)}}(\theta_p)_0 \end{bmatrix}.
$$

Hence, from Lemma 2, we conclude that the agents in V_{out} reach a consensus if $\lambda^{(1)} + \lambda^{(2)} = 1$, $\forall k \in \mathbb{N}_{\geq 0}$.

∎

As an immediate consequence of Theorem 4 we get

Corollary 4. Consider the network $\mathcal{G}_k^\dagger(\varepsilon) = (V, E_k^\dagger(\varepsilon))$ in (3.3) populated with receptively updating Dirichlet opinion followers and two cautiously updating Dirichlet opinion leaders. Suppose that the condition in Lemma 3 is satisfied. Let D_k denote the confidence matrix of the receptively updating opinion followers with $\|D_k\| \leq \rho < 1$, $\forall k \geq N_D$, for some $N_D \in \mathbb{N}_{\geq 0}$. Then, with a CUE-based update strategy, the following are true:

(a) All the agents reach a consensus iff the converged opinions of the two opinion leaders are equal.

(b) When the opinions of the opinion leaders are not equal, the opinion followers form an opinion cluster if the proportion of weights each receptively updating agent gives to the two opinion leaders is identical for $k \geq N_D$ and no opinion leader is given zero weight. ∎

3.4 Empirical Evaluation and Discussion

In this section, we present some of the typical results obtained through extensive simulations of scenarios where agent opinions are captured via p.m.f.s (see Section 3.4.1), Dirichlet BoEs (see Section 3.4.2), and more general DST BoEs (see Section 3.4.3). The results confirm our theoretical analysis in Section 3.2 and demonstrate the applicability of these results for the more general DST models presented in Section 3.1.

In all the simulations, the agents employ a CUE-based update strategy to update their opinions with $\alpha_i = 0.50$, $\forall i \in \overline{1, N}$, and the agents' bounds of confidence taken to be identical, i.e., $\varepsilon_i = \varepsilon$, $\forall i \in \overline{1, N}$. Note that, for $\alpha_i > 0$, $\forall i \in \overline{1, N}$, and for sufficiently large ε, all the agents satisfy the self-communicating or the strong-aperiodic property [48, 51]. Even though the agents are embedded in a static network, the agents must accommodate the bounds of confidence as the opinions are updated. In effect, this creates a dynamic network $\mathcal{G}_k^\dagger(\varepsilon)$.

For ease of visualization, the results of consensus/cluster formation are displayed using bifurcation diagrams that depict the state of consensus/cluster formation in the limit density versus ε [18].

3.4.1 Probabilistic Agent Opinions

For our simulations we embed seven agents on a graph of seven nodes and 100 agents on ER random graphs of 100 nodes. First we present the results for seven agents, and then the results for 100 agents.

3.4.1.1 Simulations with Seven Agents

The FoD of the opinion BoE of each agent is $\Theta = \{\theta_1, \theta_2, \theta_3\}$. For the results shown in Figures 3.3 to 3.6, initial opinion profile of θ_1 is selected as

$$\boldsymbol{\pi}(\theta_1)_0 = [0.80, 0.78, 0.76, 0.40, 0.80, 0.10, 0.20]^T; \tag{3.38}$$

the remaining masses are equally distributed between the opinion profiles $\boldsymbol{\pi}(\theta_2)_0$ and $\boldsymbol{\pi}(\theta_3)_0$.

No Opinion Leaders

Fig. 3.3a shows the network topology of the seven receptive agents R_i, $i \in \overline{1,7}$ and Fig. 3.3b shows the corresponding bifurcation diagram. As can be seen from Fig. 3.3b, for smaller values of ε, seven 'opinion clusters' are generated because agents are essentially isolated. As the value of ε is increased, the number of opinion clusters decreases because the agents are updating their opinions based on opinions of their neighbors who are within their confidence bounds. Eventually, for $\varepsilon > 0.46$ (approx.) consensus emerges (see Fig. 3.3b). This is expected from our analytical results in Section 3.2.1.

Even though the underlying network topology shown in Fig. 3.3a is static, opinion updating occurs in a dynamic network $\mathcal{G}_k^\dagger(\varepsilon)$. To illustrate this further, consider the case when $\varepsilon = 0.5$. As Fig. 3.3c illustrates, agent pairs (R_3, R_6) and (R_5, R_7) do not exchange opinions because their opinion distances exceed the bound of confidence $\varepsilon = 0.5$. However, after the first iteration of opinion exchanges, distances among agents change. As Fig. 3.3d illustrates, the opinion distances of all agents (including (R_3, R_6) and (R_5, R_7)) are now well within $\varepsilon = 0.5$ and, at the second iteration, all agents exchange opinions with their neighboring agents.

(A) Network topology.

(B) Bifurcation diagram for $\boldsymbol{\pi}(\theta_1)$.

(C) For $\varepsilon = 0.5$, first iteration: (R_3, R_6) and (R_5, R_7) do not exchange opinions because the corresponding opinion distances are above $\varepsilon = 0.5$.

(D) For $\varepsilon = 0.5$, second iteration: All neighbor agents exchange opinions because all corresponding opinion distances are below $\varepsilon = 0.5$.

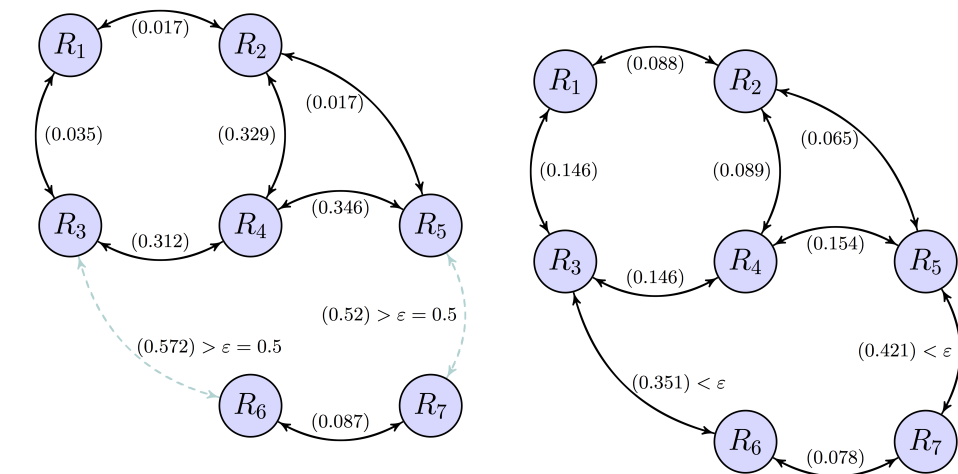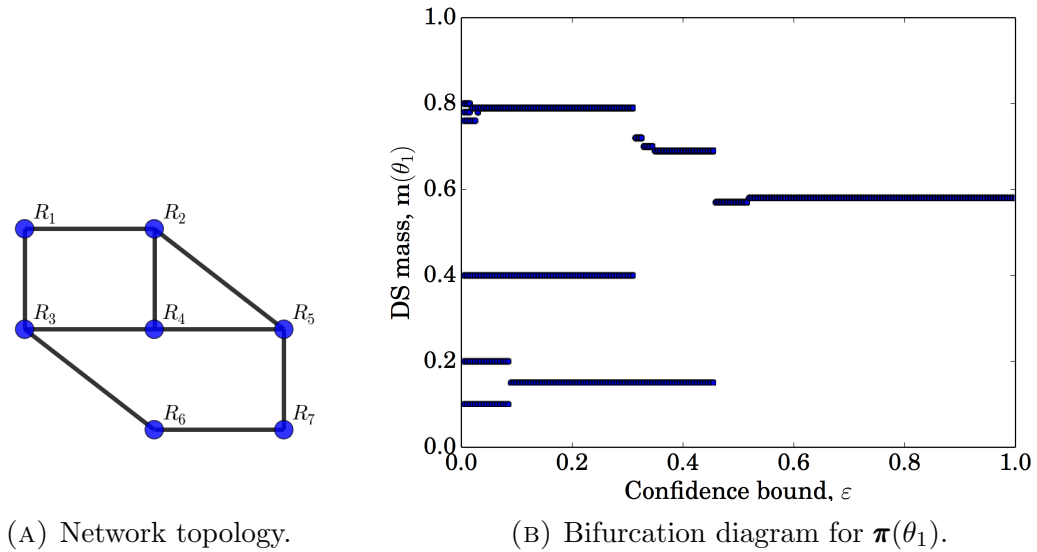FIGURE 3.3: Probabilistic agents: Simulation results for seven receptively updating agents $(R_i, i \in \overline{1,7})$ and no opinion leaders. A consensus occurs for $\varepsilon > 0.46$ (approx.). For $\varepsilon = 0.5$, Figs 3.3c and 3.3d show the 'directions' of opinion exchange at the first two iterations. Edge labels indicate the distance between the opinions of the corresponding agent pair.

Single Opinion Leader

To create this scenario, we replace the receptive agent R_1 in Fig. 3.3a with a cautious agent C_1 and obtain the graph shown in Fig. 3.4a. For this topology the bifurcation diagram for $\boldsymbol{\pi}(\theta_1)$ is depicted in Fig.3.4b. As before, for smaller values of ε, each agent forms its own 'opinion cluster'. For larger values of ε, in accordance with Corollary 1, a consensus is formed, and this consensus opinion is the opinion of the opinion leader C_1 (viz., $m(\theta_1) = 0.80$). As Fig. 3.4b indicates, this consensus begins to appear for $\varepsilon > 0.46$ (approx.). The dynamic nature of opinion exchange in the first three iterations is illustrated in Figs 3.4c, 3.4d, and 3.4e.

Two Opinion Leaders

Here we replaced the two receptive agents $\{R_1, R_7\}$ in Fig. 3.3a by the cautious agents $\{C_1, C_7\}$, respectively. Figs 3.5a and 3.5b show the corresponding network topology and bifurcation diagram for $\boldsymbol{\pi}(\theta_1)$, respectively. In accordance with Corollary 2, no consensus is reached because the two opinion leaders $\{C_1, C_7\}$ possess different opinions.

With the opinions of two opinion leaders being different, the number of opinion clusters created depends on the bound of confidence ε. For $0.31 < \varepsilon < 0.42$ (approx.), we observe two opinion clusters, the minimum number of clusters possible. We have achieved this by picking the agent BoEs carefully so that the network separates into two components, each with its own opinion leader, for the aforementioned values of ε. For these values, the network gets separated into two components $\{C_1, R_2, R_3, R, R_5\}$ and $\{C_7, R_6\}$ because $\|\mathcal{E}_3 - \mathcal{E}_6\| > \varepsilon$ and $\|\mathcal{E}_5 - \mathcal{E}_7\| > \varepsilon$, for $0.31 < \varepsilon < 0.42$ (approx.). The ensuing network generates two opinion clusters at the opinions of the two opinion leaders C_1 and C_7. For larger values of ε, some (or all) receptive agents get influenced

(A) Network topology.

(B) Bifurcation diagram for $\boldsymbol{\pi}(\theta_1)$.

(C) For $\varepsilon = 0.46$, first iteration: (R_3, R_6) and (R_5, R_7) do not exchange opinions.

(D) For $\varepsilon = 0.46$, second iteration. (R_3, R_6) now exchange opinions; (R_5, R_7) still do not exchange opinions.

(E) For $\varepsilon = 0.46$, third iteration. All agents exchange opinions with their neighbors.
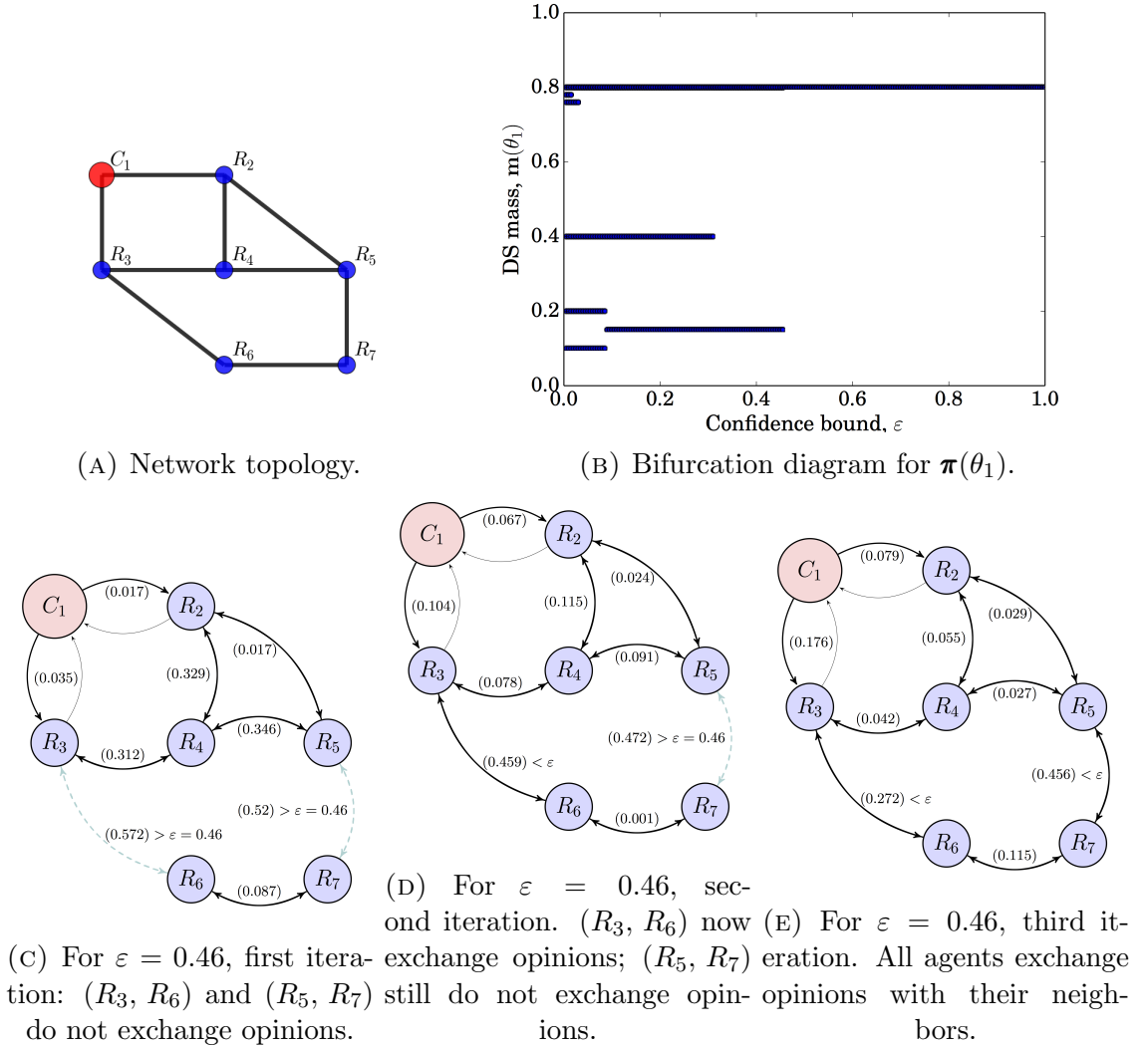
FIGURE 3.4: Probabilistic agents: Simulation results for one opinion leader (C_1) and six receptively updating agents (R_i, $i \in \overline{2,7}$). A consensus occurs for $\varepsilon > 0.46$ (approx.), at C_1's opinion (i.e., $m(\theta_1) = 0.80$). For $\varepsilon = 0.46$, Figs 3.4c, 3.4d, and 3.4e show the 'directions' of opinion exchange at the first three iterations. Edge labels indicate the distance between the opinions of the corresponding agent pair.

by both opinion leaders which creates different opinion clusters that are influenced by both opinion leaders.

As asserted in Corollary 2, when the two opinion leaders possess different opinions, the receptive agents will reach a consensus if the matrices $C_k^{(1)}$ and $C_k^{(2)}$ in Definition 3.5 satisfy $\lambda^{(1)} C_k^{(1)} \mathbf{1}_{N_{C1}} = \lambda^{(2)} C_k^{(2)} \mathbf{1}_{N_{C2}}$, $\forall k \in \mathbb{N}_{\geq 0}$. Fig. 3.6a shows a network topology which satisfies this condition, and the corresponding bifurcation diagram in Fig. 3.6b
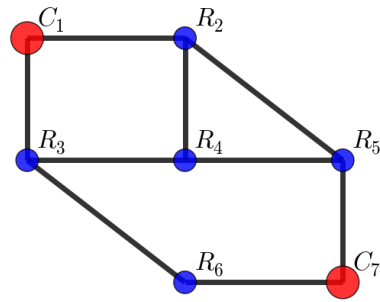
(A) Network topology.
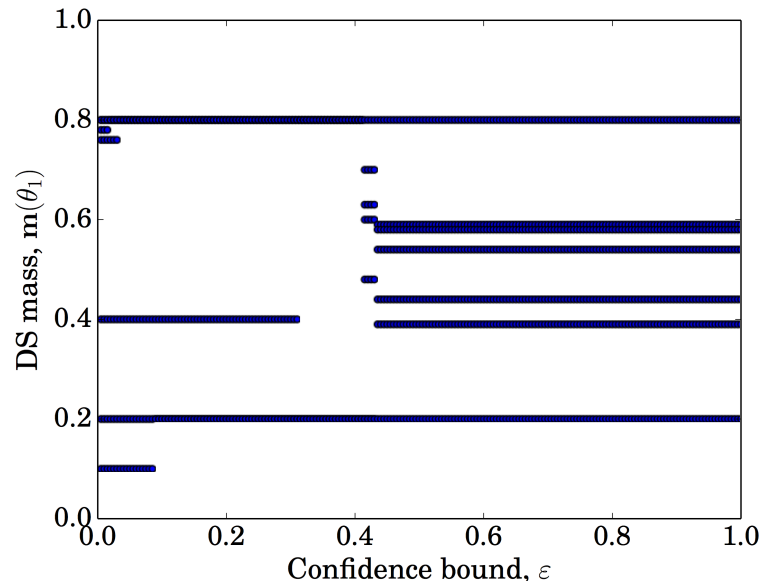
(B) Bifurcation diagram for $\boldsymbol{\pi}(\theta_1)$.

FIGURE 3.5: Simulation results for two opinion leaders (C_1 and C_7) and five receptive agents (R_i, $i \in \overline{2,6}$). With the two opinion leaders possessing different opinions, no consensus is achieved. A minimum of 2 opinion clusters are achieved for $0.31 < \varepsilon < 0.42$ (approx.). There is no consensus among the receptive agents.
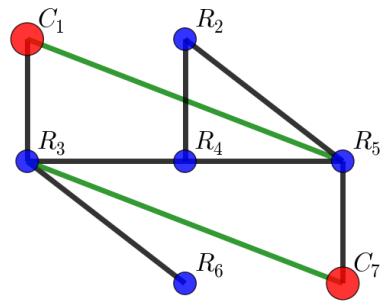
(A) Altered topology.
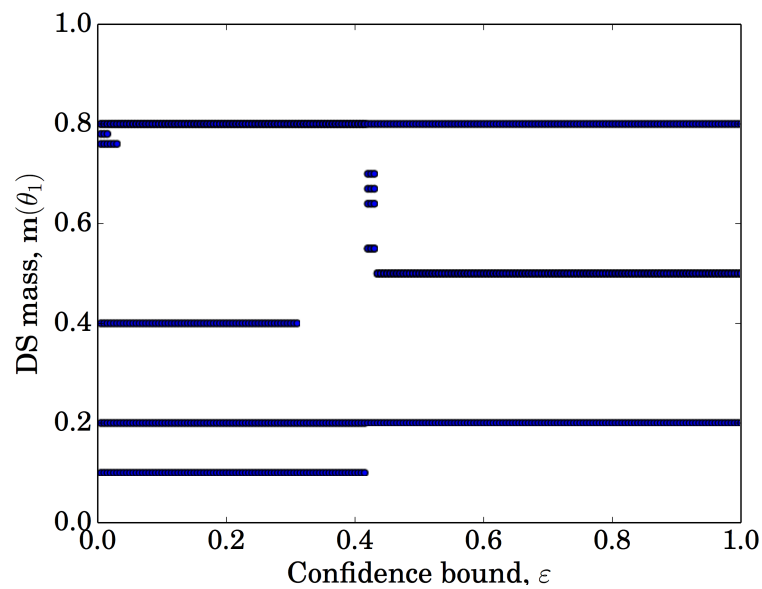
(B) Bifurcation diagram for $\boldsymbol{\pi}(\theta_1)$.

FIGURE 3.6: Simulation results for two opinion leaders (C_1 and C_7) and five receptive agents (R_i, $i \in \overline{2,6}$) embedded in a topology that generates a consensus among the receptive agents. This consensus appears for $\varepsilon > 0.43$ (approx.).

shows the appearance of a third opinion cluster around $m(\theta)_1 = 0.50$ (approx.) for $\varepsilon > 0.43$ (approx.).

3.4.1.2 Simulations with 100 Agents

Further experiments were carried out with 100 agent with completely connected graph and then Erdős-Rényi (ER) random graph model.
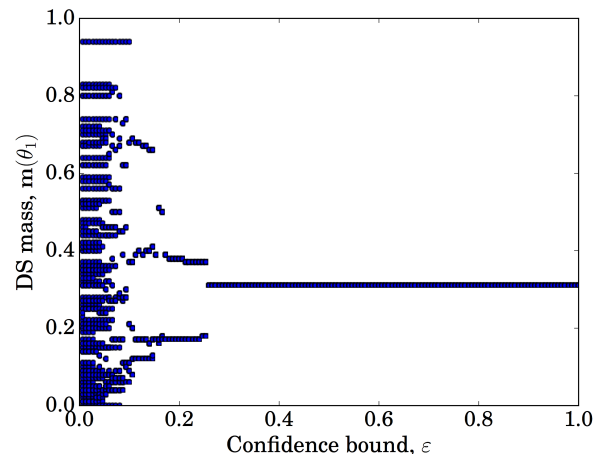
(A) No opinion leaders.
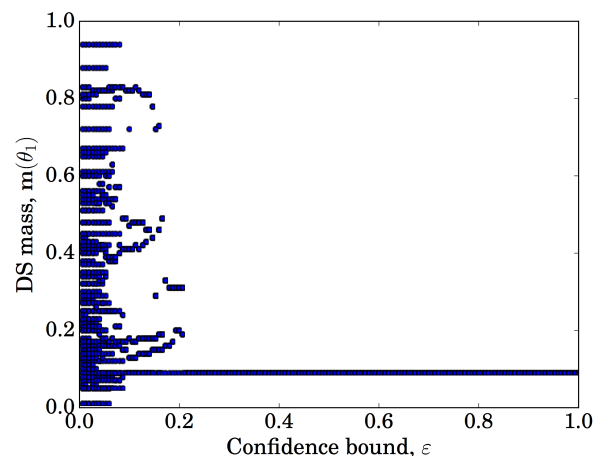
(B) One opinion leader.
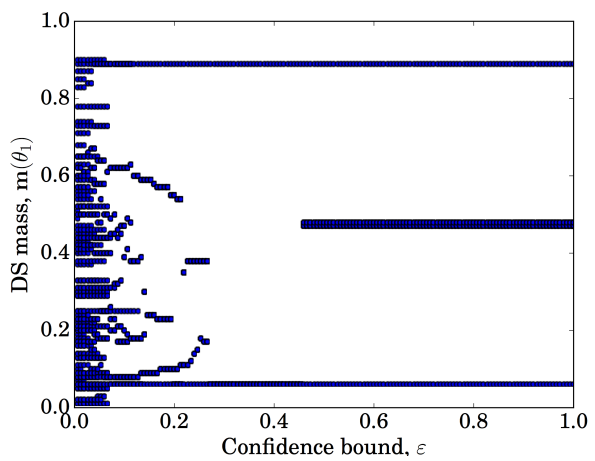
(C) Two opinion leaders.

FIGURE 3.7: Simulation results for 100 agents embedded in an Erdős-Rényi random graph with $p = 0.10$ and agent BoEs sampled from $\mathrm{Dir}(1,1,1)$. Consensus can be seen in Fig. 3.7a and 3.7b, for $\varepsilon > 0.26$ and $\varepsilon > 0.21$, respectively.

Completely Connected Graph

We have explored all the three scenarios: all receptive agents, one cautious agent, and multiple cautious agents embedded in completely connected graph with each agent having the identical FoD $\Theta = \{\theta_1, \theta_2, \theta_3\}$. An agent i's opinion is represented via a DST model with focal elements restricted to be singleton propositions, i.e., the opinion model is essentially a p.m.f. with probabilities $\{m_i(\theta_1), m_i(\theta_2), \cdots, m_i(\theta_M)\}$, where $\sum_{j=1}^{M} m_i(\theta_j) = 1$.

Completely Connected Graph: All Receptive Agents

Uniformly Distributed Case:

Most studies on real-valued opinion dynamics (with agents having a single opinion in the range $[0, 1]$) have used random and uniformly distributed initial opinion profiles or initial densities that are uniformly distributed in the opinion space [18]. In accordance with these previous works, we have conducted a trial experiment with $N = 100$ receptive agents, assuming that the initial DST mass of the opinion on θ_1 is uniformly distributed in the range $[0, 1]$, i.e., $m_i(\theta_1) = \mathcal{U}(0, 1)$, $i \in \{1, \ldots, 100\}$. The remaining mass is equally distributed among the other singletons θ_2 and θ_3. Fig. 3.8 shows the corresponding bifurcation diagram with respect to θ_1. It can be seen that, for smaller values of ε (approximately < 0.12), no consensus is formed. Indeed, the lower the value of ε, the higher the number of clusters. The cluster formation at $\varepsilon = 0.1$ and consensus at $\varepsilon = 0.3$ appear in Figs 3.9a and 3.9b, respectively.

Dirichlet Distribution Case:

Further experiments were carried out by sampling opinions from a Dirichlet distribution [52], i.e., $m_i(\theta_1, \theta_2, \theta_3; 2, 2, 2) = Dir(2, 2, 2)$, $i \in \{1, \ldots, 100\}$. The symmetric Dirichlet distribution corresponds to the case having no prior information to favor one
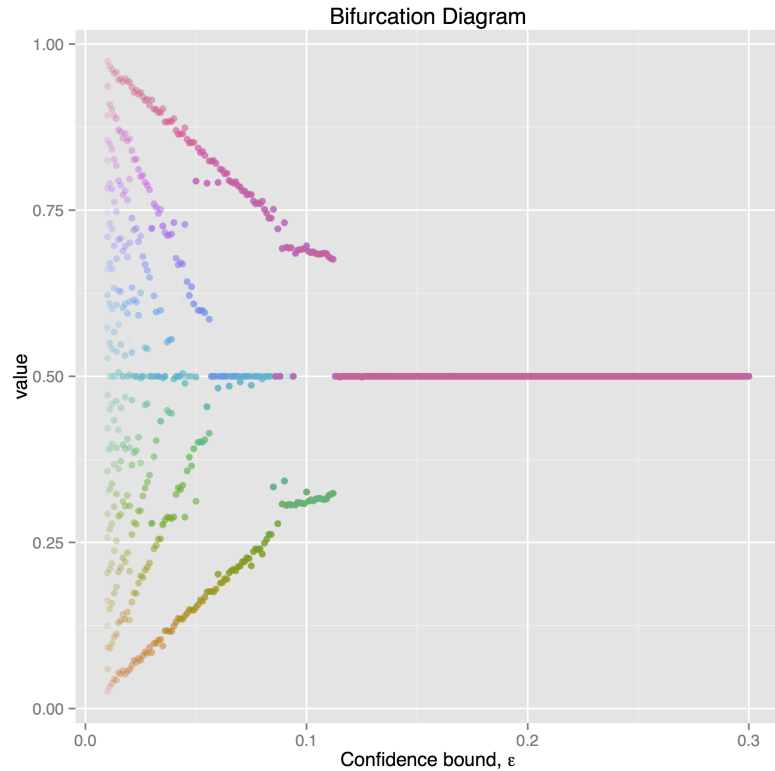
FIGURE 3.8: All receptive agents case/mass for θ_1 sampled from a uniform distribution: bifurcation diagram for θ_1. The mass values of singleton θ_1 of agents in the limit density are given for different bound of confidence values ε. The intensity of each point corresponds to the denseness of the representing cluster.

singleton over the other. The symmetric Dirichlet distribution with concentration parameter equal to one is equivalent to a uniform distribution over the open standard 2-simplex. The parameters $(2, 2, 2)$ in the Dirichlet distribution enforce a symmetric and dense distribution with a centered mode. Fig. 3.10 shows the bifurcation diagram with all receptive agents. It can be seen that consensus is reached when $\varepsilon > 0.15$ (approximately). Fig. 3.11 shows that the minimum bound of confidence ε required for consensus gets lower as the number of agents increases. This is to be expected because, as the number of agents increases, it is easier to make an ε-chain with a lower bound of confidence.
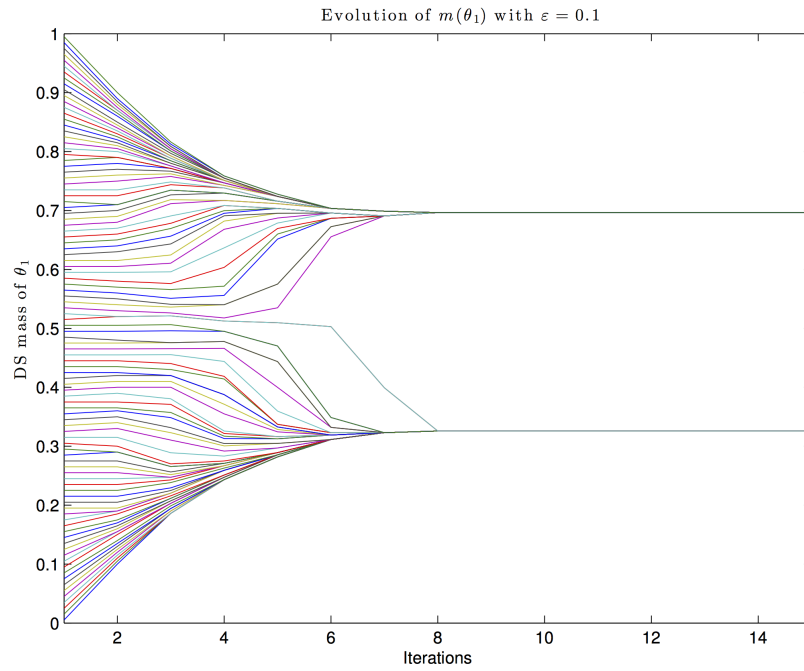
(A) Bound of confidence $\varepsilon = 0.1$.
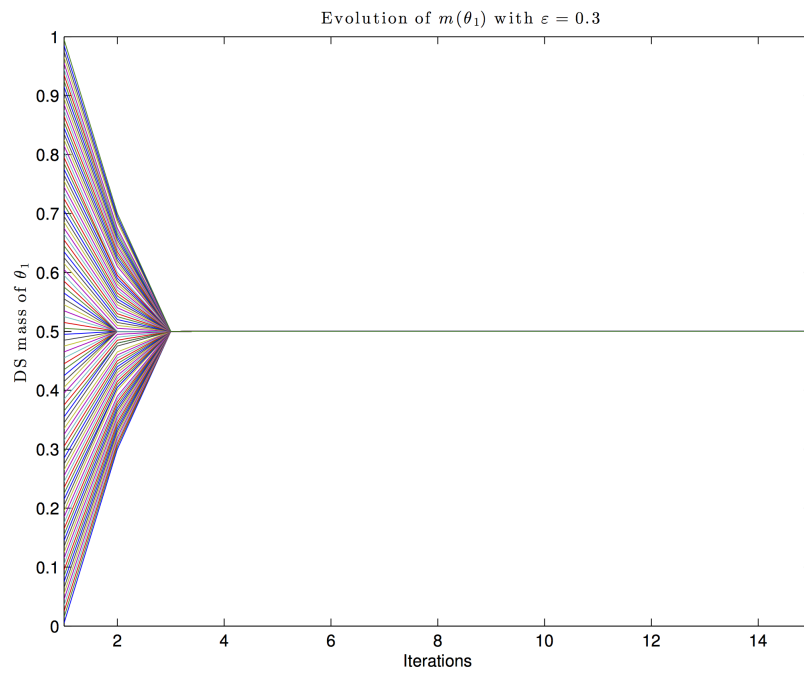
(B) Bound of confidence $\varepsilon = 0.3$

FIGURE 3.9: All receptive agents case/mass for θ_1 sampled from a uniform distribution: evolution of opinion profile of θ_1.
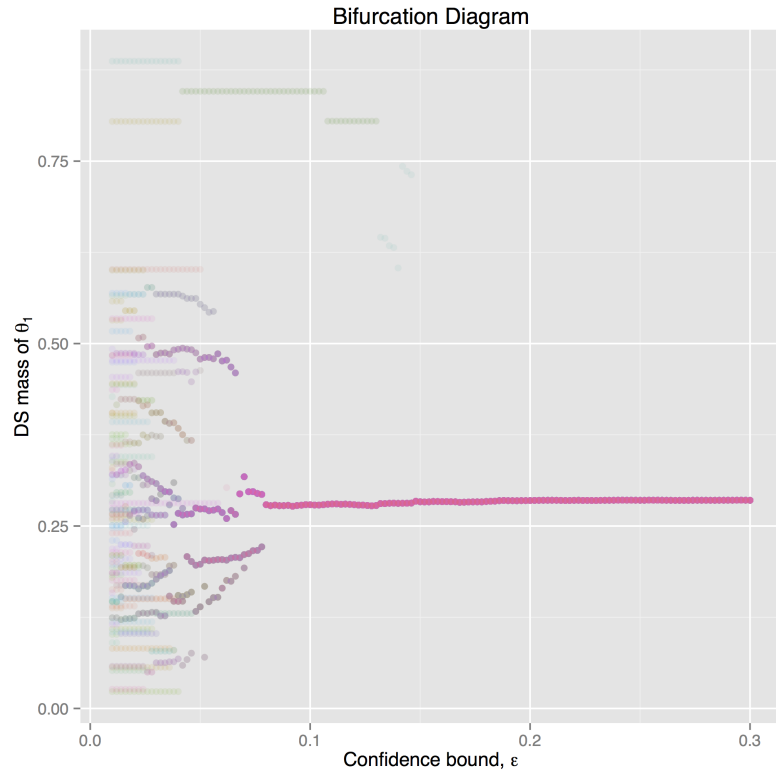
FIGURE 3.10: All receptive agents case/masses of all singletons sampled from $Dir(2,2,2)$: bifurcation diagram for θ_1.

Completely Connected Graph: One Cautious Agent

A test with one cautious agent and 99 receptive agents was carried by selecting the initial mass assignment of the cautious agent as, $m_{C1}(\theta_1) = 0.50$, $m_{C1}(\theta_2) = 0.25$ and $m_{C1}(\theta_3) = 0.25$. Masses of the receptive agents were sampled from $Dir(2,2,2)$. The bifurcation diagram with respect to θ_1 is given in Fig. 3.12, where consensus is reached when $\varepsilon > 0.13$ (approximately). As expected from the analysis, the cautious agent acts as an opinion leader and guides the consensus by influencing all the other receptive agents to converge to the cautious BoE $\mathcal{E}_{C1}$.
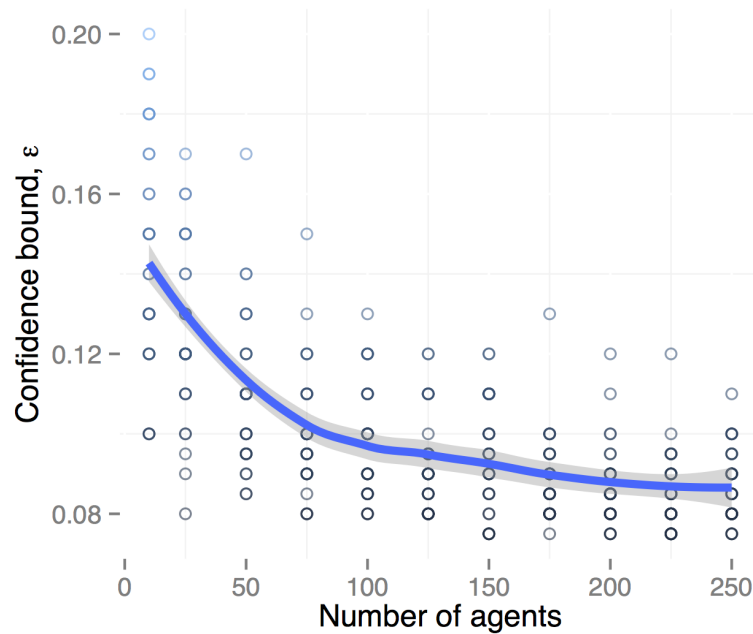
FIGURE 3.11: All receptive agents case/masses of all singletons sampled from $Dir(2,2,2)$: minimum bound of confidence required for consensus versus the number of receptive agents (solid line: estimated minimum bound of confidence, shaded: standard error).

Completely Connected Graph: Multiple (Two) Cautious Agents

Fig. 3.13 gives the bifurcation diagram for the scenario with two cautious agents $C1, C2$ and 98 receptive agents. The mass assignment for the initial state of cautious agents are $m_{C1}(\theta_1) = 0.75$, $m_{C1}(\theta_2) = 0.125$, $m_{C1}(\theta_3) = 0.125$, and $m_{C2}(\theta_1) = 0.25$, $m_{C2}(\theta_2) = 0.375$ and $m_{C2}(\theta_3) = 0.375$. The masses of the opinions of receptive agents were sampled from $Dir(2,2,2)$. As the 'stubborn' opinion leaders carry different opinions, we will not see any consensus in this scenario. The minimum number of two clusters can be observed for $0.08 < \varepsilon < 0.23$ (approximately), where receptive agents are influenced by the closest opinion leaders and cling to the closest group. Note that the singleton opinions on θ_1 have been clustered to two groups with masses belonging to either $m_{C1}(\theta_1)$ or $m_{C2}(\theta_1)$; other singletons behave similarly. Further, since the majority of the receptive agents were initially closer to the cautious agent

FIGURE 3.12: One cautious agent case/masses of receptive agents sampled from $Dir(2, 2, 2)$/masses of cautious agent $\{m_{C1}(\theta_1), m_{C1}(\theta_2), m_{C1}(\theta_3)\} = \{0.50, 0.25, 0.25\}$: bifurcation diagram for θ_1.

$C2$, as can be seen from the higher intensity line in Fig. 3.13, the group formed under $C2$ has a higher agent density compared to the group with $C1$. However, for $\varepsilon > 0.23$, the number of clusters is fixed at 3. When the receptive agents have a higher bound of confidence, they get influenced by both opinion leaders, thus forming a group where the majority of the group has opinions in the convex hull of the leader opinions.

We also make another observation. In the presence of cautious agents, the number of iterations required before reaching a fixed point is higher compared to that of a scenario with all receptive agents. Fig. 3.14 shows that, in the presence of cautious agents, it requires about 200 iterations to reach a fixed point, whereas in all receptive agents case Fig. 3.9, it has taken less than 10 iterations to reach a fixed point.

FIGURE 3.13: Two cautious agents case/masses of receptive agents sampled from $Dir(2,2,2)$/masses of cautious agents are $\{m_{C1}(\theta_1), m_{C1}(\theta_2), m_{C1}(\theta_3)\} = \{0.75, 0.125, 0.125\}$, $\{m_{C2}(\theta_1), m_{C2}(\theta_2), m_{C2}(\theta_3)\} = \{0.25, 0.375, 0.375\}$: bifurcation diagram for θ_1.

ER Random Graph

For our simulations, we embed 100 agents in random graphs of 100 nodes generated using the Erdős-Rényi (ER) random graph model with edge formation probability $p = 0.10$ [34]. It is well established that, for an ER random graph, the phase transition for network connectivity occurs when $p > \ln n/n$ [34], which is about 0.046 for $n = 100$. To be safe, we used $p = 0.10$ for generating all random graphs. Also, every random graph was tested for connectedness before starting the simulation. Therefore, for sufficiently large values of ε, the graph $\mathcal{G}_k^{\dagger}(\varepsilon)$ is essentially the same as $\mathcal{G}_k$, and thus it is connected.
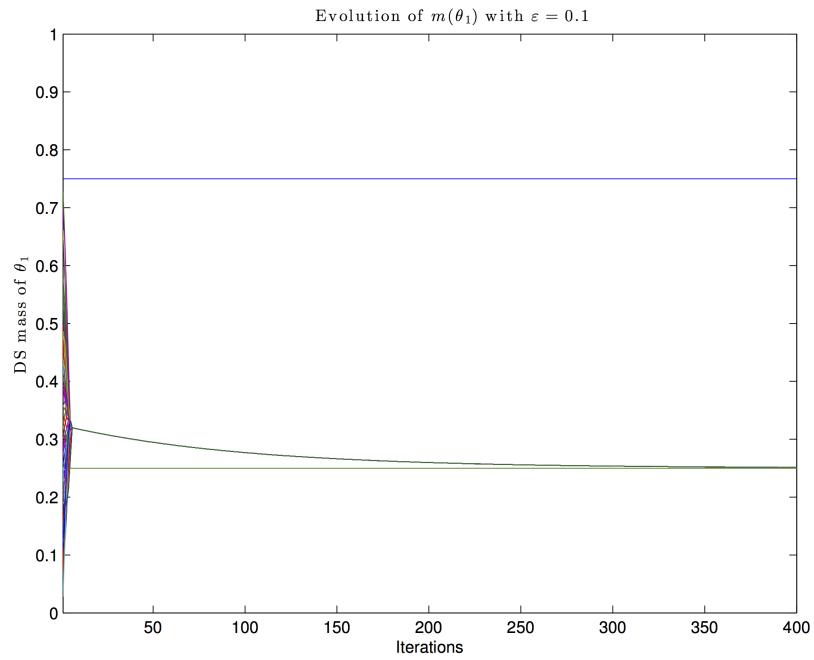
(A) $\varepsilon = 0.1$
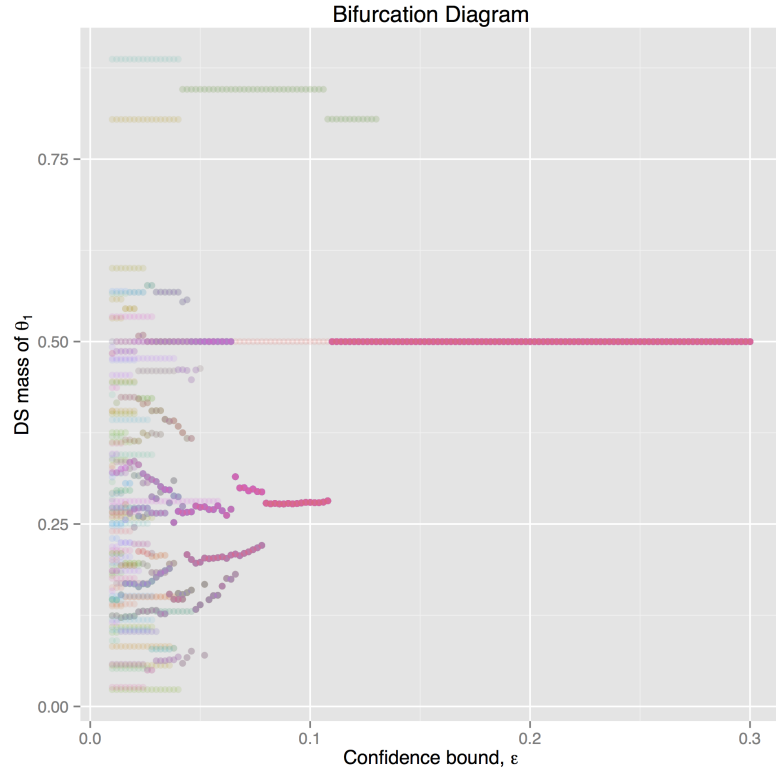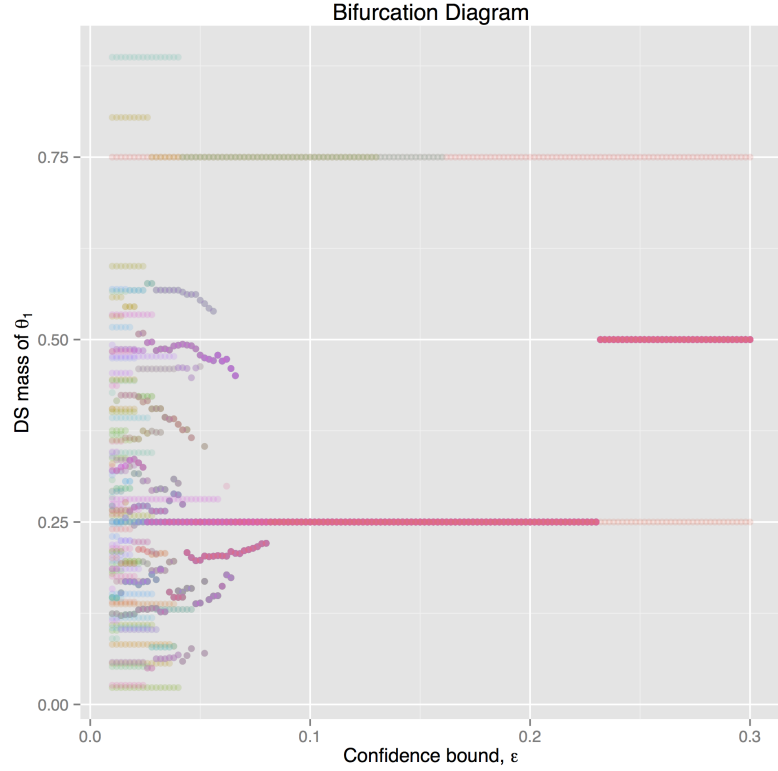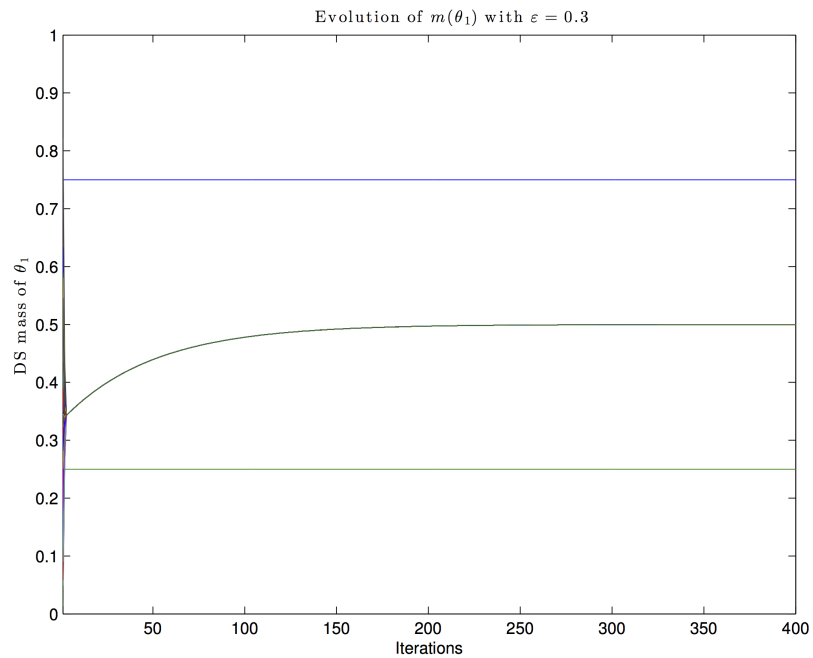

(B) $\varepsilon = 0.3$

FIGURE 3.14: Two cautious agents case/masses of receptive agents sampled from $Dir(2,2,2)$/masses of cautious agents are $\{m_{C1}(\theta_1), m_{C1}(\theta_2), m_{C1}(\theta_3)\} = \{0.75, 0.125, 0.125\}$, $\{m_{C2}(\theta_1), m_{C2}(\theta_2), m_{C2}(\theta_3)\} = \{0.25, 0.375, 0.375\}$: evolution of opinion profile of θ_1.

The BoE of each agent was sampled from the symmetric Dirichlet distribution $\mathrm{Dir}(1,1,1)$, which is equivalent to a uniform distribution over the open standard 2-simplex [52]. As Figs 3.7a and 3.7b show, for $\varepsilon > 0.26$ (approx.) and $\varepsilon > 0.21$ (approx.), a consensus appears when there are no opinions leaders and when only one opinion leader is present, respectively. In accordance with Corollary 2, Fig. 3.7c shows that there is no consensus among the 100 agents when two opinion leaders have different opinions.

3.4.2 Dirichlet Agent Opinions

Here, we repeat the experiments conducted with the 7-agent topologies in Section 3.4.1.1 but with Dirichlet agent opinions. For all the agents, we kept the same mass vectors $\pi(\theta_2)_0$ and $\pi(\theta_3)_0$ as those in Section 3.4.1.1 while we used $\pi(\Theta)_0 = 0.1$; the remaining masses were assigned to $\pi(\theta_1)_0$.

Fig. 3.15 shows the corresponding bifurcation diagrams. As is evident, and in consistent with Corollary 3, a consensus can be seen in Figs 3.15a and 3.15b for $\varepsilon > 0.51$ and $\varepsilon > 0.5$, respectively. However, in Fig. 3.15c, there is no consensus among the agents. This is consistent with Corollary 4(a) because the cautious agents do not possess the same converged opinion. The minimum number of opinion clusters appear for $0.26 < \varepsilon < 0.52$ (approx.).

Interestingly, even for higher values of ε, no consensus emerges even among the receptive agents. Indeed, one would expect that the receptive agents who are now less restrained to exchange opinions with their neighbors would form an opinion clusters of their own. In contrast, when the agents are embedded in the graph shown in Fig. 3.6a, a consensus emerges among the receptive agents for $\varepsilon > 0.46$ (approx.). See Fig. 3.16. This is because the graph topology in Fig. 3.6a satisfies the condition in Corollary 4(b).
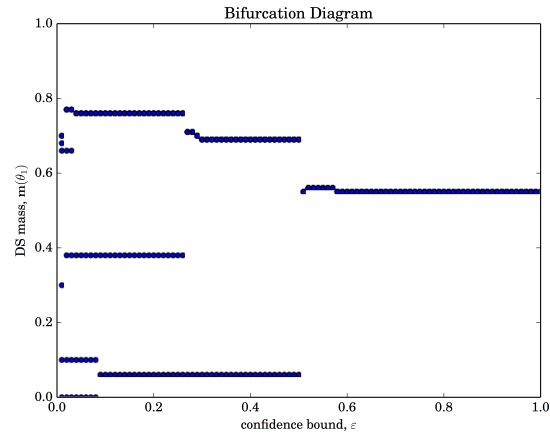
(A) No opinion leaders.
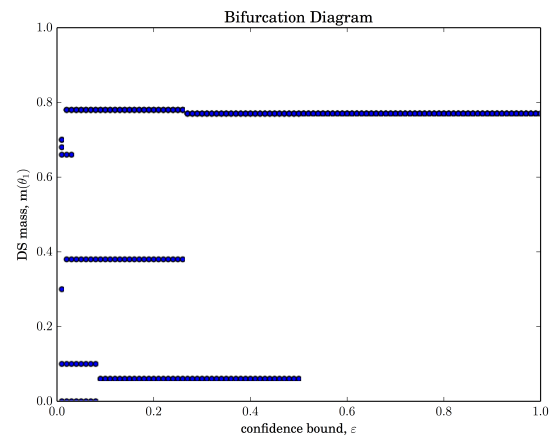
(B) One opinion leader.
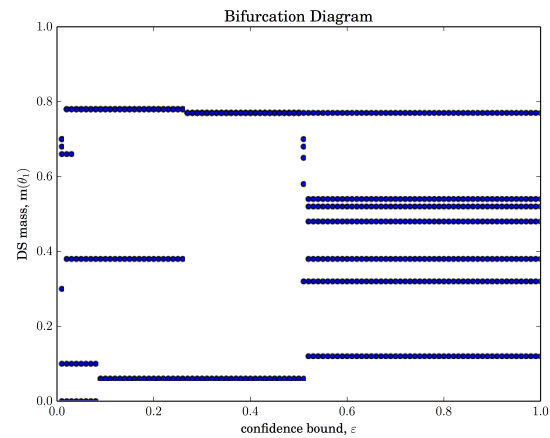
(C) Two opinion leaders.

FIGURE 3.15: Dirichlet agents: Simulation results for 7 agents with no opinion leaders, one opinion leader and two opinion leaders embedded in the graphs in Figs 3.3a, 3.4a, and 3.5a respectively. Consensus can be seen in Fig. 3.15a and 3.15b, for $\varepsilon > 0.51$ and $\varepsilon > 0.5$, respectively.
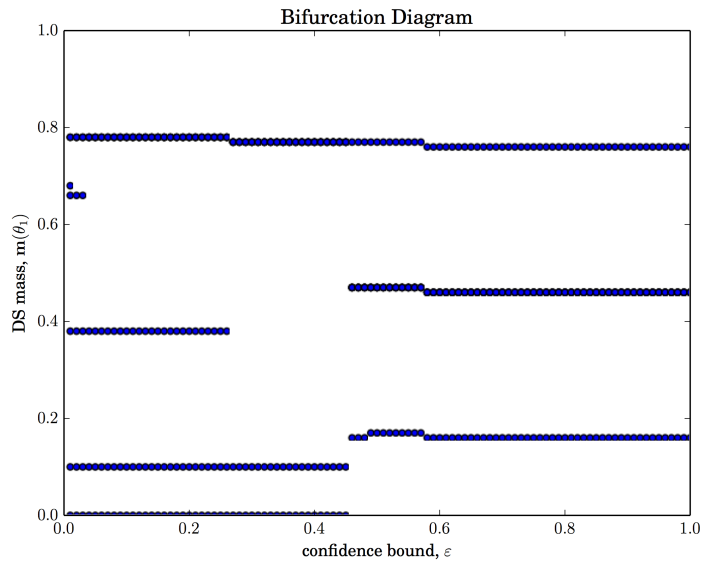
FIGURE 3.16: Dirichlet agents: Simulation results for two opinion leaders (C_1 and C_7) and five receptively updating agents (R_i, $i \in \overline{2,6}$) embedded in the graph in Fig. 3.6a generates a consensus among the receptively updating agents. This consensus appears for $\varepsilon > 0.46$ (approx.).

3.4.3 DST Agent Opinions

3.4.3.1 Simulations with Seven Agents

In this study, we assigned random DS mass assignments for the seven agents embedded within the topology in Fig. 3.6a. For this purpose, we utilized the Dirichlet distribution which has been widely employed in opinion modeling [53–55]. In particular, for each agent in each trial, the DS masses for θ_1, θ_2, θ_3, $(\theta_1\theta_2)$, $(\theta_1\theta_3)$, $(\theta_2\theta_3)$, and $\Theta = (\theta_1\theta_2\theta_3)$ were sampled from the Dirichlet distribution $\mathrm{Dir}(4,4,4,2,2,2,1)$. Figs 3.17a, 3.17b, and 3.17c show the bifurcation diagrams when the network contains no opinion leaders, one opinion leader, and two opinion leaders, respectively.

As Fig. 3.17a shows, with no opinion leaders, a consensus appears for $\varepsilon > 0.18$ (approx.). Fig. 3.17b shows bifurcation diagram when only one opinion leader is present, and we can see that a consensus appears for $\varepsilon > 0.17$ (approx.). As Fig. 3.17c
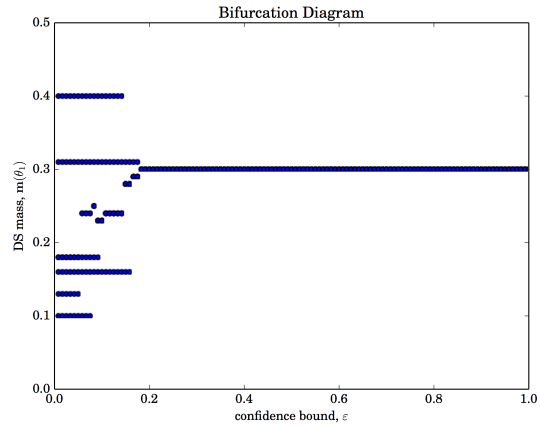
(A) No opinion leaders.
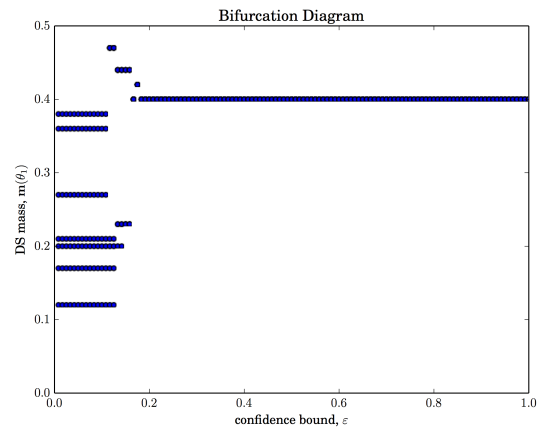
(B) One opinion leader.
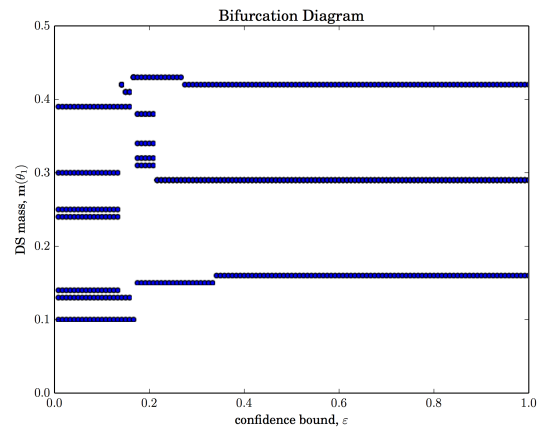
(C) Two opinion leaders.

FIGURE 3.17: General DST agents: Simulation results for seven receptively up-
dating agents embedded within the network topology in 3.6a and DST mass values
sampled from $\mathrm{Dir}(4, 4, 4, 2, 2, 2, 1)$. A consensus appears for the no opinion leader
and one opinion leaders cases in Fig. 3.17a and 3.17c for $\varepsilon > 0.18$ (approx.) and
$\varepsilon > 0.17$ (approx.), respectively.

shows, with two opinion leaders, no consensus is reached among the agents. However, it is interesting to note that three opinion clusters emerge for $\varepsilon > 0.21$ (approx.).

In essence, the observations related to consensus and opinion cluster formation when agents opinions are modeled via DST BoEs are in accordance with the results obtained with probabilistic agent opinions. The main reason for utilizing the DST framework for capturing agent opinions is its ability to capture the types of uncertainties and the nuances that are characteristic of agent states and opinions. DST agent opinions can also generate new emergent behavior which cannot be captured via probabilistic agents. For example, consider 7 receptive agents embedded within the topology in Fig. 3.3a. The initial opinions and the converged opinions for $\varepsilon = 0.30$ appear in Table 3.1. Notice that two opinion clusters have emerged: the first cluster formed by $\{R_1, \ldots, R_5\}$ converge to the probabilistic opinion $\{m_1^*(\theta_1), m_1^*(\theta_2), m_1^*(\theta_3)\} = \{0.63, 0.19, 0.18\}$; the second cluster formed by R_6 and R_7 converge to the general DST opinion $\{m_2^*(\theta_1), m_2^*(\theta_2, \theta_3)\} = \{0.15, 0.85\}$ which allows no further 'refinement' between the singletons θ_2 and θ_3. Such emergent behavior is qualitatively different than what appears in prior models [14, 18, 39–41].

TABLE 3.1: Initial and Converged Opinions (with $\varepsilon = 0.30$)

Agent	DST Mass Values				
	θ_1	θ_2	θ_3	(θ_1, θ_2)	(θ_2, θ_3)
Initial Opinions:					
R_1	0.60	0.10	0.10	0.10	0.10
R_2	0.62	0.11	0.04	0.11	0.12
R_3	0.51	0.12	0.05	0.12	0.20
R_4	0.57	0.15	0.03	0.15	0.10
R_5	0.60	0.10	0.10	0.10	0.10
R_6	0.10	–	–	–	0.90
R_7	0.20	–	–	–	0.80
Converged Opinions (with $\varepsilon = 0.30$):					
$\{R_1, \ldots, R_5\}$	0.63	0.19	0.18	–	—
$\{R_6, R_7\}$	0.15	–	–	–	0.85

3.5 Chapter Summary

In this chapter, we explore the DST framework for representing agent opinions and the formation of consensus and opinion clusters when agents residing within a network exchange and update their opinions. In particular, we examine the effect that opinion leaders have on these processes. Our opinion model explained in this chapter accounts for aspects from SJT and possesses the ability to capture a wider variety of uncertainties and nuances in agent opinions, an advantage inherited from its DST basis. Theoretical analysis, which focuses on probabilistic and Dirichlet agent opinions, provides conditions for the emergence of consensus and opinion clusters in the presence of opinion leaders. The results in this chapter show that a consensus can be formed when the number of opinion leaders is no more than one and with a sufficiently high bound of confidence of the agents. With two or more opinion leaders possessing different opinions, no consensus can be reached in general. We have also analyzed the conditions for opinion cluster formation among the opinion followers.

Chapter 4 extends the theoretical analysis to scenarios where agent opinions are captured via more general DST BoEs, adopts tools from paracontractions theory [19]. It is also noteworthy that for simulations in this chapter we have taken all agents to possess the identical bound of confidence value. When this is not the case, the opinion exchange mechanism itself would be directional (because an agent with a lower bound of confidence may update itself from its neighbor agent who may not update itself because of a higher bound of confidence value). An interesting future research problem is the study of networked agents whose bounds of confidence values are different. Another interesting issue to be addressed is the assessment of the convergence speed of our algorithms [56].

Chapter 4

Consensus/Opinion Clustering: Paracontractions theoretic view

The study of consensus from the viewpoint of nonlinear paracontracting operators allows one to account for time-varying graph topologies with asynchronous nonlinear protocols [57–59]. In this chapter, we use the paracontraction theoretic notions in analyzing the formation of not only consensus, but also opinion clustering in the presence of opinion leaders (or cautious agents).

4.1 Problem Formulation

As before, let us consider a set of N agents embedded in the directed graph $\mathcal{G}_k^\dagger(\varepsilon) = (\mathcal{V}, E_k^\dagger(\varepsilon))$ where ε denotes the bound of confidence. Remember the edges, $E_k^\dagger$ are refined in accordance to SJT, hence the graph $\mathcal{G}_k^\dagger(\varepsilon)$. The set of nodes $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$ denotes the agents and $e_{ij} \in E_k^\dagger(\varepsilon)$, represents unidirectional information exchange link from agent j to agent i. In this chapter, the discrete time index, denoted by k^d, will be taken flexible, such that k^d will represent the time instances when the multi-agent system undergoes change. The index k^d is often called as the Event-based Discrete Time Index (EDTI). We can carefully select and properly arrange k^d such

(A) Agents v_1, v_2 and v_3 updating at the same time $k = 0$

(B) $k^d = 0$ (C) $k^d = 1$ (D) $k^d = 2$
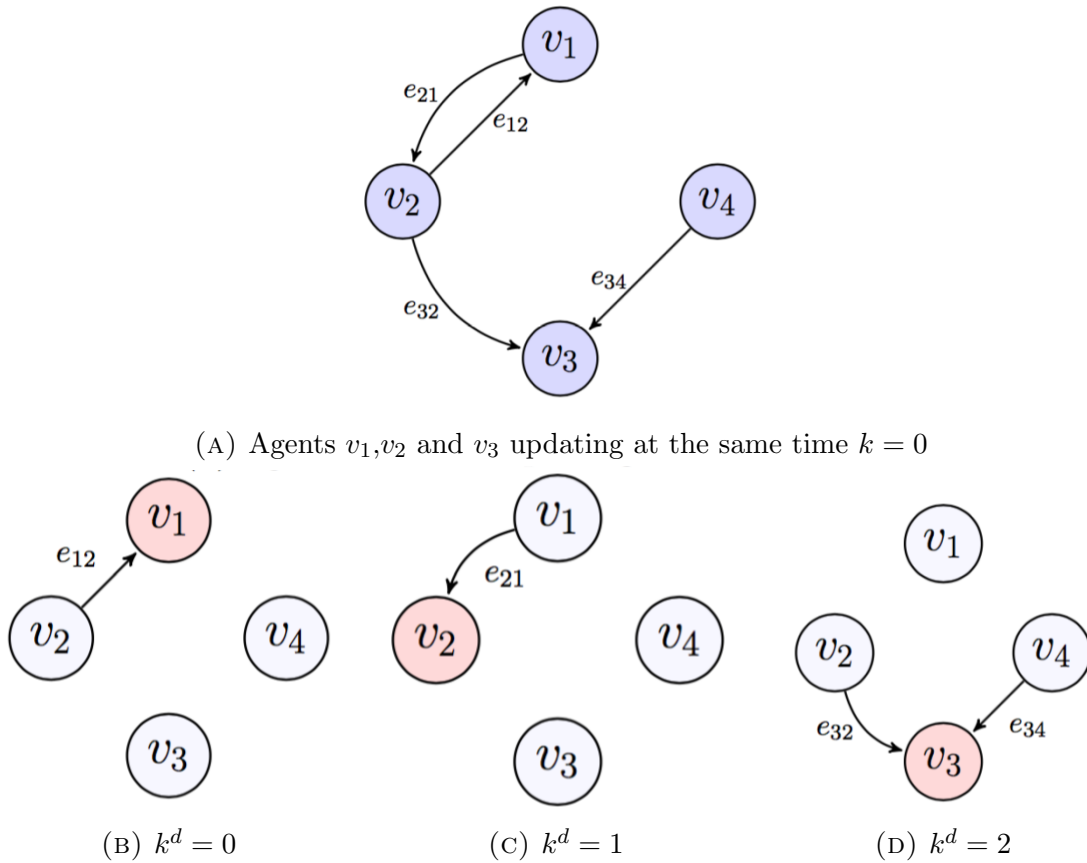
FIGURE 4.1: For a situation of four agents v_1, v_2, v_3 and v_4 exchange opinions at time $k = 0$, event-based discrete time index k^d for each agent has been given.

that only one agent v_i updates its state at a given event-based time index. Figure 4.1 shows an illustration of the selection of event-based discrete time indexing for an opinion updating scenario with four agents represented with v_1, v_2, v_3 and v_4. From the figure it can be seen that, a separate event-based discrete time index has been allocated for each individual agent update. Furthermore, for notational convenience let us denote the state of agent, $v_i \in \mathcal{V}$ at discrete time event k^d by $x_i[k^d]$.

However, in Figure 4.1, the temporal coupling of agents have not been presented enough. For instance, in Figure 4.1c it is not clear whether agent v_2 updates from a delayed state of v_1 or the updated current state of v_1. Hence, we use *iteration graphs* for better representation.

4.1.1 Iteration Graphs

The graphical representation shown in Figure 4.1 only describes the spatial coupling among agents and cannot completely represent the communication delays. Because of that, the representation in Figure 4.1 is ambiguous when used with event-based indexing as it has a weaker capability on handling temporal coupling among agents.

Iteration graphs can represent spatial as well as temporal coupling [60]. Each updating agent at a discrete event-based time index gets a vertex. Furthermore, a set of vertices are reserved for the initial conditions, with negative valued discrete time indexes corresponding to each agent. Therefore, the set of discrete time indexes for the full set of vertices in the iteration graph is, $\{-N, \cdots, -1\} \cup \mathbb{N}_0$. There will be an edge from a vertex corresponding to time index k_1^d to a vertex with k_2^d, if and only if, at the k_2^d the iteration update, the agent opinion vector corresponding to k_1^d has been used.

Figure 4.2 gives an illustration of the iteration graphs using and extended example used that of Figure 4.1. The interaction topologies of a four-agent system at time $k = 0$ and $k = 1$ is shown in Figures 4.2a and 4.2b respectively. The corresponding iteration graph is given in Figure 4.2c. The initial conditions of agents v_4, v_3, v_2 and v_1 are shown with vertices corresponding to discrete time indexes $-4, -3, -2$ and -1 respectively. Starting from $k^d = 0$, each and every individual agent update has been given. For instance, at $k^d = 0$ agent v_1 gets updated from the initial condition of agent v_2 at $k = 0$. At $k^d = 3$ agent v_2 gets updated from the opinions of agents v_1 and v_3 at $k = 1$.

(A) $k = 0$

(B) $k = 1$
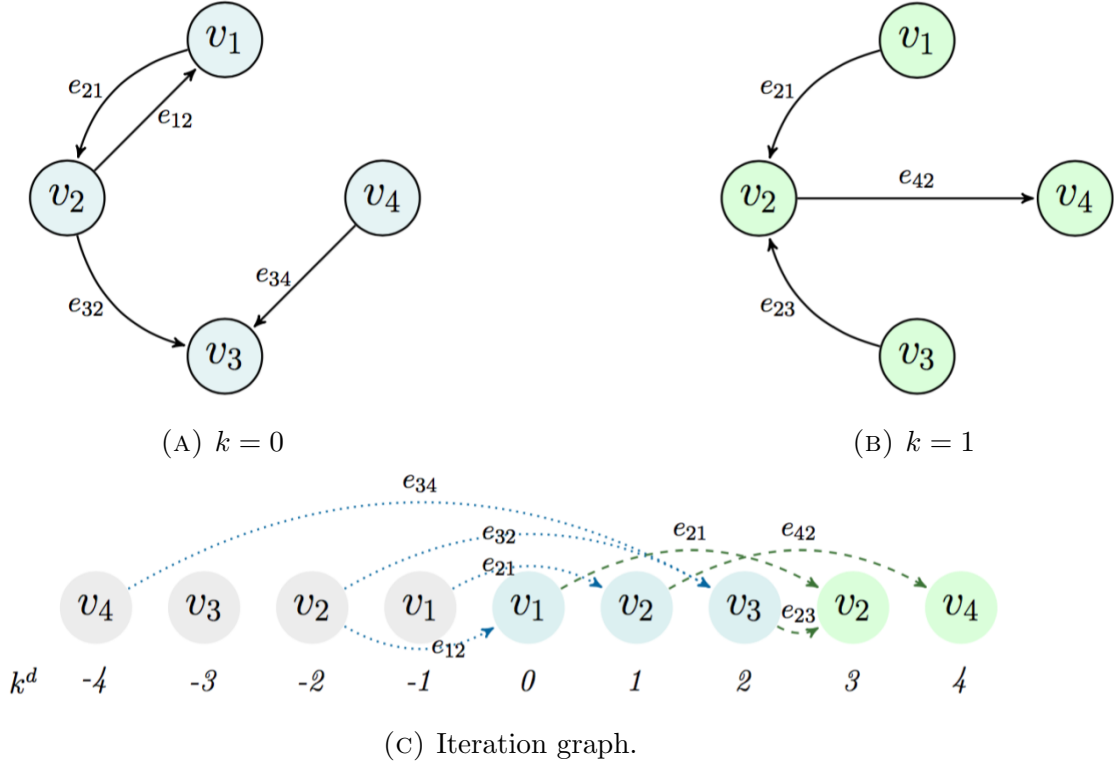
(C) Iteration graph.

FIGURE 4.2: Iteration graph four-agent system with different interaction topologies. The discrete-time indexes $-4, -3, -2$ and -1 corresponds to initial conditions. Indexes from 0 onward corresponds to updating of each individual agents.

4.1.2 Agent Interaction Topologies

The spatial connectivity among agents at different EDTIes are referred to as *agent interaction topologies*[61]. Let us denote the jth interaction topology used by agent v_i by $T_{i,j}$, $i \in \{1, \cdots, N\}$ and $j \in \{1, \cdots, n_i\}$, where N is the number of agents in the system and n_i is the number of topologies for agent v_i. Also, let the set of interaction topologies be denoted by $\mathcal{T} \equiv \{T_{i,j} | i = 1, \cdots, N; j = 1, \cdots, n_i\}$.

Under EDTIes, only a single agent gets updated (via its interaction topology) from other connected agents in the system. Thus we can associate each time index with the corresponding updating agent, hence forming a interaction topology sequence. Let us denote this topology sequence as $\{T[k^d] \in \mathcal{T} | k^d = 0, 1, 2, \cdots\}$. Now, $T[k^d]$ identifies the interaction topology of the updating agent at EDTI k^d for $k^d = 0, 1, 2, \cdots$.

Depending on the connection of each agent to other agents, we can categorize multi-agent systems as, *fully connected* vs. *partially connected* or *static* vs. *dynamic* system. In a fully connected multi-agent system, the interaction topology of each agent has incoming connections from all other agents in the system. On the other hand, in a partially connected system, at least one of the agents in the community has, at least one of the other agents, who is not connecting via and incoming edge. If the interaction topologies of agents vary over time then the system is dynamic, and static otherwise. Therefore, in a static multi-agent system, we can denote the time independent interaction topology of each agent v_i as T_i such that, $\forall i \in \{1, \cdots, N\}$, $T_{i,j} = T_i$. Again, N denotes the number of agents in the system.

4.1.3 Synchronous Versus Asynchronous Consensus Protocols

To keep the generality, let the opinion state of agent v_i at time k be noted by m-tuple, $x_i[k]$. The initial state of agent v_i is $x_i[0] \in \mathbb{R}^m$. Under this notation the consensus can be defined as in Definition 4.1.

Definition 4.1. (Opinion Clusters, Consensus) Let a system of agents $V = \{v_1, v_2, \ldots, v_N\}$ with each agent v_i having state $x_i[k]$ has subsets $\breve{S} \subseteq V$. Then for any valid norm $\| \cdot \|$,

1. Suppose $\lim_{k \to \infty} \|x_i[k] - x_j[k]\| = 0$, $\forall v_i, v_j \in \breve{S}$, then the agents in $\breve{S}$ are said to form an *opinion cluster*.

2. A *consensus* will be reached when $\breve{S} = V$.

∎

Note that, Definition 4.1 is indeed similar to Definition 3.2 in Chapter 3 except that Definition 4.1 is more general s.t. it is applicable to any m-tuple.

A *consensus protocol* is a consensus generating mechanism. We use the well established consensus protocols to study the formation of opinion clusters as well. A *synchronous* consensus protocol is a system where all the agents update their states at the same time using the latest value from their neighbors. Equation (4.1) gives a linear synchronous consensus protocol.

$$x_i[k+1] = \sum_{j=1}^{N} h_{ij}[k]x_j[k], \qquad (4.1)$$

where $h_{ij}[k]$ is non-negative and $\sum_{j=1}^{N} h_{ij}[k] = 1$, for all i and k.

However, in some situations, due to the non-existence of a central clock and unreliability of communication links, studying asynchronous consensus protocols is important. Equation (4.2) gives an example of asynchronous protocol.

$$x_i[k+1] = H_{i,j}(x_1[s^1(k)], \cdots, x_N[s^N(k)]) \qquad (4.2)$$

where $s^r(k), r = 1, \cdots, N$ are sequences from $\mathbb{N}_0$, with $s^r(k) \leq k, \forall r, k$ and $T[k], k = 0, 1, \cdots$. The difference $k - s^r(k)$ gives the *iteration delays*. The operator $H_{i,j}$ could be either linear or nonlinear. By deploying EDTIing we can arrange k^d such that only one agent update each event-based discrete time instance. Hence equation (4.2) can be re-written as in equation (4.3) with EDTIes.

$$x_i[k^d + 1] = \begin{cases} H_{i,j}(x_1[s^1(k^d)], \cdots, x_N[s^N(k^d)]) & \text{if } T[k^d] = T_{i,j}, \\ x_i[k^d] & \text{otherwise.} \end{cases} \qquad (4.3)$$

This gives us the flexibility to identify operators attached to each EDTI. For each unique operator observed during the whole updating process, let us give a unique index from a set $\mathbb{I}$. Operators belonging to various agents are defined on different products of $\mathbb{R}^m$, depending on the number of neighboring agents from whom the opinions are updated. Hence, an operator identified with index i applied on n_i neighboring agents can be denoted[1] as $H^i : \mathbb{R}^{mn_i} \to \mathbb{R}^m$. Then the pool of operators can be given as $\mathcal{H} = \{H^i | i \in \mathbb{I}\}$. Let the updating sequence of agent operator indices be denoted by $I(k^d)$, such that, if agent r updates at EDTI k^d using operator H^i, then $I(k^d) = i$, where $i \in \mathbb{I}$.

Let us consider the sequence of indices identifying the operators at each time index as follows,

Definition 4.2 (Index sequence of fusion operators $\mathcal{I}$). [19, 60] Let k^d denote the discrete event-based time index and $I[k^d] \in \mathbb{I}$ is the index of corresponding operator $H_{\triangleleft}^{I[k^d]}$ at k^d. Then the sequence of fusion operators $\mathcal{I}$ is defined as $\mathcal{I} \equiv \{I[k^d] \in \mathbb{I} | k^d = 0, 1, 2, \ldots\}$. ∎

From now onward for the rest of this chapter we only consider EDTI, unless otherwise stated. So we can simplify the notation of indices and simply use k for event based time indexing instead of k^d.

4.1.3.1 Fixed Points

Convergence problems including consensus can be considered as a special case of finding common fixed points [60] of a finite set of paracontracting operators (defined in Section 4.3). A fixed point of a multiple point operator corresponding to agent r,

---

[1]Note that the indexing of the operators H^x are different of that $H_{y,z}$.

$H^i : \mathbb{R}^{mn_i} \to \mathbb{R}^m$ is a vector $\xi_r \in \mathbb{R}^m$ which satisfies,

$$H^i(\xi_1, \xi_2, \ldots, \xi_r, \ldots, \xi_{n_i}) = \xi_r \,, \tag{4.4}$$

where $i \in \mathbb{I}$, $1 \leq r \leq n_i$ and $n_i \in \{1, 2, \ldots, N\}$.

If a particular fixed point ζ, is a fixed point for all the operators in a pool of operators $\mathcal{H}$, it is termed as a common fixed point, such that $H^i(\zeta, \ldots, \zeta) = \zeta, \forall H^i \in \mathcal{H}$. In Section 4.3 criteria for contraction will be discussed. However, for an arbitrary asynchronous iteration scheme, having common fixed points does not guarantee convergence even with contractive operators. For instance, when the operators have several common fixed points and the sequence alternates between the fixed points from time to time, clearly it will not converge. Hence the asynchronous iterations require coupling conditions for convergence, which will be discussed in the next section.

4.2 Asynchronous Iterations and Coupling Conditions

An asynchronous iteration denoted by $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$ is defined as below.

Definition 4.3. Let $\mathbb{I}$ be some set, $N \in \mathbb{N}$ is a fixed number and $\mathcal{H} = \{H^i | i \in \mathbb{I}\}$ be a pool of operators $H^i : \mathbb{D}^{n_i} \to \mathbb{D}$, where $n_i \in \{1, 2, \ldots, N\}$, $\forall i \in \mathbb{I}$ and $\mathbb{D} \subset \mathbb{R}^m$ is closed. Also let $\mathcal{X}_0 = \{x[0], x[-1] \ldots, x[-N]\} \subset \mathbb{D}$ be a set of given vectors. Then, if there are sequences $\mathcal{I} = I(k)$, $k = 0, 1, \ldots$, of elements in $\mathbb{I}$, $\mathcal{S} = \{s^1(k), \ldots, s^{n_{I(k)}}(k) | k = 0, 1, \ldots\}$ of $n_{I(k)}$-tuple from $\mathbb{N}_0 \cup \{-1, -2, \ldots, -N\}$ with $s^r(k) \leq k$ for each $k \in \mathbb{N}_0$, $r = 1, 2, \ldots, n_{I(k)}$, we call the sequence $x(k)$, $k = 0, 1, \ldots$,

given by

$$x[k+1] := H^{I(k)}\left(x[s^1(k)], x[s^2(k)], \dots, x[s^{n_{I(k)}}(k)]\right), k = 0, 1, \dots, \qquad (4.5)$$

as an asynchronous iteration, denoted by $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$. ∎

In summary, the asynchronous iteration denoted by $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$, each symbol encapsulates meaning as

$\mathcal{H}$ denotes the pool of operators

$\mathcal{X}_0$ denotes the initial conditions

$\mathcal{I}$ index sequence of updating operators

$\mathcal{S}$ sequence of delays involved

In order to satisfy the theorems of convergence that we will discuss later in Section 4.3, let us study certain condtions on $\mathcal{I}$ and $\mathcal{S}$.

Definition 4.4. Let us consider the asynchronous iteration $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$. Then

(a) $\mathcal{I}$ is *admissible* if $\forall k \in \mathbb{N}_0$, $\{I(k)\} \cup \{I(k+1)\} \cup \cdots = \mathbb{I}$ holds.

(b) $\mathcal{I}$ is an *indexwise-regulated* sequence if $\forall i \in \mathbb{I}$, $\exists c_i \in \mathbb{N}_0$, s.t. $\forall k \in \mathbb{N}_0$, $i \in \{I(k)\} \cup \{I(k+1)\} \cup \cdots \cup \{I(k+c_i)\}$

(c) $\mathcal{I}$ is *regulated* if $\exists c \in \mathbb{N}_0$ s.t. $\forall k \in \mathbb{N}_0$, $\{I(k)\} \cup \{I(k+1)\} \cup \cdots \cup \{I(k+c)\} = \mathbb{I}$, i.e., $\mathbb{I}$ has to be finite.

(d) $\mathcal{S}$ is called *admissible* if for $k \to \infty$, $s^r(k) \to \infty$, $\forall r = 1, \dots, n_{I(k)}$.

(e) $\mathcal{S}$ is said to be regulated if $s := \max_{k,r}(k - s^r(k))$ exists.

■

As already mentioned, even if there are common fixed points with contracting operators (see Section 4.3) the sequence could still be divergent. Hence we will consider *confluence conditions* on an asynchronous iterations $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$ to converge.

Definition 4.5. Let $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$ be an asynchronous iteration, with directed graph $\mathcal{G}_k^\dagger(\varepsilon) = (V, E_k^\dagger(\varepsilon))$, whose vertices V are given by $V = \mathbb{N}_0 \cup \{-1, -2, \ldots, -N\}$, and whose edges are given by $e_{k,k_0} \in E_k^\dagger(\varepsilon)$ iff $\exists r, \ 1 \leq r \leq n_{I(k_0-1)}$, s.t. $s^r(k_0 - 1) = k$, $\forall k \geq -N, \ k_0 \geq 1$. Now $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$ is called confluent, if there are numbers $u_0, b \in \mathbb{N}$ and a sequence $\{b_k \in \mathbb{N} | k = u_0, u_0 + 1, \ldots\}$ s.t. $\forall k \geq u_0$ following is true:

(a) for every vertex $k_0 \geq k$ there is a directed path from b_k to k_0 in iteration graph $\mathcal{G}_k^\dagger(\varepsilon)$,

(b) $k - b_k \leq b$,

(c) $\mathcal{S}$ is regulated,

(d) for every $i \in \mathbb{I}$, there are numbers $c_i \in \mathbb{N}$, so that $\forall k \geq u_0$, there exists a vertex $v_k^i \in V$, which is a successor of b_k and a predecessor of b_{k+c_i} and for which $i(v_k^i - 1) = i$.

■

4.3   Criteria of Contraction

In order to find a common fixed point in a pool of operators $\mathcal{H}$, let us analyze criteria of contraction, strictly non-expansive, (c)-paracontracting. We use the notation of $X = (x^1, x^2, \ldots, x^N)$ for an element of $\mathbb{R}^{mN}$, where $x^r \in \mathbb{R}^m$, $r = 1, 2, \ldots, N$.

Definition 4.6. Let $\mathcal{H}$ be a pool of operators then,

i.) If $\forall i \in \mathbb{I}$, $X, Y \in \mathbb{D}^{n_i}$, $\mathbb{D}^{n_i} \subset \mathbb{R}^{mn_i}$, $0 < \omega < 1$ and a valid norm $\| \cdot \|$

$$\|H^i(X) - H^i(Y)\| \leq \omega \max_r \|x^r - y^r\|,$$

then $\mathcal{H}$ is called *contractive* on $\mathbb{D}$.

ii.) If $\forall i$ in $\mathbb{I}$, $X, Y \in \mathbb{D}^{n_i}$ and a valid norm $\| \cdot \|$,

$$\|H^i(X) - H^i(Y)\| < \max_r \|x^r - y^r\|$$

or

$$H^i(X) - H^i(Y) = x^r - y^r$$

$\forall r \in \{1, \ldots, n_i\}$, then $\mathcal{H}$ is called *strictly non-expansive* on $\mathbb{D}$.

iii.) If $\forall i \in \mathbb{I}$, $X \in \mathbb{D}^{n_i}$ and a valid norm $\| \cdot \|$, H^i is continuous on $\mathbb{D}^{n_i}$, then $\mathcal{H}$ is *(c)-paracontracting* on $\mathbb{D}$, if for any fixed point $\xi \in \mathbb{R}^m$ of H^i,

$$\|H^i(X) - \xi\| < \max_r \|x^r - \xi\|, \ r \in \{1, \ldots, n_i\},$$

or $X = (x, \ldots, x)$ and x is a fixed point of H^k.

iv.) If for every vector $x \in \mathbb{D}$ there is a valid norm $\| \cdot \|$ on $\mathbb{R}^m$, and $\xi \in \mathbb{R}^m$ is a fixed point of exactly the nonempty subset $\{H^i | i \in \mathbb{I}\}$ of $\mathcal{H}$, and if this entails for all $i \in \mathbb{I}$ and $X \in \mathbb{D}^{n_i}$

$$\|H^i(X) - \xi\| \leq \omega^i(X) \max_r \|x^r - \xi\|,$$

where the set of functionals $\{\omega^k_\xi(\cdot) : \mathbb{D} \rightarrow [0,1] | i \in \mathbb{I}\}$ is equicontinuous (i.e. when all continuous functions variation over a given neighborhood is equal) on

$\mathbb{D}$, & $\forall X \in \mathbb{D}^{n_i}$, for which

$$\mathbb{I}_{X,\xi} = \{i \in \mathbb{I}_\xi | \exists r \in \{1, 2, \ldots, n_i\} : H^i(x^1, x^2, \ldots, x^{n_i}) \neq x^r\},$$

is not empty, if

$$\sup_{i \in \mathbb{I}_{X,\xi}} \omega_\xi^i(x^1, \ldots, x^{n_1}) < 1$$

holds, then $\mathcal{H}$ is called *(n)-paracontracting* on $\mathbb{D}$.

■

4.3.1   Convergence Theorems

Now we can state the convergence theorems as in [57].

Theorem 4.7. *Let $\|\cdot\|$ be a strictly convex vector norm on R^m and $\mathcal{H}$ be a finite (c)-paracontracting pool on $\mathbb{R}^m$ and let $\mathcal{I}$ be regulated. Then the asynchronous iteration $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$ converges if and only if common fixed point of $\mathcal{H}$ exists.* ■

From Theorem 4.7 it is clear that $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$ converges, it converges to a common fixed point of $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$.

Theorem 4.8. *Let $\mathcal{H}$ be a (n)-paracontracting pool $\mathbb{D} \subset \mathbb{R}^m$, and assume that $\mathcal{H}$ has a common fixed point $\zeta \in \mathbb{D}$. Then a confluent asynchronous iteration $(\mathcal{H}, \mathcal{X}_0, \mathcal{I}, \mathcal{S})$ converges to a common fixed point of $\mathcal{H}$.* ■

4.4 Non-linear Asynchronous Consensus Protocol

Here the originally formulated opinion dynamics system will be formulated as an asynchronous iteration problem. Then after verifying the paracontracting property of the pool of operators and the coupling conditions, Theorem 4.7 or Theorem 4.8 can be applied [60].

4.4.1 Formulation of Asynchronous Iteration Problem

In (4.5) the opinion vector $x \in \mathbb{D} \subset \mathbb{R}^m$ is updated at every iteration step and all components of x have the same delay. However, the asynchronous updating in (4.3) has different x_i, $i \in \{1, 2, \ldots, N\}$ updating at each step. In order to represent (4.3) in the form of (4.5) we can introduce an auxiliary system with new states $y[k]$. Once the consensus problem is formulated as confluent asynchronous iterations, we can apply the associated convergence theorems.

4.4.1.1 Formulating the Asynchronous Consensus Problem as Asynchronous Iterations

We can rewrite equation (4.3) using the index sequence of updating sets $\mathcal{I}$ as in (4.6).

$$
x_i[k+1] = \begin{cases} H^i(x_1[s^1(k)], x_2[s^2(k)], \ldots, x_N[s^N(k)]) & \text{if } i = I(k), \\ x_i[k] & \text{if } i \neq I(k), \end{cases} \tag{4.6}
$$

where $s^r(k)$, $k = 0, 1, \ldots$, $r = 1, 2, \ldots, N$, with $s^r(k) \leq k \; \forall r, k$ and $I(k)$, $k = 0, 1, \ldots$ is a sequence of operators.

Without loss of generality, let us assume $s^r(k)$, $k = 0, 1, \ldots$, are selected such that all $x_r[s^r(k)]$ in (4.6) themselves are updated at time $s^r(k)$. Further assume that, the initial condition of agents, i.e., $x_r[0]$, $r = 1, 2, \ldots, N$ are mapped to $x[-r]$ such that,

$$x[-r] := x_r[0] \quad \forall r = 1, 2, \ldots, N.$$

The asynchronous iteration $(\mathcal{H}, \mathcal{Y}_0, \mathcal{I}, \mathcal{S})$ can be taken as,

$$y[k+1] = H^{I(k)}(y[\tilde{s}^1(k)], y[\tilde{s}^2(k)], \ldots, y[\tilde{s}^{n_{I(k)}}(k)]), \quad k = 0, 1, \ldots \tag{4.7}$$

where $\mathcal{H} = \{H^{I(k)} | k = 0, 1, \ldots\}$ as in (4.6), $\mathcal{I} = I(k), k = 0, 1, \ldots$, $\mathcal{S} = \{\tilde{s}^i(k) | k = 0, 1, \ldots; i = 1, 2, \ldots, N_{I(k)}\}$, with $\tilde{s}^i(k) := s^{N_{I(k)}(i)}(k), \forall k \in \mathbb{N}_0, i = 1, \ldots, N_{I(k)}$, and $\mathcal{Y}_0$ is $y[-r] := x_r[-r], r = 1, \ldots, N$. Hence from (4.7) $y[k+1] = x_{I(k)}[k+1], \forall k \in \mathbb{N}_0$.

4.4.2   Verification of the Paracontracting Property of the Pool of CUE-based Operators

4.4.2.1   The Pool of CUE-based Operators

Let $\mathscr{E}_\Theta \equiv \{\mathcal{E} | \mathcal{E} = \{\Theta, \mathcal{F}, m(\cdot)\}\}$ denote the set of all possible BoEs defined on Θ. Then the set of N BoEs corresponding to the N agents can be represented as $\mathcal{E}_i = \{\Theta, \mathcal{F}_i, m_i(\cdot)\} \in \mathscr{E}_\Theta$, $i = 1, 2, \ldots, N$. Then the CUE operator $H^i_\lhd : \mathscr{E}_\Theta^N \mapsto \mathscr{E}_\Theta$ that updates $\mathcal{E}_i$ with all $\mathcal{E}_j$, $j \in \{1, 2, \ldots, m\} \setminus \{i\}$ can be written as,

$$H^i_\lhd(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N) \equiv \mathcal{E}_i \lhd (\mathcal{E}_1 \bowtie \mathcal{E}_2 \bowtie \cdots \bowtie \mathcal{E}_{i-1} \bowtie \mathcal{E}_{i+1} \bowtie \cdots \bowtie \mathcal{E}_N),$$

where CUE is as mentioned in Section 2.1.5. We follow the claims and proofs in [19] on the paracontracting property of CUE operators. The proof of Claim 4.9 is

an extension of the proof presented in [19] giving special attention to fixed points associated with cautious agents.

Claim 4.9. *The operator $H_\lhd^i : \mathscr{E}_\Theta^N \mapsto \mathscr{E}_\Theta$ is paracontractive on $\mathscr{E}_\Theta$ with respect to any p-norm, $\| \cdot \| : \mathscr{E}_\Theta \mapsto \mathbb{R}$ given by*

$$\|\mathcal{E}\| = \left(\sum_{B \subseteq \Theta} |m(B)|^p \right)^{\frac{1}{p}}. \tag{4.8}$$

$\square$

Proof. Consider a set of N BoEs $\mathcal{E}_i \in \mathscr{E}_\Theta$, $i = 1, 2, \ldots, N$ and arbitrary fixed point $\mathcal{E}^\star \in fix(H_\lhd^i)$. In order to show the paracontractivity of the operators $H_\lhd^i$, we need to prove

$$\|H_\lhd^i(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N) - \mathcal{E}^\star\| < \max_j \|\mathcal{E}_j - \mathcal{E}^\star\|,$$

or

$$\mathcal{E}_j = \mathcal{E}^\star, \quad \text{otherwise, for } j = 1, 2, \ldots, N.$$

Let $\mathcal{B}_j \subseteq \mathcal{F}_j$ where $\forall B \in \mathcal{B}_j$, $\nexists C \in \mathcal{F}_j$ such that $C \subset B$. Now for any $B \in \mathcal{B}_j$, with a conditioning event $A \in \mathcal{F}_j$,

$$\sum_{A \in \mathcal{F}_j} m_j(A) m_j(B|A) = \sum_{B \subseteq A \in \mathcal{F}_j} m_j(A) m_j(B|A) + \sum_{B \supset A \in \mathcal{F}_j} m_j(A) m_j(B|A).$$

It can be shown that when $A \subset B$, $m(B|A) = 0$, either by applying Conditional Core Theorem (CCT) [26] or with an analysis as in Appendix C.1.

$$\sum_{A \in \mathcal{F}_j} m_j(A) m_j(B|A) = \underbrace{\sum_{A \in \mathcal{F}_i, A \supset B} m_j(A) m_j(B|A)}_{\geq 0}$$

$$+ m_j(B) \underbrace{m_j(B|B)}_{=1}$$

$$+ \sum_{A \in \mathcal{F}_j, A \subset B} m_j(A) \underbrace{m_j(B|A)}_{=0}.$$

$$\sum_{A \in \mathcal{F}_j} m_j(A) m_j(B|A) \geq m_j(B), \quad \forall B \in \mathcal{B}_j. \tag{4.9}$$

Now $\forall B \in \mathcal{F}$ and for any $p \in \mathbb{R}_{>0}$,

$$\sum_{B \in \mathcal{F}} \left| \sum_{A \in \mathcal{F}_j} m_j(A) m_j(B|A) - m \right|^p < \sum_{B \in \mathcal{F}} |m_j(B) - m(B)|^p,$$

because of (4.9), $\sum_{B \in \mathcal{F}} m(B) = 1$ and the fact that $\mathcal{E}_j \neq \mathcal{E}$. We also know that,

$$\sum_{B \subseteq \Theta} \sum_{A \in \mathcal{F}_j} m_j(A) m_j(B|A) = 1,$$

and

$$\sum_{A \in \mathcal{F}_j} m_j(A) m_j(C|A) \leq m_j(C), \quad \forall C \notin \mathcal{B}_j.$$

Furthermore, $\forall C \notin \mathcal{F}$, $\sum_{C \notin F} m(C) = 0$,

$$\sum_{C \notin \mathcal{F}} \left| \sum_{A \in \mathcal{F}_j} m_j(A) m_j(C|A) - \underbrace{m(C)}_{m(C)=0, \forall C \notin \mathcal{F}} \right|^p$$

$$= \sum_{C \notin \mathcal{F}} \left| \sum_{A \in \mathcal{F}_j} m_j(A) m_j(C|A) \right|^p$$

$$\leq \sum_{C \notin \mathcal{F}} |m_j(C)|^p$$

$$= \sum_{C \notin \mathcal{F}} |m_j(C) - m(C)|^p.$$

Hence,

$$\sum_{C \notin \mathcal{F}} \left| \sum_{A \in \mathcal{F}_j} m_j(A) m_j(C|A) - m(C) \right|^p \leq \sum_{C \notin \mathcal{F}} |m_j(C) - m(C)|^p. \qquad (4.10)$$

In [19] the contractive property of $H_{\triangleleft}^i$ has been clearly shown for widely considered all receptive agent scenario. Let us further analyze the contractive property of the CUE operators with special attention to cautious agents. First let us consider the case with N agents where there is one cautious agent and rest of the $(N-1)$ agents are receptive[2].

Without loss of generality, let the index of the cautious agent be given by c and the receptive agents are indexed with $\{1, 2, \ldots, N\} \setminus c$. Then for the CUE updates of the receptive agents we can write (4.11).

---

[2]We simply refer agents who update opinions using receptive strategy as *receptive agents* and those who update with cautious strategy as *cautions agents.*

$$\|H_{\triangleleft}^r\left(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N\right) - \mathcal{E}\|^p$$

$$= \sum_{B \subseteq \Theta} \left| \lambda_r m_r(B) + \sum_{\substack{j=1 \\ j \neq r}} \sum_{A \in \mathcal{F}_j} \kappa_{j,r} m_j(A) m_j(B|A) - m^\star(B) \right|^p \tag{4.11}$$

for $r = \{1, 2, \ldots, N\} \setminus c$, where λ_r and $\kappa_{j,r}$ are such that, $\lambda_r + \sum_{j \neq r} \kappa_{j,r} = 1$, and $m^\star(B)$ corresponds to a fixed point.

For CUE updates of cautious agents equation (4.12) holds.

$$\|H_{\triangleleft}^c\left(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N\right) - \mathcal{E}\|^p$$

$$= \sum_{B \subseteq \Theta} \left| \lambda_c m_c(B) + \sum_{\substack{j=1 \\ j \neq c}} \sum_{A \in \mathcal{F}_j} \mu_j m_c(A) m_j(B|A) - m^\star(B) \right|^p \tag{4.12}$$

where c is the index of the cautious agent and $\lambda_c + \sum_{j \neq c} \mu_j = 1$, and $m^\star$ corresponds to a fixed points as before.

In [19] it has been shown that (4.11) would lead to

$$\|H_{\triangleleft}^r\left(\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_N\right) - \mathcal{E}\| \tag{4.13}$$

$$< \max_k \left[\sum_{B \subseteq \Theta} |m_r(B) - m^\star(B)|^p \right]^{\frac{1}{p}} \tag{4.14}$$

$$= \max_k \|\mathcal{E}_k - \mathcal{E}^\star\|^p. \tag{4.15}$$

Following a similar analysis, we can show that for the pool of operators to be para-contractive and for agents to reach a consensus, the fixed point indeed should be in accordance with the opinion of the cautious agent. From (4.12),

$$\|H_{\lhd}^c\left(\mathcal{E}_1,\mathcal{E}_2,\ldots,\mathcal{E}_N\right)-\mathcal{E}\|^p$$

$$= \sum_{B\subseteq\Theta}\left|\lambda_c\big(m_c(B)-m^\star(B)\big)+\sum_{\substack{j=1\\j\neq c}}^{N}\sum_{A\in\mathcal{F}_j}\mu_j m_c(A)m_j(B|A)-(1-\lambda_c)\,m^\star(B)\right|^p$$

$$\text{let }\max_j(m_j(B|A))=m_k(B|A),\text{ then}$$

$$\leq \max_j\sum_{B\subseteq\Theta}\left|\lambda_c(m_c(B)-m^\star(B))+\sum_{\substack{j=i\\j\neq c}}^{N}\sum_{A\in\mathcal{F}_j}\mu_j m_c(B)m_k(B|A)-(1-\lambda_c)m^\star(B)\right|^p$$

$$= \max_j\sum_{B\subseteq\Theta}\left|\lambda_c(m_c(B)-m^\star(B))+\underbrace{\sum_{\substack{j=1\\j\neq c}}^{N}\mu_j}_{=1-\lambda_c}\,m_c(B)\sum_{A\in\mathcal{F}}m_k(B|A)-(1-\lambda_c)m^\star(B)\right|^p$$

$$\leq \max_j\sum_{B\subseteq\Theta}\left\{\left|\lambda_c\big(m_c(B)-m^\star(B)\big)\right|+\left|(1-\lambda_c)\Big(\sum_{A\in\mathcal{F}}m_c(B)m_k^\star(B|A)-m^\star(B)\Big)\right|\right\}$$

$$\leq \sum_{B\in\Theta}\left\{\left|\lambda_c\big(m_c(B)-m^\star(B)\big)\right|+(1-\lambda_c)\left|m_c(B)-m^\star(B)\right|\right\}$$

$$= \sum_{B\in\Theta}\left|m_c(B)-m^\star(B)\right|$$

$$\leq \|\mathcal{E}_c-\mathcal{E}^\star\|^p. \tag{4.16}$$

$$\blacksquare$$

Hence in order for the pool of operators to be paracontractive and for agents to reach a consensus under confluence conditions, the fixed point has to be on the opinion of the cautious agent after a sufficiently large number of iterative updates. For instance, if the cautious agent does not change its opinion, then to achieve a consensus all the agents has to converge to the opinion of the cautious agent.

4.4.3 Verification of Confluence Conditions

In order to avert divergence of an iterating process, one can verify the confluence conditions as given in Definition 4.5.

Claim 4.10. *[60] Consider an asynchronous iteration $(\mathcal{H}, \mathcal{Y}_0, \mathcal{I}, \mathcal{S})$ as given in (4.7) along which the sequence of iteration graphs $\mathcal{G}_1^\dagger(\varepsilon), \mathcal{G}_2^\dagger(\varepsilon), \dots$ is repeatedly jointly rooted with period ℓ. If,*

1. *When updating each agent v_{i_0} always uses its own latest state to update its current state, i.e. $s^{i_0}(k) = \max\{k_0 \leq k | I(k_0 - 1) = i_0\}$, $\forall k > \min\{k_0 \in \mathbb{N}_{\geq 0} | I(k_0) = i_0\}$ with $I_k = i_0$,*

2. *$\mathcal{I} = I(k)$, $k = 0, 1 \dots$, is regulated,*

3. *$k - s^r(k) \leq s$, $\forall k \in \mathbb{N}_{\geq 0}$, $r = 1, 2, \dots N$, for an $s \in \mathbb{N}_{\geq 0}$,*

then, $(\mathcal{H}, \mathcal{Y}_0, \mathcal{I}, \mathcal{S})$ is confluent. $\square$

Proof. See the proof of Lemma 3 in [60]. ∎

4.4.4 Nonlinear Consensus Protocol

In [19] convergence to consensus has been studies under several network configurations as

- Synchronous, fully connected,

- Synchronous, static, partially connected,

- Synchronous, dynamic, partially connected and

- Asynchronous, fully connected networks

with remarks for asynchronous, dynamic, partially connected networks. We utilize those remarks given for asynchronous, dynamic, partially connected networks to study the consensus/opinion cluster formation with multiple cautious agents (opinion leaders). But before that, let us analyze fixed points associated with cautious agents.

Following claim gives conditions under which a cautious agent becomes a fixed point.

Claim 4.11. *A cautious agent with disjoint focal elements will not change its BoE as far as it is not receiving evidence with lower cardinality elements than it currently possesses.* □

Proof. Consider set of N BoEs $\mathcal{E}_r = \{\Theta, \mathcal{F}_r, m_r(\cdot)\}$, $r = \{1, 2, \ldots, N\}$ updating opinions with each other under CUE. Without loss of generality, assume agent i is a cautious agent. In order for agent i to have disjoint focal elements $B \in \mathcal{F}_i$, such that for any two focal elements $B, C \in \mathcal{F}_i$, $B \cap C = \emptyset$. For the cautious agent not to receive any lower cardinality elements than it currently possesses, $\nexists D \in \cup_{j=1}^{N} \mathcal{F}_j$, such that $D \subset B$, $B \in \mathcal{F}_i$. Now for any $B \in \mathcal{F}_i$ of the caution agent the update equation is as (4.17).

$$Bl_i(B)_{(k+1)} = \alpha_{i,k} Bl_i(B)_k + \sum_{\substack{j=1, \\ j \neq i}}^{N} \sum_{A \in \mathcal{F}_j} \beta_{ij}(A)_k Bl_j(B|A)_k, \qquad (4.17)$$

where $\alpha_{i,k} + \sum_{j \neq i} \sum_{A \in \mathcal{F}_j} \beta_{ij}(A)_k = 1$, $\forall k = 0, 1, 2, \ldots$. Furthermore for cautious updating $\beta_{ij}(A)_k = \mu_{ij,k} m_i(A)_k$, hence we can write (4.17) as (4.18).

$$Bl_i(B)_{(k+1)} = \alpha_{i,k} Bl_i(B)_k + \sum_{\substack{j=1, \\ j \neq i}}^{N} \sum_{A \in \mathcal{F}_j} \mu_{ij,k} \underbrace{m_i(A)_k Bl_j(B|A)_k}_{\text{Cases}} \qquad (4.18)$$

Case 1. When $A = B$:

From [25] it can be easily seen that $Bl(B|A) = 1$.

Case 2. When $A \supset B$:

Using the conditionals in [25] we can see that

$Bl_j(B|A)_k = Bl_j(B)_k / (Bl_j(B)_k + Pl_j(\bar{B} \cap A)_k)$.

However for $A \supset B$, $m_i(A)_k = 0$ because $\mathcal{F}_i$ has only disjoint elements. Therefore this case will not affect the update process.

Case 3. When $A \subset B$:

Not applicable because $|A| < |B|$, i.e., A the receiving evidence has lower cardinality than the cardinality of the updating element B.

Case 4. When $A \cap B \neq \emptyset$, $A \nsubseteq B$ and $A \nsupseteq B$:

In this scenario $m_i(A)_k = 0$, hence no effect on the update.

It can be seen that only case 1 affects the update process. Then from (4.18) it can be seen that the updated belief of the cautious agent on B, i.e. $Bl_i(B)_{(k+1)}$, is just a convex combination of the previous belief $Bl_i(B)_k$ hence no change in updated beliefs. However, we picked element B arbitrarily, therefore the above arguments are valid for any element in $\mathcal{F}_i$. This indeed shows that the BoE of the cautious agent does not change with the assumptions in place. ∎

As shown in (4.16), we can associate the presence of fixed point with the cautious agent. From here onwards we use the notation such that, for the cautious agent with BoE $\mathcal{E}_c$ the associated fixed point is denoted as $\mathcal{E}_c^\star$. Hence from the Claim 4.11, it is clear that when the cautious agent's BoE $\mathcal{E}_c$ with disjoint focal elements does not receive evidence with lower cardinality elements than it currently possesses, $\mathcal{E}_c = \mathcal{E}_c^\star$.

4.4.4.1 Asynchronous, Dynamic, Partially Connected Network

The asynchronous updating for this scenario can be written as in (4.19).

$$
\mathcal{E}_j[k+1] = \begin{cases} \mathcal{H}_{\triangleleft}^{I(k)}(\mathcal{E}_{i_1}[s^{i_1}(k)], \mathcal{E}_{i_2}[s^{i_2}(k)], \dots, \mathcal{E}_{i_{n_i}}[s^{i_{n_i}}(k)]), & j = i \leftarrow I(k) \\ \mathcal{E}_j[k], & j \neq I(k) \end{cases} \tag{4.19}
$$

for all $j \in \{1, 2, \dots, N\}$ where the notation follows as mentioned in sections above.

Consider $(\mathcal{H}_{\triangleleft}, \mathcal{Y}_0, \mathcal{I}, \mathcal{S})$ as the asynchronous iteration obtained after transforming the asynchronous update given in (4.19). In [19] it has been proven that for scenarios with all receptive agents and a group of receptive agents with one cautious agent achieve consensus when the asynchronous iterations satisfy the confluence conditions. A key assumption for the asynchronous iterations to satisfy the confluence conditions, is the sequence of iteration graphs $\mathcal{G}_1^{\dagger}(\varepsilon), \mathcal{G}_2^{\dagger}(\varepsilon), \dots$ being *repeatedly jointly rooted*. Furthermore, without loss of generality, the analysis in [19] has been carried out on finite pool $\mathcal{H}_{\triangleleft}$ of CUE-based operators that contains common fixed points, and will be the case in our analysis as well.

4.4.5 Opinion Clusters with All Receptive Agents

Another interesting phenomenon that occurs, is the formation of opinion clusters among sub-groups of agents. In other words, each sub-group reaches its own "consensus" state separately. Following analysis gives conditions for the formation of opinion clusters with a group of all receptive agents.

For N receptive agents let us consider that the updating system has a sequence of interaction iteration graphs $\mathcal{G}_0^{\dagger}(\varepsilon), \mathcal{G}_1^{\dagger}(\varepsilon), \dots$ taken from the set $\mathscr{G}$ with vertex set $V =$

$v_1, v_2, \ldots, v_N$. After sufficient number of iterations q, let us assume that the sequence of interaction iteration graphs $\mathcal{G}_q^\dagger(\boldsymbol{\varepsilon}), \mathcal{G}_{q+1}^\dagger(\boldsymbol{\varepsilon}), \mathcal{G}_{q+2}^\dagger(\boldsymbol{\varepsilon}), \ldots$ can be divided in to two subsequence of interaction iteration graphs $\mathcal{G}_1^{(a)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(a)}(\boldsymbol{\varepsilon}), \ldots$ and $\mathcal{G}_1^{(b)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(b)}(\boldsymbol{\varepsilon}), \ldots$ with disjoint index sequences $\mathcal{I}^{(a)}$ and $\mathcal{I}^{(b)}$ respectively. Then we can define two asynchronous iterations using following steps,

- After iteration q we can separate the vertices V in to two disjoint sets $V^{(a)}$ and $V^{(b)}$ such that they corresponds to index sequences $\mathcal{I}^{(a)}$ be $V^{(a)}$, and $V^{(b)}$ for $\mathcal{I}^{(b)}$.

- Denote the pool of operators for $\mathcal{G}_1^{(a)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(a)}(\boldsymbol{\varepsilon}), \ldots$ as $\mathcal{H}_\triangleleft^{(a)}$, and for $\mathcal{G}_1^{(b)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(b)}(\boldsymbol{\varepsilon}), \ldots$ as $\mathcal{H}_\triangleleft^{(b)}$.

- Take the state of agents $V^{(a)}$ at time index q as the initial conditions $\mathcal{Y}_0^{(a)}$, of the sequence $\mathcal{G}_1^{(a)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(a)}(\boldsymbol{\varepsilon}), \ldots$ and the state of agents $V^{(b)}$ at q as the initial conditions $\mathcal{Y}_0^{(b)}$, of the sequence $\mathcal{G}_1^{(b)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(b)}(\boldsymbol{\varepsilon}), \ldots$.

- Let the sequence of delays involved in updating agents $V^{(a)}$ and $V^{(b)}$ be represented by $\mathcal{S}^{(a)}$ and $\mathcal{S}^{(b)}$ respectively.

Now we can identify two separate asynchronous iterations as $(\mathcal{H}_\triangleleft^{(a)}, \mathcal{Y}_0^{(a)}, \mathcal{I}^{(a)}, \mathcal{S}^{(a)})$ and $(\mathcal{H}_\triangleleft^{(b)}, \mathcal{Y}_0^{(b)}, \mathcal{I}^{(b)}, \mathcal{S}^{(b)})$. If those two iterations satisfy the conditions given in Claim 4.10 and the sequences $\mathcal{G}_1^{(a)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(a)}(\boldsymbol{\varepsilon}), \ldots$ and $\mathcal{G}_1^{(b)}(\boldsymbol{\varepsilon}), \mathcal{G}_2^{(b)}(\boldsymbol{\varepsilon}), \ldots$ are repeatedly jointly rooted, then by Theorem 4.8 there will be two separate opinion clusters each having its own consensus. It should be noted that when the individual consensus reached by separate opinion clusters, are the same in value (which is not the case in general), it can be counted under one global consensus.

4.4.6 Consensus/Opinion Clusters with One Cautious Agent Embedded in a Group of Receptive Agents

In the presence of a cautious agent the nature of the common fixed point depends on the fixed point reached by the cautious agent $\mathcal{E}^\star$, as shown in (4.16). Furthermore, Claim 4.11 gives conditions where a cautious agent does not change its opinion thus a fixed point $\mathcal{E}_c^\star$.

Let us consider that, among the N agents, one agent is cautious and other $N-1$ agents are receptive. Without loss of generality we can assume that agent corresponding to node v_c is the cautious agent. Following a similar notation as with all receptive agent scenario, we can see that the agents will reach a consensus if the sequence of interaction iteration graphs $\mathcal{G}_0^\dagger(\varepsilon), \mathcal{G}_1^\dagger(\varepsilon), \ldots$ are repeatedly jointly rooted and satisfy conditions in Claim 4.10. This has been analyzed in [19] and with the notation here, the consensus will indeed be on the fixed point of the cautious agent, i.e., $\mathcal{E}_c^\star$.

In contrast, when the sequence of interaction iteration graphs $\mathcal{G}_0(\varepsilon), \mathcal{G}_1(\varepsilon), \ldots$ are not repeatedly jointly rooted, but after finite number of iteration that sequence divides into two subsequences $\mathcal{G}_1^{(a)}(\varepsilon), \mathcal{G}_2^{(a)}(\varepsilon), \ldots$ and $\mathcal{G}_1^{(b)}(\varepsilon), \mathcal{G}_2^{(b)}(\varepsilon), \ldots$ as before, we can find opinion clusters with the given conditions.

4.4.7 Consensus/Opinion Clusters with Multiple Cautious Agents Embedded in a Group of Receptive Agents

Here we study the situation with two cautious agents in a group of receptive agents. The assertion for studying a situation with two cautious agents is, it can be generalized to analyze multiple cautious agents scenario. Let us consider N agents, with two

cautious agents and $(N-2)$ receptive agents. Without loss of generality we can assume that the two cautious agents are represented with vertices v_{c_1} and v_{c_2}.

Now for conditions mentioned in Claim 4.11 the two fixed points corresponding to cautious agents v_{c_1} and v_{c_2} are denoted as $\mathcal{E}_{c_1}^{\star}$ and $\mathcal{E}_{c_2}^{\star}$ respectively. In order for all agents to reach a consensus the fixed points corresponding to the cautious agents have to be the same, i.e., $\mathcal{E}_{c_1}^{\star} = \mathcal{E}_{c_2}^{\star}$. Furthermore, when the sequence of interaction iteration graphs $\mathcal{G}_0^{\dagger}(\varepsilon), \mathcal{G}_1^{\dagger}(\varepsilon), \ldots$ are repeatedly jointly rooted and satisfy the conditions in Claim 4.10, then all the agents will reach a consensus.

However, in general scenario it is unlikely that the two cautious agents will reach the same fixed point, which indeed leads to either polarization or clustering. Polarization can be seen as a special case of clustering where only two clusters of agents are reaching consensus individually. We can give conditions when polarization occurs with two cautious agents as follows.

After finite number of iterations, consider that the sequence of interaction iteration graphs can be divided into two subsequences $\mathcal{G}_1^{(a)}(\varepsilon), \mathcal{G}_2^{(a)}(\varepsilon), \ldots$ and $\mathcal{G}_1^{(b)}(\varepsilon), \mathcal{G}_2^{(b)}(\varepsilon), \ldots$ each embedding a cautious agent v_{c_1} and v_{c_2} respectively. When each interaction sequence $\mathcal{G}_1^{(a)}(\varepsilon), \mathcal{G}_2^{(a)}(\varepsilon), \ldots$ and $\mathcal{G}_1^{(b)}(\varepsilon), \mathcal{G}_2^{(b)}(\varepsilon), \ldots$ is repeatedly jointly rooted and satisfies the conditions in Claim 4.10, then polarization will occur with each subgroup reaching consensus on $\mathcal{E}_{c_1}^{\star}$ and $\mathcal{E}_{c_2}^{\star}$.

4.5 Chapter Summary

In this chapter we modeled the formation of consensus of agent opinions as convergence to a common fixed point of paracontraction operators. We exploited the

paracontraction nature of the CUE in opinion updating and focused on the conflu-
ence conditions which need to be satisfied for the existence of a common fixed point.
In particular, the paracontractive nature of the CUE operator under the presence of
cautious agents gave insights into the nature of consensus with opinion leaders.

Event-based discrete time indexing allows one to represent temporal coupling of agents
with iteration graphs. Iteration graphs can effectively represent communication delays
which are otherwise difficult to capture with traditional graphs. This indeed led the
analysis of opinion dynamics in more general setting having asynchronous, dynamic,
partially connected networks.

Part III

Network Generation for Consensus

Chapter 5

Random Network Generation for Consensus-Based Distributed Decision Making

5.1 Network Generation for Multi-Agent Systems

Enforcing a consensus in a continuous opinion dynamic system is pivotal in a wide range of multi-agent distributed decision-making applications. For instance, when autonomous mobile robots soldiers collaborate to complete a task, forming a consensus is important for them to cofunction as one unit. For a given agent opinion distribution it is the confidence bound of agents which governs the properties of the communication topology among agents in order to reach a consensus. If the confidence bound is less then it will demand more connections among agents to compensate for the links that get pruned due to confidence bound notions. The conditions sufficient for consensus have been discussed in Chapters 3 and 4. In this chapter, the conditions which lead to a consensus with our opinion model while accounting for the bound of confidence notions are analyzed.

The network generation mechanism has to be executed by autonomous agents each having confined prior information, without access to a central node. A naive solution

is to suggest a completely connected network where each autonomous agent makes communication connections with all other agents in the network. But the number of edges such a strategy produces is prohibitive and is in the order of $\mathcal{O}(N^2)$, where N is the number of agents. Moreover, the actual number of connections required to reach a consensus is significantly lower than the amount of edges generated by the completely connected network protocol. The network generation protocol proposed in this chapter produces edges in the order of $\mathcal{O}(N \ln N)$ and more importantly accounts for the notion of bound of confidence in SJT.

While SJT is intuitive in a social network setting one may question the utility of bound of confidence notions in a scenario like navigation of autonomous mobile robots. We can set the bound of confidence parameter to ignore a 'faulty' sensor which shows drastically different values from all its neighbor sensors. This will prevent the outlier information provided by a corrupted agent affecting the other agent opinions. The case example in Section 5.7 further clarifies the important role the bound of confidence notion may play in such a setting.

5.1.1   Previous Work on Network Generation

Networks such as *phone chain of closets* [62] which can be viewed as agents being connected in a circle (or a line) have edges in the order of $\mathcal{O}(N)$. Assigning ordered tags to each agent, one can design a protocol for autonomous agents to communicate and reach a consensus. However, a disruption in communication in two links either due to bound of confidence notions or simply due to network failures, results in the network being disconnected, thus obstructing a consensus state being achieved.

Even though the standard Erdős-Rényi (ER) random graphs [63] may not reflect certain important characteristics observed in real networks (e.g., clustering and assortative mixing) [34], random graphs are perhaps the most studied models of networks for the purposes of empirical evaluation of models and algorithms, significance testing of real data, and verification of theoretical results [34, 64, 65]. The main reason is that many important attributes of random graphs have theoretically derived expressions.

For instance, an ER random graph $\mathcal{G}(N, p_a)$ yields a strongly connected network when $p_a >> \ln N/N$, where p_a is the independent edge formation probability between a pair of agents [34]. However, when the bound of confidence of each agents' opinions are taken into consideration, the ER random may split into two or more components otherwise strongly connected. Hence finding the conditions such that the agent communication network remains connected even under bound of confidence notions is essential for the agents to reach a consensus state.

5.1.2 Contributions Under Network Generation

Our proposed ER random network generation mechanism can nurture a consensus state even when the agent interactions are dictated by confidence bound of agents. Let us consider that the agents are embedded in an ER random graph $\mathcal{G}(N, p_a)$. Then, we show the following:

- When agents who invoke bound of confidence interact with each other, the 'effective' opinion exchange network can be considered as a realization of the ER random network $\mathcal{G}(N, p_c)$, where the edge formation probability $p_c < p_a$. Indeed p_c is a function of ε and the probability density function (PDF) of the opinion distance between agents. Here, $\varepsilon \in \mathbb{R}^N$ is a vector that represents the confidence bound of each agent. Throughout this chapter, for analytical

convenience we consider the homogeneous setting where all agents have the same confidence bound ε.

- A theoretical lower bound for the edge formation probability p_a of the basic underlying ER random graph $\mathcal{G}(N, p_a)$, such that a consensus state can be nurtured for a given bound of confidence value ε even under the 'effective' opinion exchange network of $\mathcal{G}(N, p_c)$.

- We employ non-parametric kernel density estimation methods to avoid the necessity of obtaining a parametric form of the PDF of the distance between agent opinions.

The proposed method generates ER random graphs having edges in the order of $\mathcal{O}(N \ln N)$. Even though the order of edges is slightly higher compared to networks like *phone chain of closets*, our proposed method accounts for notions in SJT and can be designed to tolerate a given amount of network failures.

5.2 DST Modeling of Opinions

In the analysis that follows, we consider N agents embedded in a directed graph $\mathcal{G}$. The agent opinions are modeled as explained in Section 3.1 with DST BoEs, viz., the opinion of the i-th agent at time instant $k \in \mathbb{N}_{\geq 0}$ is taken to be captured via the BoE $\mathcal{E}_{i,k} = \{\Theta, \mathcal{F}_{i,k}, m_i(\cdot)_k\}$, $i \in \overline{1, N}$. We assume that the agent opinion BoEs are associated with the identical FoD Θ. The opinion update model accounts for bounded confidence notions as in Section 3.1.1. Opinion exchange is CUE-based as described in Section 3.1.2.

Let us consider the CUE-based fusion operation as defined in Definition 2.11 in Chapter 2. Without loss of generality we can consider a *discrete event-based time* index as

explained in Section 4.1, where only one agent updates its state at any given discrete time index k^d. One can think of this as an 'expansion' of time axis such that only one CUE update operation is carried at a particular time index k^d. Furthermore, let $\mathbb{I}$ denotes a finite index set from the positive integers, i.e., $\mathbb{I} \subset \mathbb{N}$. $\mathbb{I}$ can be considered as the set of unique indices of agents' fusion operators. In our analysis we use paracontraction notions from [19] and Chapter 4.

5.3   Sensitivity of Confidence Bound

Suppose the underlying graph $\mathcal{G}(N, p_a)$ is generated based on the ER random graph model with edge formation probability p_a. Based on the confidence bounds, the agent connections are given by graph $\mathcal{G}_k^\dagger(\boldsymbol{\varepsilon})$, as in (3.3). For two agents with BoEs $\mathcal{E}_i$ and $\mathcal{E}_j$ embedded in $\mathcal{G}_k^\dagger(\boldsymbol{\varepsilon})$, let us denote the probability that their opinions are within the confidence bound of each other as $p_b(\varepsilon)$. Using the distribution of opinions we estimate $p_b(\varepsilon)$ as

$$p_b(\varepsilon) = \mathrm{P}\left(\|\mathcal{E}_i - \mathcal{E}_j\|_J < \varepsilon\right), \tag{5.1}$$

where $\|\cdot\|_J$ is as in Definition 2.12. Hence, we can write an expression for the probability $p_b(\varepsilon)$ as

$$p_b(\varepsilon) = \mathrm{P}\left(\sqrt{0.5(\mathbf{m}_i - \mathbf{m}_j)^T \mathcal{D}(\mathbf{m}_i - \mathbf{m}_j)} < \varepsilon\right) \tag{5.2}$$

where $\mathcal{D}$ is as defined in Definition 2.12.

In [66], it has been assumed that the distance among agents does not increase after an interaction, which is referred to as *private marginal benefit*. Based on that assumption

we can find conditions to apply Claim 1 taken from [19] in order to facilitate the infrastructure for a possible consensus among agents as follows.

5.3.1 Convergence of Synchronous, Dynamic, Partially Connected Network

With discrete event-based time indexing, each agent in the multi-agent system of N agents can be considered to iteratively update its state $\mathcal{E}_i[k^d] = \{\Theta, \mathcal{F}_i[k^d], m_i(\cdot)[k^d]\}$ at discrete event-based time index k^d via some fusion operator $H_\lhd^i$. Note the difference in time index from k to k^d to represent the discrete time indices and it will be used in Section 5.3.1 mainly for the application of paracontraction theories. In all other sections the time index is represented by k. In our analysis since we consider a partially connected, synchronous, dynamic network the opinion updating process in terms of fusion operator and discrete event-based time indices can be given as

$$\mathcal{E}_i[k^d + 1] = \begin{cases} H_\lhd^{I[k^d]}(\mathcal{E}_{i_{1,k^d}}[k^d], \mathcal{E}_{i_{2,k^d}}[k^d], \ldots, \mathcal{E}_{i_{n_i,k^d}}[k^d]) & , \quad \text{for } i = I[k^d], \\ \mathcal{E}_i[k^d], & \text{otherwise,} \end{cases} \tag{5.3}$$

where $\{i_{1,k^d}, i_{2,k^d}, \ldots, i_{n_i,k^d}\} = \mathbb{I}_{i,k^d}$ is as explained in (3.4) and (3.5).

The following result is from [19]:

Claim 1 (Convergence: partially-connected network). If $\mathcal{I}$ is regulated, iterated updating in (5.3) converges as long as the graph union of agent interaction topologies of all agents is connected. ∎

Thus, essentially we need to satisfy two main conditions. The first is to make the index sequence $\mathcal{I}$ regulated, which can be achieved when all agents periodically update from

their neighbors. The other condition to make the graph union of agent interaction topologies of all agents connected, which we explore in the following section.

5.3.2 Connected Graph Union in $\mathcal{G}_k^\dagger(\varepsilon)$

Let us denote the probability of two arbitrarily selected agents being connected in $\mathcal{G}_k^\dagger(\varepsilon)$ as p_c. Let the random variable C represent the connectedness among two arbitrarily selected agents in $\mathcal{G}_k^\dagger(\varepsilon)$. $C = 1$ represents a connection and $C = 0$ represents no connection. However, the underlying graph $\mathcal{G}$ is been generated as an ER random graph with edge formation probability of p_a. Let A represent the connectedness among two arbitrarily selected agents in $\mathcal{G}$. Then A is a random variable with Bernoulli distribution,

$$P(A = 1) = p_a = 1 - P(A = 0).$$

Furthermore, let B represent the random variable where arbitrarily selected two agents' opinions are within the confidence bound of each other. Hence B also follows a Bernoulli distribution as

$$P(B = 1) = p_b(\varepsilon) = 1 - P(B = 0).$$

Now C can be written as $C = AB$ because $C = 1$ only when both A and B equal to one. However, A and B are mutually independent Bernoulli random variables, hence the product C is also Bernoulli [67]. Then C is a random variable with Bernoulli distribution,

$$P(C = 1) = p_c = 1 - P(C = 0),$$

where $p_c = p_a p_b(\varepsilon)$. In other words, we can think of $\mathcal{G}_k^\dagger(\varepsilon)$ as an ER random graph generated with edge formation probability of p_c.

For an ER random graph, the threshold for the connectedness is $p_c > \ln N/N$ with probability tending to one [68], where N is the number of agents embedded in the graph $\mathcal{G}$. Thus for a given $p_b(\varepsilon)$ if we select a sufficiently large p_a such that $p_a p_b(\varepsilon) > \ln N/N$, then the initial graph $\mathcal{G}_0^\dagger(\varepsilon)$ will be connected with probability tending to one. Then based on the assumption of *private marginal benefit* we can consider that $\mathcal{G}_k^\dagger(\varepsilon)$ continues to be connected $\forall k \in \mathbb{N}_{\geq 0}$ thus satisfying the conditions for Claim 1. However, the convergence of the sequence in (5.3) indeed implies a consensus among agents [19].

One can think of the above analysis as a bottom-up approach of assessing the emergence of a consensus given confidence bound ε, opinions distribution and underlying graph $\mathcal{G}$. However, with a top-bottom approach we can reverse the analysis and find out the required structure of the underlying graph $\mathcal{G}$ in order to yield a consensus among agents as discussed next.

5.4 Network Generation for Consensus

From the analysis in Section 5.3 it can be seen that under the assumption of private marginal benefit, if we can generate an initial graph $\mathcal{G}$ such that we satisfy the condition of $p_c > \ln N/N$ in $\mathcal{G}_0^\dagger(\varepsilon)$ then there will be consensus among agent opinions almost surely. Suppose the PDF of the agent opinion distances are given by $f_z(z)$ where Z is

$$Z = \sqrt{0.5(\mathbf{m}_i - \mathbf{m}_j)^T \mathcal{D}(\mathbf{m}_i - \mathbf{m}_j)}, \qquad\qquad i,j \in \overline{1,N}. \qquad (5.4)$$

Using $f_z(z)$ we can estimate $p_b(\varepsilon)$ as

$$p_b(\varepsilon) = \int_0^\varepsilon f_z(z)\, \mathrm{d}z. \tag{5.5}$$

Hence if we can satisfy the condition

$$p_a > \frac{\ln N}{N p_b(\varepsilon)}, \tag{5.6}$$

a consensus among agent opinions could emerge. Note that in case of the value of minimum probability p_a is greater than 1 we simply omit this result, because it is not possible to generate an underlying graph to reach a consensus under such condition. For instance, when the agents' confidence bounds are almost zero, i.e., the agents are not willing to update from other agents, it is not possible for a consensus to emerge among agents in general, even with a completely connected underlying network $\mathcal{G}$. This is because, when bound of confidence notions are taken into account $\mathcal{G}_k^\dagger(\varepsilon)$ will not be connected.

Next we consider a special case of uniformly distributed agent opinions for analytical convenience followed by a more general representation of agent opinions.

5.4.1 Consensus with Uniformly Distributed Agent Opinions

Consider N agents possessing opinions which are uniformly distributed. Let $\Theta = \{\theta_1, \theta_2\}$ be the FoD of agent opinions. Then the $m_i(\theta_1)$ of each agent $i \in \overline{1, N}$ has a distribution as

$$f\big(m_i(\theta_1)\big) = \begin{cases} 1, & \text{for } 0 \leq m(\theta_1) \leq 1, \\ 0, & \text{otherwise,} \end{cases} \tag{5.7}$$

and $m_i(\theta_2) = 1 - m_i(\theta_1)$. In other words, one can see this as sampling from the beta distribution Beta(1,1). Then following (5.4) the random variable Z for this case can be simplified as

$$Z = |m_i(\theta_1) - m_j(\theta_1)|, \qquad\qquad i, j \in \overline{1, N}. \qquad (5.8)$$

Applying (5.8) in (5.5) we get,

$$\begin{aligned} p_b(\varepsilon) &= \mathrm{P}(|m_i(\theta_1) - m_j(\theta_1)| < \varepsilon), \\ &= \mathrm{P}(-\varepsilon < m_i(\theta_1) - m_j(\theta_1) < \varepsilon). \end{aligned} \qquad (5.9)$$

But we know that $m_i(\theta_1)$ and $m_j(\theta_2)$ are independent and uniformly distributed random variables. Hence,

$$f(m_i(\theta_1) - m_j(\theta_1)) = f(m_i(\theta_1)) * f(-m_j(\theta_1)), \qquad (5.10)$$

where $f(m_i(\theta_1))$, $f(-m_j(\theta_1))$ take the form in (5.7). Let $z_{ij} = m_i(\theta_1) - m_j(\theta_1)$ and from the convolution in (5.10) we get

$$f(z_{ij}) = \begin{cases} z_{ij} + 1, & \text{for } -1 \le z_{ij} < 0, \\[2mm] 1 - z_{ij}, & \text{for } 0 \le z_{ij} \le 1. \end{cases} \qquad (5.11)$$

Using (5.11) in (5.9) we get

$$\begin{aligned} p_b(\varepsilon) &= 2 \int_0^\varepsilon (1 - z_{ij}) \, \mathrm{d}z_{ij}, \\ &= \left(2\varepsilon - \varepsilon^2\right). \end{aligned} \qquad (5.12)$$

From (5.6), if we generate the underlying graph $\mathcal{G}$ as an ER random graph with edge formation probability p_a such that

$$p_a > \frac{\ln N}{N\varepsilon(2-\varepsilon)}, \tag{5.13}$$

then we should see a consensus among agent opinions almost surely.

5.4.2 Consensus with General DS Agent Opinions

In order to capture nuanced agent opinions it is crucial to utilize general DS BoEs for agent opinion representation. Let the FoD for all N agents be $\Theta = \{\theta_1, \theta_2, \ldots, \theta_M\}$ and each agent opinion is represented with a DS BoE $\mathcal{E}_i = \{\Theta, \mathcal{F}_i, m_i(\cdot)\}$, $i \in \overline{1, N}$. One can use Dirichlet Distribution, often denoted as $\text{Dir}(\boldsymbol{\alpha}_D)$, where $\boldsymbol{\alpha}_D \in \mathbb{R}^{2^M}_{\geq 0}$, to represent the distribution of agent opinions. For the mass vector $\mathbf{m}$ as given in (2.1), let the $\boldsymbol{\alpha}_D$ be

$$\boldsymbol{\alpha}_D = [\alpha_{\theta_1}, \ldots, \alpha_{\theta_M}, \alpha_{\theta_1\theta_2}, \ldots, \alpha_{\theta_1\theta_M}, \alpha_{\theta_1\theta_2\theta_3}, \ldots, \alpha_{\theta_1\theta_2\theta_M}, \ldots, \alpha_\Theta] \in \mathbb{R}^{2^M-1}_{[0,1]}. \tag{5.14}$$

Then for each agent BoE the opinions can be sampled from the Dirichlet distribution

$$f(\mathbf{m}; \boldsymbol{\alpha}_D) = \frac{\Gamma\left(\sum_{A \subseteq \Theta} \alpha_A\right)}{\prod_{A \subseteq \Theta} \Gamma(\alpha_A)} \prod_{A \subseteq \Theta} (m(A))^{\alpha_A - 1}. \tag{5.15}$$

Similar to the analysis with uniformly distributed agent opinions in Section 5.4.1, one can proceed with the analysis by using convolution to directly find the probability density function of agent opinion distances $f_Z(z)$, where Z is as given in (5.4). However, we have utilized non-parametric kernel density estimation methods to directly

estimate $f_Z(z)$ as explained in 5.5. Using $f_Z(z)$, $p_b(\varepsilon)$ can be found as in (5.5), hence a lower bound for p_a as in (5.6).

5.5 Non-Parametric Estimation of Agent Opinion Distance Distribution

Nonparametric (NP) p.d.f. estimation techniques possesses the ability to detect structures which sometimes remain undetected by traditional parametric estimation techniques [69]. With parametric form of the agent opinions distribution estimation we have to assume a parametric form of the opinion distribution. For instance, one may assume a multivariate Gaussian distribution and try to estimate the distribution parameters to fit the observed opinion data. However, the assumed parametric distribution might not be the best form to represent the underlying distribution and worst of all the underlying distribution might not be confined to a standard parametric form. We utilize non-parametric kernel density estimation methods where we do not have to assume the functional form of the distribution [70]. However we have to select a proper kernel and a bandwidth for kernel density estimation with non-parametric techniques.

Using the notation in [70] we can elaborate more on nonparametric p.d.f. estimation. Consider a particular data point z. The a kernel $K(z)$ has to satisfy, $K(z) \geq 0$, $\forall z \in \mathbb{R}$, and $\int K(z)dz = 1$. A kernel which satisfies this requirement is given in (5.16).

$$K_h(z_i, z) = k\left(\frac{z_i - z}{h}\right) \tag{5.16}$$

where, h refers to as the bandwidth (which is different to the meaning in spectral analysis) and $k(\cdot)$ could be any univariate kernel, such as, uniform, Epanechnikov, biweight, triweight, Gaussian etc . . . .

While the selection of the kernel is an important choice, its overall effect on the results is limited when compared to the effect that the bandwidth can have. The two popular data-driven method for bandwidth selection are least-squares cross-validation (LSCV) and likelihood cross-validation (LCV). We have selected Gaussian kernel with cross validation based bandwidth for the non-parametric estimation of the PDF $f_Z(z)$ [71].

Let $f_Z(z)$ denote the p.d.f. of z and estimator of p.d.f. by $\hat{f}_Z(z)$. Assume that we were given a data-set with n instances, $\{z_1, z_2 \cdots, z_n\}$. Then the estimated p.d.f. $\hat{f}_Z(z)$ can be found as:

$$\hat{f}_Z(z) = \frac{1}{nh} \sum_{i=1}^{n} K_h(z_i, z). \tag{5.17}$$

5.6 Empirical Evaluations and Discussion

In this section we present simulations of agent networks where agent opinions are captured with DS BoEs. The agents in all the simulations employ a CUE-based opinion update strategy with $\alpha_i = 0.5$, $\forall i \in \overline{1, N}$. The agents' bounds of confidence are taken as identical, i.e., $\varepsilon_i = \varepsilon$, $\forall i \in \overline{1, N}$. Note that, even though the agents are embedded in an underlying static network, the agents have to accommodate the bounds of confidence as the opinions are updated, hence creating a dynamic network $\mathcal{G}_k^{\dagger}(\varepsilon)$.

We have selected 400 agents with identical FoDs $\Theta = \{\theta_1, \theta_2\}$. The opinions of the agents have been sampled from a Dirichlet distribution $f(\mathbf{m}; \boldsymbol{\alpha}_D)$ as in (5.15), where
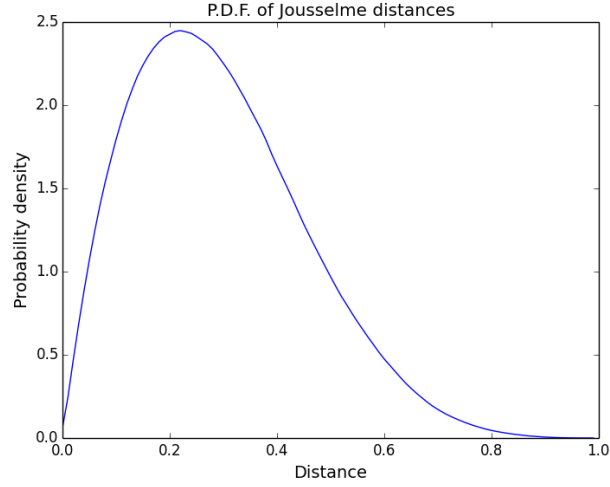
FIGURE 5.1: PDF of opinion distances when the agents opinions are distributed with Dirichlet distribution $f(\mathbf{m}; \boldsymbol{\alpha}_D)$ as in (5.15), where $\boldsymbol{\alpha}_D = [1, 1, 1]$.

$\boldsymbol{\alpha}_D = [1, 1, 1]$ and $\mathbf{m} = [m(\theta_1), m(\theta_2), m(\theta_1\theta_2)]$. The following three steps explain the estimation of edge formation probability p_a for $\mathcal{G}_0^\dagger(\varepsilon)$.

Step I Estimating the p.d.f. of opinion distances

As explained in Section 5.5 non-parametric kernel density estimation method [70] has been utilized to find $f_Z(z)$. For the chosen opinion distribution $f(\mathbf{m}; \boldsymbol{\alpha}_D)$, the probability density function of agent opinion distances $f_Z(z)$ is shown in Fig. 5.1.

Step II Calculate the probability $p_b(\varepsilon)$

After estimating the PDF of opinion distances $f_Z(z)$ we can calculate the probability $p_b(\varepsilon)$ of two randomly chosen agent opinions are within the bound of confidence ε using (5.5).

Step III Estimating the probability p_a

Using $p_b(\varepsilon)$ the minimum edge formation probability p_a for $\mathcal{G}_0^\dagger(\varepsilon)$ to be connected can be estimated using (5.6), where $N = 400$.
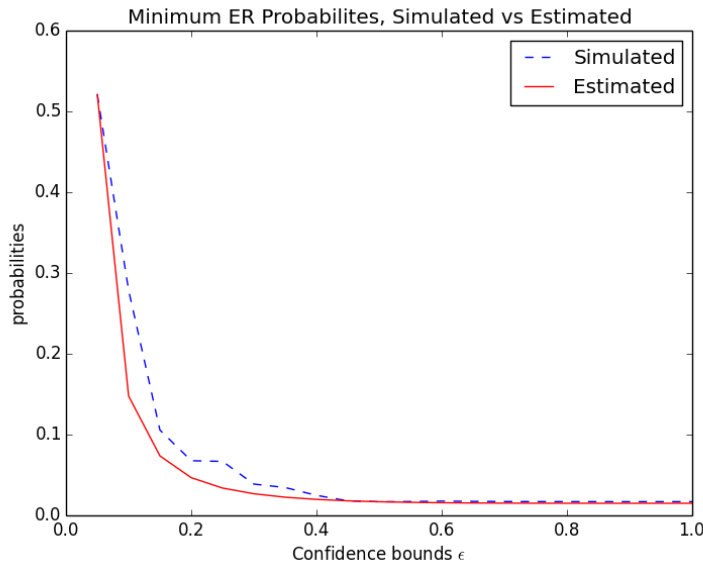
FIGURE 5.2: Comparison of estimated minimum edge formation probabilities p_a of $\mathcal{G}$ for in ER random graph with the minimum values obtained with simulations for $\mathcal{G}_0^\dagger(\varepsilon)$ to be a connected graph. The agents opinions are distributed with Dirichlet distribution $f(\mathbf{m}; \boldsymbol{\alpha}_D)$ as in (5.15), where $\boldsymbol{\alpha}_D = [1, 1, 1]$.

We have simulated the scenario with 400 agents with opinions sampled from $f(\mathbf{m}; \boldsymbol{\alpha}_D)$ to find the minimum edge formation probability p_a for different confidence bounds ε. Fig. 5.2 shows a comparison of minimum edge formation probability p_a found with estimations and simulations. As can be seen from Fig. 5.2 the minimum $p_b(\varepsilon)$ for connectedness of $\mathcal{G}_0^\dagger(\varepsilon)$ is closer to the estimated value specially when the bound of confidence ε is higher.

With the connectedness of $\mathcal{G}_0^\dagger(\varepsilon)$ and the assumption of *private marginal benefit* we can assume that the graph union of agent interaction topologies of all agents is connected. When the agent updating sequence $\mathcal{I}$ is regulated we can apply Claim 1. Claim 1 states a convergence of sequence in (5.3) and as discussed in 5.3.2 the convergence in (5.3) implied a consensus. Hence, if we generate the underlying graph $\mathcal{G}$ such that even after the bound of confidence notions the graph union of all agent interaction topologies remain connected, then a possible consensus should emerge. Fig. 5.3 shows
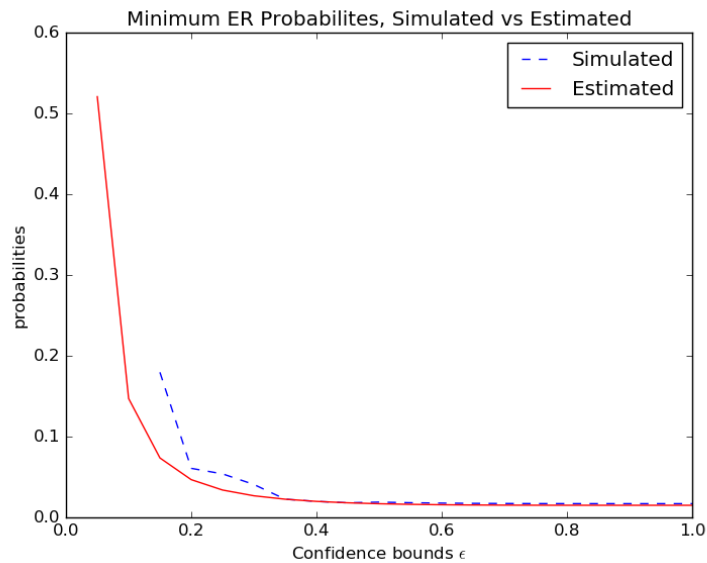
FIGURE 5.3: Comparison of estimated minimum edge formation probabilities p_a of $\mathcal{G}$ for in ER random graph with the minimum values obtained with simulations for agents to reach a consensus. The agents opinions are distributed with Dirichlet distribution $f(\mathbf{m}; \boldsymbol{\alpha}_D)$ as in (5.15), where $\boldsymbol{\alpha}_D = [1,1,1]$.

the comparison of estimated and simulated values of minimum ER edge formation probability p_a for the agents to reach a consensus.

Fig. 5.3 shows that the simulated minimum value of p_a for consensus is closer to the theoretical estimation, particularly for larger values of confidence bound ε. However, for smaller value of ε the minimum values obtained for p_a from simulations tends to be larger than the theoretically estimated values. We hypothesize that one reason behind the difference between simulated and estimated values is the assumption of *private marginal benefits*. As per the observations the *private marginal benefit* assumption seems to be valid for larger values of confidence bounds ε. However, further analysis has to be done to assess the validity of this hypothesis specially for lower values of the confidence bounds.

5.7 Case Example: Autonomous Mobile Robot Navigation

5.7.1 Scenario

Consider a platoon of autonomous mobile robot soldiers who are required to stay together and navigate as a group. Hence, the group must establish as consensus opinion regarding the direction to navigate. To keep the case example simpler let us assume the robots are navigating toward the North by default and based on their sensor information they only need to take the decision either to turn toward the East or the West. We use the DS framework to model opinions, because we need to handle the situation where the agents may not have clear sensor information hence do not have a firm opinion on the direction initially. Let the FoD of agents be denoted by $\Theta = \{\theta_E, \theta_W\}$. $m_i(\theta_E)_k$ and $m_i(\theta_W)_k$ represent the mass of the opinion of i-th agent at time instance k on turning to East and West respectively. The doubleton $m_i(\theta_E\theta_W)_k$ captures the ambiguity of the i-th agent at time instance k on whether to turn East or West.

Let us assume the platoon consists of $N = 250$ autonomous robots. Communicating through a completely connected underlying network is costly in terms of the number of connections (62250 edges in total) and indeed unnecessary. Let us go through some possible network topologies that the platoon can establish on their own. Before that let us briefly discuss the importance of confidence bound and fault tolerance in a scenario like this.

5.7.2 Role of Confidence Bound

While the confidence bound notion is intuitive for opinion exchange among humans, one may ask, what role does bound of confidence notion play for mobile robots or any other distributed sensor scenario? Furthermore, if the objective is to establish a consensus opinion, what purpose does it serve to deploy the bound of confidence notion among the mobile robots? To answer that, let us consider a case of having an agent with faulty sensors, thus generating a drastically different "opinion". If this "errorneous" opinion is used by the neighboring agents within their update processes, the converged or consensus state will certainly be affected. However, by deploying a bound of confidence on agent updates, the system can avoid obvious outliers.

5.7.3 Fault Tolerance

Fault tolerant design is crucial for a mission critical system to continue operating properly even in an event of failure of some communication links or some agents. For instance, let us consider the agents are embedded in a circular topology. While it is difficult to design a protocol for autonomous agents to circularly connect themselves considering the bound of confidences (which depend on the dynamic opinions), the topology is extremely vulnerable to failures. An attacker would only need to turn down two mobile agents or two communication links to separate the system into two groups making the group to reach a consensus improbable. Hence, the circular topology is clearly not a fault tolerant design.

To surpass the vulnerability with circular design, one might propose a lattice like underlying network. However, an attacker could still break the group into small partitions by targeting selected agents. The attacker can achieve this by destroying agents in a line. Likewise any predictive standard connected topology can be separated with

a predefined attack strategy. But what if we generate the network with a random strategy? Then, it would be difficult for the attacker to separate the group with a preplanned strategy. Let us now discuss how we can utilize the proposed method in this chapter to build a random topology among autonomous agents accounting for the bound of confidence and fault tolerance.

5.7.4   Design with Proposed Method

Let us assume that we need the platoon of 250 mobile robots to be properly functional, even with the loss of 50 agents. Furthermore, suppose the probability distribution $f_Z(z)$ has been estimated as explained in Section 5.4 using training data. The level of confidence bound ε for mobile agents is a design parameter.

For this example, let us pick the confidence bound as $\varepsilon = 0.4$ and the agents opinions are sampled from a Dirichlet distribution $f(\mathbf{m}, [1, 1, 1])$. Then using (5.5) we can estimate p_b the probability of randomly picked two agents' opinions are within bound of confidence. For $\varepsilon = 0.4$ the value of $p_b(\varepsilon = 0.4) = 0.75$. Since we need to design the system to function properly even with a loss of 50 agents, we use $N = 200$ and the value of $p_b(\varepsilon)$ in (5.6) and find the minimum edge formation probability p_a to be 0.0353 (approx.).

The autonomous agents could form the network as a ER random graph with edge formation probability $p_a = 0.0353$ (or slightly higher for robustness). The network will have communication links around 2198 (total number of links $\times p_a$). Due to the random nature of the network generation, it is not possible for an attacker to separate the group with a predefined plan based on the structure of the graph. Assuming the failures/attacks occur for randomly selected agents, the agent system will continue to

reach a possible consensus and navigate as one group even in the loss of some agents (up to 50, approximately for this scenario).

5.8   Chapter Summary

In this chapter, we have presented a network generation method to facilitate a possible consensus among agents with iterating opinion exchanges. Utilizing non-parametric techniques on prior data (or using the knowledge of agent opinion distribution), an estimation of the probability of two random agents' opinion within a certain confidence $p_b(\varepsilon)$ can be found. Along with probability p_b, using the knowledge on the edge formation probability for phase transition of Erdős-Rényi random graphs p_c, we can estimate the edge formation probability p_a of the original static underlying agent network $\mathcal{G}$. With the assumptions in place, we were able to compare the estimated and simulated results for p_a. The validity of the proposed mechanism was demonstrated for relatively larger values of confidence bounds ε.

Equipped with this network generation mechanism, in a distributed multi-agent environment, autonomous agents could form edges with probability p_a as a ER graph. The decision of the group will be taken based on the consensus opinion of all agents. A case example presented in Section 5.7 explains how the proposed method can be used in a scenario of the navigation of autonomous mobile robot soldiers. The proposed method accounts for bound of confidence notions and can be designed to tolerate faults.

Chapter 6

Conclusion and Future Work

6.1 Conclusion

The main subject matter being explored in this dissertation is the formation of a consensus and/or clustering when networked agents exchange their opinions. We primarily employed DS theoretic models to capture agent opinions. This framework is better suited to capture a wider variety of uncertainties, such as those encountered in soft evidence sources (e.g., human-generated nuanced opinions). Of particular importance among the results presented in this dissertation is the incorporation of SJT theoretic notions, including the bound of confidence, in the opinion exchange process. Under bound of confidence agents were considered to exchange opinions only with neighboring agents with 'closer' opinions.

The work in Chapter 3 explores the cases where agent opinions are captured via p.m.f.s and Dirichlet BoEs, both special cases of the DST framework. We have provided analytical results based on matrix theory giving conditions under which a consensus will be reached in the presence of opinion leaders. The work in Chapter 3 assumes an asynchronous network, i.e., there are no delays in the update process of agent opinions.

The work in Chapter 4 employs the paracontractions theoretic view of the network and the opinion exchange mechanism. This allows most of the simplifying assumptions made in Chapter 3 to be relaxed, thus allowing asynchronous updating and agents opinions that are represented via general DST BoEs.

The work in Chapter 5 proposes a method that can be used to generate a random network such that a possible consensus state can be sustained even when bound of confidence notions are employed by the embedded networked agents. The fault tolerance and outlier removal capabilities makes the proposed network generation model applicable in a wide range of scenarios which are prone to preplanned attacks and severe conditions.

6.2   Future Work

6.2.1   Opinion Dynamics

Our analysis made the assumption that all agents possess identical FoDs. To make the analysis more general the scenario having agents with non-identical FoDs [72] should be studied. Furthermore in regard to the bound of confidences of agents, we considered the case of homogeneous confidence bound where all the agents possess the same confidence bound. Future work can extend the analysis to handle the heterogeneous case where each agent may possess a different confidence bound [73].

In Chapter 4.1 with the paracontractions theoretic view of consensus and opinion clusters, bound of confidence has been modeled as another parameter which controls agent connectedness, hence the confluence conditions. However, this does not directly capture the relationship of confidence bound with the opinion updating equation. Further analysis could be done incorporating bound of confidence into the conditional

operator itself so that the pool of paracontracting operators $\mathcal{H}_\lhd$ encapsulates the bounded confidence notions. Then one can apply the convergence theorems in [19] with the same confluence conditions, where bound of confidence notions are taken not under confluence conditions but under paracontracting operators.

6.2.2 Influencing an Opinion

For future work it is beneficial to study the problem of influencing a network of agents with a desired opinion preferably in the least amount of iterations. Suppose we are trying to spread an opinion or market a product over a group of individuals and the resources are limited such that we can target only one or few individuals. Then naturally the question arises as which individual should be targeted in order to propagate the opinion via the interactions that individual has with other members of the group.

The notion of influencing a network has been analyzed in economics community particularly in *viral marketing*. Some researchers model it as a influence maximization problem [10, 11] whereas some in micro-finance, model it as finding the node with highest diffusion centrality [74]. Furthermore, finding influential nodes has attracted sociophysicist and other researchers who studies opinion formation on social networks [9, 75–77].

For instance, in viral marketing it is important to select individuals who can spread the ideas over a network as fast as possible. Had we picked an isolated individual or an individual with less connections to other individuals, the spreading of the idea would not be very effective. Instead if the selected individual has connections to other individuals in the network via different paths, then it is likely that the idea would spread more rapidly over the entire network.

6.2.2.1 Previous Work

Optimal Opinion Control

In [4] influencing a network with a *strategic agent* has been modeled an an *optimal control problem*. They consider a campaign of certain length, where the strategic agent tries to influence the ongoing dynamics among normal agents with strategically placed opinions in such a way that, by the end of the campaign, as many normal agents as possible end up with opinions in a certain interval of the opinion space. That model considers the notions of SJT, in particular, bounded confidence. However, there is a subtle difference between the influence model in [4] and our objective. In [4] the strategic agent change the its opinion time to time in order to get the ultimate desired outcome. For instance, at the beginning the influencing agent could pose an opinion closer to neighbors in-order to be within their bound of confidence ranges. When time progress the influencing agent alter its option towards the ultimate opinion that needs to be spread across the network. In contrast, our objective is to place the cautious agent (tantamount to the strategic agent in [4]) with a fixed option that needs to be spread, in the most productive location in the connected random network, based on ex ante estimates. The randomness of the underlying networks discerns our work further from the problem addressed in [4] where the underlying network (before accounting for bounded confidence notions) is completely connected.

PageRank Related Methods

The network topology where agents are embedded plays an important role in deciding the location of most influential agent. As briefly explained in Section 2.2.3, the centrality measures quantify and order the influential vertices. One such popular centrality measure is PageRank. PageRank model of opinion formation on social

networks has been proposed in [75, 77]. In [75] they find that the society elite, corresponding to the top PageRank nodes, can impose its opinion on a significant fraction of the society.

Epidemiological Models

Epidemic models are mainly used to analyze the transmission of communicable diseases through individuals. One can model of spreading of an opinion as spreading of a diseases with epidemiological models [12]. Two popular epidemiological models are SIR (Susceptible-Infected-Recovered) and SIS (Susceptible-Infected-Susceptible). In [78] k-shell decomposition analysis is used to locate the influential spreaders. In k-shell decomposition analysis each node is assigned an integer index or coreness k_s, which represents the location of the node according to successive layers (k shells) in the network. In [78] it further claims the robustness of the k_s index measure stating the node rankings are not influenced significantly in the case of incomplete information.

However, we have an open question whether we can model opinions with epidemic models. One argument behind that doubt is, for instance, in SIR model the transition from recovery to susceptible is modeled merely with a transition probability which does not capture the all the notions of opinion exchanges including the notion of SJT, to the best of our knowledge.

6.2.2.2 Opinion Influencing

In our opinion dynamic models, the receptive agents are embedded in an underlying random network and the opinion leader, i.e., cautious agent, with the desired opinion attempts to influence the opinion of receptive agents to within a certain interval in

the opinion space. The opinions are represented with DST BoEs. For analytical convenience we assume that there is a resource constraint that we have only one cautious agent and that an agent can make a connection with only one receptive agent in the network. Our intent is to find the best receptive agent to make the connection with the cautious agent, such that the desired opinion to be influenced can be propagated in the least number of opinion update iterations.

At first we can consider the situation with one opinion leader. If two or more opinion leaders are present the setting takes a more game theoretical context[4], in which opinion leaders have to take into account that there are others that try to influence the desired opinions as well. In game theoretic terms opinion leaders not only have to play with receptive agents but also play a game against (or with) each other[1].

6.2.2.3   Simulations

A preliminary simulation has been done in order to assess the appropriateness of existing network centrality measures in order to find the most influential agent in a group of 20 agents. Each agent's opinion is modeled with DST BoEs $\mathcal{E}_i = \{\Theta, \mathcal{F}_i, m\}$, where $\Theta = \{\theta_1, \theta_2, \theta_3\}$ and $i = \{1, 2, \ldots, 20\}$. The mass assignment for each agent is given in Table 6.1 and the graph with connections among agents is shown in Figure 6.1.

The opinion update of each agent is modeled with CUE and all the agents given in Table 6.1 are considered as receptive agents. The opinion to be influenced throughout the network is chosen as θ_1 where we stop the opinion updating iterations when all agents agreeing for opinions that are closer to θ_1. In order to 'inject' the opinion to the network we attach a cautious agent $\mathcal{E}_{21} = \{\Theta, \mathcal{F}_{21}, m\}$, with $m(\theta_1) = 1$ to any single selected agent in the network which we find as most influential in terms of spreading the opinion of interest, i.e., θ_1.

---

[1][4] suggests concepts like Nash equilibrium to find solutions to such complicated games.

TABLE 6.1: Simulation setup.

Agent	Focal element						
	θ_1	θ_2	θ_3	$\theta_1\theta_2$	$\theta_1\theta_3$	$\theta_2\theta_3$	Θ
1	0.8	—	—	—	—	0.2	—
2	0.8	—	—	—	—	0.2	—
3	0.6	0.2	0.1	0.1	—	—	—
4	0.7	—	—	0.2	0.1	—	—
5	0.7	—	—	—	—	—	0.3
6	0.6	—	—	0.2	—	0.2	—
7	0.7	—	—	—	—	0.1	0.2
8	0.6	—	—	0.3	—	—	0.1
9	0.4	—	—	0.2	0.2	—	0.2
10	0.3	—	0.1	0.2	0.2	—	0.2
11	0.3	—	0.2	0.3	—	0.1	0.1
12	0.2	0.1	0.2	0.1	0.1	—	0.3
13	0.2	—	0.3	0.2	—	—	0.3
14	0.2	—	0.5	—	—	0.2	0.1
15	0.2	—	0.5	0.1	—	0.1	0.1
16	0.2	—	0.6	—	—	—	0.2
17	0.1	—	0.6	—	—	0.2	0.1
18	0.1	—	0.7	—	—	0.1	0.1
19	0.1	—	0.7	—	—	0.2	—
20	0.1	—	0.8	—	—	—	0.1

For instance, suppose we choose agent four as the point of entry to the network and attach agent 21 with agent 4 in order to drive the agents to the opinion of 21st agent. Figure 6.2a shows the initial attachment of 21st agent to the 20 agent network via agent 4. The bound of confidence level has been chosen as $\varepsilon = 0.9$ and the iterative updating is continued until all agents reach opinions with distances (as mentioned in Section 2.1.6) less 0.1 of the opinion of 21st agent.

Even though 4th agent is the node possessing highest degree in terms of direct connections it is not the best agent to connect 21st agent, in order to spread the opinion of 21st agent with minimum number of iterations. Table 6.2 gives the number of iterations for all first 20 agents to reach opinions with distance less than 0.1 of the opinion of the 21st agent. As seen from the table, under the given circumstances it
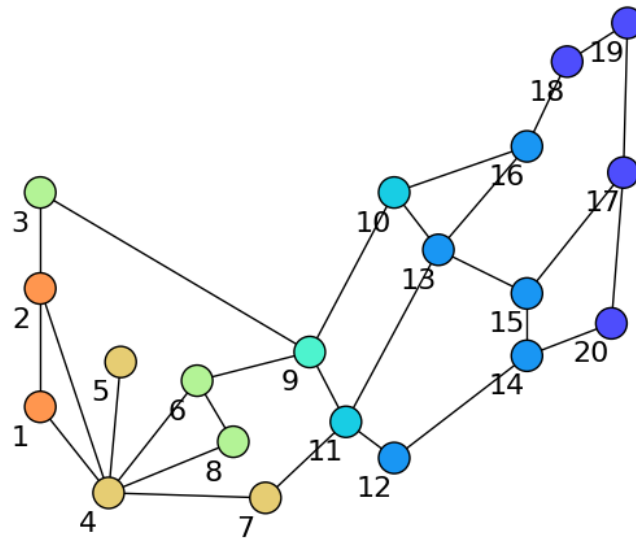
FIGURE 6.1: The graph showing the connection of 20 agents.

is agent 11 who should be picked to propagate the desired opinion with minimum number of iterations (counting to 145) as depicted in Figure 6.3.

6.2.2.4 Conclusion: Opinion Influence

The ultimate objective is to come up with an *opinion influencing centrality measure*, which gives a higher rank for agents who can influence the network of agents faster for a given opinion. For this purpose, we believe that the following factors may play a critical role:

- Geodesic distance to other agents in the network.

- Impact of bound of confidence on the geodesic distance to other agents.
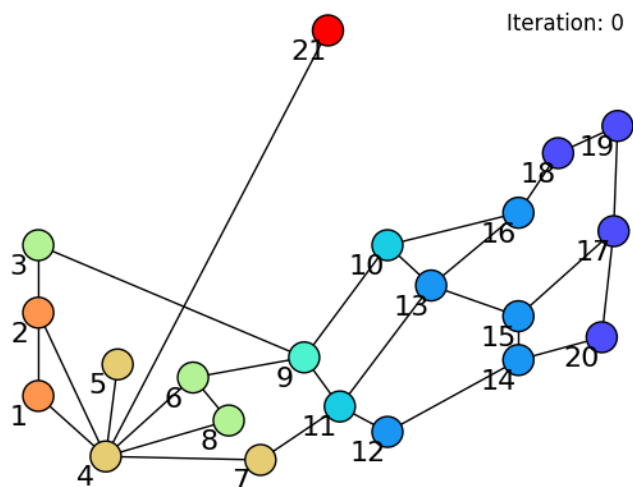
(A) The initial state.
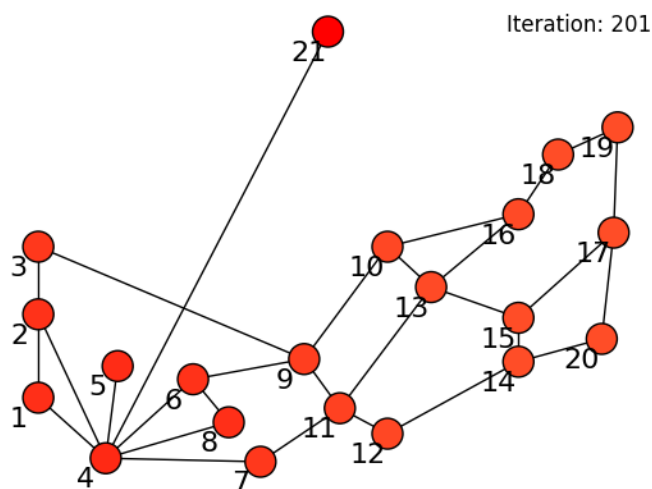
(B) After 201 update iterations.

FIGURE 6.2: Influencing the network via agent 4.

- Whether some following agents have already possess the opinion that needs to be influenced. For instance, if the majority of closely connected agents already possess the opinion we need to influence, then it would rather be effective to influence a small set of agents who possess different opinions.

TABLE 6.2: Simulation Results on Network Influence.

Agent	# iter	Rank and Centrality				
		Degree	PageRank	Betweenness	Closeness	Katz
11	**145**	4	0.075	**0.36**	**0.442**	0.231
13	159	4	0.07	0.246	0.404	0.23
9	162	4	0.063	0.287	0.422	0.228
10	168	3	0.059	0.152	0.388	0.228
16	184	3	0.072	0.137	0.345	0.23
15	194	3	0.049	0.136	0.339	0.223
12	197	2	0.047	0.0117	0.352	0.223
4	201	**6**	**0.107**	0.252	0.352	**0.235**
7	202	2	0.045	0.184	0.38	0.223
14	205	3	0.061	0.092	0.317	0.226
6	207	3	0.056	0.106	0.358	0.225
17	216	3	0.046	0.063	0.279	0.221
3	220	2	0.038	0.064	0.333	0.22
2	226	3	0.043	0.035	0.297	0.221
18	236	2	0.028	0.067	0.284	0.217
20	244	2	0.039	0.014	0.264	0.22
8	247	2	0.032	0	0.288	0.219
19	249	2	0.023	0.023	0.25	0.216
1	253	2	0.024	0	0.271	0.217
5	317	1	0.022	0	0.264	0.216

Ideally, one prefers to do all the centrality calculations with initial setup and select the best agent to influence a network at all time. However, this is not necessarily the case due to the dynamic nature of agent interactions. One can periodically check for best agent to influence and it will clearly affect the computation time. The topology of the agent network along with the initial distribution of agent opinions plays a central role in the opinion influence centrality measure.
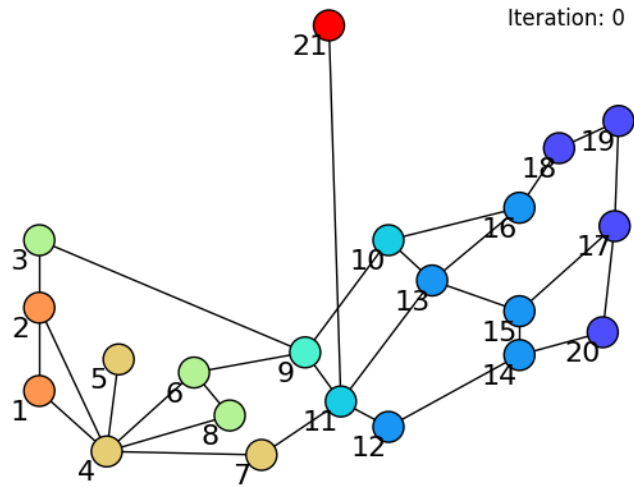
(A) The initial state.
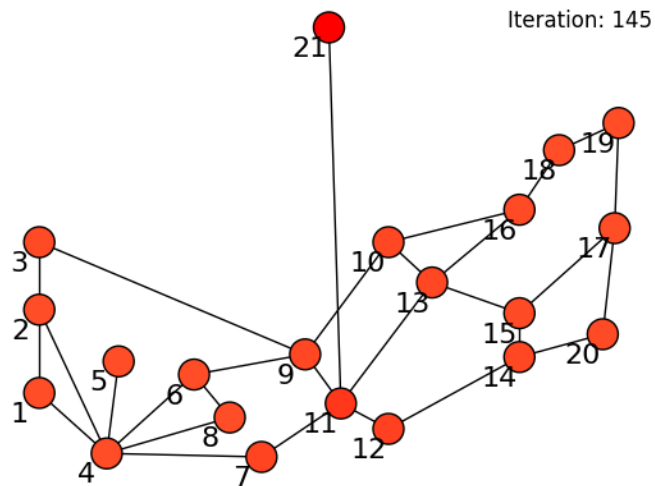
(B) After 145 iterations.

FIGURE 6.3: Influencing the network via agent 11.

Appendix A

Alternative Analysis of Agent Opinions Under Bounded Confidence for Chapter 3

Section A.1 gives an alternative proof for the consensus formation under bounded confidence primarily based on the work of [39] and [13].

A.1 Consensus Formation Under Bounded Confidence

Consider our N BoEs $\{\mathcal{E}_1, \ldots, \mathcal{E}_N\}$ defined on the same FoD $\Theta = \{\theta_i, \ldots, \theta_M\}$. Note that $m_i(\theta_w)_k$ denotes the opinion on singleton $\theta_w \in \Theta$ of agent $i \in \{1, \ldots, N\}$ at discrete-time $k \in \mathbb{N}$. The vector $\mathbf{m}(\theta_w)_k = [m_1(\theta_w)_k, \ldots, m_N(\theta_w)_k]^T \in \mathbb{R}^N_{\geq 0}$ is the opinion profile of singleton opinion θ_w at time k.

Suppose agent i updates $\mathcal{E}_i$ by taking into account the opinions of all agents j whose BoEs $\mathcal{E}_j$ lie within the distance ε_i from agent i's own opinion, i.e., $\|\mathcal{E}_i - \mathcal{E}_j\| \leq \varepsilon_i$. Here, $\varepsilon_i > 0$ is the bound of confidence of agent i based on the selected norm. The opinion update is modeled via the CUE in (2.5) which, in terms of masses, can be

expressed as

$$m_i(B)_{(k+1)} = \alpha_{i,k} m_i(B)_k + \sum_{j \neq i} \sum_{A \in \mathcal{F}_{j,k}} \beta_{ij}(A)_k m_j(B|A)_k, \qquad \text{(A.1)}$$

where $i, j \in \{1, \ldots, N\}$, $\alpha_{i,k}$ and $eq : CUEmPar_A \beta_{ij}(\cdot)_k$ are non-negative real numbers satisfying

$$\alpha_{i,k} + \sum_{j \neq i} \sum_{A \in \mathcal{F}_{j,k}} \beta_{ij}(A)_k = 1. \qquad \text{(A.2)}$$

For mathematical convenience, we assume a homogeneous bounded confidence model, i.e., all agents are taken to have the same confidence range $\varepsilon_i \equiv \varepsilon$. In addition, we also assume a static network (in the sense that the communication links are static). The analytical results below apply to the case where the DST models contain only singleton focal elements (i.e., the p.m.f. case).

Definition 1. For agent $i \in \{1, \ldots, N\}$, the set of neighborhood agents at discrete-time instance k is

$$I_k(i) = \{j = 1, \ldots, N : \|\mathcal{E}_i - \mathcal{E}_j\| \leq \varepsilon\}.$$

The number of neighbor agents of agent i is $\tau_{i,k} = |I_k(i)|$. $\qquad \blacksquare$

For the case of DST models that has only singleton focal elements, one can show that the opinion update can be written as the following discrete-time dynamical system:

$$\mathbf{m}(\theta_w)_{k+1} = \mathbf{A}_k \mathbf{m}(\theta_w)_k, \qquad \text{(A.3)}$$

where the initial conditions are denoted as $\mathbf{m}(\theta_w)_0 \in (\mathbb{R}_{\geq 0})^N$ and the $N \times N$ matrix $\mathbf{A}_k = \{a_{ij,k}\}$ is defined as

$$
a_{ij,k} = \begin{cases} \alpha_{i,k}, & \text{for } i = j; \\ \gamma_{i,k}, & \text{for } j \in I_x(i); \\ 0, & \text{otherwise.} \end{cases} \tag{A.4}
$$

Here, $\gamma_{i,k} = (1 - \alpha_{i,k})/\tau_{i,k}$.

We also use the following notion [13, 39]:

Definition 2. The range $\mathcal{R}$ of a singleton opinion profile θ_w at discrete-time k is

$$
\mathcal{R}(\mathbf{m}(\theta_w)_k) = \max_{1 \leq i,j \leq N} (m_i(\theta_w)_k - m_j(\theta_w)_k),\ \theta_w \in \Theta. \tag{A.5}
$$

∎

In [39], sufficient conditions for consensus have been given using a certain ε-profile. However, this previous work only considers real-valued agent opinions. Here, we extend the result to p.m.f.s under certain assumptions on DST BoEs.

Similar to the real-valued opinions case in [13, 39], let us assume that our BoEs can be arranged as an *ordered list* according to the distance relative to a particular 'reference' BoE. Without loss of generality, we take $\mathcal{E}_1$ as the reference BoE and re-label all the BoEs as $\mathcal{E}_1, \ldots, \mathcal{E}_N$ so that $\|\mathcal{E}_2 - \mathcal{E}_1\| \leq \ldots \leq \|\mathcal{E}_N - \mathcal{E}_1\|$. If this order does not change when the opinions of agents get updated each time, then it is called an *order preserving* arrangement. Furthermore, if each singleton opinion profile can be arranged in the same order as the BoE indices as ascending or descending mass values, then it is referred to as a *strictly order preserved* arrangement.

Definition 3. Under the strictly order preserved arrangement, the BoE setting is said to have a ε-chain if each agent has distance to its neighbors less than the bound of confidence ε such that

$$\|\mathcal{E}_{i+1} - \mathcal{E}_i\| \leq \varepsilon, \ \forall i = 1, \ldots, N-1. \tag{A.6}$$

∎

A.1.1 All Receptive Agents

We first explore consensus when all agents are receptive. For this purpose, we consider a strictly order preserved arrangement and analyze fixed points of the agent opinions. For such an arrangement, the diagonal and off-diagonal entries of the transition matrix $\mathbf{A}_k$ are positive (Proposition 3 in [13]). Furthermore, a product of $(N-1)$ such matrices, say $\mathbf{B}_\zeta = \{b_{ij,\zeta}\}$, is positive and given by

$$\mathbf{B}_\zeta = \mathbf{A}_{(N-1)(\zeta+1)-1} \cdots \mathbf{A}_{(N-1)\zeta} > 0, \tag{A.7}$$

i.e., $b_{ij,\zeta} > 0$, for all i, j, and $\zeta \in \mathbb{N}$.

For all singleton opinion profiles $\mathbf{m}(\theta_w)$, $\theta_w \in \Theta$, the updates at discrete-time $(k+1)$, such that $k \in (N-1) \cdot \mathbb{N}$, can be given as

$$\mathbf{m}(\theta_w)_{k+1} = \mathbf{B}_\zeta \mathbf{B}_{\zeta-1} \cdots \mathbf{B}_0 \mathbf{m}(\theta_w)_0, \tag{A.8}$$

where $\zeta = (k+1)/(N-1) - 1$.

From [39], for a non-negative row stochastic matrix $\mathbf{C}$, we have the following result:

Lemma 4. [39] When a matrix $\mathbf{C}$ is row stochastic, then, for all $\mathbf{m}(\theta_w) \in \mathbb{R}^N_{\geq 0}$,

$$\mathcal{R}\left(\mathbf{C}\mathbf{m}(\theta_w)_k\right) \leq \left(1 - \min_{1 \leq i,j \leq N} \sum_{\ell=1}^{N} \min\{a_{i\ell,k}, a_{j\ell,k}\}\right) \mathcal{R}(\mathbf{m}(\theta_w)_k). \quad \text{(A.9)}$$

■

As the matrices $\mathbf{B}_\zeta$ s are positive row-stochastic, from Lemma 4 we have

Lemma 5. For all positive row-stochastic matrices $\mathbf{B}_\zeta$ s,

$$\mathcal{R}(\mathbf{B}_\zeta \mathbf{m}(\theta_w)_{k+1}) \leq \lambda \mathcal{R}(\mathbf{m}(\theta_w)_{k+1}), \text{ for some } \lambda < 1. \quad \text{(A.10)}$$

■

From Lemma 5, we know that $\mathcal{R}(\mathbf{B}_\zeta \mathbf{m}(\theta_w)_{k+1}) < \mathcal{R}(\mathbf{m}(\theta_w)_{k+1})$. Hence, from Corollary 7 in [13], we can form a sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_{\tilde{k}}))_{\tilde{k}}$ that converges to 0. Due to the row stochasticity of $\mathbf{A}_k$, from Lemma 4, we can show that the sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k$ is monotonically decreasing and indeed a Cauchy sequence. But, $(\mathcal{R}(\mathbf{m}(\theta_w)_{\tilde{k}}))_{\tilde{k}}$ is a subsequence of the sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k$. Hence the sequence $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k$ converges to 0.

When $(\mathcal{R}(\mathbf{m}(\theta_w)_k))_k \to 0$, $\forall \theta_w \in \Theta$, we reach a consensus. In summary, we have shown that consensus will be reached for a strictly order preserved arrangement of BoEs with an ε-chain. However, when a 'crack' appears in the ε-chain, the agents get divided into independent groups, and the above result applies to each subgroup yielding a fixed point with separate cluster points for each independent group.

A.1.2 One Cautious Agent

When a cautious agent, $\mathcal{E}_{C1}$, $C1 \in \{1, \ldots, N\}$, is present with all the others being receptive agents, the transition matrix $\mathbf{A}_k$ at time k can be given as in

$$\mathbf{A}_k = \begin{pmatrix} P_k & \mathbf{u}_k & S_k \\ \mathbf{0}^T & 1 & \mathbf{0}^T \\ R_k & \mathbf{v}_k & Q_k \end{pmatrix}, \tag{A.11}$$

where

$$P_k, Q_k, R_k, S_k = \text{square matrices of appropriate size;}$$

$$\mathbf{u}_k, \mathbf{v}_k = \text{vectors of appropriate size;}$$

$$\mathbf{0} = \text{zero vectors of appropriate size.}$$

The matrix in (A.11) is row-stochastic. Hence, Lemma 4 can be applied. Further, for the all singleton scenario, as the opinion of the cautious agent does not change, it can be seen that

$$\min_{y \in I_k(C1)} [m_y(\theta_w)_{k+1}] \leq m_{C1}(\theta_w)_{k+1}$$

$$\leq \max_{z \in I_k(C1)} [m_z(\theta_w)_{k+1}], \tag{A.12}$$

for all $y, z \in I_k(C1)$, $\theta_w \in \Theta$. Hence the neighboring receptive agents of cautious agent $C1$ converge to the opinion of the leader, viz., $C1$. When the conditions for the ε-chain are satisfied, a consensus will be reached. The fixed point is given by the masses of the cautious agent's singleton opinions. Again, when a crack appears in the ε-chain, agents get divided into independent groups. However, the group that contains the cautious agent converges to the cautious agent's opinion.

A.1.3 Multiple Cautious Agents

If there are more than one cautious agent, clearly there will be no consensus (unless of course all the cautious agents have the same opinion). In a strictly order preserved arrangement with an ε-chain, the receptive agents who get updated from only one cautious agent converge to that particular cautious agent's opinion. The analysis is similar to the scenario with one cautious agent (see Section A.1.2).

When there is more than one cautious agent in the bound of confidence of a receptive agent, two possibilities could happen.

(a) If the receptive agent continues to have more than one cautious agent for iterations to come, then it will converge to a fixed point. The fixed point is in the convex hull of the set of points corresponding to neighboring cautious agents.

(b) Even if the receptive agent's neighborhood initially contains more than one cautious agent, this neighborhood could later contain fewer cautious agents. This situation can be addressed as in A.1.2 or item (a) above.

Appendix B

Application of Opinion Dynamic Model in Chapter 3

We have applied our opinion model to capture the political opinion dynamics of students in an MIT dormitory during the 2008 presidential election. The political opinions have been extracted from the main Social Evolution dataset[79].

B.1 Social Evolution Dataset

The social evolution experiment has been designed to study the adoption political opinions, diet, exercise, obesity, eating habits, epidemiological contagion, depression and stress, dorm political issues, interpersonal relationships, and privacy. The dataset contains the spatio-temporal patters in everyday life of a population of 30 freshmen, 20 sophomores, 10 juniors, 10 seniors and 10 graduate student tutors.

B.1.1 Political Opinions

The political opinions have been captured using three monthly web-based surveys, once each in Septermber, October, and November 2008 (immediately after the US

presidential election) [80]. The political opinions captured in the surveys are explained in Table B.1.

TABLE B.1: Captured Political Opinions

Political Opinions	Possible Responses
Political interest	Very interested
	Somewhat interested
	Slightly interested
	Not at all interested
Liberal or conservative	Extremely liberal
	Liberal
	Slightly liberal
	Slightly conservative
	Conservative
	Extremely conservative
	Moderate middle of the road
Preferred party	Democrat
	Republican
	Member of another party
	Independent
Preferred party details	Strong Democrat
	Democrat
	Not very strong Democrat
	Not very strong Republican
	Republican
	Strong Republican
	Neither
Likely candidate to vote (before election)	Definitely Barak Obama
	Probably Barak Obama
	Probably John McCain
	Definitely John McCain
	Other candidate
	Undecided
Voted candidate (after election)	Barak Obama
	John McCain
	Other candidate

B.1.2 Opinion Modelling

Based on the political opinions of the first survey in September, initial student opinions were modeled with DS BoEs. The FoD was taken as $\Theta = \{\theta_1, \theta_2, \theta_3\}$ where $\theta_1 \equiv$ 'Vote for Obama', $\theta_2 \equiv$ 'Vote for McCain' and $\theta_3 \equiv$ 'Vote neither'. Note the higher cardinality proposition like $\theta_1\theta_2\theta_3$ represent nuanced opinions. For our preliminary simulations we selected masses as given in Table B.2.

TABLE B.2: Initial DS mass assignment on political opinions

Survey Response	DST Mass Values			
	θ_1	θ_2	θ_3	Θ
'Definitely Barack Obama'	0.97	–	–	0.03
'Probably Barack Obama'	0.75	–	–	0.25
'Definitely John McCain'	–	0.97	–	0.03
'Probably John McCain'	–	0.75	–	0.25
'Other candidate'	–	–	0.75	–
'Undecided'	–	–	–	1.00

B.1.3 Network Topology and Opinion Updating

Using the survey results of social evolution dataset the underline network topology where agents, i.e., students, embedded was estimated. We used survey information on

- 'Close friend'

- 'Socialize twice per week'

- 'Political discussant'

- 'Facebook tagged photos'

- 'Blog, LiveJournal, Twitter'

to built the agent interaction topology. However a proper modeling of the co-evolution of behaviors and social relationships using mobile phone data can be found in [81]. Opinion exchanges were modeled using conditional update equation (CUE) as explained in Section 2.1.5. The opinion update strategy of a particular student was decided based on the 'preferred party details' of the political opinion surveys. Students with political preferences as 'strong democrats' and 'strong republicans' were considered as 'cautious' agents while all other students as 'receptive' agents. The inertia of CUE was selected as $\alpha = 0.5$ for all agents. Furthermore, the bound of confidence $\varepsilon = 0.8$ was considered for all agents.

B.1.4   Analysis of the Results

The number of iteration the opinion exchange needs to be done to explain the voting by the time of election date has to be properly investigated. However, by trial and error, for this scenario we analyzed the result after 8 iterations of opinion exchanges among agents. Clearly the number of iterations depend on the 'inertia' used in CUE.

To report the performance we use DS theoretic performance measures based on the definition given for DS theoretic precision and recall in [82]. However, to give a performance we need to identify 'positive' and 'negative' samples. For our discussion, let us assume data with 'vote for Obama' as 'positive' samples and all others as 'negative' samples. Moreover, let the positive data points be denoted as 'a' and negative points as 'b'. Hence for this selection of 'positive' and 'negative' samples 'a' corresponds to θ_1 and 'b' corresponds to $\theta_1\theta_2$.

Let s denote a data point. Using pignistic transformations [83] we can then obtain the pignistic probabilities $\widehat{BetP}(x)^s$, $x = \{a, b\}$, for each sample s. Let the set of positive samples be denoted by $\mathcal{S}_{(+)}$ and set of negative by $\mathcal{S}_{(-)}$. Under this setting the 'true

positives'(TP), 'false positives'(FP), 'false negatives'(FN) and 'true negatives'(TN) can be given as:

$$TP_{DS} \ \sum_{s \in \mathcal{S}_{(+)}} \widehat{BetP}(a)$$

$$FP_{DS} \ \sum_{s \in \mathcal{S}_{(-)}} \widehat{BetP}(a)$$

$$FN_{DS} \ \sum_{s \in \mathcal{S}_{(+)}} \widehat{BetP}(b)$$

$$TN_{DS} \ \sum_{s \in \mathcal{S}_{(-)}} \widehat{BetP}(b)$$

Now the *DS F-measure* is given in (B.1) as

$$F_{\beta(DS)} = \frac{(1 + \beta^2) \cdot TP_{DS}}{(1 + \beta^2) \cdot TP_{DS} + \beta^2 \cdot FN_{DS} + FP_{DS}}. \tag{B.1}$$

Under the given settings we got $F_{1(DS)} = 0.83$ (approx.). However further tuning has to be done in order to improve the performance of the model.

B.1.5   Conclusion of the Analysis

The preliminary analysis tested the usage of our opinion model in political opinions of social evolution dataset. The model allowed us to 'predict' the opinion dynamics given the initial opinions and network topology. However, further tuning has to be done to properly assess the opinion parameters as well as the network topologies.

Appendix C

Lemmas and Proofs of Chapter 4

Claim C.1. *Consider the BoE $\mathcal{E} = \{\Theta, \mathcal{F}, m(\cdot)\}$ and $A, B \in \mathcal{F}$. For $A \subset B$,*
$m(B|A) = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proof. We know that, $Bl(B|A) = \frac{Bl(AB)}{Bl(A\cup B) + Pl(A\cup\bar{B})}$.

But when $A \subset B$, $A \cup B = A$ and $A \cup \bar{B} = \emptyset$.

Hence,

$$Bl(B|A) = 1 \qquad\qquad\qquad\qquad\text{(C.1)}$$

and

$$Bl(A|A) = 1. \qquad\qquad\qquad\qquad\text{(C.2)}$$

By (C.1) and (C.2) we can see that $m(B) = 0$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ ■

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