2010-04-22

Casson-Lin Type Invariants for Links

Eric Harper
University of Miami, harper@math.miami.edu

UNIVERSITY OF MIAMI

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By

Eric Harper

A DISSERTATION

Submitted to the Faculty
of the University of Miami
in partial fulfillment of the requirements for
the degree of Doctor of Philosophy

Coral Gables, Florida

May 2010

UNIVERSITY OF MIAMI

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Approved:

Nikolai Saveliev, Ph.D. Professor of Mathematics	Terri A. Scandura, Ph.D. Dean of the Graduate School
Ludmil Katzarkov, Ph.D. Professor of Mathematics	Kenneth Baker, Ph.D. Professor of Mathematics
Orlando Alvarez, Ph.D. Professor of Physics	

HARPER, ERIC (Ph.D., Mathematics)
Casson-Lin Type Invariants for Links (May 2010)

Abstract of a dissertation at the University of Miami.

Dissertation supervised by Professor Nikolai Saveliev.
No. of pages in text. (45)

In 1992, Xiao–Song Lin constructed an invariant h of knots in the 3–sphere via a signed count of the conjugacy classes of irreducible $SU(2)$–representations of the fundamental group of the knot exterior with trace-free meridians. Lin showed that h equals one-half times the knot signature. Using methods similar to Lin's, we construct an invariant of two-component links in the 3-sphere. Our invariant is a signed count of conjugacy classes of projective $SU(2)$–representations of the fundamental group of the link exterior with a fixed 2-cocycle and corresponding non-trivial second Stiefel–Whitney class. We show that our invariant is, up to a sign, the linking number.

We further construct, for a two-component link in an integral homology sphere, an instanton Floer homology whose Euler characteristic is, up to sign, the linking number between the components of the link. We relate this Floer homology to the Kronheimer–Mrowka instanton Floer homology of knots. We also show that, for two-component links in the 3-sphere, the Floer homology does not vanish unless the link is split.

Acknowledgements

I would like to thank my advisor, Dr. Nikolai Saveliev, for his immense efforts in making this dissertation possible. He introduced me to the topic and without his teaching, infinite patience, and never-ending encouragement, this project surely would not have been possible.

Many thanks are in order for committee member Dr. Ken Baker and for my colleague Chris Harris for all of the discussions we had regarding this work. Also, I am grateful for the many helpful suggestions Dr. Daniel Ruberman made throughout the course of this work.

I am grateful to have Dr. Ludmil Katzarkov and Dr. Orlando Alvarez as members of my committee. The faculty and staff at the University of Miami have always gone above and beyond the call of duty in assisting me as a graduate student and for that I owe them much gratitude.

My wife and family deserve special thanks for their constant support. I am deeply indebted to them for all of their help along the way.

Contents

Chapter 1

Introduction

One of the characteristic features of the fundamental group of a closed 3-manifold is that its representation variety in a compact Lie group tends to be finite, in a properly understood sense. This has been a guiding principle for defining invariants of 3-manifolds ever since Casson defined his λ-invariant for integral homology 3-spheres in 1985. Casson's invariant is (roughly) a signed count of the conjugacy classes of irreducible $SU(2)$ representations of the fundamental group, with signs determined by a Heegaard splitting.

Among the numerous generalizations of Casson's invariant that have appeared since, we will single out the line of development that started with the invariant of Xiao–Song Lin [20]. This invariant is defined for knots in S^3 via a signed count of the conjugacy classes of irreducible $SU(2)$ representations of the fundamental group of the knot exterior. The knot exterior is a 3-manifold with non-empty boundary so the above finiteness principle only applies after one imposes a proper boundary condition. Lin's choice of the boundary condition, namely, that all the knot meridians are represented by trace-free $SU(2)$ matrices, resulted in an invariant $h(K)$ of knots

$K \subset S^3$. Lin further showed that $h(K)$ in fact equals one-half the knot signature of K. The signs in Lin's construction were determined by a braid representation for the knot.

Austin (unpublished) and Heusener and Kroll [12] extended Lin's construction by letting the meridians of the knot be represented by $SU(2)$ matrices with a fixed trace which need not be zero. More specifically, for any $\alpha \in (0, \pi)$, their invariant $h^\alpha(K)$ counts the conjugacy classes of irreducible representations $\rho : \pi_1(S^3 - K) \to SU(2)$ such that $\operatorname{tr}(\rho(\mu)) = 2\cos(\alpha)$ for all meridians μ of the knot K. Let F be a Seifert surface for K and S its Seifert matrix, then the matrix

$$H(e^{2\mathbf{i}\alpha}) = (1 - e^{2\mathbf{i}\alpha})S + (1 - e^{-2\mathbf{i}\alpha})S^T$$

is Hermitian. Heusener and Kroll [12] showed that, unless $e^{2\mathbf{i}\alpha}$ is a root of the Alexander polynomial of K, one-half times the signature of $H(e^{2\mathbf{i}\alpha})$ is equal to the invariant $h^\alpha(K)$. The signature of $H(e^{2\mathbf{i}\alpha})$ is known as the Tristram-Levine knot signature, or the equivariant knot signature. Note that $h^\alpha(K) = h(K)$ in the case of $\alpha = \pi/2$.

In Chapter 1, we extend Lin's construction from knots to two-component links L in S^3. In essence, we replace the count of $SU(2)$ representations with a count of *projective* $SU(2)$ representations of $\pi_1(S^3 - L)$, in the sense of [23].

A projective representation is, up to an element of the center, a representation. In fact, every projective $SU(2)$ representation gives rise to an $SO(3)$ representation by composing it with the adjoint representation $\operatorname{ad} : SU(2) \to SO(3)$.

More concretely, let $G = \pi_1(S^3 - L)$ and let $\{\pm 1\} = \mathbb{Z}_2$ be the center of $SU(2)$. Then a projective $SU(2)$ representation is map $\rho : G \to SU(2)$ such that $\rho(gh) = \pm\rho(g)\rho(h)$ for all $g, h \in G$. Associated with a projective representation is its 2-cocycle

$c : G \times G \to \mathbb{Z}_2$ given by $c(g,h) = \rho(gh)\rho(h)^{-1}\rho(g)^{-1}$. The cocycle c defines an element $[c] = w_2(\operatorname{ad}\rho)$ in the group cohomology $H^2(G, \mathbb{Z}_2)$, called the second Stiefel–Whitney class of the representation $\operatorname{ad}\rho : G \to SU(2) \to SO(3)$. The representation $\operatorname{ad}\rho$ lifts to an $SU(2)$ representation if and only if $w_2(\operatorname{ad}\rho) = 0$.

We fix a 2-cocycle c such that $[c] \neq 0 \in H^2(G, \mathbb{Z}_2)$. We define our invariant $h(L)$ as (roughly) a signed count of the conjugacy classes of projective $SU(2)$–representations with associated 2-cocycle c. The signs are derived from a braid representation of L. The two main theorems of Chapter 1 are then as follows.

Theorem 1. *For any two-component link $L \subset S^3$, the integer $h(L)$ is a well-defined invariant of L, up to sign.*

Theorem 2. *For any two-component link $L = \ell_1 \cup \ell_2$ in S^3, one has*

$$h(L) \;=\; \pm \operatorname{lk}(\ell_1, \ell_2).$$

It is worth mentioning that our choice of the 2-cocycle c imposes the trace-free condition on us. This is in contrast to Lin's construction, where the choice of the boundary condition seemed somewhat arbitrary. This also means one should not expect to extend our construction, along the lines of Heusener and Kroll's work [12], to projective $SU(2)$ representations with non-zero trace boundary condition.

Shortly after Casson introduced his invariant for homology 3-spheres, Taubes [25] gave a gauge theoretic description of it in terms of a signed count of flat $SU(2)$ connections. After Lin's work, but before Heusener and Kroll, a gauge theoretic interpretation of the Lin invariant was given by Herald [11]. He used this interpretation to define an extension of the Lin invariant, now known as the Herald–Lin invariant, to knots in arbitrary homology spheres, with arbitrary fixed-trace (possibly non-zero) boundary condition. A construction of the Herald–Lin invariant not relying on gauge theory was later given by Kroll in his doctoral thesis [17].

An attractive feature of the gauge theoretic approach is that it can be used to produce ramified versions of the above invariants. Floer [8] introduced the instanton homology theory for integral homology spheres whose Euler characteristic is twice the Casson invariant. Collin and Steer [5] defined an instanton Floer homology of knots using 3-orbifolds singular along a knot in a homology sphere. The Euler characteristic of their Floer homology theory is the Herald–Lin invariant. Weiping Li [18] defined a symplectic Floer homology of knots with the same Euler characteristic.

More recently, the papers of Lim [19] and Kronheimer-Mrowka [14] have drawn attention to a different version of the instanton Floer homology of knots. The interest in this theory, which first appeared in Floer [9], is explained by its conjectured relationship with the Seiberg–Witten and Heegaard Floer homologies. This is evidenced, for instance, by the fact that the Alexander polynomial of the knot can be expressed in terms of its instanton Floer homology; see [14] and [19].

In Chapter 2 we define an instanton Floer homology for links of two components in an integral homology sphere. We use a slight variant of the construction of Floer [9] (which was later developed by Braam and Donaldson [3]). First, we furl up the link exterior by gluing one boundary component to the other in such a way that the resulting manifold is a homology $S^1 \times S^2$. We show that such a gluing always exists and then define the instanton Floer homology of the link as the instanton Floer homology of the resulting homology $S^1 \times S^2$. Lastly, it follows from excision that the Floer homology is independent of the gluing map used to furl up the link exterior.

We show that the Euler characteristic of this Floer homology is the linking number between the components of the link. A similar result for two-component links in the

3-sphere was announced by Braam and Donaldson [3, Example 3.13] with an outline of the proof that relied on the Floer exact triangle. Our approach is more direct and its main advantage is that it yields the result for links in arbitrary homology spheres.

We provide several examples of calculations of Floer homology groups of links and also observe that they appear on the relative term of the Floer exact triangle for knots related by a single skein move.

Chapter 2

A Casson-Lin type link invariant

2.1 Braids and representations

Let F_n be a free group of rank $n \geq 2$, with a fixed generating set $x_1, \ldots, x_n$. We will follow conventions of [21] and define the n-string braid group $\mathcal{B}_n$ to be the subgroup of $\mathrm{Aut}(F_n)$ generated by the automorphisms $\sigma_1, \ldots, \sigma_{n-1}$, where the action of σ_i is given by

$$\begin{array}{lll} \sigma_i : \; x_i & \mapsto & x_{i+1} \\ x_{i+1} & \mapsto & (x_{i+1})^{-1}\, x_i\, x_{i+1} \\ x_j & \mapsto & x_j, \quad j \neq i, i+1. \end{array}$$

The natural homomorphism $\mathcal{B}_n \to S_n$ onto the symmetric group on n letters, $\sigma \mapsto \bar{\sigma}$, maps each generator σ_i to the transposition $\bar{\sigma}_i = (i, i+1)$. A useful observation is that, for any $\sigma \in \mathrm{Aut}(F_n)$, one has

$$\sigma(x_i) = w\, x_{\bar{\sigma}^{-1}(i)}\, w^{-1} \tag{2.1}$$

for some word $w \in F_n$. One can also observe that σ preserves the product $x_1 \cdots x_n$, that is,

$$\sigma(x_1 \cdots x_n) = x_1 \cdots x_n \tag{2.2}$$

2.1.1 $SU(2)$ representations

Consider the Lie group $SU(2)$ of unitary two-by-two matrices with determinant one, that is, of the complex matrices

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}$$

such that $u\bar{u} + v\bar{v} = 1$. We will often identify $SU(2)$ with the group $Sp(1)$ of unit quaternions via

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \quad \mapsto \quad u + vj \in \mathbb{H}.$$

Let $R_n = \operatorname{Hom}(F_n, SU(2))$ be the space of $SU(2)$ representations of F_n, and identify it with $SU(2)^n$ by sending a representation $\alpha : F_n \to SU(2)$ to the vector $(\alpha(x_1), \ldots, \alpha(x_n))$ of $SU(2)$ matrices. The above representation $\mathcal{B}_n \to \operatorname{Aut}(F_n)$ then gives rise to the representation

$$\rho : \mathcal{B}_n \longrightarrow \operatorname{Diff}(R_n) \tag{2.3}$$

via $\rho(\sigma)(\alpha) = \alpha \circ \sigma^{-1}$. We will abbreviate $\rho(\sigma)$ to σ. We will also denote $X = (X_1, \ldots, X_n) \in R_n$ and write $\sigma(X) = (\sigma(X)_1, \ldots, \sigma(X)_n)$.

Example. For any $(X_1, \ldots, X_n) \in R_n$, we have $\sigma_1(X_1, X_2, X_3, \ldots, X_n) =$

$(X_1X_2X_1^{-1}, X_1, X_3, \ldots, X_n)$.

2.1.2 Extension to the wreath product $\mathbb{Z}_2 \wr \mathcal{B}_n$

The wreath product $\mathbb{Z}_2 \wr \mathcal{B}_n$ is the semidirect product of $\mathcal{B}_n$ with $(\mathbb{Z}_2)^n$, where $\mathcal{B}_n$ acts on $(\mathbb{Z}_2)^n$ by permuting the coordinates, $\sigma(\varepsilon_1, \ldots, \varepsilon_n) = (\varepsilon_{\bar{\sigma}(1)}, \ldots, \varepsilon_{\bar{\sigma}(n)})$. Thus the elements of $\mathbb{Z}_2 \wr \mathcal{B}_n$ are the pairs $(\varepsilon, \sigma) \in (\mathbb{Z}_2)^n \times \mathcal{B}_n$, with the group multiplication law

$$(\varepsilon, \sigma) \cdot (\varepsilon', \sigma') \;=\; (\varepsilon\sigma(\varepsilon'),\, \sigma\sigma'),$$

using the multiplicative notation for $(\mathbb{Z}_2)^n$. The representation (2.3) can be extended to a representation

$$\rho :\; \mathbb{Z}_2 \wr \mathcal{B}_n \longrightarrow \mathrm{Diff}(R_n) \tag{2.4}$$

by defining

$$\rho(\varepsilon, \sigma)(X) = \varepsilon \cdot \sigma(X) = (\varepsilon_1\sigma(X)_1, \ldots, \varepsilon_n\sigma(X)_n),$$

where ε_i are viewed as elements of the center $\mathbb{Z}_2 = \{\pm 1\}$ of $SU(2)$. That (2.4) is a representation follows by a direct calculation after one observes that, because of (2.1),

$$\sigma(X)_i = AX_{\bar{\sigma}(i)}A^{-1} \quad \text{for some } A \in SU(2). \tag{2.5}$$

Again, we will abuse notations and write simply $\varepsilon\sigma$ for both (ε, σ) and $\rho(\varepsilon, \sigma)$.

Example. For any $(X_1, \ldots, X_n) \in R_n$ and any $(\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{Z}_2)^n$, we have $(\varepsilon\sigma_1)(X_1, X_2, X_3, \ldots, X_n) = (\varepsilon_1\, X_1X_2X_1^{-1}, \varepsilon_2\, X_1, \varepsilon_3\, X_3, \ldots, \varepsilon_n\, X_n)$. Also $\sigma_1(\varepsilon X) = \sigma_1(\varepsilon)\sigma_1(X) = (\varepsilon_2X_1X_2X_1^{-1}, \varepsilon_1X_1, \varepsilon_3X_3, \ldots, \varepsilon_nX_n)$.

2.1.3 Braids and link groups

The closure $\hat{\sigma}$ of a braid $\sigma \in \mathcal{B}_n$ is a link in S^3 whose exterior has the fundamental group

$$\pi_1(S^3 - \hat{\sigma}) = \langle\, x_1, \dots, x_n \mid x_i = \sigma(x_i),\ i = 1, \dots, n\,\rangle,$$

where each x_i represents a meridian of $\hat{\sigma}$. One can easily see that the fixed points of the diffeomorphism $\sigma : R_n \to R_n$ are representations $\pi_1(S^3 - \hat{\sigma}) \to SU(2)$. A fixed point $\alpha = (\alpha(x_1), \dots, \alpha(x_n))$ of the diffeomorphism $\varepsilon\sigma : R_n \to R_n$ gives rise to a projective representation $\alpha : \pi_1(S^3 - \hat{\sigma}) \to SU(2)$ which becomes a representation $\operatorname{ad}\alpha : \pi_1(S^3 - \hat{\sigma}) \to SO(3)$ after composing it with the adjoint representation $\operatorname{ad} : SU(2) \to SO(3)$. Depending on ε, the representation $\operatorname{ad}\alpha$ may or may not lift to an $SU(2)$ representation, the only obstruction being the second Stiefel–Whitney class $w_2(\operatorname{ad}\alpha) \in H^2(\pi_1(S^3 - \hat{\sigma}); \mathbb{Z}_2)$.

2.2 Definition of $h(\varepsilon\sigma)$

According to Alexander [1], every link in S^3 is the closure $\hat{\sigma}$ of a braid σ. Given a two component link $\hat{\sigma}$, we will associate with it, for a carefully chosen ε, an integer $h(\varepsilon\sigma)$. Later in Section 2.3 we will prove that h is an invariant of the link $\hat{\sigma}$.

2.2.1 Choice of ε

The number of components of the link $\hat{\sigma}$ is exactly the number of cycles in the permutation $\bar{\sigma}$. We will be interested in two component links, that is, the closures of braids σ with

$$\bar{\sigma} = (i_1 \dots i_m)(i_{m+1} \dots i_n) \quad \text{for some } \ 1 \leq m \leq n-1. \tag{2.6}$$

Given such a braid σ, choose a vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{Z}_2)^n$ such that

$$\varepsilon_{i_1} \cdots \varepsilon_{i_m} = \varepsilon_{i_{m+1}} \cdots \varepsilon_{i_n} = -1. \tag{2.7}$$

This choice of ε is dictated by the following two considerations. First, we wish to preserve condition (2.2) in the form

$$(\varepsilon\sigma)(X)_1 \cdots (\varepsilon\sigma)(X)_n = X_1 \cdots X_n, \tag{2.8}$$

and second, we want the fixed points α of the diffeomorphism $\varepsilon\sigma : R_n \to R_n$ to have non-zero $w_2\,(\mathrm{ad}\,\alpha)$.

2.2.1 Lemma. *Let α be a fixed point of $\varepsilon\sigma : R_n \to R_n$ with ε as in (2.7) then $w_2\,(\mathrm{ad}\,\alpha) \neq 0$.*

Proof. The class $w_2\,(\mathrm{ad}\,\alpha)$ is the obstruction to lifting $\mathrm{ad}\,\alpha$ to an $SU(2)$ representation. Extend α arbitrarily to a function $\alpha : \pi_1(S^3 - \hat{\sigma}) \to SU(2)$ lifting $\mathrm{ad}\,\alpha$ then $w_2\,(\mathrm{ad}\,\alpha)$ will vanish if and only if there is a function $\eta : \pi_1(S^3 - \hat{\sigma}) \to \mathbb{Z}_2 = \{\pm 1\}$ such that $\eta \cdot \alpha$ is a representation. Suppose that such a function exists, and denote $\eta(x_i) = \eta_i = \pm 1$. Also, assume without loss of generality that $\bar{\sigma} = (1 \dots m)(m+1 \dots n)$. It follows from (2.5) that in order to satisfy the relations $X_i = (\varepsilon\sigma)(X)_i$ we must have $\eta_1 = \varepsilon_1 \eta_2 = \varepsilon_1 \varepsilon_2 \eta_3 = \ldots = \varepsilon_1 \cdots \varepsilon_m \eta_1 = -\eta_1$, a contradiction with $\eta_1 = \pm 1$. $\square$

The above result concerning $w_2\,(\mathrm{ad}\,\alpha)$ can be sharpened using the following algebraic topology lemma.

2.2.2 Lemma. *Let $\widehat{\sigma}$ be a link of two components. If $\widehat{\sigma}$ is non-split then $H^2(\pi_1(S^3 - \hat{\sigma});\mathbb{Z}_2) = \mathbb{Z}_2$; otherwise, $H^2(\pi_1(S^3-\hat{\sigma});\mathbb{Z}_2) = 0$.*

Proof. If $\widehat{\sigma}$ is non-split then $S^3 - \widehat{\sigma}$ is a $K(\pi,1)$ by the Sphere Theorem, hence $H^2(\pi_1(S^3-\widehat{\sigma});\mathbb{Z}_2) = H^2(S^3-\widehat{\sigma};\mathbb{Z}_2) = \mathbb{Z}_2$. If $\widehat{\sigma}$ happens to be split then $K(\pi_1(S^3 - \widehat{\sigma}),1)$ has the homotopy type of a one-point union of two circles and the result again follows. $\square$

2.2.3 Corollary. *Let $\widehat{\sigma}$ be a split link of two components, and let ε be chosen as in (2.7). Then the diffeomorphism $\varepsilon\sigma : R_n \to R_n$ has no fixed points.*

2.2.2 The zero-trace condition

A naive way to define $h(\varepsilon\sigma)$ would be as the intersection number of the graph of $\varepsilon\sigma : R_n \to R_n$ with the diagonal in the product $R_n \times R_n$. One can observe though that, in addition to this intersection not being transversal, its points $(X,X) = (X_1,\ldots,X_n,X_1,\ldots,X_n)$ have the property that $\mathrm{tr}\ X_1 = \ldots = \mathrm{tr}\ X_n = 0$. This can be seen as follows.

Assume without loss of generality that $\bar{\sigma} = (1\ldots m)(m+1\ldots n)$. Then the relations $X = \varepsilon\sigma(X)$ together with (2.5) imply that

$$\begin{aligned}
X_1 &= \varepsilon_1\sigma(X)_1 = \varepsilon_1 A_1\cdot X_{\bar{\sigma}(1)}\cdot A_1^{-1} = \varepsilon_1 A_1 X_2 A_1^{-1}\\
&= \varepsilon_1 A_1\cdot\varepsilon_2\,\sigma(X)_2\cdot A_1^{-1} = \varepsilon_1\varepsilon_2\,A_1A_2\cdot X_{\bar{\sigma}(2)}\cdot A_2^{-1}A_1^{-1} = \ldots\\
&= \varepsilon_1\cdots\varepsilon_m\,(A_1\cdots A_m)\cdot X_1\cdot(A_1\ldots A_m)^{-1}.
\end{aligned}$$

Since trace is conjugation invariant and $\varepsilon_1 \dots \varepsilon_m = -1$, we conclude that tr $X_1 = \dots =$ tr $X_m = 0$. Similarly, tr $X_{m+1} = \dots =$ tr $X_n = 0$.

Therefore, in our definition we will restrict ourselves to the subset of R_n consisting of $X = (X_1, \dots, X_n)$ with tr $X_1 = \dots =$ tr $X_n = 0$. The non-transversality problem will be addressed below by factoring out the conjugation symmetry and lowering the dimension of the ambient manifold.

2.2.3 The definition

The subset of $SU(2)$ consisting of the matrices with zero trace is a conjugacy class in $SU(2)$ diffeomorphic to S^2. Define

$$Q_n = \{(X_1, \dots, X_n) \in R_n \mid \text{ tr } X_i = 0\,\} \subset R_n,$$

so that Q_n is a manifold diffeomorphic to $(S^2)^n$. Also define

$$H_n = \{(X_1, \dots, X_n, Y_1, \dots, Y_n) \in Q_n \times Q_n \mid X_1 \cdots X_n = Y_1 \cdots Y_n\,\}.$$

This is no longer a manifold due to the presence of *reducibles.* We call a point $(X_1, \dots, X_n, Y_1, \dots, Y_n) \in Q_n \times Q_n$ reducible if all X_i and Y_j commute with each other or, equivalently, if there is a matrix $A \in SU(2)$ such that AX_iA^{-1} and AY_iA^{-1} are all diagonal matrices, $i = 1, \dots, n$. The subset $S_n \subset Q_n \times Q_n$ of reducibles is closed.

2.2.4 Lemma. *$H_n^* = H_n - S_n$ is an open manifold of dimension $4n - 3$.*

Proof. Let us consider the open manifold $(Q_n \times Q_n)^* = Q_n \times Q_n - S_n$ of dimension

$4n$ and the map $f : (Q_n \times Q_n)^* \to SU(2)$ given by

$$f(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = X_1 \cdots X_n Y_n^{-1} \cdots Y_1^{-1}. \tag{2.9}$$

According to Lemma 1.5 of [20], this map has $1 \in SU(2)$ as a regular value. Since $H_n^* = f^{-1}(1)$, the result follows. $\square$

Because of (2.5) and the fact that multiplication by $-1 \in SU(2)$ preserves the zero-trace condition, the representation (2.4) gives rise to a representation

$$\rho : \ \mathbb{Z}_2 \wr \mathcal{B}_n \longrightarrow \mathrm{Diff}(Q_n). \tag{2.10}$$

Given $\varepsilon\sigma \in \mathbb{Z}_2 \wr \mathcal{B}_n$ such that (2.6) and (2.7) are satisfied, consider two submanifolds of $Q_n \times Q_n$: one is the graph of $\varepsilon\sigma : Q_n \to Q_n$,

$$\Gamma_{\varepsilon\sigma} = \{\, (X, \varepsilon\sigma(X)) \mid X \in Q_n \,\},$$

and the other the diagonal,

$$\Delta_n = \{\, (X, X) \mid X \in Q_n \,\}.$$

Note that both $\Gamma_{\varepsilon\sigma}$ and Δ_n are subsets of H_n: this is obvious for Δ_n, and follows from equation (2.8) for $\Gamma_{\varepsilon\sigma}$.

2.2.5 Proposition. *The intersection $\Gamma_{\varepsilon\sigma} \cap \Delta_n \subset H_n$ consists of irreducible representations.*

Proof. Assume without loss of generality that $\bar{\sigma} = (1 \ldots m)(m+1 \ldots n)$, and suppose that $(X, X) = (X_1, \dots, X_n, X_1, \dots, X_n) \in \Gamma_{\varepsilon\sigma} \cap \Delta_n$ is reducible. Then all of the X_i commute with each other and, in particular, $\sigma(X) = (X_{\bar\sigma(1)}, \dots, X_{\bar\sigma(n)})$. The equality $X = \varepsilon\sigma(X)$ then implies that $X_1 = \varepsilon_1 X_{\bar\sigma(1)} = \varepsilon_1 X_2 = \varepsilon_1\varepsilon_2 X_{\bar\sigma(2)} = \dots = \varepsilon_1 \cdots \varepsilon_m X_1 = -X_1$, a contradiction with $X_1 \in SU(2)$. □

Let $\Gamma^*_{\varepsilon\sigma} = \Gamma_{\varepsilon\sigma} \cap H^*_n$ and $\Delta^*_n = \Delta_n \cap H^*_n$ be the irreducible parts of $\Gamma_{\varepsilon\sigma}$ and Δ_n, respectively. They are both open submanifolds of H^*_n of dimension $2n$.

2.2.6 Corollary. *The intersection $\Delta^*_n \cap \Gamma^*_{\varepsilon\sigma} \subset H^*_n$ is compact.*

Proof. Proposition 2.2.5 implies that $\Delta^*_n \cap \Gamma^*_{\varepsilon\sigma} = \Delta_n \cap \Gamma_{\varepsilon\sigma}$, and the latter intersection is obviously compact as it is the intersection of two compact subsets of H_n. □

The group $SO(3) = SU(2)/\{\pm 1\}$ acts freely by conjugation on H^*_n, Δ^*_n, and $\Gamma^*_{\varepsilon\sigma}$. Denote the resulting quotient manifolds by

$$\widehat{H}_n = H^*_n/SO(3), \quad \widehat{\Delta}_n = \Delta^*_n/SO(3), \quad \text{and} \quad \widehat{\Gamma}_{\varepsilon\sigma} = \Gamma^*_{\varepsilon\sigma}/SO(3).$$

The dimension of $\widehat{H}_n$ is $4n - 6$, and $\widehat{\Delta}_n$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ are submanifolds, each of dimension $2n - 3$. Since the intersection $\widehat{\Delta}_n \cap \widehat{\Gamma}_{\varepsilon\sigma}$ is compact, one can isotope $\widehat{\Gamma}_{\varepsilon\sigma}$ into a submanifold $\widetilde{\Gamma}_{\varepsilon\sigma}$ using an isotopy with compact support so that $\widehat{\Delta}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma}$ consists of finitely many points. Define

$$h(\varepsilon\sigma) = \#_{\widehat{H}_n} (\widehat{\Delta}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma})$$

as the algebraic intersection number, where the orientations of $\widehat{H}_n$, $\widehat{\Delta}_n$ and $\widetilde{\Gamma}_{\varepsilon\sigma}$ are described in the following subsection. It is obvious that $h(\varepsilon\sigma)$ does not depend on the perturbation of $\widehat{\Gamma}_{\varepsilon\sigma}$ so we will simply write

$$h(\varepsilon\sigma) = \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma} \,\rangle_{\widehat{H}_n}.$$

2.2.4 Orientations

Orient the copy of $S^2 \subset SU(2)$ cut out by the trace zero condition arbitrarily, and endow $Q_n = (S^2)^n$ and $Q_n \times Q_n$ with product orientations. The diagonal Δ_n and the graph $\Gamma_{\varepsilon\sigma}$ are naturally diffeomorphic to Q_n via projection onto the first factor, and they are given the induced orientations. Note that if we reverse the orientation of S^2, then the orientation of Q_n is reversed if n is odd. Hence the orientations of both Δ_n and $\Gamma_{\varepsilon\sigma}$ are reversed if n is odd, while the orientation of $Q_n \times Q_n = (S^2)^{2n}$ is preserved regardless of the parity of n.

Orient $SU(2)$ by the standard basis $\{i, j, k\}$ in its Lie algebra $\mathfrak{su}(2)$, and orient $H_n^* = f^{-1}(1)$ by applying the base–fiber rule to the map (2.9). The adjoint action of $SO(3)$ on $S^2 \subset SU(2)$ is orientation preserving, hence the $SO(3)$ quotients $\widehat{H}_n$, $\widehat{\Delta}_n$, and $\widehat{\Gamma}_{\varepsilon\sigma}$ are orientable. We orient them using the base–fiber rule. The discussion in the previous paragraph shows that reversing orientation on S^2 may reverse the orientations of $\widehat{\Delta}_n$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ but that it does not affect the intersection number $\langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma} \,\rangle_{\widehat{H}_n}$.

2.3 The link invariant h

In this section, we will prove Theorem 1. This will be accomplished by proving that $h(\varepsilon\sigma)$ is independent first of ε and then of σ.

2.3.1 Independence of ε

We will first show that, for a fixed σ whose closure $\hat{\sigma}$ is a link of two components, $h(\varepsilon\sigma)$ is independent of the choice of ε as long as ε satisfies (2.7).

2.3.1 Proposition. *Let ε and ε' be such that (2.7) is satisfied. Then $h(\varepsilon\sigma) = h(\varepsilon'\sigma)$.*

Proof. Assume without loss of generality that $\bar{\sigma} = (1\dots m)(m+1\dots n)$ and let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ and $\varepsilon' = (\varepsilon_1', \dots, \varepsilon_n')$. Define $\delta = (\delta_1, \dots, \delta_n)$ as the vector in $(\mathbb{Z}_2)^n$ with coordinates

$$\delta_1 = 1 \quad \text{and} \quad \delta_{k+1} \;=\; \delta_k\,\varepsilon_k\,\varepsilon_k' \quad \text{for} \quad k = 1, \dots, n-1,$$

and define the involution $\tau : Q_n \to Q_n$ by the formula

$$\tau(X) = \delta X = (\delta_1 X_1, \delta_2 X_2, \dots, \delta_n X_n).$$

Recall that $Q_n = (S^2)^n$ so that τ is a diffeomorphism which restricts to each of the factors S^2 as either the identity or the antipodal map. In particular, τ need not be orientation preserving.

The map $\tau\times\tau : Q_n\times Q_n \to Q_n\times Q_n$ obviously preserves the irreducibility condition and commutes with the $SO(3)$ action. It gives rise to an orientation preserving automorphism of $\widehat{H}_n$ which will again be called $\tau\times\tau$. It is clear that $(\tau\times\tau)(\widehat{\Delta}_n) = \widehat{\Delta}_n$. It is also true that $(\tau\times\tau)(\widehat{\Gamma}_{\varepsilon\sigma}) = \widehat{\Gamma}_{\varepsilon'\sigma}$, which can be seen as follows. Given a pair $(\delta X, \delta\varepsilon\sigma(X))$ whose conjugacy class belongs to $(\tau\times\tau)(\widehat{\Gamma}_{\varepsilon\sigma})$, write it as

$$(\delta X, \delta\varepsilon\sigma(X)) = (\delta X, \delta\varepsilon\sigma(\delta\delta X)) = (\delta X, \delta\varepsilon\sigma(\delta)\,\sigma(\delta X))$$

using the multiplication law in the group $\mathbb{Z}_2 \wr \mathcal{B}_n$. The conjugacy class of this pair belongs to $\Gamma_{\varepsilon'\sigma}$ if and only if $\delta\varepsilon\sigma(\delta) = \varepsilon'$. That this condition holds can be verified directly from the definition of δ.

Recall that the orientations of $\widehat{\Delta}_n$, $\widehat{\Gamma}_{\varepsilon\sigma}$, and $\widehat{\Gamma}_{\varepsilon'\sigma}$ are induced by that of Q_n. Therefore, the maps $\tau\times\tau : \widehat{\Delta}_n \to \widehat{\Delta}_n$ and $\tau\times\tau : \widehat{\Gamma}_{\varepsilon\sigma} \to \widehat{\Gamma}_{\varepsilon'\sigma}$ are either both orientation preserving or both orientation reversing depending on whether $\tau : Q_n \to Q_n$ preserves or reverses orientation. Hence we have

$$\begin{aligned} h(\varepsilon\sigma) = \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma} \,\rangle_{\widehat{H}_n} &= \langle\, (\tau\times\tau)(\widehat{\Delta}_n), (\tau\times\tau)(\widehat{\Gamma}_{\varepsilon\sigma}) \,\rangle_{(\tau\times\tau)(\widehat{H}_n)} \\ &= \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon'\sigma} \,\rangle_{\widehat{H}_n} = h(\varepsilon'\sigma). \end{aligned}$$

□

From now on, we will drop ε from the notation and simply write $h(\sigma)$ for $h(\varepsilon\sigma)$ assuming that a choice of ε satisfying (2.7) has been made.

2.3.2 Independence of σ

In this section, we will show that $h(\sigma)$ only depends on the link $\hat{\sigma}$, not on a particular choice of braid σ, by verifying that h is preserved under Markov moves. We will follow the proof of [20, Theorem 1.8] which goes through with little change once the right ε are chosen.

Recall that two braids $\alpha \in \mathcal{B}_n$ and $\beta \in \mathcal{B}_m$ have isotopic closures $\hat{\alpha}$ and $\hat{\beta}$ if and only if one braid can be obtained from the other by a finite sequence of Markov moves; see for instance [2]. A type 1 Markov move replaces $\sigma \in \mathcal{B}_n$ by $\xi^{-1}\sigma\xi \in \mathcal{B}_n$ for any $\xi \in \mathcal{B}_n$.

A type 2 Markov move means replacing $\sigma \in \mathcal{B}_n$ by $\sigma_n^{\pm 1}\sigma \in \mathcal{B}_{n+1}$, or the inverse of this operation.

2.3.2 Proposition. *The invariant $h(\sigma)$ is preserved by type 1 Markov moves.*

Proof. Let ξ, $\sigma \in \mathcal{B}_n$ and assume as usual that $\bar{\sigma} = (1 \dots m)(m+1 \dots n)$. Then

$$\overline{\xi^{-1}\sigma\xi} = (\bar{\xi}(1) \dots \bar{\xi}(m))(\bar{\xi}(m+1) \dots \bar{\xi}(n))$$

has the same cycle structure as $\bar{\sigma}$. To compute $h(\xi^{-1}\sigma\xi)$, we will make a choice of $\varepsilon \in (\mathbb{Z}_2)^n$ which satisfies the condition (2.7) with respect to the braid $\xi^{-1}\sigma\xi$, that is, $\varepsilon_{\bar{\xi}(1)} \cdots \varepsilon_{\bar{\xi}(m)} = \varepsilon_{\bar{\xi}(m+1)} \cdots \varepsilon_{\bar{\xi}(n)} = -1$.

The braid ξ gives rise to the map $\xi : Q_n \to Q_n$. It acts by permutation and conjugation on the S^2 factors in Q_n hence it is orientation preserving (we use the fact that S^2 is even dimensional). It induces an orientation preserving map $\xi \times \xi : Q_n \times Q_n \to Q_n \times Q_n$, which preserves the irreducibility condition and commutes with the $SO(3)$ action. Equation (2.2) then ensures that we have a well defined orientation preserving automorphism $\xi \times \xi : \widehat{H}_n \to \widehat{H}_n$.

That this automorphism preserves the diagonal, $(\xi \times \xi)(\widehat{\Delta}_n) = \widehat{\Delta}_n$, is obvious. Concerning the graphs, let $(X, \varepsilon\xi^{-1}\sigma\xi(X)) \in \widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi}$ then

$$\begin{aligned}&(\xi \times \xi)(X, \varepsilon\xi^{-1}\sigma\xi(X)) \\ &\qquad = (\xi(X), \xi(\varepsilon\xi^{-1}\sigma\xi(X))) = (\xi(X), \xi(\varepsilon)\sigma(\xi(X))) \in \widehat{\Gamma}_{\xi(\varepsilon)\sigma}.\end{aligned}$$

Therefore, $(\xi \times \xi)(\widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi}) = \widehat{\Gamma}_{\xi(\varepsilon)\sigma}$. Since $\xi : Q_n \to Q_n$ is orientation preserving, the above identifications of the diagonals and graphs via $\xi \times \xi$ are also orientation preserving.

Observe that $\xi(\varepsilon)_i = \varepsilon_{\bar{\xi}(i)}$ hence $\xi(\varepsilon)$ satisfies (2.7) with respect to σ and thus can be used to compute $h(\sigma)$. The following calculation now completes the argument:

$$\begin{aligned} h(\xi^{-1}\sigma\xi) = \langle \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi} \rangle_{\widehat{H}_n} &= \langle (\xi \times \xi)(\widehat{\Delta}_n), (\xi \times \xi)(\widehat{\Gamma}_{\varepsilon\xi^{-1}\sigma\xi}) \rangle_{(\xi\times\xi)(\widehat{H}_n)} \\ &= \langle \widehat{\Delta}_n, \widehat{\Gamma}_{\xi(\varepsilon)\sigma} \rangle_{\widehat{H}_n} = h(\sigma). \end{aligned}$$

□

2.3.3 Proposition. *The invariant $h(\sigma)$ is preserved by type 2 Markov moves.*

Proof. Given $\sigma \in \mathcal{B}_n$ and ε satisfying (2.7), change σ to $\sigma_n\sigma \in \mathcal{B}_{n+1}$ and let $\varepsilon' = \sigma_n(\varepsilon, 1)$. If $X = (X_1, \dots, X_n)$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ then

$$\begin{aligned} (\sigma_n\sigma)(X, X_{n+1}) &= \sigma_n(\sigma(X), X_{n+1}) \\ &= (\sigma(X)_1, \dots, \sigma(X)_{n-1},\, \sigma(X)_n X_{n+1} \sigma(X)_n^{-1},\, \sigma(X)_n)) \end{aligned}$$

and $\varepsilon' = (\varepsilon_1, \dots, \varepsilon_{n-1}, 1, \varepsilon_n)$. In particular, ε' satisfies (2.7) with respect to $\sigma_n\sigma$. Consider the embedding $g : Q_n \times Q_n \to Q_{n+1} \times Q_{n+1}$ given by

$$g(X_1, \dots, X_n, Y_1, \dots, Y_n) = (X_1, \dots, X_n, Y_n, Y_1, \dots, Y_n, Y_n)$$

One can easily see that $g(H_n) \subset H_{n+1}$ and that g commutes with the conjugation, thus giving rise to an embedding $\hat{g} : \widehat{H}_n \to \widehat{H}_{n+1}$. A straightforward calculation using the above formulas for $\sigma_n\sigma$ and ε' then shows that

$$\hat{g}\,(\widehat{\Delta}_n) \subset \widehat{\Delta}_{n+1}, \quad \hat{g}\,(\widehat{\Gamma}_{\varepsilon\sigma}) \subset \widehat{\Gamma}_{\varepsilon'\sigma_n\sigma} \quad \text{and} \quad \hat{g}\,(\widehat{\Delta}_n \cap \widehat{\Gamma}_{\varepsilon\sigma}) = \widehat{\Delta}_{n+1} \cap \widehat{\Gamma}_{\varepsilon'\sigma_n\sigma}.$$

Now, one can achieve all the necessary transversalities and match the orientations in exactly the same way as in the second half of the proof of [20, Theorem 1.8]. This shows that $h(\sigma_n\sigma) = h(\sigma)$. The proof of the equality $h(\sigma_n^{-1}\sigma) = h(\sigma)$ is similar. $\square$

2.4 The invariant $h(\sigma)$ as the linking number

In this section we will prove Theorem 2, that is, show that for any link $\hat{\sigma} = \ell_1 \cup \ell_2$ of two components, one has

$$h(\sigma) \;=\; \pm\,\mathrm{lk}\,(\ell_1, \ell_2).$$

Our strategy will be to show that the invariant $h(\sigma)$ and the linking number $\mathrm{lk}\,(\ell_1, \ell_2)$ change according to the same rule as we change a crossing between two strands from two different components of $\hat{\sigma} = \ell_1 \cup \ell_2$ (the link $\hat{\sigma}$ will need to be oriented for that, although a particular choice of orientation will not matter). After changing finitely many such crossings, we will arrive at a split link, for which both the invariant $h(\sigma)$ and the linking number $\mathrm{lk}\,(\ell_1, \ell_2)$ vanish; see Corollary 2.2.3. The change of crossing as above obviously changes the linking number by ± 1. To calculate the effect of the crossing change on $h(\sigma)$, we will follow [20] and reduce the problem to a calculation in the pillowcase $\widehat{H}_2$.

2.4.1 The pillowcase

We begin with a geometric description of $\widehat{H}_2$ as a pillowcase, compare with [20, Lemma 1.2]. Remember that

$$H_2 = \{(X_1, X_2, Y_1, Y_2) \in Q_2 \times Q_2 \mid X_1 X_2 = Y_1 Y_2\,\}.$$

We will use the identification of $SU(2)$ with $Sp(1)$ when convenient. Since X_2 is trace free, we may assume that $X_2 = i$ after conjugation. Conjugating by $e^{i\varphi}$ will not change X_2 but, for an appropriate choice of φ, will make X_1 into

$$X_1 = \begin{pmatrix} ir & u \\ -u & -ir \end{pmatrix},$$

where both r and u are real, and u is also non-negative. Since $r^2 + u^2 = 1$ we can write $r = \cos\theta$ and $u = \sin\theta$ for a unique θ such that $0 \le \theta \le \pi$. In the quaternionic language, $X_1 = i\,e^{-k\theta}$ with $0 \le \theta \le \pi$. Similarly, the condition $\operatorname{tr}(Y_2) = \operatorname{tr}(Y_1^{-1}X_1X_2) = 0$ implies that $Y_1 = i\,e^{-k\psi}$, this time with $-\pi \le \psi \le \pi$. To summarize,

$$X_1 = ie^{-k\theta},\ \ X_2 = i,\ \ Y_1 = ie^{-k\psi},\ \ Y_2 = ie^{-k(\psi-\theta)}.$$

The 2-manifold $\widehat{H}_2$ is then parameterized by the rectangle $[0,\pi]\times[-\pi,\pi]$, with proper identifications along the edges and with the reducibles removed. The reducibles occur when both θ and ψ are multiples of π hence we end up with a 2-sphere with the points $A = (0,0)$, $B = (\pi,0)$, $A' = (0,\pi)$, and $B' = (\pi,\pi)$ removed; see Figure 2.1. According to [20], the orientation on the front sheet of $\widehat{H}_2$ coincides with the standard orientation on the (θ,ψ) plane.

Example. Let $\sigma = \sigma_1^2$ so that $\widehat{\sigma} = \ell_1 \cup \ell_2$ is the Hopf link with $\operatorname{lk}(\ell_1,\ell_2) = \pm 1$. To calculate $h(\sigma)$, we let $\varepsilon = (-1,-1)$, the only available choice satisfying (2.7), and consider the submanifolds $\widehat{\Delta}_2$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ of $\widehat{H}_2$. We have, in quaternionic notations, $\widehat{\Delta}_2 = \{(ie^{-k\theta}, i, ie^{-k\theta}, i)\}$, which is the diagonal $\psi = \theta$ in the pillowcase. A straightforward calculation shows that $\widehat{\Gamma}_{\varepsilon\sigma} \subset \widehat{H}_2$ is given by $\psi = 3\theta - \pi$. As can be seen in Figure 2.1, the intersection $\widehat{\Delta}_2 \cap \widehat{\Gamma}_{\varepsilon\sigma}$ consists of one point coming with a sign. Hence

$h(\sigma_1^2) = \pm 1$, which is consistent with the fact that $\mathrm{lk}\,(\ell_1, \ell_2) = \pm 1$.

Example. Let $\sigma = \sigma_1^{2n}$ then arguing as above one can show that $\widehat{\Gamma}_{\varepsilon\sigma} \subset \widehat{H}_2$ is given by $\psi = (2n+1)\theta - \pi$. In this case there are n intersection points all of which come with the same sign. This shows that $h(\sigma_1^{2n}) = \pm n$, which is again consistent with the fact that $\mathrm{lk}\,(\ell_1, \ell_2) = \pm n$.

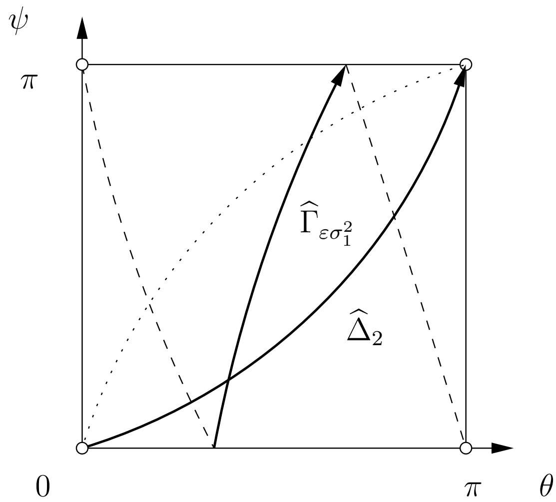

Figure 2.1: The pillowcase and the Hopf link

2.4.2 The difference cycle

Given a two component link $\hat{\sigma}$, fix an orientation on it. A particular choice of orientation will not matter because we are only interested in identifying $h(\sigma)$ with the linking number $\mathrm{lk}\,(\ell_1, \ell_2)$ up to sign. We wish to change one of the crossings between the two components of $\hat{\sigma}$. Using a sequence of first Markov moves, we may assume that the first two strands of σ belong to two different components of $\hat{\sigma}$, and that the crossing change occurs between these two strands. Furthermore, we may assume that the crossing change makes σ into $\sigma_1^{\pm 2}\sigma$, where the sign depends on the type of the crossing we change. Note that the braid $\sigma_1^{\pm 2}\sigma$ has the same permutation type as σ; in particular, its closure is a link of two components. In fact, if we let $\sigma' = \sigma_1^{-2}\sigma$ then

$$h(\sigma_1^{-2}\sigma) - h(\sigma) = h(\sigma') - h(\sigma_1^2\sigma') = -\left(h(\sigma_1^2\sigma') - h(\sigma')\right),$$

hence we only need to understand the difference $h(\sigma_1^2\sigma) - h(\sigma)$. Let us fix $\varepsilon = (-1,-1,1,\ldots,1)$. Since σ_1^2 and ε commute, we have

$$\begin{aligned} h(\sigma_1^2\sigma) - h(\sigma) &= \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma_1^2\sigma}\,\rangle - \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle \\ &= \langle\, \widehat{\Gamma}_{\sigma_1^{-2}}, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle - \langle\, \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle = \langle\, \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n, \widehat{\Gamma}_{\varepsilon\sigma}\,\rangle, \end{aligned}$$

where all intersection numbers are taken in $\widehat{H}_n$. This leads us to consider the *difference cycle* $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$ which is carried by $\widehat{H}_n$. The next step in our argument will be to reduce the analysis of the above intersection to an intersection theory in the pillowcase $\widehat{H}_2$.

2.4.3 The pillowcase reduction

Let us consider the subset $V_n \subset H_n$ consisting of all $(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \in H_n$ such that $X_k = Y_k$ for $k = 3,\ldots,n$. Equivalently, V_n consists of all $(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \in Q_n \times Q_n$ such that $(X_1,X_2,Y_1,Y_2) \in H_2$ and $X_k = Y_k$ for all $k = 3,\ldots,n$, and therefore can be identified as

$$V_n = H_2 \times \Delta_{n-2} \;\subset\; (Q_2 \times Q_2) \times (Q_{n-2} \times Q_{n-2}).$$

2.4.1 Lemma. *The quotient* $\widehat{V}_n = (H_2^* \times \Delta_{n-2})/SO(3)$ *is a submanifold of* $\widehat{H}_n$ *of* *dimension* $2n-2$.

Proof. Since H_2^* and Δ_{n-2} are smooth manifolds of dimensions five and $2n-4$, respectively, and their product contains no reducibles, the statement follows. □

2.4.2 Lemma. *The difference cycle* $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$ *is carried by* $\widehat{V}_n$.

Proof. Observe that neither $\widehat{\Gamma}_{\sigma_1^{-2}}$ nor $\widehat{\Delta}_n$ are in fact subsets of $\widehat{V}_n$. However, their portions that do not fit in $\widehat{V}_n$,

$$\widehat{\Gamma}_{\sigma_1^{-2}} - (\widehat{\Gamma}_{\sigma_1^{-2}} \cap \widehat{V}_n) \quad \text{and} \quad \widehat{\Delta}_n - (\widehat{\Delta}_n \cap \widehat{V}_n),$$

are exactly the same – namely, they consist of the equivalence classes of $2n$–tuples $(X_1, \dots, X_n, X_1, \dots, X_n)$ such that X_1 commutes with X_2. These cancel in the difference cycle $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$, thus making it belong to $\widehat{V}_n$. □

One can isotope $\widehat{\Gamma}_{\varepsilon\sigma}$ into $\widetilde{\Gamma}_{\varepsilon\sigma}$ using an isotopy with compact support so that $\widetilde{\Gamma}_{\varepsilon\sigma}$ is transverse to $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n$. The latter means precisely that $\widetilde{\Gamma}_{\varepsilon\sigma}$ stays away from $(S_2 \times \Delta_{n-2})/SO(3)$ and is transverse to both $\widehat{\Gamma}_{\sigma_1^{-2}}$ and $\widehat{\Delta}_n$; a precise argument can be found in [12, page 491]. We further extend this isotopy to make $\widetilde{\Gamma}_{\varepsilon\sigma}$ transverse to $\widehat{V}_n$ so that their intersection is a naturally oriented 1-dimensional submanifold of $\widehat{H}_n$.

The natural projection $p : V_n \to H_2$ induces a map $\hat{p} : \widehat{V}_n \to \widehat{H}_2$. Use a further small compactly supported isotopy of $\widetilde{\Gamma}_{\varepsilon\sigma}$ if necessary to make $\hat{p}\,(\widehat{V}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma})$ into a 1-submanifold of $\widehat{H}_2$. The proofs of Lemmas 2.2 and 2.3 in Lin [20] then go through with little change to give us the following identity

$$\langle\, \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_n,\ \widetilde{\Gamma}_{\varepsilon\sigma} \,\rangle_{\widehat{H}_n} = \langle\, \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_2,\ \hat{p}\,(\widehat{V}_n \cap \tilde{\Gamma}_{\varepsilon\sigma}) \,\rangle_{\widehat{H}_2}.$$

2.4.4 Computation in the pillowcase

We begin by studying the behavior of $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ near the corners of $\widehat{H}_2$.

2.4.3 Proposition. *There is a neighborhood around A' in the pillowcase $\widehat{H}_2$ inside which $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ is a curve approaching A'.*

Proof. Let us consider the submanifold

$$\Delta'_n = \{(X_1, X_2, X_3, \ldots, X_n; Y_1, Y_2, X_3, \ldots, X_n)\} \subset Q_n \times Q_n$$

and observe that $V_n \cap \Gamma_{\varepsilon\sigma} = \Delta'_n \cap \Gamma_{\varepsilon\sigma}$. We will show that the intersection of Δ'_n with $\Gamma_{\varepsilon\sigma}$ is transversal at $(\mathbf{i}, \varepsilon\,\mathbf{i}) = (i, \ldots, i; -i, -i, i, \ldots, i)$. This will imply that $\Delta'_n \cap \Gamma_{\varepsilon\sigma}$ a submanifold of dimension four in a neighborhood of $(\mathbf{i}, \varepsilon\,\mathbf{i})$ and, after factoring out the $SO(3)$ symmetry, that $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ is a curve approaching $A' = p\,(\mathbf{i}, \varepsilon\,\mathbf{i})$.

Note that $\dim \Delta'_n = 2n + 4$ hence the dimension of $T_{(\mathbf{i},\varepsilon\,\mathbf{i})}(\Delta'_n \cap \Gamma_{\varepsilon\sigma}) = T_{(\mathbf{i},\varepsilon\,\mathbf{i})}\Delta'_n \cap T_{(\mathbf{i},\varepsilon\,\mathbf{i})}\Gamma_{\varepsilon\sigma}$ is at least four. Therefore, checking the transversality amounts to showing that this dimension is exactly four. Write

$$T_{(\mathbf{i},\varepsilon\mathbf{i})}(\Delta'_n) = \{(u_1, \ldots u_n; v_1, v_2, u_3, \ldots, u_n)\} \subset T_{(\mathbf{i},\varepsilon\mathbf{i})}(Q_n \times Q_n)$$

and

$$T_{(\mathbf{i},\varepsilon\mathbf{i})}(\Gamma_{\varepsilon\sigma}) = \{(u_1, \ldots, u_n; d_{\mathbf{i}}(\varepsilon\sigma)(u_1, \ldots, u_n)\} \subset T_{(\mathbf{i},\varepsilon\mathbf{i})}(Q_n \times Q_n).$$

Then $T_{(\mathbf{i},\varepsilon\mathbf{i})}(\Delta_n') \cap T_{(\mathbf{i},\varepsilon\,\mathbf{i})}(\Gamma_{\varepsilon\sigma})$ consists of the vectors $(u_1,\ldots,u_n) \in T_{\mathbf{i}}\,Q_n = T_i\,S^2 \oplus \ldots \oplus T_i\,S^2$ that solve the matrix equation

$$\begin{bmatrix} d_{\mathbf{i}}(\sigma) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} * \\ * \\ u_3 \\ \vdots \\ u_n \end{bmatrix}; \tag{2.11}$$

since $\varepsilon = (-1,-1,1,\ldots,1)$, we can safely replace $[d_{\mathbf{i}}(\varepsilon\sigma)]$ by $[d_{\mathbf{i}}(\sigma)]$. It is shown in [21] that $[d_{\mathbf{i}}(\sigma)]$ is the Burau matrix of σ with parameter equal to -1. It is a real matrix acting on $T_{\mathbf{i}}\,Q_n = \mathbb{C}^n$, hence all we need to show is that the space of $(u_1,\ldots,u_n) \in \mathbb{R}^n$ solving (2.11) has real dimension two. Let us write

$$\begin{bmatrix} d_{\mathbf{i}}(\sigma) \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is a 2×2 matrix and D is an $(n-2)\times(n-2)$ matrix. Equation (2.11) is equivalent to

$$\begin{bmatrix} C \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 - D \end{bmatrix} \begin{bmatrix} u_3 \\ \vdots \\ u_n \end{bmatrix},$$

so the proposition will follow as soon as we show that $1 - D$ is invertible. $\square$

The invertibility of $1-D$ is a consequence of the following two lemmas.

2.4.4 Lemma. *Let $\sigma \in \mathcal{B}_n$ then the Burau matrix of σ with parameter -1 and the permutation matrix of $\bar{\sigma}$ are the same modulo 2.*

Proof. According to [2], the Burau matrix of σ with parameter t is the matrix

$$\frac{\partial \sigma(x_i)}{\partial x_j}\Big|_{x_i=t}$$

where x_i are generators of the free group and ∂ is the derivative in the Fox free differential calculus; see [10]. Applying the Fox calculus we obtain:

$$\begin{aligned}\frac{\partial \sigma(x_i)}{\partial x_j} &= \frac{\partial (w x_{\bar{\sigma}(i)} w^{-1})}{\partial x_j} = \frac{\partial w}{\partial x_j} + w\left(\frac{\partial (x_{\bar{\sigma}(i)} w^{-1})}{\partial x_j}\right)\\ &= \frac{\partial w}{\partial x_j} + w\left(\frac{\partial x_{\bar{\sigma}(i)}}{\partial x_j} + x_{\bar{\sigma}(i)}\frac{\partial w^{-1}}{\partial x_j}\right) = \frac{\partial w}{\partial x_j} + w\frac{\partial x_{\bar{\sigma}(i)}}{\partial x_j} - w x_{\bar{\sigma}(i)} w^{-1}\frac{\partial w}{\partial x_j},\end{aligned}$$

where w is a word in the x_i. After evaluating at $t=-1$ and reducing modulo 2, the above becomes simply $\partial x_{\bar{\sigma}(i)}/\partial x_j$, which is the permutation matrix of σ. □

2.4.5 Lemma. *Let $\sigma \in \mathcal{B}_n$ be such that $\widehat{\sigma}$ is a two component link. Then $1-D$ is invertible.*

Proof. Our assumption in this section has been that $\bar{\sigma} = (1,\cdots)(2,\cdots)$. We may further assume that

$$\bar{\sigma} = (1,3,4,\ldots,k)\,(2,k+1,k+2,\ldots,n)$$

by applying a sequence of first Markov moves fixing the first two strands of σ. The matrix $D \mod 2$ is obtained by crossing out the first two rows and first two columns in the permutation matrix of $\bar{\sigma}$; see Lemma 2.4.4. This description implies that $D \mod 2$ is upper diagonal, and hence so is $(1-D) \mod 2$. The diagonal elements of the latter matrix are all equal to one, therefore, $\det(1-D) = 1 \mod 2$ so $1-D$ is invertible. $\square$

Remark. The orientation of the component of $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$ limiting to A' can be read off its description near A' given in the proof of Proposition 2.4.3. In particular, this orientation is independent of the choice of σ.

2.4.6 Proposition. *There are neighborhoods around A and B' in the pillowcase $\widehat{H}_2$ which are disjoint from $\hat{p}\,(\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma})$.*

Proof. The arguments for A and B' are essentially the same so we will only give the proof for A. Assuming the contrary we have a curve in $\widehat{V}_n \cap \widehat{\Gamma}_{\varepsilon\sigma}$ limiting to a reducible representation in $V_n \cap \Gamma_{\varepsilon\sigma}$. After conjugating if necessary, this representation must have the form

$$(i, i, e^{i\varphi_3}, \ldots, e^{i\varphi_n}, i, i, e^{i\varphi_3}, \ldots, e^{i\varphi_n}).$$

Using the fact that $\varepsilon = (-1, -1, 1, \ldots, 1)$ and arguing as in the proof of Proposition 2.2.5, we arrive at the contradiction $i = -i$. $\square$

2.4.5 Proof of Theorem 2

According to Proposition 2.4.3, near A', the 1-submanifold $\hat{p}\,(\widehat{V}_n \cap \widetilde{\Gamma}_{\varepsilon\sigma})$ is a curve approaching A'.

According to Proposition 2.4.6, the other end of this curve must approach B. Therefore,

$$h(\sigma_1^2\sigma) - h(\sigma) \;=\; \langle \widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_2,\, \hat{p}\,(\widehat{V}_n \,\cap\, \widetilde{\Gamma}_{\varepsilon\sigma})\rangle_{\widehat{H}_2}$$

is the same as the intersection number of an arc going from A' to B with the difference cycle $\widehat{\Gamma}_{\sigma_1^{-2}} - \widehat{\Delta}_2$. This number is either 1 or -1 but it is the same for all σ; see Remark 2.4.4. This is sufficient to prove that $h(\sigma)$ is the linking number up to an overall sign.

Chapter 3

Instanton Floer homology for links

3.1 Furling up link exteriors

Let $L = \ell_1 \cup \ell_2$ be an oriented link of two components in an oriented integral homology sphere Σ. Consider the link exterior $X = \Sigma - N(L)$ and furl it up by gluing the boundary components of X together via an orientation reversing diffeomorphism $\varphi : T^2 \to T^2$. The resulting closed oriented manifold will be denoted X_φ.

3.1.1 Lemma. *The gluing map φ can be chosen so that X_φ has the integral homology of $S^1 \times S^2$.*

Proof. An orientation reversing diffeomorphism φ is determined up to isotopy by the homomorphism $\varphi_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ it induces on the fundamental groups of the two boundary components of X. Let μ_1, λ_1 be the canonical oriented meridian–longitude pair on one boundary component of X, and μ_2, λ_2 on the other. With respect to this choice of bases, φ_* is given by an integral matrix

$$\varphi_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with} \quad ad - bc = -1.$$

Now $H_1(X_\varphi)$ has generators t, μ_1, μ_2 and relations

$$\begin{aligned} \mu_2 &= a\mu_1 + b\lambda_1, \\ \lambda_2 &= c\mu_1 + d\lambda_1, \end{aligned}$$

where $\lambda_1 = n\mu_2$ and $\lambda_2 = n\mu_1$ with $n = \mathrm{lk}\,(\ell_1, \ell_2)$. In particular, X_φ has the integral homology of $S^1 \times S^2$ exactly when

$$\det \begin{pmatrix} a & bn-1 \\ c-n & dn \end{pmatrix} = bn^2 - 2n + c \; = \; \pm 1. \tag{3.1}$$

It is clear that one can always find φ such that this is the case. □

Remark. This construction appeared in Brakes [4] and Woodard [26] under the name of "sewing-up link exteriors", and was generalized by Hoste in [13].

3.2 Floer homology of homology $S^1 \times S^2$

In this section, we give a brief overview of the instanton Floer homology of 3-manifolds whose integral homology is that of $S^1 \times S^2$. We will (roughly) follow Floer [8], [9], Donaldson [6], and Braam–Donaldson [3].

Let $SO(3)$ be the Lie group of 3×3 real orthogonal matrices with determinant one. View $SO(3)$ as the quotient $SO(3) = SU(2)/\{\pm 1\}$. Let Y be a homology $S^1 \times S^2$, that is a closed oriented 3–manifold which has the integral homology of $S^1 \times S^2$.

Consider a principal $SO(3)$–bundle P over Y. Such bundles are classified by their second Stiefel–Whitney class $w_2(P) \in H^2(Y, \mathbb{Z}_2)$. Since $H^2(Y, \mathbb{Z}_2) = \mathbb{Z}_2$, P can be in one of exactly two isomorphism classes of bundles. We will only be interested in bundles P with $w_2(P) \neq 0$. Note that an $SO(3)$ bundle P lifts to an $SU(2)$ bundle precisely when $w_2(P) = 0$.

A *connection* on P is a smooth G-equivariant horizontal distribution of 3–planes in the tangent bundle TP. The connections on P form an affine space which will be denoted $\mathcal{A}$. The underlying vector space of $\mathcal{A}$ can be identified with $\Omega^1(Y, \operatorname{ad} P)$, the space of $\operatorname{ad} P$ valued one-forms on Y, where $\operatorname{ad} P$ is the vector bundle $P \times_{\mathrm{ad}} \mathfrak{so}(3)$ associated to the adjoint representation $\mathrm{ad} : SO(3) \to \mathrm{Aut}(\mathfrak{so}(3))$.

Given a connection $A \in \mathcal{A}$, any smooth path $\gamma : [0, 1] \to Y$ can be lifted to a horizontal path $\tilde{\gamma} : [0, 1] \to P$. The path $\tilde{\gamma}$ is determined uniquely by the lift $p = \tilde{\gamma}(0)$ of $\gamma(0)$. If $\gamma(0) = \gamma(1)$ then $\tilde{\gamma}(1)$ differs from $\tilde{\gamma}(0)$ by an element of $SO(3)$. This defines a map $\mathrm{hol}_{A,p} : \Omega(Y, \gamma(0)) \to SO(3)$, where $\Omega(Y, \gamma(0))$ is the monoid of loops in Y based at $\gamma(0)$. This map is called the *holonomy* of A at p. The image of the *holonomy* map of A at p is a subgroup of $SO(3)$, called the *holonomy group* of A. Any other lift of $\gamma(0)$ corresponds to conjugating this subgroup by an element of $SO(3)$. Therefore the holonomy group of A is well-defined up to $SO(3)$–conjugation.

The *curvature* $F_A \in \Omega^2(Y, \operatorname{ad} P)$ is a differential two-form with values in the vector bundle $\operatorname{ad} P$. In a local trivialization, $F_A = dA + A \wedge A$. A connection A is called *flat* if $F_A = 0$. In this case, the holonomy map $\mathrm{hol}_{A,p}$ factors through the fundamental group $\pi_1(Y, \gamma(0))$, so a flat connection A defines a homomorphism $\mathrm{hol}_{A,p} : \pi_1(Y, \gamma(0)) \to SO(3)$.

The automorphism group of the bundle P is called the *gauge group* $\mathcal{G}$. It is identified with the group of smooth sections of $\operatorname{Ad} P$, where $\operatorname{Ad} P$ is the fiber bundle $P \times_{Ad} SO(3)$ associated with the action of $SO(3)$ on itself by conjugation.

The gauge group $\mathcal{G}$ acts on $\mathcal{A}$ by sending a connection A to its pullback g^*A, where g is an automorphism of the bundle. Locally, the action is given by $g^*A = g^{-1}dg + g^{-1}Ag$. The stabilizer of A is isomorphic to the centralizer of its holonomy group. A connection A is called *reducible* when its stabilizer is non-trivial, and *irreducible* otherwise.

Endow Y with a Riemannian metric. In an appropriate Sobolev completion, $\mathcal{A}$ is a Hilbert manifold. Furthermore, reducible connections form a closed subset of $\mathcal{A}$ of infinite codimension, so that the space of irreducible connections $\mathcal{A}^*$ is a Hilbert manifold. Let $\mathcal{B} = \mathcal{A}^*/\mathcal{G}$ be the orbit space. In an appropriate Sobolev completion of $\mathcal{G}$, $\mathcal{B}$ is a Hilbert manifold.

It is well known that any compact oriented 3–manifold, such as Y, bounds a smooth compact oriented 4–manifold W. Extend A to a connection A' over W. Then the functional

$$\mathbf{cs}(A) = \frac{1}{8\pi^2}\int_W \operatorname{tr}\left(F_{A'} \wedge F_{A'}\right)$$

is well-defined on $\mathcal{B}$ up to an integer. It is called the *Chern–Simons functional*, $\mathbf{cs} : \mathcal{B} \to \mathbb{R}/\mathbb{Z}$. The instanton Floer homology of Y is, in essence, the Morse homology of $\mathbf{cs}$. A more precise definition follows.

The metric on Y induces the *Hodge star operator* $* : \Omega^k(Y, \operatorname{ad} P) \to \Omega^{3-k}(Y, \operatorname{ad} P)$. The tangent space to $\mathcal{B}$ at $[A]$ can be identified with the kernel of the operator $-{*d_A*} = d_A^* : \Omega^1(Y, \operatorname{ad} P) \to \Omega^0(Y, \operatorname{ad} P)$, which is the formal adjoint to the covariant derivative d_A.

Up to a constant, the L^2–gradient of $\mathbf{cs}$ is $\nabla(\mathbf{cs})(A) = *F_A$. Therefore, the critical points of **cs** are the gauge orbits of flat connections. It is well known that using holonomy one can identify the set of the gauge equivalence classes of flat connections on P with the subset of the $SO(3)$–representation variety $\mathrm{Hom}\,(\pi_1 Y, SO(3))/SO(3)$ consisting of representations with non-trivial second Stiefel–Whitney class. Under this identification, irreducible connections correspond to irreducible representations. Note that the condition $w_2(P) \neq 0$ implies that all flat connections on P are irreducible.

The *Hessian* of **cs** at a critical point $[A]$ is, up to a constant, given by $*d_A : \ker d_A^* \to \ker d_A^*$. The kernel of this operator can be identified with $H^1_A(Y, \mathrm{ad}\, P)$, the first cohomology of Y with coefficients in the bundle $\mathrm{ad}\, P$ endowed with the flat connection A. Therefore, a critical point $[A]$ is *non-degenerate* when $H^1_A(Y, \mathrm{ad}\, P) = 0$.

We assume for the sake of simplicity that all critical points of **cs** are non-degenerate. If not, we generically perturb **cs** so that all of its critical points are non-degenerate and proceed with all constructions using the perturbed **cs**.

In order to define the index of a critical point, we introduce the operator

$$K_A = \begin{pmatrix} 0 & d_A^* \\ d_A & -*\,d_A \end{pmatrix} : (\Omega^0 \oplus \Omega^1)(Y, \mathrm{ad}\, P) \to (\Omega^0 \oplus \Omega^1)(Y, \mathrm{ad}\, P).$$

For any connection A, this is a self-adjoint Fredholm operator. Its eigenvalues form a discrete subset of the real line, unbounded in both directions.

Let $A(0)$ and $A(1)$ be flat connections and $A(t)$ a continuous family of connections from $A(0)$ to $A(1)$. We consider the discrete set of spectral curves of $K_{A(t)}$ connecting the eigenvalues of $K_{A(0)}$ to the eigenvalues of $K_{A(1)}$. The non-zero eigenvalues of $A(0)$ and $A(1)$ are bounded away from zero. Define the *spectral flow* of $K_{A(t)}$ as the number of eigenvalues, counted with multiplicities, that cross the t-axis from below to above minus the number of eigenvalues that cross from above to below.

Fix a flat connection C which we will use to compare all other flat connections with. Given a flat connection A on P, define its relative Floer index $\mu(A) = \mathrm{sf}(A, C) \mod 4$, where $\mathrm{sf}(A, C)$ is the spectral flow of the family of operators $K_{A(t)}$ corresponding to a path $A(t)$ of connections from A to C. By the index theorem, this index is well-defined for $\mathcal{G}$–equivalence classes of connections modulo 4.

As long as all critical points of **cs** are non-degenerate, they are isolated, and the compactness of $\mathrm{Hom}\,(\pi_1 Y, SO(3))/SO(3)$ implies that there are finitely many of them. Let $IC_n(Y)$ be the free abelian group generated by gauge equivalence classes of flat connections A on P with $\mu(A) = n \mod 4$.

In order to define the boundary operator ∂, consider the infinite cylinder $\mathbb{R} \times Y$ and the pull-back bundle, again called P. A connection A on P is called ASD if its curvature satisfies the ASD equation

$$*F_A + F_A = 0.$$

It is said to have *finite Yang-Mills action* if

$$\|F_A\|_{L^2}^2 = \int_{\mathbb{R}\times Y} \mathrm{tr}\,(F_A \wedge *F_A) < \infty.$$

For any critical points $[A]$ and $[B]$ of **cs**, define $\mathcal{M}(A, B)$ to be the moduli space of ASD connections in P with finite Yang-Mills action that limit to A at negative infinity and limit to B at positive infinity. After a small perturbation of the ASD equation if necessary, $\mathcal{M}(A, B)$ is a smooth oriented manifold of dimension $\dim \mathcal{M}(A, B) = \mu(A) - \mu(B) \mod 4$. It is canonically oriented by a choice of orientation on $H^1(Y, \mathbb{R})$

(this orientation is called the *homology orientation*).

Translation along $\mathbb{R}$ defines a free action on $\mathcal{M}(A,B)$ whose quotient is a manifold $\widehat{\mathcal{M}}(A,B)$ of dimension

$$\dim \widehat{\mathcal{M}}(A,B) = \mu(A) - \mu(B) - 1 \mod 4.$$

The zero-dimensional part of $\widehat{\mathcal{M}}(A,B)$ is known to be compact and hence consists of finitely many points with signs. Denote their count by $\#\widehat{\mathcal{M}}(A,B)$ and define the boundary operator $\partial : IC_n(Y) \to IC_{n-1}(Y)$ by the formula

$$\partial(A) = \sum_B \#\widehat{\mathcal{M}}(A,B) \cdot B.$$

The square of the boundary operator ∂ is zero, hence the abelian groups $IC_n(Y)$ along with the boundary operator ∂ form a chain complex whose homology, denoted $I_*(Y)$, is the instanton Floer homology of Y.

As a final remark, one uses a somewhat subtle cobordism argument to show that I_* is independent of the choices of metric and perturbation.

3.3 The instanton Floer homology of links

Given an oriented link L of two components in an oriented integral homology sphere Σ, choose the gluing map φ so that $H_*(X_\varphi) = H_*(S^1 \times S^2)$ and let

$$I_*(\Sigma, L) = I_*(X_\varphi).$$

Here, $I_*(X_\varphi)$ is the instanton Floer homology constructed as in Section 3.2 from $SO(3)$ connections with $w_2 \neq 0$. This Floer homology has a relative $\mathbb{Z}/4$ grading. We will refer to $I_*(\Sigma, L)$ as the *instanton Floer homology* of the two-component link $L \subset \Sigma$.

3.3.1 Theorem. *Let $L = \ell_1 \cup \ell_2$ be an oriented two-component link in an oriented integral homology sphere Σ. Then $I_*(\Sigma, L)$ is independent of the choice of φ, and its Euler characteristic is given by*

$$\chi(I_*(\Sigma, L)) \;=\; \pm\,\mathrm{lk}\,(\ell_1 \cup \ell_2).$$

Proof. The first statement follows from the excision principle of Floer [9], see also [3, Proposition 3.5]. Since X_φ is a homology $S^1 \times S^2$, we know that $\chi(I_*(X_\varphi)) = \pm 1/2\,\Delta''(1)$, where $\Delta(t)$ is the Alexander polynomial of X_φ normalized so that $\Delta(1) = 1$ and $\Delta(t) = \Delta(t^{-1})$ (a direct proof of this result can be found in [22]). To calculate $\Delta(t)$, let $\widetilde{X}_\varphi$ be the infinite cyclic cover of X_φ. Then $H_1(\widetilde{X}_\varphi)$, as a $\mathbb{Z}[t, t^{-1}]$–module, has generators μ_1, μ_2 and relations

$$\begin{aligned} t\mu_2 &= a\mu_1 + b\lambda_1 \\ t\lambda_2 &= c\mu_1 + d\lambda_1, \end{aligned}$$

where $\lambda_1 = n\mu_2$ and $\lambda_2 = n\mu_1$. Therefore,

$$\Delta(t) = \det \begin{pmatrix} a & bn - t \\ c - nt & dn \end{pmatrix} = -nt^2 + (bn^2 + c)t - n,$$

up to a unit in $\mathbb{Z}[t, t^{-1}]$. After taking (3.1) into account, we obtain

$$\Delta(t) = \pm(-nt + (2n \pm 1) - nt^{-1})$$

so that

$$1/2\, \Delta''(1) = \pm n = \pm\, \mathrm{lk}\,(\ell_1, \ell_2).$$

$\square$

Remark. The requirement that X_φ have homology of $S^1 \times S^2$ was only needed to make the discussion more elementary, and in general can be omitted. Braam and Donaldson [3, Proposition 3.5] show that *any* choice of φ gives an admissible object X_φ in Floer's category, and that the properly defined Floer homology of X_φ is independent of φ.

3.4 Non-triviality

For two-component links $L = \ell_1 \cup \ell_2$ with linking number $\mathrm{lk}\,(\ell_1, \ell_2) \neq 0$, the instanton homology $I_*(\Sigma, L)$ must be non-trivial by Theorem 3.3.1. In this section, we will give a sufficient condition for $I_*(\Sigma, L)$ to be non-trivial even when $\mathrm{lk}\,(\ell_1, \ell_2) = 0$.

Let Y be a closed, irreducible, oriented 3-manifold, and let $0 \neq v \in H^2(Y; \mathbb{Z}/2)$. In the proof of [16, Theorem 3], Kronheimer and Mrowka show that the instanton Floer homology of Y constructed from $SO(3)$ connections with $w_2 = v$ must be non-trivial. This implies that $I_*(\Sigma, L)$ is non-trivial whenever X_φ is irreducible.

3.4.1 Lemma. *Let X be the exterior of a 2-component link in a 3-manifold. If X is irreducible, then X_φ is irreducible.*

Proof. We suppose that X_φ is reducible. Then there exists an embedded sphere S^2 such that S^2 does not bound a 3-ball. If we further suppose that S^2 does not intersect the gluing torus T^2, then S^2 lifts to an embedded sphere in X. It must bound a 3-ball there so we have a contradiction.

Suppose that the intersection $S^2 \cap T^2$ is non-empty. We may assume that the intersection is transversal so that it consists of finitely many circles. First, we consider those circles which are null-homotopic in the torus. Such a circle bounds a disc on the torus. We thicken the disc and cut along it to produce two spheres. At least one of these spheres does not bound a 3-ball. We repeat this process until all circles that are null-homotopic in the torus have been removed.

Now choose a circle γ in the intersection that is innermost in S^2. This circle is non-trivial in the torus and bounds a disc on S^2 that is embedded in X_φ and disjoint from T^2 except for along the boundary. Lift this disc to an embedded disc in X whose boundary lies on one boundary torus of X. Compress the boundary torus to get an embedded sphere in X that separates the boundary components of X. This contradicts the irreducibility of X.

□

3.4.2 Theorem. *Let $L = \ell_1 \cup \ell_2$ be an oriented two-component link in an integral homology sphere Σ such that the link exterior is irreducible. Then $I_*(\Sigma, L)$ is non-trivial.*

Since link exteriors are irreducible for non-split links in the 3-sphere, we obtain the following result.

3.4.3 Corollary. *For all non-split two-component links in S^3, the Floer homology $I_*(S^3, L)$ is non-trivial.*

On the other hand, for any split link $L \subset \Sigma$, we have $I_*(\Sigma, L) = 0$ since w_2 must evaluate non-trivially on $S^2 \subset X_\varphi$, which is impossible because there are no non-trivial flat $SO(3)$ connections on S^2 due to the fact that $\pi_1(S^2) = 1$.

3.5 A surgery description

Let $L \subset \Sigma$ be an oriented two-component link in an oriented homology sphere. In what follows we will give a surgery description of a manifold Y such that $I_*(\Sigma, L) = I_*(Y)$. This will allow us to calculate $I_*(\Sigma, L)$ for several examples.

Attach a band from one component of L to the other matching orientations, and call the resulting knot k. Introduce a small circle γ going once around the band with linking number zero. Frame γ by zero and k by ± 1. Any manifold obtained from Σ by performing surgery on the framed link $k \cup \gamma$ will be called Y. According to [3, Proposition 3.5], see also [13], the manifold Y is diffeomorphic to $X_{\varphi'}$ for a choice of gluing map φ'. Since Y has the integral homology of $S^1 \times S^2$, the map φ' must be as in Lemma 3.1.1. The independence of the choice of φ then implies that

$$I_*(Y) = I_*(X_{\varphi'}) = I_*(X_\varphi) = I_*(\Sigma, L).$$

Our main tool for calculating $I_*(Y)$ will be the Floer exact triangle of [9], see also [3]. Let γ be a knot in an integral homology sphere Σ. Denote by $\Sigma - \gamma$ the integral homology sphere obtained by (-1)–surgery along γ, and by $Y = \Sigma + 0 \cdot \gamma$ the homology $S^1 \times S^2$ obtained by 0–surgery along γ. The instanton Floer homology of the three manifolds are then related by the Floer exact triangle of total degree -1:

$$\begin{tikzcd} & I_*(Y) \arrow[dl] & \\ I_*(\Sigma) \arrow[rr] & & I_*(\Sigma - \gamma) \arrow[ul] \end{tikzcd}$$

Example. Let $L_n \subset S^3$ be the Hopf link with linking number n as in Figure 3.1 (where $n = 2$). Then $I_*(S^3, L) = I_*(Y)$ for the manifold Y as in Figure 3.2.

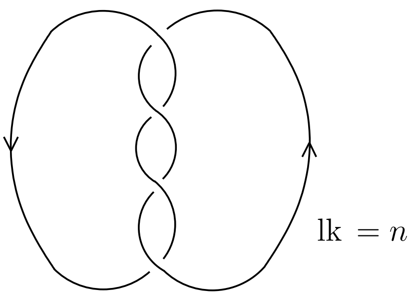

Figure 3.1:

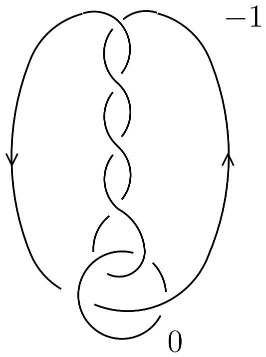

Figure 3.2:

Apply the Floer exact triangle with $\Sigma = S^3$ and γ the zero framed circle in Figure 3.2. Since $I_*(S^3) = 0$, we have an isomorphism $I_*(Y) = I_*(S^3 - \gamma)$, where the manifold $S^3 - \gamma$ has surgery description as shown in Figure 3.3.

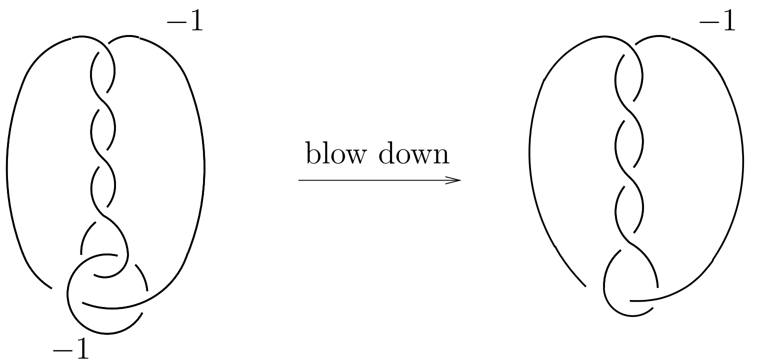

Figure 3.3:

After the blow down, we see that $S^3 - \gamma$ is the result of (-1)–surgery on a twist knot, hence is diffeomorphic to the Brieskorn homology sphere $\Sigma(2,3,6n+1)$ with reversed orientation; see for instance [24, Figure 3.19]. Therefore (cf. [7])

$$I_*(S^3, L_n) \;=\; \begin{cases} (\,\mathbb{Z}^{n/2}, \quad 0, \quad \mathbb{Z}^{n/2}, \quad 0\,), & \text{if } n \text{ is even,} \\ (\,\mathbb{Z}^{(n-1)/2}, \; 0, \; \mathbb{Z}^{(n+1)/2}, \; 0\,), & \text{if } n \text{ is odd.} \end{cases}$$

3.6 Relation with the Floer homology of knots

The instanton Floer homology $KHI_*(k)$ for knots in integral homology spheres was defined in [14] as follows. Let k be a knot in an integral homology sphere Σ with exterior $X = \Sigma - N(k)$, and let F be a punctured 2-torus. Glue the manifolds $F \times S^1$ and X together along their boundaries by matching S^1 with the meridian of k and ∂F with the longitude. The instanton Floer homology of the resulting manifold is by definition $KHI_*(k)$.

Let k_+ and k_- be knots in Σ related by a single crossing change, and let k_0 be the corresponding link, see Figure 3.4.

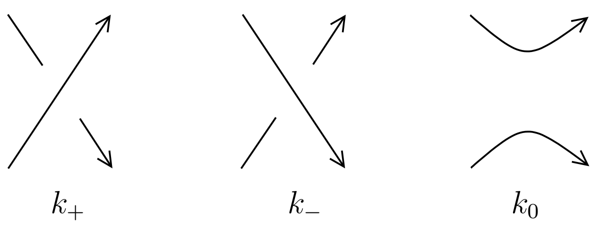

Figure 3.4:

The instanton Floer homology of these can be included into the following Floer exact triangle, see Braam–Donaldson [3, Proposition 3.11] :

$$\begin{tikzcd}
& I_*(\Sigma, k_0) \otimes H_*(T^2) \arrow[dl] & \\
KHI_*(k_+) \arrow[rr] & & KHI_*(k_-) \arrow[ul]
\end{tikzcd}$$

Observe that $\chi(KHI_*(k)) = 1$ for all knots $k \subset \Sigma$, see Kronheimer–Mrowka [14, Theorem 1.1] and also Braam–Donaldson [3, Example 3.13]. This is consistent with the above exact triangle because $\chi(I_*(\Sigma, k_0) \otimes H_*(T^2)) = \chi(I_*(\Sigma, k_0)) \cdot \chi(T^2) = 0$.

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