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Algebraic Density Property of Homogeneous Spaces

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UNIVERSITY OF MIAMI

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By

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Let X be an affine algebraic variety with a transitive action of the algebraic automorphism group. Suppose that X is equipped with several fixed point free non-degenerate SL_2-actions satisfying some mild additional assumption. Then we prove that the Lie algebra generated by completely integrable algebraic vector fields on X coincides with the set of all algebraic vector fields. In particular, we show that apart from a few exceptions this fact is true for any homogeneous space of form G/R where G is a linear algebraic group and R is a proper reductive subgroup of G.

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Contents

Chapter 1

Introduction

1.1 Andersén-Lempert theory

The notion of the density property appeared for the first time, even if not mentioned explicitly, in the work of Andersén and Lempert on the group $\text{Aut}_{hol}(\mathbb{C}^n)$ of holomorphic automorphisms of $\mathbb{C}^n$ $(n \geq 2)$ [2]. It's a basic fact of one variable complex analysis that, if $n = 1$, $\text{Aut}_{hol}(\mathbb{C})$ consists of maps of the type $z \mapsto az + b$ $(a \neq 0)$. In particular, $\text{Aut}_{hol}(\mathbb{C})$ has a natural structure of two-dimensional complex manifold. If $n \geq 2$, we can allow a and b to be holomorphic functions of the first $n-1$ variables, obtaining the so-called overshears.

1.1.1 Definition. An overshear is an automorphism of $\mathbb{C}^n$ that, in some linear coordinates $(z_1, ..., z_n)$, can be written as $(z_1, z_2, ..., z_n) \mapsto (z_1, z_2, ..., z_{n-1}, f(z_1, z_2, ..., z_{n-1})z_n + h(z_1, z_2, ...z_{n-1}))$, where f and g are holomorphic functions, and f is nowhere zero.

Therefore $\text{Aut}_{hol}(\mathbb{C}^n)$ becomes infinite-dimensional for $n \geq 2$, and Andersén and Lempert proved the following amazing fact.

Theorem 1.1.2. *(Andersen and Lempert, 1992) The subgroup $G(\mathbb{C}^n)$ of $\text{Aut}_{hol}(\mathbb{C}^n)$ generated by overshears is dense (with respect to the compact open topology) in $\text{Aut}_{hol}(\mathbb{C}^n)$.*

The proof is based on some results of control theory of ordinary diffferential equations and (as noticed by Rosay [9]) on the following fact, which was also proven by Andersén and Lempert.

Theorem 1.1.3. *For $n \geq 2$, the vector space generated by completely integrable holomorphic vector fields of $\mathbb{C}^n$ is dense in the space of all holomorphic vector fields.*

The reader can convice herself/himself that Theorem 1.1.3 implies Theorem 1.1.2 by observing that an overshear $(z_1, z_2, ..., z_n) \mapsto (z_1, z_2, ..., z_{n-1}, f(z_1, z_2, ..., z_{n-1})z_n + h(z_1, z_2, ...z_{n-1}))$ is the phase flow at time one of the complete vector field $(h + log(f)z_n)\frac{\partial}{\partial z_n}$. Therefore Theorem 1.1.3 can be interpreted as the "infinitesimal version" of Theorem 1.1.2.

Other corollaries of Theorem 1.1.3

Beside Theorem 1.1.2, Theorem 1.1.3 implies many other facts. The first one is a more general version of 1.1.2, also proved in the original paper of Andersén and Lempert, on the approximations of holomorphic automorphisms of open subsets of $\mathbb{C}^n$ by global ones. Recall that an open subset D of $\mathbb{C}^n$ is said to have the Runge property if every holomorphic function on D can be approximated, uniformly on compact subsets, by entire functions.

Theorem 1.1.4. *Let D be a starshaped domain, and $f : D \to f(D)$ be a biholomorphic map such that $f(D)$ has the Runge property. Then f can be approximated, uniformly on compact subsets, by elements of $G(\mathbb{C}^n)$.*

Next, recall that that an embedding $f : \mathbb{C}^k \to \mathbb{C}^n$ is called rectifiable if there is an automorpchism ψ of $\mathbb{C}^n$ such that $\psi \circ f$ is a linear map. The Andersén Lempert theory was crucial for the proof of the following result.

Theorem 1.1.5. *There exits a non-rectifiable proper holomorphic embedding of $\mathbb{C}^k$ in $\mathbb{C}^n$ ($n \geq 2$, $k < n$);*

In the algebraic contest, namely if we require f and ψ to be polynomial maps, it has been proven that every algebraic embedding is rectifiable for $n \geq 2k + 2$ []. For other values of k and n, the rectifiability of every algebraic embedding is a conjecture, which was posed by Abhyankar and Sataye in the case $k = n - 1$. The conjecture has been proven so far only for $n = 2$, where it is known as the Abhyankar-Moh-Susuki Theorem.

The work of Andersén and Lempert has also been applied in determining the existence of non-linearizable $\mathbb{C}^*$-actions.

Theorem 1.1.6. *There exist non-linearizable holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^n$, for $n \geq 5$.*

The result, which was proven by Kutzschebauch, is in contrast with the algebraic case, where Koras and Russell proved that all algebraic $\mathbb{C}^*$-actions on $\mathbb{C}^3$ are linearizable for $n \leq 3$, while in higher dimensions the conjecture remains open.

Finally, we have the famous Fatou-Bieberbach fenomenon.

Theorem 1.1.7. *There exists a proper open subset of* $\mathbb{C}^n$ *biholomorphic to* $\mathbb{C}^n$ *(*$n \geq 2$*).*

Furthermore, the Andersén-Lempert theory implies that such domains, called Fatou-Bieberbach domains, might have very ugly boundaries. For example, for $n = 2$, there are domains for wich the Hausdorff dimension of the boundary is equal to 4.

1.2 Density property

It is natural to ask whether there are other complex manifolds that enjoy the properties listed in the previous theorems. In order to look for such classes of manifolds, Varolin introduced the notion of density property.

1.2.1 Definition. (Density Property)

A complex manifold X is said to have the density property if the Lie algebra $\mathrm{Lie}_{\mathrm{hol}}(X)$ generated by complete holomorphic vector fields on X is dense in the Lie algebra $\mathrm{VF}_{\mathrm{hol}}(X)$ of all holomorphic vector fields on X.

Theorem 1.1.3 says that $\mathbb{C}^n$ has the density property ($n \geq 2$). Varolin and Toth proved the density property for complex semisimple algebraic groups and for quotient of complex semisimple algebraic groups with trivial center [25, 26].

Varolin [27] shows that the proofs in [2, 9] can be adapted to manifolds with the density property to show the following result.

Theorem 1.2.2. *Let X be a Stein manifold of dimension n with the density property. Then:*

(1) for any $x \in X$, there exists an injective, non-surjective holomorphic map $f : \mathbb{C}^n \to X$ such that $f(0) = x$;

(2) for any $x \in X$, there exists an injective, non-surjective holomorphic map $f : X \to X$ such that $f(x) = x$.

1.2.3 Remark. It's natural to expect that Theorem 1.1.5 will also hold, but no proofs are known of this fact.

The notion of density propery has important applications in complex analysis, the most remarkable one being the validity of the Oka-Grauert-Gromov principle on the existence of holomorphic sections of a submersion.

Theorem 1.2.4. *Let $h : W \to X$ be a holomorphic submersion of Stein manifolds, such that the fibers have the density property. Then the embedding of the space of the holomorphic sections into the space of continous sections is a weak homotopy equivalence.*

The theorem is based on the existence of a fiber dominating Gromov's spray [11] for the submersion.

The present thesis answer a natural question arising from the results of Kaliman and Kutzschebauch on complex linear algebraic groups, that is whether affine homogeneous manifolds have the density property (see next section). It is therefore worth mentioning the following theorem, which shows that homogeneous spaces are the natural objects of our study.

Theorem 1.2.5. *If X is a manifold with the density property, then it is homogeneous with respect to $\mathrm{Aut}_{hol}(X)$.*

Proof. Consider the subgroup $G \subset \mathrm{Aut}_{hol}(X)$ generated by the flows of the complete holomorphic vector fields of X, and fix $x_0 \in X$. The open orbit theorem [14] implies that the orbit Gx_0 is an open submanifold of X. Since X has the density property, the complete integrable vector fields form a basis on x_0, hence Gx_0 has the same dimension of X and it is open in X. Now it is clear that X is homogeneous respect to the subgroup G of $\mathrm{Aut}_{hol}(X)$: indeed, for any two points x and y, we can cover a path joining them by open subset generated by G. □

1.3 Algebraic density property

More recently, Kaliman and Kutzschebauch extended the results on the density property by introducing techniques coming from algebraic geometry. Let's start by giving a refined version of Defintion 1.2.1, which is more suitable to the contest of affine geometry.

1.3.1 Definition. (Algebraic Density Property) An affine algebraic manifold X is said to have the algebraic density property if the Lie algebra $\mathrm{Lie}_{\mathrm{alg}}(X)$ generated by complete algebraic vector fields on it coincides with the Lie algebra $\mathrm{VF}_{\mathrm{alg}}(X)$ of all algebraic vector fields on it.

Since $\mathrm{VF}_{\mathrm{alg}}(X)$ is dense in $\mathrm{VF}_{\mathrm{hol}}(X)$ the algebraic density property implies the density property. Kaliman and Kutzschebauch [16] introduced new criteria which provide an elegant proof of the algebraic density property for almost all complex linear algebraic groups (Theorem 1.3.2). Their idea, which leads to the notion of compatibility condition (see Chapter 2), is to look for "good types" of complete fields, in particular locally nilpotent derivations, that generate (as a Lie algebra) the set of all algebraic vector fields. Relying on the fact that linear algebraic groups are endowed with a lot of $\mathbb{C}_+$-actions, they were able to prove the following remarkable result.

Theorem 1.3.2. *Let G be a complex linear algebraic group whose connected component is different from a torus or $\mathbb{C}_+$. Then G has the algebraic density property.*

1.3.3 Remark. $\mathbb{C}_+$ and $\mathbb{C}^*$ do not have the density property, since the only complete vector fields on them are, respectively, of the form $(a + bz)\frac{\partial}{\partial z}$ or $az\frac{\partial}{\partial z}$, for $a, b \in \mathbb{C}$. The case $(\mathbb{C}^*)^n$ is still open. Since the criteria of compatibility requires the use of $\mathbb{C}_+$-actions, there is no possible way to adapt the tecniques of Kaliman and Kutzschebauch to the study of this difficult problem.

My thesis [5] is an extension of the results of Kaliman and Kutzschebauch, and those of Toth and Varolin, since it proves the algebraic density property for a large class of quotients of algebraic groups. The result is based on the following tecnhical theorem, which constitutes the main result of my thesis work.

Theorem 1.3.4. *(Main Result of My Thesis) Let X be a smooth complex affine algebraic variety whose group of algebraic automorphisms is transitive. Suppose that X is equipped with N nondegenerate, fixed point free SL_2-actions ($N \geq 1$), such that at some point $x_0 \in X$ the tangent spaces of the corresponding SL_2-orbits through x_0 generate the whole space $T_{x_0}X$. Then X has the algebraic density property.*

The theorem is based on the fact that SL_2 contains two subgroups isomorphic to $\mathbb{C}_+$, that induce particularly nice complete vector fields, for which the compatibility criterion can be applied. Beside the compatibility, the main tool was found in Luna's Slice Theorem, that provides a nice geometrical description of varieties equipped with actions of reductive groups.

In order to find nontrivial examples of homogeneous spaces having sufficiently many fixed point free SL_2-actions, we make use of some notions of Lie group theory. With the important collaboration of Dr. Alexander Dvorsky (University of Miami), we were able to show that Theorem 1.3.4 has a wide range of applications

Theorem 1.3.5. *(Donzelli, Dvorsky, Kaliman, 2008)*

Let G be complex algebraic group and R be a reductive subgroup of G. Then G/R has the algebraic density property, if it is not isomorphic to $\mathbb{C}_+$ or $(\mathbb{C}^*)^n$ or $\mathbb{Q}$-homology plane with fundamental group isomorphic to $\mathbb{Z}_2$;

The thesis is organized as follows. In Chapter 2 we present the preliminary background, Chapter 3 is devoted to the proof of Theorem 1.3.4, while Theorem 1.3.5 will be proved in Chapter 4. The reader interested in the applications of Theorem 1.3.4 might skip Chapter 3. Chapter 4 is not self contained, since it uses an important theorem, proved in the Appendix of [5]. Finally Chapter 5 discusses some work in progress.

Conventions

All varieties will be defined over the field of complex numbers. Therefore, we will intend the classical Lie groups to be defined over $\mathbb{C}$. Given an affine variety X, the ring of its regular functions will be denoted by $\mathbb{C}[X]$. The set of maximal ideals of a ring, endowed with the Zariski topology, will be denoted by $\operatorname{Spec}(R)$. Recall that $X = \operatorname{Spec}(R)$ is affine if and oly if R is a finitely generated $\mathbb{C}$-algebra, and the Hilbert nullstelleansatz says that $X = \operatorname{Spec} \mathbb{C}[X]$. An affine variety will be sometimes treated as a complex space, and the words "algebraic" or "holomorphic" will be used to distinguish between the two contests.

Chapter 2

Preliminaries

The proof of the algebraic density property for homogeneous spaces makes use essentially of two main tools: the Luna's slice theorem and the compatibility criterion. Luna's theorem is an important result in geometric invariant theory, and the expert reader might skip the related section. We describe how the compatibility criterion is used to prove the algebraic density property of an affine manifold, and we will refer to [16] for the proofs. We also list some properties of actions of reductive groups on algebraic varieties, that will be used throughout the paper.

2.1 Actions of algebraic groups

Algebraic groups

A complex linear algebraic group G is a Zariski closed subset of GL_n, with group multiplication law induced by the one of GL_n. We need to recall some definitions from Lie group theory, that will play an important role in our paper.

2.1.1 Definition. A complex linear algebraic group G is called reductive if there exists a (maximal) compact subgroup K, such that the complexification of K is isomorphic to G

Instead of defining the notion of complexification, let's describe it with the following example.

2.1.2 Example. $\mathbb{C}^*$ is reductive. In fact the unit circle S^1 is a maximal compact subgroup of $\mathbb{C}^*$, which can be embedded in $SL_2(\mathbb{R})$ as the closed subset of matrices of the type

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

where $x^2 + y^2 = 1$.

The complex solutions of the defining equation of S^1 in $SL_2(\mathbb{R})$ define $\mathbb{C}^*$ (embedded in $SL_2(\mathbb{C})$).

The complex dimension of G reductive is equal to the real dimension of one of its maximal compact subgroups.

2.1.3 Example. $\mathbb{C}_+$ (the group of complex numbers with addition) is not reductive. In fact, it does not contain nontrivial compact subgroups.

An important properties of reductive groups, on which is based Luna's slice theorem, is stated in the following proposition ([20]).

2.1.4 Proposition. *For a complex reductive group, every linear representation is completely reducible.*

Given a linear algebraic group G, we can construct its Lie algebra $\mathrm{Lie}(G)$. A Lie algebra is said to be simple if it is not abelian and it does not contain non trivial ideals.

2.1.5 Definition. A linear algebraic group G is said to be simple if its Lie algebra is simple. G is semisimple if it is isomorphic to the quotient of direct product of simple groups by a (finite) central subgroup.

SL_n is simple and reductive, while GL_n is a reductive group that is not semisimple. On the contrary, a semisimple group is always reductive. In general, a reductive group is isomorphic to the quotient, by a discrete normal subgroup, of a product of a semisimple group and a torus $(\mathbb{C}^*)^n$.

Group actions and quotient varieties

Let G be a linear algebraic group. A G-action on an affine algebraic variety X is an action of G on X which is also a morphism of varieties; in simple words, if we describe X and G as zero sets of some affine spaces $\mathbb{C}^n$ and $\mathbb{C}^m$, the action is given by polynomial functions in $n+m$ variables. Sometimes a G-action on an affine variety will be called algebraic G-action, in order to distinguish it from a holomorphic G-action on X. A G-action on X naturally defines a linear action of G on $\mathbb{C}[X]$: if $p(x) \in \mathbb{C}[X]$, then $g.p(x) = p(g^{-1}x)$. The algebraic quotient is then defined as $X//G = \mathrm{Spec}\,\mathbb{C}[X]^G$, being $\mathbb{C}[X]^G$ the subring of G-invariant regular functions of X.

By $\pi : X \to X//G$ we denote the natural quotient morphism generated by the embedding $\mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$. This definition of quotient is natural from the point of view of the Hilbert Nullstelleansatz, and more important it satisfies the universal property of quotient of algebraic varieties: any morphism from X constant on orbits of G factors through π. It's not true in general that $\mathbb{C}[X]^G$ is a finitely generated $\mathbb{C}$-algebra, i.e. that $\text{Spec}(X//G)$ is affine. For reductive G this and other important fact are indeed true, as listed in the next proposition (e.g., see [23], [22], [4], [10]).

2.1.6 Proposition. *Let G be a reductive group.*

(1) The quotient $X//G$ is an affine algebraic variety which is normal in the case of a normal X and the quotient morphism $\pi : X \to X//G$ is surjective.

(2) The closure of every G-orbit contains a unique closed orbit and each fiber $\pi^{-1}(y)$ (where $y \in X//G$) contains also a unique closed orbit O. Furthermore, $\pi^{-1}(y)$ is the union of all those orbits whose closures contain O.

(3) In particular, if every orbit of the G-action on X is closed then $X//G$ is isomorphic to the orbit space X/G.

(4) The image of a closed G-invariant subset under π is closed.

If X is a complex algebraic group, and G is a closed subgroup acting on X by multiplication, all the orbits, being of the same dimension, are closed. If G is reductive, the previous proposition implies that the quotient X/G is affine. The next proposition (Matsushima's criterion) shows that the converse is also true.

2.1.7 Proposition. *Let G be a complex reductive group, and H be a closed subgroup of G. Then the quotient space G/H is affine if and only if H is reductive.*

Besides reductive groups actions in this paper, a crucial role will be played by $\mathbb{C}_+$-actions. In general algebraic quotients in this case are not affine but only quasi-affine [31]. However, we shall use later the fact that for the natural action of any $\mathbb{C}_+$-subgroup of SL_2 generated by multiplication one has $SL_2//\mathbb{C}_+ \cong \mathbb{C}^2$. In Section 2.3 we describe the relation between $\mathbb{C}_+$-actions complete vector fields and locally nilpotent derivations.

2.2 Luna's slice theorem

It's well known that if a Lie group G acts freely on a smooth manifold X, then X is the total space of a G-principal bundle over the orbit space X/G, and X is locally isomorphic to a product $U \times G$. Luna' slice Theorem provides a similar description of the geometry of an affine G-variety, for a not necessarily free G-action, on a Zariski open neighborhood of a closed orbit.

The facts about Luna's slice theorem are taken from [4, 22]. Let's recall some terminology first. Suppose that $f : X \to Y$ is a G-equivariant morphism (i.e. it preserves the G-actions) of affine algebraic G-varieties X and Y. Then the induced morphism $f_G : X//G \to Y//G$ is well defined and the following diagram is commutative.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X//G & \xrightarrow{f_G} & Y//G \end{array} \tag{2.1}$$

2.2.1 Definition. A G-equivariant morphism f is called strongly étale if

(1) The induced morphism $f_G : X//G \to Y//G$ is étale

(2) The quotient morphism $\pi_G : X \to X//G$ induces a G- isomorphism between X and the fibred product $Y \times_{Y//G} (X//G)$.

A morphism of smooth complex algebraic varieties is called étale if it is a local biholomorphism. From the properties of étale maps ([4]) it follows that f is étale (in particular, quasi-finite).

Let H be an algebraic subgroup of G, and Z an affine H-variety. We denote $G \times_H Z$ the quotient of $G \times Z$ by the action of H given by $h(g, z) = (gh^{-1}, hz)$. The left multiplication on G generates a left action on $G \times_H Z$. The next lemma is an obvious consequence of 2.1.6.

2.2.2 Lemma. *Let X be an affine G-variety and G be reductive. Then the H-orbits of $G \times X$ are all isomorphic to H. Therefore the fibers of the quotient morphism $G \times X \to G \times_H X$ coincide with the H-orbits.*

The isotropy group of a point $x \in X$ will be denoted by G_x. Recall also that an open set U of X is called saturated if $\pi_G^{-1}(\pi_G(U)) = U$. Geometrically speaking, an open set U is saturated if and only if it is G-invariant and the closure of an orbit contained in U is contained in U. We are ready to state the Luna slice theorem.

Theorem 2.2.3. *Let G be a reductive group acting on an affine algebraic variety X, and let $x \in X$ be a point in a closed G-orbit. Then there exists a locally closed affine algebraic subvariety V (called a slice) of X containing x such that*

(1) V is G_x-invariant;

(2) the image of the G-morphism $\varphi : G \times_{G_x} V$ induced by the action is a saturated open set U of X;

(3) the restriction $\varphi : G \times_{G_x} V \to U$ is strongly étale.

Given a saturated open set U, we will denote $\pi_G(U)$ by $U//G$. It follows from 2.1.6 that $U//G$ is open. This theorem implies that the following diagram is commutative

$$\begin{array}{ccc} G \times_{G_x} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V//G_x & \longrightarrow & U//G \end{array} \tag{2.2}$$

and $G \times_{G_x} V \simeq U \times_{U//G} V//G_x$.

The proof of the theorem flows along the following idea. The tangent space T_xGx of the orbit at x is invariant under the differential of the G_x-action (2.1.4). Since the orbit of x is closed, and G is reductive 2.1.7, it follows that G_x is reductive, therefore there exists a G-invariant complement N of T_xGx in T_xX. Therefore $T_xX = T_xG_x \oplus N$, and we prove the theorem at level of the tangent spaces. Taking a small euclidean neighborrhood O of N, indeed, we see that X is biholomorphic to $U \times G$. If we want to extend O to a Zariski open subset $V \subset N$, one has to consider the fact that the orbit Gx might intersect N at more than one point. If G_x is trivial, the multiplication by G gives rise to a $n : 1$ map $G \times V \to U$, where n is the number of points of intersection of Gx with N. If we want to take into account a non trivial isotropy group G_x, we get a similar map after passing to the quotient $G \times_{G_x} V$.

2.3 Locally nilpotent derivations and $\mathbb{C}_+$-actions

We refer to [9, 30] for a detailed exposition of the theory of locally nilpotent derivations, with its application to affine geometry.

2.3.1 Definition. A derivation on an affine algebraic variety X is a $\mathbb{C}$-linear map $\delta : \mathbb{C}[X] \to \mathbb{C}[X]$, satisfying the Leibniz rule: $\forall\ f,\ g \in \mathbb{C}[X]$, $\delta(fg) = \delta(f)g + f\delta(g)$.

The set of derivations $\mathrm{Der}(X)$ on X is in one-to-one correspondence with the set $\mathrm{VF}_{\mathrm{alg}}(X)$ of algebraic vector field on X. Indeed, if X is embedded in $X \subset \mathbb{C}^n_{(z_1,\ldots z_n)}$ and $p_i = \delta(z_i) \in \mathbb{C}[X]$, then $\delta = p_1\frac{\partial}{\partial z_1} + \ldots + p_n\frac{\partial}{\partial z_n}$. Conversely, given such a vector field v, we can define the derivation on $\mathbb{C}[X]$ by knowing the polynomials $p_i = \delta(z_i) \in \mathbb{C}[X]$. A holomorphic vector field on X is called completely integrable (or, in short, complete) if its phase flow defines a $\mathbb{C}_+$-action, which turns out to be holomorphic. We remark that even if a complete vector field is algebraic, it does not necessarily define an algebraic $\mathbb{C}_+$-action.

2.3.2 Example. Let $X = \mathbb{C}$; the vector field $z\frac{\partial}{\partial z}$ is algebraic and complete, and the $\mathbb{C}$-action on X is $t.z = e^t z$, for $t \in \mathbb{C}$.

In the previous example, we see that for fixed t, the automorphism defined by the action is actually algebraic, and the holomorphic action factors through an algebraic $\mathbb{C}^*$-action via the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$.

2.3.3 Definition. A derivation on an affine variety is called semisimple if its phase flow defines an algebraic $\mathbb{C}^*$-action.

The next example shows that the automorphism defined at fixed time by a complete algebraic vector field does not need to be algebraic.

2.3.4 Example. On $\mathbb{C}^2$, let $v = -x^2y\frac{\partial}{\partial x} + xy^2\frac{\partial}{\partial y}$; one can check that the action of $t \in \mathbb{C}_+$ is $t.(x, y) = (e^{-txy}x, e^{txy}y)$.

It's not difficult to prove that the $\mathbb{C}_+$-action defined by a complete algebraic vector field v is algebraic if and only if the corresponding derivation is, as defined next, locally nilpotent.

2.3.5 Definition. A derivation δ on X is called locally nilpotent if for any $f \in \mathbb{C}[X]$ there is a non negative integer N such that $\delta^N(f) = 0$.

As an easy example, $\delta = \frac{\partial}{\partial z}$ is a locally nilpotent derivation on $\mathbb{C}$, whose flow is the translation $z \mapsto z + t$. If $p \in \mathbb{C}[z]$ is a polynomial of degree N, then it's obvious that $\delta^N p \neq 0$, while $\delta^{N+1}p = 0$. In the same way, a locally nilpotent derivation on X defines a degree function on the ring $\mathbb{C}[X]$.

2.3.6 Definition. Let δ be a locally nilpotent derivation on X, and let $p \in \mathbb{C}[X]$. Then we define $deg_\delta(p)$, the degree of p with respect to δ, to be the largest integer N such that $\delta^N(p) \neq 0$ and $\delta^{N+1}(p) = 0$.

We conclude the section with a useful fact.

2.3.7 Proposition. *Let δ be a locally nilpotent derivation on X, and $f \in \mathbb{C}[X]$. If $\delta(f) = 0$, then $f\delta$ is locally nilpotent; if $\deg_\delta(f) = 1$, then $f\delta$ is complete.*

2.4 The compatibility criterion

In [16] the authors give a short proof of the algebraic density property of $\mathbb{C}^n$, based on a nice trick, and then show how to extend the idea to tackle the algebraic density property for homogeneous affine manifolds endowed with many $\mathbb{C}_+$ actions.

2.4.1 Definition. Let δ_1 and δ_2 be nontrivial algebraic vector fields on an affine algebraic manifold X such that δ_1 is a locally nilpotent derivation on $\mathbb{C}[X]$, and δ_2 is either also locally nilpotent or semi-simple. That is, δ_i generates an algebraic action of H_i on X where $H_1 \simeq \mathbb{C}_+$ and H_2 is either $\mathbb{C}_+$ or $\mathbb{C}^*$. We say that δ_1 and δ_2 are semi-compatible if the vector space $\mathrm{Span}(\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2)$ generated by elements from $\mathrm{Ker}\,\delta_1 \cdot \mathrm{Ker}\,\delta_2$ contains a nonzero ideal in $\mathbb{C}[X]$. A semi-compatible pair is called compatible if in addition one of the following condition holds

(1) when $H_2 \simeq \mathbb{C}^*$ there is an element $a \in \mathrm{Ker}\,\delta_2$ such that $\deg_{\delta_1}(a) = 1$, i.e. $\delta_1(a) \in \mathrm{Ker}\,\delta_1 \setminus \{0\}$;

(2) when $H_2 \simeq \mathbb{C}_+$ (i.e. both δ_1 and δ_2 are locally nilpotent) there is an element a such that $\deg_{\delta_1}(a) = 1$ and $\deg_{\delta_2}(a) \leq 1$.

2.4.2 Remark. If $[\delta_1, \delta_2] = 0$ then condition (1) and condition (2) with $a \in \mathrm{Ker}\,\delta_2$ hold automatically.

2.4.3 Example. Consider SL_2 with two natural $\mathbb{C}_+$-subgroups: namely, the subgroup H_1 (resp. H_2) of the lower (resp. upper) triangular unipotent matrices. Denote by

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

an element of SL_2. Then the left multiplication generate actions of H_1 and H_2 on SL_2 with the following associated locally nilpotent derivations on $\mathbb{C}[SL_2]$

$$\delta_1 = a_1 \frac{\partial}{\partial b_1} + a_2 \frac{\partial}{\partial b_2}$$

$$\delta_2 = b_1 \frac{\partial}{\partial a_1} + b_2 \frac{\partial}{\partial a_2} \, .$$

Clearly, $\operatorname{Ker} \delta_1$ is generated by a_1 and a_2 while $\operatorname{Ker} \delta_2$ is generated by b_1 and b_2. Hence δ_1 and δ_2 are semi-compatible. Furthermore, taking $a = a_1 b_2$ we see that condition (2) of Definition 2.4.1 holds, i.e. they are compatible.

2.4.4 Definition. A finite subset M of the tangent space $T_x X$ at a point x of a complex algebraic manifold X is called a generating set if the image of M under the action of the isotropy group (of algebraic automorphisms) of x generates $T_x X$.

A pair of compatible derivation guarantees the existence of a $\mathbb{C}[X]$ submodule $\mathcal{M}$ of $\mathrm{VF}_{\mathrm{alg}}(X)$, contained in $\mathrm{Lie}_{\mathrm{alg}}(X)$. For example, suppose that for the compatible pair $\{\delta_1, \delta_2\}$ there exists a regular function a with $\deg_{\delta_1}(a) = 1$ and $\deg_{\delta_2}(a) = 0$. Then one can show that $\mathcal{M} = b\mathbb{C}[X]\delta_2$, where $b = \delta_1(a)$, is contained in $\mathrm{Lie}_{\mathrm{alg}}(X)$. The manifold X has the algebraic density property iff $\mathcal{M} = \mathrm{Lie}_{\mathrm{alg}}(X)$. Therefore, in order to extend the module $\mathcal{M}$ to the entire module of algebraic vector fields, we will need sufficiently many pairs of compatible derivations.

This explains the origin of the next theorem.

Theorem 2.4.5. *Let X be a smooth homogeneous (with respect to $\mathrm{Aut}_{alg} X$) affine algebraic manifold with finitely many pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}_{k=1}^m$ such that for some point $x_0 \in X$ vectors $\{\delta_2^k(x_0)\}_{k=1}^m$ form a generating set at x_0. Then $\mathrm{Lie}_{\mathrm{alg}}(X)$ contains a nontrivial $\mathbb{C}[X]$-module and X has the algebraic density property.*

As an application of this theorem we have the following.

2.4.6 Proposition. *Let X_1 and X_2 be smooth homogeneous (with respect to algebraic automorphism groups) affine algebraic varieties such that each X_i admits a finite number of integrable algebraic vector fields $\{\delta_i^k\}_{k=1}^{m_i}$ whose values at some point $x_i \in X_i$ form a generating set and, furthermore, in the case of X_1 these vector fields are locally nilpotent. Then $X_1 \times X_2$ has the algebraic density property.*

An SL_2-action on an affine varieties X will induce two locally nilpotent derivations, associated with the subgroups H_1 and H_2 from Example 2.4.3. Our task is to prove that if the action is non-degenarate and fixed-point-free then the derivations are compatible. The existence of sufficiently many of such actions as in Theorem 3.1.2, corresponds to having a collection of pairs $\{\delta_1^k, \delta_2^k\}_{k=1}^m$, such that $\{\delta_2^k(x_0)\}_{k=1}^m$ form a generating set.

We end this section with some technical results, that will be used throughout the proof of Theorem 3.1.2. First let's keep in mind that we will use the following geometrical reformulation of Definition 2.4.1.

2.4.7 Proposition. *Suppose that* H_1 *and* H_2 *are as in Definition* 2.4.1*,* X *is a normal affine algebraic variety equipped with nontrivial algebraic* H_i*-actions where* $i = 1, 2$ *(in particular, each* H_i *generates an algebraic vector field* δ_i *on* X*). Let* $X_i = X//H_i$ *and* $\rho_i : X \to X_i$ *the quotient morphisms. Set* $\rho = (\rho_1, \rho_2) : X \to Y := X_1 \times X_2$ *and* Z *equal to the closure of* $\rho(X)$ *in* Y*. Then* δ_1 *and* δ_2 *are semi-compatible iff* $\rho : X \to Z$ *is a finite birational morphism.*

The next two lemmas, taken again from [16], describe conditions under which quasi-finite morphisms preserve semi-compatibility.

2.4.8 Lemma. *Let* $G = SL_2$ *and* X, X' *be normal affine algebraic varieties equipped with non-degenerate* G*-actions. Suppose that subgroups* H_1 *and* H_2 *of* G *are as in Example* 2.4.3*, i.e. they act naturally on* X *and* X'*. Let* $\rho_i : X \to X_i := X//H_i$ *and* $\rho'_i : X' \to X'_i := X'//H_i$ *be the quotient morphisms and let* $p : X \to X'$ *be a finite* G*-equivariant morphism, i.e. we have commutative diagrams*

$$\begin{array}{ccc} X & \xrightarrow{\rho_i} & X_i \\ \downarrow p & & \downarrow q_i \\ X' & \xrightarrow{\rho'_i} & X'_i \end{array}$$

for $i = 1, 2$*. Treat* $\mathbb{C}[X_i]$ *(resp.* $\mathbb{C}[X'_i]$*) as a subalgebra of* $\mathbb{C}[X]$ *(resp.* $\mathbb{C}[X']$*). Let* $\mathrm{Span}(\mathbb{C}[X_1] \cdot \mathbb{C}[X_2])$ *contain a nonzero ideal of* $\mathbb{C}[X]$*. Then* $\mathrm{Span}(\mathbb{C}[X'_1] \cdot \mathbb{C}[X'_2])$ *contains a nonzero ideal of* $\mathbb{C}[X']$*.*

2.4.9 Lemma. *Let the assumption of Lemma* 2.4.8 *hold with two exceptions: we do not assume that* G*-actions are non-degenerate and instead of the finiteness of* p *we suppose that there are a surjective étale morphism* $r : M \to M'$ *of normal affine algebraic varieties equipped with trivial* G*-actions and a surjective* G*-equivariant* *morphism* $\tau' : X' \to M'$ *such that* X *is isomorphic to fibred product* $X' \times_{M'} M$ *with* $p : X \to X'$ *being the natural projection (i.e.* p *is surjective étale). Then the conclusion of Lemma* 2.4.8 *remains valid.*

The last result from [16] that we need allows us to switch from local to global compatibility.

2.4.10 Proposition. *Let* X *be an* SL_2*-variety with associated locally nilpotent derivations* δ_1 *and* δ_2*,* Y *be a normal affine algebraic variety equipped with a trivial* SL_2*-action, and* $r : X \to Y$ *be a surjective* SL_2*-equivariant morphism. Suppose that for any* $y \in Y$ *there exists an étale neighborhood* $g : W \to Y$ *such that the vector fields induced by* δ_1 *and* δ_2 *on the fibred product* $X \times_Y W$ *are semi-compatible. Then* δ_1 *and* δ_2 *are semi-compatible.*

Chapter 3

Density property and SL_2-actions

Suppose that G is a complex algebraic group, containing a reductive subgroup R, and a subgroup Γ isomorphic to SL_2. Then the action of Γ via left multiplication induces an action of Γ on the space G/R of the left cosets, which is affine by Proposition 2.1.7. It is clear from Example 2.4.3 that we can extract, from the SL_2-actions, a pair of locally nilpotent derivations on X. In this Chapter, which constitutes the main result of this thesis, we prove that if the SL_2-action is fixed point free and nondegenerate, then the derivations are compatible (Theorem3.2.1). This result is the main tool for determining the algebraic density propery for homogeneous spaces, and it will be applied in Theorem 3.1.2 and in Theorem 4.1.1 of Chapter 4

3.1 Statement of the main theorem

3.1.1 Notation. We suppose that H_1, H_2, δ_1 and δ_2 are as in Example 2.4.3. Note that if SL_2 acts algebraically on an affine algebraic variety X then we have automatically the $\mathbb{C}_+$-actions of H_1 and H_2 on X that generate locally nilpotent vector fields on X, which by abuse of notation will be denoted by the same symbols δ_1 and δ_2. If X admits N SL_2-actions, we denote by $\{\delta_1^k, \delta_2^k\}_{k=1}^N$ the corresponding collection of pairs of locally nilpotent derivations on $\mathbb{C}[X]$.

Theorem 1.3.4 is equivalent to the following theorem.

Theorem 3.1.2. *Let X be a smooth complex affine algebraic variety, whose group of algebraic automorphisms is transitive. Suppose that X is equipped with N fixed point free non-degenerate actions of SL_2-groups $\Gamma_1, \dots, \Gamma_N$. Let $\{\delta_1^k, \delta_2^k\}_{k=1}^N$ be the corresponding pairs of locally nilpotent vector fields. If $\{\delta_2^k(x_0)\}_{k=1}^N \subset T_{x_0}X$ is a generating set at some point $x_0 \in X$ then X has the algebraic density property.*

3.1.3 Remark. Note that we can choose any nilpotent element of the Lie algebra of SL_2 as δ_2 (or, as it's the same, any $\mathbb{C}_+$-subgroup of SL_2.) Since the adjoint representation on the Lie algebra of SL_2 is irreducible, the nilpotent elements generate, as a vector space, the Lie algebra of SL_2 (or, $\mathbb{C}_+$-actions "acts along the three indepenent direction of SL_2".) Thus we can reformulate Theorem 3.1.2 as follows: a smooth complex affine algebraic variety X with a transitive group of algebraic automorphisms has the algebraic density property provided it admits "sufficiently many" fixed point free non-degenerate SL_2 -actions, where "sufficiently many" means that at some point $x_0 \in X$ the tangent spaces of the corresponding SL_2-orbits through x_0 generate the whole space $T_{x_0}X$.

3.2 Proof of the main theorem

By virtue of Theorem 2.4.5, the main result will be a consequence of the following.

Theorem 3.2.1. *Let X be a smooth complex affine algebraic variety equipped with a fixed point free non-degenerate SL_2-action that induces a pair of locally nilpotent vector fields $\{\delta_1, \delta_2\}$. Then these vector fields are compatible.*

The proof will make use of Luna's slice theorem, in order to reduce the determination of the compatibility to a local problem (in the Zariski topology). The next lemma is an easy but important one, since Luna's theorem dictates the geometry of a variety around its closed orbits.

3.2.2 Lemma. *Let the assumption of Theorem* 3.2.1 *hold and $x \in X$ be a point contained in a closed SL_2-orbit. Then the isotropy group of x is either finite, or isomorphic to the diagonal $\mathbb{C}^*$-subgroup of SL_2, or to the normalizer of this $\mathbb{C}^*$-subgroup (which is the extension of $\mathbb{C}^*$ by $\mathbb{Z}_2$).*

Proof. By Matsushima's criterion (Proposition 2.1.7) the isotropy group G_x must be reductive and it cannot be SL_2 itself since the action has no fixed points. The only two-dimensional reductive group is $\mathbb{C}^* \times \mathbb{C}^*$ ([7]), which is not contained in SL_2. Since finite subgroups are reductive,we are left to consider the one-dimensional reductive subgroups. A one-dimensional reductive subgroup of SL_2 must contain a one-dimensional torus $\mathbb{C}^*$ (indeed, since such a group can not be semisimple, its semisimple part must be zero-dimensional). We can assume that $\mathbb{C}^*$ is diagonal, since all maximal tori are conjugated. If G_x is not $\mathbb{C}^*$, then it is an extension of it. The normalizer of $\mathbb{C}^*$ which is its extension by $\mathbb{Z}_2$ generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is reductive. If we try to find an extension of $\mathbb{C}^*$ by another finite subgroup that contains an element B not from the normalizer then $\mathbb{C}^*$ and $B\mathbb{C}^*B^{-1}$ meet at the identical matrix. In particular, the reductive subgroup must be at least two-dimensional, which is not possible. $\square$

Part 1: The existence of a of Definition 2.4.1

Let's start proving theorem 3.2.1, by showing first the existence of the regular function g with the property (2) required by Definition 2.4.1.

3.2.3 Proposition. *Let X, δ_1, δ_2 be as in Theorem 3.2.1. Then there exists a regular function $g \in \mathbb{C}[X]$ such that $\deg_{\delta_1}(g) = \deg_{\delta_2}(g) = 1$.*

Proof. Let $x \in X$ be a point of a closed SL_2-orbit. Luna's slice Theorem yields diagram (2.2) with $G = SL_2$ and G_x being one of the subgroups described in Lemma 3.2.2. That is, we have the natural morphism $\varphi : SL_2 \times V \to U$ that factors through the étale morphism $SL_2 \times_{G_x} V \to U$ where V is the slice at x. First, consider the case when G_x is finite: since G_x acts freely on $SL_2 \times V$, φ itself is étale. Furthermore, replacing V by its Zariski open subset and U by the corresponding Zariski open SL_2-invariant subset one can suppose that φ is also finite: indeed, by the equivariant Zariski's main theorem ([4], Theorem 3.3) we can factor φ through an open immersion $SL_2 \times V \hookrightarrow X'$ and a finite (dominant) morphism $X' \to U$. Set $f = a_1 b_2$ where a_i, b_i are as in Example 2.4.3. Note that each δ_i generates a natural locally nilpotent vector field $\tilde{\delta}_i$ on $SL_2 \times V$ such that $\mathbb{C}[V] \subset \operatorname{Ker} \tilde{\delta}_i$ and $\varphi_*(\tilde{\delta}_i)$ coincides with the vector field induced by δ_i on X. Treating f as an element of $\mathbb{C}[SL_2 \times V]$ we have $\deg_{\tilde{\delta}_i}(f) = 1$, $i = 1, 2$. For every $h \in \mathbb{C}[SL_2 \times V]$ we define a function $\hat{h} \in \mathbb{C}[U]$ by $\hat{h}(u) = \sum_{y \in \varphi^{-1}(u)} h(y)$ (see Remark below). One can check that if $h \in \operatorname{Ker} \tilde{\delta}_i$ then $\delta_i(\hat{h}) = 0$. Hence $\delta_i^2(\hat{f}) = 0$ but we also need $\delta_i(\hat{f}) \neq 0$ which is not necessarily true. Thus multiply f by $\beta \in \mathbb{C}[V]$. Since $\beta \in \operatorname{Ker} \tilde{\delta}_i$ we have $\delta_i(\widehat{\beta f})(u) = \sum_{y \in \varphi^{-1}(u)} \beta(\pi_V(y)) \tilde{\delta}_i(f)(y)$. Note that $\tilde{\delta}_i(f)(y_0)$ is not zero at a general $y_0 \in SL_2 \times V$ since $\tilde{\delta}_i(f) \neq 0$. By a standard application of the Nullstellensatz we can choose β with prescribed values at the finite set $\varphi^{-1}(u_0)$ where $u_0 = \varphi(y_0)$. Hence we can assure that $\delta_i(\widehat{\beta f})(u_0) \neq 0$, i.e. $\deg_{\delta_i}(\widehat{\beta f}) = 1$. There is still one problem: $\widehat{\beta f}$ is regular on U but necessarily not on X. In order to fix it we set $g = \alpha \widehat{\beta f}$ where α is a lift of a nonzero function on $X//G$ that vanishes with high multiplicity on $(X//G) \setminus (U//G)$. Since $\alpha \in \operatorname{Ker} \delta_i$ we still have $\deg_{\delta_i}(g) = 1$ which concludes the proof in the case of a finite isotropy group.

For a one-dimensional isotropy group note that f is $\mathbb{C}^*$-invariant with respect to the action of the diagonal subgroup of SL_2. That is, f can be viewed as a function on $SL_2 \times_{\mathbb{C}^*} V$. Then we can replace morphism φ with morphism $\psi : SL_2 \times_{\mathbb{C}^*} V \to U$ that factors through the étale morphism $SL_2 \times_{G_x} V \to U$. Now ψ is also étale and the rest of the argument remains the same.

□

3.2.4 Remark. Let $\pi : X \to Y$ be an étale and finite map, and let $f \in \mathbb{C}[X]$. Then the function $\hat{f}$, defined by averaging the values on the fibers of π (as done in the proof in the above proposition) is indeed a regular function of Y. In fact, since π is étale, X can be defined locally as a subvariety of $Y \times \mathbb{C}$, given by the zero of a single polynomial $p \in \mathbb{C}[Y][t]$ of degree n, that splits into linear factors, say $p = c(a_1 - t)...(a_n - t)$, where c and a_i are regualar functions of Y. The fact that π is finite implies that the degre of the defining polynomial is always equal to n (and n is the cardinality of all the fibers). Therefore we can write $\hat{f}(y) = f(y, a_1(y)) + ... + f(y, a_n(y))$. If π is not finite, the function $\hat{f}$ is rational, since some of the linear factors of the defining polynomials can became infinite. Consider, for example, $X = \{(xy-1)x = 0\} \subset \mathbb{C}^2$, $Y = \mathbb{C}$, and $\pi : X \to Y$ be $\pi(x, y) = y$. Then, for $x = f \in \mathbb{C}[X]$, $\hat{f} = \frac{1}{y}$ has a pole at $y = 0$.

Part 2: Semicompatibility

In order to finish the proof of Theorem 3.2.1, we need to show semi-compatibility of vector fields δ_1 and δ_2 on X. Let U be a saturated set as in diagram (2.2) with $G = SL_2$. Since U is SL_2-invariant, it is H_i-invariant (where H_i is from Notation 3.1.1), and the restriction of δ_i to U is a locally nilpotent derivation which we denote again by δ_i. Moreover, being saturated, U contains a closed orbit, and since we can cover X by saturated open subsets, Proposition 2.4.10 implies the following.

3.2.5 Lemma. *If for saturated open set U as before the locally nilpotent vector fields δ_1 and δ_2 are semi-compatible on U then they are semi-compatible on X.*

3.2.6 Notation. Suppose further that H_1 and H_2 act on $SL_2 \times V$ by left multiplication on first factor. The locally nilpotent vector fields associated with these actions of H_1 and H_2 are, obviously, semi-compatible since they are compatible on SL_2 (see Example 2.4.3). Consider the SL_2-equivariant morphism $G \times V \to G \times_{G_x} V$ where V, $G = SL_2$, and G_x are as in diagram (2.2). By definition $G \times_{G_x} V$ is the quotient of $G \times V$ with respect to the G_x-action whose restriction to the first factor is the multiplication from the right. Hence H_i-action commutes with G_x-action and, therefore, one has the induced H_i-action on $G \times_{G_x} V$. Following the patten of Notation 3.1.1 we denote the associated locally nilpotent derivations on $G \times_{G_x} V$ again by δ_1 and δ_2. That is, the SL_2-equivariant étale morphism $\varphi : G \times_{G_x} V \to U$ transforms vector field δ_i on $G \times_{G_x} V$ into vector field δ_i on U.

From Lemma 2.4.9 and Luna's slice theorem we have immediately the following.

3.2.7 Lemma. *If the locally nilpotent vector fields δ_1 and δ_2 are semi-compatible on $G \times_{G_x} V$ then they are semi-compatible on U.*

Now need to prove the compatibility condition on saturated open sets for all isotropy groups listed in Lemma 3.2.2. The case of finite isotropy group comes for free from 2.4.8.

3.2.8 Proposition. *If the isotropy group G_x is finite δ_1 and δ_2 are semi-compatible on $G \times_{G_x} V$.*

Now we have to tackle semi-compatibility in the case of one-dimensional isotropy subgroup G_x using Proposition 2.4.7 as a main tool. We start with the case of $G_x = \mathbb{C}^*$, which acts on the slice V, fixing x. Since the SL_2 action is non-degenerate, $\dim U//SL_2 = \dim X//SL_2 \leq \dim X - 3$, thus we have the following lemma.

3.2.9 Lemma. *The isotropy group at a generic point of the slice V under the $\mathbb{C}^*$ action is finite.*

Proof. Suppose the contrary: then V will consist entirely of fixed points, and $V//\mathbb{C}^* \cong V$. This is impossible, since there would be an étale map $V \to U//SL_2$, but $\dim V = \dim X - 2$, while $\dim U//SL_2 \leq \dim X - 3$. □

3.2.10 Notation. Consider the diagonal $\mathbb{C}^*$-subgroup of SL_2, i.e. elements of form

$$s_\lambda = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

The action of s_λ on $v \in V$ will be denoted by $\lambda.v$. When we speak later about the $\mathbb{C}^*$-action on V we mean exactly this action. Set $Y = SL_2 \times V$, $Y' = SL_2 \times_{\mathbb{C}^*} V$, $Y_i = Y//H_i$, $Y_i' = Y'//H_i$. Denote by $\rho_i : Y \to Y_i$ the quotient morphism of the H_i-action and use the similar notation for Y', Y_i'. Set $\rho = (\rho_1, \rho_2) : Y \to Y_1 \times Y_2$ and $\rho' = (\rho_1', \rho_2') : Y' \to Y_1' \times Y_2'$.

Note that $Y_i \simeq \mathbb{C}^2 \times V$ since $SL_2//\mathbb{C}_+ \simeq \mathbb{C}^2$. Furthermore, looking at the kernels of δ_1 and δ_2 from Example 2.4.3, we see for

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \in SL_2$$

the quotient maps $SL_2 \to SL_2//H_1 \simeq \mathbb{C}^2$ and $SL_2 \to SL_2//H_2 \simeq \mathbb{C}^2$ are given by $A \mapsto (a_1, a_2)$ and $A \mapsto (b_1, b_2)$ respectively. Hence the morphism $\rho : SL_2 \times V = Y \to Y_1 \times Y_2 \simeq \mathbb{C}^4 \times V \times V$ is given by

$$\rho(a_1, a_2, b_1, b_2, v) = (a_1, a_2, b_1, b_2, v, v)\,. \tag{3.1}$$

As we mentioned before, to define $Y' = SL_2 \times_{\mathbb{C}^*} V$ we let $\mathbb{C}^*$ act on SL_2 via right multiplication. Since H_1 and H_2 act on SL_2 from the left, there are well-defined $\mathbb{C}^*$-actions on Y_1 and on Y_2 and a torus $\mathbb{T}$-action on $Y_1 \times Y_2$, where $\mathbb{T} = \mathbb{C}^* \times \mathbb{C}^*$.

Namely,

$$(\lambda, \mu).(a_1, a_2, b_1, b_2, v, w) = (\lambda a_1, \lambda^{-1} a_2, \mu b_1, \mu^{-1} b_2, \lambda.v, \mu.w) \tag{3.2}$$

for $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$ and $(\lambda, \mu) \in \mathbb{T}$.

Since the $\mathbb{C}^*$-action on Y and the action of H_i, $i = 1, 2$ are commutative, the following diagram is also commutative.

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Y_1 \times Y_2 \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ Y' & \xrightarrow{\rho'} & Y_1' \times Y_2', \end{array} \tag{3.3}$$

where q (resp. p) is the quotient map with respect to the $\mathbb{T}$-action (resp. $\mathbb{C}^*$-action). It is also worth mentioning that the $\mathbb{C}^*$-action on Y induces the action of the diagonal of $\mathbb{T}$ on $\rho(Y)$, i.e. for every $y \in Y$ we have $\rho(\lambda.y) = (\lambda, \lambda).\rho(y)$.

3.2.11 Lemma. *Let $Z = \rho(Y)$ in diagram (3.3) and Z' be the closure of $\rho'(Y')$.*

(i) The map $\rho : Y \to Z$ is an isomorphism and Z is the closed subvariety of $Y_1 \times Y_2 = \mathbb{C}^4 \times V \times V$ that consists of points $(a_1, a_2, b_1, b_2, v, w) \in Y_1 \times Y_2$ satisfying the equations $a_1 b_2 - a_2 b_1 = 1$ and $v = w$.

(ii) Let T be the $\mathbb{T}$-orbit of Z in $Y_1 \times Y_2$ and $\bar{T}$ be its closure. Then T coincides with the $(\mathbb{C}^* \times 1)$-orbit (resp. $(1 \times \mathbb{C}^*)$-orbit) of Z. Furthermore, for each $(a_1, a_2, b_1, b_2, v, w) \in \bar{T}$ one has $\pi(v) = \pi(w)$ where $\pi : V \to V//\mathbb{C}^*$ is the quotient morphism.

(iii) The restriction of diagram (3.3) yields the following

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Z \subset \bar{T} \\ \big\downarrow{\scriptstyle p} & & \big\downarrow{\scriptstyle q} \\ Y' & \xrightarrow{\rho'} & q(Z) \subset Z' \end{array} \tag{3.4}$$

where $Y' = Y//\mathbb{C}^* = Y/\mathbb{C}^*$, q *is the quotient morphism of the* $\mathbb{T}$*-action (i.e.* $Z' = \bar{T}//\mathbb{T}$*), and* $q(Z) = \rho'(Y')$.

Proof. The first statement is an immediate consequence of formula (3.1). The beginning of the second statement follows from the fact that the action of the diagonal $\mathbb{C}^*$-subgroup of $\mathbb{T}$ preserves Z. This implies that for every $t = (a_1, a_2, b_1, b_2, v, w) \in T$ points $v, w \in V$ belong to the same $\mathbb{C}^*$-orbit and, in particular, $\pi(v) = \pi(w)$. This equality holds for each point in $\bar{T}$ by continuity.

In diagram (3.3) $Y' = Y//\mathbb{C}^* = Y/\mathbb{C}^*$ because of Proposition 2.1.6 (3) and Lemma 2.2.2, and the equality $q(Z) = \rho'(Y')$ is the consequence of the commutativity of that diagram. Note that $\bar{T}$ is $\mathbb{T}$-invariant. Hence $q(\bar{T})$ coincides with Z' by Proposition 2.1.6 (4). Being the restriction of the quotient morphism, $q|_{\bar{T}} : \bar{T} \to Z'$ is a quotient morphism itself (e.g., see [4]) which concludes the proof.

□

3.2.12 Lemma. *There is a rational* $\mathbb{T}$*-quasi-invariant function* f *on* $\bar{T}$ *such that for* $t = (a_1, a_2, b_1, b_2, w, v) \in T$ *one has*

(1) $\frac{1}{f(t)} a_1 b_2 - f(t) a_2 b_1 = 1$ *and* $w = f(t).v$*;*

(2) the set $\bar{T} \setminus T$ *is contained in* $(f)_0 \cup (f)_\infty$*;*

(3) f *generates a regular function on a normalization* T_N *of* T

Proof. By Lemma 3.2.11 (ii) any point $t = (a_1, a_2, b_1, b_2, w, v) \in T$ is of form $t = (\lambda, 1).z_0$ where $z_0 \in Z$ and $\lambda \in \mathbb{C}^*$. Hence formula (3.2) implies that $w = \lambda.v$ and $\lambda^{-1} a_1 b_2 - \lambda a_2 b_1 = 1$. The last equality yields two possible values (one of which can be ∞ or 0 if any of numbers a_1, a_2, b_1, or b_2 vanish)

$$\lambda_\pm = \frac{-1 \pm \sqrt{1 + 4a_1 a_2 b_1 b_2}}{2a_2 b_1}$$

and we assume that

$$\lambda = \lambda_- = \frac{-1 - \sqrt{1 + 4a_1a_2b_1b_2}}{2a_2b_1},$$

i.e. $w = \lambda_-.v$. Note that $\lambda_+.v = w$ as well only when

$$\tau = \frac{\lambda_+}{\lambda_-} = \frac{-1 + \sqrt{1 + 4a_1a_2b_1b_2}}{-1 - \sqrt{1 + 4a_1a_2b_1b_2}}$$

is in the isotropy group of v. Let's show now that we can define a branch of the multi-valued function $\lambda_\pm$. The generic $\mathbb{C}^*$-orbit on the slice V is one-dimensional, that is it has finite isotropy group ($\mathbb{C}^*$ is abelian, so all points on the same orbit have the same isotropy group). Denote by B be the proper closed subset in V of fixed points, and $V^0 = V \setminus B$. Also, consider the variety $I \subset V^0 \times V^0 \times \mathbb{C}^*$ given by $I = \{(v, w, \lambda) : \lambda.v = w\}$: the projection $\pi : I \to V^0 \times V^0$, is a quasi-finite morphism, and after removing a proper closed subset from V^0 we can assume that it is unramified. Therefore the functions $\lambda_\pm$ defines local holomorphic sections of π, and they are locally defined holomorphic functions on T. Now, since the closure of the set S of points such that $\tau.v = v$ and $v \in V^0$ is a proper subvariety of T, we have only one choice for λ on its complement. Its extension to $\bar{T}$, which is denoted by f, satisfies (1).

Let $t_n \in T$ and $t_n \to t \in \bar{T}$ as $n \to \infty$. By Lemma 3.2.11 (ii) t_n is of form $t_n = (f(t_n)a_1^n, \frac{1}{f(t_n)}a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n)$ where

$$\begin{pmatrix} a_1^n & a_2^n \\ b_1^n & b_2^n \end{pmatrix} \in SL_2 \text{ and } v = \lim_{n\to\infty} v_n\,.$$

If sequences $\{f(t_n)\}$ and $\{1/f(t_n)\}$ are bounded then switching to a subsequence one can suppose that $f(t_n) \to f(t) \in \mathbb{C}^*$, $w = f(t)v$, and $t = (f(t)a'_1, \frac{1}{f(t)}a'_2, b'_1, b'_2, f(t).v, v)$ where

$$\begin{pmatrix} a'_1 & a'_2 \\ b'_1 & b'_2 \end{pmatrix} \in SL_2\,,$$

i.e. $t \in T$. Hence $\bar{T} \setminus T$ is contained in $((f)_0 \cup (f)_\infty)$ which is (2).

Function f is regular on $T \setminus S$ by construction. Consider $t \in S$ with w and v in the same non-constant $\mathbb{C}^*$-orbit, i.e. $w = \lambda.v$ for some $\lambda \in \mathbb{C}^*$. Then $w = \lambda'.v$ if and only if only λ' belongs to the coset Γ of the isotropy subgroup of v in $\mathbb{C}^*$. For any sequence of points t_n convergent to t one can check that $f(t_n) \to \lambda \in \Gamma$ by continuity, i.e. f is bounded in a neighborhood of t. Let $\nu : T_N \to T$ be a normalization morphism. Then function $f \circ \nu$ extends regularly to $\nu^{-1}(t)$ by the Riemann extension theorem. The set of point of S for which v is a fixed point of the $\mathbb{C}^*$-action is of codimension at least 2 in T. By the Hartogs' theorem $f \circ \nu$ extends regularly to T_N which concludes (3).

$\square$

3.2.13 Remark. Consider the rational map $\kappa : T \to Z$ given by $t \mapsto (\frac{1}{f(t)}, 1).t$. It is regular on $T \setminus S$ and if $t \in T \setminus S$ and $z \in Z$ are such that $t = (\lambda, 1).z$ then $\kappa(t) = z$. In particular κ sends $\mathbb{T}$-orbits from T into $\mathbb{C}^*$-orbits of Z. Furthermore, morphism $\kappa_N = \kappa \circ \nu : T_N \to Z$ is regular by the same reason as function $f \circ \nu$ is.

3.2.14 Lemma. *Let $E_i = \{t = (a_1, a_2, b_1, b_2, w, v) \in \bar{T} | b_i = 0\}$ and $\bar{T}^b$ coincide with $T \cup ((f)_0 \setminus E_2) \cup ((f)_\infty \setminus E_1)$. Suppose that $\bar{T}^b_N$ is a normalization of $\bar{T}^b$. Then there is a regular extension of $\kappa_N : T_N \to Z$ to a morphism $\bar{\kappa}^b_N : \bar{T}^b_N \to Z$.*

Proof. Since the set $(f)_0 \cap (f)_\infty$ is of codimension 2 in $\bar{T}$, the Hartogs' theorem implies that it suffices to prove the regularity of $\bar{\kappa}_N^b$ on the normalization of $\bar{T}^b \setminus ((f)_0 \cap (f)_\infty)$. Furthermore, by the Riemann extension theorem it is enough to construct a continuous extension of κ from $T \setminus S$ to $\bar{T}^b \setminus (S \cup ((f)_0 \cap (f)_\infty))$.

By Lemma 3.2.12 (2) we need to consider this extension, say, at $t = (a_1, a_2, b_1, b_2, w, v) \in (f)_0 \setminus (f)_\infty$. Let $t_n \to t$ as $n \to \infty$ where

$$t_n = (f(t_n)a_1^n, \frac{1}{f(t_n)}a_2^n, b_1^n, b_2^n, f(t_n).v_n, v_n) \in T$$

with $a_1^n b_2^n - a_2^n b_1^n = 1$ and $f(t_n) \to 0$. Perturbing, if necessary, this sequence $\{t_n\}$ we can suppose every $t_n \notin S$, i.e. $\kappa(t_n) = (a_1^n, a_2^n, b_1^n, b_2^n, v_n, v_n)$. Note that $\lim v_n = v$, $b_k = \lim b_k^n$, $k = 1, 2$ and $a_2^n \to 0$ since a_2 is finite. Hence $1 = a_1^n b_2^n - a_2^n b_1^n \approx a_1^n b_2$ and $a_1^n \to 1/b_2$ as $n \to \infty$. Now we get a continuous extension of κ by putting $\kappa(t) = (1/b_2, 0, b_1, b_2, v, v)$. This yields the desired conclusion.

□

3.2.15 Remark. If we use the group $(1 \times \mathbb{C}^*)$ instead of the group $(\mathbb{C}^* \times 1)$ from Lemma 3.2.11 (ii) in our construction this would lead to the replacement of f by f^{-1}. Furthermore for the variety $\bar{T}^a = T \cup ((f)_0 \setminus \{a_1 = 0\}) \cup ((f)_\infty \setminus \{a_2 = 0\})$ we obtain a morphism $\bar{\kappa}_N^a : \bar{T}_N^a \to Z$ similar to $\bar{\kappa}_N^b$.

The next fact will be crucial for the application of Hartog's Theorem.

3.2.16 Lemma. *The complement $\bar{T}^0$ of $\bar{T}^a \cup \bar{T}^b$ in $\bar{T}$ (which is $\bar{T}^0 = (\bar{T} \setminus T) \cap \bigcup_{i \neq j} \{a_i = b_j = 0\}$) has codimension at least 2.*

Proof. Let's show first why $\bar{T}^0 = (\bar{T} \setminus T) \cap \bigcup_{i \neq j} \{a_i = b_j = 0\}$. If $t \in \bar{T}^0$, then $t \notin T$, and by (2) of Lemma 3.2.12, t must be either in $(f)_0$ or in $(f)_\infty$. If $t \in (f)_0$, then since $t \notin (f)_0 \setminus E_2$, t belongs to E_2, that is $b_2 = 0$. The previous remark shows that in this case we also have $a_1 = 0$. If $t \in (f)_\infty$, with the same reasoning we show that $b_1 = a_2 = 0$.

Let $t_n \to t = (a_1, a_2, b_1, b_2, w, v)$ be as in the proof of Lemma 3.2.14. Since for a general point of the slice V the isotropy group is finite after perturbation we can suppose that each v_n is contained in a non-constant $\mathbb{C}^*$-orbit $O_n \subset V$. Treat v_n and $f(t_n).v_n$ as numbers in $\mathbb{C}^* \simeq O_n$ such that $f(t_n).v_n = f(t_n)v_n$. Let $|v_n|$ and $|f(t_n).v_n|$ be their absolute values. Then one has the annulus $A_n = \{|f(t_n).v_n| < \zeta < |v_n|\} \subset O_n$, i.e. $\zeta = \eta v_n$ where $|f(t_n)| < |\eta| < 1$ for each $\zeta \in A_n$. By Lemma 3.2.11 (iii) $\pi(v) = \pi(w)$ but by Lemma 3.2.12 (3) the $\mathbb{C}^*$-orbit $O(v)$ and $O(w)$ are different unless $w = v$ is a fixed point of the $\mathbb{C}^*$-action. In any case, by Proposition 2.1.6 (2) the closures of these orbits meet at a fixed point $\bar{v}$ of the $\mathbb{C}^*$-action.

Consider a compact neighborhood $W = \{u \in V | \varphi(u) \leq 1\}$ of $\bar{v}$ in V where φ is a plurisubharmonic function on V that vanishes at $\bar{v}$ only. Note that the sequence $\{(\lambda, \mu).t_n\}$ is convergent to $(\lambda a_1, a_2/\lambda, \mu b_1, b_2/\mu, \lambda.w, \mu.v)$. In particular, replacing $\{t_n\}$ by $\{(\lambda, \mu).t_n\}$ with appropriate λ and μ we can suppose that the boundary ∂A_n of any annulus A_n is contained in W for sufficiently large n. By the maximum principle $\bar{A}_n \subset W$. The limit $A = \lim_{n\to\infty} \bar{A}_n$ is a compact subset of W that contains both v and w, and also all points $\eta.v$ with $0 < |\eta| < 1$ (since $|f(t_n)| \to 0$). Unless $O(v) = \bar{v}$ only one of the closures of sets $\{\eta.v | \, 0 < |\eta| < 1\}$ or $\{\eta.v | \, |\eta| > 1\}$ in V is compact and contains the fixed point $\bar{v}$ (indeed, otherwise the closure of $O(v)$ is a complete curve in the affine variety V). The argument before shows that it is the first one.

That is, $\mu.v \to \bar{v}$ when $\mu \to 0$. Similarly, $\lambda.w \to \bar{v}$ when $\lambda \to \infty$. It is not difficult to check now that the dimension of the set of such pairs (w, v) is at most $\dim V$.

Consider the set $(\bar{T} \setminus T) \cap \{a_1 = b_2 = 0\}$. It consists of points $t = (0, a_2, b_1, 0, w, v)$ and, therefore, its dimension, is at most $\dim V + 2$. Thus it has codimension at least 2 in $\bar{T}$ whose dimension is $\dim V + 4$. This yields the desired conclusion.

□

The next technical fact may be somewhere in the literature, but unfortunately we did not find a reference and our proof is a bit artificial.

3.2.17 Proposition. *Let a reductive group G act on an affine algebraic variety X and $\pi : X \to Q := X//G$ be the quotient morphism such that one of closed G-orbits O is contained in the smooth part of X. Suppose that $\nu : X_N \to X$ and $\mu : Q_N \to Q$ are normalization morphisms, i.e. $\pi \circ \nu = \pi_N \circ \mu$ for some morphism $\pi_N : X_N \to Q_N$. Then $Q_N \simeq X_N//G$ for the induced G-action on X_N and π_N is the quotient morphism.*

Proof. Let $\psi : X_N \to X_N//G$ be the quotient morphism, and observe that $X//G$ is normal by (1) of 2.1.6. By the universal property of quotient morphims there exists a morphism $\varphi : X_N//G \to Q_N$ such that $\pi_N = \varphi \circ \psi$. We need to show that that φ is an isomorphism, or equivalently, being Q and $X_N//G$ are both normal, that it is finite and birational The points of Q (resp. $X_N//G$) are nothing but the closed G-orbits in X (resp. X_N) by Proposition 2.1.6, and above each closed orbit in X we have only a finite number of closed orbits in X_N because ν is finite. Hence $\mu \circ \varphi : X_N//G \to Q$ and, therefore, $\varphi : R \to Q_N$ are at least quasi-finite. There is only one closed orbit O_N in X_N above orbit $O \subset \operatorname{reg} X$. Thus φ is injective in a neighborhood of $\psi(O_N)$. That is, φ is birational and by the Zariski Main theorem it is an embedding, that is birational.

It remains to show that φ is proper. Recall that G is a complexification of a its compact subgroup $G^{\mathbb{R}}$ and there is a so-called Kempf-Ness real algebraic subvariety

$X^{\mathbb{R}}$ of X such that the restriction $\pi|_{X^{\mathbb{R}}}$ is nothing but the standard quotient map $X^{\mathbb{R}} \to X^{\mathbb{R}}/G^{\mathbb{R}} = Q$ which is automatically proper (e.g., see [23]). Set $X_N^{\mathbb{R}} = \nu^{-1}(X^{\mathbb{R}})$. Then the restriction of $\pi \circ \nu$ to $X_N^{\mathbb{R}}$ is proper being the composition of two proper maps. On the other hand the restriction of $\mu \circ \pi_N = \pi \circ \nu$ to $X_N^{\mathbb{R}}$ is proper only when morphism φ, through which it factors (ψ is surjective), is proper which concludes the proof. □

3.2.18 Proposition. *Morphism $\rho' : Y' \to Z'$ from diagram* 3.4 *is finite birational.*

Proof. Morphism ρ' factors through $\rho'_N : Y' \to Z'_N$ where $\mu : Z'_N \to Z'$ is a normalization of Z' and the statement of the proposition is equivalent to the fact that ρ'_N is an isomorphism. Set $Z'(b) = q(\bar{T}^b)$ and $Z'(a) = q(\bar{T}^a)$. Note that $Z' \setminus (Z'(a) \cup Z'(b))$ is in the q-image of the $\mathbb{T}$-invariant set $\bar{T}^0$ from Lemma 3.2.16. Hence $Z' \setminus (Z'(b) \cup Z'(a))$ is of codimension 2 in Z' and by the Hartogs' theorem it suffices to prove that ρ'_N is invertible over $Z(b)'$ (resp. $Z'(a)$).

By Remark 3.2.13 $\bar{\kappa}_N^b$ sends each orbit of the induced $\mathbb{T}$-action on $\bar{T}_N^b$ onto a $\mathbb{C}^*$-orbit in Z. Thus the composition of $\bar{\kappa}_N^b$ with $p : Z \simeq Y \to Y'$ is constant on $\mathbb{T}$-orbits and by the universal property of quotient spaces it must factor through the quotient morphism $q_N^b : \bar{T}_N^b \to Q$. By Proposition 3.2.17 $Q = Z'_N(b)$ where $Z'_N(b) = \mu^{-1}(Z'(b))$. That is, $p \circ \bar{\kappa}_N^b = \tau^b \circ q_N^b$ where $\tau^b : Z'_N(b) \to Y'$. Our construction implies that τ^b is the inverse of ρ'_N over $Z'_N(b)$. Hence ρ'_N is invertible over $Z'_N(b)$ which concludes the proof.

□

Proposition 3.2.3 and Proposition 3.2.18 imply the compatibility in the case $G_x \cong \mathbb{C}^*$.

3.2.19 Proposition. *If the isotropy group G_x is isomorphic to $\mathbb{C}^*$, then δ_1 and δ_2 are compatible on U.*

Finally, we are left with the case of $G_x \cong \hat{\mathbb{C}}^*$, the normalizer of $\mathbb{C}^*$ in SL_2.

3.2.20 Proposition. *If G_x is isomorphic to the normalizer of $\mathbb{C}^*$ in SL_2, the vector fields δ_1 and δ_2 are compatible.*

Proof. We just proved that the vector fields are compatible on $Y = SL_2 \times_{\mathbb{C}^*} V$. Since $\hat{\mathbb{C}}^*$ normalizes $\mathbb{C}^*$, the action of $\hat{\mathbb{C}}^*/\mathbb{C}^* \cong \mathbb{Z}_2$ on Y is well defined. Since the quotient map $Y \to Y/\mathbb{Z}_2 = SL_2 \times_{\hat{\mathbb{C}}^*} V$ is étale, we are done by Proposition 2.4.8 □

Conclusion of the proof

For each pair $\{\delta_1,\delta_2\}$ of derivations induced by a SL_2-action, Proposition 3.2.3 gives the existence of the regular function g with degree 1 with respect to both derivations; Propositions 3.2.18, 3.2.19, 3.2.20, with the aid of Proposition 2.4.7 imply the semicompatibility. Therefore Theorem3.2.1 holds. Since we have "sufficiently many" SL_2-actions, as explained in remark 3.1.3, Theorem 3.2.1 and Theorem 2.4.5 implies Theorem 3.1.2.

The next theorem shows that the condition of the SL_2-action of being fixed point-free is essential for the criterion of compatibility.

Theorem 3.2.21. *Let SL_2 act on X, and $\hat{x} \in X$ be a fixed point. Then δ_1, δ_2 are not a compatible derivations on $\mathbb{C}[X]$.*

Proof. We have the same setting described in Notation 3.2.10, with the exception that $\mathbb{C}^*$ is replaced by SL_2. In this case the slice V coincide with the saturated open set $U = SL_2.V$, and $Y' = SL_2 \times_{SL_2} V \cong U$. $SL_2 \times SL_2$ acts on $Y_1 \times Y_2$, and Z is a closed subset of $Y_1 \times Y_2$ invariant with respect to the action of the diagonal subgroup $\Delta \subset SL_2 \times SL_2$.

Let $\underline{a} = (a_1, a_2)$, $\underline{b} = (b_1, b_2)$. The action of $(\sigma, \tau) \in SL_2 \times SL_2$ on a point $(\underline{a}, \underline{b}, v, w) \in Y_1 \times Y_2$ is given by $(\sigma, \tau).(\underline{a}, \underline{b}, v, w) = (\underline{a}\sigma^{-1}, \underline{b}\tau^{-1}, \sigma.v, \tau.w)$.

We show that $\rho' : Y' \to Z'$ is not finite. Let $t = (\underline{a}, \underline{b}, \hat{x}, \hat{x}) \in Z$. For generic $\underline{a}$ and $\underline{b}$, the closed subset B of SL_2 given by the equation

$$\det \begin{pmatrix} \underline{a}\sigma^{-1} \\ \underline{b} \end{pmatrix} = 1$$

is two-dimensional.

Therefore $\tilde{B} = \{(\sigma, 1).t : \sigma \in B\}$ is a two-dimensional subvariety of Z. All the points of $\tilde{B}$ belong to the same $SL_2 \times SL_2$-orbit, but they belong to pairwise different closed Δ-orbits. Therefore $p(\rho^{-1}(\tilde{B}))$ is two-dimensional, but $\rho'(p(\rho^{-1}(\tilde{B}))) = p(\tilde{B})$ consists of 1 point which means that $\rho' : Y' \to Z'$ is not finite. Therefore, according to Proposition 2.4.7 the pair of derivations are not compatible on U, which are not compatible on $\mathbb{C}[X]$, since the map $\rho : X \to X_1 \times X_2$ restricts to $\rho' : Y' \to Y_1' \times Y_2'$.

□

3.2.22 Remark. (1) We do not know if the condition about the absence of fixed points is essential for Theorem 3.1.2. In examples we know the presence of fixed points is not an obstacle for the algebraic density property. Say, for $\mathbb{C}^n$ with $2 \leq n \leq 4$ any algebraic SL_2-action is a representation in a suitable polynomial coordinate system (see, [21]) and, therefore, has a fixed point; but the validity of the algebraic density property is a consequence the Andersén-Lempert work.

(2) The simplest case of a degenerate SL_2-action is presented by the homogeneous space $SL_2/\mathbb{C}^*$ where $\mathbb{C}^*$ is the diagonal subgroup. Let

$$A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$$

be a general element of SL_2. Then the ring of invariants of the $\mathbb{C}^*$-action is generated by $u = a_1a_2$, $v = b_1b_2$, and $z = a_2b_1 + 1/2$ (since $a_1b_2 = 1 + a_2b_1 = 1/2 + z$). Hence $SL_2/\mathbb{C}^*$ is isomorphic to a hypersurface S in $\mathbb{C}^3_{u,v,z}$ given by the equation $uv = z^2 - 1/4$. In particular, it has the algebraic density property by [15].

(3) However, the situation is more complicated if we consider the normalizer T of the diagonal $\mathbb{C}^*$-subgroup of SL_2 (i.e. T is an extension of $\mathbb{C}^*$ by $\mathbb{Z}_2$). Then $\mathcal{P} = SL_2/T$ is isomorphic to $S/\mathbb{Z}_2$ where the $\mathbb{Z}_2$-action is given by $(u, v, z) \to (-u, -v, -z)$. It can be shown that this surface $\mathcal{P}$ is the only $\mathbb{Q}$-homology plane which is simultaneously a Danilov-Gizatullin surface (i.e. it has a trivial Makar-Limanov invariant (see [6])), and its fundamental group is $\mathbb{Z}_2$. We doubt that $\mathcal{P}$ has algebraic density property, and it would be the first example of an homogeneous affine manifold (of dimension at least two) without the density property.

Chapter 4

Applications

In this Chapter we show that Theorem 3.1.2 can be applied to a wide class of quotient of linear algebraic groups. In particular, we obtain an almost complete analog of Theorem 1.3.2, where now we can replace the linear algebraic group G by a quotient of it with respect to a reductive subgroup.

4.1 Density property of homogeneous spaces

Theorem 4.1.1. *Let G be a linear algebraic group and R be its proper reductive subgroup such that the homogeneous space G/R is different from $\mathbb{C}_+$, a torus, or the $\mathbb{Q}$ -homology plane with fundamental group $\mathbb{Z}_2$ (i.e. the surface $\mathcal{P}$ from Remark* 3.2.22 *(3)). Then G/R has the algebraic density property.*

4.1.2 Notation. *Let G denote a complex algebraic group, R a reductive subgroup, and Γ a SL_2-subgroup, acting on the space $X = G/R$ of left cosets via left multiplication.*

4.1.3 Lemma. *The isotropy group of $\pi(a) \in X$ is isomorphic to $\Gamma \cap aRa^{-1}$. Therefore, the action has no fixed points iff $a^{-1}\Gamma a$ is not contained in R for any $a \in G$ and it is non-degenerate if $\Gamma^a := a^{-1}\Gamma a \cap R \simeq \Gamma \cap aRa^{-1}$ is finite for some $a \in R$.*

Theorem 3.1.2 implies the following.

4.1.4 Proposition. *Let G be an algebraic group and $\Gamma_1, \ldots, \Gamma_k$ be its SL_2-subgroups such that at some $x \in G$ the set $\{\delta_2^i(x)\}$ is a generating one (where (δ_1^i, δ_2^i) is the corresponding pair of locally nilpotent vector fields on G generated by the natural Γ_i-action). Suppose that for each $i = 1, \ldots, k$ and any $a \in G$ the group $\Gamma_i^a := a^{-1}\Gamma_i a$ is not isomorphic to Γ_i, and furthermore Γ_i^a is finite for some a. Then $G//R$ has the algebraic density property.*

We will use the following techical fact whose proof requires some non trivial facts of Lie group theory. The proof of this theorem can be found in the Appendix of [5].

Theorem 4.1.5. *Let G be a simple Lie group with Lie algebra different from $\mathfrak{sl}_2$ and R be its proper reductive subgroup. Then there exists an SL_2-subgroup Γ in G such that Γ^a is not isomorphic to Γ for any $a \in G$, and, furthermore, Γ^a is finite for some $a \in G$.*

4.2 Proof of Theorem 4.1.1: G semisimple

Let G be a semisimple group. Then G is isomorphic to a direct product of simple groups, say $G = G_1 \oplus G_2 ... \oplus G_n$, where G_i are all simple. Let R be a reductive subgroup of G, and consider a SL_2-subgroup Γ_1 of G_1; denote by $a = (a_1, ..., a_n)$, for $a_i \in G_i$, an element of G, and by e_i the identity of G_i. Then, according to Lemma 4.1.3, the isotropy group of the natural Γ_1-action on G/R at a point $\pi(a)$ is isomorphic to $a_1^{-1}\Gamma_1 a_1 \times e_2 \ldots \times e_n \cap R \cap G_1 \times e_2 \ldots \times e_n = a_1^{-1}\Gamma a_1 \cap R \cap G_1$. In order to apply Theorem 4.1.5, we need that $R \cap G_1 \neq G_1$, and the next lemma allows us to exclude this possibilty.

4.2.1 Lemma. *If $G_i \cap R = G_i$, then $G/R \cong G/(\pi_1 \times \pi_{i-1} \times \pi_{i+1} \times \ldots \times \pi_n)(R)$*

Proof. Without loss of generality we can assume $i = 1$. Let $G/R \to G/(\pi_2 \times \ldots \pi_n)(R)$ be the map induced by the projection $G \to G_2 \times \ldots \times G_n$. Then it's an easy check that ψ is well defined and surjective. To check injectivity, use the fact that, since $G_1 \times e_2 \times \ldots e_n \subset R$ by hypothesis, then $G_1 \times r' \subset R$ for all $r' \in (\pi_2 \times \ldots \pi_n)(R)$. □

4.2.2 Corollary. *Let $X = G/R$ be an affine homogeneous space of a semi-simple Lie group G. Suppose that X is different from a $\mathbb{Q}$-homology plane with $\mathbb{Z}_2$ as a fundamental group. Then X is equipped with N pairs $\{\delta_1^k, \delta_2^k\}_{k=1}^N$ of compatible derivations such that the collection $\{\delta_2^k(x_0)\}_{k=1}^N \subset T_{x_0}X$ is a generating set at some point $x_0 \in X$. In particular, X has the algebraic density property by Theorem* 2.4.5.

Proof. Note that R is reductive by Proposition 2.1.7 (Matsushima's theorem). Then X is isomorphic to a quotient of form G/R where $G = G_1 \oplus \ldots \oplus G_N$, each G_i is a simple Lie group and R is not necessarily connected. However, we can suppose that R is connected by virtue of Proposition 2.4.8. Let $R_k = R \cap G_k$, which we can assume to be different from G_k (Lemma4.2.1). Let's show first that R_k is reductive. Given the projection $\pi' : R \to G_1 \times G_{k-1} \times G_{k+1} \ldots \times G_n$, $\pi'(R)$ is reductive being the image of a reductive group. Since $R_k = \ker \pi'$, it is reductive by Matsushima's theorem 2.1.7.c

Assume first that none of G_i's is isomorphic to SL_2. By Theorem 4.1.5 one can choose an SL_2-subgroup $\Gamma_k < G_k$ such that the natural Γ_k-action on $G_k/\mathbb{R}_k$ and, therefore, on G/R is fixed point free and non-degenerate. Denote by δ_1^k and δ_2^k the corresponding pair of locally nilpotent derivations for the Γ_k-action. Since the adjoint representation is irreducible for a simple Lie group, $\{\delta_2^k(e)\}_{k=1}^N$ is a generating set of the tangent space T_eG at $e = e_1 \oplus \ldots \oplus e_N \in G$, where e_k is a unit of G_k, and the desired conclusion in this case follows from Theorem 3.1.2.

Suppose now that $G_k \cong SL_2$ for some k. If R_k is discrete, then the action of Γ_k is nondegenerate and fixed point free. Therefore we can suppose that R_k is of positive dimension but then, by our assumption that $G_k \neq R_k$ and by Lemma 3.2.2, R_k must be isomorphic to a torus $\mathbb{C}^*$. In this case the Γ_k-action is degenerate, but we can perturb Γ_k in the following way.

Choose an isomorphism $\varphi_k : \Gamma_k \to \Gamma_1$ such that $\varphi(R_k)$ intersect R_1 only in a finite group. (start with any such ismomorphism, and then act on Γ_1 by conjugation to obtain the desired one). Then we have an SL_2-group $\Gamma^{\varphi_k} = \{(\varphi_k(\gamma), \gamma) | \gamma \in \Gamma_k\} < \Gamma_1 \times \Gamma_k$ acting naturally on $G_1 \times G_k$ and, therefore, on G in a non-degenerate way. Moreover, it's a set theory exercise to check that the action is also fixed point free.

In particular, by Theorem 3.2.1 we get pairs of compatible locally nilpotent derivations $\tilde{\delta}_1^{\varphi_k}$ and $\tilde{\delta}_2^{\varphi_k}$ corresponding to such actions. Set $G' = G_2 \oplus \ldots \oplus G_N$ and $e' = e_2 \oplus \ldots \oplus e_N \in G'$. Since the adjoint representation is irreducible for a simple Lie group the orbit of the set $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$ under conjugations generates a subspace of S of T_eG such that the restriction of the natural projection $T_eG \to T_{e'}G'$ to S is surjective. In order to enlarge $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N$ to a generating subset of T_eG consider an isomorphism $\psi_2 : \Gamma_2 \to \Gamma_1$ different from φ_2 and such that the Γ^{ψ_2}-action is non-degenerate and fixed point free. Denote the corresponding compatible locally nilpotent derivations by $\tilde{\delta}_1^{\psi_2}$ and $\tilde{\delta}_2^{\psi_2}$ on $G_1 \oplus G_2$ (and also by abusing notation on G). Note that the vectors $\tilde{\delta}_2^{\varphi_2}(e_1 \oplus e_2)$ and $\tilde{\delta}_2^{\psi_2}(e_1 \oplus e_2)$ can be assumed different with an appropriate choice of

ψ_2. Hence these two vectors form a generating subset of $T_{e_1 \oplus e_2} G_1 \oplus G_2$. Taking into consideration the remark about S we see that $\{\tilde{\delta}_2^{\varphi_k}(e)\}_{k=2}^N \cup \{\tilde{\delta}_2^{\psi_2}(e)\}$ is a generating subset of $T_e G$. Now pushing these SL_2-actions to X we get the desired conclusion.

$\square$

4.3 Proof of Theorem 4.1.1: the general case

Let's assume now that G is any linear algebraic group, and R any reductive subgroup of R. Since all components of G/R are isomorphic as varieties we can suppose that G is connected. Furthermore, by Corollary 4.2.2 and Remark 3.2.22 (2) we are done with a semi-simple G.

Let us consider first the case of a reductive but not semi-simple G. Then the center $Z \simeq (\mathbb{C}^*)^n$ of G is nontrivial. Let S be the semi-simple part of G. Assume for the time being that G is isomorphic as group to the direct product $S \times Z$ and consider the natural projection $\tau : G \to Z$. Set $Z' = \tau(R) = R/R'$ where $R' = R \cap S$. Since we are going to work with compatible vector fields we can suppose that R is connected by virtue of Lemma 2.4.8. Then Z' is a subtorus of Z and also R' is reductive by Proposition 2.1.7. Hence $G/R = (G/R')/Z'$ and $G/R' = S/R' \times Z$. Note that there is a subtorus Z'' of Z such that $Z'' \simeq Z/Z'$ and $Z' \cdot Z'' = Z$. (Let $G \subset GL_n$; then T and T' can be diagonalized, and if T' contains an element i-th diagonal element not equal to one, then by connectedness it contains all i-th diagonal elements; the complement T" is consists of the set of diagonal elements which can not be different to one on T''). Hence G/R is isomorphic to $\varrho^{-1}(Z'') \simeq S/R' \times Z''$ where $\varrho : G/R' \to Z$ is the natural projection. Note that both factors are nontrivial since otherwise G/R is either a torus or we are in the semi-simple case again. Thus X has the algebraic density property by Proposition 2.4.6 with S/R' playing the role of X_1 and Z'' of X_2. In particular, we have a finite set of pairs of compatible vector fields $\{\delta_1^k, \delta_2^k\}$ as in Theorem 2.4.5. Furthermore, one can suppose that the fields δ_1^k correspond to one parameter subgroups of S isomorphic to $\mathbb{C}_+$ and δ_2^k to one parameter subgroups of Z isomorphic to $\mathbb{C}^*$. In the general case G/R is the factor of X with respect to the natural action of a finite (central) normal subgroup $F < G$. Since F is central the fields δ_1^k, δ_2^k induce completely integrable vector fields $\tilde{\delta}_1^k$, $\tilde{\delta}_2^k$ on G/R while $\tilde{\delta}_2^k(x_0)$ is a generating set for some $x_0 \in G/R$. By Lemma 2.4.8 the pairs $\{\tilde{\delta}_1^k, \tilde{\delta}_2^k\}$ are compatible and the density property for G/R follows again from Theorem 2.4.5.

In the case of a general linear algebraic group G different from a reductive group, $\mathbb{C}^n$, or a torus $(\mathbb{C}^*)^n$ consider the nontrivial unipotent radical $\mathcal{R}_u$ of G. It is automatically an algebraic subgroup of G ([3], p. 183). By Mostow's theorem [19] (see also [3], p. 181) G contains a (Levi) maximal closed reductive algebraic subgroup G_0 such that G is the semi-direct product of G_0 and $\mathcal{R}_u$, i.e. G is isomorphic as affine variety to the product $\mathcal{R}_u \times G_0$. Furthermore, any other maximal reductive subgroup is conjugated to G_0. Hence, replacing G_0 by its conjugate, we can suppose that R is contained in G_0. Therefore G/R is isomorphic as an affine algebraic variety to the $G_0/R \times \mathcal{R}_u$ and we are done now by Proposition 2.4.6 with $\mathcal{R}_u$ playing the role of X_1 and G_0/R of X_2.

4.3.1 Remark. (1) The algebraic density property implies, in particular, that the Lie algebra generated by completely integrable algebraic (and, therefore, holomorphic) vector fields is infinite-dimensional, i.e. this is true for homogeneous spaces from Theorem 4.1.1. For Stein manifolds of dimension at least two that are homogeneous spaces of holomorphic actions of a connected complex Lie groups the infinite dimensionality of such algebras was also established by Huckleberry and Isaev [13].

(2) Note that as in [16] we proved actually a stronger fact for a homogeneous space $X = G/R$ from Theorem 4.1.1. Namely, it follows from the construction that the Lie algebra generated by vector fields of form $f\sigma$, where σ is either locally nilpotent or semi-simple and $f \in \operatorname{Ker} \sigma$ for semi-simple σ and $\deg_\sigma f \leq 1$ in the locally nilpotent case, coincides with $\operatorname{AVF}(X)$.

Chapter 5

Density property of fibrations

The previous two chapters give, at least in the cases of Theorem 4.1.1, a positive answer to the following question: let $\pi : W \to X$ be a locally trivial fibration (in the étale topology) with affine fiber F; if F and W have the density property, does X have the density property? In this Chapter instead we investigate methods for determining whether W has the density property assuming that the fiber F and the base space X have density property. We are able to give only a particular answer, mainly for two reasons. First, we need to assume the existence of particularly nice complete vector fields on W, that (at a generic point of W) are tranversal to the fibers of π. Secondly, even after this assumption, we obtain some results only in the case of fibrations that are locally trivial in the Zariski topology, while for the étale case there are some obstacles.

5.1 Zarisky locally trivial fibrations

Let W, X, F be affine manifolds. A morphism $\pi : W \to X$ is called a locally trivial fibration (in the Zariski topology) with fiber F, if there exists an affine open covering $\{V_\alpha\}$ of X, and isomorphisms $\varphi_\alpha : \pi^{-1}(V_\alpha) := W_\alpha \to V_\alpha \times F$, commuting with the projections onto V_α. We can assume that each V_α is the complement of the divisor of a regular function $f_\alpha \in \mathbb{C}[X]$. The fiber at a point $w \in W$ will be denoted by F_w.

We will make use of the following criterion [16].

Theorem 5.1.1. *Let X be an affine manifold, and $\mathcal{M}$ a $\mathbb{C}[X]$-submodule of TX. If $\mathcal{M} \subset \mathrm{Lie}_{\mathrm{alg}}(X)$, and for all $x \in X$ the fiber $\mathcal{M}_x$ coincides with T_xX , then X has the algebraic density property.*

The proof follows immediately from ([12], Chapter II, excercise 5.8), and it is left to the reader. We obtain immediately the following corollary.

5.1.2 Corollary. *If X and Y have the algebraic density property, then $X \times Y$ has the algebraic density property.*

Proof. From the hypothesis it follows that $\mathbb{C}[Y]\,\mathrm{VF}_{\mathrm{alg}}(X)$ and $\mathbb{C}[X]\,\mathrm{VF}_{\mathrm{alg}}(Y)$ are submodules of $\mathrm{Lie}_{\mathrm{alg}}(X\times Y)$. Thus $\mathcal{M} = \mathbb{C}[Y]\,\mathrm{VF}_{\mathrm{alg}}(X)+\mathbb{C}[X]\,\mathrm{VF}_{\mathrm{alg}}(Y) \subset \mathrm{Lie}_{\mathrm{alg}}(X\times Y)$, and $\mathcal{M}_{(x,y)} = T_{(x,y)}X\times Y$ for all $(x, y) \in X\times Y$. The result follows from Theorem 5.1.1. □

In order to extend the corollary to a non trivial fibration $\pi : W \to X$ as described above, we assume the existence of enough locally nilpotent derivations of W which are generically transversal to the fibers.

Theorem 5.1.3. *Let W and F be affine algebraic manifolds, and $\pi : W \to X$ be a Zarisky locally trivial fibration over an affine manifold X. Let F have the algebraic density property. Suppose that (i) for all $t \in F$ there is a derivation δ non vanishing on t, where δ is either locally nilpotent or it defines a $\mathbb{C}^*$-action with a closed one-dimensional orbit through t and (ii) there exists a finite collection Ξ of derivations $\mathbb{C}[W]$ which are either locally nilpotent, of semisimple with generic closed one-dimensional orbit, and such that for all $w \in W$, $\mathrm{Span}\{v_w : v \in \Xi\} + T_w F_w = T_w W$.*

Then W has the algebraic density property.

Proof. We are going to construct a $\mathbb{C}[W]$-submodule $\mathcal{M}$ of vector fields, satisfying the hypothesis of Theorem 5.1.1. A vector field δ will be called vertical if $\delta(w) \in F_w$ for all $w \in W$.

5.1.4 Proposition. *Let $\mathcal{V}$ be a $\mathbb{C}[W]$-module generated by vertical vector fields. If F has the algebraic density property, then $\mathcal{V} \subset \mathrm{Lie}_{\mathrm{alg}}(W)$.*

Proof. Let $\delta \in \mathcal{V}$. Write $\delta_{|W_\alpha} = \varphi_{\alpha *}^{-1} \varphi_{\alpha *} \delta$.

Since φ_α is an isomorphism between fibers, $\varphi_{\alpha *}\delta := \delta_\alpha$ is vertical. Choose a collection of regular functions $g_k \in \mathbb{C}[F]$ such that $F_k = F \setminus (g_k)$ is an affine cover of F, and TF_k is trivial. For each k there are vector fields e_i^k, $i = 1, ..., n = dim(F)$, forming a basis at each point of F_k. Thus, for each k there are regular functions $c_i^k \in \mathbb{C}[V_\alpha \times F]$ such that $\delta_{\alpha|_{V_\alpha \times F_k}} = \sum c_i^k e_i^k$. Moreover, there is an integer N, such that $g_k^N e_i^k$ extends to a section of TF, for all i, k. Choose regular functions $h_k \in \mathbb{C}[F]$ such that $\sum g_k^N h_k = 1$. Then $\delta_\alpha = \sum g_k^N h_k c_i^k e_i^k$. Since F has the algebraic density property, and $\mathbb{C}[V_\alpha \times F] = \mathbb{C}[V_\alpha] \otimes \mathbb{C}[F]$, we see that $\delta_\alpha \in \mathrm{Lie}_{\mathrm{alg}}(V_\alpha \times F)$. More precisely, the above constructions shows that δ_α, and consequently $\delta_{|W_\alpha}$, can be written as a linear combination of commutators of complete vertical algebraic vector fields. The same application of the Hilbert Nullstelleansatz shows that $\delta \in \mathrm{Lie}_{\mathrm{alg}}(W)$. Indeed, since $\delta(f_\alpha) = 0$, for some integer N we can assme that $f_\alpha^N \delta \in \mathrm{Lie}_{\mathrm{alg}}(W)$; there are $h_\alpha \in \mathbb{C}[X]$ such that $\sum f_\alpha^N h_\alpha = 1$; since $\delta(h_\alpha) = 0$, then $\delta = \sum f_\alpha^N h_\alpha \delta \in \mathrm{Lie}_{\mathrm{alg}}(W)$.

□

In particular, ler $\mathcal{V}$ be the $\mathbb{C}[W]$-module of all vertical derivations. Since $\mathbb{C}[F]$-module of derivations of the smooth affine variety F generate $T_t F$ for all $t \in F$, using the local triviality of W we see that for each $w \in W$ there is a collection of vertical derivations generating $T_w F_w$. Since $\mathcal{V} \subset \mathrm{Lie}_{\mathrm{alg}}(W)$, we have proven that W has the "vertical" algebraic density property, that is the Lie subalgebra $\mathcal{V}$ of vertical vector fields coincides with the Lie algebra generated by complete vertical fields.

We now proceed to extend $\mathcal{V}$ to a module $\mathcal{M}$ whose fibers coincide with the whole tangent space of W at each point w of W. Given a derivation satisfying (i), extend it to some $W_\alpha \cong V_\alpha \times F$, and then regularized it to obtain a vertical derivation δ on W. If δ is locally nilpotent there is a Zarisky open set $U \subset W$ and a variety U' such that U is equivariantly isomorphic to $U' \times \mathbb{C}_+$ [8]. If δ is semisimple, and the corresponding $\mathbb{C}^*$-action has a one-dimensional closed orbit, then by Luna's slice Theorem [4] there exists a $\mathbb{C}^*$-invariant Zarisky open set U such that all its orbits are closed and one-dimensional. In both cases, the orbits locally coincide with the fibers of the quotient maps defined by the $\mathbb{C}_+$ or $\mathbb{C}^*$-actions.

5.1.5 Lemma. *For $v \in \Xi$ as in the hypothesis (ii) of the theorem, there exists a regular function $f \in \operatorname{Ker} v_i$, such that $\delta(f) \neq 0$.*

Proof. Let K (resp. H) be the group actions defined by v (resp. δ), where K and H are isomorphic either to $\mathbb{C}^*$ or $\mathbb{C}_+$. There exist an open set U equivariantly isomorphic to $U//K \times K$, and an open set V equivarianlty isomorphic to $V//H \times H$.

We have the quotient morphisms

$$\begin{array}{ccc} W & \xrightarrow{p} & W//H \\ \big\downarrow{\scriptstyle q} & & \\ W//K & & \end{array}$$

such that the fibers or p and q are generically one-dimensional.

We denote by (f) the zero set of a regular function on a variety. Let $w \in U \cap V$, so that the K and H orbits are one-dimensional. Then

$$p^{-1}(p(w)) = p^{-1}\left(\bigcap_{f:f(p(w))=0}(f)\right) = \bigcap_{f\in\operatorname{Ker} v:f(w)=0}(f)$$

and

$$q^{-1}(q(w)) = q^{-1}\left(\bigcap_{f:f(q(w))=0}(f)\right) = \bigcap_{f\in\operatorname{Ker}\delta:f(w)=0}$$

By hypothesis we can assume that δ and v are transversal ar w, thus $p^{-1}(p(w)) \cap p^{-1}(p(w)) = w$. The result follows by contradiction.

$\square$

5.1.6 Remark. The conclusion of Lemma 5.1.5 is valid also if W is locally trivial in the étale topology

The existence of such a function allows us to perturb δ to a non vertical complete vector field, as shown by the next lemma.

5.1.7 Lemma. *Let X be a complex manifold. Let δ and v be holomorphic vector fields, and f a holomorphic function in the kernel of v, $x_0 \in X$ be such that $f(x_0) = 0$. Then* *the phase flow at time 1 of the vector field $f\sigma$ defines a linear action on the tangent space at x_0 given by $\delta \to \delta_{x_0} + \delta_{x_0}(f)\sigma_{x_0}$.*

Proof. Choose a system of coordinates $(z_1, ..., z_n)$ centered at w_0 such that $\delta = \frac{d}{dz_1}$. Then the result follows from a simple calculation.

□

Let $w \in W$ and $\delta \in LND(W)$ such that $\delta_w \neq 0$. For each $v \in \Xi$, let $f_v \in \mathbb{C}[W]$ as in Lemma 5.1.5. Then the group of automorphism generated by $\{f_v v : v \in \Xi\}$ applied to $\mathcal{V}$ a module $\mathcal{M} \subset \mathrm{Lie}_{\mathrm{alg}}(W)$, such that $\mathcal{M}_w = T_w W$. Repeating the construcion for all $w \in W$, we can assume that $\mathcal{M}_w = T_w W$ for all $w \in W$, and the theorem is proven.

□

5.1.8 Remark. If K or H as in Lemma 5.1.5 define holomorphic $\mathbb{C}^*$-actions, we can prove the holomorphic version of the previous lemma, which will imply that W has the density property. Indeed, the quotient of a holomorphic action of a reductive group exists [24]. The obstacle arises when we are dealing with holomorphic $\mathbb{C}_+$-actions (where the quotient is only defined on a dense open set, if v has a second integral [27]).

5.2 Applications

We would like to include some applications of Theorem 5.1.3, even though they are already covered by Theorem 4.1.1.

5.2.1 Example. Since GL_n is SL_n-equivariantly isomorphis to $SL_n \times \mathbb{C}^*$, if R is a proper subgroup of SL_n such that SL_n/R has the algebraic density property,

$GL_n//R$ has the algebraic density property. In particular Example 5.2.3 implies that $GL_n//SL_k$ has the algebraic density property, for $k \leq n-1$ (the $k=0$ is studied in [16]).

5.2.2 Example. Let S be a semisimple group, and R be a reductive subgroup, acting on S via left multiplication. Let F be an affine manifold with the algebraic density property. Then $S//R \times F$ has the algebraic density property. Indeed, the collection Ξ of vector fields as in Theorem 5.1.3 is obtained as follows: let $\delta \in \mathrm{LND}(S)$, corresponding to a $\mathbb{C}_+$-subgroup acting from the left on S. Since the adjoint representation acts irreducibly the T_eS, after conjugation on δ we obtain a collection of left invariant locally nilpotent derivations spanning T_eS, which descend to $S//R$ giving the desired collection Ξ.

5.2.3 Example. In [15] the authors prove that an hypersurface in $\mathbb{C}^{n+2}$ given by the equation $uv = p(x_1, ..., x_n)$ has the algebraic density property, whenever p is a nonzero polynomial with smooth zero fiber. This result applies in particular to the homogeneous space $X = SL_n//SL_{n-1}$, where SL_{n-1} is embedded in the lower right $(n-1) \times (n-1)$ block.

Indeed for a matrix $[x_{ij}] \in SL_n$, let $A_{ij} = (-1)^{i+j} det M_{ij}$, where M_{ij} is the minor corresponding to X_{ij}. The quotient space is $SL_n//SL_{n-1} = \mathrm{Spec}(\mathbb{C}[SL_n]^{SL_{n-1}})$, and not diffucult to see that $\mathbb{C}[SL_n]^{SL_{n-1}} = \frac{\mathbb{C}[x_{11}, x_{12}, ... x_{1n}, A_{11}, A_{12}, ..., A_{1n}]}{\langle x_{11}A_{11} + x_{12}A_{12} + ... + x_{1n}A_{1n} - 1 \rangle}$.

Using this fact, and Theorem 5.1.3, we can prove that $SL_n//SL_{n-k}$ has the algebraic density property, for $k = 0, ..., n$. The case $k = n$ is studied in [16], so let's assume $k = 1, ..., n-1$.

We have that $\pi : SL_n \to SL_n//SL_{n-1}$ is a locally trivial fibration. Indeed, for $V_n = \{M_i \neq 0\}$ for $i = 1, ..., n$, $\pi^{-1}(V_n)$ is (equivariantly) isomorphic to $V_n \times SL_{n-1}$, The structure of locally trivial SL_n-fibration on $SL_n \to SL_n//SL_{n-1}$ induces a structure of locally trivial $(SL_{n-1}//SL_{n-k})$-fibration on $SL_n//SL_{n-k} \to SL_n//SL_{n-1}$. The result can be thus proven by induction on n (the existence of the collection Ξ has estabilished in the previous example).

The same construction can be applied to $SO(n)$. Different authors [15, 26] proved that $SO(n)//SO(n-1)$ has the algebraic density property. The fibration description shows that $SO(n)//SO(n-k)$ has the algebraic density property.

5.3 Work in progress

The weakness of Theorem 5.1.3 stays in the fact that in order to apply Lemma 5.1.7 we need to assume the existence of "horizontal" derivations. We would like to prove the generalization of Theorem 5.1.2 to non trivial fibrations.

5.3.1 Question. *Let $\pi : W \to X$ be a fibration, locally trivial in the étale topology, with fiber F. Suppose that F, X, W are affine homogeneous manifolds. If X and F has the density property, then W has the density property.*

In order to prove our conjecture, we would like to lift locally nilpotent derivations on X to nice complete fields on W. Untill today, we can prove that a lift exists.

5.3.2 Proposition. *Let v be an algebraic vector field on X. Then there is an algebraic vector field $\hat{v}$ on W, call the lift of v, such that $\pi_* \hat{v} = v$. If v is complete, then $\hat{v}$ is complete.*

Proof. Denote by v_α be lift of v to $V_\alpha \times F$, which is zero along the fiber. Multiply $\varphi_*^{-1} v_\alpha$ by f_α^N, in order to remove a possible polar behaviour of the divisor $X \backslash \pi^{-1}(V_\alpha)$.

Then, as usual, let g_α such that $\sum_\alpha f_\alpha^N g_\alpha = 1$. Then we let the reader to check that $\hat{v} = \sum_\alpha f_\alpha^N g_\alpha \varphi_*^{-1} v_\alpha$ is a lift of v.

Let's now prove that $\hat{v}$ is complete. Let $\gamma : \mathbb{C} \to X$ an integral curve of the vector field v starting at $x \in X$. We can cover $\mathbb{C}$ by open sets U_k over which the integral curve $\hat{\gamma_k} : U_k \to W$ of $\hat{v}$ (lifting γ) exists. By choosing a point $z_k \in U_k$ for each k, and $w_k \in \pi^{-1}(x_k)$ for each k, the curve $\hat{\gamma_k}$ is uniquely determined. Since the action of G is transitive and free on each fiber, there are (uniquely determined) holomorphic functions $g_{kj} : U_k \cap U_j \to G$, such that $\hat{\gamma_k} = g_{kj}\hat{\gamma_j}$ on the intersections $U_k \cap U_j$.Since $\hat{v}$ is G-equivariant, g_{ij} must be constant functions. Therefore we obtain a 1-cocycle $\{g_{ij}\}$ of the constant sheaf G. Since $H^1(\mathbb{C}, G) = \{1\}$, there are constant functions $z_k : U_k \to G$, such that $g_{ij} = z_j - z_i$ on U_{ij}. Redefining $\hat{\gamma_k}$ to $\hat{\gamma_k} + z_k$, we obtain a globally defined curve $\hat{\gamma} : \mathbb{C}_+ \to W$. By multiply $\hat{\gamma}$ by a suitable element of g, we obtain the integral curve of $\hat{v}$ starting at any point we want.

□

The proposition tells that, at least, we can obtain a collection of complete vector fields, by lifting a collection of locally nilpotent derivations on X. Therefore, let $\hat{v}$ be the lift to W of a locally nilpotent derivation on X: our next step is to try to give an affirmative answer to one of the three questions.

5.3.3 Questions.

1. *Given $\hat{v}$, can we define a quotient of X by a the $\mathbb{C}_+$-action defined by $\hat{v}$, in order to be able to apply Lemma* 5.1.5*?*

2. *Can we modify the construction of the lift of v in such a way that $\hat{v}$ will turn out to be a locally nilpotent derivations?*

3. *Suppose that $\hat{v}$ is indeed locally nilpotent; can pair $\hat{v}$ with a vertical locally nilpotent derivation δ, such that $\hat{v}, \delta$ are compatible?*

In particular, a positive answer to (3) would be a good achievement, since we will be able to prove the density property of fibrations with one-dimensional fibers. All of this is a work in progress.

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