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Quasipositive Surfaces and Convex Surface Theory

Moses Koppendrayer
University of Miami, kankan-moses@hotmail.com

UNIVERSITY OF MIAMI

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Moses Koppendrayer

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Moses Koppendrayer

Approved:

Kenneth Baker, Ph.D.
Professor of Mathematics

Nikolai Saveliev, Ph.D.
Professor of
Mathematics

Drew Armstrong, Ph.D.
Associate Professor of Mathematics

Guillermo J. Prado, Ph.D.
Dean of the Graduate School

John Etnyre, Ph.D.
Professor of Mathematics
Georgia Tech

KOPPENDRAYER, MOSES (Ph.D., Mathematics)
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Quasipositive surfaces originally arose in the study of complex plane curves in the ’80s. They were originally defined by Lee Rudolph in a topological manner, as the Bennequin surface of a strongly quasipositive link. Lee Rudolph and others have studied these surfaces using tools from algebraic topology. In more recent years many connections have been made between quasipositive surfaces and contact geometry including a definition of quasipositive surfaces as the ribbon surface of a Legendrian graph. Beyond this classification many things are known about these surfaces, for instance they satisfy the Bennequin and Slice-Bennequin inequalities, yet many things remain a mystery.

Outside of fibered knots it is not known whether the strong quasipositivity of a link guarantees the quasipositivity of a minimal genus Seifert surface. In the case of fibered knots where there is a unique minimal genus Seifert surface, the strong quasipositivity of a link guarantees that this minimal gensu Seifert surface is quasipositive. Outside of this case strongly quasipositive links can have distinct non-isotopic minimal genus Seifert surfaces. Baker and Motegi raised the question of whether strongly quasipositive links can have minimal genus Seifert surfaces which are not quasipositive.

This question can be refined with the contact characterization of quasipositive surfaces. If a link bounds a surface which is the ribbon of a Legendrian graph must every minimal genus Seifert surface be isotopic to the ribbon of a Legendrian graph. As stated this question doesn’t take advantage of the contact structure. The boundary of a Legendrian ribbon can be given a natural structure as a transverse link, to use

the contact structure we can ask the question: if a link bounds a surface which is the ribbon of a Legendrian graph must every minimal genus Seifert surface be isotopic to the ribbon of a Legendrian graph via an isotopy preserving the transverse link type of its boundary?

The answer to this question is no. In this dissertation we construct a transverse link in the universally tight contact structure on $L(4,1)$ which bounds a Legendrian ribbon and another surface which cannot be isotoped into a Legendrian ribbon preserving the transverse link type of its boundary. In order to prove the non-existence of this isotopy we use convex surface theory. In particular we analyze the bypasses near the surface to find that there can be no such transverse isotopy.

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I would like to thank all of the great educators from my life before grad school, in particular Ms. Matthew who taught me to love proofs the first time I encountered them and Dr. Froncek who showed me so much of the beauty of math. Lastly I would like to thank my friends for listening to my complaints and for believing in me.

Contents

Chapter 1

Introduction

Strongly quasipositive links arose in the study of complex plane curves by Lee Rudolph. They are a subclass of the class of oriented links obtained by intersecting complex curves with S^3 [Rud83]. He defined strongly quasipositive links as closures of a certain class of braid. One tool used to study these links is the surfaces that they bound. A quasipositive surface is defined to be the standard Seifert surface of a strongly quasipositive braid presentation of a strongly quasipositive link. This property is topological, essentially it is equivalent to a surface being constructed from a stack of disks with negatively twisted vertical bands connecting them. Rudolph only considered strongly quasipositive links in S^3, but the notion can be generalized to other 3-manifolds using open book decompositions. While quasipositive surfaces were originally defined in relation to strongly quasipositive knots, now people often view strongly quasipositive knots as the boundaries of quasipositive surfaces.

Contact structures are also related to complex geometry. For instance the standard contact structure on S^3 arises as the boundary of the unit 4-ball in $\mathbb{C}^2$ with its symplectic structure. Thus it is perhaps not surprising that there is a connection between quasipositive surfaces and contact geometry. This connection has been well established already. Firstly, there are transverse representatives of strongly quasipositive knots which realize the Bennequin bound. The Bennequin bound states that the self linking of a transverse knot in the standard contact structure in S^3 is bounded above by the negative Euler characteristic of the knot. Quasipositive surfaces realize the Euler characteristic of their boundary, hence when their boundary has maximal self-linking number they realize the Bennequin bound. Hedden proved another connection, that is, in the case of fibered knots the quasipositivity of the fiber surface can be detected by a contact invariant (the Ozsváth Szabó contact invariant) of the contact structure induced by the open book decomposition of the fibration. [Hed10].

The connection of quasipositive surfaces and contact geometry has been further demonstrated; in a tight contact 3-manifold quasipositive surfaces are equivalent to ribbons of Legendrian graphs. This was prove in S^3 by Baader and Ishikawa [BI09] and in the more general case by Hayden [Hay17a], Hedden, and Baykur, Etnyre, Hedden, Kawamuro and Van Horn-Morris (see remark in [Hay17a]).

Quasipositive surfaces are minimal genus Seifert surfaces for their boundary link, but it is unknown whether every minimal genus Seifert surface of a strongly quasipositive knot must be a quasipositive surface. Baker and Motegi raise this question in [BM17]. Reframing the question with the Legendrian ribbon characterization of quasipositive surfaces we have the following:

Question 1.0.1. *Can every minimal genus Seifert surface for a strongly quasipositive knot or link be represented by the ribbon of a Legendrian graph?*

In order to study ribbons of Legendrian graphs we find that a ribbon of a Legendrian graph is equivalent to a subsurface of a convex surface. This approach allows us to use the powerful tools from convex surface theory to study quasipositive surfaces.

Theorem 1.0.2. *Let R be a surface in a contact manifold (M, ξ). The surface is the ribbon of a Legendrian graph iff it is the closure of a component of Σ_+ for some convex surface Σ.*

The subsurface Σ_+ is the 'positive' side of the dividing curves of Σ. Both as a Legendrian ribbon and as a component of Σ_+ the boundary of a surface is naturally a transverse link in the contact structure. This raises a further question:

Question 1.0.3. *Given a transverse representative of a strongly quasipositive link which bounds a Legendrian ribbon, must every minimal genus Seifert surface for that link be isotopic to a Legendrian ribbon with the same the transverse link type on the boundary?*

There is a slight subtlety to whether we want the isotopy of the surface to be a transverse isotopy of the boundary or whether we just want there to be a transverse isotopy of the link independent of the surface. This dissertation answers Question 1.0.3 in the first sense.

Theorem 1.0.4. *There exists a transverse link in a universally tight contact structure on a lens space which bounds a Legendrian ribbon and another surface which cannot be isotoped to be the ribbon of a Legendrian graph through an isotopy which restricts to a transverse isotopy of its boundary.*

In the process of proving this result we find it useful to prove the following result.

Theorem 1.0.5. *The universally tight contact structures on the lens space $L(4,1)$ admit a contact vector field which is everywhere tangent to a Seifert fibration of the manifold.*

This result raises a question of which other Seifert fibered spaces have a similar property. We expect that the tight contact structure on S^3 as well as universally tight contact structures on all lens spaces share this property.

Chapter 2 recalls many theorems and definitions which are crucial background in contact geometry, convex surface theory and quasipositive surface theory. For background on knot theory there are many standard references for instance [Rol90].

The results in foliation theory used in this work can largely be found in [Etn] and [Gei08]. Chapter 3 contains the proof of Theorem 1.0.2 as well as the proofs of some properties of convex Heegaard decompositions and a discussion of the operation of folding. Chapters 4 through 7 contain the proof of Theorem 1.0.4 with the proof of Theorem 1.0.5 appearing in Chapter 6. Chapter 8 contains a few ideas for extending the results of this dissertation to surfaces with a knot as boundary instead of a link or to a larger class of manifolds.

Chapter 2

Background

Throughout this work we will consider 3-dimensional, closed, orientable manifolds unless otherwise specified.

2.1 Contact Structures

In this section I will define several important notions from contact topology and state relevant theorems for my thesis. There are numerous resources which cover this material in more detail and contain more proofs. Among these resources are class notes by John Etnyre [Etn01] and a textbook by Hansjörg Geiges [Gei08].

Definition 2.1.1. A *contact structure* ξ on an orientable odd dimensional manifold M is a maximally non-integrable plane distribution. We denote the pair by (M, ξ).

Contact structures can be defined on any orientable odd dimensional manifold, but for this thesis I'm only interested in 3 dimensional manifolds so I'll only discuss them on 3 manifolds. A contact structure on M^3 can locally be defined as the kernel of a one form α with $\alpha \wedge d\alpha \neq 0$. If a contact structure can be defined globally by a single one form then it is called coorientable. For example the standard contact structure on R^3 is given by $\xi = ker(dz - ydx)$ and is hence coorientable. For the remainder of this work I will only consider coorientable contact structures.

The maximal non-integrability condition on a contact structure means that any smooth subsurface C of (M, ξ) cannot have an open neighborhood where $T_xC = \xi_x$. This contrasts with smooth one dimensional complexes which can always be isotoped to be tangent to the contact structure.

Definition 2.1.2. A graph G embedded in a contact manifold (M, ξ) is called *Legendrian* if it is tangent to the contact structure at every point. In particular if G contains no vertices then it is a *Legendrian* link. These graphs are considered equivalent if they are Legendrian isotopic, that is isotopic through graphs which are all Legendrian.

Definition 2.1.3. The *Thurston-Bennequin number* of a Legendrian knot K, denoted $tb(K)$ measures the twisting of the contact structure relative to a surface that K bounds.

Associated to a Legendrian graph is a ribbon surface.

Definition 2.1.4. A *ribbon of a Legendrian graph*, or *Legendrian ribbon G* is a surface R s.t.

1) $G \subset R$ and at every point x in G, we have $T_xR = T_x\xi$.

2) G is a topological spine of R, that is R deformation retracts onto G.

3) There are no other points in R where its tangent space corresponds with the contact plane.

4) The boundary is a transverse link.

5) The annuli between the Legendrian graph and the boundary components are foliated by arcs going from the graph to the boundary.

The standard definition of a ribbon of a Legendrian graph consists of only the first 3, but adding the last two don't topologically restrict the set of surfaces which are Legendrian ribbons. We can always find a ribbon with the last two properties by taking perhaps a smaller neighborhood of the Legendrian graph.

Instead of considering links that are tangent to the contact structure, we could consider links that are transverse to the contact structure.

Definition 2.1.5. A *transverse link* is a link L embedded in a contact manifold (M, ξ) such that at every point in the link, it is transverse to the contact structure. These links are equivalent if there is a transverse isotopy between them, that is an isotopy through transverse links.

Note that the coorientation of a cooriented contact structure gives transverse knots and hence the components of transverse links a natural orientation.

Definition 2.1.6. The *characteristic foliation* of a surface Σ in a contact manifold (M, ξ) is the singular foliation induced by $\xi \cap S$. We denote this characteristic foliation by Σ_ξ.

The singularities of this foliation are the points x where $T_x S = \xi_x$. For instance the characteristic foliation of a ribbon of a legendrian graph has every point in the graph as a singularity and no other singularities. A singularity is positive if the orientation of $T_x S$ corresponds with the orientation of ξ_x.

Locally all contact structures look the same, in fact Darboux's theorem states that any point in any contact structure on M^{2n+1} has a neighborhood contactomorphic to a neighborhood of the origin in the standard contact structure on R^{2n+1}. Thus there are no local invariants for contact structures. There are however global invariants.

Definition 2.1.7. An *overtwisted disk* is a disk that is bounded by a Legendrian unknot U with $tb(U) = 0$. That is, a disk whose boundary is everywhere tangent to the contact structure and which can be isotoped so that every point on the boundary is a singularity of the characteristic foliation.

Contact structures are split into two kinds by the existence or non-existence of such disks. A contact structure containing and overtwisted disk is called an overtwisted (OT) contact structure. A contact structure not containing an overtwisted contact structure is called tight. Overtwisted contact structures on 3-manifolds were classified by Eliashberg. This classification shows an infinite family of overtwisted contact structures on S^3, one for each integer. There is however only a single tight contact structure given by the compactification of the standard contact structure on R^3. Since Eliashberg's classification of OT contact structures there has been more interest in tight contact structures.

Theorem 2.1.8 ([Eli89]). *Overtwisted contact structures on a three manifold M are in bijection with the homotopy classes of 2-plane distributions on M.*

2.2 Convex Surface Theory

Since contact structures are maximally non-integrable, surfaces cannot be everywhere tangent to the contact structure. However this does not prevent them being of use in studying contact structures. One tool in this vein is convex surface theory. This theory was introduced by Giroux [Gir91] and has been expanded on by many people. Notable among these is Honda who introduced the notion of the bypass see [Hon00]. For a more complete reference to this theory see for example [Hon00], [Etn] or [Gei08].

Definition 2.2.1. A *contact vector field* in a contact manifold (M, ξ) is a vector field s.t. the flow of the contact vector field preserves the contact structure.

Definition 2.2.2. A *convex surface* in a contact manifold (M, ξ) is a surface Σ such that there is a contact vector field transverse to it.

Convex surfaces are an important and powerful tool for understanding contact structures. Convex surfaces determine the contact structure in a small neighborhood, all of the structure of a convex surface can be encoded with a very minimal amount of information, and they are a useful variety of surface at which to decompose contact manifolds.

The contact vector field transverse to a convex surface allows us to determine the contact structure within a neighborhood of the convex surface.

Lemma 2.2.3. *A surface Σ is convex iff there is an I neighborhood of Σ on which the contact structure is invariant in the I direction.*

Definition 2.2.4. The *dividing set* of a convex surface are a collection of closed curves or properly embedded arcs. They consists of the points on the surface where the contact vector field lies within the contact structure.

Definition 2.2.5. The set of points in a contact manifold for which a contact vector field lie within the contact structure forms a surface. This surface is called the *characteristic surface* of that contact vector field.

Note that the dividing set of a convex surface is the intersection of that surface with the characteristic surface of the contact vector field transverse to it.

Definition 2.2.6. A multicurve Γ is said to *divide* a singular foliation on a surface Σ if

1) $\Sigma\backslash\Gamma$ consists of two components Σ_+ and Σ_-

2) Γ is transverse to the foliation

3) there is a volume form on Σ and a vector field u that directs Σ_ξ such that the divergence of the foliation is positive in Σ_+ and negative in Σ_- and such that the foliation flows from Σ_+ to Σ_- along Γ.

Theorem 2.2.7. *[Gir91] Let Σ be an orientable surface in (M,ξ) with ∂M Legendrian if it is non-empty. Then Σ is convex iff Σ_ξ has dividing curves. If Σ is convex the dividing set will be dividing curves for its characteristic foliation.*

Let Σ be a convex surface. The dividing curves of Σ divide Σ into two subsurfaces Σ_+ and Σ_-. As a result of the divergence condition, Σ_+ contains all of the positive elliptic singularities of Σ_ξ, which are sources of the foliation and Σ_- which contains all of the negative elliptic singularities (the sinks) of Σ_ξ. In a sense the dividing curves of a surface are all that is necessary to determine the contact structure nearby, as reflected in the following theorem of Giroux.

Theorem 2.2.8 (Giroux Flexibility, [Gir91])**.** *Let Σ be a compact surface embedded in (M, ξ), with Legendrian boundary if $\partial\Sigma \neq \emptyset$, and let Γ_Σ be the dividing set of Σ and F be the characteristic foliation. Then for any foliation F' divided by Γ_Σ there is an isotopy, fixed on Γ_Σ, of Σ through convex surfaces (divided by Γ_Σ) such that at the beginning of the isotopy $\Sigma_\xi = F$ and at the end of the isotopy $\Sigma_\xi = F'$. Furthermore this isotopy can be taken to be fixed outside any neighborhood of Σ.*

This theorem together with Lemma 2.2.3 essentially tell us the contact structure in a neighborhood of Σ. A particularly useful application of this theorem is the Legendrian Realization Principle (LRP). A special case of which was proved by Kanda and later the full version by Honda.

Definition 2.2.9. A graph is called *non-isolating* if it intersects the dividing curves transversely and if every component of the complement of the graph intersects the dividing curves.

Theorem 2.2.10 (Legendrian Realization Principle, [Hon00], [Kan98])**.** *Let Σ be a convex surface embedded in (M, ξ) and G a properly embedded non-isolating graph. Then there is an isotopy of Σ, rel boundary, to a surface Σ' such that G is contained in Σ'_ξ.*

One consequence of this theorem is a way to find OT disks. If there is a disconnected dividing set on a convex surface Σ and there is a dividing curve which bounds a disk in Σ there is a push-off of the dividing curve which is non-isolating. Thus we can Legendrian realize this curve and it is the boundary of an OT disk in Σ. This fact is part of Giroux' Criterion.

Theorem 2.2.11 (Giroux's Criterion, [Gir91])**.** *The vertically invariant contact structure in the neighborhood of a contact structure is tight iff the surface is S^2 and the dividing set is connected or the surface is not S^2 and the dividing set does not contain any curves which bound disks in the surface.*

One particular case of non-isolating graph that will be relevant is a Legendrian divide.

Definition 2.2.12. A *Legendrian divide* is a Legendrian curve of singularities in a convex surface in between two parallel dividing curves.

The dividing curves tell us many things about a convex surface, one is the twisting of Legendrian curves relative to the surface.

Definition 2.2.13. For a Legendrian curve L in a surface Σ the *twisting number* of the curve measures the twisting of the contact structure relative to Σ along the curve. It is denoted by $tw(L, \Sigma)$. In order to ensure $tw(L, \Sigma)$ is an integer it is enough for L to be closed or for both endpoints of L to occur at singularities of Σ_ξ.

In the special case when L is the boundary of a convex surface the twisting number is the same as the Thurston-Bennequin number. On a convex surface $tw(L) = -1/2|(L \cap \Gamma_\Sigma)|$. In particular if L is a Legendrian knot and Σ is a surface that it bounds, then $tb(L) = -1/2|(L \cap \Gamma_\Sigma)|$. This fact has the corollary that if Σ is a convex surface with Legendrian boundary then $tw(\gamma, \Sigma) \leq 0$ for each component γ of $\partial\Sigma$. The converse of this corollary is also true.

Theorem 2.2.14 ([Gir91], [Kan98])**.** *Any closed surface Σ is C^∞ close to a convex surface. Any surface Σ with Legendrian boundary is C^∞ close in the interior and C^0 close on the boundary to a convex surface if $tw(\gamma, \Sigma) \leq 0$ around each boundary component γ.*

The collection of closed surfaces and convex surfaces with Legendrian boundary is the classical collection of convex surfaces to work with. However Etnyre and Van Horn-Morris prove the following theorem:

Theorem 2.2.15 ([EVHM11]). *Suppose that Σ is a surface with boundary positively transverse to the contact structure. If the characteristic foliation of Σ is Morse-Smale, then there is a contact vector field V which is transverse to Σ and Γ_Σ is disjoint from the boundary of Σ. By definition, this implies that Σ is convex.*

It is worth noting that not all classical results in convex surface theory are true for convex surfaces with transverse boundary. The only result necessary for the following arguments is the existence of a vertically invariant neighborhood in the contact structure which does hold.

Lemma 2.2.16 ([Hon00],[Kan97]). *Let Σ and Σ' be convex surfaces and let $\partial\Sigma'$ be contained in Σ and intersecting Γ_Σ. Then the dividing sets of Σ and Σ' are interleaved along $\partial\Sigma'$. That is as you follow the Legendrian curve around you alternately meet Γ_Σ and Γ'_Σ.*

Theorem 2.2.17 ([Hon00]). *Let Σ and Σ' be convex surfaces which intersect transversely along a Legendrian boundary component L with $tw(L,\Sigma) < 0$. Then as above the dividing curves are interleaved. We can take a neighborhood of L and make its boundary, T, convex. Let S be the surface obtained by taking the union of Σ and Σ', then by replacing the intersection of S with the neighborhood of L with the closure of one of the components of $T\backslash S$ we can smooth the corners of the resulting surface to be convex. The effect on the dividing curves will be to veer right, if you are looking from the component of T which was not used.*

2.2.1 Bypasses

The isotopies of convex surfaces I've discussed so far do not change the dividing set of the surface. However there are topologically isotopic surfaces with different dividing sets. To study the changes in dividing sets Honda developed the notion of a bypass ([Hon00]).

Definition 2.2.18. The *attaching arc* for a bypass is a Legendrian arc in a convex surface Σ which meets Γ_Σ at both endpoints of the arc and once in the middle.

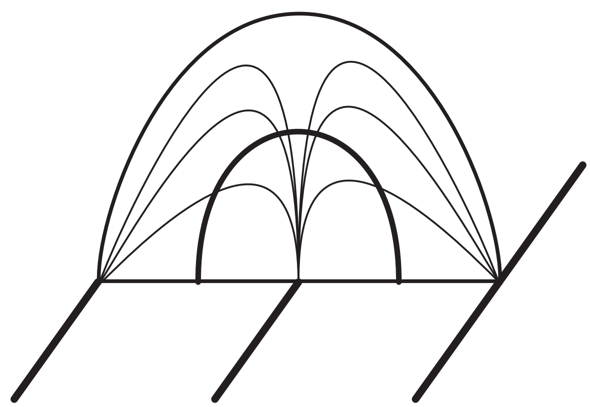

Figure 2.1: After applying Giroux Flexibility we can take the foliation of a bypass to look like this. The bold curves are dividing curves on the respective surfaces. The lighter curves correspond to Legendrian leaves in the foliation of the bypass disk.

Definition 2.2.19. A *bypass* on a convex surface Σ along attaching arc δ is a convex disk whose boundary consists of δ and another arc intersecting Σ only at its endpoints. This disk's boundary has twisting number -1. The endpoints of δ are corners of the disk and are elliptic singularities in the characteristic foliation of the disk. See Figure 2.1

Performing a bypass or taking a bypass is the action of isotoping a neighborhood of the attaching arc across a small ball containing the bypass disk. If the surface is closed we can think of performing a bypass as isotoping the entire surface slightly within the vertically invariant neighborhood guaranteed by its convexity and then further isotoping a small neighborhood of the attaching arc across the bypass disk.

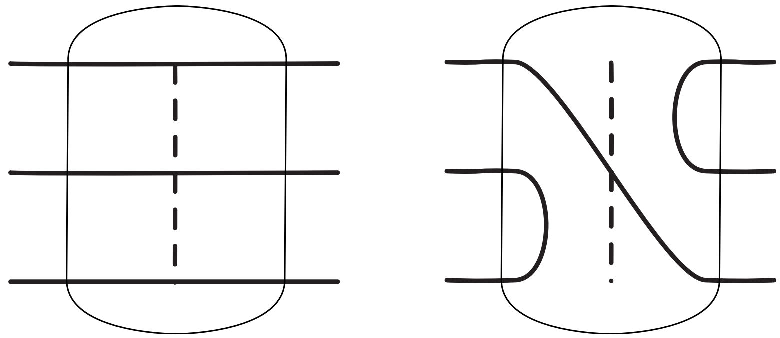

Figure 2.2: In the picture on the left, the horizontal arcs are the dividing curves of Σ and the vertical dashed line is the attaching arc of the bypass. The picture on the right is the surface after performing the bypass. The isotopy fixes the convex structure outside of a neighborhood of the attaching arc. In this diagram the bypass is attached to the front of Σ. In order to see the effect of a bypass on the back, flip the diagram horizontally.

Theorem 2.2.20 ([Hon00])**.** *Let Σ be a convex surface and D a bypass attached along δ. Then inside any neighborhood of $\Sigma \cup D$ there is an I-neighborhood of $\Sigma \cup D$ such that one side is Σ and the other side is Σ' where Γ_Σ is related to Γ'_Σ by the diagram in Figure 2.2.*

A natural question to ask is how abundant bypasses are. Given an attaching arc on a convex surface we know what the effect of a bypass along that arc would be, but the bypass disk may or may not exist within the contact structure. The following are a collection of some of the means of finding bypasses within a contact manifold.

Definition 2.2.21. A *trivial bypass* is one for which the effect of the bypass on the dividing set is trivial.

Theorem 2.2.22 (Right to Life Principle, [Hon02])**.** *Let δ be a potential attaching arc for a bypass on a convex surface Σ. If attaching a bypass along δ would be trivial and one of the arcs of $\delta \backslash \Gamma_\Sigma$ bounds a disk along with an arc of $\Gamma_\Sigma \backslash \delta$ then the bypass exists.*

Lemma 2.2.23 ([Hon00])**.** *Suppose that Σ and S are convex surfaces, $\partial S \subset \Sigma$ is Legendrian and Γ_S has at least two components. If there is a dividing curve on S*

which is boundary parallel, then it its endpoints are next to three dividing curves in Σ*. There exists a bypass disk in* S *along the arc of* ∂S *between the three dividing curves of* Σ *adjacent to the boundary parallel dividing curve in* S*.*

Lemma 2.2.24. *When you take a bypass you move a neighborhood of the bypass disk across a surface. There is an inverse bypass with attaching arc on the new convex surface that undoes moves the ball back across the surface and undoes the change on the dividing curves.*

Let me sketch the proof of this lemma. If we look at the neighborhood of the bypass disk from Figure 2.2 we have a tight 3-ball. The potential attaching arc connecting the three dividing curves along the top disk is a trivial bypass in that tight 3-ball and hence exists by 2.2.22.

Lemma 2.2.25 (Bypass Sliding Lemma, [HKM03])**.** *Let* R *be a rectangle in a convex surface with sides* a, b, c*, and* d *where* a *is the attaching arc of a bypass. The sides* b *and* d *are in the dividing set and* c *is a Legendrian arc that intersects the dividing set of the convex surface efficiently. Then there exists a bypass along the arc* c *on the same side of the convex surface as the bypass along* a*.*

In a sense bypasses are all that we need in order to discuss isotopies of surfaces within a contact manifold (M, ξ). This follows from the Isotopy Discretization Principle.

Theorem 2.2.26 ([Col97])**.** *Let* (M, ξ) *be a contact manifold and let* Σ *and* Σ' *be convex surfaces either closed or with identical Legendrian boundary which are isotopic. Then there is a sequence of bypasses which take* Σ *to* Σ'*.*

2.2.2 Convex Decompositions

One way to analyze contact structures is to decompose them along convex surfaces into smaller manifolds with boundary. Honda proves a gluing theorem for contact manifolds using this idea ([Hon00]). The machinery he uses is that of state transitions.

Definition 2.2.27. Given a convex decomposition of a manifold M along $N \subset M$ into a manifold M', a *configuration* is a pair (Γ_N, ξ) where ξ is a contact structure on $(M', \Gamma_{\partial M'})$ with $\Gamma_{\partial M'}$ coming from $\Gamma_{\partial M}$ and Γ_N via edge rounding. We say that a configuration (Γ_N, ξ) is potentially allowable if ξ is tight on M'.

Definition 2.2.28. A *state transition* is a move from one configuration to another. A state transition from $C = (\Gamma_N, \xi)$ to another configuration $C' = (\Gamma'_N, \xi')$ is allowable if: C is potentially allowable, C' is obtained from C by a single nontrivial bypass along N, and if C' is obtained from C by a bypass implies that ξ' comes from ξ by moving a single $N \times I$ from one side of N to the other. A configuration C is allowable if every C' which can be obtained from C via state transitions is potentially allowable.

Definition 2.2.29. Define $\mathcal{C}$ to be a graph where the vertices are configurations and an edge exists between two configurations if there is a state transition from either one to the other. Thus C is allowable if every C' in its connected component of $\mathcal{C}$ is potentially allowable. Define $\mathcal{C}_0$ to be the graph obtained by taking only allowable configurations.

Theorem 2.2.30 ([Hon02])**.** *Let M be a compact, oriented, irreducible 3-manifold, and let M' be the manifold obtained by decomposing M along an incompressible surface $N \subset M$. We fix $\Gamma_\partial M$ if M is a manifold with boundary and if N has boundary let its boundary be Legendrian with negative twisting number for each component of its boundary. Then the set of tight contact structures on M (with dividing set Γ_M if $\partial M \neq \emptyset$) is in one to one correspondence with $\pi_0(C_0)$.*

Theorem 2.2.31 ([HKM02])**.** *Consider a tight contact $T^2 \times I$ with convex boundary with dividing curves of the same slope on both boundary tori and a properly embedded convex annulus A with one boundary component on each boundary torus. If there is a nonseparating dividing curve on the annulus then the contact structure on $T^2 \times I$ is determined by Γ_A in particular it is contactomorphic to a contact structure on $S^1 \times A$ where each A fiber is convex with dividing set Γ_A.*

2.3 Quasipositive Surfaces

Quasipositive surfaces are a special class of surface embedded in S^3 which originally arose in the study of complex curves. Lee Rudolph introduced them and studied them extensively, see [Rud05] for a more detailed exposition of these surfaces.

Definition 2.3.1. A *strongly quasipositive braid* in S^3 is a braid that consists of only positive braid letters σ_i and conjugates of positive braid letters of the form $(\sigma_i\sigma_{i+1}...\sigma_{j-1})\sigma_j(\sigma_i\sigma_{i+1}...\sigma_{j-1})^{-1}$ using the standard generators of the braid group.

The closures of these braids bound surfaces constructed by taking a disk for each level and attaching a negatively twisted band between the corresponding disks for each positive braid letter or conjugate of each positive braid letter. Essentially these surfaces can be created by taking a stack of disks and attaching negatively twisting bands between adjacent or non-adjacent disks.

Definition 2.3.2. Surfaces in S^3 obtained in this manner from strongly quasipositive (SQP) links are called *quasipositive* (QP).

2.4 Connection of Quasipositive surfaces and Contact Topology

Baader and Ishikawa demonstrated a connection between quasipositive surfaces and contact structures by proving the following theorem:

Theorem 2.4.1 ([BI09])**.** *Each quasipositive surface is topologically isotopic to the ribbon of a Legendrian graph and each ribbon of a Legendrian graph is in the isotopy class of a quasipositive surface.*

The notion of quasipositivity can be extended to other manifolds by considering braids inside of open book decompositions of those manifolds ([Hay17b],[IK17]). Both this result and the following theorem have also been obtained by Hedden and Baykur, Etnyre, Hedden, Kawamuro and Van Horn-Morris (see remark 1.1 in [Hay17a]). This way of extending the notion of quasipositivity to other manifolds maintains the connection to contact surface theory.

Theorem 2.4.2 ([Hay17a])**.** *A link in an open book is strongly quasipositive iff it bounds the ribbon of a Legendrian graph in the corresponding contact structure.*

Chapter 3

Preliminaries

Theorem 3.0.1. *Let R be surface in a contact manifold (M, ξ). This surface is the ribbon of a Legendrian graph iff it is the closure of a component of Σ_+ for some convex surface Σ. Furthermore if R is a component of Σ_- we can realize R as a ribbon of a Legendrian graph by reversing the orientation of Σ.*

Proof. Suppose that R is a component of Σ_+ for some convex surface Σ. By using Giroux Flexibility we can choose a foliation on S such that a topological spine of R is a Legendrian graph realized with every point a positive singularity and such that there are no other singularities in R. We can do this by choosing a foliation of Σ such that the topological spine of every component of Σ_+ and of Σ_- is a graph of singularities of the appropriate sign. The remainder of the surface is a collection of annuli with sources along one boundary component and sinks along the other. We foliate these annuli with arcs going from the sources to the sinks. If R is a component of Σ_- then we can first reverse the orientation of Σ and proceed as above.

Conversely suppose that R is the ribbon of a Legendrian graph G. We first isotope R to alter the characteristic foliation of R near G so that the vertices are elliptic singularities and the edges alternate between hyperbolic and elliptic singularities along their length. This can be accomplished by twisting R slightly along segments of the edges of G to break the tangency of R to the contact structure. We can take the isotopy small enough to not introduce new tangencies because transversality is an open condition. All elliptic singularities are sources so they can't be adjacent in the foliation. Similarly if two hyperbolic singularities were adjacent we can look at the singularity at the origin of the leaf between them. The stable separatrices of this singularity come from the boundary of the ribbon. This contradicts the fact that the foliation flows out of the ribbon along the boundary. Elliptic and hyperbolic singularities must alternate along the edges of G. This foliation is of Morse-Smale type so we can make R convex with dividing curves disjoint from ∂R by Theorem 2.2.15.

Take another copy R' of R with the opposite orientation within a vertically invariant neighborhood guaranteed by the convexity of R together with annuli connecting their boundaries. We can make this new surface, Σ, convex since it is a closed surface as follows. Since a topological spine of R is a Legendrian graph of positive singularities and flowlines between them the spine of R must be contained in Σ_+. Similarly the spine of R' is containted in Σ_-. Thus when we make Σ convex there must be a set of dividing curves parallel to ∂R. Now we can isotope the dividing curves onto ∂R. (If necessary we can again apply Giroux Flexibility to remove any singularities in between ∂R and the dividing set because the foliation with arcs from ∂R to Γ_Σ is also divided by Γ_Σ.) Thus R is a component of Σ_+. $\square$

This lemma proves a little bit more, that is when we have a Legendrian ribbon R we can make R the only component of Σ_+.

Definition 3.0.2. A *quasipositive presentation* of a quasipositive surface S is a component of Σ_+ or Σ_- for some convex surface Σ which is topologically isotopic to S.

Lemma 3.0.3. *Let* (M, ξ) *be a compact, oriented, irreducible contact 3-manifold with no properly embedded overtwisted disks. Let* M' *be the manifold obtained by decomposing* M *along a properly embedded surface* $N \subset M$*. If* M *is a manifold with boundary, let its boundary be convex with dividing set* $\Gamma_{\partial M}$*. If* N *has boundary assume its boundary to be Legendrian with negative twisting number for each component. Then the set of tight contact structures on* M *(with dividing set* Γ_M *if* $\partial M \neq \emptyset$*) is in one to one correspondence with* $\pi_0(C_0)$*.*

This is almost identical to Honda's Gluing Theorem (Theorem 2.2.30). The difference is that we can exchange the assumption that N is incompressible for the assumption that M has no properly embedded overtwisted disks.

Proof. The proof of Honda is essentially as follows (see Theorem 2.2.30): if we consider any potentially allowable configuration in $\mathcal{C}$ and the glued manifold is overtwisted then any OT disk D must intersect N since (M', ξ) is tight. Then after a finite number of bypasses, we can move N off of D. Thus some other configuration in the same connected component of $\mathcal{C}$ is not potentially allowable, thus the original configuration is not allowable and is hence not in $\mathcal{C}_0$. Then he shows that if you have two configurations in the same connected component that they give you the same contact structure on the glued manifold.

The meat of the first part of the proof is the claim that you can use bypasses to disentangle the overtwisted disk from N. The intersection $N \cap D$ consists of closed curves and arcs. The closed curves bound disks in both N and D as N is incompressible and as M is irreducible the pair of disks bounds a ball in $M \backslash N$. This ball can be pushed across using just an isotopy or a series of bypasses. An arc of intersection that is outermost on D bounds a disk in $(M \backslash N) \cap D$. If we take a neighborhood of the arc of intersection in N together with a disk in $M \backslash N$ such that together they bound a ball containing the disk in D that the arc bounds we again have a ball on one side of N that we want to push to the other. A technical note is that we want the boundary of the neighborhood of the arc in N to be Legendrian, so it must intersect Γ_N as ξ is tight in a neighborhood of N.

In fact we can relax the incompressible restriction by proceeding as follows. For each outermost closed curve of intersection on D there is an arc in D connecting that closed curve to ∂D such that the endpoint of this arc is not on the boundary of M. A neighborhood of this arc is a disk. We want to proceed as above for the arcs of intersection to move this small disk across N effectively doing a finger move to push part of the closed curve off of D turning it into an arc of intersection. A priori the neighborhood of this arc in N need not intersect Γ_N, but we can do a second finger move to push the boundary of the neighborhood to intersect Γ_N so that we can realize it as a Legendrian curve. □

3.1 Convex Heegaard Decompositions

Definition 3.1.1. A *convex Heegaard decomposition* for a contact manifold is the collection of a convex Heegaard surface together with convex representatives of each α and β disks.

In this section I will prove a collection of lemmas about bypasses in relation to convex Heegaard decompositions of irreducible tight contact manifolds.

Lemma 3.1.2. *Suppose that Σ is a convex surface, S is another convex surface with $\partial S \subset \Sigma$ with the interior of S disjoint from Σ, and B is a bypass with attaching arc on Σ. If the attaching arc of B is disjoint from S and the rest of B intersects S in a collection of arcs and simple closed curves which bound disks in S, then there are a series of bypasses on S and isotopies of S and B after which B is disjoint from S.*

This might seem like a restrictive assumption on S, but in the context of this work I'll only use it in the case where the contact manifold is a tight solid torus and S is a disk. This lemma is really a corollary of the proof of Honda's Gluing Theorem found in [Hon02], but let us go through the proof in detail anyway.

Proof. Consider the intersection of B with S. It is composed of some collection of arcs and circles which bound disks. Because $\partial S \subset \Sigma$ every arc of intersection must have its endpoints in ∂B. In particular there cannot be any arc of intersection contained in a disk in B bounded by a circle of intersection. Thus if there are any circles of intersection, then there must be one which is innermost on B.

Firstly let us consider an innermost on B circle of intersection which is disjoint from the dividing sets of either surface. This circle of intersection bounds a disk in each surface which together cobound a ball. By doing a finger move moving the circle of intersection to intersect the dividing set of S twice we can find a larger ball which has a product structure since it is a tight ball with a single arc of its dividing curve in each hemisphere. Thus by uniqueness of the contact structure on a tight 3 ball it must have a product structure. We can perform an isotopy of S across B using this product ball as a guide. Note that this finger move may intersect B again and indeed B may intersect the disk that the curve bounds in S, but taking the isotopy of the ball across S eliminates the curve in question and does not create new intersections. Thus it decreases the number of intersections of B and S.

Let C be an innermost on B circle of intersection which intersects Γ_S twice. Then C bounds a disk in S with a single dividing curve in it and a disk in B with a single dividing curve. Thus the two disks bound a tight 3 ball with a single dividing curve. Thus there is a unique contact structure on this ball and we can isotope the disk in S across that ball to eliminate C as an intersection of S and B.

If there is an innermost on B circle of intersection C which intersects Γ_S more than twice, then there are at least two arcs of Γ_B on the disk bounded by C in B. Thus we can realize a bypass on S along one of these arcs. In this manner we can perform a number of bypasses until there is only a single arc of Γ_B which reduces to the previous case.

Now that we have eliminated all circles of intersection we are left with only arcs of intersection so there must be an arc which is outermost. Consider an arc A which is outermost on B and a neighborhood U of this arc in S. If ∂U doesn't intersect Γ_S then we can do a finger move of on U to make it intersect Γ_S.

Now suppose that ∂U meets Γ_S, thus we can make it Legendrian. Now there is a disk D' with $\partial D' = \partial U$ so that D' together with U bounds a ball containing the outermost disk of B cut off by A. If U intersects the dividing set of S once, then viewing D' as being tangent to S at its boundary we see that the contact structure has a product structure on the ball. Thus we can isotope S across the ball preserving the convex structure. If the neighborhood of A intersects the dividing set of S more than once then $\Gamma_{D'}$ has more than one component so we can find a bypass on S in D'. After taking finitely many there is only a single dividing curve so we can do an isotopy that moves S off of B near A, reducing the number of intersections. $\square$

Corollary 3.1.3. *Suppose that Σ is a convex surface bounding a handlebody and D and D' are properly embedded convex disks in that handlebody. Further suppose that $\partial D \cap \partial D' = \emptyset$, then there exists a series of bypasses on D and isotopies after which $D \cap D' = \emptyset$.*

Proof. Follows directly from the proof of the previous lemma. $\square$

Lemma 3.1.4. *Given a convex meridianal disk D, and any bypass whose attaching arc meets 3 distinct parallel dividing curves there is a bypass meeting the same curves such that its attaching arc is disjoint from the meridianal disk. If the interior of the meridianal disk is not disjoint from the interior of the bypass there is a series of bypasses on D to make it disjoint.*

Proof. The proof of the first part follows from Honda's Bypass Sliding Lemma (see Lemma 2.2.25. Suppose that the minimum number of intersections of the attaching arc among equivalent bypasses with ∂D is greater than one. Consider a bypass with the minimum number, for example see Figure 3.1. Call the attaching arc δ and its endpoints p and q. Additionally call the dividing curves that p and q lie on γ_p and γ_q. There is an intersection of ∂D with δ that is closest to p, call it r. WLOG r is on the arc from p to the middle dividing curve. Because ∂D intersects the dividing set efficiently, the boundary of D has an arc from r to γ_p and one from r to γ_q. Label the point where it intersects γ_p as s. The arc of the dividing curve sp, the arc of the disk sr and the arc of the attaching arc pr cobound a disk see Figure 3.1. Slide the segment pq a little towards s to make it disjoint from δ, then slide the arc pr across the disk. Now apply the Bypass Sliding Lemma to find a bypass whose attaching arc has fewer intersections with ∂D.

For the second part notice that since D is a disk any circles of intersection of the bypass and D bound a disk in D. Thus this configuration satisfies the conditions of lemma 3.1.2 so we can find an equivalent meridianal disk with the desired properties. $\square$

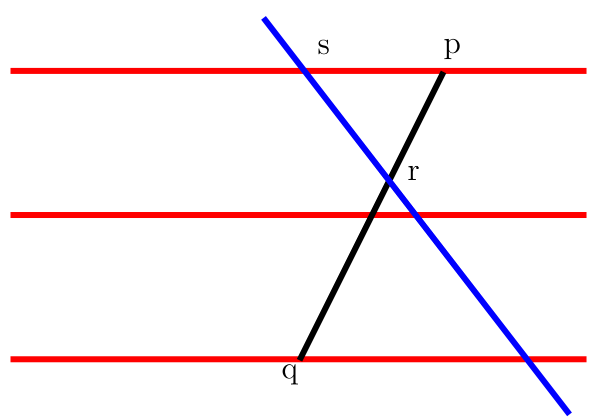

Figure 3.1: The configuration of the attaching arc δ, its endpoints p and q, the intersection of δ with ∂D which is r, and s the intersection of ∂D with γ_p.

3.2 Folds and Fold Bypasses

In this section we will discuss the notion of folding and the more general notion of fold bypasses. Folding has been discussed by several people before, for example see the proof of theorem 3.1 in [Etn] or [Hon00].

Definition 3.2.1. Consider a closed non-isolating Legendrian curve C in a convex surface Σ such that $C \cap \Gamma_\Sigma = \emptyset$. This curve C has an annular neighborhood A' which does not intersect Γ_Σ. Take a slightly smaller annular neighborhood A strictly contained in A'. There is an isotopy of Σ fixed outside of A' contained in a vertically invariant product neighborhood of A' such that the image of Σ consists of 2 copies of A and one copy of $-A$ and 4 annuli connecting them to $\partial S \backslash int(A')$ and each other as in Figure 3.2. This operation is called a *fold* of Σ along C. Note that there are two folds along C, one as shown and one where we mirror the image horizontally.

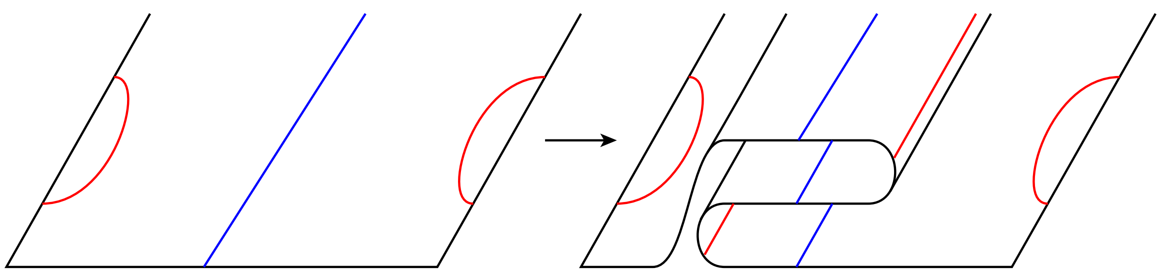

Figure 3.2: The operation of 'folding' corresponds to taking an annulus along a Legendrian divide, as in the diagram on the left, and replacing it by a 'folded' annulus as in the diagram on the right. The curves in the flat regions of the annulus are Legendrian divides while the curves on the curved regions of the annulus are dividing curves.

Proposition 3.2.2. *The effect of a fold along C is to add two dividing curves parallel to C.*

Proof. Let Σ be a convex surface. Without loss of generality let $C \subset R_+$. Since C doesn't intersect the dividing set of Σ it has twisting number 0 relative to Σ. Thus we can isotope Σ so that C is a Legendrian divide. Since the isotopy is fixed outside of

A' there is already a convex structure on $\Sigma \backslash A'$. Following the proof of theorem 2.2.14 (see e.g. [Etn]) in order to realize Σ as a convex surface take a neighborhood of each of the three circles of singularities coming from the three copies of C and add it to R_i where i is the sign of the singularities. We now have three 'stacked' copies of A, the top and bottom copies of which have copies of C which have positive singularities, and the middle copy of C has negative singularities. Thus on the image of A' we have two annular subsets of R_+ and one of R_- in the order plus, minus, plus as we move across the image of A'. Since A' was originally a subset of R_+ the top and bottom of copies of R_+ are already separated from the $\Sigma \backslash A'$. We do however get two new dividing curves between the annular subsets of R_+ and R_-. □

Let us now look at a model of a fold along a Legendrian divide of a torus. Let us consider the line $y = 0, z = 1$ as a Legendrian divide on the cylinder $x^2 + z^2 = 1$ in the standard contact structure on $\mathbb{R}^3$. This cylinder is convex and the line along the top is clearly a Legendrian divide. The foliation of the surface is by closed compressing curves for the cylinder. Each of these curves bound a disk, in fact we can foliate the solid cylinder by disks whose boundaries coincide with the foliation of the cylinder. Further we can take these disks so that the vector field ∂_x is transverse to them. The flow of this vector field preserves the contact structure on $\mathbb{R}^3$ so each of the foliating disks is naturally convex as it has a vertically invariant neighborhood by Lemma 2.2.3. The characteristic surface for this contact vector field is the plane given by $y = 0$.

Theorem 3.2.3. *Suppose that C is a Legendrian divide in a convex surface S and that C is between two parallel dividing curves. Then for a convex surface Σ with $\partial\Sigma \subset S$ whose boundary intersects C transversely, the effect on Γ_Σ of performing a fold along C is to add a boundary parallel dividing curve over one of the new dividing curves in Γ_S.*

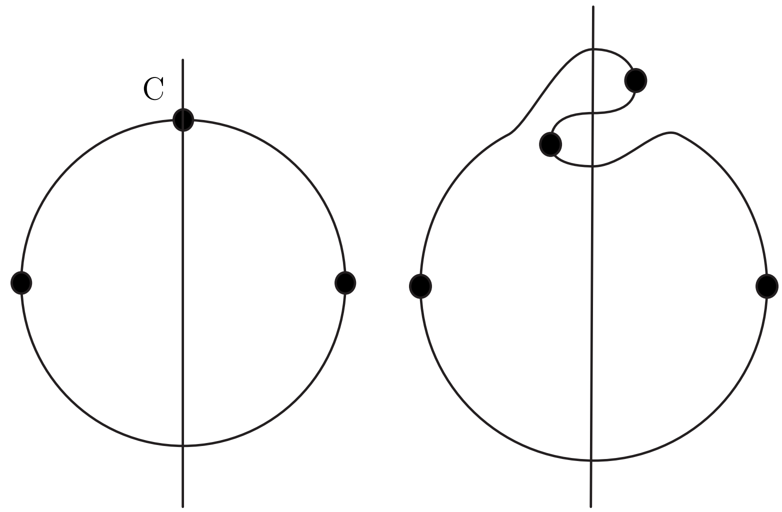

Figure 3.3: The cross section of the standard model of a fold. The left picture shows the Legendrian divide C as well as the unlabelled dividing curves. The right picture shows only the four dividing curves on T. The vertical line is the characteristic surface of the vector field ∂_x.

Proof. Let us perform a fold and look at the local model. In the course of performing a fold we make the surface convex again. A cross section is shown in Figure 3.3. Now let us look at a component of Σ that intersects C transversely. Locally it is transverse to ∂_x so we can use ∂_x as a model for its contact vector field locally. Thus when the fold is performed the boundary parallel curve shown in Figure 3.3 appears so there is a boundary parallel arc centered on one of the new dividing curves. □

Consider now folds on a convex torus bounding a solid torus. A fold is an isotopy, so by the Isotopy Discretization Principle Theorem 2.2.26 it can be accomplished by a series of bypasses. In fact, Honda classified bypasses on a torus see [Hon00]. The only bypass that increases the number of dividing curves is one in which the attaching arc only involves one of the dividing curves of the torus and is not trivial. This bypass as seen in Figure 3.4 replaces a single dividing curve with three.

Definition 3.2.4. A *fold bypass* is a bypass which replaces a single dividing curve with 3 dividing curves such that the central dividing curve of the inverse bypass is parallel to one of the other dividing curves.

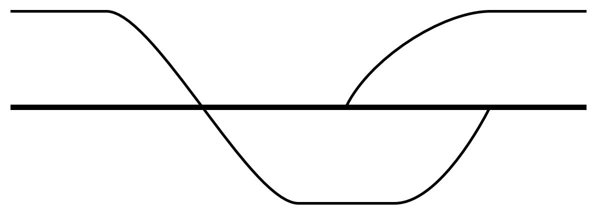

Figure 3.4: The attaching arc of a fold bypass on a convex torus. The bold curve is the dividing curve and the lighter curve is the attaching arc.

Lemma 3.2.5. *A topological fold along a Legendrian divide can be realized by a single fold bypass.*

Proof. After the operation of folding there is a bypass on each side of the folded surface, one inside each of the lobes of the fold, see Figure 3.2.3. Thus there is a bypass centered on one of the new dividing curves on one side, and a bypass centered on the other new dividing curve on the other side. Taking either of these bypasses will absorb the two new dividing curves into one of the original dividing curves parallel to the Legendrian divide. The inverse of this bypass is then a fold bypass on that dividing curve. □

Let us now consider the effect of a fold bypass on the α and β disks of a convex Heegaard decomposition for a contact manifold. What we want to know is when we perform a bypass of this type, what happens to the dividing set of one of the convex α or β disks whose boundary intersects the dividing curve the bypass is on transversely.

Lemma 3.2.6. *Suppose that there is a bypass B in the β handlebody with attaching arc in a convex Heegaard surface Σ. Suppose that the attaching arc of B intersects the boundary of a β disk transversely and that the signs of all the intersections are the same. Suppose further that the meridianal disk D remains a disk in the β handlebody after performing the bypass. The effect on the dividing set of the meridianal disk is to do a finger move of a section of the boundary through a single dividing curve for each intersection of ∂D with the dividing curve the bypass intersects in Σ.*

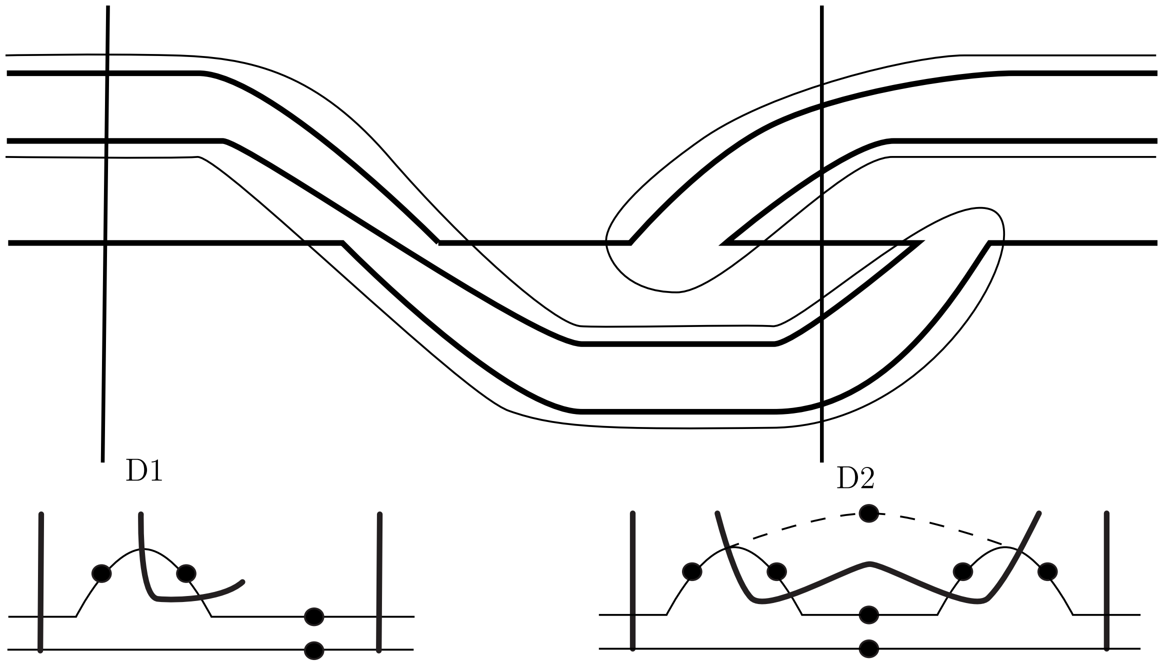

Figure 3.5: The top picture is of the effect of performing a fold bypass on a convex Heegaard surface. The bold curves are the dividing set of the torus and the vertical segments are arcs of the boundaries of convex β disks D_1 and D_2. The bottom pictures are collars of the arcs of ∂D_1 and ∂D_2 that appear in the top picture. Effectively a cross section of a Fold Bypass. The light arcs are the surface before and after performing the bypass and the circles are the dividing curves on the convex surface. The bold curves are dividing curves on D_i. The dotted curve is the surface after taking the trivial bypass along the boundary of D_2.

Proof. Since the attaching arc intersects the meridianal disk in the same direction for all intersections we can arrange the bypass attaching arc to intersect the boundary of the β disk as shown in the top of Figure 3.5. Consider an annular neighborhood of the attaching arc in Σ. Let's look at it from the side of the β handlebody. After performing the bypass Σ has the same foliation outside of a neighborhood of the attaching arc as it stayed within a vertically invariant neighborhood of its original location. A disk within Σ has had its dividing curves changed according to the effect of adding a bypass on the front of the attaching arc. Convex β disks fall into two categories. Disks which intersect the attaching arc once on one side of the dividing curve and disks which intersect the dividing set on each side of the dividing curve. Let's look at the effect of taking the bypass on the first kind of disk. The meridianal disk D has remained fixed throughout this process so in order for there to be two additional dividing curves along the top of the the bypass neighborhood there must be an arc of Γ_D which intersects it as in the bottom left of Figure 3.5. For the second kind of disk, there are now five dividing curves after performing the bypass in the place of one as seen in the bottom right of Figure 3.5. Thus for each piece of $\Sigma \subset D$ that moved there must be an intersection with the dividing set of D. As seen in the top picture of Figure 3.5 there is a trivial bypass along the arc of the meridianal disk in between the central 3 dividing curves. Performing this bypass would slide Σ a little farther into D in the center and eliminate two dividing curves. Thus the two arcs in Γ_D must be connected as shown in the lower right picture of Figure 3.5.

□

Note that one can always find β disks which remain β disks after performing the bypass by taking bypasses with attaching arcs in the disk to remove intersections which do not intersect its boundary according to Lemma 3.1.2.

Now let us consider simpler case of the effect on an α disk instead.

Lemma 3.2.7. *Suppose that there is a fold bypass B in the β handlebody with attaching arc in a convex Heegaard surface Σ. Suppose that the attaching arc of B intersects the boundary of an α disk transversely and coherently. The effect of taking the bypass is to add a boundary parallel dividing curve centered on one of the three dividing curves coming from the original.*

Proof. The boundary of the α disk is transverse to C and all of the intersections have the same sign as in the proof of lemma 3.2.6. We know that the effect on the β handlebody is to move a $S^1 \times D^2$ with a single dividing curve on each meridional disk from the β handlebody to the α handlebody. There is an extension of the α disk through that solid torus transverse to the contact vector field invariant in the S^1

direction. Thus we see a boundary parallel dividing curve on the α disk centered on one of the new dividing curves. $\square$

In spirit this operation is the opposite to the operation in corollary 3.2.6 in that we do a finger move of the boundary into a portion of the characteristic surface on the other side. This is the analogy between folds and fold bypasses. A fold bypass takes the convex surface Σ and pushes it onto the characteristic surface on one side to make a bypass on one side centered on one of the new dividing curves.

Chapter 4

Heegaard Decompositions of $L(4, 1)$

The notation from this chapter and particularly the notation developed for Figure 4.1 and Proposition 4.0.2 will be used for the rest of this work. Suppose that we have a torus with a given meridian and longitude basis. A curve with slope $\frac{p}{q}$ on that torus is given by p-times the longitude and q-times the meridian, i.e. by the ordered pair (q, p) as $(1, 0)$ represents the meridian and $(0, 1)$ represents the longitude. Given a curve (q, p) let (q', p') be the curve given by the conditions that $pq' - qp' = 1, p > p' > 0$, and $q \geq q' > 0$. In particular the curve represented by the ordered pair $(1, n)$ corresponds to the slope $\frac{n}{1}$ and (q', p') would be $(1, n - 1)$.

Honda classifies tight contact structures on the lens space $L(p, q)$. Specifically he proves the following as part of his Theorem 2.1 in [Hon00].

Theorem 4.0.1 ([Hon00])**.** *There exist exactly* $|(r_0 + 1)(r_1 + 1) \dots (r_k + 1)|$ *tight contact structures on the lens space* $L(p, q)$ *up to isotopy, where* $r_0, \dots, r_k$ *are the coefficients of the continued fraction expansion of* $-\frac{p}{q}$.

The proof of this theorem can be sketched as follows. Firstly you can find a decomposition of $L(p, q)$ into two solid tori V_0 and V_1 along a convex Heegaard torus with two dividing curves. Relative to a choice of longitude for ∂V_0 the slope is $s(V_0) = \infty$ and relative to a choice of longitude for ∂V_1 the slope is $s(V_1) = -\frac{p'}{q'}$.

Since there is a meridianal disk for V_0 whose boundary only intersects the dividing curves of the torus twice there is a unique tight contact structure on V_0. An upper bound on the number of tight contact structures on $L(p,q)$ is thus given by the number of tight contact structures on V_1.

The second part of the proof is to prove that each of these possible contact structures is realized by a tight contact structure. This is done by constructing a link in S^3 with Legendrian representatives with different rotation numbers for each component. Legendrian surgery on each of these links gives a distinct tight contact structure on $L(p,q)$. A lower bound is thus given by the number of choices of rotation numbers for the components of the link. After checking that the lower bound is equal to the upper bound the number of tight contact structures is found.

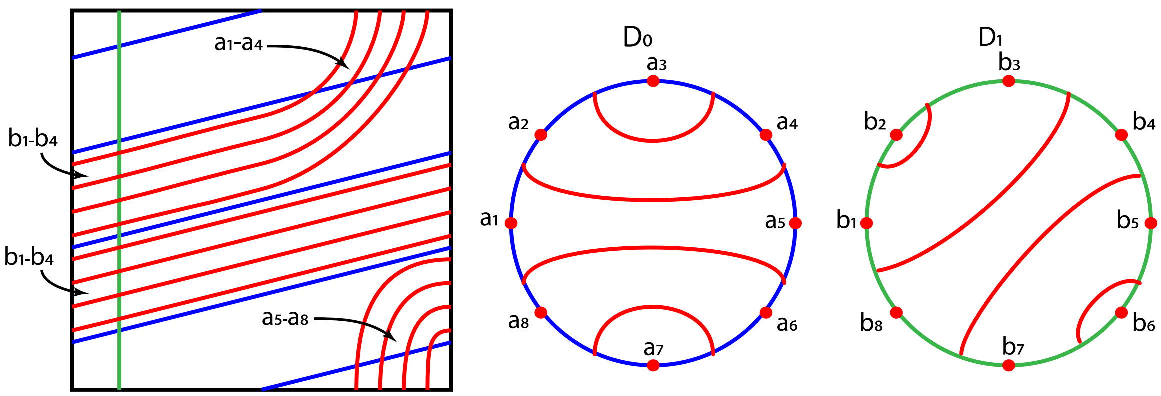

Figure 4.1: Convex Heegaard decomposition of $L(4,1)$: ∂D_1 is shown by the vertical green line to the left and D_1 can be thought of as 'behind' the torus. ∂D_0 is shown by the blue curve and D_0 can be thought of as 'in front' of the torus. The 4 dividing curves are shown in red, the intersections of the dividing set with ∂D_0 are labeled with a_i the intersections of the dividing set with ∂D_1 are labeled with b_i.

Figure 4.1 shows a convex Heegaard decomposition for a contact structure on $L(4,1)$. The torus T is obtained by identifying the edges of the square and is shown with the boundaries of the meridianal disks and the 4 dividing curves. The meridianal disks D_0 and D_1 for the solid tori V_0 and V_1 respectively are shown to the right with their dividing curves.

Proposition 4.0.2. *The two universally tight contact structure on $L(4,1)$ have a convex Heegaard decomposition as shown in Figure 4.1.*

Proof. By the proof of Theorem 4.0.1, given a Lens space, $L(p,q)$, and a convex Heegaard Torus inside it whose dividing curves have slope ∞ to one side and $-\frac{p'}{q'}$ to the other, V_1 there is a tight contact structure on $L(p,q)$ for each tight contact structure on V_1. Now lets look in the lens space $L(4,1)$. Let T be a convex torus with dividing curves of slope ∞ to one side, V_0 and dividing curves of slope -3 on the other, V_1. We can find a meridianal disk, D_1 of V_1 whose boundary meets the dividing set 6 times, thus if we realize D_1 as a convex disk there will be 3 dividing curves on it (there cannot be any closed curves in the dividing set because the contact structure would be overtwisted). The 3 dividing curves can be in any of 5 configurations; up to rotational symmetry of the disk they fall into two types shown in Figure 4.2. The contact structures on the solid tori are distinguished by the relative Euler class of the meridianal disks. For convex surfaces with Legendrian boundary the relative Euler class can be computed by $\chi(\Sigma_+) - \chi(\Sigma_-)$ (see Proposition 4.5 of [Hon00]). In this example this comes down to the number of positive regions on D_1 minus the number of negative regions. The configuration on the left of Figure 4.3 gives two contact structures as the difference can be ± 2 while the configuration on the right only gives one as the difference is always 0. The two contact structures coming from the disk on the left are universally tight while the one coming from the disk on the right is virtually overtwisted.

Consider the two contact structures on V_1 corresponding to the convex meridianal disks appearing in the left diagram of Figure 4.2. If we choose D_0 to be a meridianal disk of V_0 such that its boundary intersects Γ_T and ∂D_1 efficiently, up to relabelling the convex Heegaard decompositions for the Lens space will appear as shown in Figure 4.3. Each of the decompositions can be modified to be the desired decomposition shown in Figure 4.1 through a series of bypasses, isotopies and folds as we will see.

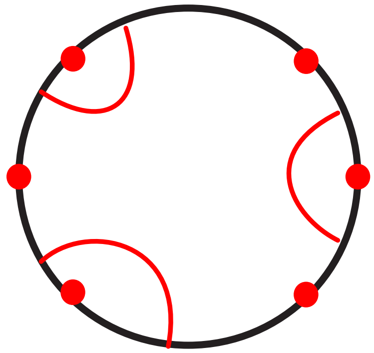

Figure 4.2: The two types of configurations for a solid torus with convex boundary with 2 dividing curves with slope -3. The one on the left has two distinct rotations while the one on the right has 3.

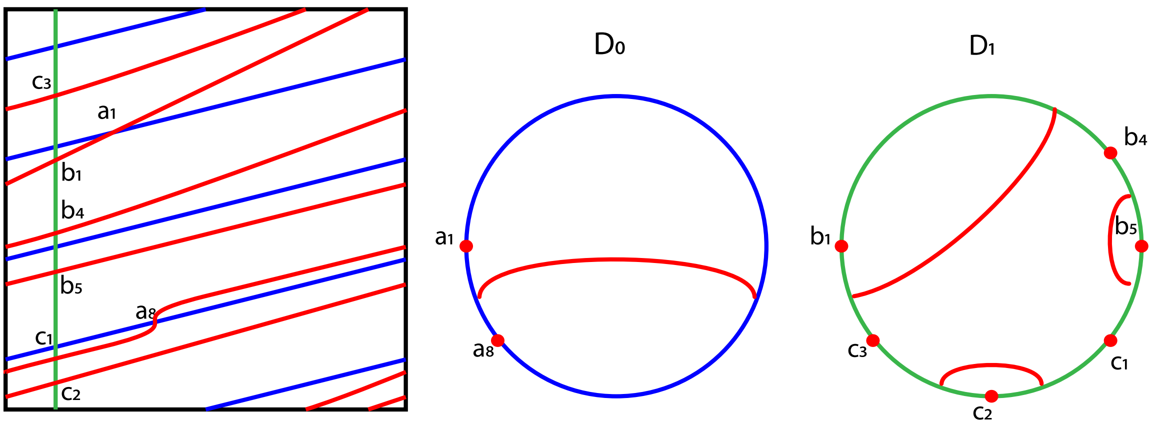

Figure 4.3: The convex Heegaard decomposition of the two universally tight contact structures on $L(4,1)$. The difference between them is obscured in this picture because Σ_+ and Σ_- are not distinguished. Notation is the same as in Figure 4.1 except 3 of the intersections between the dividing set of T and ∂D_1 are c_1, c_2 and c_3.

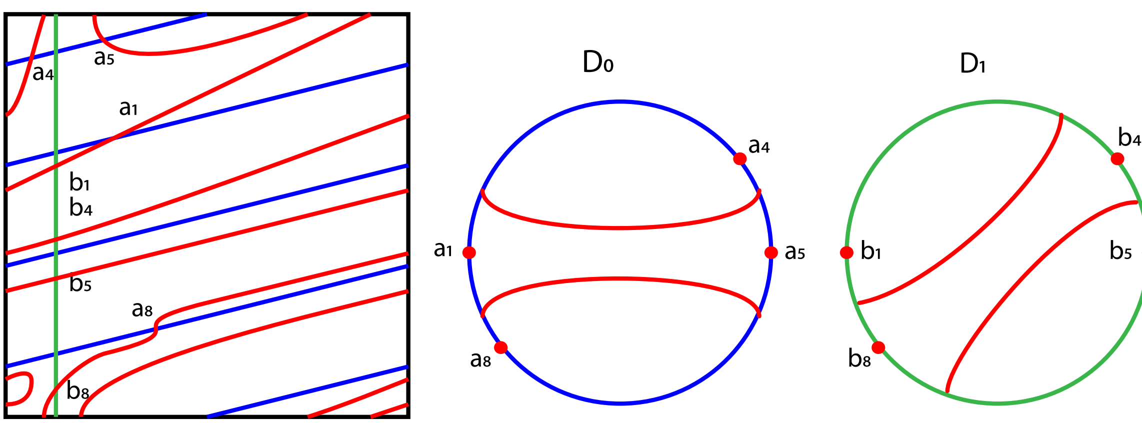

Figure 4.4: The Heegaard decomposition obtained performing the bypass along $[c_1c_2c_3]$ in the Heegaard decomposition in Figure 4.3.

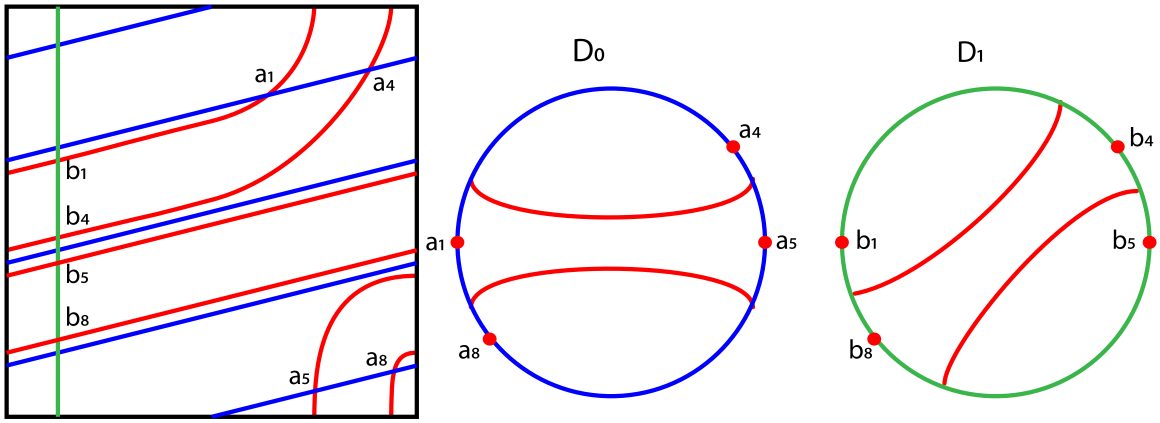

Figure 4.5: This convex Heegaard decomposition can be achieved after doing an isotopy of the Heegaard decomposition in Figure 4.4.

The first step in rearranging this to be the desired configuration is to perform the bypass in D_1 whose attaching arc is the segment $[c_1c_2c_3]$. The result of the bypass is shown in Figure 4.4. After performing the bypass ∂D_0 intersects Γ_T in four points. There are two possible dividing sets for D_0 one with a dividing curve over a_5 one without. The inverse of the bypass we just performed has an attaching arc along $[a_1, a_8, a_5]$. We can find a meridianal disk with boundary intersecting the dividing set efficiently close to the attaching arc of the bypass on each side of the bypass which are disjoint from the bypass by applying Lemma 3.1.2. If the inverse bypass is not in the dividing set of either of the meridianal disks then we can perform the bypass centered on a_5 and the inverse bypass and find an OT disk in the convex torus. Since the contact structure on the lens space is tight the dividing sets on D_0 must be as shown. After some isotopy the configurations appear as in Figure 4.5.

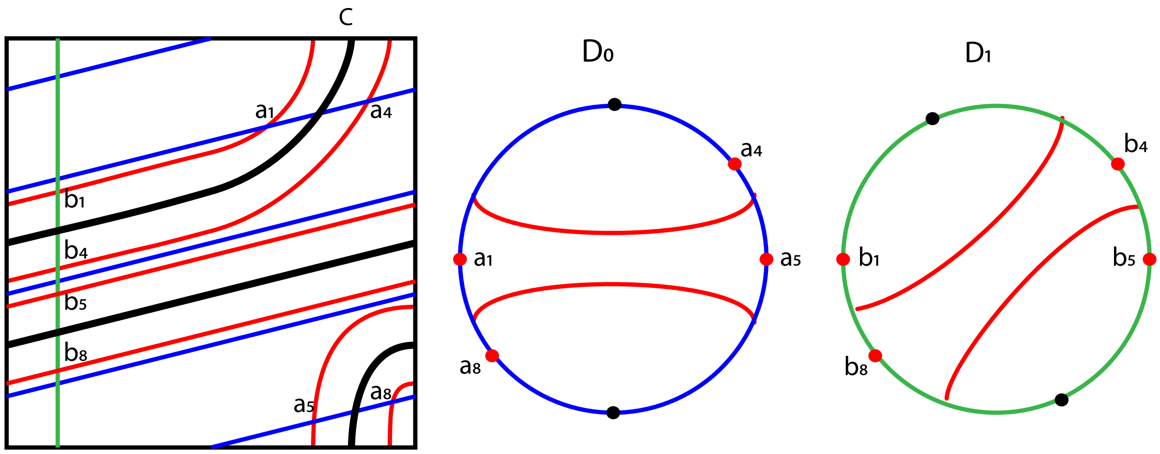

Figure 4.6: The new bold curve is the curve C which we fold along to get the decomposition shown in Figure 4.1. In choosing C we can guarantee that it intersects ∂D_0 and ∂D_1 at the unlabeled dots on their boundaries.

The next step is to perform a fold along the curve C, shown in Figure 4.6. The effect of this fold is to add a new boundary parallel dividing curve over a_3 and a_7. A similar argument holds for D_1. This gives us the decomposition shown in Figure 4.1 as desired. □

Chapter 5

Bypasses existing in V_i

For this chapter we will continue with the notation of the previous chapter. Since the convex Heegaard decompositions of V_0 and V_1 are combinatorially the same, whichever restrictions coming from the Heegard decomposition on the bypasses existing on V_i will be the same as well. Thus it is sufficient to analyze the bypasses existing within a single solid torus with the given Heegaard decomposition and both of the V_i will have the same structure. For the remainder of this chapter let us focus on a solid torus with convex boundary T which is contactomorphic to V_0.

The attaching arc of a bypass can meet either one, two, or three dividing curves.

Lemma 5.0.1. *If the attaching arc of a bypass on T meets the same dividing curve at one endpoint and the center and the arc between them cobounds a disk with the segment of the dividing curve between them, then the bypass must be trivial.*

Proof. By Honda's classification of bypasses on a torus [Hon02] if a bypass meets the same dividing curve in the center and one endpoint the only possibility for it to not be trivial is for it to be a fold bypass. However if the arcs cobound a disk then it cannot be a fold bypass. □

In particular, for any tight manifold with torus boundary T, if $|\Gamma_T| > 2$, any bypass along T^2 involving exactly two curves must be trivial. Since $|\Gamma_T| > 2$ we cannot have an attaching arc which meets the same curve at its endpoints and a different in its center, thus the segment of the attaching arc between the two intersections must cobound a disk with the segment of the dividing curve between them. We can then apply Lemma 5.0.1. We are left to consider bypasses that meet exactly one or three dividing curves.

In order to discuss the bypasses inside a solid torus let's introduce terminology. Since all dividing curves on a torus are parallel, for a bypass whose attaching arc meets 3 different dividing curves, knowing the dividing curve in the center of the attaching arc determines the attaching arc.

Definition 5.0.2. We will say a bypass with attaching arc in a convex torus is *centered* at a dividing curve γ if the attaching arc of the bypass meets 3 dividing curves and γ is the middle of the 3.

Lemma 5.0.3. *Let D' be a meridianal disk whose boundary intersects Γ_T efficiently and let D' be disjoint from D inside V. Then if D' also intersects Γ_T efficiently the dividing set on D' is equivalent to the dividing set on D.*

Proof. Without loss of generality let's use the notation from V_0.

There is already a bypass centered at γ_3. After performing this bypass, any other bypass centered at γ_3 becomes trivial, hence it also exists.

If there is a bypass centered on γ_2 or γ_4, we can perform that bypass and one centered on γ_3 and we will see a closed dividing curve which bounds a disk in the torus. This contradicts the tightness of the contact structure.

Observe that any bypass with attaching arc on D is either trivial or involves 3 dividing curves. Additionally note that any bypass involving 3 dividing curves after sliding the attaching arc must be between one of the outermost arcs and the two dividing curves which are not outermost as seen in Figure 5.1. If a bypass is performed along this arc it will create a bypass centered on γ_2 or γ_4 depending on

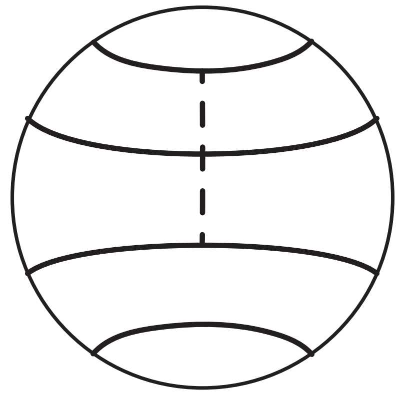

Figure 5.1: Up to bypass sliding and rotation of the disk the attaching arc of any non-trivial bypass on a meridianal disk D of the solid torus V must appear as shown here.

the side of the surface it is performed on. Thus by tightness, any convex structure on the disk obtained by performing a single bypass on the disk cannot exist in V.

Suppose there was another meridianal disk D' disjoint from D. In particular $\partial D \cap \partial D' = \emptyset$ and both intersect Γ_T efficiently they cobound an annulus in T which has dividing set consisting of arcs from one component to the other, thus we can use Giroux Flexibility to foliate the annulus with boundary parallel arcs, thus we can isotope the boundaries of the disks to be identical. Then applying the Isotopy Discretization Principle there exists a series of bypasses from D to D'. However there can be no first bypass of this sequence which is non-trivial, thus there cannot be a disjoint meridianal disk whose dividing set is not equivalent to the dividing set on D. $\square$

In fact more is true, we can relax the assumption on disjoint disks to only requiring the disks have disjoint boundaries and we achieve the same result.

Lemma 5.0.4. *Let D' be a meridianal disk whose boundary intersects Γ_T efficiently and let $\partial D'$ be disjoint from ∂D inside T, then the dividing set on D' is equivalent to the dividing set on D.*

Proof. The proof of this is achieved by finding a series of disks inside V going from D to D' such that at each step the disks are disjoint from the previous. This sequence of disks can by found by applying Corollary 3.1.3. We can then apply Lemma 5.0.3 to each step to show that the dividing sets are the same. □

Lemma 5.0.5. *The only bypasses involving 3 distinct dividing curves which can exist within V_i are centered on the dividing curves which already have bypasses centered on them in D_i.*

Proof. WLOG let us look in V_0. As we've seen above a bypass centered on γ_2 or γ_4 would result in an overtwisted disk in T so those bypasses cannot exist.

If there were a bypass centered at γ_1 we can make the bypass disjoint from D_0 by a lemma above, then we can extend the bypass to a meridianal disk and make that disk convex preserving the bypass centered at γ_1. This contradicts lemma 5.0.3. □

Chapter 6

Contact Structure and Seifert Fibration

Throughout this chapter we will continue to use the notation from Chapter 4.

Definition 6.0.1. A contact structure on a contact $S^1 \times \Sigma$ is S^1 *invariant* if there is a contact vector field which is everywhere tangent to the S^1 fibers.

Definition 6.0.2. A contact structure on a Seifert fibered space is called *Seifert fibered invariant* (or *Sf-invariant*) if there is a contact vector field whose flowlines agree with the Seifert fibration.

Lemma 6.0.3. *There is a Seifert fibration of V_i for which the contact structures induced from the universally tight contact structures of $L(4,1)$ are Sf-invariant.*

Proof. We can foliate $S^1 \times D^2$ by D^2 fibers. Consider the set of D^2 fibers which can be made convex with the same contact dividing set as a given meridianal disk. It is certainly open since given a convex disk there is a vertically invariant neighborhood of the disk in which there are other disks which have the same dividing set. Let D now be the first disk which cannot be made convex with the same dividing curves and D' the last disk in the same interval as D which cannot be made convex with the given dividing curves.

Claim: D' coincides with D. Suppose not, then there would be some open interval of disks in between them. A disk from within this interval disk can be made convex while still being between them and has the same dividing set as the original D^2 fiber. Thus D' coincides with D.

We now have D as a single disk in between two open sets where the contact structure coincides. Because contact structures are continuous, the contact structure must be the same at D as well. This implies that D is convex with the same dividing curves.

Since all of the meridianal disks are convex and each has a neighborhood which is vertically invariant, by compactness we can find a single contact vector field which is transverse to each of the disks. By examining the characteristic surface within the solid torus we can determine what this vector field is. We know that the boundary of the characteristic surface spins around the boundary torus at the same rate as the dividing curves on the torus because dividing sets of transverse surfaces interleave. Within a neighborhood of each disk the vector field looks like the I-direction of a $D^2 \times I$ neighborhood of the disk. In fact the vector field on the entire solid torus is given by taking a vector field in the I direction of $D^2 \times I$ and then twisting the top and gluing to the bottom. The amount of twisting is precisely given by the slope of the dividing curves. Thus this vector field integrates to be a Seifert fibration of the solid torus. Note that when restricted to the boundary torus the vector field integrates to be a foliation of the torus by simple closed curves including and parallel to the dividing curves. $\square$

Lemma 6.0.4. *Performing a bypass which reduces the number of dividing curves on a convex torus with 4 or more dividing curves corresponds to moving a copy of $T^2 \times I$ whose contact structure is S^1 invariant in the direction of the dividing curves on one of the convex boundary tori components from one side of the convex surface the bypass is on to the other.*

Proof. Since the torus has at least 4 dividing curves performing the bypass cannot change the slope of the dividing set on the torus. Thus both boundary tori of the $T^2 \times I$ have the same slope. There is a horizontal annulus on which the dividing set has at least two arcs from one torus to the other as well as the dividing curve corresponding to the bypass that was taken. Thus by Theorem 2.2.31 the contact structure on the entire $T^2 \times I$ is determined by the dividing set on A and in particular is S^1 invariant. This puts us in a very similar situation to the one in lemma 6.0.3. Proceeding as before we find a contact vector field corresponding to a Seifert fibration which when restricted to each boundary component integrates to curves parallel to the dividing curves. $\square$

Lemma 6.0.5. *A fold bypass corresponds to moving a copy of $T^2 \times I$ with a contact structure that is invariant in the direction of the dividing curves from one side to the other.*

Proof. The inverse bypass of a fold bypass is a bypass which reduces the number of dividing curves on the torus. Thus by lemma 6.0.4 the $T^2 \times I$ is S^1 invariant in the direction of the dividing curves. $\square$

Lemma 6.0.6. *After doing some isotopy of T during which the number of its dividing curves do not decrease below 4, the contact structure on V_i will still be Sf-invariant.*

Proof. The isotopy in question can be accomplished by bypasses by theorem 2.2.26. Each of these bypasses corresponds to attaching a non-rotational $T^2 \times I$ with the same dividing slope on each boundary copy of T^2 to the boundary (or taking one off). Thus by [HKM02] the contact structure on that layer is S^1 invariant. Since for all of the pieces we're considering the boundary components have vector field which foliate the torus with curves parallel to the dividing curves combining these surfaces or separating them can be well defined up to multiplying the vector fields by some scalar function. Thus the Sf-invariance is preserved by adding or subtracting bypasses of these types and so the contact structure on V_i remains Sf-invariant. □

Theorem 1.0.5. *The universally tight contact structures on the lens space $L(4,1)$ admit a contact vector field which is everywhere tangent to a Seifert fibration of the manifold.*

Proof. The lens space $L(4,1)$ is composed of two solid tori glued along a convex torus. By lemma 6.0.3 each of these solid tori have a contact vector field corresponding to a Seifert Fibration of the solid torus. In particular on the boundary these vector fields integrate to a foliation by curves parallel to the dividing curves. Thus as above after multiplying one by a scalar function if necessary, we can glue the vector fields to be a single vector field. This vector field integrates to a Seifert Fibration of $L(4,1)$ thus the contact structure is S^1 invariant. □

Chapter 7

Convex Presentation of Quasipositive Annuli

A dividing curve is a transverse knot embedded in the convex surface. Performing a bypass is a topological isotopy of the convex surface and hence a topological isotopy of the transverse knot which carries the dividing curve. One can ask whether this topological isotopy can be a transverse isotopy of the transverse knots in the surface.

Lemma 7.0.1. *Performing a bypass on a torus that involves 3 distinct dividing curves cannot be a transverse isotopy of the transverse knot carrying the central dividing curve.*

Proof. WLOG if we look at the foliation of a bypass disk D, the central dividing curve γ passes through the bypass disk at a positive elliptic singularity. Thus there is a natural orientation of the dividing curve to make it positively transverse. The arc at the other end of the bypass a is Legendrian and has only negative singularities. We now think of 'wiggling' D to investigate the nearby contact structure while leaving the bypass disk fixed. We'll call the 'wiggled' D by the name D' instead. Since the arc on the boundary of D is Legendrian with all negative singularities and 0 twisting relative to D we can isotope D so that the arc becomes a line of negative singularities in the characteristic foliation. The bypass in question will isotope γ through a. At the moment when γ crosses a it cannot preserve it's transverse knot type since it must be simultaneously positively and negatively oriented with respect to the contact structure. □

Contrast this with lemma 7.0.3.

Lemma 7.0.2. *Suppose we have a rectangle in a convex surface contained within Σ_+ or Σ_- with characteristic foliation along the boundary as follows: transverse to the boundary along the top flowing into the rectangle, transverse to the boundary along the bottom flowing out of the rectangle and along the left and the right the flowlines are all the way from the top to the bottom. Then we can isotope the surface to have the entire rectangle foliated by flowlines from the top to the bottom.*

Proof. WLOG let the rectangle be within Σ_+, if it is not, reverse the orientation of the contact vector field. First isotope the rectangle to have only generic singularities and to have the minimum number of singularites in its interior while leaving the foliation along the boundary the same. The set of leaves going from top to bottom is a closed set so if there are leaves of the foliation which do not go from top to bottom there is a last leaf which does go from top to bottom. This leaf must have a singularity in it, since the foliation flows into the singularity from the top and out of the singularity on the bottom it cannot be elliptic, so it is a hyperbolic singularity. The flowline from the top is one of the stable separatrices of the singularity and the flowline to the bottom is one of the unstable separatrices. There are two more separatrices and they alternate, there is an unstable separatrix flowing out of the singularity above a stable separatrix flowing into the hyperbolic singularity. The unstable separatrix must flow into the boundary of the rectangle or into another singularity because the divergence of the vector field giving the foliation is positive within Σ_+. If the unstable separatrix flows into the boundary then the unstable separartrix must either come from another singularity or from a limit cycle. There can't be a limit cycle since the contact structure is tight. Therefore from the original singularity there must be another singularity joined by a flowline. If this singularity is elliptic we can cancel them and reduce the number of singularities which contradicts the minimality assumption. If this singularity is hyperbolic we can repeat a similar argument. Thus after a finite number of steps we either have a cycle of hyperbolic singularities connected to each other, or we have a hyperbolic singularity connected to an elliptic singularity. In the second case we can cancel the singularities to reduce their number which is a contradiction to the minimality assumption. If we have a cycle of hyperbolic singularities then we have a disk whose boundary is Legendrian, but not a Legendrian unknot. Generically a flowline of the characteristic foliation within Σ_+ must flow through the dividing set since there are no sink elliptic singularities within Σ_+, only a finite number can flow into hyperbolic singularities and the divergence of the foliation is positive. Thus there cannot be a closed cycle within Σ_+ in a tight contact manifold. Thus the minimum number of singularities is zero so the entire rectangle is foliated by leaves from top to bottom. □

Lemma 7.0.3. *Performing a bypass on a torus that meets 3 distinct dividing curves can be taken to be a transverse isotopy of the transverse knots carrying the dividing curves which the bypass is not centered on. Furthermore the new dividing curve is transversely isotopic to each of the transverse knots in the convex torus resulting from the bypass.*

Proof. Let us call the bypass disk B and call its attaching arc δ. As in the proof of the previous lemma we can realize $\partial B \backslash \delta$ as an arc of singularities. The bypass can be performed by isotoping a small neighborhood of δ across a ball constructed by taking a neighborhood of a vertically invariant neighborhood of B and rounding the corners.

Consider one of the outside dividing curves, γ, define $\gamma \cap N(\delta) = C$. If C remains transverse to the contact structure throughout the bypass then we have a transverse isotopy of γ. Being transverse to a contact structure is an open condition so any isotopy of C so long as it is sufficiently small will be a transverse isotopy. Thus we can find an isotopy of C that moves it off of D and off of T on the side the bypass occurs on. Since we just need to move T across B in a small neighborhood we can choose such a neighborhood such that the image of C is contained in the new T. We still need to make T convex again, but the new T is C^∞ close to a convex surface, so we can take that isotopy in a small fashion such that C remains transverse.

After the isotopy we have C being transverse to the foliation of T as well as a dividing curve γ_1 which is also transverse to the foliation of T. Since C is a segment of γ and γ_1 contains $\gamma \backslash Int(C)$ the two curves are tangent at their intersection.

Thus we have a rectangle on T whose foliation along the boundary is as in the previous lemma thus we can isotope it so that the foliation is from C to the dividing curve. Thus the portion of the dividing curve that is disjoint from the transverse knot can be transversely isotoped to it in T. $\square$

Lemma 7.0.4. *In a tight contact $S^1 \times D^2$ with an Sf-invariant contact structure, if there is a bypass in the solid torus which reduces the number of dividing curves, that bypass exists within a meridianal disk.*

Proof. By the Sf-invariance we can find a single contact vector field transverse to all of the meridianal disks. The characteristic surface of this contact vector field intersects the meridianal disks in their dividing curves. If a bypass in the solid torus reduces the number of dividing curves then there must have been at least 4 dividing curves to begin. Thus the bypass must be an S^1 invariant layer which agrees with the Sf-invariance of the contact structure along the boundary. Thus the bypass must come from a boundary parallel annular leaf of the characteristic surface being pushed across the contact structure in a disk. Thus the bypass appears in a meridianal disk. $\square$

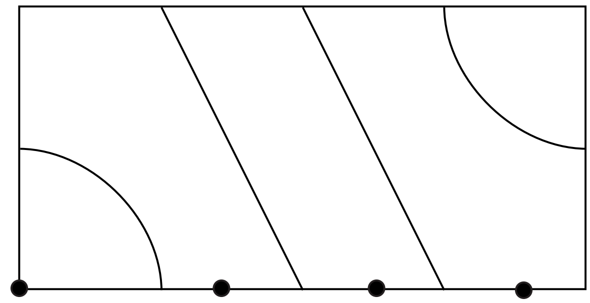

Figure 7.1: Diagram showing the meridianal disk D_0 with B along the vertical sides and A along the top. The transverse link L intersects the disk at the four corners. The dividing curves in T are labeled on the bottom, but they occur symetrically along the top as well.

Consider the convex Heegaard decomposition constructed earlier and shown in Figure 4.1. Let γ_i be the dividing curve that passes through a_i and b_i for $i \in 1, 2, 3, 4$. Then γ_2 and γ_3 cobound two annuli. One of these two annuli, A, has interior dividing curves and one, B, does not.

Theorem 7.0.5. *Let T be the convex Heegaard torus for $L(4,1)$ shown in Figure 4.1. Let A be the annulus bounded by γ_2 and γ_3 which contains the dividing curves γ_1 and γ_4 and let B be the other annulus. Any isotopy of A into a quasipositive presentation does not preserve the transverse link type of its boundary, L.*

Proof. Any isotopy of A extends to an isotopy of the torus T. Suppose there was an isotopy of A which put it into a quasipositive presentation. By the Isotopy Discretization Principle (Theorem 2.2.26) this isotopy of T can be realized by performing a series of bypasses on it. At some point there is a first bypass in either V_0 or V_1 after which A has no interior dividing curves, WLOG let it be in V_0. By Lemmas 6.0.6 and 7.0.4 this bypass must occur within a meridianal disk of V_0. In fact, since A always has interior dividing curves up until this bypass, there are always at least 4 dividing curves on T. Thus the bypasses preserve the SF-invariance of the contact structure. Hence the contact structure of V_0 is entirely encoded in a meridianal disk. Before the isotopy the dividing curves on the meridianal disk appear as in Figure 7.1.

The moves to the diagram that are allowed are removing boundary parallel curves by taking a bypass, introducing boundary parallel curves by taking a fold bypass on the torus located in the other solid torus, or doing a finger move of a dividing curve to the boundary of the disk breaking it into two by doing a fold bypass in V_i. There are three kinds of boundary parallel arcs that can arise in this process. The first kind are bypasses centered on one of the dividing curves on the boundary of A. The second kind of bypass is one which appears by doing a fold bypass in V_1. The third kind is one which is formed by performing a fold bypass in V_0. Performing a bypass over a bypass of the first kind cannot be a transverse isotopy of L by Lemma 7.0.1. Performing a bypass on a dividing curve of the second or third kind would undo the fold bypass which created the boundary parallel curve. If we additionally keep track of the bypasses on D_1 we see that all we can do without taking a bypass centered on a component of L is to perform fold bypasses and undo them which never decreases the number of dividing curves below 4. In particular there are always two dividing curves on D_0 which go from the top of the diagram to the bottom, therefore there will always be dividing curve interior to A. Thus the isotopy takes a bypass centered over a component of L which implies that the isotopy is not a transverse isotopy of L. □

This example proves Theorem 1.0.4.

Theorem 1.0.4. *There exists a transverse link in a universally tight contact structure on a lens space which bounds a Legendrian ribbon and another surface which cannot be isotoped to be the ribbon of a Legendrian graph through an isotopy which restricts to a transverse isotopy of its boundary.*

Proof. Consider the universally tight contact structures on $L(4,1)$. Take the pair of curves γ_2 and γ_3 to be a transverse link L. They bound two surfaces, B which is a Legendrian ribbon and A which is not. By theorem 7.0.5 there is no isotopy of A into a quasipositive presentation which is a transverse isotopy of L. Quasipositive presentations are equivalent to Legendrian ribbons so there is no isotopy to a Legendrian ribbon which is a transverse isotopy of the boundary. $\square$

Chapter 8

Future Directions

This construction is only presented in $L(4,1)$ in this dissertation, but there are several directions in which one could proceed further. Firstly one could try to extend this construction in to a wider variety of lens spaces. Any lens space containing a Klein bottle should contain a similar construction with perhaps the number of dividing curves on the meridianal disk increasing. Another avenue to proceed would be to try to construct an example where the boundary of the surface is a knot instead of a link. One way to approach this would be to attach another convex torus to T by plumbing a quasipositively presented annulus in the new torus to the already quasipositively presented annulus in T so that the link formed by γ_2 and γ_3 becomes a knot. Quasipositivity is preserved under plumbing quasipositive surfaces so the resulting closed genus 2 surface could be split into two subsurfaces, one a Legendrian ribbon and another which was not separated by a knot in the dividing set.

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