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Symmetric Twisted Differentials and the Quadric Algebra

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SYMMETRIC TWISTED DIFFERENTIALS AND THE QUADRIC ALGEBRA

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We study the relationships of the algebra of symmetric twisted differentials, the algebra generated by tangentially homogeneous polynomials and the quadric algebra of a smooth projective subvariety whose codimension is small relative to its dimension. It is conjectured that these three algebras coincide when the dimension of $X \subset \mathbb{P}^N$ satisfies $n > 2/3(N-1)$ and we prove this for complete intersections and subvarieties of codimension two. The connection between these three algebras leads to questions about the local projective differential geometry of X, trisecant varieties and the linear system of quadrics through X.

to my family

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CHAPTER 1

Introduction

Throughout this work $X \subset \mathbb{P}^N$ will be a smooth complex subvariety of the complex projective space $\mathbb{P}^N$. We will assume it is non-degenerate. The focus will be on X for which the codimension is small relative to the dimension, in particular, $\dim(X) = n$ will satisfy the inequality $n > 2/3(N-1)$. The goal is to investigate the connection between three apparently unrelated algebras associated with X: the algebra of symmetric twisted differentials, the algebra generated by the quadrics through X, and the algebra generated by tangentially homogeneous polynomials relative to X. It is conjectured that these algebras coincide in the dimensional range $n > 2/3(N-1)$ and we prove this for codimension one and two and for complete intersections in general. The proof of the equivalence of these three algebras leads to interesting questions about the local projective differential geometry of X as well as the classical question of the number of linearly independent quadrics through X.

1.1 Background

Let $X \subset \mathbb{P}^N$ be as above. In [BO08], Bogomolov and De Oliveira investigated the non-vanishing of the space $H^0(X, S^m[\Omega_X] \otimes \mathcal{O}_X(k))$ of *symmetric twisted differentials.*

The context for this work was the following theorem of Schneider [Sch92] from the early nineties showing that any smooth subvariety $X \subset \mathbb{P}^N$ of dimension $n > N/2$ has no symmetric differentials of order m even if twisted by $\mathcal{O}_X(k)$ for $k < m$:

Theorem 1.2 *(Schneider) Let $X \subset \mathbb{P}^N$ be a smooth projective subvariety with dimension $n > N/2$. Then if $k < m$,*

$$H^0(X, S^m[\Omega^1_X] \otimes \mathcal{O}_X(k)) = 0$$

Bogomolov and De Oliveira viewed the border case $k = m$ as special and were able to give a geometric characterization of the space $H^0(X, S^m[\Omega_X(1)])$ in the range $n > 2/3(N-1)$. Moreover, using this characterization they were able to study the local invariance of the dimensions $h^0(X, S^m\Omega_X \otimes K_X)$ in smooth families, answering a question of M. Paun.

The range $n > 2/3(N-1)$ is special in projective geometry. For instance, every smooth subvariety in this range is linearly normal [Zak93] and it is conjectured that every smooth subvariety in the range $n > 2/3N$ is a complete intersection [Har74]. The siginificance for the present work is the properties of the tangent map $\tau : \mathbb{T}X \to \mathbb{P}^N$ that appear in the range $n > 2/3(N-1)$. In particular:

Theorem 1.3 *(Bogomolov-De Oliveira) Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimension $n > 2/3(N-1)$. Then $\tau : \mathbb{T}X \to \mathbb{P}^N$ is surjective with connected fibers.*

Roughly speaking, these properties of τ allow one to pull back a special subclass of homogeneous polynomials on $\mathbb{P}^N$ to the tangent spaces of X and obtain symmetric twisted differentials on X. Their characterization of this subclass is the following theorem:

Theorem 1.4 *(Bogomolov-De Oliveira) Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimension $n > 2/3(N-1)$ then there is an isomorphism of vector spaces:*

$$H^0(X, S^m\Omega_X(1)) \cong \{P \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}^N}(m)) : V(P) \cap \mathbb{T}_x X \textit{ is a cone with vertex } x \textit{ for all } x \in X\}$$

In other words, given a homogeneous polynomial of degree m in $\mathbb{P}^N$, it pulls back to a symmetric twisted differential if and only if the intersection of its zero locus with each projective tangent space $\mathbb{T}_x X$ is a cone with vertex x. Note that an example of such a polynomial is a quadric vanishing on X and an understanding of this subspace of polynomials in general is one of the main goals of this thesis. Along these lines, they have the following result for codimension one and two:

Theorem 1.5 *(Bogomolov-DeOliveira) Let $X \subset \mathbb{P}^N$ be a smooth subvariety of codimension less than or equal two. Then*

$$H^0(X, S^m\Omega_X(1)) = 0$$

if and only if X is not contained in a quadric hypersurface.

Note that if X is a hypersurface, then this result says that X only has symmetric twisted differentials if $X = Q$ is a quadric and in which case there is a graded isomorphism of graded algebras:

$$\bigoplus_{m=0}^{\infty} H^0(Q, S^m\Omega_Q(1)) \cong \mathbb{C}[Q]$$

However, there was no corresponding isomorphism for codimension two which led to the following question which we phrase as a conjecture:

Conjecture 1.6 *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimension $n > 2/3(N-1)$ and $n > 1$ then there is a graded isomorphism of graded algebras:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \cong \mathbb{C}[Q_0, ..., Q_r]$$

where $\mathbb{C}[Q_0, ..., Q_r]$ is the $\mathbb{C}$-algebra generated by the linear system of quadrics through X.

The primary aim of this thesis is to prove this conjecture and other results surrounding it. Our main results in this direction are a verification of the conjecture for complete intersections and for codimension one and two varieties in general.

An important remark to be made is that while the the algebra $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ has historically been studied in the context of manifolds with ample cotangent bundle, and hence in the high codimensional range $n < N/2$ when the algebra is big, our work lies on the opposite side of the spectrum in low codimensions when the algebra is in some sense as small as possible. In this light, the above conjecture should be viewed as a statement about the linear system of quadrics through X and properties of the rational map they define when the codimension of X is small relative to its dimension.

1.7 Summary of Results

We introduce the notion of tangentially homogeneous polynomials and the $\mathbb{C}$-algebra they generate, $TH(X)$:

Definition 1.8 *Let $x \in X$. A degree m homogeneous polynomial P is tangentially homogeneous at x if the dehomogenization of P in a neighborhood of x is homogeneous relative to x when restricted to T_xX. It is tangentially homogeneous relative to X if it is tangentially homogeneous at every $x \in X$.*

$TH(X)$ is a subalgebra of $\mathbb{C}[Z_0, ..., Z_N]$ containing $\mathbb{C}[Q_0, ..., Q_r]$. Thus, to X we associate the three algebras $TH(X)$, $\mathbb{C}[Q_0, ..., Q_r]$ and $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$. The goal of this thesis is to investigate the relationship of these three algebras for varieties whose codimension is small relative to their dimension. Their relationship can be summarized by the following diagram where i is inclusion and τ is the tangent map of X:

$$
\begin{tikzcd}
\mathbb{C}[Q_0, ..., Q_r] \arrow[rr, "\tau^* \circ i"] \arrow[dr, hook, "i"'] & & \bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \\
& TH(X) \arrow[ur, "\tau^*"'] &
\end{tikzcd}
$$

The relationship between $TH(X)$ and $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ induced by the pull back of the tangent map τ is in fact an isomorphism in the range $n > 2/3(N-1)$:

Theorem 1.9 *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimension $n > 2/3(N-1)$. Then there is a graded isomorphism of $\mathbb{C}$-algebras:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \cong TH(X)$$

induced by the tangent map τ.

The relationship between $\mathbb{C}[Q_0, ..., Q_r]$ and $TH(X)$ is more delicate and although it is expected that the two algebras coincide in the dimensional range $n > 2/3(N-1)$, the equivalence is only currently understood for complete intersections and varieties with codimension less than or equal to two and dimension $n > 2/3(N-1)$. It is under this assumption that one can establish a correspondence between the defining equations of X and the quadrics of the projective second fundamental form at a point $x \in X$ which in turn can be used to compute the dimension of the image of τ. We have the following result:

Theorem 1.10 *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection for which the tangent map $\tau : \mathbb{T}X \to \mathbb{P}^N$ is surjective. Then there is a graded isomorphism of $\mathbb{C}$-algebras:*

$$TH(X) \cong \mathbb{C}[Q_0, ..., Q_r]$$

where $\{Q_0, ..., Q_r\}$ *is a basis for* $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$.

Note that surjectivity of τ is guaranteed in the range $n > 2/3(N-1)$ and we have the following equivalence for complete intersections:

Corollary 1.11 *Let* $X \subset \mathbb{P}^N$ *be a smooth complete intersection with dimension* $n > 2/3(N-1)$. *Then we have graded isomorphisms of the three algebras:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \cong TH(X) \cong \mathbb{C}[Q_0, ..., Q_r]$$

This result is of course weaker than one would hope since there certainly exist varieties with dimension $n > 2/3(N-1)$ that are not complete intersections. For instance, the six dimensional Grassmannian of lines in $\mathbb{P}^4$ can be embedded $G(1,4) \hookrightarrow \mathbb{P}^9$ as a non-complete intersection (it is the intersection of five quadric hypersurfaces.) Our belief in the conjecture is justified though by the following result for codimension two subvarieties:

Theorem 1.12 *Let* $X \subset \mathbb{P}^N$ *be a smooth subvariety of codimension two then we have graded isomorphisms of the three algebras*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \cong TH(X) \cong \mathbb{C}[Q_0, ..., Q_r]$$

If one believes the Hartshorne conjecture, then in light of the theorem for complete intersections, one would expect the difficulty to arise in the range $2/3(N-1) < n < 2/3N$. Indeed, much of the proof of theorem 1.12 can be reduced to the case of the Segre three-fold $\Sigma_{1,2}$, which is the image of the embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ and lies in the range $2/3(N-1) < n < 2/3N$ for $N = 5$. In this situation, one can compute the dimensions of the graded pieces of $\bigoplus_{m=0}^{\infty} H^0(\Sigma_{1,2}, S^m\Omega_{\Sigma_{1,2}}(1))$ and $\mathbb{C}[Q_0, Q_1, Q_2]$ directly:

Theorem 1.13 *Let $X = \Sigma_{1,2}$ be the Segre three-fold given by the embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ and let $\{Q_0, Q_1, Q_2\}$ be a basis for $H^0(\mathbb{P}^N, \mathcal{I}_{\Sigma_{1,2}}(2))$ Then there is a graded isomorphism*

$$\bigoplus_{m=0}^{\infty} H^0(\Sigma_{1,2}, S^m\Omega_{\Sigma_{1,2}}(1)) \cong \mathbb{C}[Q_0, Q_1, Q_2]$$

Note that since $\Sigma_{1,2}$ has dimension $n > 2/3(N-1)$ we in fact have an isomorphism of all three algebras since $TH(\Sigma_{1,2}) \cong \bigoplus_{m=0}^{\infty} H^0(\Sigma_{1,2}, S^m\Omega_{\Sigma_{1,2}}(1))$.

The proof of theorem 1.12 requires the introduction of the variety $C_X X$ of trisecant lines that are tangent and meet X in at least two distinct points. The significance of this subvariety is that tangentially homogeneous polynomials must vanish on it. Note that in general $C_X X$ is a subvariety of $S_3(X)$, the variety of trisecant lines of X. In the dimensional range $n > 2/3(N-1)$ we prove that these varieties coincide:

Theorem 1.14 *Let $X \subset \mathbb{P}^N$ be a smooth subvariety of codimension two and $n \geq 3$. Then*

$$C_X X = S_3(X)$$

The value of this arises from the fact that trisecant varieties of codimension two subvarieties are well understood, see for instance [Kwa01]. We are able to use this classification to understand tangentially homogeneous polynomials and prove theorem 4.4.

An interesting observation about this classicifation of trisecant varieties for X with $c = 2$ and $n \geq 3$ is that they always coincide with the base locus of the linear system of quadrics through X which we call the quadric envelope of X and denote

$QE(X)$. We conjecture that this holds in general for smooth subvarieties in the range $n > 2/3(N-1)$. In this direction we verify it for complete intersections:

Theorem 1.15 *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n > 2/3(N-1)$. Then*

$$S_3(X) = QE(X)$$

Our belief is that establishing this for any smooth variety with $n > 2/3(N-1)$ will allow us to drop the condition of complete intersection in theorem 4.4. Roughly speaking, it will allow us to pass from X to its quadric envelope $QE(X)$ and adapt the proof of theorem 4.4. It seems this strategy will require an understanding of when the quadric algebra is freely generated. At the moment this has only been verified for complete intersections with surjective tangent map.

CHAPTER 2

Preliminaries

In this thesis $X \subset \mathbb{P}^N$ will always mean a smooth complex subvariety of the complex projective space $\mathbb{P}^N$. We will typically use n to denote its dimension and c its codimension. We will denote by $\mathcal{I}_X$ the ideal sheaf of X and $I(X)$ the homogeneous ideal of X. We will write $I(X) = \langle F_1, ..., F_k \rangle$ where F_i are homogeneous polynomials to denote a set of generators for $I(X)$.

2.1 The Tangent Spaces T_xX, $\mathbb{T}_xX$ and $\widehat{T}_xX$

A smooth projective variety has a few different notions of tangent space. If $x \in X$ is contained in the affine open set $U \subset \mathbb{P}^N$ then $X \cap U$ is an affine subvariety of U and thus has a notion of tangent space at x. We denote this tangent space T_xX. These tangent spaces define a bundle over X which we denote TX which is isomorphic to the tangent bundle of X when considered as a complex manifold.

In addition, if we consider the Gauss map $\gamma_X : X \to G(n+1, N+1)$ and the universal subbundle $\mathcal{S} \subset G(n+1, N+1) \times \mathbb{C}^{N+1}$, then the *extended tangent bundle* *on* X is defined as:

$$\widehat{T}_X := \gamma_X^* \mathcal{S}$$

If $\hat{x} \subset \mathbb{C}^{N+1}$ is the line through x then the tangent spaces to the affine cone $\widehat{X}$ are constant along $\hat{x}$ and we define

$$\widehat{T}_x X := T_z \widehat{X}$$

where $z \in \hat{x}$ is any point on the line defined by x. The extended tangent bundle $\widehat{T}_X$ is then the bundle of these extended tangent spaces.

We define the *projective tangent bundle of* X to be the projectivization of the extended tangent bundle:

$$\mathbb{T}_X := \mathbb{P}(\widehat{T}_X)$$

The projectivizations $\mathbb{T}_x X := \mathbb{P}(\widehat{T}_x X)$ we call the projective tangent spaces of X. Note that if $I(X) = \langle F_1, ..., F_k \rangle$, the k equations $\sum_{j=0}^{N} \frac{\partial F_i}{\partial z_j}(x) z_j = 0$ define an n-dimensional linear subspace of $\mathbb{P}^N$ which is the projective tangent space $\mathbb{T}_x X$.

The relationships between the three tangent spaces $T_x X$, $\mathbb{T}_x X$ and $\widehat{T}_x X$ are as follows:

$$\mathbb{P}(\widehat{T}_x X) = \mathbb{T}_x X$$

$$\mathbb{T}_x X \cap U \cong T_x X$$

$$\widehat{T}_x X / \hat{x} \cong T_x X$$

Remark 2.2 *where* U *is an open neighborhood of* x. *The last isomorphism arises from the differential of the quotient map* $q : \mathbb{C}^{N+1} \setminus \{0\} \to \mathbb{P}^N$ *restricted to* $\widehat{T}_x X$. *Specifically, if a point* $z \in \hat{x}$ *is chosen, the differential at* z *of* q *restricted to* $\widehat{T}_x X$ *defines a map* $dq_z : \widehat{T}_x X \to T_x X$ *whose kernel is the line through* x. *In other words,* *at each $x \in X$, if one chooses $z \in \hat{x}$ we have a short exact sequence of vector spaces:*

$$0 \to \hat{x} \xrightarrow{i} \widehat{T}_x X \xrightarrow{dq_z} T_x X \to 0$$

An interesting observation to be made here is that this sequence not come from a sequence of vector bundles. In other words although for each $x \in X$ we have $\widehat{T}_x X/\hat{x} \cong T_x X$, there is no corresponding isomorphism of the vector bundles $\widehat{T}_x X/\mathcal{O}_X(-1)$ and T_X. This is because at each x, the isomorphism $\widehat{T}_x X/\hat{x} \cong T_x X$ requires a choice of $z \in \hat{x}$ and such a choice cannot be made globally as there are no non-trivial global sections of $\mathcal{O}_X(-1)$.

2.3 The Tangent Map and the Tangent Variety

The projective tangent bundle is a subbundle of the trivial vector bundle on X:

$$\mathbb{T}_X \subset X \times \mathbb{P}^N$$

and so comes with two projections:

$$\begin{tikzcd} \mathbb{T}_X \arrow[r, "\tau"] \arrow[d, "\pi"'] & \mathbb{P}^N \\ X & \end{tikzcd}$$

We call τ the *tangent map* of X. Its image is the subvariety of $\mathbb{P}^N$ swept out by the projective tangent spaces $\mathbb{T}_x X$ and is called the *tangent variety* of X which we will denote $Tan(X)$:

$$Tan(X) = \bigcup_{x \in X} \mathbb{T}_x X$$

Since $Tan(X)$ is the union of an $n-$dimensional family of $n-$dimensional linear subspaces of $\mathbb{P}^N$, the expected dimension of $Tan(X)$ is $2n$. From the tangent map τ we see that

$$\dim Tan(X) = 2n - \delta$$

where δ is the dimension of the generic fiber of τ and is typically referred to as the *tangent defect* of X. Note that for a generic variety of dimension $n > N/2$ one expects $Tan(X) = \mathbb{P}^N$ in other words, the tangent spaces of X should fill up $\mathbb{P}^N$. Moreover, one would also expect that once the dimension is "large enough" one would have $Tan(X) = \mathbb{P}^N$ for all X. This is in fact true for varieties with dimension $n > 2/3(N-2)$ and follows from two theorems of Zak, [Zak93].

Theorem 2.4 *(Zak) Let $X \subset \mathbb{P}^N$ be a smooth subvariety then one of the following holds:*

i) dim $Tan(X) = 2n$ and dim Sec(X)=2n+1

ii) Tan(X)=Sec(X)

Here $Sec(X)$ is the secant variety of X defined as the join of X with itself. One always has $Tan(X) \subset Sec(X)$ for smooth X. As a consequence of the theorem, note that if $n > N/2$ then one has $Tan(X) = Sec(X)$. Now, we also have Zak's theorem on linear normality:

Theorem 2.5 *(Zak) Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimension $n > 2/3(N-2)$. Then $Sec(X) = \mathbb{P}^N$.*

Thus, together these theorems imply that $Tan(X) = \mathbb{P}^N$ for smooth nondegenerate X with dimension $n > 2/3(N-2)$ i.e. τ is surjective. The results of this thesis rely heavily on one additional property of the tangent map τ concerning its fibers. Given $y \in \mathbb{P}^N$, we have

$$\tau^{-1}(y) = \{(x, y) : y \in \mathbb{T}_x X\}$$

and so $\tau^{-1}(y)$ projects via π to the subvariety on X where the tangent spaces pass through y. The following theorem says that in the dimensional range $n > 2/3(N-1)$ this subvariety is connected for general $y \in \mathbb{P}^N$:

Theorem 2.6 *(Bogomolov- De Oliveira) Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimension $n > 2/3(N-1)$ and $n > 1$. Then $\tau : \mathbb{T}_X \to \mathbb{P}^N$ is surjective with connected fibers.*

This geometry is one of our main tools for understanding symmetric twisted differentials. It allows us to pull back a certain subclass of homogeneous polynomials on $\mathbb{P}^N$ to sections of $S^m\Omega_X(1)$.

2.7 Symmetric Twisted Differentials: $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$

2.7.1 Symmetric powers of a vector bundle

Let $\pi : E \to X$ be a vector bundle over X of rank r. We can define a new vector bundle S^mE whose fiber $(S^mE)_x$ over a point $x \in X$ is the m-th symmetric power of E_x. If $\{U_i\}$ is an open cover of X and $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n$ trivializations, then φ_i induces trivializations $\tilde{\varphi}_i : S^mE|_{U_i} \to U_i \times \mathbb{C}^{\binom{n+1+m}{m}}$. Note that if $\varphi_1 \in H^0(X, S^{m_1}E)$

and $\varphi_2 \in H^0(X, S^{m_2}E)$ then there is a product $\varphi_1\varphi_2 \in H^0(X, S^{m_1+m_2}E)$. This multiplication gives us a graded algebra of global sections

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m E)$$

2.7.2 The projective bundle of a vector bundle

Let $\mathbb{P}(E) \to X$ denote the projective bundle of lines of E i.e. the $\mathbb{P}^{r-1}$ bundle over X whose fiber over $x \in X$ is the projective space of lines through the origin of E_x. $\mathbb{P}(E)$ has a tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(E)}(1) \arrow[d] & & \\
\mathbb{P}(E) \arrow[dr, "\pi"] & & S^m E \arrow[dl, "\pi"'] \\
& X &
\end{tikzcd}$$

and we have isomorphisms induced via pushforward:

$$\pi_* \mathcal{O}_{\mathbb{P}(E)}(m) \cong S^m E \tag{2.1}$$

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \cong H^0(X, S^m E) \tag{2.2}$$

2.7.3 Section ring of a line bundle

Let L be a line bundle over X and $L^{\otimes m}$ the $m-th$ tensor power of L. If $s_1 \in H^0(X, L^{\otimes m_1})$ and $s_2 \in H^0(X, L^{\otimes m_2})$ then $s_1 \otimes s_2 \in H^0(X, L^{\otimes m_1+m_2})$ giving rise a to graded $\mathbb{C}$-algebra referred to as the section ring of L:

$$\bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$$

For the case $L = \mathcal{O}_{\mathbb{P}(E)}(1)$, the isomorphism 2.2 gives:

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m E) \cong \bigoplus_{m=0}^{\infty} H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \tag{2.3}$$

where $\bigoplus_{m=0}^{\infty} H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m))$ is the section ring of the line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$.

2.7.4 Iitaka dimension of a line bundle

For each m, the complete linear system $|L^{\otimes m}|$ defines a rational mapping

$$\phi_m : X \dashrightarrow \mathbb{P}(H^0(X, L^{\otimes m}))$$

The Iitaka dimension of L is defined to be the maximal dimension of the images of these rational maps:

$$\kappa(X, L) := max\{dim(\phi_m(X))\}$$

Note that one always has $\kappa(X, L) \leq dim(X)$. The Iitaka dimension measures the asymptotic growth of the dimensions $h^0(X, L^{\otimes m})$ in the following sense: set $\kappa = \kappa(X, L)$ then there exist constants $a, A > 0$ such that

$$a \cdot m^{\kappa} \leq h^0(X, L^{\otimes m}) \leq A \cdot m^{\kappa}$$

and we view this as a measure of the growth of the section ring $\bigoplus_{m=0}^{\infty} H^0(X, L^{\otimes m})$. In this way, by setting $L = \mathcal{O}_{\mathbb{P}(E)}(1)$, we obtain a notion of the growth of the graded algebra $\bigoplus_{m=0}^{\infty} H^0(X, S^m E)$ via the isomorphism 2.3 and the Iitaka dimension of $\mathcal{O}_{\mathbb{P}(E)}(1)$.

2.7.5 The algebra of symmetric twisted differentials

We now consider the above ideas for the vector bundle $\Omega_X(1) := \Omega_X \otimes \mathcal{O}_X(1)$ where Ω_X is the cotangent bundle of X. An element of the vector space $H^0(X, S^m\Omega_X(1))$ we call a symmetric twisted differential of degree m. As above we have the corresponding graded algebra:

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$$

which we call the algebra of symmetric twisted differentials. We have the isomorphism

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \cong \bigoplus_{m=0}^{\infty} H^0(\mathbb{P}(\Omega_X(1)), \mathcal{O}_{\mathbb{P}(\Omega_X(1))}(m))$$

and we measure the growth of this algebra by the Iitaka dimension of the line bundle $\mathcal{O}_{\mathbb{P}(\Omega_X(1))}(1)$, $\kappa(\mathcal{O}_{\mathbb{P}(\Omega_X(1))}(1))$.

2.7.6 Examples

Example 2.8 *Let $l : \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ be a line. Then $\Omega_l = \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\mathcal{O}_l(1) = \mathcal{O}_{\mathbb{P}^1}(1)$. Hence $S^m\Omega_l(1) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $H^0(l, S^m\Omega_l(1)) = 0$ for all $m > 0$. In other words, lines do not have nontrivial symmetric twisted differentials and we have*

$$\bigoplus_{m=0}^{\infty} H^0(l, S^m\Omega_l(1)) = 0$$

with $\kappa = -\infty$.

Example 2.9 *Consider a smooth quadric plane curve $Q \cong \mathbb{P}^1 \subset \mathbb{P}^2$. We have $\mathcal{O}_Q(1) = \mathcal{O}_{\mathbb{P}^1}(2)$ and $\Omega_Q = \mathcal{O}_{\mathbb{P}^1}(-2)$. Thus $H^0(Q, S^m\Omega_Q(1)) = H^0(Q, S^m[\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(2)]) = H^0(Q, \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}$. In other words, the only symmetric twisted differentials on smooth conics are constants and we have*

$$\bigoplus_{m=0}^{\infty} H^0(Q, S^m\Omega_X(1)) = \mathbb{C}$$

with $\kappa = 0$.

Example 2.10 *More generally, consider a smooth plane curve C of degree $d > 2$. $\mathcal{O}_C(1)$ will be a degree d line bundle on C and the genus of C will be $g = \frac{1}{2}(d-1)(d-2)$ making Ω_C a line bundle of degree $(d-1)(d-2)-2$. Thus $S^m\Omega_C(1)$ is a line bundle of degree $m((d-1)(d-2)-2+d) = m(d^2-2d) = md(d-2)$. By Riemann-Roch, the dimensions $h^0(C, S^m\Omega_C(1))$ grow like $md(d-2)$ i.e. $\kappa = 1$ for $d > 2$.*

Example 2.11 *Consider the twisted cubic* $\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^3$ *since* C *is a degree three curve with genus zero. We have* $\Omega_C = \mathcal{O}_{\mathbb{P}^1}(-2)$ *and* $\mathcal{O}_C(1) = \mathcal{O}_{\mathbb{P}^1}(3)$. *Thus,* $S^m\Omega_C(1) = \mathcal{O}_{\mathbb{P}^1}(m)$. *Since* $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = \binom{m+1}{1}$ *we have*

$$\bigoplus_{m=0}^{\infty} H^0(C, S^m\Omega_C(1)) \cong \mathbb{C}[x, y]$$

with $\kappa = 1$.

Note that what makes these computations possible is that Ω_X is a line bundle when X a curve and so $S^m\Omega_X(1)$ is a line bundle as well. When $n > 1$ these computations become much more difficult. In chapter five we compute $S^m\Omega_X(1)$ explicitly for a quadric hypersurface in $\mathbb{P}^3$ and the segre three fold in $\mathbb{P}^5$.

2.12 Tangentially Homogeneous Polynomials: $TH(X)$

Let $\{U_i\}$ be the open covering of $\mathbb{P}^N$ where $U_i \cong \mathbb{C}^N$ is defined by the equation $z_i \neq 0$. If P is a homogeneous polynomial of degree m on $\mathbb{C}^{N+1}$ then $\frac{P}{z_i^m}$ defines a polynomial function on U_i. We refer to $\frac{P}{z_i^m}$ as the *dehomogenization of* P in the neighborhood U_i. We will often denote this dehomogenization as $\tilde{P}$. $\tilde{P}$ can be expanded about the point x and we denote this expansion $\tilde{P}^x$.

Definition 2.13 *Let* $x \in X$. *A degree* m *homogeneous polynomial* P *is tangentially homogeneous at* x *if the dehomogenization of* P *in an open neighborhood of* x *is homogeneous relative to* x *when restricted to* T_xX. *It is tangentially homogeneous relative to* X *if it tangentially homogeneous at every* $x \in X$.

The vector space of all tangentially homogenous polynomials relative to X of degree m is denoted by $TH^{(m)}(X)$ and the graded algebra generated by tangentially homogeneous polynomials relative to X we denote $TH(X)$:

$$TH(X) := \bigoplus_{m=0}^{\infty} TH^{(m)}(X)$$

The following are two basic facts about tangentially homogeneous polynomials used repeatedly throughout this work.

Let $X \subset \mathbb{P}^N$ be a smooth subvariety and $P \in TH^{(m)}(X)$ with $m \neq 0$. Then $P \in I(X)$.

Proof: Let $P \in TH^{(m)}(X)$. Since P is homogeneous of degree m on $T_x X$ in particular it vanishes at x and we have $X \subset V(P)$. Thus $P \in I(X)$. ∎

Let $X \subset P^N$ be a smooth subvariety whose tangent map is surjective. If $P, Q \in \mathbb{C}[X_0, ..., X_N]$ are so that $PQ \in TH(X)$, then P and Q are both in $TH(X)$.

Proof: Without loss of generality suppose $\tilde{P}^x \mid_{T_x X}$ was not homogeneous at the general point $x \in X$. Note that since $Tan(X) = \mathbb{P}^N$, we know that Q does not vanish on $\mathbb{T}_x X$. It follows that $\widetilde{PQ}^x \mid_{T_x X} = \tilde{P}^x \mid_{T_x X} \cdot \tilde{Q}^x \mid_{T_x X}$ is not homogeneous since the product of a non-zero inhomogeneous polynomial with a non-zero polynomial cannot be homogeneous.

∎

2.13.1 Examples

Example 2.14 *Constant polynomials are tautologically tangentially homogenous relative to any $X \subset \mathbb{P}^N$ and form the zeroth graded piece of $TH(X)$:*

$$TH^{(0)}(X) = \mathbb{C}$$

Example 2.15 *The homogeneous polynomials F in the ideal of the tangent variety of X i.e. $F \in I(Tan(X))$ are tangentially homogeneous relative to X.*

Example 2.16 *A linear polynomial L is tangentially homogeneous relative to X if and only if $X \subset V(L)$.*

Example 2.17 *The key example of tangentially homogeneous polynomials that are not constant on the tangent variety $Tan(X)$, are the quadratic polynomials $Q \in I(X)$. This holds, since $\forall x \in X$, $\tilde{Q}^x|^{(0)}_{T_xX} = 0$ and $\tilde{Q}^x|^{(1)}_{T_xX} = 0$ making $\tilde{Q}^x|_{T_xX}$ homogeneous of degree two.*

Remark 2.18 *If X is such that the tangent map τ is not surjective then the algebra $TH(X)$ is not finitely generated. For instance, if X is a smooth quadric inside the hyperplane $\{X_0 = 0\} \subset \mathbb{P}^4$ then $TH(X) = (X_0) + \mathbb{C}[Q]$ which is not finitely generated.*

2.19 The Quadric Algebra: $\mathbb{C}[Q_0, ..., Q_r]$

Consider the ideal sheaf sequence for X twisted by $\mathcal{O}_{\mathbb{P}^N}(2)$:

$$0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbb{P}^N}(2) \to \mathcal{O}_X(2) \to 0$$

and the corresponding long exact sequence of cohomology:

$$0 \to H^0(\mathbb{P}^N, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(2)) \to H^0(X, \mathcal{O}_X(2)) \to \cdots$$

The vector space $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$ corresponds to degree two homogeneous polynomials that vanish on X. We will typically use the letter r to denote the dimension the projectivization of this vector space:

$$r = h^0(\mathbb{P}^N, \mathcal{I}_X(2)) - 1$$

The vector space $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$ defines a linear system on $\mathbb{P}^N$ and thus a rational map:

$$\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^r$$

We also define a subalgebra of $\mathbb{C}[Z_0, ..., Z_N]$ generated by the quadric polynomials vanishing on X:

Definition 2.20 *Let $X \subset \mathbb{P}^N$ be a projective subvariety and $\{Q_0, ..., Q_r\}$ a basis for $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$. These quadrics generate a graded algebra $\mathbb{C}[Q_0, ..., Q_r]$ which we refer to as the quadric algebra of X. Note that this definition does not depend on a choice of basis for $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$.*

Since $\mathbb{C}[Q_0, ..., Q_r]$ is finitely generated there exists some ideal $I \subset \mathbb{C}[x_0, ..., x_r]$ such that $\mathbb{C}[Q_0, ..., Q_r] \cong \mathbb{C}[x_0, ..., x_r]/I$. If we consider the rational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^r$ then $\mathbb{C}[Q_0, ..., Q_r]$ is equal to the homogeneous coordinate ring of the image $\phi(\mathbb{P}^N)$ viewed as a $\mathbb{C}$-algebra. In other words, there is a graded isomorphism of $\mathbb{C}$-algebras:

$$\mathbb{C}[Q_0, ..., Q_r] \cong \mathbb{C}[x_0, ..., x_r]/I(\phi(\mathbb{P}^N)) \tag{2.4}$$

where $I(\phi(\mathbb{P}^N))$ is the ideal of the image of the rational map ϕ.

We can decompose $\mathbb{C}[Q_0, ..., Q_r]$ into graded components:

$$\mathbb{C}[Q_0, ..., Q_r] = \bigoplus_{m \in 2\mathbb{Z}_{\geq 0}} S^{\frac{m}{2}}[\mathbb{C}Q_0 \oplus \cdots \oplus \mathbb{C}Q_r]$$

and there is function $\mathbb{N} \to \mathbb{N}$ associating to each m the dimension of the vector space $S^{\frac{m}{2}}[\mathbb{C}Q_0 \oplus \cdots \oplus \mathbb{C}Q_r]$. By the isomorphism 2.4, this is just the hilbert function of the subvariety $\phi_(\mathbb{P}^N)$. Recall that this function is equal to a polynomial in m for large m where the degree of the polynomial is equal to the dimension of the image $\phi(\mathbb{P}^N)$. In other words, for large m the algebra $\mathbb{C}[Q_0, ..., Q_r]$ grows like m^n where n is the dimension of $\phi(\mathbb{P}^N)$.

2.20.1 Examples

Example 2.21 *Consider a line $l \subset \mathbb{P}^3$. The vector space $H^0(\mathbb{P}^3, \mathcal{I}_l(2))$ is seven-dimensional, spanned by say, $\{Q_0, ..., Q_6\}$. The linear system spanned by these quadrics defines a rational map $\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^6$. The image is a projection of the degree two Veronese embedding $\nu_2 : \mathbb{P}^3 \hookrightarrow \mathbb{P}^9$, the map defined by the complete linear system of quadrics on $\mathbb{P}^3$. Thus $\phi(\mathbb{P}^3)$ is the blowup of $\mathbb{P}^3$ along l and so has dimension 3. Thus the dimension of the graded pieces of $\mathbb{C}[Q_0, ..., Q_6]$ grow like m^3 for large m.*

Example 2.22 *As a more interesting example consider the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. The image of this embedding is denoted $\Sigma_{1,2}$. The linear system of quadrics* *in* $\mathbb{P}^5$ *through* $\Sigma_{1,2}$ *is spanned by the three quadrics*

$$Q_0 = z_0 z_4 - z_1 z_3 \quad Q_1 = z_1 z_5 - z_2 z_4 \quad Q_2 = z_0 z_5 - z_2 z_3$$

which defines a rational map $\phi : \mathbb{P}^5 \dashrightarrow \mathbb{P}^2$. *We would like to compute the dimension of the image of this map. To do this we need to compute the rank of the jacobian matrix at a generic point. Using the explicit equations listed above we compute the rank at the point* $(0,0,0,1,0)$ *to be* 2. *Since rank is lower semicontinuous, there is an open neighborhood of* $(0,0,0,1,0)$ *such that the rank is two. Thus the rational map* ϕ *is dominant and the homogeneous coordinate ring of the image is isomorphic to* $\mathbb{C}[z_0, z_1, z_2]$. *In other words, the quadrics* $\{Q_0, Q_1, Q_2\}$ *through* $\Sigma_{1,2}$ *do not satisfy any polynomial relations and the quadric algebra is the free algebra generated by* $\{Q_0, Q_1, Q_2\}$*:*

$$\mathbb{C}[Q_0, Q_1, Q_2] \cong \mathbb{C}[Z_0, Z_1, Z_2]$$

Example 2.23 *Consider the linear system of quadrics through the twisted cubic in* $\mathbb{P}^3$. *They define a dominant rational map* $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. *Thus, these quadrics generate a free algebra.*

An interesting question is under what conditions $\mathbb{C}[Q_0, ..., Q_r]$ is free. Note that this is equivalent to the rational map $\phi : \mathbb{P}^N \dashrightarrow \mathbb{P}^r$ defined by $\{Q_0, ..., Q_r\}$ being dominant. We prove later that the quadric algebra of a smooth complete intersection X with $Tan(X) = \mathbb{P}^N$ has a free quadric algebra. Briefly, the idea is that any polynomial relation of the $\{Q_0, ..., Q_r\}$ restricts to a polynomial relation of $\{Q_0|_{T_xX}, ..., Q_r|_{T_xX}\}$. However, since X is a complete intersection these quadrics define a sub system of the linear system of quadrics defining the second fundamental form at x and hence cannot satisfy any polynomial relations by lemma 4.5.

CHAPTER 3

Symmetric Twisted Differentials and Tangentially Homogeneous Polynomials

Using theorem 2.6 Bogomolov and De Oliveira in [BO08] gave a geometric characterization of symmetric one-twisted differentials. In this section we give an alternative characterization of these differentials identifying them with homogeneous polynomials which are tangentially homogeneous relative to X. Following [BO08] we consider first the bundle $\widetilde{\Omega}_X(1)$ of differential one-forms on the affine cone $\widehat{X}$ and the corresponding symmetric differentials $H^0(X, S^m\widetilde{\Omega}_X(1))$. Homogeneous polynomials on $\mathbb{P}^N$ can be pulled back via the tangent map to sections of $S^m\widetilde{\Omega}_X(1)$ and bijectivity of this pullback is guaranteed when τ is both surjective and connected. Thus in this context $H^0(X, S^m\widetilde{\Omega}_X(1))$ can be identified with homogeneous polynomials on $\mathbb{P}^N$ and we try to understand the inclusion $H^0(X, S^m\Omega_X(1)) \hookrightarrow H^0(X, S^m\widetilde{\Omega}_X(1))$ arising from the Euler sequence on X.

3.1 Preliminaries

3.1.1 Symmetric Powers and Projective Bundles

Recall that for a vector bundle $\pi : E \to X$ we have isomorphisms

$$\pi^* \mathcal{O}_{\mathbb{P}(E)}(m) \cong S^m E \tag{3.1}$$

$$H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \cong H^0(X, S^m E) \tag{3.2}$$

These isomorphisms will allow us to work with line bundles instead of vector bundles and to see the tangent map in our calculations. We will need the following basic fact about projective bundles:

Lemma 3.2 *Let $\pi_1 : E \to X$ be a vector bundle over X and $\pi_2 : L \to X$ a line bundle over X. Then there is a natural isomorphism of projective bundles $\phi : \mathbb{P}(E \otimes L) \xrightarrow{\cong} \mathbb{P}(E)$.*

The tangent map will be used to pull back the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$ and its sections to the tangent bundle $\mathbb{P}(\widehat{T}_X)$ and we will need the following lemma:

Lemma 3.3 *Let $f : X \to Y$ be a morphism and L a line bundle on Y. Then f induces a map of sections $f^* : H^0(Y, L) \to H^0(X, f^* L)$. Furthermore, if f is both surjective and connected then f^* induces an isomorphism of sections.*

Proof: By the projection formula we have:

$$f_*(f^*L) \cong L \otimes f_*\mathcal{O}_X$$

If f is both surjective and connected then $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ and the previous isomorphism becomes

$$f_*(f^*L) \cong L$$

Thus $H^0(X, f^*L) \cong H^0(Y, f_*(f^*L)) \cong H^0(Y, L)$. ■

Morphisms $X \to \mathbb{P}^N$ are equivalent to the specification of a line bundle L on X. If E is a vector bundle on X then the analogue of this for morphisms $Y \to \mathbb{P}(E)$ is the folowing. Let $p : Y \to X$. Then giving a line subbundle $L \hookrightarrow p^*E$ of the pullback of E is equivalent to specifying a map $f : Y \to \mathbb{P}(E)$ over X:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathbb{P}(E) \arrow[dl, "\pi"] \\ & X & \end{tikzcd}$$

Under this correspondence $L = f^*\mathcal{O}_{\mathbb{P}(E)}(1)$. The following is a special case of this that arises frequently in our arguments.

Lemma 3.4 *Let $f : E \to F$ be a morphism of vector bundles on X. Then there is an induced rational map $f : \mathbb{P}(E) \to \mathbb{P}(F)$ of projective bundles defined outside the projective bundle $\mathbb{P}(ker\ f) \subset \mathbb{P}(E)$. Moreover, we have $f^*\mathcal{O}_{\mathbb{P}(F)}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$.*

Proof: Since f is a morphism of vector bundles we have the commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow{id} & X \end{array}$$

and thus the induced maps $f_x : E_x \to F_x$ on the fibers where f_x are linear maps of vector spaces for all $x \in X$. Linearity implies that the f_x define rational maps $f_x : \mathbb{P}(E_x) \dashrightarrow \mathbb{P}(F_x)$ defined outside ker f_x. Thus f induces a map $f : \mathbb{P}(E) \dashrightarrow \mathbb{P}(F)$ defined outside the projectivization of the kernel bundle, $\mathbb{P}(\ker f)$ such that the diagram

$$\begin{array}{ccc} \mathbb{P}(E)/\mathbb{P}(\ker\ f) & \xrightarrow{f} & \mathbb{P}(F) \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow{id} & X \end{array}$$

commutes. Now consider the pullback of the tautological line bundle $f^*\mathcal{O}_{\mathbb{P}(F)}(1)$ to $\mathbb{P}(E)$. In a fiber, f was induced by a linear map $E_x \to F_x$ and thus $f_x : \mathbb{P}(E) \dashrightarrow \mathbb{P}(F)$ is either an inclusion of a linear subspace into a projective space or a linear projection from one projective space to a linear subspace. In either case, we see that the pullback f^* restricted to a fiber gives an isomorphism $f_x^*\mathcal{O}_{\mathbb{P}(F)}(1)|_{F_x} \cong \mathcal{O}_{\mathbb{P}(E)}(1)|_{E_x}$ and thus $f_x^*\mathcal{O}_{\mathbb{P}(F)}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$.

■

Let E be a vector bundle on X which is a subbundle of $\bigoplus^{N+1} L$ where L is a line bundle on X,

$$0 \to E \xrightarrow{i} \bigoplus^{N+1} L$$

Let $\mathbb{P}(E)$ and $\mathbb{P}(\bigoplus^{N+1} L)$ be the corresponding projective bundles. The map i induces an inclusion:

$$i : \mathbb{P}(E) \hookrightarrow \mathbb{P}(\bigoplus^{N+1} L)$$

and an isomorphism

$$i^*\mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} L)}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$$

via lemma 3.4. By lemma 3.2 there is a natural isomorphism $\phi : \mathbb{P}(\bigoplus^{N+1} L) \to \mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)$ for which $\phi^*\mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} \mathcal{O})}(1) \cong \mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} L)}(1) \otimes \pi^* L^{-1}$.

The projective bundle $\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)$ is the product $X \times \mathbb{P}^N$, if p_2 denotes the projection onto the secnod factor, then $\mathcal{O}_{\mathbb{P}(\bigoplus^{N+1} \mathcal{O}_X)}(1) \cong p_2^*\mathcal{O}_{\mathbb{P}^N}(1)$. Concluding, the inclusion i induces a map $f_i = p_2 \circ \phi \circ i : \mathbb{P}(E) \to \mathbb{P}^N$ and the isomorphism

$$f_i^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L^{-1}$$

As in lemma 3.3, f_q induces a map on sections

$$f_i^* : H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^* L^{-1})$$

where injectivity and surjectivity of f_i^* depend upon surjectivity and connectivity of f_i. An important special case for us is the following.

Let E be a vector bundle on a smooth projective variety X. If E is a subbundle of a trivial vector bundle

$$0 \to E \xrightarrow{i} \bigoplus^{N+1} \mathcal{O}_X$$

and the induced map $f_i : \mathbb{P}(E) \to \mathbb{P}^N$ is surjective with connected fibers, then f_i induces an isomorphism

$$H^0(X, S^m E) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$$

via the pullback f_i^*. Indeed, since the map $f_i : \mathbb{P}(E) \to \mathbb{P}^N$ arises from the inclusion i of vector bundles, we have an isomorphism $\mathcal{O}_{\mathbb{P}(E)}(m) \cong f_i^*\mathcal{O}_{\mathbb{P}^N}(m)$ by lemma 3.4. Moreover, since f_i is both surjective and connected, by lemma 3.3 there is an isomorphism $H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$ which by proposition **??** is equivalent to an isomorphism

$$H^0(X, S^m E) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$$

To make this more concrete, if we choose a basis $\{e_0, ..., e_N\}$ for $\mathbb{C}^{N+1}$ this gives a global frame $\{s_0, ..., s_N\}$ for the trivial bundle $X \times \mathbb{C}^{N+1}$. A section of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$ which we view as an element of $S^m[\mathbb{C}e_0^* \oplus \cdots \oplus \mathbb{C}e_N^*]$ then pulls back via the projection $p_2 : X \times \mathbb{P}^N \to \mathbb{P}^N$ to an element of $H^0(X \times \mathbb{P}^N, \mathcal{O}_{X \times \mathbb{P}^N}(m))$ which we view as an element of $S^m[\mathbb{C}s_0^* \oplus \cdots \oplus \mathbb{C}s_N^*]$. Now, the dual map $q : X \times \mathbb{C}^{N+1} \to E^* \to 0$ maps this element to an element of $S^m[\mathbb{C}q(s_0^*) \oplus \cdots \oplus \mathbb{C}q(s_N^*)]$ which we finally view as an element of $H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m))$.

3.4.1 The Euler Sequence on X

The relationship between $\Omega_X(1)$ and $\widetilde{\Omega}(1)$ and the inclusion $H^0(X, S^m\Omega_X(1)) \subset H^0(X, S^m\widetilde{\Omega}(1))$ arises from the Euler sequence on X and many of our arguments in general come from the geometric idea it encodes. Let us first recall the construction of this sequence on $\mathbb{P}^N$. Let $G := G(k, V)$ denote the Grassmannian of k-dimensional subspaces of a vector space V over $\mathbb{C}$ and $\mathcal{V} := G \times V$ the trivial vector bundle on G where the fiber over a point $[\Lambda] \in G$ is the vector space V. We denote by $\mathcal{S}$ the subbundle of $\mathcal{V}$ whose fiber $\mathcal{S}_{[\Lambda]}$ over a point $[\Lambda]$ is the subspace Λ itself. $\mathcal{S}$ is called the universal subbundle on G. The quotient $\mathcal{Q} := \mathcal{V}/S$ is called the universal quotient bundle. Note that for the case $k = 1$ we have $G = \mathbb{P}(V)$ and $\mathcal{S} = \mathcal{O}_{\mathbb{P}(V)}(-1)$. The relavance of these ideas for us comes from the following fact expressing the tangent bundle of G in terms of $\mathcal{S}$ and $\mathcal{Q}$.

Theorem 3.5 *The tangent bundle T_G to the Grassmannian $G := G(k, V)$ is isomorphic to $Hom_G(\mathcal{S}, \mathcal{Q})$.*

See for example, [cite]. Suppose now that $G = \mathbb{P}^N$ and consider the universal sequence

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-1) \to \bigoplus^{N+1} \mathcal{O}_{\mathbb{P}^N} \to \mathcal{Q} \to 0$$

and tensor with $\mathcal{O}_{\mathbb{P}^N}(1)$ to obtain

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \bigoplus^{N+1} \mathcal{O}_{\mathbb{P}^N}(1) \to \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{Q} \to 0$$

By the identification $T_{\mathbb{P}^N} \cong \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^N}(-1), \mathcal{Q})) \cong \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{Q}$ we arrive at the *euler sequence*

$$0 \to \mathcal{O}_{\mathbb{P}^N} \to \bigoplus^{N+1} \mathcal{O}_{\mathbb{P}^N}(1) \to T_{\mathbb{P}^N} \to 0$$

This sequence identifies an element of $\hat{x}^* \otimes \mathbb{C}^{N+1}$ with an element of $T_x\mathbb{P}^N$ via the identification $T_x\mathbb{P}^N \cong \mathrm{Hom}(\hat{x}, \mathbb{C}^{N+1}/\hat{x}) \cong \hat{x}^* \otimes \mathbb{C}^{N+1}/\hat{x}$. An element of $\hat{x}^* \otimes \mathbb{C}^{N+1}$ can be thought of as vector field on $\mathbb{C}^{N+1}$ that descends to $\mathbb{P}^N$ via the quotient $\mathbb{C}^{N+1} \backslash \{0\} \to \mathbb{P}^N$ and the kernel of this correspondence consists of the radial or *euler* fields on $\mathbb{C}^{N+1}$.

We will also need the dual sequence:

$$0 \to \Omega_{\mathbb{P}^N} \to \bigoplus^{N+1} \mathcal{O}_{\mathbb{P}^N}(-1) \to \mathcal{O}_{\mathbb{P}^N} \to 0$$

Now suppose $X \subset \mathbb{P}^N$ is a smooth projective subvariety of dimension n. The Euler sequences can be restricted to X:

$$0 \to \mathcal{O}_X \to \bigoplus^{N+1} \mathcal{O}_X(1) \to T_{\mathbb{P}^N}|_X \to 0$$

$$0 \to \Omega_{\mathbb{P}^N}|_X \to \bigoplus^{N+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$$

We can also consider the euler sequence *on* X. Corresponding to theorem 3.5 we have the following characterization of the tangent bundle of X:

Lemma 3.6 *Let $X \subset \mathbb{P}^N$ be a smooth subvariety. Then $T_X \cong Hom(\mathcal{O}_X(-1), \widehat{T}_X/\mathcal{O}_X(-1))$.*

Now, corresponding to the univsersal subbundle $0 \to \mathcal{O}_X(-1) \to \bigoplus^{N+1} \mathcal{O}_{\mathbb{P}^N}$, we have on X the subbundle $0 \to \mathcal{O}_X(-1) \to \widehat{T}_X$ and a universal sequence on X:

$$0 \to \mathcal{O}_X(-1) \to \widehat{T}_X \to \mathcal{Q}_X \to 0$$

By the lemma we have $T_X = \mathrm{Hom}(\mathcal{O}_X(-1), \mathcal{Q}_X) \cong \mathcal{O}_X(1) \otimes \mathcal{Q}_X$ and so by tensoring the above sequence by $\mathcal{O}_X(1)$ we obtain the *euler sequence on X*

$$0 \to \mathcal{O}_X \to \widehat{T}_X \otimes \mathcal{O}_X(1) \to T_X \to 0$$

and its dual

$$0 \to \Omega_X \to \widehat{\Omega}_X \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$$

We define $\widetilde{T}_X := \widehat{T}_X \otimes \mathcal{O}_X(1)$ and $\widetilde{\Omega}_X := \widehat{\Omega}_X \otimes \mathcal{O}_X(-1)$ and rewrite these sequences:

$$0 \to \mathcal{O}_X \to \widetilde{T}_X \to T_X \to 0$$

and

$$0 \to \Omega_X \to \widetilde{\Omega}_X \to \mathcal{O}_X \to 0$$

3.6.1 Conormal Sequence on X

Given a smooth projective subvariety $X \subset \mathbb{P}^N$ we have the normal sequence

$$0 \to T_X \to T_{\mathbb{P}^N}|_X \to N_{X/\mathbb{P}^N} \to 0$$

arising from the pushforward of the inclusion $X \hookrightarrow \mathbb{P}^N$. Moreover, if we pull back the universal sequence

$$0 \to \mathcal{S} \to \mathcal{V} \to \mathcal{Q} \to 0$$

via the gauss map γ we have $0 \to \widehat{T}_X \to \bigoplus^{N+1} \mathcal{O}_X \to \gamma^*\mathcal{Q} \to 0$ and after twisting by $\mathcal{O}_X(1)$:

$$0 \to \widetilde{T}_X \to \bigoplus^{N+1} \mathcal{O}_X(1) \to \gamma^*\mathcal{Q} \otimes \mathcal{O}_X(1) \to 0$$

3.6.2 A Fundamental Commutative Diagram

Putting everything together we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & \widetilde{\mathcal{T}}_X \arrow[r] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d] & \bigoplus^{N+1} \mathcal{O}_X(1) \arrow[r] \arrow[d] & \mathcal{T}_{\mathbb{P}^N}|_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \gamma^*\mathcal{Q} \otimes \mathcal{O}_X(1) \arrow[r] \arrow[d] & \mathcal{N}_{X/\mathbb{P}^N} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Note that this implies $\gamma^*\mathcal{Q} \otimes \mathcal{O}_X(1) \cong N_{X/\mathbb{P}^N}$ and we arrive at the following fundamental commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] \arrow[d, "id"'] & \widetilde{\mathcal{T}}_X \arrow[r] \arrow[d] & \mathcal{T}_X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X \arrow[r] & \bigoplus^{N+1} \mathcal{O}_X(1) \arrow[r] \arrow[d] & \mathcal{T}_{\mathbb{P}^N}|_X \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{N}_{X/\mathbb{P}^N} \arrow[r, "id"'] \arrow[d] & \mathcal{N}_{X/\mathbb{P}^N} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

as well as its dual diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N^*_{X/\mathbb{P}^N} \arrow[r, "id"'] \arrow[d] & N^*_{X/\mathbb{P}^N} \arrow[d] & & \\
0 \arrow[r] & \Omega_{\mathbb{P}^N}|_X \arrow[r] \arrow[d] & \bigoplus^{N+1} \mathcal{O}_X(-1) \arrow[r, "c"'] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \Omega_X \arrow[r, "e"'] \arrow[d] & \widetilde{\Omega}_X \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Once we twist this diagram by $\mathcal{O}_X(1)$ we will be able to understand $H^0(X, S^m\widetilde{\Omega}_X(1))$ via the surjection $\bigoplus^{N+1} \mathcal{O}_X(-1) \to \widetilde{\Omega}_X(1) \to 0$ and consequently $H^0(X, S^m\Omega_X(1))$ via the inclusion $0 \to \Omega_X \to \widetilde{\Omega}_X$.

3.7 Theorem for $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ and $TH(X)$

In this section we establish the equivalence of the algebra of symmetric twisted differentials and the algebra generated by tangentially homogeneous polynomials for smooth subvarieties with dimension $n > 2/3(N-1)$.

Theorem 3.8 *Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth subvariety with dimension satisfying $n > 2/3(N-1)$ and $n > 1$. Then there is a graded isomorphism of algebras induced by the tangent map:*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq TH(X)$$

Proof: By twisting the above commutative diagram by $\mathcal{O}_X(1)$ we obtain:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & N^*_{X/\mathbb{P}^N}(1) \arrow[r, "id"'] \arrow[d] & N^*_{X/\mathbb{P}^N}(1) \arrow[d] & & \\
0 \arrow[r] & \Omega_{\mathbb{P}^N}|_X(1) \arrow[r] \arrow[d] & \bigoplus^{N+1} \mathcal{O}_X \arrow[r, "c"'] \arrow[d] & \mathcal{O}_X(1) \arrow[r] \arrow[d, "id"] & 0 \\
0 \arrow[r] & \Omega_X(1) \arrow[r, "e"'] \arrow[d] & \widetilde{\Omega}_X(1) \arrow[r] \arrow[d] & \mathcal{O}_X(1) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and we see that we have an inclusion

$$H^0(X, S^m\Omega_X(1)) \subset H^0(X, S^m\widetilde{\Omega}_X(1))$$

To understand this inclusion we will projectivize the vector bundles $\Omega_X(1)$ and $\widetilde{\Omega}_X(1)$ to obtain

$$H^0(\mathbb{P}(T_X(-1)), \mathcal{O}_{\mathbb{P}(T_X(-1))}(m)) \subset H^0(\mathbb{P}(\widetilde{T}_X(-1)), \mathcal{O}_{\mathbb{P}(\widetilde{T}_X(-1))}(m))$$

Moreover, if we make the identifications $\widetilde{T}_X(-1)) \cong \widehat{T}_X$ and $\mathbb{P}(\widehat{T}_X) = \mathbb{T}_X$, then this inclusion becomes:

$$H^0(\mathbb{P}(T_X(-1)), \mathcal{O}_{\mathbb{P}(T_X(-1))}(m)) \subset H^0(\mathbb{T}_X, \mathcal{O}_{\mathbb{T}_X}(m))$$

The relevant commutative diagram is then dualization of the one above:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_X(-1) \arrow[r] \arrow[d, "id"'] & \widehat{T}_X \arrow[r] \arrow[d] & \mathcal{T}_X(-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-1) \arrow[r] & \bigoplus^{N+1} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{T}_{\mathbb{P}^N}|_X(-1) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{N}_{X/\mathbb{P}^N}(-1) \arrow[r, "id"'] \arrow[d] & \mathcal{N}_{X/\mathbb{P}^N}(-1) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

The projectivization of the inclusion $\widehat{T}_X \to \bigoplus^{N+1} \mathcal{O}_X$ is the natural inclusion of projective bundles:

$$0 \to \mathbb{T}_X \xrightarrow{i} X \times \mathbb{P}^N$$

Since this map was induced by a map of vector bundles, by lemma 3.4 we have $i^*\mathcal{O}_{X\times\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{T}_X}(1)$. If we denote the second projection $p_2 : X \times \mathbb{P}^N \to \mathbb{P}^N$ then we also have $p_2^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{X\times\mathbb{P}^N}(1)$ since in each fiber of $X \times \mathbb{P}^N$, p_2 is the identity map $id : \mathbb{P}^N \to \mathbb{P}^N$. Lastly, by composing i with p_2 we obtain the tangent map $\tau : \mathbb{T}_X \to \mathbb{P}^N$ and the pullback of τ induces the isomorphism $\tau^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{T}_X}(1)$. Now, our assumption is that the dimension n of X satisfies the inequality $n > 2/3(N-1)$ and thus by theorem 2.6, τ is surjective with connected fibers. Hence, by lemma 3.3, τ induces the isomorphism of global sections:

$$\tau^* : H^0(\mathbb{T}_X, \mathcal{O}_{\mathbb{T}_X}(m)) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \tag{3.3}$$

In other words, $H^0(X, S^m\widetilde{\Omega}_X(1)) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$. To summarize, this isomorphism is obtained by pulling back homogeneous polynomials of degree m on $\mathbb{C}^{N+1}$ to the bundle of tangent spaces $\widehat{T}X$ via the inclusion $0 \to \widehat{T}_X \to X \times \mathbb{C}^{N+1}$.

The idea now is to understand which of these polynomials descend via the Euler map $\widehat{T}_X \to T_X(-1)$. In other words, we would like to understand the image of the inclusion :

$$p^* : H^0(\mathbb{P}(T_X(-1)), \mathcal{O}_{\mathbb{P}(T_X(-1))}(m)) \hookrightarrow H^0(\mathbb{T}_X, \mathcal{O}_{\mathbb{T}_X}(m))$$

arising from the pullback of the map p:

$$0 \to \mathcal{O}_X(-1) \to \widehat{T}_X \xrightarrow{p} T_X(-1) \to 0 \tag{3.4}$$

Let P be an element in this image. By the isomorphism 3.3, P is a homogeneous polynomial of degree m on $\mathbb{P}^N$. In a fiber over $x \in X$ the exact sequence 3.4 becomes

$$0 \to \hat{x} \to \widehat{T}_x X \to T_x X \to 0$$

and we see that $P|_{\widehat{T}_x X}$ is in fact a homogeneous polynomial on the quotient $\widehat{T}_x X/\hat{x} \cong T_x X$. After a linear change of coordinates we can assume that $\widehat{T}_x X$ is spanned by $\{\frac{\partial}{\partial z_0}, ..., \frac{\partial}{\partial z_n}\}$ and that $\hat{x} = \langle \frac{\partial}{\partial z_0} \rangle$. It follows that $P|_{\widehat{T}_x X}$ is a homogeneous polynomial in the variables $\{dz_1, ..., dz_n\}$. Since $T_x X = U_0 \cap \mathbb{T}_x X$ where $U_0 \subset \mathbb{P}^N$ is defined by the equation $z_0 \neq 0$, $P|_{\widehat{T}_x X}$ defines a function $\frac{P|_{\widehat{T}_x X}}{z_0^m}$ on the open subset $T_x X \subset \mathbb{T}_x X$. Since, $P|_{\widehat{T}_x X}$ did not involve the variable dz_0, we obtain a homogeneous polynomial on $T_x X$ which is exactly the dehomogenization of the restriction $P|_{\widehat{T}_x X}$ in the open set $\mathbb{T}_x X \cap U_0$. Now, by the commutative diagram:

$$\begin{array}{ccc} \mathbb{T}_x X & \hookrightarrow & \mathbb{P}^N \\ \uparrow & & \uparrow \\ T_x X & \hookrightarrow & U_0 \end{array}$$

if we instead pull back P via the chain of inclusions $T_xX \hookrightarrow U_0 \hookrightarrow \mathbb{P}^N$ we still obtain something homogeneous and so P is tangentially homogeneous at x in the sense of definition 2.13 i.e. $P \in TH(X)$.

∎

Remark 3.9 *We see from the proof that the pullback of a tangentially homogeneous polynomial P to each $\widehat{T}_xX$ descends to a homogeneous polynomial on the quotient $\widehat{T}_xX/\hat{x}$. Moreover, once a point $z \in \hat{x}$ is chosen we can identify $\widehat{T}_xX/\hat{x}$ with the tangent space T_xX and we see P is in fact homogeneous on T_xX. Note, however, that there can be no global choice of $z \in \hat{x}$ as there are no non-trivial global sections of $\mathcal{O}_X(-1)$. In other words, although $T_xX \cong \widehat{T}_xX/\hat{x}$ there is no corresponding isomorphism of vector bundles $T_X \cong \widehat{T}_X/\mathcal{O}_X(-1)$. Thus, almost paradoxically, a tangentially homogeneous polynomial pulls back to a homogeneous polynomial on each tangent space T_xX but does not pull back to a homogeneous polynomial on the bundle T_X i.e. a section of $H^0(\mathbb{P}(T_X), \mathcal{O}_{\mathbb{P}(T_X)}(m))$.*

In the next chapter we turn to question of characterizing $TH(X)$. While it is clear that $\mathbb{C}[Q_0, ..., Q_r] \subset TH(X)$, $TH(X)$ can in general be larger. For instance, we saw in chapter one that curves of degree greater than two in $\mathbb{P}^2$ have many symmetric twisted differentials. Surprisingly, the relationship between $\mathbb{C}[Q_0, ..., Q_r]$ and $TH(X)$ is also connected to the tangent map τ and the tangent variety $Tan(X)$.

CHAPTER 4

Tangentially Homogeneous Polynomials and the Quadric Algebra

In the previous section the equivalence of the algebra of symmetric twisted differentials and the algebra generated by tangentially homogeneous polynomials was established. What is needed now is an understanding of the connection between the algebra generated by tangentially homogeneous polynomials and the algebra generated by quadrics vanishing on X. The goal of this section is to establish this connection for complete intersections. Recall the following definition:

Definition 4.1 *A subvariety $X \subset \mathbb{P}^N$ of dimension $n \geq 1$ is a complete intersection if there exist $N-n$ homogeneous polynomials $f_i \in \mathbb{C}[Z_0, ..., Z_N]_{d_i}$ of degree $d_i \geq 1$, generating the homogeneous ideal $I(X) \subset \mathbb{C}[Z_0, ..., Z_N]$.*

We'll see that the assumption of complete intersection allows us to conclude local properties of the defining equations of X. The framework for doing this is the connection between the projective second fundamental form of X and the dimension of the tangent variety $Tan(X)$. Roughly speaking, the dimension of the tangent variety is determined by how much X bends at a general point x, which in turn is determined by the number of algebraic relations in the degree two parts of the local equations of X restricted to the tangent space T_xX.

4.2 Preliminaries

4.2.1 The Projective Second Fundamental Form

We will need a detailed understanding of the projective second fundamental form II_x of X at a point $x \in X$. Let us begin by recalling the necessary background.

Let $G(k, N)$ be the Grassmannian of k-planes in $\mathbb{C}^N$ and let $V \in G(k, N)$. Recall that there is an identification $T_V G(k, N) \cong \mathrm{Hom}(V, \mathbb{C}^N/V)$. Loosely speaking, tangential movement away from V is equivalent to the specification of a normal vector for each $v \in V$. To make this identification explicit, let $\alpha(t)$ be a curve in $G(k, N)$ with $\alpha(0) = V$. We would like to show how this curve defines an element $\varphi_\alpha \in \mathrm{Hom}(V, \mathbb{C}^N/V)$. Let $v \in V$ and let $\beta(t)$ be a curve in $\mathbb{C}^N$ such that $\beta(t) \in \alpha(t)$ for all t and $\beta(0) = v$. Then we define $\varphi_\alpha(v)$ to be the image of $\frac{d}{dt}\beta(0)$ under the quotient map $\mathbb{C}^N \to \mathbb{C}^N/V$. The correspondence $\alpha \mapsto \varphi_\alpha$ is well defined since if another curve $\beta'(t)$ is chosen then $\beta(t) - \beta'(t) = u(t) \in \alpha(t)$. Since $u(0) = \beta(0) - \beta'(0) = v - v = 0$, we can write $u(t) = t\tilde{u}(t)$ and so $\frac{d}{dt}\beta(0) - \beta'(t) = \tilde{u}(0) \in V$.

To define the projective second fundamental form of a projective subvariety $X \subset \mathbb{P}^N$ at a point $x \in X$ we apply these ideas to the Gauss map $\gamma : X \to G(n+1, N+1)$ which associates to each $x \in X$ the $n+1$-dimensional subspace $\widehat{T}_xX$. At x we have the differential of this map

$$d\gamma(x) : T_xX \to T_{\widehat{T}_xX} G(n+1, N+1)$$

Using the identification $T_V G(k, n) \cong \mathrm{Hom}(V, \mathbb{C}^n/V)$ we can rewrite this map

$$d\gamma(x) : T_xX \to \mathrm{Hom}(\widehat{T}_xX, \mathbb{C}^{N+1}/\widehat{T}_xX)$$

We can make this more explicit in the following way. Without loss of generality assume $x \in U_0 \subset \mathbb{P}^N$. We can choose coordinates $(z_0, ..., z_N)$ on $\mathbb{C}^{N+1}$ so that $x = [1 : 0 : \cdots : 0]$ and $\widehat{T}_xX$ is spanned by $\{\frac{\partial}{\partial z_c}, ..., \frac{\partial}{\partial z_N}\}$. Let $\{\frac{\partial}{\partial x_{c+1}}, ..., \frac{\partial}{\partial x_N}\}$ be a basis for T_xX. We can locally parameterize X by T_xX:

$$f : (x_{c+1}, ..., x_N) \mapsto (f_1, ..., f_c, x_{c+1}, ..., x_N) \in X \cap U_0$$

where $f_i \in \mathcal{O}(T_xX)$ and $f(0) = x$. The tangent space $T_{f(x')}X$ is spanned by:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_i}(x') \\ \vdots \\ \frac{\partial f_c}{\partial x_i}(x') \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad for \quad i = c+1, ..., N$$

We can also view f this as a map into $\hat{X}$ via the map $U_0 \hookrightarrow \{z_0 = 1\} \subset \mathbb{C}^{N+1}$:

$$f : (x_{c+1}, ..., x_N) \mapsto (f_1, ..., f_c, 1, x_{c+1}, ..., x_N) \in \widehat{X}.$$

The tangent space $\widehat{T}_{f(x')}X$ for $x' = (x_{c+1}, ..., x_N) \in T_xX$ is then spanned by:

$$\frac{\partial}{\partial z_c}(x') := \begin{bmatrix} f_1(x') \\ \vdots \\ f_c(x') \\ 1 \\ x_{c+1} \\ \vdots \\ x_N \end{bmatrix} \quad and \quad \frac{\partial}{\partial z_i}(x') := \begin{bmatrix} \frac{\partial f_1}{\partial x_i}(x') \\ \vdots \\ \frac{\partial f_c}{\partial x_i}(x') \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad for \quad i = c+1, ..., N$$

Moreover, we can choose the parameterization f so that $\frac{\partial}{\partial z_i}(0) = \frac{\partial}{\partial z_i}$. To understand the image of a tangent vector $\frac{\partial}{\partial x_i} \in T_x X$, $i = c+1, ..., N$, under the differential $d\gamma(x)$, we take a holomorphic curve $\alpha_i(t) \in X \cap U_0$ such that $\frac{d}{dt}\alpha_i(0) = \frac{\partial}{\partial x_i}$. For instance, we can take α_i to be the composition $\alpha_i : \Delta \to T_x X \to X \cap U_0$ given by:

$$\alpha : t \mapsto te_i \mapsto f(te_i) \in X$$

where $e_i = \frac{\partial}{\partial x_i} \in T_x X$. The image of this curve $\gamma(\alpha_i(t)) \subset Gr(n+1, N+1)$ is the one-dimensional family of tangent spaces $\{\widehat{T}_{\alpha_i(t)} X\}$ and $d\gamma(x)(\frac{\partial}{\partial x_i}) = \frac{d}{dt}\gamma \circ \alpha_i(0) \in T$. To understand $\varphi_i := \frac{d}{dt}\gamma \circ \alpha_i(0) \in \mathrm{Hom}(\widehat{T}_x X, \mathbb{C}^{N+1}/\widehat{T}_x X)$, we need to say how φ_i acts on the vectors $\frac{\partial}{\partial z_j} \in \widehat{T}_x X$ for $j = c, ..., N$. As explained in the first paragraph of this section, we need to choose a curve $\beta_j(t)$ in $\mathbb{C}^{N+1}$ such that $\beta_j(t) \in \widehat{T}_{\alpha_i(t)} X$ and

$\beta_j(0) = \frac{\partial}{\partial z_j}$. For $j = c+1, ..., N$ we can take the curve:

$$\beta_j(t) = \begin{bmatrix} \frac{\partial f_1}{\partial x_j}(te_i) \\ \vdots \\ \frac{\partial f_c}{\partial x_j}(te_i) \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{N+1}$$

and we have

$$\varphi_i(\frac{\partial}{\partial z_j}) = \frac{d}{dt}\beta_j(0) = \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_i x_j}(0) \\ \vdots \\ \frac{\partial^2 f_c}{\partial x_i x_j}(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{N+1}/\widehat{T}_x X \tag{4.1}$$

Now, for $j = c$ we have the vector $\frac{\partial}{\partial z_c} \in \widehat{T}_x X$ and we can take the curve:

$$\beta_c(t) = \begin{bmatrix} f_1(te_i) \\ \vdots \\ f_c(te_i) \\ 1 \\ \vdots \\ t \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{N+1}$$

and we have

$$\varphi_i(\frac{\partial}{\partial z_c}) = \frac{d}{dt}\beta_c(0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_i}(0) \\ \vdots \\ \frac{\partial f_c}{\partial x_i}(0) \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \widehat{T}_x X$$

In other words, the tangent vector pointing in the direction of the line $\hat{x}$ is in the kernel of every homomorphism $\varphi_i \in Hom(\widehat{T}_x X, \mathbb{C}^{N+1}/\widehat{T}_x X)$ in the image of $d\gamma(x)$. Thus, the differential gives a map $d\gamma(x) : T_x X \to \mathrm{Hom}(\widehat{T}_x X/\hat{x}, \mathbb{C}^{N+1}/\widehat{T}_x X)$ and with identifications $\widehat{T}_x X/\hat{x} \cong T_x X$ and $\mathbb{C}^{N+1}/\widehat{T}_x X \cong N_x X$ a map

$$d\gamma(x) : T_x X \to \mathrm{Hom}(T_x X, N_x X)$$

Summarizing, the image of a tangent vector v is the specification of the normal component of first order motion of each point in T_xX as you move along X in the tangential direction v. Now, with the identification $\mathrm{Hom}(T_xX, N_xX) \cong T_xX^\vee \otimes N_xX$ we arrive at a bilinear map

$$II_x : T_xX \otimes T_xX \to N_xX$$

known as the projective second fundamental form of X at x where given a pair of tangent directions $u \otimes v$, II_x specifies how v moves in the normal direction when one moves tangentially along X in the direction of u. By equation 4.2.1 we have

$$\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} \mapsto \begin{bmatrix} \frac{\partial^2 f_1}{\partial x_i \partial x_j}(0) \\ \vdots \\ \frac{\partial^2 f_c}{\partial x_i \partial x_j}(0) \end{bmatrix} \in N_xX \tag{4.2}$$

for $i, j \in \{c+1, ..., N\}$. For $v, u \in T_xX$ we have

$$v \otimes u \mapsto \begin{bmatrix} \sum_{i,j}(u_i v_j + u_j v_i)\frac{\partial^2 f_1}{\partial x_i \partial j_j}(x) \\ \vdots \\ \sum_{i,j}(u_i v_j + u_j v_i)\frac{\partial^2 f_c}{\partial x_i \partial j_j}(x) \end{bmatrix} \in N_xX \tag{4.3}$$

Note that by the implicit function theorem the functions $f_1, ..., f_c$ are holomorphic and we have equality of mixed partial second derivatives. Thus $II_x : T_xX \otimes T_xX \to N_xX$ is a *symmetric* bilinear form:

$$II_x : S^2 T_xX \to N_xX$$

4.2.2 The Second Fundamental Form and the Tangent Variety

The relevance of II_x for us comes from an observation originally due to Terracini relating II_x with the dimension of the tangent variety $Tan(X)$:

Let $X \subset \mathbb{P}^N$ be a smooth subvariety and $\tau : \mathbb{T}X \to \mathbb{P}^N$ the tangent map. Let $(x, l) \in \mathbb{T}_X$ be a point on a generic fiber of τ and $\hat{v}$ any vector in the direction of l. Let $v \in \widehat{T}_x X/\hat{x}$ be its image in the quotient. Then $\dim(\tau(\mathbb{T}X)) = 2n - \dim \ker II_x(v, \_)$ where $(x, \tilde{v})$ is a point on a generic fiber.

Proof: This can be proved using the method of moving frames, see [GH79]. However, the full force of this theory is not needed and we give an alternative proof here.

The dimension of $\tau(X)$ is $2n$ minus the dimension of a generic fiber of τ and the dimension of a fiber is equal to the dimension of the kernel of the differential at a point along the fiber. Let $(x, l) \in \mathbb{T}X \subset X \times \mathbb{P}^N$ be a point along a generic fiber of τ. We have the following commutative diagram:

$$\begin{array}{ccc} \widehat{T}_X & \xrightarrow{\hat{\tau}} & \mathbb{C}^{N+1} \\ \downarrow & & \downarrow \\ \mathbb{P}(\widehat{T}_X) & \xrightarrow{\tau} & \mathbb{P}^N \end{array}$$

Note that $T_{(x,l)}\mathbb{T}X \cong T_x X \oplus T_x X$. Suppose $u \oplus w \in \ker d\tau_{(x,l)}$ and let $\beta(t) \subset \mathbb{T}_X$ such that $\beta(0) = (x, l)$ and $\frac{d}{dt}\beta(0) = u \oplus w$. Let $\hat{\beta} \subset \widehat{T}_X$ be a lifting of β such that $\hat{\beta}(0) = (x, \hat{v})$. We have

$$\pi_2 \circ \hat{\tau} \circ \hat{\beta}(t) = \tau \circ \beta(t)$$

and

$$\frac{d}{dt}\pi_2 \circ \hat{\tau} \circ \hat{\beta}(0) = \frac{d}{dt}\tau \circ \beta(0) = 0$$

which gives

$$d\pi_2(\hat{v})(\frac{d}{dt}\hat{\tau} \circ \hat{\beta}(0))) = 0.$$

and so $\frac{d}{dt}\hat{\tau} \circ \hat{\beta}(0)$ points in the direction of $\hat{v}$ and in particular $\frac{d}{dt}\hat{\tau} \circ \hat{\beta}(0) \in \widehat{T}_x X$. Thus $II_x(v, w) = 0$ i.e. $w \in ker(II_x(v, \_))$. Since τ is injective on fibers of $\mathbb{T}_X$, we have:

$$\text{ker } d\tau_{(x,l)} \cong d\pi_{(x,l)}(\text{ker } d\tau_{(x,l)}) \subset ker(II_x(v\_))$$

Thus, $dim\, ker(d\tau_{(x,l)}) \leq dim\, ker(II_x(v, \_))$.

Now suppose $w \in \text{ker} II_x(v, \_)$ and choose a curve $\beta(t) \subset \mathbb{T}_X$ through (x, l) such that $d(\beta'(0)) = 0 \oplus w$ and let $\hat{\beta}$ be a lifting of β such that $\hat{\beta}(0) = \hat{v}$. As before we have:

$$\frac{d}{dt}\tau \circ \beta(0) = \frac{d}{dt}\pi_2 \circ \hat{\tau} \circ \hat{\beta}(0) = d\pi_2(\hat{v})(\frac{d}{dt}\hat{\tau} \circ \hat{\beta}(0))) = d\pi_2(\hat{v})(II_x(w, v))$$

Since $II_x(v, w) = 0 \in N_x X$, $\frac{d}{dt}\hat{\tau} \circ \hat{\beta}(0)) \in \widehat{T}_x X$, say $\frac{d}{dt}\hat{\tau} \circ \hat{\beta}(0)) = \hat{u}$. Let $u = d\pi_2(\hat{v})(\hat{u})$ then we have $\frac{d}{dt}\tau \circ \beta(0) = u \in T_x X$. We can adjust our original curve so that $\beta'(0) = -u \oplus w$. Then by linearity of the derivative we have $\frac{d}{dt}\tau \circ \beta(0) = -u + u = 0$ and we've established $\text{ker } II_x(v, \_) \subset d\pi_{(x,l)}(\text{ker } d\tau_{(x,l)})$. Thus

$$dim\, ker(d\tau_{(x,l)}) \geq dim\, ker(II_x(v, \_))$$

■

4.2.3 The Second Fundamental Form and the Local Defining Equations of X

We will need a characterization of proposition 4.2.2 in terms of the generators of the local ideal of X. To do this we consider the dual map $II_x^* : N_x^*X \to S^2T_x^*X$. By equation 4.2, the image of II_x^* is spanned by forms

$$\left\{ \sum_{i,j=c+1}^{N} \frac{\partial^2 f_1}{\partial x_i \partial x_j}(0) dx_i dx_j, ..., \sum_{i,j=c+1}^{N} \frac{\partial^2 f_c}{\partial x_i \partial x_j}(0) dx_i dx_j \right\}$$

where $f : (x_{c+1}, ..., x_N) \mapsto (f_1, ..., f_c, x_{c+1}, ..., x_N)$ is a local parameterization of X. Thus the image of II_x^* is spanned by the quadrics $\{f_1^{(2)}, ..., f_c^{(2)}\}$ on X. Note that if F_k is a local defining equation of X then $dF_k = \sum_{i=1}^{c} \frac{\partial F_k}{\partial x_i}(x) dx_i \in N_x^*X$ and the image under II_x^* is

$$\frac{\partial F_k}{\partial x_1}(x) \cdot \sum_{i,j=c+1}^{N} \frac{\partial^2 f_1}{\partial x_i \partial x_j}(0) dx_i dx_j + \cdots + \frac{\partial F_k}{\partial x_c}(x) \cdot \sum_{i,j=c+1}^{N} \frac{\partial^2 f_c}{\partial x_i \partial x_j}(0) dx_i dx_j \tag{4.4}$$

Consider the the expansion of F_k about x:

$$F_k = F_k(x) + \sum_{i=1}^{N} \frac{\partial F_k}{\partial x_i}(x) x_i + \sum_{i,j=1}^{N} \frac{\partial^2 F_k}{\partial x_i \partial x_j}(x) x_i x_j + \cdots$$

If we denote $\psi_{F_k} = \sum_{i=1}^{N} \frac{\partial F_k}{\partial x_i}(x) x_i$, then equation 4.4 is the degree two part of the composition $\psi_{F_k} \circ f$ i.e. $\psi_{F_k}|_X^{(2)}$.

If $\{F_1, ..., F_c\}$ are local generators of the ideal of X then since $\{[dF_1, ..., dF_c\}$ is a basis for N_x^*X, we see that the image of II_X^* is spanned by the quadrics:

$$\{\psi_{F_1}|_X^{(2)}, ..., \psi_{F_c}|_X^{(2)}\} \subset H^0(\mathbb{P}(T_xX), \mathcal{O}_{\mathbb{P}(T_xX)}(2)) \tag{4.5}$$

We define $|II_x|$ to be the linear system on T_xX defined by these quadrics and $ii_x : \mathbb{P}(T_xX) \dashrightarrow \mathbb{P}(N_xX)$ the rational map defined by this linear system. We have the following proposition relating ii_x and II_x:

Let $x \in X$ then II_xX is the derivative of ii_x. More precisely $dii_x(v)(\_) = \frac{1}{2}II_x(v, \_)$ for a tangent vector $v \in T_xX$.

Proof: First, we have $ii_x(v) = II_x(v, v)$:

$$\begin{aligned}
ii_x(v) &= \left[\sum_{i,j=c+1}^{N} v_i v_j \frac{\partial f_1}{\partial x_i \partial x_j}, ..., \sum_{i,j=c+1}^{N} v_i v_j \frac{\partial f_c}{\partial x_i \partial x_j} \right] \\
&= \left[\sum_{i,j=c+1}^{N} \frac{1}{2}(v_i v_j + v_j v_i) \frac{\partial f_1}{\partial x_i \partial x_j}, ..., \sum_{i,j=c+1}^{N} \frac{1}{2}(v_i v_j + v_j v_i) \frac{\partial f_c}{\partial x_i \partial x_j} \right] \\
&= \frac{1}{2} II_x(v, v)
\end{aligned}$$

Now, let $v(t)$ be a curve in T_xX with $v(0) = v$. We compute the derivative of $ii_x(v(t))$ at $t = 0$:

$$
\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} ii_x(v(t)) &= \frac{d}{dt}\bigg|_{t=0} II_x(v(t), v(t)) \\
&= \frac{d}{dt}|_{t=0}\left[\sum_{i,j=c+1}^{N} v_i(t)v_j(t)\frac{\partial f_1}{\partial x_i \partial x_j}, ..., \sum_{i,j=c+1}^{N} v_i(t)v_j(t)\frac{\partial f_c}{\partial x_i \partial x_j}\right] \\
&= \left[\sum_{i,j=c+1}^{N} (v_i'(0)v_j(0) + v_i(0)v_j'(0))\frac{\partial f_1}{\partial x_i \partial x_j}, ..., \sum_{i,j=c+1}^{N} (v_i'(0)v_j(0) + v_i(0)v_j'(0))\frac{\partial f_c}{\partial x_i \partial x_j}\right] \\
&= \left[\sum_{i,j=c+1}^{N} v_i'(0)v_j(0)\frac{\partial f_1}{\partial x_i \partial x_j}, ..., \sum_{i,j=c+1}^{N} v_i'(0)v_j(0)\frac{\partial f_c}{\partial x_i \partial x_j}\right] \\
&+ \left[\sum_{i,j=c+1}^{N} v_i(0)v_j'(0)\frac{\partial f_1}{\partial x_i \partial x_j}, ..., \sum_{i,j=c+1}^{N} v_i(0)v_j'(0)\frac{\partial f_c}{\partial x_i \partial x_j}\right] \\
&= 2II_x(v(0), v'(0))
\end{aligned}
$$

■

4.3 Theorem for $TH(X)$ and $\mathbb{C}[Q_0, ..., Q_r]$

In this section we use the local differential geometry described above to establish an equivalence of tangentially homogeneous polynomials and the quadric algebra for complete intersections.

Theorem 4.4 *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate complete intersection with $Tan(X) = \mathbb{P}^N$. Then:*

$$TH(X) \cong \mathbb{C}[Q_0, ..., Q_r]$$

Proof: Let $c = \text{codim}(X)$ and X be of multi-degree $(d_1, ..., d_c)$, $d_1 \geq d_2 \geq ... \geq d_c$ where $I(X) = (F_1, ..., F_c)$ and $\deg F_i = d_i$. With this notation we aim to show that if $P \in \mathbb{C}[X_0, ..., X_N]^h_{TX}$, then $P \in \mathbb{C}[F_k, ..., F_c]$, where $k = \min\{i \mid \; d_i = 2\}$. Note that $\{F_k, ..., F_c\}$ form a basis for $H^0(\mathbb{P}^N, I_X(2))$. This follows from the fact that X is non-degenerate and hence there are no degree one generators.

Let $P \in \mathbb{C}[X_0, ..., X_N]^h_{TX}$ have degree d. By proposition 2.12 we have $P \in I(X)$. This allows us the following representation of P in terms of the defining equations of X:

$$P = \sum_{(i_1,...,i_c) \in I} G_{i_1...i_c} F_1^{i_1} ... F_c^{i_c} \tag{4.6}$$

where I is some finite index set, $G_{i_1...i_c} \notin I(X)$ and $\deg(G_{i_1...i_c}) = d - (i_1 d_1 + \cdots + i_c d_c)$. Indeed, since $P \in I(X)$, there exist homogeneous polynomials $G_1, ..., G_c$ such that $P = G_1 F_1 + \cdots G_c F_c$. Now, if $G_i \in I(X)$ then it can again be split using $F_1, ..., F_c$. By iterating this process we arrive at equation 4.6 in a finite number of steps.

Now, we would like to use this representation to understand what the condition of being tangentially homogeneous imposes on P at a point $x \in X$. To do this, we should consider the Taylor expansion at x in $T_x X$ of the dehomogenization $\tilde{P}$. Since $\tilde{F}_i^x|^{(0)}_{T_x X} = 0$ and $\tilde{F}_i^x|^{(1)}_{T_x X} = 0$ $\forall i \in \{1, ..., c\}$, we have

$$P|_{T_x X} = \sum_{i_1 + \cdots + i_c = ld(P)/2} G^{(0)}_{i_1...i_c}|_{T_x X} (F_1|^{(2)}_{T_x X})^{i_1} ... (F_c|^{(2)}_{T_x X})^{i_c} + \text{higher order terms}$$

where $ld(P) = 2\min\{i_1 + \cdots + i_c | (i_1, ..., i_c) \in I\}$ is the lowest degree present in the expansion of $\tilde{P}^x|_{T_xX}$. To proceed we show that the collection $\{F_1^x|_{T_xX}^{(2)}, ..., F_c^x|_{T_xX}^{(2)}\} \subset S^2[(T_xX)^*] = H^0(\mathbb{P}_l(T_xX), \mathcal{O}(2))$ is algebraically independent.

Lemma 4.5 *Let $X^{(n)} \subset \mathbb{P}^N$ be a non-degenerate complete intersection, $X = V(F_1, ..., F_c)$, with $n > 2/3(N-2)$. Then if $x \in X$ is general the collection:*

$$\{\tilde{F_1}^x|_{T_xX}^{(2)}, ..., \tilde{F_c}^x|_{T_xX}^{(2)}\} \subset S^2[(T_xX)^*] = H^0(\mathbb{P}_l(T_xX), O(2))$$

is algebraically independent.

Proof: (Proof of Lemma) At each $x \in X$ we have the projective second fundamental form arising from the differential of the Gauss map:

$$II_x : S^2[T_xX] \to N_xX$$

The functions $\tilde{F_1}^x, ..., \tilde{F_c}^x$ generate the local ring of X and hence the differentials $d\tilde{F_1}^x, ..., d\tilde{F_c}^x$ give a basis for the conormal space N_x^*X. Thus the image of the dual map $II_x^* : N_x^*X \to S^2T_x^*X$ has image spanned by the quadrics $\{\psi_{\tilde{F_1^x}}|_X^{(2)}, ..., \psi_{\tilde{F_c^x}}|_X^{(2)}\}$ in T_xX (by 4.5.) As explained in the previous section, these quadrics define a linear system $|II_x|$ and a rational map $ii_x : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{c-1}$. Note that for each defining equation F_i, we have $\tilde{F_i}^x|_X \equiv 0$ in a neighborhood of x. In particular, the degree two part, $\tilde{F_i}^x|_X^{(2)}$, must vanish and we have

$$\psi_{\tilde{F_i^x}}|_X^{(2)} + \tilde{F_i}^x|_{T_xX}^{(2)} \equiv 0$$

Thus $\psi_{\tilde{F_i}^x}|_X^{(2)} = -\tilde{F_i}^x|_{T_xX}^{(2)}$ and the set of quadrics $\{\tilde{F_1}^x|_{T_xX}^{(2)}, ..., \tilde{F_c}^x|_{T_xX}^{(2)}\}$ defines the same linear system on T_xX as $\{\psi_{\tilde{F_i}^x}|_X^{(2)}, ..., \psi_{\tilde{F_c}^x}|_X^{(2)}\}$ and hence define the same rational map $ii_x : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$. Now by propositions 4.2.2 and 4.2.3,

$$N = 2n - \dim \ker(dii_x(v)) \tag{4.7}$$

for generic x and $v \in T_xX$ since the dimensional hypothesis $n > 2/3(N-2)$ implies surjectivity of the tangent map τ. Suppose $\{\tilde{F}_1|_{T_xX}^{(2)}, ..., \tilde{F}_c|_{T_xX}^{(2)}\}$ were algebraically dependent, i.e. $H(\tilde{F}_1|_{T_xX}^{(2)}, ..., \tilde{F}_c|_{T_xX}^{(2)}) \equiv 0$ for some homogeneous polynomial H of c variables. It follows that $ii_x(\mathbb{P}^{n-1})$ is contained in the hypersurface H and in particular, dim $(ii_x(\mathbb{P}^{n-1})) < c-1$ or equivalently dim $ii_x^{-1}(y) > n-c$ for a generic $y \in \mathbb{P}^{c-1}$. By equation 4.7, we would then have the contradiction

$$N = 2n - \dim\, ii_x^{-1}(y) < 2n - (n-c) = N$$

■

To proceed with the proof of the theorem we conclude from the lemma that:

$$\sum_{i_1+\cdots+i_c=ld(P)/2} G^{(0)}_{i_1...i_c}|_{T_xX}(F_1|_{T_xX}^{(2)})^{i_1}...(F_c|_{T_xX}^{(2)})^{i_c} \not\equiv 0$$

since $\{\tilde{F_1}^x|_{T_xX}^{(2)}, ..., \tilde{F_c}^x|_{T_xX}^{(2)}\}$ are algebraically independent at the general x and the defining condition $G_{i_1...i_c} \notin I(X)$ forces $\tilde{G}^x_{i_1...i_c}|_{T_xX}^{(0)} \neq 0$ at general $x \in X$. The assumption that P is tangentially homogeneous means that $\tilde{P}^x|_{T_xX}$ is homogeneous of degree d, and so:

$$d = ld(P)$$

Moreover, if $(i_1, ..., i_c) \in I$ then

$$d = 2 \cdot \frac{ld(P)}{2} \leq 2(i_1 + \cdots + i_c) \leq d$$

Thus $i_1+\cdots+i_c = ld(P)/2$ for all $(i_1, ..., i_c) \in I$ and we have both $d_1 i_1+\cdots+d_c i_c \leq d$ and $2i_1 + \cdots + 2i_c = d$. Since $d_k = \cdots d_c = 2$ we obtain

$$(d_1 - 2)i_1 + \cdots + (d_{k-1} - 2)i_{k-1} \leq 0$$

Since $d_i - 2 > 0$ for $i = 1, ..., k-1$ we conclude $i_1 = \cdots = i_{k-1} = 0$ for $(i_1, ..., i_c) \in I$. In other words,

$$P = \sum_{i_k+...+i_c=d/2} c_{i_k...i_c} F_k^{i_k} ... F_c^{i_c} \in \mathbb{C}[F_k, ..., F_c]$$

as desired.

■

4.6 Freeness of the Quadric Algebra

An interesting consequence of lemma 4.5 is that the quadric algebra $\mathbb{C}[Q_0, ..., Q_r]$ of a smooth complete intersection subvariety with $Tan(X) = \mathbb{P}^N$, must be free:

Theorem 4.7 *Let $X \subset \mathbb{P}^N$ be smooth with $Tan(X) = \mathbb{P}^N$. Then $\mathbb{C}[Q_0, ..., Q_r]$ is free.*

Proof: Since $Tan(X) = \mathbb{P}^N$, X must be non-degenerate. It follows that the homogeneous ideal of X contains no degree one polynomials and thus any generating set for I_X must contain a basis for $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$. Suppose $\{Q_0, ..., Q_r\}$ is such a basis. These quadrics generate $\mathcal{Q}_X$ and so if $\mathcal{Q}_X$ was not free, we would have at least one polynomial relation in the Q_i and moreover, this relation would pull back to a polynomial relation of the restrictions $Q_i|_{T_xX}$ on the generic T_xX. However, the $Q_i|_{T_xX}$ extend to a basis of $|II_x|$ at x since they are linearly independent on T_xX. This contradicts lemma 4.5. ■

CHAPTER 5

Symmetric Twisted Differentials and the Quadric Algebra

To summarize what we have so far, there is an equivalence of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ and $TH(X)$ when $n > 2/3(N-1)$ and an equivalence of $TH(X)$ and $\mathbb{C}[Q_0, ..., Q_r]$ when $n > 2/3(N-1)$ and X is a complete intersection. In this section we show that in some cases it is possible to relate $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ and $\mathbb{C}[Q_0, ..., Q_r]$ directly without going through the intermediate algebra $TH(X)$. This is because in the dimensional range $n > 2/3(N-1)$ we are always guaranteed an inclusion of graded algebras:

$$\mathbb{C}[Q_0, ..., Q_r] \hookrightarrow \bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \tag{5.1}$$

and thus an equivalence can be established if one is able to compute the dimensions of the grade pieces, for instance, when Ω_X decomposes into line bundles and we have an explicit list of equations for $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$.

Remark 5.1 *The relationship between $\mathbb{C}[Q_0, ..., Q_r]$ and $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ provides an interesting perspective on the question of the number of linearly independent* *quadrics vanishing on a smooth variety whose codimension is small relative to its dimension. In the range $n > 2/3N$ one expects $h^0(\mathbb{P}^N, \mathcal{I}_X(2)) \leq c$ if one believes Hartshorne's conjecture. An interesting question in the context of our work is then what sort of bound exists for $h^0(\mathbb{P}^N, \mathcal{I}_X(2))$ for varieties in the range $2/3(N-1) < n < 2/3N$. For instance, the segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ has codimension two but three linearly independent quadrics vanishing on it. The classical bound for a smooth variety of codimension c is due to Castelnuovo and proved by Zak [?]:*

$$h^0(\mathbb{P}^N, \mathcal{I}_X(2)) \leq \binom{c+1}{2} \tag{5.2}$$

Thus, in a fixed $\mathbb{P}^N$ in the range $n > 2/3(N-1)$, one might expect a better bound for varieties close to the boundary $(2/3(N-1))$ when c is as large as possible. We remark here that if the quadric algebra is freely generated and the dimension of X satisfies $n > 2/3(N-1)$ then

$$h^0(\mathbb{P}^N, \mathcal{I}_X(2)) \leq \kappa(X, \Omega_X(1))$$

Moreover, since $\kappa(X, \Omega_X(1)) \leq \kappa(X, \widetilde{\Omega}_X(1)) \leq N-1$ we would obtain the following bound on the number of linearly independent quadrics through X:

$$h^0(\mathbb{P}^N, \mathcal{I}_X(2)) \leq N-1 \tag{5.3}$$

For varieties in the range $2/3(N-1) < n < 2/3N$, the bound $N-1$ grows like N and $\binom{c+1}{2}$ like N^2 and thus 5.3 would give a better bound then 5.2 for large N.

5.2 The Quadric Hypersurface in $\mathbb{P}^3$

Suppose $X = Q$ is the quadric hypersurface $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$. Note $\mathbb{C}[Q_0, ..., Q_r] = \mathbb{C}[Q]$ and the m-th graded component is just $\mathbb{C}[Q^m]$. To compute $\bigoplus_{m=0}^{\infty} H^0(Q, S^m\Omega_Q(1))$, we consider the pullback $\sigma^* S^m\Omega_Q(1)$ to $\mathbb{P}^1 \times \mathbb{P}^1$. Since σ is an embedding we have $H^0(Q, S^m\Omega_Q(1)) \cong H^0(\mathbb{P}^1 \times \mathbb{P}^1, \sigma^* S^m\Omega_Q(1))$. Note that we have

$$\sigma^* S^m\Omega_Q(1) = S^m[\sigma^*\Omega_Q \otimes \sigma^*\mathcal{O}_Q(1)]$$

To compute $\sigma^*\Omega_Q$ and $\sigma^*\mathcal{O}_Q(1)$, consider the two projections

$$\begin{tikzcd} \mathbb{P}^1 \times \mathbb{P}^1 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathbb{P}^1 \\ \mathbb{P}^1 & \end{tikzcd}$$

Since σ is an embedding we have $\sigma^*\Omega_Q \cong \Omega_{\mathbb{P}^1\times\mathbb{P}^1} = \Omega_{\mathbb{P}^1} \oplus \Omega_{\mathbb{P}^1} = \pi_1^*\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \pi_2^*\mathcal{O}_{\mathbb{P}^1}(-2)$. Note that the projections induce an isomorphism $Pic(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$ via pullback we can write $\Omega_{\mathbb{P}^1\times\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2, 0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0, -2)$ and $\sigma^*\mathcal{O}_Q(1) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 1)$. We have:

$$\begin{aligned} \sigma^* S^m\Omega_Q(1) &= S^m\Big[(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,0) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-2)) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\Big] \\ &= S^m\Big[\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,1) \oplus \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1)\Big] \\ &= \bigoplus_{i+j=m} \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-i+j, i-j) \end{aligned}$$

and so

$$H^0(\mathbb{P}^1 \times \mathbb{P}^1, \sigma^* S^m\Omega_Q(1)) = \bigoplus_{i+j=m} H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-i+j, i-j)) \tag{5.4}$$

Note that $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l)) \cong \mathbb{C}$ if at least one of k or l is negative. Indeed, suppose $k < 0$ then any non-trivial section of $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l)$ pulls back to a non-trivial section on each fiber of π_1. However, the pull back of the line bundle $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(k,l))$ to each fiber is trivial and has no non-constant sections. Thus, by 5.4 we see that the only terms $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-i+j, i-j))$ that contribute something non-trivial are those for which $i = j$. This gives

$$H^0(Q, S^m\Omega_Q(1)) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \mathbb{C} & \text{if } m \text{ is even} \end{cases}$$

Thus, the dimensions of the $m-th$ graded pieces of the algebras $\bigoplus_{m=0}^{\infty} H^0(Q, S^m\Omega_Q(1))$ and $\mathbb{C}[Q]$ coincide and the inclusion 5.1 implies:

$$\bigoplus_{m=0}^{\infty} H^0(Q, S^m\Omega_Q(1)) \cong \mathbb{C}[Q]$$

Moreover, since a hypersurface in $\mathbb{P}^3$ lies in the range $n > 2/3(N-1)$ we have an equivalence of all three algebras $\bigoplus_{m=0}^{\infty} H^0(Q, S^m\Omega_Q(1))$, $TH(Q)$ and $\mathbb{C}[Q]$.

5.3 The Segre Threefold

Let $\Sigma_{1,2}$ denote the image of the segre embedding $\mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. The ideal of $\Sigma_{1,2}$ is generated by three quadrics which we will denote $\{Q_0, Q_1, Q_2\}$. Thus, $\Sigma_{1,2}$ is an example of a subvariety in the range $2/3(N-1) < n < 2/3(N))$ that is not a complete intersection and cannot be handled by theorem 4.4.

Theorem 5.4 *Let $\Sigma_{1,2}$ be the Segre three-fold $\sigma : \mathbb{P}^1 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ and $\{Q_0, Q_1, Q_2\}$ a basis for $H^0(\mathbb{P}^5, \mathcal{I}_{\Sigma_{1,2}}(2))$. Then*

$$\bigoplus_{m=0}^{\infty} H^0(\Sigma_{1,2}, S^m\Omega_{\Sigma_{1,2}}(1)) \cong \mathbb{C}[Q_0, Q_1, Q_2]$$

Proof: Let $p_i : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^i$ be the natural projections. Using the embedding σ and $\sigma^*\mathcal{O}_{\mathbb{P}^N}(1) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,1)$, we have that:

$$\sigma^*(\Omega_{\Sigma_{1,2}}(1)) \cong \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-1,1) \oplus p_2^*(\Omega^1_{\mathbb{P}^2}) \otimes \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(1,1)$$

and hence:

$$H^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)]) \simeq H^0(\mathbb{P}^1 \times \mathbb{P}^2, \bigoplus_{i=0}^{m} O_{\mathbb{P}^1\times\mathbb{P}^2}(-m+2i, m) \otimes p_2^*(S^i[\Omega_{\mathbb{P}^2}])) \quad (5.5)$$

The summands $H^0(\mathbb{P}^1 \times \mathbb{P}^2, O_{\mathbb{P}^1\times\mathbb{P}^2}(-m+2i, m) \otimes p_2^*(S^i[\Omega_{\mathbb{P}^2}]))$ of the right side of 5.5 vanish:

i) if $i < m/2$, on the fibers of p_2, $p_2^{-1}(t) = \mathbb{P}^1$ the bundle

$$O_{\mathbb{P}^1\times\mathbb{P}^2}(-m+2i, m) \otimes p_2^*(S^i[\Omega^1_{\mathbb{P}^2}])|_{\mathbb{P}^1} \simeq O(-m+2i) \oplus ... \oplus O(-m+2i)$$

has no nontrivial sections on $\mathbb{P}^1$.

ii) if $i > m/2$, on the fibers of p_1, $p_1^{-1}(t) = \mathbb{P}^2$, we have the bundle:

$$\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^2}(-m+2i, m) \otimes p_2^*(S^i[\Omega_{\mathbb{P}^2}])|_{\mathbb{P}^2} \simeq S^i\Omega_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(m)$$

which has no notrivial sections by the following lemma by setting $X = \mathbb{P}^2$:

Lemma 5.5 *Let $X \subset \mathbb{P}^N$ be a smooth subvariety such that for the general $x \in X$ the lines in X passing through x fill the embedded the embedded tangent space $\mathbb{T}_x X$. Then*

$$H^0(X, S^i\Omega^1_X \otimes \mathcal{O}_X(m)) = 0 \quad \text{if } m < 2i$$

Proof: A symmetric differential $w \in H^0(X, S^i\Omega^1_X \otimes \mathcal{O}_{\mathbb{P}^2}(m)$ defines at $x \in X$ where $w(x) \neq 0$ a hypersurface:

$$Z_w(x) \subset T_x X$$

consisting of all tangent vectors in the zero locus of $w(x)$, where $w(x)$ is viewed as an homogenous polynomial of degree i on $T_x X$ (with values in $\mathcal{O}_X(m)(x) \cong \mathbb{C}$).

If there is a nontrivial differential w, then by hypothesis at a general point $x \in X$ there is a line $i_l : l \hookrightarrow X$ through x such that $i_{l*}(x)(T_x l) \not\subset Z_w(x)$. This implies

$$0 \neq i_l^* w \in H^0(l, S^i\Omega_l \otimes \mathcal{O}_l(m))$$

contradicting $H^0(l, S^i\Omega_l \otimes \mathcal{O}_l(m)) \simeq H^0(\mathbb{P}^1, \mathcal{O}(-2i + m)) = 0$ when $m < 2i$.

$\blacksquare$

Thus, in 5.5 we are left only with the terms for which $-m+2i=0$. Of course, these terms only exist if m is even and we have:

$$H^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)]) = \begin{cases} 0 \ \text{ if } m \text{ is odd} \\ \\ H^0(\mathbb{P}^2, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]) \ \text{ if } m \text{ is even} \end{cases}$$

What remains is to show $H^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)]) = \tau^*(S^{\frac{m}{2}}[Q_0, Q_1, Q_2])$. Recall that we always have

$$\tau^*(S^{\frac{m}{2}}[Q_0, Q_1, Q_2]) \subset H^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)])$$

since quadrics are tangentially homogeneous relative to X. Moreover, as we computed in example 2.22, $\mathbb{C}[Q_0, Q_1, Q_2]$ is free and so $\dim t^*(S^{\frac{m}{2}}[Q_0, Q_1, Q_2]) = \binom{\frac{m}{2}+2}{2}$ (the tangent map is surjective). As a consequence, the result will follow if:

$$h^0(\Sigma_{1,2}, S^m[\Omega^1_{\Sigma_{1,2}}(1)]) = h^0(\mathbb{P}^2, S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]) \leq \binom{\frac{m}{2}+2}{2}$$

To see that this is true, consider the ideal sequence for $\mathbb{P}^1 \subset \mathbb{P}^2$ tensored by $S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]$:

$$0 \to S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)] \otimes \mathcal{O}(-1) \to S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)] \to S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1} \to 0$$

and the long exact sequence of cohomology:

$$0 \to H^0(\mathbb{P}^2, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)] \otimes \mathcal{O}(-1)) \to H^0(\mathbb{P}^2, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]) \to H^0(\mathbb{P}^1, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1}) \to$$

By lemma 5.5, $H^0(\mathbb{P}^2, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)] \otimes \mathcal{O}(-1)) = 0$ and we have an inclusion

$$0 \to H^0(\mathbb{P}^2, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]) \to H^0(\mathbb{P}^1, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1})$$

and hence

$$h^0(\mathbb{P}^2, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]) \leq h^0(\mathbb{P}^1, S^{\frac{m}{2}}[\Omega_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1})$$

Now, since $\Omega_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}$ we have $S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(\frac{m}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(\frac{m}{2} - 1) \oplus ... \oplus \mathcal{O}$ and hence

$$h^0(\mathbb{P}^1, S^{\frac{m}{2}}[\Omega^1_{\mathbb{P}^2}(2)]|_{\mathbb{P}^1}) = \binom{\frac{m}{2}+2}{2}$$

■

5.6 Iitaka Dimension of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ and the dual defect of X

Note that the inclusions:

$$\mathbb{C}[Q_0, ..., Q_r] \subset \bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \subset \bigoplus_{m=0}^{\infty} H^0(X, S^m\widetilde{\Omega}_X(1))$$

and the isomorphism $\bigoplus_{m=0}^{\infty} H^0(X, S^m\widetilde{\Omega}_X(1)) \cong H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$ (which always hold in the range $n > 2/3(N-1)$) imply that both the Iitaka dimension of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$ and the dimension of the image of the rational map $\mathbb{P}^N \overset{[Q_0:\cdots:Q_r]}{\dashrightarrow} \mathbb{P}^r$ are bounded by N. In this section we investigate the consquences of an isomorphism of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)$ and $\mathbb{C}[Q_0, ..., Q_r]$ in the dimensional range $n > 2/3(N-1)$. The existence of such an isomorphism implies that the Iitaka dimension of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)$, which is equal to the dimension of the image of the rational map $\mathbb{P}(T_X(-1)) \dashrightarrow \mathbb{P}^r$ defined by the linear system $\mathbb{P}(H^0(\mathbb{P}(T_X(-1)), \mathcal{O}_{\mathbb{P}(T_X(-1))}(2)) \cong \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{I}_X(2)))$ is in turn equal to the dimension of the rational map on $\mathbb{P}^N$ defined by $\{Q_0, ..., Q_r\}$. Thus, we wish to understand the rational map:

$$\mathbb{P}(T_X(-1)) \overset{[Q_0:\cdots:Q_r]}{\dashrightarrow} \mathbb{P}^r$$

The second fundamental form gives the following map of locally free sheaves:

$$S^2T_X \to N_{X/\mathbb{P}^N}$$

Twisting by $\mathcal{O}_X(-2)$ gives

$$S^2(T_X(-1)) \to N_{X/\mathbb{P}^N}(-2)$$

and any choice of hyperplane H gives a map to $N_{X/\mathbb{P}^N}(-1)$ via multiplication by H:

$$S^2(T_X(-1)) \to N_{X/\mathbb{P}^N}(-2) \xrightarrow{\cdot H} N_{X/\mathbb{P}^N}(-1)$$

Finally, projectivizing gives:

$$\mathbb{P}(S^2(T_X(-1))) \to \mathbb{P}(N_{X/\mathbb{P}^N}(-2)) \to \mathbb{P}(N_{X/\mathbb{P}^N}(-1))$$

Consider the diagram:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}(T_X(-1))}(2) \arrow[d] & & \\
\mathbb{P}(T_X(-1)) \arrow[dr, "\pi"] & & S^2(T_X(-1)) \arrow[dl, "\pi"] \\
& X &
\end{tikzcd}$$

The isomorphism $\pi_*\mathcal{O}_{\mathbb{P}(T_X(-1))}(m) \cong S^m(T_X(-1))$ implies $\pi^*\pi_*\mathcal{O}_{\mathbb{P}(T_X(-1))}(m) \subset \pi^*S^m(T_X(-1))$ and so this line subbundle defines a rational map:

$$\phi : \mathbb{P}(T_X(-1)) \dashrightarrow \mathbb{P}(S^2(T_X(-1)))$$

such that $\phi^*\mathcal{O}_{\mathbb{P}(S^2(T_X(-1)))}(1) = \mathcal{O}_{\mathbb{P}(T_X(-1))}(2)$. Thus, we have:

$$\mathbb{P}(T_X(-1)) \dashrightarrow \mathbb{P}(S^2(T_X(-1))) \overset{II}{\dashrightarrow} \mathbb{P}(N_{X/\mathbb{P}^N}(-2)) \overset{\cdot H}{\dashrightarrow} \mathbb{P}(N_{X/\mathbb{P}^N}(-1))$$

The bundle $\mathbb{P}(N_{X/\mathbb{P}^N}(-1))$ has a natural identification:

$$\mathbb{P}(N_{X/\mathbb{P}^N}(-1)) \cong \{(x, H) : \mathbb{T}_x X \subset H\} \subset X \times \mathbb{P}^{N*}$$

The image of the second projection is the dual variety of X denoted X^*, it is the variety of tangent hyperplanes to X. For any X one always has $n \geq dim(X^*) \leq N-1$. Composing with the second projection we finally arrive the following rational map:

$$\phi : \mathbb{P}(T_X(-1)) \overset{|\mathcal{O}_{\mathbb{P}(T_X(-1))}(2)|}{\dashrightarrow} X^*$$

where $\phi^*\mathcal{O}_{X^*}(1) = \mathcal{O}_{\mathbb{P}(T_X(-1))}(2)$.

Thus, the conjectured isomorphism implies that the Iitaka dimension of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)$ is bounded by $dim(X^*)$ which in turn implies the image of the rational map $\mathbb{P}^N \dashrightarrow \mathbb{P}^r$ defined by $\{Q_0, ..., Q_r\}$ is bounded by $dim(X^*) \leq N-1$. Note that if the quadric algebra $\mathbb{C}[Q_0, ..., Q_{e_2(X)}]$ is free, then this gives the bound $e_2(X) \leq dim(X^*) \leq N-1$. It is unclear at the moment when one can expect $\mathbb{C}[Q_0, ..., Q_{e_2(X)}]$ to be free in the dimensional range $n > 2/3(N-1)$. However, as a consequence of theorem 4.7, we do know that the quadric algebra is free for smooth varieties defined by quadratic equations in the range $n \geq 2/3N$.

CHAPTER 6

Tangentially Homogeneous Polynomials and Trisecant Lines

Let $P \in TH(X)^{(m)}$. Since the restriction $P|_{\mathbb{T}_x X}$ is in the image of the pullback of the projection

$$p : \mathbb{T}_x X \dashrightarrow \mathbb{P}(T_x X)$$

it follows that $V(P) \cap \mathbb{T}_x X$ is a cone with vertex x for each $x \in X$. We see then that tangentially homogeneous polynomials correspond to hypersurfaces that contain many lines. To make this more precise, we introduce the global tangent cone variety $C_X X$ of trisecant lines that are tangent and meet X in at least two distinct points.

6.1 Global Tangent Cone Varieties, Trisecant Varieties, and Quadrics

Definition 6.2 *Let $X \subset \mathbb{P}^N$ be a subvariety and $l \in \mathbb{G}(1, N)$, l is of X-type $(d_1, ..., d_k)$ if $l \cap X = \{x_1, ..., x_k\}$ set theoretically and $\text{length}_{x_i}(X \cap l) = d_i$. The convention $d_1 \geq ... \geq d_k$ will be used. Set $\Sigma_{X,(d_1,...,d_k)} = \{l \in \mathbb{G}(1, N) | l \text{ of X-type } (d_1, ..., d_k)\}$.*

Definition 6.3 *Let $X \subset \mathbb{P}^N$ be a subvariety, set:*

i) $\Sigma_{X,3} = \{(x, l) \in \mathbb{P}^N \times \mathbb{G}(1, N) | x \in l, l \in \Sigma_{X,(d_1,...,d_k)}, \sum_{i=1}^{k} d_i \geq 3\}$

ii) $\Sigma_{X,3,t} = \{(x,l) \in \mathbb{P}^N \times \mathbb{G}(1,N) | x \in l, l \in \Sigma_{X,(d_1,\ldots,d_k)}, \sum_{i=1}^k d_i \geq 3, d_1 \geq 2\}$

iii) $\Sigma_{X,3,st} = \overline{\{(x,l) \in \mathbb{P}^N \times \mathbb{G}(1,N) | x \in l, l \in \Sigma_{X,(d_1,\ldots,d_k)}, \sum_{i=1}^k d_i \geq 3, d_1 \geq 2, k \geq 2\}}$

Let $p_1 : \mathbb{P}^N \times \mathbb{G}(1,N) \to \mathbb{P}^N$ be the natural projection into the 1st factor. Then we have:

i) $S_3(X) = p_1(\Sigma_{X,3})$*, the trisecant variety of X, i.e. the union of all trisecant lines to X.*

ii) $S_3^t(X) = p_1(\Sigma_{X,3,t})$*, the tangent-trisecant variety of X, i.e. the union of all tangent trisecant lines to X.*

iii) $C_X X = p_1(\Sigma_{X,3,st})$*, the global tangent cone variety of X, i.e. the closure of union of all tangent trisecant lines to X that meet X at least at two distinct points.*

The projections of $\Sigma_{X,3}$, $\Sigma_{X,3,t}$ and $\Sigma_{X,3,st}$ into $\mathbb{G}(1,N)$ via the 2nd natural projection are denoted respectively by $[\Sigma_{X,3}]$, $[\Sigma_{X,3,t}]$ and $[\Sigma_{X,3,st}]$. Note that in general, one always has the inclusions:

$$C_X X \subset S_3^t(X) \subset S_3(X)$$

The main results of this section is that for X with dimension $n > 2/3(N-1)$, one has $C_X X = S_3^t(X) = S_3(X)$. An important tool will be the following lemma of Bogomolov and De Oliveira:

Lemma 6.4 *(Bogomolov-De Oliveira) Let $X \subset \mathbb{P}^N$ and $\subset \mathbb{P}^N \times T$ be a family of lines in $\mathbb{P}^N$ over an irreducible projective curve T such that all lines pass through a* *fixed $z \notin X$ and whose union is not a line. If the general lines meets X at least twice, then one of the lines must meet X with multiplicity at least two at some point.*

We can deduce from this the following theorem:

Theorem 6.5 *Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth subvariety of codimension two with dimension $n \geq 3$. Then:*

$$C_X X = S_3(X)$$

Proof:

Note that $S_3(X)$ is irreducible, see [Kwa01]. The proof of the theorem proceeds in two steps:

Step 1: $S_3^t(X) = S_3(X)$.

Since a tangent trisecant line is in particular a trisecant line one always has $S_3^t(X) \subset S_3(X)$ and so we need to establish the inclusion $S_3(X) \subset S_3^t(X)$. Let $z \in S_3(X)$, $\mathbb{P}^{N-1} \subset \mathbb{P}^N \setminus \{z\}$ be a hyperplane, and $p_z : \mathbb{P}^N \to \mathbb{P}^{N-1}$ be the projection with center z into $\mathbb{P}^{N-1}$. The subvariety of $\mathbb{G}(1, N)$ consisting of the lines through z can be naturally identified with $\mathbb{P}^{N-1}$ and we set $[\Sigma_{X,3}]_z, [\Sigma_{X,3,t}]_z \subset \mathbb{P}^{N-1}$ to be respectively the subvariety of trisecant, tangent-trisecant lines to X through z. To establish $S_3(X) \subset S_3^t(X)$ it is sufficient to show $[\Sigma_{X,3}]_z \cap [\Sigma_{X,3,t}]_z \neq \emptyset$. Note that this will follow from lemma 6.4 if we can show there is a positive dimensional subvariety of $[\Sigma_{X,3}]_z$.

We note that the case $z \in X$ can be resolved by a simple dimensional argument. The dimensional hypothesis on X implies $N \geq 5$ and hence $\dim X \cap \mathbb{T}_z X \geq N-4 > 0$ guaranteeing the existence of tangent-trisecant lines through z. So assume from now on that $z \in S_3(X) \setminus X$.

The projection $p_z : X \to \mathbb{P}^{N-1}$ is a finite map with $p_z(X) \subset \mathbb{P}^{N-1}$ irreducible of dimension n. Let $y \in [\Sigma_{X,3}]_x$, i.e. y is a triple point of $p_z(X)$, and x_1, x_2, x_3 be points in $l_y \cap X$, where l_y is the line corresponding to y (we can assume that the x_i are distinct, otherwise $y \in [\Sigma_{X,3,t}]_x$). Denote by Z_k the local irreducible components of $p_z(X)$ at y. There is an open neighborhood U of y such that each point x_i has a neighborhood surjecting onto the $Z_{k_i} \cap U$. This implies $Z_{k_1} \cap Z_{k_2} \cap Z_{k_3} \cap U \subset [\Sigma_{X,3}]_x$ and by the intersection inequality $\dim Z_{k_1} \cap Z_{k_2} \cap Z_{k_3} \cap U > 3(2/3(N-1)) - 2(N-1) > 0$ there is a positive dimensional family of trisecant lines through z. Hence $[\Sigma_{X,3}]_z$ intersects $[\Sigma_{X,3,t}]_z$ by lemma 6.4.

Step 2: $C_X X = S_3^t(X)$

Note that one always has $S_3^t(X) = C_X X \cup S_3^{tt}(X)$ where $S_3^{tt}(X)$ is the variety of trisecant lines meeting X at only one point. However, suppose l is a line that meets X at the point x with multiplicity three. Then l is in the tangent cone $C_x(T_x X \cap X)$ and thus is the limit of secant lines $\overline{xy}$ where $y \in \mathbb{T}_x X \cap X$ and so is contained in $C_X X$. Thus $S_3^{tt}(X) \subset C_X X$ and we have $C_X X = S_3^t(X)$.

■

Thus we have the following geometric characterization of tangentially homogeneous polynomials:

Corollary 6.6 *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimension $n > 2/3(N-1)$. Then every homogeneous polynomial $P \in TH(X)$ must vanish on the trisecant variety $S_3(X)$.*

Proof: Any $P \in TH(X)$ must vanish on $C_X X$. By theorem 6.5 we have $C_X X = S_3(X)$ and so P vanishes on $S_3(X)$. ∎

Clearly this characterization of tangentially homogeneous polynomials is valuable insofar as one understands the variety $S_3(X)$. As constructed, $S_3(X)$ has many lines, and it is possible in some situations to use classification results about varieties with many lines to understand $S_3(X)$. For instance, in codimension two there are the following three possibilies [kwak]:

1. $S_3(X) = \mathbb{P}^{n+2}$
2. dim $S_3(X) = n+1$ and $S_3(X) = Q$ is a quadric hypersurface.
3. dim $S_3(X) \leq n$ and X is one of the following:
   (a) complete intersection of two quadrics.
   (b) cone over a twisted cubic curve in $\mathbb{P}^3$.
   (c) cubic scroll surface in $\mathbb{P}^4$
   (d) segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$

In the range $n > 2/3(N-1)$, codimension two subvarieties must have dimension greater or equal to three and so we can rule out cases b) and c). In the next chapter this classification will allow us to establish the equivalence of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$, $TH(X)$ and $\mathbb{C}[Q_0, ..., Q_r]$ for codimension two subvarieties.

6.7 Trisecant Variety and the Quadric Envelope

An interesting observation about the above classification is that for smooth subvarieties in codimension two in the range $n > 2/3(N-1)$, $S_3(X)$ is the intersection of quadrics containing X. Based on this evidence we introduce the conjecture:

Conjecture 6.8 *Let $X \subset \mathbb{P}^N$ be a smooth nondegenerate subvariety with dimenension $n > 2/3(N-1)$ and let $QE(X)$ denote the intersection of quadrics vanishing on X. If $h^0(\mathbb{P}^N \mathcal{I}_X(2)) = 0$ we define $QE(X) = \mathbb{P}^N$. Then*

$$S_3(X) = QE(X)$$

Note that by Bezout's theorem one always has the inclusion $S_3(X) \subset QE(X)$ and so the question is whether every point $z \in QE(X)$ lies on some trisecant line. In this section we establish this for complete intersection varieties. First, we need the following lemma.

Lemma 6.9 *Let $H \subset \mathbb{P}^N$ be a hypersurface and $z \in \mathbb{P}^N$ a point not contained in H. Consider $\mathbb{P}^N$ blown up at z as a projective bundle over $\mathbb{P}^{N-1}$, with the blow up map σ_z and its natural projection:*

$$\begin{tikzcd} \mathbb{P}(O_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \arrow[r, "\sigma_z"] \arrow[d, "p_{\sigma_z}"'] & \mathbb{P}^N \\ \mathbb{P}^{N-1} & \end{tikzcd}$$

Denote $\widehat{X} := \sigma_z^{-1}(X)$. Then there exists a base change

$$\begin{tikzcd} \mathbb{P}(f^* O_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_Y) \arrow[r, "\hat{f}"] \arrow[d, "p"'] & \mathbb{P}(O_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \arrow[d] \\ Y \arrow[r, "f"] & \mathbb{P}^{N-1} \end{tikzcd}$$

where $f : Y \to \mathbb{P}^{N-1}$ is a finite map and $\hat{f}^{-1}(\widehat{H})$ is a union of d sections of p.

Proof: We first consider the base change via the projection $p_{\sigma_z}|_H : \widehat{H} \to \mathbb{P}^{N-1}$:

$$\begin{tikzcd} \mathbb{P}(p_{\sigma_z}|_H^* O_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_Y) \arrow[r, "\widehat{p_{\sigma_z}|_H}"] \arrow[d, "p"'] & \mathbb{P}(O_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \arrow[d] \\ H \arrow[r, "p_{\sigma_z}|_H"] & \mathbb{P}^{N-1} \end{tikzcd}$$

The preimage $\hat{p}_{\sigma_z}|_H^{-1}(\widehat{H})$ defines a multisection p however, the diagonal map $h \mapsto (h, h)$ gives a section of p and thus $\hat{p}_{\sigma_z}|_H^{-1}(\widehat{H})$ decomposes into the union of a section and a multisection of degree one less. Repeating this process a finite number of times gives the lemma. ■

Theorem 6.10 *Let $X^{(n)} \subset \mathbb{P}^N$ be a smooth complete intersection with $n \geq 2/3(N - 2 - r)$ where $r = h^0(\mathbb{P}^N, \mathcal{I}_X(2))$. Then:*

$$S_3(X) = QE(X)$$

where $QE(X)$ denotes the quadric envelope of X.

Proof: Let X be a complete intersection, $I_X = (F_1, ..., F_c)$, of multi-degree $(d_1, ..., d_c)$, where $c =$codim(X), $d_1 \geq d_2 \geq ... \geq d_c$. Let $k =$min$\{i|d_i = 2\}$ if $\{i|d_i = 2\} \neq \emptyset$, the quadric envelope $QE(X) = V(F_k, ..., F_c)$ if $\{i|d_i = 2\} \neq \emptyset$, otherwise $QE(X) = \mathbb{P}^N$. The irreducibility of $QE(X)$ is forced by the smoothness of X.

Note that in the dimensional range $n > 2/3(N-1)$ we always have

$$X \subset Trisec(X) \subset QE(X)$$

The first inclusion follows from the implication $n > 2/3(N-1) \implies \dim \mathbb{T}_x X \cap X \geq 1$ and the second from Bezout's theorem. Thus if X is a complete intersection of quadrics we have $X = QE(X)$ which forces $Trisec(X) = QE(X)$. Consider then the case $X \neq QE(X)$ and let z be a general point of $QE(X)$ in the sense that $z \notin [V(F_1) \cup ... \cup V(F_{k-1})] \cap QE(X)$, hence in particular $z \notin X$. Consider $\mathbb{P}^N$ blown up at z as a projective bundle over $\mathbb{P}^{N-1}$, with the blow up map σ_z and its natural projection:

$$\begin{tikzcd} \mathbb{P}(O_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \arrow[r, "\sigma_z"] \arrow[d, "p_{\sigma_z}"'] & \mathbb{P}^N \\ \mathbb{P}^{N-1} & \end{tikzcd}$$

The pre-image $\sigma^{-1}(z) = Im(s_E)$, where s_E is the section of p_{σ_z} corresponding to the subbundle $O(1) \oplus 0 \subset O(1) \oplus O$. The section s_E is rigid, but the projection p_z has sections which move, corresponding to the hyperplanes in $\mathbb{P}^N$ not meeting z. The complement $\mathbb{P}(O(1) \oplus O) \setminus Im(s_E)$ is the total space of the line bundle $O(1)$ over $\mathbb{P}^{N-1}$, $Tot(O(1))$.

$$\begin{tikzcd}
\mathbb{P}(O(1) \oplus O) \setminus Im(s_E) = Tot(O(1)) \arrow[r, "\sigma_z"] \arrow[d, "p_z"'] & \mathbb{P}^N \setminus \{z\} \\
\mathbb{P}^{N-1} &
\end{tikzcd}$$

The map $\sigma_z : Tot(O(1)) \to \mathbb{P}^N \setminus \{z\}$ is a biregular map and $p_z := p_{\sigma_z}|_{Tot(O(1))}$ corresponds via the biregular map σ_z to the projection $p_z : \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-1}$ with center z (the fibers of $p_z : Tot(O(1)) \to \mathbb{P}^{N-1}$ correspond to the lines through z in $\mathbb{P}^N$ punctured at z).

By the generality condition on $z \in QE(X)$,

$$H_i := V(F_i) \subset \mathbb{P}^N \setminus \{z\} \qquad \forall i = 1, ..., k-1$$

and hence correspond via σ_z to multi-sections $H_i' = \sigma_z^{-1}(H_i)$ of degree d_i of $O(1)$ over $\mathbb{P}^{N-1}$, in the sense that $p_{\sigma_z} : H_i' \to \mathbb{P}^{N-1}$ is a finite surjective map. The pre-images $\sigma_z^{-1}(H_i)$, where $i = k, ..., c$ are no longer multi-sections since $z \in V(F_i)$, $i = k, ..., c$. Moreover, an essential feature due to Bezout's theorem is that if $x \in X_z$, then:

$$l_{x,z} \subset H_i \qquad \forall i = k, ..., c$$

since $length(l_{x,z} \cap H_i) \geq 3$ and $\deg H_i = 2$, for $i = k, ..., c$. Note that there is a $N-3$ dimensional family of lines in H_i passing through z for $i = k, ..., c$. Let $Z_i \subset \mathbb{P}^{N-1}$ for $i = k, ..., c$ denote these lines and $W := Z_k \cap \cdots \cap Z_c$.

We proceed by doing a base change of $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathbb{P}^{N-1})$ via the projection $p_z|_{H_1'} : H_1' \to \mathbb{P}^{N-1}$:

$$\begin{array}{ccc} \mathbb{P}(\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \times_{\mathbb{P}^{N-1}} H'_1 & \xrightarrow{p_1} & \mathbb{P}(\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_z} \\ H'_1 & \xrightarrow{p_z|_{H'_1}} & \mathbb{P}^{N-1} \end{array}$$

Note that $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \times_{\mathbb{P}^{N-1}} H'_1 \cong \mathbb{P}(\pi^*|_{H'_1}\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{H'_1})$ and we have

$$\begin{array}{ccc} \mathbb{P}(\pi^*|_{H'_1}\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{H'_1}) & \xrightarrow{p_1} & \mathbb{P}(\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_z} \\ H'_1 & \xrightarrow{p_z|_{H'_1}} & \mathbb{P}^{N-1} \end{array}$$

The key observation here is that $p_1^{-1}(H'_1)$ now is the union of a section and a multisection where the degree of the multisection is one less than the degree of H'_1. Moreover, the preimages $p_2^{-1}(H'_2), ..., p_2^{-1}(H'_{k-1})$ remain multisections of p_1. Repeating the process we arrive in a finite number of steps at:

$$\begin{array}{ccc} \mathbb{P}(f^*\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_Y) & \xrightarrow{\hat{f}} & \mathbb{P}(\mathcal{O}_{\mathbb{P}^{N-1}}(1) \oplus \mathcal{O}_{\mathbb{P}^{N-1}}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_z} \\ Y & \xrightarrow{f} & \mathbb{P}^{N-1} \end{array}$$

where $f : Y \to \mathbb{P}^{N-1}$ is a finite surjective map and the preimages $\hat{f}^{-1}(V(H_i))$ are the union of sections of p:

$$\hat{f}^{-1}(\widehat{V(H_i)}) = D_{i_1} + \cdots + D_{i_{d_i}}$$

where the D_{i_j} are divisors corresponding to sections s_{i_j} of the ample line bundle $f^*\mathcal{O}_{\mathbb{P}^{N-1}}(1)$ over Y:

$$D_{i_j} = s_{i_j}(Y)$$

Note that $\hat{f}^{-1}(\widehat{QE(X)})$ contains $p^{-1}(f^{-1}(W))$ and $f^{-1}(W)$ is of codimension $2(c+1-k)$ on Y.

A general point $z \in QE(X)$, as chosen before, belongs to Trisec(X) if and only if there is a fiber of p_{σ_z}, $p_{\sigma_z}^{-1}(x)$, containing at least three points in the intersection of the quasi-sections $\widehat{V(H_k)}, ..., \widehat{V(H_c)}$ and the multisections $\widehat{V(H_1)}, ..., \widehat{V(F_{k-1})}$. Note that this implies in particular that $x \in W = Z_k \cap \cdots \cap Z_c$ with the Z_i as described above, since otherwise the line corresponding to x would meet one of the quadrics $V(H_i)$, $i = k, ..., c$ in at most two points.

The condition for $z \in$ Trisec(X) after base change translates to: there is a $y \in Y$ such that $y \in f^{-1}(W)$ and

$$\#\left(p^{-1}(y) \cap \bigcap_{i=1}^{k-1} (D_{i_1} \cup \cdots \cup D_{i_{d_i}}\right) \geq 3$$

This is guaranteed, in particular, if y is in the following subset of Y:

$$T := \{t \in f^{-1}(W) | (s_{1_l} - s_{j_l})(t) = 0 \text{ for } j = 2, ..., k-1 \text{ and } l = 1, 2, 3\}$$

Now, since the $s_{1,l} - s_{jl}$ are sections of the ample line bundle $f^*\mathcal{O}_{\mathbb{P}^{N-1}}(1)$, their zeros define Weil divisors on Y i.e. codimension one subvarieties of Y and together cut out a subvariety of codimension at most $3(k-2)$. Thus dim $(T) \geq \dim(Y) - \mathrm{cod}(f^{-1}(W)) - 3(k-2)$ and this is greater than or equal to zero if $n \geq 2/3(N-2-r)$.

■

CHAPTER 7

Symmetric Twisted Differentials, Tangentially Homogeneous Polynomials and the Quadric Algebra

In this final chapter we summarize our current understanding of the equivalence of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$, $TH(X)$ and $\mathbb{C}[Q_0, ..., Q_r]$ in the dimensional range $n > 2/3(N-1)$. The main result is the equivalence of the three algebras for codimension two subvarieties with dimension $n \geq 3$. The proof of this equivalence uses the global tangent cone variety $C_X X$ introduced in the previous chapter and the fact that it coincides with the trisecant variety $S_3(X)$ in codimension two.

7.1 Hypersurfaces

The hypersurface case was actually known to Bogomolov and De Oliveira in [BO08] and was stated in theorem 1.5. As noted above, it also follows from corollary 7.12 since hypersurfaces are complete intersections. Thus if H is a hypersurface of degree greater than two all three algebras are trivial. If $H = Q$ is a quadric hypersurface then

$$\bigoplus_{m=0}^{\infty} H^0(Q, S^m\Omega_Q(1)) \cong TH(Q) \cong \mathbb{C}[Q]$$

7.2 Codimension Two

The strategy here is to break into cases based on the dimension of the trisecant variety $S_3(X)$. There are three possibilities for a codimension two subvariety with dimension $n \geq 3$:

1. dim $S_3(X) = N$
2. dim $S_3(X) = N - 1 = n + 1$ and $S_3(X)$ is a quadric hypersurface.
3. dim $S_3(X) = N - 2 = n$ and there are two possibilities for X:
   (a) X is the complete intersection of quadrics.
   (b) X is the segre threefold $\Sigma_{1,2}$

Theorem 7.3 *Let $X \subset \mathbb{P}^N$ be a nondegenerate smooth subvariety with codim$(X) = 2$ and dimension $n \geq 3$. Then we have graded isomorphisms*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \cong TH(X) \cong \mathbb{C}[Q_0, ..., Q_r]$$

where $\{Q_0, ..., Q_r\}$ is any basis of $H^0(\mathbb{P}^N, I_X(2))$.

Proof: We break this into cases according to the dimension of $S_3(X)$.

Case 0: dim $S_3(X) = N$

In this case $S_3(X) = \mathbb{P}^N$ and hence the tangentially homogeneous polynomials relative to X of positive degree must be trivial. Therefore

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \cong TH(X) \cong \mathbb{C}[Q_0, ..., Q_r] \cong \mathbb{C}$$

as desired.

Case 1: $\dim S_3(X) = N - 1$

In this case $S_3(X) = V(Q)$ with Q spanning $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$. An irreducible tangentially homogeneous polynomial relative to X of positive degree H must vanish on $V(Q)$ and hence $H = Q$. If $\mathrm{cod}(X) = 2$ and $n \geq 3$ the tangent map of X is surjective, and so by proposition 2.12 the product of two homogeneous polynomials is a tangentially homogeneous polynomial relative to X if and only if both factors are. As a consequence the algebra $TH(X)$ is generated by the irreducible tangentially homogeneous polynomials giving in this case the graded isomorphism:

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega^1_X(1)]) \simeq TH(X) = \mathbb{C}[Q]$$

as desired.

Case 2: $\dim S_3(X) = N - 2 = n$

As mentioned previously, the only smooth subvarieties with this property are complete intersections of two quadrics and the Segre Threefold. In the first case we can apply theorem 4.4. The case of the Segre Threefold was confirmed in theorem 5.4.

■

7.4 The Range $n > 2/3N$

The Harshorne conjecture asserts that every smooth subvariety in this range is a complete intersection. If this conjecture were confirmed, our theorem 4.4 for complete intersections would confirm the equivalence of $\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1))$, $TH(X)$ and $\mathbb{C}[Q_0, ..., Q_{e_2(X)}]$ in this range. At present, the Hartshorne conjecture has only been confirmed for quadratic manifolds. A subvariety is called quadratic if its homogeneous ideal is generated by degree two polynomials. The following theorem of Ionescu and Russo [IR13] confirms the Hartshorne conjecture for smooth quadratic subvarieties allowing us to apply theorem 4.4 for this class of varieties.

Theorem 7.5 *(Ionescu-Russo) Let $X \subset \mathbb{P}^N$ be a smooth quadratic subvariety with dimension $n > 2/3N$. Then X is a complete intersection.*

As a corollary we have an equivalence of our three algebras for quadratic manifolds in the range $n > 2/3N$:

Corollary 7.6 *Let $X \subset \mathbb{P}^N$ be a quadratic manifold with dimension $n > 2/3N$ and ideal $I(X) = \langle Q_0, ..., Q_r \rangle$. Then*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m\Omega_X(1)) \cong TH(X) \cong \mathbb{C}[Q_0, ..., Q_{e_2(X)}]$$

Proof: By theorem 7.5 X must be a complete intersection and so we can apply theorem 4.4. ■

7.7 The Range $n \geq 2/3N$

Smooth subvarieties for which $n = 2/3N$ are referred to as Hartshorne manifolds. The following result of Ionescu and Russo classifies the quadratic Hartshorne manifolds:

Theorem 7.8 *(Ionescu-Russo) Let $X \subset \mathbb{P}^N$ be a smooth quadratic subvariety of dimension $n = 2/3N$. Then X is either a complete intersection, $G(1,4)$ or S_{10}.*

Here $G(1,4)$ is the grassmannian of lines in $\mathbb{P}^4$ and S_{10} is the ten dimensional spinor variety. To confirm our conjecture for quadratic manifolds in the range $n \geq 2/3N$ would require only the two cases $G(1,4)$ and S_10. Note that $G(1,4)$ has five linearly independent quadrics through it while S_{10} has ten. We state these two cases as conjectures:

Conjecture 7.9 *Let $G := G(1,4) \subset \mathbb{P}^9$ be the grassmannian of lines in $\mathbb{P}^4$ and let $\{Q_0, ..., Q_4\}$ be a basis for $H^0(\mathbb{P}^N, \mathcal{I}_G(2))$. Then there is a graded isomorphism of* *algebras:*

$$\bigoplus_{m=0}^{\infty} H^0(G, S^m\Omega_G(1)) \cong \mathbb{C}[Q_0, Q_1, Q_2, Q_3, Q_4]$$

Conjecture 7.10 *Let $S_{10} \subset \mathbb{P}^{15}$ be the ten dimensional spinor variety and let $\{Q_0, ..., Q_9\}$ be a basis for $H^0(\mathbb{P}^N, \mathcal{I}_{S_{10}}(2))$. Then there is a graded isomorphism of algebras:*

$$\bigoplus_{m=0}^{\infty} H^0(S_{10}, S^m\Omega_{S_{10}}(m)) \cong \mathbb{C}[Q_0, Q_1, Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q_8, Q_9]$$

At present, it is not clear to what extent these computations will be possible.

7.11 The Range $n > 2/3(N-1)$

By theorems 4.4 and 4.7 we have the following corollary:

Corollary 7.12 *Let $X \subset \mathbb{P}^N$ be either a smooth nondegenerate strict complete intersection with dimension $n > 2/3(N-1)$ or a smooth nondegenerate subvariety of codimension two and $n \geq 3$ then*

$$\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega_X(1)]) \cong \mathbb{C}[Q_0, ..., Q_r] \cong TH(X)$$

where $\{Q_0, ..., Q_r\}$ is a basis for $H^0(\mathbb{P}^N, \mathcal{I}_X(2))$.

Proof: Since $n > 2/3(N-1)$, we have the correspondence $\bigoplus_{m=0}^{\infty} H^0(X, S^m[\Omega_X(1)]) \cong TH(X)$ by theorem 4.7. Moreover, by theorem 2.6 the tangent map is surjective when $n > 2/3(N-1)$ and theorem 4.4 plus the assumption of complete intersection implies the equivalence $TH(X) \cong \mathbb{C}[Q_0, ..., Q_r]$.

■

This is the most general statement we can make in the range $n > 2/3(N-1)$. At the moment, our justification for the truth of this equivalence for non-complete intersections in the range $n > 2/3(N-1)$ is the verification for codimension two subvarieties and the Segre threefold $\Sigma_{1,2} \subset \mathbb{P}^5$.

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