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Extension Problem for Flexible Varieties

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EXTENSION PROBLEM FOR FLEXIBLE VARIETIES

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We consider n - dimensional ($n \geq 4$) flexible quasi-affine varieties X, that are varieties on which the group generated by all one-parameter unipotent subgroups of $\mathrm{Aut}(X)$ acts transitively. We prove that for any subvariety Γ of X, isomorphic to a line, every SL_n - automorphism of the normal bundle of Γ is induced by a global automorphism of X. We also extend this result also to automorphisms of jet bundles on Γ.

To my family

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CHAPTER 1

Preliminaries

1.1 Notations

- $\mathbb{A}^n_{\Bbbk}$ - the n dimensional affine space over the base field $\Bbbk$
- $\mathbb{C}$ - the field of complex numbers
- $\mathrm{Aut}(X)$ - the group of automorphisms of the variety X
- $\mathrm{SAut}(X)$ - the special automorphism group of the variety X
- $\Bbbk[X]$ - the ring of regular functions of the variety X.
- $\Bbbk(X)$ - the field of rational functions of the variety X.
- X_{reg} - the Zariski open subset of regular points of the variety X
- $\mathrm{Der}(A)$ - the set of derivations on the ring A
- $\mathrm{Der}_A(B)$ - the set of derivations on the ring B that vanish on a subring A of the ring B
- $\mathrm{LND}(A)$ - the set of all locally nilpotent derivations on the ring A
- $\mathrm{LND}_A(B)$ - the set of locally nilpotent derivations on the ring B that vanish on a subring A of the ring B.
- δ - stands for locally nilpotent derivation

- $\ker(\delta)$ - the kernel of the locally nilpotent derivation δ
- $\mathbb{G}_a(\mathbb{k})$ - the algebraic $(\mathbb{k}, +)$ action
- A^G - the subring (respectively subfield) of the invariant elements of the ring (respectively field) A under the action of the group G
- $X \to X /\!\!/ G$ - the algebraic quotient
- $X \to X/G$ - the geometric quotient
- $M_n(R)$ - the $n \times n$ matrices with entries in the ring R
- $\mathrm{GL}_n(R)$ - the general linear group over the ring R
- $\mathrm{SL}_n(R)$ - the special linear group over the ring R
- $\mathrm{E}_n(R)$ - the group of standard elementary transformations over the ring R
- $\mathrm{SL}_n(A, \mathfrak{q})$ - the principal congruence subgroup corresponding to the ideal $\mathfrak{q}$ of the ring A
- $\mathrm{E}_n(A, \mathfrak{q})$ - the group of $\mathfrak{q}-$ standard elementary transformations corresponding to the the ideal $\mathfrak{q}$ of the ring A
- $\mathcal{C}_{\mathfrak{q}}$ - the group of universal Mennicke symbols corresponding to the ideal $\mathfrak{q}$ of the ring A
- $\mathrm{Spec}(A)$ - the spectrum of the ring A
- $\mathrm{Hol}(M)$ - the ring of holomorphic functions on a complex manifold M
- $\mathrm{div}(\vec{F})$ - the divergence of the vector forms $\vec{F}$
- $\mathscr{F}_j$ - the set of n tuples of homogeneous j forms in n variables, with coefficients in a ring (respectively field)
- $\mathscr{F}_j^0$ - the set of n tuples of homogeneous j forms in n variables, with coefficients in a ring (respectively field), and with divergence zero

1.2 Introduction

In this thesis the extension problem for curves with infinitesimal neighborhoods in flexible varieties is considered. In general, the extension problem can be formulated in the following way: given Y_1, Y_2 - subvarieties of a variety X,

$$\varphi : Y_1 \xrightarrow{\sim} Y_2$$

- an isomorphism, is φ a restriction of a global automorphism of X?

The extension problem started from the work of Abhyankar-Moh-Suzuki where the following fact was established:

Theorem 1.2.1. *[AM75], [Suz74] Given two plane curves isomorphic to a line, one can be transferred to another by a global automorphism of* $\mathbb{A}^2_{\Bbbk}$.

Later on, Jelonek proved the following result for a smooth subvariety Y in $\mathbb{A}^n_{\Bbbk}$:

Theorem 1.2.2. *[Jel87] Let Y be a closed subvariety in $\mathbb{A}^n_{\Bbbk}$ of dimension* $\dim(Y) = k$, *and*

$$\varphi : Y \to \mathbb{A}^n_{\Bbbk}$$

be an embedding. If $n \geq 4k+2$ then there exists an automorphism of $\mathbb{A}^n_{\Bbbk}$, that restricts to φ on Y.

The more general case for non-smooth subvarieties was done in the works of S.Kaliman and V. Srinivas [Sri91]:

Theorem 1.2.3. *[Kal91] [Sri91] Let $\varphi : Y_1 \to Y_2$ be an isomorphism of two closed subvarieties of $\mathbb{A}^n_{\Bbbk}$. If $n > \max(2\dim(Y_1) + 1, \dim(TY_1))$, where TY_1 is the Zariski tangent bundle of Y_1, then φ extends to an automorphism of $\mathbb{A}^n_{\Bbbk}$.*

We want to consider the extension for varieties X, that are different from affine spaces $\mathbb{A}^n_{\Bbbk}$. Such varieties should possess a rich automorphism group. We consider one big class of such varieties, which are called flexible varieties.

Through out this paper, we fix an algebraically closed field $\mathbb{k}$ of characteristic zero. Flexible varieties have the following characterization. Let X be a variety over $\mathbb{k}$, and consider the group $\mathrm{Aut}(X)$ of automorphisms of the variety X. The subgroup of $\mathrm{Aut}(X)$, which is the group generated by subgroups of $\mathrm{Aut}(X)$ that are isomorphic to the flows of all $\mathbb{G}_a-$ actions, is called the special automorphism group and is denoted by $\mathrm{SAut}(X)$. A flexible variety is a quasi-affine variety of dimension greater or equal to 2, such that the special automorphism group $\mathrm{SAut}(X)$ acts transitively on X_{reg}, with X_{reg} being the Zariski open subset of regular points of X.

For a flexible variety different from $\mathbb{A}^n_{\mathbb{k}}$, the first result was obtained by Van Santen (formerly Stampfli) where he considered a variety isomorphic to $\mathrm{SL}_n(\mathbb{k})$ and proved the following theorem:

Theorem 1.2.4. *[Sta17] Let $f, g : \mathbb{k} \to \mathrm{SL}_n(\mathbb{k})$ be embeddings. If $n \geq 3$ then f and g are the same up to an algebraic automorphism of $\mathrm{SL}_n(\mathbb{k})$. If $n = 2$ then f and g are the same up to a holomorphic automorphism of $\mathrm{SL}_n(\mathbb{k})$.*

We also want to consider the extension problem in the case of subvarieties with their infinitesimal neighborhoods (so called non-reduced case).

Example 1.2.5. The simplest example of the non-reduced subvariety is the following. The point $p \in \mathbb{A}^1_z$, given by the equation $\{z = 0\}$ is a reduced subvariety (the origin). In the same time, we can consider the point $p' \in \mathbb{A}^1_z$ given by the equation $\{z^2 = 0\}$. Geometrically p' also defines the origin, but it is a non-reduced subvariety (the so called "fat" point).

For the non-reduced case, the only known result for the extension problem, is the case where the subvariety consists of a finite number of points in a flexible variety. Let X be a flexible variety, $p \in X$ - a point in X, $\mu = \mu_p$ - the maximal ideal of p in $\mathbb{k}[X]$. Consider μ/μ^{m+1}, which is called the $m-$ jet at the point p. Note that for $m = 1$, μ/μ^2 is the cotangent space at the point p and its dual is the tangent space at p. This case of a finite number of points is considered in the paper of I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, M. Zaidenberg

Theorem 1.2.6. *[AFK⁺13] Let X be a flexible algebraic variety and $x_1, \dots, x_N$ be a set of distinct points in X. There exists $\alpha \in \mathrm{SAut}(X)$ with prescribed automorphisms of m - jets*

$$\frac{\mu_j}{\mu_j^{m+1}} \xrightarrow{\sim} \frac{\mu_j}{\mu_j^{m+1}}$$

at these points, which preserves a local volume form.

E.g.: for prescribed $\mathrm{SL}_n(\Bbbk)$ automorphism for tangent (respectively cotangent) spaces at these points, there is always a global automorphism of the flexible variety, that fixes these points and induces these $\mathrm{SL}_n(\Bbbk)$ automorphism.

The natural question we ask is the following.

Question 1.2.7. Can we replace the points x_j by curves Γ_j, with vanishing ideals I_j, and the corresponding cotangent spaces μ_j/μ_j^2 at the points x_j, respectively the $m-$ jets μ_j/μ_j^{m+1}, by conormal bundles I_j/I_j^2 of the curves Γ_j, respectively $m-$ jet bundles I_j/I_j^{m+1}?

The following two main theorems below are proven in this thesis, and the first one of them gives a positive answer to the question 1.2.7 above. Let X be a flexible variety with dimension $\dim(X) \geq 4$, and Γ_j be lines in X.

Theorem 1.2.8. *Given the automorphisms of jet - bundles*

$$\bar{\varphi}_j : \frac{I_j}{I_j^{m+1}} \xrightarrow{\sim} \frac{I_j}{I_j^{m+1}}, \quad j = 1, \dots, N,$$

that leave every point of curves Γ_j fixed, and with $\mathrm{Jac}(\bar{\varphi}_j) = 1 \mod I_j^m$, there exists a global automorphism $\alpha \in \mathrm{SAut}(X)$ which induces $\bar{\varphi}_j$ and leaves every point of curves Γ_j fixed.

E.g.: Every $\mathrm{SL}_n(\Bbbk[\Gamma_j])$ automorphism of normal (respectively conormal) bundles of curves Γ_j, that leave every point of Γ_j fixed, there exists a global automorphism of X that induces these automorphisms.

Theorem 1.2.9. *Consider the isomorphisms*

$$\phi_0 : \Gamma_1 \xrightarrow{\sim} \Gamma_2,$$

$$\bar{\varphi} : \frac{I_1}{I_1^{m+1}} \xrightarrow{\sim} \frac{I_2}{I_2^{m+1}},$$

that preserves global volume form $\mod I_1^m$. *Suppose* ϕ_0 *is induced by a global automorphism* Φ *of* X. *Note that* Φ *need not induce* φ. *Nevertheless, there exists another global automorphism* $\alpha \in \mathrm{SAut}(X)$, *which simultaneously induces* ϕ_0 *and* $\bar{\varphi}$.

Remark 1.2.10. The theorems above are proven not only for curves isomorphic to lines, but also for any curves Γ_j, that are once punctured curves with trivial normal bundles, whose coordinate rings satisfy the condition $\mathrm{SL}_n\Big(\mathbb{k}[\Gamma_j]\Big) = \mathrm{E}_n(\mathbb{k}[\Gamma_j])$. Here $\mathrm{E}_n(\mathbb{k}[\Gamma_j])$ are elementary transformations with coefficients in the ring $\mathbb{k}[\Gamma_j]$. Example $\Gamma_j \simeq \mathbb{A}^1_{\mathbb{k}}$.

1.3 Preliminaries

Synopsis In this section we recall some relevant notions. In particular, the notion of locally nilpotent derivation (LND) and an algebraic $\mathbb{G}_a(\Bbbk)-$action. The Theorem 1.3.16 describes the equivalence of these two notions. Locally nilpotent derivations and $\mathbb{G}_a(\Bbbk)-$actions allow us to define the notion of a flexible variety, and flexible varieties are the main objects we work with in this thesis. Certain important properties of flexible varieties are described as well, and a number of examples show that flexible varieties represent a huge class of objects. In addition, we recall the notion of a Mennicke symbol which is closely related to the congruence subgroup problem and the Theorem of Bass, Milnor and Serre (the Theorem 1.3.56). This theorem plays a very important role in the proof of the main theorems of the thesis (the Theorem 1.2.8 and the Theorem 1.2.9.) Finally, we briefly recall some facts from the Representation Theory, which are also relevant to the proofs of the main theorems of the thesis.

1.3.1 Locally Nilpotent Derivations

Definition 1.3.1. Let A be a ring. A mapping $\partial : A \to A$ which is an endomorphism of additive group of A and satisfies the Leibniz rule

$$\partial(ab) = \partial(a)b + a\partial(b)$$

is called a **derivation** on A.

Notation 1.3.2. The set of all derivations on a ring A is denoted $\mathrm{Der}(A)$.

Below are some definitions and theorems related to locally nilpotent derivations that can be found in [Fre06, Dai03]

Notation 1.3.3. For a derivation $\partial \in \mathrm{Der}(A)$ denote by $\mathrm{Nil}(\partial) \subset A$ the following subset of A

$$\mathrm{Nil}(\partial) := \{b \in B | \exists m \in \mathbb{N},\, \partial^m(b) = 0\}.$$

Definition 1.3.4. Let A be a ring and $\delta \in \mathrm{Der}(A)$ be a derivation on A. δ is called a **locally nilpotent derivation** (LND) if for every element a in a ring A there exists a natural number $n \in \mathbb{N}$ such that $\delta^n(a) = 0$. Using notation 1.3.3 this means that $\mathrm{Nil}(\delta) = A$.

Notation 1.3.5. The set of all locally nilpotent derivations on A is denoted $\mathrm{LND}(A)$.

Definition 1.3.6. Given $\delta \in \mathrm{LND}(A)$, the **kernel** of δ is

$$\ker(\delta) := \{a \in A | \delta(a) = 0\}.$$

Notation 1.3.7. For $A \subset B$ - rings, the set of locally nilpotent derivations on B that vanish on A is denoted $\mathrm{LND}_A(B)$:

$$\mathrm{LND}_A(B) := \{\delta \in \mathrm{LND}(B) | A \subset \ker(\delta)\}.$$

Definition 1.3.8. Let A be a ring. A degree function on ring A is a map

$$\deg : A \to \mathbb{N} \cup \{-\infty\} \text{ satisfying}$$

$$\begin{aligned} &a) \deg(a) = 0 \quad \text{if and only if} \quad a = 0 \\ &b) \deg(ab) = \deg(a) + \deg(b) \\ &c) \deg(a+b) \leq \deg(a) + \deg(b) \end{aligned}$$

for all $a, b \in A$.

Definition 1.3.9. Given $\delta \in \mathrm{LND}(A)$ it defines a degree function via

$$\begin{aligned} &\deg_\delta(a) = \max\{n \in \mathbb{N} | \delta^n(a) \neq 0\} \quad \text{if} \quad a \neq 0 \\ &\deg_\delta(0) = -\infty \end{aligned}$$

Lemma 1.3.10. *Let B be a ring, $\delta_1, \delta_2 \in \mathrm{LND}(B)$. If $\delta_2 \circ \delta_1 = \delta_1 \circ \delta_2$ then $\delta_1 + \delta_2 \in \mathrm{LND}(B)$.*

Denote $\delta^n := \delta \circ \delta \circ \ldots \circ \delta$ the n-times composition of the derivation δ with itself. It simply follows by induction that

$$\delta^n(ab) = \sum_{i=0}^{n} \binom{n}{i} \delta^i(a)\delta^{n-i}(b) \tag{1.1}$$

In particular, the formula (1.1) implies that for $\partial \in \mathrm{Der}(A)$, $\mathrm{Nil}(\partial)$ is a subring of A.

Definition 1.3.11. Given a locally nilpotent derivation $\delta \in \mathrm{LND}(A)$ and $f \in \ker(\delta)$, $f\delta$ is called a **replica** of δ. The formula (1.1) implies that the replica $f\delta$ is again a locally nilpotent derivation.

Example 1.3.12. Let $A = \mathbb{k}[x, y]$, then the derivation

$$\delta = (x + y)\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

is locally nilpotent since

$$\delta^2(x) = \delta(x + y) = (x + y) - (x - y) = 0$$

and

$$\delta^2(y) = \delta(-(x + y)) = -(x + y) + (x - y) = 0$$

Example 1.3.13. Let $B = A[x_1, \ldots, x_n]$. Then an A-derivation $\partial : B \to B$ is called **triangular** if $\partial(x_i) \in A[x_1, \ldots, x_{i-1}]$ for $i = 1, \ldots, n$. Every triangular derivation is locally nilpotent: by induction $x_i \in \mathrm{Nil}(\partial)$, $i = 1, \ldots, n$, hence $\mathrm{Nil}(\partial) = A$.

Definition 1.3.14. Let $\mathbb{k}$ be an algebraically closed field of characteristic 0 and X - a $\mathbb{k}$ - variety. An algebraic $\mathbb{G}_a(\mathbb{k})-$**action**, also called a $(\mathbb{k}, +)$ action, is a morphism

$$\alpha : \mathbb{k} \times X \to X \tag{1.2}$$

such that

$$\alpha(0, x) = x \quad \forall x \in X$$

$$\alpha(t_1 + t_2, x) = \alpha(t_1, \alpha(t_2, x)) \quad \forall t_1, t_2 \in k,\, \forall x \in X$$

$\mathbb{G}_a(\Bbbk)$ is an algebraic group via

$$\mathbb{G}_a(\Bbbk) = \operatorname{Spec}(\Bbbk[t])$$

with the algebraic group structure

$$\mathbb{G}_a(\Bbbk) \times \mathbb{G}_a(\Bbbk) \to \mathbb{G}_a(\Bbbk)$$

defined by the $\Bbbk-$ algebra homomorphism

$$\phi : \Bbbk[t] \to \Bbbk[x, y],$$
$$\phi(t) = x + y.$$

And the action (1.2) of the group $\mathbb{G}_a(\Bbbk)$ on the variety X, with $A = \Bbbk[X]$, in terms of coordinate rings, is given by a $\Bbbk-$algebra homomorphism

$$\rho : A \to A[t], \tag{1.3}$$

satisfying

1. $$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[rr, bend right, "\mathrm{id}"'] & A[t] \arrow[r, "\epsilon_0"] & A \end{tikzcd}, \tag{1.4}$$

   with ϵ_0 being evaluation at $t = 0$

2. $$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "\rho"] & A[t] \arrow[d, "\tilde{\phi}"] \\ A[t] \arrow[r, "\psi"] & A[x, y] \end{tikzcd} \tag{1.5}$$

   where $\tilde{\phi}$ is defined by

   $$\tilde{\phi}|_A = \mathrm{id}_A, \quad \tilde{\phi}(t) = x + y,$$

   and ψ is defined by

   $$\psi(t) = x, \quad \psi|_A = \tilde{\phi} \circ \rho.$$

Example 1.3.15. The translation

$$\mathbb{k} \times \mathbb{A}^n_{\mathbb{k}} \to \mathbb{A}^n_{\mathbb{k}},$$

defined by

$$(t, x_1, \ldots, x_n) \mapsto (x_1 + t, \ldots, x_n + t),$$

is an example of a $\mathbb{G}_a(\mathbb{k})-$ action.

Given a $\mathbb{k}-$ algebra A and $\delta \in \mathrm{LND}_{\mathbb{k}}(A)$, for each $t \in \mathbb{k}$ one can define the exponential map

$$e^{t\delta} : A \to A \tag{1.6}$$

by

$$e^{t\delta}(a) = a + \frac{t\delta(a)}{1!} + \frac{(t\delta)^2(a)}{2!} + \cdots \tag{1.7}$$

The sum terminated since for some $n \in \mathbb{N} : \delta^n(a) = 0$. Obviously

$$e^{t\delta}(a+b) = e^{t\delta}(a) + e^{t\delta}(b),$$

and

$$\begin{aligned} e^{t\delta}(ab) &= \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{i=0}^{n} \binom{n}{i} \delta^i(a)\delta^{n-i}(b) = \\ &= \sum_{i=0}^{\infty} \frac{t^i \delta^i(a)}{i!} \sum_{j=0}^{\infty} \frac{t^j \delta^j(b)}{j!} = e^{t\delta}(a)e^{t\delta}(b), \end{aligned}$$

which gives $e^{t\delta}$ is a ring homomorphism. Moreover, since

$$e^{t_1\delta}(e^{t_2\delta}(a)) = \sum_{m=0}^{\infty} \frac{t_1^m}{m!} \delta^m \left(\sum_{n=0}^{\infty} \frac{t_2^n}{n!} \delta^n(a) \right) =$$

$$= \sum_{k=0}^{\infty} \left(\sum_{m+n=k} \frac{k!}{m!n!} t_1^m t_2^n \right) \frac{\delta^k(a)}{k!} = \sum_{k=0}^{\infty} (t_1 + t_2)^k \frac{\delta^k(a)}{k!} = e^{(t_1+t_2)\delta}(a),$$

it follows that (1.6) is an isomorphism with $e^{-t\delta}$ being it's inverse.

Theorem 1.3.16. *[Dai03] Let A be a $\mathbb{k}-$ algebra, then there exists a bijection:*

$$\mathrm{LND}_{\mathbb{k}}(A) \longleftrightarrow \textit{set of } \mathbb{G}_a(\mathbb{k}) \textit{ actions on } \mathrm{Spec}(A). \tag{1.8}$$

The correspondence (1.8) is the following

1. Given $\delta \in \mathrm{LND}_{\Bbbk}(A)$ and $t \in \Bbbk$ define a $\Bbbk$ - algebra homomorphism

$$\rho : A \to A[t],$$

with

$$\rho(a) = e^{t\delta}(a).$$

This gives a $\mathbb{G}_a(\Bbbk)$ action.

2. Vice versa, given an algebraic action (1.3) define a derivation δ by

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[rrr, bend right, "\delta"'] & A[t] \arrow[r, "d/dt"] & A[t] \arrow[r, "\epsilon_0"] & A \end{tikzcd},$$

which is locally nilpotent since d/dt is.

Definition 1.3.17. Given a group G acting on a variety X, $A = \Bbbk[X]$,

$$A^G := \{a \in A | \forall g \in G, g.a = a\} \tag{1.9}$$

is called the **ring of invariants** of the action G.

In case of the $G := \mathbb{G}_a(\Bbbk)$ action associated with a locally nilpotent derivation δ, since $\Bbbk$ has characteristic zero, it is obvious that

$$\Bbbk[X]^G = \ker(\delta),$$

since for $f \in \Bbbk[X]$ it follows

$$f \in \Bbbk[X]^G \iff e^{t\delta}(f) = f, \forall t \in \Bbbk, \iff f \in \ker(\delta).$$

1.3.2 Flexible varieties

Definition 1.3.18. Given a group G action on a space X, and a natural number $m \in \mathbb{N}$, the action is called $m-$**transitive**, if for any pair of m-tuples $(x_1, \dots, x_m)$ and $(x'_1, \dots, x'_m)$ of distinct points in X, there exists $g \in G$ such that

$$g.x_i = x'_i$$

Definition 1.3.19. The action of G on X is called **infinite transitive** if it is m-transitive for every $m \in \mathbb{N}$.

Definition 1.3.20. Let X be a variety, and consider the group $\mathrm{Aut}(X)$ of automorphisms of the variety X. The subgroup of $\mathrm{Aut}(X)$, which is the group generated by the elements of all $\mathbb{G}_a-$ actions on X, is called the **special automorphism group** of the variety X. It is denoted by $\mathrm{SAut}(X)$.

Notation 1.3.21. Given Y a closed subvariety of X, denote by $\mathrm{SAut}\,(X/Y)$ to be the subgroup of $\mathrm{SAut}(X)$ consisting of elements that fix Y pointwise.

Definition 1.3.22. A quasi-affine algebraic variety X, with $\dim(X) \geq 2$, is said to be **flexible** if for every point $x \in X_{reg}$, X_{reg} - the set of regular points of X, the tangent space $T_x X$ is spanned by the tangent vectors to the orbits $G.x$, $G \in \mathrm{SAut}(X)$.

The following theorem gives equivalent definitions of the notion of flexibility.

Theorem 1.3.23. *[AFK+13] Let X be a smooth affine algebraic variety with* $\dim(X) \geq 2$. *The following are equivalent*

1. *X is flexible.*

2. *The group* $\mathrm{SAut}\,X$ *acts transitively on X.*

3. *The group* $\mathrm{SAut}\,X$ *acts infinitely transitively on X.*

Given a variety X, $A = \Bbbk[X]$, with a group G acting on it, the inclusion $\iota : A^G \hookrightarrow A$ induces a morphism $\rho : X \to Q$, with $Q = \Bbbk[A^G]$, called the **algebraic quotient.**

Notation 1.3.24. The algebraic quotient is denoted as

$$X \to X /\!\!/ G.$$

In the case of an affine variety X, its ring of regular functions $A = \Bbbk[X]$ is a finitely generated algebra over $\Bbbk$. A special case of the Hilbert's 14th problem asks whether the A^G is a finitely generated algebra over $\Bbbk$ [Dai07]. In the case when G is a reductive group (e.g. $\mathrm{GL}_n, \mathrm{SL}_n$), then A^G is a finitely generated $\Bbbk-$ algebra. This is the theorem of Hilbert and Mumford [Spr06]. In 1959 Nagata constructed a counterexample to Hilbert's conjecture [Nag59]. In 2000 in the paper of Freudenberg [Fre00] it was shown that a kernel of a certain locally nilpotent derivation in $\mathbb{A}^6_{\Bbbk}$ is not a finitely generated $\Bbbk-$ algebra. Thus in general for an affine variety X it's algebraic quotient $X /\!\!/ G$ is not necessarily affine. So we are interested in not only affine but also quasi-affine varieties.

Definition 1.3.25. Quasi-affine algebraic variety is a Zariski open subset of an affine algebraic variety.

In the work of Winkelmann, it is proved that $Y = \mathrm{Spec}(A^G)$ is always at least quasi-affine [Win03].

Definition 1.3.26. Given a variety X and a group G acting on it, the morphism of varieties

$$\pi : X \to Y$$

is called a geometric quotient, if

1. for every $y \in Y$ the fiber $\pi^{-1}(y)$ is an orbit of the action
2. the topology of Y is the quotient topology
3. for any Zariski open subset $V \subset Y$ we have an isomorphism

$$\pi^\star : \Bbbk[V] \xrightarrow{\sim} \Bbbk[\pi^{-1}(V)]^G.$$

Notation 1.3.27. The geometric quotient is denoted as

$$X \to X/G.$$

Given a group G acting on a variety X, it induces an action on the field of rational functions $\Bbbk(X)$ by

$$g.f(x) = f(g^{-1}.x),\ x \in X, g \in G, f \in \Bbbk(X)$$

Definition 1.3.28. Given a group G acting on a variety X,

$$\Bbbk(X)^G := \{f \in \Bbbk(X) | \forall g \in G, g.f = f\} \tag{1.10}$$

is called the **field of rational invariants.**

Since every rational function $f \in \Bbbk(X)^G$ is constant on the orbits, it induces a function on the orbit space:

$$\bar{f}(\mathcal{O}) := f(x),\ x \in \mathcal{O},$$

where $\mathcal{O}$ is an orbit.

Definition 1.3.29. A rational invariant $f \in \Bbbk(X)^G$ is said to **separate orbits** $\mathcal{O}_1$ and $\mathcal{O}_2$ if

$$\bar{f}(\mathcal{O}_1) \neq \bar{f}(\mathcal{O}_2).$$

Definition 1.3.30. A subset $S \subset \Bbbk(X)^G$ of the field of rational invariants is said to **separate general orbits**(or **separate orbits in general position**) if there exists nonempty Zariski open subset $U \subset X$ such that for every x_1, x_2, with non-equal orbits $\mathcal{O}_{x_1} \neq \mathcal{O}_{x_2}$, there exists $f \in \Bbbk(X)^G$ with

$$\bar{f}(\mathcal{O}_{x_1}) \neq \bar{f}(\mathcal{O}_{x_2})$$

Theorem 1.3.31. *(Rosenlicht) [PV94] Given an algebraic action*

$$\alpha : G \times X \to X$$

on the irreducible variety X there exists a finite set of rational invariants that separate general orbits.

In other words, Rosenlicht's theorem says that on some nonempty Zariski open subset $U \subset X$ we have the geometric quotient

$$U \to U/G$$

Definition 1.3.32. Given a category C and an object X in C with a group G acting on it

$$\alpha : G \times X \to X,$$

a **categorical quotient** is an object $Y \in C$ together with a morphism

$$\pi : X \to Y$$

such that

(i) the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\alpha"] \arrow[dr, "\pi \circ p_2"'] & X \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

commutes, with p_2 being the second projection.

(ii) any other morphism $X \to Z$ satisfying (i) factors through π :

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[dr] & Y \arrow[d] \\ & Z. \end{tikzcd}$$

Given a flexible variety X, by Rosenlicht's theorem there exist rational functions $f_1, \ldots, f_m \in \Bbbk(X)$ that separate orbits in general orbits. These functions give a rational map

$$\rho : X \to Q.$$

Since we are considering $G_a(\Bbbk)$ - action, the functions f_i must be regular. If we suppose that they are not regular, then the map ρ above will have indeterminacy subset. At this subset the closures of general fibers of ρ must meet. For a $G_a(\Bbbk)$-action the fibers, which are the orbits of the action, are lines. Hence, since they are not closed their closures must be a projective lines. This is impossible since X a quasi-affine variety.

Definition 1.3.33. The morphism

$$\rho : X \to Q \tag{1.11}$$

given by regular functions $\Bbbk[X]^{G_a}$ that separate general orbits, and with Q normal, is called a **partial quotient.**

Remark 1.3.34. In case if the functions $f_1, \ldots, f_m$ generate the subring $\Bbbk[X]^{G_a}$, the morphism (1.11) is the categorical quotient.

Flexible varieties represent a huge class of objects. Here are some examples that demonstrate it.

Example 1.3.35. Let $X = \mathbb{A}^n_{\Bbbk}$, with $n \geq 2$. Then X is flexible since the group $\operatorname{SAut} X$ acts transitively on X.

The following example is the Theorem of Gromov and Winkelmann [Gro13] [Win90]

Example 1.3.36. Let Y be a closed subvariety in $\mathbb{A}^n_{\Bbbk}$, with $\operatorname{codim}_{\mathbb{A}^n_{\Bbbk}} Y \geq 2$. Then the pointwise stabilizer subgroup $\operatorname{SAut}(\mathbb{A}^n_{\Bbbk}/Y)$ of Y in $\operatorname{SAut}(\mathbb{A}^n_{\Bbbk})$ acts transitively on $\mathbb{A}^n_{\Bbbk} - Y$. This in particular implies that $\mathbb{A}^n_{\Bbbk} - Y$ is flexible.

The next example is a generalization of Gromov - Winkelmann's Theorem. It is a theorem in the paper [FKZ13]

Example 1.3.37. Let X be a smooth quasi-affine variety with $\dim X \geq 2$, and Y be a closed subvariety of X with $\operatorname{codim}_X Y \geq 2$. And let X be a flexible variety. Then the pointwise stabilizer subgroup $\operatorname{SAut}(X/Y)$ of Y in $\operatorname{SAut}(X)$ acts infinitely transitively on $X - Y$, which of course implies that $X - Y$ is flexible as well.

Definition 1.3.38. Let X be an affine variety and $f \in \Bbbk[X]$ be a non-constant regular function. The affine variety

$$\operatorname{Susp}(X, f) \coloneqq \left\{(u, v, x) \in \Bbbk^2 \times X \;\middle|\; uv - f(x) = 0\right\}$$

Example 1.3.39. A suspension over a flexible affine variety is again flexible [AZK12].

Definition 1.3.40. A normal algebraic variety is called **toric** if it admits a regular action of algebraic torus $T \simeq (\Bbbk^\star)^n$ with an open orbit.

Definition 1.3.41. An affine toric variety X is called **non-degenerate** if the only invertible regular functions on X are nonzero constants.

Example 1.3.42. Any non-degenerate affine toric variety is flexible [AZK12].

1.3.3 Mennicke symbol

Let A be a Dedekind ring and $\mathfrak{q}$ a nonzero ideal in A. Consider the set of pairs

$$W_{\mathfrak{q}} = \{(a,b) \in A^2 \mid a = 1 \mod \mathfrak{q},\ b = 0 \mod \mathfrak{q}, \text{ and } (a) + (b) = (1)\}$$

where (a) and (b) are the corresponding ideals of elements a and b, (1) - the unit ideal.

Definition 1.3.43. Two pairs (a_1, b_1) and (a_2, b_2) are called $\mathfrak{q}$ equivalent, denoted

$$(a_1, b_1) \sim_{\mathfrak{q}} (a_2, b_2),$$

if one can be obtain from other by a finite number transformations of the form:

$$(a,b) \mapsto (a, b + ra) \quad \text{with} \quad r \in \mathfrak{q},$$
$$(a,b) \mapsto (a + tb, b) \quad \text{with} \quad t \in A.$$

Definition 1.3.44. Let C be a group. A **Mennicke symbol** on $W_{\mathfrak{q}}$ is a function $[\,] : W_{\mathfrak{q}} \to C$, notation

$$(a,b) \mapsto \begin{bmatrix} b \\ a \end{bmatrix},$$

satisfying the following two conditions:

$$MS\,1: \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1, \quad \text{and} \quad \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} = \begin{bmatrix} b_2 \\ a_2 \end{bmatrix} \quad if \quad (a_1, b_1) \sim_{\mathfrak{q}} (a_2, b_2);$$
$$MS\,2: \ If \quad (a_1, b_1), (a_2, b_2) \in W_{\mathfrak{q}} \quad \text{then} \quad \begin{bmatrix} b_1 b_2 \\ a_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ a_1 \end{bmatrix} \begin{bmatrix} b_2 \\ a_1 \end{bmatrix}$$

Definition 1.3.45. The free group on $W_{\mathfrak{q}}$ modulo relations $MS\,1 - MS\,2$, denoted by $\mathcal{C}_{\mathfrak{q}}$, is called the **universal Mennicke symbol.**

Lemma 1.3.46 (Lam, Mennicke-Newman)**.** *[BMS67] If $[\,] : W_{\mathfrak{q}} \to C$ is a Mennicke symbol, (a_1, b), $(a_2, b) \in W_{\mathfrak{q}}$, then*

$$\begin{bmatrix} b \\ a_1 a_2 \end{bmatrix} = \begin{bmatrix} b \\ a_1 \end{bmatrix} \begin{bmatrix} b \\ a_2 \end{bmatrix} \tag{1.12}$$

Let $\mathbb{k}$ be an algebraically closed field of characteristic 0. Let $A = \mathbb{k}[z]$ be the ring of polynomials in one variable z over the field $\mathbb{k}$, and $q(z)$ be a polynomial in $\mathbb{k}[z]$. Consider the ideal $\mathfrak{q} = (q(z))$ generated by the polynomial $q(z)$. Consider the Mennicke symbol $\mathcal{C}_{\mathfrak{q}}$.

Lemma 1.3.47. *The following hold:*

(a)

$$\begin{bmatrix} f(z)q(z) \\ 1 \end{bmatrix} = 1, \quad f(z) \in \mathbb{k}[z]$$

(b)

$$\begin{bmatrix} 0 \\ 1 + f(z)q(z) \end{bmatrix} = 1, \quad f(z) \in \mathbb{k}[z];$$

(c)

$$\begin{bmatrix} cf_2(z)q(z) \\ 1 + f_1(z)q(z) \end{bmatrix} = \begin{bmatrix} f_2(z)q(z) \\ 1 + f_1(z)q(z) \end{bmatrix}, \quad c \in \mathbb{k}, c \neq 0;$$

Proof. (a) By $MS\,2$ and $MS\,1$

$$\begin{bmatrix} f(z)q(z) \\ 1 \end{bmatrix} = \begin{bmatrix} f(z)q(z) - f(z)q(z) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1;$$

(b) denote

$$g := \begin{bmatrix} 0 \\ 1 + f(z)q(z) \end{bmatrix}, \quad \text{then by} \quad MS\,2 \quad \text{we have} \quad g^2 = g \implies g = 1;$$

(c) First note that

$$\begin{bmatrix} \frac{1}{c}q(z) \\ 1 + f_1(z)q(z) \end{bmatrix} = \begin{bmatrix} \frac{1}{c}q(z) \\ 1 + f_1(z)q(z) - (cf_1(z)) * (\frac{1}{c}q(z)) \end{bmatrix} = \\ = \begin{bmatrix} \frac{1}{c}q(z) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1,$$

hence

$$\begin{bmatrix} cf_2(z)q(z) \\ 1 + f_1(z)q(z) \end{bmatrix} = \begin{bmatrix} cf_2(z)q(z) \\ 1 + f_1(z)q(z) \end{bmatrix} \begin{bmatrix} \frac{1}{c}q(z) \\ 1 + f_1(z)q(z) \end{bmatrix} = \\ = \begin{bmatrix} f_2(z)q^2(z) \\ 1 + f_1(z)q(z) \end{bmatrix} = \begin{bmatrix} f_2(z)q(z) \\ 1 + f_1(z)q(z) \end{bmatrix} \begin{bmatrix} q(z) \\ 1 + f_1(z)q(z) \end{bmatrix} = \\ = \begin{bmatrix} f_2(z)q(z) \\ 1 + f_1(z)q(z) \end{bmatrix}.$$

$\square$

Proposition 1.3.48 (Exercise). *Let $A = \mathbb{k}[x]$ and $\mathfrak{q} = (z) \subset \mathbb{k}[z]$. Then the group of Mennicke symbols $\mathcal{C}_{\mathfrak{q}}$ for the ideal $\mathfrak{q}$ is trivial.*

Proof. Let $(a, b) \in W_{\mathfrak{q}}$. Then $a = 1 + zf_1(z)$, $b = zf_2(z)$, with $f_1(z), f_2(z) \in \mathbb{k}[z]$.

Step 1: $f_1(z) = g(z)f_2(z) + r(z)$ with $\deg(r(z)) < \deg(f_2(z))$ or $r(z) \equiv 0$.

$$\begin{aligned} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} zf_2(z) \\ 1 + zf_1(z) \end{bmatrix} = \begin{bmatrix} zf_2(z) \\ 1 + z(g(z)f_2(z) + r(z)) \end{bmatrix} = \\ = \begin{bmatrix} zf_2(z) \\ 1 + zg(z)f_2(z) + zr(z) - g(z)(zf_2(z)) \end{bmatrix} = \begin{bmatrix} zf_2(z) \\ 1 + zr(z) \end{bmatrix}. \end{aligned} \tag{1.13}$$

If $r(z) \equiv 0$ then by Lemma 1.3.47 *a*)

$$\begin{bmatrix} b \\ a \end{bmatrix} = 1$$

and we are done. Otherwise, performing (1.13) we can always achieve

$$\deg(f_1(z)) < \deg(f_2(z)).$$

Step 2: write

$$f_2(z) = b_{n_2} z^{n_2} + \ldots + b_0,$$

$$f_1(z) = a_{n_1} z^{n_1} + \ldots + a_0,$$

then by Step 1 $n := n_2 - n_1 \geq 1$ and

$$\frac{b_{n_2}}{a_{n_1}} z^n \in (z).$$

Now

$$\begin{aligned} \begin{bmatrix} zf_2(z) \\ 1 + zf_1(z) \end{bmatrix} = \begin{bmatrix} zf_2(z) - \frac{b_{n_2}}{a_{n_1}} z^n \left(1 + zf_1(z)\right) \\ 1 + zf_1(z) \end{bmatrix} = \\ \begin{bmatrix} z(b_{n_2} z^{n_2} + \ldots + b_0) - \frac{b_{n_2}}{a_{n_1}} z^{n_2 - n_1}(1 + z(a_{n_1} z^{n_1} + \ldots + a_0)) \\ 1 + zf_1(z) \end{bmatrix} = \\ = \begin{bmatrix} z(c_{n_2 - 1} z^{n_2 - 1} + \ldots c_0) + z\left(-\frac{b_{n_2}}{a_{n_1}} z^{n_2 - 1 - n_1}\right) \\ 1 + zf_1(z) \end{bmatrix} = \begin{bmatrix} z\tilde{f}_2(z) \\ 1 + zf_1(z) \end{bmatrix} \end{aligned}$$

with $\deg(\tilde{f}_2(z)) < \deg(f_2(z))$. If it happens that $\tilde{f}_2 \equiv 0$ then we are done by Lemma 1.3.47 *b*). We repeat this performance and reduce the degree of $f_2(z)$ until we achieve $\deg(f_2(z)) \leq \deg(f_1(z))$.

Step 3: We repeat subsequently Step 1 and Step 2 until we achieve $\deg(f_2) = \deg(f_1) = 0$, then

$$\begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} b_0 z \\ 1 + a_0 z \end{bmatrix} = \begin{bmatrix} b_0 z \\ 1 \end{bmatrix} = 1.$$

$\square$

Let $M_n(A)$ be the ring of $n \times n$ matrices with coefficients in the ring A.

Definition 1.3.49. The **standard elementary matrices** over the ring A are defined as:

$$e_{ij} = \mathrm{Id}_n + r\epsilon_{ij},\ i \neq j,\ r \in A \tag{1.14}$$

where ϵ_{ij} is the matrix with 1 at the position (i, j) and 0 everywhere else.

Remark 1.3.50. Matrices ϵ_{ij} satisfy obvious relation

$$\epsilon_{ij}\epsilon_{km} = \delta_{jk}\epsilon_{im}$$

with δ_{jk} being the Kronecker symbol.

Notation 1.3.51. The group generated by all standard elementary matrices (1.14) is denoted by $\mathrm{E}_n(A)$.

Definition 1.3.52. The group generated by standard elementary matrices of the form

$$e_{ij} = \mathrm{Id}_n + r\epsilon_{ij},\ i \neq j,\ r \in \mathfrak{q}$$

is called the group of standard $\mathfrak{q}$ **- elementary** matrices, and is denoted by $\mathrm{E}_n(A, \mathfrak{q})$.

Remark 1.3.53. The group $\mathrm{E}_n(A, \mathfrak{q})$ is a normal subgroup of $\mathrm{E}_n(A)$.

Proposition 1.3.54. *[Bas06] Let $n \geq 3$. Given two ideals $\mathfrak{q}_1$ and $\mathfrak{q}_2$ in the ring A, it follows*

$$[\mathrm{E}_n(A, \mathfrak{q}_1), \mathrm{E}_n(A, (\mathfrak{q}_2)] \supset \mathrm{E}_n(A, \mathfrak{q}_1\mathfrak{q}_2).$$

Here $[,]$ stands for the commutator.

Definition 1.3.55. Given an ideal $\mathfrak{q}$ of the ring A, the subgroup $\mathrm{SL}_n(A, \mathfrak{q})$ of the group $\mathrm{SL}_n(A)$, defined as

$$\mathrm{SL}_n(A, \mathfrak{q}) := \ker\{\mathrm{SL}_n(A) \to \mathrm{SL}_n(A/\mathfrak{q})\},$$

is called the $\mathfrak{q}-$ **principal congruence subgroup.**

The following theorem is due to Bass, Milnor and Serre.

Theorem 1.3.56. *[BMS67] Let A be a Dedekind ring, $\mathfrak{q}$ - an ideal in A, and $n \geq 3$. Then*

$$\mathrm{E}_n(A, \mathfrak{q}) = [\mathrm{SL}_n(A), \mathrm{SL}_n(A, \mathfrak{q})]. \tag{1.15}$$

Furthermore, if the group of Mennicke symbols $\mathcal{C}_{\mathfrak{q}}$ for the ideal $\mathfrak{q}$ is trivial, then

$$\mathrm{E}_n(A, \mathfrak{q}) = \mathrm{SL}_n(A, \mathfrak{q}).$$

Here $[,]$ stands for the commutator.

Corollary 1.3.57. *Let $\Bbbk$ be an algebraically closed field of characteristic 0, and n - a natural number, $n \geq 3$. Then*

$$\mathrm{E}_n(\Bbbk[z], (z)) = \mathrm{SL}_n(\Bbbk[z], (z)) \tag{1.16}$$

Proof. The proof follows immediately from Proposition 1.3.48 and Theorem 1.3.56 □

Corollary 1.3.58. *Given a Dedekind ring A, if the Mennicke symbol for this ring is trivial (meaning it is trivial for the unit ideal), then for $n \geq 3$*

$$\mathrm{E}_n(A) = \mathrm{SL}_n(A).$$

1.3.4 Some Facts From Representation Theory

Let G be a group, A a ring, and M a free module over A of finite rank. Denote $\mathrm{Aut}(M)$ to be the group of A - module automorphisms of M.

Definition 1.3.59. A group homomorphism

$$\phi : G \to \mathrm{Aut}(M) \tag{1.17}$$

is called a **representation** of G on M.

Definition 1.3.60. Given M a representation of G, a submodule N of M is a **subrepresentation** if it is stable under G. That is, for every element g in G the automorphism $\phi(g)$ on M restricts to an automorphism of N.

Definition 1.3.61. A representation M of G is called **irreducible** if it has no proper subrepresentations.

Let $u_1, \ldots, u_n$ be indeterminants and $F^1(u_1, \ldots, u_n), \ldots, F^n(u_1, \ldots, u_n)$ be polynomials in $u_1, \ldots, u_n$ with coefficients in A. We denote by $F^i_j(u_1, \ldots, u_n)$ the homogeneous part of $F^i(u_1, \ldots, u_n)$ of degree j. We make the following notations

- $\vec{u} = (u_1, \ldots, u_n)$
- $\vec{F}(\vec{u}) = (F^1(\vec{u}), \ldots, F^n(\vec{u}))$
- $$\mathrm{div}(\vec{F}(\vec{u})) = \frac{\partial F^1(\vec{u})}{\partial u_1} + \ldots + \frac{\partial F^n(\vec{u})}{\partial u_n}$$
- $\vec{F}_j(\vec{u}) = \left(F^1_j(\vec{u}), \ldots, F^n_j(\vec{u})\right)$
- $\mathscr{F}_j = \left\{\left(F^1_j(\vec{u}), \ldots, F^n_j(\vec{u})\right)\right\}$ - the set of n-tuples of homogeneous j-forms in n variables $u_1, \ldots, u_n$, with coefficients in A.
- $$\mathscr{F}^0_j := \left\{\vec{F}_j(\vec{u}) \in \mathscr{F}_j \,\middle|\, \mathrm{div}(\vec{F}_j(\vec{u})) = 0\right\}$$

Remark 1.3.62. The sets $\mathscr{F}_j$ and $\mathscr{F}^0_j$ are $A-$ modules in an obvious way.

Let us treat $\vec{u}$ as a column vector and the elements of $\mathscr{F}_j$ (respectively $\mathscr{F}_j^0$) as the row vectors. For $m \in \mathbb{N}$ we have the following $\mathrm{SL}_n(A)$ action on $\mathscr{F}_m$ (respectively $\mathscr{F}_m^0$) defined by

$$a.\vec{F}_m(\vec{u})) \mapsto \vec{F}_m(a\vec{u})a^{-1},$$

with $a \in \mathrm{SL}_n(A)$ and $\vec{F}_m \in \mathscr{F}_m$ (respectively $\vec{F}_m \in \mathscr{F}_m^0$) This is indeed an action since for $a, b \in \mathrm{SL}_n(A)$ it follows

$$(ab).(\vec{F}_m(\vec{u})) = \vec{F}_m(ab\vec{u})(ab)^{-1} = \vec{F}_m(ab\vec{u})b^{-1}a^{-1} =$$
$$= a.(\vec{F}_m(b\vec{u})b^{-1}) = a.(b.\vec{F}_m(\vec{u})).$$

Let us consider this action of $\mathrm{SL}_n(A)$ on $\vec{F}_m \in \mathscr{F}_m^0$ and write it as a group homomorphism

$$\phi : \mathrm{SL}_n(A) \to \mathrm{Aut}(\mathscr{F}_m^0). \tag{1.18}$$

Most results in Representation Theory are proven when A is a field and M is a finite dimensional vector space. The following proposition is a result for this case:

Proposition 1.3.63. *[FH13] The representation* (1.18) *with $A = \Bbbk$ being a field:*

$$\phi : \mathrm{SL}_n(\Bbbk) \to \mathrm{Aut}(\mathscr{F}_m^0) \tag{1.19}$$

is irreducible.

CHAPTER 2

The Extension Problem for Conormal and Jet Bundles

2.1 The Extension Problem

Synopsis In this section we consider and state the extension problem for smooth curves Y and Z, with their corresponding $m-$th infinitesimal neighborhoods Y_m and Z_m, in a flexible variety X, and prove some auxiliary propositions. In Proposition 2.1.1 we show that the isomorphism $Y_m \simeq Z_m$ between the $m-$th infinitesimal neighborhoods yields to the isomorphisms between all infinitesimal neighborhoods of lower orders: $Y_j \simeq Z_j$, $j = 0, \ldots, m-1$. In the case of the first infinitesimal neighborhood, which corresponds to $m = 1$, we have the extension problem for the conormal bundles I_Y/I_Y^2 and I_Z/I_Z^2. The Proposition 2.1.6 allows us to reduce the problem to the one, in which we consider only one curve Γ. For the sake of convenience of the proof in later chapters, we state the extension problem in terms of the normal bundle of this curve Γ, which is represented by the diagram (2.6). In the case of the infinitesimal neighborhoods of higher orders, that correspond to $m > 1$, we have the extension problem for the $m-$th jet bundles I_Y/I_Y^{m+1} and I_Z/I_Z^{m+1}. And the Proposition 2.1.11 allows us to reduce this problem to the one, in which we again consider only one curve.

Let X be a flexible variety with a coordinate ring $\Bbbk[X]$. And let Y, Z be smooth closed reduced curves in X, with the defining prime ideals I_Y and I_Z correspondingly, i.e.

$$Y = \operatorname{Spec}\left(\frac{\Bbbk[X]}{I_Y}\right), \quad Z = \operatorname{Spec}\left(\frac{\Bbbk[X]}{I_Z}\right)$$

For a natural number $m \in \mathbb{N}$ denote

$$Y_m \coloneqq \operatorname{Spec}\left(\frac{\Bbbk[X]}{I_Y^{m+1}}\right), \quad Z_m \coloneqq \operatorname{Spec}\left(\frac{\Bbbk[X]}{I_Z^{m+1}}\right),$$

in particular $Y_0 = Y$, $Z_0 = Z$. Consider an isomorphism of these infinitesimal neighborhoods:

$$Y_m \simeq Z_m.$$

In terms of the coordinate rings, this isomorphism is given by

$$\varphi_m : \frac{\Bbbk[X]}{I_Y^{m+1}} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z^{m+1}}. \tag{2.1}$$

Denote $\psi_m \coloneqq \varphi_m^{-1}$.

Proposition 2.1.1. *The isomorphism φ_m induces the isomorphisms*

$$\varphi_i : \frac{\Bbbk[X]}{I_Y^{i+1}} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z^{i+1}}, \tag{2.2}$$

for $i = 0, \dots, m-1$.

Proof. For the ideals I_Y^i / I_Y^{m+1} and I_Z^i / I_Z^{m+1} of $\frac{\Bbbk[X]}{I_Y^{i+1}}$ and $\frac{\Bbbk[X]}{I_Z^{i+1}}$ correspondingly, let

$$\pi_Y : \frac{\Bbbk[X]}{I_Y^{m+1}} \to \frac{\Bbbk[X]/I_Z^{m+1}}{I_Z^{i+1}/I_Z^{m+1}},$$

$$\pi_Z : \frac{\Bbbk[X]}{I_Z^{m+1}} \to \frac{\Bbbk[X]/I_Z^{m+1}}{I_Z^{i+1}/I_Z^{m+1}},$$

be the natural projection and consider the diagram

$$\begin{tikzcd}
\frac{\Bbbk[X]}{I_Y^{m+1}} \arrow[r, "\varphi_m", "\simeq"'] \arrow[dr] & \frac{\Bbbk[X]}{I_Z^{m+1}} \arrow[d, "\pi_Z"] \\
& \frac{\Bbbk[X]/I_Z^{m+1}}{I_Z^{i+1}/I_Z^{m+1}}
\end{tikzcd}$$

Since the ideals I_Y and I_Z are prime, the ideals I_Y/I_Y^{m+1} and I_Z/I_Z^{m+1} are the nilradicals of $\Bbbk[X]/I_Z^{m+1}$ and $\Bbbk[X]/I_Y^{m+1}$ correspondingly. The isomorphism φ_m maps bijectively nilradical I_Y/I_Y^{m+1} to nilradical I_Z/I_Z^{m+1}, and hence maps bijectively the corresponding $i-$th power

$$(I_Y/I_Y^{m+1})^i = I_Y^i/I_Y^{m+1}$$

of nilradical I_Y/I_Y^{m+1} to the corresponding $i-$th power

$$(I_Z/I_Z^{m+1})^i = I_Z^i/I_Z^{m+1}$$

of nilradical I_Z/I_Z^{m+1}. This implies that the homomorphism $\pi_Z \circ \varphi_m$ induces an isomorphism $\bar{\varphi}$ defined and shown on the diagram below:

$$\begin{tikzcd}
\frac{\Bbbk[X]}{I_Y^{m+1}} \arrow[r, "\varphi_m", "\simeq"'] \arrow[d, "\pi_Y"] \arrow[dr] & \frac{\Bbbk[X]}{I_Z^{m+1}} \arrow[d, "\pi_Z"] \\
\frac{\Bbbk[X]/I_Y^{m+1}}{I_Y^{i+1}/I_Y^{m+1}} \arrow[r, "\bar{\varphi}", "\simeq"'] & \frac{\Bbbk[X]/I_Z^{m+1}}{I_Z^{i+1}/I_Z^{m+1}}
\end{tikzcd}$$

By the third isomorphism theorem for rings, we have

$$\frac{\Bbbk[X]/I_Y^{m+1}}{I_Y^{i+1}/I_Y^{m+1}} \simeq \frac{\Bbbk[X]}{I_Y^{i+1}},$$

$$\frac{\Bbbk[X]/I_Z^{m+1}}{I_Z^{i+1}/I_Z^{m+1}} \simeq \frac{\Bbbk[X]}{I_Z^{i+1}}.$$

This implies the existence of the isomorphisms (2.2), defined on the diagram below:

$$\begin{tikzcd}
\frac{\Bbbk[X]/I_Y^{m+1}}{I_Y^{i+1}/I_Y^{m+1}} \arrow[r, "\tilde{\varphi}_m", "\simeq"'] \arrow[d, "\simeq"] & \frac{\Bbbk[X]/I_Z^{m+1}}{I_Z^{i+1}/I_Z^{m+1}} \arrow[d, "\simeq"] \\
\frac{\Bbbk[X]}{I_Y^{i+1}} \arrow[r, "\varphi_i", "\simeq"'] & \frac{\Bbbk[X]}{I_Z^{i+1}}
\end{tikzcd}$$

□

2.1.1 Case of Conormal Bundles

Let $m = 1$, and consider the isomorphism

$$\varphi_1 : \frac{\Bbbk[X]}{I_Y^2} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z^2}.$$

By Proposition 2.1.1 we have the induced isomorphism

$$\varphi_0 : \frac{\Bbbk[X]}{I_Y} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z},$$

on the curves. I_Y/I_Y^2 is a sheaf of modules over the ring $\Bbbk[X]/I_Y$ in the obvious way: given $f \in \Bbbk[X]$ and $u \in I_Y$

$$(f + I_Y) \cdot (u + I_Y^2) \mapsto fu + I_Y^2$$

is well defined. Similarly for I_Z/I_Z^2.

Remark 2.1.2. Since Y and Z are smooth, for any natural number m, I_Y/I_Y^m and I_Z/I_Z^m are not just sheaf of modules, but in fact are bundles. And for $m = 1$, I_Y/I_Y^2 and I_Z/I_Z^2 are the conormal bundles of the curves Y and Z correspondingly.

Since φ_1 gives a bijective correspondence between I_Y/I_Y^2 and I_Z/I_Z^2, it induces an isomorphism of bundles between them:

$$\bar{\varphi} : \frac{I_Y}{I_Y^2} \xrightarrow{\sim} \frac{I_Z}{I_Z^2}, \tag{2.3}$$

in the following way:

$$\bar{\varphi}((f + I_Y) \cdot (u + I_Y^2)) = \varphi_0(f + I_Y)\varphi_1(u + I_Y^2). \tag{2.4}$$

We are interested in the following problem:

Problem 2.1.3. Given the isomorphism φ_1, for which the induced isomorphism φ_0 is a restriction of a global automorphism of X :

$$\Phi : \Bbbk[X] \xrightarrow{\sim} \Bbbk[X],$$

is it the case that the isomorphism $\bar{\varphi}$ is also induced by a global automorphism of ambient space X?

Notation 2.1.4. By analogy with the notations above, given a global automorphism

$$\Phi : \Bbbk[X] \xrightarrow{\sim} \Bbbk[X],$$

we denote by Φ_0 and Φ_1 the induced isomorphisms

$$\Phi_i : \frac{\Bbbk[X]}{I_Y^{i+1}} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z^{i+1}},\ i = 0, 1,$$

on the curves and their first infinitesimal neighborhoods correspondingly. And we denote by $\bar{\Phi}$ the induced isomorphism

$$\bar{\Phi} : \frac{I_Y}{I_Y^2} \xrightarrow{\sim} \frac{I_Z}{I_Z^2}.$$

between the conormal bundles of Y and Z.

The Problem 2.1.3 can be reduced to the one, in which we consider only one curve. Let Γ be a smooth curve in X, with the defining prime ideal I, and consider the Problem:

Problem 2.1.5. Given the automorphism

$$\alpha_1 : \frac{\Bbbk[X]}{I^2} \xrightarrow{\sim} \frac{\Bbbk[X]}{I^2},$$

which induces the identity map on Γ :

$$\alpha_0 : \frac{\Bbbk[X]}{I} \xrightarrow{\mathrm{id}} \frac{\Bbbk[X]}{I},$$

and induces the isomorphism

$$\bar{\alpha} : \frac{I}{I^2} \xrightarrow{\sim} \frac{I}{I^2} \tag{2.5}$$

of the conormal bundle of Γ, is it the case that $\bar{\alpha}$ is induced by a global automorphism of X?

Proposition 2.1.6. *A positive solution to the Problem 2.1.5 yields to a positive solution of the Problem 2.1.3.*

Proof. Let

$$\varphi_1 : \frac{\Bbbk[X]}{I_Y^2} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z^2},$$

be an isomorphism on the first infinitesimal neighborhoods of the curves Y and Z, for which the isomorphism

$$\varphi_0 : \frac{\Bbbk[X]}{I_Y} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z},$$

of the curves is induced by a global automorphism

$$\Phi : \Bbbk[X] \xrightarrow{\sim} \Bbbk[X],$$

i.e. $\varphi_0 = \Phi_0$. Consider the automorphism $\bar{\alpha} \coloneqq \bar{\Phi}^{-1} \circ \bar{\varphi}$ of the bundle I_Y/I_Y^2 :

$$\begin{tikzcd} \frac{I_Y}{I_Y^2} \arrow[r, "\bar{\varphi}"'] \arrow[rr, bend left, "\bar{\alpha}"] & \frac{I_Z}{I_Z^2} \arrow[r, "\bar{\Phi}^{-1}"'] & \frac{I_Y}{I_Y^2}. \end{tikzcd}$$

Suppose that $\bar{\alpha}$ is induced by a global automorphism

$$H : \Bbbk[X] \xrightarrow{\sim} \Bbbk[X],$$

i.e. $\bar{\alpha} = \bar{H}$. Then

$$\bar{\varphi} = \bar{\Phi} \circ \bar{\alpha} = \bar{\Phi} \circ \bar{H} = \overline{(\Phi \circ H)}$$

meaning that $\bar{\varphi}$ is induced by a global automorphism $\Phi \circ H$. It remains to note that

$$\Phi_0 = \varphi_0 = \Phi_0 \circ \alpha_0 \implies \Phi_0^{-1} \circ \Phi_0 = \Phi_0^{-1} \circ \Phi_0 \circ \alpha_0 \implies \mathrm{id} = \alpha_0.$$

$\square$

Since the dual of I/I^2 is the normal bundle, we can state the problem 2.1.5 in terms of the normal bundle $\mathrm{N}\Gamma$ of the curve Γ :

Problem 2.1.7. Given the commutative diagram

$$\begin{tikzcd} \mathrm{N}\Gamma \arrow[r, "\theta_N"] \arrow[d, "\pi_\Gamma"] & \mathrm{N}\Gamma \arrow[d, "\pi_\Gamma"] \\ \Gamma \arrow[r, "id"] & \Gamma \end{tikzcd} \tag{2.6}$$

with θ_N being an automorphism of the normal bundle $\mathrm{N}\Gamma$ and π_Γ being the natural projection, is it the case that θ_n is induced by a global automorphism of X?

It will be proven in Chapter 2.4 that for a certain class of curves (in particular for lines), given an $\mathrm{SL}_n(\Bbbk[\Gamma])$ automorphism (2.5), or θ_N above, it follows that these automorphisms are induced by a global automorphism of X.

2.1.2 Case of General Jet Bundles

Let $m > 1$ and consider the isomorphism

$$\varphi_m : \frac{\Bbbk[X]}{I_Y^{m+1}} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z^{m+1}}, \tag{2.7}$$

which in turn by Proposition 2.1.1 induces isomorphisms

$$\varphi_i : \frac{\Bbbk[X]}{I_Y^{i+1}} \xrightarrow{\sim} \frac{\Bbbk[X]}{I_Z^{i+1}}, \tag{2.8}$$

for $i = 0, \dots, m-1$. We have the sheaf of modules structure for I_Y/I_Y^{m+1} over the ring $\Bbbk[X]/I_Y$ in the following way. Let $u_1, \dots, u_k$ be the generators of the ideal I_Y :

$$I_Y = (u_1, \dots, u_k).$$

Then every element of I_Y/I_Y^{m+1} can be viewed as a polynomial $F(u_1, \dots, u_k)$, in variables $u_1, \dots, u_k$, with coefficients in the ring $\Bbbk[X]/I_Y$, of degree less then or equal to m, and without a constant term. $F(u_1, \dots, u_k)$ uniquely decomposes into it's homogeneous parts:

$$F(u_1, \dots, u_k) = F_1(u_1, \dots, u_k) + \dots + F_m(u_1, \dots, u_k),$$

with $F_j(u_1, \dots, u_k)$ - homogeneous polynomial in $u_1, \dots, u_k$ of degree j. Since I_Y^j/I_Y^{j+1} is a sheaf of modules over the ring $\Bbbk[X]/I_Y$, we have an isomorphism

$$\frac{I_Y}{I_Y^{m+1}} \simeq \frac{I_Y}{I_Y^2} \oplus \frac{I_Y^2}{I_Y^3} \oplus \dots \oplus \frac{I_Y^m}{I_Y^{m+1}} \tag{2.9}$$

which defines a $\Bbbk[X]/I_Y$ sheaf of modules structure for I_Y/I_Y^{m+1}. Similarly for I_Z/I_Z^{m+1}. In analogy with (2.4), the isomorphisms (2.8) and the isomorphism (2.9) imply that we have an isomorphism

$$\bar{\varphi} : \frac{I_Y}{I_Y^{m+1}} \xrightarrow{\sim} \frac{I_Z}{I_Z^{m+1}}.$$

Definition 2.1.8. Given a smooth subvariety Y of a variety X, with the defining ideal I_Y, $\frac{I_Y}{I_Y^{m+1}}$ is called the $m-$**th jet bundle** of Y.

Consider the following problem

Problem 2.1.9. Given the isomorphism φ_m (defined in (2.7)), for which the induced isomorphism φ_0 on the curves is a restriction of a global automorphism of X, is it the case that the isomorphism $\bar{\varphi}$ is also induced by a global automorphism of X?

As with the case $m = 1$, we can reduce this problem to the one in which we consider only one subvariety:

Problem 2.1.10. Given the automorphism

$$\alpha_m : \frac{\Bbbk[X]}{I^{m+1}} \xrightarrow{\sim} \frac{\Bbbk[X]}{I^{m+1}},$$

which induces the identity map

$$\alpha_0 : \frac{\Bbbk[X]}{I} \xrightarrow{\text{id}} \frac{\Bbbk[X]}{I},$$

is the automorphism

$$\bar{\alpha} : \frac{I}{I^{m+1}} \xrightarrow{\sim} \frac{I}{I^{m+1}}$$

induced by a global automorphism of X?

Proposition 2.1.11. *A positive solution to the Problem 2.1.10 yields to a positive solution to the Problem 2.1.9.*

Proof. The proof is just the repetition of the proof of the Proposition 2.1.6. $\square$

Next, the isomorphism

$$\bar{\alpha} : \frac{I}{I^{m+1}} \xrightarrow{\sim} \frac{I}{I^{m+1}}$$

is induced by some isomorphism

$$\breve{\alpha} : I \xrightarrow{\sim} I.$$

In other words, we have a commutative diagram

$$\begin{tikzcd} I \arrow[r, "\breve{\alpha}"] \arrow[d, "\pi_m"] & I \arrow[d, "\pi_m"] \\ \frac{I}{I^{m+1}} \arrow[r, "\bar{\alpha}"] & \frac{I}{I^{m+1}} \end{tikzcd}$$

with π_m being the natural projections. After proving that the Problem 2.1.5 has a positive solution, we will use this result and prove in Chapter 2.5 that the Problem 2.1.10 has a positive solution, provided

$$\mathrm{Jac}(\breve{\alpha}) = 1 \mod I^m.$$

By abuse of notation, we may write (as in Theorem 1.2.8)

$$\mathrm{Jac}(\bar{\alpha}) = 1 \mod I^m.$$

Remark 2.1.12. The propositions 2.1.6 and 2.1.11 imply that once we prove the Theorem 1.2.8 even for one curve Γ, then the Theorem 1.2.9 will be proven as well.

2.2 Local Elementary Transformations

Synopsis In this section we consider a smooth curve Γ with a trivial normal bundle $\mathrm{N}\Gamma$. This means that $\mathrm{N}\Gamma$ is a free module over the coordinate ring $A = \Bbbk[\Gamma]$ of our curve Γ, and we have a global bases $\{v_1, \ldots, v_n\}$ for $\mathrm{N}\Gamma$. We consider two regular functions q_1 and q_2 in the ring A with non-intersecting zeros $V(q_1)$ and $V(q_2) : V(q_1) \cap V(q_2) = \emptyset$. On the Zariski open neighborhoods $U_i = \Gamma - V(q_i)$, we consider vectors $\{u_1^{(i)}, \ldots, u_n^{(i)}\}$ that form basis of $\mathrm{N}\Gamma$ on U_i, $i = 1, 2$. The main result of the chapter is the following: having q_i- standard elementary transformations $\mathrm{E}_n(A, (q_i))$ in the basis $\{u_1^{(i)}, \ldots, u_n^{(i)}\}$, we can generate all the standard elementary transformations $\mathrm{E}_n(A)$ for the global bases $\{v_1, \ldots, v_n\}$.

Let Γ be a smooth curve in a flexible variety X, with $\dim(X) \geq 4$. Denote $A \coloneqq \Bbbk[\Gamma]$ to be the ring of regular functions of our curve Γ. Since Γ is smooth, it follows that A is a Dedekind domain. We suppose that the normal bundle $\mathrm{N}\Gamma$ of our curve Γ is trivial, and we denote it by $V \coloneqq \mathrm{N}\Gamma$. Hence V is a free A module of rank $n \geq 3$. Given a regular function $q \in A$ in the ring A, denote $S_{(q)} := S^{-1}A$ to be the localization of A with respect to the multiplicative system $S = \{q^l\}_{l\geq 0}$, and similarly $V_{(q)} := S^{-1}V$ to be the localization of V. Let the vectors $\{v_1, \ldots, v_n\}$ form a basis of V and denote $\bar{v} := \{v_1, \ldots, v_n\}$. The vectors $\{u_1, \ldots, u_n\} \subset V$ form a basis for the $A_{(q)}$ - module $V_{(q)}$ if and only if there exists $n \times n$ matrix C with entries in A such that $\bar{u} = C\bar{v}$, where $\bar{u} = \{u_1, \ldots, u_n\}$, and C is an invertible(over $A_{(q)}$) matrix with entries in A.

Lemma 2.2.1. *The following holds: for some natural number $m \in \mathbb{N}$ we have*

$$C^{-1}\mathrm{SL}_n(A, (q))C \supset \mathrm{SL}_n(A, (q^m)) \tag{2.10}$$

Proof. Since the matrix C belongs to $\mathrm{GL}_n(A_{(q)})$ it follows that $\det(C) = 0$ on the zeros of q with some fixed power, say m_1, i.e. $\det(C) = q^{m_1}$. Take a natural number m, with $m > m_1$, say $m = m_1 + 1$. We have

$$C^{-1} = \frac{1}{\det C}C^\star, \quad \text{with} \quad C^\star - \text{the adjoint matrix to the matrix} \quad C,$$

and for every $a \in \mathrm{SL}_n(A,(q^m))$ we write a as

$$a = \mathrm{Id}_n + q^m b,$$

where b is an $n \times n$ matrix with coefficients in A. Now

$$\begin{aligned} CaC^{-1} &= \frac{1}{\det C} CaC^{\star} = \frac{1}{\det C} C(\mathrm{Id}_n + (a - \mathrm{Id}_n))C^{\star} = \\ &= \mathrm{Id}_n + C\frac{1}{\det C}(a - \mathrm{Id}_n)C^{\star} = \mathrm{Id}_n + q(CbC^{\star}) = \\ &= \mathrm{Id}_n \mod (q) \in \mathrm{SL}_n(A,(q)). \end{aligned}$$

Hence

$$C\mathrm{SL}_n(A,(q^m))C^{-1} \subset \mathrm{SL}_n(A,(q)),$$

and as a result

$$C^{-1}\mathrm{SL}_n(A,(q))C \supset \mathrm{SL}_n(A,(q^m))$$

□

Let $S_1 = \{x_1, \dots, x_{k_1}\}$ and $S_2 = \{y_1, \dots, y_{k_2}\}$ be disjoint sets consisting of finitely many points on our curve Γ, given by zeros of regular functions $q_1, q_2 \in A : S_1 = V(q_1)$, $S_2 = V(q_2)$, $S_1 \cap S_2 = \emptyset$. Suppose $\bar{u}^{(i)} = \{u_1^{(i)}, \dots, u_n^{(i)}\} \subset V$ form a basis for the localization $V_{(q_i)}$. Then $\bar{u}_i = C_i \bar{v}$, with C_i being an invertible(over $A_{(q_i)}$) matrix with entries in A. Change of basis in $V_{(q_i)}$ is given by a q_i-elementary transformation $e \in \mathrm{E}_n(A, q_i)$ on $\bar{u}_i$. Such a transformation yields to a transformation $C_i^{-1} e C_i$ of our global basis $\bar{v}$, $i = 1, 2$. Consider the group

$$G =< C_1^{-1}\mathrm{E}_n(A,(q_1))C_1, C_2^{-1}\mathrm{E}_n(A,(q_2))C_2 > \tag{2.11}$$

generated by all such transformations.

Theorem 2.2.2. *Let $n \geq 3$. Then there exists a natural number $l \in \mathbb{N}$ such that*

$$G \supset \mathrm{E}_n(A,(q_i^l)),\ i = 1, 2. \tag{2.12}$$

Proof. By Theorem 1.3.56 we have

$$\mathrm{E}_n(A,(q_i)) = [\mathrm{SL}_n(A), \mathrm{SL}_n(A,(q_i))],\ i = 1, 2.$$

Hence

$$G \supset C_i^{-1} E_n(A,(q_i))C_i = C_i^{-1}[\mathrm{SL}_n(A), \mathrm{SL}_n(A,(q_i))]C_i. \tag{2.13}$$

Given a commutator $[a,b] = aba^{-1}b^{-1}$ of matrices a and b and a matrix c, one has

$$c^{-1}[a,b]c = c^{-1}aba^{-1}b^{-1}c = (c^{-1}ac)(c^{-1}bc)(c^{-1}a^{-1}c)(c^{-1}b^{-1}c) =$$
$$= (c^{-1}ac)(c^{-1}bc)(c^{-1}ac)^{-1}(c^{-1}bc)^{-1} = [c^{-1}ac, c^{-1}bc].$$

Hence

$$C_i^{-1}[\mathrm{SL}_n(A), \mathrm{SL}_n(A,(q_i))]C_i = [C_i^{-1}\mathrm{SL}_n(A)C_i, C_i^{-1}\mathrm{SL}_n(A,(q_i))C_i].$$

And the formula (2.13) continues

$$\begin{aligned} &G \supset [C_i^{-1}\mathrm{SL}_n(A)C_i, C_i^{-1}\mathrm{SL}_n(A,(q_i))C_i] \supset \\ &\supset [C_i^{-1}\mathrm{SL}_n(A,(q_i))C_i, C_i^{-1}\mathrm{SL}_n(A,(q_i))C_i]. \end{aligned} \tag{2.14}$$

By Lemma 2.2.1 there exist natural numbers $m_1, m_2 \in \mathbb{N}$ such that

$$C_1^{-1}\mathrm{SL}_n(A,(q_1))C_1 \supset \mathrm{SL}_n(A,(q_1^{m_1})),$$
$$C_2^{-1}\mathrm{SL}_n(A,(q_2))C_2 \supset \mathrm{SL}_n(A,(q_2^{m_2})).$$

Put $m = \max(m_1, m_2)$, then

$$C_i^{-1}\mathrm{SL}_n(A,(q_i))C_i \supset \mathrm{SL}_n(A,(q_i^m)),\ i = 1,2$$

Now the formula (2.14) gives

$$G \supset [\mathrm{SL}_n(A,(q_i^m)), \mathrm{SL}_n(A,(q_i^m))] \supset [\mathrm{E}_n(A,(q_i^m)), \mathrm{E}_n(A,(q_i^m))]. \tag{2.15}$$

By Proposition 1.3.54

$$[\mathrm{E}_n(A,(q_i^m)), \mathrm{E}_n(A,(q_i^m))] \supset \mathrm{E}_n(A,(q_i^{2m})), \tag{2.16}$$

which gives

$$G \supset \mathrm{E}_n(A,(q_i^{2m})).$$

Putting $l = 2m$ the Theorem is proved. □

Theorem 2.2.3. *Let $n \geq 3$, then*

$$G \supset \mathrm{E}_n(A). \tag{2.17}$$

Proof. Let $f \in A$ be any regular function and consider any elementary matrix

$$e = \mathrm{Id}_n + f\epsilon_{ij} \in \mathrm{E}_n(A).$$

By Theorem 2.2.2 there exists a natural number $l \in \mathbb{N}$ such that

$$G \supset \mathrm{E}_n(A, (q_i^l)),\ i = 1, 2. \tag{2.18}$$

Since $V(q_1) \cap V(q_2) = \emptyset$ by Nullstellensatz there exist $f_1, f_2 \in A$ such that

$$f_1 q_1^l + f_2 q_2^l = f. \tag{2.19}$$

Then

$$\begin{aligned} e = \mathrm{Id}_n + f\epsilon_{ij} &= \mathrm{Id}_n + f_1 q_1^m \epsilon_{ij} + f_2 q_2^m \epsilon_{ij} = \\ &= (\mathrm{Id}_n + f_1 q_1^m \epsilon_{ij})(\mathrm{Id}_n + f_2 q_2^m \epsilon_{ij}) \in G. \end{aligned} \tag{2.20}$$

□

Corollary 2.2.4. *If the coordinate ring A of the curve Γ has the property*

$$\mathrm{E}_n(A) = \mathrm{SL}_n(A),$$

then

$$G \supset \mathrm{SL}_n(A).$$

Corollary 2.2.5. *If Γ is isomorphic to a line then the group G generates* $\mathrm{SL}_n(A)$.

Proof. In this case $A = \Bbbk[\Gamma] = \Bbbk[z]$ - the polynomial ring in one variable z. Since $\Bbbk[z]$ is Euclidean

$$\mathrm{E}_n(A) = \mathrm{SL}_n(A)$$

and the result follows. □

Remark 2.2.6. Even if A is a principal ideal domain(PID), in general $\mathrm{E}_n(A)$ is not equal to $\mathrm{SL}_n(A)$.

Definition 2.2.7. A PID A is called **special** if $\mathrm{E}_n(A) = \mathrm{SL}_n(A)$.

Theorem 2.2.8. *[Lam06] Let A be a special PID and $S \subset A$ be a multiplicatively closed subset. Then the localization $S^{-1}A$ is a special PID.*

Definition 2.2.9. A curve which is birationally equivalent to a line is called a **rational curve**.

Corollary 2.2.10. *If Γ is a rational curve with $A = \Bbbk[\Gamma]$, then the group G generates* $\mathrm{SL}_n(A)$.

Proof. A is a localization of the polynomial ring in one variable. $\square$

2.3 One Property of Pullbacks

Synopsis In this section we prove an auxiliary result that we will use in Chapter 2.4, where we solve the extension problem for the case of the conormal bundle. More specifically, we consider a smooth once punctured curve Γ in our flexible variety X, and a partial quotient morphism ρ_δ, corresponding to some locally nilpotent derivation δ. We can remove a finitely many points S from Γ, and obtain an isomorphism of $\Gamma - S$ onto its image under the morphism ρ_δ. The main result, which is the Theorem 2.3.1, says the following: every regular function on Γ, that vanishes on S with high enough multiplicity, is a pullback of a regular function on the curve Γ_δ, which is the image of Γ under the partial quotient morphism ρ_δ.

Let Γ be a smooth once punctured curve in a flexible variety X, and

$$\rho_\delta : X \to Q_\delta$$

be a partial quotient morphism associated with a locally nilpotent derivation δ. Denote

$$\Gamma_\delta \coloneqq \rho_\delta(\Gamma)$$

to be the image of Γ. We have the following commutative diagram

$$\begin{tikzcd} \Gamma \arrow[r, hook] \arrow[d, "\rho_\delta"] & X \arrow[d, "\rho_\delta"] \\ \Gamma_\delta \arrow[r, hook] & Q_\delta \end{tikzcd} \tag{2.21}$$

After removing a finitely many points, denote this subset of points by S, from our curve Γ, we have an isomorphism

$$\rho_\delta\Big|_{\Gamma^\star} : \Gamma^\star \xrightarrow{\sim} \Gamma^\star_\delta,$$

where $\Gamma^\star = \Gamma - S$ and $\Gamma^\star_\delta = \rho_\delta(\Gamma^\star)$ are the Zariski open and dense subsets of Γ and Γ_δ correspondingly. Thus we are given the following commutative diagram:

$$\begin{tikzcd} \Gamma^* \arrow[r, hook] \arrow[d, "\simeq"] & \Gamma \arrow[r, hook] \arrow[d, "\rho_\delta"] & X \arrow[d, "\rho_\delta"] \\ \Gamma^*_\delta \arrow[r, hook] & \Gamma_\delta \arrow[r, hook] & Q_\delta \end{tikzcd} \tag{2.22}$$

Our aim is to prove the following theorem:

Theorem 2.3.1. *There exists a natural number m_0 such that for every $m \geq m_0$ and every regular function $f \in \Bbbk[\Gamma]$, that vanishes at every point $x \in S = \Gamma - \Gamma^\star$ with multiplicity m, it follows that*

$$f \in \rho^\star \left(\Bbbk[\Gamma_\delta]\right) \tag{2.23}$$

First consider the case when $\Bbbk = \mathbb{C}$ is the field of complex numbers. We start with the following proposition, which is given here because of lack of references.

Proposition 2.3.2. *Let*

$$\rho : \Delta \to V$$

be a bijective holomorphic map, where $\Delta = \{z : |z| < 1\}$ - unit disc on the complex plane and $V \subset \mathbf{C}^n$ is a closed affine curve with y being the only singular point of V. Then there exists a natural number m_0 such that for every $m \geq m_0$ and every holomorphic function $f \in \mathrm{Hol}(\Delta)$ with the property that $f(0) = 0$ with multiplicity m, it follows that that

$$f = \tilde{f} \circ \rho$$

with $\tilde{f} \in \mathrm{Hol}(V)$ being holomorphic.

Proof. The map ρ induces the inclusion of rings

$$\rho^\star : \mathrm{Hol}(V) \hookrightarrow \mathrm{Hol}(\Delta).$$

Let $\tilde{\mu} \subset \mathrm{Hol}(V)$ be the maximal ideal of the point y and

$$\mu := \rho^\star(\tilde{\mu})$$

be its image in $\mathrm{Hol}(\Delta)$. Consider the semigroup $\Pi \subset \mathbb{N}$, defined by the following property:

$$\forall l \in \Pi\, \forall f \in \mathrm{Hol}(\Delta), f(0) = 0 \,\text{with order}\, l, \implies f \in \mu$$

We have to prove the following:

$$\exists m_0 \in \Pi\, \forall m \geq m_0 \implies m \in \Pi$$

<u>Case 1:</u> $\gcd(\Pi) = 1$.

Then there exist $r, l \in \Pi$ with $\gcd(r, l) = 1$. Every $m \geq m_0 := rl$ can be written as $m = ar + bl$ with $a, b \in \mathbb{N}$. Let $f \in \mathrm{Hol}(\Delta)$ and $f(0) = 0$ with multiplicity m. Then locally around 0

$$f(z) = z^m g(z),$$

with $g(0) \neq 0$. Write

$$f(z) = z^{(ar+bl)} g(z) = (z^r)^a (z^l g_1(z))^b,$$

with $g_1(z) = (g(z))^{1/b}$. We have

$$z^r = \rho^\star(\tilde{f}_r), \quad z^l g_1(z) = \rho^\star(\tilde{f}_l),$$

for some $\tilde{f}_r, \tilde{f}_l \in \tilde{\mu}$. Finally

$$\rho^\star\left(\tilde{f}_r^a \tilde{f}_l^b\right) = \left(\rho^\star(\tilde{f}_r)\right)^a \left(\rho^\star(\tilde{f}_l)\right)^b = (z^r)^a (z^l g_1(z))^b = f(z).$$

<u>Case 2:</u> $\gcd(\Pi) = d > 1$.

Let $\varphi_1(z)$ have zero of order k at the origin, then $k = k_0 d$ for some $k_0 \in \mathbb{N}$ and

$$\varphi_1(z) = z^k h(z), \quad \text{with} \quad h(z) \in \mathrm{Hol}(\Delta) \quad \text{and} \quad h(0) \neq 0.$$

Now

$$\rho(z) = (\varphi_1(z), \ldots, \varphi_n(z)),$$

with $\varphi_i(z) \in \mathrm{Hol}(\Delta)$, $i = 1, \ldots, n$. We can assume that $\varphi_1(z) = z^k$. This is because in some open disc Δ_ϵ with sufficiently small radius ϵ around the origin $h(z) \neq 0$ for every $z \in \Delta_\epsilon$. By the inverse function theorem $z = g(\xi)$ - a biholomrphic function and we can take the domain for ξ to be $\Delta = \{\xi : |\xi| < 1\}$. Then for

$$\rho_1(\xi) := \rho \circ g(\xi) = (\psi_1(\xi), \ldots, \psi_n(\xi))$$

we have $\rho(z) = \rho_1(\xi)$ and $\psi_1(\xi) = \xi^k$. Now for at leas one $j \in 2, \ldots, n$,

$$\varphi_j(z) = a^j_{dp} z^{dp} + \ldots + a^j_i z^i + \ldots$$

has nonzero coefficient $a_i^j \neq 0$ with index i which is not divisible by d because otherwise we would have that ρ is a function of z^d and this would contradict to the bijectivity of ρ. Hence there exist functions in μ which have in its Taylor's series nonzero coefficients with index non-divisible by d. Consider one such a function

$$f(z) = a_{ds_0} z^{ds_0} + \ldots + a_i z^i + \ldots \in \mu$$

with index t for which $a_t \neq 0$, d does not divide t and $t - ds_0$ is minimal. Without loss of generality we can put $a_{ds_0} = 1$ since the function $f(z)/a_{ds_0} \in \mu$ will have the same index t with the same properties as above. Now consider the function

$$f(z)^{k_0} - \varphi_1^{s_0}(z) \in \mu$$

We have two possibilities: either a_t is the next nonzero coefficient after a_{ds_0} in $f(z)$ in which case

$$f(z)^{k_0} - \varphi_1^{s_0}(z) = (z^{ds_0} + a_t z^t + \ldots)^{k_0} - z^{dk_0 s_0} = a_t z^{ds_0(k_0-1)+t} + \ldots$$

has zero of order $ds_0(k_0-1)+t$ which is not divisible by d and we have a contradiction, or a_{ds_1} is the next nonzero coefficient after a_{ds_0} in $f(z)$ in which case

$$\begin{aligned} f(z)^{k_0} - \varphi_1^{s_0}(z) &= (z^{ds_0} + a_{ds_1} z^{ds_1} \ldots + a_t z^t + \ldots)^{k_0} - z^{dk_0 s_0} = \\ &= a_{ds_1} z^{ds_0(k_0-1)+ds_1} + \ldots + a_t z^{ds_0(k_0-1)+t} + \ldots \end{aligned}$$

and

$$ds_0(k_0-1) + t - ds_0(k_0-1) - ds_1 = t - ds_1 < t - ds_0$$

- a contradiction. This proves the proposition. $\square$

Theorem 2.3.3. *[Kal91] Let V be an algebraic variety and f - a rational function on V. If f is holomorphic in a neighborhood of a point $p \in V$ then f is regular at p.*

Now we are ready for the proof of the Theorem 2.3.1

proof of Theorem 2.3.1. Let $x_\delta \in \Gamma_\delta - \Gamma_\delta^\star$, then $\rho^{-1}(x_\delta) = \{x_1, \ldots, x_k\}$ since Γ is once punctured. Choose a ball $B \subset \mathbb{C}^n$ so that $x_\delta \in B \cap (\Gamma_\delta - \Gamma_\delta^\star)$ and

$$\Gamma_\delta \cap B = \sum_{i=1}^{k} V_i,$$

with each V_i being an irreducible analytic set. That is, V_i is a bijective holomorphic image of the unit disc Δ on a complex plane, so we have the normalizations

$$\nu_i : \Delta \to V_i$$

We can choose analytic neighborhoods U_i around of each point $x_i, i = 1, \ldots, k$, so that

$$\lambda_i : U_i \to \Delta$$

is a biholomorphism. By the universal property of normalization these biholomorphisms can be chosen so that

$$\rho_\delta\Big|_{U_i} = \nu_i' := \nu_i \circ \lambda_i : U_i \to V_i.$$

Consider functions $\tilde{g}_1, \ldots, \tilde{g}_k \in Hol(\Gamma_\delta \cap B)$ such that

$$\tilde{g}_j\Bigg|_{\bigcup\limits_{i=1, i\neq j}^{k} V_i} = 0 \quad \text{and} \quad \tilde{g}_j\Bigg|_{V_j} \neq 0.$$

Then the holomorphic functions

$$g_j := (\nu_j')^\star(\tilde{g}_j)$$

have zero of some order n_j at the point x_j. By the proposition 2.3.2

$$\exists m_j \in \mathbb{N}\, \forall m \geq m_j\, \forall h \in \mathrm{Hol}(U_j),$$

such that $h(x_j) = 0$ with multiplicity m. It follows that

$$h = (\nu_j')^\star(\tilde{h}),$$

with $\tilde{h} \in \mathrm{Hol}(V_j)$. Define

$$m_0 = max\{m_1 + n_1, \dots, m_k + n_k\}.$$

Consider the function $f \in \Bbbk[\Gamma]$ which vanishes on $\Gamma - \Gamma^\star$ with multiplicity $m \geq m_0$. The inclusion

$$\rho_\delta^\star : \mathbb{C}[\Gamma_\delta] \hookrightarrow \mathbb{C}[\Gamma]$$

and the isomorphism

$$\rho_\delta\Big|_{\Gamma^\star} : \Gamma^\star \xrightarrow{\sim} \Gamma_\delta^\star$$

imply that

$$f = \rho_\delta^\star(f'), \quad \text{with} \quad f' \in \mathbb{C}(\Gamma_\delta).$$

Consider

$$f_j := f/g_j \in Hol(U_j),$$

then $f_j(x_j) = 0$ with multiplicity $m - n_j \geq m_j$, hence

$$f_j = (\nu_j')^\star(\tilde{f}_j), \quad \text{with} \quad \tilde{f}_j \in \mathrm{Hol}(V_j).$$

Define

$$\tilde{f} := \sum_{i=1}^{k} \tilde{g}_j \tilde{f}_j,$$

then

$$\tilde{f} = (\rho_\delta)_\star(f) = f'.$$

The Theorem 2.3.1 now implies that f' is regular at x_δ. Since we have finitely many $x_\delta \in \Gamma_\delta - \Gamma_\delta^\star$ our theorem is proved for the case $\Bbbk = \mathbb{C}$. The case of the general $\Bbbk$ follows from the Lefshitz principle. $\square$

2.4 Solution of the Extension Problem for the Conormal Bundles

Synopsis In this chapter, using the results of Chapters 2.2 and 2.3, we give the solution to the extension problem for the conormal bundle of the curve Γ.

Let Γ be a smooth once punctured curve with a trivial normal bundle $\mathrm{N}\Gamma$ in a flexible variety X, with $\dim(X) \geq 4$. For convenience we put $\dim(X) = n+1$, $n \in \mathbb{N}$. Denote $A := \Bbbk[\Gamma]$ to be the ring of regular functions of our curve Γ, and $\bar{v} := \{v_1, \ldots, v_n\}$ to be a global basis for the normal bundle $\mathrm{N}\Gamma$. Consider some partial quotient morphism

$$\rho : X \to Q$$

Choose a point $z_o \in \Gamma$. The following follows from the paper [Kal17] (the Theorem 4.2 in this paper): there exists an algebraic family of locally nilpotent derivations $\{\delta_\alpha | \alpha \in \mathcal{A}\}$ such that for every $\alpha \in \mathcal{A}$ there exists a smooth neighborhood U'_α of $z_\alpha \coloneqq \rho_\alpha(z_o)$ in $Q_\alpha \coloneqq \rho_\alpha(X)$ such that we have the natural isomorphism

$$\rho_\alpha^{-1}(U'_\alpha) \simeq U'_\alpha \times \mathbb{C}.$$

Since our curve is once punctured, it follows that

$$\rho_\alpha(\Gamma) = \overline{\rho_\alpha(\Gamma)}$$

with $\overline{\rho_\alpha(\Gamma)}$ being the Zariski closure of $\rho_\alpha(\Gamma)$. Denote

$$\Gamma_\alpha \coloneqq \rho_\alpha(\Gamma), \quad V'_\alpha = \Gamma_\alpha \cap U'_\alpha, \quad V_\alpha = \rho_\alpha^{-1}(V'_\alpha) \cap \Gamma.$$

The morphism

$$\rho_\alpha\Big|_{V_\alpha} : V_\alpha \to V'_\alpha$$

is an isomorphism. We can choose $\alpha_1, \ldots, \alpha_{n+1} \in \mathcal{A}$ such that the corresponding locally nilpotent derivations $\{\delta_{\alpha_i}\Big|_{z_o} | i = 1, \ldots, n+1\}$ form a basis of the tangent space $T_{z_o}(X)$ of X at the point z_0.

For simplicity we will denote

$$V_i := V_{\alpha_i},\ \delta_i := \delta_{\alpha_i},\ \rho_i := \rho_{\alpha_i},\ \Gamma_i := \Gamma_{\alpha_i}.$$

Set

$$\Gamma^\star := \bigcap_{i=1}^{n+1} V_i, \quad \Gamma_i^* = \rho_i(\Gamma^*).$$

Then we have an isomorphism

$$\rho_i\Big|_{\Gamma^*} : \Gamma^* \to \Gamma_i^*$$

for each $i = 1, \ldots, n+1$. Now $\delta_1, \ldots, \delta_{n+1}$ generate sections of the normal bundle

$$\mathrm{N}\Gamma = \mathrm{TX}\Big|_{\Gamma} / \mathrm{T}\Gamma$$

of the curve Γ, and reducing Γ^* further if necessary, we can suppose that

$$\bar{\delta} := \{\delta_1, \ldots, \delta_n\}$$

form a basis of $\mathrm{N}\Gamma\Big|_{\Gamma^*}$. Then

$$\bar{\delta} = C_1 \bar{v},$$

where C_1 is an $n \times n$ matrix with coefficients c_{ij} in the ring A.

Remark 2.4.1. $C_1\Big|_{\Gamma^*}$ is invertible.

For each locally nilpotent derivation δ_i consider the corresponding partial quotient morphism

$$\rho_i : X \to Q_i$$

Let $\tilde{f}, \tilde{h}$ be regular functions in the ring $\Bbbk[Q_i]$ with $\tilde{f}\Big|_{\Gamma_i} = 0$. Denote

$$h' := \rho_i^*(\tilde{h}), \quad f' := \rho_i^*(\tilde{f})$$

be their pullbacks in the ring $\Bbbk[X]$, and denote

$$h := \iota^\star(h'), \quad f := \iota^\star(f')$$

the pullbacks under the inclusion

$$\iota : \Gamma \hookrightarrow X$$

Since

$$\rho_i^*(\Bbbk[Q_i]) = \ker \delta_i$$

then $h'f'\delta_i$ is a locally nilpotent derivation. Hence we can consider the vector flows for the automorphisms

$$\Phi_i = \exp(h'f'\delta_i). \tag{2.24}$$

The following Lemma describes the map of the tangent space of X at a certain point $p \in X$ under the automorphism $\Phi = \exp(f\delta)$:

Lemma 2.4.2. *[AFK+13] Let* $\delta \in \mathrm{LND}_{\Bbbk}(A)$, $X = \mathrm{Spec}(A)$, $p \in X$, $f \in \ker \delta$, $f(p) = 0$, *and* $\Phi = \exp(f\delta)$. *Then for every* $w \in T_pX$:

$$d_p\Phi(w) = w + df(w)\delta(p). \tag{2.25}$$

Using the formula (2.25), we can describe the phase flows of the automorphisms (2.24). Say for $i = 1$, we have:

$$\begin{aligned}
&\delta_1 \to \delta_1 \\
&\delta_2 \to \delta_2 + t[h'\delta_2(f') + f'\delta_2(h')]\delta_1 \\
&\vdots \\
&\delta_n \to \delta_n + t[h'\delta_n(f') + f'\delta_n(h')]\delta_1.
\end{aligned}$$

Taking the restriction to the curve Γ and taking into the account that $\tilde{f}\Big|_{\Gamma_i} = 0$, which implies $f = 0$, we obtain

$$\begin{aligned}
&\delta_1 \to \delta_1 \\
&\delta_2 \to \delta_2 + th\Big[\delta_2(f')\Big|_{\Gamma}\Big]\delta_1 \\
&\vdots \\
&\delta_n \to \delta_n + th\Big[\delta_n(f')\Big|_{\Gamma}\Big]\delta_1.
\end{aligned}$$

Remark 2.4.3. The advantage of this flow is that it fixes every point of Γ, but not the action on the normal bundle $\mathrm{N}\Gamma$.

Now let $\tilde{f}_1, \ldots, \tilde{f}_k$, $k \geq n$, be generators of the defining ideal of Γ_1 in $\Bbbk[Q_1]$, and let $\tilde{h}_1, \ldots, \tilde{h}_k$ be any regular functions in the ring $\Bbbk[Q_1]$. As above, we denote by

$$h'_j := \rho_1^*(\tilde{h}_j), \quad f'_j := \rho_1^*(\tilde{f}_j),$$

their pullbacks in the ring $\Bbbk[X]$ and by

$$h_j := \iota^\star(h'_j), \quad f_j := \iota^\star(f'_j)$$

the pullbacks in the coordinate ring $\Bbbk[\Gamma]$. Then the phase flow of the composition

$$\Phi_1 \circ \cdots \circ \Phi_k = \exp(h'_1 f'_1 \delta_1) \circ \ldots \circ \exp(h'_k f'_k \delta_1)$$

on the curve Γ is given by:

$$\begin{aligned} \delta_1 &\to \delta_1 \\ \delta_2 &\to \delta_2 + t \sum_{j=1}^{k} h_j \left[\delta_2(f'_j)\Big|_\Gamma \right] \delta_1 \\ &\vdots \\ \delta_n &\to \delta_n + t \sum_{j=1}^{k} h_j \left[\delta_n(f'_j)\Big|_\Gamma \right] \delta_1. \end{aligned} \tag{2.26}$$

Reducing Γ^* further we can suppose that the matrix

$$D_1 := \left(\left[\delta_i(f'_j)\Big|_\Gamma \right] \right)_{i=2,\ldots,n}^{j=1,\ldots,n-1}$$

is invertible over Γ^*. For a given vector

$$\vec{d} = F \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix}, \quad \text{with} \quad d_i \in A, i = 1, \ldots, n-1,$$

and $F \in A$, such that F vanishes on the points $\Gamma - \Gamma^\star$ with high multiplicity m, define

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \end{bmatrix} := \operatorname{adj}(D_1) F \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix}$$

From here it follows that each function h_i vanishes on $\Gamma - \Gamma^\star$ with some multiplicity $m_i \geq m$. It follows throm the Theorem 2.3.1 of Chapter 2.3, that each such function is a pullback(under $\rho_i \circ \iota$) of a function $\tilde{h}_i \in \Bbbk[Q_1]$. Then

$$D_1 \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n-1} \end{bmatrix} = \det(D_1) F \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix}$$

and by (2.26) we have

$$\begin{aligned} &\delta_1 \to \delta_1 \\ &\delta_2 \to \delta_2 + t(F \det(D_1)) d_1 \delta_1 \\ &\vdots \\ &\delta_n \to \delta_n + t(F \det(D_1)) d_{n-1} \delta_1. \end{aligned}$$

In particular if we take

$$\begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix} = g \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{bmatrix} = g \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

with g being any function from A, we will have for $\bar{\delta}$ the standard elementary transformation of the form

$$\bar{\delta} \to (Id + g \det(D_1) \epsilon_{k1}) \bar{\delta}, \quad k = 2, \ldots, n.$$

In the same way we can consider the phase flows for $i = 2, \dots, n$ and we will obtain the standard elementary transformations

$$\bar{\delta} \to (Id + g\det(D_i)\epsilon_{ki})\bar{\delta}, \quad k = 1, \dots, \hat{i}, \dots, n.$$

All these together imply that we have all standard elementary transformations

$$\bar{\delta} \to \left(Id + g \left(\prod_{k=1}^{n} \det(D_k) \right) \epsilon_{ij} \right) \bar{\delta},$$

Considering all removed points and removing more if necessary, we may assume that the removed points are given by zeros $V(q_1)$ of a regular function $q_1 \in A$. Thus we have for $\bar{\delta}$ all standard elementary transformations $\mathrm{E}_n(A, (q_1))$ with coefficients in the principal ideal (q_1).

Remark 2.4.4.

$$\prod_{i=1}^{n} \det D_i \quad \text{divides} \quad q_1.$$

Remark 2.4.5.

$$\det C_1 \quad \text{divides} \quad q_1.$$

Next, we are going to use the following theorem, which is a special case of the Theorem 6.1 in the paper [Kal17], under our assumptions for X and Γ :

Theorem 2.4.6. *[Kal17], Given a partial quotient morphism*

$$\rho_\delta : X \to Q,$$

associated with a locally nilpotent derivation δ, and a finite subset of points S in Γ, there exists a connected algebraic family $\mathcal{A}$ of automorphisms, such that for a general element $\alpha \in \mathcal{A}$ and for the closure $\bar{\Gamma}'_\alpha$ of $\Gamma'_\alpha = \rho_\delta \circ \alpha(\Gamma)$ in Q, one can find a neighborhood V'_δ of $\rho_\delta(\alpha(S))$ such that for $V_\delta = \rho_\delta^{-1}(V'_\delta) \cap \alpha(\Gamma)$, the morphism

$$\rho_\delta\Big|_{V_\delta} : V_\delta \to V'_\delta$$

is an isomorphism.

This theorem and all the reasoning of this chapter above, imply that for a set of finite points $S = V(q_1)$, we can find a neighborhood $\Gamma - V(q_2)$, with $q_2 \in A$, $V(q_1) \cap V(q_2) = \emptyset$, and locally nilpotent derivations $\partial_1, \ldots, \partial_n$, which form a basis for the normal basis $\mathrm{N}\Gamma$ on this neighborhood $\Gamma - V(q_2)$. Thus for $\bar{\partial} = \{\partial_1, \ldots, \partial_n\}$ we have the q_2- standard elementary transformations $\mathrm{E}_n(A, (q_2))$. Now by the Theorem 2.2.3, we can generate all the elementary transformations $\mathrm{E}_n(A)$ of our global bases $\bar{v}$, and we have proved the following theorem:

Theorem 2.4.7. *Let Γ be a smooth once punctured curve with a trivial normal basis in a flexible variety X, with $\dim(X) \geq 4$. Denote by I the ideal of Γ in $\Bbbk[X]$, and suppose that the coordinate ring $A = \Bbbk[\Gamma]$ satisfies the condition $\mathrm{E}_n(A) = \mathrm{SL}_n(A)$. Then every $\mathrm{SL}_n(A)$ automorphism of the normal bundle $\mathrm{N}\Gamma$, equivalently every $\mathrm{SL}_n(A)$ automorphism*

$$\bar{\alpha} : \frac{I}{I^2} \xrightarrow{\sim} \frac{I}{I^2}$$

of the conormal bundle I/I^2 of Γ, that induces the identity map on Γ, is induced by a global automorphism of the ambient space X.

Corollary 2.4.8. *Let Γ be a line in a flexible variety X, with $\dim(X) \geq 4$. Then every $\mathrm{SL}_n(A)$ automorphism*

$$\bar{\alpha} : \frac{I}{I^2} \xrightarrow{\sim} \frac{I}{I^2}$$

of the conormal bundle of Γ, that induces the identity map on Γ, is induced by a global automorphism of the ambient space X.

2.5 Solution of the Extension Problem for Jet Bundles

Synopsis In this last chapter, we prove the extension problem for the case of a general jet bundle. We summaries our result, which lead to the proof of the two main theorems of the theses: the Theorem 1.2.8 and the Theorem 1.2.9.

Definition 2.5.1. Let X be a quasi-affine algebraic variety and $Y \subset X$ be a closed subvariety. Let $k \in \mathbb{N}$ be the minimal natural number for which there exist regular functions $f_1, \ldots, f_k \in \Bbbk[X]$ that generate the ideal I_Y of the subvariety Y :

$$I_Y = (f_1, \ldots, f_k).$$

Y is called a **strict complete intersection** if

$$k = \dim(X) - \dim(Y).$$

Theorem 2.5.2. *(Serre) [Kem80] Let $\mathcal{F}$ be a quasi-coherent sheaf on an affine scheme X. Then $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$.*

Let X be a flexible variety of dimension $\dim(X) = n + 1 \geq 4$. Consider a smooth once punctured curve Γ in X, with a defining ideal

$$I = (f_1, \ldots, f_k),$$

whose conormal bundle I/I^2 is trivial. First we prove the following theorem.

Theorem 2.5.3. *Let X be a quasi-affine algebraic variety and $Y \subset X$ be a closed subvariety witha a trivial conormal bundle I_Y/I_Y^2. There exists a Zariski open neighborhood U of X, containing Y, in which Y is a strict complete intersection:*

$$J_Y = (h_1, \ldots, h_k),$$

with J_Y - the ideal of Y in U, $k = \operatorname{codim}_X Y$. Furthermore, we can choose $h_1, \ldots, h_k$ to be in the coordinate ring $\Bbbk[X]$.

Proof. Given a cover $\{U_i\}$ of X, denote $I_i = I_Y|_{U_i}$ and $I_{il} = I_Y|_{U_i\cap U_l}$. Since the conormal bundle is trivial, there exists a cover $\{U_i\}$ of X, there exist $f_i^j \in \Bbbk[U_i]$, $i = 1, \dots, N$, $j = 1, \dots, k$, such that $f^i \in I_i$ and

$$f_i^j + I_i^2 = v^j\Big|_{U_i} + I_i^2.$$

Hence

$$f_i^j\Big|_{U_i\cap U_l} - f_l^j\Big|_{U_i\cap U_l} \in I_{il}^2$$

Consider $g_{il}^j \in \Bbbk[U_i \cap U_j]$ with

$$g_{il}^j \coloneqq f_i^j\Big|_{U_i\cap U_l} - f_l^j\Big|_{U_i\cap U_l}.$$

Hence $g_{il}^j \in J_{ij}^2$ and $\{g_{il}^j\}$ is a 1-cocycle. By Theorem 2.5.2 there exist a collection $g_i^j \in \Bbbk[U_i]$ such that $g_i^j \in I_i^2$ and

$$g_{il}^j = g_i^j\Big|_{U_i\cap U_l} - g_l^j\Big|_{U_i\cap U_l}.$$

Now define the functions in $h_i^j \in \Bbbk[U_i]$ as

$$h_i^j = f_i^j - g_i^j.$$

Note that $h_i^j \in I_i$, and $\{h_l^j\}$ agree on $\{U_i \cap U_l\}$, and moreover

$$h^j\Big|_{U_i} + I_i^2 = f_i^j + I_i^2 = v^j\Big|_{U_i} + I_i^2.$$

Hence $\{h_i^j\}$ define a global function $h^j \in \Bbbk[X]$ with $h^j \in I_Y$, and moreover

$$h^j + I_Y^2 = v^j + I_Y^2. \tag{2.27}$$

Which means that $\{h^1 + I_Y^2, \dots, h^k + I_Y^2\}$ generate I_Y/I_Y^2. Nevertheless $\{h^1, \dots, h^k\}$ may not generate I_Y since their zero locus

$$Y' = V(h^1, \dots, h^k).$$

may have points outside of Y :

$$Y' = Y \cup Z,$$

with Z being the other components of Y'. Since $\{h^1+I_Y^2, \ldots, h^k+I_Y^2\}$ generate I_Y/I_Y^2, it follows that Y and Z do not intersect:

$$\Phi^{-1}(0) = Y' = Y \sqcup Z.$$

Next, consider a function $q \in \Bbbk[X]$ which is 0 on Z and 1 on Y, and define a neighborhood U of X as

$$U = X - q^{-1}(0).$$

The map

$$\varphi : U \xrightarrow{(h^1|_U,\ldots,h^k|_U)} \Bbbk^n$$

is a submersion and $\Gamma = \varphi^{-1}(0)$, which implies that the ideal J_Y of Y in U is generated by $h^1|_U, \ldots, h^k|_U$. $\square$

This theorem implies that we have the isomorphism $J_Y/J_Y^{m+1} \simeq I_Y/I_Y^{m+1}$. And since we proved the case of the conormal bundle I_Y/I_Y^2 in the previous chapter, the isomorphism $J_Y/J_Y^{m+1} \simeq I_Y/I_Y^{m+1}$ allows us to treat our curve Γ in question as a strict complete intersection curve.

Next, in addition to the notations from Chapter 2.1, we will use the following notations

- $R := \Bbbk[X]$
- $A := \Bbbk[\Gamma] = \frac{\Bbbk[X]}{I}$
- $R_m := \frac{\Bbbk[X]}{I^{m+1}}$, in particular $R_0 = A$
- $\bar{I}_m := \frac{I}{I^{m+1}}$
- $\bar{J}_m := \frac{I^m}{I^{m+1}}$
- $\mathrm{Aut}^\Gamma(\bar{I}_m) := \left\{\bar{\alpha} \in \mathrm{Aut}(\bar{I}_m) \,\middle|\, \alpha_0 = \mathrm{id}_A\right\}$
- $\mathrm{Aut}^\Gamma(R) := \left\{\alpha \in \mathrm{Aut}(R) \,\middle|\, \alpha_0 = \mathrm{id}_A\right\}$
- $G(\bar{I}_m) := \left\{\bar{\alpha} \in \mathrm{Aut}^\Gamma(\bar{I}_m) \,\middle|\, \exists \beta \in \mathrm{Aut}^\Gamma(R) \text{ with } \bar{\beta} = \bar{\alpha}\right\}$

Consider the $A-$ module isomorphisms

$$\begin{aligned} I &\simeq \bar{I}_1 \oplus \bar{J}_2 \oplus \ldots \oplus \bar{J}_j \oplus \ldots, \\ \bar{I}_m &\simeq \bar{I}_1 \oplus \bar{J}_2 \oplus \ldots \oplus \bar{J}_m, \end{aligned} \tag{2.28}$$

together with the natural projections defined by them. Every $\bar{\alpha} \in \mathrm{Aut}^{\Gamma}(\bar{I}_m)$ is induced by some isomorphism $\breve{\alpha} : I \xrightarrow{\sim} I$. Consider the commutative diagram

$$\begin{tikzcd} I \arrow[r, "\breve{\alpha}"] \arrow[d, "\pi_m"'] & I \arrow[d, "\pi_m"] & \\ \bar{I}_m \arrow[r, "\bar{\alpha}"] & \bar{I}_m \arrow[r, "\eta_j"] & \bar{J}_j \end{tikzcd} \tag{2.29}$$

with π_m and η_j being the natural projections coming from (2.28).

Remark 2.5.4. It follows from the diagram above that the image of $\pi_m \circ \breve{\alpha}$ does not depend on the choice of $\breve{\alpha}$ and is completely determined by $\bar{\alpha}$. In particular, for $\bar{\alpha}^{-1}$ we can choose $\breve{\alpha}^{-1}$ as the corresponding representative.

Since I is a free $A-$ module on n generators $u_1, \ldots, u_n$, $I \simeq A^n$, $\breve{\alpha}$ and $\pi_m \circ \breve{\alpha}$ is described by where they sends the generators, and in the view of the isomorphism (2.28) we get:

$$\breve{\alpha}(u_i) = H^i(u_1, \ldots, u_n)$$

with H^i - a polynomial in $u_1, \ldots, u_n$ with coefficients in A,

$$\pi_m \circ \breve{\alpha}(u_i) = F^i(u_1, \ldots, u_n),$$

with $F^i = H^i \mod I^{m+1}$ - a polynomial of degree less than or equal to m,

$$\eta_j \circ \pi_m \circ \breve{\alpha}(u_i) = F^i_j(u_1, \ldots, u_n). \tag{2.30}$$

with $F^i_j(u_1, \ldots, u_n)-$ the homogeneous component of $F^i(u_1, \ldots, u_n)$ of degree j. Denote

$$\mathrm{Jac}(\alpha) = \det \left(\frac{\partial H^i}{\partial u_j} \right)$$

- the jacobian of the Jacobi matrix

$$\mathcal{J}\vec{H}(\vec{u}) = \left[\partial H^i / \partial u_j \right]_{i=1,\ldots,n}^{j=1,\ldots,n}.$$

Additionally, we are going to use the notations introduced in the subsection 1.3.4.

Remark 2.5.5. For

$$\vec{F}(\vec{u}) = \vec{H}(\vec{u}) \mod I^{m+1}$$

it follows

$$\mathcal{J}\vec{F}(\vec{u}) = \mathcal{J}\vec{H}(\vec{u}) \mod I^{m},$$

hence

$$\mathrm{Jac}(\alpha) \mod I^m = \det\left(\frac{\partial F^i}{\partial u_j}\right) \mod I^m$$

Our aim is to prove the following theorem

Theorem 2.5.6. *Let* Γ *be a smooth once punctured curve with a trivial normal basis in a flexible variety* X*, with* $\dim(X) \geq 4$*, and suppose that* $\mathrm{E}_n(A) = \mathrm{SL}_n(A)$*. Then given* $\bar{\alpha} \in \mathrm{Aut}^\Gamma(\bar{I}_m)$ *with*

$$\mathrm{Jac}(\breve{\alpha}) = 1 \mod I^m \tag{2.31}$$

it follows that $\bar{\alpha} \in G(\bar{I}_m)$*, i.e.* $\bar{\alpha}$ *is induced by a global automorphism.*

First, from now on, we consider the elements of $\bar{\alpha} \in \mathrm{Aut}^\Gamma(\bar{I}_m)$ that satisfy the condition (2.31). Second, note that for $m = 1$ the polynomials $F^1(\vec{u}), \ldots, F^n(\vec{u})$ are linear, hence $\mathcal{J}\vec{F}(\vec{u})$ is a constant matrix a with coefficients in A. And $\mathrm{Jac}(\breve{\alpha}) = 1 \mod I^m$ implies $a \in \mathrm{SL}_n(A)$. Thus for $m = 1$ the theorem 2.5.6 is just the Theorem 2.4.7 in Chapter 2.4. Before we start the proof of this theorem, we state an obvious corollary:

Corollary 2.5.7. *Let* Γ *be a line in a flexible variety* X*, with* $\dim(X) \geq 4$*. Then given* $\bar{\alpha} \in \mathrm{Aut}^\Gamma(\bar{I}_m)$ *with*

$$\mathrm{Jac}(\breve{\alpha}) = 1 \mod I^m \tag{2.32}$$

it follows that $\bar{\alpha}$ *is induced by a global automorphism of the ambient space* X*.*

Now we start the proof of the Theorem 2.5.6 by the sequence of contractions, lemmas, propositions and theorems presented below. Consider the projections

$$\theta_j : \mathrm{Aut}^\Gamma(\bar{I}_m) \to \{\text{ maps } \mathscr{F}_1 \to \mathscr{F}_j\},$$

defined from (2.30) with

$$\theta_j(\bar{\alpha}) : (u_1, \dots, u_n) \mapsto (F_j^1(\vec{u}), \dots, F_j^n(\vec{u})).$$

For $m = 1$ given $\bar{\alpha} \in \mathrm{Aut}^{\Gamma}(\bar{I}_1)$, $\theta_1(\bar{\alpha})$ is given by a matrix $a \in \mathrm{SL}_n(A)$, which defines the isomorphism

$$\bar{\alpha} : \frac{I}{I^2} \to \frac{I}{I^2}$$

in terms of the bases $\{u_1 + I^2, \dots, u_n + I^2\}$. For $m \geq 2$ consider $\mathrm{Aut}^{\Gamma}_m(\bar{I}_m) \subset \mathrm{Aut}^{\Gamma}(\bar{I}_m)$ defined as

$$\mathrm{Aut}^{\Gamma}_m(\bar{I}_m) \coloneqq \left\{ \bar{\alpha} \in \mathrm{Aut}^{\Gamma}(\bar{I}_m) \;\middle|\; \theta_1(\bar{\alpha}) = \vec{u} \text{ and } \theta_j(\bar{\alpha}) = 0 \text{ for } 2 \leq j \leq m-1 \right\}$$

In other words,

$$\breve{\alpha}(\vec{u}) \mod I^{m+1} = \vec{F}(\vec{u}) = \vec{u} + \vec{F}_m(\vec{u}) \tag{2.33}$$

Lemma 2.5.8. *Given* $\bar{\alpha} \in \mathrm{Aut}^{\Gamma}_m(\bar{I}_m)$ *with* $\breve{\alpha}(\vec{u})$ *as in* (2.33)

$$\mathrm{Jac}(\breve{\alpha}) = 1 \mod I^m$$

if and only if $\mathrm{div}(\vec{F}_m) = 0$.

Proof.

$$\mathcal{J}\vec{F}(\vec{u}) = Id + \mathcal{J}\vec{F}_m(\vec{u}). \tag{2.34}$$

Since

$$\frac{\partial F^i_m(\bar{u})}{\partial u_j}\frac{\partial F^k_m(\bar{u})}{\partial u_l} \in I^{2m-2},$$

and $I^{2m-2} \subset I^m$ for $m \geq 2$, we have

$$\mathrm{Jac}(\breve{\alpha}) \mod I^m = \det(Id + \mathcal{J}\vec{F}_m(\bar{u})) \mod I^m =$$
$$= 1 + \mathrm{tr}(\mathcal{J}\vec{F}_m(\vec{u})) = 1 + \mathrm{div}(\vec{F}_m),$$

and the result follows. □

Proposition 2.5.9. *Let* $\bar{\alpha} \in \mathrm{Aut}^{\Gamma}(\bar{I}_m)$ *and* $\bar{\gamma} \in \mathrm{Aut}^{\Gamma}_m(\bar{I}_m)$. *Then we have the following* *(a)* $\theta_j(\bar{\alpha} \circ \bar{\gamma}) = \theta_j(\bar{\alpha})$ *for* $j \leq m-1$,

(b) $\theta_m(\bar{\alpha} \circ \bar{\gamma}) = \theta_m(\bar{\alpha}) + \theta_1(\bar{\alpha}) \circ \theta_m(\bar{\gamma})$,

(c) $\theta_m(\bar{\gamma} \circ \bar{\alpha}) = \theta_m(\bar{\alpha}) + \theta_m(\bar{\gamma} \circ \theta_1(\bar{\alpha}))$.

Proof. Given

$$\breve{\alpha}(\vec{u}) \mod I^{m+1} = \vec{F}(\vec{u}) = \vec{F}_1(\vec{u}) + \ldots + \vec{F}_m(\vec{u})$$

and

$$\breve{\gamma}(\vec{u}) \mod I^{m+1} = \vec{u} + \vec{P}_m(\vec{u})$$

we have

$$(\breve{\alpha} \circ \breve{\gamma})(\vec{u}) \mod I^{m+1} = \vec{F}(\vec{u} + \vec{P}_m(\vec{u})) \mod I^{m+1} =$$

$$= \vec{F}_1(\vec{u} + \vec{P}_m(\vec{u})) + \sum_{j=2}^{m} \vec{F}_j(\vec{u} + \vec{P}_m(\vec{u})) \mod I^{m+1} =$$

$$= \vec{F}_1(\vec{u}) + \vec{F}_1(\vec{P}_m(\vec{u})) + \sum_{j=2}^{m} \vec{F}_j(\vec{u}) =$$

$$= \vec{F}_1(\vec{u}) + \ldots + \vec{F}_{m-1}(\vec{u}) + (\vec{F}_m(\vec{u}) + \vec{F}_1(\vec{P}_m(\vec{u})))$$

which proves parts (a) and (b). Next,

$$(\breve{\gamma} \circ \breve{\alpha})(\vec{u}) \mod I^{m+1} = \vec{F}(\vec{u}) + \vec{P}_m(\vec{F}_1(\vec{u}) + \ldots + \vec{F}_m(\vec{u})) \mod I^{m+1} =$$

$$= \bar{F}(\vec{u}) + \vec{P}_m(\vec{F}_1(\vec{u})),$$

which proves part (c). □

Corollary 2.5.10. *If* $\bar{\alpha}, \bar{\gamma} \in \mathrm{Aut}^{\Gamma}_m(\bar{I}_m)$, *then*

$$\theta_m(\bar{\alpha} \circ \bar{\gamma}) = \theta_m(\bar{\gamma} \circ \bar{\alpha}) = \theta_m(\bar{\alpha}) + \theta_m(\bar{\gamma}).$$

Proof. Since $\theta_1(\bar{\alpha})$ is the identity, we have

$$\theta_m(\bar{\alpha} \circ \bar{\gamma}) = \theta_m(\bar{\alpha}) + \theta_1(\bar{\alpha}) \circ \theta_m(\bar{\gamma}) = \theta_m(\bar{\alpha}) + \theta_m(\bar{\gamma}).$$

□

Proposition 2.5.11. *Given* $\bar{\alpha} \in \operatorname{Aut}^{\Gamma}(\bar{I}_m)$ *and* $\bar{\gamma} \in \operatorname{Aut}^{\Gamma}_m(\bar{I}_m)$, *it follows that*

$$\bar{\alpha}^{-1} \circ \bar{\gamma} \circ \bar{\alpha} \in \operatorname{Aut}^{\Gamma}_m(\bar{I}_m),$$

and moreover

$$\theta_m(\bar{\alpha}^{-1} \circ \bar{\gamma} \circ \bar{\alpha}) = \theta_1^{-1}(\bar{\alpha}) \circ \theta_m(\bar{\gamma} \circ \theta_1(\bar{\alpha})). \tag{2.35}$$

Proof. First note that for any $\bar{\alpha}, \bar{\beta} \in \operatorname{Aut}^{\Gamma}(\bar{I}_m)$, it follows that

$$\theta_1(\bar{\alpha} \circ \bar{\beta}) = \theta_1(\bar{\alpha}) \circ \theta_1(\bar{\beta}).$$

which in particular implies

$$\theta_1(\bar{\alpha}^{-1}) = \theta_1^{-1}(\bar{\alpha}).$$

Next, denote $\bar{\delta} \coloneqq \bar{\alpha}^{-1} \circ \bar{\gamma} \circ \bar{\alpha}$, then

$$\bar{\alpha} \circ \bar{\delta} = \bar{\gamma} \circ \bar{\alpha} \tag{2.36}$$

For $j = 1$

$$\theta_1(\bar{\delta}) = \theta_1(\bar{\alpha}^{-1}) \circ \theta_1(\bar{\gamma}) \circ \theta_1(\bar{\alpha}) = \theta_1^{-1}(\bar{\alpha}) \circ \operatorname{id} \circ \theta_1(\bar{\alpha}) = \operatorname{id}.$$

Next consider

$$\breve{\alpha}(\vec{u}) = \vec{F}(\vec{u}) = \vec{F}_1(\vec{u}) + \ldots + \vec{F}_m(\vec{u}),$$
$$\breve{\alpha}^{-1}(\vec{u}) = \vec{H}(\vec{u}) = \vec{H}_1(\vec{u}) + \ldots + \vec{H}_m(\vec{u}),$$
$$\breve{\gamma}(\vec{u}) = \vec{u} + \vec{P}_m(\vec{u}),$$

then for $j = 2, \ldots, m-1$,

$$\theta_j(\bar{\delta}) = \theta_j(\bar{\alpha}^{-1} \circ \bar{\gamma} \circ \bar{\alpha}) = \theta_j\Big(\vec{H}[\vec{F}(\vec{u}) + \vec{P}_m(\vec{F}(\vec{u}))]\Big) =$$
$$= \theta_j\Big(\vec{H}[\vec{F}(\vec{u})]\Big) = \theta_j(\vec{u}) = 0,$$

therefore $\bar{\delta} \in \operatorname{Aut}^{\Gamma}_m(R_m)$. Finally, applying the Proposition 2.5.9 part (b) to $\bar{\alpha} \circ \bar{\delta}$ and part (c) to $\bar{\gamma} \circ \bar{\alpha}$, and using the equality (2.36) we obtain

$$\theta_m(\bar{\alpha}) + \theta_1(\bar{\alpha}) \circ \theta_m(\bar{\delta}) = \theta_m(\bar{\alpha}) + \theta_m(\bar{\gamma} \circ \theta_1(\bar{\alpha})) \implies$$
$$\implies \theta_m(\bar{\delta}) = \theta_1^{-1}(\bar{\alpha}) \circ \theta_m(\bar{\gamma} \circ \theta_1(\bar{\alpha})).$$

$\square$

Remark 2.5.12. As mentioned before, $\theta_1(\bar{\alpha})$ is given by a matrix $a \in \mathrm{SL}_n(A)$, hence the formula (2.35) the representation (1.18).

Lemma 2.5.13. *There exists $\bar{\alpha} \in \mathrm{Aut}_m^\Gamma(\bar{I}_m) \cap G(\bar{I}_m)$, $\bar{\alpha} \neq id$, such that for every $h \in A$ and $\bar{\alpha}_h \in \mathrm{Aut}_m^\Gamma(\bar{I}_m)$, defined by $\theta_m(\bar{\alpha}_h) = h\theta_m(\bar{\alpha})$, it follows that $\bar{\alpha}_h \in \mathrm{Aut}_m^\Gamma(\bar{I}_m) \cap G(\bar{I}_m)$.*

Proof. Consider a locally nilpotent derivation δ, and a regular function $f \in \ker \delta$ that vanishes on Γ with multiplicity one, and a global automorphism

$$\Phi = \exp(f^m \delta).$$

We have

$$\Phi(u_i) = u_i + f^m \delta(u_i) = u_i + g_i f^m \mod I^{m+1}$$

with $g_i \in A$, and for at least one i, $g_i \notin I$. For

$$\vec{\Phi}_m(\vec{u}) = (g_1 f^m, \ldots, g_n f^m)$$

we have

$$\operatorname{div}(\vec{\Phi}_m(\vec{u})) = g_1 \frac{f^m}{\partial u_1} + \ldots + g_n \frac{f^m}{\partial u_n} = \delta(f^m) = 0.$$

Hence $\bar{\alpha}$, induced by Φ, satisfies $\bar{\alpha} \in \mathrm{Aut}_m^\Gamma(\bar{I}_m) \cap G(\bar{I}_m)$, $\bar{\alpha} \neq id$. Now for any $h \in A$, conditions on $\bar{\alpha}_h$ imply

$$\breve{\alpha}_h(u_i) = u_i + h g_i f^m \mod I^{m+1}$$

which is induced by $\Phi_h := \exp(h f^m \delta)$, and the result follows. □

We will need the following two well known propositions from the commutative algebra, that are presented below, to prove the main theorem of this chapter.

Proposition 2.5.14. *[Ati18] Let B be a local ring with maximal ideal μ. Let M be a finitely generated B module and $x_i, i = 1, \ldots, n$, be elements of M whose images in $M/\mu M$ form a basis of this vector space. Then x_i generate M.*

Proposition 2.5.15. *[Ati18] Let*

$$\varphi : N \to M$$

be an A - module homomorphism. The following are equivalent

(i) φ is an isomorphism

(ii) $\varphi_\rho : N_\rho \to M_\rho$ is an isomorphism for each prime ideal ρ.

(iii) $\varphi_\mu : N_\mu \to M_\mu$ is an isomorphism for each maximal ideal μ.

Denote $M := \mathscr{F}_m^0$, and let N be the submodule of M, whose elements come from global automorphisms.

Theorem 2.5.16. *The following holds*

$$N = M.$$

Proof. Consider the representation (1.18). For every point x on the curve Γ with maximal ideal μ_x in the ring A we have the induced representation of the localization M_{μ_x}

$$\phi_1 : \mathrm{SL}_n(A_{\mu_x}) \to \mathrm{Aut}(M_{\mu_x}), \tag{2.37}$$

which in turn induces the residual representation

$$\phi_2 : \mathrm{SL}_n\left(\frac{A_{\mu_x}}{\mu_x A_{\mu_x}}\right) \to \mathrm{Aut}\left(\frac{M_{\mu_x}}{\mu_x M_{\mu_x}}\right). \tag{2.38}$$

Construct $\bar{\alpha}$ as in Lemma 2.5.13, with $\theta_m(\bar{\alpha})(\vec{u}) = \vec{\Phi}_m(\vec{u})$, and

$$\Phi(u_i) = u_i + f^m \delta(u_i) \mod I^{m+1}$$

For each point $x \in \Gamma$ we can take a locally nilpotent derivation δ so that $\vec{\Phi}_m(\vec{u})$ gives a nonzero element in $M_{\mu_x}/\mu_x M_{\mu_x}$. Since the $A_{\mu_x}/\mu_x A_{\mu_x}$ is a field, the proposition 1.3.63 gives us that the representation (2.38) is irreducible. Hence,

$$\frac{N_{\mu_x}}{\mu_x N_{\mu_x}} = \frac{M_{\mu_x}}{\mu_x M_{\mu_x}},$$

and the Proposition 2.5.14 implies that

$$N_{\mu_x} = M_{\mu_x}$$

for every $x \in \Gamma$. From here the proposition 2.5.15 gives

$$N = M.$$

□

Theorem 2.5.17. *Given $\vec{F}_m(\vec{u}) \in \mathscr{F}_m^0$ there exists $\bar{\tau} \in \mathrm{Aut}_m^\Gamma(\bar{I}_m) \cap G(\bar{I}_m)$ such that*

$$\theta_m(\bar{\tau}) = \vec{F}_m(\vec{u})$$

Proof. Let $\bar{\alpha}$ and $\bar{\alpha}_h$, for $h \in A$, be as in Lemma 2.5.13. By Theorem 2.5.16 there exist $h_1, \ldots, h_k \in A$, and $a_1, \ldots, a_k \in \mathrm{SL}_n(A)$ such that

$$\vec{F}_m(\vec{u}) = h_1 a_1.\theta_m(\bar{\alpha}) + \ldots h_k a_k.\theta_m(\bar{\alpha}),$$

There exists global automorphisms that induce some $\bar{\gamma}_i \in \mathrm{Aut}^\Gamma(\bar{I}_m) \cap G(\bar{I}_m)$ with $\theta_1(\bar{\gamma}_i) = a_i$. Define

$$\bar{\tau} = (\bar{\gamma}_1^{-1} \circ \bar{\alpha}_{h_1} \circ \bar{\gamma}_1) \circ \ldots \circ (\bar{\gamma}_k^{-1} \circ \bar{\alpha}_{h_k} \circ \bar{\gamma}_k)$$

Note that $\bar{\tau} \in G(\bar{I}_m)$, and $\bar{\tau} \in \mathrm{Aut}_m^\Gamma(\bar{I}_m)$ by Proposition 2.5.11. Moreover

$$\theta_m(\bar{\tau}) = \theta_m(\bar{\gamma}_1^{-1} \circ \bar{\alpha}_{h_1} \circ \bar{\gamma}_1) + \ldots + \theta_m(\bar{\gamma}_k^{-1} \circ \bar{\alpha}_{h_k} \circ \bar{\gamma}_k) =$$
$$= h_1 a_1.\theta_m(\bar{\alpha}) + \ldots h_k a_k.\theta_m(\bar{\alpha}) = \theta_m(\bar{\beta}).$$

□

Corollary 2.5.18. $\mathrm{Aut}_m^\Gamma(\bar{I}_m) \subset G(\bar{I}_m)$.

Theorem 2.5.19. *The following holds*

$$\mathrm{Aut}^\Gamma(\bar{I}_m) \subset G(\bar{I}_m)$$

Proof. We know that the result is true for $m = 1$. From here we proceed by induction. Suppose

$$\mathrm{Aut}^\Gamma(\bar{I}_{m-1}) \subset G(\bar{I}_{m-1})$$

and consider $\bar{\beta} \in \mathrm{Aut}^\Gamma(\bar{I}_m)$. By induction hypothesis there exists a global automorphism that induces some

$$\bar{\gamma} \in \mathrm{Aut}^\Gamma(\bar{I}_m) \cap G(\bar{I}_m)$$

with

$$\theta_j(\bar{\gamma}) = \theta_j(\bar{\beta}),\ \ j = 1, \ldots, m-1.$$

Define

$$\bar{\lambda} := \bar{\beta} \circ \bar{\gamma}^{-1} \implies \bar{\lambda} \circ \bar{\gamma} = \bar{\beta}$$

Since

$$\theta_j(\bar{\gamma}) = \theta_j(\bar{\beta}),\ \ j = 1, \ldots, m-1,$$

it follows that $\bar{\lambda} \in \mathrm{Aut}_m^\Gamma(\bar{I}_m)$. By Corollary 2.5.18 $\bar{\lambda} \in G(\bar{I}_m)$, hence

$$\bar{\beta} = \bar{\lambda} \circ \bar{\gamma} \in G(\bar{I}_m).$$

□

Thus we have proved the Theorem 2.5.6, which is a special case of the Theorem 1.2.8 when we consider only one curve Γ in X. And the Theorem 1.2.8 in its general statement, where we consider several curves, follows immediately by a simple induction argument with combination with the Example 1.3.37.

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