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New Approaches to Spacetime Rigidity and Splitting

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UNIVERSITY OF MIAMI

NEW APPROACHES TO SPACETIME RIGIDITY AND SPLITTING

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We take a new approach to Lorentzian splitting geometry, revamping and generalizing the classical notion of 'horosphere' from hyperbolic geometry. We begin with a broad definition of Lorentzian *sphere*, which, in particular, gives an achronal boundary. Using an achronal decomposition of Penrose, we define the *achronal limit* of a sequence of monotonic achronal boundaries, and then a *horosphere* as an achronal limit of spheres whose centers approach infinity. As achronal limits are themselves achronal boundaries, our horospheres are C^0 hypersurfaces by construction. In particular, this resolves, in an elegant and geometric way, the poor regularity of the Busemann function in the Lorentzian setting. Moreover, we show that such horospheres exhibit intrinsic support mean convexity properties, and using the maximum principle of [2], several splitting results are given under the timelike convergence condition, including applications to a conjecture of R. Bartnik (inspired by S.-T. Yau), related to the rigidity of the Hawking-Penrose singularity theorems. In particular, we construct two concrete examples, the *ray* and *Cauchy* horospheres, and give a proof of the conjecture in terms of the latter, under the additional assumption that a certain 'max-min' condition hold on its base Cauchy surface. Finally, turning attention to spacetimes with positive cosmological constant, we develop a notion of *limit mean convexity* and corresponding 'maximum principle' for achronal limits, and use these to prove a rigid singularity result for asymptotically de Sitter spacetimes.

Dedicated to

Basilia and Arturo Amor

and Arturo Amor Jr.

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Chapter 1

Introduction

In the early 1980's, S.-T. Yau posed the 'Lorentzian splitting problem' in [30], i.e., the problem of establishing a Lorentzian analogue of the Cheeger-Gromoll splitting theorem of Riemannian geometry. Approaches to this have involved the introduction of the Lorentzian Busemann function, and a study of its level sets, the now standard Lorentzian 'horospheres' (cf. [6], [11], [16], and [22]). While these tools have seen very effective application, the Busemann function in the Lorentzian setting suffers characteristically from poor analytic properties, and its regularity theory is considerably more complicated and less complete than in the Riemannian case, (cf. [18]).

The Lorentzian splitting problem was motivated, in part, by the question of rigidity in the Hawking-Penrose singularity theorems. The positive resolution of the splitting theorem did not, however, settle this question. This is due, again, to a distinctively Lorentzian issue, namely that, while one can in context produce a causal line, there is a danger that, during construction, this line does not remain 'upright', i.e., *timelike*, and tips over to a *null* line.

In 1988, Bartnik realized the rigidity question concretely as Conjecture 2 in [4], which may stated as follows:

Conjecture 1 (Bartnik Splitting Conjecture)**.** *Let (M, g) be a spacetime and suppose that:*

1) M has a compact Cauchy surface S.

2) M satisfies the timelike convergence condition, i.e., $\mathrm{Ric}(X, X) \geq 0$ *for all timelike X.*

3) M is timelike geodesically complete.

Then M splits, i.e., $(M, g) \approx (\mathbb{R} \times N, -dt^2 + h)$*, for some compact Riemannian manifold* (N, h).

In fact, the first result towards this conjecture appeared in [14], four years prior. There, the splitting is established under an additional 'no observer horizons' condition. This was in turn improved upon in [4].

Using only conditions 1) and 3) of Conjecture 1, it is easy to construct a complete *causal* line. If the line is timelike, then by condition 2), the Lorentzian splitting theorem takes over and splits the spacetime. However, in [9], a spacetime is constructed which is timelike geodesically complete, with compact Cauchy surfaces, but which contains no timelike lines. Consequently, for such a strategy to work, the curvature condition should somehow play an active role in the construction of the line itself.

Having proven to be powerful splitting tools, it is natural to approach Conjecture 1 via Busemann functions. We recall that, fixing any unit speed timelike geodesic ray, $\gamma : [0, \infty) \to M$, the associated Busemann function b_γ is the limit function

$$b_\gamma(x) = \lim_{k \to \infty} \big[k - d(x, \gamma(k))\big]$$

The geometry is now somewhat buried, but the 0-level set of the pre-Busemann function $b_k(x) = k - d(x, \gamma(k))$ is the past point sphere of radius k from $\gamma(k)$, i.e., $\{b_k = 0\} = S_k^-(\gamma(k))$. These are the 'prehorospheres' associated to the standard horosphere $\{b_\gamma = 0\}$. The convergence, in general, of these sets, however, is a rather murky question in the Lorentzian setting. The analytic difficulties for the Lorentzian Busemann function may be said to begin with the reversal of the triangle inequality in the Lorentzian case. As an interesting consolation, however, this reversed triangle inequality nonetheless implies a type of 'causal monotonicity' for the prehorospheres $\{S_k^-(\gamma(k))\}$. In approaches to Conjecture 1, we have also considered a variation of the standard Busemann function. This variation gave certain advantages over the standard construction, but was ultimately seen to suffer from similar regularity issues. Again, however, the associated prehorospheres were observed to obey a certain causal monotonicity.

The approach taken here is to forget the functions, and focus on the spheres. The classical 'horosphere' of hyperbolic geometry is the limit of a monotonic sequence of spheres, which meets infinity. The philosophy here is that the correct notion of horosphere should indeed, in the first place, be a limit sphere, in our case, 'centered' at past or future infinity. Moreover, in hopes of developing a robust and flexible splitting tool, we were inspired to keep this notion very general. This is somewhat akin to a paper of Wu generalizing Busemann functions in the Riemannian setting (cf. [29]).

Below, we give a broad definition of *sphere*, whose center may be a point, any compact set, or more generally, a set which exhibits *causal completeness* (cf. [15]). Focusing on the globally hyperbolic case, such spheres are achronal boundaries. Achronal boundaries satisfy many important structural properties, and in particular, are edgeless, achronal C^0 hypersurfaces. Moreover, as shown by Penrose in [25], every achronal boundary 'causally separates' spacetime in a useful and unique way. This separating property gives a notion of monotonicity for achronal boundaries and, in turn, allows for an elegant notion of *achronal* *limit* for monotonic sequences of such boundaries. We then define a *horosphere* as an achronal limit of monotonic spheres, with centers approaching infinity, or more specifically, with radii $\to \infty$. As achronal limits are themselves achronal boundaries, our spheres and horospheres are edgeless, achronal C^0 hypersurfaces.

Exploiting the auxiliary structure of spheres, namely that they come with intrinsic 'radial' maximal segments tied to each point, we show that, taking limits, a past horosphere S^-_∞, for example, admits a future 'radial' S^-_∞-ray from each point. Such radial rays endow horospheres with intrinsic support mean convexity properties, and using the maximum principle of [2], we give various splitting results for abstract horospheres under the timelike convergence condition.

In addition, we construct two important concrete classes of horospheres. The monotonicity of $\{S^-_k(\gamma(k))\}$ mentioned above is used to recast the standard Lorentzian Busemann horosphere, $\{b_\gamma = 0\}$, as the *ray horosphere*, $S^-_\infty(\gamma)$. This resolves, for one, the issue of having to contend with the tedious regularity and convexity theory of the set $\{b_\gamma = 0\}$. We also consider an analogous construction, with a Cauchy surface S in place of the ray γ, to produce the *Cauchy horosphere*, $S^-_\infty(S)$. Among several applications to Conjecture 1, we use $S^-_\infty(S)$ to give a proof of the conjecture in the case that a certain 'max-min' condition hold on S.

Finally, turning attention to the positive cosmological constant case, we develop a notion of *limit mean convexity*, and use the Dirichlet solutions of Bartnik in [3] to establish a corresponding '(pseudo)-maximum principle' for achronal limits. We use these to establish, in particular, a rigid singularity result for asymptotically de Sitter spacetimes, related to results in [8] and [1].

1.1 Spacetime Cheat Sheet

For completeness and convenience, we compile in this preliminary section some basic Lorentzian geometry and causal theory needed below. We note the standard references: [21], [25], [23], [28], [5]. Because of their central role in the theory to follow, two topics are left out of this introductory section. Achronal boundaries are introduced in Section 2.1. The Lorentzian distance function is introduced in Section 3.1.

1.1.1 Lorentzian Manifolds and Spacetimes

The simplest and most important spacetime is *Minkowski space*, $\mathbb{M}^{n+1}$. This may be defined as $\mathbb{R}^{n+1} \approx \mathbb{R} \times \mathbb{R}^n \approx \{(t = x^0, x^1, ..., x^n)\}$, together with the scalar product

$$\langle x, y \rangle = -x^0 y^0 + \sum_{i=1}^{n} x^i y^i$$

We think of the first coordinate $t = x^0$ as designating a moment in time and the remaining coordinates $(x^1, ..., x^n)$ as designating a location in space.

Lorentzian geometry is based on Minkowski space in the same way that Riemannian geometry is based on Euclidean space. In particular, by a *smooth Lorentzian manifold* $M = (M^{n+1}, g)$ we mean a smooth $(n+1)$-dimensional manifold, with $n \geq 1$, furnished with a smooth Lorentzian metric, g, i.e., a smooth, nondegenerate symmetric $(0, 2)$-tensor of constant index $\nu = 1$. (More generally, for a *semi-Riemannian manifold*, we allow any dimension and any constant index $0 \leq \nu \leq \dim M$.) At each point $p \in M$ of a Lorentzian manifold, the tangent space T_pM admits an orthonormal basis $[e_0, e_1, ..., e_n]$, with respect to which the metric g_p at p is given by

$$g_p(X,Y) = -X^0Y^0 + \sum_{i=1}^{n} X^iY^i$$

for all $X, Y \in T_pM$. Hence, each tangent space of a Lorentzian manifold M is isomorphic to Minkowski space; $(T_pM, g_p) \approx \mathbb{M}^{n+1}$.

The causal character of a vector $X \in TM$ is defined as follows:

$$\begin{array}{lll} X \text{ is } \textit{timelike} & \text{if} & g(X,X) < 0 \\ X \text{ is } \textit{null} & \text{if} & g(X,X) = 0 \\ X \text{ is } \textit{spacelike} & \text{if} & g(X,X) > 0 \end{array}$$

We say X is *causal* if it is either timelike or null.

A Lorentzian manifold is *time-orientable* if it admits a smooth, everywhere-timelike vector field, a choice, T, of which is called a *time-orientation.* The natural time-orientation on $\mathbb{M}^{n+1}$, for example, is ∂_t. A general Lorentzian manifold is either time-orientable or has a double-cover which is.

Definition 1.1.1. By a *spacetime*, we will mean a smooth, connected, time-oriented Lorentzian manifold $M = (M^{n+1}, g, T)$.

Unless otherwise indicated, M shall henceforth denote a spacetime.

1.1.2 Causal Curves, Futures, Pasts, and Diamonds

The time-orientation T on a spacetime M is interpreted as pointing towards the 'future' and determines the time-orientation of a causal vector X as follows:

$$\begin{array}{lll} X \text{ is } \textit{future-directed} & \text{if} & g(T,X) \leq 0 \\ X \text{ is } \textit{past-directed} & \text{if} & g(T,X) \geq 0 \end{array}$$

We say that a smooth curve β is *timelike*, (resp. *null*, *causal*, *spacelike*) if its tangent vector field β' is everywhere timelike (resp. null, causal, spacelike). More

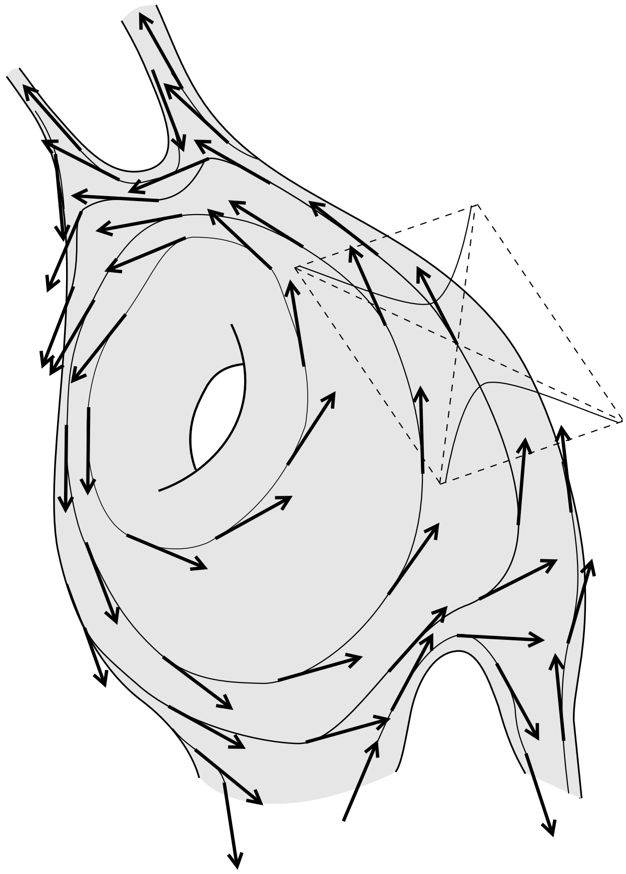


Figure 1.1: A spacetime M with time orientation T.

generally, by a *future-directed (resp. past-directed) timelike (resp. null, causal) curve*, we shall mean a piecewise-smooth curve $\alpha : I \to M$, whose tangents, including one-sided tangents at any break points or endpoints, are all future-directed (resp. past-directed) timelike (resp. null, causal). Clearly, α is future-directed iff $-\alpha$ is past-directed. We will often refer simply to a 'causal curve α', for example, ignoring its time-orientation.

A future-directed (causal) curve $\alpha : (c, d) \to M$, with $-\infty \leq c < d \leq \infty$, is *future-inextendible* if $\lim_{t \to d^-} \alpha(t)$ does not exist and *past-inextendible* if $\lim_{t \to c^+} \alpha(t)$ does not exist. A causal curve is *inextendible* (or *endless*) if it is both future- and past-inextendible.

Any semi-Riemannian manifold, and hence every spacetime M admits a unique torsion-free, metric-compatible connection, ∇, called the *Levi-Civita connection.* A curve α in M is called a *geodesic* if it is non-accelerating, i.e., if $\alpha'' = \nabla_{\alpha'}\alpha' = 0$. By a *pregeodesic* we mean a curve which admits a geodesic reparameterization. The geodesics, (strictly speaking, the images thereof), in Minkowski space, $\mathbb{M}^{n+1}$, are precisely the straight lines segments. It follows from the definition and metric-compatibility, that every geodesic in a spacetime is either timelike, null, or spacelike.

Recall that a subset U of a vector space is convex if, for any two points $v, w \in U$, the (unique) line segment $\overline{vw}$ joining them is contained entirely in U. Analogously, given a semi-Riemannian manifold M, we say a subset $U \subset M$ is *convex* if any two points $p, q \in U$ are joined by a unique geodesic segment contained entirely in U, in which case we denote this segment by $\overline{pq}$. Every semi-Riemannian manifold, and hence every spacetime admits arbitrarily small convex neighborhoods around each of its points. Note that $\mathbb{M}^{n+1}$ is itself convex in the sense of semi-Riemannian manifolds.

The *length* of a causal curve segment, $\alpha : [a, b] \to M$, where α is piecewise smooth with breaks $a = t_0 < t_1 < ... < t_m = b$, is given by $L(\alpha) = \sum_{i=1}^{m} \int_{t_{i-1}}^{t^i} |\alpha'| dt$. The following result is fundamental to causal theory and says, in particular, that causal geodesics locally *maximize* length:

Proposition 1.1.2. *Let $U \subset M$ be a convex neighborhood.*

(1) If there is a timelike (resp. causal) curve in U from p to q, then $\overline{pq}$ is timelike (resp. causal).

(2) If $\overline{pq}$ is timelike, then $L(\overline{pq}) \geq L(\alpha)$ for any causal curve segment in U joining p and q, with equality iff α is a reparametrization of $\overline{pq}$.

The causal curves of a spacetime M define two fundamental 'orders' on M. For $p, q \in M$, we write:

$p \ll q$ if there is a future-directed timelike curve segment from p to q

$p \leq q$ if there is a future-directed causal curve segment from p to q

We define the *chronological future* and *past* of a point $p \in M$, respectively, by

$$\begin{aligned} I^+(p) &= \{q \in M : p \ll q\} \\ I^-(p) &= \{q \in M : q \ll p\} \end{aligned}$$

and the *causal future* and *past* of $p \in M$, respectively, by

$$\begin{aligned} J^+(p) &= \{q \in M : p \leq q\} \\ J^-(p) &= \{q \in M : q \leq p\} \end{aligned}$$

For a general subset $S \subset M$, we define

$$I^{\pm}(S) = \bigcup_{p \in S} I^{\pm}(p), \quad J^{\pm}(S) = \bigcup_{p \in S} J^{\pm}(p)$$

Proposition 1.1.3. *If $q \in J^+(p) \setminus I^+(p)$, then any causal curve from p to q is a null pregeodesic.*

Corollary 1.1.4. *The relations $\ll$ and $\leq$ are transitive. Further, if either $p \ll q \leq r$ or $p \leq q \ll r$, then $p \ll r$.*

Given any open set $U \subset M$ and $p \in U$, we define $I^+(p, U)$ to be the set of points $q \in U$ such that there is a future-directed timelike curve α within U from p to q. In other words, $I^+(p, U)$ is the chronological future of p in the open Lorentzian submanifold $U \subset M$. $I^-(p, U)$ and $J^{\pm}(p, U)$ are defined similarly.

Proposition 1.1.5. *For a convex neighborhood U of p, we have:*

(1) $I^{\pm}(p, U)$ *is open in U and hence also in M.*

(2) $J^{\pm}(p, U)$ *is the closure in U of* $I^{\pm}(p, U)$.

For any subset $S \subset M$ we have:

(3) $\mathrm{int}\, J^{\pm}(S) = I^{\pm}(S)$. *In particular,* $I^{\pm}(S)$ *is open.*

(4) $\partial J^{\pm}(S) = \partial I^{\pm}(S)$. *Hence,* $J^{\pm}(S) \subset \overline{I^{\pm}(S)}$ *with equality iff* $J^{\pm}(S)$ *is closed.*

For any $p, q \in M$, we will refer to $I^{+}(p) \cap I^{-}(q)$ as a *timelike diamond*, and to $J^{+}(p) \cap J^{-}(q)$ as a *causal diamond.* Note that timelike diamonds are always open.

1.1.3 Limit Curves

A C^0 curve $\alpha : I \to M$ is said to be *future-directed causal* if for all $t_0 \in I$, there is an open interval neighborhood $t_0 \in I_0 \subset I$ and a convex normal neighborhood U_0 of $\alpha(t_0)$, with $\alpha(I_0) \subset U_0$, such that, for all $[t_1, t_2] \in I_0$, $\alpha(t_2) \in I^{+}(\alpha(t_1), U_0)$.

The Limit Curve Lemma is a fundamental tool in causal theory. We state it here as follows. We note that every smooth manifold admits a complete Riemannian metric.

Lemma 1.1.6 (Limit Curve Lemma)**.** *Fix a complete Riemannian metric h on M. Let $\alpha_k : [0, \infty) \to M$ be a sequence of future-inextendible causal curves parameterized with respect to h arc length. Given any limit point p of the sequence of base points $\{\alpha_k(0)\}$, there is a future-inextendible C^0 causal curve $\alpha : [0, \infty) \to M$ from $\alpha(0) = p$, and a subsequence $\{\alpha_{k_j}\}$ converging locally uniformly to α, i.e., on any compact parameter interval $[0, T]$, $\alpha_{k_j} : [0, T] \to M$ converges uniformly to $\alpha : [0, T] \to M$. (We note that, in general, α is not parameterized with respect to h arc length.)*

We shall also need the related fact that the (Lorentzian) arc length functional, (which extends to C^0 causal curves), is upper semicontinuous in the following sense:

Proposition 1.1.7. *If a sequence of causal curves $\alpha_k : [a,b] \to M$ converge uniformly to a causal curve $\alpha : [a,b] \to M$, then $L(\alpha) \geq \limsup L(\alpha_k)$.*

The Limit Curve Lemma may be used to prove the following extension of Proposition 1.1.3:

Proposition 1.1.8. *Let $S \subset M$ be closed. Then each $x \in \partial I^+(S) \setminus S$ lies on a null geodesic contained in $\partial I^+(S)$ which either has a past endpoint on S or is past-inextendible in M.*

1.1.4 Strong Causality and Global Hyperbolicity

A spacetime M is said to be *causal* (resp. *chronological*) if it admits no nontrivial closed causal (resp. timelike) curves. Consequently, M is causal if and only if $(M, \leq)$ is a partially ordered set. M is said to be 'strongly causal' if it admits no 'almost-closed' causal curves. This is made precise using the notion of causal convexity. We say an (arbitrary) subset $V \subset M$ is *causally convex* if every causal curve with endpoints in V is contained entirely in V. Equivalently, $V \subset M$ is causally convex if no causal curve meets V in disconnected sets. It is easy and convenient to reformulate this in terms of diamonds:

Lemma 1.1.9. *An (arbitrary) subset $V \subset M$ is causally convex iff V contains all of its diamonds, i.e., iff $J^+(p) \cap J^-(q) \subset V$ whenever $p, q \in V$.*

Causal convexity is often defined only for open neighborhoods, but as we will see, this property is meaningful for other kinds of sets, (cf. Lemmas 1.1.13 and 4.3.1). Note that any (timelike or causal) diamond is causally convex.

We say *strong causality holds at* $p \in M$ if p admits arbitrarily small causally convex open neighborhoods. A subset $H \subset M$ is *strongly causal* if strong causality holds at each point $p \in H$. We note the following:

Lemma 1.1.10 (Escape Lemma)**.** *If $K \subset M$ is compact and strongly causal, then every future-inextendible (resp. past-inextendible) curve must eventually leave K and never return.*

We say a causal curve α from p to q is *maximal* if $L(\alpha) \geq L(\beta)$ for all causal curves β joining p to q. (This will be rephrased in Section 3.1.) Strong causality is related to the existence of maximal geodesic segments.

Lemma 1.1.11. *Let $p \leq q$ and suppose that $J^+(p) \cap J^-(q)$ is strongly causal and compact. Then there is a maximal causal geodesic joining p to q.*

Motivated by Lemma 1.1.11, we distinguish an important class of spacetimes:

Definition 1.1.12. A spacetime M is said to be *globally hyperbolic* if it is strongly causal and has compact causal diamonds, $J^+(p) \cap J^-(q)$.

Hence, by Lemma 1.1.11, any two causally related points in a globally hyperbolic spacetime are joined by a maximal causal geodesic segment.

We say a subset $\mathcal{H} \subset M$ is *globally hyperbolic* if it is strongly causal and, for all $p, q \in \mathcal{H}$, the causal diamond $J^+(p) \cap J^-(q)$ is compact and contained in $\mathcal{H}$. Hence, while global hyperbolicity is not inherited by arbitrary subsets, we have the following:

Lemma 1.1.13. *If $\mathcal{H} \subset M$ is globally hyperbolic and $V \subset \mathcal{H}$ is causally convex, then V is globally hyperbolic.*

Global hyperbolicity has many important consequences. For the moment we note the following:

Proposition 1.1.14. *Let M be globally hyperbolic and $A, B \subset M$ are compact.*

(1) $J^{\pm}(A)$ is closed. In particular, $J^{\pm}(A) = \overline{I^{\pm}(A)}$.

(2) $J^{+}(A) \cap J^{-}(B)$ is compact.

Lemma 1.1.15. *If M is globally hyperbolic, the relation $p \leq q$ is closed on M, i.e., if $p_k \to p$ and $q_k \to q$, with $p_k \leq q_k$ for all k, then $p \leq q$.*

1.1.5 Achronal Sets and Edge Points

Achronality will play a central role in the theory to follow. We say a set $A \subset M$ is *achronal (acausal)* if no two points of A can be joined by a timelike (causal) curve. The *edge* of an achronal set $A \subset M$ is defined as follows: A point $p \in \overline{A}$ is in edge A, if every neighborhood U of p contains a timelike curve from $I^{-}(p, U)$ to $I^{+}(p, U)$ which does not meet A. We note that an achronal set consists entirely of topological boundary points (in the manifold topology). Nonetheless, the edge of an achronal set is a type of boundary.

Lemma 1.1.16. *If A is achronal, then $\overline{A} = A \cup \operatorname{edge} A$.*

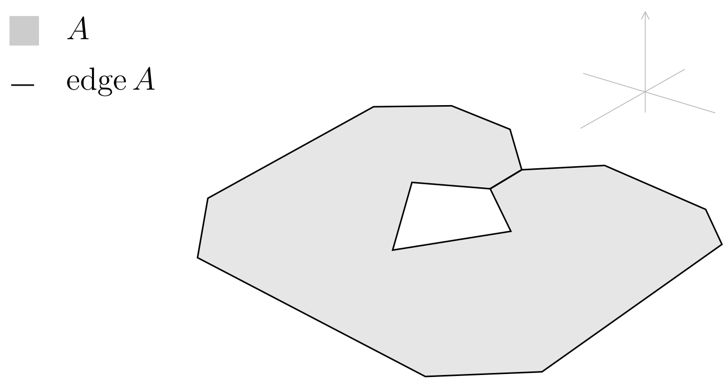


Figure 1.2: An achronal set A which 'almost kisses' itself.

The following consequence of achronality will be especially important below:

Proposition 1.1.17. *If $\emptyset \neq A \subset M$ is achronal, then A is a C^0 hypersurface iff $A \cap \operatorname{edge} A = \emptyset$. In particular, any edgeless achronal set is a closed C^0 hypersurface.*

Moreover, we have the following specialization of Proposition 1.1.8:

Corollary 1.1.18. *Let $S \subset M$ be closed and achronal. Then each $x \in \partial I^+(S) \setminus S$ lies on a null geodesic contained in $\partial I^+(S)$ which either has a past endpoint on $\operatorname{edge} S$ or is past-inextendible in M.*

Proof. As this fact, (in this precise form), does not appear to be immediately available in most of the standard references, we include a proof. Fixing $x \in \partial I^+(S) \setminus S$, by Proposition 1.1.8, we may suppose x is the future endpoint of a null geodesic $\eta : [0,1] \to \partial I^+(S)$, with $\eta(0) = s \in S$ and $\eta(1) = x$. Furthermore, we may suppose η leaves S immediately, i.e., $\eta \cap S = \{\eta(0)\} = \{s\}$. We want to show $s \in \operatorname{edge} S$. Suppose otherwise, that there is a neighborhood U of s such that each timelike curve in U from $I^-(s,U)$ to $I^+(s,U)$ meets S. Let $p \in I^-(s,U)$ and $q \in I^+(s,U)$. Then $I^+(p,U) \cap I^-(q,U)$ is an open neighborhood of s and hence meets $\eta(\epsilon)$ for some $\epsilon > 0$. Hence, there is a timelike curve α^- in U from p to $\eta(\epsilon)$ and a timelike curve α^+ in U from $\eta(\epsilon)$ to q. The concatenation, $\alpha = \alpha^- + \alpha^+$ is a timelike curve in U from $p \in I^-(s,U)$ to $q \in I^+(s,U)$, and hence must meet S, i.e., either α^- or α^+ meets S. Suppose the latter. Then, since $\eta(\epsilon) \notin S$, we have $\alpha^+ \setminus \{\eta(\epsilon)\}$ meets S at some point $s' \in S$. But then, we have $s \leq \eta(\epsilon) << s'$, and, hence $s << s'$, by Corollary 1.1.4, which violates the achronality of S. Hence α^- meets S, in particular, $\alpha^- \setminus \{\eta(\epsilon)\}$ meets S. But this implies $\eta(\epsilon) \in I^+(S)$, and hence $x \in I^+(S)$, contradicting $x \in \partial I^+(S)$. Hence $s \in \operatorname{edge} S$. $\square$

1.1.6 Cauchy Developments and Cauchy Surfaces

Note that we may write the causal past and future of a set $S \subset M$, $J^-(S)$ and $J^+(S)$, also called the *past* and *future domains of influence of* S, respectively, as follows:

$$\begin{aligned} J^+(S) &= \{q \in M : \text{ some past-directed causal curve from } q \text{ meets } S\} \\ J^-(S) &= \{p \in M : \text{ some future-directed causal curve from } p \text{ meets } S\} \end{aligned}$$

We may define the *(total) domain of influence of* S as $J(S) = J^+(S) \cup J^-(S)$. (Similarly, we let $I(S) = I^-(S) \cup I^+(S)$).

For $S \subset M$ achronal, we define the *past* and *future domain of dependence of* S, respectively, by

$$\begin{aligned} D^+(S) &= \{q \in M : \text{ every past-inextendible causal curve from } q \text{ meets } S\} \\ D^-(S) &= \{p \in M : \text{ every future-inextendible causal curve from } p \text{ meets } S\} \end{aligned}$$

The *(total) domain of dependence of* S is $D(S) = D^+(S) \cup D^-(S)$. The words '*Cauchy development*' are often used in place of 'domain of dependence'. Hence, $D^+(S)$ is also called the future Cauchy development of S, for example, and $D(S)$ is the (full) Cauchy development of S.

The *past* and *future Cauchy horizons of* S are defined, respectively, by

$$\begin{aligned} H^+(S) &= \overline{D^+(S)} \setminus I^-(D^+(S)) \\ H^-(S) &= \overline{D^-(S)} \setminus I^+(D^-(S)) \end{aligned}$$

The *(total) Cauchy horizon of* S is $H(S) = H^+(S) \cup H^-(S)$.

We have the following:

Proposition 1.1.19. *Let $S \subset M$ be achronal.*

(1) $S \subset D^{\pm}(S) \subset S \cup I^{+}(S) \subset J^{\pm}(S)$.

(2) $H^{\pm}(S)$ *is achronal.*

(3) $\operatorname{edge} H^{\pm}(S) \subset \operatorname{edge} S$, *with equality holding iff S is closed.*

(4) $H^{\pm}(S) \setminus \operatorname{edge} S \subset J^{\pm}(S)$.

(5) $\partial D^{\pm}(S) = S \cup H^{\pm}(S)$ *and* $\partial D(S) = H(S)$.

As an immediate corollary to Propositions 1.1.17 and 1.1.19, we have:

Corollary 1.1.20. *If $S \subset M$ is achronal, then $H^{\pm}(S) \setminus \operatorname{edge} S$, if nonempty, is an edgeless achronal C^0 hypersurface.*

The following is an analog of Proposition 1.1.18 for Cauchy horizons:

Proposition 1.1.21. *Suppose $S \subset M$ is achronal. Then any point $p \in H^{+}(S) \setminus \operatorname{edge} S$ is the future endpoint of a null geodesic contained entirely in $H^{+}(S)$ which either has a past endpoint on $\operatorname{edge} S$ or is past-inextendible.*

Regarding the structure of Cauchy developments, we have:

Proposition 1.1.22. *Let $S \subset M$ be achronal. Then $\operatorname{int} D(S)$, if nonempty, is globally hyperbolic. Moreover, if S is an acausal hypersurface, then $D(S)$ is open.*

Moreover, we have:

Proposition 1.1.23. *Let $\emptyset \neq S \subset M$ be achronal.*

(1) $D^{-}(S) = J^{-}(S) \iff H^{-}(S) = \emptyset$, *in which case* $S = \partial I^{-}(S)$.

(2) $M = D(S) = J(S) \iff H(S) = \emptyset$, *in which case* $\partial I^{-}(S) = S = \partial I^{+}(S)$.

Definition 1.1.24. Let $\emptyset \neq S \subset M$ be achronal. We say S is a *past Cauchy surface* if $H^{-}(S) = \emptyset$. *Future Cauchy surfaces* are defined time-dually. We say S is a (full) *Cauchy surface* if $H(S) = \emptyset$, or equivalently, if $D(S) = M$.

Hence, a nonempty achronal set is a Cauchy surface iff it is both a past and future Cauchy surface. As we will show in Section 2.1, it follows from Proposition 1.1.23 that (past, future) Cauchy surfaces are edgeless. Hence, by Proposition 1.1.17, we have:

Corollary 1.1.25. *Any (past, future) Cauchy surface is an edgeless achronal C^0 hypersurface.*

The definition of 'Cauchy surface' has many variations in the literature. Often, they are defined by (some version of) the following:

Proposition 1.1.26. *Let $S \subset M$ be achronal. Then S is a Cauchy surface for M iff every inextendible causal curve in M meets S. In this case, every inextendible causal curve also meets $I^{-}(S)$ and $I^{+}(S)$.*

We note also the following property (related to causal completeness below):

Proposition 1.1.27. *Let S be a past Cauchy surface. Then, for every $p \in J^{-}(S)$, the set $J^{+}(p) \cap J^{-}(S)$ is compact.*

Using Propositions 1.1.21 and 1.1.27 and Lemma 1.1.10, we have:

Proposition 1.1.28. *Any compact, edgeless achronal set $S \subset M$ must be a Cauchy surface for M.*

We have the following basic topological facts regarding Cauchy surfaces:

Theorem 1.1.29. *Let S be a Cauchy surface for M and let X be any smooth complete timelike vector field on M. Then M splits topologically as $M \approx \mathbb{R} \times S$, where the t-curves are precisely the (timelike) integral curves of X. Furthermore, any two Cauchy surfaces for a spacetime are homeomorphic.*

Finally, we note the following fundamental result relating Cauchy developments and global hyperbolicity:

Theorem 1.1.30 (Geroch)**.** *A spacetime M is globally hyperbolic iff it admits a Cauchy surface.*

1.1.7 Completeness and Singularities

A geodesic $\gamma : I \to M$ is *complete* if $I = \mathbb{R}$. We say M is *geodesically complete* if every geodesic in M extends to a complete geodesic. Less stringently, we say M is *future timelike (resp. null) geodesically complete* if every future-directed timelike (resp. null) geodesic γ is extendible as such to $\gamma : I \to M$ with $[a, \infty) \subset I$, for some $a \in \mathbb{R}$. Past timelike and null completeness is understood time-dually. We say M is *timelike (resp. null) geodesically complete* if it is both future and past timelike (resp. null) geodesically complete.

By a *singularity* in M, we will usually mean the existence of a (future or past) incomplete timelike or null geodesic, in which case we may say M is *singular*.

Chapter 2

Achronal Limits

2.1 Pasts, Futures, and Achronal Boundaries

In this section we review some of the basic theory of achronal boundaries as developed by Penrose in [25]. In particular, we recall (and give a new proof of) an 'achronal decomposition' result in [25], Proposition 2.1 below, which is perhaps somewhat less standard, but will play a central role in the theory to follow.

To begin, we say $P \subset M$ is a *past set* if $P = I^-(S)$ for some $S \subset M$, and define *future sets*, F, time-dually. The following is immediate from the definitions and Proposition 1.1.5:

Lemma 2.1.1. *Past and future sets are open. The union of past sets is a past set, and similarly for future sets.*

For (nonsentimental) reasons which will be clear below, we will often focus attention on pasts, though, as always, time-dual statements exist for those not mentioned explicitly.

We first note the following:

Lemma 2.1.2 (Past Sets). *Let $P \subset M$. The following are equivalent:*

i) P *is a past set.*

ii) $P = I^-(P)$.

iii) P *is open and* $I^-(P) \subset P$.

iv) $M \setminus \overline{P}$ *is a future set.*

Proof. Noting that, for any set $S \subset M$, $\operatorname{int} S \subset I^{\pm}(S)$, the equivalence of i), ii), and iii) is straighforward. i) $\Longleftrightarrow$ iv): Suppose that P is a past set and let $x \notin \overline{P}$. Were $I^+(x)$ to meet $\overline{P}$, then $I^+(x)$ would meet P and we would have $x \in I^-(P) = P$. Hence, $I^+(x) \subset M \setminus \overline{P}$. Since x was arbitrary, we have $I^+(M \setminus \overline{P}) \subset M \setminus \overline{P}$ and hence, by the time dual of i) $\Longleftrightarrow$ iii), $M \setminus \overline{P}$ is a future set. The converse is the time dual. $\square$

Definition 2.1.3. The boundary of a past or future set, if non-empty, is called an *achronal boundary.*

Hence, an achronal boundary is a set of the form $\emptyset \neq A = \partial I^{\pm}(S)$, for some $S \subset M$. By Lemma 2.1.2, if $A = \partial I^{\pm}(S)$ for some $S \subset M$, then $A = \partial I^{\mp}(M \setminus \overline{I^{\pm}(S)})$. Hence, there is no distinction between the boundary of a past or a future, and all achronal boundaries may, for example, be expressed in the form $A = \partial I^+(S)$.

Lemma 2.1.4. *Let* $A = \partial I^+(S)$.

1) $I^+(A) \subset I^+(S)$.

2) $I^-(A) \subset M \setminus \overline{I^+(S)}$.

Proof. Let $a \in A = \partial I^+(S)$. 1) If $b \in I^+(a)$, then $I^-(b)$ is an open neighborhood of a and hence meets $I^+(S)$, and thus $b \in I^+(S)$. This shows $I^+(A) \subset I^+(S)$. 2) Now fix any $c \in I^-(a)$. By 1), we know $c \notin A$. Were $c \in I^+(S)$, then a, being in the future of c, would be in the future of S, contradicting $A \cap I^+(S) = \emptyset$. Hence $c \in M \setminus \overline{I^+(S)}$. Thus we have $I^-(A) \subset M \setminus \overline{I^+(S)}$. □

It follows from Lemma 2.1.4 that every achronal boundary is achronal and edgeless. Hence we have the following corollary to Proposition 1.1.17:

Corollary 2.1.5. *Achronal boundaries are edgeless achronal C^0 hypersurfaces.*

We now take a new approach to the achronal decomposition result given in [25], Proposition 2.1 below, which here follows from a few additional observations on the structure of past sets. For convenience, we begin by rephrasing Lemma 2.1.4 as follows:

Corollary 2.1.6. *Let P be a past set and $x \in \partial P$. Then $I^-(x) \subset P$.*

We have the following regarding the structure of pasts:

Lemma 2.1.7. *Let $\emptyset \neq P \neq M$ be a past set in a (connected) spacetime M and let $\{\mathcal{P}_i\}_{i\in I}$ be the connected components of P. Then each $\mathcal{P}_i$ is a past set and ∂P is the disjoint union of the boundaries $\{\partial \mathcal{P}_i\}_{i\,\in I}$, each of which is nonempty;*

$$\partial P = \bigcup_{i \in I} \partial \mathcal{P}_i$$

Proof. Since $I^-(p)$ is (path) connected for any $p \in M$, it follows that $I^-(\mathcal{P}_i) \subset \mathcal{P}_i$. Since $\mathcal{P}_i$ is necessarily open, it follows from Lemma 2.1.2 that $\mathcal{P}_i$ is a past set. Since M is connected, $\partial \mathcal{P}_i \neq \emptyset$, and since the sets $\mathcal{P}_i$ are pairwise disjoint, so are their boundaries, by Corollary 2.1.6. Now, of course we have $\partial \mathcal{P}_i \subset \overline{\mathcal{P}_i} \subset \overline{P}$. But since $\partial\mathcal{P}_i \cap \mathcal{P}_j \neq \emptyset$ is impossible, we must have $\partial\mathcal{P}_i \subset \partial P$. Hence $\cup_{i\in I}\partial\mathcal{P}_i \subset \partial P$. Conversely, let $p \in \partial P$. Let $\alpha : [0,\epsilon) \to M$ be a past timelike curve segment from $p = \alpha(0)$. Then $\alpha \setminus \{\alpha(0)\}$ is connected and contained in P and hence must be contained in some $\mathcal{P}_i$. It follows that $p \in \partial\mathcal{P}_i$. □

The following gives a uniqueness property for achronal boundaries:

Lemma 2.1.8 (Achronal Boundary Uniqueness)**.** *Let P and Q be nonempty past sets with $\partial P = \partial Q$. Then $P = Q$.*

Proof. If $\partial P = \partial Q = \emptyset$, then $P = Q = M$. Hence, suppose $\partial P = \partial Q \neq \emptyset$. Let $\{\mathcal{P}_i\}_{i\in I}$ be the connected components of P and $\{\mathcal{Q}_j\}_{j\in J}$ the connected components of Q. Pick any P-component, $\mathcal{P}_0 := \mathcal{P}_{i_0}$. Then by Lemma 2.1.7, we have $\emptyset \neq \partial\mathcal{P}_0 \subset \partial P$, and since $\partial P = \partial Q$, there is a $\mathcal{Q}_0 := \mathcal{Q}_{j_0}$ with $\partial\mathcal{P}_0 \cap \partial\mathcal{Q}_0 \neq \emptyset$. Then by Lemma 2.1.6, $\mathcal{P}_0 \cap \mathcal{Q}_0 \neq \emptyset$. But since $\partial\mathcal{P}_0$ can not meet $\mathcal{Q}_0$, and vice versa, we have $\mathcal{P}_0 = \mathcal{Q}_0$. This shows $P \subset Q$, and by symmetry, we have $P = Q$. □

Combining Lemmas 2.1.2, 2.1.4, and 2.1.8, we have:

Proposition 2.1.9 (Achronal Decomposition)**.** *Given any achronal boundary A, there exists a unique past set $P = P_A$ and a unique future set $F = F_A$ such that $\partial P = A = \partial F$. The triple $\{P, A, F\}$ forms a (mutually disjoint) partition of M,*

$$M = P \cup A \cup F,$$

with $I^+(A) \subset F$ and $I^-(A) \subset P$. Any curve from P to F must meet A, in a unique point if the curve is timelike.

Henceforth, given an achronal boundary A, we shall often speak of the sets $P = P_A$ and $F = F_A$ without explicit reference to Proposition 2.1. We note that every Cauchy surface S is an achronal boundary with $P = I^-(S)$ and $F = I^+(S)$.

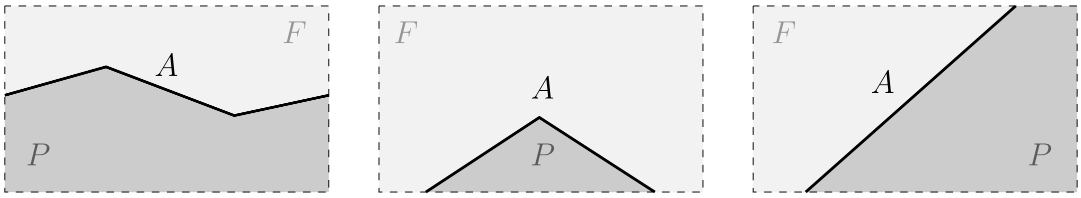


Figure 2.1: Achronal decompositions.

2.2 Achronal Limits

As may be seen in Figure 2.1, the inclusions $I^-(A) \subset P$ and $I^+(A) \subset F$ in Proposition 2.1 may be strict, in general.

Definition 2.2.1 (Proper Boundaries)**.** We say that an achronal boundary A is *past proper* provided $P = I^-(A)$, or equivalently, if $\partial I^-(A) = A$. *Future proper* achronal boundaries are defined time-dually. We say A is *proper* if it is both past and future proper.

Lemma 2.2.2. *Let A and B be achronal boundaries with associated unique past and future sets $\{P_A, F_A\}$ and $\{P_B, F_B\}$, respectively.*

1) $P_A \subset P_B \iff F_B \subset F_A$

2) If A is past proper, then $J^-(A) = A \cup I^-(A) = \overline{I^-(A)} = \overline{P_A}$.

3) If A and B are past proper, then

$$P_A \subset P_B \iff J^-(A) \subset J^-(B) \iff A \subset J^-(B)$$

Proof. 1) By Proposition 2.1, $P_A \subset P_B \Longleftrightarrow B \cup F_B \subset A \cup F_A$. But since F_B is open, if $F_B \cap A \neq \emptyset$, then $F_B \cap P_A \neq \emptyset$, which implies $F_B \cap P_B \neq \emptyset$, a contradiction. So $F_B \subset F_A$. The converse is the time dual of this. 2) $A \cup I^-(A) \subset J^-(A) \subset \overline{I^-(A)} = \partial I^-(A) \cup I^-(A) = A \cup I^-(A)$. 3) Follows from $I^-(A) \subset I^-(B) \iff J^-(A) \subset J^-(B)$, which follows from 2) by taking closures and interiors. $\square$

We will say a sequence of achronal boundaries $\{A_k\}$ is *monotonic* if either $\{P_k\}$ is increasing, i.e., $P_k \subset P_{k+1}$ or $\{P_k\}$ is decreasing, i.e., $P_{k+1} \subset P_k$. Note that by Lemma 2.2.2, $\{P_k\}$ is increasing iff $\{F_k\}$ is decreasing, and vice-versa.

Definition 2.2.3 (Achronal Limits)**.** Let $\{A_k\}$ be a monotonic sequence of achronal boundaries.

1) If $\{P_k\}$ is increasing, we define the *future achronal limit*, A_∞, of $\{A_k\}$ by

$$A_\infty = \partial\left(\bigcup_k P_k\right)$$

2) If $\{F_k\}$ is increasing, we define the *past achronal limit*, A_∞, of $\{A_k\}$ by

$$A_\infty = \partial\left(\bigcup_k F_k\right)$$

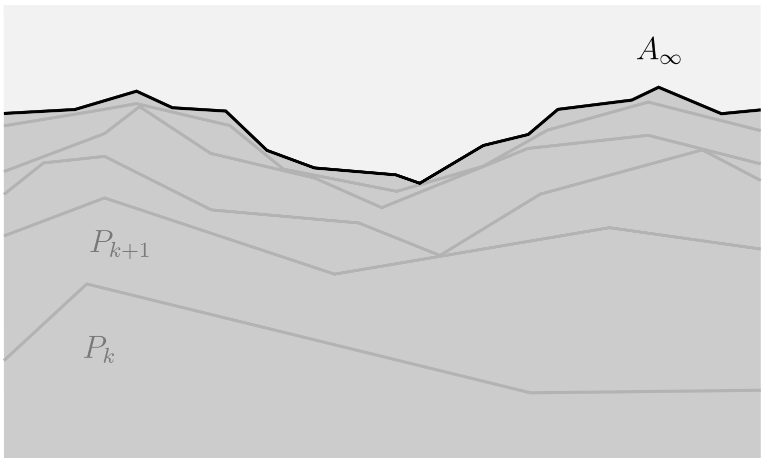


Figure 2.2: A future achronal limit.

We note, again, that when $\{P_k\}$ is increasing, for example, we have equivalently that $\{F_k\}$ is decreasing. In this case, the future achronal limit is given by either:

$$A_\infty = \partial\Big(\bigcup_k P_k\Big) = \partial\Big(\bigcap_k \overline{F}_k\Big)$$

Similarly, when $\{P_k\}$ is increasing, the past achronal limit is given by:

$$A_\infty = \partial\Big(\bigcup_k F_k\Big) = \partial\Big(\bigcap_k \overline{P}_k\Big)$$

Moreover, when $\{P_k\}$ is increasing, we note that, by Lemma 2.1.1, $P_\infty := \cup_k P_k$ is an (open) past set, and hence $A_\infty = \partial P_\infty$, (if nonempty), is an achronal boundary. Similarly, when $\{P_k\}$ is decreasing, $A_\infty = \partial F_\infty$, with $F_\infty = \cup_k F_k$ an (open) future set. In particular, we have the following:

Corollary 2.2.4. *Any nonempty achronal limit, A_∞, is an achronal boundary, and hence an edgeless achronal C^0 hypersurface.*

We say a sequence of points $\{x_k\}$ is *future causal* if $x_k \leq x_{k+1}$. If such a sequence converges, we call $x = \lim x_k$ the *future (causal) limit* of $\{x_k\}$. *Past causal sequences* and *limits* are understood time-dually.

Proposition 2.2.5 (Sequential Characterization of Achronal Limits)**.** *Let A_∞ be the future (resp. past) achronal limit of $\{A_k\}$. Then any limit point of a sequence $a_k \in A_k$ is contained in A_∞. Moreover, fixing any point $a \in A_\infty$ and any timelike curve α through a, a is the future (resp. past) causal limit of the (eventual) sequence $a_k = \alpha \cap A_k$.*

Proof. Suppose $a \in M$ is a limit point of a sequence $a_k \in A_k$, with $a_{k_j} \to a$. Let U be any neighborhood of a. For large j, we have $a_{k_j} \in U$. Then for large j, since U meets $A_{k_j} = \partial P_{k_j}$, it meets P_{k_j} and hence also $\cup_k P_k = P_\infty$. Also, U intersects $I^+(a)$ at some point y, and since $I^-(y)$ is open and contains a, it contains a_{k_j} for all sufficiently large j. Thus, $y \in I^+(A_{k_j}) \subset F_{k_j}$ for all large j, and consequently, $y \notin \cup_k P_k = P_\infty$. It follows that $a \in \partial P_\infty = A_\infty$. Hence, A_∞ contains all limit points of sequences $a_k \in A_k$.

Now let $a \in A_\infty$ and let $\alpha : I \to M$ be any future pointing timelike curve with $0 \in I$ and $\alpha(0) = a$. Fix $T > 0$ with $-T \in I$. We have $\alpha|_{[-T,0)} \subset I^-(A_\infty) \subset P_\infty = \cup_k P_k$ and $a \notin P_k$. Thus, for all sufficiently large k, α is a timelike curve from $\alpha(-T) \in P_k$ to $\alpha(0) = a \in A_k \cup F_k$. It follows then from Proposition 2.1 that for each sufficiently large k, there is a unique $t_k \in (-T, 0]$ such that $a_k := \alpha(t_k) \in A_k$. By achronality, $a_k = \alpha \cap A_k$. The fact that $\{P_k\}$ is increasing implies that $\{t_k\}$ must be (weakly) increasing in k, and hence that $\{a_k\}$ is future increasing. Suppose that $t_k \not\to 0$, i.e., that $t_k \leq 2\delta < 0$. Then $\alpha(\delta) \in I^-(A_\infty) \subset P_\infty$, and hence $\alpha(\delta) \in P_k$ for large k. On the other hand, $\alpha(\delta) \in I^+(A_k) \subset F_k$ for large k, which is not possible since $P_k \cap F_k = \emptyset$. So we have $t_k \to 0$, and thus $a_k = \alpha(t_k) \to \alpha(0) = a$. □

2.3 Hausdorff Closed Limits

We show in this section that achronal limits are a special case of so-called *Hausdorff closed limits*. In fact, the latter not only provide a convenient framework for, but also allow for a generalization of much of the theory to follow. The definitions below were introduced by Hausdorff in [20], and used, with some small variations in [7] and [5]. See also [24].

Definition 2.3.1 (Hausdorff Closed Limits)**.** Let $\{S_k\}$ be an infinite sequence of nonempty subsets of a topological space X. The *Hausdorff upper* and *lower limits* of $\{S_k\}$ may be defined, respectively, by

$$S_\infty^{\mathrm{up}} = \overline{\lim}\{S_k\} = \{p : \text{each neighborhood of } p \text{ meets infinitely many } S_k\text{'s}\}$$

$$S_\infty^{\mathrm{low}} = \underline{\lim}\{S_k\} = \{p : \text{each neighborhood of } p \text{ misses only finitely many } S_k\text{'s}\}$$

Hence, $\underline{\lim}\{S_k\} \subset \overline{\lim}\{S_k\}$. In the case of equality, the common limit is called the *Hausdorff closed limit* of $\{S_k\}$, which we denote by $S_\infty = \lim\{S_k\}$.

It is straightforward to check that S_∞^{up} and S_∞^{low} are closed. Hence, when it exists, the Hausdorff closed limit S_∞ is indeed closed. Moreover, it is easy to verify the following characterizations:

Lemma 2.3.2 (Sequential Characterization of Hausdorff Limits)**.** *Let $\{S_k\}$ be a sequence of subsets of a metric space X.*

(1) $S_\infty^{\mathrm{up}} = \overline{\lim}\{S_k\}$ *is precisely the set of limit points of sequences $s_k \in S_k$.*

(2) $S_\infty^{\mathrm{low}} = \underline{\lim}\{S_k\}$ *is precisely the set of limits of sequences $s_k \in S_k$.*

In particular, if S_∞ exists, then any limit point of a sequence $s_k \in S_k$ is in S_∞ and every point in S_∞ is the limit of some (convergent) sequence $\widetilde{s}_k \in S_k$.

Hence, by Proposition 2.2.5, achronal limits are Hausdorff closed limits:

Corollary 2.3.3. *The Hausdorff closed limit of a monotonic sequence of achronal boundaries $\{A_k\}$ exists and coincides with its achronal limit. Hence, we may write either as $A_\infty = \lim\{A_k\}$.*

The following fails for arbitrary Hausdorff lower limits:

Lemma 2.3.4 (Local Uniform Convergence)**.** *Let S_∞^{up} be the Hausdorff upper limit of $\{S_k\}$ and let K be any compact set. For any neighborhood U of $S_\infty^{\mathrm{up}} \cap K$, there is a $k_0 \in \mathbb{N}$, such that, for all $k \geq k_0$,*

$$S_k \cap K \subset U \cap K.$$

Proof. Otherwise, for each $j \in \mathbb{N}$, we can find $s_j \in S_{k_j} \cap K$ with $s_j \notin U$. As $\{s_j\} \subset K$, the sequence $\{s_j\}$ has a limit point $s_\infty \in K$. But by Lemma 2.3.2, we have $s_\infty \in S^{\mathrm{up}}_\infty$, which contradicts the fact that $\{s_j\}$ never enters the neighborhood U. $\square$

Corollary 2.3.5. *Hausdorff closed limits, and hence also achronal limits, converge locally uniformly (as in Lemma 2.3.4).*

Chapter 3

Spheres and Horospheres

Starting in Section 3.3, we will assume global hyperbolicity throughout the remainder, though we will continue to state this explicitly at times.

3.1 Distance and Maximal Curves

For $p, q \in M$, we denote by $\Omega^c_{p,q}$ the set of future-directed causal curve segments from p to q. Motivated by Proposition 1.1.2, the *(Lorentzian) distance* between $p, q \in M$ is defined by

$$d(p,q) := \begin{cases} \sup\{L(\alpha) : \alpha \in \Omega^c_{p,q}\} & q \in J^+(p) \\ 0 & q \notin J^+(p) \end{cases}$$

Unlike Riemannian distance, d is not continuous, or even finite-valued, in general. Moreover, d fails every property of being a metric (in the sense of metric spaces). We do, however, have the following:

Proposition 3.1.1. *$d : M \times M \to [0, \infty]$ is lower semicontinous and satisfies the reverse triangle inequality: For $x \le y \le z$,*

$$d(x, y) + d(y, z) \le d(x, z) \tag{3.1.1}$$

Much of the theory below is based on considering, more generally, distance to, from, or between subsets of M. For $S \subset M$, we define the *distance to S* by

$$d(p, S) \ := \ \sup\{d(p, s) : s \in S\}$$

Similarly, the *distance from S* is defined by $d(S, q) := \sup\{d(s, q) : s \in S\}$. More generally, given two subsets $A, B \subset M$, we define the *distance between A and B* by $d(A, B) := \sup\{d(a, b) : a \in A, b \in B\}$.

Distance-realizing curves play a central role below. A causal curve segment α from p to q is *maximal* if $L(\alpha) = d(p, q)$, i.e., if α is a 'longest' causal curve from p to q. We first note the following important consequence of Proposition 1.1.2:

Corollary 3.1.2. *Any maximal causal curve is a pregeodesic, that is, (up to parametrization), such curves are either timelike or null geodesics.*

The proof of Corollary 3.1.2 involves a local 'corner-cutting' argument based on Proposition 1.1.2. Since this type of argument is used so many times below, we give an example here:

Lemma 3.1.3 (Corner-Cutting)**.** *Suppose that α is a maximal future causal curve segment from o to p, and β is a maximal future causal curve segment from p to q. If, for example, the concatenation $\alpha + \beta$ fails to glue together to form a geodesic from o to q, then there is a longer causal curve from o to q.*

Proof. Let U be a convex neighborhood of p. Let $p_- \in \alpha \cap U$ and $p_+ \in \beta \cap U$, with $p_- << p << p_+$. Note that the concatenation $\alpha + \beta$ must fail to be a geodesic by having a 'sharp corner' at p. Hence, by Proposition 1.1.2, there is a curve in U from p_- to p_+ which is strictly longer than that provided by $\alpha + \beta$. Following α up from o, but then 'cutting the corner' from p_- to p_+ along this new segment, and thereafter continuing up along β to q gives a strictly longer curve from o to q. □

We note that, as demonstrated by a null spiral segment on the Lorentzian cylinder, $(\mathbb{R} \times \mathbb{S}, -dt^2 + d\theta^2)$, a non-maximal geodesic may split into two maximal halves. Hence, a concatenation as in Lemma 3.1.3 may form a geodesic, yet still fail to maximize.

Continuing with distance-realizing curves, we say a causal curve segment α from $S \subset M$ to $q \in J^+(S)$, which we will call an *S-segment*, is *maximal* (as an S-segment) if $L(\alpha) = d(S, q)$, i.e., if α is a 'longest' causal curve from S to q. Note that this implies that α realizes the distance from S to any of its points. We use the same language for curves joining $p \in J^-(S)$ to S.

By a *future ray* we mean a future-inextendible causal curve $\gamma : [0, d) \to M$, each segment of which is maximal. More generally, for any $S \subset M$, a *future S-ray* is a future-inextendible causal curve $\gamma : [0, d) \to M$ from $\gamma(0) \in S$, each segment of which is maximal as an S-segment. Past S-rays are understood time-dually. Note that γ is a ray iff it is a $\{\gamma(0)\}$-ray. Finally, by a causal *line* we mean an inextendible causal curve, each segment of which is maximal.

We will make use of the following observations regarding maximal segments from achronal boundaries:

Lemma 3.1.4. *Let A be an achronal boundary and $\eta : [0, T] \to M$, $\eta(0) \in A$, a future-directed null curve segment. If η is maximal as an A-segment, then $\eta \subset A$. Consequently, any null future A-ray is necessarily contained in A.*

Proof. Let P and F be the unique past and future sets associated to A, as in Proposition 2.1. Then $\eta \subset J^+(A) \subset A \cup F$. Suppose that $\eta(c) \in F$ for some $c \in (0, b]$. Letting $p \in I^-(\eta(0)) \subset P$, we have $p \in I^-(\eta(c))$. Letting α be a timelike curve from $p \in P$ to $\eta(c) \in F$, then, as in Proposition 2.1, α must meet A. Consequently, we have $\eta(c) \in I^+(A)$, and hence $d(A, \eta(c)) > 0 = L(\eta|_{[0,c]})$, contradicting the maximality of η. Hence η must remain in A. $\square$

As a consequence to Proposition 1.1.26 and Lemma 3.1.4, we have:

Corollary 3.1.5. *If S is a Cauchy surface, then any S-ray is timelike.*

While maximal segments do not, in general, join to form maximal segments, we have the following:

Lemma 3.1.6. *Let A be an achronal boundary. Let $a \in A$ and suppose that $\alpha : [-R, 0] \to M$ is a maximal future-directed A-segment ending at $\alpha(0) = a$ and $\beta : [0, T] \to M$ is a maximal future-directed A-segment from $\beta(0) = a$. Then $\gamma = \alpha + \beta : [-R, T] \to M$ is a future-directed maximal segment. In this case, α and β are either both timelike or both null, in which case, γ is timelike or null, respectively. Consequently, if A admits both a past and future A-ray from the same point, then these join to form a line.*

Proof. If α is null, then $\alpha \subset A$ by Lemma 3.1.4. Then, by a corner cutting argument, β must also be null, with $\beta \subset A$. Hence, $\gamma(-R) \in J^-(\gamma(T))$, but since these two points are in A, we have $\gamma(-R) \notin I^-(\gamma(T))$. Then by Proposition 1.1.3, γ is a null pregeodesic. Hence, $L(\gamma) = 0 = d(\gamma(-R), \gamma(T))$, so γ is also maximal. Now suppose α is timelike. Then β must also be timelike, and since A is achronal, we must have $\alpha(-R) \in P$ and $\beta(T) \in F$, with P and F as in Proposition 2.1. Let $\sigma : [t_-, t_+] \to M$ be any causal curve from $\alpha(-R) = \gamma(-R) \in P$ to

$\beta(T) = \gamma(T) \in F$. Then, as in Proposition 2.1, σ must meet A at some (possibly non-unique) point $\sigma(t_0)$. We have:

$$L(\sigma) = L(\sigma_{[t_-,t_0]}) + L(\sigma_{[t_0,t_+]}) \leq L(\alpha) + L(\beta) = L(\gamma)$$

Hence, γ is maximal. The statement about rays joining to form a line follows. $\square$

Regarding distance and maximal segments, we have the following standard result:

Theorem 3.1.7. *Suppose M is globally hyperbolic. Then $(p,q) \mapsto d(p,q)$ is finite and continuous on $M \times M$. Furthermore, for $p \leq q$, there exists a maximal causal geodesic segment $\gamma \in \Omega^c_{p,q}$, i.e., $L(\gamma) = d(p,q)$.*

We will derive an analog to Theorem 3.1.7 for $q \to d(S,q)$ in Section 3.2.

3.1.1 The Maximal Limit Curve Lemma

The following lemma collects various standard constructions, and is the key to proving Theorem 3.4.4.

Lemma 3.1.8 (Maximal Limit Curve Lemma)**.** *Let M be a noncompact spacetime and let $\{S_k\}$ be a sequence of subsets with Hausdorff closed limit S_∞. Suppose that for each k, α_k is a maximal future S_k-segment from $x_k \in S_k$ to $y_k \in J^+(S_k)$. If the base points $\{x_k\}$ have a limit point $p \in S_\infty$ and the endpoints $y_k \to \infty$, then any limit curve α of $\{\alpha_k\}$ from p is a future S_∞-ray.*

Proof. We fix a complete Riemannian metric h on M and let $\alpha_k : [0, T_k] \to M$ be the parameterization of α_k with respect to h arc length, with $\alpha_k(0) = x_k$ and $\alpha_k(T_k) = y_k$. Extend each α_k arbitrarily so that each is future-inextendible and still parameterized with respect to h arc length, $\alpha_k : [0, \infty) \to M$. Let α :

$[0,\infty) \to M$ be a future-inextendible causal C^0 limit curve of $\{\alpha_k\}$ from $\alpha(0) = p$, as in Lemma 1.1.6, with a subsequence $\{\alpha_{k_j}\}$ converging locally uniformly to α.

Recall that '$y_k \to \infty$' iff the tail of $\{y_k\}$ lies outside every compact set, i.e., given any compact set $K \subset M$, there is an index k_0 such that, for all $k \geq k_0$, $y_k \notin K$. Note that, since h is Riemannian, this implies $T_k \to \infty$.

Let $T \geq 0$ and $z \in S_\infty$. Then there is a sequence $z_k \in S_k$ with $z_k \to z$, and for sufficiently large j, $\alpha_{k_j}(T)$ is defined. By the maximality of the α_{k_j}'s, the upper semicontinuity of Lorentzian arc length, (Prop 1.1.7), and the lower semicontinuity of Lorentzian distance, we have:

$$
\begin{aligned}
L(\alpha_{[0,T]}) &\geq \limsup L(\alpha_{k_j}) \\
&= \limsup d(S_{k_j}, \alpha_{k_j}(T)) \\
&\geq \liminf d(S_{k_j}, \alpha_{k_j}(T)) \\
&\geq \liminf d(z_{k_j}, \alpha_{k_j}(T)) \\
&\geq d(z, \alpha(T))
\end{aligned}
$$

Hence, for all $T \geq 0$, $\alpha : [0,T] \to M$ is maximal as a future S_∞-segment, and thus $\alpha : [0,\infty) \to M$ is a future S_∞-ray. $\square$

We note first that $S_\infty = \lim\{S_k\}$ may be replaced in Lemma 3.1.8 by the Hausdorff lower limit, $\underline{\lim}\{S_k\}$. In fact, that was the version given in [19] under the name 'S_k-segment Lemma'. As stated, however, Lemma 3.1.8 fails for Hausdorff upper limits. Lemma 3.1.8 captures many typical constructions, including rays from a point, asymptotes and other co-rays to a curve, as well as S-rays for more general sets S. For example, Lemma 3.1.8 may be used to establish the following standard fact:

Lemma 3.1.9. *Every compact Cauchy surface S admits a past S-ray and a future S-ray, which (by Lemma 3.1.5) are necessarily timelike.*

Proof. Fix a complete Riemannian metric h on M. Fixing any future-inextendible timelike curve $\beta : [0, \infty) \to M$, parameterized with respect to h arc length, β must enter the timelike future of S. Hence, for large k, let $\alpha_k : [0, T_k] \to M$ be a maximal S-segment to $y_k := \beta(k)$ and let $x_k = \alpha_k(0) \in S$. By Lemma 1.1.10, we have $y_k \to \infty$, and by compactness, $\{x_k\}$ has a limit point $p \in S$. Hence, applying Lemma 3.1.8 with $S_k = S = S_\infty$, any limit curve α of $\{\alpha_k\}$ from p is a future S-ray. By time-dualizing, one produces a past S-ray, η. $\square$

Remark 3.1.10. Being future maximal, the curves $\{\alpha_k\}$ and α in Lemma 3.1.8 are either timelike or null geodesics, or rather, may be parameterized as such. Hence, it is natural to compare the limit curve α with the geodesic limit of the α_k's. In [18], Galloway and Horta observed that, if α is timelike, then, roughly speaking, the two notions of limits agree. This fact will be important to the convexity and rigidity of horospheres. See Lemma 4.5.4 below for an application.

3.2 Causal Completeness and Boundedness

To get the desired generalization of Theorem 3.1.7 for $d(S, \cdot)$, we need to impose conditions on the 'center' set S. The following notion was first introduced in [15].

Definition 3.2.1 (Causal Completeness)**.** A subset $S \subset M$ is said to be *future causally complete* if for all $p \in J^+(S)$, the closure in S of $J^-(p) \cap S$ is compact. *Past causal completeness* is defined time-dually. S is *causally complete* if it is both past and future causally complete.

We note that compact sets, in particular (single) points are causally complete. Moreover, by Proposition 1.1.27, Cauchy surfaces are (past and future) causally complete. In general, if S is either future or past causally complete, then S is necessarily closed:

Lemma 3.2.2. *If S is future causally complete, then S is closed.*

Proof. Suppose otherwise, that for some sequence $\{s_k\} \subset S$, we have $s_k \to x \in M - S$. Let $y \in I^+(x)$. Then for all large k, $s_k \in J^-(y) \cap S$, and, in particular, $y \in J^+(S)$. However, the closure in S of $J^-(y) \cap S$ contains the tail of $\{s_k\}$, but not its limit, x, and hence is not compact. $\square$

In the globally hyperbolic setting, causal completeness may be rephrased as follows:

Lemma 3.2.3. *Let M be globally hyperbolic and $S \subset M$ closed. Then the following are equivalent:*

(i) S is future causally complete.

(ii) $J^-(p) \cap S$ is compact for all $p \in J^+(S)$.

(iii) $J^-(p) \cap J^+(S)$ is compact for all $p \in J^+(S)$.

Proof. Let $p \in J^+(S)$. Since S is closed, and since by Lemma 1.1.14, $J^-(p)$ is closed, we have $\mathrm{cl}_S(J^-(p) \cap S) = \mathrm{cl}_M(J^-(p) \cap S) = J^-(p) \cap S$. Hence, $(i) \iff (ii)$. The equivalence $(ii) \iff (iii)$ follows from Lemma 1.1.14 and the relations:

$$J^-(p) \cap S \subset J^-(p) \cap J^+(S) = J^-(p) \cap J^+(J^-(p) \cap S)$$

$\square$

We note the following:

Lemma 3.2.4. *Let M be globally hyperbolic. If S is future causally complete, then $J^+(S)$ is closed.*

Proof. Let $q_k \in J^+(S)$ converge to $q \in M$. Let $q_+ \in I^+(q)$. Then, for all large k, we have $q_k \in J^-(q_+)$, and in particular, $q_+ \in J^+(S)$. Since $q_k \in J^+(S)$, each $q_k \in J^+(s_k)$ for some $s_k \in S$. Then, for all large k, $s_k \in J^-(q_+) \cap S$, which is compact. Hence $\{s_k\}$ has a limit point $s_\infty \in S$. By Lemma 1.1.15, $s_k \leq q_k$ for all (large) k implies $s_\infty \leq q$, and hence, $q \in J^+(S)$. □

We are now ready to prove a useful analog of Theorem 3.1.7:

Lemma 3.2.5. *Let M be globally hyperbolic. If S is future causally complete, then $x \to d(S,x)$ is finite-valued and continuous on M, and given any $q \in J^+(S)$, there is a maximal future S-segment α from S to q, i.e., $L(\alpha) = d(S,q)$.*

Proof. Theorem 3.1.7 will be used throughout. First note that S is closed. Let $q \in J^+(S)$. By Lemma 3.2.3, $J^-(q) \cap S$ is compact. Let $x_k \in J^-(q) \cap S$ such that $d(x_k, q) \to d(S,q)$. Then $\{x_k\}$ has a limit point $p \in S$, and by continuity of d on $M \times M$, we have $d(p,q) = \lim_{j\to\infty} d(x_{k_j}, q) = d(S,q)$. In particular, $d(S,q) < \infty$. Furthermore, by Lemma 1.1.14, $p \leq q$, hence, p is joined to q by a maximal causal geodesic segment α, which must also be maximal as an S-segment, $L(\alpha) = d(p,q) = d(S,q)$. Note that, since $J^+(S)$ is closed (by Lemma 3.2.4), $d(S,\cdot)$ is continuous on the open set $M \setminus J^+(S)$, where it vanishes identically. Hence it remains to show continuity at $q \in J^+(S)$. Note that for this, it suffices to show that for any sequence $q_k \to q$, we have $\lim_{j\to\infty} d(S, q_{k_j}) = d(S,q)$, for some subsequence $\{q_{k_j}\}$, (since this would apply to a supremum-realizing sequence as well as an infimum-realizing sequence). Fix $q_+ \in I^+(q) \subset J^+(S)$. Then $J^-(q_+) \cap S$

is compact. For all large k, we have $q_k \in J^-(q_+)$ and hence $J^-(q_k) \cap S \subset J^-(q_+) \cap S$. Let $p_k \in J^-(q_+) \cap S$ with $d(p_k, q_k) = d(S, q_k)$, where p_k is chosen arbitrarily for any $q_k \notin J^+(S)$. By compactness of $J^-(q_+) \cap S$, $\{p_k\}$ has a subsequence $\{p_{k_j}\}$ converging to some $p_\infty \in J^-(q_+) \cap S$. Note that (by future causal completeness), there is a $p_0 \in S$ with $d(p_0, q) = d(S, q)$. By continuity on $M \times M$, we have $\lim_{j\to\infty} d(S, q_{k_j}) = \lim_{j\to\infty} d(p_{k_j}, q_{k_j}) = d(p_\infty, q) \le d(S, q) = d(p_0, q)$. On the other hand, we have $d(p_0, q_{k_j}) \le d(S, q_{k_j}) = d(p_{k_j}, q_{k_j})$, the limit of which gives $d(p_0, q) \le d(S, q)$. Hence $\lim_{j\to\infty} d(S, q_{k_j}) = d(S, q)$. □

Note that, for closed subsets of a globally hyperbolic spacetime, past causal completeness is inherited to the past:

Lemma 3.2.6. *Let M be globally hyperbolic. If C is past causally complete and $S \subset J^-(C)$ is closed, then S is past causally complete.*

Proof. Let $x \in J^-(S)$. Then, since $J^+(x) \cap S$ is a closed subset of $J^+(x) \cap J^-(C)$, it is compact. Hence by Lemma 3.2.3, S is past causally complete. □

In globally hyperbolic spacetimes, causal completeness has an important effect on 'causal horizons', e.g., $\partial I^+(S)$ and $H^+(S)$. We first note the following corollary to Proposition 1.1.8 and Lemma 1.1.10:

Corollary 3.2.7. *Let M be globally hyperbolic and $S \subset M$ past causally complete. Then each $x \in \partial I^-(S) \setminus S$ is the past endpoint of a null geodesic contained in $\partial I^-(S)$ which has a future endpoint on S.*

Proof. Let $x \in \partial I^-(S) \setminus S$. By Proposition 1.1.8, x is the past endpoint of a null geodesic $\eta \subset \partial I^-(S)$, which is either future-inextendible or has a future endpoint on S. Since, by (the time-dual of) Lemma 3.2.4, $J^-(S)$ is closed, we have $\eta \subset J^+(x) \cap \partial I^-(S) \subset J^+(x) \cap J^-(S)$. By the past causal completeness of S

and Lemma 3.2.3, the latter set is compact. Hence, by the 'escape' Lemma 1.1.10, η must have a future endpoint on S. $\square$

Corollary 3.2.8. *Let M be globally hyperbolic and suppose $\emptyset \neq S \subset M$ is achronal and edgeless. If S is past causally complete, then $S = \partial I^-(S)$, i.e., S is a past proper achronal boundary.*

Proof. Suppose otherwise that $x \in \partial I^-(S) \setminus S$. Then by Corollaries 1.1.18 and 3.2.7, x is the past endpoint a null geodesic with future endpoint on $\operatorname{edge} S$. But S is edgeless. Hence $S = \partial I^-(S)$. $\square$

Similar to Corollary 3.2.7, we have the following corollary to Proposition 1.1.21:

Corollary 3.2.9. *Let M be globally hyperbolic and suppose $S \subset M$ is achronal and edgeless. Then S is past causally complete iff $H^-(S) = \emptyset$.*

Proof. Suppose first that S is past causally complete. Suppose to the contrary that there is a $p \in H^-(S)$. Then by Proposition 1.1.21, p is the past endpoint of a null geodesic $\eta \subset H^-(S)$ which is future-inextendible. But this impossible by Lemma 1.1.10, since $\eta \subset J^+(p) \cap H^-(S) \subset J^+(x) \cap J^-(S)$. The converse is Proposition 1.1.27. $\square$

The following will be used throughout to 'causally control' subsets of a globally hyperbolic spacetime M:

Definition 3.2.10 (Causal Boundedness)**.** We say a subset $A \subset M$ is *future bounded* if there is a Cauchy surface S in M such that $A \subset J^-(S)$. Past boundedness is defined time-dually.

This notion is very closely related with causal completeness:

Lemma 3.2.11. *Let M be globally hyperbolic and $\emptyset \neq S \subset M$ closed. Then S is future bounded iff S is past causally complete.*

Proof. First suppose S is future bounded by a Cauchy surface Σ, i.e., $S \subset J^-(\Sigma)$. Since Σ is (past) causally complete, then S is past causally complete, by Lemma 3.2.6. Now suppose S is past causally complete. Let $A = \partial I^-(S)$. We first note that A is nonempty. To see this, observe that since $S \neq \emptyset$, we have $I^-(S) \neq \emptyset$. Hence $A = \emptyset$ iff $M = I^-(S) = J^-(S)$. If this were the case, then the future of any point would be compact, by Lemma 3.2.3, contradicting the Escape Lemma 1.1.10. Hence, $A \neq \emptyset$ is closed, achronal and edgeless. Furthermore, by Corollary 3.2.7, we have $A \subset J^-(S)$. Hence, by Lemma 3.2.6, A is also past causally complete. Then by Corollary 3.2.9, A is a past Cauchy surface, i.e., $H^-(A) = \emptyset$ and $D^-(A) = J^-(A)$. Let $\widetilde{M} = M - J^-(A)$. Let $p, q \in \widetilde{M}$, and let $x \in J^+(p) \cap J^-(q)$. Then $x \in \widetilde{M}$, since otherwise, if $x \in J^-(A)$, then $p \in J^-(x) \subset J^-(A)$. Furthermore, $J^-(A)$ is closed, by Lemma 3.2.4. It follows that $\widetilde{M}$ is an open, globally hyperbolic (sub)spacetime. Hence, by Theorem 1.1.30, $\widetilde{M}$ admits a Cauchy surface, Σ. Let, for example, $\beta : (-\infty, \infty) \to M$ be a future-directed, inextendible causal curve in M. If $\beta \subset \widetilde{M}$, then β must meet Σ. Otherwise, if β meets $J^-(A) = D^-(A)$, it must meet A, at some point $\beta(t_0) \in A$. By past causal completeness, (c.f. Lemma 3.2.3), $J^+(\beta(t_0)) \cap J^-(A)$ is compact. Hence, by the Escape Lemma, 1.1.10, β must, at some point, leave A and never return. Hence, there is some $t_1 \geq t_0$ with $\beta(t_1) \in A$ and $\beta : (t_1, \infty) \to \widetilde{M}$. Since $\lim_{t \to t_1^+} \beta(t)$ does not exist in $\widetilde{M}$, $\beta_{(t_1,\infty)}$ is inextendible in $\widetilde{M}$. Consequently, $\beta_{(t_1,\infty)}$, and hence β must meet Σ. Since β was arbitrary, Σ meets every inextendible causal curve in M. It follows that Σ is a Cauchy surface for M, (c.f. Proposition 1.1.26). Finally, fix any $x \in S$.

Let $\alpha : [0, \infty) \to M$ be any future-inextendible causal curve from $x \in S$. Then, by past causal completeness, there is some $s_0 \geq 0$, for which $\alpha(s_0) \in J^-(S)$ and $\alpha(s_0, \infty) \subset M \setminus J^-(S) \subset M \setminus J^-(A)$. Hence, the future end of α meets Σ, which means $x \in J^-(\Sigma)$. Since $x \in S$ was arbitrary, we have $S \subset J^-(\Sigma)$. $\square$

3.3 Spheres

From now on, we assume global hyperbolicity throughout.

Definition 3.3.1 (Spheres)**.** For $C \subset M$ past causally complete and $r > 0$, we define the *past sphere* from (the *center*) C of *radius* r by

$$S_r^-(C) := \{x \in J^-(C) : d(x, C) = r\}$$

Future spheres are defined time-dually, from future causally complete centers.

We note that one can similarly define a sphere of radius $r = 0$. By past causal completeness, this gives $S_0^-(C) = \partial I^-(C)$. All of the properties given below for spheres of positive radius have analogs for spheres of radius zero. Conversely, these results may be viewed as an extension of basic causal theoretic results like Proposition 1.1.8. In any case, we will restrict attention to $r > 0$.

As noted above, (single) points and Cauchy surfaces are causally complete to both the future and past. For $p \in M$, we call $S_r^\pm(p)$ a *point sphere*. For $S \subset M$ a Cauchy surface, we call $S_r^\pm(S)$ a *Cauchy sphere*. Cauchy spheres are not Cauchy surfaces in general, but see Lemma 3.3.5 below.

The simplest examples, point spheres in Minkowski space, will be surprisingly useful to keep in mind. We note from the outset that a sphere is really more than just the set $S_r^-(C)$ itself. By definition, each $p \in S_r^-(C)$ satisfies $d(p, C) = r$, and hence, by Lemma 3.2.5, p is joined to C by a maximal C-segment, which we call a 'radial segment', (see Lemma 3.3.2). We may think of this as a taught string from p to C. Hence, we may think of $S_r^-(C)$ and C as bounding a 'harp' of such strings, as in Figure 3.1.

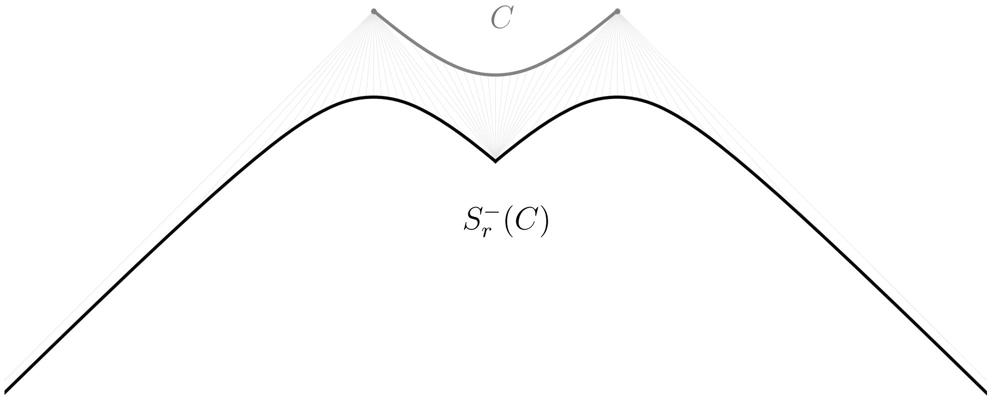


Figure 3.1: A sphere and its center form a harp of radial segments.

Lemma 3.3.2 (Radial Segments)**.** *Each $x \in S_r^-(C)$ is joined to C by a future timelike geodesic segment of length r which maximizes both the distance to C and from $S_r^-(C)$. We refer to these as* radial segments*.*

Proof. By past causal completeness, any $x \in S_r^-(C) \subset J^-(C)$ is joined to C by a maximal C-segment of length r. If such a segment did not also maximize the distance from $S_r^-(C)$, then using a 'corner-cutting' argument, we could construct a curve from $S_r^-(C)$ to C of length strictly larger than r, contradicting the definition of $S_r^-(C)$. $\square$

We note that, as in Figure 3.1, there may be several radial segments emanating from the same point $x \in S_r^-(C)$, and possibly also several such segments ending at the same point $c \in C$. It follows from corner-cutting, however, that radial segments never touch at interior points.

We list several properties of spheres:

Lemma 3.3.3 (Past Spheres). *Let $C \subset M$ be past causally complete and $r > 0$ and suppose $S = S_r^-(C) \neq \emptyset$. Then S is a closed, edgeless acausal C^0 hypersurface. Furthermore, S is past causally complete. Thus, $H^-(S) = \emptyset$ and $S = \partial I^-(S)$.*

Proof. By Lemma 3.2.5, $d(\cdot, C)$ is continuous, hence S is closed. Then by Lemma 3.2.6, S is past causally complete. It follows more or less by definition that S is achronal and edgeless. Hence S is a C^0 hypersurface by Corollary 1.1.17, and by Corollaries 3.2.9 and 3.2.8, $H^-(S) = \emptyset$ and $S = \partial I^-(S)$. To see that $S = S_r^-(C)$ is further acausal, let $y \in S$ and $z \in J^-(y) - \{y\}$. By the past causal completeness of C, let β be a maximal timelike C-segment from C to y. Then, by "cutting the corner" in a neighborhood of y, one can produce a causal curve from C to z which is strictly longer than β. Hence, $d(z, C) > L(\beta) = d(y, C) = r$, so $z \notin S$. □

Lemma 3.3.4 (Past Sphere Foliation). *Let $C \subset M$ be past causally complete. Then the timelike past of C is foliated by its past spheres,*

$$I^-(C) = \bigcup_{r>0} S_r^-(C)$$

and we have

$$S_a^-(S_r^-(C)) = S_{r+a}^-(C)$$

Hence, for $r \leq s$, we have $d(S_s^-(C), S_r^-(C)) = s - r$, (when nonempty), and every radial $S_s^-(C)$-segment restricts to a radial $S_r^-(C)$-segment.

Proof. Recall that, by Lemma 3.3.3, each sphere $S_r^-(C)$ is past causally complete. Hence, $S_a^-(S_r^-(C))$ is well-defined. To show that 'radius is additive', first let

$x \in S_a^-(S_r^-(C))$. By Lemma 3.2.5, x is joined to some $y \in S_r^-(C)$ by a timelike segment of length a. As y is similarly joined to C by a segment of length r, we have $d(x, C) \geq r + a$. Then, letting α be a maximal future C-segment from x to C, α must pass through $S_r^-(C)$. The portion of α before $S_r^-(C)$ is bounded in length by a and the portion after by r, thus $d(x, C) \leq r + a$, so $d(x, C) = r + a$, i.e., $x \in S_{r+a}^-(C)$. Now let $x \in S_{r+a}^-(C)$. Then there is a maximal C-segment α from x to C of length $r + a$. As any portion of α ending at C must also be C-maximal, the point $x' \in \alpha$ from which the remaining portion of α has length r is a maximal C-segment of length r, and hence $x' \in S_r^-(C)$, so $d(x, S_r^-(C)) \geq d(x, x') = a$. But, of course, $d(x, S_r^-(C)) \leq a$, since otherwise, one could produce a curve from x to C of length greater than $r + a$. Thus, $d(x, S_r^-(C)) = a$, i.e., $x \in S_a^-(S_r^-(C))$. Hence, $S_a^-(S_r^-(C)) = S_{r+a}^-(C)$. The rest of the statement follows from this. For example, for $s > r$,

$$d(S_s^-(C), S_r^-(C)) = d(S_{s-r}^-(S_r^-(C)), S_r^-(C)) = s - r$$

□

The following will also be used below:

Lemma 3.3.5 (Compact Cauchy Spheres)**.** *Suppose S is a compact Cauchy surface for M. If M is future timelike geodesically complete, then every future Cauchy sphere $S_r^+(S)$ from S is a compact Cauchy surface.*

Proof. Let $S_r := S_r^+(S)$. By Proposition 1.1.28 it suffices to show that S_r is compact. Fix a complete Riemannian metric h on M. Since S_r is closed, if it is noncompact, there is a sequence $x_k \in S_r$ with $x_k \to \infty$. Let $\alpha_k : [0, T_k] \to M$ be future maximal S-segment from $\alpha_k(0) \in S$ to $\alpha_k(T_k) = x_k$, parameterized with respect to h arc length. Hence, $T_k \to \infty$. Since S is compact, $\alpha_k(0)$ has a limit point $p \in S$ and, fixing any causal limit curve $\alpha : [0, \infty) \to M$ of $\{\alpha_k\}$ from $p = \alpha(0)$, as in Lemma 1.1.6, we have that α is an S-ray, by Lemma 3.1.8. Hence, by Lemma 3.1.5, α is timelike, and hence future complete, by assumption. Thus, α must meet $S_r = S_r^+(S)$ at some point $\alpha(T_r)$. But this means $\alpha(T_r + 1) \in I^+(S_r)$. Then, $\alpha_k(T_r + 1)$ (sub)converges to $\alpha(T_r + 1)$, which contradicts $\alpha_k(T_r + 1) \leq \alpha_k(T_k) \in S_r$, for large k. $\square$

3.4 Horospheres

Let $\{S_k^- = S_{r_k}^-(C_k)\}$ be a sequence of (nonempty) past spheres, with each C_k past causally complete. By Lemma 3.3.3, each S_k^- is a past proper achronal boundary, and hence has associated past and future sets, P_k and F_k, as in Proposition 2.1, with $P_k = I^-(S_k^-)$, and by Lemma 2.2.2, $\overline{P_k} = J^-(S_k^-)$. Recall that we say $\{S_k^-\}$ is monotonic if either $\{P_k\}$ is increasing or decreasing, or equivalently, if $\{F_k\}$ is decreasing or increasing, respectively.

Definition 3.4.1 (Horospheres)**.** Let $\{S_k^- = S_{r_k}^-(C_k)\}$ be a monotonic sequence of past spheres with radii $r_k \to \infty$.

1) If $\{P_k\} = \{I^-(S_k^-)\}$ is increasing, we obtain the future achronal limit:

$$S_\infty^- = \partial\Big(\bigcup_k P_k\Big) = \partial\Big(\bigcup_k I^-(S_k^-)\Big)$$

2) If $\{P_k\}$, or equivalently $\{J^-(S_k^-)\}$ is decreasing, and hence $\{F_k\}$ increasing, we obtain the past achronal limit:

$$S_\infty^- = \partial\Big(\bigcup_k F_k\Big) = \partial\Big(\bigcap_k J^-(S_k^-)\Big)$$

In either case, (if nonempty), we refer to $S_\infty^- = \lim\{S_k^-\}$ as the *past horosphere* associated to the sequence of *prehorospheres*, $\{S_k^-\}$. *Future horospheres*, S_∞^+, are constructed time-dually, namely, as (past or future) achronal limits of future spheres, $\{S_k^+\}$.

Since horospheres are achronal boundaries by construction, (being nonempty by definition), we have the following immediate corollary to Proposition 1.1.17:

Corollary 3.4.2. *Horospheres are edgeless, achronal C^0 hypersurfaces.*

In particular, horospheres are achronal limits, and hence Hausdorff closed limits. Thus, by either Proposition 2.2.5 or Lemma 2.3.2, we have:

Corollary 3.4.3. *Let S_∞^- be a past horosphere, with prehorospheres $\{S_k^-\}$. Then any limit point of a sequence $s_k \in S_k^-$ is in S_∞^- and every point in S_∞^- is the limit of some (convergent) sequence $\widetilde{s}_k \in S_k^-$.*

We think of S_∞^- itself as a sphere, in particular, as a past sphere of 'infinite radius centered at infinity'. In fact, much of the horosphere theory developed here is motivated by this simple analogy. We begin with an analog of Lemma 3.3.2.

Theorem 3.4.4 (Radial Rays)**.** *Let S_∞^- be a past horosphere. Then S_∞^- admits a future (timelike or null) S_∞^--ray from each point. We call these* radial rays.

Proof. The main ingredient is the Maximal Limit Curve Lemma, Lemma 3.1.8. Let $p \in S_\infty$. Then there is a sequence $p_k \in S_k$ with $p_k \to p$, and by Lemma 3.3.2, a maximal future radial segment α_k from p_k to $q_k \in C_k$ of length $L(\alpha_k) = r_k$. Recall that we assume M is globally hyperbolic. Hence, Lorentzian distance is continuous, as in Theorem 3.1.7. It follows that $q_k \to \infty$. Using Lemma 1.1.6 to produce a limit curve α of $\{\alpha_k\}$ from p, Lemma 3.1.8 ensures α is a future S_∞^--ray. $\square$

Hence, just as a sphere and its center form a 'harp' of maximal radial segments, a horosphere S^-_∞ is one end of a harp of radial S^-_∞-rays, which are strung, at the other end, to infinity.

In general, some (or all) of the radial rays from a horosphere S^-_∞ may be null. By Lemma 3.1.4, such null rays lay flat against S^-_∞. On the other hand, any timelike radial S^-_∞-ray must leave (the achronal set) S^-_∞ immediately. The following, which may be seen as an analog to Lemma 3.3.3, addresses this issue:

Theorem 3.4.5 (Bounded Horospheres)**.** *Suppose a past horosphere S^-_∞ is future bounded. Then S^-_∞ admits a timelike future S^-_∞-ray from each point, and in general, every (future or past) S^-_∞-ray is timelike. Consequently, S^-_∞ is acausal. Furthermore, S^-_∞ is past causally complete, and hence $S^-_\infty = \partial I^-(S^-_\infty)$ and $H^-(S^-_\infty) = \emptyset$, i.e., S^-_∞ is a past Cauchy surface.*

Proof. Fix a Cauchy surface $\Sigma \subset M$ with $S^-_\infty \subset J^-(\Sigma)$. By Theorem 3.4.4, there is a future S^-_∞-ray, γ_p, from each $p \in S^-_\infty$, which is either timelike or null. By Lemma 3.1.4, if γ_p is null, then $\gamma_p \subset S^-_\infty$. But by Proposition 1.1.26, γ_p must meet $I^-(\Sigma)$, and hence must leave S^-_∞. Thus, each future S^-_∞-ray is necessarily timelike. Now suppose η is a past S^-_∞-ray from $p \in S^-_\infty$. Fixing a future S^-_∞-ray, γ_p, from p, by Lemma 3.1.6, η and γ_p join together to form a line, and, as γ_p is necessarily timelike, hence also must η be. Hence every (future or past) S^-_∞-ray is timelike and S^-_∞ admits a timelike future S^-_∞-ray from each point. That S^-_∞ is acausal then follows by 'corner-cutting'. Let $y \in S^-_\infty$ and let $\gamma_y : [0, \ell) \to M$ be a future S^-_∞-ray from y, necessarily timelike, parameterized with respect to (Lorentzian) arc length. Suppose there is some $z \in S^-_\infty$ with $y \neq z \leq y$. Since S^-_∞ is achronal, $z \in \partial I^-(y)$ and z is joined to y by a null geodesic. But then, for any $0 < a < \ell$, we can find a curve from z to $\gamma_y(a)$ of length greater than

a, by 'cutting the corner' near y. But this contradicts the maximality of $\gamma_y|_{[0,a]}$ as an S_∞^--segment. Hence, S_∞^- is acausal. Note that S_∞^- is closed by definition and hence past causally complete, by either Lemma 3.2.6 or Lemma 3.2.11. Then $H^-(S) = \emptyset$ by Lemma 3.2.9 and $S_\infty^- = \partial I^-(S_\infty^-)$ by Lemma 3.2.8. □

3.4.1 Ray and Cauchy Horospheres

In this section we construct two important concrete classes of horospheres. The *ray horosphere* is built as a limit of point spheres with centers taken along a ray, and mimics the standard Busemann level-set construction. The *Cauchy horosphere* is built instead from a Cauchy surface, S, and its sequence of future Cauchy spheres.

Ray Horospheres

Suppose $\gamma : [0, \infty) \to M$ is a future complete, unit speed timelike geodesic ray. Define the sequence of *ray prehorospheres* by:

$$S_k^- := S_k^-(\gamma(k))$$

By Lemma 3.3.3, each S_k^- is a past proper achronal boundary, and hence comes with corresponding unique past and future sets, $P_k = I^-(S_k^-)$ and F_k, with $\partial P_k = S_k^- = \partial F_k$ and $M = P_k \cup S_k^- \cup F_k$, as in Proposition 2.1. As in Lemma 2.2.2, we have $\overline{P}_k = J^-(S_k^-)$.

We now observe that, as a result of the reverse triangle inequality, the sequence $\{S_k^-\}$ is monotonic, with increasing pasts, $\{I^-(S_k^-)\}$, as in Figure 3.2. To see this, let $x \in S_k^- = S_k^-(\gamma(k))$. Then $x \le \gamma(k) \le \gamma(k+1)$, and hence, by the reverse triangle inequality, (3.1.1),

$$d(x, \gamma(k+1)) \ge d(x, \gamma(k)) + d(\gamma(k), \gamma(k+1)) = k + 1$$

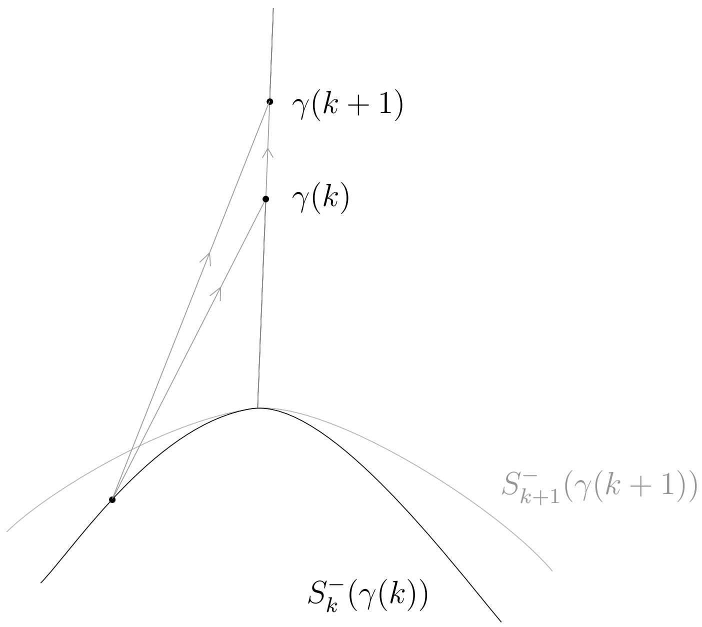


Figure 3.2: Ray horosphere monotonicity.

Consequently, $x \in J^-(S^-_{k+1}(\gamma(k+1))) = J^-(S^-_{k+1})$. Hence, $S^-_k \subset J^-(S^-_{k+1})$, and as in Lemma 2.2.2, the sequences $\{J^-(S^-_k)\}$ and $\{P_k\} = \{I^-(S^-_k)\}$ are increasing. Hence we have a well-defined achronal limit, $S^-_\infty(\gamma) = \lim\{S^-_k\}$, with $S^-_\infty(\gamma)$ non-empty, as $\gamma(0) \in S^-_k$ for all k implies $\gamma(0) \in S^-_\infty$. We repeat some of these facts in the following definition:

Definition 3.4.6 (Ray Horosphere)**.** Let $\gamma : [0, \infty) \to M$ be a future complete, unit speed timelike geodesic ray. Then the sequence of *ray prehorospheres*, $\{S^-_k\} := \{S^-_k(\gamma(k))\}$ is monotonic, with increasing pasts $\{P_k\} = \{I^-(S^-_k)\}$ and we define the *ray horosphere* associated to γ to be the future achronal limit,

$$S^-_\infty(\gamma) := \partial\left(\bigcup_k I^-(S^-_k)\right)$$

We list several properties:

Proposition 3.4.7 (Ray Horospheres)**.** *$S^-_\infty(\gamma)$ is a closed, edgeless achronal C^0 hypersurface which admits a future $S^-_\infty(\gamma)$-ray from each point and $S^-_\infty \subset \overline{I^-(\gamma)}$. If $S^-_\infty(\gamma)$ is future bounded by a Cauchy surface, then all $S^-_\infty(\gamma)$-rays are timelike and $S^-_\infty(\gamma)$ is an acausal past Cauchy surface, i.e., $H^-(S^-_\infty(\gamma)) = \emptyset$, with $S^-_\infty \subset I^-(\gamma)$. In general, γ is itself a timelike S^-_∞-ray. In particular, $\gamma(0) \in S^-_\infty(\gamma) \neq \emptyset$.*

Proof. We observed above that $\gamma(0) \in S^-_\infty$. To see why γ is an $S^-_\infty(\gamma)$-ray, suppose otherwise, that for some $y \in S^-_\infty(\gamma)$ and some $t > 0$, we have $d(y, \gamma(t)) > t$. Then, fixing any integer $k_0 > t$, we have, for all $k \geq k_0$, $d(y, \gamma(k)) \geq d(y, \gamma(t)) + d(\gamma(t), \gamma(k)) > t + (k - t) = k$. Hence, $y \in I^-(S^-_k)$ for all large k. This contradicts $y \in S^-_\infty(\gamma)$. That $S^-_\infty(\gamma) \subset \overline{I^-(\gamma)}$ follows from the fact that for all k, $S^-_k = S^-_k(\gamma(k)) \subset I^-(\gamma(k)) \subset I^-(\gamma)$ and, say, Corollary 3.4.3. The rest follows from Theorems 3.4.4 and 3.4.5. $\square$

The following is an easy consequence of the definitions and Corollary 3.4.3:

Lemma 3.4.8. *Let $\gamma : [0, \infty) \to M$ be a future complete S-ray from a Cauchy surface S. Then, for all k, we have $S^-_k(\gamma(k)) \subset J^-(S)$, and hence, also $S^-_\infty(\gamma) \subset J^-(S)$. In particular, $S^-_\infty(\gamma)$ is future bounded by S.*

Cauchy Horospheres

For this construction, we begin with a Cauchy surface, $S \subset M$. The idea is to replace the sequence of center points, $\{\gamma(k)\}$, in the construction of the ray horosphere, with the sequence of future Cauchy spheres $\{S^+_k(S)\}$, then similarly, take a limit of past spheres from this sequence. In view, for example, of Conjecture 1, we will focus on the case that M is future timelike geodesically complete, and S

is compact. Hence, by Lemma 3.3.5, each $S_k^+(S)$ is a Cauchy surface. In particular, each $S_k^+(S)$ is past causally complete.

We define the sequence of *Cauchy prehorospheres* by

$$\widetilde{S}_k := S_k^-(S_k^+(S))$$

Again, each $\widetilde{S}_k$ is a past proper achronal boundary, with corresponding past and future sets, $\widetilde{P}_k = I^-(\widetilde{S}_k)$ and $\widetilde{F}_k$, with $\partial\widetilde{P}_k = \widetilde{S}_k = \partial\widetilde{F}_k$ and $M = \widetilde{P}_k \cup \widetilde{S}_k \cup \widetilde{F}_k$, and the closures of the pasts are given by $\overline{\widetilde{P}_k} = J^-(\widetilde{S}_k)$.

Like the ray prehorospheres, the sequence of Cauchy prehorospheres is monotonic, but in the opposite direction, as in Figure 3.3.

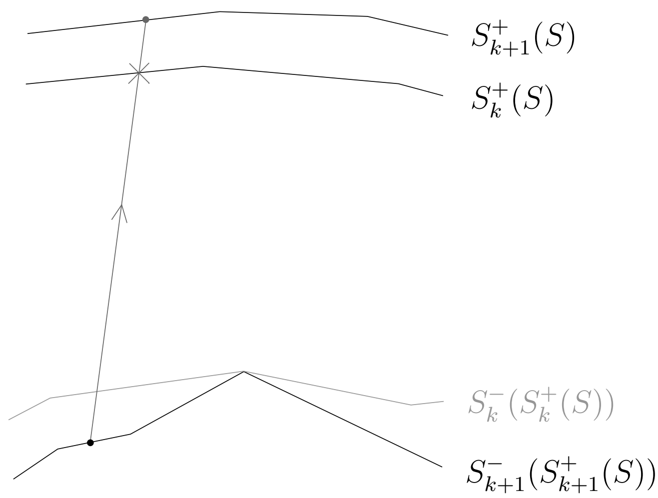


Figure 3.3: Cauchy horosphere monotonicity.

To see this, let $x \in \widetilde{S}_{k+1}$. Hence, $d(x, S_{k+1}^+(S)) = k+1$. Let α be any future timelike curve from x to $S_{k+1}^+(S)$ which realizes this distance. Then there is a unique point $x_k := \alpha \cap S_k^+(S)$. Let α_k^- be the portion of α before x_k and α_k^+ the portion after. By (the time-dual of) Lemma 3.3.4, $d(S_k^+(S), S_{k+1}^+(S)) = 1$, and hence $L(\alpha_k^+) \leq 1$. Thus, we have:

$$L(\alpha_k^-) = L(\alpha) - L(\alpha_k^+) \geq (k+1) - 1 = k$$

Hence, $d(x, S_k^+(S)) \geq L(\alpha_k^-) \geq k$, which implies $x \in J^-(\widetilde{S}_k) = J^-(S_k^-(S_k^+(S)))$. Since x was arbitrary, this shows $\widetilde{S}_{k+1} \subset J^-(\widetilde{S}_k)$. By Lemma 2.2.2, this is equivalent to $\{J^-(\widetilde{S}_k)\}$ and $\{\widetilde{P}_k\} = \{I^-(\widetilde{S}_k)\}$ decreasing, or $\{\widetilde{F}_k\}$ increasing. Hence, we have a well-defined achronal limit, $S_\infty^-(S) = \lim\{\widetilde{S}_k\}$. Moreover, letting γ be any future S-ray, (by Lemma 3.1.9), we have $\gamma(0) \in \widetilde{S}_k$ for all k and hence $\gamma(0) \in S_\infty^-(S) \neq \emptyset$. Again, we repeat some of this in the following definition, and in Proposition 3.4.10 below:

Definition 3.4.9 (Cauchy Horosphere)**.** Suppose M is future timelike geodesically complete and admits a compact Cauchy surface S. Then the sequence of *Cauchy prehorospheres*, $\{\widetilde{S}_k\} := \{S_k^-(S_k^+(S))\}$ is monotonic, with decreasing pasts $\{\widetilde{P}_k\} = \{I^-(\widetilde{S}_k)\}$, or equivalently, increasing associated future sets, $\{F_k\}$, and we define the *Cauchy horosphere* associated to S to be the past achronal limit,

$$S_\infty^-(S) := \partial\Big(\bigcup_k \widetilde{F}_k\Big) = \partial\Big(\bigcap_k J^-(\widetilde{S}_k)\Big)$$

The definitions and discussion above, along with Theorem 3.4.5, give:

Proposition 3.4.10 (Cauchy Horospheres)**.** *Let M be future timelike geodesically complete with compact Cauchy surface S. Then the Cauchy prehorospheres, $\widetilde{S}_k$, and hence also the Cauchy horosphere $S_\infty^-(S)$ are future bounded by S, i.e., $S_\infty^-(S) \subset J^-(S)$. Consequently, $S_\infty^-(S)$ is an acausal past Cauchy surface, i.e., $H^-(S_\infty^-) = \emptyset$, which admits a future complete timelike $S_\infty^-(S)$-ray from each point. Furthermore, for any future S-ray γ, we have $\gamma(0) \in S \cap S_\infty^-(S)$. In particular, $S_\infty^-(S) \neq \emptyset$.*

Chapter 4

Convexity and Rigidity

4.1 Support Mean Curvature

Before we get to the more general notion of support mean curvature, we begin by recalling the smooth model case. Let $\Sigma^n \subset (M^{n+1}, g)$ be a smooth, spacelike hypersurface, with future timelike unit normal field u. We denote the second fundamental form of Σ by B, i.e., for $X, Y \in T_pM$, $B(X,Y) = g(\nabla_X u, Y) = -g(\nabla_X Y, u)$. Recall that B is symmetric, with $B(X,u) = 0$. The mean curvature H of Σ is the trace (along Σ) of its second fundamental form, $H = \mathrm{tr}_\Sigma B$. Hence, $H : \Sigma \to \mathbb{R}$ and fixing any local orthonormal basis $[E_1, ..., E_n]$ for $T_p\Sigma$, we have

$$H(p) = \sum_{i=1}^{n} g(\nabla_{E_i} u, E_i)|_p$$

We say a spacelike hypersurface Σ is *maximal* if its mean curvature vanishes, i.e., if $H_\Sigma = 0$.

Example 4.1.1 (Minkowski Point Spheres and Time Slices)**.** Past and future point spheres in $\mathbb{M}^{n+1}$ are smooth spacelike hypersurfaces with constant mean curvature given by

$$H(S_r^+(p)) = \frac{n}{r} \qquad \text{and} \qquad H(S_r^-(p)) = -\frac{n}{r}$$

Every time slice, $S_t := \{t\} \times \mathbb{R}^n \subset \mathbb{M}^{n+1}$, is maximal; $H(S_t) = 0$. (Note that each such slice may be viewed as a Cauchy sphere from the 0-slice.)

We now review the generalization of these concepts to 'rough' spacelike hypersurfaces via the notion of mean curvature *in the sense of support hypersurfaces*, (cf. [12]).

First, by a C^0 *spacelike hypersurface* $S \subset M$, we mean a set which is locally acausal-and-edgeless, i.e., for every $p \in S$, there is a neighborhood U of p such that $S \cap U$ is acausal and edgeless in U. In this case, necessarily $\text{edge}\, S \cap S = \emptyset$, and it follows from Proposition 1.1.17 that S is a C^0 hypersurface. We note that a C^0 spacelike hypersurface S may fail to be acausal and/or edgeless globally. On the other hand, any (globally) acausal and edgeless set $S \subset M$ is, of course, a C^0 spacelike hypersurface. In particular, spheres and bounded horospheres are C^0 spacelike hypersurfaces.

Given two C^0 spacelike hypersurfaces $S, S' \subset M$ which meet at a point $p \in S \cap S'$, we say S' *is locally to the past of* S *near* p if for some neighborhood U of p in which S is acausal and edgeless, we have $S' \cap U \subset J^-(S, U)$. In this case, we also call S' a *past support hypersurface for* S *at* p.

Definition 4.1.2 (Support Mean Curvature)**.** Let $S \subset M$ be a C^0 spacelike hypersurface. With $a \in \mathbb{R}$, we say S has *support mean curvature* $\geq a$ *at* $p \in S$ if for all $\epsilon > 0$, there is a smooth (at least C^2) past support spacelike hypersurface $S_{p,\epsilon}$ for S at p with mean curvature $H_{p,\epsilon}$ satisfying

$$H_{p,\epsilon}(p) \geq a - \epsilon$$

Similarly, we say a C^0 spacelike hypersurface S has *support mean curvature* $\leq a$ *at* $p \in S$ if for all $\epsilon > 0$, there is a smooth future support spacelike hypersurface $S_{p,\epsilon}$ for S at p with mean curvature $H_{p,\epsilon}$ satisfying $H_{p,\epsilon}(p) \leq a + \epsilon$. We say a C^0 spacelike hypersurface S has *support mean curvature* $\geq a$ *(resp.* $\leq a$*)* if S has support mean curvature $\geq a$ (resp. $\leq a$) at every point $p \in S$.

Example 4.1.3. The C^0 spacelike hypersurface $\{t = \frac{1}{2}|x|\}$ in $\mathbb{M}^{1+1}$ has support mean curvature ≥ 0.

4.1.1 Maximal Segments

We have noted above that a sphere occurs naturally as a 'harp' of maximal radial segments, and similarly, a horosphere as a harp of radial rays. (cf. Lemma 3.3.2 and Theorem 3.4.4). Here we make the dual observation, that a maximal segment occurs as a 'flower' of support spheres, and that when the segment extends to a ray, what grows is a 'support horosphere', as in figure 4.1. In fact, this special case is precisely the construction of a ray horosphere.

Lemma 4.1.4 (Blooming Petal Structure)**.** *Let M be a globally hyperbolic spacetime and fix $p \in M$. Let $\alpha : [0, b) \to M$ be a past-inextendible, unit speed timelike geodesic from $\alpha(0) = p$, and for each $t \in (0, b)$, set $S_t^+ := S_t^+(\alpha(t))$. Then α is maximal up to $t = r$ iff $p \in S_t^+$ for all $t < r$. Furthermore, in this case, the family $\{S_t^+\}_{t<r}$ is monotonic, with increasing futures, and $\bigcap_{t<r} S_t^+ = \{p\}$.*

Proof. α is maximal up to $t = r$ iff, for all $t < r$, we have $d(\alpha(t), \alpha(0)) = t$, or equivalently, $\alpha(0) \in S_t^+(\alpha(t)) = S_t^+$. Monotonicity follows exactly as for the ray horosphere. Similarly, the fact that the petals only touch at the stem follows from corner-cutting. See, for example, Figure 3.2. $\square$

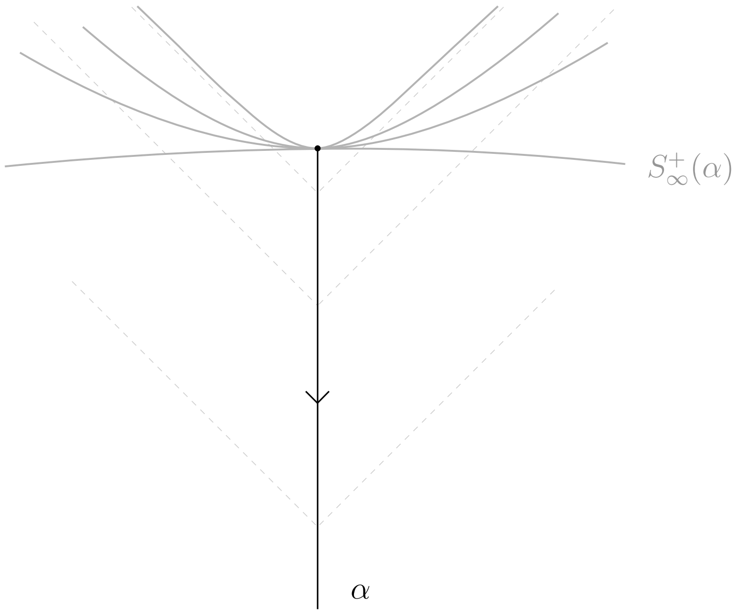


Figure 4.1: A fully bloomed flower of (support) spheres.

Lemma 4.1.4 generalizes to S-segments, for arbitrary S. We state the following version specialized for C^0 spacelike hypersurfaces.

Lemma 4.1.5 (Maximal Segments and Support Spheres)**.** *Let M be a globally hyperbolic spacetime, and $S \subset M$ a C^0 spacelike hypersurface. Let $\alpha : [0, b) \to M$ be a past-inextendible, unit speed timelike geodesic from $p = \alpha(0) \in S$, and for each $t \in (0, b)$, set $S^+_t := S^+_t(\alpha(t))$. Then $\alpha|_{[0,r]}$ is maximal as an S-segment iff $S \cap J^+(S^+_t) = S \cap S^+_t = \{p\}$, for all $t < r$. In this case, for each $t < r$, S^+_t restricts locally to a smooth future support hypersurface for S at p.*

Proof. The equivalence follows from the definitions and corner-cutting. Suppose $\alpha|_{[0,r]}$ is maximal. Then, for all $t < r$, $\alpha(t)$ is not a cut point from $\alpha(0)$. Consequently, distance from $\alpha(t)$ is smooth near $p = \alpha(0)$, and also its level set $S^+_t = S^+_t(\alpha(t)) = \{x : d(\alpha(t), x) = t\}$ is smooth and spacelike near $p \in S^+_t$. (For a discussion of timelike cut loci, see, for example, [27]). That S never enters the timelike future of S_t^+ implies that S_t^+ is locally to the future of S near p. $\square$

Let, for example, S be an acausal C^0 spacelike hypersurface and shoot a past geodesic α from $p \in S$. Lemma 4.1.5 says that α is maximal so long as its flower of support spheres has 'room to bloom'. Once one of its petals touches $S - \{p\}$, α ceases to maximize. Conversely, when α is known to maximize up to a distance r, we get a (local) smooth support sphere of radius *almost*-r at the base point p.

By standard Raychaudhuri comparison analysis, lower 'timelike' Ricci curvature bounds imply mean curvature bounds for smooth spacelike point spheres. This gives the following result:

Lemma 4.1.6 (Maximal Segments and Support Mean Convexity)**.** *Let M^{n+1} be a globally hyperbolic spacetime, and S a C^0 spacelike hypersurface. Let α be a past-inextendible, unit-speed timelike geodesic from $p = \alpha(0) \in S$.*

0) Suppose that $\mathrm{Ric}(X, X) \geq 0$, for all timelike vectors X. If $\alpha : [0, r] \to M$ is maximal as an S-segment, then S has support mean curvature $\leq \frac{n}{r}$ at p. If α is a complete past S-ray, then S has support mean curvature ≤ 0 at p.

λ) Now suppose (only) that $\mathrm{Ric}(X, X) \geq -\lambda^2 n$, for all timelike unit vectors X, with $\lambda > 0$. If $\alpha : [0, r] \to M$ is maximal as an S-segment, then S has support mean curvature $\leq \lambda n \coth(\lambda r)$ at p. If α is a complete past S-ray, then S has support mean curvature $\leq \lambda n$ at p.

Proof. Suppose that $\alpha|_{[0,r]}$ is a maximal S-segment. Then, by Lemma 4.1.5, for each $0 < t < r$, the future sphere S_t^+ restricts to a smooth, spacelike support hypersurface for S at p. By standard Riccati comparison theory, (cf. Theorem 4.2 in [10], which holds equally well in the Lorentzian case), S_t^+ has (smooth) mean curvature $\leq n/t$ near p in case 0), and (smooth) mean curvature $\leq \lambda n \coth(\lambda t)$

near p in case λ). The first part of both cases then follows by taking t arbitrarily close to r. When α is a complete past S-ray, we may take $t \to \infty$ in both cases to complete the proof. $\square$

For convenience, the following is stated for past spheres. Using Lemmas 4.1.6 and 3.3.2, we get:

Corollary 4.1.7 (Support Mean Convexity of Spheres)**.** *Let M^{n+1} be a globally hyperbolic spacetime. Let $C \subset M$ be past causally complete and $r > 0$, and suppose that $S_r^-(C)$ is nonempty.*

0) If $\mathrm{Ric}(X, X) \geq 0$ for all timelike vectors X, then $S_r^-(C)$ has support mean curvature $\geq -n/r$.

λ) If $\mathrm{Ric}(X, X) \geq -\lambda^2 n$ for all timelike unit vectors X, with $\lambda > 0$, then $S_r^-(C)$ has support mean curvature $\geq -\lambda n \coth(\lambda r)$.

In the appropriate setting, the extension to horospheres is straightforward: 'simply take $r \to \infty$'. However, for the support version of the maximum principle, Theorem 4.1.10 below, a bit more is needed. In [2], a somewhat stronger version of support mean curvature is given, which includes 'one-sided Hessian bounds'. To avoid a detour at this point, we leave a discussion of this technical condition to Section 4.5 in the appendix. As explained in [2], however, the issue typically boils down to having a well-behaved set of support normals. In Lemma 4.5.5, we show that this is the case for (appropriate) horospheres. In conjunction with Corollary 4.1.7, this gives:

Proposition 4.1.8 (Support Mean Convexity of Horospheres)**.** *Let M^{n+1} be globally hyperbolic and suppose S_∞^- is a past horosphere such that all future S_∞^--rays* *are timelike and future complete, (for example, if M is future timelike geodesically complete and S_∞^- is future bounded).*

0) If $\mathrm{Ric}(X, X) \geq 0$ *for all timelike vectors X, then S_∞^- has support mean curvature* ≥ 0 *with one-sided Hessian bounds.*

λ) If $\mathrm{Ric}(X, X) \geq -\lambda^2 n$ *for all timelike unit vectors X, with* $\lambda > 0$*, then S_∞^- has support mean curvature* $\geq -\lambda n$ *with one-sided Hessian bounds.*

Proof. The support mean curvature bounds are immediate from Lemma 4.1.6 and Theorem 3.4.4. The one-sided Hessian bounds follow from Lemma 4.5.5 in the appendix and, for example, Proposition 3.5 in [2]. □

4.1.2 Support Maximum Principle

Smooth spacelike hypersurfaces satisfy the following maximum principle, (cf. [12] and references therein):

Theorem 4.1.9 (Smooth Maximum Principle)**.** *Suppose $\Sigma_1, \Sigma_2 \subset M$ are smooth spacelike hypersurfaces such that, for some $a \in \mathbb{R}$,*

i) Σ_2 is locally to the future of Σ_1 near $p \in \Sigma_1 \cap \Sigma_2$.

ii) Σ_1 has mean curvature $H_1 \geq a$.

iii) Σ_2 has mean curvature $H_2 \leq a$.

Then for some neighborhood U of p, $\Sigma_1 \cap U = \Sigma_2 \cap U$ and this intersection is a smooth spacelike hypersurface with $H = a$.

In [2], the following 'rough' version of Theorem 4.1.9 was established. (Again, we refer the reader to Section 4.5 for the definition of 'support mean curvature with one-sided Hessian bounds'.)

Theorem 4.1.10 (Support Maximum Principle [2]). *Suppose* $S_1, S_2 \subset M$ *are* C^0 *spacelike hypersurfaces such that, for some* $a \in \mathbb{R}$,

i) S_2 *is locally to the future of* S_1 *near* $p \in S_1 \cap S_2$.

ii) S_1 *has support mean curvature* $\geq a$ *with one-sided Hessian bounds.*

iii) S_2 *has support mean curvature* $\leq a$.

Then for some neighborhood U *of* p, $S_1 \cap U = S_2 \cap U$ *and this intersection is a smooth spacelike hypersurface with* $H = a$.

4.2 Rigidity Under Timelike Convergence

In this section we establish several splitting results for spacetimes satisfying the timelike convergence condition:

$$\mathrm{Ric}(X, X) \geq 0, \quad \text{for all timelike vectors } X \tag{4.2.1}$$

We begin with a half-splitting result, an extension (in essence) of Theorem C in [17] to C^0 spacelike hypersurfaces:

Proposition 4.2.1 (Half-Splitting). *Let* M *be a globally hyperbolic, future timelike geodesically complete spacetime which satisfies the timelike convergence condition,* (4.2.1). *Suppose* S *is a connected, acausal, future causally complete* C^0 *spacelike hypersurface with support mean curvature* ≤ 0. *If* S *admits a future* S*-ray, then* S *is a smooth, maximal, geodesically complete spacelike hypersurface and* $(J^+(S), g)$ *splits via the normal exponential map,*

$$(J^+(S), g) \;\approx\; ([0, \infty) \times S, -dt^2 \oplus h),$$

where h *is the induced metric on* S.

Proof. Since S is future causally complete, it is closed, and hence, being a C^0 hypersurface, must be edgeless. Thus, by (the time dual of) Lemma 3.2.9, S is a future Cauchy surface and $J^+(S) = D^+(S)$. Then, by restricting attention to the open subspacetime $D(S)$ if necessary, we may assume without loss of generality that S is a (full) Cauchy surface for M. Fix a future S-ray, γ. Then γ is timelike, by Corollary 3.1.5, and hence future complete by assumption. Hence the past ray horosphere $S^-_\infty(\gamma)$ from γ is well-defined, and as in Lemma 3.4.8, is future bounded by S. Thus, $S^-_\infty(\gamma)$ admits a future complete timelike $S^-_\infty(\gamma)$-ray from each point, and by Lemma 4.1.8, has support mean curvature ≥ 0 with one-sided Hessian bounds. Let S^- be the connected component of $S^-_\infty(\gamma)$ containing $\gamma(0)$. Hence $S \cap S^-$ is closed and nonempty. Since $S^- \subset S^-_\infty(\gamma) \subset J^-(S)$, the support maximum principle, Theorem 4.1.10, implies that $S \cap S^-$ is also open in both S and S^-, and hence, $S = S^-$, and further that $S = S^-$ is a smooth spacelike hypersurface with mean curvature $H = 0$. Note that $S^- \subset S^-_\infty(\gamma)$ admits a future complete timelike $S^-_\infty(\gamma)$-ray from each point, which is necessarily a S^--ray. Hence, S admits a future complete timelike S-ray from each point, and since S is smooth, these are precisely the future normal geodesics from S. Then, as in Lemma 4.7.1 in the appendix, $(N^+(S), g)$ splits as $([0,\infty) \times S, -dt^2 + h)$, where the normal future $N^+(S)$ is the future image of the normal exponential map E of S. Then, adapting Theorem 3.68 in [5], using the warped product structure and future causal completeness of S, we get that S is geodesically complete. Then, since geodesics in a product are products of geodesics, and by the geodesic completeness of S, no future-directed causal geodesic can ever leave $([0,\infty) \times S, -dt^2 + h)$. It follows that $J^+(S) = N^+(S)$. $\square$

Using Proposition 4.2.1 we get a full splitting result for horospheres:

Theorem 4.2.2 (Horosphere Splitting)**.** *Let M be a globally hyperbolic, timelike geodesically complete spacetime satisfying the timelike convergence condition, i.e.,* $\mathrm{Ric}(X,X) \geq 0$*, for all timelike vectors X, and suppose S_∞^- is a past horosphere which is future bounded. If S_∞^- admits a past S_∞^--ray, then S_∞^- is a smooth spacelike, geodesically complete Cauchy surface along which M splits, i.e.,*

$$(M,g) \;\approx\; (\mathbb{R} \times S_\infty^-, -dt^2 + h),$$

where h is the induced metric on S_∞^-.

Proof. Since S_∞^- is future bounded, it is an acausal, past causally complete C^0 spacelike hypersurface, (by Corollary 3.4.2 and Theorem 3.4.5). By Proposition 4.1.8, S_∞^- has support mean curvature ≥ 0 (with one-sided Hessian bounds). Let S^- be the connected component of S_∞^- which contains $\gamma(0)$. Then by (the time dual of) Proposition 4.2.1, S^- is a smooth, maximal, geodesically complete spacelike hypersurface, and $(J^-(S^-), g) \;\approx\; ((-\infty, 0], -dt^2 + h)$. Since S^- is smooth, (and since every S_∞^--ray is also an S^--ray), it follows that the future radial rays from S^-, (as in Theorem 3.4.4), are precisely the normal geodesics from S^-, which, by assumption, are complete. Then, as in Lemma 4.7.1, $N^+(S^-)$ splits as desired, but again by this product structure and the geodesic completeness of S^-, it follows that $N^+(S^-) = J^+(S^-)$. Hence, the normal image of S^- coincides with its domain of influence, $N(S^-) = J(S^-)$, and splits as $(\mathbb{R} \times S^-, -dt^2 + h)$. As this product is geodesically complete, (by the product structure and the geodesic completeness of S^-), it follows by Propositions 1.1.21 and 1.1.23 that $H(S^-)$ is empty, and hence $D(S^-) = J(S^-) = M$. In particular, S^- is a Cauchy surface for M. Finally, suppose $x \in S_\infty^- \setminus S^-$. Then $x \in I^\pm(S^-)$, since S^- is a Cauchy surface, but this violates the achronality of S_∞^-. Hence, $S^- = S_\infty^-$. $\square$

4.2.1 Compact Horospheres

As a consequence of Proposition 1.1.28, we have the following:

Lemma 4.2.3. *Let S_∞ be a (future or past) horosphere. Then the following are equivalent:*

i) S_∞ is compact.

ii) S_∞ is a compact Cauchy surface.

iii) S_∞ is both future and past bounded by compact Cauchy surfaces.

Proof. Suppose, for example, that S_∞^- is compact. Then S_∞^- is a Cauchy surface by Proposition 1.1.28, and hence is future and past bounded by itself. Conversely, if $S_\infty^- \subset J^+(S^-) \cap J^-(S^+)$, for some compact Cauchy surfaces S^- and S^+, then S_∞^- must be compact, by Lemma 1.1.14. $\square$

Recall that Lemma 3.1.9 ensures that a compact Cauchy surface S admits both a future and past S-ray. Combining this with Lemma 4.2.3, we have the following corollary to Theorem 4.2.2:

Corollary 4.2.4 (Compact Horospheres)**.** *Let M be a globally hyperbolic, timelike geodesically complete spacetime which satisfies the timelike convergence condition,* (4.2.1). *Let S_∞ be a (future or past) horosphere in M. If S_∞ is compact, then S_∞ is a smooth spacelike Cauchy surface along which M splits, i.e.,*

$$(M, g) \;\approx\; (\mathbb{R} \times S_\infty, -dt^2 + h),$$

where h is the induced metric on S_∞.

4.2.2 Cauchy Horospheres

As can be seen from Corollary 4.2.4, compactness is a particularly consequential property for horospheres. In this section, we specialize to Cauchy horospheres and give a sufficient condition for the compactness of $S^-_\infty(S)$ via a 'max-min condition' on its base Cauchy surface S.

We first note that since $S^-_\infty(S)$ is automatically future bounded by S, as in Proposition 3.4.10, we have the following corollary to Lemma 4.2.3:

Corollary 4.2.5. *Let M be future timelike geodesically complete with compact Cauchy surface S. Then $S^-_\infty(S)$ is a compact Cauchy surface iff it is past bounded.*

Definition 4.2.6 (Max-Min Condition)**.** Let M be future timelike geodesically complete with compact Cauchy surface S. For each $k \in \mathbb{N}$, let $S_k := S^+_k(S)$. We say the *max-min condition* holds on S if there is an $R > 0$, such that for all $k \in \mathbb{N}$,

$$\max_{x \in S} d(x, S_k) - \min_{x \in S} d(x, S_k) < R$$

We note that, by definition of $S_k = S^+_k(S)$, we have $\max_{x \in S} d(x, S_k) = k$. The max-min condition is easily seen to hold for any Cauchy surface in a Lorentzian warped product $(\mathbb{R} \times N, -dt^2 + f^2(t)h)$, with $f : \mathbb{R} \to (0, \infty)$ and (N, h) compact Riemannian. In particular, it holds for any Cauchy surface in de Sitter space.

Lemma 4.2.7. *Let M be timelike geodesically complete with compact Cauchy surface S. If the max-min condition holds on S, then $S^-_\infty(S)$ is past bounded and hence is a compact Cauchy surface.*

Proof. Suppose that $\max_{x \in S} d(x, S_k) - \min_{x \in S} d(x, S_k) < R$, for some $R > 0$. Note that $S^-_R(S)$ is a compact Cauchy surface by Lemma 3.3.5. We will show that $\widetilde{S}_k \subset J^+(S^-_R(S))$. Suppose otherwise, that there is some $x_1 \in \widetilde{S}_k$ and $x_2 \in S^-_R(S)$, with $x_1 << x_2$. By definition of $S^-_R(S)$, there is a timelike curve of length R from x_2 to $x_3 \in S$. Then, there is a timelike curve from x_3 to $x_4 \in S^+_k(S)$ of length at least $\min_{x \in S} d(x, S^+_k(S))$. Concatenating these curves, we get a curve from $x_1 \in \widetilde{S}_k = S_k^-(S_k^+(S))$ to $x_4 \in S_k^+(S)$ of length strictly greater than $R + \min_{x\in S} d(x, S_k^+(S))$, and hence, $\max_{x\in S} d(x, S_k^+(S)) \geq d(x_1, S_k^+(S)) > R + \min_{x\in S} d(x, S_k^+(S))$, a contradiction. The conclusion follows from Lemma 4.2.3. □

Combining Lemma 4.2.7 and Corollary 4.2.4 we have:

Theorem 4.2.8 (Max-Min Splitting). *Let M be a timelike geodesically complete spacetime satisfying the timelike convergence condition, and suppose that S is a compact Cauchy surface for M. If the max-min condition holds on S, then $S_\infty^-(S)$ is a smooth, compact spacelike Cauchy surface along which M splits, i.e.,*

$$(M, g) \;\approx\; (\mathbb{R} \times S_\infty^-(S), -dt^2 + h),$$

where h is the induced metric on $S_\infty^-(S)$.

In [13], a splitting result is obtained under the 'S-ray condition', that some Cauchy surface S admits a future ray S-ray γ for which $S \subset I^-(\gamma)$. We note the following:

Lemma 4.2.9. *Let M be timelike geodesically complete, $S \subset M$ a compact Cauchy surface, and $S_\infty^- = S_\infty^-(S)$ its associated Cauchy horosphere. If γ is a timelike future S-ray such that $S \subset I^-(\gamma)$, then the max-min condition holds on S.*

Proof. Parameterize γ with respect to arc length. Since $S \subset I^-(\gamma)$ and S is compact, we have $S \subset I^-(\gamma(k_0))$ for some $k_0 \in \mathbb{N}$. Then, for any $x \in S \subset I^-(\gamma(k_0))$, and $k_0 \leq k$, the reverse triangle inequality gives $d(x, \gamma(k_0)) + (k - k_0) \leq d(x, \gamma(k))$, and rewriting, we get $k - d(x, \gamma(k)) \leq k_0 - d(x, \gamma(k_0))$. As the right hand side is a continuous function on the compact set S, it is bounded above by some $0 \leq R$, and we get, $k - d(x, \gamma(k)) \leq R$, for all $k_0 \leq k$. Since

$d(x, \gamma(k)) \leq d(x, S_k^+(S))$, we have $k \leq d(x, S_k^+(S)) + R$. Taking the minimum over $x \in S$, we get $k \leq m_k + R$. $\square$

4.2.3 Lines

In this section we present an alternative proof to Theorem 4.2.2, when specialized to Cauchy horospheres, based on the Lorentzian splitting theorem, ([11], [16], [22]).

The version of the Lorentzian splitting theorem we will use is the following:

Theorem 4.2.10 (Lorentzian Splitting Theorem)**.** *Let M be a globally hyperbolic spacetime which satisfies the timelike convergence condition,* $\mathrm{Ric}(X, X) \geq 0$*, for all timelike vectors X. If M admits a complete timelike line, then (M, g) splits as $(\mathbb{R} \times N, -dt^2 + h)$, where (N, h) is a complete Riemannian manifold.*

We will also use the following, which shows in particular, that the compactness of $S_\infty^-(S)$ in Theorem 4.2.12 below is necessary for splitting:

Lemma 4.2.11. *Suppose $(M, g) \approx (\mathbb{R} \times N, -dt^2 + h)$, where (N, h) is a compact Riemannian manifold. Then for any Cauchy surface S in M, its associated Cauchy horosphere $S_\infty^-(S)$ is compact.*

Proof. Since N is a Cauchy surface for M, S must also be compact. So $S_\infty^-(S)$ is well defined. Since S is compact, it is past bounded by some t-slice, which we may take to be N itself, i.e., $S \subset J^+(N)$. Since $\widetilde{N}_k = N_k^-(N_k^+(N)) = N$, it follows that $\widetilde{S}_k \subset J^+(N) \cap J^-(S)$, and hence also $S_\infty^-(S) \subset J^+(N) \cap J^-(S)$. Thus $S_\infty^-(S)$ is a compact Cauchy surface, as in Lemma 4.2.3. $\square$

Combining this with Theorem 3.4.4 and Lemma 3.1.6 yields the following:

Theorem 4.2.12. *Let M be a future timelike geodesically complete spacetime, satisfying* $\mathrm{Ric}(X, X) \geq 0$ *for all timelike vectors X. Suppose M admits a compact Cauchy surface S and that its associated Cauchy horosphere $S^-_\infty(S)$ admits a complete past $S^-_\infty(S)$-ray. Then $S^-_\infty(S)$ is a smooth, compact spacelike Cauchy surface, along which M splits.*

Proof. Let $\eta : [0, \infty) \to M$ be a complete past geodesic $S^-_\infty(S)$-ray. By Theorem 3.4.5, η must be timelike. By Theorem 3.4.4, $S^-_\infty(S)$ admits a future complete timelike $S^-_\infty(S)$-ray $\gamma : [0, \infty) \to M$, from $\gamma(0) = \eta(0)$, and by Lemma 3.1.6, γ and η join to form a complete timelike line. The Lorentzian splitting theorem then splits M as: $(M, g) \approx (\mathbb{R} \times N, -dt^2 + h)$, with (N, h) a complete Riemannian manifold. As N must be a Cauchy surface for M, we have that N is compact. Hence, by Lemma 4.2.11, $S^-_\infty(S)$ is compact. Consequently, the time coordinate t achieves a maximum on $S^-_\infty(S)$, which, without loss of generality, we may take to be $t = 0$. Hence, $S^-_\infty(S) \subset J^-(N)$, and there is at least one point $p \in N \cap S^-_\infty(S)$. Let S^- be the connected component of $S^-_\infty(S)$ which contains p. As N is maximal in M, i.e., has mean curvature $H = 0$, and since $S^-_\infty(S)$ has mean curvature ≥ 0 (with one-sided Hessian bounds), it follows from Theorem 4.1.10 that the intersection $N \cap S^-$ is open in both N and S^-. Since, in general, Cauchy surfaces are connected, so must N be. It follows that $N = S^-$. Hence, S^- is a Cauchy surface, from which it follows that $N = S^- = S^-_\infty(S)$. $\square$

4.3 Limit Mean Curvature

In this section we develop a notion of weak mean convexity for achronal limits, $A_\infty = \lim\{A_k\}$, which may not have their own (useful) support surfaces, but which may nonetheless benefit from the support surfaces of their limiting boundaries, A_k. This is analogous to what is used in [8].

We first observe that achronal boundaries are causally convex:

Lemma 4.3.1. *Let A be an achronal boundary. If α is a causal curve segment with endpoints on A, then $\alpha \subset A$ (and α is necessarily a null pregeodesic segment).*

Proof. Let P and F be the past and future sets associated to A, as in Proposition 2.1. Since $I^+(A) \subset F$, we have $J^+(A) \subset \overline{F} = A \cup F$, and similarly, $J^-(A) \subset A \cup P$. Suppose $\alpha : [0,1] \to M$ is future causal with $\alpha(0), \alpha(1) \in A$. Then $\alpha \subset J^+(\alpha(0)) \subset J^+(A) \subset A \cup F$. But similarly, $\alpha \subset J^-(\alpha(1)) \subset J^-(A) \subset A \cup P$. Since P and F are disjoint, we have $\alpha \subset A$. (Since A is achronal, α is a null pregeodesic segment, by Proposition 1.1.3.) □

The following consequence of Lemma 4.3.1 is used in Definition 4.3.3 below:

Corollary 4.3.2. *An achronal boundary is a C^0 spacelike hypersurface (i.e., locally acausal-and-edgeless) iff it is (globally) acausal.*

The following is similar to Definition 1 in [8]:

Definition 4.3.3 (Limit Mean Curvature)**.** Let A_∞ be the (future or past) achronal limit of a sequence of achronal boundaries, $\{A_k\}$, each of which is acausal. We say A_∞ has *limit mean curvature* $\geq a$ (resp. $\leq a$) if A_k has support mean curvature $\geq a_k$ (resp. $\leq a_k$) and $a_k \to a$.

By Corollary 4.1.7, and its time dual, we have:

Lemma 4.3.4 (Limit Mean Convexity of Horospheres)**.** *Let M^{n+1} be globally hyperbolic and let S^-_∞ be a past horosphere and S^+_∞ a future horosphere. Then we have the following:*

0) If $\mathrm{Ric}(X,X) \geq 0$ *for all timelike vectors* X*, then* S_∞^- *has limit mean curvature* ≥ 0 *and* S_∞^+ *has limit mean curvature* ≤ 0*.*

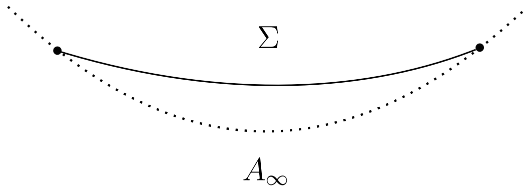
λ) If $\mathrm{Ric}(X,X) \geq -\lambda^2 n$ *for all timelike unit vectors* X*, with* $\lambda > 0$*, then* S_∞^- *has limit mean curvature* $\geq -\lambda n$ *and* S_∞^+ *has limit mean curvature* $\leq \lambda n$*.*

Remark 4.3.5. Note that, contrary to Proposition 4.1.8, there is no completeness assumption in Lemma 4.3.4. This feature of limit mean curvature becomes critical in the singular setting, where one must necessarily do without the assumption of, say, (full) future or past completeness.

4.3.1 A Limit Mean Convexity Lemma

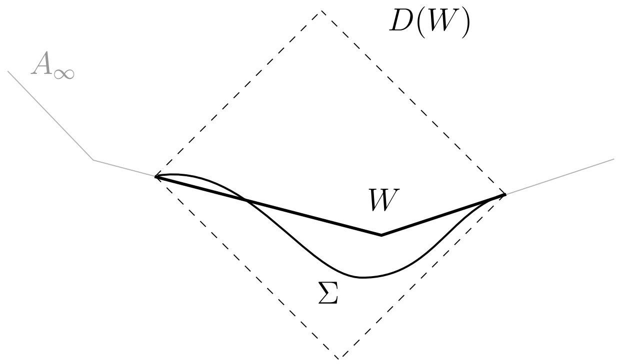
The following is a key component in the proof of the 'limit maximum principle', Lemma 4.3.9, below.

Lemma 4.3.6 (Limit Mean Convexity Lemma)**.** *Let* M^{n+1} *be a globally hyperbolic spacetime such that, for some* $\lambda \geq 0$, $\mathrm{Ric}(X,X) \geq -\lambda^2 n$ *for all timelike unit vectors* X*. Let* $A_\infty \subset M$ *be an achronal limit with limit mean curvature* $\geq \lambda n$*, (resp.* $\leq \lambda n$*), and suppose that* W *is a domain in* A_∞ *with* $\overline{W}$ *acausal and* $\overline{D(W)}$ *compact. Let* $\Sigma \subset D(W)$ *be a smooth, achronal spacelike hypersurface with* $\mathrm{edge}\,\Sigma = \mathrm{edge}\,W$ *and mean curvature* $H_\Sigma = \lambda n$*. Then* $\Sigma \subset J^+(W)$*. In particular,* $\Sigma \subset J^+(A_\infty)$*, (resp.* $\Sigma \subset J^-(W) \subset J^-(A_\infty)$*):*



Proof. We consider the case $\lambda > 0$. The proof for $\lambda = 0$ is completely analogous, with only minor modifications.

Suppose to the contrary that Σ meets $I^-(W)$. Hence, the picture, schematically, is as below. We encourage the reader to picture also $\{A_k\}$, the sequence of achronal boundaries approaching A_∞:



The idea of the proof is as follows. We perturb (most of) Σ to get a smooth hypersurface with mean curvature *strictly less than* λn. That A_∞ has limit mean curvature $\geq \lambda n$, means A_k has support mean curvature $\geq \lambda n + c_k$, with $c_k \to 0$. Then, 'sliding down' a past support hypersurface for A_k, for large enough k, gives a past support hypersurface for the perturbed Σ, with mean curvature arbitrarily close to λn, producing a contradiction. (The curvature condition is used to control the mean curvature during sliding.) This will involve a bit of careful setup first.

To begin, we first note that $D(W)$ is open by Proposition 1.1.22. Since $\Sigma, W \subset \overline{D(W)}$, the closures $\overline{\Sigma}$ and $\overline{W}$ are compact, and hence the distance $\ell := d(\overline{\Sigma}, \overline{W}) \geq d(\Sigma, W) > 0$ is realized by points $p \in \overline{\Sigma}$ and $q \in \overline{W}$. But since $\overline{\Sigma} = \Sigma \cup \text{edge}\,\Sigma$ and $\overline{W} = W \cup \text{edge}\,W$, and since $\text{edge}\,\Sigma = \text{edge}\,W$, we must have $p \in \Sigma$ and $q \in W$ and thus,

$$\ell = d(p,q) = d(\Sigma, W)$$

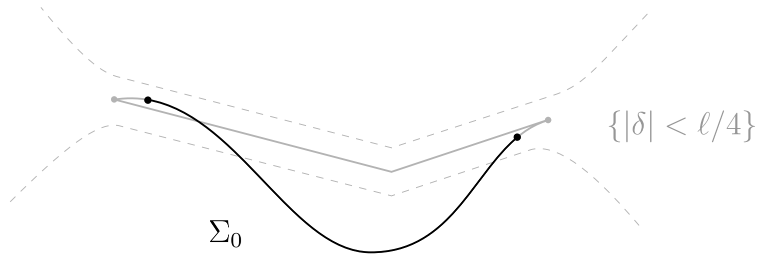
Since $\overline{W}$ is acausal and compact, its 'signed distance function',

$$\delta(x) := d(\overline{W}, x) - d(x, \overline{W}),$$

is continuous on all of M, and we have:

$$\delta(x) = \begin{cases} + & x \in I^+(\overline{W}) \\ 0 & x \notin I^-(\overline{W}) \cup I^+(\overline{W}) \\ - & x \in I^-(\overline{W}) \end{cases}$$

Hence, for any $a > 0$, the set $\{|\delta| < a\}$ is an open neighborhood of $\overline{W}$ (and by achronality, all of A_∞). Using a (proper) Whitney Embedding, and Sard's Theorem, Σ admits an exhaustion by smooth compact domains. Then, using the fact that $\Sigma \cap \{|\delta| \geq \ell/4\} = \overline{\Sigma} \cap \{|\delta| \geq \ell/4\}$ is compact, let $\Sigma_0 \subset \Sigma$ be a smooth compact domain with $\partial\Sigma_0 \subset \{|\delta| < \ell/4\}$ and $p \in \Sigma_0$. Hence, still $d(\Sigma_0, W) = d(p, q) = \ell$.



For sufficiently small $f \in C^\infty(\Sigma_0)$, with $f|_{\partial\Sigma_0} = 0$, let $\mathcal{H}(f)$ denote the mean curvature of the surface $\Sigma_f : x \to \exp_x f N_x$ where N is the future unit normal to Σ_0. The mean curvature operator $\mathcal{H}$ has linearization, (cf. [3]):

$$\mathcal{H}'(0) = \triangle - (\mathrm{Ric}(N, N) + |B|^2)$$

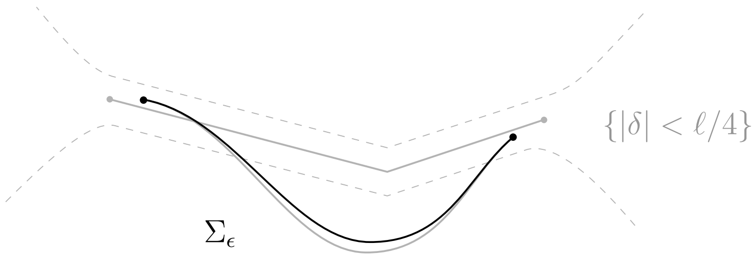
where B denotes the second fundamental form of Σ_0. Since,

$$\mathrm{Ric}(N,N) + |B|^2 \geq -\lambda^2 n + \frac{H^2}{n} = -\lambda^2 n + \lambda^2 n = 0,$$

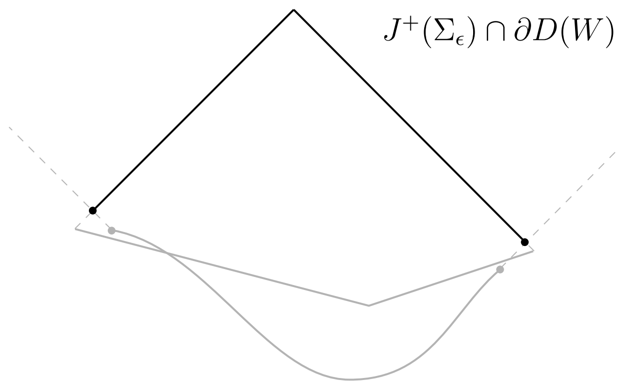
$\mathcal{H}'(0)$ is invertible. Thus, by the inverse function theorem, for sufficiently small $\epsilon > 0$, there exists a smooth compact spacelike hypersurface $\Sigma_\epsilon \subset D(W)$, with $\partial\Sigma_\epsilon = \partial\Sigma_0$ and mean curvature $H_{\Sigma_\epsilon} = \lambda n(1-\epsilon)$, and such that

$$\ell_\epsilon := d(\Sigma_\epsilon, \overline{W}) = d(p_\epsilon, q_\epsilon) \geq \frac{7}{8}\ell$$

for some $p_\epsilon \in \operatorname{int}\Sigma_\epsilon$ and $q_\epsilon \in \overline{W}$.



Applying Proposition 1.1.26 to $D(W)$, for example, one observes that $J^+(\Sigma_\epsilon) \cap \partial D(W) \subset I^+(W) \subset I^+(\overline{W})$. Furthermore, since Σ_ϵ and $\overline{D(W)}$ are compact, so is $J^+(\Sigma_\epsilon) \cap \partial D(W)$ (using Proposition 1.1.14).



Hence, the signed distance function δ of $\overline{W}$ achieves a positive minimum $\delta_0 > 0$ on $J^+(\Sigma_\epsilon) \cap \partial D(W)$. Let $\delta_1 := \min\{\delta_0, \ell/4\}$. By Lemma 2.3.4, we may choose k_1 sufficiently large so that

$$A_k \cap \overline{D(W)} \;\subset\; \{|\delta| < \delta_1\} \cap \overline{D(W)}, \;\; \text{for all } k \geq k_1$$

Thus, for all $k \geq k_1$, we have

$$\begin{aligned} J^+(\Sigma_\epsilon) \cap \big(A_k \cap \partial D(W)\big) \;&\subset\; \big(J^+(\Sigma_\epsilon) \cap \partial D(W)\big) \,\cap\, \big(A_k \cap \partial D(W)\big) \\ &\subset\; \{\delta \geq \delta_0\} \,\cap\, \{|\delta| < \delta_1\} \\ &\subset\; \{\delta \geq \delta_1\} \,\cap\, \{|\delta| < \delta_1\} \\ &=\; \emptyset \end{aligned}$$

Hence for large k, Σ_ϵ can only see $A_k \cap \overline{D(W)}$ within the interior $D(W)$:

$$J^+(\Sigma_\epsilon) \cap \big(A_k \cap \overline{D(W)}\big) \subset A_k \cap D(W), \;\; \text{for all } k \geq k_1\,. \tag{4.3.2}$$

We now show that, for large k, the distance between the compact sets Σ_ϵ and $A_k \cap \overline{D(W)}$ remains bounded away from 0 and ∞, and is realized by points $p_k \in \operatorname{int}\Sigma_\epsilon$ and $q_k \in A_k \cap D(W)$. Let $\sigma : [0, \ell_\epsilon] \to M$ be a future-directed maximal timelike unit-speed geodesic segment from $\sigma(0) \in \Sigma_\epsilon$ to $\sigma(\ell_\epsilon) \in \overline{W}$, realizing the distance $d(\Sigma_\epsilon, \overline{W}) = \ell_\epsilon$. Since $\overline{W} \subset A_\infty$, σ is a timelike curve from Σ_ϵ to A_∞. To cover both cases, A_∞ is a past/future achronal limit, extend σ slightly to the future to a timelike curve, $\sigma : [0, L] \to M$, with $\ell_\epsilon < L$. Then, as in Proposition 2.2.5, there is an integer $k_\epsilon \geq k_1$ such that (the extended) σ meets A_k for all $k \geq k_\epsilon$. Hence, for $k \geq k_\epsilon \geq k_1$, we have $\sigma \cap A_k \cap \overline{D(W)} \subset \{|\delta| < \delta_1\} \subset \{|\delta| < \ell/4\}$, and it follows that:

$$d(\Sigma_\epsilon, A_k \cap \overline{D(W)}) \geq \ell_\epsilon - \frac{\ell}{4} \geq \frac{7\ell}{8} - \frac{\ell}{4} = \frac{5\ell}{8}$$

Now, for each $k \geq k_\epsilon$, by compactness, we may find points $p_k \in \Sigma_\epsilon$ and $q_k \in A_k \cap \overline{D(W)}$ such that $\ell_k := d(p_k, q_k) = d(\Sigma_\epsilon, A_k \cap \overline{D(W)})$. But since $k_\epsilon \geq k_1$, it follows from (4.3.2) that we must have $q_k \in A_k \cap D(W)$. Furthermore, since $\partial\Sigma_\epsilon \subset \{|\delta| < \ell/4\}$, it follows that we must have $p_k \in \operatorname{int}\Sigma_\epsilon$. Then, letting $\ell_W := d(\Sigma_\epsilon, \overline{D(W)})$, we have, for all $k \geq k_\epsilon$,

$$\ell_k = d(p_k, q_k) = d(\Sigma_\epsilon, A_k \cap \overline{D(W)}) = d(\operatorname{int}\Sigma_\epsilon, A_k \cap D(W)),$$

with,

$$\frac{5\ell}{8} \leq \ell_k \leq \ell_W$$

Again, the idea of the last part of the proof, which will also be used in Lemmas 4.3.9 and 4.4.2 below, is to take the support hypersurfaces for A_k at q_k, and 'slide them down' to support hypersurfaces for Σ_ϵ at p_k. Let $V_k \subset J^-(A_k) \cap D(W)$ be a (small) smooth spacelike past support hypersurface for A_k at q_k. Since A_∞ has limit mean curvature $\geq \lambda n$, by choosing $k \geq k_\epsilon$ sufficiently large, we can take $H_{V_k}(q_k) \geq \lambda n(1 - \frac{1}{2}\epsilon_k)$, for $\epsilon_k > 0$ arbitrarily small. Let $\sigma_k : [0, \ell_k] \to M$ be a maximal past directed unit speed timelike geodesic from $\sigma_k(0) = q_k \in A_k$ to $\sigma_k(\ell_k) = p_k \in \Sigma_\epsilon$. Since σ_k maximizes the distance to A_k, and $V_k \subset J^-(A_k)$, then σ_k also maximizes the distance to V_k. Consequently, V_k has no focal points along σ_k, except possibly the endpoint $\sigma_k(\ell_k)$. We may, in fact, push this (potential) focal point into the past by 'bending' V_k slightly to the past, keeping p_k fixed. To carry this out, one can, for example, let $\widehat{V}_k \subset J^-(V_k)$ be a small spacelike paraboloid (in appropriate coordinates) from V_k which opens to the past from $q_k \in V_k \cap \widehat{V}_k$. This gives a strict inequality on the corresponding second fundamental forms, and one may adapt Proposition 2.3 in [10], for example, to see that this inequality ensures that the first focal point, if any, along σ_k comes strictly later, (further in the past), for $\widehat{V}_k$ than for V_k. Furthermore, by taking this paraboloid to be sufficiently flat, (relative to V_k), we can ensure, for example, that $H_{\widehat{V}_k}(p_k) \geq \lambda n(1-\epsilon_k)$.

It follows then that the past normal exponential map E of $\widehat{V}_k$ is a diffeomorphism on some neighborhood of $[0,\ell_k] \times \{p_k\}$, and hence, for some neighborhood $\widetilde{V}_k$ of p_k in $\widehat{V}_k$, the past slice $E(\{t\} \times \widetilde{V}_k)$ is a smooth spacelike hypersurface for all $t \in [0,\ell_k]$. Letting $\theta(t)$ denote the mean curvature of this slice at $\sigma_k(t)$, the Raychaudhuri equation, together with the curvature condition, give:

$$\theta'(t) - \frac{\theta^2(t)}{n} \geq \mathrm{Ric}(\partial_t, \partial_t) \geq \lambda^2 n$$

Using the initial condition, $\theta(0) \geq \lambda n(1-\epsilon_k)$, a standard comparison argument gives: $\theta(t) \geq \lambda n \tanh(c_k - t)$ for all $t \in [0,\ell_k]$, where $c_k := \tanh^{-1}(1-\epsilon_k)$, (cf. [10]). Hence, letting $V_k' := E(\{\ell_k\} \times \widetilde{V}_k)$, then the mean curvature of V_k' satisfies:

$$H_{V_k'}(p_k) \geq \lambda n \tanh(c_k - \ell_k) \geq \lambda n \tanh(c_k - \ell_W)$$

Furthermore, for every $x \in V_k'$, we have, $d(x, A_k) \geq d(x, \widehat{V}_k) \geq \ell_k$, by construction. Hence, V_k' can not meet $I^+(\Sigma_\epsilon)$. Consequently, V_k' serves as a smooth past support hypersurface for Σ_ϵ at p_k. But by taking ϵ_k sufficiently small, we can make $c_k - \ell_W$ arbitrarily large so as to ensure that $H_{V_k'}(p_k) > \lambda n(1-\epsilon) = H_{\Sigma_\epsilon}(p_k)$, contradicting the basic inequality $B_{\Sigma_\epsilon}(p_k) \geq B_{V_k'}(p_k)$. □

4.3.2 A Limit Maximum Principle

We will use the following notation below. By a *(timelike) diamond neighborhood*, I_p, around $p \in M$, we mean a diamond $I_p := I^+(p_-) \cap I^-(p_+)$, for some $p_- << p << p_+$. We denote the corresponding causal diamond by J_p, i.e., $J_p := J^+(p_-) \cap$

$J^-(p_+)$. Hence, always $p \in I_p \subset J_p$, so that $p \in \operatorname{int} J_p$.

Because causally convex neighborhoods contain their diamonds, (c.f. Lemma 1.1.9), we have the following:

Lemma 4.3.7. *If strong causality holds at $p \in M$, then for any neighborhood U of p, there is a timelike diamond neighborhood I_p of p with $p \in I_p \subset J_p \subset U$.*

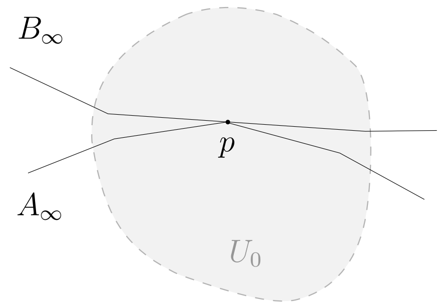
Moreover, one may establish an 'avoidance lemma' for small diamonds. That is, if J is a sufficiently small diamond and $x \notin J$, then there is an inextendible curve through x which misses J. Consequently, such diamonds are 'closed' under Cauchy developments:

Lemma 4.3.8 (Dependence-Trapping Diamonds)**.** *Let M be globally hyperbolic and fix $p \in M$. Then for any neighborhood U of p in M, there is a diamond neighborhood I_p of p, with $p \in I_p \subset J_p \subset U$ such that for any achronal set $A \subset J_p$, we have $D(A_p) \subset J_p$.*

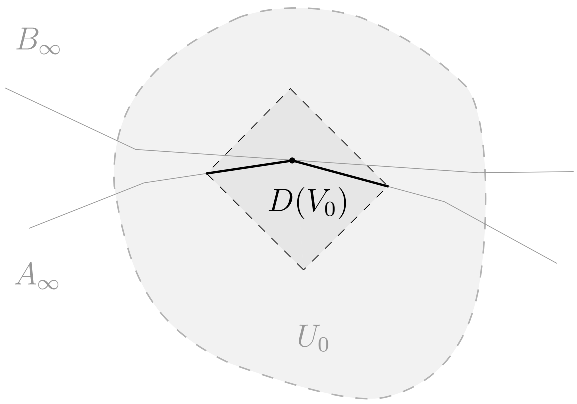
We are ready to establish the following 'maximum principle' for limit mean curvature:

Lemma 4.3.9 (Limit Maximum Principle)**.** *Let (M^{n+1}, g) be a globally hyperbolic spacetime satisfying, for some $\lambda \geq 0$, $\mathrm{Ric}(X,X) \geq -\lambda^2 n$ for all timelike unit vectors X. Let A_∞ and B_∞ be two achronal limits meeting at $p \in A_\infty \cap B_\infty$ such that, near p, both achronal limits are acausal, with B_∞ locally to the future of A_∞ (see proof). If A_∞ has limit mean curvature $\geq \lambda n$ and B_∞ has limit mean curvature $\leq \lambda n$, then for some neighborhood U of p in M, $A_\infty \cap U = B_\infty \cap U$ is a smooth, acausal spacelike hypersurface with $H = \lambda n$.*

Proof. Explicitly, we assume that there is a neighborhood U_0 of p in M such that $A_\infty \cap U_0$ and $B_\infty \cap U_0$ are acausal, and $B_\infty \cap U_0 \subset J^+(A_\infty \cap U_0)$;



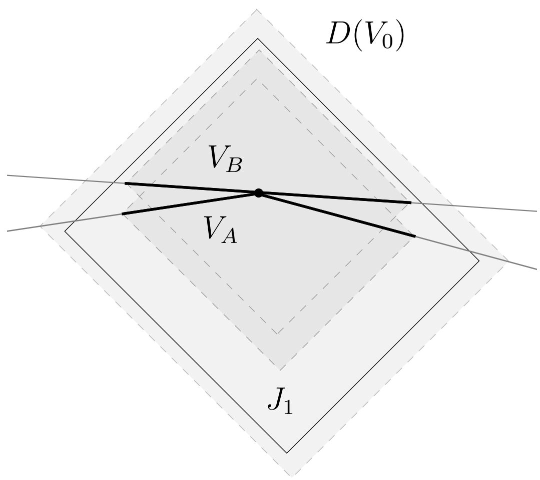
As in Lemma 4.3.6, the proof involves some careful setup. We first establish a domain of dependence within U_0. Using Lemma 4.3.8, let I_0 be a 'dependence-trapping' diamond neighborhood of p with $p \in I_0 \subset J_0 \subset U_0$. Let V_0 be a domain in A_∞ around p with $V_0 \subset A_\infty \cap I_0$. Then $D(V_0) \subset J_0 \subset U_0$.



Since $V_0 \subset A_\infty \cap I_0 \subset A_\infty \cap U_0$, we have that V_0 is acausal. Hence, by Proposition 1.1.22, $D(V_0)$ is an open, globally hyperbolic subspacetime, with Cauchy surface V_0. Recall that our (ambient) spacetime M is equipped with a time orientation, T, a smooth, global timelike vector field. Let h be a complete Riemannian metric on $D(V_0)$. Let T' be the restriction of T to $D(V_0)$ and set $T_0 := T'/||T'||_h$. It follows that T_0 is a smooth, complete timelike vector field on $D(V_0)$. Hence, as in Proposition 1.1.29, we have a homeomorphism $\Phi : \mathbb{R} \times V_0 \to D(V_0)$, where, for each $q_0 \in V_0$, the t-curve, $\phi_{q_0}(t) = \Phi(t, q_0)$ is the T_0-integral curve through q_0. This map will be used throughout the proof to relate the various (achronal) sets in $D(V_0)$ with which we will work.

The rest of the setup involves a pair of nested diamonds in $D(V_0)$. First, using Lemma 4.3.7, let I_1 be a diamond around p with $p \in I_1 \subset J_1 \subset D(V_0)$. By global hyperbolicity, J_1 is compact. Furthermore, since J_1 is causally convex (contains all of its diamonds), it is a globally hyperbolic subset of M, (cf. Lemma 1.1.13). Now, using Lemma 4.3.8, let I_2 be a smaller, 'dependence-trapping' diamond around p with $p \in I_2 \subset J_2 \subset I_1 \subset J_1$. Hence, we have $J_2 \subset\subset J_1$.

Now let V_B be a small domain around p in B_∞, with V_B homeomorphic to an open ball in $\mathbb{R}^n$, and $V_B \subset B_\infty \cap I_2$. Hence, $D(V_B) \subset J_2 \subset\subset J_1$. The projection $\pi_2 \circ \Phi^{-1}|_{V_B} : V_B \to V_0$ is continuous, and one-to-one, by achronality. It follows by invariance of domain that its image, $V_A := \pi_2 \circ \Phi^{-1}|_{V_B}(V_B)$, is a domain in V_0 around p. By shrinking V_B if necessary, (as a ball), we may suppose also $V_A \subset A_\infty \cap I_2$. Hence also $D(V_A) \subset\subset J_1$. We can forget about J_2 at this point and just remember $D(V_A), D(V_B) \subset\subset J_1$.

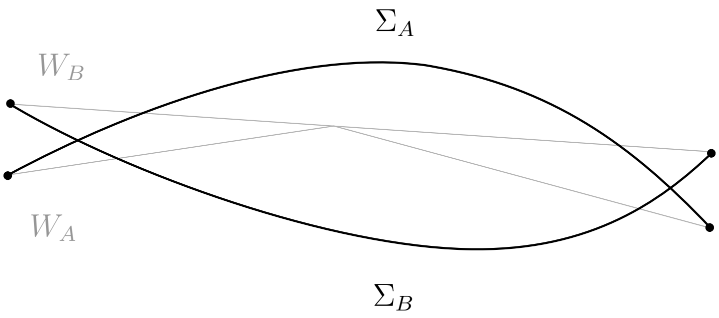


We emphasize that the points of V_A and V_B are in one-to-one correspondence via the (timelike) integral curves of T_0. Hence, fixing any $q \in V_A$, there is a unique point $q' \in V_B$ on the T_0-integral curve through q. (Including the possibility $q' = q$.) We will denote this kind of correspondence via the integral curves of T_0 by $V_B \approx^{T_0} V_A$, and will use it below on other sets. Note that, in this case, since B_∞ is to the future of A_∞ in U_0, the point $q' \in V_B$ above, either equals q, or is a future point on the integral curve through q.

We will show $V_A = V_B$. We have $p \in V_A \cap V_B$. Fix $x \in V_A - \{p\}$. Then, since V_A is homeomorphic to V_B, which is homeomorphic to a (hyper)-ball, we can choose a domain W_A in V_A with $\overline{W}_A \subset V_A$ and $x \in \overline{W}_A - W_A = \operatorname{edge} W_A$. Let W_B be the corresponding domain in V_B, i.e., $W_B \approx^{T_0} W_A$. In fact, since $\overline{W}_A \subset V_A$ and $\overline{W}_B \subset V_B$, we have also $\operatorname{edge} W_B \approx^{T_0} \operatorname{edge} W_A$.

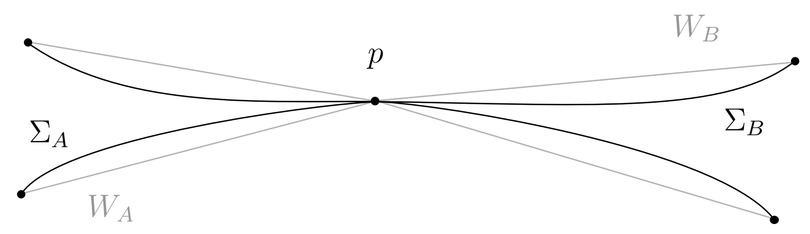
Since $D(W_A) \subset D(V_A) \subset\subset J_1$, with J_1 globally hyperbolic, it follows that (W_A, J_1) is a 'standard data set' as defined by Bartnik in [3]. Then, since $\overline{W}_A$ is acausal, [3, Theorem 4.1] produces a smooth, achronal spacelike hypersurface $\Sigma_A \subset D(W_A)$ of constant mean curvature $H_{\Sigma_A} = \lambda n$, with $\operatorname{edge} \Sigma_A = \operatorname{edge} W_A$ and $\Sigma_A \approx^{T_0} W_A$. By Lemma 4.3.6, we have $\Sigma_A \subset J^+(W_A)$. Similarly, now using (W_B, J_1) as the 'standard data set', [3, Theorem 4.1] and Lemma 4.3.6 give a smooth, achronal spacelike hypersurface $\Sigma_B \subset J^-(W_B)$ of constant mean curvature $H_{\Sigma_B} = \lambda n$, with $\operatorname{edge} \Sigma_B = \operatorname{edge} W_B$ and $\Sigma_B \approx^{T_0} W_B$. Note that, since $\operatorname{edge} W_A = \operatorname{edge}\Sigma_A$ and $\operatorname{edge} W_B = \operatorname{edge}\Sigma_B$, we have $\operatorname{edge} \Sigma_B \approx^{T_0} \operatorname{edge} \Sigma_A$.

We now show that Σ_B cannot enter $I^-(\Sigma_A)$. Suppose otherwise and let $p_B \in \overline{\Sigma}_B$ and $q_A \in \overline{\Sigma}_A$ such that $\ell = d(\overline{\Sigma}_B, \overline{\Sigma}_A) = d(p_B, q_A) > 0$. By the achronality of Σ_A and Σ_B, and the causal relations on the boundaries, it follows that $p_B \notin \operatorname{edge} \Sigma_B$ and $q_A \notin \operatorname{edge} \Sigma_A$, so $p_B \in \Sigma_B$, $q_A \in \Sigma_A$ and $\ell = d(\overline{\Sigma}_B, \overline{\Sigma}_A) = d(\Sigma_B, \Sigma_A) = d(p_B, q_A)$.

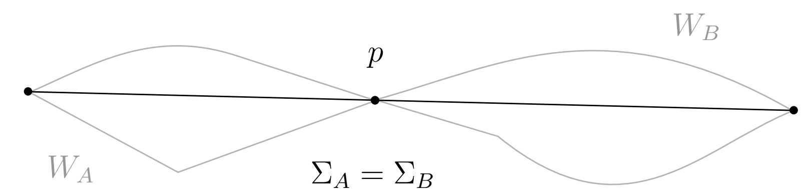


Hence, the past sphere $S_\ell^- = S_\ell^-(\overline{\Sigma}_A)$ meets Σ_B at $p_B \in \Sigma_B \cap S_\ell^-$. Fix an arbitrary intersection point $z \in \Sigma_B \cap S_\ell^-$. Since S_ℓ^- is acausal and edgeless, $D(S_\ell^-)$ is an open neighborhood of z. If Σ_B entered $I^-(S_\ell^-)$, we could produce a curve from Σ_B to S_ℓ^- of positive length, and hence a curve from Σ_B to Σ_A of length strictly greater than ℓ. Hence, Σ_B cannot enter $I^-(S_\ell^-)$, and near $z \in \Sigma_B \cap S_\ell^- \subset D(S_\ell^-)$, we have Σ_B locally to the future of S_ℓ^-. Furthermore, for such a z, there is some $y \in \overline{\Sigma}_A$ such that $\ell = d(z, \overline{\Sigma}_A) = d(z, y)$. But since $y \in \operatorname{edge}\Sigma_A \subset J^-(\operatorname{edge}\Sigma_B)$ would lead to a violation of the achronality of Σ_B, we have $y \in \Sigma_A$. Then, by an argument similar to that in Lemma 4.3.6, (starting with Σ_A as a past support hypersurface for itself, (bending to the past), and sliding down to S_ℓ^-), we can show that S_ℓ^- has support mean curvature $\geq \lambda n$ at $z \in \Sigma_B \cap S_\ell^-$. Let S^- be the connected component of S_ℓ^- which contains p_B. It follows from Theorem 4.1.10, that the intersection $\Sigma_B \cap S^-$ is open in Σ_B. Since S^- is closed, this intersection is also closed in Σ_B. Since Σ_B is homeomorphic to the (connected) domain W_B, Σ_B is connected, and hence, $\Sigma_B \cap S^- = \Sigma_B$, i.e., $\Sigma_B \subset S^-$. But since S^- is closed, this implies $\operatorname{edge}\Sigma_B \subset S^- \subset I^-(\overline{\Sigma}_A)$, which again leads to an achronality violation.

Hence, Σ_B does not meet $I^-(\Sigma_A)$. It follows that Σ_A and Σ_B are 'sandwiched' between W_A and W_B, with $p \in \Sigma_A \cap \Sigma_B$, and Σ_B to the future of Σ_A.



The (smooth) maximum principle then gives that $\Sigma_A \cap \Sigma_B$ is open in both Σ_A and Σ_B. Suppose $\Sigma_A \cap \Sigma_B$ is not closed in Σ_A. Then $\Sigma_A \cap \Sigma_B$ has a limit point $p_0 \in \Sigma_A \setminus \Sigma_B$. Then $p_0 \in \overline{\Sigma}_B \setminus \Sigma_B = \operatorname{edge} \Sigma_B$. But since $\operatorname{edge} \Sigma_B \subset J^+(\operatorname{edge} \Sigma_A)$, $p_0 \in \Sigma_A \cap \operatorname{edge} \Sigma_B$ leads to an achronality violation. Hence, $\Sigma_A \cap \Sigma_B$ is closed in Σ_A, and by connectedness, $\Sigma_A \subset \Sigma_B$. By symmetry, it follows that $\Sigma_A = \Sigma_B$. Hence, $\operatorname{edge} W_A = \operatorname{edge} \Sigma_A = \operatorname{edge} \Sigma_B = \operatorname{edge} W_B$. Thus, the above procedure 'sews' A_∞ and B_∞ together along the edges of W_A and W_B.



In particular, we have $x \in \operatorname{edge} W_A = \operatorname{edge} W_B \subset V_B$. Since $x \in V_A - \{p\}$ was arbitrary, (and since $p \in V_A \cap V_B$), we have $V_A \subset V_B$, and since $V_A \approx^{T_0} V_B$, this means $V_A = V_B$. It follows that $W_A = W_B = \Sigma_B = \Sigma_A$. Hence, near p, A_∞ and B_∞ agree and are smooth and spacelike, with mean curvature λn. $\square$

4.4 Rigidity under $\Lambda > 0$

In this section we consider spacetimes (M^{n+1}, g), with $n > 2$, which obey the Einstein equation,

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = 8\pi T_{ij}\,, \tag{4.4.3}$$

with positive cosmological constant $\Lambda > 0$, where the energy-momentum tensor T_{ij} is assumed to satisfy the strong energy condition,

$$(T_{ij} - \frac{1}{n-1}Tg_{ij})X^i X^j \geq 0 \tag{4.4.4}$$

for all timelike vectors X, where $T = T_i^{\ i}$.

Setting $\Lambda = n(n-1)\lambda^2/2$, the strong energy condition (4.4.4) is equivalent to,

$$\mathrm{Ric}(X, X) \geq -\lambda^2 n, \ \ \text{for all timelike unit vectors } X \tag{4.4.5}$$

We begin with the following past singularity result, which is in some sense analogous to Proposition 4.2.1.

Proposition 4.4.1 (Limit Mean Rigidity). *Let M^{n+1} be a globally hyperbolic spacetime such that, for some $\lambda \geq 0$, $\mathrm{Ric}(X, X) \geq -\lambda^2 n$ for all timelike unit vectors X. Let $S_\infty \subset M$ be a past causally complete achronal limit and suppose that S_∞ is acausal with limit mean curvature $\geq \lambda n$. Suppose also that S_∞ admits a past S_∞-ray, γ, and let S_∞^0 be the connected component of S_∞ containing $\gamma(0)$. Then either S_∞ admits a past incomplete timelike S_∞-ray, or S_∞^0 is a smooth, geodesically complete spacelike past Cauchy surface with mean curvature $H = \lambda n$ and $J^-(S_\infty^0)$ splits as:*

$$(J^-(S_\infty^0), g) \approx ((-\infty, 0] \times S_\infty^0, -dt^2 + e^{2\lambda t}h),$$

where h denotes the induced metric on S_∞^0.

Proof. Observe that, since S_∞ is acausal, every S_∞-ray, future or past, is timelike. Assume all past S_∞-rays are past complete. Hence γ is timelike and past complete. Construct $S^+_\infty(\gamma)$. By Lemma 4.3.4, $S^+_\infty(\gamma)$ has limit mean curvature $\leq \lambda n$. Let S^+ be the connected component $S^+_\infty(\gamma)$ containing $\gamma(0)$. Hence, the intersection $S^0_\infty \cap S^+$ is nonempty and closed. Since S^0_∞ is acausal, it follows that S^+ must also be locally acausal and to the future of S^0_∞ near any intersection point $x \in S^0_\infty \cap S^+$. Hence, by Lemma 4.3.9, $S^0_\infty = S^+$ is a smooth spacelike hypersurface with mean curvature $H = \lambda n$. Hence, by (both parts of) Lemma 4.7.1, we have that the normal past, $N^-(S^0_\infty)$ splits as $((-\infty, 0] \times S^0_\infty, -dt^2 + e^{-2\lambda t}h)$. Then, again, by adapting the proof of Theorem 3.68 in [5], using the past causal completeness of S^0_∞ and the warped product structure, we get that S^0_∞ is geodesically complete. Hence, it follows from Theorem 3.69 in [5] that $N^-(S^0_\infty) = J^-(S^0_\infty)$. $\square$

4.4.1 Asymptotically de Sitter Spacetimes

Recall that de Sitter space, $\mathbb{S}^n_1 := \{x \in \mathbb{M}^{n+1} : g(x,x) = 1\}$, the unit 'pseduosphere' in Minkowksi space, is the simply connected Lorentzian space form of constant curvature 1. In particular, de Sitter satisifes (4.4.5) with $\lambda = 1$. Furthermore, we note that de Sitter space is both timelike and null geodesically complete.

de Sitter space admits the following warped product structure:

$$\mathbb{S}^n_1 \approx (\mathbb{R} \times \mathbb{S}^{n-1}, -dt^2 + \cosh^2 t \, g_{\mathbb{S}^{n-1}})$$

Each t-slice, $S_t := \{t\} \times \mathbb{S}^{n-1} \subset \mathbb{S}^n_1$, has constant mean curvature:

$$H_t := H(S_t) = n\,\frac{(\cosh t)'}{\cosh t} = n \tanh t$$

Hence, the 'waistline', $S_0 = \{t = 0\}$, is maximal, and the mean curvature of the slice S_t increases to n (resp. decreases to $-n$), as t approaches ∞ (resp. $-\infty$). In fact, a brief computation shows,

$$H_t = n + O(e^{-2t}) \tag{4.4.6}$$

Theorem 4.4.3 below will show, in effect, that if the mean curvature converges to n any faster, there will be past singularities.

We begin the following lemma:

Lemma 4.4.2. *Let M^{n+1} be a future timelike geodesically complete spacetime satisfying* $\mathrm{Ric}(X,X) \geq -n$ *for all timelike unit vectors X. Suppose S is a compact Cauchy surface for M such that each future Cauchy sphere $S_k^+(S)$ has support mean curvature $\geq a_k$, where, letting $n_k := \min\{n, a_k\}$, we have*

$$n_k = n + o(e^{-2k}) \tag{4.4.7}$$

Then the Cauchy horosphere $S_\infty^-(S)$ has limit mean curvature $\geq n$.

Proof. Let $S_k := S_k^+(S)$ and recall that the sequence of Cauchy prehorospheres is defined by $\widetilde{S}_k := S_k^-(S_k^+(S))$. Fix any $\widetilde{x}_k \in \widetilde{S}_k$. By definition, $d(\widetilde{x}_k, S_k) = k$, and $\widetilde{x}_k$ is joined to some $x_k \in S_k$ by a past-directed S_k-maximal unit speed timelike geodesic segment $\alpha_k : [0,k] \to M$, with $\alpha_k(0) = x_k$ and $\alpha_k(k) = \widetilde{x}_k$.

Let Σ_k be a smooth past support hypersurface for S_k at x_k with mean curvature $H_{\Sigma_k}(x) \geq a_k - \frac{1}{2}e^{-3k}$. Perturbing Σ_k slightly to the past, keeping x_k fixed, as in Lemma 4.3.6, we obtain a smooth past support hypersurface $\widehat{\Sigma}_k$ for S_k at x_k with mean curvature $H_{\widehat{\Sigma}_k}(x) \geq a_k - e^{-3k} \geq n_k - e^{-3k}$, such that $\widehat{\Sigma}_k$ has no focal points along α_k, and hence, such that the past normal exponential map E from $\widehat{\Sigma}_k$ is smooth on $[0,k] \times \widetilde{\Sigma}_k$, for some neighborhood $\widetilde{\Sigma}_k$ of x_k in $\widehat{\Sigma}_k$. Letting $\theta_k(t)$ be the mean curvature of the slice $E(\{t\} \times \widetilde{\Sigma}_k)$ at $\alpha_k(t)$, then $\theta = \theta_k(t)$ satisfies the

Raychauduri inequality $\theta' \geq \mathrm{Ric}(\alpha_k', \alpha_k') + \theta^2/n$. By the curvature condition, since α_k is unit speed timelike, this gives $\theta' \geq \theta^2/n - n$, or, letting $\Theta := \theta/n$,

$$\Theta'(t) \geq \Theta^2(t) - 1, \quad \Theta(0) \geq \frac{n_k - e^{-3k}}{n}$$

Since $|(n_k - e^{-3k})/n| < 1$, the elementary comparison solution is $\tanh(b_k - t)$, with

$$b_k = \tanh^{-1}\left(\frac{n_k - e^{-3k}}{n}\right) = \frac{1}{2}\ln\left(\frac{n + n_k - e^{-3k}}{n - n_k + e^{-3k}}\right)$$

Thus we have,

$$\begin{aligned}
\theta_k(k) &\geq n\tanh(b_k - k) \\
&= n\,\frac{e^{2b_k} - e^{2k}}{e^{2b_k} + e^{2k}} \\
&= n\,\frac{(n + n_k - e^{-3k}) - (n - n_k + e^{-3k})e^{2k}}{(n + n_k - e^{-3k}) + (n - n_k + e^{-3k})e^{2k}} \\
&= n\,\frac{(n + n_k - e^{-3k}) - (n - n_k)e^{2k} - e^{-k}}{(n + n_k - e^{-3k}) + (n - n_k)e^{2k} + e^{-k}} \quad =: \widetilde{\theta}_k
\end{aligned}$$

Note that using the asymptotic assumption (4.4.7), we have $\lim_{k\to\infty}\widetilde{\theta}_k = n$. Because $\widehat{\Sigma}_k \subset J^-(S_k)$, it follows that the slice $E(\{k\} \times \widetilde{\Sigma}_k)$ is a smooth past support hypersurface for $\widetilde{S}_k$ at $\alpha_k(k) = \widetilde{x}_k$. Since $\widetilde{x}_k$ was arbitrary, we have shown that $\widetilde{S}_k$ has mean curvature $\geq \widetilde{\theta}_k$ in the support sense. Since $\widetilde{\theta}_k \to n$, the conclusion follows. $\square$

Theorem 4.4.3. *Let M^{n+1} be a future timelike geodesically complete spacetime satisfying* $\mathrm{Ric}(X,X) \geq -n$ *for all timelike unit vectors X. Suppose S is a compact Cauchy surface for M such that each future Cauchy sphere $S_k^+(S)$ has support mean curvature $\geq a_k$, where, letting $n_k := \min\{n, a_k\}$, we have $n_k = n + o(e^{-2k})$. Let $S_\infty^-(S)$ be the past Cauchy horosphere associated to S and suppose that $S_\infty^-(S)$ admits a past $S_\infty^-(S)$-ray γ. Then either*

(1) $S_\infty^-(S)$ admits a past incomplete timelike $S_\infty^-(S)$-ray, or

(2) $S_\infty^-(S)$ is a smooth, compact spacelike Cauchy surface with mean curvature $H = n$ and, letting h denote the induced metric on $S_\infty^-(S)$, M splits as:

$$(M, g) \approx (\mathbb{R} \times S_\infty^-(S), -dt^2 + e^{2t}h)$$

In either case, M is past timelike incomplete. In the latter case, M is also past null incomplete.

Proof. Since $S_\infty^-(S)$ is inherently future bounded by S, it is acausal and all $S_\infty^-(S)$-rays, future or past, are timelike. We will suppose every past S_∞^--ray is complete and show (2). By Lemma 4.4.2, $S_\infty^-(S)$ has limit mean curvature $\geq n$. Then, letting S^- be the connected component of $S_\infty^-(S)$ which contains $\gamma(0)$, Proposition 4.4.1 gives that S^- is a smooth, geodesically complete, spacelike past Cauchy surface, with mean curvature $H = n$, and:

$$(J^-(S^-), g) \approx ((-\infty, 0] \times S^-, -dt^2 + e^{2t}h)$$

Note that the future radial rays from S_∞^- are all timelike and future complete. Since S^- is smooth, there must only be one such ray from each point $p \in S^-$, and it must be the future normal geodesic from $p \in S^-$. Hence, the future normal exponential map E is a diffeomorphism onto the future image $N^+(S^-) =$

$E([0,\infty) \times S^-)$. The standard comparison argument via the Raychaudhuri equation gives $H_t \le n$ for the future normal slice $N_t := E(\{t\} \times S^-)$. But the usual argument does not give $H_t = n$. To get the splitting to the future, we will identify N_t with a portion of (what is essentially) the Cauchy horosphere associated to the Cauchy surface $S_t := S_t^+(S)$. Like S_∞^-, this horosphere will inherit limit mean curvature $\ge n$ from the sequence $\{S_k^+(S)\}$, and we can run our arguments again to get $H = n$ for this horosphere, (locally), and hence $H_t = n$ for the slice N_t.

As in (the time dual of) Lemma 3.3.4, we have $S_k^+(S) = S_{k-t}^+(S_t^+(S)) = S_{k-t}^+(S_t)$, and hence, $S_{k-t}^-(S_k^+(S)) = S_{k-t}^-(S_{k-t}^+(S_t))$. The same monotonicity argument for the usual Cauchy prehorospheres shows that the sequence $\{J^-(S_{k-t}^-(S_k^+(S)))\} = \{J^-(S_{k-t}^-(S_{k-t}^+(S_t)))\}$ is decreasing. Letting $\widetilde{S}_{k-t} := S_{k-t}^-(S_k^+(S))$, consider the horosphere

$$S_{\infty-t}^- := \partial\Big(\bigcap_k J^-(\widetilde{S}_{k-t})\Big)$$

Recall that the (usual) Cauchy prehorospheres, $\widetilde{S}_k$, are constructed as follows: shoot S k units to the future to $S_k^+(S)$, then pull this back k units to $\widetilde{S}_k = S_k^-(S_k^+(S))$. The new prehorospheres $\widetilde{S}_{k-t} = S_{k-t}^-(S_k^+(S))$ take S, shoot it k units to the future, but then pull back only $k - t$ units. Hence, roughly speaking, $S_{\infty-t}^-$ is constructed by shooting S '∞ units' to the future, and then pulling it back by only '$\infty - t$ units'. Alternatively, since $\widetilde{S}_{k-t} = S_{k-t}^-(S_k^+(S)) = S_{k-t}^-(S_{k-t}^+(S_t))$, we can also view this (essentially) as the Cauchy horosphere associated to the Cauchy surface $S_t = S_t^+(S)$.

In any case, we want to show $N_t \subset S_{\infty-t}^-$. We first note that, as with the usual prehorospheres, $\widetilde{S}_{k-t} = S_{k-t}^-(S_{k-t}^+(S_t))$ is future bounded by S_t. Let $x_\infty \in S^- \subset S_\infty^-$ and fix a sequence $x_k \in \widetilde{S}_k$ with $x_k \to x_\infty$. Since $x_k \in \widetilde{S}_k = S_k^-(S_k^+(S))$, there is a future maximal unit speed timelike geodesic segment, $\sigma_k : [0,k] \to M$, joining

$\sigma_k(0) = x_k$ to $\sigma_k(k) \in S_k^+(S)$. Then $x_{k-t} := \sigma_k(t) \in S_{k-t}^-(S_k^+(S)) = \widetilde{S}_{k-t}$. Letting $x_{-1} \in I^-(x_\infty)$, we have $x_{k-t} \in J^+(x_{-1}) \cap J^-(S_t)$, for large k. Hence, the sequence $\{x_{k-t}\}$ has a limit point, $x_{\infty-t}$, which must be contained in $S_{\infty-t}^-$, by Proposition 2.2.5. Since $S_{\infty-t}^-$ is future bounded, it admits a timelike future $S_{\infty-t}^-$-ray η from $x_{\infty-t}$. Since $t = d(\sigma_k(0), \sigma_k(t)) = d(x_k, x_{k-t}) \to d(x_\infty, x_{\infty-t})$, there is a maximal geodesic segment $\beta : [0, t] \to M$ from x_∞ to $x_{\infty-t}$. Finally, since $d(\widetilde{S}_k, \widetilde{S}_{k-t}) = t$, we have $d(S_\infty^-, S_{\infty-t}^-) = t$. It follows that the concatenation $\sigma = \beta + \eta$ is an S_∞^--ray from $x_\infty \in S^-$. Since σ is also an S^--ray, paramterizing σ as a unit speed geodesic, we have $\sigma(t) = x_{\infty-t}$. This shows $N_t \subset S_{\infty-t}^-$.

Replacing e^{2k} by $e^{2(k-t)} = e^{2k-2t}$ in the calculation in Lemma 4.4.2, that is, sliding the past support hypersurface for $S_k^+(S)$ down for a time $k - t$ instead of k, shows that $S_{\infty-t}^-$ has limit mean curvature $\geq n$. Recall that $N_t \subset S_{\infty-t}^-$ has (smooth) mean curvature $H_t \leq n$. Hence, working locally, and viewing N_t as the (constant) achronal limit of itself, Lemma 4.3.9 gives N_t has constant mean curvature $H_t = n$. Since $t > 0$ was arbitrary, all future normal slices have constant mean curvature $H = n$. Plugging this back into the Raychaudhuri equation, the characterization of the equality case gives that each slice N_t is totally umbillic with $B_t = h$, where h is the induced metric on N_t. Then, as in Lemma 4.7.1, this gives $N^+(S^-) \approx ([0, \infty) \times S^-, -dt^2 + e^{2t}h)$. As in Remark 3.71 of [5], and the related discussion, which cites also [26], this warped product structure means that $N^+(S^-)$ is future null and timelike geodesically complete. Hence, any future causal geodesic starting from S^- can never leave $N^+(S^-)$. Since any $y \in J^+(S^-)$ is joined to some $s \in S^-$ by a future causal geodesic segment from $s \in S^-$, we have $y \in N^+(S^-)$. Hence, $J^+(S^-) = N^+(S^-)$, and $J(S^-) = N(S^-)$, with

$$(J(S^-), g) \approx ((-\infty, \infty) \times S^-, -dt^2 + e^{2t}h)$$

In particular, $H^+(S^-) \subset J^+(S^-) \subset N^+(S^-)$, but by Theorem 3.69 in [5], S^- is a Cauchy surface for $N(S^-) = J(S^-)$. Hence, $H^+(S^-) = \emptyset$. Recalling that also $H^-(S^-) = \emptyset$, we have that S^- is a Cauchy surface for M. By achronality, this means $S^-_\infty = S^-$, which gives the conclusion. $\square$

Appendix

4.5 One-Sided Hessian Bounds

The following is the second half of Definition 3.3 in [2]:

Definition 4.5.1 (Support Mean Curvature with One-Sided Hessian Bounds)**.** Let S be a C^0 spacelike hypersurface and $a \in \mathbb{R}$. We say S has *support mean curvature* $\geq a$ *with one-sided Hessian bounds* if, fixing any compact subset $K \subset S$, there is a compact set $\widehat{K} \subset TM$ and a constant $C_K > 0$ such that for all $q \in K$ and all $\epsilon > 0$, there is a C^2 past support hypersurface $S_{q,\epsilon}$ for S at q such that

i) The future unit normal field, $\eta_{q,\epsilon}$, of $S_{q,\epsilon}$ satisfies: $\eta_{q,\epsilon}(q) \in \widehat{K}$

ii) The mean curvature, $H_{q,\epsilon}$, of $S_{q,\epsilon}$ satisfies: $H_{q,\epsilon}(q) \geq a - \epsilon$

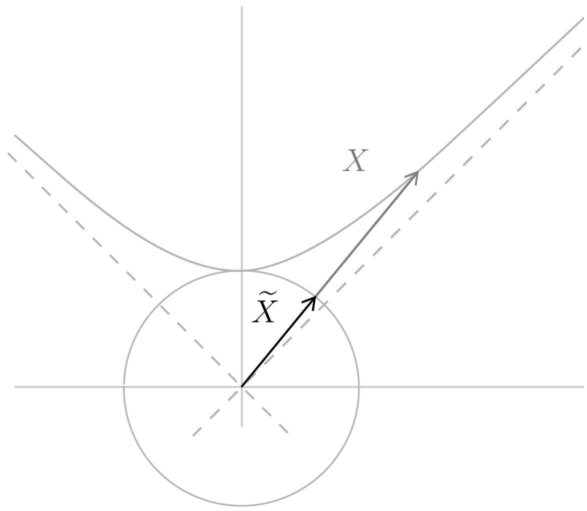
iii) The second fundamental form, $B_{q,\epsilon}$, of $S_{q,\epsilon}$ satisfies: $B_{q,\epsilon}(q) \geq -C_K$

As discussed in [2], when the support surfaces $S_{q,\epsilon}$ are smooth point past spheres, one-sided Hessian bounds boil down to the support normals being 'locally compact', or equivalently, that no sequence of such normals 'tips over'. We make this precise in the next subsection.

4.5.1 Tipping Over

Note that while it is impossible for a sequence of timelike unit vectors to have a null limit vector, such a sequence may 'become null' in the sense that the sequence of directions may approach a null direction. To make this precise, fix a Riemannian metric h on M. For each $X \in TM$, we define the *direction* of X, (with respect to h), by

$$\widetilde{X} := \frac{X}{||X||_h}$$



Definition 4.5.2. We will say a sequence $\{X_k\}$ of timelike unit vectors *tips over at* $p \in M$ if $\pi(X_k) = p_k \to p$ and $g(\widetilde{X}_k, \widetilde{X}_k) \to 0$, where $\pi : TM \to M$ is the standard projection.

We say a set of vectors $\mathcal{Z} \subset TM$ is *locally compact* if, over any compact $K \subset M$, the subset $\mathcal{Z}_K := \mathcal{Z} \cap \pi^{-1}(K)$ is compact.

Lemma 4.5.3. *Let $\mathcal{N}$ be a set of future timelike unit vectors. If $\mathcal{N}$ is locally compact, then no sequence in $\mathcal{N}$ tips over. If $\mathcal{N}$ is closed, the converse holds.*

Proof. First suppose that $\mathcal{N}$ is locally compact. Suppose otherwise that there is a sequence $X_k \in \mathcal{N}$ which tips over at some $p \in M$. Let U be a neighborhood of p with $K = \overline{U}$ compact. Then $\mathcal{N}_K$ contains the tail of $\{X_k\}$ and is compact by assumption. Hence, $\{X_k\}$ has a limit point (vector) $X_\infty \in \mathcal{N}_K$. Hence, for some subsequence $\{X_{k_j}\}$, we have $X_{k_j} \to X_\infty$ and hence also $||X_{k_j}||_h \to ||X_\infty||_h$. Furthermore, since $X_k \neq 0 \neq X_\infty$, we have

$$\widetilde{X}_{k_j} = \frac{X_{k_j}}{||X_{k_j}||_h} \longrightarrow \frac{X_\infty}{||X_\infty||_h} = \widetilde{X}_\infty$$

Then, $0 = \lim_{j\to\infty} g(\widetilde{X}_{k_j}, \widetilde{X}_{k_j}) = g(\widetilde{X}_\infty, \widetilde{X}_\infty)$, which means X_∞ is null, contradicting $X_\infty \in \mathcal{N}_K \subset \mathcal{N}$. Hence, no sequence in $\mathcal{N}$ tips over.

Now, suppose $\mathcal{N}$ is closed and contains no sequence which tips over. Suppose that $\mathcal{N}$ is not locally compact. Hence, for some compact $K \subset M$, the subset $\mathcal{N}_K$ contains a sequence $\{X_k\}$ which has no limit point in $\mathcal{N}_K$. The direction sequence $\{\widetilde{X}_k\}$ is contained in the h-unit bundle over K, and hence must have a limit point, $\widetilde{X}_\infty$, which must be future pointing causal. If $\widetilde{X}_\infty$ is timelike, then $X_\infty := ||\widetilde{X}_\infty||_g^{-1}\ \widetilde{X}_\infty$ is timelike unit and, fixing a subsequence $\widetilde{X}_{k_j}$ with $\lim_{j\to\infty} \widetilde{X}_{k_j} = \widetilde{X}_\infty$, we have:

$$X_{k_j} = \frac{\widetilde{X}_{k_j}}{||\widetilde{X}_{k_j}||_g} \longrightarrow \frac{\widetilde{X}_\infty}{||\widetilde{X}_\infty||_g} = X_\infty$$

Hence, X_∞ is a limit point of $\{X_k\}$. But since $\mathcal{N}$ is closed by assumption, so is $\mathcal{N}_K = \mathcal{N} \cap \pi^{-1}(K)$. Hence, $X_\infty \in \mathcal{N}_K$, a contradiction. Suppose then that $\widetilde{X}_\infty$ is null. But then, $0 = g(\widetilde{X}_\infty, \widetilde{X}_\infty)$ must be a limit point of $\{g(\widetilde{X}_k, \widetilde{X}_k)\}$, which means a subsequence of $\{X_k\}$ tips over at some $p \in K$. Hence, $\mathcal{N}$ must be locally compact after all. $\square$

4.5.2 Horospheres

We will need the following fact, extracted and specialized from [18]:

Lemma 4.5.4 (Timelike Maximal Limit Curves vs Geodesic Limits)**.** *Let $\gamma_k : [0,\infty) \to M$ be a sequence of future complete, unit speed timelike geodesic rays. Let h be a complete Riemannian metric on M and let $\alpha_k : [0,\infty) \to M$ be the reparameterization of γ_k with respect to h arc length. Suppose that $\alpha : [0,\infty) \to M$ is a timelike limit curve of $\{\alpha_k\}$, and without loss of generality, suppose that (the full sequence), $\{\alpha_k\}$ converges locally uniformly to α. Then, α is maximal and hence may be reparameterized as a unit speed timelike geodesic ray, $\gamma : [0,T) \to M$, with $T \in (0,\infty]$, which then satisfies $\gamma_k'(0) \to \gamma'(0)$.*

Lemma 4.5.5 (Horosphere Support Normals)**.** *Let M be a globally hyperbolic spacetime and suppose that S_∞^- is a past horosphere such that all future S_∞^--rays are timelike and future complete. Let $\mathcal{N}$ be the set of the initial tangent vectors of all future S_∞^--rays, parameterized as unit speed geodesics. Then $\mathcal{N}$ is closed and no sequence in $\mathcal{N}$ tips over. Hence, $\mathcal{N}$ is locally compact.*

Proof. Note that $\mathcal{N}$ is a set of timelike unit vectors. Let $\{\gamma_k'(0)\}$ be a sequence from $\mathcal{N}$ with $\lim_{k\to\infty} \gamma_k'(0) = X_\infty \in TM$. Fixing a complete Riemannian metric h on M, let α_k be the reparameterization of γ_k with respect to h arc length and let α_∞ be a limit curve of $\{\alpha_k\}$. Since each α_k is a future S_∞^--ray, so is α_∞, by the Maximal Limit Curve Lemma 3.1.8, (applied using the constant sequence $S_k = S_\infty^-$). Hence, α_∞ is timelike. Then, reparameterizing α_∞ as a geodesic, γ_∞, by Lemma 4.5.4, $\gamma_\infty'(0)$ is a limit point of $\{\gamma_k'(0)\}$. Hence, $X_\infty = \gamma_\infty'(0) \in \mathcal{N}$. This shows that $\mathcal{N}$ is closed. A similar argument will show that no sequence in $\mathcal{N}$ can tip over. Suppose otherwise, that there is a sequence $\{\gamma_k'(0)\} \in \mathcal{N}$ which tips over at $p \in S_\infty^-$. Then, as above, letting $\alpha_\infty \approx \gamma_\infty$ be a limit curve of $\{\alpha_k\} \approx \{\gamma_k\}$, $\gamma'_\infty(0)$ is a limit point of $\{\gamma'_k(0)\}$. But then $\gamma'_\infty(0)/||\gamma'_\infty(0)||_h$ is a timelike limit point of $\{\gamma'_k(0)/||\gamma'_k(0)||_h\}$, hence $\{\gamma'_k(0)\}$, in fact, does not tip over. By Lemma 4.5.3, $\mathcal{N}$ is locally compact. $\square$

4.6 The Raychaudhuri Equation

The following version of the Raychaudhuri equation is convenient for most of our applications:

Lemma 4.6.1 (Raychaudhuri to the Future)**.** *Suppose that $S \subset M^{n+1}$ is a smooth spacelike hypersurface with future unit normal field N, and that for some $a, b > 0$, the normal exponential map E is a diffeomorphism from $(-a, b) \times S$ onto a (normal) neighborhood of S. For each $p \in S$, let $\gamma_p(t) := \exp_p(tN) = E(t,p)$ denote the t-curve through p. For each $t \in (-a,b)$, denote the corresponding t-slice by $N_t := E(\{t\} \times S)$. Let B and H denote the second fundamental form and mean curvature, respectively, of the $\{t\}$-slices, with respect to the future unit normal, $\partial_t := E_*(\partial_t)$. Fix any $p \in S$ and let $\theta(t) = H(\gamma_p(t))$ be the mean curvature of the slice N_t at $\gamma_p(t)$. Then we have:*

$$\theta'(t) + \frac{\theta^2(t)}{n} + \mathrm{Ric}(\partial_t, \partial_t)_{\gamma_p(t)} \ \leq \ \theta'(t) + |B|^2_{\gamma_p(t)} + \mathrm{Ric}(\partial_t, \partial_t)_{\gamma_p(t)} \ = \ 0 \qquad (4.6.8)$$

with equality at t iff, at the point $\gamma_p(t)$, B is a multiple of the induced metric h_t of the slice N_t.

Remark 4.6.2 (Raychaudhuri to the Past). In fact, we mostly use the Raychaudhuri equation to the past. Using the reverse setup, (i.e., $N \to -N$), so that $\gamma_p(t)$ is now past-directed, and the future unit normal field of the slices is $-\partial_t$, we have:

$$\theta' - \frac{\theta^2}{n} - \mathrm{Ric}(\partial_t, \partial_t) \;\geq\; \theta' - |B|^2 - \mathrm{Ric}(\partial_t, \partial_t) \;=\; 0, \tag{4.6.9}$$

with the same characterization of the equality case.

4.7 Normal Half-Splitting Lemma

Lemma 4.7.1 (Normal Half-Splitting)**.** *Let M^{n+1} be a globally hyperbolic spacetime and suppose $S \subset M$ is a smooth spacelike hypersurface such that all future normal geodesics from S are future complete S-rays. Then, letting h denote the induced metric on S, we have the following:*

0) If M satisfies $\mathrm{Ric}(X,X) \geq 0$ for all timelike vectors X, and S has mean curvature $H = 0$, then the normal future $(N^+(S), g)$ is isometric (via the normal exponential map) to

$$([0,\infty) \times S, -dt^2 + h)$$

λ) If M satisfies $\mathrm{Ric}(X,X) \geq -\lambda^2 n$ for all timelike unit vectors X, with $\lambda > 0$, and S has mean curvature $H = -\lambda n$, then the normal future $(N^+(S), g)$ is isometric (via the normal exponential map) to

$$([0,\infty) \times S, -dt^2 + e^{-2t\lambda} h)$$

Proof. The proof is fairly standard. Because the future normal geodesics to S are future complete S-rays, the future normal exponential map gives a diffeomorphism, $N^+(S) \approx [0,\infty) \times S$. Let $N_t := E(\{t\} \times S)$ be the future normal t-slice. By the Gauss lemma, we have

$$(N^+(S), g) \approx ([0,\infty) \times S, -dt^2 + h_t)$$

where, for each $t \geq 0$, h_t is a metric on $N_t \approx S$. In particular, ∂_t is a future unit normal for N_t. Fix $p \in S$ and let $\gamma_p(t) = \exp_p(tN)$ be the future-directed normal geodesic through p. Then, as in Lemma 4.6.8, we have:

$$\theta'(t) + \frac{\theta^2(t)}{n} + \mathrm{Ric}(\partial_t, \partial_t) \;\leq\; \theta'(t) + |B|^2 + \mathrm{Ric}(\partial_t, \partial_t) \;=\; 0,$$

where B and θ are the second fundamental form and mean curvature, respectively, of the slice N_t at $\gamma_p(t)$.

0) First suppose $\mathrm{Ric}(X, X) \geq 0$ for all timelike X. Then, $\theta' \leq -\theta^2/n$. Since $\theta(0) = 0$, this implies that $\theta(t) \leq 0$ for all $t \geq 0$. Suppose that for some $t_0 > 0$, we have $\theta(t_0) < 0$. Then we have $\theta(t) < 0$, for all $t > t_0$. In particular, for such t, we have $\theta(t) \neq 0$, and dividing by $-\theta^2$ and integrating, we get $\theta^{-1}(t) \geq (t + C_0)/n$, or $\theta(t) \leq \dfrac{n}{t + C_0}$, for some constant C_0 and all $t > t_0$. Since $\theta(t_0) < 0$, we must have $C_0 < -t_0$. But this implies $\lim_{t \to -C_0^+} \theta(t) = -\infty$, contradicting the fact that N_{-C_0} is smooth. Hence we must have $\theta(t) = 0$ for all $t \geq 0$. Since γ_p was arbitrary, this shows that each slice N_t is maximal. Plugging this back into the Raychaudhuri equation, we get that the second fundamental form of N_t vanishes. Let $X = \partial_x$ and $Y = \partial_y$ be coordinate vector fields on S. Then $\partial_t\, h_t(X, Y) = h_t(\nabla_t X, Y) + h_t(X, \nabla_t Y) = h_t(\nabla_X \partial_t, Y) + h_t([\partial_t, X], Y) + h_t(X, \nabla_Y \partial_t) + h_t(X, [\partial_t, Y]) = 0$, where the bracket terms vanish because ∂_t, X, and Y are all coordinate vector fields, and the remaining terms vanish because $B = 0$. Hence $h_t(X, Y) = h_0(X, Y) = h(X, Y)$, which gives the splitting.

λ) Now suppose that $\mathrm{Ric}(X, X) \geq -\lambda^2 n$, for all timelike unit vectors X, with $\lambda > 0$. Thus, dividing by $\lambda^2 n$ and setting $\Theta(t) := \frac{\theta(t)}{\lambda n}$, we have $\Theta(0) = -1$ and the Raychaudhuri equation gives: $\Theta'(t) \leq 1 - \Theta^2(t)$. Then, by Riccati comparison theory, (cf. [10]), we get $\Theta(t) \leq -1$ for all $t \geq 0$. Suppose that $\Theta(t_0) = -(1+\delta)$ for some $t_0, \delta > 0$. Then again by Riccati comparison, we get $\Theta(t) \leq \coth(t + C_0)$ for all $t > t_0$, where $C_0 = \coth^{-1}(\Theta(t_0)) < 0$, which leads to a blow up at $t = -C_0 > 0$, contradicting the smoothness of Θ on $[0, \infty)$. Hence, for all $t \geq 0$, we have $\Theta(t) = -1$, i.e., $\theta(t) = -\lambda n$. Plugging this back into the Raychaudhuri equation, we get that $\lambda^2 n \leq |B|^2 \leq \lambda^2 n$. By the characterization of the equality case, (c.f. Lemma 4.6.8), follows that $B = -\lambda h_t$. Again, letting $X = \partial_x$ and $Y = \partial_y$ be coordinate vector fields on S, we have $\partial_t\, h_t(X,Y) = h_t(\nabla_t X, Y) + h_t(X, \nabla_t Y) = h_t(\nabla_X \partial_t, Y) + h_t([\partial_t, X], Y) + h_t(X, \nabla_Y \partial_t) + h_t(X, [\partial_t, Y]) = -2\lambda h_t(X,Y)$. Hence $h_t(X,Y) = e^{-2\lambda t} h_0(X,Y) = e^{-2\lambda t} h(X,Y)$, which gives the splitting. $\square$

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