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The Index Bundle for a Family of Dirac-Ramond Operators

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UNIVERSITY OF MIAMI

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String theoretic considerations imply the existence of a Dirac-like operator, known as the Dirac-Ramond operator, on the free loop space of a closed string manifold. We study the index bundle of the Dirac-Ramond operator associated with a family $\pi : Z \to X$ of closed spin manifolds. We work instead with a formal version of the operator, the twisted Dirac operator $\partial\!\!\!/ \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}}$. Its index bundle is an element of $K(X)[[q]]$. In the case where the total space Z is a string manifold, we show that the Chern character of this index bundle has certain modular properties. We then use the modularity to derive some explicit formulas for the Chern character of this index bundle. We also show that these formulas identify the index bundle with an $L(E_8)$ bundle in a special case.

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Chapter 1

Introduction

In the 1980's several genera valued in the ring of modular forms were introduced. The elliptic genera originated in work of Ochanine [22] and Landweber and Stong [19], and were soon after given an interpretation through physics. By extending the path integral proof of the Atiyah-Singer index theorem to a certain supersymmetric nonlinear sigma model, it was shown by Alvarez, Killingback, Mangano, and Windey in [1] and [2] that the elliptic genera could be viewed as the equivariant index of a certain twisted Dirac like operator on the free loop space. They also showed that the index of the untwisted version of this operator, known as the Dirac-Ramond operator, could be computed; this produced another genus called the string genus. The string genus is also known as the Witten genus because independently around the same time Witten greatly further elucidated the relationships between quantum field theory, genera, and index theory in [28] and [29]. More recently, Alvarez and Windey have shown that their earlier work can be extended to the case of families of Dirac-Ramond operators. The index theorem proved in [3] for the Dirac-Ramond operator is the elliptic analogue of the original cohomological version of the Atiyah-Singer index theorem for a family of Dirac operators [6].

No one has given a general mathematical construction for the Dirac-Ramond operator on the full loop space, though there are some partial results (see e.g. [25] for the case when the manifold is flat). Each manifold can be embedded in its loop space via constant maps, and the Dirac-Ramond operator can be defined rigorously on the normal bundle given by this embedding as in [24] or [26]. It is well known (see [15], for instance) that the index of this operator is given by a certain formal sum of twisted $\widehat{A}$-genera. From this perspective, the index of the Dirac-Ramond operator can also be obtained by considering the operator as a formal sum of twisted Dirac operators, or equivalently as the usual Dirac operator twisted by a formal sum of bundles. This is the viewpoint we will take below in the family case, defining the index bundle of the Dirac-Ramond operator to be the formal sum of index bundles from the appropriate twisted Dirac operators. Note that such an object has been considered by Liu and Ma in [21] and subsequent work where they achieved considerable rigidity results.

The outline of this thesis is as follows. In Chapter 2 we discuss the necessary ingredients for constructing a family of Dirac operators and the index bundle of a family. We start by defining Clifford algebras, and spin groups, and studying their representation theory. This gives rise to spinor bundles on Riemannian manifolds with a spin structure, and the Dirac operator. The index theory for a single operator and a family of operators is then discussed. In Chapter 3 we introduce the Dirac-Ramond operator as a formal sum of twisted Dirac operators. After defining a family of these operators as the formal sum of families of the appropriate twisted Dirac operators, we obtain the index bundle as a certain formal power series with (virtual) vector bundle coefficients. We then show derive a formula for the Chern character of this bundle and show that it agrees with that obtained by Alvarez and Windey. In Chapter 4, we review some of the theory of modular forms. We then investigate the modular properties of the index bundle obtained from a family of Dirac-Ramond operators. In the case of certain families of string manifolds, we show that the Chern character of the family index takes values in cohomology with coefficients in the ring of (quasi)modular forms. This was also done in [3], but here we obtain the result by a new method. We show by way of an example how one can use modularity to generate relations between the index bundles associated to the various operators used in defining the Dirac-Ramond operator. These sorts of relations are similar to the "anomaly cancellation formulas" which arise in physics. Some results of the same type, but on the level of differential forms, were derived using elliptic genera in [14]. We make heavy use of the theory of Jacobi-like forms in order to derive an explicit formula describing the Chern character of the index bundle for the Dirac-Ramond operator in terms of the components of the Chern character for some twisted Dirac operators and Eisenstein series. In Chapter 5, we apply the above formalism in the case where the manifold has dimension 8 and the parameterizing space has dimension less than 16. We then use the formulas from Section 3 to show that under certain conditions the index of the family of Dirac-Ramond operators is equal in $(K(X) \otimes \mathbb{Q})[[q]]$ to a vector bundle associated with the basic representation of the loop group for E_8.

Chapter 2

The Dirac Operator

In the following, by manifold we will always mean a smooth connected manifold without boundary.

2.1 The Spin Group

Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional vector space over a field $\mathbb{K}$ ($\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) with a symmetric bilinear form.

Definition 2.1. *Let $T(V) = \oplus_{n=0}^{\infty} V^{\otimes k} = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus ...$ denote the tensor algebra of V and $I(V) \subset T(V)$ be the two-sided ideal generated by all elements of the form $\langle v, v\rangle \cdot 1 + v \otimes v$. The quotient $\mathrm{Cl}(V, \langle\cdot,\cdot\rangle) := T(V)/I(V)$ is called the Clifford algebra of V with respect to $\langle\cdot,\cdot\rangle$.*

We will often suppress the mention of the bilinear form, writing $\mathrm{Cl}(V)$ instead of $\mathrm{Cl}(V, \langle\cdot,\cdot\rangle)$, when it is clear from context which bilinear form is being used. One can see that $\mathbb{K}$ and V are canonically identified with subspaces of $\mathrm{Cl}(V)$. Moreover, $\mathrm{Cl}(V)$ is a $\mathbb{K}$-algebra under the multiplication induced by $\otimes$, which we will often omit. The Clifford algebra has the following universal property.

Proposition 2.2. *Let V be a vector space over $\mathbb{K}$ with a symmetric bilinear form $\langle\cdot,\cdot\rangle$ and let $i : V \to \mathrm{Cl}(V)$ be the inclusion map. If A is any unital $\mathbb{K}$-algebra and $\phi : V \to A$ satisfies $\phi(v)^2 = -\langle v, v\rangle$ for all $v \in V$, then there is a unique algebra homomorphism $\tilde{\phi} : \mathrm{Cl}(V) \to A$ such that $\phi = \tilde{\phi} \circ i$.*

As a vector space, $\mathrm{Cl}(V)$ is isomorphic to $\Lambda(V)$ via the map $\mathrm{Cl}(V) \to \Lambda(V)$ defined on generators by

$$e_{i_1}...e_{i_k} \mapsto e_{i_1} \wedge ... \wedge e_{i_k}.$$

where $i_1 < ... < i_k$ and $0 \leq k \leq n$. Thus the space $\mathrm{Cl}(V)$ has dimension 2^n. Note that the map is not an isomorphism of algebras unless $\langle\cdot,\cdot\rangle = 0$.

We will restrict from now on to the case when V is a real vector space of dimension n and $\langle\cdot,\cdot\rangle$ is an inner product. An orthonormal basis $e_1, ..., e_n$ for V then satisfies $e_i^2 = -1$ and $e_ie_j = -e_je_i$ for $i \neq j$. We will also be interested in $\mathrm{Cl}_{\mathbb{C}}(V) := \mathrm{Cl}(V \otimes \mathbb{C}, \langle\cdot,\cdot\rangle_{\mathbb{C}})$, where $\langle\cdot,\cdot\rangle_{\mathbb{C}}$ is the extension of $\langle\cdot,\cdot\rangle$ by complex bilinearity. It is straightforward to check that $\mathrm{Cl}_{\mathbb{C}}(V) \simeq \mathrm{Cl}(V) \otimes \mathbb{C}$. There is the following fact which can be found, for instance, in [20] or [23]:

Proposition 2.3. *Let V be a real vector space of dimension $2r$. Then $\mathrm{Cl}_{\mathbb{C}}(V)$ has just one irreducible representation of (complex) dimension 2^r.*

Proof. We will only show existence of the representation. In order to do so we will use the construction of what is known in physics as a fermionic Fock space. Let $J : V \to V$ be a complex structure on V that is compatible with $\langle\cdot,\cdot\rangle$. That is, J satisfies $J^2 = -\mathrm{Id}$ and $\langle Jx, Jy\rangle = \langle x, y\rangle$ for all $x, y \in V$. Then $V \otimes \mathbb{C} = V_i \oplus V_{-i}$, where $V_{\pm i}$ is the $\pm i$ eigenspace of $J \otimes 1$. Note that $V_{\pm i}$ are isotropic subspaces since $\langle x, y\rangle = \langle Jx, Jy\rangle = -\langle x, y\rangle$ for all $x, y \in V_{\pm i}$. Because of this, the Clifford subalgebra $\Delta := \mathrm{Cl}(V_i) \subset \mathrm{Cl}_{\mathbb{C}}(V)$ is isomorphic to $\Lambda(V_i)$. We will define the representation $\rho : \mathrm{Cl}_{\mathbb{C}}(V) \to \mathrm{End}_{\mathbb{C}}(\Delta)$ by defining it first on V and then showing that it satisfies the defining Clifford relation. It will follow then by Proposition 2.2 that ρ extends to a representation of the full Clifford algebra. For $u \in V_i$, w of V_{-i}, and $\psi \in \Delta$ define

$$\rho(u)\psi = \sqrt{2}u \wedge \psi$$
$$\rho(w)\psi = -\sqrt{2}\iota_w \psi.$$

Here ι_w is the interior product, an antiderivation which is characterized by

$$\iota_w(u \wedge v) = \iota_w(u) \wedge v + (-1)^{\deg(u)} u \wedge \iota_w(v) \quad \forall u, v \in \Delta \tag{2.1a}$$

and

$$\iota_w(u) = \langle w, u \rangle_{\mathbb{C}} \quad \forall u \in V_i \subset \Delta. \tag{2.1b}$$

Thus for an arbitrary element $v = u + w \in V \otimes \mathbb{C}$ with $u \in V_i$ and $w \in V_{-i}$ we have

$$\rho(v)\psi = \sqrt{2}(u \wedge \psi - \iota_w \psi)$$

for all $\psi \in \Delta$. The operators $u\wedge$ and ι_w are known as creation and annihilation operators, respectively, and they satisfy the following relations:

$$(u\wedge)^2 = 0 \tag{2.2a}$$
$$(\iota_w)^2 = 0 \tag{2.2b}$$
$$\iota_w u \wedge + u \wedge \iota_w = \langle w, u \rangle_{\mathbb{C}}. \tag{2.2c}$$

The first identity is obvious and the last two follow from (2.1a) and (2.1b). For instance,

$$\begin{aligned}(\iota_w u \wedge + u \wedge \iota_w)\psi &= \iota_w(u) \wedge \psi - u \wedge \iota_w(\psi) + u \wedge \iota_w(\psi) \\ &= \langle w, u \rangle_{\mathbb{C}} \psi\end{aligned}$$

for all $u \in V_i$, $w \in V_{-i}$, and $\psi \in \Delta$. Hence

$$\rho(v)^2\psi = (\sqrt{2}u \wedge -\sqrt{2}\iota_w)^2\psi = -2\langle w, u\rangle_{\mathbb{C}}\psi = -\langle v, v\rangle_{\mathbb{C}}\psi$$

so that ρ extends to a representation of $\mathrm{Cl}_{\mathbb{C}}(V)$ on $\Delta \simeq \Lambda(V_i)$. From the construction it is clear the the representation is irreducible. $\square$

The algebra $\mathrm{Cl}(V)$ has a natural $\mathbb{Z}/2\mathbb{Z}$ grading $\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V)$ where the even (resp. odd) part is the span of all elements $e_{i_1}...e_{i_s}$ where s is even (resp. odd). Notice that for any $0 \neq v \in \mathrm{Cl}(V)$, $v \cdot (-v/||v||^2) = 1$. We can now make the following definition.

Definition 2.4. *Let $\langle\cdot,\cdot\rangle$ denote the usual Euclidean inner product on $\mathbb{R}^n$. The spin group* $\mathrm{Spin}(n)$ *is the multiplicative group generated by all $v \in \mathrm{Cl}^0(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ such that $||v||^2 = 1$.*

For any $x \in \mathbb{R}^n$ with $||x||^2 = 1$ and any $v \in \mathbb{R}^n$ we have in $\mathrm{Cl}(\mathbb{R}^n)$

$$xv(-x^{-1}) = xvx = v - 2x\langle x, v\rangle \in \mathbb{R}^n. \tag{2.3}$$

Note that this has a geometric interpretation as a reflection of v in a hyperplane orthogonal to x. Since all elements of $\mathrm{Spin}(n)$ are products of an even number of unit vectors in $\mathbb{R}^n$ we have the Lie group homomorphism

$$\tau : \mathrm{Spin}(n) \to \mathrm{SO}(n)$$

which is obtained by iterating (2.3) to become

$$\tau(x_1...x_{2k})(v) := x_1...x_{2k}vx_{2k}...x_1 = x_1...x_{2k}v(x_1...x_{2k})^{-1}. \tag{2.4}$$

Proposition 2.5. *The map* $\tau : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ *is a two-sheeted covering map. Furthermore,* $\mathrm{Spin}(n)$ *is the universal cover of* $\mathrm{SO}(n)$ *for* $n \geq 3$.

Since τ is a covering map, the induced map $\tau_* : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ must be a Lie algebra isomorphism. We will describe this map more explicitly.

The usual commutator bracket $[a, b] = ab - ba$ on the Clifford Algebra $\mathrm{Cl}(\mathbb{R}^n)$ gives it the structure of a Lie algebra. Let E_{ij} denote the matrix with 1 in the (i, j) place and 0 elsewhere. The matrices $E_{ij} - E_{ji}$ form a basis for the space of skew adjoint matrices $\mathfrak{so}(n)$. One can check that the map

$$E_{ij} - E_{ji} \mapsto \frac{1}{2} e_i e_j \tag{2.5}$$

is an injective Lie algebra homomorphism of $\mathfrak{so}(n)$ into $\mathrm{Cl}(\mathbb{R}^n)$. In fact, by dimension counting we see that this map defines a Lie algebra isomorphism between $\mathfrak{so}(n)$ and the Lie subalgebra $\mathrm{Cl}_2(\mathbb{R}^n) \subset \mathrm{Cl}(\mathbb{R}^n)$ spanned, as a vector space, by the elements $e_i e_j$ for $i \neq j$.

Notice that $\mathfrak{spin}(n) = \mathrm{Cl}_2(\mathbb{R}^n)$. This is evident because for any $i < j$ we can define $\gamma : (-1, 1) \to \mathrm{Spin}(n)$ by $\gamma(t) = (\cos(t)e_i + \sin(t)e_j)(-\cos(t)e_i + \sin(t)e_j) = \cos(2t) + \sin(2t)e_i e_j$. Then $\gamma(0) = 1$ and $\gamma'(0) = e_i e_j$. Hence the collection of all $e_i e_j$ for all $i < j$ is a subset of $\mathfrak{spin}(n)$. Since $\mathrm{Cl}_2(\mathbb{R}^n)$ is then a subspace of $\mathfrak{spin}(n)$, and both spaces have the same dimension, they coincide.

Lemma 2.6. *Let* $e_i \wedge e_j$ *for* $i < j$ *denote the generators* $E_{ij} - E_{ji}$ *of the Lie algebra* $\mathfrak{so}(n)$. *Then the Lie algebra isomorphism* $\tau_* : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ *induced from (2.4) is given in the basis* $\{e_i e_j\}_{i<j}$ *by*

$$\tau_*(e_i e_j) = 2e_i \wedge e_j.$$

Proof. First note that for any $v, w \in \mathbb{R}^n$, if we define $v \wedge w : \mathbb{R}^n \to \mathbb{R}^n$ by $(v \wedge w)(x) = \langle v, x\rangle w - \langle w, w\rangle v$ then as $v \wedge w$ is a skew-symmetric endomorphism of $\mathbb{R}^n$ it defines an element of $\mathfrak{so}(n)$. One easily see that $e_i \wedge e_j$ is, as a matrix, the $E_{ij} - E_{ji}$ described above.

Recall from (2.4) that the map τ was defined for $g \in \mathrm{Spin}(n)$ by $\tau(g)x = gxg^{-1}$ for any $x \in \mathbb{R}^n$. The derivative of τ then satisfies $\tau_*(\cdot)(x) = [\cdot, x]$. On a basis element $e_ie_j \in \mathfrak{spin}(n)$ we see

$$\begin{aligned}\tau_*(e_ie_j)(x) &= e_ie_jx - xe_ie_j \\ &= e_ie_jx + (e_ix + 2\langle e_i, x\rangle)e_j \\ &= e_ie_jx + 2e_j\langle e_i, x\rangle - e_i(e_jx + 2\langle e_j, x\rangle) \\ &= 2(e_i \wedge e_j)(x)\end{aligned}$$

□

An immediate consequence of this is that for any $v, w \in \mathbb{R}^n$,

$$\tau_*^{-1}(v \wedge w) = \frac{1}{4}[v, w]. \tag{2.6}$$

Since $\mathrm{Spin}(2r)$ is in $\mathrm{Cl}(\mathbb{R}^{2r})$ and hence in $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{2r})$, the representation of $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{2r})$ by endomorphisms of Δ from Proposition 2.3 restricts to a representation of $\mathrm{Spin}(2r)$. The element $\Gamma := i^r e_1 ... e_{2r} \in \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{2r})$ satisfies $\Gamma^2 = 1$. Hence $\Delta = \Delta^+ \oplus \Delta^-$ where $\Delta^{\pm}$ is the ± 1 eigenspace of Γ. Since also $x\Gamma = \Gamma x$ for all $x \in \mathrm{Cl}^0(\mathbb{R}^{2r})$ the elements of $\mathrm{Spin}(2r)$ preserve this decomposition. That is, Δ^+ and Δ^- each provide a representation of $\mathrm{Spin}(2r)$. These are called the half-spin representations of positive and negative chirality, respectively. To see that they are irreducible, we could notice from the proof of Proposition 2.3 that the representation Δ was identified with $\Lambda(V_i)$, where V_i was the $+i$ eigenspace of a complex structure $J \otimes 1$ on $\mathbb{C}^{2n}$. The decomposition $\Delta = \Delta^+ \oplus \Delta^-$ is then $\Lambda(V_i) = \Lambda^{\text{even}}(V_i) \oplus \Lambda^{\text{odd}}(V_i)$. And it is clear from the construction that $\text{Spin}(2n)$ acts irreducibly on each component.

2.2 Spin Manifolds

We now wish to consider a parameterized version of the linear algebra which was developed in the previous section. Once some groundwork has been laid out we will be able to define the Dirac operator. This section follows closely parts of chapter II in [20].

Let $\pi : E \to M$ be an oriented rank n real vector bundle over a manifold M. Assume the bundle is equipped with a Euclidean structure $g : E \oplus E \to \mathbb{R}$. Let $P_{\text{SO}}(E)$ be the principal bundle of oriented orthonormal frames on E.

Definition 2.7. *A spin structure on E is a principal $\text{Spin}(n)$-bundle $P_{\text{Spin}(n)}(E)$ together with a two-sheeted covering map*

$$\xi : P_{\text{Spin}(n)}(E) \to P_{\text{SO}(n)}(E)$$

such that $\xi(pg) = \xi(p)\tau(g)$ for all $p \in P$ and $g \in \text{Spin}(n)$. Here $\tau : \text{Spin}(n) \to \text{SO}(n)$ is the double covering defined in (2.4). Note that a morphism of principal bundles defined in this way is automatically an isomorphism of principal bundles.

Two spin structures $P_{\text{Spin}(n)}(E)$ and $P'_{\text{Spin}(n)}(E)$ on E are isomorphic if there is a $\text{Spin}(n)$ equivariant map such that the following diagram commutes

$$\begin{tikzcd}
P_{\text{Spin}(n)}(E) \arrow[rr] \arrow[dr, "\xi"'] & & P'_{\text{Spin}(n)}(E) \arrow[dl, "\xi'"] \\
& P_{\text{SO}(n)}(E) &
\end{tikzcd}$$

There is the following result regarding existence and classification for spin structures on a given vector bundle.

Theorem 2.8. *Let $E \to M$ be an oriented vector bundle over a manifold M. There exists a spin structure on E if and only if the second Stiefel-Whitney class $w_2(E)$ vanishes. Furthermore, if $w_2(E) = 0$ then the distinct spin structures on E are in bijective correspondence with the elements of the set $H^1(M, \mathbb{Z}_2)$.*

From now on, we will primarily be interested in the case of the the vector bundle $E = TM \to M$, where M is an oriented $2r$-dimensional Riemannian manifold. If a spin structure exists on TM, then M with a choice of spin structure is called a spin manifold. We will talk about spin structures on M when really meaning spin structures on TM and write $P_{\mathrm{SO}(2r)}(M)$ in place of $P_{\mathrm{SO}(2r)}(TM)$. and similarly for $P_{\mathrm{Spin}(2r)}(M)$.

Definition 2.9. *Let $\pi : P \to M$ be a principal G-bundle and $\rho : G \to \mathrm{GL}(V)$ be a representation of G on the vector space V. There is an action of G on $P \times V$ given by $g \cdot (p, v) = (pg^{-1}, \rho(g)v)$. The orbit space of this action is denoted by $P \times_\rho V$. Then the associated vector bundle is defined to be $\pi_\rho : P \times_\rho V \to M$ where the map π is defined by*

$$\pi_\rho[(p, v)] = \pi(p).$$

.

Note that trivializing neighborhoods $\{U_\alpha\}$ for P also serve as trivializing neighborhoods for $P \times_\rho V$. Furthermore, if P has transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, then $P \times_\rho V$ has transition functions $\rho \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(V)$.

Let ρ_{2r} be the defining representation of $\mathrm{SO}(2r)$ on $\mathbb{R}^{2r}$. The action of $\mathrm{SO}(2r)$ on $\mathbb{R}^{2r}$ extends to an action on $\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{2r})$ producing a representation $\mathrm{Cl}_{\mathbb{C}}(\rho_{2r}) : \mathrm{SO}(2r) \to \mathrm{GL}(\mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{2r}))$. This leads to the following definition.

Definition 2.10. *Let M be an oriented $2r$-dimensional Riemannian manifold. The Clifford bundle on M is the complex vector bundle*

$$\mathrm{Cl}_{\mathbb{C}}(M) := P_{\mathrm{SO}(2r)}(TM) \times_{\mathrm{Cl}_{\mathbb{C}}(\rho_{2r})} \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{2r}).$$

The bundle TM is a bundle which assigns to each point in M a vector space with an inner product and $\mathrm{Cl}_{\mathbb{C}}(M)$ is the bundle which assigns to each point the corresponding Clifford algebra. In fact, $\mathrm{Cl}_{\mathbb{C}}(M)$ could equally well have been constructed as

$$(\bigoplus_{k=0}^{\infty} TM^{\otimes^k})/I(TM)$$

where $I(TM)$ is the subbundle of $\bigoplus_{k=0}^{\infty} TM^{\otimes^k}$ which has as fiber at each point $x \in M$ the ideal generated by all elements $v \otimes v + ||v||^2$ for $v \in TM_x$.

We now have bundles of Clifford algebras and, as mentioned above, we are interested in bundles of modules for these algebras.

Definition 2.11. *Let M be a $2r$-dimensional spin manifold. The (complex) spinor bundle on M is the bundle*

$$S = P_{\mathrm{Spin}(2r)}(TM) \times_{\mu} \Delta.$$

Here $\mu : \mathrm{Spin}(2r) \to \mathrm{GL}(\Delta)$ is the representation obtained by restricting the representation $\rho : \mathrm{Cl}_{\mathbb{C}}(\mathbb{R}^{2r}) \to \mathrm{End}(\Delta)$ from Proposition 2.3.

The bundle S is a bundle of modules for the bundle of Clifford algebras $\mathrm{Cl}_{\mathbb{C}}(M)$. Notice that the sections of $\mathrm{Cl}_{\mathbb{C}}(M)$ form an algebra and the sections of S are a module over this algebra.

Recall the decomposition into half-spin representations $\Delta = \Delta^+ \oplus \Delta^-$ where Δ^+ and Δ^- are the $+1$ and -1 eigenspaces of $\Gamma = i^m e_1 ... e_{2r}$. We can form a global section of $\mathrm{Cl}_{\mathbb{C}}(TM)$ defined as $\Gamma(x) = i^m e_{1_x} ... e_{2r_x}$ where $e_{1_x}, ..., e_{2r_x}$ is a positively oriented orthonormal basis for TM_x. One can check that the section Γ is independent of the choice of basis and hence well defined. We conclude there is a decomposition $S = S^+ \oplus S^-$. Alternatively, these bundles could have been defined by

$$S^{\pm} = P_{\mathrm{Spin}(2r)}(TM) \times_{\rho^{\pm}} \Delta^{\pm}.$$

The last piece needed to define the Dirac operator is a connection on the spin bundle S (and hence on the subbundles $S^{\pm}$). First we recall the following concepts.

Definition 2.12. *A covariant derivative ∇ on a vector bundle $E \to M$ is a linear map $\Gamma(E) \to \Gamma(T^*M \otimes E)$ satisfying*

$$\nabla(f\psi) = df \otimes \psi + f\nabla\psi$$

for all $f \in C^\infty(M)$ and $\psi \in E$.

Definition 2.13. *Let $P \to M$ be a principal G-bundle. A connection ω on P is a Lie algebra valued 1-form $\omega \in \Omega^1(M) \otimes \mathfrak{g}$. such that*

$$\omega\left(\frac{d}{dt}|_{t=0} p \ \exp(tV)\right) = V \quad \textit{for all } V \in \mathfrak{g}, \textit{ and}$$
$$g^*\omega = Ad_{g^{-1}}\omega \quad \textit{for all } g \in G \textit{ acting on } P.$$

For any oriented vector bundle $E \to M$ of rank $2r$ equipped with a Euclidean structure, a connection on the principal bundle $P_{\mathrm{SO}(2r)}(E)$ induces a covariant derivative on E, and conversely, any covariant derivative on E, satisfying a certain compatibility condition with the metric on E, induces a connection on $P_{\mathrm{SO}(2r)}(E)$ (see [20, Proposition 4.4]). Furthermore, a connection on any principal G-bundle $P \to M$ induces a covariant derivative on any vector bundle over M obtained from P as an associated vector bundle.

Let $\omega \in \Omega^1(P, \mathfrak{so}(2r))$ be the connection on $P_{\mathrm{SO}(2r)}(M)$ induced by the Levi-Civita connection on TM. We define a new connection $\widetilde{\omega} \in \Omega^1(P, \mathfrak{spin}(2r))$ called the spin connection which is the lift of ω to $\mathfrak{spin}(2r)$. That is, $\widetilde{\omega}$ satisfies

$$\omega = \tau_* \widetilde{\omega}$$

where $\tau : \mathrm{Spin}(2r) \to \mathrm{SO}(2r)$ is the map defined in (2.4). The connection on the spinor bundle S is then the connection induced by the spin representation.

Making use of Lemma 2.6, one can see that in a local trivialization of TM, if the Levi-Civita connection induces the covariant derivative

$$d + \sum_{i<j} \Omega_{ij} e_i \wedge e_j$$

where Ω_{ij} is a skew symmetric matrix of 1-forms, then the connection on the spinor bundle S with respect to the induced local trivialization (see the comments after Definition 2.9) will be

$$d + \frac{1}{2} \sum_{i<j} \Omega_{ij} e_i e_j.$$

Denote this connection by ∇.

Definition 2.14. *Let M be a spin manifold and $S \to M$ be the spin bundle constructed above. The Dirac operator on a spin manifold M is the operator*

$$C^\infty(S) \to C^\infty(T^*M \otimes S) \to C^\infty(S)$$

where the first arrow is given by the covariant derivative ∇ and the second arrow is Clifford multiplication (after using the metric to identify TM and T^*M).

If $e_1, ..., e_n$ is an oriented orthonormal basis for T_xM then

$$\partial\!\!\!/ \psi = \sum_i e_i \nabla_{e_i} \psi.$$

at $x \in M$.

Note that since Clifford multiplication maps $\Delta^{\pm}$ to $\Delta^{\mp}$ the Dirac operator decomposes as $\partial\!\!\!/ = \partial\!\!\!/^{+} \oplus \partial\!\!\!/^{-}$ where $\partial\!\!\!/^{\pm} : C^{\infty}(S^{\pm}) \to C^{\infty}(S^{\mp})$. There is a Hermitian metric $(\cdot, \cdot)$ on S which one can use (together with the volume form dV on M) to define the Hilbert space

$$L^2(S) = \{\psi \in \Gamma(S) : ||\psi||^2 = \int_M (\psi, \psi) dV < \infty\}.$$

The space $C^{\infty}(S)$ is dense in $L^2(S)$. We will also be interested in the Sobolev space

$$L^2_1(S) = \{\psi \in \Gamma(S) : \int_M ||\nabla \psi|| dV < \infty\}$$

where $|| \cdot ||$ is the norm on $T^*M \otimes S$ induced by the metric on M and the metric on S. It can be shown that the Dirac operator then defines a self-adjoint operator $\partial\!\!\!/ : L^2_1(S) \to L^2(S)$. As the index theory for self-adjoint operators is not very interesting (their index always being zero), we will focus only on the chiral Dirac operators $\partial\!\!\!/^{+}$ in what follows, and denote it by $\partial\!\!\!/$ instead.

We will also be interested in twisted (chiral) Dirac operators. Let $E \to M$ be a complex Hermitian vector bundle with a connection ∇^E. The bundle $S \otimes E$ is a also a bundle of modules for $\mathrm{Cl}_{\mathbb{C}}(M)$ since for $\psi \in S$, $\sigma \in E$, and $v \in \mathrm{Cl}_{\mathbb{C}}(M)$ we have

$$v \cdot (\psi \otimes \sigma) := (v \cdot \psi) \otimes \sigma.$$

We can also equip the bundle $S \otimes E$ with the tensor product connection $\nabla^{S \otimes E}$

$$\nabla^{S \otimes E} = \nabla \otimes 1_E + 1_S \otimes \nabla^E.$$

The twisted Dirac operator

$$\partial\!\!\!/^E : C^\infty(S^+ \otimes E) \to C^\infty(T^*M \otimes S^+ \otimes E) \to C^\infty(S^- \otimes E)$$

is defined again by covariant differentiation followed by Clifford multiplication.

2.3 The Index Theorem

Let M be a compact, oriented, even-dimensional manifold (recall all manifolds are assumed to be without boundary) of dimension d with vector bundles E and F over M. Let $D : C^\infty(E) \to C^\infty(F)$ be a differential operator of order m. For $f \in C^\infty(E)$ we have in local coordinates

$$(Df)_i = \sum_{|\alpha|=m} a^\alpha_{ij} \frac{\partial^m f_j}{\partial z^\alpha} + \text{lower order terms}$$

Here we are using the multi-index notation: α denotes the d-tuple $(\alpha_1, ..., \alpha_d)$, $|\alpha| = \sum_i \alpha_i$, and $\partial/\partial z^\alpha = \partial^m/\partial z^{\alpha_1} ... \partial/\partial z^{\alpha_d}$. Clearly a^α_{ij} is symmetric in the α indices and it is not difficult to show that for each i, j it transforms as an element of $\mathrm{Sym}^m(T^*M)$. Taking into account the i, j indices a defines the symbol of D

$$\sigma(D) \in \mathrm{Sym}^m(T^*M) \otimes Hom(E, F).$$

An operator D is elliptic if $\sigma(D)(\eta, ...\eta) \in Hom(E, F)$ is invertible for all nonzero $\eta \in T^*M$.

The Dirac operator,

$$\partial\!\!\!/ : C^\infty(S^+) \to C^\infty(S^-)$$

is a first order operator whose symbol corresponds to the map

$$\sigma(\not{\partial}) : T^*M \times S^+ \to S^-$$

given by Clifford multiplication on spinors. For any $\eta \in T^*M$ we have $\sigma(\not{\partial})(\eta)^2\psi = -||\eta||^2\psi$ for all $\psi \in S^+$, so $\sigma(\not{\partial})(\eta)$ is invertible whenever $\eta \neq 0$ and hence $\not{\partial}$ is elliptic. It is easy to see that the above goes through with minor modifications to show that twisted Dirac operators are also elliptic.

As noted in the previous section, the Dirac operator extends to an operator $\not{\partial} : L^2_1(S^+) \to L^2(S^-)$. Similarly, after equipping a coefficient bundle E with a Hermitian metric, there is the twisted Dirac operator $\not{\partial}^E : L^2(S^+ \otimes E) \to L^2(S^- \otimes E)$. It is a classical result that any elliptic operator on a compact manifold is Fredholm, therefore Dirac operators and twisted Dirac operators are Fredholm operators. The index of a Fredholm operator P is defined as

$$\operatorname{Ind} P := \dim\ker P - \dim\operatorname{coker} P. \tag{2.7}$$

The famed Atiyah-Singer Index Theorem [5] gives the formula for the index of a twisted Dirac operator

$$\operatorname{Ind} \not{\partial}^E = \int_M \widehat{A}(M)\operatorname{ch}(E). \tag{2.8}$$

To briefly explain the formula we let $x_1, ..., x_{d/2}$ denote the roots for the tangent bundle TM. The x_i's are defined so that the total Pontryagin class $p(M) = 1 + p_1(M) + p_2(M) + ...$ satisfies

$$p(M) = \prod_{i=1}^{d/2}(1 + x_i^2).$$

Then $\widehat{A}(p_1, p_2, ...) = \prod_{i=1}^{d/2} \frac{x_i/2}{\sinh(x_i/2)} \in H^*(M, \mathbb{Q})$. Specifically, the first few terms are

$$\hat{A}(M) = 1 - \frac{1}{24}p_1(M) + \frac{1}{5760}\left(7p_1(M)^2 - 4p_2(M)\right) + ...$$

If $y_1, ..., y_r$ are the formal Chern roots for the vector bundle $E \to M$, then the Chern character is defined by

$$\mathrm{ch}(E) := \sum_{i=1}^{r} e^{x_i} = r + c_1(E) + \frac{1}{2}(c_1(E)^2 - 2c_2(E)) + ...$$

2.4 The Index of a Family

Let $\pi : Z \to X$ be a fiber bundle with fibers Y, where X, Y, and Z are compact oriented manifolds of dimension n, m and $d = m + n$, respectively. The following is true whether we use (co)homology with coefficients in $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}$. By Poincaré duality we have isomorphisms $PD : H^i(X) \to H_{n-i}(X)$ and $PD : H^i(Z) \to H_{d-i}(Z)$. The map $\int_Y : H^i(Z) \to H^{i-m}(X)$, is known as integration over the fiber (or the Gysin homomorphism), is then defined by the following diagram

$$\begin{tikzcd} H^i(Z) \arrow[r, "\int_Y"] \arrow[d, "PD"'] & H^{i-m}(X) \arrow[d, "PD"] \\ H_{d-i}(Z) \arrow[r, "\pi_*"] & H_{d-i}(X) \end{tikzcd}$$

In de Rham cohomology the map $\int_Y$ *is* the integration of differential forms over the fiber.

We note two properties of the map $\int_Y$. First, $H^*(X)$ is clearly a module over itself and the ring homomorphism $\pi^* : H^*(X) \to H^*(Z)$ makes $H^*(Z)$ into a module over $H^*(X)$ as well. The first property is that $\int_Y$ is a module homomorphism, that is

$$\int_Y \pi^*(\omega) \wedge \eta = \omega \int_Y \eta \tag{2.9}$$

for all $\omega \in H^*(X)$, $\eta \in H^*(Z)$. Second, if $\eta \in H^d(Z)$ we have

$$\int_Z \eta = \int_X \int_Y \eta. \tag{2.10}$$

Here $\int_Z$ means evaluation on the fundamental class (or equivalently, integration over the fiber applied to the fiber bundle $Z \to \{pt\}$) and similarly for $\int_X$. Notice that if we are working with de Rham cohomology and $\int_Y$ is given by integration of differential forms, then this last property is merely the Fubini theorem.

In the following we will be interested in families of (twisted) Dirac operators parameterized by a compact manifold X.

Definition 2.15. *By a family of compact spin manifolds, we will mean a triple* $\mathcal{F} = (\pi, Z, X)$*, where* Z *and* X *are compact manifolds and* $\pi : Z \to X$ *has the structure of a fiber bundle with structure group* $\mathrm{Diff}_S(Y)$*. Here,* Y *is some compact spin manifold equipped with a spin structure and* $\mathrm{Diff}_S(Y)$ *denotes the spin structure-preserving diffeomorphisms of* Y*.*

Given a family of compact spin manifolds $\mathcal{F} = (\pi, Z, X)$ with fibers $Y_x := \pi^{-1}(X)$, define the vertical vector bundle $V := \ker(\pi_* : TZ \to TX)$. This bundle has fibers which are tangent to the fibers of $\pi : Z \to X$. This is easy to see: since if $i : Y_x \hookrightarrow Z$ denotes the embedding we have the following commutative diagram,

$$\begin{tikzcd} TY_x \arrow[r, "i_*"] \arrow[d] & V \arrow[d] \\ Y_x \arrow[r, hook, "i"] & Z \end{tikzcd}$$

and i_* is an isomorphism on the fibers, so the bundle $i^*V \to Y_x$ is isomorphic to the bundle $TY_x \to Y_x$.

The constructions from Section 2.2 applies to each of the spin manifolds Y_x. Hence there are bundles $\mathrm{Cl}_{\mathbb{C}}(Y_x)$ and $S_x = S^+ \oplus S^-$ for each $x \in X$. Given a vector bundle

$E \to Z$, it provides a coefficient bundle by restriction $E|_{Y_x} \to Y_x$. That is, we can form a family of Dirac operators with coefficients in E. We will denote this family by $\not{\partial}^E$.

The index of a single Fredholm operator is an integer. The index of a family P of elliptic operators $P_x : C^\infty(E_x) \to C^\infty(F_x)$ parameterized by a compact space X is an element of $K(X)$ defined as follows. For each $x \in X$, one has the finite dimensional subspaces $\ker P_x$ and $\operatorname{coker} P_x$ of $C^\infty(E_x)$ and $C^\infty(F_x)$, respectively. If these vector spaces gave rise to the vector bundles $\ker P$ and $\operatorname{coker} P$ over X, the desired generalization of (2.7) would be

$$\operatorname{Ind} P := \ker P - \operatorname{coker} P \in K(X). \tag{2.11}$$

This need not be the case, however, as there is the possibility that, as x varies, the dimensions of $\ker P_x$ and $\operatorname{coker} P_x$ may jump. It is only the difference in their dimensions that is fixed by invariance of the index under continuous perturbations. Thus $\ker P$ and $\operatorname{coker} P$ need not be vector bundles. However, Atiyah and Singer have shown in [6, Section 2] that one can define the index of a family in a way which reduces to (2.11) when $\dim \ker P$ is constant. We quote the result.

Theorem 2.16. *Let $\mathcal{F} = (\pi, Z, X)$ be a family of compact manifolds and $\mathcal{E}$ and $\mathcal{F}$ be two smooth vector bundles over Z. Let $P_x : \Gamma(E_x) \to \Gamma(F_x)$ be a family of elliptic operators. Then there exists a finite set of sections $(s_1, ..., s_q)$ of F over X such that for each $x \in X$ the map $\tilde{P}_x : \mathbb{C}^q \oplus \Gamma(E_x) \to \Gamma(F_x)$ given by $\tilde{P}_x(t_1, ..., t_q; \phi) = \sum t_j s_j|_{Z_x} + P_x(\phi)$ is surjective for all $x \in X$. The vector spaces $\ker \tilde{P}_x$ form a vector bundle over X, and the element*

$$[\ker \tilde{P}] - [\mathbb{C}^q] \in K(X)$$

depends only on the operator P.

The original formula for the Chern character of the index bundle for a family of elliptic operators is given in [6, Theorem 5.1]. For the case of a family of Dirac operators twisted by a coefficient bundle $E \to Z$ the formula is

$$\operatorname{ch}(\operatorname{Ind} \partial\!\!\!/^{V_n}) = \int_Y \widehat{A}(V)\operatorname{ch}(E) \in H^*(X, \mathbb{Q}) \tag{2.12}$$

where $\int_Y : H^*(Z, \mathbb{Q}) \to H^{*-m}(X, \mathbb{Q})$ is the integration over the fibers map, (see [7, Theorem 4.17] or [20, Corollary 15.5]).

Chapter 3

The Dirac-Ramond Operator

3.1 The Index of a Single Operator

Let M be a compact spin manifold of even dimension d. For any vector bundle $W \to M$ we have a sequence of vector bundles $\{W_n\}$ defined by the generating series

$$\bigotimes_{j=1}^{\infty} S_{q^j} W_{\mathbb{C}} = \sum_{n=0}^{\infty} q^n W_n \tag{3.1}$$

where $W_{\mathbb{C}}$ denotes the complexification of W and $S_t(W_{\mathbb{C}}) = \mathbb{C} + tW_{\mathbb{C}} + t^2 S^2(W_{\mathbb{C}}) + ...$ is a formal power series with vector bundle coefficients. One has, for instance,

$$\begin{aligned}
W_0 &= \mathbb{C} \\
W_1 &= W_{\mathbb{C}} \\
W_2 &= S^2 W_{\mathbb{C}} \oplus W_{\mathbb{C}} \\
W_3 &= S^3 W_{\mathbb{C}} \oplus (W_{\mathbb{C}} \otimes W_{\mathbb{C}}) \oplus W_{\mathbb{C}} \\
&\vdots
\end{aligned}$$

We can use these bundles as coefficients for the usual chiral Dirac operator $\partial\!\!\!/$ on M to obtain the twisted Dirac operators $\partial\!\!\!/^{W_n} : C^\infty(S^+ \otimes W_n) \to C^\infty(S^- \otimes W_n)$ where S^+ and S^- are the spinor bundles of positive and negative chirality, respectively. By (2.8), the index of these operators are

$$\operatorname{Ind} \partial\!\!\!/^{W_n} = \int_M \widehat{A}(TM)\,\operatorname{ch}(W_n) \quad \text{for } n = 0, 1, 2, ... \tag{3.2}$$

Now we define the Dirac-Ramond operator on M as the formal series of operators

$$D\!\!\!\!/ = \sum_{n=0}^{\infty} q^n \partial\!\!\!/^{TM_n}.$$

Equivalently, if we allow for "q-vector bundles," that is, formal power series in q whose coefficients are vector bundles, then we can view it as the the twisted Dirac operator,

$$D\!\!\!\!/ : C^\infty(S^+ \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}}) \to C^\infty(S^- \otimes \bigotimes_{n=1}^{\infty} S_{q^n} TM_{\mathbb{C}}).$$

Extending the definition of index to each coefficient of the formal operator $D\!\!\!\!/$ as

$$\operatorname{Ind} D\!\!\!\!/ = \sum_{n=0}^{\infty} q^n \operatorname{Ind} \partial\!\!\!/^{TM_n}, \tag{3.3}$$

the calculation of this index is straightforward. We will essentially do it below when we prove the family index theorem. After using

$$\operatorname{ch}(\operatorname{Sym}_t(TM_{\mathbb{C}})) = \prod_{i=1}^{d/2} \frac{1}{(1 - te^{x_i})(1 - te^{-x_i})}, \tag{3.4}$$

where $x_1, ..., x_{d/2}$ are the formal roots for TM the result can be written in any of the following ways:

$$\begin{aligned}\operatorname{Ind} \not{D} &= \frac{q^{d/24}}{\eta(q)^d} \int_M \prod_{i=1}^{d/2} x_i \frac{\theta'(0,q)}{\theta(x_i,q)} \\ &= \frac{q^{d/24}}{\eta(q)^d} \int_M \prod_{i=1}^{d/2} \frac{x_i}{\sigma(x_i,q)} e^{G_2(q)p_1(M)} \\ &= \frac{q^{d/24}}{\eta(q)^d} \int_M \prod_{i=1}^{d/2} \exp\left(\sum_{n=1}^{\infty} \frac{2}{2n!} G_{2n}(q) x_i^{2n}\right).\end{aligned} \tag{3.5}$$

Here $\theta(x, q)$ is the Jacobi theta function, $\sigma(x, q)$ is the Weierstrass sigma function, and $G_{2n}(q)$ is the Eisenstein series of weight $2n$. The definitions of all of these functions and some of the relations between them can be found in Appendix A. With the assumption that M is spin the equation (3.3) implies that (3.5) is an element of $\mathbb{Z}[[q]]$. The expression on the RHS of (3.5) makes sense for any compact oriented manifold and more generally lies in $\mathbb{Q}[[q]]$. The factor of

$$\frac{q^{d/24}}{\eta(q)^d} = \prod_{j=1}^{\infty} (1-q^j)^{-d} \tag{3.6}$$

is not very interesting. We will often choose to omit it, defining

$$\operatorname{ind} \not{D} := \frac{\eta(q)^d}{q^{d/24}} \operatorname{Ind} \not{D}. \tag{3.7}$$

This normalization makes $\operatorname{ind} \not{D}$ the q-expansion of a modular form of weight $d/2$ when $p_1(M) = 0$ and a quasimodular form of the same weight otherwise. The second line of (3.5) indicates that $\operatorname{ind} \not{D}$ is an elliptic analogue of the index of the usual Dirac operator: $\sinh(x/2)$ is a periodic function on $i\mathbb{R}$ with simple zeros at every point in the lattice $2\pi i\mathbb{Z}$, while the Weierstrass sigma function $\sigma(x, \tau)$ is a (quasiperiodic) extension of this to the complex plane and the two dimensional lattice $2\pi i(\mathbb{Z} \oplus \tau\mathbb{Z})$. Here we have used the change of variables $q = e^{2\pi i\tau}$, common in the study of modular forms, where τ is an element of the upper half complex plane. Based on this similarity we will define the following for a rank r real vector bundle W over M with roots $w_1, ..., w_{r/2}$

$$\widehat{a}(W, q) = \prod_{i=1}^{r/2} \frac{w_i}{\sigma(w_i, q)} \in H^*(M, \mathbb{Q}[[q]]) = H^*(M, \mathbb{Q})[[q]].$$

In the case when M is a string manifold (that is, M is a manifold with $p_1(M) = 0$)[1], then we can write

$$\operatorname{ind} \not{D} = \int_M \widehat{a}(TM, q). \tag{3.8}$$

Example 3.1. Let M be an 8-dimensional manifold M with $p_1(M) = 0$, the (normalized) index of the Dirac-Ramond operator ind $\not{D}$ is a modular form of weight 4 (see Section 4.1). The space of weight 4 modular forms is spanned by the Eisenstein series $E_4(q)$. Thus we need only compare the terms which are constant in q. Taking $q \to 0$ in (3.3) shows that on M

$$\operatorname{Ind} \not{D}(q = 0) = \operatorname{Ind} \not{\partial}.$$

Since the factor (3.6) takes the value 1 at $q = 0$, then the normalized index (3.7) also satisfies ind $\not{D}(q = 0) = \operatorname{Ind} \not{\partial}$. From (4.7), we see that $E_4(q)$ is normalized so that its constant term is 1. Therefore,

$$\operatorname{ind} \not{D} = (\operatorname{Ind} \not{\partial}) E_4(q). \tag{3.9}$$

The index of the Dirac-Ramond operator on M is then

[1]Since we are working only with rational cohomology, we ignore the subtler condition that $p_1(M)/2$ should equal 0 in integral cohomology.

$$\begin{aligned}\operatorname{ind} \not{D} &= (\operatorname{Ind} \not{\partial})\frac{q^{1/3}}{\eta(q)^8}E_4(q) \qquad (3.10)\\ &= (\operatorname{Ind} \not{\partial})(qj(q))^{1/3} = (\operatorname{Ind} \not{\partial})(1 + 248q + 4124q^2 + ...).\end{aligned}$$

Here $j(q)$ is the j-invariant for elliptic curves.

3.2 A Family of Dirac-Ramond Operators

Let $\mathcal{F} = (\pi, Z, X)$ be a family of compact spin manifolds with fibers diffeomorphic to Y. Recall there is the vertical vector bundle $V \to Z$ given by $V := \ker \pi_*$. Applying the generating sequence (3.1) to V we get a sequence of vector bundles V_n over Z. Given a family of compact spin manifolds, we will define a family of Dirac-Ramond operators to be the q series with the coefficient of q^n being the family of Dirac operators twisted by V_n

$$\not{D} := \sum_{n=0}^{\infty} q^n \not{\partial}^{V_n}.$$

We are thus led to our main object of study.

Definition 3.2. *Let $\mathcal{F} = (\pi, Z, X)$ be a family of compact spin manifold. Let V_n be the sequence of vector bundles obtained as above. The index of the family of Dirac-Ramond operators is defined to be*

$$\operatorname{Ind} \not{D} := \sum_{n=0}^{\infty} q^n \operatorname{Ind} \not{\partial}^{V_n} \in K(X)[[q]]. \qquad (3.11)$$

The Chern character $\mathrm{ch} : K(X) \to H^*(X, \mathbb{Q})$ extends naturally to a map $K(X)[[q]] \to H^*(X, \mathbb{Q})[[q]]$. In particular,

$$\mathrm{ch}(\mathrm{Ind}\, \not{D}) = \sum_{n=0}^{\infty} q^n \,\mathrm{ch}(\mathrm{Ind}\, \not{\partial}^{V_n}) \in H^*(X, \mathbb{Q})[[q]].$$

We can calculate this Chern character easily by applying the usual Atiyah-Singer index theorem for families (2.12) coefficient by coefficient.

Proposition 3.3. *Let $\mathcal{F} = (\pi, Z, X)$ be a family of compact spin manifold whose fibers $Y_x := \pi^{-1}(x)$ are of even dimension m. Let $V \to Z$ denote the vertical bundle. Then*

$$\mathrm{ch}(\mathrm{Ind}\, \not{D}) = \frac{q^{m/24}}{\eta(q)^m} \int_Y \hat{a}(V, q) e^{G_2(q) p_1(V)}. \tag{3.12}$$

Proof. The equation (3.12) is an equation in $H^*(X, \mathbb{Q})[[q]]$. We establish (3.12) by showing equality for each coefficient of q^n. On the LHS, we get the coefficient of q^n by applying the Chern character to $\mathrm{Ind}\, \not{\partial}^{V_n}$ which is $\int_Y \widehat{A}(V)\,\mathrm{ch}(V_n)$, see (2.12). Let $y_1, ..., y_{m/2}$ be the formal variables for the vertical bundle V. Making use of (3.4) and the formulas from the appendix we have

$$\begin{aligned}
\frac{q^{m/24}}{\eta(q)^m} \hat{a}(V, q) e^{G_2(q) p_1(V)} &= \prod_{i=1}^{m/2} \frac{q^{1/12}}{\eta(q)^2} \frac{y_i}{\sigma(y_i, q)} e^{G_2(q) y_i^2} = \prod_{i=1}^{m/2} \frac{q^{1/12}}{\eta(q)^2} y_i \frac{\theta'(0, q)}{\theta(y_i, q)} \\
&= \prod_{i=1}^{m/2} \frac{y_i/2}{\sinh(y_i/2)} \prod_{j=1}^{\infty} \frac{1}{(1 - q^j e^{y_i})(1 - q^j e^{-y_i})} \\
&= \hat{A}(V)\,\mathrm{ch}\left(\bigotimes_{j=1}^{\infty} S_{q^j}(V_{\mathbb{C}}) \right).
\end{aligned}$$

After integrating over the fiber we can identify the coefficient of q^n as $\int_Y \widehat{A}(V)\,\mathrm{ch}(V_n)$ and the result follows. □

We are interested in the formal definition (3.11) because our formula (3.12) matches that of [3]. In the spirit of that paper, from now on, given a family of Dirac-Ramond operators we will denote the Chern character of the index bundle by

$$\mathrm{Sch}(\mathcal{F};q) := \mathrm{ch}\left(\sum_{n=0}^{\infty} q^n \,\mathrm{Ind}\,\partial\!\!\!/^{V_n}\right) = \sum_{n=0}^{\infty} q^n \,\mathrm{ch}(\mathrm{Ind}\,\partial\!\!\!/^{V_n}) \in H^*(X,\mathbb{Q})[[q]], \tag{3.13}$$

using the letter "S" because this Chern character is of a "stringy" version of the usual index bundle. By the proposition we have

$$\mathrm{Sch}(\mathcal{F};q) = \frac{q^{m/24}}{\eta(q)^m}\int_Y \hat{a}(V,q)e^{G_2(q)p_1(V)}. \tag{3.14}$$

At times we will prefer to use instead

$$\mathrm{sch}(\mathcal{F};q) := \frac{\eta(q)^m}{q^{m/24}}\,\mathrm{Sch}(q) = \int_Y \hat{a}(V,q)e^{G_2(q)p_1(V)}. \tag{3.15}$$

Similar to (3.7), the latter definition will have better modular properties as we will see in the next section.

Chapter 4

Modular Properties of the Index Bundle

From now on we restrict ourselves to so called string families of compact spin manifolds $\mathcal{F} = (\pi, Z, X)$, that is, families of compact spin manifolds with $p_1(Z) = 0$. It follows that each of the manifolds $Y_x = \pi^{-1}(x)$ is a string manifold. To see this, let $i : Y_x \hookrightarrow Z$ be the inclusion map. Then the restriction i^*TZ splits as $TY_x \oplus N_x$, where $N_x \to Y$ is the normal bundle to Y_x in TZ. Since $\pi : Z \to X$ is locally trivial, N_x is trivial. Therefore, $p_1(N_x) = 0$ and hence

$$p_1(Y_x) = i^* p_1(TZ) = 0.$$

One can always split the tangent bundle to Z, though non-canonically, as $TZ = V \oplus \pi^*(TX)$. Consequently, $p_1(V) = -\pi^* p_1(X)$ and thus

$$\int_Y p_1(V)^n = (-1)^n \int_Y \pi^* p_1(X)^n = (-1)^n p_1(X)^n \int_Y 1 = 0 \quad \text{for all } n \in \mathbb{N}. \tag{4.1}$$

After a short review on modular forms we will show that if $\mathcal{F}$ is a string family and X is a string manifold, then $\mathrm{sch}(\mathcal{F}; q)$ is an element of the cohomology of X with coefficients in modular forms. If X is not string then the coefficients will be quasi-modular forms. As a consequence of this we will be able to derive many relationships between the homogeneous components of $\mathrm{ch}(\mathrm{Ind}\,\partial\!\!\!/^{V_n})$ for various n. Finally, we will obtain a formula for the case when X is not string which expresses $\mathrm{Sch}(\mathcal{F};q)$ entirely in terms of modular forms, the components of $\mathrm{ch}(\mathrm{Ind}\,\partial\!\!\!/^{V_n})$ for various n, and $p_1(X)$.

4.1 Modular Forms

In this section we briefly recall some definitions and some basic facts related to modular forms and quasimodular forms. Most all our definitions of specific functions are consistent with [15]. Two other excellent references for this section are [18] and [31]. We start with a basic definition.

Definition 4.1. *Let $\mathfrak{h} = \{\tau \in \mathbb{C} | \operatorname{im}(\tau) > 0\}$ denote the upper half plane. A holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ is called a (level 1) modular form of weight $k \in \mathbb{Z}$ if f is holomorphic at $i\infty$ and*

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau) \quad \textit{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}). \tag{4.2}$$

The *level 1* aspect of these modular forms is that they transform as (4.2) under the full group $\mathrm{SL}(2,\mathbb{Z})$. One could define spaces of modular forms which transform as (4.2) only with respect to a fixed subgroup of $\mathrm{SL}(2,\mathbb{Z})$, but that will not be necessary in the following. To develop a small amount of feel for why one might be interested in functions with the property (4.2), notice that if f is a modular form of even weight $k \geq 0$ then $f(\tau)(d\tau)^{k/2}$ is invariant under transformations $\tau \mapsto \tau' = \frac{a\tau+b}{c\tau+d}$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$. This follows simply by noting $d(\frac{a\tau+b}{c\tau+d}) = \frac{d\tau}{(c\tau+d)^2}$. Since the quotient space $\mathrm{SL}(2,\mathbb{Z})\backslash\mathfrak{h}$ parameterizes the isomorphism classes of elliptic curves over $\mathbb{C}$, a modular form of weight k can be interpreted as a section of the $2k$th power of the canonical bundle over the moduli space of elliptic curves. Note that to check that a function satisfies (4.2), it is enough to check it on the generators for $\mathrm{SL}(2,\mathbb{Z})$

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

To understand the *holomorphic at* $i\infty$ condition, we note that the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ belongs to $\mathrm{SL}(2,\mathbb{Z})$. The property (4.2) applied to this matrix shows that f satisfies $f(\tau+1) = f(\tau)$ for all $\tau \in \mathfrak{h}$. Hence f is periodic with period 1. Here and in the following we set $q = e^{2\pi i\tau}$. The function f then has the Fourier expansion,

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i\tau} = \sum_{n=0}^{\infty} a_n q^n \quad \text{for all } \tau \in \mathfrak{h}. \tag{4.3}$$

The map $\tau \to e^{2\pi i\tau} = q$ sends the upper half plane into the unit disk with $i\infty$ getting mapped to 0; the condition *holomorphic at* $i\infty$ can then be interpreted as the requirement that the Fourier expansion of f is how we have written it, containing no negative powers of q. In the following sections we will frequently write $f(q)$ instead of $f(\tau)$. This is a slight abuse of notation as more correctly one would put $\tilde{f}(q) = f(g(q))$ where g is any right inverse of the map $\tau \to e^{2\pi i\tau} = q$, but this is more cumbersome and the practice of using f to denote both functions f and $\tilde{f}$ is common in the literature. In fact, this presents very little problem as we will often not interpret $f(q)$ as a function at all. Instead we will identify $f(q)$ as the element of $\mathbb{C}[[q]]$ defined by (4.3). We will actually be interested in those modular forms whose Fourier expansion identify them with elements of $\mathbb{Q}[[q]]$.

Definition 4.2. *Let* $\mathcal{M}^k \subset \mathbb{Q}[[q]]$ *denote the set of*

$$f(q) = \sum_{n=0}^{\infty} a_n q^n$$

such that

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i \tau}$$

is a modular form of weight k. *Define* $\mathcal{M} = \oplus_{k=0}^{\infty} \mathcal{M}^k$.

It is easy to see that $\mathcal{M}$ has the structure of a $\mathbb{Q}$-vector space and also a graded ring. We have taken the direct sum only for k in $\mathbb{Z}_{\geq 0}$ because, as we will see shortly, if f is a modular form of negative weight then $f \equiv 0$. In fact, the sum could really be taken over only positive even integers. This follows by applying (4.2) in the case of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ to see that if f is a modular form of weight k then

$$f(\tau) = (-1)^k f(\tau).$$

Thus if k is odd, then $f \equiv 0$. Therefore $\mathcal{M}^{2k+1} = \{0\}$ for all $k \in \mathbb{Z}$.

Clearly constant functions are modular forms of weight 0. Before producing any other examples we quote the following result which plays a large role in determining the possible modular forms for a given weight. Denote the orbit space $\mathrm{SL}(2, \mathbb{Z}) \backslash \mathfrak{h}$ by $\mathcal{R}$.

Theorem 4.3. *Let* f *be a nonzero modular form of weight* k. *For* $p \in \mathfrak{h}$, *let* $ord_p(f)$ *denote the order of the zero of* f *at* p. *Let* $ord_\infty(f)$ *denote the smallest* n *such that the coefficient* a_n *in* (4.3) *is nonzero. Then*

$$\frac{1}{2} ord_i(f) + \frac{1}{3} ord_{e^{2\pi i/3}}(f) + \sum_{p \in \mathcal{R} \backslash \{i, e^{2\pi i/3}\}} ord_p(f) + ord_\infty(f) = \frac{k}{12}. \tag{4.4}$$

There are many consequences of this formula. Since f is assumed holomorphic, there can be no modular forms of negative weight or of weight 2. If f is a modular form of weight 0, and c is any value that f takes, then $f(\tau) - c$ is also a modular form of weight 0. The RHS of (4.4) is zero while the LHS is nonzero; we must conclude $f(\tau) - c \equiv 0$.

Hence the only modular forms of weight 0 are constants. For $k = 4, 6, 8, 10$, or 14 we see that the space of modular forms of weight k is also one-dimensional; for if f and g have weight k then the equation (4.4) implies that they have a unique common zero. Since the quotient f/g is then a modular form of weight 0 and hence constant it must be the case that $f = \lambda g$.

To produce generators for these spaces we introduce the following Eisenstein series for $k > 2$

$$\mathbb{G}_k(\tau) = \sum_{m,n}{}' \frac{1}{(m\tau + n)^k}, \tag{4.5}$$

where the "primed" summation means it is taken over all integers m, n except when both m and n are zero. Since $k > 2$, the double summation converges absolutely and one can check that it defines a holomorphic function on $\mathfrak{h}$. Since

$$\lim_{\tau \to i\infty} \sum_{m,n}{}' \frac{1}{(m\tau + n)^k} = \sum_{n \neq 0} \frac{1}{n^k} = 2\zeta(k)$$

is finite, the Fourier expansion of $\mathbb{G}_k$ has no negative powers of q. Clearly $\mathbb{G}_k(\tau+1) = \mathbb{G}_k(\tau)$, and we also easily see

$$\mathbb{G}_k(-1/\tau) = \sum_{m,n}{}' \frac{1}{(-\frac{m}{\tau} + n)^k} = \tau^k \mathbb{G}_k(\tau).$$

Hence $\mathbb{G}_k(\tau)$ is a modular form of weight k for all $k > 2$. Of course we know from above that the $\mathbb{G}_k$ vanish for all odd k. One can then obtain the Fourier expansion

$$\mathbb{G}_{2k}(\tau) = 2\zeta(2k) \left(1 - \frac{4k}{B_{2k}} \sum_{n=0}^{\infty} \sigma_{2k-1}(n) q^n \right).$$

Here B_{2k} is the $2k$th Bernoulli number and $\sigma_{2k-1}(n)$ is the arithmetic function given by $\sigma_{2k-1}(n) = \sum_{d|n} d^{2k-1}$. The (normalized) Eisenstein series of weight $2k$ is given by

$$E_{2k}(q) = \frac{1}{2\zeta(2k)}\mathbb{G}(\tau) = 1 - \frac{4k}{B_{2k}}\sum_{n=1}^{\infty}(\sum_{d|n} d^{2k-1})q^n \tag{4.6}$$

and is an element of $\mathcal{M}^{2k}$.

For instance,

$$\begin{aligned} E_4(q) &= 1 + 240q + 2160q^2 + 6720q^3 + ... \\ E_6(q) &= 1 - 504q - 16632q^2 - 122976q^3 - ... \end{aligned} \tag{4.7}$$

There is one further normalization which we will also find useful.

$$G_{2k}(q) = -\frac{B_{2k}}{4k}E_{2k}(q) \in \mathcal{M}^{2k}.$$

For instance, $G_4(q) = \frac{1}{240}E_4(q)$ and $G_6(q) = -\frac{1}{504}E_6(q)$. The ring structure of the space of modular forms is easy to describe. It turns out the following holds.

Theorem 4.4. *The graded ring $\mathcal{M}$ is a polynomial ring with generators E_4 and E_6. That is, $\mathcal{M} \simeq \mathbb{Q}[E_4, E_6]$.*

A certain class of modular forms which will be important in the following are *cusp* forms, those modular forms whose q-expansion vanishes at $q = 0$. As a function of τ this translates to vanishing at $i\infty$ and we see from (4.4) that this is not possible until weight $k = 12$. In that case there is the well-known discriminant function which can be written in terms of its q-expansion as

$$\Delta(q) := \frac{E_4(q)^3 - E_6(q)^2}{1728} = q - 24q^2 + 252q^3 + ... \in \mathcal{M}^{12}.$$

If we denote the space of weight k cusp forms (with rational q-expansion) by $\mathcal{S}^k$ then there is the following exact sequence for all even k

$$0 \longrightarrow \mathcal{S}^k \longrightarrow \mathcal{M}^k \xrightarrow{q \mapsto 0} \mathbb{Q} \longrightarrow 0$$

where the first map is given by inclusion and the second by evaluating at $q = 0$. Since these are vector spaces the sequence is split. A section $\mathbb{Q} \to \mathcal{M}^k$ is given, after noticing in (4.6) that $E_k(0) \neq 0$ when k is even, by $\lambda \mapsto \lambda E_k(q)$. Hence

$$\mathcal{M}^k = \mathbb{Q}E_k \oplus \mathcal{S}^k.$$

An immediate observation is that $\dim \mathcal{S}^k = \dim \mathcal{M}^k - 1$. Since also $\dim \mathcal{M}^{k-12} = \dim \mathcal{M}^k - 1$ we can be a bit more explicit by noticing that the injective linear map $\mathcal{M}^{k-12} \to \mathcal{S}^k$ defined by $f(q) \mapsto \Delta(q)f(q)$ must then be an isomorphism of vector spaces. Therefore

$$\mathcal{M}^k = \mathbb{Q}E_k \oplus \Delta(q)\mathcal{M}^{k-12}.$$

The equation (4.6) which defined the Fourier expansion of the Eisenstein series $E_{2k}(\tau)$ for $2k > 2$ still defines an element of $\mathbb{Q}[[q]]$ in the case $2k = 2$. We define

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n = 1 - 24q - 72q^2 - 96q^3 - ... \tag{4.8}$$

and $G_2(q) = -\dfrac{1}{24}E_2(q)$. The RHS of (4.8) with $q = e^{2\pi i\tau}$ converges rapidly and defines a holomorphic function $E_2(\tau)$ on $\mathfrak{h}$. In fact, if we are careful about the order of summation we have

$$\begin{aligned} E_2(\tau) &= \frac{1}{2\zeta(2)} \sum_{m=-\infty}^{\infty} \sideset{}{'}\sum_{n=-\infty}^{\infty} \frac{1}{(m\tau+n)^2} \\ &= 1 + \frac{6}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau+n)^2}. \end{aligned}$$

One might naturally wonder if this defines a modular form of weight 2. However, we saw above that there are no modular forms of weight 2. The function $E_2(\tau)$ is what is referred to as a quasimodular form. Heuristically, it transforms under the action of $\mathrm{SL}(2,\mathbb{Z})$ as a weight 2 modular form should, except with a "correction" term. More precisely, we have for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2(\tau) + \frac{12c}{2\pi i}(c\tau+d) \tag{4.9}$$

and correspondingly

$$G_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 G_2(\tau) - \frac{c}{4\pi i}(c\tau+d). \tag{4.10}$$

The space of quasimodular forms can be defined more directly, but we will put them as the following.

Definition 4.5. *The ring of quasimodular forms is the subring $\widetilde{\mathcal{M}} \subset \mathbb{Q}[[q]]$ of polynomials in E_2 with coefficients in the ring of modular forms $\mathcal{M}$. That is,*

$$\widetilde{\mathcal{M}} = \mathcal{M}[E_2] = \mathbb{Q}[E_2, E_4, E_6].$$

Considering E_2 as a weight two element, $\widetilde{\mathcal{M}} = \oplus_{k=0}^{\infty}\widetilde{\mathcal{M}}^k$ has the structure of a graded ring and elements of $\mathcal{M}^k$ are called weight k quasimodular forms.

The ring $Q[[q]]$ is equipped with a derivation $D = q\frac{d}{dq}$. We will be interested in calculating how this derivation acts when restricted to the subring $\widetilde{\mathcal{M}} \subset \mathbb{Q}[[q]]$. Notice that under the map into holomorphic functions $\widetilde{\mathcal{M}} \to \mathrm{Hol}(\mathfrak{h})$ induced by $q = e^{2\pi i\tau}$ we have $D = \frac{1}{2\pi i}\frac{d}{d\tau}$. Applying $\frac{1}{2\pi i}\frac{d}{d\tau}$ to the defining transformation property (4.2) we see that given a modular form f of weight k, its derivative $Df := f'$ transforms as

$$f'\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k+2}f'(\tau) + \frac{k}{2\pi i}(c\tau+d)^{k+1}f(\tau).$$

Comparing this with (4.9) we see that the expression

$$\vartheta_k f := f' - \frac{k}{12}E_2 f$$

defines a modular form of weight $k+2$. The derivation ϑ_k is known as the Serre derivative. Pulled back to $\mathcal{M}$ we have

$$\vartheta_k : \mathcal{M}^k \to \mathcal{M}^{k+2}$$

defined by

$$\vartheta_k f(q) = Df(q) - \frac{k}{12}E_2(q)f(q).$$

Note that $\vartheta_4 E_4(q) \in \mathcal{M}^6 = \mathbb{Q}E_6(q)$. As the constant term in $\vartheta_4 E_4(q)$ is easily seen to be $-\frac{1}{3}$ we have $\vartheta_4 E_4(q) = -\frac{1}{3}E_6(q)$. Similarly, $\vartheta_6 E_6(q) = -\frac{1}{2}E_8(q) = -\frac{1}{2}E_4(q)^2$. It follows easily from this that

$$\begin{aligned} DE_4(q) &= \frac{E_2(q)E_4(q) - E_6(q)}{3} \\ DE_6(q) &= \frac{E_2(q)E_6(q) - E_4(q)^2}{2}. \end{aligned} \tag{4.11}$$

By differentiating (4.9) we can see that $E_2' - \frac{1}{12}E_2^2$ is a modular form of weight 4. We have thus also obtained

$$DE_2(q) = \frac{E_2(q)^2 - E_4(q)}{12}. \tag{4.12}$$

It follows from (4.11) and (4.12) that the operator D is a map

$$D = q\frac{d}{dq} : \widetilde{\mathcal{M}}^k \to \widetilde{\mathcal{M}}^{k+2}.$$

It is worth mentioning that $\Delta(q)$ satisfies $\vartheta_{12}\Delta(\tau) = 0$ and hence

$$D\Delta(q) = \Delta(q)E_2(q).$$

The last function related to modular forms which needs to be introduced is the Dedekind eta function

$$\eta(q) = q^{1/24}\prod_{n=0}^{\infty}(1-q^n).$$

The function $\eta(\tau) = e^{2\pi i\tau/24}\prod_{n=1}^{\infty}(1-e^{2\pi i n\tau})$ is not a level 1 modular form, however it is a weight 1/2 modular form with respect to a subgroup of $\mathrm{SL}(2,\mathbb{Z})$, but this will not be necessary for us. However,

$$\eta(q)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \Delta(q).$$

4.2 The Case When The Parameterizing Space is String

Viewing $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ as $\mathbb{Q}$-vector spaces we have $H^*(X,\mathcal{M}) \simeq H^*(X,\mathbb{Q}) \otimes \mathcal{M}$ and similarly $H^*(X,\widetilde{\mathcal{M}}) \simeq H^*(X,\mathbb{Q}) \otimes \widetilde{\mathcal{M}}$. Both $H^*(X,\mathcal{M})$ and $H^*(X,\widetilde{\mathcal{M}})$ are naturally identified with subrings of $H^*(X,\mathbb{Q}[[q]]) \simeq H^*(X,\mathbb{Q})[[q]]$. The integration over the fibers map above was extended to $H^*(Z,\mathbb{Q})[[q]] \to H^{*-m}(X,\mathbb{Q})[[q]]$ coefficient by coefficient. It is not hard to see that this restricts to the map

$$\int_Y : H^*(Z, \widetilde{\mathcal{M}}^*) \to H^{*-m}(X, \widetilde{\mathcal{M}}^*)$$
$$\omega \otimes f(q) \mapsto \left(\int_Y \omega\right) \otimes f(q).$$

In [3], the assumption $p_1(X) = 0$ was present throughout and the following theorem and its corollary were already noticed, though the modularity of the graded components was shown by other means.

Theorem 4.6. *Let $\mathcal{F} = (\pi, Z, X)$ be a string family of compact spin manifolds with fibers $Y_x = \pi^{-1}(x)$ of dimension m. If we expand $\mathrm{sch}(\mathcal{F}; q)$ into its homogeneous components as an element of $H^*(X, \mathbb{Q}[[q]])$,*

$$\mathrm{sch}(\mathcal{F}; q) = \mathrm{sch}_0(\mathcal{F}; q) + \mathrm{sch}_1(\mathcal{F}; q) + ...$$

then $\mathrm{sch}_j(\mathcal{F}; q) \in H^{2j}(X, \widetilde{\mathcal{M}}^{\frac{m}{2}+j})$, and if $p_1(X) = 0$ then $\mathrm{sch}_j(\mathcal{F}; q) \in H^{2j}$ $(X, \mathcal{M}^{\frac{m}{2}+j})$. Moreover,

$$\mathrm{sch}_0(\mathcal{F}; q) = \mathrm{ind}\, \not{D}_Y,$$

i.e. $\mathrm{sch}_0(\mathcal{F}; q)$ is the string, or Witten, genus of Y.

Proof. Since $p_1(Z) = 0$

$$\mathrm{sch}(\mathcal{F}; q) = e^{-G_2(q)p_1(X)} \int_Y \hat{a}(V, q). \tag{4.13}$$

The function $\widehat{a}(z, \tau) = \dfrac{z}{\sigma(z, \tau)} = \exp\left(\displaystyle\sum_{n=2}^{\infty} \frac{2}{2n!} G_{2n}(\tau) z^{2n}\right)$ is even, has $\widehat{a}(0, \tau) = 1$, and is homolomorphic at $z = 0$. Thus it has a Taylor series expansion

$$\widehat{a}(z, \tau) = 1 + f_2(\tau)z^2 + f_4(\tau)z^4 + ... \tag{4.14}$$

We have

$$\widehat{a}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=\exp\left(\sum_{n=2}^{\infty}\frac{2}{2n!}G_{2n}\left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{z}{c\tau+d}\right)^{2n}\right)=\widehat{a}(z,\tau).$$

Thus

$$\widehat{a}\left(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\right)=1+f_2\left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{z}{c\tau+d}\right)^2+f_4\left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{z}{c\tau+d}\right)^4+...$$

is equal to (4.14). Equating coefficients of z^{2n} it follows that $f_{2n}\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^{2n}f_{2n}(\tau)$. Hence (4.14) can be considered as a formal power series with modular form coefficients. After identifying modular forms with their q-expansions we see that the coefficient f_{2n} lies in $\mathcal{M}^{2n}$. In the language of Appendix A, $\sigma(x,\tau)$ is a Jacobi-like form of weight 0 and index 0. We can now consider the Weierstrass sigma function σ as an element $\sigma(z,q)\in\mathcal{M}^*[[z]]$. Letting $y_1,...,y_{m/2}$ denote the roots for the vertical bundle V, it follows that

$$\begin{aligned}\prod_{n=1}^{m/2}\widehat{a}(y_n,q)&=\prod_{n=1}^{m/2}(1+f_2(q)y_n^2+f_4(q)y_n^4+...)\\&=1+f_2(q)p_1(V)+f_4(q)p_1(V)^2+(f_2(q)^2-2f_4(q))p_2(V)+...\end{aligned}$$

That is, if we let $\prod_{n=1}^{m/2}\widehat{a}(y_n,q)=1+a_2(q)+a_4(q)+...$ be the decomposition into homogeneous components in cohomology, then $a_{2n}(q)\in H^{4n}(Z,\mathcal{M}^{2n})$. Now,

$$\mathrm{sch}_j(\mathcal{F};q)=\int_Y a_{\frac{m}{2}+j}(q) \tag{4.15}$$

and integration over the fiber defines a map $H^{m+2j}(Z,\mathcal{M}^{\frac{m}{2}+j})\to H^{2j}(X,\mathcal{M}^{\frac{m}{2}+j})$. Thus the $p_1(X)=0$ part of the theorem follows. If $p_1(X)\neq 0$ then the factor $e^{-G_2(q)p_1(X)}$ will produce cohomology classes with coefficients that are polynomials in $G_2(q)$ with coefficients in $\mathcal{M}$, which is precisely $\widetilde{\mathcal{M}}$. Note that since $G_2(q)$ has modular weight 2 and $p_1(X)$ has cohomological degree 4, a homogeneous cohomology class of degree $2j$ in $\mathrm{sch}(\mathcal{F};q)$ will still have quasimodular weight $m/2+j$. The last part follows by evaluating on a point in X. $\square$

The (quasi)modularity forces many relations between the characteristic classes of the index bundles at each level. When the dimension m of the manifold Y and the degree in cohomology are small we have the following result.

Corollary 4.7. *Assume the setup of the previous theorem with $p_1(X)=0$. Let j be such that $\frac{m}{2}+j \leq 14$ and $\frac{m}{2}+j \neq 12$. Then, for each n there is $c(j,n) \in \mathbb{Q}$ such that*

$$\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_n}) = c(j,n)\,\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/) \quad \in H^{2j}(X,\mathbb{Q}).$$

Proof. From the previous theorem we know that the degree $2j$ component $\mathrm{sch}_j(\mathcal{F};q)$ of $\mathrm{sch}(\mathcal{F};q)$ is an element of $H^{2j}(X,\mathcal{M}^{\frac{m}{2}+j}) = H^{2j}(X,\mathbb{Q})\otimes\mathcal{M}^{\frac{m}{2}+j}$. For $m/2+j = 4,6,8,10,14$, have $\mathcal{M}^{m/2+j} = \mathbb{Q}E_{\frac{m}{2}+j}(q)$. Thus

$$\mathrm{sch}_j(\mathcal{F};q) = \omega E_{\frac{m}{2}+j}(q), \quad \text{for some } \omega \in H^{2j}(X,\mathbb{Q}). \tag{4.16}$$

Now also,

$$\mathrm{sch}_j(\mathcal{F};q) = \frac{\eta(q)^m}{q^{m/24}}\,\mathrm{Sch}_j(\mathcal{F};q) = \prod_{n=1}^{\infty}(1-q^n)^m \sum_{i=0}^{\infty}\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_i})q^i \tag{4.17}$$

Comparing the q^0 term in (4.16) and (4.17) we see $\omega = \mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/)$. Hence

$$\begin{aligned}\sum_{i=0}^{\infty}\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_i})q^i &= \mathrm{Sch}_j(\mathcal{F};q) \\ &= \frac{q^{m/24}}{\eta(q)^m}\,\mathrm{sch}_j(q) = \mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/)q^{m/24}\frac{E_{\frac{m}{2}+j}(q)}{\eta(q)^m}.\end{aligned} \tag{4.18}$$

The proportionality factor $c(j,n)$ are then extracted from $E_{\frac{m}{2}+j}(q)\prod_{n=1}^{\infty}(1-q^n)^{-m}$ by taking the coefficient of q^n. Since $\mathcal{M}^{\frac{m}{2}+j}=\{0\}$ for all other values of $m/2+j$ up to 14 except 12, the result follows (trivially) for these values. $\square$

The above does not apply in the case of $m/2+j=12$ since $\mathcal{M}^{12}$ is 2 dimensional. However, the methods of the proof above can be extended. Essentially, one sees that if the dimension of $\mathcal{M}^{m/2+j}$ is $s>1$, then for all $n\geq s$, $\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_n})$ will be a linear combination of $\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_0})$,...,$\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_{s-1}})$.

We will illustrate all of this very explicitly when dim $Y=8$ for low degrees in cohomology. Assume also that $p_1(X)=0$ and that dim X is divisible by 4. In this situation every time a cohomology class of degree $2k$ appears, it will be multiplied by a modular form of weight $4+k$. From Corollary 4.7 we know

$$\begin{aligned}\mathrm{sch}_{\leq 6}(\mathcal{F};q) = {} & \mathrm{ch}_0(\mathrm{Ind}\,\partial\!\!\!/)E_4(q)+\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/)E_6(q)+\mathrm{ch}_4(\mathrm{Ind}\,\partial\!\!\!/)E_8(q) \\ & +\mathrm{ch}_6(\mathrm{Ind}\,\partial\!\!\!/)E_{10}(q).\end{aligned} \tag{4.19}$$

The degree 0 component of the Chern character is just the (virtual) rank of the index bundle. Set $\nu_i:=\mathrm{rk}\,\mathrm{Ind}\,\partial\!\!\!/^{V_i}$. In particular ν_0 is the virtual rank of the index bundle of the untwisted Dirac operator, i.e. the index of the usual Dirac operator on Y. Using (4.18) for $k=0$ one obtains

$$\nu_0 q^{1/3}\frac{E_4(q)}{\eta(q)^8}=\sum_{i=0}^{\infty}\nu_i q^i.$$

We can use

$$\nu_0\frac{E_4(q)}{\eta(q)^8}=\nu_0 j(q)^{1/3}=\nu_0 q^{-1/3}(1+248q+4124q^2+...). \tag{4.20}$$

Then comparing the right hand sides of the previous two equations we get relations like

$$\begin{aligned} \mathrm{rk}\,\mathrm{Ind}\,\partial\!\!\!/^{V_{\mathbb{C}}} &= 248\,\mathrm{rk}\,\mathrm{Ind}\,\partial\!\!\!/ \\ \mathrm{rk}\,\mathrm{Ind}\,\partial\!\!\!/^{S^2V_{\mathbb{C}}\oplus V_{\mathbb{C}}} &= 4124\,\mathrm{rk}\,\mathrm{Ind}\,\partial\!\!\!/ \\ &\vdots \end{aligned} \tag{4.21}$$

Putting $k = 2$ in (4.18) gives

$$\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/)q^{1/3}\frac{E_6(q)}{\eta(q)^8} = \sum_{i=0}^{\infty}\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/^{V_i})q^i.$$

Noting also that $q^{1/3}\dfrac{E_6(q)}{\eta(q)^8} = (1 - 496q - 20620q^2 + ...)$ gives

$$\begin{aligned} \mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/^{V_{\mathbb{C}}}) &= -496\,\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/) \\ \mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/^{S^2V_{\mathbb{C}}\oplus V_{\mathbb{C}}}) &= -20620\,\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/) \\ &\vdots \end{aligned} \tag{4.22}$$

There are similar relations between the cohomology classes in degree 8 and 12 that one could write out. The degree 16 cohomology classes, however, are not all proportional. This is because they have coefficients in $\mathcal{M}^{12}$, which is 2 dimensional. We use the basis $\{E_4(q)^3 - 728\Delta(q), \Delta(q)\}$ of $\mathcal{M}^{12}$. We have

$$\begin{aligned} \mathrm{Sch}_8(\mathcal{F}; q) &= \alpha q^{1/3}\frac{E_4(q)^3 - 728\Delta(q)}{\eta(q)^8} + \beta q^{1/3}\frac{\Delta(q)}{\eta(q)^8} \\ &= \alpha(1 + 196732q^2 + ...) + \beta(q - 16q^2 + ...). \end{aligned}$$

which is to be compared with

$$\sum_{i=0}^{\infty}\mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_i})q^i = \mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_0}) + \mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_1})q + ...$$

and the values of α and β are easily read off so that (4.19) can be extended by including

$$\mathrm{sch}_8(\mathcal{F};q) = \mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/)(E_4(q)^3 - 728\Delta(q)) + \mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_{\mathbb{C}}})\Delta(q).$$

We could expand this in powers of q. Rather than the classes $\mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_n})$ being proportional for all $n \in \mathbb{N}$, as before, we would see that $\mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_n})$ for $n \geq 2$ is a linear combinations of $\mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/)$ and $\mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_1})$.

Moving to degree 20 in cohomology will give weight 14 modular forms. Here again the Corollary 4.7 can be applied and the next term is simply in $\mathrm{sch}(\mathcal{F};q)$ is

$$\mathrm{sch}_{10}(\mathcal{F};q) = \mathrm{ch}_{10}(\mathrm{Ind}\,\partial\!\!\!/)E_{14}(q).$$

The method for all higher degrees is a straightforward generalization of the degree 16 case. Let r denote the dimension of $\mathcal{M}^k$, then there is a basis $\{f_0(q),\ f_1(q)\Delta(q),\ \dots,\ f_{r-1}(q)\Delta(q)^{r-1}\}$ for $\mathcal{M}^k$ with $f_i(0) = 1$ and, of course, $\Delta(q)^i = q^i + \dots$. Simple linear algebra can be used to find in $\mathcal{M}^k$ a basis $\{\phi_0(q), \dots, \phi_{r-1}(q)\}$ instead, which satisfies $q^{1/3}\dfrac{\phi_i(q)}{\eta(q)^8} = q^i + \mathcal{O}(q^r)$. Working in this basis it is easy to see

$$\mathrm{Sch}_{k-4}(\mathcal{F};q) = \frac{q^{1/3}}{\eta(q)^8}\left(\mathrm{ch}_{k-4}(\mathrm{Ind}\,\partial\!\!\!/^{V_0})\phi_0(q) + \dots + \mathrm{ch}_{k-4}(\mathrm{Ind}\,\partial\!\!\!/^{V_{r-1}})\phi_{r-1}(q)\right).$$

We restate the last part of the example in more general terms.

Proposition 4.8. *Let $Z \to X$ be a string family of compact spin manifolds where each $Y_x = \pi^{-1}(X)$ has even dimension m and $p_1(X) = 0$. Let $s_j = \dim \mathcal{M}^{\frac{m}{2}+j}$, and $\{\phi_0(q), \dots, \phi_{s_j-1}(q)\}$ be the basis for $\mathcal{M}^{\frac{m}{2}+j}$ which satisfies $q^{m/24}\dfrac{\phi_i(q)}{\eta(q)^m} = q^i + \mathcal{O}(q^{s_j})$. Then*

$$\mathrm{Sch}_j(\mathcal{F};q) = \frac{q^{m/24}}{\eta(q)^m}\left(\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_0})\phi_0(q) + \dots + \mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_{s_j-1}})\phi_{s_j-1}(q)\right).$$

Before moving on we wish to point out some connection with the preceding and anomaly cancellation. Thinking of the index bundle for the Dirac operator as the formal difference $\operatorname{Ind}\not{\partial} = \ker\not{\partial} - \operatorname{coker}\not{\partial}$ in $K(X)$, one can define the determinant line bundle

$$\det\not{\partial} = \det(\ker\not{\partial}) \otimes \det(\operatorname{coker}\not{\partial})^* \in K(X).$$

As in defining the index bundle, this is not strictly true as the dimension of each space $\ker\not{\partial}$ and $\operatorname{coker}\not{\partial}$ may individually jump. However the determinant line bundle $\det\not{\partial} \to X$ can still be defined (see [12]) and one has

$$c_1(\det\not{\partial}) = c_1(\operatorname{Ind}\not{\partial}) = \operatorname{ch}_1(\operatorname{Ind}\not{\partial}) \in H^2(X, \mathbb{Q}). \tag{4.23}$$

In physics, this characteristic class is referred to as an anomaly. The Proposition 4.8 produces many "anomaly cancellation formulas." Examples of these formulas are

$$c_1(\det\not{\partial}^{V_n}) = \alpha(n) c_1(\det\not{\partial}), \quad \text{for some } \alpha(n) \in \mathbb{Q}. \tag{4.24}$$

which follow directly from Corollary 4.7 whenever $\dim Y \leq 24$, except when $\dim Y = 20$. The equation (4.24) holds nontrivially when $\dim Y = 6, 10, 14, 18$, or 22. The operator $\not{\partial}^{V_1} = \not{\partial}^{V_{\mathbb{C}}} : C^\infty(S^+ \otimes V_{\mathbb{C}}) \to C^\infty(S^- \otimes V_{\mathbb{C}})$ is almost what is known as the Rarita-Schwinger operator. If $\dim Y = 6$ and $p_1(Z) = p_1(X) = 0$ [1] then

$$c_1(\det\not{\partial}^{V_{\mathbb{C}}}) = 246 c_1(\det\not{\partial})$$

If $\dim Y = 20$ or $\dim Y > 24$ one needs to appeal more directly to Proposition 4.8. For instance, in the case that $\dim Y = 20$ the proposition gives

$$\operatorname{Sch}_1(\mathcal{F}; q) = \frac{q^{5/6}}{\eta(q)^{20}} \left(\operatorname{ch}_1(\operatorname{Ind}\not{\partial})(E_4(q)^3 - 740\Delta(q)) + \operatorname{ch}_1(\operatorname{Ind}^{V_1})\Delta(q) \right)$$

[1] It is sufficient to require just that $p_1(V) = 0$ instead.

from which it follows that

$$c_1(\det \partial\!\!\!/^{S^2V_{\mathbb{C}}\oplus V_{\mathbb{C}}}) = 196870c_1(\det \partial\!\!\!/) - 4c_1(\det \partial\!\!\!/^{V_{\mathbb{C}}}).$$

4.3 Some Computational Motivation

Now we will drop the assumption that $p_1(X) = 0$ (but maintain $p_1(Z) = 0$). The results will now be quasimodular rather than modular. The dimensions of the space of quasimodular forms grow much quicker than for modular forms as one goes to higher weights. Because of this one might expect much less rigidity in the structure of the index bundle for a family of Dirac-Ramond operators. However, we will see that this is not the case. In the next section we will state and prove a theorem which generalizes (4.8) in the case $p_1(X) \neq 0$. The formula within the theorem is very complicated and in this section we will demonstrate the formula in some special cases. In the case where the fiber Y has dimension 8 some direct computation with the formula (3.15) (see Appendix B) shows the following

$$\begin{aligned}
&\mathrm{sch}_{\leq 6}(\mathcal{F};q) = \\
&\nu_0\Bigg(E_4(q) + \frac{1}{(4)_1}E_4'(q)\left(\frac{p_1(X)}{2}\right) + \frac{1}{2!(4)_2}E_4''(q)\left(\frac{p_1(X)}{2}\right)^2 \\
&\qquad\qquad + \frac{1}{3!(4)_3}E_4'''(q)\left(\frac{p_1(X)}{2}\right)^3\Bigg) \\
&+ \mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/)\Bigg(E_6(q) + \frac{1}{(6)_1}E_6'(q)\left(\frac{p_1(X)}{2}\right) + \frac{1}{2!(6)_2}E_6''(q)\left(\frac{p_1(X)}{2}\right)^2\Bigg) \\
&+ \mathrm{ch}_4(\mathrm{Ind}\,\partial\!\!\!/)\Bigg(E_8(q) + \frac{1}{(8)_1}E_8'(q)\left(\frac{p_1(X)}{2}\right)\Bigg) \\
&+ \mathrm{ch}_6(\mathrm{Ind}\,\partial\!\!\!/)E_{10}(q)
\end{aligned} \tag{4.25}$$

where $(k)_n = k(k+1)...(k+n-1)$ is the Pochhammer symbol and $f'(q) = q\frac{d}{dq}f(q)$. Note that all of the terms with a derivative are of order q, so when $q \to 0$ one obtains

$$\mathrm{Sch}_{\leq 6}(\mathcal{F};0) = \nu_0 + \mathrm{ch}_2(\mathrm{Ind}\,\not{\partial}) + \mathrm{ch}_4(\mathrm{Ind}\,\not{\partial}) + \mathrm{ch}_6(\mathrm{Ind}\,\not{\partial}) = \mathrm{ch}_{\leq 6}(\mathrm{Ind}\,\not{\partial}),$$

as expected from (3.13). Of course, the $p_1(X) \to 0$ limit reduces to the previous case (4.19).

Using (4.25), $\mathrm{Sch}(\mathcal{F};q) = \mathrm{sch}(\mathcal{F};q)\prod_{n=1}^{\infty}(1-q^n)^{-8}$, and the q expansion of the Eisenstein series the following relations are obtained

$$\begin{aligned}
\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial}^{V_{\mathbb{C}}}) &= -496\,\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial}) + 30\nu_0 p_1(X) \\
\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial}^{S^2V_{\mathbb{C}}\oplus V_{\mathbb{C}}}) &= -20620\,\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial}) + 780\nu_0 p_1(X) \\
\mathrm{ch}_4(\mathrm{Ind}\,\not{\partial}^{V_{\mathbb{C}}}) &= 488\,\mathrm{ch}_4(\mathrm{Ind}\,\not{\partial}_z) - 42p_1(X)\,\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial}) + \frac{3}{2}\nu_0 p_1(X)^2 \\
\mathrm{ch}_4(\mathrm{Ind}\,\not{\partial}^{S^2V_{\mathbb{C}}\oplus V_{\mathbb{C}}}) &= 65804\,\mathrm{ch}_4(\mathrm{Ind}\,\not{\partial}) - 3108p_1(X)\,\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial}) + 66\nu_0 p_1(X)^2.
\end{aligned} \tag{4.26}$$

The next relevant degree is 16, where the modular forms become weight 12 and now have an extra dimension; we expect something interesting to happen. The result can be written

$$\begin{aligned}
\mathrm{sch}_8(\mathcal{F};q) = {}& \mathrm{ch}_8(\mathrm{Ind}\,\not{\partial})(E_4(q)^3 - 728\Delta(q)) + \mathrm{ch}_8(\mathrm{Ind}\,\not{\partial}^{V_{\mathbb{C}}})\Delta(q) \\
&+ \frac{\nu_0}{4!\cdot(4)_4}\left(E_4^{(4)}(q) - 240\Delta(q)\right)\left(\frac{p_1(X)}{2}\right)^4 \\
&+ \frac{\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial})}{3!\cdot(6)_3}\left(E_6^{(3)}(q) + 504\Delta(q)\right)\left(\frac{p_1(X)}{2}\right)^3 \\
&+ \frac{\mathrm{ch}_4(\mathrm{Ind}\,\not{\partial})}{2!\cdot(8)_2}\left(E_8^{(2)}(q) - 480\Delta(q)\right)\left(\frac{p_1(X)}{2}\right)^2 \\
&+ \frac{\mathrm{ch}_6(\mathrm{Ind}\,\not{\partial})}{(10)_1}\left(E_{10}'(q) + 264\Delta(q)\right)\left(\frac{p_1(X)}{2}\right).
\end{aligned} \tag{4.27}$$

What we see is that the pattern of coefficients in (4.25) continues, but there are extra terms. Notice that each of the terms $E_4^{(4)}(q) - 240\Delta(q)$,...,$E'_{10}(q) + 264\Delta(q)$ are all order q^2. This makes sense since after multiplying by $\prod_{n=1}^{\infty}(1-q^n)^{-8}$ the second term will give $\mathrm{ch}_8(\mathrm{Ind}\,\partial\!\!\!/^{V_{\mathbb{C}}})q$ and this is entirely what the coefficient of q in $\mathrm{Sch}_8(q)$ should be.

When $\dim Y = 6$, one obtains similar results. The same calculations as above show that in this case

$$
\begin{aligned}
&\mathrm{sch}_{\leq 7}(\mathcal{F};q) = \\
&\mathrm{ch}_1(\mathrm{Ind}\,\partial\!\!\!/)\times \qquad\qquad (4.28)\\
&\left(E_4(q) + \frac{1}{(4)_1}E'_4(q)\left(\frac{p_1(X)}{2}\right) + \frac{1}{2!(4)_2}E''_4(q)\left(\frac{p_1(X)}{2}\right)^2 + \frac{1}{3!(4)_3}E'''_4(q)\left(\frac{p_1(X)}{2}\right)^3\right)\\
&+\mathrm{ch}_3(\mathrm{Ind}\,\partial\!\!\!/)\left(E_6(q) + \frac{1}{(6)_1}E'_6(q)\left(\frac{p_1(X)}{2}\right) + \frac{1}{2!(6)_2}E''_6(q)\left(\frac{p_1(X)}{2}\right)^2\right)\\
&+\mathrm{ch}_5(\mathrm{Ind}\,\partial\!\!\!/)\left(E_8(q) + \frac{1}{(8)_1}E'_8(q)\left(\frac{p_1(X)}{2}\right)\right)\\
&+\mathrm{ch}_7(\mathrm{Ind}\,\partial\!\!\!/)E_{10}(q)
\end{aligned}
$$

4.4 A General Formula

Noticing the similarity of the coefficients in (4.25) and (A.6) we are led to formulate a general theorem which puts $\mathrm{sch}(\mathcal{F};q)$ as something akin to a "Jacobi-like form in $\frac{p_1(X)}{2}$". To make this more precise we first define a map

$$
\begin{aligned}
\mathcal{J}^{+}_{\mathbb{Q},k,\lambda} &\to H^*(X,\widetilde{\mathcal{M}})\\
F(z,q) = \sum_{j=0}^{\infty}\chi_{2j}(q)z^{2j} &\mapsto \sum_{j=0}^{\infty}\chi_{2j}(q)\left(\frac{p_1(X)}{2}\right)^j
\end{aligned}
$$

The map is well defined since X is finite dimensional. To maybe abuse notation, we will denote the image of an element $F(z,q)$ under this map by $F\left(\frac{1}{2}p_1(X),q\right)$ and call it a cohomological Jacobi-like form, or CJLF for short. Using this, we see that (4.25) can be restated as

$$\begin{aligned}\mathrm{sch}_{\leq 6}(\mathcal{F};q) =&\nu_0\widetilde{E_4}\left(\frac{p_1(X)}{2},q\right)_{\leq 12}+\mathrm{ch}_2(\mathrm{Ind}\,\not{\partial})\widetilde{E_6}\left(\frac{p_1(X)}{2},q\right)_{\leq 8} \\ &+\mathrm{ch}_4(\mathrm{Ind}\,\not{\partial})\widetilde{E_8}\left(\frac{p_1(X)}{2},q\right)_{\leq 4}+\mathrm{ch}_6(\mathrm{Ind}\,\not{\partial})\widetilde{E_{10}}\left(\frac{p_1(X)}{2},q\right)_{\leq 0}\end{aligned} \tag{4.29}$$

where for any CJLF $F\left(\frac{1}{2}p_1(X),q\right)$ we denote its projection onto degree at most n in cohomology by $F\left(\frac{1}{2}p_1(X),q\right)_{\leq n}$. Compare (4.29) to (4.19). The general formula for $\mathrm{sch}(\mathcal{F};q)$ given in the theorem below can be better understood by reexamining what we have so far, and what we get by moving to the next (nonzero) degree in cohomology. The terms that one gets from Proposition 4.8 in the $p_1(X)=0$ case, i.e. those in (4.19), have modular form coefficients. Taking the Cohen-Kuznetsov lift of these modular forms and promoting them to CJLF's gives (4.29). Moving to degree 16 in cohomology we see from (4.27) that we get two more modular terms guaranteed by Proposition 4.8 and four other terms having modular (cusp) form coefficients which cancel the coefficient of q in all terms containing $p_1(X)$ in (4.29) (making them order q^2). The pattern then repeats each time you move to the next relevant degree in cohomology, there will be some new terms which arise from Proposition 4.8, say s of them, and another term which cancels the first s coefficient of all other terms containing a $p_1(X)$.

To put things more formally we first need the following.

Lemma 4.9. *Let $\phi\in\mathcal{M}^k$ and put $s_{2j}=\dim\mathcal{M}^{k+2j}$. Then there is a unique element*

$$\phi^\natural(z,q)=\sum_{j=0}^{\infty}\chi_{2j}(q)z^{2j}\in\mathcal{J}^+_{\mathbb{Q},k,1}$$

such that $\chi_0(q) = \phi(q)$ *and for* $j > 0$

$$\chi_{2j}(q) \in q^{s_{2j}}\mathbb{Q}[[q]] \cap \widetilde{\mathcal{M}}^{k+2j}.$$

Proof. As in Theorem A.6, a sequence of modular forms $\{f_{2\ell}\}_{\ell \geq 0}$ such that $f_{2\ell} \in \mathcal{M}^{k+2\ell}$ uniquely defines an element $\phi^{\natural}(z,q) \in \mathcal{J}^{+}_{\mathbb{Q},k,\lambda}$ via

$$\phi^{\natural}(z,q) = \widetilde{f}_0(z,q) + z^2\widetilde{f}_2(z,q) + z^4\widetilde{f}_4(z,q) + ...$$

We set $f_0(q) = \phi(q)$ and define $f_{2\ell}(q)$ for $\ell > 0$ recursively by requiring that

$$f_{2\ell}(q) + \sum_{n=0}^{j-1} \frac{f_{2n}^{(\ell-n)}(q)}{(\ell-n)!(k+2n)_{\ell-n}} \in q^{s_{2\ell}}\mathbb{Q}[[q]].$$

This recursive equation has a unique solution for each j since the equation puts $s_{2\ell}$ independent conditions on the $f_{2\ell}$ and $\mathcal{M}^{k+2\ell}$ is $s_{2\ell}$ dimensional. $\square$

One can see that $0^{\natural}(z,q) = 0$; a less trivial example follows.

Example 4.10.

$$E_4^{\natural}(z,q) = \widetilde{E_4}(z,q) - z^8\frac{240}{4!(4)_4}\widetilde{\Delta}(z,q) - z^{12}\left(\frac{240}{6!(4)_6} - \frac{240}{4!(4)_4}\frac{1}{2!(12)_2}\right)\widetilde{\Delta E_4}(z,q) + ...$$

Comparing the example with (4.29) and (4.27) we see that, up to degree 16, the coefficient of ν_0 in $\mathrm{sch}(\mathcal{F};q)$ is the CJLF $E_4^{\natural}\left(\frac{1}{2}p_1(X),q\right)$. The following theorem asserts that this extends to all degrees in cohomology and that all the other components of the Chern characters of the index bundles of the various twisted Dirac operators that show up in $\mathrm{sch}(q)$ also have coefficients that are a CJLF $f^{\natural}\left(\frac{1}{2}p_1(X),q\right)$ for some modular form f.

Theorem 4.11. *Let $\mathcal{F} = (\pi, Z, X)$ be a string family of compact spin manifolds where each $Y_x = \pi^{-1}(x)$ has even dimension m. Let $s_j = \dim \mathcal{M}^{\frac{m}{2}+j}$. Then*

$$\mathrm{Sch}(\mathcal{F}; q) = \tag{4.30}$$
$$\frac{q^{m/24}}{\eta(q)^m} \sum_{j=0}^{\infty} \mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_0})\phi^{\natural}_{j,0}\left(\frac{p_1(X)}{2}, q\right) + ... + \mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_{s_j-1}})\phi^{\natural}_{j,s_j-1}\left(\frac{p_1(X)}{2}, q\right)$$

where for each j the collection $\phi_{j,0}(q), ..., \phi_{j,s_j-1}(q)$ is given by Proposition 4.8.

Proof. Consider the polynomial rings $S = \mathbb{Q}[p_1, ..., p_{m/2}]$ in the indeterminates p_i of weight $4i$. We regard S as a subspace of the polynomial ring $\mathbb{Q}[y_1, ..., y_{m/2}]$, with indeterminates y_i of weight $2i$, via the degree preserving injection $p_i \mapsto \sigma_i(y_1^2, ..., y_{m/2}^2)$, where σ_i is the ith elementary symmetric function. Using the series expansion of the exponential function one obtains

$$\psi(z, q) = \prod_{i=1}^{m/2} \frac{zy_i}{\sigma(zy_i, q)} = \prod_{i=1}^{m/2} \exp\left(\sum_{n=2}^{\infty} \frac{2}{2n!} G_{2n}(q)(y_i)^{2n} z^{2n}\right)$$

as an element of $(\mathbb{Q}[[q]] \otimes S)[[z]]$. From the proof of Theorem 4.6, we see that $\psi(z, q)$ actually lies in a smaller space. Namely, let $S = S^0 \oplus S^4 \oplus ...$ be the decomposition into homogeneous subspaces and set $\mathcal{R} = \bigoplus_{j=0}^{\infty} \mathcal{M}^{2j} \otimes S^{4j}$; then $\psi(z, q) \in \mathcal{R}[[z]]$ and the coefficient of z^n is in $\mathcal{M}^{2n} \otimes S^{4n}$.

Let $r \in \{0, 2\}$ be the reduction of m modulo 4. Expanding

$$\psi(z, q) = 1 + z^2 a_2(q; p_1) + z^4 a_4(q; p_1, p_2) + ...$$

we define

$$\Psi(z,q) \tag{4.31}$$
$$= \frac{1}{z^{m/2}}\left[\psi(z,q) - \left(1 + z^2 a_2(q;p_1) + ... + z^{(m+r)/2-2} a_{(m+r)/2-2}(q;p_1,...)\right)\right] e^{G_2(q)p_1z^2}$$
$$= e^{G_2(q)p_1z^2} \sum_{j=(m+r)/4}^{\infty} a_{2j}(q;p_1,...)z^{2j-m/2}$$

Notice that the coefficient of z^n in $\sum_{j=(m+r)/4}^{\infty} a_{2j}(q;p_1,...)z^{2j-m/2}$ is an element of $\mathcal{M}^{\frac{m}{2}+n} \otimes S^{m+2n}$. Set $\widetilde{\mathcal{R}} = \bigoplus_{j=0}^{\infty} \widetilde{\mathcal{M}}^{2j} \otimes S^{4j}$; then we have $\Psi(z,q) \in \widetilde{\mathcal{R}}[[z]]$ and if

$$\Psi(z,q) = \chi_0(q;p_1,...) + \chi_1(q;p_1,...)z + \chi_2(q;p_1,...)z^2 + ...$$

then $\chi_n(q;p_1,...) \in \widetilde{\mathcal{M}}^{\frac{m}{2}+n} \otimes S^{m+2n}$.

Using Example A.2, we see from (4.31) that for any $(y_1,...,y_{m/2}) \in C^{m/2}$ it follows that Ψ is in either $\mathcal{J}^{+}_{m/2,-\frac{p_1}{2}}$ or $\mathcal{J}^{-}_{m/2,-\frac{p_1}{2}}$ depending on whether $r = 0$ or $r = 2$, respectively. We can then form the modular combinations as in (A.8)

$$\xi_n(q;p_1,...) = \sum_{0\le j\le n/2} \frac{(\frac{p_1}{2})^j(\frac{m}{2}+n-j-2)!}{j!(\frac{m}{2}+n-2)!}\chi^{(j)}_{n-2j}(q;p_1,...) \in \mathcal{M}^{m/2+n} \otimes S^{m+2n} \tag{4.32}$$

and, as above, $\chi^{(j)} = \left(q\dfrac{d}{dq}\right)^j \chi$. Then as in (A.7) we have

$$\Psi(z,q) = \widetilde{\xi}_0(z,q) + z\widetilde{\xi}_1(z,q) + z^2\widetilde{\xi}_2(z,q) + ...$$

where

$$\widetilde{\xi}_n(z,q) = \sum_{\nu=0}^{\infty} \frac{(-\frac{p_1}{2})^{\nu}\xi_n^{(\nu)}(q;p_1,...)}{\nu!(m/2+n)_{\nu}} z^{2n} \in \widetilde{\mathcal{R}}[[z]]$$

is the Cohen-Kuznetsov lift of $\widetilde{\xi}_n$ with index $-\frac{p_1}{2}$. From this we obtain the important formula

$$\Psi(z,q)=\sum_{n=0}^{\infty}z^n\widetilde{\xi}_n(z,q)=\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\mu}\frac{\xi_\nu^{(\mu-\nu)}(q;p_1,...)}{(\mu-\nu)!(m/2+\nu)_{\mu-\nu}}\left(-\frac{p_1}{2}\right)^{\mu-\nu}z^{2\nu-\mu}. \tag{4.33}$$

Now we take $z=1$ and use the identity in (4.33) with the p_i's replaced by Pontryagin classes for the vertical bundle $V\to Z$. The assumption $p_1(Z)=0$ gives $p_1(V)=-\pi^*p_1(X)$ and since each $a_{2j}(q;p_1,...)$ is degree $4j$ in cohomology we have

$$\int_Y a_{2j}(q;p_1,...)e^{G_2(q)p_1(V)}=e^{-G_2(q)p_1(X)}\int_Y a_{2j}(q;p_1,...)=0$$

for $2j\le(m+r)/2-2$. Therefore,

$$\int_Y\psi(1,q)e^{G_2(q)p_1(V)}=\int_Y\Psi(1,q).$$

We have thus obtained

$$\begin{aligned}\mathrm{sch}(\mathcal{F};q)&=\int_Y\prod_{i=1}^{m/2}\frac{y_i}{\sigma(y_i,q)}e^{G_2(q)y_i^2}=\int_Y\Psi(1,q)\\&=\int_Y\widetilde{\xi}_0(1,q)+\widetilde{\xi}_1(1,q)+\widetilde{\xi}_2(1,q)+...\in H^*(X,\widetilde{\mathcal{M}}^*)\end{aligned} \tag{4.34}$$

Note that $\int_Y\xi_j(q;p_1,...)\in H^{2j}(X,\mathcal{M}^{\frac{m}{2}+j})$. Now, we will proceed by induction to show that for each j_0

$$\int_Y\xi_{j_0}(q;p_1,...)=\sum_{j=0}^{j_0}\sum_{i=0}^{s_j-1}\mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_i})\left(\frac{p_1(X)}{2}\right)^{\lfloor(j_0-j)/2\rfloor}f_{j,i}^{j_0-j}(q) \tag{4.35}$$

where for each $j\le j_0$ the collection of $\phi_{j,i}(q):=f_{j,i}^0(q)\in\mathcal{M}^{\frac{m}{2}+j}$, for $i=1,...,s_j-1$, are given by Proposition 4.8 and $f_{j,i}^{j_0-j}(q)\in\mathcal{M}^{\frac{m}{2}+j_0}$ are defined so that when $j_0-j=$

2ℓ they satisfy

$$f^{2\ell}_{j,i}(q) + \sum_{n=0}^{\ell-1} \frac{D^{\ell-n} f^{2n}_{j,i}(q)}{(\ell-n)!(m/2+j+2n)_{\ell-n}} \in q^{s_{j_0}}\mathbb{Q}[[q]].$$

When $j_0 - j$ is odd the $f^{j_0-j}_{j,i}$'s might as well be taken to be zero. The reason for this is that $\mathrm{sch}_j(\mathcal{F}, q)$, $\mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_i})$, and the ξ_j all vanish for all odd j or all even j depending on whether $r = 0$ or $r = 2$, respectively. It is for this reason that the appearance of the floor function in (4.35) and the following is not all that significant. To see why (4.35) will imply the theorem, we combine it with (4.33) and (4.34) to see

$$\begin{aligned}
\mathrm{sch}_{\le j_0}(q) &= \left(\int_Y \sum_{j=0}^{\infty} \widetilde{\xi}_j(1,q)\right)_{\le 2j_0} \\
&= \sum_{\mu=0}^{j_0}\sum_{\nu=0}^{\mu} \frac{1}{(\mu-\nu)!(m/2+\nu)_{\mu-\nu}} \left(\frac{p_1(X)}{2}\right)^{\mu-\nu} \int_Y \xi_\nu^{(\mu-\nu)}(q, p_1, \ldots) \\
&= \sum_{\mu=0}^{j_0}\sum_{\nu=0}^{\mu}\sum_{j=0}^{\nu}\sum_{i=0}^{s_j-1} \mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_i}) \left(\frac{p_1(X)}{2}\right)^{\mu-\nu+\lfloor(\nu-j)/2\rfloor} \frac{D^{\mu-\nu} f^{\nu-j}_{j,i}(q)}{(\mu-\nu)!(m/2+\nu)_{\mu-\nu}}.
\end{aligned}$$

After setting $2\ell = \nu - j$ and $\beta = \mu - \nu$ this becomes

$$\begin{aligned}
&\sum_{j=0}^{j_0}\sum_{i=0}^{s_j-1} \mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_i}) \sum_{\ell=0}^{\lfloor(j_0-j)/2\rfloor} \left(\frac{p_1(X)}{2}\right)^{\ell} \sum_{\beta=0}^{\lfloor(j_0-j)/2\rfloor-\ell} \frac{D^\beta f^{2\ell}_{j,i}(q)}{\beta!(m/2+j+2\ell)_\beta} \left(\frac{p_1(X)}{2}\right)^{\beta} \\
&= \sum_{j=0}^{j_0}\sum_{i=0}^{s_j-1} \mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_i}) \sum_{\ell=0}^{\lfloor(j_0-j)/2\rfloor} \left(\widetilde{f}^{\ell}_{j,i}\left(\frac{p_1(X)}{2}, q\right)\right)_{\le 2(j_0-j-2\ell)} \left(\frac{p_1(X)}{2}\right)^{\ell} \\
&= \sum_{j=0}^{j_0} \mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_0}) \phi^\natural_{j,0}\left(\frac{p_1(X)}{2}, q\right)_{\le 2(j_0-j)} + \ldots + \mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_{s_j-1}}) \phi^\natural_{j,s_j-1}\left(\frac{p_1(X)}{2}, q\right)_{\le 2(j_0-j)}
\end{aligned}$$

For $j = 0$, (4.35) is an equation in $H^0(X, \mathcal{M}^{\frac{m}{2}})$. As in Proposition 4.8, one can solve $\mathrm{sch}_0(\mathcal{F}; q) = \int_Y \xi_0(q; p_1, \ldots) = \sum_{i=0}^{s_0-1} \mathrm{ch}_0(\mathrm{Ind}\, \partial\!\!\!/^{V_i})\phi_{0,i}(q)$. This is just the computation of the index of the Dirac-Ramond operator on Y in terms of the index of the Dirac operator and the indices of the first $s_0 - 1$ twisted Dirac operators.

Now suppose that the formula (4.35) holds for $\nu < j_0$. From (4.33) and (4.34) we have

$$\begin{aligned} \mathrm{sch}_{j_0}(\mathcal{F}, q) &= \int_Y \sum_{\nu=0}^{j_0} \frac{\xi_j^{(j_0-\nu)}(q; p_1, ...)}{(j_0-\nu)!(m/2+\nu)_{j_0-\nu}} \left(-\frac{p_1}{2}\right)^{j_0-\nu} \qquad (4.36)\\ &= \sum_{\nu=0}^{j_0} \frac{1}{(j_0-\nu)!(m/2+\nu)_{j_0-\nu}} \left(\frac{p_1(X)}{2}\right)^{j_0-\nu} \int_Y \xi_\nu^{(j_0-\nu)}(q; p_1, ...). \end{aligned}$$

Applying the induction hypothesis (4.35) to the terms with $\nu < j_0$ gives

$$\begin{aligned} &\Upsilon := \mathrm{sch}_{j_0}(\mathcal{F}; q) - \int_Y \xi_{j_0}(q; p_1, ...) \\ &= \sum_{\nu=0}^{j_0-1} \frac{1}{(j_0-\nu)!(m/2+\nu)_{j_0-\nu}} \sum_{j=0}^{\nu} \sum_{i=0}^{s_j-1} \mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_i}) D^{j_0-\nu} f_{j,i}^{\nu-j}(q) \left(\frac{p_1(X)}{2}\right)^{\lfloor (j_0-j)/2 \rfloor} \\ &= \sum_{j=0}^{j_0-1} \sum_{i=0}^{s_j-1} \mathrm{ch}_j(\mathrm{Ind}\, \partial\!\!\!/^{V_i}) \times \\ &\sum_{\ell=0}^{\lfloor (j_0-1-j)/2 \rfloor} \frac{D^{j_0-j-2\ell} f_{j,i}^{2\ell}(q)}{(j_0-j-2\ell)!(m/2+j+2\ell)_{j_0-j-2\ell}} \left(\frac{p_1(X)}{2}\right)^{\lfloor (j_0-j)/2 \rfloor} \qquad (4.37) \end{aligned}$$

where in the third line we set $2\ell = \nu - j$.

From the definition we have

$$\mathrm{Sch}_{j_0}(\mathcal{F}; q) = \sum_{\nu=0}^{\infty} q^\nu \,\mathrm{ch}_{j_0}(\mathrm{Ind}\, \partial\!\!\!/^{V_\nu}).$$

Multiplying this equation by $\dfrac{\eta(q)^m}{q^{m/24}}$ and combining it with (4.37) gives

$$\int_Y \xi_{j_0}(q; p_1, ...) = \frac{\eta(q)^m}{q^{m/24}} \sum_{\nu=0}^{\infty} q^\nu \,\mathrm{ch}_{j_0}(\mathrm{Ind}\, \partial\!\!\!/^{V_\nu}) - \Upsilon \in H^{2j_0}(X, \mathcal{M}^{\frac{m}{2}+j_0}).$$

We proceed as in Proposition 4.8 and find a basis for $\mathcal{M}^{\frac{m}{2}+j_0}$ of the form $\{\phi_{j_0,0}(q),$

..., $\phi_{j_0,s_{j_0}-1}(q)\}$ which satisfies $q^{m/24}\dfrac{\phi_{j_0,i}(q)}{\eta(q)^m} = q^i + \mathcal{O}(q^{s_{j_0}})$. Then

$$\begin{aligned}
&\frac{\eta(q)^m}{q^{m/24}}\sum_{i=0}^{\infty} q^i \operatorname{ch}_{j_0}(\operatorname{Ind} \partial\!\!\!/^{V_i}) = \\
&\operatorname{ch}_{j_0}(\operatorname{Ind} \partial\!\!\!/^{V_0})\phi_{j_0,0}(q) + ... + \operatorname{ch}_{j_0}(\operatorname{Ind} \partial\!\!\!/^{V_{s_{j_0}-1}})\phi_{j_0,s_{j_0}-1}(q) \pmod{q^{s_{j_0}}}.
\end{aligned}$$

Let $\Omega \in H^{2j_0}(X, \mathcal{M}^{\frac{m}{2}+j_0})$ denote the RHS of the previous equation. Then

$$\left(\frac{\eta(q)^m}{q^{m/24}}\sum_{i=0}^{\infty} q^i \operatorname{ch}_{j_0}(\operatorname{Ind} \partial\!\!\!/^{V_i}) - \Omega\right) - \Upsilon$$

is equal to $-\Upsilon$ up to order $q^{s_{j_0}-1}$ as an element of $H^*(X,\mathbb{Q})[[q]]$. For each (j,i) we can find an $f_{j,i}^{j_0-j}(q) \in \mathcal{M}^{\frac{m}{2}+j_0}$ such that

$$f_{j,i}^{j_0-j}(q) = -\sum_{\ell=0}^{\lfloor (j_0-1-j)/2\rfloor} \frac{D^{j_0-j-2\ell} f_{j,i}^{2\ell}(q)}{(j_0-j-2\ell)!(m/2+j+2\ell)_{j_0-j-2\ell}} \pmod{q^{s_j}}$$

which is uniquely defined since $\mathcal{M}^{\frac{m}{2}+j_0}$ is s_{j_0} dimensional. Thus, we have

$$\begin{aligned}
\Lambda &:= \sum_{j=0}^{j_0-1}\sum_{i=0}^{s_j-1} \operatorname{ch}_j(\operatorname{Ind} \partial\!\!\!/^{V_i}) \left(\frac{p_1(X)}{2}\right)^{j_0-j} f_{j,i}^{j_0-j}(q) \\
&= -\sum_{j=0}^{j_0-1}\sum_{i=0}^{s_j-1} \operatorname{ch}_j(\operatorname{Ind} \partial\!\!\!/^{V_i}) \times \\
&\sum_{\ell=0}^{\lfloor (j_0-1-j)/2\rfloor} \frac{D^{j_0-j-2\ell} f_{j,i}^{2\ell}(q)}{(j_0-j-2\ell)!(m/2+j+2\ell)_{j_0-j-2\ell}} \left(\frac{p_1(X)}{2}\right)^{\lfloor (j_0-j)/2\rfloor} \pmod{q^{s_{j_0}}}
\end{aligned}$$

Now, since $\displaystyle\int_Y \xi_{j_0} = \Omega - \Lambda \pmod{q^{s_{j_0}}}$ and each quantity is in $H^{2j_0}(X, \mathcal{M}^{\frac{m}{2}+j_0})$ we have equality for all orders of q, i.e.

$$\int_Y \xi_{j_0}(q, p_1, ...) =$$
$$\sum_{i=0}^{s_{j_0}} \mathrm{ch}_{j_0}(\mathrm{Ind}\,\partial\!\!\!/^{V_i})\phi_{j_0,i}(q) + \sum_{j=0}^{j_0-1}\sum_{i=0}^{s_j-1} \mathrm{ch}_j(\mathrm{Ind}\,\partial\!\!\!/^{V_i})\left(\frac{p_1(X)}{2}\right)^{\lfloor (j_0-j)/2 \rfloor} f_{j,i}^{j_0-j}(q)$$

and this is the induction step we wanted to show in (4.35). $\square$

Chapter 5

The E_8 Bundle

Our formal version of the Dirac-Ramond operator should arise as the restriction of an actual operator on loop space. And it is believed (see [9], for instance) that the actual operator on loop space should fit into the framework of a $\mathrm{Diff}(S^1)$ equivariant K-theory of loop space. It is desirable then for the index bundle $\operatorname{Ind} \not{D} := \sum_{n=0}^{\infty} q^n \operatorname{Ind} \not{\partial}^{V_n}$ to be the restriction of a "Virasoro equivariant" vector bundle on loop space. A nice class of algebras whose representations also furnish representations for the Virasoro algebra are affine Lie algebras. Below we will show that under some stringent conditions we can identify the index bundle with a bundle associated to a representation of affine E_8. In the following we will assume all spaces X and Y are compact and any Lie group G is compact, simply connected and (semi)simple.

5.1 Classifying Spaces

For every group G there is a classifying space BG with a principal G bundle $EG \to BG$ such that for any X the set of principal G bundles over X corresponds naturally to the set of homotopy classes of maps from X to BG, denoted $[X, BG]$. The correspondence is given by

$$[X, BG] \ni f \mapsto (f^*EG \to X).$$

Suppose the cohomology ring of BG satisfies $H^*(BG, \mathbb{Z}) \simeq \mathbb{Z}[c_1, ..., c_n]$. Then characteristic classes for the universal bundle EG are given simply by $c_i(EG) = c_i$ and for all other principal bundles $f^*(EG)$ they are given by naturality

$$c_i(f^*(EG)) = f^*(c_i(EG)) = f^*(c_i) \in H^*(X, \mathbb{Z}). \tag{5.1}$$

Related to classifying spaces, we will also make use of the Eilenberg-MacLane spaces $K(G, n)$ which are constructed so that

$$\pi_i(K(G, n)) = \begin{cases} G & \text{if } i = n \\ 0 & \text{otherwise} \end{cases}$$

Because of this, one can obtain the singular cohomology of X with coefficients in G by using $H^n(X, G) \simeq [X, K(G, n)]$.

5.2 Principal E_8 Bundles

The homotopy groups for E_8 are known to satisfy $\pi_i(E_8) = 0$ for $1 \leq i \leq 14$ except for $\pi_3(E_8) \simeq \mathbb{Z}$. So for $i \leq 15$ the only nonzero homotopy group of BE_8 is $\pi_4(BE_8) \simeq \mathbb{Z}$. This makes BE_8 an "approximate" $K(\mathbb{Z}, 4)$. That is, for manifolds X of dimension at most 14 the cellular approximation theorem gives an isomorphism

$$[X, BE_8] \simeq [X, K(\mathbb{Z}, 4)] \simeq H^4(X, \mathbb{Z}). \tag{5.2}$$

This has the effect that, in low dimensions, principal E_8 bundles over X are in bijective correspondence with the elements of its fourth cohomology. Let $P \to X$ be a principal E_8 bundle over X. If $EE_8 \to BE_8$ denotes the universal principal E_8 bundle over the classifying space BE_8, then $P = \gamma^* EE_8$ for some $\gamma : X \to BE_8$. The bijective correspondence in (5.2) associates the principal bundle P to the cohomology class $\omega_\gamma = \gamma^*(u)$ where u is the generator of $H^4(BE_8, \mathbb{Z})$.

The adjoint representation of E_8 is a 248 dimensional unitary representation. Let $\rho : E_8 \to U(248)$ denote this representation. In fact, if we compose ρ with the determinant map then we get a map from E_8 into $U(1)$. Since E_8 is simple, the kernel of this map must be all of E_8. Thus we actually have image$(\rho) \subset SU(248)$ and we see that P is also a principal $SU(248)$ bundle. The goal is now to compute the Chern classes of this bundle.

To obtain the Chern classes we need a map $X \to BSU(248)$. The representation ρ induces a map $B\rho : BE_8 \to BSU(248)$. The map we need then is given by the composition $B\rho \circ \gamma : X \to BE_8 \to BSU(248)$. Since $H^2(BSU(248), \mathbb{Z}) = 0$ we trivially have $c_1(P) = 0$. Now,

$$c_2(P) = (B\rho \circ \gamma)^*(c_2) = \gamma^* B\rho^* c_2.$$

Since $H^4(BSU(248), \mathbb{Z})$ and $H^4(BE_8, \mathbb{Z})$ are both canonically isomorphic to $\mathbb{Z}$, any homomorphism between them is determined by a single integer. The integer induced by the adjoint representation is known as the Dynkin index of E_8 and has been computed to be 60 (see [27] and references therein). We restate all this in the following proposition.

Proposition 5.1. *Let $\rho : E_8 \to SU(248)$ be the adjoint representation and c_2 and u be the generators of $H^4(BSU(248), \mathbb{Z})$ and $H^4(BE_8, \mathbb{Z})$, respectively. Then*

$$B\rho^* : H^4(BSU(248), \mathbb{Z}) \to H^4(BE_8, \mathbb{Z})$$
$$c_2 \mapsto 60u \tag{5.3}$$

We now see that

$$c_2(P) = 60\omega^*(u).$$

Before we proceed we need the following.

Lemma 5.2. *Given a spin manifold X, $p_1(X)$ is even. That is $\frac{p_1(X)}{2} \in H^4(X, \mathbb{Z})$.*

Proof. The proof relies on the fact that $c_{2i}(TX \otimes \mathbb{C})(\bmod 2) = w_{4i}(TX \oplus TX)$. Also, note that

$$w(TX \oplus TX) = w(TX)w(TX) = (1 + w_1(TX) + w_2(TX) + w_3(TX) + w_4(TX))^2.$$

Squaring out the RHS and reducing modulo 2 gives $w_4(TX \oplus TX) = w_2(TX)^2$. We then have

$$p_1(X)(\text{mod } 2) = -c_2(TX \otimes \mathbb{C})(mod\ 2) = w_4(TX \oplus TX) = w_2(TX)^2$$

If X is spin the result follows.

$\square$

Since we then have $\frac{p_1(X)}{2} \in H^4(X, \mathbb{Z})$, we can choose $\omega_\gamma = -\frac{p_1(X)}{2}$ and therefore $c_2(P) = -30p_1(X)$. Let $W = P \times_\rho \mathbb{C}^{248}$ be the complex vector bundle over X associated to the representation ρ. The Chern character of W is the same as the Chern character for P, since W can also be viewed as being associated to P as a $SU(248)$ bundle using the standard representation. Working with formal Chern variables $x_1, ..., x_{248}$ we see

$$x_1^2 + ... + x_{248}^2 = (x_1 + ... + x_{248})^2 - 2\sum_{i<j} x_i x_j = c_1(W)^2 - 2c_2(W) = -2c_2(W)$$

and thus

$$\mathrm{ch}_2(W) = \frac{1}{2}(x_1^2 + ... + x_{248}^2) = -c_2(W) = 30p_1(X).$$

It is a result of Atiyah-Hirzebruch [4] that for any compact simple group G, the completed representation ring $R(G)^\wedge$ is isomorphic to $K(BG)$. Making use of the Chern character we then also have an isomorphism between $R(E_8)^\wedge \otimes \mathbb{Q}$ and $H^{**}(BE_8, \mathbb{Q})$, the completion of the cohomology ring of BE_8 (see [8, Section 6.1]). Under this identification a representation Λ is identified with $\mathrm{ch}(EE_8 \times_\Lambda \mathbb{C}^r)$ where r is the dimension of the representation Λ. From some character calculations in [27] we can see that

$$\mathrm{ch}(EE_8 \times_\rho \mathbb{C}^{248}) = 248 + 60u + 6u^2 + ... \tag{5.4}$$

Using γ to pull back to X we obtain,

$$\mathrm{ch}(W) = 248 + 30p_1(X) + \frac{3}{2}p_1(X)^2 + ... \tag{5.5}$$

5.3 The Basic Representation of Affine E_8

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra and $\langle \cdot, \cdot \rangle$ be the Killing form on $\mathfrak{g}$. We will also take $\mathfrak{g}$ to be simply laced, i.e. of type A_n,D_n, or E_n. The affine Lie algebra $\widehat{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ is

$$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

where $d = t\dfrac{d}{dt}$ and the bracket is defined by

$$\begin{aligned}&[f(t)\otimes X+\alpha K+\mu d, g(t)\otimes Y+\beta K+\nu d]\\ =&f(t)g(t)\otimes[X,Y]+\langle X,Y\rangle \text{Res}_{\,t=0}\left(\frac{df}{dt}(t)g(t)\right)K+\mu(dg)(t)\otimes Y-\nu(df)(t)\otimes X.\end{aligned}$$

Via the identification $t = e^{i\theta}$ the first summand appearing in $\widehat{\mathfrak{g}}$ can be thought of as loops on $\mathfrak{g}$ with finite Fourier expansion. Writing G for the compact simply connected Lie group corresponding to $\mathfrak{g}$, these loops exponentiate to polynomial loops on G. The second summand provides a central extension of this algebra. In terms of θ, the operator d is $-i\dfrac{d}{d\theta}$. This operator exponentiates to rigid rotations on the circle and so the third summand provides a semidirect product with the infinitesimal generator of such transformations. Of course, $\mathfrak{g}$ is a subalgebra via the identification of $\mathfrak{g}$ with $1\otimes\mathfrak{g}$.

Of all the irreducible representations of $\widehat{\mathfrak{g}}$ there is a nontrivial one which is simplest in some ways. This representation $V(\Lambda_0)$ is known as the basic representation and contains a highest weight vector v_0 satisfying

$$Kv_0 = v_0 \text{ and } \left(\mathbb{C}[t]\otimes\mathfrak{g}\oplus\mathbb{C}d\right)v_0 = 0.$$

Let $W_n = \{v \in V(\Lambda_0) | dv = -nv\}$. Since $[\mathfrak{g}, d] = 0$ each W_n is a representation for $\mathfrak{g}$. The character of the basic representation $V(\Lambda_0)$ is given by (see for instance [17], or more in the case at hand [16, equation 2])

$$\text{char }(V(\Lambda_0))(q, z_1, ..., z_r) = q^{r/24}\frac{\Theta_{\mathfrak{g}}(q, z_1, ..., z_r)}{\eta(q)^r} \tag{5.6}$$

where r is the rank of $\mathfrak{g}$, $(z_1, ..., z_r)$ represents a point (after choosing a basis) in the

Cartan subalgebra, and the theta function is defined on the root lattice Q by

$$\Theta_{\mathfrak{g}}(q, z_1, ..., z_r) = \sum_{\gamma \in Q} e^{2\pi i \langle \gamma, \overrightarrow{z} \rangle} q^{||\gamma||^2/2}. \tag{5.7}$$

We now specify all of this to the case where $\mathfrak{g} = E_8$. The representation $V(\Lambda_0)$ breaks up in terms of the W_n's as a sequence of finite dimensional representations for E_8 (the algebra or the group) as

$$V(\Lambda_0) = 1 + W_1 q + W_2 q^2 + ... \tag{5.8}$$

where 1 denotes the trivial one-dimensional representation. It is a fact that W_1 is the adjoint representation ρ for E_8. Recall from the previous section there is an E_8 bundle P over X corresponding to the cohomology class $-\frac{p_1(X)}{2}$. For each $n > 0$ define the associated vector bundles $\underline{W}_n = P \times_{\rho_n} W_n$ over X and put

$$\mathcal{V} = 1_{\mathbb{C}} + \underline{W}_1 q + \underline{W}_2 q^2 + ... \in K(X)[[q]] \tag{5.9}$$

where $1_{\mathbb{C}} \to X$ is the trivial one-dimensional complex vector bundle.

Using (5.5) we see that

$$\mathrm{ch}(\mathcal{V}) = 1 + (248 + 30 p_1(X)) q + ... \tag{5.10}$$

and the rest of the terms are at least degree 8 in cohomology or at least degree 2 in q.

The proof of the following Lemma was pointed out to me by Antun Milas. The result can also be found in [13].

Lemma 5.3. *There is a basis for the E_8 root lattice such that*

$$\Theta_{E_8}(q, z_1, ..., z_8) = \tag{5.11}$$
$$\frac{1}{2}\left(\prod_{i=1}^{8}\theta_2(2\pi i z_i, q) + \prod_{i=1}^{8}\theta_3(2\pi i z_i, q) + \prod_{i=1}^{8}\theta_4(2\pi i z_i, q) + \prod_{i=1}^{8}\theta(2\pi i z_i, q)\right)$$

where $\theta_2(z, q), \theta_3(z, q), \theta_4(z, q)$ are the three classical (even) Jacobi theta functions(see Appendix A).

Proof. Let $\epsilon_1, ..., \epsilon_8$ be the orthonormal basis for the usual Euclidean lattice $\mathbb{Z}^8$. The E_8 lattice consists of the set

$$\{z_1\epsilon_1 + ... + z_8\epsilon_8 \in \mathbb{Z}^8 | z_1 + ... + z_8 = 0 \bmod 2\} \tag{5.12a}$$

together with a "shifted version" of this

$$\{z_1\epsilon_1 + ... + z_8\epsilon_8 \in (\mathbb{Z} + \tfrac{1}{2})^8 | z_1 + ... + z_8 = 0 \bmod 2\}. \tag{5.12b}$$

We can sum the theta function (5.7) over (5.12a) and (5.12b) separately. Using (A.1c) and (A.1d) the sum over (5.12a) is

$$\prod_{n=1}^{8}\theta_3(2\pi i z_n, \tau) + \prod_{n=1}^{8}\theta_4(2\pi i z_n, \tau).$$

using (A.1a) and (A.1b) the sum over (5.12b) is

$$\prod_{n=1}^{8}\theta_2(2\pi i z_n, \tau) + \prod_{n=1}^{8}\theta(2\pi i z_n, \tau).$$

The equation (5.11) now follows. □

Now, let

$$H(\tau, z_1, ..., z_8) = \frac{1}{2} e^{G_2(\tau)(z_1^2+...+z_8^2)} \left(\prod_{i=1}^{8} \theta_2(z_i, \tau) + \prod_{i=1}^{8} \theta_3(z_i, \tau) + \prod_{i=1}^{8} \theta_4(z_i, \tau) \right).$$

Notice that since $\theta(0, \tau) = 0$, it follows that $\theta(z, \tau) = \mathcal{O}(z)$ and hence

$$\Theta_{E_8}(\tau, z_1, ..., z_8) = e^{-G_2(\tau)((2\pi i z_1)^2+...+(2\pi i z_8)^2)} H(\tau, 2\pi i z_1, ..., 2\pi i z_8) + \mathcal{O}(z^8).$$

Using the classical transformation formulas for the Jacobi theta functions and for G_2 one sees that

$$H\left(\frac{a\tau + b}{c\tau + d}, \frac{z_1}{c\tau + d}, ..., \frac{z_8}{c\tau + d} \right) = (c\tau + d)^4 H(\tau, z_1, ..., z_8). \tag{5.13}$$

Let e_i be the ith elementary symmetric polynomial in $z_1^2, ..., z_8^2$. Since each theta function in $H(\tau, z_1, ..., z_8)$ is even, it can be expanded in terms of the elementary symmetric functions e_i's

$$H(\tau, z_1, ..., z_8) = a_0(\tau) + a_{1,1}(\tau)e_1 + a_{2,1}(\tau)e_1^2 + a_{2,2}(\tau)e_2 + ...$$

It follows from (5.13) that each $a_{i,j}$ is a modular form of weight $4 + 2i$. For $i < 4$, the space of modular forms of weight $4 + 2i$ is one dimensional. So to determine $a_{i,j}$ one need only calculate its constant term. This can be done very easily using Mathematica. One finds

$$H(\tau, z_1, ..., z_8) =$$
$$E_4(\tau) - \frac{1}{12} E_6(\tau) \frac{1}{2} e_1 + \frac{1}{2! \cdot 12^2} E_8(\tau) (\frac{1}{2} e_1)^2 - \frac{1}{3! \cdot 12^3} E_{10}(\tau) (\frac{1}{2} e_1)^3 + \mathcal{O}(z^8).$$

Then

$$\frac{1}{2}\left(\prod_{i=1}^{8}\theta_2(z_i,\tau)+\prod_{i=1}^{8}\theta_3(z_i,\tau)+\prod_{i=1}^{8}\theta_4(z_i,\tau)\right) \bmod z^8$$
$$= e^{-G_2(\tau)e_1}\left(E_4(\tau)-\frac{1}{12}E_6(\tau)\frac{1}{2}e_1+\frac{1}{2!\cdot 12^2}E_8(\tau)(\frac{1}{2}e_1)^2-\frac{1}{3!\cdot 12^3}E_{10}(\tau)(\frac{1}{2}e_1)^3\right)$$
$$= E_4(\tau)+\frac{1}{4}E_4'(\tau)\left(\frac{e_1}{2}\right)+\frac{1}{2!\cdot 4\cdot 5}E_4''(\tau)\left(\frac{e_1}{2}\right)^2+\frac{1}{3!\cdot 4\cdot 5\cdot 6}E_4'''(\tau)\left(\frac{e_1}{2}\right)^3 \quad (5.14)$$

Using (5.6) together with (5.14) we get

$$\mathrm{ch}_{\leq 12}(V(\Lambda_0)) = \quad (5.15)$$
$$\frac{q^{1/3}}{\eta(q)^8}\left(E_4(\tau)+\frac{1}{4}E_4'(\tau)\left(\frac{e_1}{2}\right)+\frac{1}{2!\cdot 4\cdot 5}E_4''(\tau)\left(\frac{e_1}{2}\right)^2+\frac{1}{3!\cdot 4\cdot 5\cdot 6}E_4'''(\tau)\left(\frac{e_1}{2}\right)^3\right)$$

With the identification $R(E_8)^\wedge\otimes\mathbb{Q}\simeq H^{**}(BE_8,\mathbb{Q})$, $\mathrm{ch}(V(\Lambda_0))$ is an element of $H^{**}(BE_8,\mathbb{Q})[[q]]$. Proceeding as in the proof of Theorem 4.6 one could see further that $\frac{\eta(q)^8}{q^{1/3}}\mathrm{ch}(V(\Lambda_0))$ is an element of $H^{**}(BE_8,\widetilde{\mathcal{M}}^*)$. Now we use the map $\gamma: X\to BE_8$ corresponding to $-\frac{p_1(X)}{2}$ (see Section 5.2) to pull back (5.15) to $H^*(X,\mathbb{Q})[[q]]$ and compare the degree 4 element of cohomology appearing as the coefficient of q with that in (5.5). We see that $\gamma^*e_1=p_1(X)$. Thus

$$\mathrm{ch}(\mathcal{V}) = \frac{q^{1/3}}{\eta(q)^8}\{E_4(q)+\frac{1}{4}E_4'(q)\left(\frac{p_1(X)}{2}\right) \quad (5.16)$$
$$+\frac{1}{2!\cdot 4\cdot 5}E_4''(q)\left(\frac{p_1(X)}{2}\right)^2+\frac{1}{3!\cdot 4\cdot 5\cdot 6}E_4'''(q)\left(\frac{p_1(X)}{2}\right)^3+\ldots\in H^*(X,\mathbb{Q})[[q]]$$

An application of Theorem 4.11, or more directly (4.25), now gives the following result.

Theorem 5.4. *Let $\mathcal{F}=(\pi,Z,X)$ be a string family of compact spin manifolds having fibers $Y_x=\pi^{-1}(x)$ of dimension 8. Suppose also that X is a compact spin manifold of dimension less than 16. If $\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/)=\mathrm{ch}_4(\mathrm{Ind}\,\partial\!\!\!/)=\mathrm{ch}_6(\mathrm{Ind}\,\partial\!\!\!/)=0$ then the Chern* *character of the index bundle for the family of Dirac-Ramond operators satisfies*

$$\mathrm{Sch}(\mathcal{F};q) = \mathrm{ch}(\mathbb{C}^{\nu_0} \otimes \mathcal{V}) \tag{5.17}$$

where ν_0 is the index of the Dirac operator on Y and $\mathcal{V} \in K(X)[[q]]$ is constructed as above.

The fact that the RHS of (5.16) shows up in (4.28) allows us also to say something about the case when there is a family of 6 dimensional spin manifolds. However, there is an awkward issue with the factor (3.6). As (5.16) has the factor $\Pi_{n=1}^{\infty}(1-q^n)^{-8}$ and (4.28) should be multiplied by $\Pi_{n=1}^{\infty}(1-q^n)^{-6}$, we must correct by a factor of $\Pi_{n=1}^{\infty}(1-q^n)^{2}$. Recall from the end of Section 4.2 the determinant line bundle $\det \bar{\partial} \to X$. We use this to state the result as follows.

Theorem 5.5. *Let $Z \to X$ be a string family of compact spin manifolds having fibers $Y_x = \pi^{-1}(x)$ of dimension 6. Suppose also that X is a compact spin manifold of dimension less than 16. If $\mathrm{ch}_3(\mathrm{Ind}\,\bar{\partial}) = \mathrm{ch}_5(\mathrm{Ind}\,\bar{\partial}) = \mathrm{ch}_7(\mathrm{Ind}\,\bar{\partial}) = 0$ then the Chern character of the index bundle for the family of Dirac-Ramond operators satisfies*

$$\mathrm{Sch}(q) = c_1(\det \bar{\partial}) \cdot \mathrm{ch}(\mathcal{V}) \prod_{n=1}^{\infty}(1-q^n)^2 \tag{5.18}$$

where $\mathcal{V} \in K(X)[[q]]$ is constructed as above. If $1_{\mathbb{C}}$ denotes the trivial complex rank one vector bundle over X, then this can also be put as

$$\mathrm{Sch}(q) = \mathrm{ch}\left((\det \bar{\partial} - 1_{\mathbb{C}}) \otimes \mathcal{V}\right) \prod_{n=1}^{\infty}(1-q^n)^2. \tag{5.19}$$

Appendix A

Elliptic Functions and Jacobi-Like Forms

We will first recall some functions important in the study of elliptic function theory. Let $\mathfrak{h}$ denote the complex upper half plane and, as usual, $q = e^{2\pi i\tau}$. The (odd) Jacobi theta function $\theta : \mathbb{C} \times \mathfrak{h} \to \mathbb{C}$ is defined by

$$\theta(z,\tau) = 2q^{1/8}\sinh(z/2)\prod_{n=1}^{\infty}(1-q^n e^z)(1-q^n e^{-z})(1-q^n).$$

Observe that

$$\theta'(0,\tau) = \eta(\tau)^3$$

where $\eta(\tau)$ is the Dedekind eta function defined in Section 4.1. The other three (even) Jacobi theta functions are

$$\theta_2(z,\tau) = 2q^{1/8}\cosh(z/2)\prod_{n=1}^{\infty}(1+q^n e^z)(1+q^n e^{-z})(1-q^n)$$
$$\theta_3(z,\tau) = \prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}}e^z)(1+q^{n-\frac{1}{2}}e^{-z})(1-q^n)$$
$$\theta_4(z,\tau) = \prod_{n=1}^{\infty}(1-q^{n-\frac{1}{2}}e^z)(1-q^{n-\frac{1}{2}}e^{-z})(1-q^n).$$

We also list their series expansions

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^n q^{\frac{1}{2}n^2} e^{nz} \tag{A.1a}$$

$$\theta_2(z,\tau) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} q^{\frac{1}{2}n^2} e^{nz} \tag{A.1b}$$

$$\theta_3(z,\tau) = \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{nz} \tag{A.1c}$$

$$\theta_4(z,\tau) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{nz}. \tag{A.1d}$$

Another important elliptic function is the Weierstrass sigma function, which is typically defined by

$$\sigma(z,\tau) = z \prod_{0\neq\gamma\in 2\pi i(\mathbb{Z}+\tau\mathbb{Z})} \left(1 - \frac{z}{\gamma}\right) e^{\frac{z}{\gamma}+\frac{1}{2}(\frac{z}{\gamma})^2}.$$

For our purposes, this is not a useful expression. We make more use of it in the following identities

$$\begin{aligned}\sigma(z,\tau) &= \frac{\theta(z,\tau)}{\theta'(0,q)} e^{G_2(\tau)z^2} \\ &= z \exp\left(-\sum_{n=2}^{\infty} \frac{2}{2n!} G_{2n}(\tau) z^{2n}\right).\end{aligned} \tag{A.2}$$

Let $\mathrm{Hol}(\mathfrak{h})$ denote the space of holomorphic functions on $\mathfrak{h}$. We will make extensive use of the following

Definition A.1. *A Jacobi-like form of weight k and index λ is an element of $F(z,\tau) \in$ $\mathrm{Hol}(\mathfrak{h})[[z]]$ such that*

$$F\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \exp\left(\frac{c\lambda}{c\tau+d}\frac{z^2}{2\pi i}\right) F(z,\tau) \tag{A.3}$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$. *We will denote the collection of Jacobi-like forms of weight* k *and index* λ *by* $\mathcal{J}_{k,\lambda}$.

Jacobi-like forms satisfy one of the two transformations properties which essentially characterize Jacobi forms. The foundations for Jacobi forms were laid out in [11] and the generalization of Jacobi forms to Jacobi-like forms was introduced in [10] and [30].

Example A.2. Given a modular form $f \in \mathcal{M}^k$ one can use (4.10) to verify that $F(z,\tau) = z^n f(\tau) e^{-G_2(\tau)\lambda z^2}$ is a Jacobi-like form of weight $k-n$ and index $\lambda/2$.

As with modular forms, Jacobi-like forms can be defined on subgroups of the modular group as well, but that will not be necessary for us here. Since Jacobi-like forms are invariant under the transformation $\tau \mapsto \tau+1$ they have a Fourier expansion in terms of $e^{2\pi i\tau}$ and hence can also be viewed as elements of $\mathbb{C}[[q]][[z]]$. We will denote by $\mathcal{J}_{\mathbb{Q},k,\lambda}$ the elements of $\mathcal{J}_{k,\lambda}$ which define elements of $\mathbb{Q}[[q]][[z]]$.

Proposition A.3. *Given a Jacobi-like form* $F(z,\tau) = \sum_{j=0}^{\infty} \chi_j(\tau) z^j \in \mathcal{J}_{k,\lambda}$. *Each* χ_j *is a modular form of weight* $k+j$ *if* $\lambda = 0$ *and a quasimodular form of the same weight otherwise.*

Proof. Let $F(z,\tau) = \displaystyle\sum_{n=0}^{\infty} \chi_n(\tau) z^n \in \mathcal{J}_{k,\lambda}$. Set $H(z,\tau) = e^{2G_2(\tau)\lambda z^2} F(z,\tau)$. Then

$$\begin{aligned} &H\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) \\ &= \exp\left(\left((c\tau+d)^2\, G_2(\tau) - \frac{c}{4\pi i}(c\tau+d)\right) 2\lambda \left(\frac{z}{c\tau+d}\right)^2\right) \times \\ &\qquad\qquad\qquad (c\tau+d)^k \exp\left(\frac{c\lambda}{c\tau+d}\frac{z^2}{2\pi i}\right) F(z,\tau) \\ &= (c\tau+d)^k H(z,\tau). \end{aligned}$$

Now expand $H(z,\tau)$ as a power series $\sum_{n=0}^{\infty} \widetilde{\chi}_n(\tau)z^n$. The previous equation becomes

$$\sum_{n=0}^{\infty} \widetilde{\chi}_n\left(\frac{a\tau+b}{c\tau+d}\right)\left(\frac{z}{c\tau+d}\right)^n = (c\tau+d)^k \sum_{n=0}^{\infty} \widetilde{\chi}_n(\tau)z^n$$

so that each $\widetilde{\chi}_n$ is a modular form of weight $k+n$. Then $F(z,\tau) = e^{-2G_2(\tau)\lambda z^2} \sum_{n=0}^{\infty} \widetilde{\chi}_n(\tau)z^n$, and it follows from this that each χ_n is a quasimodular form of weight $k+n$. □

Hence when the χ_j's have rational q-expansion, i.e. $F(z,\tau)$ is an element of $\mathcal{J}_{\mathbb{Q},k,\lambda}$, then $F(z,\tau)$ canonically determines an element $F(z,q)$ of $\widetilde{\mathcal{M}}^*[[z]]$. As with modular forms we will use these notions interchangeably. Since $\widetilde{\mathcal{M}}^{k+j}$ is trivial when $k+j$ is odd, the $\chi_j(\tau)$ are necessarily zero for half of the j's. Thus $\mathcal{J}_{k,\lambda}$ is the direct sum of the two subspaces

$$\mathcal{J}^+_{k,\lambda} := \{F(z,\tau) \in \mathcal{J}_{k,\lambda} | F(z,\tau) = \sum_{j=0}^{\infty} \chi_{2j}(\tau)z^{2j}\}$$
$$\mathcal{J}^-_{k,\lambda} := \{F(z,\tau) \in \mathcal{J}_{k,\lambda} | F(z,\tau) = \sum_{j=0}^{\infty} \chi_{2j+1}(\tau)z^{2j+1}\}.$$

It is easy to verify that the following map is an isomorphism:

$$\begin{aligned} \mathcal{J}^+_{k,\lambda} &\to \mathcal{J}^-_{k-1,\lambda} \\ F(z,\tau) &\mapsto zF(z,\tau). \end{aligned} \tag{A.4}$$

Given an $F(z,\tau) \in \mathcal{J}_{k,\lambda}$, evaluating (A.3) at $z=0$ shows that $F(0,\tau)$ is a modular form of weight k. Thus there is a map $\mathcal{J}^+_{\mathbb{Q},k,\lambda} \to \mathcal{M}^k$.

Definition A.4. *Given $f \in \mathcal{M}^k$ the Cohen-Kuznetsov series (or lift) of f with index λ is given by*

$$\widetilde{f}(z,\tau) = \sum_{n=0}^{\infty} \frac{\lambda^n f^{(n)}(\tau)}{n!(k)_n} z^{2n} \in \mathrm{Hol}(\mathfrak{h})[[z]] \tag{A.5}$$

where $(k)_n = (k+n-1)!/(k-1)! = k(k+1)...(k+n-1)$ is the Pochhammer symbol and $f^{(n)}(\tau) := \left(\frac{1}{2\pi i}\frac{d}{d\tau}\right)^n f(\tau)$. If no mention is made of the index, it will be assumed that $\lambda = 1$.

The derivation $D := \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}$, was defined in Section 4.1 where it was also pointed out that D defines a map $D : \widetilde{\mathcal{M}}^* \to \widetilde{\mathcal{M}}^{*+2}$.

Example A.5.

$$\begin{aligned}
\widetilde{E}_4(z,q) &= E_4(q) + \frac{1}{4}E_4'(q)z^2 + \frac{1}{2!\cdot 4\cdot 5}E_4''(q)z^4 + \frac{1}{3!\cdot 4\cdot 5\cdot 6}E_4'''(q)z^6 + ... \\
&= E_4(q) + \frac{1}{12}\left(E_4(q)E_2(q) - E_6(q)\right)z^2 \\
&+ \frac{1}{288}\left(E_4(q)^2 - 2E_6(q)E_2(q) + E_4(q)E_2(q)^2\right)z^4 + ...
\end{aligned} \tag{A.6}$$

It is shown in Section 3 of [11] that if $f \in \mathcal{M}^k$ and $\widetilde{f}(z,\tau)$ is its Cohen-Kuznetsov lift with index λ, then $\widetilde{f}(z,\tau)$ satisfies (A.3) and so $\widetilde{f}(z,\tau) \in \mathcal{J}^+_{\mathbb{Q},k,\lambda}$. This justifies the term *lift*, as the map $f \mapsto \widetilde{f}$ provides a section for the map $\mathcal{J}^+_{\mathbb{Q},k,m} \to \mathcal{M}^k$. In fact, all elements of $\mathcal{J}_{\mathbb{Q},k,\lambda}$ can be constructed from Cohen-Kuznetsov lifts via the following.

Theorem A.6. *Given a Jacobi-like form $F(z,\tau) \in \mathcal{J}_{\mathbb{Q},k,\lambda}$ there is a corresponding sequence of modular forms $\xi_0, \xi_1, \xi_2, ...$ such that $\xi_n \in \mathcal{M}^{k+n}$ and*

$$F(z,\tau) = \widetilde{\xi_0}(z,\tau) + z\widetilde{\xi_1}(z,\tau) + z^2\widetilde{\xi_2}(z,\tau) + ... = \sum_{n=0}^{\infty} z^n \widetilde{\xi_n}(z,\tau) \tag{A.7}$$

and $\widetilde{\xi_n}(z,\tau)$ is, as above, the Cohen-Kuznetsov lift of ξ_n with index λ. Conversely, for any sequence of modular forms $\xi_0, \xi_1, \xi_2, ...$ satisfying $\xi_n \in \mathcal{M}^{k+n}$, the equation (A.7) defines an $F(z,\tau) \in \mathcal{J}_{\mathbb{Q},k,\lambda}$.

Remark A.7. Again, since $\mathcal{M}^{k+j} = \{0\}$ whenever $k+j$ is odd, half of the ξ_j's necessarily vanish. Note that in terms of the usual expansion $\sum_{j=0}^{\infty}\chi_j(q)z^j$ of an $F(z,\tau) \in \mathcal{J}_{\mathbb{Q},k,\lambda}$, the sequence of modular forms are given by

$$\xi_n(\tau) = \sum_{0\leq j\leq n/2} \frac{(-\lambda)^j(k+n-j-2)!}{j!(k+n-2)!}\chi_{n-2j}^{(j)}(\tau). \tag{A.8}$$

See Section 3 of [11] for the proof.

Given $F(z,\tau) = \sum_{n=0}^{\infty}\chi_{2n}(\tau)z^{2n} \in \mathcal{J}_{0,\lambda}^{+}$ then $\chi_0(\tau)$ is a modular form of weight 0 and hence constant. Assume $\chi_0(\tau) = 1$. Then $\prod_{i=1}^{m/2} F(y_i,\tau)$ is expressible in terms of the elementary symmetric functions $p_1, ..., p_{m/2}$ in the variables $y_1^2, ..., y_{m/2}^2$. It follows then that if we write

$$\prod_{i=1}^{m/2} F(y_i,\tau) = 1 + a_{1,1}(\tau)p_1 + a_{2,1}(\tau)p_1^2 + a_{2,2}(\tau)p_2 + ... \tag{A.9}$$

then each $a_{i,j}(\tau)$ is a modular form of weight $2i$ if $\lambda = 0$ and a quasimodular form of the same weight otherwise.

Appendix B

The Computation

In this section we show how the computations in Section 4.3 were done and hope to elucidate the proof in Section 4.4. We assume the setup from the previous sections. Namely, we have a string family $Z \to X$ parameterizing the compact spin manifolds $Y_x = \pi^{-1}(x)$ and $V \to Z$ is the vertical bundle. The string condition on Z allows us to make much use of (4.1). For simplicity, we will restrict to when the dimension of Y is 8. All, computations below were done with the help of Mathmematica.

We recall the formula for the $\widehat{A}$-class. For a vector bundle V with Pontryagin classes $p_1(V), ..., p_{m/2}(V)$ we have

$$\hat{A}(V) = 1 - \frac{1}{24}p_1(V) + \frac{7p_1(V)^2 - 4p_2(V)}{5760} + \frac{-31p_1(V)^3 + 44p_1(V)p_2(V) - 16p_3(V)}{967680}$$
$$+ \frac{381p_1(V)^4 - 904p_1(V)^2p_2(V) + 208p_2(V)^2 + 512p_1(V)p_3(V) - 192p_4(V)}{464486400} + ... \tag{B.1}$$

We will start with the case when dim $Y = 8$. Then by the usual Atiyah-Singer index theorem for families of Dirac operators we have

$$\begin{aligned}
\mathrm{ch}_0(\mathrm{Ind}\,\partial\!\!\!/) &= \int_Y -\frac{4p_2(V)}{5760} \\
\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/) &= \int_Y \frac{44p_1(V)p_2(V) - 16p_3(V)}{967680} \\
&\vdots
\end{aligned} \tag{B.2}$$

Recall that in the notation above we have $V_1 = V_{\mathbb{C}}$. Some manipulations with formal Chern variables show

$$\begin{aligned}
\mathrm{ch}(V_{\mathbb{C}}) = 8 + p_1(V) + \frac{p_1(V)^2 - 2p_2(V)}{12} + \frac{p_1(V)^3 - 3p_1(V)p_2(V) + 3p_3(V)}{360} \\
+ \frac{p_1(V)^4 - 4p_1(V)^2p_2(V) + 2p_2(V)^2 + 4p_1(V)p_3(V) - 4p_4(V)}{20160} + \ldots
\end{aligned} \tag{B.3}$$

After multiplying this by (B.1), the index theorem (2.12) gives

$$\begin{aligned}
\mathrm{ch}_0(\mathrm{Ind}\,\partial\!\!\!/^{V_1}) &= \int_Y -\frac{31p_2(V)}{180} \\
\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/^{V_1}) &= \int_Y \frac{-13p_1(V)p_2(V) + 62p_3(V)}{7560} \\
&\vdots
\end{aligned}$$

Notice that if $p_1(X) = 0$ then $p_1(V) = -\pi^* p_1(X) = 0$ and then $\mathrm{ch}_0(\mathrm{Ind}\,\partial\!\!\!/^{V_1}) = 248\,\mathrm{ch}_0(\mathrm{Ind}\,\partial\!\!\!/)$ and $\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/^{V_1}) = -496\,\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/)$ agreeing with (4.21) and (4.22), respectively. However, when $p_1(V) = -\pi^* p_1(X) \neq 0$ we have

$$\begin{aligned}
\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/^{V_{\mathbb{C}}}) &= \int_Y -496\frac{44p_1(V)p_2(V) - 16p_3(V)}{967680} + 30\frac{-4p_2(V)}{5760}(-p_1(V)) \\
&= -496\int_Y \frac{44p_1(V)p_2(V) - 16p_3(V)}{967680} + 30p_1(X)\int_Y -\frac{4p_2(V)}{5760} \\
&= -496\,\mathrm{ch}_2(\mathrm{Ind}\,\partial\!\!\!/) + 30\,\mathrm{ch}_0(\mathrm{Ind}\,\partial\!\!\!/)p_1(X),
\end{aligned}$$

agreeing with (4.26).

To see where (4.25) comes from we recall

$$\begin{aligned}
\widehat{a}(V,\tau)e^{G_2(\tau)p_1(V)} &= \prod_{i=1}^{m/2} \frac{y_i}{\sigma(y_i,\tau)} e^{G_2(\tau)y_i^2} \\
&= \prod_{i=1}^{m/2} \exp(\sum_{n=1}^{\infty} \frac{2}{2n!} G_{2n}(\tau) y_i^{2n}) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{2}{2n!} G_{2n}(\tau)(y_1^{2n} + ... + y_{m/2}^{2n})\right)
\end{aligned}$$

and after making use of the Newton identities which express the power sums in the basis of elementary symmetric polynomials this can be written

$$\begin{aligned}
&\widehat{a}(V,\tau)e^{G_2(\tau)p_1(V)} \\
&\qquad = \exp\Bigg(-\frac{E_2(q)}{24} p_1(V) + \frac{E_4(q)}{2880}(p_1(V)^2 - 2p_2(V)) + \\
&\qquad\qquad \frac{E_6(q)}{181440}(p_1(V)^3 - 3p_1(V)p_2(V) + 3p_3(V)) + ...\Bigg) \qquad \text{(B.4)}
\end{aligned}$$

We will write p_i for $p_i(V)$ in the following. Expanding this out we get something of the form

$$\begin{aligned}
&\widehat{a}(V,\tau)e^{G_2(\tau)p_1(V)} = \\
&\qquad F(p_1,q) - \frac{E_4(q)p_2}{1440} + \frac{84E_2(q)E_4(q)p_1p_2 + 48E_6(q)p_1p_2 - 48E_6(q)p_3}{2903040} + ...
\end{aligned}$$

where $F(p_1, q)$ is some expression only depending on powers of p_1. Hence,

$$
\begin{aligned}
\mathrm{sch}_{\leq 2}(q) &= \int_Y \widehat{a}(V,\tau) e^{G_2(\tau)p_1(V)} \\
&= \int_Y -\frac{E_4(q)p_2}{1440} + \frac{84E_2(q)E_4(q)p_1p_2 + 48E_6(q)p_1p_2 - 48E_6(q)p_3}{2903040} \\
&= \int_Y -\frac{p_2}{1440}\left(E_4(q) + \frac{1}{4}\frac{E_2(q)E_4(q) - E_6(q)}{3}\left(-\frac{p_1}{2}\right)\right) + \\
&\qquad \frac{44p_1(V)p_2(V) - 16p_3(V)}{967680}E_6(q) \\
&= \mathrm{ch}_0(\mathrm{Ind}\,\not{\partial})\left(E_4(q) + \frac{1}{4}DE_4(q)\left(\frac{p_1(X)}{2}\right)\right) + \mathrm{ch}_2(\mathrm{Ind}\,\not{\partial})E_6(q). \qquad \text{(B.5)}
\end{aligned}
$$

Continuing this to higher degrees in cohomology gives (4.25).

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