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On the Combinatorics of the Free Lie Algebra With Multiple Brackets

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UNIVERSITY OF MIAMI

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By

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This thesis is concerned with the connection between Lie algebras with multiple brackets and the topology of partially ordered sets. From a partially ordered set (poset) one obtains a simplicial complex, called the order complex, whose faces are the chains of the poset. There is a long tradition of using topological properties of the order complex to study various geometric and algebraic structures.

It is a classical result that the multilinear component of the free Lie algebra is isomorphic (as a representation of the symmetric group) to the top (co)homology of the order complex of the proper part of the poset of partitions Π_n tensored with the sign representation. We generalize this result in order to study the multilinear component of the free Lie algebra on n generators with multiple compatible Lie brackets. We consider the poset of weighted partitions Π_n^w, introduced by Dotsenko and Khoroshkin in their study of a certain pair of dual operads and we introduce a new poset of weighted partitions Π_n^k that allows us to generalize the result. The maximal intervals of Π_n^w provide a generalization of the lattice Π_n of partitions, which we show possesses many of the well-known properties of Π_n; the new poset Π_n^k is a generalization of both Π_n and Π_n^w. Indeed, $\Pi_n^1 \simeq \Pi_n$ and $\Pi_n^2 \simeq \Pi_n^w$.

An important combinatorial tool for studying the topology of the order complex is provided by the theory of shellability. We prove that the poset Π_n^k with a top element added is EL-shellable and hence Cohen-Macaulay. This enables us in the case $k = 2$ to use the poset theoretic Möbius function to recover results of Dotsenko-Khoroshkin and Liu giving the dimension of the multilinear component

of the free doubly bracketed Lie algebra $\mathcal{L}ie_2(n)$ as n^{n-1}. We show that the Möbius invariant of each maximal interval of Π_n^w is given up to sign by the number of rooted trees on node set $\{1, 2, \ldots, n\}$ having a fixed number of descents. Moreover, we construct a nice combinatorial basis for the homology of these intervals consisting of fundamental cycles indexed by such rooted trees, generalizing Björner's NBC basis for the homology of Π_n. We also show that the characteristic polynomial of Π_n^w has a nice factorization analogous to that of Π_n.

EL-shellability and other properties of the more general poset Π_n^k enable us to answer questions posed by Liu on free multibracketed Lie algebras. In particular, we obtain various dimension formulas and multicolored generalizations of the classical Lyndon and comb bases for the multilinear component of the free Lie algebra. We obtain and rely on an interesting bijection between the colored Lyndon trees and the colored combs. This bijection is a generalization of the classical bijection between the classical Lyndon trees and combs.

The multilinear component of the free multibracketed Lie algebra decomposes in a natural way into more refined components according to the number of brackets of each type used in its generators. Indeed, for a weak composition $\mu = (\mu_1, \mu_2, \ldots)$ we consider the component $\mathcal{L}ie(\mu)$ whose generators contain μ_j brackets of type j for each j. We prove that the generating function of $\dim \mathcal{L}ie(\mu)$ is an e-positive symmetric function, that is, it has positive coefficients in the basis of elementary symmetric functions. We give various combinatorial descriptions of the e-coefficients in terms of leaf-labeled binary trees and in terms of the Stirling permutations introduced by Gessel and Stanley.

We also use poset theoretic techniques to obtain a plethystic formula for the Frobenius characteristic of the representation of the symmetric group on the multilinear component of the free multibracketed Lie algebra.

To Maria Nelly and Sandra.

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This dissertation represents the accomplishment of a goal that I set several years ago: to learn and discover more about the world of mathematics. The realization of this goal is the product of all the people that have touched my life with their teachings and with their perspectives.

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Table of Contents

List of Figures

Chapter 1

Introduction

There is a long tradition of using topological properties of the order complex of partially ordered sets to study various geometric and algebraic structures. This thesis is concerned with the connection between Lie algebras with multiple brackets and the topology of a family of partially ordered sets of weighted set partitions. We start by discussing classical results on the multilinear component of the free Lie algebra and the poset of partitions Π_n and we proceed to describe how this work generalizes the classical results to the free multibracketed Lie algebras.

1.1 The free Lie algebra

Recall that a *Lie bracket* on a vector space V is a bilinear binary product $[\cdot,\cdot] : V \times V \to V$ such that for all $x, y, z \in V$,

$$[x,y] + [y,x] = 0 \qquad \text{(Antisymmetry)}, \qquad (1.1.1)$$

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0 \qquad \text{(Jacobi Identity)}. \qquad (1.1.2)$$

Throughout this paper let $\mathbf{k}$ denote an arbitrary field. The *free Lie algebra on* $[n] := \{1, 2, \ldots, n\}$ (over the field $\mathbf{k}$) is the $\mathbf{k}$-vector space generated by the elements of $[n]$ and all the possible bracketings involving these elements subject only to the relations (1.1.1) and (1.1.2). Let $\mathcal{L}ie(n)$ denote the *multilinear* component of the free Lie algebra on $[n]$, i.e., the subspace generated by bracketings that contain each element of $[n]$ exactly once. We call these bracketings *bracketed permutations*. For example $[[2,3],1]$ is a bracketed permutation in $\mathcal{L}ie(3)$, while $[[2,3],2]$ is not. For any set S, the symmetric group $\mathfrak{S}_S$ is the group of permutations of S. In particular we denote by $\mathfrak{S}_n := \mathfrak{S}_{[n]}$ the group of permutations of the set $[n]$. The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathcal{L}ie(n)$ making it into an $\mathfrak{S}_n$-module. A permutation $\tau \in \mathfrak{S}_n$ acts on the bracketed permutations by replacing each letter i by $\tau(i)$. For example $(1,2)\ [[[3,5],[2,4]],1] = [[[3,5],[1,4]],2]$. Since this action respects the relations (1.1.1) and (1.1.2), it induces a representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(n)$. It is a classical result that

$$\dim \mathcal{L}ie(n) = (n-1)!.$$

Although the $\mathfrak{S}_n$-module $\mathcal{L}ie(n)$ is an algebraic object it turns out that the information needed to completely describe this object is of combinatorial nature.

1.2 The poset of partitions Π_n

A *(set) partition* of $[n]$ is a disjoint collection $\{B_1,\ldots,B_t\}$ of subsets (called *blocks*) of $[n]$ such that $\cup_{j=1}^{t} B_j = [n]$. We will very often use the notation $B_1|B_2|\cdots|B_t$ to denote a partition of $[n]$. For two partitions $\{A_1,A_2,...,A_s\}$ and $\{B_1,B_2,...,B_t\}$ of $[n]$ we define the order relation

$$\{A_1,A_2,...,A_s\} \leq \{B_1,B_2,...,B_t\}$$

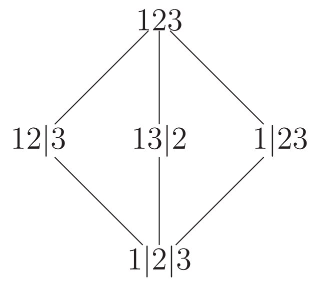


Figure 1.1: Π_3

if every block A_j is contained in some block B_i and we say that $\{A_1, A_2, ..., A_s\}$ is a *refinement* of $\{B_1, B_2, ..., B_t\}$. We denote by Π_n the *partially ordered set* (or *poset* for short) of partitions of $[n]$ ordered by refinement, see Figure 1.1 for the Hasse diagram of Π_3 (the set brackets and commas have been omitted in the figure).

The poset Π_n has a bottom element

$$\hat{0} := \{\{1\}, \{2\}, \ldots, \{n\}\}$$

and a maximal element $\hat{1} = \{[n]\}$. The covering relations are given by

$$\{A_1, A_2, ..., A_s\} \lessdot \{B_1, B_2, ..., B_t\}$$

if $\{B_1, B_2, ..., B_t\}$ is obtained from $\{A_1, A_2, ..., A_s\}$ by *merging* exactly two blocks, i.e., $\{B_1, B_2, ..., B_t\} = \{A_1, A_2, ..., A_s\} \setminus \{A_i, A_j\} \cup \{A_i \cup A_j\}$ for two different blocks A_i and A_j. For example $143|27|5|8 \lessdot 143|257|8$ since the block 257 is obtained by merging the blocks 27 and 5 while the rest of the blocks remain equal.

It is well-known that the Möbius invariant of Π_n is given by

$$\mu_{\Pi_n}(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!,$$

and that the characteristic polynomial (see [40, Example 3.10.4]) by

$$\chi_{\Pi_n}(x) = (x-1)(x-2)\dots(x-n+1). \tag{1.2.1}$$

To every poset P one can associate a simplicial complex $\Delta(P)$ (called the *order complex*) whose faces are the chains (totally ordered subsets) of P. See Appendix A for a review of poset topology and poset (co)homology.

The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n and this action induces isomorphic representations of $\mathfrak{S}_n$ on the unique nonvanishing reduced simplicial homology $\widetilde{H}_{n-3}(\overline{\Pi}_n)$ and cohomology $\widetilde{H}^{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$ of the proper part $\overline{\Pi}_n := \Pi_n \setminus \{\hat{0}, \hat{1}\}$ of Π_n. It is a classical result that

$$\mathcal{L}ie(n) \simeq_{\mathfrak{S}_n} \widetilde{H}_{n-3}(\overline{\Pi}_n) \otimes \mathrm{sgn}_n, \tag{1.2.2}$$

where sgn_n is the sign representation of $\mathfrak{S}_n$.

Equation (1.2.2) was observed by Joyal [28] by comparing a computation of the character of $\widetilde{H}_{n-3}(\overline{\Pi}_n)$ by Hanlon and Stanley (see [38]), to an earlier formula of Brandt [11] for the character of $\mathcal{L}ie(n)$. Joyal [28] gave a proof of the isomorphism using his theory of species. The first purely combinatorial proof was obtained by Barcelo [2] who provided a bijection between known bases for the two $\mathfrak{S}_n$-modules (Björner's NBC basis for $\widetilde{H}_{n-3}(\overline{\Pi}_n)$ and the Lyndon basis for $\mathcal{L}ie(n)$). Later Wachs [44] gave a more general combinatorial proof by providing a natural bijection between generating sets of $\widetilde{H}^{n-3}(\overline{\Pi}_n)$ and $\mathcal{L}ie(n)$, which revealed the strong connection between the two $\mathfrak{S}_n$-modules. Connections between Lie type structures and various types of partition posets have been studied in other places in the literature, see for example [3], [4], [24], [21], [17], [43], [12], [30].

The moral of equation (1.2.2) is that we can describe $\mathcal{L}ie(n)$ and understand its algebraic properties by studying and applying poset theoretic techniques to the combinatorial object Π_n. This observation will play a central role throughout this thesis.

1.3 Free doubly bracketed Lie algebra

Two Lie brackets $[\bullet,\bullet]_1$ and $[\bullet,\bullet]_2$ on a vector space V are said to be *compatible* if any linear combination of the brackets is also a Lie bracket on V, that is, satisfies relations (1.1.1) and (1.1.2). As pointed out in [14, 29], compatibility is equivalent to the *mixed Jacobi* condition: for all $x, y, z \in V$,

$$\begin{aligned}&[x,[y,z]_2]_1 + [z,[x,y]_2]_1 + [y,[z,x]_2]_1 + \qquad \text{(Mixed Jacobi)} \\ &[x,[y,z]_1]_2 + [z,[x,y]_1]_2 + [y,[z,x]_1]_2 = 0.\end{aligned} \tag{1.3.1}$$

Let $\mathcal{L}ie_2(n)$ be the multilinear component of the free Lie algebra on $[n]$ with two compatible brackets, that is, the multilinear component of the $\mathbf{k}$-vector space generated by (mixed) bracketings of elements of $[n]$ subject only to the five relations given by (1.1.1) and (1.1.2), for each bracket, and (1.3.1). For each i, let $\mathcal{L}ie_2(n,i)$ be the subspace of $\mathcal{L}ie_2(n)$ generated by bracketed permutations with exactly i brackets of the first type and $n-1-i$ brackets of the second type. The symmetric group $\mathfrak{S}_n$ acts naturally on $\mathcal{L}ie_2(n)$ and since this action preserves the number of brackets of each type, we have the following decomposition into $\mathfrak{S}_n$-submodules:

$$\mathcal{L}ie_2(n) = \bigoplus_{i=0}^{n-1} \mathcal{L}ie_2(n,i). \tag{1.3.2}$$

Note that interchanging the roles of the two brackets makes evident the $\mathfrak{S}_n$-module isomorphism

$$\mathcal{L}ie_2(n,i) \simeq_{\mathfrak{S}_n} \mathcal{L}ie_2(n,n-1-i)$$

for every i. Also note that in particular $\mathcal{L}ie(n)$ is isomorphic to the submodules $\mathcal{L}ie_2(n,i)$ when $i=0$ or $i=n-1$.

It was conjectured by Feigin and proved independently by Dotsenko-Khoroshkin [14] and Liu [29] that

$$\dim \mathcal{L}ie_2(n) = n^{n-1}. \tag{1.3.3}$$

In [29] Liu proves the conjecture by constructing a combinatorial basis for $\mathcal{L}ie_2(n)$ indexed by rooted trees giving as a byproduct the refinement

$$\dim \mathcal{L}ie_2(n,i) = |\mathcal{T}_{n,i}|, \tag{1.3.4}$$

where $\mathcal{T}_{n,i}$ is the set of rooted trees on vertex set $[n]$ with i descending edges (a parent with a greater label than its child).

The Dotsenko-Khoroshkin proof [14, 15] of Feigin's conjecture was operad-theoretic; they used a pair of functional equations that apply to Koszul operads to compute the $SL_2 \times \mathfrak{S}_n$-character of $\mathcal{L}ie_2(n)$. They also proved that the dimension generating polynomial has a nice factorization:

$$\sum_{i=0}^{n-1} \dim \mathcal{L}ie_2(n,i)t^i = \prod_{j=1}^{n-1}((n-j)+jt). \tag{1.3.5}$$

Since, as was proved by Drake [16], the right hand side of (1.3.5) is equal to the generating function for rooted trees on node set $[n]$ according to the number of descents of the tree, it follows that for each i, the dimension of $\mathcal{L}ie_2(n,i)$ equals the number of rooted trees on node set $[n]$ with i descents. (Drake's result is a refinement of the well-known result that the number of trees on node set $[n]$ is n^{n-1}.)

1.4 The poset of weighted partitions Π_n^w

Although Dotsenko and Khoroshkin [14] did not use poset theoretic techniques in their ultimate proof of (1.3.3), they introduced the poset of weighted partitions Π_n^w as a possible approach to establishing Koszulness of the operad associated with $\mathcal{L}ie_2(n)$, a key step in their proof. In this thesis we explore properties for Π_n^w analogous to the ones for Π_n described in Section 1.2.

A weighted partition of $[n]$ is a set $\{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a partition of $[n]$ and $v_i \in \{0, 1, 2, ..., |B_i| - 1\}$ for all i. The *poset of weighted partitions* Π_n^w is the set of weighted partitions of $[n]$ with order relation given by $\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_s}\} \leq \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$ if the following conditions hold:

- $\{A_1, A_2, ..., A_s\} \leq \{B_1, B_2, ..., B_t\}$ in Π_n
- if $B_k = A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_l}$ then $v_k - (w_{i_1} + w_{i_2} + ... + w_{i_l}) \in \{0, 1, ..., l-1\}$.

Equivalently, we can define the covering relation by

$$\{A_1^{w_1}, A_2^{w_2}, ..., A_s^{w_s}\} \lessdot \{B_1^{v_1}, B_2^{v_2}, ..., B_t^{v_t}\}$$

if the following conditions hold:

- $\{A_1, A_2, \ldots, A_s\} \lessdot \{B_1, B_2, \ldots, B_t\}$ in Π_n
- if $B_k = A_i \cup A_j$, where $i \neq j$, then $v_k - (w_i + w_j) \in \{0, 1\}$
- if $B_k = A_i$ then $v_k = w_i$.

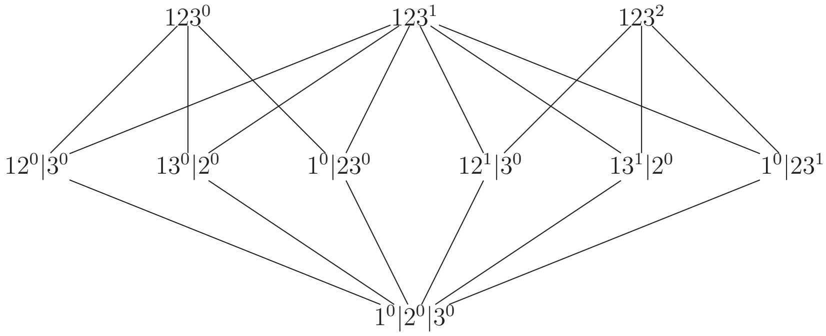


Figure 1.2: Weighted partition poset for $n = 3$

As an example, Π_3^w is illustrated in Figure 1.2.

The poset Π_n^w has a minimum element

$$\hat{0} := \{\{1\}^0, \{2\}^0, \ldots, \{n\}^0\}$$

and n maximal elements

$$\{[n]^0\},\ \{[n]^1\}, \ldots, \{[n]^{n-1}\}.$$

We write each maximal element $\{[n]^i\}$ as $[n]^i$. Note that for all i, the maximal intervals $[\hat{0}, [n]^i]$ and $[\hat{0}, [n]^{n-1-i}]$ are isomorphic to each other, and the two maximal intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$ are isomorphic to Π_n.

The basic properties of Π_n mentioned in Section 1.2 have nice weighted analogs for the intervals $[\hat{0}, [n]^i]$. For instance, the $\mathfrak{S}_n$-module isomorphism (1.2.2) can be generalized. The symmetric group acts naturally on each $\mathcal{L}ie_2(n, i)$ and on each open interval $(\hat{0}, [n]^i)$.

It follows from operad theoretic results of Vallette [43] and Dotsenko-Khoroshkin [15] that the following $\mathfrak{S}_n$- module isomorphism holds:

$$\mathcal{L}ie_2(n, i) \simeq_{\mathfrak{S}_n} \widetilde{H}^{n-3}((\hat{0}, [n]^i)) \otimes \operatorname{sgn}_n . \tag{1.4.1}$$

Note that this reduces to (1.2.2) when $i = 0$ or $i = n - 1$.

In this thesis we give an alternative proof of (1.4.1) by presenting an explicit bijection between natural generating sets of $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and $\mathcal{L}ie_2(n, i)$, which reveals the connection between these modules and generalizes the bijection that Wachs [44] used to prove (1.2.2). With (1.4.1), we take a different path to proving the Liu and Dotsenko-Khoroshkin formula (1.3.5), one that employs poset theoretic techniques.

An *EL-labeling* of a poset (defined in Section 4.1) is a labeling of the edges of the Hasse diagram of the poset that satisfies certain requirements. Such a labeling has important topological and algebraic consequences, such as the determination of the homotopy type of each open interval of the poset. The so called ascent-free maximal chains give a basis for cohomology of the open intervals. A poset that admits an EL-labeling is said to be *EL-shellable*. See [6], [8] and [46] for further information.

We prove that the augmented poset of weighted partitions

$$\widehat{\Pi_n^w} := \Pi_n^w \cup \{\hat{1}\}$$

is EL-shellable by providing an interesting weighted analog of the Björner-Stanley EL-labeling of Π_n (see [6]). In fact our labeling restricts to the Björner-Stanley EL-labeling on the intervals $[\hat{0}, [n]^0]$ and $[\hat{0}, [n]^{n-1}]$. A consequence of shellability is that $\widehat{\Pi_n^w}$ is Cohen-Macaulay, which implies a result of Dotsenko and Khoroshkin [15], obtained through operad theory, that all maximal intervals $[\hat{0}, [n]^i]$ of Π_n^w are Cohen-Macaulay. (Two prior attempts [14, 41] to establish Cohen-Macaulayness of $[\hat{0}, [n]^i]$ are discussed in Remark 4.1.8.) The ascent-free chains of our EL-labeling provide a generalization of the Lyndon basis for cohomology of $\overline{\Pi}_n$ (i.e. the basis for cohomology that corresponds to the classical Lyndon basis for $\mathcal{L}ie(n)$).

Theorem 1.4.1 (Theorem 4.1.4, Corollary 4.1.7 Theorem 4.1.9)**.** *The poset* $\widehat{\Pi_n^w} := \Pi_n^w \cup \{\hat{1}\}$ *is EL-shellable and hence Cohen-Macaulay. Consequently, for each* $i = 0, \dots, n-1$*, the order complex* $\Delta((\hat{0}, [n]^i))$ *has the homotopy type of a wedge of* $|\mathcal{T}_{n,i}|$ *spheres.*

Direct computation of the Möbius function of Π_n^w, which exploits the recursive nature of Π_n^w and makes use of the compositional formula, shows that $(-1)^{n-1}\sum_{i=0}^{n-1}\mu_{\Pi_n^w}(\hat{0}, [n]^i)t^i$ equals the right hand side of (1.3.5). From this computation and the fact that $\widehat{\Pi_n^w}$ is EL-shellable (and thus the maximal intervals of Π_n^w are Cohen-Macaulay), we conclude that

$$\sum_{i=0}^{n-1} \dim \tilde{H}_{n-3}((\hat{0}, [n]^i))t^i = \prod_{j=1}^{n-1}((n-j) + jt). \tag{1.4.2}$$

The Liu and Dotsenko-Khoroshkin formula (1.3.5) is a consequence of this and (1.4.1).

By (1.4.2) and Drake's result mentioned above, the dimension of $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ is equal to the number of rooted trees on $[n]$ with i descents. We construct a nice combinatorial basis for $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ consisting of fundamental cycles indexed by such rooted trees, which generalizes Björner's NBC basis for $\tilde{H}_{n-3}(\overline{\Pi}_n)$. Our proof that these fundamental cycles form a basis relies on Liu's [29] generalization for $\mathcal{L}ie_2(n, i)$ of the classical Lyndon basis for $\mathcal{L}ie(n)$ and our bijective proof of (1.4.1). Indeed, our bijection enables us to transfer bases for $\mathcal{L}ie_2(n, i)$ to bases for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and vice verse. We first transfer Liu's generalization of the Lyndon basis to $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ and then use the natural pairing between homology and cohomology to prove that our proposed homology basis is indeed a basis. (We also obtain an alternative proof that Liu's generalization of the Lyndon basis is a basis along the way.) By transferring the basis for $\tilde{H}^{n-3}((\hat{0}, [n]^i))$ that comes from the ascent-free chains of our EL-labeling to $\mathcal{L}ie_2(n,i)$, we obtain a different generalization of the Lyndon basis that has a somewhat simpler description than that of Liu's generalized Lyndon basis.

We also show that the Möbius invariant of the augmented poset of weighted partitions $\widehat{\Pi_n^w}$ is given by

$$\mu_{\widehat{\Pi_n^w}}(\hat{0},\hat{1}) = (-1)^n(n-1)^{n-1}$$

and that the characteristic polynomial of Π_n^w factors nicely as

$$\chi_{\Pi_n^w}(x) = (x-n)^{n-1}. \tag{1.4.3}$$

1.5 Free multibracketed Lie algebras

Liu posed the following natural question.

Question 1.5.1 (Liu [29], Question 11.7)**.** *Is it possible to define $\mathcal{L}ie_k(n)$ for any $k \geq 1$ so that it has nice dimension formulas like those for $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$? What are the right combinatorial objects for $\mathcal{L}ie_k(n)$, if it can be defined?*

The results developed in this thesis provide an answer to this question.

Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{P}$ the set of positive integers. We say that a set B of Lie brackets on a vector space is *compatible* if any linear combination of the brackets in B is a Lie bracket. We now consider compatible Lie brackets $[\cdot,\cdot]_j$ indexed by positive integers $j \in \mathbb{P}$ and define $\mathcal{L}ie_{\mathbb{P}}(n)$ to be the multilinear component of the multibracketed free Lie algebra on $[n]$; that is, the $\mathbf{k}$-vector space generated by (mixed) bracketed permutations of $[n]$ subject only to the relations given by (1.1.1) and (1.1.2), for each bracket, and the compatibility relations for any set of brackets. For example, $[[[2,5]_2,3]_1,[1,4]_1]_3$ is a generator of $\mathcal{L}ie_{\mathbb{P}}(5)$.

A *weak composition* μ of n is a sequence of nonnegative integers $(\mu(1), \mu(2), \dots)$ such that $|\mu| := \sum_{i\geq 1} \mu(i) = n$. Let wcomp be the set of weak compositions and wcomp_n the set of weak compositions of n. For $\mu \in \text{wcomp}_{n-1}$, define $\mathcal{L}ie(\mu)$ to be the subspace of $\mathcal{L}ie_{\mathbb{P}}(n)$ generated by bracketed permutations of $[n]$ with $\mu(j)$ brackets of type j for each j. For example $\mathcal{L}ie(0,1,2,0,1)$ is generated by bracketed permutations of [5] that contain one bracket of type 2, two brackets of type 3, one bracket of type 5 and no brackets of any other type.

As before, $\mathfrak{S}_n$ acts naturally on $\mathcal{L}ie(\mu)$ by replacing the letters of a bracketed permutation. Interchanging the roles of the brackets reveals that for every $\nu, \mu \in$ wcomp, such that ν is a rearrangement of μ, we have that $\mathcal{L}ie(\nu) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(\mu)$. In particular, if μ has a single nonzero component, $\mathcal{L}ie(\mu)$ is isomorphic to $\mathcal{L}ie(n)$. If μ has at most two nonzero components then $\mathcal{L}ie(\mu)$ is isomorphic to $\mathcal{L}ie(n,i)$ for some $0 \leq i \leq n-1$.

For $\mu \in$ wcomp define its *support* $\text{supp}(\mu) = \{j \in \mathbb{P} \mid \mu(j) \neq 0\}$ and for a subset $S \subseteq \mathbb{P}$ let

$$\mathcal{L}ie_S(n) := \bigoplus_{\substack{\mu \in \text{wcomp}_{n-1} \\ \text{supp}(\mu) \subseteq S}} \mathcal{L}ie(\mu).$$

Note that $\mathcal{L}ie_k(n) := \mathcal{L}ie_{[k]}(n)$ generalizes $\mathcal{L}ie(n) = \mathcal{L}ie_1(n)$ and $\mathcal{L}ie_2(n)$.

The isomorphisms (1.2.2) and (1.4.1) provide a way to study the algebraic objects $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$ by applying poset topology techniques to Π_n and Π_n^w. In particular the dimensions of the modules can be read from the structure of the posets and the bases for the cohomology of the posets can be directly translated into bases of $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$. It is then natural to look for a poset whose cohomology allows us to analyze $\mathcal{L}ie_k(n)$.

1.6 The poset of weighted partitions Π_n^k

We introduce a more general poset of weighted partitions Π_n^k where the weights are given by weak compositions supported in $[k]$. A *(composition)-weighted partition* of $[n]$ is a set $\{B_1^{\mu_1}, B_2^{\mu_2}, ..., B_t^{\mu_t}\}$ where $\{B_1, B_2, ..., B_t\}$ is a set partition of $[n]$ and $\mu_i \in \text{wcomp}_{|B_i|-1}$ with $\text{supp}(\mu_i) \subseteq [k]$. For $\nu, \mu \in \text{wcomp}$, we say that $\mu \leq \nu$ if $\mu(i) \leq \nu(i)$ for every i. Since weak compositions are infinite vectors we can use component-wise addition and subtraction, for instance, we denote by $\nu + \mu$, the weak composition defined by $(\nu + \mu)(i) := \nu(i) + \mu(i)$.

The *poset of weighted partitions* Π_n^k is the set of weighted partitions of $[n]$ with order relation given by $\{A_1^{\mu_1}, A_2^{\mu_2}, ..., A_s^{\mu_s}\} \leq \{B_1^{\nu_1}, B_2^{\nu_2}, ..., B_t^{\nu_t}\}$ if the following conditions hold:

- $\{A_1, A_2, ..., A_s\} \leq \{B_1, B_2, ..., B_t\}$ in Π_n and,
- If $B_j = A_{i_1} \cup A_{i_2} \cup ... \cup A_{i_l}$ then $\nu_j \geq (\mu_{i_1} + \mu_{i_2} + ... + \mu_{i_l})$ and $|\nu_j - (\mu_{i_1} + \mu_{i_2} + ... + \mu_{i_l})| = l - 1$

Equivalently, we can define the covering relation $\{A_1^{\mu_1}, A_2^{\mu_2}, ..., A_s^{\mu_s}\} \lessdot \{B_1^{\nu_1}, B_2^{\nu_2}, ..., B_t^{\nu_t}\}$ by:

- $\{A_1, A_2, \ldots, A_s\} \lessdot \{B_1, B_2, \ldots, B_t\}$ in Π_n
- if $B_j = A_{i_1} \cup A_{i_2}$ then $\nu_j - (\mu_{i_1} + \mu_{i_2}) = \mathbf{e_r}$ for some $r \in [k]$, where $\mathbf{e_r}$ is the weak composition with a 1 in the r-th component and 0 in all other entries.
- if $B_k = A_i$ then $\nu_k = \mu_i$.

See Figure 1.3 below for an example of Π_n^k.

The poset Π_n^k has a minimum element

$$\hat{0} := \{\{1\}^{(0,\ldots,0)}, \{2\}^{(0,\ldots,0)}, \ldots, \{n\}^{(0,\ldots,0)}\}$$

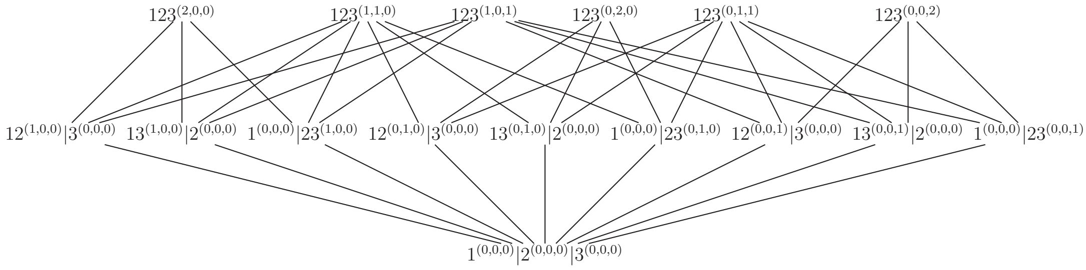


Figure 1.3: Weighted partition poset for $n = 3$ and $k = 3$

and $\binom{k+n-2}{n-1}$ maximal elements

$$\{[n]^\mu\} \text{ for } \mu \in \mathrm{wcomp}_{n-1} \text{ and } \mathrm{supp}(\mu) \subseteq [k].$$

We write each maximal element $\{[n]^\mu\}$ as $[n]^\mu$ for simplicity. Note that for every $\nu, \mu \in \mathrm{wcomp}_{n-1}$ with $\mathrm{supp}(\nu), \mathrm{supp}(\mu) \subseteq [k]$, such that ν is a rearrangement of μ, the maximal intervals $[\hat{0}, [n]^\nu]$ and $[\hat{0}, [n]^\mu]$ are isomorphic to each other. In particular, if μ has a single nonzero component, these intervals are isomorphic to Π_n. If $\mathrm{supp}(\mu) \subseteq [2]$ then these intervals are isomorphic to maximal intervals of Π_n^w. Indeed, we can think of a composition $(i, n-1-i)$ as being the weight i in Π_n^w. Hence $\Pi_n^1 \simeq \Pi_n$ and $\Pi_n^2 \simeq \Pi_n^w$. We will derive results for the more general poset Π_n^k when possible and specialize these results to Π_n^w when the results for the case $k = 2$ have nicer combinatorial formulas and interpretations than the ones for the general case.

The symmetric group acts naturally on each open interval $(\hat{0}, [n]^\mu)$. Using Wachs' technique in Chapter 2 we give an explicit isomorphism that proves the following theorem.

Theorem 1.6.1. *For* $\mu \in \mathrm{wcomp}_{n-1}$,

$$\mathcal{L}ie(\mu) \simeq_{\mathfrak{S}_n} \widetilde{H}^{n-3}((\hat{0}, [n]^\mu)) \otimes \mathrm{sgn}_n \,. \tag{1.6.1}$$

Theorem 1.6.1 is a generalization of equations (1.2.2) and (1.4.1). It reduces to equation (1.2.2) when $\mathrm{supp}(\mu) \subseteq [1]$ and to equation (1.4.1) when $\mathrm{supp}(\mu) \subseteq [2]$. We use Theorem 1.6.1 to give information about $\mathcal{L}ie(\mu)$ by studying the algebraic and combinatorial properties of the poset Π_n^k.

In [15] Dotsenko and Khoroshkin prove using operad-theoretic techniques that the operad related to $\mathcal{L}ie_k(n)$ is Koszul. This implies using Vallette's theory of operadic partition posets [43] that the maximal intervals $[\hat{0}, [n]^\mu]$ of Π_n^k are Cohen-Macaulay. In Section 4.1 we prove a stronger property.

Theorem 1.6.2. *The poset* $\widehat{\Pi_n^k} := \Pi_n^k \cup \{\hat{1}\}$ *is EL-shellable and hence Cohen-Macaulay. Consequently, for each* $\mu \in \mathrm{wcomp}_{n-1}$*, the order complex* $\Delta((\hat{0}, [n]^\mu))$ *has the homotopy type of a wedge of* $(n-3)$*-spheres.*

Using Vallette's theory, Theorem 1.6.2 gives a new proof of the fact that the operads $\mathcal{L}ie_k$ and ${}^k\mathcal{C}om$ considered in [15] are Koszul.

The set of ascent-free maximal chains of this EL-labeling provides a basis for $\widetilde{H}^{n-3}((\hat{0}, [n]^\mu))$ and hence, by the isomorphism of Theorem 1.6.1, also a basis for $\mathcal{L}ie(\mu)$. This basis is a multicolored generalization of the classical Lyndon basis for $\mathcal{L}ie(n)$. We also construct a multicolored generalization of the classical comb basis for $Lie(n)$ and use our multicolored Lyndon basis to show that our construction does indeed yield a basis for $\mathcal{L}ie(\mu)$.

We consider the generating function

$$L_n(\mathbf{x}) := \sum_{\mu \in \mathrm{wcomp}_n} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^\mu, \tag{1.6.2}$$

where $\mathbf{x}^\mu = x_1^{\mu(1)} x_2^{\mu(2)} \cdots$. Since for any rearrangement ν of μ it happens that $\mathcal{L}ie(\nu) \simeq_{\mathfrak{S}_n} \mathcal{L}ie(\mu)$ it follows that (1.6.2) belongs to the ring of symmetric functions $\Lambda_{\mathbb{Z}}$. The following theorem gives a characterization of this symmetric function.

Theorem 1.6.3. *We have*

$$\sum_{n\geq 1}\sum_{\mu\in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\,\mathbf{x}^{\mu}\frac{y^n}{n!} = \left[\sum_{n\geq 1}(-1)^{n-1}h_{n-1}(\mathbf{x})\frac{y^n}{n!}\right]^{<-1>},$$

where h_n is the complete homogeneous symmetric function and $(\cdot)^{<-1>}$ denotes the compositional inverse of a formal power series.

It follows from our construction of the multicolored Lyndon basis for $\mathcal{L}ie(\mu)$ that the symmetric function $L_n(x)$ is e-positive; i.e., the coefficients of the expansion of $L_n(x)$ in the basis of elementary symmetric functions are all nonnegative. We give various combinatorial interpretations of these coefficients in this thesis. Two of the interpretations involve binary trees and two involve the Stirling permutations introduced by Gessel and Stanley in [18]. We will now give one of the binary tree interpretations (Theorem 1.6.4). The others are given in Theorems 5.1.1 and 5.3.3.

We say that a planar labeled binary tree with label set $[n]$ is *normalized* if the leftmost leaf of each subtree has the smallest label in the subtree. See Figure 1.4 for an example of a normalized tree and Section 4.2 for the proper definitions. We denote the set of normalized binary trees with label set $[n]$ by Nor_n.

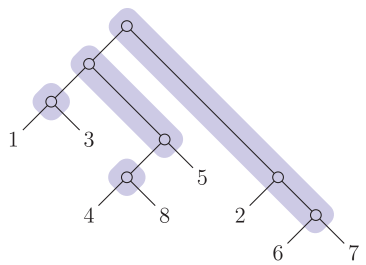


Figure 1.4: Example of a normalized tree

We associate a *type* (or integer partition) to each $\Upsilon \in \mathsf{Nor}_n$ in the following way: Let $\pi^{\mathsf{Comb}}(\Upsilon)$ be the finest (set) partition of the set of internal nodes of Υ satisfying

- for every pair of internal nodes x and y such that y is a right child of x, x and y belong to the same block of $\pi^{\mathsf{Comb}}(\Upsilon)$.

We define the *comb type* $\lambda^{\mathsf{Comb}}(\Upsilon)$ of Υ to be the (integer) partition whose parts are the sizes of the blocks of $\pi^{\mathsf{Comb}}(\Upsilon)$. In Figure 1.4 the associated partition is $\lambda^{\mathsf{Comb}}(\Upsilon) = (3, 2, 1, 1)$. The following theorem gives a direct method, alternative to Theorem 1.6.3, for computing the dimensions of $\mathcal{L}ie(\mu)$.

Theorem 1.6.4. *For all n,*

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^{\mu} = \sum_{\Upsilon \in \mathsf{Nor}_n} e_{\lambda^{\mathsf{Comb}}(\Upsilon)}(\mathbf{x}),$$

where e_λ is the elementary symmetric function associated with the partition λ.

To prove Theorem 1.6.4 we use another normalized tree type λ^{Lyn}, related to our colored Lyndon basis for $\mathcal{L}ie(\mu)$, which came from the EL-labeling of $[\hat{0}, [n]^{\mu}]$. We use the colored Lyndon basis to show that Theorem 1.6.4 holds with λ^{Comb} replaced by λ^{Lyn}. We then construct a bijection on Nor_n which takes λ^{Lyn} to λ^{Comb}. This bijection makes use of Stirling permutations and leads to two versions of Theorem 1.6.4 involving Stirling permutations.

In terms of these combinatorial objects, the dimension of $\mathcal{L}ie_k$ has a simple description as an evaluation of the symmetric function (1.6.2).

Corollary 1.6.5. *For all n and k,*

$$\dim \mathcal{L}ie_k(n) = \sum_{\Upsilon \in \mathsf{Nor}_n} e_{\lambda^{\mathsf{Comb}}(\Upsilon)}(\overbrace{1, \dots, 1}^{k \text{ times}}, 0, 0, \dots).$$

From equation (1.3.5), it follows that the polynomial $\sum_{i=0}^{n-1} \dim \mathcal{L}ie_2(n, i)\, t^i$ has only negative real roots and hence it has a property known as *γ-positivity*, i.e, when written in the basis $t^i(1+t)^{n-1-2i}$ it has positive coefficients. Note that this polynomial is actually $L_{n-1}(t, 1, 0, 0, \dots)$. The property of γ-positivity of this polynomial is a consequence of the e-positivity of $L_n(\mathbf{x})$.

A more general question is to understand the representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(\mu)$. The characters of the representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(n)$ and $\mathcal{L}ie_2(n)$ were computed in ([11, 38] and [14]). Here we consider

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \mathrm{ch}\, \mathcal{L}ie(\mu)\, \mathbf{x}^{\mu}, \tag{1.6.3}$$

where $\mathrm{ch}\, \mathcal{L}ie(\mu)$ denotes the Frobenius characteristic in variables $\mathbf{y} = (y_1, y_2, \dots)$ of the representation $\mathcal{L}ie(\mu)$. The generating function of (1.6.3) belongs to the ring Λ_R of symmetric functions in $\mathbf{y}$ with coefficients in the ring of symmetric functions $R = \Lambda_{\mathbb{Q}}$ in $\mathbf{x}$. The following result generalizes Theorem 1.6.3.

Theorem 1.6.6. *We have that*

$$\sum_{n \geq 1} \sum_{\mu \in \mathrm{wcomp}_{n-1}} \mathrm{ch}\, \mathcal{L}ie(\mu)\, \mathbf{x}^{\mu} = -\Big(-\sum_{n \geq 1} h_{n-1}(\mathbf{x}) h_n(\mathbf{y})\Big)^{[-1]},$$

where $(\cdot)^{[-1]}$ *denotes the plethystic inverse in the ring of symmetric power series in* $\mathbf{y}$ *with coefficients in the ring* $\Lambda_{\mathbb{Q}}$ *of symmetric functions in* $\mathbf{x}$.

To prove Theorem 1.6.6 we use Theorem 1.6.1 and the Whitney (co)homology technique developed by Sundaram in [42], and further developed by Wachs in [45].

The thesis is organized as follows: In Chapter 2 we describe generating sets of $\mathcal{L}ie(\mu)$ and $\widetilde{H}^{n-3}((\hat{0}, [n]^{\mu}))$ in terms of labeled binary trees with colored internal nodes. The description makes transparent the isomorphism of Theorem 1.6.1, which we prove using Wachs' technique, as in [44]. In Chapter 3 we use the recursive definition of the Möbius invariant of Π_n^k to prove an analogue of Theorem 1.6.3 for the poset Π_n^k, that together with the results of Chapter 2 imply Theorem 1.6.3. When we apply the same procedure to the special case of Π_n^w we are able to conclude further results, including a description of the Möbius invariant in terms of rooted trees. We use this description to prove the factorization formula (1.4.3) for the characteristic polynomial of Π_n^w. In Chapter 4 we prove Theorem 1.6.2, and we give a description of the ascent-free maximal chains of the EL-labeling. Theorems 1.6.3 and the version of Theorem 1.6.4 in which λ^{Comb} is replaced by λ^{Lyn}, are presented in Chapter 5 as corollaries of results in the previous chapters. In Chapter 5 we also prove Theorem 1.6.4 and we use the language of Stirling permutations to give two additional combinatorial descriptions of the dimension of $\mathcal{L}ie(\mu)$. In Chapter 6 we present the colored Lyndon basis and the colored comb basis for $\mathcal{L}ie(\mu)$ and $\widetilde{H}^{n-3}((\hat{0},[n]^{\mu}))$. We also give the basis for $\tilde{H}_{n-3}((\hat{0},[n]^{i}))$ indexed by rooted trees, and we provide results on bases for $\widetilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$ in terms of the two families of colored binary trees. We present in Chapter 7 results on Whitney numbers of the first and second kind and on Whitney cohomology. In Chapter 8 we prove Theorem 1.6.6.

Chapter 2

The isomorphism $\mathcal{L}ie(\mu) \simeq_{\mathfrak{S}_n} \widetilde{H}^{n-3}((\hat{0}, [n]^{\mu})) \otimes \mathrm{sgn}_n$

In this chapter we establish the isomorphism of Theorem 1.6.1. We will use this isomorphism to study $\mathcal{L}ie(\mu)$ by understanding the algebraic and combinatorial properties of the maximal intervals $[\hat{0}, [n]^{\mu}]$ of Π_n^k.

The generators of $\mathcal{L}ie(\mu)$ and $\widetilde{H}^{n-3}((\hat{0}, [n]^{\mu}))$ can be described in terms of trees. A *tree* is a simple connected graph that is free of cycles. A tree is said to be *rooted* if it has a distinguished node or *root*. For an edge $\{x, y\}$ in a tree T we say that x is the *parent* of y, or y is the *child* of x, if x is in the unique path from y to the root. A node that has children is said to be *internal*, otherwise we call a node without children a *leaf*. A rooted tree is said to be *planar* if for every internal node its set of children has been totally ordered. In the following we will be only considering trees that are rooted and planar and so when using the word tree we mean a planar rooted tree.

A *binary tree* is a tree for which every internal node has a left and a right child. A *colored binary tree* is a binary tree for which each internal node x has been assigned an element $\mathbf{color}(x) \in \mathbb{P}$. For a colored binary tree T with n leaves and $\sigma \in \mathfrak{S}_n$, we define the *labeled colored binary tree* (T, σ) to be the colored tree T whose jth leaf from left to right has been labeled $\sigma(j)$. For $\mu \in \text{wcomp}_{n-1}$ we denote by $\mathcal{BT}_\mu$ the set of labeled colored binary trees with n leaves and $\mu(j)$ internal nodes with color j for each j. We call these trees *μ-colored binary trees.* We will often denote a colored labeled binary tree by $\Upsilon = (T, \sigma)$. If Υ is a colored labeled binary tree, we use $\widetilde{\Upsilon}$ to denote its underlying uncolored labeled binary tree. It will also be convenient to consider trees whose label set is more general than $[n]$. For a finite subset A of positive integers with $|A| = |\mu| + 1$, let $\mathcal{BT}_{A,\mu}$ be the set of μ-colored binary trees whose leaves are labeled by a permutation of A. If $(S, \alpha) \in \mathcal{BT}_{A,\mu}$ and $(T, \beta) \in \mathcal{BT}_{B,\nu}$, where A and B are disjoint finite sets, and $j \in \mathbb{P}$ then $(S, \alpha) \overset{j}{\wedge} (T, \beta)$ denotes the tree in $\mathcal{BT}_{A \cup B, \mu+\nu+e_j}$ whose *left subtree* is (S, α), *right subtree* is (T, β), and the color of the root is j.

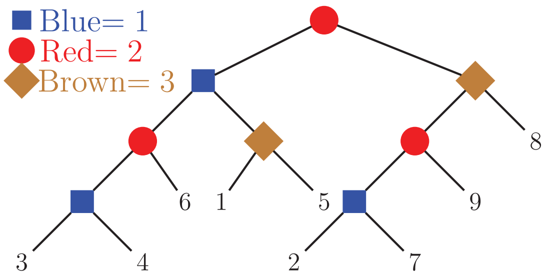
2.1 A combinatorial description of $\mathcal{L}ie(\mu)$

We give a description of the generators and relations of $\mathcal{L}ie(\mu)$. We can represent the bracketed permutations that generate $\mathcal{L}ie(\mu)$ with labeled colored binary trees. More precisely, let (T_1, σ_1) and (T_2, σ_2) be the left and right labeled subtrees of the root r of $(T, \sigma) \in \mathcal{BT}_\mu$. Then define recursively

$$[T, \sigma] = \begin{cases} \Big[[T_1, \sigma_1], [T_2, \sigma_2] \Big]_j & \text{if } \mathbf{color}(r) = j \text{ and } n > 1 \\ \sigma & \text{if } n = 1. \end{cases} \tag{2.1.1}$$

Clearly $[T, \sigma]$ is a bracketed permutation of $\mathcal{L}ie(\mu)$. See Figure 2.1.

Recall that we call a set B of Lie brackets on a vector space *compatible* if any linear combination of the brackets in B is a Lie bracket. As it turns out the description of the relations in $\mathcal{L}ie(\mu)$ are simplified by the following proposition.



$$[[[[3,4]_1,6]_2,[1,5]_3]_1,[[[2,7]_1,9]_2,8]_3]_2$$

Figure 2.1: Example of a labeled colored binary tree $(T, 346152798) \in \mathcal{BT}_{(3,3,2)}$ and $[T, 346152798] \in \mathcal{L}ie(3,3,2)$

Proposition 2.1.1. *A set of Lie brackets is compatible if and only if the brackets in the set are pairwise compatible.*

Proof. Assume that the brackets $\{[\cdot,\cdot]_j \mid j \in S\}$ are pairwise compatible. Hence for any $i, j \in S$ we have that the relation (1.3.1) holds. Now for scalars $\alpha_j \in \mathbf{k}$ and a finite subset $\{i_1, \ldots, i_k\} \subseteq S$ define

$$\langle \cdot,\cdot \rangle = \sum_{j=1}^{k} \alpha_j [\cdot,\cdot]_{i_j}.$$

By relations (1.1.2) and (1.3.1) and bilinearity of the brackets, we have

$$\begin{aligned}
0 &= \sum_{j=1}^{k} \alpha_j^2([x,[y,z]_{i_j}]_{i_j} + [z,[x,y]_{i_j}]_{i_j} + [y,[z,x]_{i_j}]_{i_j}) \\
&\quad + \sum_{l<j} \alpha_l\alpha_j([x,[y,z]_{i_j}]_{i_l} + [z,[x,y]_{i_j}]_{i_l} + [y,[z,x]_{i_j}]_{i_l} \\
&\qquad\qquad + [x,[y,z]_{i_l}]_{i_j} + [z,[x,y]_{i_l}]_{i_j} + [y,[z,x]_{i_l}]_{i_j}) \\
&= \sum_{l,j=1}^{k} \alpha_l\alpha_j[x,[y,z]_{i_l}]_{i_j} + \alpha_l\alpha_j[z,[x,y]_{i_l}]_{i_j} + \alpha_l\alpha_j[y,[z,x]_{i_l}]_{i_j} \\
&= \langle x, \langle y, z\rangle\rangle + \langle z, \langle x, y\rangle\rangle + \langle y, \langle z, x\rangle\rangle.
\end{aligned}$$

This implies that $\langle \cdot, \cdot \rangle$ satisfies relation (1.1.2). It follows from the definition that $\langle \cdot, \cdot \rangle$ also satisfies the relation (1.1.1) and hence it is a Lie bracket.

For the converse note, from the definition of compatibility, that all the brackets in a compatible set of Lie brackets are pairwise compatible. $\square$

Thus we see that $\mathcal{L}ie(\mu)$ is subject only to the relations (1.1.1) and (1.1.2), for each bracket j, and (1.3.1) for any pair of brackets $i \neq j \in [k]$. If the characteristic of $\mathbf{k}$ is not 2 we can even say that $\mathcal{L}ie(\mu)$ is subject only to relations (1.1.1) and (1.3.1) for any pair of brackets $i, j \in [k]$ (including $i = j$).

We denote by $\Upsilon_1 \overset{j}{\wedge} \Upsilon_2$, the labeled colored binary tree whose left subtree is Υ_1, right subtree is Υ_2 and root color is j, with $j \in \mathbb{P}$. If Υ is a labeled colored binary tree then $\alpha(\Upsilon)\beta$ denotes a labeled colored binary tree with Υ as a subtree. The following result is an easy consequence of relations (1.1.1) and (1.1.2) for each j, and (1.3.1) for each pair $i \neq j$.

Proposition 2.1.2. *The set* $\{[T, \sigma] \mid (T, \sigma) \in \mathcal{BT}_\mu\}$ *is a generating set for* $\mathcal{L}ie(\mu)$*, subject only to the relations for* $i \neq j \in \operatorname{supp}(\mu)$

$$[\alpha(\Upsilon_1 \overset{j}{\wedge} \Upsilon_2)\beta] + [\alpha(\Upsilon_2 \overset{j}{\wedge} \Upsilon_1)\beta] = 0 \tag{2.1.2}$$

$$\begin{aligned} [\alpha(\Upsilon_1 \overset{j}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta] \quad &- \quad [\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_2) \overset{j}{\wedge} \Upsilon_3)\beta] \\ &- \quad [\alpha(\Upsilon_2 \overset{j}{\wedge} (\Upsilon_1 \overset{j}{\wedge} \Upsilon_3))\beta] \\ &= \quad 0 \end{aligned} \tag{2.1.3}$$

$$[\alpha(\Upsilon_1 \overset{j}{\wedge} (\Upsilon_2 \overset{i}{\wedge} \Upsilon_3))\beta] \quad + \quad [\alpha(\Upsilon_1 \overset{i}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta] \tag{2.1.4}$$

$$
\begin{aligned}
- \quad [\alpha((\Upsilon_1 {}^{j}_{\wedge} \Upsilon_2) {}^{i}_{\wedge} \Upsilon_3)\beta] \quad &- \quad [\alpha((\Upsilon_1 {}^{i}_{\wedge} \Upsilon_2) {}^{j}_{\wedge} \Upsilon_3)\beta] \\
- \quad [\alpha(\Upsilon_2 {}^{j}_{\wedge} (\Upsilon_1 {}^{i}_{\wedge} \Upsilon_3))\beta] \quad &- \quad [\alpha(\Upsilon_2 {}^{i}_{\wedge} (\Upsilon_1 {}^{j}_{\wedge} \Upsilon_3))\beta] \\
&= \quad 0.
\end{aligned}
$$

2.2 A generating set for $\widetilde{H}^{n-3}((\hat{0}, [n]^{\mu}))$

The top dimensional cohomology of a pure poset P, say of length ℓ, has a particularly simple description. Let $\mathcal{M}(P)$ denote the set of maximal chains of P and let $\mathcal{M}'(P)$ denote the set of chains of length $\ell - 1$. We view the coboundary map δ as a map from the chain space of P to itself, which takes chains of length d to chains of length $d+1$ for all d. Since the image of δ on the top chain space (i.e. the space spanned by $\mathcal{M}(P)$) is 0, the kernel is the entire top chain space. Hence top cohomology is the quotient of the space spanned by $\mathcal{M}(P)$ by the image of the space spanned by $\mathcal{M}'(P)$. The image of $\mathcal{M}'(P)$ is what we call the coboundary relations. We thus have the following presentation of the top cohomology

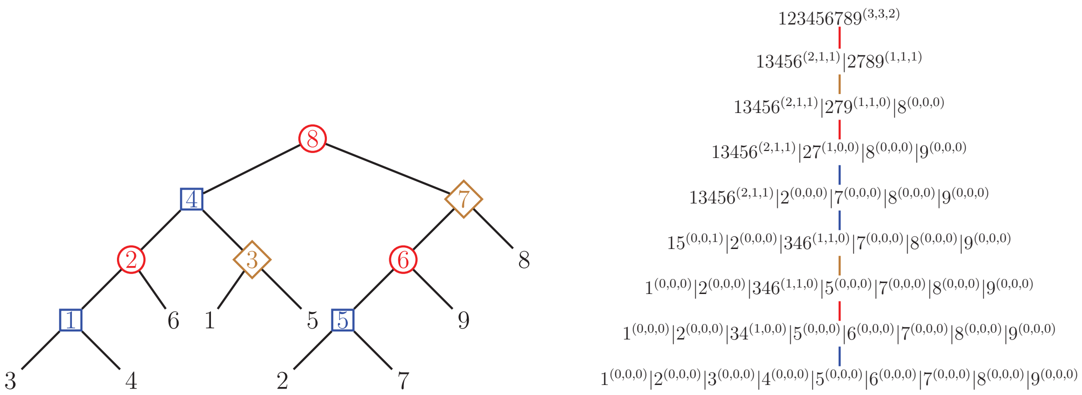
$$\widetilde{H}^{\ell}(P) = \langle \mathcal{M}(P) | \text{ coboundary relations} \rangle.$$

Recall that $\widetilde{H}^{\ell}(P)$ denotes the ℓth reduced cohomology of the order complex $\Delta(P)$ of P. The reader can visit Section A.3 in the Appendix for a more extensive treatement on poset cohomology.

Recall that the *postorder listing* of the internal nodes of a binary tree T is defined recursively as follows: first list the internal nodes of the left subtree in postorder, then list the internal nodes of the right subtree in postorder, and finally list the root. The postorder listing of the internal nodes of the binary tree of Figure 2.1 is illustrated in Figure 2.2a.

Given s blocks $A_1^{\mu_1}, A_2^{\mu_2}, \ldots, A_s^{\mu_s}$ in a weighted partition α and $\nu \in \mathrm{wcomp}_{s-1}$, by ν-*merge* these blocks we mean remove them from α and replace them by the block $(\bigcup A_i)^{\sum \mu_i + \nu}$. For $(T, \sigma) \in \mathcal{BT}_{A,\mu}$, let $\pi(T, \sigma) = A^{\mu}$.

Definition 2.2.1. For $(T, \sigma) \in \mathcal{BT}_{\mu}$ and $t \in [n-1]$, let $T_t = L_t {}^{j_t}_{\wedge} R_t$ be the subtree of (T, σ) rooted at the tth node listed in postorder. The chain $\mathrm{c}(T, \sigma) \in \mathcal{M}([\hat{0}, [n]^{\mu}])$ is the one whose rank t weighted partition is obtained from the rank $t-1$ weighted partition by $\mathbf{e_{j_t}}$-merging the blocks $\pi(L_t)$ and $\pi(R_t)$. See Figure 2.2b.



(a) $(T, \sigma) \in \mathcal{BT}_{(3,3,2)}$ (b) $\mathrm{c}(T, \sigma)$

Figure 2.2: Example of postorder (internal nodes) of the binary tree of Figure 2.1 and the chain $\mathrm{c}(T, \sigma)$

Not all maximal chains in $\mathcal{M}([\hat{0}, [n]^{\mu}])$ can be described as $\mathrm{c}(T, \sigma)$. For some maximal chains postordering of the internal nodes is not enough to describe the process of merging the blocks. We need a more flexible construction in terms of linear extensions (cf. [44]). Let $v_1, \ldots, v_{n-1}$ be the postorder listing of the internal nodes of T and let $j_i = \mathbf{color}(v_i)$ for all $i \in [n-1]$. A listing $v_{\tau(1)}, v_{\tau(2)}, \ldots, v_{\tau(n-1)}$ of the internal nodes such that each node precedes its parent is said to be a *linear extension* of T. We will say that the permutation τ induces the linear extension. In particular, the identity permutation ε induces postorder which is a linear extension. Denote by $e(T)$ the set of permutations that induce linear extensions of the internal nodes of T. For each $\tau \in e(T)$, we extend the construction of $\mathrm{c}(T, \sigma)$ by letting $\mathrm{c}(T, \sigma, \tau)$ be the chain in $\mathcal{M}([\hat{0}, [n]^{\mu}])$ whose rank t weighted partition is obtained from the rank $t-1$ weighted partition by $e_{j_{\tau(t)}}$-merging the blocks $\pi(L_{\tau(t)})$ and $\pi(R_{\tau(t)})$, where $L_i \overset{j_i}{\wedge} R_i$ is the subtree rooted at v_i. In particular, $\mathrm{c}(T,\sigma) = \mathrm{c}(T,\sigma,\varepsilon)$. From each maximal chain we can easily construct a binary tree and a linear extension that encodes the merging instructions along the chain. Thus, any maximal chain can be obtained in this form.

Lemma 2.2.2 ([44, Lemma 5.1]). *Let T be a binary tree. Then*

1. $\varepsilon \in e(T)$

2. *If $\tau \in e(T)$ and $\tau(i) > \tau(i+1)$ then $\tau(i,i+1) \in e(T)$,*

where $\tau(i,i+1)$ denotes the product of τ and the transposition $(i,i+1)$ in the symmetric group.

Proof. Postorder ε is a linear extension since in postorder we list children before parents. Now, $\tau(i) > \tau(i+1)$ means that $v_{\tau(i+1)}$ is listed in postorder before $v_{\tau(i)}$, and so $v_{\tau(i+1)}$ cannot be an ancestor of $v_{\tau(i)}$. This implies that $\tau(i,i+1)$ is also a linear extension. □

For any colored labeled binary tree (T,σ), the chains obtained with any two different linear extensions are cohomologous in the sense of Lemma 2.2.3 below.

The number of inversions of a permutation $\tau \in \mathfrak{S}_n$ is defined by $\mathrm{inv}(\tau) := |\{(i,j) \mid 1 \le i < j \le n,\ \tau(i) > \tau(j)\}|$ and the sign of τ is defined by $\mathrm{sgn}(\tau) := (-1)^{\mathrm{inv}(\tau)}$. For $T \in \mathcal{BT}_{n,\mu}$, $\sigma \in \mathfrak{S}_n$, and $\tau \in e(T)$, write $\bar{c}(T,\sigma,\tau)$ for $\overline{c(T,\sigma,\tau)} := c(T,\sigma,\tau) \setminus \{\hat{0}, [n]^\mu\}$ and $\bar{c}(T,\sigma)$ for $\overline{c(T,\sigma)} := c(T,\sigma) \setminus \{\hat{0}, [n]^\mu\}$.

Lemma 2.2.3 (cf. [44, Lemma 5.2]). *Let $(T,\sigma) \in \mathcal{BT}_\mu$, $\tau \in e(T)$. Then in $\widetilde{H}^{n-3}((\hat{0}, [n]^\mu))$,*

$$\bar{\mathrm{c}}(T,\sigma,\tau) = \mathrm{sgn}(\tau)\bar{\mathrm{c}}(T,\sigma).$$

Proof. We proceed by induction on $\mathrm{inv}(\tau)$. If $\mathrm{inv}(\tau) = 0$ then $\tau = \varepsilon$ and the result is trivial. If $\mathrm{inv}(\tau) \geq 1$, then there is some descent $\tau(i) > \tau(i+1)$ and by Lemma 2.2.2, $\tau(i, i+1) \in \mathcal{E}(T)$. Since $\mathrm{inv}(\tau(i, i+1)) = \mathrm{inv}(\tau) - 1$, by induction we have,

$$\bar{\mathrm{c}}(T, \sigma, \tau(i, i+1)) = \mathrm{sgn}(\tau(i, i+1))\bar{\mathrm{c}}(T, \sigma) = -\,\mathrm{sgn}(\tau)\bar{\mathrm{c}}(T, \sigma).$$

We have to show then that

$$\bar{\mathrm{c}}(T, \sigma, \tau) = -\bar{\mathrm{c}}(T, \sigma, \tau(i, i+1)).$$

By the proof of Lemma 2.2.2 we know that the internal nodes $v_{\tau(i)}$ and $v_{\tau(i+1)}$ are unrelated in T and so $\pi(L_{\tau(i)})$, $\pi(R_{\tau(i)})$, $\pi(L_{\tau(i+1)})$ and $\pi(R_{\tau(i+1)})$ are pairwise disjoint sets which are all blocks of the rank $i-1$ partition in both $\bar{\mathrm{c}}(T, \sigma, \tau)$ and $\bar{\mathrm{c}}(T, \sigma, \tau(i, i+1))$. The blocks $\pi(L_{\tau(i)} {}^{j_{\tau(i)}}_{\wedge} R_{\tau(i)})$ and $\pi(L_{\tau(i+1)} {}^{j_{\tau(i+1)}}_{\wedge} R_{\tau(i+1)})$ are blocks of the rank $i+1$ partition in both $\bar{\mathrm{c}}(T, \sigma, \tau)$ and $\bar{\mathrm{c}}(T, \sigma, \tau(i, i+1))$. Hence the maximal chains $\bar{\mathrm{c}}(T, \sigma, \tau)$ and $\bar{\mathrm{c}}(T, \sigma, \tau(i, i+1))$ only differ at rank i. So if we denote by c either of these maximal chains with the rank i partition removed we get, using equation (A.3.2), a cohomology relation given by

$$\delta(c) = (-1)^i(\bar{\mathrm{c}}(T, \sigma, \tau) + \bar{\mathrm{c}}(T, \sigma, \tau(i, i+1)))$$

as desired. $\square$

We conclude that in cohomology any maximal chain $c \in \mathcal{M}(\Pi_n^k)$ is cohomology equivalent to a chain of the form $\mathrm{c}(T, \sigma)$. More precisely, in cohomology $\bar{c} = \pm\bar{\mathrm{c}}(T, \sigma)$.

We will make further use of the elementary cohomology relations that are obtained by setting the coboundary (given in (A.3.2)) of a codimension 1 chain in $(\hat{0}, [n]^\mu)$ equal to 0. There are three types of codimension 1 chains, which correspond to the three types of intervals of length 2 (see Figure 2.3). Indeed, if $\bar{c}$ is a codimension 1 chain of $(\hat{0}, [n]^\mu)$ then $c = \bar{c} \cup \{\hat{0}, [n]^\mu\}$ is unrefinable except between one pair of adjacent elements $x < y$, where $[x, y]$ is an interval of length 2. If the open interval $(x, y) = \{z_1, \ldots, z_k\}$ then it follows from (A.3.2) that

$$\delta(\bar{c}) = \pm(\bar{c} \cup \{z_1\} + \cdots + \bar{c} \cup \{z_k\}).$$

By setting $\delta(\bar{c}) = 0$ we obtain the elementary cohomology relation

$$(\bar{c} \cup \{z_1\}) + \cdots + (\bar{c} \cup \{z_k\}) = 0.$$

Type I: Two pairs of distinct blocks of x are merged to get y. The open interval (x, y) equals $\{z_1, z_2\}$, where z_1 is obtained by $\mathbf{e_r}$-merging the first pair of blocks and z_2 is obtained by $\mathbf{e_s}$-merging the second pair of blocks for some $r, s \in [k]$. Hence the Type I elementary cohomology relation is

$$\bar{c} \cup \{z_1\} = -(\bar{c} \cup \{z_2\}).$$

Type II: Three distinct blocks of x are $2\mathbf{e_r}$-merged to get y, where $r \in [k]$. The open interval (x, y) equals $\{z_1, z_2, z_3\}$, where each weighted partition z_i is obtained from x by $\mathbf{e_r}$-merging two of the three blocks. Hence the Type II elementary cohomology relation is

$$(\bar{c} \cup \{z_1\}) + (\bar{c} \cup \{z_2\}) + (\bar{c} \cup \{z_3\}) = 0.$$

Type III: Three distinct blocks of x are $(\mathbf{e_r} + \mathbf{e_s})$-merged to get y, where $r \neq s \in [k]$. The open interval (x, y) equals $\{z_1, z_2, z_3, z_4, z_5, z_6\}$, where each weighted partition z_i is obtained from x by either $\mathbf{e_r}$-merging or $\mathbf{e_s}$-merging two of the three blocks. Hence the Type III elementary cohomology relation is

$$(\bar{c} \cup \{z_1\}) + (\bar{c} \cup \{z_2\}) + (\bar{c} \cup \{z_3\}) + (\bar{c} \cup \{z_4\}) + (\bar{c} \cup \{z_5\}) + (\bar{c} \cup \{z_6\}) = 0.$$

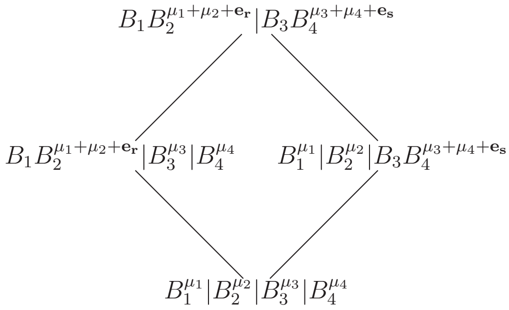
For $\Upsilon \in \mathcal{BT}_\mu$, let $I(\Upsilon)$ denote the set of internal nodes of Υ. Recall that $\Upsilon_1 \overset{j}{\wedge} \Upsilon_2$ denotes the labeled colored binary tree whose left subtree is Υ_1, right subtree is Υ_2 and root color is j, where $j \in [k]$. If Υ is a labeled colored binary tree then $\alpha(\Upsilon)\beta$ denotes a labeled colored binary tree with Υ as a subtree. The following result generalizes [44, Theorem 5.3].

Theorem 2.2.4. *The set* $\{\bar{\mathrm{c}}(T,\sigma) \mid (T,\sigma) \in \mathcal{BT}_\mu\}$ *is a generating set for* $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$*, subject only to the relations for* $i \neq j \in \mathrm{supp}(\mu)$

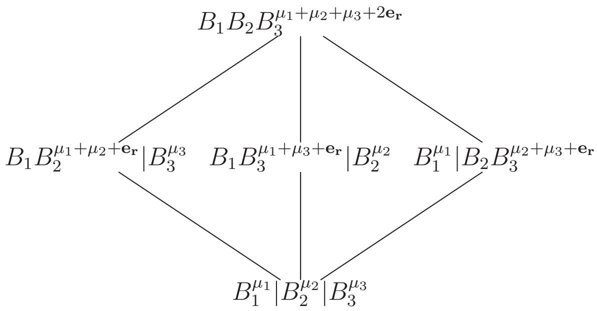
$$\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{j}{\wedge} \Upsilon_2)\beta) - (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{j}{\wedge} \Upsilon_1)\beta) = 0 \tag{2.2.1}$$

$$\begin{aligned}\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{j}{\wedge}(\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta) + (-1)^{|I(T_3)|}\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_2)\overset{j}{\wedge} \Upsilon_3)\beta) \\ + (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{j}{\wedge}(\Upsilon_1 \overset{j}{\wedge} \Upsilon_3))\beta) \\ = 0\end{aligned} \tag{2.2.2}$$

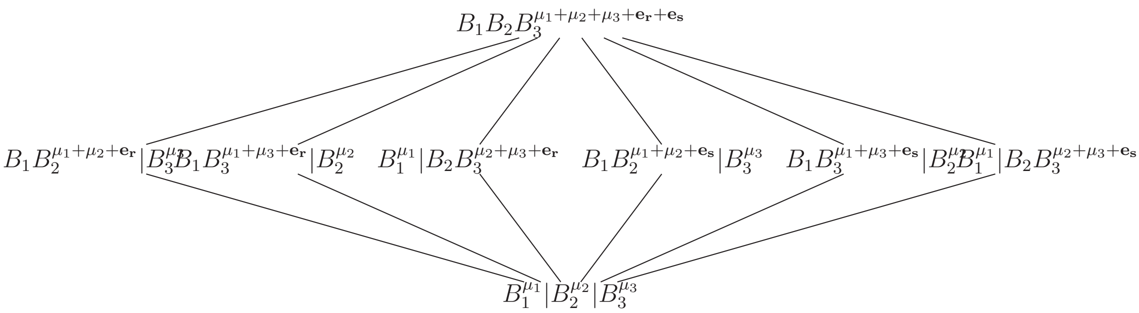
$$\begin{aligned}\bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{j}{\wedge}(\Upsilon_2 \overset{i}{\wedge} \Upsilon_3))\beta) + \bar{\mathrm{c}}(\alpha(\Upsilon_1 \overset{i}{\wedge}(\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta) \\ + \ (-1)^{|I(T_3)|}\Big(\bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_2)\overset{i}{\wedge} \Upsilon_3)\beta) + \bar{\mathrm{c}}(\alpha((\Upsilon_1 \overset{i}{\wedge} \Upsilon_2)\overset{j}{\wedge} \Upsilon_3)\beta)\Big) \\ + \ (-1)^{|I(\Upsilon_1)||I(\Upsilon_2)|}\Big(\bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{j}{\wedge}(\Upsilon_1 \overset{i}{\wedge} \Upsilon_3))\beta) + \bar{\mathrm{c}}(\alpha(\Upsilon_2 \overset{i}{\wedge}(\Upsilon_1 \overset{j}{\wedge} \Upsilon_3))\beta)\Big) \\ = 0.\end{aligned} \tag{2.2.3}$$



(a) Type I



(b) Type II



(c) Type III

Figure 2.3: Intervals of length 2

Proof. It is an immediate consequence of Lemma 2.2.3 that $\{\bar{c}(\Upsilon) | \Upsilon \in \mathcal{BT}_\mu\}$ generates $H^{n-3}((\hat{0}, [n]^\mu))$.

Relation (2.2.1): This is also a consequence of Lemma 2.2.3. Indeed, first note that

$$c(\alpha(\Upsilon_2 \overset{j}{\wedge} \Upsilon_1)\beta) = c(\alpha(\Upsilon_1 \overset{j}{\wedge} \Upsilon_2)\beta, \tau),$$

where τ is the permutation that induces the linear extension that is just like postorder except that the internal nodes of Υ_2 are listed before those of Υ_1. Since $\mathrm{inv}(\tau) = |I(\Upsilon_1)||I(\Upsilon_2)|$, relation (2.2.1) follows from Lemma 2.2.3. (Note that since Lemma 2.2.3 is a consequence only of the Type I cohomology relation, one can view (2.2.1) as a consequence only of the Type I cohomology relation.)

Relation (2.2.2): Note that the following relation is a Type II elementary cohomology relation:

$$\begin{aligned} \bar{c}(\alpha(\Upsilon_1 \overset{j}{\wedge}(\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta) \quad &+ \quad \bar{c}(\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_2) \overset{j}{\wedge} \Upsilon_3)\beta, \tau_1) \\ &+ \quad \bar{c}(\alpha(\Upsilon_2 \overset{j}{\wedge}(\Upsilon_1 \overset{j}{\wedge} \Upsilon_3))\beta, \tau_2) = 0, \end{aligned}$$

where τ_1 is the permutation that induces the linear extension that is like postorder but that lists the internal nodes of Υ_3 before listing the root of $\Upsilon_1 \wedge \Upsilon_2$, and τ_2 is the permutation that induces the linear extension that is like postorder but lists the internal nodes of Υ_1 before listing the internal nodes of Υ_2. So then $\mathrm{inv}(\tau_1) = |I(\Upsilon_3)|$ and $\mathrm{inv}(\tau_2) = |I(\Upsilon_1)||I(\Upsilon_2)|$, and using Lemma 2.2.3 we obtain relation (2.2.2).

Relation (2.2.3): Note that the following relation is a Type III elementary cohomology relation:

$$\begin{aligned} \bar{\mathrm{c}}(\alpha(\Upsilon_1 {}^j_\wedge(\Upsilon_2 {}^i_\wedge \Upsilon_3))\beta) &+ \bar{\mathrm{c}}(\alpha(\Upsilon_1 {}^i_\wedge(\Upsilon_2 {}^j_\wedge \Upsilon_3))\beta) \\ + \ \bar{\mathrm{c}}(\alpha((\Upsilon_1 {}^j_\wedge \Upsilon_2) {}^i_\wedge \Upsilon_3)\beta, \tau_1) &+ \bar{\mathrm{c}}(\alpha((\Upsilon_1 {}^i_\wedge \Upsilon_2) {}^j_\wedge \Upsilon_3)\beta, \tau_1) \\ + \ \bar{\mathrm{c}}(\alpha(\Upsilon_2 {}^j_\wedge(\Upsilon_1 {}^i_\wedge \Upsilon_3))\beta, \tau_2) &+ \bar{\mathrm{c}}(\alpha(\Upsilon_2 {}^i_\wedge(\Upsilon_1 {}^j_\wedge \Upsilon_3))\beta, \tau_2) \\ &= 0, \end{aligned}$$

where as in the previous case, τ_1 is the permutation that induces the linear extension that is like postorder but that lists the internal nodes of Υ_3 before listing the root of $\Upsilon_1 \wedge \Upsilon_2$, and τ_2 is the permutation that induces the linear extension that is like postorder but lists the internal nodes of Υ_1 before listing the internal nodes of Υ_2. So then $\mathrm{inv}(\tau_1) = |I(\Upsilon_3)|$ and $\mathrm{inv}(\tau_2) = |I(\Upsilon_1)||I(\Upsilon_2)|$, and using Lemma 2.2.3 we obtain relation (2.2.3).

To complete the “only” part of the proof, we need to show that these relations generate all the cohomology relations. We prove this in Proposition 6.1.2. $\square$

2.3 The isomorphism

The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n^k. Indeed, let $\sigma \in \mathfrak{S}_n$ act on the weighted blocks of Π_n^k by replacing each element x of each weighted block of π with $\sigma(x)$. Since the maximal elements of Π_n^k are fixed by each $\sigma \in \mathfrak{S}_n$ and the order is preserved, each open interval $(\hat{0}, [n]^\mu)$ is an $\mathfrak{S}_n$-poset. Hence (see Section A.3 of the Appendix) we have that $\tilde{H}^{n-3}((\hat{0}, [n]^\mu))$ is an $\mathfrak{S}_n$-module. The symmetric group $\mathfrak{S}_n$ also acts naturally on $\mathcal{L}ie(\mu)$. Indeed, let $\sigma \in \mathfrak{S}_n$ act by replacing letter x of a bracketed permutation with $\sigma(x)$. Since this action preserves the number of brackets of each type, $\mathcal{L}ie(\mu)$ is an $\mathfrak{S}_n$-module for each $\mu \in \mathrm{wcomp}_{n-1}$. In this section we obtain an explicit sign-twisted isomorphism between the $\mathfrak{S}_n$-modules $\tilde{H}^{n-3}((\hat{0},[n]^\mu))$ and $\mathcal{L}ie(\mu)$.

Define the *sign* of a binary tree T recursively by

$$\operatorname{sgn}(T) = \begin{cases} 1 & \text{if } I(T) = \emptyset \\ (-1)^{|I(T_2)|}\operatorname{sgn}(T_1)\operatorname{sgn}(T_2) & \text{if } T = T_1 \wedge T_2, \end{cases}$$

where $I(T)$ is the set of internal nodes of the binary tree T. The sign of a colored (labeled or unlabeled) binary tree is defined to be the sign of the binary tree obtained by removing the colors and leaf labels.

Theorem 2.3.1. *For each $\mu \in \operatorname{wcomp}_{n-1}$, there is an $\mathfrak{S}_n$-module isomorphism $\varphi : \mathcal{L}ie(\mu) \to \widetilde{H}^{n-3}((\hat{0},[n]^\mu)) \otimes \operatorname{sgn}_n$ determined by*

$$\varphi([T,\sigma]) = \operatorname{sgn}(\sigma)\operatorname{sgn}(T)\bar{c}(T,\sigma),$$

for all $(T,\sigma) \in \mathcal{BT}_\mu$.

Before proving the theorem we make a few preliminary observations. The following lemma, which is implicit in [44, Proof of Theorem 5.4], is easy to prove. For T a binary tree, $a(T)b$ denotes a binary tree with T as a subtree.

Lemma 2.3.2. *For all binary trees T_1, T_2, T_3,*

1. $\operatorname{sgn}(a(T_1 \wedge T_2)b) = (-1)^{|I(T_1)|+|I(T_2)|}\operatorname{sgn}(a(T_2 \wedge T_1)b)$

2. $\operatorname{sgn}(a((T_1 \wedge T_2) \wedge T_3)b) = (-1)^{|I(T_3)|+1}\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)$

3. $\operatorname{sgn}(a(T_2 \wedge (T_1 \wedge T_3))b) = (-1)^{|I(T_1)|+|I(T_2)|}\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)$.

For a word w denote by $l(w)$ the *length* or number of letters in w. We also have the following easy relation, which we state as a lemma.

Lemma 2.3.3. *For $uw_1w_2v \in \mathfrak{S}_n$, where u, w_1, w_2, v are subwords,*

$$\operatorname{sgn}(uw_1w_2v) = (-1)^{l(w_1)l(w_2)} \operatorname{sgn}(uw_2w_1v).$$

Proof of Theorem 2.3.1. The map ϕ maps generators onto generators and clearly respects the $\mathfrak{S}_n$ action. We will prove that the map ϕ extends to a well defined homomorphism by showing that the relations in Proposition 2.1.2 map onto to the relations in Theorem 2.2.4. Since the relations in Theorem 2.2.4 span all the relations in cohomology, this also implies that the map is an isomorphism.

For each Υ_j in the relations of Proposition 2.1.2, let w_j and T_j be such that $\Upsilon_j = (T_j, w_j)$. Let u be the permutation labeling the portion a of the tree corresponding to the preamble α, and let v be the permutation labeling the portion b of the tree corresponding to the tail β. Using Lemmas 2.3.2 and 2.3.3 we have the following.

Relation (2.1.2):

$$\begin{aligned}
\phi([\alpha(\Upsilon_2 \overset{j}{\wedge} \Upsilon_1)\beta]) &= \operatorname{sgn}(uw_2w_1v)\operatorname{sgn}(a(T_2 \wedge T_1)b)\bar{c}(\alpha(\Upsilon_2 \overset{j}{\wedge} \Upsilon_1)\beta) \\
&= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(a(T_1 \wedge T_2)b) \\
&\qquad \cdot (-1)^{l(w_1)l(w_2)+|I(T_1)|+|I(T_2)|}\bar{c}(\alpha(\Upsilon_2 \overset{j}{\wedge} \Upsilon_1)\beta) \\
&= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(a(T_1 \wedge T_2)b) \\
&\qquad \cdot (-1)^{(|I(T_1)|+1)(|I(T_2)|+1)+|I(T_1)|+|I(T_2)|}\bar{c}(\alpha(\Upsilon_2 \overset{j}{\wedge} \Upsilon_1)\beta)
\end{aligned}$$

$$= \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(a(T_1 \wedge T_2)b)$$
$$\cdot (-1)^{|I(T_1)||I(T_2)|+1}\bar{c}(\alpha(\Upsilon_2{}^j_\wedge\Upsilon_1)\beta).$$

Hence,

$$\phi([\alpha(\Upsilon_1{}^j_\wedge\Upsilon_2)\beta]) + \phi([\alpha(\Upsilon_2{}^j_\wedge\Upsilon_1)\beta]) = \operatorname{sgn}(uw_1w_2v)\operatorname{sgn}(a(T_1 \wedge T_2)b)$$
$$\cdot \big(\bar{c}(\alpha(\Upsilon_1{}^j_\wedge\Upsilon_2)\beta) - (-1)^{|I(T_1)||I(T_2)|}\bar{c}(\alpha(\Upsilon_2{}^j_\wedge\Upsilon_1)\beta)\big).$$

Relations (2.1.3) and (2.1.4):

$$\phi([\alpha((\Upsilon_1{}^j_\wedge\Upsilon_2){}^i_\wedge\Upsilon_3)\beta]) = \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a((T_1 \wedge T_2) \wedge T_3)b)$$
$$\cdot\, \bar{c}(\alpha((\Upsilon_1{}^j_\wedge\Upsilon_2){}^i_\wedge\Upsilon_3)\beta)$$

$$= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)$$
$$\cdot (-1)^{|I(T_3)|+1}\bar{c}(\alpha((\Upsilon_1{}^j_\wedge\Upsilon_2){}^i_\wedge\Upsilon_3)\beta).$$

$$\phi([\alpha(\Upsilon_2{}^j_\wedge(\Upsilon_1{}^i_\wedge\Upsilon_3))\beta]) = \operatorname{sgn}(uw_2w_1w_3v)\operatorname{sgn}(a(T_2 \wedge (T_1 \wedge T_3))b)$$
$$\cdot\ \bar{c}(\alpha(\Upsilon_2{}^j_\wedge(\Upsilon_1{}^i_\wedge\Upsilon_3))\beta)$$

$$= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)$$
$$\cdot (-1)^{l(w_1)l(w_2)+|I(T_1)|+|I(T_2)|}\bar{c}(\alpha(\Upsilon_2{}^j_\wedge(\Upsilon_1{}^i_\wedge\Upsilon_3))\beta)$$

$$= \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)$$
$$\cdot (-1)^{|I(T_1)||I(T_2)|+1}\bar{c}(\alpha(\Upsilon_2{}^j_\wedge(\Upsilon_1{}^i_\wedge\Upsilon_3))\beta).$$

Hence,

$$\begin{aligned}
&\phi([\alpha(\Upsilon_1 {}^{j}_{\wedge}(\Upsilon_2 {}^{i}_{\wedge}\Upsilon_3))\beta]) - \phi([\alpha((\Upsilon_1 {}^{j}_{\wedge}\Upsilon_2) {}^{i}_{\wedge}\Upsilon_3)\beta]) - \phi([\alpha(\Upsilon_2 {}^{j}_{\wedge}(\Upsilon_1 {}^{i}_{\wedge}\Upsilon_3))\beta]) \qquad (2.3.1)\\
&\quad = \operatorname{sgn}(uw_1w_2w_3v)\operatorname{sgn}(a(T_1 \wedge (T_2 \wedge T_3))b)\\
&\qquad \cdot \Big(\bar{\mathrm{c}}(\alpha(\Upsilon_1 {}^{j}_{\wedge}(\Upsilon_2 {}^{i}_{\wedge}\Upsilon_3))\beta) + (-1)^{|I(T_3)|}\bar{\mathrm{c}}(\alpha((\Upsilon_1 {}^{j}_{\wedge}\Upsilon_2) {}^{i}_{\wedge}\Upsilon_3)\beta)\\
&\qquad\quad + (-1)^{|I(T_1)||I(T_2)|}\bar{\mathrm{c}}(\alpha(\Upsilon_2 {}^{j}_{\wedge}(\Upsilon_1 {}^{i}_{\wedge}\Upsilon_3))\beta)\Big).
\end{aligned}$$

By setting $i = j$ in (2.3.1) we conclude that relation (2.1.3) maps to relation (2.2.2). By adding (2.3.1) with another copy of (2.3.1) after the roles of i and j have been switched, we are also able to conclude that relation (2.1.4) maps to relation (2.2.3). $\square$

Chapter 3

Möbius invariant of Π_n^w and Π_n^k

The reader is referred to the Appendix A for a review of poset topology and poset (co)homology. For further poset topology terminology not defined here the reader could also visit [40] and [46].

For $u \leq v$ in a poset P, the open interval $\{w \in P \mid u < w < v\}$ is denoted by (u, v) and the closed interval $\{w \in P \mid u \leq w \leq v\}$ by $[u, v]$. A poset is said to be *bounded* if it has a minimum element $\hat{0}$ and a maximum element $\hat{1}$. For a bounded poset P, we define the *proper part* of P as $\overline{P} := P \setminus \{\hat{0}, \hat{1}\}$. A poset is said to be *pure* (or ranked) if all its maximal chains have the same length, where the length of a chain $s_0 < s_1 < \cdots < s_\ell$ is ℓ. If P is pure and has a minimal element $\hat{0}$, we can define a rank function ρ by requiring that $\rho(\hat{0}) = 0$ and $\rho(\beta) = \rho(\alpha) + 1$ whenever $\alpha \lessdot \beta$ in P. For example Π_n^k is a pure poset with rank function given by $\rho(\alpha) = n - |\alpha|$ for every $\alpha \in \Pi_n^k$. The *length* $\ell(P)$ of a poset P is the length of its longest chain. For a bounded poset P, let $\overline{\mu}_P$ denote its Möbius function (see Appendix A for the definition). The reason for the nonstardard notation $\overline{\mu}_P$ is that we have been using the symbol μ to denote a weak composition. The rank generating function $\mathcal{F}_P(x)$ is defined by $\mathcal{F}_P(x) = \sum_{u \in P} x^{\rho(u)}$.

For $\alpha = \{A_1^{\mu_1}, \dots, A_s^{\mu_s}\} \in \Pi_n^k$, let $\mu(\alpha) = \sum_{i=1}^{s} \mu_i$. The following proposition about the structure of Π_n^k will be used in the computations below.

Proposition 3.0.4. *For all $\alpha = \{A_1^{\mu_1}, \dots, A_s^{\mu_s}\} \in \Pi_n^k$ and $\nu \in \mathrm{wcomp}_{n-1}$ such that $\nu - \mu(\alpha) \in \mathrm{wcomp}_{|\alpha|-1}$,*

1. *$[\alpha, \hat{1}]$ and $\widehat{\Pi_s^k}$ are isomorphic posets,*
2. *$[\alpha, [n]^\nu]$ and $[\hat{0}, [|\alpha|]^{\nu-\mu(\alpha)}]$ are isomorphic posets,*
3. *$[\hat{0}, \alpha]$ and $[\hat{0}, [|A_1|]^{\mu_1}] \times \cdots \times [\hat{0}, [|A_s|]^{\mu_s}]$ are isomorphic posets.*

Proposition 3.0.4 is a general statement that is satisfied by any partition poset associated to a set operad (see [43]) replacing the composition μ by an element of the given operad (see also [32]).

3.1 A formula for the Möbius invariant of the maximal intervals of Π_n^k

Recall that $\mathbf{x}^\mu = x_1^{\mu(1)} \cdots x_k^{\mu(k)}$ and $(\cdot)^{<-1>}$ denotes compositional inverse. We use the recursive definition of the Möbius function $\bar{\mu}_P$ and the compositional formula to derive the following theorem.

Theorem 3.1.1. *We have that*

$$\sum_{n\geq 1} \sum_{\substack{\mu \in \mathrm{wcomp}_{n-1} \\ \mathrm{supp}(\mu)\subseteq[k]}} \bar{\mu}_{\Pi_n^k}(\hat{0}, [n]^\mu)\mathbf{x}^\mu \frac{y^n}{n!} = \left[\sum_{n\geq 1} h_{n-1}(x_1, \dots, x_k)\frac{y^n}{n!}\right]^{<-1>},$$

where h_n is the complete homogeneous symmetric polynomial.

Proof. By the recursive definition of the Möbius function we have that

$$\begin{aligned}\delta_{n,1} &= \sum_{\substack{\mu\in\mathrm{wcomp}_{n-1}\\ \mathrm{supp}(\mu)\subseteq[k]}} \mathbf{x}^{\mu} \sum_{\hat{0}\leq\alpha\leq[n]^{\mu}} \bar{\mu}_{\Pi_n^k}(\alpha,[n]^{\mu})\\ &= \sum_{\alpha\in\Pi_n^k} \mathbf{x}^{\mu(\alpha)} \sum_{\substack{\mu\in\mathrm{wcomp}_{n-1}\\ \mu\geq\mu(\alpha)\\ \mathrm{supp}(\mu)\subseteq[k]}} \bar{\mu}_{\Pi_n^k}(\alpha,[n]^{\mu})\mathbf{x}^{\mu-\mu(\alpha)}.\end{aligned}$$

Now using Proposition 3.0.4

$$\begin{aligned}\delta_{n,1} &= \sum_{\alpha\in\Pi_n^k} \mathbf{x}^{\mu(\alpha)} \sum_{\substack{\nu\in\mathrm{wcomp}_{|\alpha|-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \bar{\mu}_{\Pi_{|\alpha|}^k}(\hat{0},[|\alpha|]^{\nu})\mathbf{x}^{\nu}\\ &= \sum_{\alpha\in\Pi_n} \prod_{i=1}^{|\alpha|} h_{|\alpha_i|-1}(x_1,...,x_k) \sum_{\substack{\nu\in\mathrm{wcomp}_{|\alpha|-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \bar{\mu}_{\Pi_{|\alpha|}^k}(\hat{0},[|\alpha|]^{\nu})\mathbf{x}^{\nu}.\end{aligned}$$

The last statement implies using the compositional formula see ([39, Theorem 5.1.4]) that the two power series are compositional inverses. □

3.2 The case $k = 2$: Π_n^w

When we let $k = 2$, $x_1 = t$ and $x_2 = 1$ in Theorem 3.1.1, we obtain an interesting product formula for the generating polynomial $\sum_{i=0}^{n-1} \bar{\mu}_{\Pi_n^w}(\hat{0},[n]^i)t^i$.

Proposition 3.2.1. *For all $n \geq 1$,*

$$\sum_{i=0}^{n-1} \bar{\mu}_{\Pi_n^w}(\hat{0},[n]^i)t^i = (-1)^{n-1}\prod_{i=1}^{n-1}((n-i)+it). \tag{3.2.1}$$

Consequently,

$$\sum_{i=0}^{n-1} \bar{\mu}_{\Pi_n^w}(\hat{0},[n]^i) = (-1)^{n-1}n^{n-1}.$$

Proof. Setting $k = 2$, $x_1 = t$ and $x_2 = 1$ in Theorem 3.1.1, we obtain that

$$U(x) = \sum_{n \geq 1} h_{n-1}(t, 1) \frac{x^n}{n!} = \sum_{n \geq 1} \frac{t^n - 1}{t - 1} \frac{x^n}{n!} = \frac{e^{tx} - e^x}{t - 1}$$

and

$$W(x) = \sum_{n \geq 1} \sum_{j=0}^{n-1} \bar{\mu}_{\Pi_n^w}(\hat{0}, [n]^j) t^j \frac{x^n}{n!}$$

are compositional inverses. It follows from [19, Theorem 5.1] that the compositional inverse of $U(x)$ is given by

$$\sum_{n \geq 1} (-1)^{n-1} \prod_{i=1}^{n-1} ((n-i) + it) \frac{x^n}{n!}.$$

(See [16, Eq. (10)].) This yields (3.2.1). □

Let T be a rooted tree on node set $[n]$. A *descent* of T is a node x that has a smaller label than its parent $p_T(x)$. We call the edge $\{x, p_T(x)\}$ a *descent edge*. We denote by $\mathcal{T}_{n,i}$ the set of rooted trees on node set $[n]$ with exactly i descents. In [16] Drake proves that

$$\sum_{i=0}^{n-1} |\mathcal{T}_{n,i}| t^i = \prod_{i=1}^{n-1} ((n-i) + it). \tag{3.2.2}$$

The following result is a consequence of this and Proposition 3.2.1.

Corollary 3.2.2. *For all* $n \geq 1$ *and* $i \in \{0, 1, \ldots, n-1\}$,

$$\bar{\mu}_{\Pi_n^w}(\hat{0}, [n]^i) = (-1)^{n-1} |\mathcal{T}_{n,i}|.$$

We can use Proposition 3.0.4 and Corollary 3.2.2 to compute the Möbius function on other intervals. A rooted forest on node set $[n]$ is a set of rooted trees whose node sets form a partition of $[n]$. We associate a weighted partition

$\alpha(F)$ with each rooted forest $F = \{T_1, \ldots, T_k\}$ on node set $[n]$, by letting $\alpha(F) = \{A_1^{w_1}, \ldots, A_k^{w_k}\}$ where A_i is the node set of T_i and w_i is the number of descents of T_i. For lower intervals we obtain the following generalization of Corollary 3.2.2.

Corollary 3.2.3. *For all* $\alpha \in \Pi_n^w$*,*

$$\bar{\mu}_{\Pi_n^w}(\hat{0}, \alpha) = (-1)^{n-|\alpha|} |\{F \in \mathcal{F}_n : \alpha(F) = \alpha\}|,$$

where $\mathcal{F}_n$ *is the set of rooted forests on node set* $[n]$*.*

Next we consider the full poset $\widehat{\Pi_n^w}$. To compute its Möbius invariant we will make use of Abel's identity (see [39, Ex. 5.31 c]),

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x(x-kz)^{k-1}(y+kz)^{n-k}. \tag{3.2.3}$$

Proposition 3.2.4.

$$\bar{\mu}_{\widehat{\Pi_n^w}}(\hat{0}, \hat{1}) = (-1)^n (n-1)^{n-1}.$$

Proof. We proceed by induction on n. If $n = 1$ then

$$\bar{\mu}_{\widehat{\Pi_1^w}}(\hat{0}, \hat{1}) = -1 = (-1)^1 (1-1)^{1-1}$$

since $\widehat{\Pi_1^w}$ is the chain of length 1.

Let $n \geq 1$ and let $\alpha \in \Pi_n^w \setminus \{\hat{0}\}$. Since the interval $[\alpha, \hat{1}]$ in $\widehat{\Pi_n^w}$ is isomorphic to $\widehat{\Pi_{|\alpha|}^w}$ (cf. Proposition 3.0.4), we can assume by induction that

$$\bar{\mu}_{\widehat{\Pi_n^w}}(\alpha, \hat{1}) = (-1)^{|\alpha|} (|\alpha| - 1)^{|\alpha|-1}.$$

Hence by the recursive definition of the Möbius function we have,

$$\begin{aligned}
\bar{\mu}_{\widehat{\Pi_n^w}}(\hat{0},\hat{1}) &= -\sum_{\alpha\in\widehat{\Pi_n^w}\setminus\hat{0}} \bar{\mu}_{\widehat{\Pi_n^w}}(\alpha,\hat{1}) \\
&= -1-\sum_{k=1}^{n-1}\sum_{\substack{\alpha\in\Pi_n^w\\|\alpha|=k}} \bar{\mu}_{\widehat{\Pi_n^w}}(\alpha,\hat{1}) \\
&= -1-\sum_{k=1}^{n-1}\sum_{\substack{\alpha\in\Pi_n^w\\|\alpha|=k}} (-1)^k(k-1)^{k-1} \\
&= -1-\sum_{k=1}^{n-1}\binom{n}{k}k^{n-k}(-1)^k(k-1)^{k-1} \quad \text{(by (7.1.6))} \\
&= -1+\sum_{k=0}^{n}\binom{n}{k}k^{n-k}(1-k)^{k-1}-(1-n)^{n-1}. \qquad (3.2.4)
\end{aligned}$$

By setting $x=1, y=0, z=1$ in Abel's identity (3.2.3), we get

$$1=\sum_{k=0}^{n}\binom{n}{k}(1-k)^{k-1}k^{n-k}.$$

Substituting this into (3.2.4) yields the result. □

Remark 3.2.5. In Section 3.3 we compute the characteristic polynomial of Π_n^w and use it to give a second proof of Proposition 3.2.4.

3.3 The characteristic polynomial of Π_n^w

For a pure poset with a $\hat{0}$ the *characteristic polynomial* is defined as

$$\chi_P(x)=\sum_{\alpha\in P}\bar{\mu}(\hat{0},\alpha)x^{\rho(P)-\rho(\alpha)}.$$

Recall that the characteristic polynomial of Π_n factors nicely (see equation (1.2.1)). We prove that the same is true for Π_n^w.

Theorem 3.3.1. *For all $n \geq 1$, the characteristic polynomial of Π_n^w is given by*

$$\chi_{\Pi_n^w}(x) := \sum_{\alpha \in \Pi_n^w} \bar{\mu}_{\Pi_n^w}(\hat{0}, \alpha) x^{n-1-\rho(\alpha)} = (x-n)^{n-1}.$$

We will need the following result.

Proposition 3.3.2 (see [39, Proposition 5.3.2])**.** *Let $\mathcal{F}_n^k$ be the number of rooted forests on node set $[n]$ with k rooted trees. Then*

$$|\mathcal{F}_n^k| = \binom{n-1}{k-1} n^{n-k}.$$

Proof of Theorem 3.3.1. We have

$$\begin{aligned}
\chi_{\Pi_n^w}(x) &= \sum_{\alpha \in \Pi_n^w} \bar{\mu}(\hat{0}, \alpha) x^{|\alpha|-1} \\
&= \sum_{k=1}^{n} \sum_{\substack{\alpha \in \Pi_n^w \\ |\alpha| = k}} \bar{\mu}(\hat{0}, \alpha) x^{k-1} \\
&= \sum_{k=1}^{n} (-1)^{n-k} |\mathcal{F}_n^k| x^{k-1} \quad \text{(by Corollary 3.2.3)} \\
&= \sum_{k=1}^{n} (-1)^{n-k} \binom{n-1}{k-1} n^{n-k} x^{k-1} \quad \text{(by Proposition 3.3.2)} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} (-n)^{n-1-k} x^k \\
&= (x-n)^{n-1}.
\end{aligned}$$

$\square$

Theorem 3.3.1 yields an easier way to calculate $\bar{\mu}_{\widehat{\Pi_n^w}}(\hat{0}, \hat{1})$.

Second proof of Proposition 3.2.4 . By the recursive definition of Möbius function,

$$\begin{aligned}
\bar{\mu}_{\widehat{\Pi_n^w}}(\hat{0},\hat{1}) &= -\sum_{\alpha\in\Pi_n^w}\bar{\mu}(\hat{0},\alpha) \\
&= -\chi_{\Pi_n^w}(1) \\
&= -(1-n)^{n-1} \\
&= (-1)^n(n-1)^{n-1}.\square
\end{aligned}$$

Chapter 4

Homotopy type of the intervals of Π_n^k

In this chapter we use the theory of EL-shellability introduced by Björner [6] and further developed by Björner and Wachs [8], to determine the homotopy type of the maximal intervals of Π_n^k.

4.1 EL-labeling

Let P be a bounded poset. An *edge labeling* is a map $\bar{\lambda} : \mathcal{E}(P) \to \Lambda$, where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of a poset P and Λ is a fixed poset. We denote by

$$\bar{\lambda}(c) = \bar{\lambda}(x_0, x_1)\bar{\lambda}(x_1, x_2) \cdots \bar{\lambda}(x_{t-1}, x_t),$$

the word of labels corresponding to a maximal chain $c = (\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{t-1} \lessdot x_t = \hat{1})$. We say that c is *increasing* if its word of labels $\bar{\lambda}(c)$ is *strictly* increasing, that is, c is increasing if

$$\bar{\lambda}(x_0, x_1) < \bar{\lambda}(x_1, x_2) < \cdots < \bar{\lambda}(x_{t-1}, x_t).$$

We say that c is *ascent-free* (or decreasing, or falling) if its word of labels $\bar{\lambda}(c)$ has no ascents, i.e. $\bar{\lambda}(x_i, x_{i+1}) \not< \bar{\lambda}(x_{i+1}, x_{i+2})$, for all $i = 0, \ldots, t-2$. *An edge-lexicographical labeling* (EL-labeling, for short) of P is an edge labeling such that in each closed interval $[x, y]$ of P, there is a unique increasing maximal chain, and this chain lexicographically precedes all other maximal chains of $[x, y]$.

A classical EL-labeling for the partition lattice Π_n is obtained as follows. Let $\Lambda = \{(i, j) \in [n-1] \times [n] \mid i < j\}$ with lexicographic order as the order relation on Λ. If $x \lessdot y$ in Π_n then y is obtained from x by merging two blocks A and B, where $\min A < \min B$. Let $\bar{\lambda}(x, y) = (\min A, \min B)$. This defines a map $\bar{\lambda} : \mathcal{E}(\Pi_n) \to \Lambda$ (Note that $\bar{\lambda}$ in this section is an edge labeling and not an integer partition). By viewing Λ as the set of atoms of Π_n, one sees that this labeling is a special case of an edge labeling for geometric lattices, which first appeared in Stanley [37] and was one of Björner's [6] initial examples of an EL-labeling. We generalize this labeling by providing one for Π_n^k that reduces to the Björner-Stanley EL-labeling when $k = 1$.

Definition 4.1.1 (Poset of labels)**.** For each $a \in [n]$, let $\Gamma_a := \{(a, b)^u \mid a < b \leq n+1,\ u \in [k]\}$. We partially order Γ_a by letting $(a, b)^u \leq (a, c)^v$ if $b \leq c$ and $u \leq v$. Note that Γ_a is isomorphic to the direct product of the chain $a + 1 < a + 2 < \cdots < n + 1$ and the chain $1 < 2 < \cdots < k$. Now define Λ_n^k to be the ordinal sum $\Lambda_n^k := \Gamma_1 \oplus \Gamma_2 \oplus \cdots \oplus \Gamma_n$ (see Figure 4.1).

Definition 4.1.2 (EL-labeling)**.** If $x \lessdot y$ in Π_n^k then y is obtained from x by $\mathbf{e_r}$-merging two blocks A and B for some $r \in [k]$, where $\min A < \min B$.

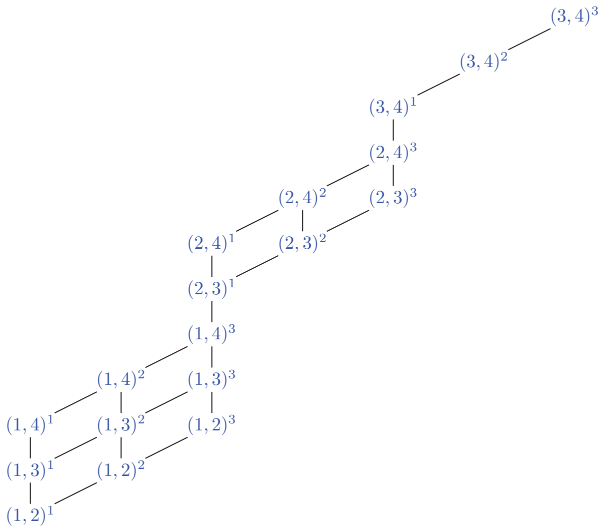


Figure 4.1: Λ_3^3

Let

$$\bar{\lambda}(x \lessdot y) = (\min A, \min B)^r.$$

This defines a map $\bar{\lambda} : \mathcal{E}(\Pi_n^k) \to \Lambda_n^k$. We extend this map to $\bar{\lambda} : \mathcal{E}(\widehat{\Pi_n^k}) \to \Lambda_n$ by letting $\bar{\lambda}([n]^\mu \lessdot \hat{1}) = (1, n+1)^1$, for all $\mu \in \mathrm{wcomp}_{n-1}$ with $\mathrm{supp}(\mu) \subseteq [k]$ (See Figure 4.2).

Remark 4.1.3. Recall that when μ has a single nonzero entry (equal to $n-1$), the interval $[\hat{0}, [n]^\mu]$ is isomorphic to Π_n. Note that $\bar{\lambda}$ reduces to the Björner-Stanley EL-labeling on those intervals.

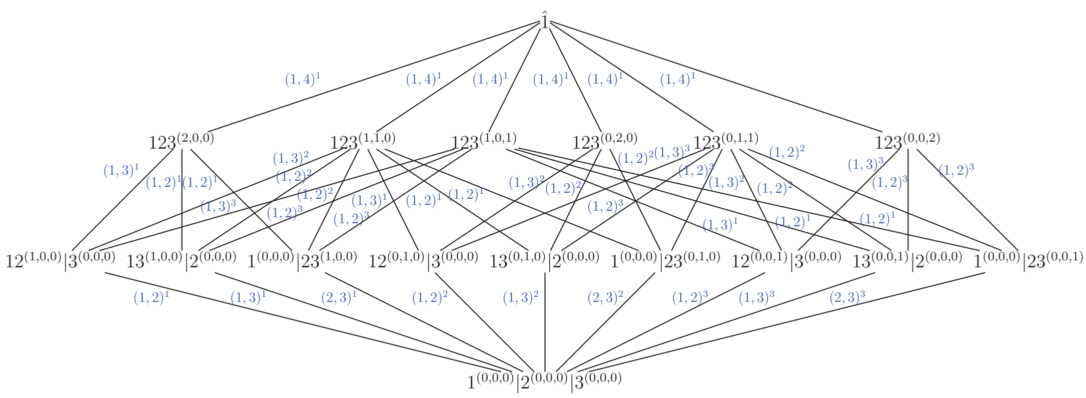


Figure 4.2: Labeling $\bar{\lambda}$ on $\widehat{\Pi_3^3}$

Theorem 4.1.4. *The labeling* $\bar{\lambda} : \mathcal{E}(\widehat{\Pi_n^k}) \to \Lambda_n$ *defined above is an EL-labeling of* $\widehat{\Pi_n^k}$.

Proof. We need to show that in every closed interval of $\widehat{\Pi_n^k}$ there is a unique increasing chain (from bottom to top), which is also lexicographically first. Let ρ denote the rank function of $\widehat{\Pi_n^k}$. We divide the proof into 4 cases:

1. *Intervals of the form* $[\hat{0}, [n]^\mu]$. Since, from bottom to top, the last step of merging two blocks includes a block that contains 1, all of the maximal chains have a final label of the form $(1, m)^u$, and so any increasing maximal chain has to have label word $(1, 2)^{u_1}(1, 3)^{u_2} \cdots (1, n)^{u_{n-1}}$ with $u_1 \leq u_2 \leq \cdots \leq u_{n-1}$ and $u_i \in [k]$ for all i. This label word is lexicographically first and the only chain with this label word is (listing only the nonsingleton blocks)

$$\hat{0} \lessdot 12^{\mathbf{e_{u_1}}} \lessdot 123^{\mathbf{e_{u_1}+e_{u_2}}} \lessdot \cdots \lessdot 123\cdots n^\mu.$$

2. *Intervals of the form* $[\hat{0}, \alpha]$ *for* $\rho(\alpha) < n-1$. Let $A_1^{\mu_1}, \ldots, A_k^{\mu_k}$ be the weighted blocks of α, where $\min A_i < \min A_j$ if $i < j$. For each i, let $m_i = \min A_i$. By the previous case, in each of the posets $[\hat{0}, A_i^{\mu_i}]$ there is only one increasing manner of merging the blocks, and the labels of the increasing chain belong to the label set Γ_{m_i}. The increasing chain is also lexicographically first. Consider the maximal chain of $[\hat{0}, \alpha]$ obtained by first merging the blocks of the increasing chain in $[\hat{0}, A_1^{\mu_1}]$, then the ones in the increasing chain in $[\hat{0}, A_2^{\mu_2}]$, and so on. The constructed chain is still increasing since the labels in Γ_{m_i} are less than the labels in $\Gamma_{m_{i+1}}$ for each $i = 1, \ldots, k-1$. It is not difficult to see that this is the only increasing chain of $[\hat{0}, \alpha]$ and that it is lexicographically first.

3. *The interval* $[\hat{0}, \hat{1}]$. An increasing chain c of this interval must be of the form $c' \cup \{\hat{1}\}$, where c' is the unique increasing chain of some interval $[\hat{0}, [n]^{\mu}]$. By Case 1, the label word of c' ends in $(1, n)^u$ for some u. For c to be increasing, u must be 1. But $u = 1$ only in the interval $[\hat{0}, [n]^{(n-1)\mathbf{e_{u_1}}}]$. Hence the unique increasing chain of $[\hat{0}, [n]^{(n-1)\mathbf{e_{u_1}}}]$ concatenated with $\hat{1}$ is the only increasing chain of $[\hat{0}, \hat{1}]$. It is clearly lexicographically first.

4. *Intervals of the form* $[\alpha, \beta]$ *for* $\alpha \neq \hat{0}$. We extend the definition of Π_n^k to Π_S^k, where S is an arbitrary finite set of positive integers, by considering partitions of S rather than $[n]$. We also extend the definition of the labeling $\bar{\lambda}$ to $\widehat{\Pi_S^k}$. Now we can identify the interval $[\alpha, \hat{1}]$ with $\widehat{\Pi_S^k}$, where S is the set of minimum elements of the blocks of α, by replacing each block A of α by its minimum element and subtracting the weight of A from the weight of the block containing A in each weighted partition of $[\alpha, \hat{1}]$. This isomorphism preserves the labeling and so the three previous cases show that there is a unique increasing chain in $[\alpha, \beta]$ that is also lexicographically first.

□

Theorem 1.6.2 is then a corollary of Theorem 4.1.4 and the following theorem linking lexicographic shellability and topology.

Theorem 4.1.5 (Björner and Wachs [9]). *Let* $\bar{\lambda}$ *be an EL-labeling of a bounded poset* P. *Then for all* $x < y$ *in* P,

1. *the open interval* (x, y) *has the homotopy type of a wedge of spheres, where for each* $r \in \mathbb{N}$ *the number of spheres of dimension* r *is the number of ascent-free maximal chains of the closed interval* $[x, y]$ *of length* $r + 2$.

2. the set

$$\{\bar{c} \mid c \text{ is an ascent-free maximal chain of } [x,y] \text{ of length } r+2\}$$

forms a basis for cohomology $\widetilde{H}^r((x,y))$*, for all* r*.*

Since the Möbius invariant of a bounded poset P equals the reduced Euler characteristic of the order complex of $\overline{P}$ (Corollary A.2.2), Proposition A.2.3 and Theorem 4.1.5 imply the following corollary.

Corollary 4.1.6. *Let* P *be a pure EL-shellable poset of length* n*. Then*

1. $\overline{P}$ *has the homotopy type of a wedge of spheres all of dimension* $n-2$*, where the number of spheres is* $|\mu_P(\hat{0},\hat{1})|$*.*

2. P *is Cohen-Macaulay, which means that* $\widetilde{H}_i((x,y)) = 0$ *for all* $x < y$ *in* P *and* $i < l([x,y]) - 2$*.*

In [15] Dotsenko and Khoroshkin use operad theory to prove that all intervals of Π_n^w are Cohen-Macaulay. The following extension of their result is a consequence of Theorem 4.1.4.

Corollary 4.1.7. *For every* $k \geq 1$*, the poset* $\widehat{\Pi_n^k}$ *is Cohen-Macaulay. In particular, the poset* $\widehat{\Pi_n^w}$ *is Cohen-Macaulay.*

Remark 4.1.8. In a prior attempt to establish Cohen-Macaulayness of each maximal interval $[\hat{0},[n]^i]$ of Π_n^w, it is argued in [14] that the intervals are totally semimodular and hence CL-shellable[1]. In [41] it is noted that this is not the case and a proposed recursive atom ordering[2] of each maximal interval $[\hat{0},[n]^i]$ is given

[1] CL-shellability is a property more general the EL-shellability, which also implies Cohen-Macaulaynes; see [8], [9] or [46]

[2] See [8], [9] or [46] for the definition of recursive atom ordering. The property of admitting a recursive atom ordering is equivalent to that of being CL-shellable.

in order to establish CL-shellability. In [41, Proof of Proposition 3.9] it is claimed that given any linear ordering $\{i_1, j_1\}, \{i_2, j_2\}, \cdots, \{i_m, j_m\}$ of the atoms of Π_n (the singleton blocks have been omitted), the linear ordering

$$\{i_1, j_1\}^0, \{i_1, j_1\}^1, \{i_2, j_2\}^0, \{i_2, j_2\}^1 \cdots \{i_m, j_m\}^0, \{i_m, j_m\}^1 \tag{4.1.1}$$

satisfies the criteria for being a recursive atom ordering of $[\hat{0}, [n]^i]$, where $1 \leq i \leq n-2$. We note here that one of the requisite conditions in the definition of recursive atom ordering fails to hold when $n = 4$ and $i = 2$. Indeed, assume (without loss of generality) that the first two atoms in the atom ordering of $[\hat{0}, [4]^2]$ given in (4.1.1) are $\{1, 2\}^0$ and $\{1, 2\}^1$. Then the atoms of the interval $[\{1, 2\}^1, [4]^2]$ that cover $\{1, 2\}^0$ are $\{1, 2, 3\}^1$ and $\{1, 2, 4\}^1$. So by the definition of recursive atom ordering one of these covers must come first in any recursive atom ordering of $[\{1, 2\}^1, [4]^2]$ and the other must come second. But this contradicts the form of (4.1.1) applied to the interval $[\{1, 2\}^1, [4]^2]$ which requires the atom $\{1, 2, 3\}^2$ to immediately follow the atom $\{1, 2, 3\}^1$ and the atom $\{1, 2, 4\}^2$ to immediately follow the atom $\{1, 2, 4\}^1$. The proof of Proposition 3.9 of [41] breaks down in the second from last paragraph.

By Theorem 4.1.4, Proposition 3.2.4, Corollary 3.2.2 and Corollary 4.1.6 we have,

Theorem 4.1.9. *For all $n \geq 1$,*

1. $\Pi_n^w \setminus \{\hat{0}\}$ *has the homotopy type of a wedge of $(n-1)^{n-1}$ spheres of dimension $n-2$,*

2. $(\hat{0}, [n]^i)$ *has the homotopy type of a wedge of $|\mathcal{T}_{n,i}|$ spheres of dimension $n-3$ for all $i \in \{0, 1, \ldots, n-1\}$.*

Moreover, we have the following result.

Corollary 4.1.10. *For $0 \le i \le n-1$,*

$$\begin{aligned} \dim \tilde{H}^{n-2}(\Pi_n^w \setminus \hat{0}) &= (n-1)^{n-1} \\ \dim \tilde{H}^{n-3}((\hat{0},[n]^i)) &= |\mathcal{T}_{n,i}| \\ \dim \bigoplus_{i=0}^{n-1} \tilde{H}^{n-3}((\hat{0},[n]^i)) &= n^{n-1}. \end{aligned}$$

Theorem 2.3.1 and Corollary 4.1.10 yield the following result.

Corollary 4.1.11 (Liu [29], Dotsenko and Khoroshkin [15])**.** *For $0 \le i \le n-1$, $\dim \mathcal{L}ie_2(n,i) = |\mathcal{T}_{n,i}|$.*

4.2 The ascent-free maximal chains from the EL-labeling

Although we do not have a simple general formula for the homotopy type of the maximal intervals $(\hat{0},[n]^{\mu})$ of Π_n^k like those appearing in Theorem 4.1.9 and Corollary 4.1.10 for the case $k=2$, we will use the ascent-free maximal chains of the EL-labeling of Theorem 1.6.2 to give a plaesant combinatorial formula.

We will describe the ascent-free maximal chains of the maximal intervals $[\hat{0},[n]^{\mu}]$ given by the EL-labeling of Theorem 4.1.4. A *Lyndon tree* is a labeled binary tree (T,σ) such that for each internal node x of T, the smallest leaf label of the subtree T_x rooted at x is in the left subtree of T_x and the second smallest label is in the right subtree of T_x. An alternative characterization of a Lyndon tree is given in Proposition 4.2.1 below.

For each internal node x of a labeled binary tree, let $L(x)$ denote the left child of x and $R(x)$ denote its right child. For each node x of a labeled binary tree

(T, σ) define its *valency* $v(x)$ to be the smallest leaf label of the subtree rooted at x. A Lyndon tree is depicted in Figure 4.3 illustrating the valencies of the internal nodes.

We say that a labeled binary tree is *normalized* if the leftmost leaf of each subtree has the smallest label in the subtree. This is equivalent to requiring that for every internal node x,

$$v(x) = v(L(x)).$$

Note that a normalized tree can be thought of simply as a labeled nonplanar binary tree (or a phylogenetic tree) that has been drawn in the plane following the convention above. We denote the set of normalized labeled binary trees on label set $[n]$ by Nor_n and the set of normalized binary trees on some arbitrary finite subset A of $\mathbb{P}$ by Nor_A. It is well-known that there are $(2n-3)!! := 1 \cdot 3 \cdots (2n-3)$ phylogenetic trees on $[n]$ and so $|\mathsf{Nor}_n| = (2n-3)!!$.

The following alternative characterization of a Lyndon tree is easy to verify.

Proposition 4.2.1. *Let (T, σ) be a labeled binary tree. Then (T, σ) is a Lyndon tree if and only if it is normalized and for every internal node x of T we have*

$$v(R(L(x)) > v(R(x)). \tag{4.2.1}$$

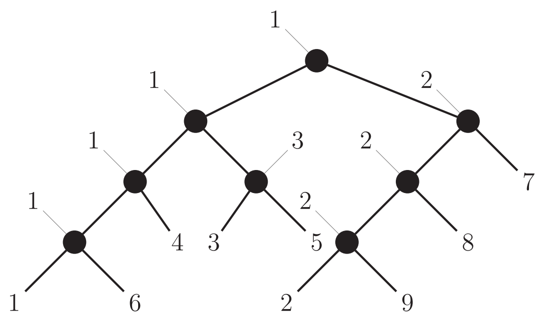


Figure 4.3: Example of a Lyndon tree. The numbers above the lines correspond to the valencies of the internal nodes

We will say that an internal node x of a labeled binary tree (T, σ) is a *Lyndon node* if (4.2.1) holds. Hence Proposition 4.2.1 says that (T, σ) is a Lyndon tree if and only if it is normalized and all its internal nodes are Lyndon nodes.

A *colored Lyndon tree* is a normalized binary tree such that for any node x that is not a Lyndon node it must happen that

$$\mathbf{color}(L(x)) > \mathbf{color}(x). \tag{4.2.2}$$

For $\mu \in \mathrm{wcomp}_{n-1}$, let Lyn_μ be the set of colored Lyndon trees in $\mathcal{BT}_\mu$ and $\mathsf{Lyn}_n = \cup_{\mu \in \mathrm{wcomp}_{n-1}} \mathsf{Lyn}_\mu$. Note that equation (4.2.2) implies that the monochromatic Lyndon trees are just the classical Lyndon trees.

The set of bicolored Lyndon trees for $n = 3$ is depicted in Figure 4.4.

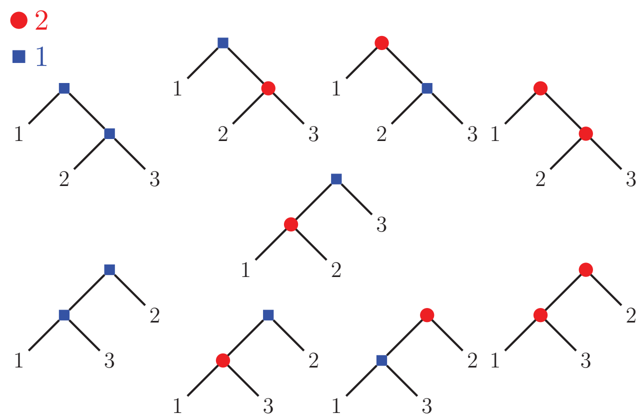


Figure 4.4: Set of bicolored Lyndon trees for $n = 3$

We will show that the ascent-free maximal chains of the EL-labeling of $[\hat{0}, [n]^\mu]$ given in Theorem 4.1.4 are of the form $c(T, \sigma, \tau)$, where $(T, \sigma) \in \mathsf{Lyn}_\mu$ and τ is the linear extension of the internal nodes of T, which we now describe: It is easy to see that there is a unique linear extension of the internal notes of $(T, \sigma) \in \mathcal{BT}_\mu$ in which the valencies of the nodes weakly decrease. Let $\tau_{T,\sigma}$ denote the permutation that induces this linear extension.

Theorem 4.2.2. *The set* $\{c(T, \sigma, \tau_{T,\sigma}) \mid (T, \sigma) \in \mathsf{Lyn}_\mu\}$ *is the set of ascent-free maximal chains of the EL-labeling of* $[\hat{0}, [n]^\mu]$ *given in Theorem 4.1.4.*

Proof. We begin by showing that $c := c(T, \sigma, \tau)$ is ascent-free whenever $(T, \sigma) \in \mathsf{Lyn}_\mu$ and $\tau = \tau_{T,\sigma}$. Let x_i be the ith internal node of T in postorder. Then by the definition of $\tau_{T,\sigma}$,

$$v(x_{\tau(1)}) \geq v(x_{\tau(2)}) \geq \cdots \geq v(x_{\tau(n-1)}), \tag{4.2.3}$$

where v is the valency. For each i, the ith letter of the label word $\bar{\lambda}(c)$ is given by

$$\bar{\lambda}_i(c) = (v(L(x_{\tau(i)})), v(R(x_{\tau(i)})))^{u_i} = (v(x_{\tau(i)}), v(R(x_{\tau(i)})))^{u_i},$$

where $u_i = \textbf{color}(x_{\tau(i)})$. Note that since (T, σ) is normalized, $v(R(x_{\tau(i)})) \neq v(R(x_{\tau(i+1)}))$ for all $i \in [n-1]$. Now suppose the word $\bar{\lambda}(c)$ has an ascent at i. Then it follows from (4.2.3) that

$$v(x_{\tau(i)}) = v(x_{\tau(i+1)}), \ \ v(R(x_{\tau(i)})) < v(R(x_{\tau(i+1)})), \ \text{ and } \ u_i \leq u_{i+1}. \tag{4.2.4}$$

The equality of valencies implies that $x_{\tau(i)} = L(x_{\tau(i+1)})$ since (T, σ) is normalized and τ is a linear extension. Hence by (4.2.4),

$$v(R(L(x_{\tau(i+1)}))) < v(R(x_{\tau(i+1)})).$$

It follows that $x_{\tau(i+1)}$ is not a Lyndon node. So by the coloring restriction on colored Lyndon trees

$$u_i = \textbf{color}(x_{\tau(i)}) = \textbf{color}(L(x_{\tau(i+1)})) > \textbf{color}(x_{\tau(i+1)}) = u_{i+1},$$

which contradicts (4.2.4). Hence the chain c is ascent-free.

Conversely, assume c is an ascent-free maximal chain of $[\hat{0}, [n]^{\mu}]$. Then $c = c(T, \sigma, \tau)$ for some bicolored labeled tree (T, σ) and some permutation $\tau \in \mathfrak{S}_{n-1}$. We can assume without loss of generality that (T, σ) is normalized. Since c is ascent-free, (4.2.3) holds. This implies that τ is the unique permutation that induces the valency-decreasing linear extension, namely $\tau_{T,\sigma}$.

If all internal nodes of (T, σ) are Lyndon nodes we are done. So let $i \in [n-1]$ be such that $x_{\tau(i)}$ is not a Lyndon node. That is

$$v(R(L(x_{\tau(i)}))) < v(R(x_{\tau(i)})).$$

Since (T, σ) is normalized and (4.2.3) holds, $L(x_{\tau(i)}) = x_{\tau(i-1)}$. Hence, $v(R(x_{\tau(i-1)})) < v(R(x_{\tau(i)}))$. Since (T, σ) is normalized we also have $v(L(x_{\tau(i-1)})) = v(L(x_{\tau(i)}))$. Since c is ascent-free we must have that

$$\mathbf{color}(x_{\tau(i-1)}) > \mathbf{color}(x_{\tau(i)}),$$

which is precisely what we need to conclude that (T, σ) is a colored Lyndon tree.

□

From Theorem 4.1.5, Theorem 4.2.2 and Corollary 4.1.6, we have the following corollary.

Corollary 4.2.3. *For all $n \geq 1$ and for all $\mu \in \mathrm{wcomp}_{n-1}$ with $\mathrm{supp}(\mu) \subseteq [k]$, the order complex $\Delta((\hat{0}, [n]^{\mu}))$ has the homotopy type of a wedge of $|\mathsf{Lyn}_{\mu}|$ spheres of dimension $n-3$. Consequently,*

$$\dim \widetilde{H}^{n-3}((\hat{0}, [n]^{\mu})) = |\mathsf{Lyn}_{\mu}|$$

and

$$\bar{\mu}_{\Pi_n^k}(\hat{0}, [n]^\mu) = (-1)^{n-1} |\mathsf{Lyn}_\mu|.$$

Chapter 5

The dimension of $\mathcal{L}ie(\mu)$

In this chapter we present various formulas for the dimension of $\mathcal{L}ie(\mu)$. We begin by using the isomorphism between $\mathcal{L}ie(\mu)$ and $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$ of Theorem 2.3.1 to transfer information on $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$ obtained in the last two chapters to $\mathcal{L}ie(\mu)$.

Theorem 5.0.4 (Theorem 1.6.3)**.** *We have*

$$\sum_{n\geq 1}\sum_{\mu\in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^\mu \frac{y^n}{n!} = \left[\sum_{n\geq 1}(-1)^{n-1}h_{n-1}(\mathbf{x})\frac{y^n}{n!}\right]^{<-1>},$$

Proof. From Corollary 4.2.3 we have that

$$\bar{\mu}_{\Pi_n^k}(\hat{0},[n]^\mu) = (-1)^{n-1}\dim \widetilde{H}^{n-3}((\hat{0},[n]^\mu)).$$

The theorem now follows from Theorems 2.3.1 and 3.1.1 when we let k get large. $\square$

We have a combinatorial description for the dimension of $\mathcal{L}ie(\mu)$.

Theorem 5.0.5. *For all* $n \geq 1$ *and* $\mu \in \mathrm{wcomp}_{n-1}$,

$$\dim \mathcal{L}ie(\mu) = |\mathsf{Lyn}_\mu|.$$

Proof. We know from Corollary 4.2.3 that $\dim \widetilde{H}^{n-3}((\hat{0}, [n]^\mu)) = |\mathsf{Lyn}_\mu|$. Hence, the isomorphism of Theorem 2.3.1 proves the theorem. $\square$

5.1 Lyndon type of a normalized tree

With a normalized tree $\Upsilon \in \mathsf{Nor}_n$ we can associate a (set) partition $\pi^{\mathsf{Lyn}}(\Upsilon)$ of the set of internal nodes of Υ, defined to be the finest partition satisfying the condition:

- for every internal node x that is not Lyndon, x and $L(x)$ belong to the same block of $\pi^{\mathsf{Lyn}}(\Upsilon)$.

For the tree in Figure 5.1, the shaded rectangles indicate the blocks of $\pi^{\mathsf{Lyn}}(\Upsilon)$.

Note that the coloring condition (4.2.2) implies that in a colored Lyndon tree Υ there are no repeated colors in each block B of the partition $\pi^{\mathsf{Lyn}}(\Upsilon)$ associated with Υ. Hence after choosing a set of $|B|$ colors for the internal nodes in B there is a unique way to assign the different colors such that the colored tree Υ is a colored Lyndon tree (the colors must decrease towards the root in each block of $\pi^{\mathsf{Lyn}}(\Upsilon)$).

Define the *Lyndon type* $\lambda^{\mathsf{Lyn}}(\Upsilon)$ of a normalized tree (colored or uncolored) Υ to be the (integer) partition whose parts are the block sizes of the partition $\pi^{\mathsf{Lyn}}(\Upsilon)$. For the tree Υ in Figure 5.1, we have $\lambda^{\mathsf{Lyn}}(\Upsilon) = (3, 2, 2, 1)$.

Let $e_\lambda(\mathbf{x})$ be the elementary symmetric function associated with the partition λ.

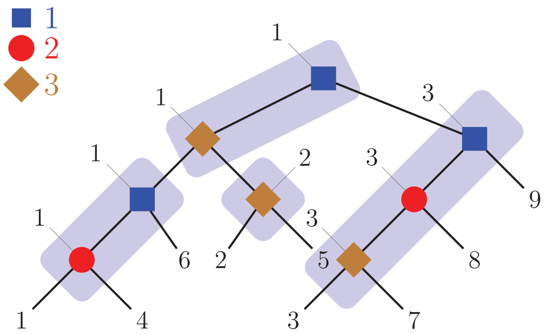


Figure 5.1: Example of a colored Lyndon tree of type (3,2,2,1). The numbers above the lines correspond to the valencies of the internal nodes

Theorem 5.1.1. *For all n,*

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^{\mu} = \sum_{\Upsilon \in \mathit{Nor}_n} e_{\lambda^{\mathsf{Lyn}}(\Upsilon)}(\mathbf{x}). \tag{5.1.1}$$

Proof. For a colored labeled binary tree Ψ we define the *content* $\mu(\Psi)$ of G as the weak composition μ where $\mu(i)$ is the number of internal nodes of Ψ that have color i. Recall that $\widetilde{\Psi}$ denotes the underlying uncolored labeled binary tree of Ψ. Note that the comments above imply that for $\Upsilon \in \mathsf{Nor}_n$, the generating function of colored Lyndon trees associated with Υ is

$$\sum_{\substack{\Psi \in \mathsf{Lyn}_n \\ \widetilde{\Psi} = \Upsilon}} \mathbf{x}^{\mu(\Psi)} = e_{\lambda^{\mathsf{Lyn}}(\Upsilon)}(\mathbf{x}). \tag{5.1.2}$$

Indeed the internal nodes in a block of size i in the partition $\pi^{\mathsf{Lyn}}(\Upsilon)$ can be colored uniquely with any set of i different colors and so the contribution from this block of $\pi^{\mathsf{Lyn}}(\Upsilon)$ to the generating function in (5.1.2) is $e_i(\mathbf{x})$.

By Theorem 5.0.5,

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^{\mu} = \sum_{\mu \in \mathrm{wcomp}_{n-1}} |\mathsf{Lyn}_{\mu}|\, \mathbf{x}^{\mu}$$

$$
\begin{aligned}
&= \sum_{\Psi \in \mathsf{Lyn}_n} \mathbf{x}^{\mu(\Psi)} \\
&= \sum_{\Upsilon \in \mathsf{Nor}_n} \sum_{\substack{\Psi \in \mathsf{Lyn}_n \\ \widetilde{\Psi} = \Upsilon}} \mathbf{x}^{\mu(\Psi)} \\
&= \sum_{\Upsilon \in \mathsf{Nor}_n} e_{\lambda^{\mathsf{Lyn}}(\Upsilon)}(\mathbf{x}),
\end{aligned}
$$

with the last equation following from (5.1.2). $\square$

5.2 Stirling permutations

A *Stirling permutation* on the set $[n]$ is a permutation of the multiset $\{1, 1, 2, 2, \cdots, n, n\}$ such that for all $m \in [n]$, all numbers between the two occurrences of m are larger than m. The set of Stirling permutations on $[n]$ will be denoted by $\mathcal{Q}_n$. For example, the permutation 12332144 is in $\mathcal{Q}_n$ but 43341122 is not since 3 is between the two ocurrences of 4. Stirling permutations were introduced by Stanley and Gessel in [18] and have been also studied by Bóna, Park, Janson, Kuba, Panholzer and others (see [10, 33, 26, 22]). For an arbitrary subset $A := \{a_1, a_2, \ldots, a_n\}$ of positive integers, we denote by $\mathcal{Q}_A$, the set of Stirling permutations of A; that is, permutations of the multiset $\{a_1, a_1, a_2, a_2, \ldots, a_n, a_n\}$, satisfying the condition above.

It is known that $|\mathcal{Q}_{n-1}| = (2n-3)!!$. So this set of Stirling permutations is equinumerous with the set Nor_n of normalized binary trees with label set $[n]$. We will present an explicit bijection between these two sets. Moreover, this bijection has some nice properties that allow us to translate the results of Chapter 4 to the language of Stirling permutations and to ultimately prove Theorem 1.6.4.

5.2.1 Type of a Stirling permutation

A *segment* u of a Stirling permutation $\theta = \theta_1\theta_2\cdots\theta_{2n}$ is a subword of θ of the form $u = \theta_i\theta_{i+1}\cdots\theta_{i+\ell}$, i.e., all the letters of u are adjacent in θ. A *block* in a Stirling permutation θ is a segment of θ that starts and ends with the same letter. For example, 455774 is a block of 12245577413366. We define $B_\theta(a)$ to be the block of θ that starts and ends with the letter a, and define $\mathring{B}_\theta(a)$ to be the segment obtained from $B_\theta(a)$ after removing the two occurrences of the letter a. For example, $B_\theta(1) = 1224557741$ in $\theta = 12245577413366$ and $\mathring{B}_\theta(1) = 22455774$.

We call (a, b) an *ascending adjacent pair* if $a < b$ and the blocks $B_\theta(a)$ and $B_\theta(b)$ are adjacent in θ, i.e., $\theta = \theta' B_\theta(a) B_\theta(b) \theta''$. An *ascending adjacent sequence* of θ of length k is a subsequence $a_1 < a_2 < \cdots < a_k$ such that (a_j, a_{j+1}) is an ascending adjacent pair for $j = 1, \ldots, k-1$. Similarly, for a Stirling permutation $\theta \in \mathcal{Q}_n$ we call (a, b) a *terminally nested pair* if $a < b$ and the block $B_\theta(b)$ is the last block in $\mathring{B}_\theta(a)$, i.e., $\mathring{B}_\theta(a) = \theta' B_\theta(b)$ for some Stirling permutation θ'. A *terminally nested sequence* of θ of length k is a subsequence $a_1 < a_2 < \cdots < a_k$ such that (a_j, a_{j+1}) is a terminally nested pair for $j = 1, \ldots, k-1$.

We can associate a *type* to a Stirling permutation $\theta \in \mathcal{Q}_n$ in two ways. We define the *ascending adjacent type* $\lambda^{\mathsf{AA}}(\theta)$, to be the partition whose parts are the lengths of maximal ascending adjacent sequences; and we define the *terminally nested type* $\lambda^{\mathsf{TN}}(\theta)$, to be the partition whose parts are the lengths of maximal terminally nested sequences. We will show that these two types are equinumerous in $\mathcal{Q}_n$.

Example 5.2.1. If $\theta = 158851244667723399$, then the maximal ascending adjacent sequences are 1239, 467, 5 and 8; then $\lambda^{\mathsf{AA}}(\theta) = (4, 3, 1, 1)$, which is a partition of $n = 9$. Also the maximal terminally nested sequences are 158, 27, 3, 4, 6 and 9; then $\lambda^{\mathsf{TN}}(\theta) = (3, 2, 1, 1, 1, 1)$, which is also a partition of $n = 9$.

It is easy to see that every Stirling permutation has a unique factorization $\theta = B_\theta(a_1)B_\theta(a_2)\cdots B_\theta(a_\ell)$ into adjacent blocks. We call this factorization the *block factorization* of θ. For example, 12245577413366 has a block factorization $1224557741 - 33 - 66$. A *Stirling factorization* of a Stirling permutation θ is a decomposition $\theta = \theta^1\theta^2\cdots\theta^\ell$, such that θ^i is a Stirling permutation for all i. Note that the block factorization of θ is the finest Stirling factorization.

Denote by $\kappa(a) := a_k$, the largest letter of the maximal terminally nested sequence $a = a_1 < a_2 < \cdots < a_k$ of $B_\theta(a)$ that contains a. In $\theta = 158851244667723399$, we have for example that $\kappa(1) = 8$, $\kappa(2) = 7$ and $\kappa(7) = 7$. We define the following two types of restricted Stirling factorizations:

- The *ascending adjacent factorization* of θ is the Stirling factorization $\theta = \theta^1\theta^2$ in which θ^1 is the shortest nonempty prefix of θ such that if $\theta^1 = \alpha B_\theta(a)$ and $\theta^2 = B_\theta(b)\beta$ then $a > b$. For example if $\theta = 133155442662$, then the ascending adjacent factorization of θ is $133155 - 442662$.

- The *terminally nested factorization* of θ is the Stirling factorization $\theta = \theta^1\theta^2$ in which θ^1 is the shortest nonempty prefix of θ such that if $\theta^1 = B_\theta(a)\alpha$ and $\theta^2 = B_\theta(b)\beta$ then $\kappa(a) > b$. In the case of $\theta = 133155442662$, the terminally nested factorization of θ is $13315544 - 2662$.

An *irreducible AA-word* is a Stirling permutation that has no nontrivial ascending adjacent factorization. It is not difficult to see that an irreducible AA-word is a Stirling permutation of the form

$$B_\theta(a_1)B_\theta(a_2)\cdots B_\theta(a_k) = a_1\tau_1a_1\,a_2\tau_2a_2\cdots a_{k-1}\tau_{k-1}a_{k-1}\,a_k\tau_ka_k, \qquad (5.2.1)$$

where $a_1 < a_2 < \cdots < a_k$ and τ_i are Stirling permutations for each i.

An *irreducible TN-word* is a Stirling permutation that has no nontrivial terminally nested factorization. It is not difficult to see that an irreducible TN-word is a Stirling permutation of the form

$$B_\theta(a)\alpha \tag{5.2.2}$$

where $\kappa(a) < a'$ for any letter a' in α. Equivalently, an irreducible TN-word is a Stirling permutation of the form

$$a_1\tau_1 a_2\tau_2 a_{k-1}\tau_{k-1} a_k a_k a_{k-1}\cdots a_2 a_1 \tau_k \tag{5.2.3}$$

where $a_1 < a_2 < \cdots < a_k$ and τ_i are Stirling permutations for each i with $a_k < a'$ for any letter a' in τ_k.

The *complete ascending adjacent (terminally nested) factorization* of θ is the factorization $\theta = \theta^1\theta^2\cdots\theta^l$ that we obtain by factoring θ into $\theta^1\theta^2$ by the ascending adjacent (resp., terminally nested) factorization and then recursively applying the same procedure to θ^2.

Let A be a subset of the positive integers. We define a map $\xi : \mathcal{Q}_A \to \mathcal{Q}_A$ recursively as follows:

1. If $\theta = mm$ then $\xi(\theta) = mm$.

2. If θ is an irreducible AA-word $a_1\tau_1 a_1 a_2\tau_2 a_2 \cdots a_{k-1}\tau_{k-1}a_{k-1}a_k\tau_k a_k$ then

$$\xi(\theta) = a_1\xi(\tau_1)a_2\xi(\tau_2)\cdots a_{k-1}\xi(\tau_{k-1})a_k a_k a_{k-1}\cdots a_2 a_1\xi(\tau_k).$$

3. If $\theta = \theta^1\theta^2\cdots\theta^l$ is the complete ascending adjacent factorization of θ then

$$\xi(\theta) = \xi(\theta^1)\xi(\theta^2)\cdots\xi(\theta^l).$$

Step (2) guarantees that ξ is well-defined. Indeed, in an irreducible AA-word of the form given in (2), we have $a_s < a_{s+1} < \cdots < a_k$ for any s. Hence, we are inserting only letters that are greater than a_s between the two occurrences of a_s.

The map ξ is in fact a bijection and it is not difficult to check that its inverse $\xi^{-1} : \mathcal{Q}_A \to \mathcal{Q}_A$ is defined by:

1. If $\theta = mm$ then $\xi^{-1}(\theta) = mm$.
2. If θ is an irreducible TN-word $a_1\tau_1 a_2\tau_2 \cdots a_{k-1}\tau_{k-1}a_k a_k a_{k-1} \cdots a_2 a_1 \tau_k$ then
$$\xi^{-1}(\theta) = a_1\xi^{-1}(\tau_1)a_1 a_2\xi^{-1}(\tau_2)a_2 \cdots a_{k-1}\xi^{-1}(\tau_{k-1})a_{k-1}a_k\xi^{-1}(\tau_k)a_k.$$
3. If $\theta = \theta^1\theta^2 \cdots \theta^l$ is the complete terminally nested factorization of θ then
$$\xi^{-1}(\theta) = \xi^{-1}(\theta^1)\xi^{-1}(\theta^2) \cdots \xi^{-1}(\theta^l).$$

Step (2) guarantees that ξ^{-1} is well-defined since in an irreducible TN-word, $a_k < b$ for any letter b in τ_k.

Example 5.2.2. Consider $\theta = 233772499468861551$. Its complete ascending factorization is $23377249946886 - 1551$; then

$$\begin{aligned}\xi(\theta) &= \xi(23377249946886 - 1551)\\ &= \xi(23377249946886) - \xi(1551)\\ &= 2\xi(3377)4\xi(99)6642\xi(88) - 11\xi(55)\\ &= 237734996642881155.\end{aligned}$$

Note that the maximal ascending adjacent sequences of θ are $(246, 37, 1, 5, 8, 9)$ which are also the maximal terminally nested sequences of $\xi(\theta)$. These observations hold in general.

Proposition 5.2.3. *The map $\xi : \mathcal{Q}_A \to \mathcal{Q}_A$ is a well-defined bijection that satisfies:*

1. *(i, j) is an ascending adjacent pair in θ if and only if (i, j) is a terminally nested pair in $\xi(\theta)$,*

2. *$\lambda^{\mathsf{TN}}(\xi(\theta)) = \lambda^{\mathsf{AA}}(\theta)$.*

Proof. Note that if $\theta = \theta^1\theta^2 \cdots \theta^l$ is the complete ascending adjacent factorization of a Stirling permutation θ, then an ascending adjacent pair can only occur within one of the factors θ^i. Similarly, if $\theta = \theta^1\theta^2 \cdots \theta^l$ is the complete terminally nested factorization of a Stirling permutation θ, then a terminally nested pair can only occur within one of the factors θ^i. Hence, without loss of generality, as a consequence of step (3) in the definitions of ξ and ξ^{-1}, we can assume that the word θ is an irreducible AA-word or an irreducible TN-word. Then the first assertion follows directly from step (2) in the definitions of ξ and ξ^{-1} and induction on the length of θ. The second assertion is an immediate consequence of the first. □

From Proposition 5.2.3 we see that λ^{AA} and λ^{TN} are equidistributed on $\mathcal{Q}_n$.

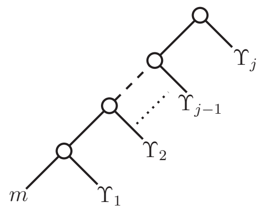
5.3 A bijection between normalized trees and Stirling permutations

Let $\hat{\mathcal{Q}}_n$ be the set of permutations $\theta \in \mathcal{Q}_n$ where $\theta_1 = \theta_{2n} = 1$. There is a natural bijection $\mathfrak{red} : \hat{\mathcal{Q}}_n \to \mathcal{Q}_{n-1}$ obtained by removing the leading and trailing 1 from

$\theta = 1\theta'1$ and then reducing the word θ' by decreasing every letter in $\{2, \dots, n\}$ by one. For example, $\mathfrak{red}(12332441) = 122133$. In greater generality, for A a subset of the positive integers, let $\hat{\mathcal{Q}}_A$ be the set of Stirling permutations of A such that both the first and last letter of the permutation is $\min A$. Define the map[1] $\tilde{\gamma} : \mathsf{Nor}_A \to \hat{\mathcal{Q}}_A$ recursively by:

1. If $\Upsilon = (\bullet, m)$ then $\tilde{\gamma}(\Upsilon) = mm$.

2. If Υ is of the form



,

then $\tilde{\gamma}(\Upsilon) = m\tilde{\gamma}(\Upsilon_1)\tilde{\gamma}(\Upsilon_2)\cdots\tilde{\gamma}(\Upsilon_{j-1})\tilde{\gamma}(\Upsilon_j)m$.

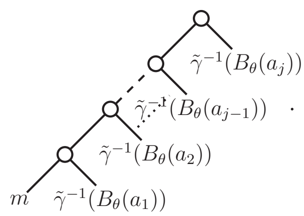
The function $\tilde{\gamma}$ is well-defined since the tree is normalized. Indeed, m is the minimal letter and we always obtain a word with values greater than m between the two occurrences of m. Proceeding by induction on the number of internal nodes of Υ, we have that the words $\gamma(\Upsilon_i)$ are Stirling permutations for each i and so it is $\gamma(\Upsilon)$.

It is not difficult to check that the inverse $\tilde{\gamma}^{-1} : \hat{\mathcal{Q}}_A \to \mathsf{Nor}_A$ can also be defined recursively by

1. If $\theta = mm$ then $\tilde{\gamma}^{-1}(mm) = (\bullet, m)$.

2. If $\theta = B_\theta(m)$ and $\mathring{B}_\theta(m) = B_\theta(a_1)B_\theta(a_2)\cdots B_\theta(a_{j-1})B_\theta(a_j)$, then

$\tilde{\gamma}^{-1}(\theta) =$



.

[1] The same map has appeared before in [13].

The tree defined in the step above is clearly normalized. So we can encode any normalized binary tree with a permutation in $\hat{\mathcal{Q}}_n$. See Figure 5.2 for an example of the bijection.

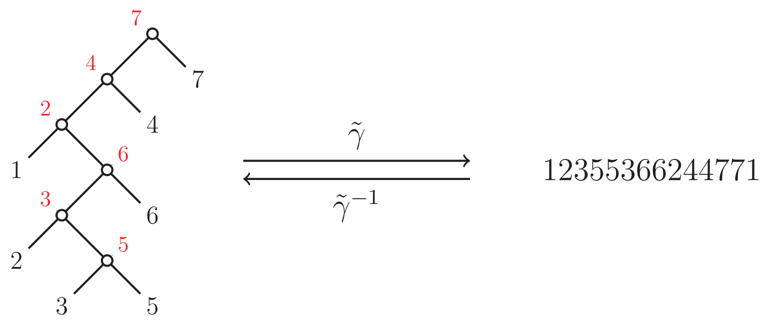


Figure 5.2: Example of the bijection $\tilde{\gamma}$

We give an alternative description of $\tilde{\gamma}$. First we extend the leaf labeling of $\Upsilon \in \mathsf{Nor}_n$ to a labeling θ that includes the internal nodes. For each internal node x, let $\theta(x)$ be the smallest leaf label in the right subtree of the subtree of Υ rooted at x; for each leaf x, let $\theta(x)$ be the leaf label of x (See Figure 5.2). Let $x_1, \ldots, x_{2n-1}$ be the listing of all the nodes of Υ (internal and leaves) in postorder and let $\theta(\Upsilon) := \theta(x_1)\theta(x_2)\ldots\theta(x_{2n-1})$.

Proposition 5.3.1. *For all $\Upsilon \in \mathsf{Nor}_n$,*

$$\tilde{\gamma}(\Upsilon) = \theta(\Upsilon)\theta(x_1)$$

where x_1 is the leftmost leaf of Υ.

Proof. If $\Upsilon = (\bullet, m)$ is a single node then $\tilde{\gamma}(\Upsilon) = mm = \theta(\Upsilon)\theta(x_1)$. If Υ has internal nodes, it can be expressed as

$$\Upsilon = (\ldots(((x_1, v(x_1)) \wedge \Upsilon_1) \wedge \Upsilon_2) \wedge \cdots \wedge \Upsilon_j)$$

(like the one in step (2) of the definition of $\tilde{\gamma}$).

Let y_i denote the parent of the root of Υ_i for each i. As a consequence of the definition of θ, we have that $\theta(y_i) = \theta(z_i)$, where z_i is the smallest leaf of Υ_i. By induction, using the definition of $\tilde{\gamma}$,

$$\begin{aligned}\tilde{\gamma}(\Upsilon) &= v(x_1)\tilde{\gamma}(\Upsilon_1)\dots\tilde{\gamma}(\Upsilon_j)v(x_1)\\ &= \theta(x_1)\theta(\Upsilon_1)\theta(y_1)\dots\theta(\Upsilon_j)\theta(y_j)\theta(x_1)\\ &= \theta(\Upsilon)\theta(x_1).\end{aligned}$$

The last step holds since the postorder traversal of Υ lists first x_1, followed by postorder traversal of Υ_1 followed by y_1, followed by postorder traversal of Υ_2 followed by y_2, and so on. $\square$

To remove the unnecessary leading and trailing ones in $\tilde{\gamma}(\Upsilon)$, we consider instead the map $\gamma : \mathsf{Nor}_n \to \mathcal{Q}_{n-1}$ defined by $\gamma(\Upsilon) := \mathfrak{red}(\tilde{\gamma}(\Upsilon))$ for each $\Upsilon \in \mathsf{Nor}_n$.

We invite the reader to recall the definition of comb type $\lambda^{\mathsf{Comb}}(\Upsilon)$ of a normalized tree Υ given in Chapter 1 before Theorem 1.6.4 and the definition of Lyndon type $\lambda^{\mathsf{Lyn}}(\Upsilon)$ given in Section 5.1. Recall also the definition of ascending adjacent and terminally nested pairs of a Stirling permutation $\theta \in \mathcal{Q}_n$, and the associated types $\lambda^{\mathsf{AA}}(\theta)$ and $\lambda^{\mathsf{TN}}(\theta)$, given in the first part of this section. We give an equivalent characterization of these pairs. An ascending adjacent pair of $\theta \in \mathcal{Q}_n$ is a pair (a, b) such that $a < b$ and in θ the second occurrence of a is the immediate predecessor of the first occurrence of b. A terminally nested pair of $\theta \in \mathcal{Q}_n$ is a pair (a, b) such that $a < b$ and in θ the second occurrence of a is the immediate successor of the second occurrence of b.

For any node (internal or leaf) x of Υ we define the (*reduced valency*) $v^{\mathfrak{r}}(x) := v(x) - 1$.

Proposition 5.3.2. *The map $\gamma : \mathsf{Nor}_n \to \mathcal{Q}_{n-1}$ is a well-defined bijection that satisfies for each $\Upsilon \in \mathsf{Nor}_n$ and internal node x of Υ,*

1. *x is a non-Lyndon node of Υ if and only if $(v^{\mathrm{r}}(R(L(x))), v^{\mathrm{r}}(R(x)))$ is an ascending adjacent pair in $\gamma(\Upsilon)$. Moreover, every ascending pair of $\gamma(\Upsilon)$ is of the form $(v^{\mathrm{r}}(R(L(x))), v^{\mathrm{r}}(R(x)))$ for some internal node x of Υ;*
2. *x has a right child $R(x)$ that is also an internal node if and only if $(v^{\mathrm{r}}(R(x)), v^{\mathrm{r}}(R(R(x))))$ is a terminally nested pair in $\gamma(\Upsilon)$. Moreover, every terminally nested pair of $\gamma(\Upsilon)$ is of the form $(v^{\mathrm{r}}(R(x)), v^{\mathrm{r}}(R(R(x))))$ for some internal node x of Υ;*
3. *$\lambda^{\mathsf{AA}}(\gamma(\Upsilon)) = \lambda^{\mathsf{Lyn}}(\Upsilon)$;*
4. *$\lambda^{\mathsf{TN}}(\gamma(\Upsilon)) = \lambda^{\mathsf{Comb}}(\Upsilon)$.*

Proof. Let $\Upsilon \in \mathsf{Nor}_n$ and let x_i be the ith node of Υ listed in postorder. We use the alternative characterization of $\tilde{\gamma}$ given in Proposition 5.3.1. We claim that:
Claim 1: The pair $(\theta(x_i), \theta(x_{i+1}))$ is an ascending adjacent pair of $\tilde{\gamma}(\Upsilon)$ if and only if x_i is a left child that is not a leaf and its parent $\mathrm{p}(x_i)$ satisfies $\theta(\mathrm{p}(x_i)) > \theta(x_i)$. (The latter condition is equivalent to $\mathrm{p}(x_i)$ being a non-Lyndon node.)
Claim 2: $(\theta(x_{i+1}), \theta(x_i))$ is a terminally nested pair of $\tilde{\gamma}(\Upsilon)$ if and only if x_i is a right child that is not a leaf.

We say that $\theta \in Q_n$ has a *first occurrence* of the letter θ_i at position i if $\theta_j \neq \theta_i$ for all $j < i$. We say that $\theta \in Q_n$ has a *second occurrence* of the letter θ_i at position i if there is a $j < i$ such that $\theta_j = \theta_i$. Before proving these claims we first observe that in the word $\tilde{\gamma}(\Upsilon) = \theta(\Upsilon)\theta(x_1)$ (Proposition 5.3.1), there is a first occurrence of a letter at position i if x_i is a leaf and a second occurrence of a letter if x_i is an internal node. The proof of the two claims follows from the following four cases that in turn are consequences of this observation.

Case 1: Let x_i be a left child that is not a leaf. Then x_{i+1} is the leftmost leaf of the right subtree of the subtree of Υ rooted at $\mathrm{p}(x_i)$. By the observation above, the position i of $\tilde{\gamma}(\Upsilon)$ contains the second occurrence of a letter while the position $i+1$ contains the first occurrence of a letter. Note that $\theta(\mathrm{p}(x_i)) = \theta(x_{i+1})$.

Case 2: Let x_i be a left child that is a leaf. Then x_{i+1} is the smallest leaf of the right subtree of the subtree of Υ rooted at $\mathrm{p}(x_i)$. Hence, positions i and $i+1$ contain first occurrences of letters in $\tilde{\gamma}(\Upsilon)$ and $\theta(x_i) < \theta(x_{i+1}) = \theta(\mathrm{p}(x_i))$.

Case 3: Let x_i be a right child that is not a leaf. Then by postorder $x_{i+1} = \mathrm{p}(x_i)$ and positions i and $i+1$ contain second occurrences of letters in $\tilde{\gamma}(\Upsilon)$. Note that $\theta(x_i) > \theta(x_{i+1})$.

Case 4: Let x_i be a right child that is a leaf. Then by postorder $x_{i+1} = \mathrm{p}(x_i)$ and by the definition of θ we have that $\theta(x_i) = \theta(\mathrm{p}(x_i)) = \theta(x_{i+1})$.

It is not difficult to see that the two claims imply (1) and (2) after applying the definitions of $\mathfrak{red}$ and $v^{\mathfrak{r}}$. Parts (3) and (4) are immediate consequences of parts (1) and (2), respectively. $\square$

We have now four different combinatorial interpretations of the coefficients of the symmetric function $L_n(\mathbf{x}) := \sum_{\mu \in \mathrm{wcomp}_n} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^\mu$ in the elementary symmetric function basis. Theorem 5.3.3 below includes Theorem 1.6.4.

Theorem 5.3.3. *For all n,*

$$\begin{aligned}
\sum_{\mu \in \mathrm{wcomp}_{n-1}} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^\mu &= \sum_{\Upsilon \in \mathsf{Nor}_n} e_{\lambda^{\mathsf{Lyn}}(\Upsilon)}(\mathbf{x}) \\
&= \sum_{\theta \in \mathcal{Q}_{n-1}} e_{\lambda^{\mathsf{AA}}(\theta)}(\mathbf{x}) \\
&= \sum_{\theta \in \mathcal{Q}_{n-1}} e_{\lambda^{\mathsf{TN}}(\theta)}(\mathbf{x}) \\
&= \sum_{\Upsilon \in \mathsf{Nor}_n} e_{\lambda^{\mathsf{Comb}}(\Upsilon)}(\mathbf{x}).
\end{aligned}$$

Proof. The first equality comes from Theorem 5.1.1, the third equality is a consequence of Proposition 5.2.3, and the second and fourth equality are consequences of Proposition 5.3.2. □

For a permutation $\theta \in \mathcal{Q}_n$ we define the *initial permutation* $\mathfrak{init}(\theta) \in \mathfrak{S}_n$ to be the subword of θ formed by the first occurrence of each of the letters in θ. For example, $\mathfrak{init}(233772499468861551) = 237496815$.

Proposition 5.3.4. *For any $\theta \in \mathcal{Q}_n$ and $\Upsilon = (T, \sigma) \in \mathsf{Nor}_n$,*

1. $\mathfrak{init}(\xi(\theta)) = \mathfrak{init}(\theta)$
2. $\sigma = \mathfrak{init}(\tilde{\gamma}(\Upsilon))$.

Proof. In the definition of ξ, the relative order of the initial occurrence of the letters is not changed; which proves (1). We consider the alternative characterization of $\tilde{\gamma}$ of Proposition 5.3.1. Recall that $\theta(x_i)$ is a first occurrence of a letter in $\theta(\Upsilon)$ if and only if x_i is a leaf of Υ. Hence, part (2) follows from the fact that postorder of the nodes of Υ restricted to the leaves is just left to right reading of the leaves. □

We have the following diagram of bijections:

$$\mathsf{Nor}_n \underset{\gamma^{-1}}{\overset{\gamma}{\rightleftarrows}} \mathcal{Q}_{n-1} \underset{\xi^{-1}}{\overset{\xi}{\rightleftarrows}} \mathcal{Q}_{n-1} \underset{\gamma}{\overset{\gamma^{-1}}{\rightleftarrows}} \mathsf{Nor}_n$$

The following theorem is a generalization of the classical bijections between Lyndon trees, combs and permutations in $\mathfrak{S}_{n-1}$. See Figure 5.3 for a complete example of the bijections.

Corollary 5.3.5. *The map $\gamma^{-1}\xi\gamma$ is a bijection on Nor_n that translates between the Lyndon type and comb type. Moreover, the bijection preserves the permutation of leaf labels for each tree.*

Proof. By Propositions 5.2.3 and 5.3.2,

$$\begin{aligned}\lambda^{\mathsf{Comb}}(\gamma^{-1}\xi\gamma(\Upsilon)) &= \lambda^{\mathsf{TN}}(\xi\gamma(\Upsilon))\\ &= \lambda^{\mathsf{AA}}(\gamma(\Upsilon))\\ &= \lambda^{\mathsf{Lyn}}(\Upsilon).\end{aligned}$$

Proposition 5.3.4 implies that the order of the leaf labels is preserved. $\square$

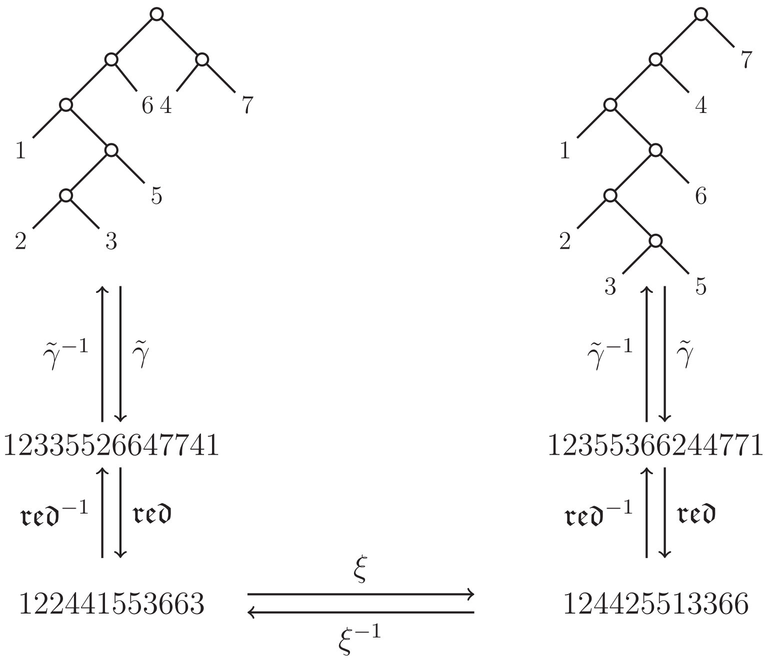


Figure 5.3: Example of the bijections $\tilde{\gamma}$, $\mathfrak{red}$ and ξ

We can combine Theorem 1.6.3 with Theorem 5.3.3 and conclude the following e-positivity result.

Theorem 5.3.6. *We have*

$$\begin{aligned}\left[\sum_{n\geq 1}(-1)^{n-1}h_{n-1}(\mathbf{x})\frac{y^n}{n!}\right]^{<-1>} &= \sum_{n\geq 1}\sum_{\Upsilon\in \mathit{Nor}_n} e_{\lambda(\Upsilon)}(\mathbf{x})\frac{y^n}{n!}\\ &= \sum_{n\geq 1}\sum_{\theta\in\mathcal{Q}_{n-1}} e_{\lambda(\theta)}(\mathbf{x})\frac{y^n}{n!},\end{aligned}$$

where $\lambda(\Upsilon)$ is either the Lyndon type or the comb type of the normalized tree Υ and $\lambda(\theta)$ is either the AA type or the TN type of the Stirling permutation θ.

In [20] the author gives another proof of Theorem 5.3.6 that does not involve poset topology and instead involves a nice interpretation of the compositional inverse of exponential generating functions given by B. Drake in [16].

5.4 A remark about colored Stirling permutations

We can also define colored Stirling permutations in analogy with the case of colored normalized binary trees. An *AA colored Stirling permutation* $\Theta = (\theta, c)$ is a Stirling permutation $\theta \in \mathcal{Q}_n$ together with a map $c : [n] \to \mathbb{P}$ such that for every occurrence of an ascending adjacent pair (a, b) in θ, c satisfies the condition $c(a) > c(b)$.

Example 5.4.1. If $\theta = 233772499468861551$, the map $c : [9] \to \mathbb{P}$ defined by the pairs $(i, c(i))$:

$$\{(1,1),(2,3),(3,3),(4,2),(5,2),(6,1),(7,1),(8,2),(9,1)\}$$

is an AA coloring, but

$$\{(1,1),(2,2),(3,3),(4,3),(5,2),(6,1),(7,1),(8,2),(9,1)\}$$

is not since 24 is an adjacent ascending pair but $c(2) = 2 < 3 = c(4)$.

In the same manner we define a *TN colored Stirling permutation* to the pair $\Theta = (\theta, c)$, where c satisfies $c(a) > c(b)$ whenever (a, b) is a terminally nested pair. For $\mu \in \mathrm{wcomp}_n$, we say that a colored Stirling permutation (θ, c) is μ-colored if $\mu(i) = |c^{-1}(i)|$ for all i. We denote by $\mathcal{Q}_\mu^{AA}$ the set of AA μ-colored Stirling permutations of $[n]$ and $\mathcal{Q}_\mu^{TN}$ the set of TN μ-colored Stirling permutations of $[n]$.

Corollary 5.4.2 (of Corollary 4.2.3)**.** *For all* $n \geq 1$ *and* $\mu \in \mathrm{wcomp}_{n-1}$

$$\bar{\mu}_{\Pi_n^k}(\hat{0}, [n]^\mu) = (-1)^{n-1}|\mathcal{Q}_\mu^{AA}| = (-1)^{n-1}|\mathcal{Q}_\mu^{TN}|.$$

Consequently,

$$\dim \widetilde{H}^{n-3}((\hat{0}, [n]^\mu)) = |\mathcal{Q}_\mu^{AA}| = |\mathcal{Q}_\mu^{TN}|.$$

Proof. Note that the bijection $\gamma : \mathsf{Nor}_n \to \mathcal{Q}_{n-1}$ extends naturally to a bijection $\mathsf{Lyn}_\mu \cong \mathcal{Q}_\mu^{AA}$ and the bijection $\xi : \mathcal{Q}_{n-1} \to \mathcal{Q}_{n-1}$ extends naturally to a bijection $\mathcal{Q}_\mu^{AA} \cong \mathcal{Q}_\mu^{TN}$. Thus the result is a corollary of Corollary 4.2.3. □

By Theorem 2.3.1,

Corollary 5.4.3. *For all* $n \geq 1$ *and* $\mu \in \mathrm{wcomp}_{n-1}$

$$\dim \mathcal{L}ie(\mu) = |\mathcal{Q}_\mu^{AA}| = |\mathcal{Q}_\mu^{TN}|.$$

Chapter 6

Combinatorial bases

In this chapter we discuss various bases for $\mathcal{L}ie(\mu)$, $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$ and $H^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$. We also present a basis for the homology $\widetilde{H}_{n-3}((\hat{0},[n]^i))$ of maximal intervals of Π_n^w that generalizes a known basis for $\widetilde{H}_{n-3}(\overline{\Pi}_n)$ due to Björner [7].

6.1 Colored Lyndon basis

Recall from Theorem 4.1.5 that the ascent-free maximal chains of the EL-labeling of $[\hat{0},[n]^\mu]$ yield a basis for the cohomology $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$. Hence, Theorem 4.2.2 gives a description of this basis in terms of colored Lyndon trees. The following result gives a simpler version of this basis. By applying the isomorphism of Theorem 2.3.1, one gets a corresponding basis for $\mathcal{L}ie(\mu)$, which reduces to the classical Lyndon basis for $\mathcal{L}ie(n)$ when μ has a single nonzero component.

Theorem 6.1.1. *The set* $\{\bar{c}(T,\sigma) \mid (T,\sigma) \in \mathsf{Lyn}_\mu\}$ *is a basis of* $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$ *and the set* $\{[T,\sigma] \mid (T,\sigma) \in \mathsf{Lyn}_\mu\}$ *is a basis for* $\mathcal{L}ie(\mu)$.

Proof. By Theorem 4.2.2 and Theorem 4.1.5, the set $\{\bar{c}(T,\sigma,\tau_{T,\sigma}) \mid (T,\sigma) \in \mathsf{Lyn}_\mu\}$ is a basis of $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$. Lemma 2.2.3 implies that we can replace $\tau_{T,\sigma}$ by any other linear extension and still obtain a basis for $\widetilde{H}^{n-3}((\hat{0},[n]^{\mu}))$. In particular, we can replace it by postorder. $\square$

Theorem 6.1.1 already implies that the set of maximal chains coming from colored Lyndon trees spans $\widetilde{H}^{n-3}((\hat{0},[n]^{\mu}))$; however, in order to complete the proof of Theorem 2.2.4, and conclude that the relations in the theorem generate all the cohomology relations, we will show that we can represent any $\bar{c} \in \widetilde{H}^{n-3}((\hat{0},[n]^{\mu}))$ as a linear combination of chains in $\{\bar{c}(T,\sigma) \mid (T,\sigma) \in \mathsf{Lyn}_{\mu}\}$ using only the relations in Theorem 2.2.4.

Proposition 6.1.2. *The relations in Theorem 2.2.4 generate all the cohomology relations in* $\widetilde{H}^{n-3}((\hat{0},[n]^{\mu}))$.

Proof. We use a "straightening" strategy using the relations of Theorem 2.2.4 in order to prove the result. Recall that for an internal node x of a normalized binary tree $\Upsilon \in \mathsf{Nor}_n$, we define the valency $v(x)$ to be the smallest of the labels in the subtree of Υ rooted at x. We define a *valency inversion* in $\Upsilon \in \mathsf{Nor}_n$ to be a pair of internal nodes (x, y) such that:

- x is in the subtree rooted at the left child of y,
- $v(R(x)) < v(R(y))$.

Let $\mathrm{valinv}(\Upsilon)$ denote the number of valency inversions in Υ. Note for example that a Lyndon tree is a normalized binary tree such that $\mathrm{valinv}(\Upsilon) = 0$.

A *coloring inversion* is a pair of internal nodes (x, y) in Υ such that

- $v(x) = v(y)$,
- x is in the subtree rooted at the left child of y,
- $\mathbf{color}(x) < \mathbf{color}(y)$.

We denote by $\mathrm{colinv}(\Upsilon)$, the number of coloring inversions in Υ.

Define the *inversion pair* of Υ to be $(\mathrm{valinv}(\Upsilon), \mathrm{colinv}(\Upsilon))$. We order these pairs lexicographically; that is, we say

$$(\mathrm{valinv}(\Upsilon), \mathrm{colinv}(\Upsilon)) < (\mathrm{valinv}(\Upsilon'), \mathrm{colinv}(\Upsilon')),$$

if either $\mathrm{valinv}(\Upsilon) < \mathrm{valinv}(\Upsilon')$ or $\mathrm{valinv}(\Upsilon) = \mathrm{valinv}(\Upsilon')$ and $\mathrm{colinv}(\Upsilon) < \mathrm{colinv}(\Upsilon')$. Note that if the inversion pair of Υ is $(0,0)$ then Υ is a colored Lyndon tree since in particular its underlying uncolored tree is a Lyndon tree.

Now let $\Upsilon \in \mathcal{BT}_\mu$ be a colored normalized binary tree that is not a colored Lyndon tree. Then Υ must have a subtree of the form: $(\Upsilon_1 \overset{i}{\wedge} \Upsilon_2) \overset{j}{\wedge} \Upsilon_3$, with $v(\Upsilon_2) < v(\Upsilon_3)$ and $i \leq j$. We will show that $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains associated with colored normalized binary trees with smaller inversion pair.

Case $i = j$**:** Using relation (2.2.2) (and relation(2.2.1)) we have that

$$\bar{c}(\alpha((\Upsilon_1 \overset{i}{\wedge} \Upsilon_2) \overset{i}{\wedge} \Upsilon_3)\beta) = \pm \bar{c}(\alpha(\Upsilon_1 \overset{i}{\wedge} (\Upsilon_2 \overset{i}{\wedge} \Upsilon_3))\beta) \pm \bar{c}(\alpha((\Upsilon_1 \overset{i}{\wedge} \Upsilon_3) \overset{i}{\wedge} \Upsilon_2)\beta).$$

(The signs in the relations of Theorem 2.2.4 are not relevant here and have therefore been suppressed.)

Let $\mathrm{p}(\Upsilon_j)$ denote the parent of the root of the subtree Υ_j in Υ. We then have that

$$\mathrm{valinv}(\alpha((\Upsilon_1 \overset{i}{\wedge} \Upsilon_2) \overset{i}{\wedge} \Upsilon_3)\beta) - \mathrm{valinv}(\alpha(\Upsilon_1 \overset{i}{\wedge} (\Upsilon_2 \overset{i}{\wedge} \Upsilon_3))\beta) \geq 1,$$

since the pair $(\mathrm{p}(\Upsilon_2), \mathrm{p}(\Upsilon_3))$ and any other valency inversion between an internal node of Υ_1 and $\mathrm{p}(\Upsilon_3)$ are valency inversions in the former tree but not in the later and no other change occurs to the set of valency inversions. We also have that

$$\operatorname{valinv}(\alpha((\Upsilon_1{}^{i}_{\wedge}\Upsilon_2){}^{i}_{\wedge}\Upsilon_3)\beta) - \operatorname{valinv}(\alpha((\Upsilon_1{}^{i}_{\wedge}\Upsilon_3){}^{i}_{\wedge}\Upsilon_2)\beta) \geq 1,$$

since the pair $(\mathrm{p}(\Upsilon_2), \mathrm{p}(\Upsilon_3))$ and any other valency inversion between an internal node of Υ_2 and $\mathrm{p}(\Upsilon_3)$ are valency inversions in the former tree but not in the later and no other change occurs to the set of valency inversions.

Case $i < j$: Using relation (2.2.3) (and relation (2.2.1)) we have that

$$\begin{aligned}
\bar{c}(\alpha((\Upsilon_1{}^{i}_{\wedge}\Upsilon_2){}^{j}_{\wedge}\Upsilon_3)\beta) = &\pm \bar{c}(\alpha((\Upsilon_1{}^{j}_{\wedge}\Upsilon_2){}^{i}_{\wedge}\Upsilon_3)\beta) \\
&\pm \bar{c}(\alpha(\Upsilon_1{}^{j}_{\wedge}(\Upsilon_2{}^{i}_{\wedge}\Upsilon_3))\beta) \\
&\pm \bar{c}(\alpha(\Upsilon_1{}^{i}_{\wedge}(\Upsilon_2{}^{j}_{\wedge}\Upsilon_3))\beta) \\
&\pm \bar{c}(\alpha((\Upsilon_1{}^{i}_{\wedge}\Upsilon_3){}^{j}_{\wedge}\Upsilon_2)\beta) \\
&\pm \bar{c}(\alpha((\Upsilon_1{}^{j}_{\wedge}\Upsilon_3){}^{i}_{\wedge}\Upsilon_2)\beta).
\end{aligned}$$

Just as in the previous case, all the labeled colored trees on the right hand side of the equation, except for the first, have a smaller number of valency inversions than that of the tree in the left hand side. The first labeled colored tree $\bar{c}(\alpha((\Upsilon_1{}^{j}_{\wedge}\Upsilon_2){}^{i}_{\wedge}\Upsilon_3)\beta)$ has the same number of valency inversions as that of $c(\alpha((\Upsilon_1{}^{i}_{\wedge}\Upsilon_2){}^{j}_{\wedge}\Upsilon_3)\beta)$. However the coloring inversion number is reduced by one and so the inversion pair is reduced.

From the two cases above we conclude that if $\Upsilon \in \mathcal{BT}_\mu$ is a colored normalized binary tree then $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains, associated to colored normalized binary trees, of smaller inversion pair. Hence by induction on the inversion pair, $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of the form $\bar{c}(\Upsilon')$ where $\Upsilon' \in \mathsf{Lyn}_\mu$. Also since by relation (2.2.1) any $\Upsilon \in \mathcal{BT}_\mu$ is of the form $\pm\bar{c}(\Upsilon')$, where Υ' is a colored normalized binary tree, the same is true for any $\Upsilon \in \mathcal{BT}_\mu$.

Since the set $\{\bar{c}(\Upsilon) \mid \Upsilon \in \mathcal{BT}_\mu\}$ is a spanning set for $\widetilde{H}^{n-3}((\hat{0}, [n]^\mu))$, we have shown using only the relations in Theorem 2.2.4 that $\{\bar{c}(\Upsilon) \mid \Upsilon \in \mathsf{Lyn}_\mu\}$ is also a spanning set for $\widetilde{H}^{n-3}((\hat{0}, [n]^\mu))$. The fact that $\{\bar{c}(\Upsilon) \mid \Upsilon \in \mathsf{Lyn}_\mu\}$ is a basis (Theorem 6.1.1), proves the result. $\square$

6.2 Colored comb basis

A *colored comb* is a normalized colored binary tree that satisfies the following coloring restriction: for each internal node x whose right child $R(x)$ is not a leaf,

$$\mathbf{color}(x) > \mathbf{color}(R(x)). \tag{6.2.1}$$

Let Comb_n be the set of colored combs in $\mathcal{BT}_n$ and Comb_μ be the set of the μ-colored ones. Note that in a monochromatic comb every right child has to be a leaf and hence they are the classical left combs that yield a basis for $\mathcal{L}ie(n)$ (see [44, Proposition 2.3]). Figure 6.1 illustrates the bicolored combs for $n = 3$.

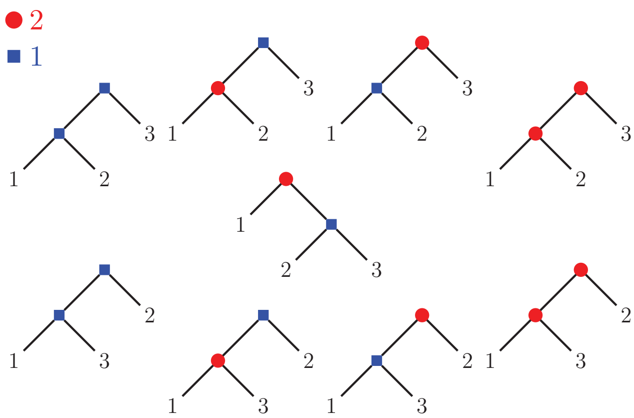


Figure 6.1: Set of bicolored combs for $n = 3$

Remark 6.2.1. Note that the coloring condition (6.2.1) is closely related to the comb type of a normalized tree defined in Chapter 1 before Theorem 1.6.4. The coloring condition implies that in a colored comb Υ there are no repeated colors in each block B of the partition $\pi^{\mathsf{Comb}}(\Upsilon)$ associated to Υ. So after choosing $|B|$ different colors for the internal nodes of Υ in B, there is a unique way to assign the colors such that Υ is a colored comb (the colors must decrease towards the right in each block of $\pi^{\mathsf{Comb}}(\Upsilon)$). In Figure 6.2 this relation is illustrated.

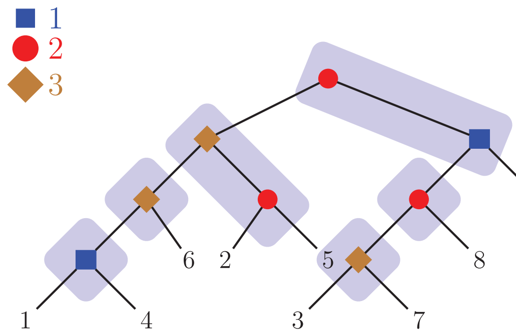


Figure 6.2: Example of a colored comb of comb type $(2, 2, 1, 1, 1, 1)$

Theorem 6.2.2. *There is a bijection*

$$\mathit{Lyn}_\mu \cong \mathit{Comb}_\mu.$$

Proof. This is a consequence of Corollary 5.3.5. Indeed, the bijection $\gamma^{-1}\xi\gamma$ that translates between the Lyndon type and comb type on Nor_n extends naturally to a bijection $\mathsf{Lyn}_\mu \cong \mathsf{Comb}_\mu$. $\square$

We obtain the following corollary from Corollary 4.2.3.

Corollary 6.2.3. *For all $n \geq 1$ and $\mu \in \mathrm{wcomp}_{n-1}$,*

$$\bar{\mu}_{\Pi_n^k}(\hat{0}, [n]^\mu) = (-1)^{n-1}|\mathit{Comb}_\mu|$$

and

$$\dim \widetilde{H}^{n-3}((\hat{0},[n]^{\mu})) = |\mathsf{Comb}_{\mu}|.$$

By Theorem 5.0.5 or Theorem 2.3.1 we have,

Corollary 6.2.4. *For all $n \geq 1$ and $\mu \in \mathrm{wcomp}_{n-1}$*

$$\dim \mathcal{L}ie(\mu) = |\mathsf{Comb}_{\mu}|.$$

Proposition 6.2.5. *The set $\{\bar{c}(T,\sigma) : (T,\sigma) \in \mathsf{Comb}_{\mu}\}$ spans $\tilde{H}^{n-3}((\hat{0},[n]^{\mu}))$, for all $\mu \in \mathrm{wcomp}_{n-1}$.*

Proof. We prove this result by "straightening" via the relations in Theorem 2.2.4. Define the *weight* $w(T)$ of a colored binary tree T to be

$$w(T) = \sum_{x \in I(T)} r(x),$$

where $I(T)$ is the set of internal nodes of T and $r(x)$ is the number of internal nodes in the right subtree of x. We say that a node y of T is a *right descendent* of a node x if y is a descent of x that can be reached from x along a path of right edges. Next we define an *inversion* of T to be a pair of internal nodes (x,y) of T such that y is a right descendent of x and $\mathbf{color}(x) < \mathbf{color}(y)$. Let $\mathrm{inv}(T)$ be the number of inversions of T. The *weight-inversion pair* of T is $(w(T),\mathrm{inv}(T))$. We order these pairs lexicographically, that is we say $(w(T),\mathrm{inv}(T)) < (w(T'),\mathrm{inv}(T'))$ if either $w(T) < w(T')$ or $w(T) = w(T')$ and $\mathrm{inv}(T) < \mathrm{inv}(T')$. For $\Upsilon = (T,\sigma) \in \mathcal{B}T_n$, let $w(\Upsilon) := w(T)$ and $\mathrm{inv}(\Upsilon) := \mathrm{inv}(T)$. Also define the weight-inversion pair of Υ to be that of T.

It follows from (2.2.1) that the chains of the form $\bar{c}(\Upsilon)$, where Υ is a normalized colored binary tree in $\mathcal{BT}_{\mu}$, span $\tilde{H}^{n-3}((\hat{0},[n]^{\mu}))$. Hence to prove the result we need only show that if $\Upsilon \in \mathcal{BT}_\mu$ is a normalized colored binary tree that is not a colored comb then $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of the form $\bar{c}(\Upsilon')$, where Υ' is a normalized colored binary tree in $\mathcal{BT}_\mu$ such that $(w(\Upsilon'), \mathrm{inv}(\Upsilon')) < (w(\Upsilon), \mathrm{inv}(\Upsilon))$ in lexicographic order. It will then follow by induction on the weight-inversion pair that $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of the form $\bar{c}(\Upsilon')$, where $\Upsilon' \in \mathsf{Comb}_\mu$.

Now let $\Upsilon \in \mathcal{BT}_\mu$ be a normalized colored binary tree that is not a colored comb. Then Υ must have a subtree of one of the following forms: $\Upsilon_1 \overset{j}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3)$ or $\Upsilon_1 \overset{i}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3)$ with $i < j \in [k]$. We will show that in all these cases $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains with a smaller weight-inversion pair.

Case 1: Υ has a subtree of the form $\Upsilon_1 \overset{j}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3)$. We can therefore express Υ as $\alpha(\Upsilon_1 \overset{j}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta$. Using relation (2.2.2) (and relation (2.2.1)) we have that

$$\bar{c}(\alpha(\Upsilon_1 \overset{j}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta) = \pm \bar{c}(\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_2) \overset{j}{\wedge} \Upsilon_3)\beta) \pm \bar{c}(\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_3) \overset{j}{\wedge} \Upsilon_2)\beta).$$

(The signs in the relations of Theorem 2.2.4 are not relevant here and have therefore been suppressed.)

It is easy to see that

$$\begin{aligned} w(\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_2) \overset{j}{\wedge} \Upsilon_3)\beta) &= w(\alpha((\Upsilon_1 \overset{j}{\wedge} \Upsilon_3) \overset{j}{\wedge} \Upsilon_2)\beta) \\ &= w(\alpha(\Upsilon_1 \overset{j}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3))\beta) - |I(\Upsilon_3)| - 1. \end{aligned}$$

Hence $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of smaller weight, and therefore of smaller weight-inversion pair.

Case 2: Υ has a subtree of the form $\Upsilon_1 \overset{i}{\wedge} (\Upsilon_2 \overset{j}{\wedge} \Upsilon_3)$. Using relation (2.2.3) (and relation (2.2.1)) we have that

$$\begin{aligned}
\bar{c}(\alpha(\Upsilon_1{}^{i}_{\wedge}(\Upsilon_2{}^{j}_{\wedge}\Upsilon_3))\beta) &= \pm\bar{c}(\alpha(\Upsilon_1{}^{j}_{\wedge}(\Upsilon_2{}^{i}_{\wedge}\Upsilon_3))\beta) \\
&\quad \pm\bar{c}(\alpha((\Upsilon_1{}^{i}_{\wedge}\Upsilon_2){}^{j}_{\wedge}\Upsilon_3)\beta) \\
&\quad \pm\bar{c}(\alpha((\Upsilon_1{}^{j}_{\wedge}\Upsilon_2){}^{i}_{\wedge}\Upsilon_3)\beta) \\
&\quad \pm\bar{c}(\alpha((\Upsilon_1{}^{i}_{\wedge}\Upsilon_3){}^{j}_{\wedge}\Upsilon_2)\beta) \\
&\quad \pm\bar{c}(\alpha((\Upsilon_1{}^{j}_{\wedge}\Upsilon_3){}^{i}_{\wedge}\Upsilon_2)\beta).
\end{aligned}$$

Just as in Case 1, all the labeled colored trees on the right hand side of the equation, except for the first, have weight smaller than that of $\alpha(\Upsilon_1{}^{i}_{\wedge}(\Upsilon_2{}^{j}_{\wedge}\Upsilon_3))\beta$. The first labeled colored tree $\alpha(\Upsilon_1{}^{j}_{\wedge}(\Upsilon_2{}^{i}_{\wedge}\Upsilon_3))\beta$ has the same weight as that of $\alpha(\Upsilon_1{}^{i}_{\wedge}(\Upsilon_2{}^{j}_{\wedge}\Upsilon_3))\beta$. However the inversion number is reduced, that is

$$\mathrm{inv}(\alpha(\Upsilon_1{}^{j}_{\wedge}(\Upsilon_2{}^{i}_{\wedge}\Upsilon_3))\beta) = \mathrm{inv}(\alpha(\Upsilon_1{}^{i}_{\wedge}(\Upsilon_2{}^{j}_{\wedge}\Upsilon_3))\beta) - 1.$$

Hence the weight-inversion pair for the first colored labeled tree is less than that of $\Upsilon := \alpha(\Upsilon_1{}^{i}_{\wedge}(\Upsilon_2{}^{j}_{\wedge}\Upsilon_3))\beta$ just as it is for the other colored labeled trees on the right hand side of the equation. We conclude that $\bar{c}(\Upsilon)$ can be expressed as a linear combination of chains of smaller weight-inversion pair. $\square$

Theorem 6.2.6. *$\{\bar{c}(T,\sigma) \mid (T,\sigma) \in \mathsf{Comb}_\mu\}$ is a basis for $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$ and $\{[T,\sigma] \mid (T,\sigma) \in \mathsf{Comb}_\mu\}$ is a basis for $\mathcal{L}ie(\mu)$.*

Proof. Corollary 6.2.3 and Proposition 6.2.5 imply that the set $\{\bar{c}(T,\sigma) \mid (T,\sigma) \in \mathsf{Comb}_\mu\}$ is a basis for $\widetilde{H}^{n-3}((\hat{0},[n]^\mu))$. Then, Theorem 2.3.1 implies that $\{[T,\sigma] \mid (T,\sigma) \in \mathsf{Comb}_\mu\}$ is a basis for $\mathcal{L}ie(\mu)$. $\square$

6.3 Bases for cohomology of the full weighted partition poset

In this section we use colored combs and colored Lyndon trees to construct bases for $\tilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$.

For a chain c in Π_n^k, let

$$\breve{c} := c \setminus \{\hat{0}\}.$$

The codimension 1 chains of $\Pi_n^k \setminus \{\hat{0}\}$ are of the form $\breve{c}$, where c is either

1. unrefinable in some maximal interval $[\hat{0}, [n]^\mu]$ except between one pair of adjacent elements $x < y$, where $[x, y]$ is an interval of length 2 in $[\hat{0}, [n]^\mu]$, or
2. unrefinable in $[\hat{0}, x]$, where x is a weighted partition of $[n]$ consisting of exactly two blocks.

The former case yields the cohomology relations of Types I, II and III given in Section 2.2, with $\bar{c}$ replaced by $\breve{c}$. The latter case yields the additional cohomology relation:

Type IV: The two blocks of x are $\mathbf{e_r}$-merged to get a single-block partition z_r. The open interval $(x, \hat{1})$ is equal to $\{z_1, z_2, \ldots, z_k\}$, see Figure 6.3. Hence the Type IV elementary cohomology relation is

$$(\breve{c} \cup \{z_1\}) + (\breve{c} \cup \{z_2\}) + \cdots + (\breve{c} \cup \{z_k\}) = 0.$$

The reader can verify, using the cohomology relations of Type I (with $\bar{c}$ replaced by, $\breve{c}$), that the proof of Lemma 2.2.3 goes through for $\tilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$. Hence $\tilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$ is generated by chains of the form $\breve{c}(\Upsilon)$ where $\Upsilon \in \mathcal{BT}_n$. The reader can also check, using the relations of Types I, II, and III, that the relations

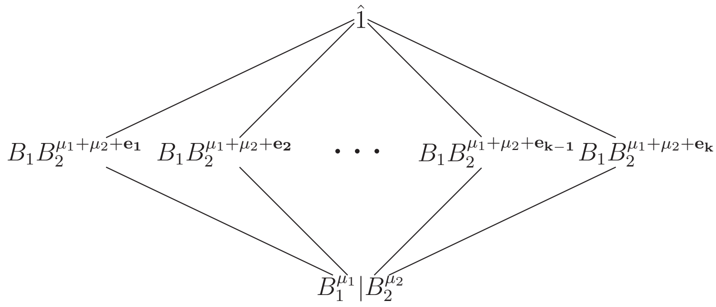


Figure 6.3: Type IV cohomology relation

in Theorem 2.2.4 hold (with $\bar{c}$ replaced by $\breve{c}$). It follows from the cohomology relation of Type IV that

$$\breve{c}(\Upsilon_1 {}^1_\wedge \Upsilon_2) + \breve{c}(\Upsilon_1 {}^2_\wedge \Upsilon_2) + \cdots + \breve{c}(\Upsilon_1 {}^k_\wedge \Upsilon_2) = 0. \tag{6.3.1}$$

for all $A \subseteq [n]$ and for all $\Upsilon_1 \in \mathcal{BT}_A$ and $\Upsilon_2 \in \mathcal{BT}_{[n]\setminus A}$.

Denote the root of a colored binary tree T by $\mathrm{root}(T)$, and define

$$\mathcal{BT}_n^k := \bigcup_{\substack{\mu \in \mathrm{wcomp}_{n-1} \\ \mathrm{supp}(\mu) \subseteq [k]}} \mathcal{BT}_\mu,$$

$$\mathsf{Comb}_n^k := \bigcup_{\substack{\mu \in \mathrm{wcomp}_{n-1} \\ \mathrm{supp}(\mu) \subseteq [k]}} \mathsf{Comb}_\mu,$$

$$\mathsf{Lyn}_n^k := \bigcup_{\substack{\mu \in \mathrm{wcomp}_{n-1} \\ \mathrm{supp}(\mu) \subseteq [k]}} \mathsf{Lyn}_\mu.$$

Theorem 6.3.1. *The set*

$$\{\breve{c}(T,\sigma) \mid (T,\sigma) \in \mathsf{Lyn}_n^k,\ \mathbf{color}(\mathrm{root}(T)) \neq 1\}$$

is a basis for $\widetilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$.

Proof. From the EL-labeling of Theorem 4.1.4 we have that all the maximal chains of $\widehat{\Pi_n^k}$ have last label $(1, n+1)^1$. Then for a maximal chain to be ascent-free it must have a second to last label of the form $(1, a)^j$ for $a \in [n]$ and $j \in [k] \setminus \{1\}$. By Theorem 4.2.2, we see that the ascent-free chains correspond to colored Lyndon trees such that the color of the root is different from 1. It therefore follows from Theorem 4.1.5 and Lemma 2.2.3 (with $\bar{c}$ replaced by $\breve{c}$) that the set is a basis for $\tilde{H}^{n-2}(\Pi_n^k \setminus \hat{0})$. $\square$

Proposition 6.3.2. *The set*

$$\{\breve{c}(T, \sigma) \mid (T, \sigma) \in \mathit{Comb}_n^k, \ \mathbf{color}(\mathrm{root}(T)) \neq k\}$$

spans $\widetilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$.

Proof. We prove, by induction on the number $k(\Upsilon)$ of internal nodes of Υ with color k, that if Υ is a normalized tree in $\mathcal{BT}_n^k$ then $\breve{c}(\Upsilon)$ can be expressed as a linear combination of chains of the form $\breve{c}(\Upsilon')$, where Υ' is a colored comb such that $\mathbf{color}(\mathrm{root}(\Upsilon')) \in [k-1]$. Note that if $k(\Upsilon) = 0$, using the relations in Theorem 2.2.4 (with $\bar{c}$ replaced by $\breve{c}$), we can use the straightening algorithm in the proof of Proposition 6.2.5 to express $\breve{c}(\Upsilon)$ as a linear combination of chains of the form $\breve{c}(\Upsilon')$, where Υ' is a colored comb with $k(\Upsilon') = 0$. Now let Υ be any normalized tree in $\mathcal{BT}_n^k$ with $k(\Upsilon)$ internal nodes colored k. Again, using the relations in Theorem 2.2.4, we can express $\breve{c}(\Upsilon)$ as a linear combination of chains of the form $\breve{c}(\Upsilon')$, where Υ' is a colored comb. If Υ' has its root colored k, we can use relation (6.3.1) to express $\breve{c}(\Upsilon')$ as a linear combination of chains associated with trees Υ'_i that look like Υ' except that $\mathbf{color}(\mathrm{root}(\Upsilon'_i)) = i \in [k-1]$. Then $k(\Upsilon'_i) = k(\Upsilon) - 1$ and by induction, each $\breve{c}(\Upsilon_i)$ is a linear combination of chains associated with colored combs whose roots have a color in $[k-1]$. The same is thus true for each $\check{c}(\Upsilon')$ and for $\check{c}(\Upsilon)$. Hence $\{\check{c}(T,\sigma) \mid (T,\sigma) \in \mathsf{Comb}_n^k,\ \mathrm{col}(\mathrm{root}(T)) \neq k\}$ spans. $\square$

Theorem 6.3.3. *The set*

$$\{\check{c}(T,\sigma) \mid (T,\sigma) \in \mathit{Comb}_n^2,\ \mathbf{color}(\mathrm{root}(T)) = 1\}$$

is a basis of $\widetilde{H}^{n-2}(\Pi_n^w \setminus \{\hat{0}\})$.

Proof. Note that Corollaries 4.1.10 and 6.2.3 together imply that $|\mathsf{Comb}_n^2| = n^{n-1}$ from every n. To construct a bicolored comb $T \in \mathsf{Comb}_n^2$, such that $\mathbf{color}(\mathrm{root}(T)) = 1$, we can choose the right subtree, which is a leaf, in $n-1$ different ways, and the left subtree, which is a bicolored comb in Comb_{n-1}^2, in $(n-1)^{n-2}$ different ways. Since by Corollary 4.1.10 we know that $(n-1)^{n-1}$ is the dimension of $\widetilde{H}^{n-2}(\Pi_n^w \setminus \{\hat{0}\})$, the theorem follows from Proposition 6.3.2. $\square$

Conjecture 6.3.4. *The set*

$$\{\check{c}(T,\sigma) \mid (T,\sigma) \in \mathit{Comb}_n^k,\ \mathbf{color}(\mathrm{root}(T)) \neq k\}$$

forms a basis for $\widetilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$.

We can combine Theorem 6.3.1 and the exact same idea of the proof of Proposition 6.1.2 to show that the set $B = \{\check{c}(T,\sigma) \mid (T,\sigma) \in \mathsf{Lyn}_n^k,\ \mathbf{color}(\mathrm{root}(T)) \neq k\}$ spans $\tilde{H}^{n-2}(\Pi_n^k \setminus \{\hat{0}\})$ by using only the relations of Theorem 2.2.4 and relation (6.3.1). We conclude that these are the only relations in a presentation of $\tilde{H}^{n-3}(\Pi_n^k \setminus \{\hat{0}\})$ since B is a basis. We summarize with the following result.

Theorem 6.3.5. *The set* $\{\check{c}(\Upsilon) : \Upsilon \in \mathcal{BT}_n^k\}$ *is a generating set for* $\tilde{H}^{n-3}(\Pi_n^k \setminus \{\hat{0}\})$, *subject only to the relations of Theorem 2.2.4 (with* $\bar{c}$ *replaced by* $\check{c}$*) and relation (6.3.1).*

6.4 Splitting tree basis for homology for the $k = 2$ case

In this section we construct a basis for homology of each maximal interval $(\hat{0}, [n]^i)$ of Π_n^w, which generalizes Bjorner's NBC basis for Π_n. Our basis consists of certain naturally constructed fundamental cycles. To prove that these fundamental cycles form a basis, we make use of a generalization of the Lyndon basis due to Liu [29], which is different from our colored Lyndon basis. The basis we use is actually a twisted version of the one in [29] and has an easier description; the two bases are related by a simple bijection.

For convenience, in this section we refer to the colors 1 and 2 as blue and red respectively.

6.4.1 Liu's bicolored Lyndon basis

We need to define a different valency from that of the previous sections. This valency is referred to in [29] as the *graphical root*. Recall that given an internal node x of a binary tree, $L(x)$ denotes the left child of x and $R(x)$ denotes the right child. For each node x of a bicolored labeled binary tree (T, σ), define its *valency* $v(x)$ recursively as follows:

$$v(x) = \begin{cases} \text{label of } x & \text{if } x \text{ is a leaf} \\ \min\{v(L(x)), v(R(x))\} & \text{if } x \text{ is a blue internal node} \\ \max\{v(L(x)), v(R(x))\} & \text{if } x \text{ is a red internal node.} \end{cases}$$

A *Liu-Lyndon tree* is a bicolored labeled binary tree (T, σ) such that for each internal node x of T,

1. $v(L(x)) = v(x)$

2. if x is blue and $L(x)$ is blue then

$$v(R(L(x))) > v(R(x))$$

3. if x is red then $L(x)$ is red or is a leaf; in the former case,

$$v(R(L(x))) < v(R(x)).$$

Note that condition (1) is equivalent to the condition that $v(L(x)) < v(R(x))$ if x is blue and $v(L(x)) > v(R(x))$ if x is red. Note also that every subtree of a Liu-Lyndon tree is a Liu-Lyndon tree. The set of Liu-Lyndon trees for $n = 3$ is depicted in Figure 6.4.

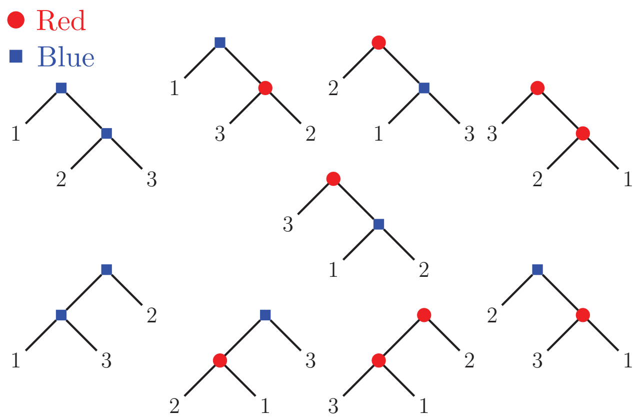


Figure 6.4: Set of Liu-Lyndon trees for $n = 3$

Let $\mathsf{Liu}_{n,i}$ be the set of Liu-Lyndon trees in $\mathcal{BT}_{n,i}$. When $i = 0$, all internal nodes are blue and it follows from the definition that $\mathsf{Liu}_{n,0}$ is the set of Lyndon trees on n leaves. When $i = n - 1$, all internal nodes are red and it follows from the definition that $\mathsf{Liu}_{n,n-1}$ consists of labeled binary trees obtained from Lyndon trees by replacing each label j by label $n + 1 - j$.

In [29] Liu proves that $\{[T,\sigma] : (T,\sigma) \in \mathsf{Liu}_{n,i}\}$ is a basis for $\mathcal{L}ie_{n,i}$ by using a perfect pairing between $\mathcal{L}ie_{n,i}$ and another module that she constructs. In the next section, we will use the natural pairing between cohomology and homology of $(\hat{0},[n]^i)$ to prove this result.

We will need a bijection of Liu [29]. Let A be a finite subset of the positive integers and let $0 \leq i \leq |A|-1$. Extend the definitions of $\mathcal{T}_{n,i}$ and $\mathsf{Liu}_{n,i}$ by letting $\mathcal{T}_{A,i}$ be the set of rooted trees on node set A with i descents and $\mathsf{Liu}_{A,i}$ be the set of Liu-Lyndon trees with leaf label set A and i red internal nodes. Define $\psi : \mathcal{T}_{A,i} \to \mathsf{Liu}_{A,i}$ recursively as follows: if $|A| = 1$, let $\psi(T)$ be the labeled binary tree whose single leaf is labeled with the sole element of A. Now suppose $|A| > 1$ and $r_T \in A$ is the root of T. Let x be the smallest child of r_T that is larger than r_T. If no such node exists let x be the largest child of r_T. Let T_x be the subtree of T rooted at x and let $T \setminus T_x$ be the subtree of T obtained by removing T_x from T. Now let

$$\psi(T) = \psi(T \setminus T_x) \overset{\mathrm{col}}{\wedge} \psi(T_x),$$

where

$$\mathrm{col} = \begin{cases} \text{blue} & \text{if } x > r_T \\ \text{red} & \text{if } x < r_T. \end{cases}$$

It will be convenient to refer to descent edges of T (i.e., edges $\{x, p_T(x)\}$, where $x < p_T(x)$) as red edges, and nondescent edges (i.e., edges $\{x, p_T(x)\}$, where $x > p_T(x)$) as blue edges. Hence ψ takes blue edges to blue internal nodes and red edges to red internal nodes. Consequently $\psi(T) \in \mathcal{BT}_{A,i}$ if $T \in \mathcal{T}_{A,i}$. By induction we see that the valuation of the root of $\psi(T)$ is equal to the label of the root of T. It follows from this that $\psi(T) \in \mathsf{Liu}_{A,i}$. It is not difficult to describe the inverse of ψ and thereby prove the following result.

Proposition 6.4.1 ([29]). *For all finite sets A and $0 \leq i \leq |A|$, the map*

$$\psi : \mathcal{T}_{A,i} \to \mathsf{Liu}_{A,i}$$

is a well-defined bijection.

Remark 6.4.2. It follows from Corollary 4.1.10, Theorem 6.2.6, Theorem 6.1.1 and Proposition 6.4.1 that

$$|\mathcal{T}_{n,i}| = |\mathsf{Comb}_{(i,n-1-i)}| = |\mathsf{Lyn}_{(i,n-1-i)}| = |\mathsf{Liu}_{n,i}|.$$

It would be desirable to find nice bijections between the given sets like that of Proposition 6.4.1 and Theorem 6.2.2 when we let $\mu = (i, n-1-i)$. We leave open the problem of finding a bijection between $\mathcal{T}_{n,i}$ and $\mathsf{Comb}_{(i,n-1-i)}$ or $\mathsf{Lyn}_{(i,n-1-i)}$.

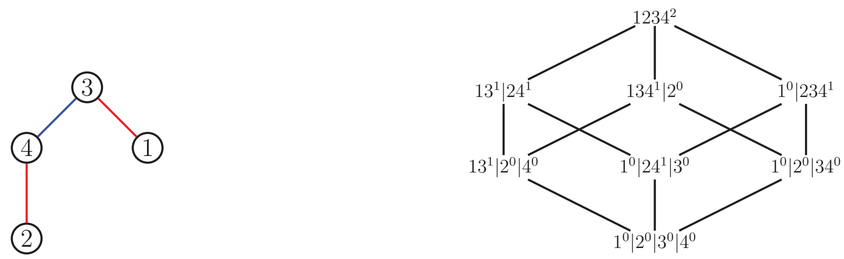
6.4.2 The tree basis for homology

We now present a generalization of Björner's NBC basis for homology of $\overline{\Pi}_n$ (see [7, Proposition 2.2]). Recall that in Section 3.2, we associated a weighted partition $\alpha(F)$ with each forest $F = \{T_1, \ldots, T_k\}$ on node set $[n]$, by letting

$$\alpha(F) = \{A_1^{w_1}, \ldots, A_k^{w_k}\},$$

where A_i is the node set of T_i and w_i is the number of descents of T_i.

Let T be a rooted tree on node set $[n]$. For each subset E of the edge set $E(T)$ of T, let T_E be the subgraph of T with node set $[n]$ and edge set E. Clearly T_E is a forest on $[n]$. We define Π_T to be the induced subposet of Π_n^w on the set $\{\alpha(T_E) : E \subseteq E(T)\}$. See Figure 6.5 for an example of Π_T. The poset Π_T is clearly isomorphic to the boolean algebra $\mathcal{B}_{n-1}$. Hence $\Delta(\overline{\Pi_T})$ is the barycentric subdivision of the boundary of the $(n-2)$-simplex. We let ρ_T denote a fundamental cycle of the spherical complex $\Delta(\overline{\Pi_T})$, that is, a generator of the unique nonvanishing integral simplicial homology of $\Delta(\overline{\Pi_T})$. Note that $\rho_T = \sum_{c \in \mathcal{M}(\Pi_T)} \pm \bar{c}$.



(a) T (b) Π_T

Figure 6.5: Example of a tree T with two descent edges (red edges) and the corresponding poset Π_T

The set $\{\rho_T : T \in \mathcal{T}_{n,0}\}$ is precisely the interpretation of the Björner NBC basis for homology of $\overline{\Pi}_n$ given in [44, Proposition 2.2], and the set $\{\rho_T : T \in \mathcal{T}_{n,n-1}\}$ is a variation of this basis. Björner's NBC basis is dual to the Lyndon basis $\{\bar{c}(\Upsilon) : \Upsilon \in \mathsf{Lyn}_n\}$ for cohomology of $\overline{\Pi}_n$ (using the natural pairing between homology and cohomology). While it is not true in general that $\{\rho_T : T \in \mathcal{T}_{n,i}\}$ is dual to any of the generalizations of the bases given in the previous sections, we are able to prove that it is a basis by pairing it with the Liu-Lyndon basis for cohomology.

Theorem 6.4.3. *The set* $\{\rho_T : T \in \mathcal{T}_{n,i}\}$ *is a basis for* $\tilde{H}_{n-3}((\hat{0}, [n]^i))$ *and the set* $\{\bar{c}(\Upsilon) : \Upsilon \in \mathsf{Liu}_{n,i}\}$ *is a basis for* $\tilde{H}^{n-3}((\hat{0}, [n]^i))$.

Our main tool in proving this theorem is Proposition A.3.2 (of the Appendix), which involves the bilinear form $\langle , \rangle$ defined in (A.3.1). In order to apply Proposition A.3.2 we need total orderings of the sets $\mathcal{T}_{n,i}$ and $\mathsf{Liu}_{n,i}$. Recall Liu's bijection $\psi : \mathcal{T}_{n,i} \to \mathsf{Liu}_{n,i}$ given in Proposition 6.4.1. We will show that any linear extension $\{T_1, T_2, \ldots, T_{|\mathcal{T}_{n,i}|}\}$ of a certain partial ordering on $\mathcal{T}_{n,i}$ provided by Liu

[29] yields a matrix $\langle \rho_{T_j}, \bar{c}(\psi(T_k))\rangle_{1\leq j,k\leq |\mathcal{T}_{n,i}|}$ that is upper-triangular with diagonal entries equal to ± 1. Theorem 6.4.3 will then follow from Proposition A.3.2 and Theorem 4.1.9 (2).

We define Liu's partial ordering $\leq_{\mathrm{Liu}}$ of $\mathcal{T}_{A,i}$ recursively. For $|A| \leq 2$, the set $\mathcal{T}_{A,i}$ has only one element. So assume that $|A| \geq 3$ and that $\leq_{\mathrm{Liu}}$ has been defined for all $\mathcal{T}_{B,j}$ where $|B| < |A|$. Let $T, T' \in \mathcal{T}_{A,i}$. We say that $T \preceq T'$ if there exist edges e of T and e' of T' such that the following conditions hold

- e and e' have the same color,
- e' contains the root of T',
- $\alpha(T_{E(T)\setminus\{e\}}) = \alpha(T'_{E(T')\setminus\{e'\}})$
- $T_1 \leq_{\mathrm{Liu}} T'_1$,
- $T_2 \leq_{\mathrm{Liu}} T'_2$,

where T_1 and T_2 are the connected components (trees) of the forest obtained by removing e from T, and T'_1 and T'_2 are the corresponding connected components (trees) of the forest obtained by removing e' from T'.

Now define $\leq_{\mathrm{Liu}}$ to be the transitive closure of the relation $\preceq$ on $\mathcal{T}_{A,i}$. It follows from [29, Lemma 8.12] that this relation is the same as the relation $\leq_{\mathrm{op}}$ that was defined in [29, Definition 7.11] and was proved to be a partial order in [29, Lemma 7.13].

Lemma 6.4.4. *Let $T, T' \in \mathcal{T}_{n,i}$ and let $\psi : \mathcal{T}_{n,i} \to \mathsf{Liu}_{n,i}$ be the bijection of Proposition 6.4.1. If $c(\psi(T')) \in \mathcal{M}(\Pi_T)$ then $T \leq_{Liu} T'$.*

Proof. First note that if $\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2$ is a bicolored labeled binary tree such that $c(\Upsilon_1 \overset{\mathrm{col}}{\wedge} \Upsilon_2)$ is a maximal chain in Π_T then there is an edge e of T whose color equals col and whose removal from T yields a forest whose connected components (trees) T_1 and T_2 satisfy: $c(\Upsilon_1)$ is a maximal chain in Π_{T_1} and $c(\Upsilon_2)$ is a maximal chain in Π_{T_2}.

Now recalling the definition of ψ, let x be the child of the root $r_{T'}$ of T', for which

$$\psi(T') = \psi(T' \setminus T'_x) \overset{\text{col}}{\wedge} \psi(T'_x),$$

where col equals the color of the edge $\{x, r_{T'}\}$. Let e be the edge of T whose removal yields the subtrees T_1 and T_2 such that $c(\psi(T' \setminus T'_x)) \in \mathcal{M}(\Pi_{T_1})$ and $c(\psi(T'_x)) \in \mathcal{M}(\Pi_{T_2})$. Then the color of e is the same as that of the edge $\{x, r_{T'}\}$. By induction we can assume that

$$T_1 \leq_{\text{Liu}} T' \setminus T'_x \quad \text{and} \quad T_2 \leq_{\text{Liu}} T'_x.$$

Since e and $e' := \{x, r_{T'}\}$ satisfy the conditions of the definition of $\preceq$, we have $T \preceq T'$, which implies the result. $\square$

Proof of Theorem 6.4.3. Let $T_1, \ldots, T_m$ be any linear extension of $\leq_{\text{Liu}}$ on $\mathcal{T}_{n,i}$, where $m = |\mathcal{T}_{n,i}|$. It follows from Lemma 6.4.4 that the matrix $M := \langle \rho_{T_j}, \bar{c}(\psi(T_k)) \rangle_{1 \leq j,k \leq m}$ is upper-triangular, where $\langle , \rangle$ is the bilinear form defined in (A.3.1). Since $c(\psi(T))$ is a maximal chain of Π_T for all $T \in \mathcal{T}_{n,i}$, the diagonal entries of M are equal to ± 1. Hence M is invertible over $\mathbb{Z}$ or any field. The result now follows from Propositions 6.4.1 and A.3.2 and Theorem 4.1.9 (2). $\square$

Remark 6.4.5. Theorems 2.3.1 and 6.4.3 yield an alternative proof of Liu's result that $\{[T, \sigma] : (T, \sigma) \in \mathsf{Liu}_{n,i}\}$ is a basis for $\mathcal{L}ie_{n,i}$.

Chapter 7

Whitney numbers and Whitney (co)homology

In this chapter we discuss weighted Whitney numbers and Whitney (co)homology of Π_n^w and Π_n^k.

7.1 Whitney numbers and weighted uniformity

Let P denote a pure poset with a minimum element $\hat{0}$. Denote by $Int(P)$ the set of closed intervals $[x, y]$ in the poset P. For some unitary commutative ring R (for example $\mathbf{k}[x]$ or $\mathbf{k}[x_1, \ldots, x_k]$) we say that a *weight function* $\varpi_P : Int(P) \to R$ is *P-compatible* if

- for any $\alpha \in P$, $\varpi_P(\alpha, \alpha) = 1$ and,
- $\theta \leq \alpha \leq \beta$ in P implies $\varpi_P(\theta, \beta) = \varpi_P(\theta, \alpha)\varpi_P(\alpha, \beta)$.

Equivalently, let $\mathbf{k}[Int(P)]$ be the unitary commutative algebra over $\mathbf{k}$ generated by intervals $[\alpha, \beta] \in Int(P)$ subject to the relations:

- $[\alpha, \alpha] = 1$ for any $\alpha \in P$, and

- $[\theta, \beta] = [\theta, \alpha][\alpha, \beta]$ for all $\theta \leq \alpha \leq \beta$ in P.

Then a P-compatible weight function is just an algebra homomorphism $\varpi_P : \mathbf{k}[Int(P)] \to R$. The poset Π_n^k has a natural Π_n^k-compatible weight function $\varpi_{\Pi_n^k}$. Indeed, we define the map $\varpi_{\Pi_n^k} : \mathbf{k}[Int(P)] \to \mathbf{k}[x_1, \ldots, x_k]$ by letting $\varpi_{\Pi_n^k}(\hat{0}, \hat{0}) = 1$ and $\varpi_{\Pi_n^k}(\hat{0}, \alpha) = x_1^{w(1)} \cdots x_k^{w(k)}$ for any $\alpha = \{A_1^{\mu_1}, \ldots, A_s^{\mu_s}\} \in \Pi_n^k$, with $w = \mu(\alpha) = \sum_{i=1}^{s} \mu_i$. This extends to any interval $[\alpha, \beta]$, by setting $\varpi_{\Pi_n^k}(\alpha, \beta) = \dfrac{\varpi_{\Pi_n^k}(\hat{0}, \beta)}{\varpi_{\Pi_n^k}(\hat{0}, \alpha)}$ (clearly a monomial in $\mathbf{k}[x_1, \ldots, x_k]$), and to $\mathbf{k}[Int(P)]$ by linearity.

The *weighted Whitney numbers* $w_j(P, \varpi_P)$ and $W_j(P, \varpi_P)$ of the first and second kind are defined as:

$$w_j(P, \varpi_P) = \sum_{\substack{\alpha \in P \\ \rho(\alpha) = j}} \overline{\mu}_P(\hat{0}, \alpha) \varpi_P(\hat{0}, \alpha),$$
$$W_j(P, \varpi_P) = \sum_{\substack{\alpha \in P \\ \rho(\alpha) = j}} \varpi_P(\hat{0}, \alpha).$$

Note that if ϖ_P is the trivial P-compatible function defined by $\varpi_P(\alpha, \alpha') = 1$ for all $\alpha \leq \alpha' \in P$, then $w_j(P) := w_j(P, \varpi_P)$ and $W_j(P) := W_j(P, \varpi_P)$ are the classical Whitney numbers of the first and second kind respectively.

Recall that for each $\alpha \in \Pi_n^k$, we have $\rho(\alpha) = n - |\alpha|$. For a partition $\lambda \vdash n$, with $\ell(\lambda) = r$ and where a part of size i occurs $m_i(\lambda)$ times, let $\lambda \setminus (1^r)$ denote the partition obtained from λ by decreasing each of its parts by 1. Recall the symmetric function

$$L_n(\mathbf{x}) := \sum_{\mu \in \mathrm{wcomp}_n} \dim \mathcal{L}ie(\mu)\, \mathbf{x}^{\mu}, \tag{7.1.1}$$

and for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, define

$$L_\lambda(\mathbf{x}) := L_{\lambda_1}(\mathbf{x}) \cdots L_{\lambda_r}(\mathbf{x}).$$

Note that $L_\lambda(\mathbf{x})$ is a homogeneous symmetric function of degree $|\lambda|$. Define $m(\lambda)! := \prod_{s=1}^{n} m_s(\lambda)!$.

Proposition 7.1.1. *For all $n \geq 1$, the weighted Whitney numbers of Π_n^k are given by*

$$w_r(\Pi_n^k, \varpi_{\Pi_n^k}) = (-1)^r \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=n-r}} \binom{n}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda \smallsetminus (1^{n-r})}(x_1, \ldots, x_k), \tag{7.1.2}$$

$$W_r(\Pi_n^k, \varpi_{\Pi_n^k}) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=n-r}} \binom{n}{\lambda} \frac{1}{m(\lambda)!} h_{\lambda \smallsetminus (1^{n-r})}(x_1, \ldots, x_k), \tag{7.1.3}$$

where h_λ denotes the complete homogeneous symmetric function associated with the partition λ.

Proof. We want to construct a weighted partition α that has underlying (unweighted) set partition $\pi \in \Pi_n$. For a block of size s in π, any monomial $x_1^{\mu(1)} \cdots x_k^{\mu(k)}$ with $|\mu| = s - 1$ is a valid weight, so the contribution of this block corresponds to the complete homogeneous symmetric polynomial $h_{s-1}(x_1, \ldots, x_k)$. Then π has a contribution of $h_{\lambda(\pi) \smallsetminus (1^{|\pi|})}$, where $\lambda(\pi)$ denotes the integer partition whose parts are equal to the block sizes of π, proving equation (7.1.3).

By Proposition 3.0.4, intervals of the form $[\hat{0}, \alpha]$ are isomorphic to products of maximal intervals of smaller copies of Π_n^k. Following a similar argument as in (7.1.3), using the fact that the Möbius function is multiplicative and Theorem 2.3.1, equation (7.1.2) follows. $\square$

Proposition 7.1.2. *For all $n \geq 1$, the Whitney numbers of Π_n^w are given by*

$$w_r(\Pi_n^w) = (-1)^r \binom{n-1}{r} n^r \tag{7.1.4}$$

$$W_r(\Pi_n^w) = \binom{n}{r} (n-r)^r. \tag{7.1.5}$$

Proof. Equation (7.1.4) follows from Theorem 3.3.1. To prove (7.1.5) let $R_n(r) = \{\alpha \in \Pi_n^w | \, \rho(\alpha) = r\}$. We need to show

$$|R_n(r)| = \binom{n}{n-r} (n-r)^r. \tag{7.1.6}$$

A weighted partition in $R_n(r)$ can be viewed as a partition of $[n]$ into $n-r$ blocks, with one element of each block marked (or distinguished). To choose such a partition, we first choose the $n-r$ marked elements. There are $\binom{n}{n-r}$ ways to choose these elements and place them in $n-r$ distinct blocks. To each of the remaining r elements we allocate one of these $n-r$ blocks. We can do this in $(n-r)^r$ ways. Hence (7.1.6) holds. □

Definition 7.1.3. A pure poset P of length ℓ with minimum element $\hat{0}$ and with rank function ρ, is said to be *uniform* if there is a family of posets $\{P_i \mid 0 \leq i \leq \ell\}$ such that for all $x \in P$, the upper order ideal $I_x := \{y \in P \mid x \leq y\}$ is isomorphic to P_i, where $i = \ell - \rho(x)$.

We refer to $(P_0, \ldots, P_\ell)$ as the associated *uniform sequence.* It follows from Proposition 3.0.4 that $P = \Pi_n^k$ is uniform with $P_i = \Pi_{i+1}^k$ for $i = 0, \ldots, n-1$.

Note that a P-compatible weight function ϖ_P induces, for any $x \in P$, an I_x-compatible weight function ϖ_{I_x}, the restriction of ϖ_P to $\mathbf{k}[Int(I_x)]$. For a uniform poset P, we say that a P-compatible weight function ϖ_P is *uniform* if for any two elements $x, y \in P$ such that $\rho(x) = \rho(y)$ and for any poset isomorphism $f : I_x \to$

I_y, the induced weight functions ϖ_{I_x} and ϖ_{I_y} satisfy $\varpi_{I_x}(z, z') = \varpi_{I_y}(f(z), f(z'))$ for all $z \leq z' \in I_x$. For example, the Π_n^k-compatible weight function $\varpi_{\Pi_n^k}$ defined before is uniform. It is clear that for a uniform poset P with associated uniform sequence $(P_0, \ldots, P_\ell)$ and uniform P-compatible weight function ϖ_P there is a well-defined induced P_i-compatible weight function ϖ_{P_i} for each i. The following proposition is a weighted version of a variant of [40, Exercise 3.130 (a)].

Proposition 7.1.4. *Let P be a uniform poset of length ℓ, with associated uniform sequence $(P_0, \ldots, P_\ell)$ and a uniform P-compatible weight function ϖ_P. Then the matrices $[w_{i-j}(P_i, \varpi_{P_i})]_{0\leq i,j\leq \ell}$ and $[W_{i-j}(P_i, \varpi_{P_i})]_{0\leq i,j\leq \ell}$ are inverses of each other.*

Proof. For a fixed $\alpha \in P$ with $\rho(\alpha) = \ell - i$ we have by the recursive definition of the Möbius function and the uniformity of P

$$\begin{aligned}
\delta_{i,j} &= \sum_{\substack{\beta\in P\\ \rho(\beta)=\ell-j}} \varpi_P(\alpha,\beta) \sum_{x\in[\alpha,\beta]} \overline{\mu}_P(\alpha,x)\\
&= \sum_{s=0}^{\ell} \sum_{\substack{x\in P\\ \rho(x)=\ell-s}} \overline{\mu}_P(\alpha,x)\varpi_P(\alpha,x) \sum_{\substack{\beta\geq x\\ \rho(\beta)=\ell-j}} \varpi_P(x,\beta)\\
&= \sum_{s=0}^{\ell} \sum_{\substack{\tilde{x}\in P_i\\ \rho(\tilde{x})=i-s}} \overline{\mu}_{P_i}(\hat{0},\tilde{x})\varpi_{P_i}(\hat{0},\tilde{x}) \sum_{\substack{\tilde{\beta}\in P_s\\ \rho(\beta)=s-j}} \varpi_{P_s}(\hat{0},\tilde{\beta})\\
&= \sum_{s=0}^{\ell} w_{i-s}(P_i,\varpi_{P_i})W_{s-j}(P_s,\varpi_{P_s}). \square
\end{aligned}$$

From the uniformity of the pair $(\Pi_n^k, \varpi_{\Pi_n^k})$ and Proposition 7.1.1, we have the following consequence of Proposition 7.1.4.

Corollary 7.1.5. *The matrices $A = \left[(-1)^{i-j} \sum_{\substack{\lambda\vdash i\\ \ell(\lambda)=j}} \binom{i}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda\setminus(1^j)}(\mathbf{x})\right]_{0\leq i,j\leq n-1}$ and $B = \left[\sum_{\substack{\lambda\vdash i\\ \ell(\lambda)=j}} \binom{i}{\lambda} \frac{1}{m(\lambda)!} h_{\lambda\setminus(1^j)}(\mathbf{x})\right]_{0\leq i,j\leq n-1}$ are inverses of each other.*

When $x_1 = x_2 = 1$ and $x_i = 0$ for $i \geq 3$, these matrices have a simpler form. From the uniformity of Π_n^w and Proposition 7.1.2, we have the following consequence of Proposition 7.1.4.

Theorem 7.1.6. *The matrices* $A = [(-1)^{i-j}\binom{i-1}{j-1}i^{i-j}]_{1\leq i,j\leq n}$ *and* $B = [\binom{i}{j}j^{i-j}]_{1\leq i,j\leq n}$ *are inverses of each other.*

This result is not new and an equivalent dual version (conjugated by the matrix $[(-1)^j\delta_{i,j}]_{1\leq i,j\leq n}$) was already obtained by Sagan in [34], also by using essentially Proposition 7.1.4, but with a completely different poset. So we can consider this to be a new proof of that result (see also [27]).

It can be shown that when $x_1 = 1$ and $x_i = 0$ for $i \geq 2$, Corollary 7.1.5 reduces to the following classical result since $\Pi_n^1 = \Pi_n$.

Theorem 7.1.7 (see [40])**.** *Let* $\mathbf{s}(i,j)$ *and* $\mathbf{S}(i,j)$ *denote respectively, the Stirling numbers of the first and of the second kind. The matrices* $A = [\mathbf{s}(i,j)]_{1\leq i,j\leq n}$ *and* $B = [\mathbf{S}(i,j)]_{1\leq i,j\leq n}$ *are inverses of each other.*

7.1.1 Relation with the poset of pointed partitions

Chapoton and Vallette [12] consider another poset that is quite similar to the poset of weighted partitions, namely the poset of pointed partitions. A pointed partition of $[n]$ is a partition of $[n]$ in which one element of each block is distinguished. The covering relation is given by

$$\{(A_1,a_1),(A_2,a_2),...,(A_s,a_s)\} \lessdot \{(B_1,b_1),(B_2,b_2),...,(B_t,b_t)\},$$

where a_i is the distinguished element of A_i and b_i is the distinguished element of B_i for each i, if the following conditions hold:

- $\{A_1, A_2, \ldots, A_s\} \lessdot \{B_1, B_2, \ldots, B_t\}$ in Π_n

- if $B_k = A_i \cup A_j$, where $i \neq j$, then $b_k \in \{a_i, a_j\}$
- if $B_k = A_i$ then $b_k = a_i$.

Let Π_n^p be the poset of pointed partitions of $[n]$. It is easy to see that there is a rank preserving bijection between Π_n^w and Π_n^p. It follows that both posets have the same Whitney numbers of the second kind. Since both posets are uniform, it follows from Proposition 7.1.4 that both posets have the same Whitney numbers of the first kind and thus the same characteristic polynomial. The following result of Chapoton and Vallette [12] is therefore equivalent to Theorem 3.3.1.

Corollary 7.1.8 (Chapoton and Vallette [12]). *For all $n \geq 1$, the characteristic polynomial of Π_n^p is given by*

$$\chi_{\Pi_n^p}(x) = (x-n)^{n-1}. \tag{7.1.7}$$

Consequently,

$$\mu_{\widehat{\Pi_n^p}}(\hat{0}, \hat{1}) = (-1)^n (n-1)^{n-1}.$$

One can also compute the Möbius function for all intervals of Π_n^p from (7.1.7). Indeed, since all n maximal intervals are isomorphic to each other, the Möbius invariant can be obtained from (7.1.7) by setting $x = 0$ and then dividing by n. This yields for all i,

$$(-1)^n \mu_{\widehat{\Pi_n^p}}(\hat{0}, ([n], i)) = n^{n-2},$$

which is the number of trees on node set $[n]$. The Möbius function on other intervals can be computed from this since all intervals of Π_n^p are isomorphic to products of maximal intervals of "smaller" posets of pointed partitions.

7.2 Whitney (co)homology

Whitney homology was introduced by Baclawski in [1] giving an affirmative answer to a question of Rota about the existence of a homology theory on the category of posets where the Betti numbers for geometric lattices are given by the Whitney numbers of the first kind. Whitney homology was later used to compute group representations on the homology of Cohen-Macaulay posets by Sundaram [42] and generalized to the non-pure case by Wachs [45] (see also [46]).

Whitney cohomology (over the field $\mathbf{k}$) of a poset P with a minimum element $\hat{0}$ can be defined for each integer r as follows:

$$WH^r(P) := \bigoplus_{x \in P} \widetilde{H}^{r-2}((\hat{0}, x); \mathbf{k}).$$

In the case of a Cohen-Macaulay poset this formula becomes

$$WH^r(P) := \bigoplus_{\substack{x \in P \\ \rho(x)=r}} \widetilde{H}^{r-2}((\hat{0}, x); \mathbf{k}). \tag{7.2.1}$$

Note that

$$\dim WH^r(P) = |w_r(P)|. \tag{7.2.2}$$

where $w_r(P)$ is the classical rth Whitney number of the first kind.

Define $\wedge^r \mathcal{L}ie_k(n)$ to be the multilinear component of the rth exterior power of the free Lie algebra on $[n]$ with k compatible brackets. From the definition of $\wedge^r \mathcal{L}ie_k(n)$ and equation (7.1.1) we can derive the following proposition.

Proposition 7.2.1. *For $0 \leq r \leq n-1$ and $k \geq 1$,*

$$\dim \wedge^r \mathcal{L}ie_k(n) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=r}} \binom{n}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda \setminus (1^r)}(1^k). \tag{7.2.3}$$

Consequently, if $\wedge \mathcal{L}ie_k(n)$ is the multilinear component of the exterior algebra of the free Lie algebra with k compatible brackets on n generators then

$$\dim \wedge \mathcal{L}ie_k(n) = \sum_{\lambda \vdash n} \binom{n}{\lambda} \frac{1}{m(\lambda)!} L_{\lambda \setminus (1^{\ell(\lambda)})}(1^k). \tag{7.2.4}$$

Equivalently,

$$\dim \wedge \mathcal{L}ie_k(n) = n![x^n] \exp\left(\sum_{i \geq 1} L_{i-1}(1^k) \frac{x^i}{i!} \right), \tag{7.2.5}$$

where $[x^n]F(x)$ denotes the coefficient of x^n in the formal power series $F(x)$ and $\exp(x) = \sum_{n \geq 1} \frac{x^n}{n!}$ (see [39, Theorem 5.1.4]).

Note that since, by equation (7.2.2), $\dim WH^r(\Pi_n^k)$ equals the signless rth Whitney number of the first kind $|w_r(\Pi_n^k)|$, Propositions 7.1.1 and 7.2.1 imply that the dimensions of $\dim WH^{n-r}(\Pi_n^k)$ and $\wedge^r \mathcal{L}ie_k(n)$ are equal.

Corollary 7.2.2. *For $0 \leq r \leq n-1$ and $k \geq 1$,*

$$\dim \wedge^r \mathcal{L}ie_k(n) = \dim WH^{n-r}(\Pi_n^k). \tag{7.2.6}$$

$$\dim \wedge \mathcal{L}ie_k(n) = \dim WH(\Pi_n^k). \tag{7.2.7}$$

where $WH(\Pi_n^k) := \oplus_{r \geq 0} WH^r(\Pi_n^k)$.

If a group G acts as a group of automorphisms on a poset P, this action induces a representation of G on $WH^r(P)$ for every r. Thus the action of $\mathfrak{S}_n$ on Π_n^k turns $WH^r(\Pi_n^k)$ into an $\mathfrak{S}_n$-module for each r. Moreover, the symmetric group $\mathfrak{S}_n$ acts naturally on $\wedge^r \mathcal{L}ie_k(n)$ giving it the structure of an $\mathfrak{S}_n$-module. We will present an equivariant verion of Corollary 7.2.2 in Theorem 7.2.3 below.

In [3] Barcelo and Bergeron proved the following $\mathfrak{S}_n$-module isomorphism for the poset of partitions:

$$WH^{n-r}(\Pi_n) \simeq_{\mathfrak{S}_n} \wedge^r \mathcal{L}ie_1(n) \otimes \mathrm{sgn}_n \,. \tag{7.2.8}$$

In [44] Wachs shows that an extension of her correspondence between generating sets of $\widetilde{H}^{n-3}(\overline{\Pi}_n)$ and $\mathcal{L}ie(n) \otimes \mathrm{sgn}_n$ can be used to prove this result. We use the same technique to prove:

$$WH^{n-r}(\Pi_n^k) \simeq_{\mathfrak{S}_n} \wedge^r \mathcal{L}ie_k(n) \otimes \mathrm{sgn}_n \,. \tag{7.2.9}$$

A *colored binary forest* is a sequence of colored binary trees. Given a colored binary forest F with n leaves and $\sigma \in \mathfrak{S}_n$, let (F, σ) denote the *labeled* colored binary forest whose ith leaf from left to right has label $\sigma(i)$. Let $\mathcal{BF}_{n,r}$ be the set of labeled colored binary forests with n leaves and r trees. If the jth labeled colored binary tree of (F, σ) is (T_j, σ_j) for each $j = 1, \ldots r$ then define

$$[F, \sigma] := [T_1, \sigma_1] \wedge \cdots \wedge [T_r, \sigma_r],$$

where now $\wedge$ denotes the wedge product operation in the exterior algebra. The set $\{[F, \sigma] : (F, \sigma) \in \mathcal{BF}_{n,r}\}$ is a generating set for $\wedge^r \mathcal{L}ie_k(n)$.

The set $\mathcal{BF}_{n,r}$ also provides a natural generating set for $WH^{n-r}(\Pi_n^k)$. For $(F, \sigma) \in \mathcal{BF}_{n,r}$, let $c(F, \sigma)$ be the unrefinable chain of Π_n^k whose rank i partition is obtained from its rank $i-1$ partition by col_i-merging the blocks L_i and R_i, where

col_i is the color of the ith postorder internal node v_i of F, and L_i and R_i are the respective sets of leaf labels in the left and right subtrees of v_i.

We have the following generalization of Theorem 2.3.1 and [44, Theorem 7.2]. The proof is similar to that of Theorem 2.3.1 and is left to the reader.

Theorem 7.2.3. *For each r, there is an $\mathfrak{S}_n$-module isomorphism*

$$\phi : \wedge^r \mathcal{L}ie_k(n) \to WH^{n-r}(\Pi_n^k) \otimes \mathrm{sgn}_n$$

determined by

$$\phi([F,\sigma]) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(F)\bar{c}(F,\sigma), \qquad (F,\sigma) \in \mathcal{BF}_{n,r},$$

where if F is the sequence $T_1, \dots, T_r$ of colored binary trees then

$$\mathrm{sgn}(F) := (-1)^{I(T_2)+I(T_4)+\cdots+I(T_{2\lfloor r/2\rfloor})}\,\mathrm{sgn}(T_1)\,\mathrm{sgn}(T_2)\dots\mathrm{sgn}(T_r).$$

When $k = 1$ it is proved in [3] that

$$\dim \wedge\mathcal{L}ie(n) = \dim WH(\Pi_n) = n!.$$

The case $k = 2$ can be derived from the discussion above.

Corollary 7.2.4. *For $0 \le r \le n-1$,*

$$\dim \wedge^{n-r}\mathcal{L}ie_2(n) = \dim WH^r(\Pi_n^w) = \binom{n-1}{r} n^r.$$

Moreover if $\wedge\mathcal{L}ie_2(n)$ is the multilinear component of the exterior algebra of the *free Lie algebra on n generators and $WH(\Pi_n^w) = \oplus_{r\geq 0} WH^r(\Pi_n^w)$ then*

$$\dim \wedge \mathcal{L}ie_2(n) = \dim WH(\Pi_n^w) = (n+1)^{n-1}.$$

Proof. Since $\dim WH^r(\Pi_n^w)$ equals the signless rth Whitney number of the first kind $|w_r(\Pi_n^w)|$, the result follows from Theorem 7.2.3, equation (7.1.4), and the binomial formula. $\square$

For a result that is closely related to Corollary 7.2.4, see [5, Theorem 2].

Chapter 8

The Frobenius characteristic of $\mathcal{L}ie(\mu)$

In this chapter we prove Theorem 1.6.6. We use a technique developed by Sundaram [42], and further developed by Wachs [45], to compute group representations on the (co)homology of Cohen-Macaulay posets using Whitney (co)homology. We introduce and develop first the concepts and results necessary to prove Theorem 1.6.6 in Sections 8.1 and 8.2; we give a proof of the theorem in Section 8.3. For information not presented here about symmetric functions, plethysm and the representation theory of the symmetric group see [31], [35], [25] and [39, Chapter 7].

8.1 Wreath product modules and plethysm

In the following we follow closely the exposition and the results in [45].

Let R be a commutative ring containing $\mathbb{Q}$ and let Λ_R denote the ring of symmetric functions with coefficients in R with variables $(y_1, y_2, \dots)$. The *power-*

sum symmetric functions p_k are defined by $p_0 = 1$ and

$$p_k = y_1^k + y_2^k + \cdots \text{ for } k \in \mathbb{P}.$$

For a partition $\lambda \vdash n$, p_λ denotes the power-sum symmetric function associated to λ, i.e., $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_{\ell(\lambda)}}$, where $\ell(\lambda)$ is the number of nonzero parts of λ. It is well-known that the set $\{p_\lambda \mid \lambda \vdash n\}$ is a basis for the component Λ_R^n of Λ_R consisting of homogeneous symmetric functions of degree n.

Let $\mathbb{Q}[[z_1, z_2, \dots]]$ be the ring of formal power series in variables $(z_1, z_2, \dots)$. If $g \in \mathbb{Q}[[z_1, z_2, \dots]]$ then *plethysm* $p_k[g]$ of p_k and g is defined as:

$$p_k[g] = g(z_1^k, z_2^k, \dots). \tag{8.1.1}$$

The definition of plethysm is then extended to p_λ multiplicatively and then to all of Λ_R linearly with respect to R.

It follows from (8.1.1), that if $f \in \Lambda_R^n$ and $g \in \mathbb{Q}[[z_1, z_2, \dots]]$, the following identity holds:

$$f[-g] = (-1)^n \omega(f)[g]. \tag{8.1.2}$$

where $\omega(\cdot)$ is the involution in Λ_R that maps $p_i(\mathbf{y})$ to $(-1)^{i-1} p_i(\mathbf{y})$.

For (perhaps empty) integer partitions ν and λ such that $\nu \subseteq \lambda$ (that is $\nu(i) \leq \lambda(i)$ for all i), let $S^{\lambda/\nu}$ denote the Specht module of shape λ/ν and $s_{\lambda/\nu}$ the Schur function of shape λ/ν. Recall that $s_{\lambda/\nu}$ is the image in the ring of symmetric functions of the specht module $S^{\lambda/\nu}$ under the Frobenius characteristic map ch, i.e., $\operatorname{ch} S^{\lambda/\nu} = s_{\lambda/\nu}$.

We will use the following standard results in the theory of symmetric functions and the representation theory of the symmetric group, respectively.

Proposition 8.1.1 (cf. [31] and [45]). *Let ν be a non empty integer partition and let $\{f_i\}_{i\geq 1}$ be a sequence of formal power series $f_i \in \mathbb{Z}[z_1, z_2, \dots]$ such that the sum $\sum_i f_i$ exists as a formal power series. Then*

$$s_\lambda \left[\sum_{i\geq 1} f_i\right] = \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_{j-1}\subsetneq\nu_j=\lambda \\ j\geq 1}} \prod_{i=1}^{j} s_{\nu_i/\nu_{i-1}}[f_i].$$

Proposition 8.1.2 (cf. [25] and [45]). *Let $\lambda \vdash \ell$ and let $(m_1, m_2, \dots, m_t)$ be a sequence of nonnegative integers whose sum is ℓ. Then the restriction of the $\mathfrak{S}_\ell$-module S^λ to the Young subgroup $\times_{i=1}^{t}\mathfrak{S}_{m_i}$ decomposes into a direct sum of outer tensor products of $\mathfrak{S}_{m_i}$-modules as follows,*

$$S^\lambda \downarrow^{\mathfrak{S}_\ell}_{\times\mathfrak{S}_{m_i}} = \bigoplus_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\lambda \\ |\nu_i|-|\nu_{i-1}|=m_i}} \bigotimes_{i=1}^{t} S^{\nu_i/\nu_{i-1}}.$$

Recall that the *wreath product* of the symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$, denoted $\mathfrak{S}_m[\mathfrak{S}_n]$, is the normalizer of the Young subgroup $\overbrace{\mathfrak{S}_n \times \cdots \times \mathfrak{S}_n}^{m \text{ times}}$ of $\mathfrak{S}_{mn}$. Each element of $\mathfrak{S}_m[\mathfrak{S}_n]$ can be represented as an $(m+1)$-tuple $(\sigma_1, \dots, \sigma_m; \tau)$ with $\tau \in \mathfrak{S}_m$ and $\sigma_i \in \mathfrak{S}_n$ for all $i \in [m]$.

From an $\mathfrak{S}_n$-module W we can construct a representation $\widetilde{W^{\otimes m}}$ of $\mathfrak{S}_m[\mathfrak{S}_n]$ on the vector space $W^{\otimes m} := \overbrace{W \otimes \cdots \otimes W}^{m \text{ times}}$ with action given by

$$(\sigma_1, \dots, \sigma_m; \tau)(w_1 \times \cdots \times w_m) := \sigma_1 w_{\tau^{-1}(1)} \times \cdots \times \sigma_m w_{\tau^{-1}(m)},$$

and from an $\mathfrak{S}_m$-module V we can construct a representation $\widehat{V}$ of $\mathfrak{S}_m[\mathfrak{S}_n]$ with action given by

$$(\sigma_1, \dots, \sigma_m; \tau)(v) := \tau v,$$

called the *pullback* of V from $\mathfrak{S}_m$ to $\mathfrak{S}_m[\mathfrak{S}_n]$. The *wreath product module* $V[W]$ of the $\mathfrak{S}_m$-module V and the $\mathfrak{S}_n$-module W is the $\mathfrak{S}_m[\mathfrak{S}_n]$-module

$$V[W] := \widetilde{W^{\otimes m}} \otimes \widehat{V}, \tag{8.1.3}$$

where $\otimes$ denotes inner tensor product.

Proposition 8.1.3 ([31]). *Let V be an $\mathfrak{S}_m$-module and W an $\mathfrak{S}_n$-module. Then*

$$\begin{aligned} \operatorname{ch}\left((V \otimes W)\uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}\right) &= \operatorname{ch} V \operatorname{ch} W, \\ \operatorname{ch}\left(V[W]\uparrow_{\mathfrak{S}_m[\mathfrak{S}_n]}^{\mathfrak{S}_{mn}}\right) &= \operatorname{ch} V[\operatorname{ch} W], \end{aligned}$$

where $\uparrow_*^*$ denotes induction.

8.2 Weighted integer partitions

Now let Φ be a finitary totally ordered set and let $||\cdot|| : \Phi \to \mathbb{P}$ be a map. We call a finite multiset $\tilde{\lambda} = (\tilde{\lambda}_1 \geq_\Phi \tilde{\lambda}_2 \geq_\Phi \cdots \geq_\Phi \tilde{\lambda}_j)$ of Φ a Φ-*partition* of *length* $\ell(\tilde{\lambda}) := j$. We also define $|\tilde{\lambda}| := \sum_j ||\tilde{\lambda}_j||$ and say that $\tilde{\lambda}$ is a Φ-partition of n if $|\tilde{\lambda}| = n$. Denote the set of Φ-partitions by $\mathsf{Par}(\Phi)$ and the set of Φ-partitions of length ℓ by $\mathsf{Par}_\ell(\Phi)$. For $\phi \in \Phi$, we denote by $m_\phi(\tilde{\lambda})$, the number of times ϕ appears in $\tilde{\lambda}$.

Let V be an $\mathfrak{S}_\ell$-module, W_ϕ be an $\mathfrak{S}_{||\phi||}$-module for each $\phi \in \Phi$ and $\tilde{\lambda}$ a Φ-partition with ℓ parts. Note that $\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$ is a finite product since $\tilde{\lambda}$ is a finite multiset. The module

$$\bigotimes_{\phi\in\Phi} \widetilde{W_\phi^{\otimes m_\phi(\tilde{\lambda})}} \otimes \widehat{V}^{\tilde{\lambda}},$$

is the inner tensor product of two $\widetilde{\times_{\phi\in\Phi}}\mathfrak{S}_{m_\phi(\tilde\lambda)}[\mathfrak{S}_{||\phi||}]$-modules. The first module is the outer tensor product $\bigotimes_{\phi\in\Phi}\widetilde{W_\phi^{\otimes m_\phi(\tilde\lambda)}}$ of the $\mathfrak{S}_{m_\phi(\tilde\lambda)}[\mathfrak{S}_{||\phi||}]$-modules $\widetilde{W_\phi^{\otimes m_\phi(\tilde\lambda)}}$ (cf. Section 8.1) and the second module is the pullback $\widehat{V}^{\tilde\lambda}$ of the restricted representation $V\downarrow^{\mathfrak{S}_\ell}_{\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde\lambda)}}$ to $\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde\lambda)}[\mathfrak{S}_{||\phi||}]$ through the product of the natural homomorphisms $\mathfrak{S}_{m_\phi(\tilde\lambda)}[\mathfrak{S}_{||\phi||}]\to\mathfrak{S}_{m_\phi(\tilde\lambda)}$ given by $(\sigma_1,\ldots,\sigma_{m_\phi(\tilde\lambda)};\tau)\mapsto\tau$.

The following theorem generalizes [45, Theorem 5.5] and the proof follows the same idea.

Theorem 8.2.1. *Let V be an $\mathfrak{S}_\ell$-module and W_ϕ be an $\mathfrak{S}_{||\phi||}$-module for each $\phi\in\Phi$. Then*

$$\sum_{\tilde\lambda\in\mathsf{Par}_\ell(\Phi)}\mathrm{ch}\left(\left(\bigotimes_{\phi\in\Phi}\widetilde{W_\phi^{\otimes m_\phi(\tilde\lambda)}}\otimes\widehat{V}^{\tilde\lambda}\right)\Bigg\uparrow^{\mathfrak{S}_{|\tilde\lambda|}}_{\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde\lambda)}[\mathfrak{S}_{||\phi||}]}\right)z_{\tilde\lambda}=\mathrm{ch}(V)\left[\sum_{\phi\in\Phi}\mathrm{ch}(W_\phi)z_\phi\right],$$

where z_ϕ are indeterminates with $z_{\tilde\lambda}:=z_{\tilde\lambda_1}\cdots z_{\tilde\lambda_\ell}$.

Proof. Note that restriction, induction, pullback, ch and plethysm in the outer component are all linear and inner tensor product is bilinear. Thus it is enough to prove the theorem for V equal to an irreducible $\mathfrak{S}_\ell$-module S^η (the Specht module associated to the partition $\eta\vdash\ell$). Since the set Φ is a finitary totally ordered set, we can denote by ϕ^i, the ith element in the total order of Φ. Consider $\tilde\lambda\in\mathsf{Par}_\ell(\Phi)$ and let $t:=\max\{i\mid\phi^i\in\tilde\lambda\}$. Now using Proposition 8.1.2 and the definition of a wreath product module in equation (8.1.3) yields

$$\begin{aligned}\bigotimes_{i=1}^{t}\widetilde{W_{\phi^i}^{\otimes m_{\phi^i}(\tilde\lambda)}}\otimes\widehat{S^\eta}^{\tilde\lambda}&=\bigotimes_{i=1}^{t}\widetilde{W_{\phi^i}^{\otimes m_{\phi^i}(\tilde\lambda)}}\otimes\left(\bigoplus_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\|\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde\lambda)}}\bigotimes_{i=1}^{t}\widehat{S^{\nu_i/\nu_{i-1}}}\right)\\&=\bigoplus_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\|\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde\lambda)}}\left(\bigotimes_{i=1}^{t}\widetilde{W_{\phi^i}^{\otimes m_{\phi^i}(\tilde\lambda)}}\otimes\bigotimes_{i=1}^{t}\widehat{S^{\nu_i/\nu_{i-1}}}\right)\end{aligned}$$

$$= \bigoplus_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \bigotimes_{i=1}^{t} \left(\widetilde{W_{\phi^i}^{\otimes m_{\phi^i}(\tilde{\lambda})}} \otimes \widehat{S^{\nu_i/\nu_{i-1}}} \right)$$

$$= \bigoplus_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \bigotimes_{i=1}^{t} S^{\nu_i/\nu_{i-1}}[W_{\phi^i}].$$

We induce and then apply the Frobenius characteristic map ch. Then using Proposition 8.1.3 and the transitivity property of induction of representations, we have that

$$\mathrm{ch}\left(\left(\bigotimes_{i=1}^{t} \widetilde{W_{\phi^i}^{\otimes m_{\phi^i}(\tilde{\lambda})}} \otimes \widehat{S^{\eta}}^{\tilde{\lambda}} \right) \Big\uparrow_{\times_{i=1}^{t}\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})}[\mathfrak{S}_{||\phi^i||}]}^{\mathfrak{S}_{|\tilde{\lambda}|}} \right)$$

$$= \mathrm{ch}\left(\bigoplus_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \left(\bigotimes_{i=1}^{t} S^{\nu_i/\nu_{i-1}}[W_{\phi^i}] \right) \Big\uparrow_{\times_{i=1}^{t}\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})}[\mathfrak{S}_{||\phi^i||}]}^{\mathfrak{S}_{|\tilde{\lambda}|}} \right)$$

$$= \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \mathrm{ch}\left(\left(\bigotimes_{i=1}^{t} S^{\nu_i/\nu_{i-1}}[W_{\phi^i}] \right) \Big\uparrow_{\times_{i=1}^{t}\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})}[\mathfrak{S}_{||\phi^i||}]}^{\mathfrak{S}_{|\tilde{\lambda}|}} \right)$$

$$= \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \mathrm{ch}\left(\left(\left(\bigotimes_{i=1}^{t} S^{\nu_i/\nu_{i-1}}[W_{\phi^i}] \right) \Big\uparrow_{\times_{i=1}^{t}\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})}[\mathfrak{S}_{||\phi^i||}]}^{\times_{i=1}^{t}\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})||\phi^i||}} \right) \Big\uparrow_{\times_{i=1}^{t}\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})||\phi^i||}}^{\mathfrak{S}_{|\tilde{\lambda}|}} \right)$$

$$= \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \mathrm{ch}\left(\left(\bigotimes_{i=1}^{t} \left(S^{\nu_i/\nu_{i-1}}[W_{\phi^i}] \Big\uparrow_{\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})}[\mathfrak{S}_{||\phi^i||}]}^{\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})||\phi^i||}} \right) \right) \Big\uparrow_{\times_{i=1}^{t}\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})||\phi^i||}}^{\mathfrak{S}_{|\tilde{\lambda}|}} \right)$$

$$= \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \prod_{i=1}^{t} \mathrm{ch}\left(S^{\nu_i/\nu_{i-1}}[W_{\phi^i}] \Big\uparrow_{\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})}[\mathfrak{S}_{||\phi^i||}]}^{\mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})||\phi^i||}} \right)$$

$$= \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \prod_{i=1}^{t} s_{\nu_i/\nu_{i-1}}[\mathrm{ch}\, W_{\phi^i}].$$

Now note that $s_{\nu_i/\nu_{i-1}}[\mathrm{ch}\, W_\phi z_\phi] = s_{\nu_i/\nu_{i-1}}[\mathrm{ch}\, W_\phi] z_\phi^{|\nu_i|-|\nu_{i-1}|}$ and that $s_{\nu_i/\nu_{i-1}} = s_\emptyset = 1$ if $\nu_i = \nu_{i-1}$. And using Proposition 8.1.1, we obtain

$$\sum_{\tilde{\lambda}\in \mathsf{Par}_\ell(\Phi)} \mathrm{ch}\left(\left(\widetilde{\bigotimes_{i\geq 1} W_{\phi^i}^{\otimes m_{\phi^i}(\tilde{\lambda})}} \otimes \widehat{S^\eta}^{\tilde{\lambda}}\right)\Bigg\uparrow_{\times_{i=1}^t \mathfrak{S}_{m_{\phi^i}(\tilde{\lambda})}[\mathfrak{S}_{||\phi^i||}]}^{\mathfrak{S}_{|\tilde{\lambda}|}}\right) z_{\tilde{\lambda}}$$

$$\begin{aligned}
&= \sum_{\tilde{\lambda}\in \mathsf{Par}_\ell(\Phi)} \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_t=\eta\\ |\nu_i|-|\nu_{i-1}|=m_{\phi^i}(\tilde{\lambda})}} \prod_{i=1}^t s_{\nu_i/\nu_{i-1}}[\mathrm{ch}\, W_{\phi^i} z_{\phi^i}]\\
&= \sum_{\substack{\emptyset=\nu_0\subseteq\nu_1\subseteq\cdots\subseteq\nu_{j-1}\subsetneq\nu_j=\eta\\ j\geq 1}} \prod_{i=1}^j s_{\nu_i/\nu_{i-1}}[\mathrm{ch}\, W_{\phi^i} z_{\phi^i}]\\
&= s_\eta\left[\sum_{i\geq 1} \mathrm{ch}\, W_{\phi^i} z_{\phi^i}\right]\\
&= \mathrm{ch}\, S^\eta\left[\sum_{\phi\in\Phi} \mathrm{ch}\, W_\phi z_\phi\right].\square
\end{aligned}$$

8.3 Using Whitney (co)homology to compute (co)homology

The technique of Sundaram [42] to compute characters of G-representations on the (co)homology of pure G-posets is based on the following result:

Lemma 8.3.1 ([42] Lemma 1.1)**.** *Let P be a bounded poset of length $\ell \geq 1$ and let G be a group of automorphisms of P. Then the following isomorphism of virtual G-modules holds*

$$\bigoplus_{i=0}^{\ell} (-1)^i WH^i(P) \cong_G 0.$$

Recall that if a group G of automorphisms acts on the poset P, this action induces a representation of G on $WH^r(P)$ for every r. From equation (7.2.1), when P is Cohen-Macaulay, $WH^r(P)$ breaks into the direct sum of G-modules

$$WH^r(P) \cong_P \bigoplus_{\substack{x \in P/\sim \\ \rho(x)=r}} \widetilde{H}^{r-2}((\hat{0}, x); \mathbf{k}) \big\uparrow_{G_x}^{G}, \tag{8.3.1}$$

where $P/\sim$ is a set of orbit representatives and G_x the stabilizer of x.

Let $\mu \in \mathrm{wcomp}_{n-1}$. We want to apply Lemma 8.3.1 to the dual poset $[\hat{0}, [n]^\mu]^*$ of the maximal interval $[\hat{0}, [n]^\mu]$, which by Theorem 1.6.2 is Cohen-Macaulay. In order to compute $WH^r([\hat{0}, [n]^\mu]^*)$, by equation (8.3.1), we need to specify a set of orbit representatives for the action of $\mathfrak{S}_n$ on $[\hat{0}, [n]^\mu]^*$. For this we consider the set

$$\Phi = \{\phi \in \mathrm{wcomp} \mid \mathrm{supp}(\phi) \subseteq [k]\}$$

and the map $||\phi|| := |\phi| + 1$ for $\phi \in \mathrm{wcomp}$ (cf. Section 8.2). We fix any finitary total order on Φ. For any Φ-partition $\tilde{\lambda}$ of n of length ℓ we denote by $\alpha_{\tilde{\lambda}}$, the weighted partition $\{A_1^{\tilde{\lambda}_1}, \dots, A_\ell^{\tilde{\lambda}_\ell}\}$ of $[n]$ whose blocks are of the form

$$A_i = \left[\sum_{j=1}^{i} ||\tilde{\lambda}_j||\right] \setminus \left[\sum_{j=1}^{i-1} ||\tilde{\lambda}_j||\right].$$

Recall that for $\nu, \mu \in \mathrm{wcomp}$, we say that $\mu \leq \nu$ if $\mu(i) \leq \nu(i)$ for every i and we denote by $\nu + \mu$, the weak composition defined by $(\nu+\mu)(i) := \nu(i) + \mu(i)$. Let

$$\mathsf{Par}^\mu(\Phi) := \{\tilde{\lambda} \in \mathsf{Par}(\Phi) \mid |\tilde{\lambda}| = ||\mu||\,, \sum_i \tilde{\lambda}_i \leq \mu\}.$$

It is not difficult to see that $\{\alpha_{\tilde{\lambda}} \mid \tilde{\lambda} \in \mathsf{Par}^\mu(\Phi)\}$ is a set of orbit representatives for the action of $\mathfrak{S}_n$ on $[\hat{0}, [n]^\mu]^*$. Indeed, any weighted partition $\beta \in [\hat{0}, [n]^\mu]^*$ can be obtained as $\beta = \sigma\alpha_{\tilde{\lambda}}$ for suitable $\tilde{\lambda} \in \mathsf{Par}^{\mu}(\Phi)$ and $\sigma \in \mathfrak{S}_n$. It is also clear that $\alpha_{\tilde{\lambda}} \neq \sigma\alpha_{\tilde{\lambda}'}$ for $\tilde{\lambda} \neq \tilde{\lambda}' \in \mathsf{Par}^{\mu}(\Phi)$ and for every $\sigma \in \mathfrak{S}_n$. Observe that the partition $\alpha_{\tilde{\lambda}}$ has stabilizer $\times_{\phi}\mathfrak{S}_{m_{\phi}(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$. By equation (8.3.1) applied to $[\hat{0}, [n]^{\mu}]^{*}$,

$$WH^{r}([\hat{0}, [n]^{\mu}]^{*}) \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\tilde{\lambda} \in \mathsf{Par}^{\mu}(\Phi) \\ \ell(\tilde{\lambda}) = r}} \widetilde{H}^{r-3}((\alpha_{\tilde{\lambda}}, [n]^{\mu}))\Big\uparrow_{\times_{\phi\in\Phi}\mathfrak{S}_{m_{\phi}(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]}^{\mathfrak{S}_n}. \tag{8.3.2}$$

Note that if $r = 2$ then the open interval $(\alpha_{\tilde{\lambda}}, [n]^{\mu})$ is the empty poset. Hence $\widetilde{H}^{r-3}((\alpha_{\tilde{\lambda}}, [n]^{\mu}))$ is isomorphic to the trivial representation of $\times_{\phi\in\Phi}\mathfrak{S}_{m_{\phi}(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$. If $r = 1$ then $\alpha_{\tilde{\lambda}} = [n]^{\mu}$. In this case we use the convention that $\widetilde{H}^{r-3}((\alpha_{\tilde{\lambda}}, [n]^{\mu}))$ is isomorphic to the trivial representation of $\times_{\phi\in\Phi}\mathfrak{S}_{m_{\phi}(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$ (see also Section A.3 of the Appendix).

We apply Lemma 8.3.1 together with equation (8.3.2) to obtain the following result.

Lemma 8.3.2. *For $n \geq 1$ and $\mu \in \mathrm{wcomp}_{n-1}$ we have the following $\mathfrak{S}_n$-module isomorphism*

$$\mathbf{1}_{\mathfrak{S}_n}\delta_{n,1} \cong_{\mathfrak{S}_n} \bigoplus_{\substack{\tilde{\lambda} \in \mathsf{Par}^{\mu}(\Phi) \\ |\tilde{\lambda}| = n}} (-1)^{\ell(\tilde{\lambda})-1}\widetilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}}, [n]^{\mu}))\Big\uparrow_{\times_{\phi\in\Phi}\mathfrak{S}_{m_{\phi}(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]}^{\mathfrak{S}_n}, \tag{8.3.3}$$

where $\mathbf{1}_{\mathfrak{S}_n}$ denotes the trivial representation of $\mathfrak{S}_n$.

Lemma 8.3.3. *For all $\tilde{\lambda} \in \mathsf{Par}(\Phi)$ with $|\tilde{\lambda}| = n$ and $\nu \in \mathrm{wcomp}_{\ell(\tilde{\lambda})-1}$, the following $\times_{\phi\in\Phi}\mathfrak{S}_{m_{\phi}(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$-module isomorphism holds:*

$$\widetilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}}, [n]^{\nu + \sum \tilde{\lambda}_j})) \cong \left(\bigotimes_{\phi\in\Phi} \widetilde{(\mathbf{1}_{\mathfrak{S}_{||\phi||}})^{\otimes m_{\phi}(\tilde{\lambda})}}\right) \otimes \widetilde{H}^{\ell(\tilde{\lambda})-3}\widehat{((\hat{0}, [\ell(\tilde{\lambda})]^{\nu}))}^{\tilde{\lambda}}.$$

Proof. The poset $[\hat{0},[\ell(\tilde{\lambda})]^{\nu}]$ is a $\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$-poset with the action given by the pullback through the product of the natural homomorphisms $\mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}] \rightarrow \mathfrak{S}_{m_\phi(\tilde{\lambda})}$. There is a natural poset isomorphism between $[\alpha_{\tilde{\lambda}},[n]^{\nu+\sum\tilde{\lambda}_j}]$ and $[\hat{0},[\ell(\tilde{\lambda})]^{\nu}]$. Indeed, for a weighted partition $\{B_1^{\mu_1},\ldots,B_t^{\mu_t}\} \geq \alpha_{\tilde{\lambda}} = \{A_1^{\tilde{\lambda}_1},\ldots,A_\ell^{\tilde{\lambda}_\ell}\}$, each weighted block $B_j^{\mu_j}$ is of the form $B_j = A_{i_1}\cup A_{i_2}\cup\ldots\cup A_{i_s}$ and $\mu_j = u_j + \sum_k \tilde{\lambda}_{i_k}$, where $|u_j| = s-1$ and $\sum_j u_j \leq \nu$. Let

$$\Gamma : [\alpha_{\tilde{\lambda}},[n]^{\nu+\sum\tilde{\lambda}_j}] \rightarrow [\hat{0},[\ell(\tilde{\lambda})]^{\nu}]$$

be the map such that $\Gamma(\{B_1^{\mu_1},\ldots,B_t^{\mu_t}\})$ is the weighted partition in which each weighted block $B_j^{\mu_j}$ is replaced by $\{i_1,i_2,\ldots,i_s\}^{u_j}$. The map Γ is an isomorphism of posets that commutes with the action of $\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$. The isomorphism of $\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$-posets induces an isomorphism of the $\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$-modules $\widetilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}},[n]^{\nu+\sum\tilde{\lambda}_j}))$ and $\widetilde{H}^{\ell(\tilde{\lambda})-3}\widehat{((\hat{0},[\ell(\tilde{\lambda})]^{\nu}))}^{\tilde{\lambda}}$. The result follows since $\bigotimes_{\phi\in\Phi}\widetilde{(\mathbf{1}_{\mathfrak{S}_{||\phi||}})}^{\otimes m_\phi(\tilde{\lambda})}$ is the trivial representation of $\times_{\phi\in\Phi}\mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$. □

Let R be the ring of symmetric functions $\Lambda_{\mathbb{Q}}$ in variables $\mathbf{x} = (x_1,x_2,\ldots)$. There is a natural inner product in Λ_R defined for arbitrary partitions λ and ν, by

$$\langle p_\lambda, p_\nu\rangle = \delta_{\lambda,\nu},$$

and then extended linearly to Λ_R. This inner product defines a notion of convergence. Indeed, for a sequence of symmetric functions $f_n \in \Lambda_R$, $n \geq 1$, we say that $\{f_n\}_{n\geq 1}$ converges if for every partition ν there is a number N such that $\langle f_n, p_\nu\rangle = \langle f_m, p_\nu\rangle$ whenever $n,m \geq N$. We use $\widehat{\Lambda_R}$ to denote the completion of the ring of Λ_R with respect to this topology. It is not difficult to verify that $\widehat{\Lambda_R}$ consists of the class of formal power series in two sets of variables, $\mathbf{x} = (x_1,x_2,\ldots)$ and $\mathbf{y} = (y_1,y_2,\ldots)$, that can be expressed as $\sum_\lambda c_\lambda(\mathbf{x})p_\lambda(\mathbf{y})$, where $c_\lambda(\mathbf{x}) \in \Lambda_{\mathbb{Q}}$.

Given a formal power series $F(\mathbf{y}) = \sum_\lambda c_\lambda(\mathbf{x})p_\lambda(\mathbf{y})$ in $\widehat{\Lambda_R}$ and a formal power series $g \in \mathbb{Q}[[z_1, z_2, \dots]]$, we can extend the definition of plethysm from symmetric functions to formal power series in $\widehat{\Lambda_R}$ by

$$F[g] := \sum_\lambda c_\lambda(\mathbf{x})p_\lambda[g].$$

The reader can check that $\widehat{\Lambda_R}$, together with plethysm and the plethystic unit $p_1(\mathbf{y})$, has the structure of a monoid.

Let $G(\mathbf{y})$ and $F(\mathbf{y})$ be in $\widehat{\Lambda_R}$. The power series $G(\mathbf{y})$ is said to be a *plethystic inverse* of $F(\mathbf{y})$ with respect to $\mathbf{y}$, if $F(\mathbf{y})[G(\mathbf{y})] = p_1(\mathbf{y})$. It is straightforward to show that if this is the case, then $F(\mathbf{y})$ is unique and also $G(\mathbf{y})[F(\mathbf{y})] = p_1(\mathbf{y})$. Thus $G(\mathbf{y})$ and $F(\mathbf{y})$ are said to be *plethystic inverses* of each other with respect to $\mathbf{y}$, and we write $G(\mathbf{y}) = F^{[-1]}(\mathbf{y})$. Note that

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \mathrm{ch}\, \widetilde{H}^{n-3}((\hat{0}, [n]^\mu))\, \mathbf{x}^\mu = \sum_{\lambda \vdash n-1} \mathrm{ch}\, \widetilde{H}^{n-3}((\hat{0}, [n]^\lambda))\, m_\lambda(\mathbf{x}) \in \Lambda_R,$$

where m_λ is the monomial symmetric function associated to the partition λ. Hence the left hand side of equation (8.3.4) below is in $\widehat{\Lambda_R}$.

Theorem 8.3.4. *We have*

$$\sum_{n\geq 1}(-1)^{n-1} \sum_{\mu \in \mathrm{wcomp}_{n-1}} \mathrm{ch}\, \widetilde{H}^{n-3}((\hat{0}, [n]^\mu))\, \mathbf{x}^\mu = \Big(\sum_{n \geq 1} h_{n-1}(\mathbf{x})h_n(\mathbf{y})\Big)^{[-1]} \tag{8.3.4}$$

Proof. For presentation purposes let us use temporarily the notation $G_{\tilde{\lambda}} := \times_{\phi \in \Phi} \mathfrak{S}_{m_\phi(\tilde{\lambda})}[\mathfrak{S}_{||\phi||}]$. We also use the convention that

$$\widetilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}}, [n]^\mu)) = 0$$

whenever $\alpha_{\tilde{\lambda}} \nleq [n]^\mu$. Applying the Frobenius characteristic map ch (in $\mathbf{y}$ variables) to both sides of equation 8.3.3, multiplying by $\mathbf{x}^\mu$ and summing over all $\mu \in \mathrm{wcomp}_{n-1}$ with $\mathrm{supp}(\mu) \subseteq [k]$ yields

$$\begin{aligned} h_1(\mathbf{y})\delta_{n,1} &= \sum_{\substack{\mu\in\mathrm{wcomp}_{n-1}\\ \mathrm{supp}(\mu)\subseteq[k]}} \mathbf{x}^\mu \,\mathrm{ch}\left(\bigoplus_{\substack{\tilde{\lambda}\in\mathsf{Par}^\mu(\Phi)\\ |\tilde{\lambda}|=n}} (-1)^{\ell(\tilde{\lambda})-1}\widetilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}},[n]^\mu))\big\uparrow_{G_{\tilde{\lambda}}}^{\mathfrak{S}_n}\right)\\ &= \sum_{\substack{\mu\in\mathrm{wcomp}_{n-1}\\ \mathrm{supp}(\mu)\subseteq[k]}} \mathbf{x}^\mu \sum_{\substack{\tilde{\lambda}\in\mathsf{Par}^\mu(\Phi)\\ |\tilde{\lambda}|=n}} (-1)^{\ell(\tilde{\lambda})-1}\,\mathrm{ch}\left(\widetilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}},[n]^\mu))\big\uparrow_{G_{\tilde{\lambda}}}^{\mathfrak{S}_n}\right)\\ &= \sum_{\substack{\tilde{\lambda}\in\mathsf{Par}(\Phi)\\ |\tilde{\lambda}|=n}} (-1)^{\ell(\tilde{\lambda})-1} \sum_{\substack{\nu\in\mathrm{wcomp}_{\ell(\tilde{\lambda})-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \mathbf{x}^{\nu+\sum\tilde{\lambda}_r}\,\mathrm{ch}\left(\widetilde{H}^{\ell(\tilde{\lambda})-3}((\alpha_{\tilde{\lambda}},[n]^{\nu+\sum\tilde{\lambda}_r}))\big\uparrow_{G_{\tilde{\lambda}}}^{\mathfrak{S}_n}\right). \end{aligned}$$

Using the shorthand notation

$$H_n^\mu := \widetilde{H}^{n-3}((\hat{0},[n]^\mu)),$$

Lemma 8.3.3 and summing over all $n \geq 1$ we have

$$\begin{aligned} &h_1(\mathbf{y})\\ &= \sum_{\tilde{\lambda}\in\mathsf{Par}(\Phi)} (-1)^{\ell(\tilde{\lambda})-1} \sum_{\substack{\nu\in\mathrm{wcomp}_{\ell(\tilde{\lambda})-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \mathbf{x}^{\nu+\sum\tilde{\lambda}_r}\,\mathrm{ch}\left(\left(\widetilde{\bigotimes_{\phi\in\Phi}(\mathbf{1}_{\mathfrak{S}_{||\phi||}})^{\otimes m_\phi(\tilde{\lambda})}\otimes \widehat{H^\nu_{\ell(\tilde{\lambda})}}}^{\tilde{\lambda}}\right)\Bigg\uparrow_{G_{\tilde{\lambda}}}^{\mathfrak{S}_{|\tilde{\lambda}|}}\right)\\ &= \sum_{\ell\geq1}(-1)^{\ell-1} \sum_{\tilde{\lambda}\in\mathsf{Par}_\ell(\Phi)} \sum_{\substack{\nu\in\mathrm{wcomp}_{\ell-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \mathbf{x}^{\nu+\sum\tilde{\lambda}_r}\,\mathrm{ch}\left(\left(\widetilde{\bigotimes_{\phi\in\Phi}(\mathbf{1}_{\mathfrak{S}_{||\phi||}})^{\otimes m_\phi(\tilde{\lambda})}\otimes \widehat{H^\nu_{\ell}}}^{\tilde{\lambda}}\right)\Bigg\uparrow_{G_{\tilde{\lambda}}}^{\mathfrak{S}_{|\tilde{\lambda}|}}\right)\\ &= \sum_{\ell\geq1}(-1)^{\ell-1} \sum_{\substack{\nu\in\mathrm{wcomp}_{\ell-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \mathbf{x}^{\nu} \sum_{\tilde{\lambda}\in\mathsf{Par}_\ell(\Phi)} \mathbf{x}^{\sum\tilde{\lambda}_r}\,\mathrm{ch}\left(\left(\widetilde{\bigotimes_{\phi\in\Phi}(\mathbf{1}_{\mathfrak{S}_{||\phi||}})^{\otimes m_\phi(\tilde{\lambda})}\otimes \widehat{H^\nu_{\ell}}}^{\tilde{\lambda}}\right)\Bigg\uparrow_{G_{\tilde{\lambda}}}^{\mathfrak{S}_{|\tilde{\lambda}|}}\right). \end{aligned}$$

Now we use Theorem 8.2.1 with $z_\phi = \mathbf{x}^\phi$,

$$\begin{aligned}
h_1(\mathbf{y}) &= \sum_{\ell\geq 1}(-1)^{\ell-1} \sum_{\substack{\nu\in \mathrm{wcomp}_{\ell-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \mathbf{x}^\nu \,\mathrm{ch}(H_\ell^\nu) \left[\sum_{\substack{\phi\in\mathrm{wcomp}\\ \mathrm{supp}(\phi)\subseteq[k]}} h_{||\phi||}(\mathbf{y})\mathbf{x}^\phi\right] \\
&= \left(\sum_{\ell\geq 1}(-1)^{\ell-1} \sum_{\substack{\nu\in \mathrm{wcomp}_{\ell-1}\\ \mathrm{supp}(\nu)\subseteq[k]}} \mathrm{ch}(H_\ell^\nu)\mathbf{x}^\nu\right) \left[\sum_{j\geq 1} h_j(\mathbf{y}) h_{j-1}(x_1,\ldots,x_k)\right].
\end{aligned}$$

The last step uses the definition of the complete homogeneous symmetric polynomial $h_{j-1}(x_1,\ldots,x_k)$. To complete the proof we let k get arbitrarily large. □

Proof of Theorem 1.6.6. We have

$$\begin{aligned}
p_1(\mathbf{y}) &= \left(\sum_{n\geq 1}(-1)^{n-1} \sum_{\mu\in\mathrm{wcomp}_{n-1}} \mathrm{ch}\,\widetilde{H}^{n-3}((\hat{0},[n]^\mu))\,\mathbf{x}^\mu\right)\left[\sum_{n\geq 1} h_{n-1}(\mathbf{x})h_n(\mathbf{y})\right] \\
&= \left(-\sum_{n\geq 1}\sum_{\mu\in\mathrm{wcomp}_{n-1}} \omega\left(\mathrm{ch}\,\widetilde{H}^{n-3}((\hat{0},[n]^\mu))\right)\,\mathbf{x}^\mu\right)\left[-\sum_{n\geq 1} h_{n-1}(\mathbf{x})h_n(\mathbf{y})\right] \\
&= \left(-\sum_{n\geq 1}\sum_{\mu\in\mathrm{wcomp}_{n-1}} \mathrm{ch}\left(\widetilde{H}^{n-3}((\hat{0},[n]^\mu))\otimes_{\mathfrak{S}_n}\mathrm{sgn}_n\right)\,\mathbf{x}^\mu\right)\left[-\sum_{n\geq 1} h_{n-1}(\mathbf{x})h_n(\mathbf{y})\right] \\
&= \left(-\sum_{n\geq 1}\sum_{\mu\in\mathrm{wcomp}_{n-1}} \mathrm{ch}\,\mathcal{L}ie(\mu)\,\mathbf{x}^\mu\right)\left[-\sum_{n\geq 1} h_{n-1}(\mathbf{x})h_n(\mathbf{y})\right].
\end{aligned}$$

The first two equalities above follow from Theorem 8.3.4 and from equation (8.1.2), respectively. Recall that for an $\mathfrak{S}_n$-module V we have that

$$\mathrm{ch}(V\otimes_{\mathfrak{S}_n}\mathrm{sgn}_n) = \omega(\mathrm{ch}\,V),$$

which proves the third equality. Finally the last equality makes use of Theorem 2.3.1. □

In the case $k = 2$, Theorem 1.6.6 specializes to the following result when $x_1 = t$, $x_2 = 1$ and $x_i = 0$ for $i \geq 3$.

Theorem 8.3.5. *For $n \geq 1$,*

$$\sum_{n\geq 1}\sum_{i=0}^{n-1} \operatorname{ch}\mathcal{L}ie_2(n,i)\, t^i = -\Big(-\sum_{n\geq 1}\frac{t^n-1}{t-1}h_n(\mathbf{y})\Big)^{[-1]}.$$

For a closely related result obtained using operad theoretic arguments, see [14]. One should also be able to approach Theorem 1.6.6 via operad theory.

And we obtain a well-known classical result when $x_1 = 1$, $x_i = 0$ for $i \geq 2$.

Theorem 8.3.6. *For $n \geq 1$,*

$$\sum_{n\geq 1} \operatorname{ch}\mathcal{L}ie(n) = -\Big(-\sum_{n\geq 1} h_n(\mathbf{y})\Big)^{[-1]}.$$

We show that Theorem 1.6.6 reduces to Theorem 1.6.3 after applying an appropiate specialization. Recall that $R = \Lambda_{\mathbb{Q}}$ and consider the map $E : \widehat{\Lambda_R} \rightarrow R[[y]]$ defined by:

$$E(p_i(\mathbf{y})) = y\delta_{i,1}$$

for $i \geq 1$ and extended multiplicatively, linearly and taking the corresponding limits to all of $\widehat{\Lambda_R}$. It is not difficult to check that E is a ring homomorphism since E is defined on generators. Moreover, we show in the following proposition that the specialization E maps plethysm in $\widehat{\Lambda_R}$ to composition in $R[[y]]$.

Proposition 8.3.7. *For all $F, G \in \widehat{\Lambda_R}$,*

$$E(F[G]) = E(F)(E(G)).$$

Proof. Using the definition of plethysm and using the convention $\mathbf{x}^k = (x_1^k, x_2^k, \dots)$,

$$\begin{aligned} p_\nu(\mathbf{y})\left[\sum_\lambda c_\lambda(\mathbf{x})p_\lambda(\mathbf{y})\right] &= \prod_{i=1}^{\ell(\nu)} p_{\nu_i}(\mathbf{y})\left[\sum_\lambda c_\lambda(\mathbf{x})p_\lambda(\mathbf{y})\right] \\ &= \prod_{i=1}^{\ell(\nu)}\sum_\lambda c_\lambda(\mathbf{x}^{\nu_i})p_\lambda(\mathbf{y}^{\nu_i}) \\ &= \prod_{i=1}^{\ell(\nu)}\sum_\lambda c_\lambda(\mathbf{x}^{\nu_i})\prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(\mathbf{y}^{\nu_i}) \\ &= \prod_{i=1}^{\ell(\nu)}\sum_\lambda c_\lambda(\mathbf{x}^{\nu_i})\prod_{j=1}^{\ell(\lambda)} p_{\lambda_j\nu_i}(\mathbf{y}). \end{aligned} \tag{8.3.5}$$

Note that $E(p_{\lambda_j\nu_i}(\mathbf{y})) = y\delta_{\lambda_j\nu_i,1} = y\delta_{\lambda_j,1}\delta_{\nu_i,1}$. Then if ν has at least one part ν_i of size greater than 1, equation (8.3.5) implies

$$E\left(p_\nu(\mathbf{y})\left[\sum_\lambda c_\lambda(\mathbf{x})p_\lambda(\mathbf{y})\right]\right) = 0 = E(p_\nu(\mathbf{y}))\left(E\left(\sum_\lambda c_\lambda(\mathbf{x})p_\lambda(\mathbf{y})\right)\right),$$

since $E(p_\nu(\mathbf{y})) = 0$. If $\nu = (1^m)$, then

$$\begin{aligned} E\left(p_\nu(\mathbf{y})\left[\sum_\lambda c_\lambda(\mathbf{x})p_\lambda(\mathbf{y})\right]\right) &= E\left(\prod_{i=1}^{\ell(\nu)}\sum_\lambda c_\lambda(\mathbf{x}^{\nu_i})\prod_{j=1}^{\ell(\lambda)} p_{\lambda_j\nu_i}(\mathbf{y})\right) \\ &= E\left(\prod_{i=1}^{m}\sum_\lambda c_\lambda(\mathbf{x})\prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(\mathbf{y})\right) \end{aligned}$$

$$= \left(E \left(\sum_{\lambda} c_\lambda(\mathbf{x}) \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(\mathbf{y}) \right) \right)^m$$
$$= E(p_{(1^m)}(\mathbf{y})) \left(E \left(\sum_{\lambda} c_\lambda(\mathbf{x}) \prod_{j=1}^{\ell(\lambda)} p_{\lambda_j}(\mathbf{y}) \right) \right).$$

We just proved that $E\left(p_\nu(\mathbf{y})\left[G\right]\right) = E\left(p_\nu(\mathbf{y})\right)\left[E(G)\right]$ for any $G \in \widehat{\Lambda_R}$. The proof of the proposition follows by extending this result to all of $\widehat{\Lambda_R}$ by linearity and taking limits. $\square$

Since $E(p_1(\mathbf{y})) = y$, we conclude that E is a monoid homomorphism

$$(\widehat{\Lambda_R}, \text{plethysm}, p_1) \to (R[[y]], \text{composition}, y).$$

The specialization E can be better understood under the definition of the Frobenius characteristic map. Let V be a representation of $\mathfrak{S}_n$ and χ^V its character, then

$$\operatorname{ch}(V) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^V(\sigma) p_{\lambda(\sigma)}(\mathbf{y}),$$

where $\lambda(\sigma)$ is the cycle type of the permutation $\sigma \in \mathfrak{S}_n$.

We have that

$$E(\operatorname{ch} V) = \frac{1}{n!} E\left(\sum_{\sigma \in \mathfrak{S}_n} \chi^V(\sigma) p_{\lambda(\sigma)}(\mathbf{y}) \right) = \chi_V(id)\frac{y^n}{n!} = \dim V \frac{y^n}{n!}.$$

In particular since $h_n(\mathbf{y}) = \operatorname{ch}(\mathbf{1}_{\mathfrak{S}_n})$, the Frobenius characteristic of the trivial representation of $\mathfrak{S}_n$, we have that $E(h_n(\mathbf{y})) = \dfrac{y^n}{n!}$. Therefore Theorem 1.6.6 reduces to Theorem 1.6.3 after we apply the specialization E.

Theorem 1.6.6 gives an implicit description of the character for the representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(\mu)$; Theorem 8.3.5 gives a description of the character of $\mathcal{L}ie_2(n, i)$. Dotsenko and Khoroshkin in [14] computed an explicit product formula for the $SL_2 \times \mathfrak{S}_n$-character of $\mathcal{L}ie_2(n)$. From this one can get the coefficients (as polynomials in t) of p_λ in the symmetric function $\sum_{i=0}^{n-1} \operatorname{ch} \mathcal{L}ie_2(n,i)\, t^i$.

Question 8.3.8. *Can we find explicit character formulas for the representation of $\mathfrak{S}_n$ on $\mathcal{L}ie(\mu)$ for general $\mu \in \mathrm{wcomp}_{n-1}$? What are the multiplicities of the irreducibles?*

Since $\sum_{\mu \in \mathrm{wcomp}_{n-1}} \operatorname{ch} \mathcal{L}ie(\mu)\, \mathbf{x}^\mu$ is a symmetric function in $\mathbf{x}$ with coefficients that are symmetric functions in $\mathbf{y}$, we can write

$$\sum_{\mu \in \mathrm{wcomp}_{n-1}} \operatorname{ch} \mathcal{L}ie(\mu)\, \mathbf{x}^\mu = \sum_{\lambda \vdash n-1} C_\lambda(\mathbf{y}) e_\lambda(\mathbf{x}),$$

where $C_\lambda(\mathbf{y})$ is a homogeneous symmetric function of degree n with coefficients in $\mathbb{Z}$.

By Theorem 5.3.3, $E(C_\lambda(\mathbf{y}))$ equals the number $c_{n,\lambda}$ of trees $\Upsilon \in Nor_n$ of comb type (or Lyndon type) $\lambda(\Upsilon) = \lambda$. We propose the following conjecture.

Conjecture 8.3.9. *The coefficients $C_\lambda(\mathbf{y})$ are Schur positive.*

The conjecture basically asserts that $C_\lambda(\mathbf{y})$ is the Frobenius characteristic of a representation of dimension $c_{n,\lambda}$. An approach to proving the conjecture is to find such a representation.

Appendix A

Notation and techniques

We recall basic notation and techniques in poset topology as well as background information about the ring of symmetric functions.

A.1 Partially ordered sets

A *partially ordered set* or *poset* is a pair $(P, \leq)$, where P is a set and $\leq$ is a relation on P satisfying for every $x, y, z \in P$ the following properties:

- $x \leq x$ (Reflexivity),
- $x \leq y$ and $y \leq x$ implies $x = y$ (Antisymmetry), and
- $x \leq y$ and $y \leq z$ implies $x \leq z$ (Transitivity).

By abuse of notation we normally refer to P meaning both the poset and its underlying set when the context makes the difference clear. For poset terminology not defined here see [40], [46]. The notation $x < y$ is used as a shorthand for $x \leq y$ and $x \neq y$ and $x \geq y$ is used for $y \leq x$. We say that y *covers* x or that x *is covered by* y (denoted by $x \lessdot y$) if $x < y$ and there is no $z \in P$ such that $x < z < y$. We denote $\mathcal{E}(P)$ to the set of all covering relations of P. We can think of a covering

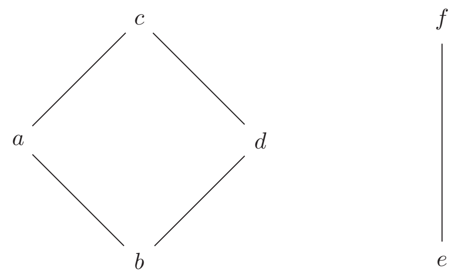


Figure A.1: Example of the Hasse diagram of a poset.

relation $x \lessdot y$ as an arrow from x pointed to y. The *Hasse diagram* of a poset is the directed graph $(P, \mathcal{E}(P))$ whose vertex set is P and directed edge set is $\mathcal{E}(P)$. As a convention, a Hasse diagram is drawn so that the edges are directed upward, that is, if $x < y$ in P then y is drawn higher than x. The Hasse diagram of the poset on the set $\{a, b, c, d, e, f\}$ whose covering relations are $b \lessdot a, b \lessdot d, a \lessdot c, b \lessdot c$ and $e \lessdot f$ is illustrated in Figure A.1. The *dual poset* P^* is the poset with the same underlying set as P but with $x \leq_{P^*} y$ whenever $y \leq_P x$. For $u \leq v$ in a poset P, the open interval $\{w \in P : u < w < v\}$ is denoted by (u, v) and the closed interval $\{w \in P : u \leq w \leq v\}$ by $[u, v]$. A poset is said to be *bounded* if it has a minimum element $\hat{0}$ (i.e., $\hat{0} \leq x$ for all $x \in P$) and a maximum element $\hat{1}$ (i.e., $x \leq \hat{1}$ for all $x \in P$). For a bounded poset P, we define the *proper part* of P as the *induced subposet* (a subset of P with all the relations) $\overline{P} := P \setminus \{\hat{0}, \hat{1}\}$. A poset is said to be *pure* (graded or ranked) if all its maximal chains have the same length, where the length of a chain $s_0 < s_1 < \cdots < s_n$ is n. The *length* $l(P)$ of a poset P is the length of its longest chain. For a graded poset P with a minimum element $\hat{0}$, the rank function $\rho : P \to \mathbb{N}$ is defined by $\rho(s) = l([\hat{0}, s])$. The rank generating function $\mathcal{F}_P(x)$ is defined by $\mathcal{F}_P(x) = \sum_{u \in P} x^{\rho(u)}$.

We say that a map $f : P \to Q$ between posets P and Q is *order preserving* or a *poset map* if $x \leq_P y$ implies $f(x) \leq_Q f(y)$. In particular, an order preserving bijection $f : P \to Q$ whose inverse is order preserving is called an *isomorphism*

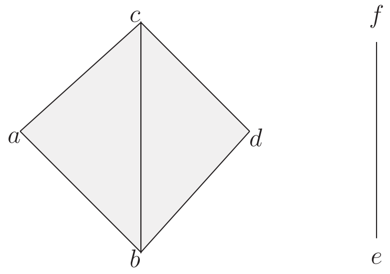


Figure A.2: Example of the order complex of the poset in Figure A.1

of posets and P and Q are said to be isomorphic posets. In general, we would be interested in poset properties that are preserved under isomorphisms also known as *invariants*. For example, the rank generating function $\mathcal{F}_P(x)$ is a poset invariant.

A *(finite) simplicial complex* is a pair $(V, \mathcal{C})$ where V is a finite set and $\mathcal{C}$ is a class of subsets of V satisfying:

- $\{x\} \in \mathcal{C}$ for every $x \in V$, and
- If $A \subseteq B$ and $B \in \mathcal{C}$ then $A \in \mathcal{C}$.

To every poset P we can associate a simplicial complex $\Delta(P)$ with vertex set P and whose faces are the chains of P. $\Delta(P)$ is called the *order complex* of P and it is the fundamental link between posets and topology. In Figure A.2 the order complex of the poset in Figure A.1 is illustrated. Note that the points correspond to the six chains of length 0 ($\{a, b, c, d, e, f\}$), the edges to the six chains of length 1 ($\{b < a, b < d, b < c, a < c, d < c, e < f\}$) and the shaded regions to the two chains of length 2 ($\{b < a < c, b < d < c\}$).

A.2 The Möbius function

Let $\mathbf{k}$ be a field. Denote by $Int(P)$ the set of closed intervals $[x, y]$ in the poset P.

The *Möbius function* is the function $\bar{\mu} = \bar{\mu}_P : Int(P) \to \mathbb{C}$ defined recursively as:

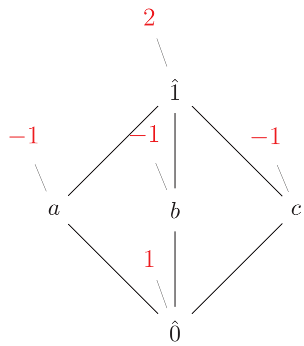


Figure A.3: Example of $\bar{\mu}(\hat{0}, t)$

$$\begin{aligned} &\bar{\mu}(x, x) = 1, \quad \text{for all } x \in P \\ &\bar{\mu}(x, y) = -\sum_{x \le z < y} \bar{\mu}(x, z), \quad \text{for all } x < y \in P. \end{aligned} \tag{A.2.1}$$

Or equivalently,

$$\begin{aligned} &\bar{\mu}(x, x) = 1, \quad \text{for all } x \in P \\ &\bar{\mu}(x, y) = -\sum_{x < z \le y} \bar{\mu}(z, y), \quad \text{for all } x < y \in P. \end{aligned} \tag{A.2.2}$$

For a bounded poset P, we define the Möbius invariant

$$\bar{\mu}(P) = \bar{\mu}_P(\hat{0}, \hat{1}).$$

In the Figure A.3 the Möbius numbers $\bar{\mu}(\hat{0}, t)$ for all $t \in P$ are shown in red. Note that in the example $\bar{\mu}(P) = \bar{\mu}_P(\hat{0}, \hat{1}) = 2$.

The importance of the Möbius function in poset topology is highlighted by the strong connection with a topological invariant given in Corollary A.2.2 below.

Theorem A.2.1 (P. Hall [23]). *Let P be a finite bounded poset and let c_i denote the number of chains of length i in $\overline{P}$ with $c_{-1} = 1$ (the empty chain). Then*

$$\bar{\mu}_P(\hat{0}, \hat{1}) = -c_{-1} + c_0 - c_1 + c_2 - c_3 + \cdots . \tag{A.2.3}$$

Theorem A.2.1 can be restated using the order complex of $\overline{P}$.

Corollary A.2.2.

$$\bar{\mu}_P(\hat{0}, \hat{1}) = \widetilde{\chi}(\Delta(\overline{P})) \tag{A.2.4}$$

where $\widetilde{\chi}(\Delta)$ denotes the reduced Euler characteristic of the simplicial complex Δ.

For the basic example of Figure A.3 note that the poset $\overline{P}$ is formed by 3 incomparable elements. Hence we have $c_0 = 3$ and $-c_{-1} + c_0 = -1 + 3 = 2 = \bar{\mu}(\hat{0}, \hat{1})$. Also $\Delta(\overline{P})$ is the simplicial complex formed by 3 disjoint points whose reduced Euler characteristic $\widetilde{\chi}(\Delta(\overline{P})) = 2$.

The following consequence of the Euler-Poincaré formula is a standard result in topology.

Proposition A.2.3. *If a simplicial complex Δ has the homotopy type of a wedge of m spheres of dimension d, then*

$$\widetilde{\chi}(\Delta) = (-1)^d m.$$

For a pure poset with a $\hat{0}$ the *characteristic polynomial* is defined as

$$\chi_P(x) = \sum_{\alpha \in P} \bar{\mu}(\hat{0}, \alpha) x^{\rho(P) - \rho(\alpha)}.$$

For the example of Figure A.3 we have $\chi_P(x) = x^2 - 3x + 2 = (x-1)(x-2)$.

A.3 Homology and Cohomology of a Poset

We give a brief review of poset (co)homology with group actions. For further information see [46].

Let P be a finite poset of length ℓ. The reduced simplicial (co)homology of P is defined to be the reduced simplicial (co)homology of its order complex $\Delta(P)$. We will review the definition here by dealing directly with the chains of P, and not resorting to the order complex of P.

Let $\mathbf{k}$ be an arbitrary field. The (reduced) chain and cochain complexes

$$\cdots \underset{\delta_r}{\overset{\partial_{r+1}}{\rightleftarrows}} C_r(P) \underset{\delta_{r-1}}{\overset{\partial_r}{\rightleftarrows}} C_{r-1}(P) \underset{\delta_{r-2}}{\overset{\partial_{r-1}}{\rightleftarrows}} \cdots$$

are defined by letting $C_r(P)$ be the $\mathbf{k}$-module generated by the chains of length r in P. Note that $C_{-1}(P)$ is generated by the empty chain, and $C_r(P) = (0)$ if $r < -1$ or $r > \ell$. The boundary maps $\partial_r : C_r(P) \to C_{r-1}(P)$ are defined on chains by

$$\partial_r(\alpha_0 < \alpha_1 < \cdots < \alpha_r) = \sum_{i=0}^{r} (-1)^i (\alpha_0 < \cdots < \hat{\alpha_i} < \cdots < \alpha_r)$$

where $\hat{\alpha_i}$ means that the element α_i is omitted from the chain.

Define the bilinear form $\langle , \rangle$ on $\bigoplus_{r=-1}^{\ell} C_r(P)$ by

$$\langle c, c' \rangle = \delta_{c,c'}, \tag{A.3.1}$$

where c, c' are chains of P, and extend by linearity. This allows us to define the coboundary map $\delta_r : C_r(P) \to C_{r+1}(P)$ by

$$\langle \delta_r(c), c' \rangle = \langle c, \partial_{r+1}(c') \rangle.$$

Equivalently,

$$\delta_r(\alpha_0 < \cdots < \alpha_r) = \sum_{i=0}^{r+1} (-1)^i \sum_{\alpha \in (\alpha_{i-1}, \alpha_i)} (\alpha_0 < \cdots < \alpha_{i-1} < \alpha < \alpha_i < \cdots < \alpha_r), \tag{A.3.2}$$

where $\alpha_{-1} = \hat{0}$ and $\alpha_{r+1} = \hat{1}$ of the augmented poset $\hat{P}$ in which a minimum element $\hat{0}$ and a maximum element $\hat{1}$ have been adjoined to P.

Let $r \in \mathbb{Z}$. Define the cycle space $Z_r(P) := \ker \partial_r$ and the boundary space $B_r(P) := \operatorname{im} \partial_{r+1}$. Homology of the poset P in dimension r is defined by

$$\tilde{H}_r(P) := Z_r(P)/B_r(P).$$

Define the cocycle space $Z^r(P) := \ker \delta_r$ and the coboundary space $B^r(P) := \operatorname{im} \delta_{r-1}$. Cohomology of the poset P in dimension r is defined by

$$\tilde{H}_r(P) := Z^r(P)/B^r(P).$$

For $x \leq y$, consider the open interval (x, y) of P. Note that if y covers x then (x, y) is the empty poset whose only chain is the empty chain. Therefore $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = 0$ unless $r = -1$, in which case $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = \mathbf{k}$. If $y = x$ then we adopt the convention that $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = 0$ unless $r = -2$, in which case $\tilde{H}_r((x, y)) = \tilde{H}^r((x, y)) = \mathbf{k}$.

Proposition A.3.1. *Let P be a finite poset of length ℓ whose order complex has the homotopy type of a wedge of m spheres of dimension $\ell - 2$. Then $\tilde{H}_{\ell-2}(P)$ and $\tilde{H}^{\ell-2}(P)$ are isomorphic free $\mathbf{k}$-modules of rank m.*

The following proposition gives a useful tool in identifying bases for top homology and top cohomology modules.

Proposition A.3.2 (see [46, Theorem 1.5.1], [36, Proposition 6.4]). *Let P be a finite poset of length ℓ whose order complex has the homotopy type of a wedge of m spheres of dimension ℓ. Let $\{\rho_1, \rho_2, ..., \rho_m\} \subseteq Z_\ell(P)$ and $\{\gamma_1, \gamma_2, ..., \gamma_m\} \subseteq Z^\ell(P)$. If the matrix $(< \rho_i, \gamma_j >)_{i,j\in[m]}$ is invertible over $\mathbf{k}$ then the sets $\{\rho_1, \rho_2, ..., \rho_m\}$ and $\{\gamma_1, \gamma_2, ..., \gamma_m\}$ are bases for $\tilde{H}_\ell(P; \mathbf{k})$ and $\tilde{H}^\ell(P; \mathbf{k})$ respectively.*

Let G be a finite group. A G-poset is a poset P together with a G-action on its elements that preserves the partial order; i.e., $x < y \implies gx < gy$ in P.

Now assume that $\mathbf{k}$ is a field. Let P be a G-poset of length ℓ. Since $g \in G$ takes r-chains to r-chains, g acts as a linear map on the chain space $C_r(P)$ (over $\mathbf{k}$). It is easy to see that for all $g \in G$ and $c \in C_r(P)$,

$$g\partial_r(c) = \partial_r(gc) \text{ and } g\delta_r(c) = \delta_r(gc).$$

Hence g acts as a linear map on the vector spaces $\tilde{H}_r(P)$ and on $\tilde{H}^r(P)$. This implies that whenever P is a G-poset, $\tilde{H}_r(P)$ and $\tilde{H}^r(P)$ are G-modules. The bilinear form $\langle \cdot, \cdot \rangle$, induces a pairing between $\tilde{H}_r(P)$ and $H^r(P)$, which allows one to view them as dual G-modules. For $G = \mathfrak{S}_n$ we have the $\mathfrak{S}_n$-module isomorphism

$$\tilde{H}_r(P) \simeq_{\mathfrak{S}_n} \tilde{H}^r(P) \tag{A.3.3}$$

since dual $\mathfrak{S}_n$-modules are isomorphic.

Example A.3.3. The symmetric group $\mathfrak{S}_n$ acts naturally on Π_n by permuting the letters of $[n]$ and this action induces isomorphic representations of $\mathfrak{S}_n$ on the unique nonvanishing reduced simplicial homology $\tilde{H}_{n-3}(\overline{\Pi}_n)$ of the order complex $\Delta(\overline{\Pi}_n)$ and on the unique nonvanishing simplicial cohomology $\tilde{H}^{n-3}(\overline{\Pi}_n)$.

A.4 The ring of symmetric functions

For information not presented here about symmetric functions and the representation theory of the symmetric group see [31], [35], [25] and [39, Chapter 7].

Let R be a commutative ring containing $\mathbb{Q}$ and $\mathbf{y} = \{y_1, y_2, \dots\}$ an infinite set of variables. Let $R[[\mathbf{y}]] := R[[y_1, y_2, \dots]]$ denote the *ring of formal power series* in the $\mathbf{y}$ variables. We call a *monomial*, a term of the form

$$\mathbf{y}^\mu := y_1^{\mu_1} y_2^{\mu_2} \cdots ,$$

where $\mu \in \mathrm{wcomp}$. If $|\mu| = n$ we say that $\mathbf{y}^\mu$ has *degree* n.

Example A.4.1. If $\mu = (0, 3, 1, 0, 2)$ then $\mathbf{y}^\mu = y_2^3 y_3 y_5^2$ is a monomial of degree 6.

We say that $f(\mathbf{y}) \in R[[\mathbf{y}]]$ is *homogeneous of degree* n if every monomial in f has the same degree n.

Example A.4.2.

$$\sum_{\substack{i \geq 1 \\ j \geq 1}} y_i^2 y_j = y_1^3 + y_1^2 y_2 + y_2^2 y_1 + \cdots$$

is homogeneous of degree 3.

Now let $f(\mathbf{y}) \in R[[\mathbf{y}]]$ be homogeneous of degree n. We say that f is a *symmetric function* if

$$f(x_1, x_2, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, \dots)$$

for every permutation $\sigma \in \mathfrak{S}_{\mathbb{P}}$. Let Λ_R^n denote the vector space of homogeneous symmetric functions of degree n and let

$$\Lambda_R := \Lambda_R^0 \oplus \Lambda_R^1 \oplus \Lambda_R^2 \oplus \cdots .$$

Note that multiplication in $R[[\mathbf{y}]]$ gives a map

$$\Lambda_R^i \times \Lambda_R^j \to \Lambda_R^{i+j}$$

that makes Λ_R into a graded ring called the *ring of symmetric functions*. It is known that the dimension of each homogeneous component Λ_R^n equals the number of (number) partitions of n.

We describe known bases for Λ_R^n. For a partition $\lambda \vdash n$ the *monomial symmetric function* m_λ is defined by

$$m_\lambda := \sum_{\mu \in \mathrm{wcomp}_\lambda} \mathbf{y}^\mu,$$

where wcomp_λ is the set of rearrangements of λ. For every $n \geq 0$, the *elementary symmetric function* e_n, the *complete homogeneous symmetric function* h_n and the *power sum symmetric function* p_n are defined by

$$\begin{aligned}
e_0 = h_0 = p_0 &:= 1,\\
e_n &:= \sum_{i_1 < i_2 < \cdots < i_n} y_{i_1} y_{i_2} \cdots y_{i_n} = m_{(1^n)},\\
h_n &:= \sum_{\lambda \vdash n} m_\lambda,\\
p_n &:= \sum_{i \geq 1} y_i^n,
\end{aligned}$$

where $(1^n) := \overbrace{(1, 1, \ldots, 1)}^{n \text{ times}}$. For a family of symmetric functions $u_0, u_1, \ldots,$

$$\begin{array}{|c|c|c|}
\hline
1 & 1 & 2 \\
\hline
2 & 2 & \multicolumn{1}{c}{} \\
\cline{1-2}
3 & \multicolumn{2}{c}{} \\
\cline{1-1}
4 & \multicolumn{2}{c}{} \\
\cline{1-1}
\end{array}$$

Figure A.4: Example of a SSYT of shape $\lambda = (3, 2, 1, 1)$ and content $\mu = (2, 3, 1, 1)$.

we denote by u_λ, the symmetric function defined multiplicatively as $u_\lambda := u_{\lambda_1} u_{\lambda_2} \cdots u_{\lambda_{\ell(\lambda)}}$; the symmetric functions e_λ, h_λ and p_λ are defined in this manner.

Theorem A.4.3. *The sets*

$$\{m_\lambda \mid \lambda \vdash n\},\ \{e_\lambda \mid \lambda \vdash n\},\ \{h_\lambda \mid \lambda \vdash n\} \ \textit{ and } \{p_\lambda \mid \lambda \vdash n\}$$

are bases for Λ_R^n.

There is yet another important basis for the space of homogeneous symmetric functions. For a partition $\lambda \vdash n$, the *Ferrers diagram* of λ is a two dimensional left justified arrangement of cells (rows and columns) where row i contains λ_i cells for each i. A *SemiStandard Young Tableau* (SSYT) is a filling of the Ferrers diagram of λ with positive integers that are weakly increasing in the rows and strictly increasing in the columns. In Figure A.4 a SSYT T of *shape* $\mathrm{sh}(T) = (3, 2, 1, 1)$ and *content* $\mu(T) = (2, 3, 1, 1)$ (two 1's, three 2's, one 3 and one 4) is illustrated.

For a SSYT T of shape λ and content μ let $\mathbf{y}^T := \mathbf{y}^\mu$ (in Figure A.4 $\mathbf{y}^T = y_1^2 y_2^3 y_3 y_4$). For $\lambda \vdash n$, the *Schur symmetric function* s_λ is defined by

$$s_\lambda := \sum_{\substack{T \text{ a SSYT} \\ \mathrm{sh}(T) = \lambda}} \mathbf{y}^T. \tag{A.4.1}$$

Theorem A.4.4. *The set* $\{s_\lambda \mid \lambda \vdash n\}$ *is a basis for* Λ_R^n.

It is well-known (see for example [25]) that the irreducible representations of the symmetric group $\mathfrak{S}_n$ are also indexed by number partitions $\lambda \vdash n$. For a partition $\lambda \vdash n$ we denote by S^λ the irreducible *Specht module* indexed by λ.

Let V be a representation of $\mathfrak{S}_n$ and χ^V its character, the *Frobenius characteristic map* is defined

$$\mathrm{ch}(V) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^V(\sigma) p_{\lambda(\sigma)}(\mathbf{y}),$$

where $\lambda(\sigma)$ is the cycle type of the permutation $\sigma \in \mathfrak{S}_n$.

The map ch is a ring isomorphism between the ring of virtual representations of $\mathfrak{S}_n$ and the ring of symmetric functions. The definition of both the Specht modules and the Schur functions can be extended to Ferrers diagrams with skew shapes. For (perhaps empty) integer partitions ν and λ such that $\nu \subseteq \lambda$ (that is $\nu(i) \leq \lambda(i)$ for all i), let $S^{\lambda/\nu}$ denote the Specht module of shape λ/ν and $s_{\lambda/\nu}$ the Schur function of shape λ/ν. Then $s_{\lambda/\nu}$ is the image in the ring of symmetric functions of the (not necesssary irreducible) specht module $S^{\lambda/\nu}$ under the Frobenius characteristic map ch.

Theorem A.4.5. *For every pair of partitions $\mu \subseteq \lambda$,*

$$\mathrm{ch}\, S^{\lambda/\nu} = s_{\lambda/\nu}.$$

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