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Universal Classification of Topological Categories

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UNIVERSITY OF MIAMI

UNIVERSAL CLASSIFICATION OF TOPOLOGICAL CATEGORIES

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The main purpose of this dissertation is to construct, for various well known families of topological categories and some of their generalizations, a member of the family that is universal in the sense that every member of the family is isomorphic to the pullback, along its so called classifying functor, of the said universal family member. This is carried out by first constructing a topological category that is universal for the family of all topological categories and then by defining various family universal categories by describing their classifying functors. A further refinement is made by placing restrictions on the classifying functors themselves, thus obtaining various "restricted" families of topological categories along with their corresponding "restricted universal categories". These constructions and results are first described in the more general setting of *horizontal structures.* We will show that all horizontal structures can be obtained by pulling back the *universal horizontal structure* along an appropriate classifying functor and as a consequence, by restriction, every topological category can be realized as the pullback, along its classifying functor, of the universal topological category.

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CONTENTS

CHAPTER 1

INTRODUCTION

Topological functors were introduced by Herrlich (in terms of the existence of certain initial lifts in [10]) and by Wyler (in terms of contravariant functors to the category of complete lattices in [21]) among others in the early 1970s as a result of axiomatizing the properties that many categories share: topological and pretopological spaces [16], filter and stack convergence spaces [18], limit spaces [17], bornological spaces [1] etc. The most important feature of these categories is the existence of final (and initial) structures, in particular, for topological spaces the formation of induced (and coinduced) topologies, which can be viewed as *lifting properties* of the underlying set functor U : Top $\longrightarrow$ Sets. In general, a functor $E \longrightarrow B$ between categories E and B is said to be a topological functor and E a topological category over B, if it satisfies certain *horizontal lifting* conditions. Several aspects of topological spaces can be extended to topological categories, which can then serve as"realms" in which to formulate and test various topological notions.

Many of the familiar examples of topological categories, nearness and uniform spaces [9], grill spaces [18], along with the examples listed above result from generalizing some particular aspect of the notion of topological space, while others, such as: the categories of pairs, relations, preorders and families have other origins.

In the second chapter the 2-category of *horizontal structures* over a category A (denoted by H_A) is defined as a supercategory of topological categories over A. The so called *horizontal morphisms* of these horizontal structures behave much like those morphisms (continuous functions) $\bar{f} : (X, \tau) \longrightarrow (Y, \sigma)$ in Top where σ is the topology coinduced on Y by f, in the sense that given any other topology ρ on Y such that $f : (X, \tau) \longrightarrow (Y, \rho)$ is continuous ($f = \bar{f}$ in Sets), f factors through $\bar{f}$ in Top. Using the Grothendieck construction, H_A will be shown to be naturally equivalent to the functor category CAT^A. Horizontal structures, just as topological structures, are pullback stable; this fact will lead to the definition of a contravariant 2-functor $\mathcal{H} : \mathrm{CAT} \longrightarrow 2-\mathrm{Cat}$ which assigns H_A to a category A, and for a functor $F : A \longrightarrow B$, $\mathcal{H}(F) : H_B \longrightarrow H_A$ acts via the canonical pullback construction. The natural (strong-lax) isomorphism $\int$ based on the Grothendieck construction shows that $\mathcal{H}$ is a representable 2-functor. The representing object must be a suitable collection of categories large enough to contain the large categories used later (in particular the category of posets), but it must be a category itself and as such an object in CAT; its objects must form a proper class (the objects of CAT form a conglomerate). This universe of categories will be denoted by $\flat\mathrm{CAT}$. With $\flat\mathrm{CAT}$ the representing object for $\mathcal{H}$, the *universal object* is the image of the identity functor $1 : \flat\mathrm{CAT} \longrightarrow \flat\mathrm{CAT}$ under the functor $\int_{\flat\mathrm{CAT}}$ by the enriched version of the Yoneda Lemma. $\int_{\flat\mathrm{CAT}} 1$ will be referred to as the *universal horizontal structure* and denoted by CAT_*; the universality of CAT_* means that every horizontal structure is isomorphic to a pullback of $\mathrm{CAT}_* \xrightarrow{P} \flat\mathrm{CAT}$ along some functor. In particular, considering functors in $\mathrm{CoP}^A \subseteq \flat\mathrm{CAT}^A$, the resulting universal object is the universal topological category CoP_*; (CoP is the category of cocomplete posets with cocontinuous functors).

In the third chapter we will describe the classifying functors for some familiar topological categories and for their more general versions. The categories discussed all resemble Top, the category of topological spaces and continuous functions. The generalization of Top is achieved in two steps. First a topology will be defined on a suitable poset rather than as a subset τ of the powerset $P(X)$ of some set X; then the topology will be viewed in terms of its characteristic function $\chi_\tau : P(X) \longrightarrow 2$ which will then be replaced by a function $\mathcal{T} : A \longrightarrow M$ with appropriate properties. The elements of the poset M give the degree of membership of the elements of the poset A in the topology defined by $\mathcal{T}$. The definition of $\mathcal{T}$ will include as special cases all of the different versions of *fuzzy* topologies that appear in the literature ([4],[8],[19],[24]). The universal member of this family of topological categories will be identified through its classifying functor.

The fourth chapter deals with categories whose objects are (po)sets with some kind of convergence structure defined on them. Stack convergence spaces, filter and local filter convergence spaces, limit spaces and pretopological spaces will be described through their respective classifying functors. Each convergence structure is defined via a convergence function $q : S(X) \longrightarrow P(X)$ or $q : F(X) \longrightarrow P(X)$ where $S(X)$ and $F(X)$ denote the set of all stacks and filters on X, respectively and q associates to a stack (filter) the set of points of X to which it converges. To generalize the categories listed above, $P(X)$ will again be replaced by an appropriate poset; as a consequence stacks and filters will become down segments and ideals of the poset (due to the reverse ordering). The classifying functors will be subfunctors of one another, since they are obtained by putting restrictions on the convergence function. There will thus be obtained a restricted family universal category for each of the convergence types discussed.

CHAPTER 2

FOUNDATIONS

Horizontal Structures

Definition 2.1 cf. Definition 7.1.1 in [3] A *2-category* C is defined by the following data:

(1) a class of objects called 0-cells,

(2) for each pair a, b of 0-cells a category $C(a, b)$ (often required to be small); (the objects of $C(a, b)$ are called 1-cells and its arrows are called 2-cells),

(3) for each triple $a,\ b,\ c$ of 0-cells, a bifunctor

$$c_{abc} : C(a, b) \times C(b, c) \longrightarrow C(a, c),$$

(4) for each 0-cell a, a functor $u_a : \mathbf{1} \longrightarrow C(a, a)$.

These data are required to satisfy the usual associativity and unit axioms. Given 1-cells f, g, h in $C(a, b)$ and 2-cells $\alpha : f \Rightarrow g$ and $\beta : g \Rightarrow h$, the composition of α and β in the category $C(a, b)$ will be denoted by $\beta \odot \alpha$, and given 2-cells $\alpha : f \Rightarrow g$ in $C(a, b)$ and $\varphi : k \Rightarrow l$ in $C(b, c)$

$$\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, bend left, "k"] \arrow[r, bend right, "l"'] \arrow[r, phantom, "\Downarrow\varphi"] & c \end{tikzcd}$$

$c_{abc}(\alpha, \varphi) : kf \Rightarrow lg$ will be denoted by $\varphi * \alpha$. The composition map c_{abc} being a functor implies that the interchange law $(\psi * \beta) \odot (\varphi * \alpha) = (\beta \odot \alpha) * (\psi \odot \varphi)$ holds

for the 2-cells as pictured: $\begin{tikzcd} a \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & b \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\varphi"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\psi"] & c \end{tikzcd}$.

Depending on possible size restrictions imposed on the collection of objects and morphisms of categories, we'll adopt the following notation: CAT will denote the quasicategory of all categories (as in Definitions 3.49 and 3.50 in[1]), Cat the category of all small categories and $\flat$CAT will denote a category with categories (not necessarily small) as objects such that $\flat$CAT is an object in CAT. CAT, Cat and $\flat$CAT are 2-categories. If A is a 2-category, so are A^{op} (1-cells reversed) and A^{co} (2-cells reversed).

Given a category A, the functor category $\flat\mathrm{CAT}^A$ is a 2-category with functors $F : A \longrightarrow \flat\mathrm{CAT}$ the 0-cells, natural transformations $\alpha : F \Rightarrow G$ the 1-cells and for $\alpha, \beta : F \Rightarrow G$, the 2-cells (*modifications*) $m : \alpha \rightsquigarrow \beta$ are defined as follows: each 1-cell $\alpha : F \Rightarrow G$ is a collection of functors (arrows in $\flat$CAT) $\{\alpha_a : F(a) \longrightarrow G(a)\}_{a \in A}$ and a 2-cell $m : \alpha \rightsquigarrow \beta$ is then a collection of natural transformations $\{m_a : \alpha_a \Rightarrow \beta_a\}_{a \in A}$ such that for $f : a \longrightarrow b$ in A, $1_{Gf} * m_a = m_b * 1_{Ff}$ (see diagram below) where 1_{Ff} and 1_{Gf} are the identity natural transformations on the functors Ff and Gf respectively, and $*$ denotes the standard horizontal composition of natural transformations (cf. page 43 in [14]).

$$
\begin{tikzcd}
a \arrow[d, "f"] \\
b
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=large]
F(a) \arrow[r, bend left, "\alpha_a"] \arrow[r, bend right, "\beta_a"'] \arrow[r, phantom, "\Downarrow m_a"] \arrow[d, "Ff"'] & G(a) \arrow[d, "Gf"] \\
F(b) \arrow[r, bend left, "\alpha_b"] \arrow[r, bend right, "\beta_b"'] \arrow[r, phantom, "\Downarrow m_b"] & G(b)
\end{tikzcd}
$$

Each m_a again consists of components (arrows in the category $G(a)$) corresponding to objects x of $F(a)$: $(m_a)_x : \alpha_a(x) \longrightarrow \beta_a(x)$. (To avoid the clutter of multiple pairs of parentheses, $(m_a)_x$ will sometimes be written simply as m_{ax}: for example, the image of the arrow $(m_a)_x$ under a functor φ_a will be written as $\varphi_a(m_{ax})$ rather than $\varphi_a((m_a)_x)$). Given functors $F, G : A \longrightarrow \flat\mathrm{CAT}$ and natural transformations $\alpha, \beta, \gamma : F \Rightarrow G$, composition of arrows (2-cells) $m : \alpha \rightsquigarrow \beta$ and $n : \beta \rightsquigarrow \gamma$ in $\flat\mathrm{CAT}^A(F, G)$ is defined for $a \in A$ by $(n \odot m)_a = n_a \cdot m_a$ where $n_a \cdot m_a$ is the standard vertical composition of natural transformations (page 42 in [14]) with the component for $x \in F(a)$ given by the composition of arrows in the category $G(a)$: $(n \odot m)_{ax} = n_{ax} m_{ax} : \alpha_a(x) \longrightarrow \gamma_a(x)$. This definition implies that composition of arrows in $\flat\mathrm{CAT}^A(F, G)$ is associative. The identity 2-cell $1_\alpha : \alpha \rightsquigarrow \alpha$ on $\alpha : F \Rightarrow G$ consists of the collection of identity arrows $\{1_{\alpha_a(x)} \,|\, a \in A \text{ and } x \in F(a)\}$ in $G(a)$. Given a triple of objects F, G, M in $\flat\mathrm{CAT}^A$, the composition functor

$c_{_{FGM}} : \flat\mathrm{CAT}^A(F, G) \times \flat\mathrm{CAT}^A(G, M) \longrightarrow \flat\mathrm{CAT}^A(F, M)$ is defined for $\alpha : F \Rightarrow G$ and $\varphi : G \Rightarrow M$ by the composition of functors; for an object a in A , $c_{_{FGM}}(\alpha, \varphi)_a = \varphi_a \circ \alpha_a$. (Note: $c_{_{FGM}}(\alpha, \varphi)$ will be denoted by $\varphi\alpha$; thus $(\varphi\alpha)_a = \varphi_a \circ \alpha_a$.)

The horizontal composition $c_{_{FGM}}(m,n) = n * m$ of 2-cells m and n as pictured below

$$\begin{tikzcd} F \arrow[r, Rightarrow, bend left, "\alpha"] \arrow[r, Rightarrow, bend right, "\beta"'] \arrow[r, phantom, "\Downarrow m"] & G \arrow[r, Rightarrow, bend left, "\varphi"] \arrow[r, Rightarrow, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow n"] & M \end{tikzcd}$$

is given for an object a in A

$$\begin{tikzcd} F(a) \arrow[r, bend left, "\alpha_a"] \arrow[r, bend right, "\beta_a"'] \arrow[r, phantom, "\Downarrow m_a"] & G(a) \arrow[r, bend left, "\varphi_a"] \arrow[r, bend right, "\psi_a"'] \arrow[r, phantom, "\Downarrow n_a"] & M(a) \end{tikzcd}$$

by $(n * m)_a = n_a * m_a : \varphi_a \circ \alpha_a \Rightarrow \psi_a \circ \beta_a$ where $n_a * m_a$ is again the horizontal composition of natural transformations: for $x \in F(a)$, $(n_a * m_a)_x$ is given by the diagonal arrow of the commutative square below.

$$\begin{tikzcd} (\varphi_a \circ \alpha_a)(x) \arrow[r, "(n_a)_{\alpha_a(x)}"] \arrow[d, "\varphi_a(m_{ax})"'] & (\psi_a \circ \alpha_a)(x) \arrow[d, "\psi_a(m_{ax})"] \\ (\varphi_a \circ \beta_a)(x) \arrow[r, "(n_a)_{\beta_a(x)}"] & (\psi_a \circ \beta_a)(x) \end{tikzcd}$$

For $c_{_{FGM}}$ to be a functor, it must preserve identities and composition: Given an identity two cell $(1_\alpha, 1_\varphi)$ in $\flat\mathrm{CAT}^A(F,G) \times \flat\mathrm{CAT}^A(G,M)$, the component of $c_{_{FGM}}(1_\alpha, 1_\varphi)$ corresponding to objects a in A and x in $F(a)$ is $1_{\varphi_a(\alpha_a(x))}$, the identity arrow on the object $\varphi_a(\alpha_a(x))$ in the category $M(a)$, which is the same as the corresponding component of $1_{c_{_{FGM}}(\alpha,\varphi)} = 1_{\varphi\alpha}$. $c_{_{FGM}}$ preserving the composition of 2-cells translates for m_1, m_2, n_1 and n_2 as pictured below

$$\begin{tikzcd} F \arrow[r, Rightarrow, bend left=60] \arrow[r] \arrow[r, Rightarrow, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow m_1"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow m_2"] & G \arrow[r, Rightarrow, bend left=60] \arrow[r] \arrow[r, Rightarrow, bend right=60] \arrow[r, phantom, shift left=2.5ex, "\Downarrow n_1"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow n_2"] & M \end{tikzcd}$$

to $(m_2 \odot m_1) * (n_2 \odot n_1) = (n_1 * m_1) \odot (n_2 * m_2)$, which for an object a in A becomes the standard interchange law for the natural transformations $(m_1)_a, (m_2)_a, (n_1)_a$

and $(n_2)_a$. Given an object F in $\flat\mathrm{CAT}^A$, the unit 1-cell in $\flat\mathrm{CAT}^A(F,F)$ is the identity natural transformation $1_F : F \Rightarrow F$ consisting of the identity functors $1_{F(a)} : F(a) \longrightarrow F(a)$ for an object a in A and the unit 2-cell on 1_F consists of the identity arrows $\{1_x \,|\, x \in F(a)\}$. The associativity and unit axioms hold as a direct consequence of the definitions involved.

Definition 2.2 Let $P : E \longrightarrow B$ be a functor. Given an object $b \in B$, the *categorical fiber* of P over b is the subcategory $P^{-1}(b)$ of E defined to have as objects $e \in E$ such that $P(e) = b$ and as morphisms $f : e_1 \longrightarrow e_2$ in E such that $P(f) = 1_b$. The morphisms in the categorical fiber over any object are called vertical morphisms. $V(E)$ will denote the subcategory of E that has the same objects as E, and its morphisms are the vertical morphisms of E. ($V(E)$ *is* a subcategory of E, since composition of vertical morphisms gives vertical morphisms and the identity morphisms are vertical.)

Definition 2.3 Given a category A and subcategories R and L, (R, L) is called a *splitting* of A, if for each $f : a \longrightarrow b$ in A, f has a unique decomposition as $f = lr$ with $l \in L$ and $r \in R$.

Definition 2.4 Given $P : E \longrightarrow B$ and a splitting (H, V) of E, (E, V, H) will be called a *horizontal structure* over B if it satisfies the following properties: (i) V is a subcategory of $V(E)$, and (ii) each morphism $f : a \longrightarrow b$ in B lifts uniquely to any given domain in the fiber over a in E, with the lift $\overline{f}$ in H. The morphisms of H will be called *horizontal morphisms.* Then given a horizontal structure over

B, every morphism $t : e_1 \longrightarrow e_2$ in E factors uniquely as $t = vh$ with $v \in V$ and $h \in H$, and h is the unique horizontal lift of $P(t)$ to domain e_1.

Remarks 2.5 (i) H and V, being subcategories, contain all the identity morphisms of E; condition (ii) in the definition above then implies that $H \cap V = \{1_e\}_{e \in E}$.

(ii) For morphisms $f : a \longrightarrow b$ and $g : b \longrightarrow c$ in B, if $\bar{f}$ is the horizontal lift of f to domain e in E and $\bar{g}$ is the horizontal lift of g to $\mathrm{cod}(\bar{f})$, then $\bar{g}\bar{f} = \overline{gf}$: H being a subcategory means that $\bar{g}\bar{f}$ is in H, therefore both $\bar{g}\bar{f}$ and $\overline{gf}$ are horizontal lifts of gf to the same domain; by the uniqueness of horizontal lifts then, $\bar{g}\bar{f} = \overline{gf}$.

(iii) The horizontal lift of an isomorphism is an isomorphism: Suppose $f : a \longrightarrow b$ is an isomorphism in B, $\bar{f}$ is its horizontal lift to domain e in E and $\overline{f^{-1}}$ is the horizontal lift of f^{-1} to domain $\mathrm{cod}(\bar{f}) \stackrel{def}{=} \bar{e}$; by (ii) above, $\overline{f^{-1}}\bar{f} = \overline{f^{-1}f} = \overline{1_a}$ which means that both $\overline{f^{-1}}\bar{f}$ and 1_e are horizontal lifts of 1_a to domain e; then by the uniqueness of horizontal lifts, we have that $\overline{f^{-1}}\bar{f} = 1_e$ and by a similar argument $\bar{f}\,\overline{f^{-1}} = 1_{\bar{e}}$. Hence $\overline{f^{-1}}$ is the inverse of $\bar{f}$ in E.

Definition 2.6 Define the 2-category H_B of *horizontal structures* over B as follows: The 0-cells are horizontal structures $(E, V, H) \xrightarrow{P} B$ as in Definition 2.4. The 1-cells are functors $\Phi : E_1 \longrightarrow E_2$ such that for the 0-cells $(E_1, V_1, H_1) \xrightarrow{P_1} B$ and $(E_2, V_2, H_2) \xrightarrow{P_2} B$, $P_1 = \Phi P_2$ and that Φ preserves both the horizontal and vertical morphisms, i.e., $h \in H_1 \Rightarrow \Phi(h) \in H_2$ and $v \in V_1 \Rightarrow \Phi(v) \in V_2$. (Such functors also preserve the horizontal-vertical decomposition of arrows.) The 2-cells of H_B are natural transformations $\alpha : \Phi \Rightarrow \Psi$ such that for each object e in E_1,

$\alpha_e : \Phi(e) \longrightarrow \Psi(e)$ is a vertical morphism in V_2. For each pair $(E_1, V_1, H_1) \xrightarrow{P_1} B$ and $(E_2, V_2, H_2) \xrightarrow{P_2} B$ of 0-cells, the 1-cells $\Phi : E_1 \longrightarrow E_2$ and 2-cells $\alpha : \Phi \Rightarrow \Psi$ must form a category (will be denoted by $H_B(E_1, E_2)$). Composition of arrows in the category $H_B(E_1, E_2)$ is defined componentwise: given objects Φ, Ψ, and Υ and arrows $\alpha : \Phi \Rightarrow \Psi$ and $\beta : \Psi \Rightarrow \Upsilon$ in $H_B(E_1, E_2)$, $\begin{tikzcd} E_1 \arrow[r, bend left=60, "\Phi"] \arrow[r, "\Psi" description] \arrow[r, bend right=60, "\Upsilon"'] \arrow[r, phantom, shift left=2.5ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2.5ex, "\Downarrow\beta"] & E_2 \end{tikzcd}$, the composition $\beta \odot \alpha : \Phi \Rightarrow \Upsilon$ is defined for each object e in E_1 by $(\beta \odot \alpha)_e = \beta_e \alpha_e : \Phi(e) \longrightarrow \Upsilon(e)$. The associativity of composition of 2-cells then follows from the associativity of composition of morphisms in E_2. Since both $\alpha_e : \Phi(e) \longrightarrow \Psi(e)$ and $\beta_e : \Psi(e) \longrightarrow \Upsilon(e)$ are in V_2 and V_2 is closed under composition, $\beta_e \alpha_e$ is in V_2 as well, so $\beta \odot \alpha$ is well defined. For each triple $(E_i, V_i, H_i) \xrightarrow{P_i} B$, $i = 1, 2, 3$ of 0-cells, we must have a bifunctor $c : H_B(E_1, E_2) \times H_B(E_2, E_3) \longrightarrow H_B(E_1, E_3)$. For 1-cells, c is the usual composition of functors. For 2-cells $\alpha : \Phi_1 \Rightarrow \Psi_1$ and $\beta : \Phi_2 \Rightarrow \Psi_2$,

$$\begin{tikzcd} E_1 \arrow[r, bend left, "\Phi_1"] \arrow[r, bend right, "\psi_1"'] \arrow[r, phantom, "\Downarrow\alpha"] & E_2 \arrow[r, bend left, "\Phi_2"] \arrow[r, bend right, "\Psi_2"'] \arrow[r, phantom, "\Downarrow\beta"] & E_3 \end{tikzcd}$$

$c(\alpha, \beta) = \beta * \alpha : \Phi_2\Phi_1 \longrightarrow \Psi_2\Psi_1$ is defined by the usual horizontal composition of natural transformations, i.e., for each object e of E_1, as the diagonal of the commutative square below:

$$\begin{tikzcd} \Phi_2\Phi_1(e) \arrow[r, "\beta_{\Phi_1(e)}"] \arrow[d, "\Phi_2(\alpha_e)"'] & \Psi_2\Phi_1(e) \arrow[d, "\Psi_2(\alpha_e)"] \\ \Phi_2\Psi_1(e) \arrow[r, "\beta_{\Psi_1(e)}"] & \Psi_2\Psi_1(e) \end{tikzcd}$$

The functoriality of c follows from the standard interchange laws for the vertical and horizontal composition of natural transformations: $(\delta * \beta) \circ (\gamma * \alpha) = (\beta \circ \alpha) * (\delta \circ \gamma)$.

$$\begin{tikzcd} E_1 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & E_2 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=3, "\Downarrow\gamma"] \arrow[r, phantom, shift right=3, "\Downarrow\delta"] & E_3 \end{tikzcd}$$

As the unit 1-cells are the identity functors and the unit 2-cell on a 1-cell $\Phi : E \longrightarrow E$ is the identity natural transformation given by the identity arrows $1_e : e \longrightarrow e$ for any object e of E, the required unit axioms and associativity axioms hold.

Definition 2.7 (cf. Definition 7.2.1 in [3]) Given two 2-categories A and B, a functor $F : A \longrightarrow B$ on the underlying categories A and B is a *2-functor* if for each pair of objects a and b in A, F induces a functor $F_{ab} : A(a, b) \longrightarrow B(Fa, Fb)$ such that F_{ab} is compatible with composition and units, i.e., such that the following diagrams commute:

$$\begin{tikzcd} A(a,b) \times A(b,c) \arrow[r, "c_{abc}"] \arrow[d, "F_{ab} \times F_{bc}"'] & A(a,c) \arrow[d, "F_{ac}"] \\ B(Fa,Fb) \times B(Fb,Fc) \arrow[r, "c_{FaFbFc}"] & B(Fa,Fc) \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{1} \arrow[r, "u_a"] \arrow[dr, "u_{Fa}"'] & A(a,a) \arrow[d, "F_{aa}"] \\ & B(Fa,Fa) \end{tikzcd}$$

Definition 2.8 (cf. Definition 7.2.2 in [3]) Given two 2-categories A, B and two 2-functors $F, G : A \longrightarrow B$, a natural transformation $\alpha : F \Rightarrow G$ is a *2-natural transformation,* if for each pair of objects a, b in A, the following diagram of categories commutes:

$$\begin{array}{ccc} A(a,b) & \xrightarrow{F_{ab}} & B(Fa,Fb) \\ {\scriptstyle G_{ab}}\big\downarrow & & \big\downarrow{\scriptstyle R_\alpha} \\ B(Ga,Gb) & \xrightarrow{S_\alpha} & B(Fa,Gb) \end{array}$$

Both functors R_α and S_α above act via composition: for $f : Fa \longrightarrow Fb$,

$R_\alpha(f) = \alpha_b \circ f$ and for a 2-cell φ in $B(Fa,Fb)$, $R_\alpha(\varphi) = 1_{\alpha_b} * \varphi$; for $g : Ga \longrightarrow Gb$,

$S_\alpha(g) = g \circ \alpha_a$ and for a 2-cell γ in $B(Ga,Gb)$, $S_\alpha(\gamma) = \gamma * 1_{\alpha_a}$.

The purpose of the following (2.9 - 2.11) is to define a 2-functor

$\mathcal{K} : \mathrm{CAT}^{\mathrm{op}} \longrightarrow 2-\mathrm{Cat}$, where $2-\mathrm{Cat}$ is the category of 2-categories, 2-functors and 2-natural transformations.

Construction 2.9 Given categories A and B, a functor $F : A \longrightarrow B$ induces a 2-functor $F^* : \flat\mathrm{CAT}^B \longrightarrow \flat\mathrm{CAT}^A$ as follows: For a functor $T : B \longrightarrow \flat\mathrm{CAT}$, $F^*(T) = T \circ F$; given a pair of objects T_1, T_2 in $\flat\mathrm{CAT}^B$, F^* must define a functor $F^*_{1,2} : \flat\mathrm{CAT}^B(T_1,T_2) \longrightarrow \flat\mathrm{CAT}^A(T_1 \circ F, T_2 \circ F)$. Given a natural transformation $\alpha : T_1 \Rightarrow T_2$ with components $\{\alpha_b : T_1(b) \longrightarrow T_2(b) \,|\, b \in B\}$, $F^*_{1,2}(\alpha) : T_1 \circ F \Rightarrow T_2 \circ F$ is defined to be the natural transformation with $(F^*_{1,2}(\alpha))_a = \alpha_{Fa}$ as its component corresponding to an object a in A. $F^*_{1,2}(\alpha)$ will be denoted by α_F. Given 1-cells $\alpha,\ \beta$ and a 2-cell $m : \alpha \rightsquigarrow \beta$ in $\flat\mathrm{CAT}^B(T_1,T_2)$

$$\begin{tikzcd} T_1 \arrow[r, Rightarrow, bend left, "\alpha"] \arrow[r, Rightarrow, bend right, "\beta"'] \arrow[r, phantom, "\Downarrow m"] & T_2 \end{tikzcd}$$

with components the arrows $\{m_{bx} : \alpha_b(x) \longrightarrow \beta_b(x)\}$ in $T_2(b)$ for b in B and $x \in T_1(b)$, the corresponding 2-cell $F^*_{1,2}(m)$ (denoted below for short by m_F)

$$
\begin{tikzcd}
T_1 \circ F \arrow[r, bend left, Rightarrow, "\alpha_F"] \arrow[r, bend right, Rightarrow, "\beta_F"'] \arrow[r, phantom, "\Downarrow m_F"] & T_2 \circ F
\end{tikzcd}
$$

has components $\{(m_{Fa})_x : \alpha_{Fa}(x) \longrightarrow \beta_{Fa}(x)\}$ for a in A and x in $T_1(F(a))$. For an identity 2-cell 1_α on $\alpha : T_1 \Rightarrow T_2$, $F^*_{1,2}(1_\alpha)$ has components the identity arrows $\alpha_{Fa}(x) \xrightarrow{1} \alpha_{Fa}(x)$ in $T_2F(a)$; $F^*_{1,2}$ then preserves identities. Since the vertical composition of 2-cells is defined componentwise, by composition of arrows in a category, $F^*_{1,2}$ preserves composition as well. The compatibility of $F^*_{1,2}$ with horizontal composition is equivalent to the commutativity of the diagram below (the subscripts for the composition functor c were omitted).

$$
\begin{array}{ccc}
\flat\mathrm{CAT}^B(T_1,T_2) \times \flat\mathrm{CAT}^B(T_2,T_3) & \xrightarrow{c} & \flat\mathrm{CAT}^B(T_1,T_3) \\
{\scriptstyle F^*_{1,2}\times F^*_{2,3}}\downarrow & & \downarrow{\scriptstyle F^*_{1,3}} \\
\flat\mathrm{CAT}^A(T_1\circ F,T_2\circ F) \times \flat\mathrm{CAT}^A(T_2\circ F,T_3\circ F) & \xrightarrow{c} & \flat\mathrm{CAT}^A(T_1\circ F,T_3\circ F)
\end{array}
$$

For a 1-cell (α, φ) in $\flat\mathrm{CAT}^B(T_1,T_2) \times \flat\mathrm{CAT}^B(T_2,T_3)$, $c(\alpha,\varphi) = \varphi\alpha$ is defined via composition of functors; for an object b in B, $(\varphi\alpha)_b = \varphi_b \circ \alpha_b : T_1(b) \longrightarrow T_3(b)$ and then $F^*_{1,3}(\varphi\alpha)$ is the natural transformation with component for a in A the functor $(\varphi\alpha)_{Fa} = \varphi_{Fa} \circ \alpha_{Fa} : T_1F(a) \longrightarrow T_3F(a)$. Going the other way around the diagram gives first the natural transformation $(F^*_{1,2} \times F^*_{2,3})(\alpha,\varphi) = (\alpha_F, \varphi_F)$, whose image under the composition functor c is again $\varphi_F\alpha_F$ with components $\varphi_{Fa} \circ \alpha_{Fa}$ for an object a in A. The diagram then commutes for 1-cells.

For appropriate 2-cells m and n, the commutativity of the diagram translates to $(n * m)_F = n_F * m_F$. The components of both 2-cells corresponding to an object

a in A are natural transformations $\varphi_{Fa} \circ \alpha_{Fa} \Rightarrow \psi_{Fa} \circ \beta_{Fa}$ (where $m : \alpha \rightsquigarrow \beta$ and $n : \varphi \rightsquigarrow \psi$). By the definition of $n * m$, $(n * m)_{Fa} = n_{Fa} * m_{Fa}$. The diagram then commutes for 2-cells as well.

The unit axiom is equivalent to the commutativity of the diagram below:

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "u_T"] \arrow[dr, "u_{TF}"'] & \mathrm{CAT}^B(T,T) \arrow[d, "F^*_{TT}"] \\
& \mathrm{CAT}^A(TF,TF)
\end{tikzcd}$$

Since the unit 1-cells have as components the identity functors and the unit 2-cells have identity arrows as components both of which are preserved by F^*, the diagram above does commute. $F^* : \mathrm{CAT}^B \longrightarrow \mathrm{CAT}^A$ is then a 2-functor.

Lemma 2.10 *A natural transformation $\alpha : F \Rightarrow G$ for the functors $F, G : A \longrightarrow B$ induces a 2-natural transformation $\alpha^* : F^* \Rightarrow G^*$ where $F^*, G^* : CAT^B \longrightarrow CAT^A$ are the 2-functors defined in the previous construction.*

Proof. The component α^*_T of α^* corresponding to an object T in CAT^B must be shown to be a natural transformation (a 1-cell in CAT^A) $\alpha^*_T : TF \Rightarrow TG$. The component $(\alpha^*_T)_a$ of α^*_T corresponding to an object $a \in A$ is the functor defined by the image of the arrow $\alpha_a : F(a) \longrightarrow G(a)$ in B under the functor T; $(\alpha^*_T)_a = T(\alpha_a) : TF(a) \longrightarrow TG(a)$. Given a pair of objects T_1, T_2 in $\flat\mathrm{CAT}^B$, the following diagram must commute for α^* to be a 2-natural transformation:

$$\begin{tikzcd}
\flat\mathrm{CAT}^B(T_1,T_2) \arrow[r, "F^*_{1,2}"] \arrow[d, "G^*_{1,2}"'] & \flat\mathrm{CAT}^A(T_1F,T_2F) \arrow[d, "R_{\alpha^*}"] \\
\flat\mathrm{CAT}^A(T_1G,T_2G) \arrow[r, "S_{\alpha^*}"] & \flat\mathrm{CAT}^A(T_1F,T_2G)
\end{tikzcd}$$

where R_{α^*} and S_{α^*} are as in Definition 2.8. For a 1-cell $\varphi : T_1 \Rightarrow T_2$ and an object a in A, the commutativity of the diagram means $T_2(\alpha_a) \circ \varphi_{_{Fa}} = \varphi_{_{Ga}} \circ T_1(\alpha_a)$; this equality follows from the naturality of φ:

$$\begin{tikzcd}
T_1F(a) \arrow[r, "\varphi_{Fa}"] \arrow[d, "T_1(\alpha_a)"'] & T_2F(a) \arrow[d, "T_2(\alpha_a)"] \\
T_1G(a) \arrow[r, "\varphi_{Ga}"] & T_2G(a)
\end{tikzcd}$$

For a 2-cell m in $\mathrm{CAT}^B(T_1,T_2)$

$$\begin{tikzcd}
T_1 \arrow[r, Rightarrow, bend left, "\varphi"] \arrow[r, Rightarrow, bend right, "\psi"'] \arrow[r, phantom, "\Downarrow m"] & T_2
\end{tikzcd}$$

the commutativity of the square above translates to the commutativity of the diagram below:

$$\begin{tikzcd}
T_1F \arrow[r, Rightarrow, bend left, "\varphi_F"] \arrow[r, Rightarrow, bend right, "\psi_F"'] \arrow[r, phantom, "\Downarrow m_F"] \arrow[d, "\alpha^*_{T_1}=T_1(\alpha)"'] & T_1F \arrow[d, "\alpha^*_{T_2}=T_2(\alpha)"] \\
T_1G \arrow[r, Rightarrow, bend left, "\varphi_G"] \arrow[r, Rightarrow, bend right, "\psi_G"'] \arrow[r, phantom, "\Downarrow m_G"] & T_2G
\end{tikzcd}$$

Since $1_{T_2(\alpha)} * m_{_F} = m_{_G} * 1_{T_1(\alpha)}$ follows from the definition of 2-cells in CAT^A, α^* is a 2-natural transformation.

Proposition 2.11 *The previous two constructions define a 2-functor* $\mathcal{K} : CAT^{\mathrm{op}} \longrightarrow 2-Cat$ *with* $\mathcal{K}$ *acting on 0, 1 and 2-cells as follows:* $\mathcal{K}(A) = \flat CAT^A$, $\mathcal{K}(F) = F^*$ *and* $\mathcal{K}(\alpha) = \alpha^*$.

Proof. Given categories A and B, $\mathcal{K}$ must induce a functor $\mathcal{K}_{AB} : \mathrm{CAT}(A,B) \longrightarrow 2-\mathrm{Cat}(\flat\mathrm{CAT}^B, \flat\mathrm{CAT}^A)$ such that the following diagram commutes:

$$\begin{tikzcd}
\flat\mathrm{CAT}(A,B) \times \flat\mathrm{CAT}(B,C) \arrow[r, "c"] \arrow[d, "\mathcal{K}_{AB}\times\mathcal{K}_{BC}"'] & \flat\mathrm{CAT}(A,C) \arrow[d, "\mathcal{K}_{A,C}"] \\
\text{2-Cat}(\mathrm{CAT}^B, \mathrm{CAT}^A) \times \text{2-Cat}(\mathrm{CAT}^C, \mathrm{CAT}^B) \arrow[r, "c"] & \text{2-Cat}(\mathrm{CAT}^C, \mathrm{CAT}^A)
\end{tikzcd}$$

For 1-cells $F : A \longrightarrow B$ and $G : B \longrightarrow C$, $(GF)^* = F^*G^*$ holds since for T in CAT^C, $(GF)^*(T) = T(GF)$ and $F^*(G^*(T)) = F^*(TG) = (TG)F$; for $\varphi : T_1 \Rightarrow T_2$ in CAT^C, both $(GF)^*(\varphi)$ and $F^*G^*(\varphi)$ result in the natural transformation

$\varphi_{GF} : T_1GF \Rightarrow T_2GF$ (as defined in Construction 2.9).

Given 2-cells φ, γ (and then φ^* and γ^*) as pictured,

$$\begin{tikzcd}
A \arrow[d, bend right, "F_1"'] \arrow[d, bend left, "F_2"] \arrow[d, phantom, "\overset{\varphi}{\Rightarrow}"] & & \mathrm{CAT}^A \\
B \arrow[d, bend right, "G_1"'] \arrow[d, bend left, "G_2"] \arrow[d, phantom, "\overset{\gamma}{\Rightarrow}"] \arrow[rr, "\mathcal{K}"] & & \mathrm{CAT}^B \arrow[u, bend left, "F_1^*"] \arrow[u, bend right, "F_2^*"'] \arrow[u, phantom, "\overset{\varphi^*}{\Rightarrow}"] \\
C & & \mathrm{CAT}^C \arrow[u, bend left, "G_1^*"] \arrow[u, bend right, "G_2^*"'] \arrow[u, phantom, "\overset{\gamma^*}{\Rightarrow}"]
\end{tikzcd}$$

the commutativity of the diagram translates to the equation $(\gamma * \varphi)^* = \varphi^* * \gamma^*$. For T in CAT^C, by Lemma 2.10, the component of $(\gamma * \varphi)^*$ corresponding to T is the natural transformation $(\gamma * \varphi)^*_T : TG_1F_1 \Rightarrow TG_2F_2$ that has as the component for a in A the functor $((\gamma * \varphi)^*_T)_a : TG_1F_1(a) \longrightarrow TG_2F_2(a)$ which is the image of the arrow $(\gamma * \varphi)_a : G_1F_1(a) \longrightarrow G_2F_2(a)$ in C under the functor T: $((\gamma * \varphi)^*_T)_a = T((\gamma * \varphi)_a)$, where $(\gamma * \varphi)_a$ is the diagonal of the commutative square (in C) below.

$$\begin{tikzcd}
G_1F_1(a) \arrow[r, "\gamma_{F_1 a}"] \arrow[d, "G_1(\varphi_a)"'] \arrow[dr, dotted, "(\gamma * \varphi)_a"] & G_2F_1(a) \arrow[d, "G_2(\varphi_a)"] \\
G_1F_2(a) \arrow[r, "\gamma_{F_2 a}"'] & G_2F_2(a)
\end{tikzcd}$$

Since (for $i = 1, 2$) $F_i^* G_i^*(T) = TG_iF_i$, by the definition of F_i^* (Construction 2.9) $F_i^*(\gamma^*_T) = (\gamma^*_T)_{F_i}$ and on the other hand (by Lemma 2.10), $(\gamma^*_T)_{F_i} = T(\gamma_{F_i})$ and $\varphi^*_{TG_i} = TG_i(\varphi)$, applying the functor T to the diagram above gives the commutative square, whose diagonal, by the definition of the horizontal composition of natural transformations in 2-Cat is $((\varphi^* * \gamma^*)_T)_a$:

$$\begin{tikzcd}
TG_1F_1(a) = F_1^*G_1^*(T)(a) \arrow[r, "T(\gamma_{F_1 a}) = F_1^*(\gamma^*_T)(a)"] \arrow[d, "TG_1(\varphi_a) = \varphi^*_{TG_1}"'] \arrow[dr, dotted, "((\gamma^* * \varphi^*)_T)_a"] & TG_2F_1(a) = F_1^*G_2^*(T)(a) \arrow[d, "TG_2(\varphi_a) = \varphi^*_{TG_2}"] \\
TG_1F_2(a) = F_2^*G_1^*(T)(a) \arrow[r, "T(\gamma_{F_2 a}) = F_2^*(\gamma^*_T)(a)"'] & TG_2F_2(a) = F_2^*G_2^*(T)(a)
\end{tikzcd}$$

Thus $(\gamma * \varphi)^* = \gamma^* * \varphi^*$ since the components of the natural transformations are equal. $\mathcal{K}_{AB}$ must also honor the vertical composition of natural transformations:

$\begin{tikzcd} A \arrow[r, bend left=60, ""{name=U}] \arrow[r, ""{name=M}] \arrow[r, bend right=60, ""{name=D}] \arrow[from=U, to=M, phantom, "\Downarrow\alpha"] \arrow[from=M, to=D, phantom, "\Downarrow\beta"] & B \end{tikzcd}$. The component of $\mathcal{K}(\beta \odot \alpha) = (\beta \odot \alpha)^*$ corresponding to an object a in A and T in CAT^B is $T(\beta_a \alpha_a)$, whereas the corresponding component of $\mathcal{K}(\beta) \odot \mathcal{K}(\alpha) = \beta^* \odot \alpha^*$ is $T(\alpha_a)T(\beta_a)$; since T is a functor, we have that $T(\beta_a \alpha_a) = T(\alpha_a)T(\beta_a)$. As a direct consequence of the definitions involved, all unit axioms are satisfied as well, and therefore $\mathcal{K}$ is well defined.

Definition 2.12 (cf. Definition 7.5.1 in [3]) A 2-functor $F : A \longrightarrow B$ is called a *lax 2-functor,* if it preserves composition and identities up to coherent 2-cells, i.e., for every triple of objects a, b, c in A, there is a natural transformation δ_{abc} and for every object a in A, there is a natural transformation ε_a such that the following diagrams commute.

$$\begin{tikzcd} A(a,b) \times A(b,c) \arrow[r, "c"] \arrow[d, "F_{ab} \times F_{bc}"'] \arrow[dr, phantom, "\overset{\delta_{abc}}{\Nearrow}"] & A(a,c) \arrow[d, "F_{ac}"] \\ B(Fa,Fb) \times B(Fb,Fc) \arrow[r, "c"] & B(Fa,Fc) \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{1} \arrow[r, "u_a"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\varepsilon_a}{\Nearrow}"] & A(a,a) \arrow[d, "F_{ac}"] \\ \mathbf{1} \arrow[r, "u_{Fa}"] & B(Fa,Fa) \end{tikzcd}$$

The natural transformations δ and ε must satisfy the following coherence axioms. (The component of δ_{abc} corresponding to the pair of arrows (f, g) will be denoted by $\delta_{f,g}$ rather than $(\delta_{abc})_{(f,g)}$.)

Coherence with composition: for any triple of arrows $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ in A, we must have $\delta_{gf,h} \odot (1_{Fh} * \delta_{f,g}) = \delta_{f,hg} \odot (\delta_{g,h} * 1_{Ff})$.

Coherence with units: $\delta_{1_a,f} \odot (1_{Ff} * \varepsilon_a) = 1_{Ff}$ and $\delta_{f,1_b} \odot (\varepsilon_b * 1_{Ff}) = 1_{Ff}$ (see diagrams below).

$$\begin{tikzcd}
Fh \circ Fg \circ Ff \arrow[r, "1_{Fh} * \delta_{f,g}"] \arrow[d, "\delta_{f,g} * 1_{Ff}"'] & Fh \circ F(g \circ f) \arrow[d, "\delta_{gf,h}"] \\
F(h \circ g) \circ Ff \arrow[r, "\delta_{f,hg}"] & F(h \circ g \circ f)
\end{tikzcd}$$

$$\begin{tikzcd}
Ff \circ 1_{Fa} \arrow[r, "1_{Ff} * \varepsilon_a"] \arrow[d, "1_{Ff}"'] & Ff \circ F(1_a) \arrow[d, "\delta_{1_a,f}"] \\
Ff \arrow[r, "1_{Ff}"'] & F(f \circ 1_a)
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{Fb} \circ Ff \arrow[r, "\varepsilon_b * 1_{Ff}"] \arrow[d, "1_{Ff}"'] & F(1_b) \circ Ff \arrow[d, "\delta_{f,1_b}"] \\
Ff \arrow[r, "1_{Ff}"'] & F(1_b \circ f)
\end{tikzcd}$$

F will be called a strong-lax 2-functor when the natural transformations δ_{abc} and ε_a are natural isomorphisms for any objects a, b and c. (Such functors are also referred to as pseudo, or weak functors.)

The purpose of the constructions outlined in (2.13 - 2.19) is to define a strong-lax 2-functor $\mathcal{H} : \flat\mathrm{CAT}^{\mathrm{op}} \longrightarrow 2 - \mathrm{Cat}$.

Remark 2.13 Given a morphism $\Phi : [(E_1, V_1, H_1) \xrightarrow{P_1} B] \longrightarrow [(E_2, V_2, H_2) \xrightarrow{P_2} B]$ of horizontal structures over B, if $\bar{h} : x \longrightarrow y$ is the horizontal lift of $h : a \longrightarrow b$ to domain x in E_1, then $\Phi(\bar{h}) : \Phi(x) \longrightarrow \Phi(y)$ is the horizontal lift of h to domain $\Phi(x)$ in E_2: since Φ is assumed to preserve horizontal morphisms, $\Phi(\bar{h})$ is horizontal, and $P_1 = P_2\Phi$ implies that it covers h, so it is the horizontal lift of h to domain

$\Phi(x)$ by the uniqueness of horizontal lifts. Adopting the notation $\bar{h}$ for a lift of an arrow h to a domain in E_1, and $\bar{\bar{h}}$ for the lift of h to a domain in E_2, we have that $\Phi(\bar{h}) = \bar{\bar{h}}$.

Construction 2.14 Given a functor $F : A \longrightarrow B$ and a horizontal structure $(E, V, H) \xrightarrow{P} B$ over B, the canonical pullback of $E \xrightarrow{P} B$ along F defines a horizontal structure $(E^F, V^F, H^F) \xrightarrow{P^F} A$ over A. The objects of E^F are pairs (a, x) where a and x are objects in A and E respectively such that $F(a) = P(x)$; the morphisms are pairs (f, g) with f a morphism in A and g a morphism in E such that $F(f) = P(g)$. Composition of morphisms is defined componentwise. The horizontal structure (E^F, V^F, H^F) is given by $V^F = \{(a, x) \xrightarrow{(1_a, g)} (a, y)\}$ with $g \in V$ and by $H^F = \{(a, x) \xrightarrow{(f,h)} (b, y)\}$ where h is in H. In the pullback square below, the functors P^F and G are defined in the obvious way: $P^F(a, x) = a$, $P^F(f, g) = f$ and similarly $G(a, x) = x$, $G(f, g) = g$. It follows from the definitions that both V^F and H^F are subcategories of E^F, and clearly $V^F \subseteq V(E^F)$.

$$
\begin{tikzcd}
(E^F, V^F, H^F) \arrow[r, "G"] \arrow[d, "P^F"'] & (E, V, H) \arrow[d, "P"] \\
A \arrow[r, "F"] & B
\end{tikzcd}
$$

The unique horizontal lift of $f : a \longrightarrow b$ in A to domain (a, x) in E^F is

$(f, \overline{Ff}) : (a, x) \longrightarrow (b, \overline{x})$ where $\overline{Ff}$ is the unique horizontal lift of $Ff : F(a) \longrightarrow F(b)$ to domain x in E; thus $\overline{x} = \mathrm{Cod}(\overline{Ff})$. The unique factorization of a morphism $(a, x) \xrightarrow{(f,g)} (b, y)$ in E^F is $(f, g) = (1_b, v)(f, h)$ where vh is the unique factorization of g in E.

Remark 2.15 The canonical pullback of $E \xrightarrow{P} A$ along the identity functor $1_A : A \longrightarrow A$ results in an isomorphic copy E^{1_A} of E with objects (a, x) such that $P(x) = a$ and morphisms (f, g) such that $P(g) = f$; thus the objects and morphisms of E^{1_A} are of the form $(P(x), x)$ and $(P(g), g)$, respectively.

Lemma 2.16 *Given a morphism $\Phi : E_1 \longrightarrow E_2$ of horizontal structures over B and a functor $F : A \longrightarrow B$, Φ induces a morphism $\Phi^F : E_1^F \longrightarrow E_2^F$ of the pullback horizontal structures.*

Proof. On objects $\Phi^F(a, x) = (a, \Phi(x))$ and on morphisms $\Phi^F(f, g) = (f, \Phi(g))$. Since Φ^F preserves vertical morphisms and $P_1\Phi^F = P^F$, Φ^F is a morphism of horizontal structures, i.e., it is a one cell in H_A.

Construction 2.17 Given morphisms $\Phi, \Psi : E_1 \longrightarrow E_2$ of horizontal structures over B and a natural transformation $\alpha : \Phi \Rightarrow \Psi$, pulling α back along a functor $F : A \longrightarrow B$ gives a natural transformation $\alpha^F : \Phi^F \Rightarrow \Psi^F$. For an object (a, x) in E_1^F, $\alpha^F_{(a,x)} = (1_a, \alpha_x) : \Phi^F(a, x) = (a, \Phi(x)) \longrightarrow \Psi^F(a, x) = (a, \Psi(x))$, where $\alpha_x : \Phi(x) \longrightarrow \Psi(x)$ is the component of the natural transformation α for the object x in E_1; $(1_a, \alpha_x)$ is clearly a vertical morphism, and for $(a, x) \xrightarrow{(f,g)} (b, y)$ in E_1^F, the following diagram commutes

$$\begin{tikzcd}
(a, \Phi(x)) \arrow[r, "{(1_a, \alpha_x)}"] \arrow[d, "{(f, \Phi(g))}"'] & (a, \Psi(x)) \arrow[d, "{(f, \Psi(g))}"] \\
(b, \Phi(y)) \arrow[r, "{(1_b, \alpha_y)}"] & (b, \Psi(y))
\end{tikzcd}$$

since by the naturality of α, $\Psi(g)\alpha_x = \alpha_y\Phi(g)$.

Construction 2.18 Given a functor $F : A \longrightarrow B$, F gives rise to a 2-functor $H_F : H_B \longrightarrow H_A$ as follows: $(E, V, H) \xrightarrow{P} B$ is sent to $(E^F, V^F, H^F) \xrightarrow{P^F} A$ as in Construction 2.14. For a pair of objects $(E_i, V_i, H_i) \xrightarrow{P_i} B$, $i = 1, 2$ in H_B, H_F must define a functor $H_{F_{E_1,E_2}} : H_B(E_1, E_2) \longrightarrow H_A(E_1^F, E_2^F)$. (We'll write $H_{F_{1,2}}$ instead of $H_{F_{E_1,E_2}}$.) For a morphism $\Phi : E_1 \longrightarrow E_2$ of horizontal structures (an object in $H_B(E_1, E_2)$ as defined in 2.6), $H_F(\Phi) = \Phi^F$ as in Lemma 2.16. For a natural transformation $\alpha : \Phi \Rightarrow \Psi$ (an arrow in $H_B(E_1, E_2)$), $H_F(\alpha) = \alpha^F : \Phi^F \longrightarrow \Psi^F$ as defined in Construction 2.17. Given arrows $\Phi \overset{\alpha}{\Rightarrow} \Psi \overset{\beta}{\Rightarrow} \Upsilon$ in $H_B(E_1, E_2)$, $(\beta \odot \alpha)^F = (\beta^F \odot \alpha^F)$, since for an object (a, x) in E_1^F, $(\beta \odot \alpha)^F_{(a,x)} = (1_a, (\beta \odot \alpha)_x) = (1_a, (\beta_x \odot \alpha_x)) = (1_a, \beta_x) \odot (1_a, \alpha_x) = \beta^F_{(a,x)} \odot \alpha^F_{(a,x)} = (\beta^F \odot \alpha^F)_{(a,x)}$. For the identity 2-cell $1 : \Phi \longrightarrow \Phi$ in $H_B(E_1, E_2)$, $1^F : \Phi^F \longrightarrow \Phi^F$ is clearly the identity 2-cell in $H_A(E_1^F, E_2^F)$. $H(F)$ must also satisfy the following compatibility conditions to be a 2-functor. Compatibility with composition translates into the commutativity of the following diagram for objects E_1, E_2 and E_3 in H_B.

$$\begin{tikzcd}
H_B(E_1, E_2) \times H_B(E_2, E_3) \arrow[r, "c"] \arrow[d, "H_{F\,1,2} \times H_{F\,2,3}"'] & H_B(E_1, E_3) \arrow[d, "H_{F\,1,3}"] \\
H_A(E_1^F, E_2^F) \times H_A(E_2^F, E_3^F) \arrow[r, "c"] & H_A(E_1^F, E_3^F)
\end{tikzcd}$$

For the diagram above to commute on the object level, we must have $(\Psi\Phi)^F = \Psi^F \Phi^F$ for functors $\Phi : E_1 \longrightarrow E_2$ and $\Psi : E_2 \longrightarrow E_3$. Given an object (a, x) in E_1^F, $(\Psi\Phi)^F(a, x) = (a, \Psi\Phi(x))$ and $(\Psi^F \Phi^F)(a, x) = \Psi(a, \Phi(x)) = (a, \Psi\Phi(x))$; similarly for an arrow $(f, g) : (a, x) \longrightarrow (b, y)$ in E_1^F, since the functors Φ and Ψ act on the second coordinate, $(\Psi\Phi)^F(f, g) = \Psi^F(\Phi^F(f, g)) = (f, \Psi\Phi(g))$. For the diagram to commute for arrows $\alpha : \Phi_1 \longrightarrow \Psi_1$ and $\beta : \Phi_2 \longrightarrow \Psi_2$ we must have $\beta^F * \alpha^F = (\beta * \alpha)^F$. For an object (a,x) in E_1^F, the components of the natural transformations in question are (by Construction 2.17 and Definition 2.6) as follows: $(\beta * \alpha)^F_{(a,x)} = (1_a, (\beta * \alpha)_x) = (1_a, \Psi_2(\alpha_x) \circ \beta_{\Phi_1(x)})$ and $(\beta^F * \alpha^F)_{(a,x)} = \Psi_2^F(\alpha^F_{(a,x)}) \circ \beta^F_{\Phi_1^F(a,x)} = \Psi_2^F(1_a, \alpha_x) \circ \beta^F_{(a,\Phi_1(x))} = (1_a, \Psi_2(\alpha_x)) \circ (1_a, \beta_{\Phi_1(x)}) = (1_a, \Psi_2(\alpha_x) \circ \beta_{\Phi_1(x)})$. Compatibility with units means that for an object E in H_B, with $1_E : E \longrightarrow E$ the identity functor, $(1_E)^F = 1_{E^F}$ and that for a functor Φ in H_B, with 1_Φ the identity 2-cell on Φ we have $(1_\Phi)^F = 1_{\Phi^F}$. By Lemma 2.16, $(1_E)^F(a,x) = (a, 1_E(x)) = (a,x)$ for an object (a,x) in E^F and similarly for a morphism (f,g) in E^F, $(1_E)^F(f,g) = (f, 1_E(g)) = (f,g)$, so $(1_E)^F = 1_{E^F}$. By Construction 2.17, $(1_\Phi)^F_{(a,x)} = (1_a, (1_\Phi)_x) = (1_a, 1_{\Phi(x)})$ and $(1_{\Phi^F})_{(a,x)} = (1_a, 1_{\Phi(x)})$, we have that $(1_\Phi)^F = 1_{\Phi^F}$, and the 2-functor $H_F : H_B \longrightarrow H_A$ is well defined.

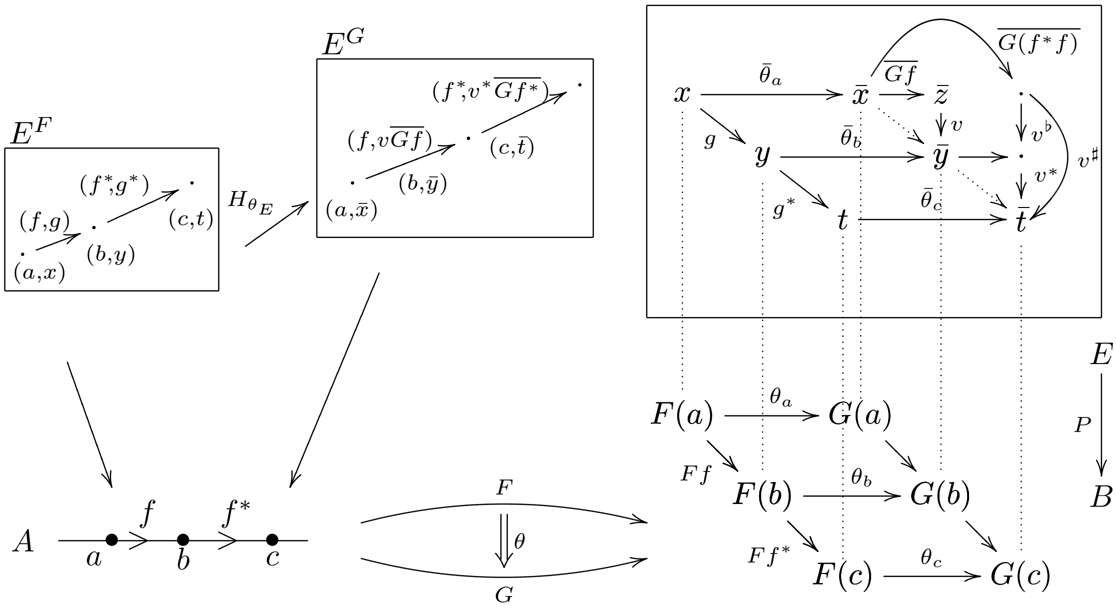
Remark 2.19 For the identity functor $1_A : A \longrightarrow A$ the corresponding 2-functor $H_{1_A} : H_A \longrightarrow H_A$ is an isomorphism of categories, since pulling back an object $E_1 \xrightarrow{P_1} A$ in H_A along the identity functor results in an isomorphic copy $E_1^{1_A}$ of E_1 (see Remark 2.15), and for horizontal structures E_1 and E_2 over A,

$H_{1_A} : H_A(E_1, E_2) \longrightarrow H_A(E_1^{1_A}, E_2^{1_A})$ gives a bijection on both the classes of 1-cells and 2-cells.

Lemma 2.20 *A natural transformation $\theta : F \Rightarrow G$ of functors $F, G : A \longrightarrow B$ defines a 2-natural transformation $H_\theta : H_F \Rightarrow H_G$ where H_F, $H_G : H_B \longrightarrow H_A$ are the 2-functors defined in Construction 2.18.*

Proof. For an object $(E, V, H) \xrightarrow{P} B$ of H_B, the corresponding arrow $H_{\theta_E} : E^F \longrightarrow E^G$ is the functor defined as follows: For an object (a,x) in E^F, $H_{\theta_E}(a,x) =$

$(a, \overline{x})$ where $\overline{x}$ is the codomain of the horizontal lift $\overline{\theta}_a : x \longrightarrow \overline{x}$ of $\theta_a : F(a) \longrightarrow G(a)$ to domain x in E. For a morphism $(f, g) : (a, x) \longrightarrow (b, y)$, $H_{\theta_E}(f, g) : (a, \overline{x}) \longrightarrow (b, \overline{y})$ is constructed as follows: let $\overline{h}$ be the horizontal lift of $h = Gf \circ \theta_a = \theta_b \circ Ff : F(a) \longrightarrow G(b)$ to domain x in E and $\overline{Gf}$ be the horizontal lift of Gf to domain $\overline{x}$. We then have both $x \xrightarrow{\overline{\theta_a}} \overline{x} \xrightarrow{\overline{Gf}} \overline{z}$ and $\overline{h}$ covering h. Since H is a subcategory and both $\overline{\theta}_a$ and $\overline{Gf}$ are in H, so is $\overline{Gf} \circ \overline{\theta}_a$ and then by the uniqueness of horizontal lifts we have $\overline{Gf} \circ \overline{\theta}_a = \overline{h} : x \longrightarrow \overline{z}$. We also have $x \xrightarrow{g} y \xrightarrow{\overline{\theta_b}} \overline{y}$ in E; let vh^* be the horizontal-vertical decomposition of $\overline{\theta_b}g$. Since $P(h^*) = P(vh^*) = P(\overline{\theta}_b g) = \theta_b P(g) = \theta_b Ff = h$, h^* is also a horizontal lift of h, so $h^* = \bar{h}$. Then $\overline{z} = \text{cod}(h^*) = \text{dom}(v)$, and we have $\overline{x} \xrightarrow{\overline{Gf}} \overline{z} \xrightarrow{v} \overline{y}$. For morphisms then, H_{θ_E} is defined by $H_{\theta_E}(f, g) = (f, v\overline{Gf}) : (a, \overline{x}) \longrightarrow (b, \overline{y})$. We must still show that H_{θ_E} preserves composition and identities. Given morphisms $(a, x) \xrightarrow{(f,g)} (b, y) \xrightarrow{(f^*,g^*)} (c, t)$ in E^F, we have that $H_\theta(f^*, g^*) \circ H_\theta(f, g) = (f^* f, v^* \overline{Gf^*}\, v\overline{Gf})$ where v^* is the vertical component of the horizontal-vertical decomposition of $\overline{\theta}_c g^*$ (see figure below) and $H_\theta(f^* f, g^* g) = (f^* f, v^\sharp \overline{G(f^* f)})$ where $v^\sharp$ is the vertical component of the horizontal-vertical decomposition of $\overline{\theta}_c g^* g$. $(\overline{\theta}_c g^* g = v^\sharp(\overline{G(f^* f)}\, \overline{\theta}_a).)$ Thus it is left to show that $v^* \overline{Gf^*}\, v\overline{Gf} = v^\sharp \overline{G(f^* f)}$. Let $v^\flat h^\flat$ be the horizontal-vertical decomposition of $\overline{Gf^*}\, v\overline{Gf}$. Then $\overline{Gf^*}\, v\overline{Gf} = v^\flat h^\flat$ means that $h^\flat$ covers $Gf^* Gf = Gf^* f$ and it is a horizontal morphism with domain $\overline{x}$; the same holds for $\overline{Gf^* f}$, so by the uniqueness of horizontal lifts, $h^\flat = \overline{Gf^* f}$. Then we have that on one hand $\overline{\theta}_c g^* g = v^* \overline{Gf^*}\, \overline{\theta}_b g = v^* \overline{Gf^*}\, v\overline{Gf}\, \overline{\theta}_a = v^* v^\flat h^\flat \overline{\theta}_a = v^* v^\flat \overline{G(f^* f)}\, \overline{\theta}_a$; on the other hand, it follows from the definition of $v^\sharp$, that $\overline{\theta}_c g^* g = v^\sharp \overline{G(f^* f)} \overline{\theta}_a$. Then by the uniqueness of horizontal-vertical decompositions, we have that $v^* v^\flat = v^\sharp$, which then shows that $v^\sharp \overline{G(f * f)} = v^* v^\flat \overline{G(f * f)} = v^* v^\flat h^\flat = v^* \overline{Gf^*}\, v\overline{Gf}$.



Given an object (a,x) in E^F, $1_{(a,x)} = (1_a, 1_x)$; we must show that $H_{\theta_E}(1_a, 1_x) = (1_a, 1_{\overline{x}})$. By the definition of H_{θ_E}, $H_{\theta_E}(1_a, 1_x) = (1_a, v\overline{1_{G(a)}})$ where v is the vertical component of the horizontal vertical decomposition of $\overline{\theta}_a\, 1_x = \overline{\theta}_a$; since $\overline{\theta}_a : x \longrightarrow \overline{x}$ can be written as $\overline{\theta}_a = 1_{\overline{x}}\overline{\theta}_a$, $v = 1_{\overline{x}}$. Moreover, $\overline{1_{G(a)}}$ being a horizontal lift is in H, and it is in V as well, since it covers the identity on $G(a)$ in B; therefore $\overline{1_{G(a)}} = 1_{\overline{x}}$ which shows that $v\overline{1_{G(a)}} = 1_{\overline{x}}$.

For H_θ to define a 2-natural transformation, the following diagram of categories must commute:

$$\begin{tikzcd}
H_B(E_1,E_2) \arrow[r, "H_{F_{(E_1,E_2)}}"] \arrow[d, "H_{G_{(E_1,E_2)}}"'] & H_A(E_1^F,E_2^F) \arrow[d, "H_A(1,H_{\theta_{E_2}})"] \\
H_A(E_1^G,E_2^G) \arrow[r, "H_A(H_{\theta_{E_1}},1)"] & H_A(E_1^F,E_2^G)
\end{tikzcd}$$

Following the object (functor) $\Phi : E_1 \longrightarrow E_2$ of $H_B(E_1,E_2)$ first horizontally and then vertically gives the functor $H_{\theta_{E_2}}\Phi^F : E_1^F \longrightarrow E_2^G$. Given an object (a,x) in

E_1^F, $\Phi^F(a,x) = (a, \Phi(x))$ and $H_{\theta_{E_2}}(a,\Phi(x)) = (a, \overline{\Phi(x)})$ where $\overline{\Phi(x)}$ is the codomain of the horizontal lift of $\theta_a : F(a) \longrightarrow G(a)$ to domain $\Phi(x)$ in E_2; following Φ in the other direction results in the functor $\Phi^G H_{\theta_{E_1}}$. Now $H_{\theta_{E_1}}(a,x) = (a,\bar{x})$ where $\bar{x}$ is the codomain of the horizontal lift of θ_a to domain x in E_1, and $\Phi^G(a,\bar{x}) = (a, \Phi(\bar{x}))$. Since by Remark 2.13 $\overline{\Phi(x)} = \Phi(\bar{x})$, we have that for objects, $H_{\theta_{E_2}}\Phi^F = \Phi^G H_{\theta_{E_1}}$. For a morphism $(f,g) : (a,x) \longrightarrow (b,y)$ in E_1^F, on one hand $H_{\theta_{E_2}}\Phi^F(f,g) = H_{\theta_{E_2}}(f,\Phi(g)) = (f, v^*\overline{\overline{Gf}})$ where v^* is the vertical component of the horizontal-vertical decomposition of $\overline{\overline{\theta_b}}\Phi(g)$ and $\overline{\overline{Gf}}$ is the horizontal lift of $Gf : G(a) \longrightarrow G(b)$ to domain $\Phi(\bar{x})$ in E_2; on the other hand $\Phi^G H_{\theta_{E_1}}(f,g) = \Phi^G(f, v\overline{Gf}) = (f, \Phi(v\overline{Gf}))$ where $\overline{Gf}$ is the horizontal lift of Gf to domain $\bar{x}$ in E_1 and v is the vertical component of the horizontal-vertical decomposition of $\overline{\theta_b}g$ in E_1. Since Φ preserves the horizontal-vertical decomposition of morphisms, the horizontal-vertical decomposition of $\bar{\bar{\theta}}_b\Phi(g) = \Phi(\bar{\theta}_b g)$ is $\Phi(v\overline{G(f)}\,\bar{\theta}_a) = \Phi(v)\overline{\overline{G(f)}}\,\bar{\bar{\theta}}_a$. Thus $\Phi(v) = v^*$ and $\Phi(v\overline{G(f)}) = v^*\overline{\overline{G(f)}}$; hence $H_{\theta_{E_2}}\Phi^F = \Phi^G H_{\theta_{E_1}}$ for morphisms as well. The commutativity of the diagram for 2-cells translates to $1_{H_{\theta_2}} * \alpha^F = \alpha^G * 1_{H_{\theta_2}}$:

$$\begin{tikzcd} E_1^F \arrow[r, bend left, "\Phi^F"] \arrow[r, bend right, "\Psi^F"'] \arrow[r, phantom, "\Downarrow\alpha^F"] & E_2^F \arrow[r, bend left, "H_{\theta_{E_2}}"] \arrow[r, bend right, "H_{\theta_{E_2}}"'] \arrow[r, phantom, "\Downarrow 1"] & E_2^G \end{tikzcd} \text{ and } \begin{tikzcd} E_1^F \arrow[r, bend left, "H_{\theta_{E_1}}"] \arrow[r, bend right, "H_{\theta_{E_2}}"'] \arrow[r, phantom, "\Downarrow 1"] & E_1^G \arrow[r, bend left, "\Phi^G"] \arrow[r, bend right, "\Psi^G"'] \arrow[r, phantom, "\Downarrow\alpha^G"] & E_2^G \end{tikzcd}.$$

Given an object (a,x) of E_1^F, the corresponding component of both composite natural transformations (by Construction 2.17) is $(1_a, \alpha_{\bar{x}}) : (a, \Phi(\bar{x}) \longrightarrow (a, \Psi(\bar{x}))$.

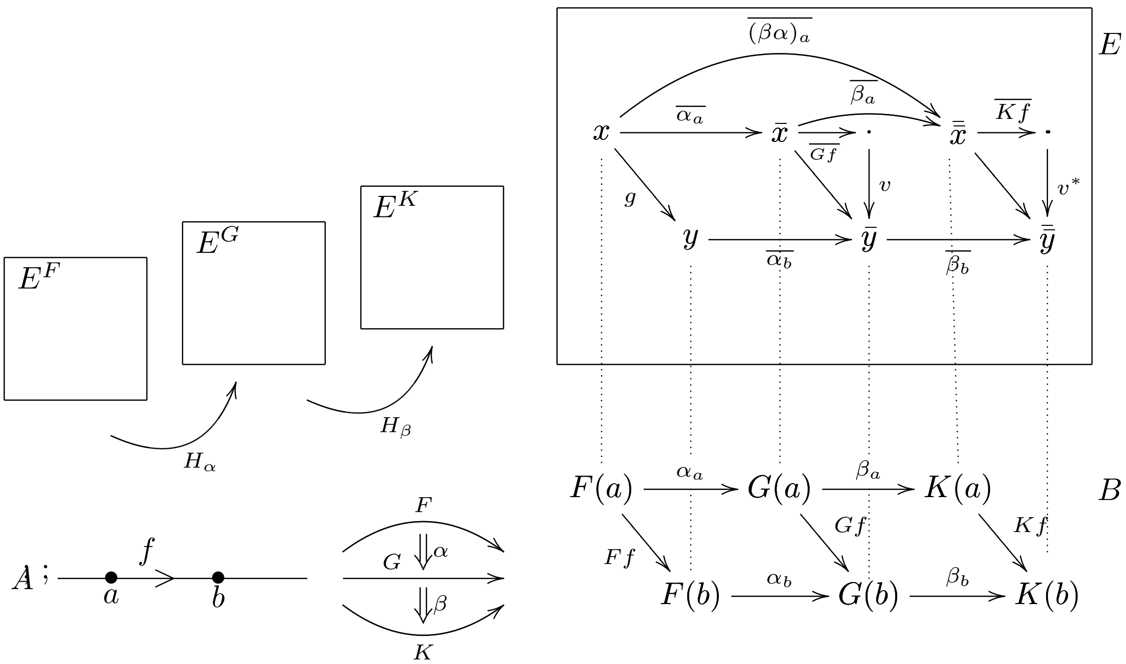
Corollary 2.21 *If the natural transformation $\theta : F \Rightarrow G$ in Lemma 2.20 is a natural isomorphism, then the components of H_θ are isomorphisms as well.*

Proof. Suppose the components $\theta_a : F(a) \longrightarrow G(a)$ of θ are invertible arrows in B; then given a horizontal structure $E \xrightarrow{P} B$, the corresponding component $H_{\theta_E} : E^F \longrightarrow E^G$ is a functor with its inverse $H_{\theta_E}^{-1} : E^G \longrightarrow E^F$ defined as follows. For an object (a, y) of E^G, $H_{\theta_E}^{-1}(a, y) = (a, \bar{y})$ with $\bar{y} = \mathrm{cod}(\overline{\theta_a^{-1}})$. Then by part (iii) of Remark 2.5, on the object level $H_{\theta_E} H_{\theta_E}^{-1} = 1_{E^G}$ and $H_{\theta_E}^{-1} H_{\theta_E} = 1_{E^F}$. Given a morphism $(f, g) : (a, x) \longrightarrow (b, y)$ in E^G $H_{\theta_E}^{-1}(f, g) \stackrel{\text{def}}{=} (f, v\overline{Ff})$ where v is the vertical component of the horizontal-vertical factorization of $\overline{\theta_b^{-1}}g$ with $\overline{\theta_b^{-1}}$ the lift of $\theta_b^{-1} : G(b) \longrightarrow F(b)$ to domain y in E. To show that $H_{\theta_E}^{-1} H_{\theta_E} = 1_{E^F}$ on the morphism level, suppose $(f, g) : (a, x) \longrightarrow (b, y)$ is a morphism in E^F; $H_{\theta_E}(f, g) = (f, v\overline{Gf})$ where v is the vertical component of the horizontal-vertical factorization of $\overline{\theta_b}g$ (θ_b is lifted to domain y to get $\overline{\theta_b}$) and $H_{\theta_E}^{-1}(f, v\overline{Gf}) = (f, v^*\overline{Ff})$ where v^* is the vertical component of the horizontal-vertical decomposition of $\overline{\theta_b^{-1}}v\overline{Gf}$. Since $\overline{\theta_b}g = v\overline{Gf}\,\overline{\theta_a}$ by the horizontal-vertical factorization of $\overline{\theta_b}g$, and $\overline{\theta_b^{-1}}v\overline{Gf} = v^*\overline{Ff}\overline{\theta_a^{-1}}$ (again using the horizontal-vertical factorization of $\overline{\theta_b^{-1}}v\overline{Gf}$), we have that $v^*Ff = \overline{\theta_b^{-1}}v\overline{Gf}\,\overline{\theta_a} = \overline{\theta_b^{-1}}\overline{\theta_b}g = g$. Thus $H_{\theta_E}^{-1} H_{\theta_E} = 1_{E^F}$ and by a similar argument, $H_{\theta_E} H_{\theta_E}^{-1} = 1_{E^G}$. Hence pulling back a horizontal structure along isomorphic functors results in isomorphic horizontal structures: $F \cong G \Rightarrow E^F \cong E^G$.

Proposition 2.22 *The preceding constructions define a strong-lax 2-functor* $\mathcal{H} : CAT^{\mathrm{op}} \longrightarrow 2\text{-}Cat$.

Proof. For zero cells, $\mathcal{H}(A) = H_A$, the 2-category of horizontal structures over A (Def. 2.6). For each pair of objects A, B in CAT, we have a functor $\mathcal{H}_{A,B} : \mathrm{CAT}(A, B) \longrightarrow 2 - \mathrm{Cat}(H_B, H_A)$ defined for $F : A \longrightarrow B$ by $\mathcal{H}_{A,B}(F) = H_F :$

$H_B \longrightarrow H_A$ as in Construction 2.18, and for $\theta : F \Rightarrow G$ ($F, G : A \longrightarrow B$) by $\mathcal{H}_{A,B}(\theta) = H_\theta$ as in Lemma 2.20. Given $F, G, K : A \longrightarrow B$, $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow K$ in $\flat\mathrm{CAT}(A, B)$, we must have $\mathcal{H}_{A,B}(\beta\alpha) = \mathcal{H}_{A,B}(\beta)\mathcal{H}_{A,B}(\alpha)$, or using the notation of Lemma 2.20, $H_{\beta\alpha} = H_\beta H_\alpha$. For an object (a, x) in E^F, $H_\alpha(a, x) = (a, \bar{x}) \in E^G$ where $\bar{x}$ is the codomain of the horizontal lift $\overline{\alpha_a}$ of $\alpha_a : F(a) \longrightarrow G(a)$ to domain x in E, and $H_\beta(a, \bar{x}) = (a, \bar{\bar{x}}) \in E^K$ where $\bar{\bar{x}}$ is the codomain of the horizontal lift $\overline{\beta_a}$ of $\beta_a : G(a) \longrightarrow K(a)$ to domain $\bar{x}$ in E. $H_{\beta\alpha}(a, x) = (a, \hat{x}) \in E^K$ where $\hat{x}$ is the codomain of the horizontal lift $\overline{(\beta\alpha)_a}$ of $(\beta\alpha)_a : F(a) \longrightarrow K(a)$ to domain x in E. Since H is assumed to be a subcategory of E, the composition of horizontal morphisms is horizontal; the uniqueness of horizontal lifts to a given domain then implies that $\overline{\beta_a}\,\overline{\alpha_a} = \overline{(\beta\alpha)_a}$ and hence $\bar{\bar{x}} = \hat{x}$. Given a morphism $(f, g) : (a, x) \longrightarrow (b, y)$ in E^F, on one hand $H_\alpha(f, g) = (f, v\overline{Gf}) : (a, \bar{x}) \longrightarrow (b, \bar{y})$ where v is the vertical component of the horizontal-vertical decomposition of $\overline{\alpha_b}g$ and $H_\beta(f, v\overline{Gf}) = (f, v^*\overline{Kf}) : (a, \bar{\bar{x}}) \longrightarrow (b, \bar{\bar{y}})$ where v^* is the vertical component of the horizontal-vertical decomposition of $\overline{\beta_b}\, v\, \overline{Gf}$. On the other hand, $H_{\beta\alpha}(f, g) = (f, \hat{v}\overline{Kf}) : (a, \bar{\bar{x}}) \longrightarrow (b, \bar{\bar{y}})$ where $\hat{v}$ is the vertical component of the horizontal-vertical decomposition of $\overline{(\beta\alpha)_b}\, g = \overline{\beta_b}\,\overline{\alpha_b} g$ (see diagram below). We must show then that $v^* = \hat{v}$: we have that $\overline{\beta_b}\,\overline{\alpha_b}\, g = \hat{v}(\overline{Kf}\,(\beta\alpha)_a) = \hat{v}\overline{Kf}\,\overline{\beta_a}\,\overline{\alpha_a}$, but also $\overline{\beta_b}\,\overline{\alpha_b}\, g = (\overline{\beta_b}\, v\, \overline{Gf})\overline{\alpha_a} = (v^*\overline{Kf}\,\overline{\beta_a})\overline{\alpha_a}$, so $v^* = \hat{v}$. For the identity 2-cell $1_F : F \longrightarrow F$ ($F : A \longrightarrow B$), we need to show that $H_{1_F} = 1_{H_F} : E^F \longrightarrow E^F$. Clearly $H_{1_F}(a, x) = (a, x)$ since $1_x : x \longrightarrow x$ is the horizontal lift of $1 : a \longrightarrow a$ to domain x in E, and similarly for $f : a \longrightarrow b$ and $g : x \longrightarrow y$, $H_{1_F}(f, g) = (f, v\overline{Ff})$ where v is the vertical component of the horizontal-vertical decomposition of $1_y g = g = v\overline{Ff}$.



For every triple of objects A, B and C in $\flat$CAT and functors $A \xrightarrow{F} B \xrightarrow{G} C$, we have a natural isomorphism $\delta_{F,G} : H_F \circ H_G \longrightarrow H_{GF}$; the component of $\delta_{F,G}$ corresponding to an object $E \xrightarrow{P} C$ of H_C is given by the isomorphism of the categories $(E^G)^F \cong E^{(GF)}$: The objects of $E^G \xrightarrow{P^G} B$ are pairs (b, x) such that $G(b) = P(x)$ and then the objects of $(E^G)^F \xrightarrow{P^F} A$ are "pairs" $(a, (b, x))$ with $F(a) = P^G(b, x) = b$, so each object $(a, (b, x)) \in (E^G)^F$ is of the form $(a, (F(a), x))$ and will be identified with $(a, x) \in E^{(GF)}$; since $F(a) = b$, $GF(a) = G(b) = P(x)$, so (a, x) is an object in $(E^G)^F$. The isomorphism works similarly for morphisms. For every object A in $\flat$CAT, we have the natural isomorphism $\varepsilon_A : 1_{H_A} \Rightarrow H_{1_A}$ that for an object E in H_A, identifies x in E with $(P(x), x)$ in E^{1_A} and similarly $g : x \longrightarrow y$ in E with $(P(g), g) : (P(x), x) \longrightarrow (P(y), y)$ in E^{1_A} (cf. Remark 2.15).

For one cells $F : A \longrightarrow B$, $\mathcal{H}(F)$ is a 2-functor defined as in Construction 2.18. For two cells $\theta : F \Rightarrow G$, $H_\theta : H_F \Rightarrow H_G$ is the 2-natural transformation with components $H_{\theta_E} : H_F(E) = E^F \longrightarrow H_G(E) = E^G$ defined as in Lemma 2.20.

Definition 2.23 (Definition 1.1 in [20]) Given a functor $F : B \longrightarrow \flat\mathrm{CAT}$, the *Grothendieck construction on F,* denoted by $\int_B F$, is the category with objects the pairs (b, x) with b an object of B and x an object of $F(b)$. A morphism $\bar{f} : (b, x) \longrightarrow (c, y)$ of $\int_B F$ is a quadruple $\bar{f} = (x, f, \alpha, y)$ such that $f : b \longrightarrow c$ is a morphism in B and $\alpha : Ff(x) \longrightarrow y$ is a morphism in $F(c)$. Composition in $\int_B F$ is defined by $(y, g, \beta, z) \circ (x, f, \alpha, y) = (x, gf, \beta \circ Fg(\alpha), z)$ for morphisms (y, g, β, z) and (x, f, α, y) in $\int_B F$, such that gf is defined in B. For an object (b, x) in $\int_B F$, $1_{(b,x)} = (x, 1_b, 1_x, x)$. The functor $U^F : \int_B F \longrightarrow B$ associated to $F : B \longrightarrow \flat\mathrm{CAT}$ is defined as $U^F(b, x) = b$ for objects and as $U^F(x, f, \alpha, y) = f$ for morphisms. For $U^F : \int_B F \longrightarrow B$, the categorical fiber $(U^F)^{-1}(b)$ over an object b is isomorphic to $F(b)$.

Remarks 2.24 (i) Given a functor $F : B \longrightarrow \mathrm{CAT}$, let $F_0 = U_0 F : B \longrightarrow \mathrm{Sets}$ where $U_0 : \mathrm{CAT} \longrightarrow \mathrm{Sets}$ is the functor which sends a category C to its set of objects. Then with $F_0 : B \longrightarrow \mathrm{Sets} \hookrightarrow \mathrm{CAT}$ (where each set is viewed as a discrete category), for $U^F : \int F_0 \longrightarrow B$ each fiber is a discrete category with the identities as the only morphisms and $\int F_0$ is the same category as the *"category of elements of F_0"* in the proof of Proposition 1 in [15].

(ii) If F factors through POS (the category of partially ordered sets and order preserving functions), i.e., $F : B \longrightarrow \mathrm{POS} \hookrightarrow \mathrm{CAT}$, then a morphism $f : b \longrightarrow c$ in B lifts to a morphism $\bar{f} : (b, x) \longrightarrow (c, y)$ in $\int_B F$ iff $Ff(x) \leq y$. This so called

lifting condition is the same when $F : B \longrightarrow \mathrm{CoP}$ (where CoP is the category of cocomplete posets; see Definition 2.39). For $\bar{f} = (x, f, \alpha, y)$, the vertical morphism $\alpha : Ff(x) \xrightarrow{\leq} y$ is unique, thus $\bar{f}$ will be written as a triple (x, f, y) and the composition rule in Definition 2.21 simplifies to $(y, g, z) \circ (x, f, y) = (x, gf, z)$.

Lemma 2.25 *For a functor $F : B \longrightarrow \flat CAT$, (H, V) defined as follows gives a splitting of the category $\int\limits_B F$. H consists of all the objects of $\int\limits_B F$ and all the morphisms of the form $(x, f, 1_{Ff(x)}, Ff(x)) : (b, x) \longrightarrow (c, Ff(x))$. V is the vertical subcategory of $\int\limits_B F$; its morphisms then are of the form $(x, 1_b, \alpha, y) : (b, x) \longrightarrow (b, y)$.*

Proof. A morphism $(x, f, \alpha, y) : (b, x) \longrightarrow (c, y)$ in $\int\limits_B F$ factors as $(x, f, \alpha, y) = (Ff(x), 1_c, \alpha, y) \circ (x, f, 1_{Ff(x)}, Ff(x))$; this factorization of (x, f, α, y) as a horizontal morphism followed by a vertical, is clearly unique.

Lemma 2.26 *The splitting of $\int\limits_B F$ given in Lemma 2.25 above defines a horizontal structure over B.*

Proof. The unique horizontal lift of $f : b \longrightarrow c$ in B to domain (b, x) in $\int\limits_B F$ is $\bar{f} = (x, f, 1_{Ff(x)}, Ff(x)) : (b, x) \longrightarrow (c, Ff(x))$.

Lemma 2.27 *A natural transformation $\alpha : F \Rightarrow G$ (where $F, G : A \longrightarrow \flat CAT$) induces a morphism $\int_A \alpha : \int_A F \longrightarrow \int_A G$ of horizontal structures (Definition 2.6).*

Proof. $\int_A \alpha$ is defined for (a, x) in $\int_A F$, $\left(\int_A \alpha\right)(a, x) = (a, \alpha_a(x))$ and for $(x, f, \varrho, y) : (a, x) \longrightarrow (b, y)$ by $\left(\int_A \alpha\right)(x, f, \varrho, y) = (\alpha_a(x), f, \alpha_b(\varrho), \alpha_b(y))$. $\int_A \alpha$ preserves identities, since for each object a in A, $\alpha_a : F(a) \longrightarrow G(a)$ is a functor, so we have that for an identity arrow $(x, 1_a, 1_x, x) : (a, x) \longrightarrow (a, x)$, $\left(\int_A \alpha\right)(x, 1_a, 1_x, x) = (\alpha_a(x), 1_a, \alpha_a(1_a), \alpha_a(x)) = (\alpha_a(x), 1_a, 1_{\alpha_a}, \alpha_a(x))$. To show that $\int_A \alpha$ preserves composition, the following equality must hold for morphisms $(x, f, r, y) : (a, x) \longrightarrow (b, y)$ and $(y, g, t, z) : (b, y) \longrightarrow (c, z)$:

$\left(\int_A \alpha\right)[(y, g, t, z) \circ (x, f, r, y)] = \left(\int_A \alpha\right)(y, g, t, z) \circ \left(\int_A \alpha\right)(x, f, r, y)$. By the definition of composition in $\int_A F$ we have that $\left(\int_A \alpha\right)[(y, g, t, z) \circ (x, f, r, y)] = \left(\int_A \alpha\right)(x, gf, t \circ Fg(r), z) = (\alpha_a(x), gf, \alpha_c(t \circ Fg(r)), \alpha_c(z))$; composing in $\int_A G$ gives $\left(\int_A \alpha\right)(y, g, t, z) \circ \left(\int_A \alpha\right)(x, f, r, y) = (\alpha_b(y), g, \alpha_c(t), \alpha_c(z)) \circ (\alpha_a(x), f, \alpha_b(r), \alpha_b(y)) = (\alpha_a(x), gf, \alpha_c(t) \circ Gg(\alpha_b(r)), \alpha_c(z)$. The two morphisms then are the same if $\alpha_c(t \circ Fg(r)) = \alpha_c(t) \circ Gg(\alpha_b(r))$; this equality follows from the naturality of α, which implies that $\alpha_c \circ Fg = Gg \circ \alpha_b$ (see diagram below).

$$\begin{tikzcd} F(b) \arrow[r, "\alpha_b"] \arrow[d, "Fg"'] & G(b) \arrow[d, "Gg"] \\ F(c) \arrow[r, "\alpha_c"'] & G(c) \end{tikzcd}$$

As a direct consequence of the definitions involved $U^F = U^G \circ \int_A$ and $\int_A \alpha$ preserves vertical morphisms; it also preserves horizontal morphisms, since

$\left(\int_A \alpha\right)(x, f, 1_{Ffx}, Ffx) = (\alpha_a(x), f, \alpha_b(1_{Ffx}), \alpha_b(y)) = (\alpha_a(x), f, 1_{\alpha_b Ffx}), \alpha_b(y))$. $\int_A \alpha : \int_A F \longrightarrow \int_A G$ is then a 1-cell in the 2-category H_A.

Construction 2.28 A category A induces a 2-functor $\int_A : \flat\mathrm{CAT}^A \longrightarrow H_A$ as follows: The image of a 0-cell $T : A \longrightarrow \flat\mathrm{CAT}$ under $\int_A$ is $\int_A T \xrightarrow{U^T} A$, the Grothendieck construction on T as given in Definition 2.23, and the image of a 1-cell $\alpha : T \Rightarrow S$ is the functor $\int_A \alpha$ defined in the previous lemma. For any pair of objects T, S in $\flat\mathrm{CAT}^A$, $\int_A$ must give a functor $\flat\mathrm{CAT}^A(T, S) \longrightarrow H_A(\int_A T, \int_A S)$

which will also be denoted by $\int_A$ to avoid multiple subscripts. The image of a 2-cell $m : \alpha \rightsquigarrow \beta$, under the functor $\int_A$ will be denoted by $\int_A m$ (see diagram below) and its component

$$\begin{tikzcd}
 & & \int_A T \arrow[dd, bend right=50, "\int_A \beta"'] \arrow[dd, bend left=50, "\int_A \alpha"] \arrow[dd, phantom, "\overset{\int_A m}{\Longrightarrow}"] \\
A \arrow[r, bend left, "T"] \arrow[r, bend right, "S"'] \arrow[r, phantom, "\alpha\Downarrow \overset{m}{\rightsquigarrow} \Downarrow\beta"] & \flat\mathrm{CAT} \arrow[r, "\int\limits_A"] & {} \\
 & & \int_A S
\end{tikzcd}$$

corresponding to an object (a, x) in $\int_A T$ is the arrow in $\int_A S$ given as follows: $\left(\int_A m\right)_{(a,x)} = (\alpha_a(x), 1_a, (m_a)_x, \beta_a(x)) : (a, \alpha_a(x)) \longrightarrow (a, \beta_a(x))$, which is a vertical morphism as required by Definition 2.5. For $\int_A : \flat\mathrm{CAT}^A(T, S) \longrightarrow H_A(\int_A T, \int_A S)$ to be a functor, it must preserve the vertical composition of 2-cells: given $m : \alpha \rightsquigarrow \beta$ and $n : \beta \rightsquigarrow \gamma$ (with $\alpha, \beta, \gamma : T \Rightarrow S$) we have on one hand for objects a and x in A and $T(a)$ respectively that $\int_A (n \odot m)_{(a,x)} = \int_A (n_{ax} \circ m_{ax}) = (\alpha_a(x), 1_a, n_{ax} \circ m_{ax}, \gamma_a(x))$ and ont the other hand $(\int_A n)_{(a,x)} \odot (\int_A m)_{(a,x)} = (\beta_a(x), 1_a, n_{ax}, \gamma_(x)) \circ (\alpha_a(x), 1_a, m_{ax}, \beta_(x)) = (\alpha_a(x), 1_a, n_{ax} \circ G(1_a)(m_{ax}), \gamma_(x))$. Since $S(1_a) = 1_{Ga}$ is the identity functor on $S(a)$, $S(1_a)(m_{ax}) = m_{ax}$ and we have that $\int_A (n \odot m) = \int_A n \odot \int_A m$ holds componentwise. $\int_A$ also preserves identity 2-cells: $\left(\int\limits_A 1_\alpha\right)_{(a,x)} = (\alpha_a(x), 1_a, (1_\alpha)_{ax}, \alpha_a(x))$ (Def 2.21) where $(1_\alpha)_{ax} = 1_{\alpha_a(x)}$.

The compatibility of $\int_A$ with composition translates into the commutativity of the following diagram:

$$\begin{tikzcd}
\flat\mathrm{CAT}^A(T, S) \times \flat\mathrm{CAT}^A(S, R) \arrow[r, "c"] \arrow[d] & \flat\mathrm{CAT}^A(T, R) \arrow[d] \\
H_A(\int\limits_A T, \int\limits_A S) \times H_A(\int\limits_A S, \int\limits_A R) \arrow[r, "c"] & H_A(\int\limits_A T, \int\limits_A R)
\end{tikzcd}$$

For a 1-cell (α, β) in $\flat\mathrm{CAT}^A(T,S) \times \flat\mathrm{CAT}^A(S,R)$ both functors $\int\limits_A \beta \circ \int\limits_A \alpha$ and $\int\limits_A \beta\alpha$ take an object (a,x) in $\int\limits_A T$ to $(a, \beta_a\alpha_a(x))$ and an arrow (x, f, ϱ, y) to $(\beta_a\alpha_a(x), f, \beta_b\alpha_b(\varrho), \beta_b\alpha_b(y))$. For a 2-cell (m,n) in $\flat\mathrm{CAT}^A(T,S) \times \flat\mathrm{CAT}^A(S,R)$ we must have that $\int\limits_A n * \int\limits_A m = \int\limits_A (n*m)$. This equality follows directly form applying the relevant definitions; for objects a in A and x in $T(a)$, the component of both 2-cells is the following arrow in $\int\limits_A R$: $(\beta_a\alpha_a(x), 1_a, (n*m)_{ax}, \eta_a\gamma_a(x)) : (a, \beta_a\alpha_a(x)) \longrightarrow (a, \eta_a\gamma_a(x))$ where α, β, γ and η are as pictured below.

$$\begin{tikzcd}[column sep=large] A \arrow[rr, bend left=45, "T"] \arrow[rr, "S" description] \arrow[rr, bend right=45, "R"'] \arrow[rr, phantom, shift left=3ex, "\alpha\Downarrow \ \overset{m}{\rightsquigarrow} \ \Downarrow\gamma"] \arrow[rr, phantom, shift right=3ex, "\beta\Downarrow \ \underset{n}{\rightsquigarrow} \ \Downarrow\eta"] & & \flat\mathrm{CAT} \arrow[r, "\int_A"] & {} \end{tikzcd}
\qquad
\begin{tikzcd}[row sep=huge] \int_A T \arrow[d, bend right=50, "\int_A \alpha"'] \arrow[d, bend left=50, "\int_A \gamma"] \arrow[d, phantom, "\overset{\int_A m}{\Longrightarrow}"] \\ \int_A S \arrow[d, bend right=50, "\int_A \beta"'] \arrow[d, bend left=50, "\int_A \eta"] \arrow[d, phantom, "\overset{\int_A n}{\Longrightarrow}"] \\ \int_A R \end{tikzcd}$$

The unit axiom also holds as a consequence of the definitions and hence

$\int_A : \flat\mathrm{CAT}^A \longrightarrow H_A$ as defined above is a 2-functor.

A functor $T : A \longrightarrow B$ is an *equivalence of 2-categories* when there is a 2-functor $S : B \longrightarrow A$ and 2-natural isomorphisms $TS \cong 1_B$ and $ST \cong 1_A$.

Lemma 2.29 *The 2-functor $\int_A$ of Construction 2.28 is an equivalence of the 2-categories $\flat CAT^A$ and H_A for any category A.*

Proof. A 2-functor $\int_A^{-1} : H_A \longrightarrow \flat\mathrm{CAT}^A$ will be defined along with natural isomorphisms θ and λ, such that $\int_A \circ \int_A^{-1} \overset{\theta}{\Rightarrow} 1_{H_A}$ and $\int_A^{-1} \circ \int_A \overset{\lambda}{\Rightarrow} 1_{\flat\mathrm{CAT}^A}$. Given $E \xrightarrow{P} A$ in H_A, the corresponding functor $T_E : A \longrightarrow \flat\mathrm{CAT}$ is defined as follows.

For an object $a \in A$, $T_E(a) = P^{-1}(a)$. Given a morphism $f : a \longrightarrow b$ in A, let $\bar{f}_x$ denote the unique horizontal lift of f to domain x and $\mathrm{cod}\bar{f}_x$ the codomain of $\bar{f}_x$. Then the functor $T_E f : P^{-1}(a) \longrightarrow P^{-1}(b)$ will be defined for an object x of $P^{-1}(a)$ by $T_E f(x) = \mathrm{cod}\bar{f}_x$. To define $T_E f$ for a morphism $t : x \longrightarrow y$ in $P^{-1}(a)$, let $g = \bar{f}_y \circ t : x \longrightarrow \mathrm{cod}\bar{f}_y$ and let $g = sh$ be the unique horizontal-vertical factorization of g in E. Since $P(g) = P(sh) = P(h)$ and also $P(g) = P(\bar{f}_y t) = f$, we have that $P(h) = f$, which means that h is a horizontal lift of f to domain x. Thus by the uniqueness of horizontal lifts, $h = \bar{f}_x$. Then define $T_E f(t) = s$. $T_E f$ preserves identities, and since the composition of horizontal morphisms is horizontal, it preserves compositions as well. For a 1-cell $\Phi : E_1 \longrightarrow E_2$ in H_A, $\int_A^{-1} \Phi : T_{E_1} \Rightarrow T_{E_2}$ must be a natural transformation with each component $(\int_A^{-1} \Phi)_a : T_{E_1}(a) \longrightarrow T_{E_2}(a)$ a functor such that the following diagram of categories commutes for all $a \xrightarrow{f} b$ in A.

$$\begin{array}{ccc}
T_{E_1}(a) & \xrightarrow{(\int_A^{-1} \Phi)_a} & T_{E_2}(a) \\
{\scriptstyle T_{E_1} f}\downarrow & & \downarrow{\scriptstyle T_{E_2} f} \\
T_{E_1}(b) & \xrightarrow{(\int_A^{-1} \Phi)_b} & T_{E_2}(b)
\end{array}$$

Defining $(\int_A^{-1} \Phi)_a$ for an object x in $T_{E_1}(a)$ by $(\int_A^{-1} \Phi)_a(x) = \Phi(x)$ and for a morphism $x \xrightarrow{t} y$ by $(\int_A^{-1} \Phi)_a(t) = \Phi(t)$, the diagram commutes since Φ preserves horizontal-vertical decompositions.

For a 2-cell $\alpha : \Phi \Rightarrow \Psi$ in H_A, the corresponding 2-cell $\int_A^{-1} \alpha : \int_A^{-1} \Phi \rightsquigarrow \int_A^{-1} \Psi$ is the collection of natural transformations $\{(\int_A^{-1} \alpha)_a : (\int_A^{-1} \Phi)_a \Rightarrow (\int_A^{-1} \Psi)_a\}_{a \in A}$ whose components are the arrows in $T_{E_2}(a)$ given by components of the 2-cell α:

For an object $x \in T_{E_1}(a) \subseteq E_1$, $(\int_A^{-1} \alpha)_{ax} = \alpha_x : \Phi(x) \longrightarrow \Psi(x)$. By Definition 2.6, the components of α are vertical morphisms; hence $\alpha_x : \Phi(x) \longrightarrow \Psi(x)$ is an arrow in $T_{E_2}(a)$; moreover, for $\int_A^{-1}\alpha$ to be a 2-cell in $\flat\mathrm{CAT}^A$, the diagram below must commute for $f : a \longrightarrow b$ in A.

$$\begin{tikzcd}
a \arrow[dd, "f"] & T_{E_1}(a) \arrow[dd, "T_1(f)"'] \arrow[rr, bend left, "(\int_A^{-1}\Phi)_a"] \arrow[rr, bend right, "(\int_A^{-1}\Psi)_a"'] \arrow[rr, phantom, "\Downarrow(\int_A^{-1}\alpha)_a"] & & T_{E_2}(a) \arrow[dd, "T_2(f)"] \\
& & & \\
b & T_{E_1}(b) \arrow[rr, bend left, "(\int_A^{-1}\Phi)_b"] \arrow[rr, bend right, "(\int_A^{-1}\Psi)_b"'] \arrow[rr, phantom, "\Downarrow(\int_A^{-1}\alpha)_b"] & & T_{E_2}(b)
\end{tikzcd}$$

The equality $1_{T_{E_2}f} * \left(\int_A^{-1} \alpha\right)_a = \left(\int_A^{-1} \alpha\right)_b * 1_{T_{E_1}f}$ of natural transformations for an object x in $T_{E_1}(a)$ translates to the equation $T_{E_2}f(\alpha_x) = \alpha_{\mathrm{cod}\bar{\bar{f}}_{\Phi(x)}}$, using the fact that for a morphism Φ of horizontal structures $\Phi(\bar{f}_x) = \bar{\bar{f}}_{\Phi(x)}$ (see Remark 2.13). (Lifts of f to E_1 and E_2 are denoted by $\bar{f}$ and $\bar{\bar{f}}$ respectively.) $T_{E_2}f(\alpha_x)$ is the vertical component of the horizontal-vertical decomposition of $\Psi(\bar{f}_x) \circ \alpha_x$; the naturality of α in the objects x and $\mathrm{cod}\bar{f}_x$ however means that $\Psi(\bar{f}_x) \circ \alpha_x = \alpha_{\mathrm{cod}\bar{\bar{f}}_{\Phi(x)}} \circ \Phi(\bar{f}_x)$, where the latter is exactly the horizontal-vertical decomposition of $\Psi(\bar{f}_x) \circ \alpha_x$; thus $T_{E_2}f(\alpha_x) = \alpha_{\mathrm{cod}\bar{\bar{f}}_{\Phi(x)}}$.

The component of the 2-natural transformation $\theta : \int_A \circ \int_A^{-1} \Rightarrow 1_{H_A}$ corresponding to an object $E \xrightarrow{P} A$ of H_A is the morphism of horizontal structures $\theta_E : \int_A T_E \longrightarrow E$ defined for objects as $\theta_E(a, x) = x$ and for morphisms as $\theta_E(x, f, t, y) = t \circ \bar{f}_x$. Since θ_E preserves both the horizontal and vertical morphisms ($\theta_E(x, f, 1, T_E f(x)) = \bar{f}_x$ and $\theta_E(x, 1, t, y) = t$) and $P\theta_E = U^{T_E}$, θ_E is a 1-cell in H_A. For θ to be a 2-natural transformation, the following diagram of categories must commute for any pair of objects E_1 and E_2. (To simplify notation, T_{E_i} will be denoted by T_i and similarly θ_{E_i} by θ_i).

$$\begin{array}{ccc} H_A(E_1,E_2) & \longrightarrow & H_A(\int\limits_A T_1, \int\limits_A T_2) \\ \downarrow & & \downarrow \\ H_A(E_1,E_2) & \longrightarrow & H_A(\int\limits_A T_1, E_2) \end{array}$$

Commutativity for a 1-cell $\Phi : E_1 \longrightarrow E_2$ means that $\Phi \circ \theta_1 = \theta_2 \circ \int\limits_A(\int\limits_A^{-1}\Phi) : \int\limits_A T_1 \longrightarrow E_2$; this equality holds, since both functors take an object (a,x) in $\int\limits_A T_1$ to $\Phi(x)$, and a morphism (x,f,t,y) to $\Phi(t \circ \bar{f}_x) = \Phi(t) \circ \overline{\overline{f}}_{\Phi(x)}$. Commutativity for a 2-cell $\varrho : \Phi \Rightarrow \Psi$ means that $1_{\alpha_2} * \int\limits_A(\int\limits_A^{-1}\varrho) = \varrho * 1_{\alpha_1}$; following the definitions of the natural transformations involved yields the (vertical) arrow $\varrho_x : \Phi(x) \longrightarrow \Psi(x)$ for the component of both natural transformations corresponding to the object (a,x).

The components $\theta_{_E} : \int\limits_A T_E \longrightarrow E$ of the natural transformation θ are invertible morphisms in H_A since we can define $\theta_{_E}^{-1} : E \longrightarrow \int\limits_A T_E$ for objects by $\theta_{_E}^{-1}(x) = (x, P(x))$ and for morphisms by $\theta_{_E}^{-1}(f : x \longrightarrow y) = (x, Pf, v, y)$ where v is the vertical component of the horizontal vertical decomposition of f in E and have $\theta_{_E}^{-1}\theta_{_E} = 1$ and $\theta_{_E}\theta_{_E}^{-1} = 1$. θ is then a natural isomorphism.

To define the 2-natural transformation $\lambda : 1_{\flat\mathrm{CAT}^A} \Rightarrow \int\limits_A^{-1} \circ \int\limits_A$, the following notation will be introduced. The image of a 0-cell T, a 1-cell $\alpha : T_1 \Rightarrow T_2$ and a 2-cell $m : \alpha \rightsquigarrow \beta$ in CAT^A under the 2-functor $\int\limits_A^{-1} \circ \int\limits_A$ will be denoted by T^*, α^* and m^*, respectively. Following the definitions of $\int\limits_A^{-1}$ and $\int\limits_A T$ we have that $T^*(a) = (U^T)^{-1}(a)$, the categorical fiber over a in $\int\limits_A T$, and for a morphism $f : a \longrightarrow b$ in A, $T^*f : (U^T)^{-1}(a) \longrightarrow (U^T)^{-1}(b)$ is the functor that acts on objects and morphisms as follows: $T^*f(a,x) = (b, Tf(x))$ and $T^*f(x, 1_a, s, x^*) = (Tf(x), 1_b, Tf(s), Tf(x^*))$.

Given a natural transformation $\alpha : T_1 \Rightarrow T_2$, $\alpha^* : T_1^* \Rightarrow T_2^*$ is defined by having as components the functors $\alpha_a^* : T_1^*(a) \longrightarrow T_2^*(a)$ with $\alpha_a^*(a,x) = (a, \alpha(x))$ and for a morphism $(x, 1_a, s, y) : (a,x) \longrightarrow (a,y)$ in the categorical fiber, $\alpha_a^*(x, 1_a, s, y) = (\alpha_a(x), 1_a, \alpha_a(s), \alpha_a(y))$.

Given a 2-cell $m : \alpha \rightsquigarrow \beta$ (for $\alpha, \beta : T_1 \Rightarrow T_2$) with components the arrows $(m_a)_x : \alpha_a(x) \longrightarrow \beta_a(x)$ in $T_2(a)$, the components of the modification $m^* : \alpha^* \rightsquigarrow \beta^*$ corresponding to an object (a,x) in $T_1^*(a)$ are the arrows $(m_a^*)_x = (\alpha_a(x), 1_a, (m_a)_x, \beta_a(x)) : (a, \alpha_a(x)) \longrightarrow (a, \beta_a(x))$ in $T_2^*(a)$.

The component $\lambda_T : T \Rightarrow T^*$ of the 2-natural transformation λ is a collection of functors $(\lambda_T)_a : T(a) \longrightarrow T^*(a)$ defined as $(\lambda_T)_a(x) = (a,x)$ for objects, and for a morphism $s : x \longrightarrow y$ as $(\lambda_T)_a(s) = (x, 1_a, s, y)$. For λ to be a 2-natural transformation, the following diagram must commute for any pair of objects T_1 and T_2 in $\flat\mathrm{CAT}^A$.

$$\begin{tikzcd}
\flat\mathrm{CAT}^A(T_1, T_2) \arrow[r, "1_{\mathrm{CAT}^A}"] \arrow[d, "\int_A^{-1} \circ \int_A"'] & \flat\mathrm{CAT}^A(T_1, T_2) \arrow[d] \\
\flat\mathrm{CAT}^A(T_1^*, T_2^*) \arrow[r] & \flat\mathrm{CAT}^A(T_1, T_2^*)
\end{tikzcd}$$

The commutativity of the diagram means for a 1-cell $\alpha : T_1 \Rightarrow T_2$ that $\alpha^* * \lambda_{T_1} = \lambda_{T_2} * \alpha$, and for a 2-cell $m : \alpha \rightsquigarrow \beta$ and an object a in A that $1_{(\lambda_{T_2})_a} * m_a = m_a^* * 1_{(\lambda_{T_1})_a}$. Both equalities hold as a direct consequence of the definitions involved; therefore λ is a 2-natural transformation. Moreover, the components of λ are invertible arrows in CAT^A with $(\lambda_T)_a^{-1} : T^* \longrightarrow T$ defined on its components the obvious way: $(\lambda_T)^{-1}(a,x) = x$ for objects and $(\lambda_T)_a^{-1}(x, 1_a, s, y) = s$ for morphisms.

The 2-functor $\int\limits_{A} : \mathrm{CAT}^{A} \longrightarrow H_{A}$ is then an equivalence of categories for any category A.

Definition 2.30 A 2-natural transformation α between two lax functors

$F, G : A \longrightarrow B$ is a *lax natural transformation* for which the diagram in Definition 2.8 commutes up to a natural transformation τ_{ab} satisfying appropriate coherence axioms. When F and G are strong-lax functors and τ_{ab} is a natural isomorphism for every pair of objects a and b, α will be called a strong-lax natural transformation.

Proposition 2.31 *There is a strong-lax 2-natural isomorphism $\int : \mathcal{K} \longrightarrow \mathcal{H}$ with its component corresponding to a category A the 2-functor $\int_A : \flat CAT^A \longrightarrow H_A$ defined in Construction 2.28.*

Proof. For a pair of objects A and B in $\flat$CAT, the following diagram must be shown to commute up to a natural isomorphism τ_{AB}.

$$\begin{tikzcd}
\flat\mathrm{CAT}(A,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\tau_{AB}\Uparrow"] & 2-\mathrm{Cat}(\flat\mathrm{CAT}^{B}, \flat\mathrm{CAT}^{A}) \arrow[d] \\
2-\flat\mathrm{Cat}(H_B, H_A) \arrow[r] & 2-\flat\mathrm{Cat}(\flat\mathrm{CAT}^{B}, H_A)
\end{tikzcd}$$

For a functor $F : A \longrightarrow B$, the corresponding component of τ_{AB} (will be denoted by τ_F rather than $(\tau_{AB})_F$) must be an arrow in $2-\mathrm{Cat}(\flat\mathrm{CAT}^B, H_A)$, i.e., a 2-natural transformation $\tau_F : H_F \circ \int_B \Rightarrow \int_A \circ F^*$. Given an object T in $\flat\mathrm{CAT}^B$, $(H_F \circ \int_B)(T) = H_F(\int_B T) = \left(\int_B T\right)^F$ is the pullback of $\int_B T$ along F. Its objects are "pairs" $(a, (F(a), x))$ with x in $TF(a)$ and its morphisms (f, g) with $f : a \longrightarrow a_*$

in A and $g : (F(a), x) \longrightarrow (F(a_*), x_*)$ in $\int_B T$ are such that g itself is formally a quadruple $g = (x, Ff, \alpha, x_*)$ with $\alpha : TFf(x) \longrightarrow x_*$. On the other hand, $\left(\int_A \circ F^*\right)(T) = \int_A (TF)$; the objects of $\int_A (TF)$ are pairs (a, x) with $x \in TF(a)$ and its morphisms are quadruples $(x, f, \alpha, x_*) : (a, x) \longrightarrow (a_*, x_*)$ with $f : a \longrightarrow a_*$ and $\alpha : TF(f)(x) \longrightarrow x_*$. $(\tau_F)_T$ (the component of τ_F corresponding to the functor $T : B \longrightarrow \flat\mathrm{CAT}$) then will identify the the object $(a, (F(a), x))$ in $\left(\int_B T\right)^F$ with (a, x) in $\int_A TF$ and the morphism $(f, (x, Ff, \alpha, x_*))$ in $\left(\int_B T\right)^F$ with (x, f, α, x_*) in $\int_A TF$. $(\tau_F)_T$ then gives a bijection both on the class of objects and the set of morphisms of the categories $\left(\int_B T\right)^F$ and $\int_A TF$; it also preserves both horizontal and vertical morphisms, so it is a functor of horizontal structures and hence defines an isomorphism of categories: $(\int_B T)^F \cong \int_A TF$.

For τ_F to be a 2-natural transformation, the following diagram must commute:

$$\begin{tikzcd}
\flat\mathrm{CAT}^B(T, S) \arrow[r, "H_F \circ \int_B"] \arrow[d, "\int_A \circ F^*"'] & H_A\left((\int_B T)^F, (\int_B S)^F\right) \arrow[d] \\
H_A\left(\int_A TF, \int_A SF\right) \arrow[r] & H_A\left((\int_B T)^F, \int_A SF\right)
\end{tikzcd}$$

Following a natural transformation $\varrho : T \Rightarrow S$, in $\flat\mathrm{CAT}^B(T, S)$ first going to the right and then down in the diagram above gives first a functor of horizontal structures $\int_B \varrho : \int_B T \longrightarrow \int_B S$ as defined in Lemma 2.27, which then induces the functor (Lemma 2.16) $(\int_B \varrho)^F : (\int_B T)^F \longrightarrow (\int_B S)^F$ of the pullbacks; $(\int_B \varrho)^F$ is then followed by $(\tau_F)_S : (\int_B S)^F \longrightarrow \int_A SF$. The resulting functor $(\tau_F)_s \circ (\int_B \varrho)^F$ takes an object $(a, (Fa, x))$ of $(\int_B T)^F$ first to $(a, (Fa, \varrho_{Fa}(x)))$ and then to

$(a, \varrho_{_{Fa}}(x))$ and takes a morphism $(f, (x, Ff, \alpha, x_*)) : (a, (Fa, x)) \longrightarrow (a_*, (Fa_*, x_*))$ to $(\varrho_{_{Fa}}(x), f, \varrho_{_{Fa_*}}(\alpha), \varrho_{_{Fa_*}}(x_*))$. Following $\varrho : T \Rightarrow S$ the other way around first gives $\int\limits_A \varrho_{_F} : \int\limits_A TF \longrightarrow \int\limits_A SF$ (as defined in Construction 2.9 and Lemma 2.27) which is then followed by $(\tau_{_F})_{_T}$; the resulting functor takes an object $(a, (Fa, x))$ of $(\int\limits_B T)^F$ first to (a, x) and then to $(a, \varrho_{Fa}(x))$ and a morphism $(f, (x, Ff, \alpha, x_*))$ first to (x, f, α, x_*) and then to $(\varrho_{_{Fa}}(x), f, \varrho_{_{Fa_*}}(\alpha), \varrho_{_{Fa_*}}(x_*))$. Thus for a 1-cell ϱ in $\flat\mathrm{CAT}^B(T, S)$, $(\tau_{_F})_{_S} \circ (\int_B \varrho)^F = (\tau_{_F})_{_T} \circ \int\limits_A \varrho_{_F}$. For a 2-cell $m : \varrho \rightsquigarrow \eta$ in $\flat\mathrm{CAT}^B(T, S)$ the commutativity of the diagram translates to $1_{(\tau_{_F})_{_S}} * (\int_B m)^F = \int_A m_{_F} * 1_{(\tau_{_F})_{_T}}$; for both horizontal compositions of 2-cells in the equality (see diagram below), the component corresponding to an object $(a, (Fa, x))$ in $(\int\limits_B T)^F$ is the arrow $(\varrho_{_{Fa}}(x), 1_a, (m_{_{Fa}})_x, \eta_{_{Fa}}(x)) : (a, \varrho_{_{Fa}}(x)) \longrightarrow (a, \eta_{_{Fa}}(x))$ in $\int_A SF$.

$$\begin{tikzcd}[column sep=huge]
(\int_B T)^F \arrow[r, bend left, "(\int_B \varrho)^F"] \arrow[r, bend right, "(\int_B \eta)^F"'] \arrow[r, phantom, "\Downarrow(\int_B m)^F"] & (\int_B S)^F \arrow[r, bend left, "(\tau_F)_S"] \arrow[r, bend right, "(\tau_F)_S"'] \arrow[r, phantom, "\Downarrow 1"] & \int_A SF
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge]
(\int_B T)^F \arrow[r, bend left, "(\tau_F)_T"] \arrow[r, bend right, "(\tau_F)_T"'] \arrow[r, phantom, "\Downarrow 1"] & \int_A TF \arrow[r, bend left, "\int_A \varrho_F"] \arrow[r, bend right, "\int_A \eta_F"'] \arrow[r, phantom, "\Downarrow \int_A m_F"] & \int_A SF
\end{tikzcd}$$

Given a natural transformation $\alpha : F \Rightarrow G$ with $F, G : A \longrightarrow B$, we must have the components of the natural transformations $1 * \alpha^*$ and $H_\alpha * 1$ (see diagram below) to be coherent with the components of τ.

$$\begin{tikzcd}
\flat\mathrm{CAT}^B \arrow[r, bend left, "F^*"] \arrow[r, bend right, "G^*"'] \arrow[r, phantom, "\Downarrow \alpha^*"] & \flat\mathrm{CAT}^A \arrow[r, bend left, "\int_A"] \arrow[r, bend right, "\int_A"'] \arrow[r, phantom, "\Downarrow 1"] & H_A
\end{tikzcd}
\text{ and }
\begin{tikzcd}
\flat\mathrm{CAT}^B \arrow[r, bend left, "\int_B"] \arrow[r, bend right, "\int_B"'] \arrow[r, phantom, "\Downarrow 1"] & H_B \arrow[r, bend left, "H_F"] \arrow[r, bend right, "H_G"'] \arrow[r, phantom, "\Downarrow H_\alpha"] & H_A
\end{tikzcd}.$$

For an object T in $\flat\mathrm{CAT}^B$ then, the following diagram must be shown to commute:

$$\begin{tikzcd}
(\int_B T)^F \arrow[r, "H_{\alpha_T}"] \arrow[d, "(\tau_F)_T"', "\cong"] & (\int_B T)^G \arrow[d, "\cong"', "(\tau_G)_T"] \\
\int_A TF \arrow[r, "\int_A \alpha^*_T"'] & \int_A TG
\end{tikzcd}$$

Following an object $(a,(Fa,x))$ (with x in $TF(a)$) of $(\int_B T)^F$ first down and then right gives: $(a,(Fa,x)) \xrightarrow{(\tau_F)_T} (a,x) \xrightarrow{\alpha^*_T} (a,(\alpha^*_T)_a(x)) = (a,T(\alpha_a)(x))$ by Lemma 2.10. The same object is taken (by Lemma 2.20) to $(a,\overline{(Fa,a)})$ where $\overline{(Fa,a)}$ is the codomain of the horizontal lift of $\alpha_a : F(a) \longrightarrow G(a)$ to domain $(F(a),a)$ in $\int_B T$; the horizontal lift of α_a (by Lemma 2.26) is $(Fa,a) \xrightarrow{(x,\alpha_a,1,T(a_a)(x))} (G(a),T(a_a)(x))$, so following the object $(a,(Fa,x)$ first right, then down gives

$(a,(Fa,x)) \xrightarrow{H_{\alpha_T}} (a,(G(a),T(\alpha_a)(x))) \xrightarrow{(\tau_G)_T} (a,T(\alpha_a)(x))$. For objects then the diagram above commutes.

Given a morphism $(a,(Fa,x)) \xrightarrow{(f,(x,Ff,\nu,y))} (b,(Fb,y))$ in $(\int_B T)^F$ (with $\nu : TFf(x) \longrightarrow y$ an arrow in $TF(b)$), on one hand $(\tau_F)_T(f,(x,Ff,\nu,y)) = (x,f,\nu,y)$ and by Lemma 2.27 $(\int_A \alpha^*_T(x,f,\nu,y) = (T(\alpha_a)(x),f,T(\alpha_b)(\nu),T(\alpha_b)(y))$. On the other hand, by Lemma 2.20, $H_{\alpha_T}((f,(x,Ff,\nu,y)) = (f,v\overline{Gf})$ where v is the vertical component of the horizontal-vertical decomposition of $\overline{\alpha_b} \circ (x,Ff,\nu,y)$. By Lemma 2.26 and Definition 2.23 we have that $\overline{\alpha_b} \circ (x,Ff,\nu,y) = (y,\alpha_b,1,T\alpha_b(y)) \circ (x,Ff,\nu,x) = (x,\alpha_b Ff,T\alpha_b(\nu),T\alpha_b(y)) = (T(\alpha_b Ff)(x),1_{Gb},T\alpha_b(\nu),T\alpha_b(y)) \circ (x,\alpha_b Ff,1,T(\alpha_b Ff)(x))$. Since α is a natural transformation, $TGf \circ T\alpha_a =$

$T\alpha_b \circ TFf$; we then have that $\overline{Gf}$, the horizontal lift of Gf to domain $(Ga, T\alpha_a(x))$ is $(T\alpha_a(x), Gf, 1, TGf \circ T\alpha_a(x)) = (T\alpha_a(x), Gf, 1, T\alpha_b \circ TFf)$ and then $v\overline{Gf} = (T(\alpha_b Ff)(x), 1_{Gb}, T\alpha_b(\nu), T\alpha_b(y)) \circ (T\alpha_a(x), Gf, 1, T(\alpha_b Ff)) = (T\alpha_a(x), Gf, T\alpha_b(\nu), T\alpha_b(y))$. The image of $(f, v\overline{Gf})$ under $(\tau_G)_T$ is then $(T\alpha_a(x), f, T\alpha_b(\nu), T\alpha_b(y))$ which agrees with $(\int_A \alpha_T^*)(\tau_F)_T(f, (x, Ff, \nu, y))$ above.

The natural transformation $\int$ is a natural isomorphism, since by Lemma 2.29 its components are invertible.

Universal Horizontal Structure

The horizontal structure $\int\limits_{\flat\mathrm{CAT}} 1$ (henceforth denoted by CAT_*) over $\flat$CAT that results from taking F in Lemmas 2.25 and 2.26 to be the identity $1 : \flat\mathrm{CAT} \longrightarrow \flat\mathrm{CAT}$ has the following description: The objects of CAT_* are pairs (C, x) with x an object of the category C, and its morphism are quadruples $(x, G, t, y) : (C, x) \longrightarrow (D, y)$ with $G : C \longrightarrow D$ a functor and $t : G(x) \longrightarrow y$ a morphism in D. The horizontal morphisms are of the form $(x, G, 1_{G(x)}, G(x)) : (C, x) \longrightarrow (D, G(x))$ and the categorical fiber over C is isomorphic to C itself. Applying Construction 2.14 to a functor $F : B \longrightarrow \mathrm{CAT}$ and to the horizontal structure $\mathrm{CAT}_* \xrightarrow{U^1} \mathrm{CAT}$, i.e., pulling back CAT_* along F, results in a horizontal structure $\mathrm{CAT}_*^F \longrightarrow B$ that is canonically isomorphic to $\int\limits_B F$. The isomorphism is given by identifying the object $(b, (Fb, x))$ in CAT_*^F to (b, x) in $\int\limits_B F$ and the morphism $(f, (x, Ff, t, y))$ in CAT_*^F with (x, f, t, y) in $\int\limits_B F \longrightarrow B$ and it follows from the natural isomorphism τ_F described in the proof of Proposition 2.31 applied to a functor $F : B \longrightarrow \mathrm{CAT}$ and taking T to be the identity functor $1 : \mathrm{CAT} \longrightarrow \mathrm{CAT}$. The Grothendieck construction on a functor $F : B \longrightarrow \mathrm{CAT}$ is then, up to an isomorphism, the pullback along F of $\mathrm{CAT}_* \longrightarrow \mathrm{CAT}$. By Corollary 2.21, pulling CAT_* back along isomorphic functors $F \cong G : B \longrightarrow \mathrm{CAT}$ results in the isomorphic horizontal structures $\mathrm{CAT}_*^F \cong \mathrm{CAT}_*^G$; hence we have that $F \cong G \Rightarrow \int_B F \cong \mathrm{CAT}_*^F \cong \mathrm{CAT}_*^G \cong \int_B G$. Conversely, every horizontal structure $E \xrightarrow{P} B$ defines a functor $T_E : B \longrightarrow \mathrm{CAT}$ (the fiber functor) such that $\int_B T_E \cong E$ by the isomorphism θ_E of horizontal structures described in detail in the proof of Lemma 2.29: an object (b, x) of $\int_B T_E$ corresponds to x in E and a morphism (x, f, t, y) in $\int_B T_E$ to $t \circ \overline{f_x}$ in E. We then have the following pullback square showing that every horizontal structure $E \xrightarrow{P} B$ is the pullback (up to an isomorphism) of $\mathrm{CAT}_* \xrightarrow{1} \mathrm{CAT}$, which therefore will be called the *Universal Horizontal Structure.*

$$\begin{tikzcd}
\int_B T_E \cong E \cong \mathrm{CAT}_*^{T_E} \arrow[r] \arrow[d] & \mathrm{CAT}_* \arrow[d] \\
B \arrow[r, "T_E"'] & \mathrm{CAT}
\end{tikzcd}$$

Topological Structures

Definition 2.32 Let A, E and B be categories.

(1) A *sink* in A with codomain a is a family of morphisms $(f_j : a_j \longrightarrow a)_{j\in J}$, indexed by a class J (which may be empty).

(2) Given a functor $U : E \longrightarrow B$, a *U-sink* is a family $(e_j, f_j : U(e_j) \longrightarrow b)_{j\in J}$.

(3) If $(\bar{f}_j : e_j \longrightarrow e)_{j\in J}$ is a sink in E and $(e_j, f_j : U(e_j) \longrightarrow b)_{j\in J}$ is a U-sink such that $U(\bar{f}_j) = f_j$ for all $j \in J$, then $(\bar{f}_j : e_j \longrightarrow e)_{j\in J}$ is called a *lift* of $(e_j, f_j : U(e_j) \longrightarrow b)_J$ with codomain e.

Definition 2.33 A lift $(\bar{f}_j : e_j \longrightarrow e)_{j\in J}$ of the U-sink $(e_j, f_j : U(e_j) \longrightarrow b)_{j\in J}$ is called a *final lift* if given a sink $(g_j : e_j \longrightarrow c)_{j\in J}$ such that there exists a morphism $h : b \longrightarrow U(c)$ with $hf_j = U(g_j)$ for all $j \in J$, then h lifts to a morphism $\bar{h} : e \longrightarrow c$ with $\bar{h}\bar{f}_j = g_j$.

The dual notion of a sink (final sink) is a source (initial source).

Remark 2.34 If $J = \emptyset$ in Definition 2.33 above, then the corresponding sink or U-sink with codomain b is an empty set of morphisms; the final lift of the empty U-sink with codomain $b \in B$ is $e \in U^{-1}(b)$ such that for any $c \in E$, if there is a morphism $h : b \longrightarrow U(c)$ in B, then h lifts to a morphism $\bar{h} : e \longrightarrow c$ in E.

Definition 2.35 A functor $U : E \longrightarrow B$ is a *topological functor* and E is a *topological category* over B if U satisfies the following conditions:

(i) U has small fibers,

(ii) U is faithful, (i.e., mono on hom sets);

(iii) U is amnestic, (i.e., if $f \in E$ is an isomorphism s.t. $U(f) = \mathrm{id}$, then $f = \mathrm{id}$);

(iv) Every U-sink has a final lift.

Remarks 2.36 (1) The first three conditions of the definition above imply that the categorical fiber of U is a poset: $U^{-1}(b)$ is a set by (i), it is a preorder by (ii) and a partial order by (iii).

(2) Condition (iv) is equivalent to the existence of initial lifts of arbitrary U-sources; see [1], Proposition 21.36.

(3) Final (and initial) lifts are unique by the faithfulness of U.

(4) Given an object $x \in U^{-1}(b)$ and a morphism $f : b \longrightarrow c$ in B, we can view $(x,\, f : b \longrightarrow c)$ as a U-sink. We'll refer to the final lift $\bar{f} : x \longrightarrow y$ of this U-sink as the *final lift of f to domain x.*

(5) In a horizontal structure $E \xrightarrow{P} B$, the unique horizontal lift of $f : b \longrightarrow c$ in B to domain x in E is the final lift of the P-sink $(x, f : b \longrightarrow c)$ since if $g : x \longrightarrow t$ is such that $P(g) = hf$ for some $h : c \longrightarrow P(t)$ and $g = vk$ is the unique horizontal-vertical decomposition of g, then lifting h to $y = \mathrm{cod}(\overline{f}_x)$ gives $\overline{h} : y \longrightarrow \mathrm{cod}(k)$ which means that g factors through $\overline{f}$ as $g = (\overline{h}v)\overline{f}$.

Example 2.37 Let **Top** denote the category of topological spaces and continuous functions. The forgetful functor $U : \mathrm{Top} \longrightarrow \mathrm{Sets}$ with $U(X,\tau) = X$ is a topological functor, and Top is a topological category over the category of Sets. For a set X, $U^{-1}(X) = \{(X,\tau) \,|\, \tau \text{ is a topology on } X\}$. Since $U^{-1}(X) \subseteq P^2(X)$, U has small fibers. Given a U-sink $((X_j,\tau_j), f_j : X_j \longrightarrow X)_J$ in Sets, its final lift is $(f_j : (X_j,\tau_j) \longrightarrow (X,\tau))_J$ where τ is the topology coinduced on X by the functions f_j, i.e., $\tau = \{U \in P(X) |\, f_j^{-1}(U) \in \tau_j \text{ for all } j \in J\}$.

Lemma 2.38 *Given a topological functor $U : E \longrightarrow B$, suppose $(\bar{f}_j : x_j \longrightarrow x)_{j\in J}$ is the final lift of the U-sink $(x_j, f_j : U(x_j) = b_j \longrightarrow b)_{j\in J}$, and $\bar{g} : x \longrightarrow y$ is the final lift of $g : b \longrightarrow c$ to domain x. Then $\bar{g}\bar{f}_j : x_j \longrightarrow y$ is the final lift of the U-sink $(x_j, gf_j : b_j \longrightarrow c)_J$.*

Proof. Suppose $(k_j : x_j \longrightarrow t)_{j\in J}$ is a sink such that $U(k_j)$ factors through gf_j for all j as $U(k_j) = h(gf_j)$ with $h : c \longrightarrow U(t)$; then $U(k_j)$ factors through f_j as well which means that $hg : b \longrightarrow U(t)$ lifts to a morphism $\overline{hg} : t \longrightarrow y$ in E with $k_j = \overline{hg}\bar{f}_j$ since $(\bar{f}_j : x_j \longrightarrow x)_{j\in J}$ is a final lift. Similarly, since $\bar{g}$ is a final lift, h lifts ot $\bar{h} : y \longrightarrow t$ in E.

Definition 2.39 Let *CoP* denote the category of cocomplete posets whose objects are cocomplete posets and whose morphisms are functions $f : (X, \leq) \longrightarrow (Y, \leq)$ that preserve order and arbitrary suprema, and let *CmP* denote the category of complete posets whose morphisms are functions preserving order and arbitrary infima. If we view a poset $(X, \leq)$ as a category, then for $(x_j)_{j\in J} \subseteq X$, $\underset{j\in J}{\vee}(x_j)$ is the colimit of (x_j) and a supremum-preserving function $f : X \longrightarrow Y$ is then a cocontinuous functor; similarly an infimum-preserving function is then a continuous functor.

Lemma 2.40 *A poset is cocomplete iff it is complete.*

Proof. Suppose $(X, \leq)$ is a cocomplete poset. For $(x_j)_{j\in J} \subseteq X$, let $\underset{j\in J}{\wedge}(x_j) = \vee\{s \in X \,|s \leq x_j \text{ for all } j \in J\}$, and similarly if $(X, \leq)$ is assumed to be complete, then we can define the supremum of an arbirary subset T of X as the infimum of the set of upper bounds of T.

Remarks 2.41 (1) In view of the above lemma, posets that are cocomplete and thus complete as well, will be called *plete* posets.

(2) Since a *complete lattice* is a poset in which every subset has an infimum and a supremum (Definition 2.1 in [6]), a plete poset is a complete lattice. Morphisms in the category of lattices (Lat) are functions that preserve both infima and suprema. Thus CoP, CmP and Lat have the same objects, but different morphisms.

Lemma 2.42 *If $f^* : (A, \leq) \longrightarrow (B, \leq)$ is a morphism in CoP, i.e., as a functor, f^* is cocontinuous, then it has a right adjoint $f_* : (B \leq) \longrightarrow (A, \leq)$ (which is then continuous), defined as $f_*(b) = \vee\{a \in A \,|\, f^*(a) \leq b\}$. We also have the dual statement: if $f_* : (B, \leq) \longrightarrow (A, \leq)$ is a morphism in CmP, then it has a left adjoint $f^* : (A, \leq) \longrightarrow (B, \leq)$ (which is then cocontinuous), defined as $f^*(a) = \wedge\{b \in B \,|\, f_*(b) \geq a\}$.*

Proof. Given a morphism $f^* : (A, \leq) \longrightarrow (B, \leq)$ in CoP, f_* (as defined above) preserves order, since for $b_1 \leq b_2$, $\{a \in A \,|\, f^*(a) \leq b_1\} \subseteq \{a \in A \,|\, f^*(a) \leq b_2\}$ and therefore $f_*(b_1) = \vee\{a \in A \,|\, f^*(a) \leq b_1\} \leq \vee\{a \in A \,|\, f^*(a) \leq b_2\} = f_*(b_2)$. f^* is left adjoint to f_* (written as $f^* \dashv f_*$) if and only if for all $a \in A$ and $b \in B$, $f^*(a) \leq b$ in B if and only if $a \leq f_*(b)$ in A, by Theorem 1 on page 93 in [14]. So suppose that $f^*(a) \leq b$; then $f_*f^*(a) \leq f_*(b)$ since f_* preserves order. By definition $f_*f^*(a) = \vee\{x \in A \,|\, f^*(x) \leq f^*(a)\}$, and since $a \in \{x \in A \,|\, f^*(x) \leq f^*(a)\}$, $a \leq \vee\{x \in A \,|\, f^*(x) \leq f^*(a)\} = f_*f^*(a) \leq f_*(b)$. To prove the converse, assume that $a \leq f_*(b)$. Then applying f^* to $a \leq f_*(b) = \vee\{x \in A \,|\, f^*(x) \leq b\}$ gives that

$f^*(a) \leq f^*(\vee\{x \in A \,|\, f^*(x) \leq b\}) = \vee\{f^*(x) \,|\, f^*(x) \leq b\} \leq b$ by cocontinuity of f^*. The dual statement can be proven similarly.

Given a 2-category K, we can define the adjunction category $\mathrm{Adj}(K)$ as having the same objects as K, and adjoint pairs $l \dashv r : a \longrightarrow b$ as morphisms, where the direction of the morphism is given by that of the right adjoint $r : a \longrightarrow b$. (A 1-cell $r : a \longrightarrow b$ is right adjoint to $l : b \longrightarrow a$ if there are 2-cells $\eta_b : 1_b \Rightarrow rl$ and $\varepsilon_a : lr \Rightarrow 1_a$ satisfying the triangle equalities.) Given $l \dashv r$ and $l' \dashv r'$, every 2-cell $\alpha : l \Rightarrow l'$ defines a 2-cell $\beta : r' \Rightarrow r$ by the following vertical composition of 2-cells: $\beta = \varepsilon'_r \cdot \alpha \cdot \eta_{r'}$ where $\eta_{r'}$ denotes the horizontal composition $\eta_b * 1_{r'}$ and similarly, $\varepsilon'_r = 1_r * \varepsilon_a$. $\mathrm{Adj}(K)$ is then a 2-category with $\alpha : l \dashv l' \Rightarrow r \dashv r'$ as 2-cells and hence we can define the following 2-funtors: $\Pi_l : \mathrm{Adj}(K) \longrightarrow K^{\mathrm{op}}$ with $\Pi_l(l \dashv r) = l$ and $\Pi_l(\alpha) = \alpha$, and $\Pi_r : \mathrm{Adj}(K) \longrightarrow K^{\mathrm{co}}$ with $\Pi_r(l \dashv r) = r$ and $\Pi_l(\alpha) = \beta$; both functors are the identity on objects.

Remark 2.43 The category POS of partially ordered sets can be viewed as a 2-category. The existence of a 2-cell $\alpha : f \Rightarrow g$ between $f, g : (A, \leq) \longrightarrow (b, \leq)$ means that for all $a \in A$, $f(a) \leq g(a)$.

If K=Plete, the category of plete posets and order preserving maps, then since left adjoints are cocontinuous and right adjoints are continuous, the functors Π_r and Π_l define the isomorphisms of 2-categories as pictured:

$$\begin{tikzcd} \mathrm{Adj(Plete)} \arrow[r, "\Pi_l"] \arrow[dr, "\Pi_l"', "\cong"] & \mathrm{Plete}^{\mathrm{op}} \\ & \mathrm{CoP}^{\mathrm{op}} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Adj(Plete)} \arrow[r, "\Pi_r"] \arrow[dr, "\Pi_r"', "\cong"] & \mathrm{Plete}^{\mathrm{co}} \\ & \mathrm{CmP}^{\mathrm{co}} \arrow[u, hook] \end{tikzcd}$$

Dropping the 2-cells, the above define isomorphisms of the categories: Adj(Plete) $\cong$ $\mathrm{Cop}^{\mathrm{op}}$ $\cong$ CmP.

Definition 2.44 A *geometric morphism* $f : A \longrightarrow B$ is a pair of functors $f^* : B \longrightarrow A$ and $f_* : A \longrightarrow B$ such that $f^* \dashv f_*$ and f^* is left exact (preserves all finite limits); f_* is called the direct image part of f and f^* the inverse image part of the geometric morphism (Definition 1 on page 348 in [15]). (The direction of a geometric morphism $f : A \longrightarrow B$ again agrees with that of the (continuous) direct image part $f_* : A \longrightarrow B$.) A geometric morphism with the special property that its inverse image part has a left adjoint is called an essential geometric morphism. (cf. page 360 in [15]).

Example 2.45 Consider a topological space (X, τ), and the resulting inclusion $i : \tau \hookrightarrow P(X)$ of posets; here $P(X)$ and τ are ordered by inclusion. Since i preserves arbitrary unions and finite intersections, it is cocontinuous and left exact, so with its right adjoint $r : P(X) \longrightarrow \tau$ (given for $\mathcal{O} \in P(X)$ by $r(\mathcal{O}) = \mathrm{int}\mathcal{O}$), they form a geometric morphism $h = (i \dashv r) : P(X) \longrightarrow \tau$.

Example 2.46 Essential geometric morphisms naturally arise whenever we consider two sets X and Y and a function $f : X \longrightarrow Y$: if we view $P(X)$ and $P(Y)$ as posets (again ordered by inclusion), then since $f^{-1} : P(Y) \longrightarrow P(X)$ preserves arbitrary unions and intersections, as a poset map it is bicontinuous and hence it has both a right and a left adjoint. Its continuous right adjoint $\forall_f : P(X) \longrightarrow P(Y)$ is defined as $\forall_f(S) = \bigcup\{T \in P(Y) |\, f^{-1}(T) \subseteq S\} = \{y \in Y |\, f^{-1}(y) \subseteq S\}$ by Lemma 2.42; $h = (f^{-1} \dashv \forall_f) : P(X) \longrightarrow P(Y)$ is then a geometric morphism. The cocontinuous left adjoint of f^{-1} is $\mathrm{Im}\, f : P(X) \longrightarrow P(Y)$ (the notation $\exists_f$ is also used for $\mathrm{Im}\, f$ (cf. Theorem 2 on page 58 in [15]); h is then an essential geometric morphism and we have $\exists_f \dashv f^{-1} \dashv \forall_f$. If $P(X)$ and $P(Y)$ are ordered by reverse inclusion (in this case the notation $P(X)^{\mathrm{op}}$ will be used) then for $(S_j)_{j\in J} \subseteq P(X)$ $\bigvee\limits_J S_j = \bigcap\limits_J S_j$ and $\bigwedge\limits_J S_j = \bigcup\limits_J S_j$, and $\mathrm{Im} f : P(X)^{\mathrm{op}} \longrightarrow P(Y)^{\mathrm{op}}$ becomes continuous since it preserves unions which now give infima, and $g = (f^{-1} \dashv \mathrm{Im} f) : P(X)^{\mathrm{op}} \longrightarrow P(Y)^{\mathrm{op}}$ again is a geometric morphism.

Definition 2.47 Let *GeoCmP* denote the category whose objects are plete posets and whose morphisms are geometric morphisms and let *Ess* denote the category with the same objects and morphisms the essential geometric morphisms.

We then have the following relationship between the different categories of posets: Ess $\subseteq$ GeoCmP $\subseteq$ CmP $\cong$ CoP$^{\mathrm{op}}$ $\cong$ Adj(Plete).

Example 2.48 The power-set functor can be viewed as a functor P : Sets $\longrightarrow$ Ess $\subseteq$ GeoCmP sending a function of sets $f : X \longrightarrow Y$ to $Pf = (f^{-1} \dashv \forall_f)$: $(P(X), \subseteq) \longrightarrow (P(Y), \subseteq)$ as in example 2.46. In general, given an essential geometric morphism $f : A \underset{f^*}{\overset{f_*}{\rightleftarrows}} B$ with $l : A \longrightarrow B$ the left adjoint of f^*, we can define a functor Φ : Ess $\longrightarrow$ GeoCmP on the object level by $\Phi(A) = A^{\mathrm{op}}$ and on the morphism level by $\Phi(f) = (A^{\mathrm{op}} \underset{(f^*)^{\mathrm{op}}}{\overset{l^{\mathrm{op}}}{\rightleftarrows}} B^{\mathrm{op}})$. Now $(f^*)^{\mathrm{op}} \dashv l^{\mathrm{op}}$ gives a geometric morphism. Composing the power-set functor with Φ gives the functor ΦP : Sets $\longrightarrow$ GeoCmP which sends a set X to $(P(X)^{\mathrm{op}})$ and the function $f : X \longrightarrow Y$ to $(\Phi P)(f) = ({f^{-1}}^{\mathrm{op}} \dashv {\exists_f}^{\mathrm{op}}) : P(X)^{\mathrm{op}} \longrightarrow P(Y)^{\mathrm{op}}$. For another example of essential geometric morphisms (between fuzzy sets) see page 62.

Remark 2.49 A cocomplete poset $(A, \leq)$ which satisfies the infinite distributive law $b \wedge (\bigvee_j a_j) = \bigvee_j (b \wedge a_j)$ is called a *frame,* and a poset map $f^* : (A, \leq) \longrightarrow (B, \leq)$ which preserves infinite joins and finite meets is called a frame morphism (cf. page 473 in Sheaves in [15]). Frames form the category Frm. A frame morphism is then cocontinuous and left exact, so with its right adjoint $f_* : B \longrightarrow A$ they give a geometric morphism $f = (f^* \dashv f_*) : B \longrightarrow A$. The opposite category of Frm is called the category of locales (Loc) and it is then a full subcategory of GeoCmp.

Theorem 2.50 *Let $\mathcal{K}^* : CAT^{\mathrm{op}} \longrightarrow 2 - Cat$ be the subfunctor of $\mathcal{K}$ defined by $\mathcal{K}^*(A) = CoP^A \subset CAT^A$. Then the component of the 2-natural transformation $\int : \mathcal{K} \longrightarrow \mathcal{H}$ corresponding to A, the 2-functor $\int_A : CoP^A \longrightarrow H_A$ (defined in 2.28), is such that for an object T in CoP^A, $\int_A T$ is a topological category.*

Proof. $U^T : \int_A T \longrightarrow A$ has small fibers since $(U^T)^{-1}(a) \cong T(a)$ which is a (po)set; U^T is clearly faithful and amnestic by Definition 2.22. Given a U^T-sink $((a_j, x_j), f_j : a_j \longrightarrow b)_J$, let $x = \bigvee_J Tf_j(x_j) \in T(a)$. Since $Tf_j(x_j) \leq x$ for all j, $\bar{f}_j = (x_j, f_j, 1_x, x) : (a_j, x_j) \longrightarrow (a, x)$ is a morphism in $\int_A T$ for all j. Then the sink $(\bar{f}_j : (a_j, x_j) \longrightarrow (a, x))$ is a final lift of the U-sink $((a_j, x_j), f_j : a_j \longrightarrow a)_J$, since if for a sink $(\bar{g}_j : (a_j, x_j) \longrightarrow (c, y))$ there is a morphism $h : a \longrightarrow c$ such that $g_j = hf_j$ for all j, then the cocontinuity of Th implies that $Th(x) = Th\left(\bigvee Tf_j(x_j)\right) = \bigvee ThTf_j(x_j) = \bigvee Tg_j(x_j) \leq y$, and hence $h : a \longrightarrow c$ lifts to a morphism $\bar{h} : (a, x) \longrightarrow (c, y)$ in $\int_A T$. Thus $U^T : \int_A T \longrightarrow A$ is a topological functor.

Note If a functor $T : A \longrightarrow$ CAT factors through POS, i.e., we have $T : A \longrightarrow$ POS $\hookrightarrow$ CAT, then $U : \int_A T \longrightarrow A$ satisfies conditions (i), (ii) of Definition 2.35, however as far as condition (iv) goes, an arbitrary U-sink does not necessarily have a final lift. If a U-sink $((a_j, x_j), f_j : a_j \longrightarrow a)_J$ has a final lift then $x = \bigvee_J Ff_j(x_j) \in T(a)$ exists and the final lift is $(\bar{f}_j : (a_j, x_j) \longrightarrow (a, x))$. The converse however is not true.

Lemma 2.51 *Every topological category over A is an object in H_A, the 2-category of horizontal structures over A.*

Proof. Given a topological category $E \xrightarrow{U} A$, a splitting (H, V) is defined as follows; $V = V(E)$ as in Definition 2.2, and and H contains a morphism $h : x \longrightarrow y$ of E iff it is the final lift of the U-sink $(x, U(h) : U(x) \longrightarrow U(y))$. H is a subcategory of E since given an object $x \in U^{-1}(a)$, 1_x is the final lift of the U-sink $(x, 1_a)$, and H is closed under the composition of morphisms. Given a morphism $g : x \longrightarrow z$ in E with $x \in U^{-1}(a)$ and $z \in U^{-1}(c)$, let $h : x \longrightarrow y$ be the final lift of the U-sink $(x, U(g) : a \longrightarrow c)$. Then, since $U(g)$ factors through $U(h)$ as $U(g) = 1_c \circ U(h)$, 1_c lifts (uniquely) to a morphism $\alpha : y \longrightarrow z$. Thus g factors as $g = \alpha \circ h$.

Definition 2.52 $\mathrm{CAT}_{\mathrm{Top}_A}$ will denote the full subcategory of H_A with objects the topological categories over A.

Theorem 2.53 *Restricting the 2-functor $\int_A^{-1} : H_A \longrightarrow CAT^A$ of Lemma 2.29 to CAT_{Top_A} gives a 2-functor $(\int_A^{-1})^* : CAT_{\mathrm{Top}_A} \longrightarrow CoP^A$.*

Proof For a topological category $E \xrightarrow{P} A$, the corresponding functor $\int_A^{-1}(E) = T_E : A \longrightarrow \mathrm{CAT}$ is defined on the object level by $T_E(a) = P^{-1}(a)$ which is a poset (see Remark (1) of 2.36) ordered by the relation $x \leq y$ iff there is a morphism $\alpha : x \longrightarrow y$ in the categorical fiber of P over a.

To show that $P^{-1}(a)$ is cocomplete, for $(x_j)_{j \in J} \subseteq P^{-1}(a)$, define $\bigvee\limits_{j \in J}(x_j)$ to be the codomain of the final lift of the P-sink $(x_j, 1_a)$. $T_{_E}(a)$ is then a cocomplete poset for all $a \in A$. (If $J = \emptyset$, then the final lift of the empty P-sink with codomain a is $x \in P^{-1}(a)$ as described in Remark 2.34; in particular x is such that for any $y \in P^{-1}(a)$, $1_a \in A$ lifts to a morphism $f : x \xrightarrow{\leq} y$. Hence the final lift of the empty P-sink with codomain a is the minimum element of the poset $P^{-1}(a)$.) To show that for a morphism $f : a \longrightarrow c$ in A $T_{_E}f : T_{_E}(a) \longrightarrow T_{_E}(c)$ is a morphism in CoP, we must show that $T_{_E}f$ preserves order and least upper bounds. $T_{_E}f$ is order preserving by its definition in the proof of Lemma 2.29. Let $(x_j)_{j \in J} \subseteq P^{-1}(a)$, and $x = \bigvee\limits_{j \in J}(x_j)$; thus $(x_j \xrightarrow{\leq} x)$ is the final lift of the P-sink $(x_j, 1_a)$. Consider the P-sink $(x_j, f1_a : a \longrightarrow c)$, and let $\bar{f}$ denote the final lift of f to domain x. Then by Lemma 2.38, $(x_j \xrightarrow{\leq} x \xrightarrow{\bar{f}} \mathrm{cod}\bar{f} = T_{_E}f(x))$ is a final lift of this P-sink. For each x_j, let $\bar{f}_j$ denote the final lift of f to domain x_j, and let $y_j = \mathrm{cod}\bar{f}_j = T_{_E}f(x_j)$. Since $(\leq \circ \bar{f}_j : x_j \longrightarrow \vee y_j)_J$ is a P-sink over $(x_j, f : a \longrightarrow c)$, lifting 1_c we get that $T_{_E}f(x) \leq \vee y_j$. On the other hand since $T_{_E}f$ preserves order, $x_j \leq x$ implies that $y_j = T_{_E}f(x_j) \leq T_{_E}f(x)$ for all j, and then $\vee y_j \leq T_{_E}f(x)$. Thus $T_{_E}f(\vee x_j) = T_{_E}f(x) = \vee y_j = \vee Ff(x_j)$, which proves that $T_{_E}f$ is cocontinuous and hence it defines a functor $A \longrightarrow \mathrm{CoP}$

Corollary 2.54 *In view of Theorems 2.50 and 2.53 we have that $\int_A$ defines an equivalence of the 2-categories CAT_{Top_A} and CoP^A such that given a topological* *category* $E \xrightarrow{P} A$, $\int_A T_E \cong E$*; in this case we say that* E *is classified by the functor* $T_E : A \longrightarrow Cop$*.*

By the corollary above, there is a one-to-one correspondence between topological functors $T : E \longrightarrow A$ and presheaves on the category A with values in Adj(Plete): Every topological category defines the (fiber)functor $T_E : A \longrightarrow$ Cop which, since $\text{Cop}^{\text{op}} \cong$ Adj(plete), can be viewed as a functor $T_E : A^{\text{op}} \longrightarrow$ Adj(Plete), i.e., a presheaf on A. Conversely, every presheaf $T : A^{\text{op}} \longrightarrow$ Adj(Plete) gives a functor $T : A \longrightarrow$ Cop which the defines the topological category $\int_A T$.

Universal Topological Category

Definition 2.55 The topological category $\int_{\text{CoP}} 1$ classified by the functor $1 : \text{CoP} \longrightarrow \text{CoP}$ is called the *universal topological category* and will be denoted by CoP_*. (cf. $\flat\text{CAT}_* = \int_{\flat\text{CAT}} 1$ on page 43.) The objects of the universal topological category are pairs (A, a) where A is a cocomplete poset and $a \in A$. A morphism $f : (A, \leq) \longrightarrow (B, \leq)$ in CoP lifts to a morphism $\bar{f} = (a, f, b) : (A, a) \longrightarrow (B, b)$ iff $f(a) \leq b$ in B.

Theorem 2.56 *Every topological category* $E \xrightarrow{P} A$ *is isomorphic to a pull-back of the universal topological category* $\int_{\text{CoP}} 1$*.*

Proof. Given a topological category $E \xrightarrow{P} A$, by Corollary 2.54 we have that $E \cong \int_A T_E$ where $T_E : A \longrightarrow \text{CoP}$ is the functor defined in 2.29. The pullback $(\text{CoP}_*)^{T_E}$

of CoP_* along T_E is isomorphic to $\int\limits_A T_E$ by the isomorphism τ of Proposition 2.31; hence $E \cong (\mathrm{CoP}_*)^{T_E}$ as shown by the pullback square below.

$$\begin{tikzcd}
E \cong \int\limits_A T_E \arrow[r] \arrow[d, "P"'] & \mathrm{CoP}_* = \int\limits_{\mathrm{CoP}} 1 \arrow[d, "U^1"] & \\
A \arrow[r, "T_E"] & \mathrm{CoP} \arrow[r, "1"] & \mathrm{CoP}
\end{tikzcd}$$

Similarly, we can view Sets_*, the category of pointed sets as the *universal discrete opfibration* ($P : E \longrightarrow B$ is called a discrete opfibration if every morphism $f : P(e) \longrightarrow c$ in B lifts uniquely to a morphism with domain e) classified by the identity functor on the category of Sets: given any discrete opfibration $P : E \longrightarrow B$, there is a functor $F : B \longrightarrow \mathrm{Sets}$ such that $E \cong \int\limits_B F$ with $\int\limits_B F$ the pull-back of Sets_* along F.

$$\begin{tikzcd}
E \cong \int\limits_B F \arrow[r] \arrow[d, "P"'] & \int\limits_{\mathsf{Sets}} 1 = \mathrm{Sets}_* \arrow[d, "U"] & \\
B \arrow[r, "F"] & \mathsf{Sets} \arrow[r] & \mathsf{Sets}
\end{tikzcd}$$

CHAPTER 3

Classification of General Topological Structures

Categories of Topological Posets

The category Top of topological spaces and continuous functions is classified by the functor F : Sets $\longrightarrow$ CoP defined as follows. For a set X, $F(X) = \{\tau \subseteq P(X) |\, \tau$ is a topology on $X\}$. $F(X)$ is a poset ordered by reverse inclusion; then it is cocomplete with $\vee\tau_j = \cap\tau_j$. For a function $f : X \longrightarrow Y$ and for $\tau \in F(X)$, $Ff(\tau) = \{U \in P(Y) |\, f^{-1}(U) \in \tau\}$. $Ff(\tau)$ defines a topology on the set Y since f^{-1} preserves arbitrary unions and intersections. F preserves suprema since $Ff(\vee\tau_j) = Ff(\cap\tau_j) = \{y \in Y |\, f^{-1}(y) \in \cap\tau_j\} = \cap\{y \in Y |\, f^{-1}(y) \in \tau_j\} = \cap Ff(\tau_j) = \vee Ff(\tau_j)$, so Ff is cocontinuous. For functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ in Sets, $F(gf) = F(g) \circ F(f)$ since $(gf)^{-1} = f^{-1}g^{-1}$. Thus the functor F is well defined and classifies a topological category $\int_{\text{Sets}} F$ by Theorem 2.50. The objects of $\int_{\text{Sets}} F$ are pairs (X, τ) where $\tau \in F(X)$, i.e., τ is a topology on X, and a function $f : X \longrightarrow Y$ in Sets lifts to a morphism $\overline{f} : (X\tau) \longrightarrow (Y, \sigma)$ in $\int_{\text{Sets}} F$ iff $Ff(\tau) \leq \sigma$ (by the *lifting condition* in Remark 2.24 (ii)), i.e., iff $\sigma \subseteq Ff(\tau)$, i.e., when for all $U \in \sigma$, $f^{-1}(U) \in \tau$, which means exactly that $f : (X, \tau) \longrightarrow (Y, \sigma)$ is continuous. Thus $\int_{\text{Sets}} F = \text{Top}$. Viewing Top as a horizontal structure over Sets, the horizontal morphisms are continuous functions $f : (X, \tau) \longrightarrow (Y, \sigma)$ with σ the coinduced topology.

Definition 3.1 A subset τ of a plete poset A is a *topology* in A, if τ is closed under arbitrary joins and finite meets.

Then for a topology τ in A, $0_A, 1_A \in \tau$ as $0_A = \vee\emptyset$ and $1_A = \wedge\emptyset$. In case $A = P(X)$ (ordered by inclusion) for some set X, the above definition gives a topology on X in the usual sense. Now we can generalize the above functor F.

Proposition 3.2 *Let $T : GeoCmp \longrightarrow CoP$ be defined as follows: For a plete poset $(A, \leq)$,*

$$T(A) = \{\tau \subseteq A |\, \tau \textit{ is a topology in } A\} \tag{1}$$

where the set $T(A)$ is ordered by reverse inclusion, and for a geometric morphism $h = (h^* \dashv h_*) : A \longrightarrow B$ and a topology τ in A,

$$Th(\tau) = \{b \in B |\, h^*(b) \in \tau\}. \tag{2}$$

T is a well-defined functor and therefore, by Theorem 2.50, it classifies a topological category $\int T$ over GeoCmP; moreover $TP = F$ (where P is the power-set functor of example 2.46) and hence Top $= \int\limits_{\text{Sets}} TP$.

Proof. $T(A)$ is ordered by reverse inclusion, so for $\{\tau_j\}_{j\in J} \subseteq T(A)$, $\vee\tau_j = \cap\tau_j$; thus $T(A)$ is a cocomplete poset, since arbitrary intersection of topologies in A is a topology. $Th(\tau)$ defines a topology in B, since h^* is cocontinuous and left exact, i.e., it preserves arbitrary suprema and finite infima (in essence it preserves topologies). Th is cocontinuous, i.e., it is a morphism in CoP, since $Th(\bigvee_j \tau_j) = Th(\bigcap_j \tau_j) = \{b \in B |\, h^*(b) \in \bigcap_j \tau_j\} = \bigcap_j \{b \in B |\, h^*(b) \in \tau_j\} = \bigcap_j Fh(\tau_j) = \bigvee_j Th(\tau_j)$. For $h : A \longrightarrow B$ and $g : B \longrightarrow C$ in GeoCmP, $T(gh) = T(g)T(h)$; this follows from the fact that adjoint situations can be composed, i.e., with $h = (h^* \dashv h_*)$ and $g = (g^* \dashv g_*)$, $g^*h^* \dashv g_*h_*$ and that g^*h^* is left exact when h^* and g^* are. Thus $\int\limits_{\mathrm{GeoCmp}} T$ is a topological category; its objects are pairs (A, τ) where τ is a topology in the plete poset A, and a morphism $h : A \longrightarrow B$ in GeoCmP lifts to a morphism $\bar{h} : (A, \tau) \longrightarrow (B, \sigma)$ in $\int T$ iff $Th(\tau) \leq \sigma$ in the poset $T(B)$ (by the lifting condition in Remark 2.24 (ii)); since $T(B)$ is ordered by reverse inclusion, $Th(\tau) \leq \sigma$ translates to $Th(\tau) \supseteq \sigma$, i.e., $\sigma \subseteq \{b \in B | \, h^*(b) \in \tau\}$.

Composing T : GeoCmp $\longrightarrow$ CoP with P : Sets $\longrightarrow$ GeoCmp, TP : Sets $\longrightarrow$ CoP classifies the topological category $\int\limits_{\mathrm{Sets}} TP$ whose objects are pairs (X, τ) where τ is a topology on the set X, and a function $f : X \longrightarrow Y$ of sets lifts to a morphism $\bar{f} : (X, \tau) \longrightarrow (Y, \sigma)$ in $\int TP$ iff $\sigma \subseteq TPf(\tau) = \{\mathcal{O} \in P(Y) | \, f^{-1}(\mathcal{O}) \in \tau\}$, i.e., when for all $\mathcal{O} \in \sigma$, $f^{-1}(\mathcal{O}) \in \tau$, which means again that f is continuous. It is then a morphism in Top and hence Top $= \int\limits_{\mathrm{Sets}} TP$. Also, $TPf(\tau) = \{\mathcal{O} \in P(Y) | \, f^{-1}(\mathcal{O}) \in \tau\}$ is exactly the topology on Y coinduced by f from τ.

The category $\int T$ of Proposition 3.2 will be denoted by TOP and can be considered as a universal category for the family of "Top-type" categories, in the sense that they can be obtained by pulling TOP back along some functor $C \longrightarrow$ GeoCmP. Every functor $Q^* : C \longrightarrow$ CoP that factors through T as $Q^* = TQ$ defines a "Top-type" topological category and is isomorphic to a pullback of TOP along Q. In particular, Top is the pullback of TOP along the functor P by the above proposition.

Note 3.3 If $\tau \subseteq A$ is a topology in A then the inclusion $\tau \overset{i}{\hookrightarrow} A$ is cocontinuous and left exact, so with its right adjoint $r : A \longrightarrow \tau$ they give a geometric morphism

$r : A \longrightarrow \tau$; (example 2.45). The counit of the adjunction $i \dashv r$ gives that $r(a) \leq a$ for all $a \in A$, and the unit gives that for all $t \in \tau$, $t \leq r(t)$. Thus $r(t) = t$ for all $t \in \tau$, which means that ri =id, i.e. r is a retraction and the unit of the adjunction is an isomorphism. Then, $G = ir : A \longrightarrow A$ is a left exact comonad on A, (G is monotonic, idempotent and $G(a) \leq a$ for all $a \in A$), with the set of coalgebras $A_G = \{x \in A \,|\, G(x) = x\} = \tau$. In fact, we have the following

Proposition 3.4 *For a subset τ of the cocomplete poset A, the following conditions are equivalent;*

- *(ι)* *τ is a topology in A;*
- *($\iota\iota$)* *$\tau \cong A_G$ for a left exact comonad G on A;*

Proof.: It follows from the note above, that $(\iota) \Rightarrow (\iota\iota)$. Given a comonad $G : A \longrightarrow A$ with A_G its set of coalgebras, A_G is closed under arbitrary joins: given $(x_j)_{j \in J} \subseteq A_G$, we have $\vee(x_j) = \vee(G(x_j)) \leq G(\vee x_j)$ since G preserves order; we also have $G(\vee x_j) \leq \vee(x_j)$, which then implies that $G(\vee x_j) = \vee(G(x_j)) = \vee(x_j)$. If G is left exact, then A_G is also closed under finite meets, since for $x, y \in A_G$, $G(x \wedge y) = G(x) \wedge G(y) = x \wedge y$. Thus A_G is a topology in A, which shows that $(\iota\iota) \Rightarrow (\iota)$.

Categories of Fuzzy Topological Posets

Fuzzy subsets of a set X were originally defined by Zadeh in [23] as functions $f : X \longrightarrow I = [0, 1]$; the set of all fuzzy subsets of a set X is then I^X. Later Goguen generalized the concept and defined an *L-fuzzy set* (or simply an L-set) on a set X as a function $\mu : X \longrightarrow L$; L^X is then the set of all L-fuzzy sets on X. L can be viewed as the *truth* set of μ and for $x \in X$, $\mu(x)$ is called the *degree of membership* of x in μ. In Goguen's papers L could be a semigroup, a poset, a lattice, a Boolean ring, a cl_∞-monoid etc. Proposition 2 in [8] says that, "L^X can be given whatever operations L has, and these operations in L^X will obey any law valid in L, which extends point by point."

In our case L will be a plete poset; L^X is then a plete poset as well with order and suprema defined pointwise as stated above: for $\mu,\ \nu \in L^X$, $\ \mu \leq \nu$ iff $\mu(x) \leq \nu(x)$ for all $x \in X$ and for $\mu_j \subseteq L^X$, $(\bigvee_{j\in J}\mu_j)(x) = \bigvee_{j\in J}(\mu_j(x))$; infima in L^X are defined similarly. A function $f : X \longrightarrow Y$ induces the inverse image mapping $f^{\leftarrow} : L^Y \longrightarrow L^X$ defined via composition: given $\nu \in L^Y$, $\ f^{\leftarrow}(\nu) = \nu f$. If $L =\mathbf{2} = (0 \leq 1)$, then $L^X = 2^X \cong P(X)$, and $f^{\leftarrow} = f^{-1}$ in the sense that if $B \in P(Y)$ and $\nu_{_B} : Y \longrightarrow \mathbf{2}$ is the characteristic function of B, then $f^{\leftarrow}\nu_{_B}(x) = \nu_{_B} f(x) = 1$ iff $f(x) \in B$, i.e., when $x \in f^{-1}(B)$; so $f^{\leftarrow}\nu_{_B}$ is the characteristic function of $f^{-1}(B)$.

As a functor, $f^{\leftarrow}$ is bicontinuous ($f^{\leftarrow}(\vee\nu_j)(x) = (\vee\nu_j)f(x) = \vee(\nu_j f(x)) = \vee f^{\leftarrow}(\nu_j)(x)$ and similarly for infima), so it has both a right and a left adjoint. Its continuous right adjoint $h_* : L^X \longrightarrow L^Y$ is defined for $\mu \in L^X$ by $h_*(\mu) = \bigvee\{\nu \in L^Y \,|\, \nu f \leq \mu\}$ by Lemma 2.42. Then $h = (f^{\leftarrow} \dashv h_*) : L^X \longrightarrow L^Y$ is a geometric morphism. Again if $L =\mathbf{2}$ as above (we'll write $\mu_{_S} : X \longrightarrow 2$ for example to denote the characteristic function of $S \subseteq X$), then the definition of $h_*(\mu)$ translates to $h_*(\mu_{_S}) = \vee\{\nu_{_T} \in 2^Y \,|\, \nu_{_T} f \leq \mu_{_S}\} = \cup\{T \in P(Y) \,|\, f^{-1}(T) \subseteq S\}$ which is exactly the definition of $\forall_f$ (Example 2.46). The left adjoint $h^* : L^X \longrightarrow L^X$ of $f^{\leftarrow}$ again by Lemma 2.42 is defined as $h^*(\mu) = \wedge\{\nu \in L^Y \,|\, \nu f \geq \mu\}$ which for $L =\mathbf{2}$ translates to $h^*(\mu_{_S}) = \wedge\{\nu_{_T} \in 2^Y \,|\, \nu_{_T} f \geq \mu_{_S}\} = \cap\{T \in P(Y) \,|\, f^{-1}(T) \supseteq S\}$; this set defines

$f(S)$ since $S \subseteq f^{-1}f(S) \;\Rightarrow\; h^*(\mu_S) \subseteq f(S)$ and $ff^{-1}(T) \subseteq T \;\Rightarrow\; h^*(\mu_S) \supseteq f(S)$. Therefore $h^*(\mu_S) = f(S)$ and then $h^*(\mu) = \mathrm{Im} f$.

Then for any plete poset L we have a functor P_L : Sets $\longrightarrow$ Ess $\subseteq$ GeoCmP that sends a set X, to $P_L(X) = L^X$ ordered as above, and a function $f : X \longrightarrow Y$, to $P_L(f) = (f^{\leftarrow} \dashv h_*) : L^X \longrightarrow L^Y$. Again, if $L =\mathbf{2}$, then P_L is isomorphic to the power-set functor P of Example 2.46.

The first definition of a *fuzzy topology* appeared in Chang's paper [4]; he applied the usual axioms of a topology to Zadeh's fuzzy subsets of a set: a Chang fuzzy topology on a set X is a function $\tau : I^X \longrightarrow 2$ satisfying certain axioms. Later Goguen replaced the unit interval with a complete lattice (with additional structure) L and introduced the concept of L-topological spaces [7]; in both cases the subsets of X making up the topology are fuzzy (or L-fuzzy) subsets, their membership in τ however is "crisp". Šostak generalized Chang's idea and defined a fuzzy topological space as a pair (X, τ) where $\tau : I^X \longrightarrow I$ satisfying the appropriate axioms; in his definitions both the subsets of X considered and their membership in the topology are fuzzy. Zhang later defined an *L-fuzzifying topology* on a set X in [24] as a function $\tau : 2^X \longrightarrow L$ satisfying certain conditions; in his case the subsets of X considered are crisp and their membership in the topology is fuzzy. All of these concepts (and the several other versions that appear in the literature) of a fuzzy topology can be unified by defining a topology as a function $\mathcal{T} : A \longrightarrow M$ satisfying certain properties (see Definition 3.7).

Definition 3.5 *An L-topological space is a pair (X, τ) where X is a set, L is a plete poset and τ is a topology in L^X as in Definition 3.1.(cf. the definition on page 736 in [7].)*

L-topological spaces are the objects of the category **L-Top.** A function $f : X \longrightarrow Y$ defines a morphism $f : (X, \tau) \longrightarrow (Y, \sigma)$ in L-Top, if for all $\nu \in \sigma$, $f^{\leftarrow}(\nu) = \nu f \in \tau$. If $L = \mathbf{2}$, then by identifying a subset U of X with $\mu^{-1}(1)$ of its characteristic function $\mu : X \longrightarrow \mathbf{2}$, (X, τ) is a topological space and $f : (X, \tau) \longrightarrow (Y, \sigma)$ is a continuous function. Thus for $L = \mathbf{2}$, L-Top $\cong$ Top. if $L = I$, then Definition 3.5 gives a Chang fuzzy topology on X.

Proposition 3.6 L-Top $= \int\limits_{\text{Sets}} TP_L$, i.e, L-Top is isomorphic to the pullback of TOP along the functor P_L.

Proof. The objects of $\int\limits_{\text{Sets}} TP_L$ are pairs (X, τ) where X is a set, and $\tau \in TP_L(X) = T(L^X) = \{\tau \subseteq L^X \mid \tau \text{ is a topology in } L^X\}$ as in Prop. 3.3. A function $f : X \longrightarrow Y$ lifts to a morphism $\bar{f} : (X, \tau) \longrightarrow (Y, \sigma)$ in $\int\limits_{\text{Sets}} TP_L$ iff $TP_L f(\tau) \leq \sigma$, i.e., iff $Th(\tau) = \{\nu \in L^Y \mid \nu f \in \tau\} \supseteq \sigma$ (recall that $T(L^X)$ is ordered by reverse inclusion), i.e., iff for all $\nu \in \sigma$, $\nu f \in \tau$, as above.

When (X, τ) is either a topological space or an L-topological space, τ is a crisp subset of either $P(X)$ or L^X; or in general we defined a topology τ in a poset A to be a crisp subset of A satisfying the required properties. We can also consider "fuzzy topologies", i.e., fuzzy subsets of a poset. More precisely identifying a topology τ in a poset A by $\mathcal{T}^{-1}(1)$ of its characteristic function $\mathcal{T} : A \longrightarrow \mathbf{2}$ and then replacing $\mathbf{2}$ by a cocomplete poset M leads to the following definition.

Definition 3.7 *Given plete posets A and M with their bottom and top elements denoted by $0_A,\ 0_M,\ 1_A$ and 1_M respectively, a function $\mathcal{T} : A \longrightarrow M$ is an M-valued fuzzy topology (or M-fuzzy topology)* in A, iff $\mathcal{T}$ satisfies the following conditions:

(FT1) $\quad \mathcal{T}(0_A) = \mathcal{T}(1_A) = 1_M,$

(FT2) $\quad \mathcal{T}(\mu \wedge \nu) \geq \mathcal{T}(\mu) \wedge \mathcal{T}(\nu)$ for all $\mu,\ \nu \in A,$

(FT3) $\quad \mathcal{T}(\vee \mu_j) \geq \wedge \mathcal{T}(\mu_j)$ for all $\mu_j \in A.$

Definition 3.8 Let M be a plete poset. An *M-fuzzy topological poset* is a pair $(A, \mathcal{T})$ where A is a plete poset and $\mathcal{T} : A \longrightarrow M$ is an M-fuzzy topology in A. M-fuzzy topological posets are the objects of the category **M-FTPoS.** A geometric morphism $h = (h^* \dashv h_*) : A \longrightarrow B$ of plete posets gives a morphism $\bar{h} : (A, \mathcal{T}) \longrightarrow (B, \mathcal{S})$ in M-FTPoS iff for all $b \in B$, $\mathcal{S}(b) \leq \mathcal{T}(h^*(b))$.

Example 3.9: (1) If M =**2** in Definition 3.7, then $\mathcal{T}^{-1}(1)$ is a (crisp) topology in A in the sense of Definition 3.1; thus when $A = P(X)$, M=**2** and $\mathcal{T} : P(X) \longrightarrow \mathbf{2}$ satisfies the conditions of Definition 3.7, then $\mathcal{T}^{-1}(1)$ is a topology on the set X in the usual sense.

(2) When $A = I^X$, the set of all fuzzy subsets of the set X, and M=**2** , then Definition 3.7 gives Chang's definition of a *fuzzy topological space;* (Definition 3.2 in [4]).

(3) When $A = L^X$ for a completely distributive lattice L, and M=**2**, $\mathcal{T} : L^X \longrightarrow \mathbf{2}$ in Definition 3.7 gives Gougen's L-topology (or L-fuzzy topology), as in the definition on page 736 in [8].

(4) If $A = [0,1]^X$ and $M = [0,1]$, then $\mathcal{T} : I^X \longrightarrow I$ gives Šostak's fuzzy topology, as in Definition 3.1 in [19].

(5) When $A = P(X)$ for some set X and M is a completely distributive lattice, then $\mathcal{T} : P(X) \longrightarrow M$ gives an *M-fuzzifying* topology, as in [24], page 135.

(6) When L and M are completely distributive lattices, X is a set and $A = L^X$, then $\mathcal{T} : A \longrightarrow M$ (of definition 3.7) defines an (L, M)-fuzzy topology on the set X, as in [25], page 4.

Proposition 3.10 *M-FTPoS is a topological category over GeoCmP; its classifying functor G_M : GeoCmP $\longrightarrow$ CoP is defined on the object level by*

$$G_M(A) = \{\mathcal{T} : A \longrightarrow M \,|\, \mathcal{T} \textit{ satisfies (FT1) - (FT3)}\}, \tag{3}$$

and for a morphism $h = (h^* \dashv h_*) : A \longrightarrow B$ in GeoCmP, $G_M h : G_M(A) \longrightarrow G_M(B)$ is defined for $\mathcal{T} : A \longrightarrow M$ by

$$G_M h(\mathcal{T}) = \mathcal{T} h^*. \tag{4}$$

Proof. $G_M(A)$ is a poset ordered as follows: $\mathcal{T}_1 \leq \mathcal{T}_2$ iff $\mathcal{T}_1(a) \geq \mathcal{T}_2(a)$ for all $a \in A$ and then suprema are defined by $(\vee\mathcal{T}_j)(a) = \wedge(\mathcal{T}_j(a))$. If $\vee\mathcal{T}_j$ satisfies (FT1) - (FT3), i.e., if $\vee\mathcal{T}_j \in G_M(A)$, then $G_M(A)$ is a cocomplete poset. $\vee\mathcal{T}_j(0_A) = \wedge(\mathcal{T}_j(0_A)) = \wedge 1_M = 1_M$ and similarly $(\vee\mathcal{T}_j)(1_A) = 1_M$, thus $\vee\mathcal{T}_j$ satisfies (FT1).

$\vee\mathcal{T}_j$ satisfies (FT2) since for all $\mu,\ \nu \in A,\ (\underset{j}{\vee}\mathcal{T}_j)(\mu\wedge\nu) = \underset{j}{\wedge}[\mathcal{T}_j(\mu\vee\nu)] \geq \underset{j}{\wedge}[\mathcal{T}_j(\mu)\wedge \mathcal{T}_j(\nu)] = [\underset{j}{\wedge}\mathcal{T}_j(\mu)]\wedge[\underset{j}{\wedge}\mathcal{T}_j(\nu)] = (\underset{j}{\vee}\mathcal{T}_j)(\mu)\wedge(\underset{j}{\vee}\mathcal{T}_j)(\nu)$. And similarly, $\vee\mathcal{T}_j$ satisfies (FT3) since $(\underset{j}{\vee}\mathcal{T}_j)(\underset{t}{\vee}\mu_t) = \underset{j}{\wedge}(\mathcal{T}_j(\underset{t}{\vee}\mu_t)) \geq \underset{j}{\wedge}(\underset{t}{\wedge}\mathcal{T}_j(\mu_t)) = \underset{t}{\wedge}(\underset{j}{\wedge}\mathcal{T}_j(\mu_t)) = \underset{t}{\wedge}[(\underset{j}{\vee}\mathcal{T}_j)(\mu_t)]$.

Next we have to show that $G_M h(\mathcal{T}) : B \longrightarrow M$ satisfies (FT1) - (FT3) whenever $\mathcal{T} : A \longrightarrow M$ does. $G_M h(\mathcal{T})(0_B) = 1_M$ since h^* is cocontinuous and $\mathcal{T}(0_A) = 1_M$, and $G_M h(\mathcal{T})(1_B) = 1_M$ since h^* is left exact and $\mathcal{T}(1_A) = 1_M$. Thus $G_M h(\mathcal{T})$ satisfies (FT1). For $\mu,\ \nu \in B$, $G_M h(\mathcal{T})(\mu\wedge\nu) = \mathcal{T}(h^*(\mu\wedge\nu))$ by (4). $\mathcal{T}(h^*(\mu\wedge\nu)) = \mathcal{T}(h^*(\mu)\wedge h^*(\nu))$ by the left exactness of h^*. Since $\mathcal{T}$ satisfies (FT2), $\mathcal{T}(h^*(\mu)\wedge h^*(\nu)) \geq \mathcal{T}(h^*(\mu))\wedge\mathcal{T}(h^*(\nu)) = G_M h\mathcal{T}(\mu)\wedge G_M h\mathcal{T}(\nu)$. Thus $G_M h\mathcal{T}$ satisfies (FT2). For $\{\mu_j\} \subseteq B,\ \ G_M h(\mathcal{T})(\vee(\mu_j)) \geq \vee G_M h(\mathcal{T})(\mu_j)$ since h^* is cocontinuous and $\mathcal{T}$ satisfies (FT3). Thus $G_M h(\mathcal{T})$ satisfies (FT3) as well, and then $G_M h(\mathcal{T}) \in G_M(B)$. The cocontinuity of $G_M h$ follows from the definition of $\vee\mathcal{T}_j$.

Thus G_M is a well-defined functor and $\int G_M$ is a topological category over GeoCmp; its objects are pairs $(A, \mathcal{T})$ where A is a plete poset and $\mathcal{T} : A \longrightarrow M$ is an M-fuzzy topology in A. A morphism $h = (h^* \dashv h_*) : A \longrightarrow B$ in GeoCmP lifts to a morphism $\bar{h} : (A, \mathcal{T}) \longrightarrow (B, \mathcal{S})$ in $\int G_M$ iff $G_M h(\mathcal{T}) \leq \mathcal{S}$, i.e., iff for all $b \in B$ $\mathcal{S}(b) \leq G_M h(\mathcal{T})(b) = \mathcal{T}(h^*(b))$. Hence $\int G_M$ =M-FTPoS.

Using the functor G_M and (G_2) again several familiar topological categories can be classified and hence M-FTPoS $= \int G_M$ can be considered as the universal fuzzy top-type category in the sense that every functor $Q^* : C \longrightarrow$ CoP that factors through G_M as $Q^* = G_M Q$ classifies a fuzzy type topological category, as it is isomorphic to a pullback of M-FTPoS along Q.

1. If M=**2**, the functor $G_M = G_2 : \mathrm{GeoCmP} \longrightarrow \mathrm{CoP}$ gives for each cocomplete poset A, $G_2(A) = \{\mathcal{T} : A \longrightarrow 2 \,|\, \mathcal{T}$ satisfies (FT1) - (FT3)$\} \cong \{\tau \subseteq A \,|\, \tau$ is a topology in $A\}$, since $\mathcal{T}$ satisfies (FT1) - (FT3) iff $\mathcal{T}^{-1}(1) = \tau$ is a topology in A; thus $G_2(A) \cong T(A)$. For $h : A \longrightarrow B$, $G_2h(\mathcal{T}) = \mathcal{T}h^*$ by (4); $(\mathcal{T}h^*)^{-1}(1) = \{b \in B \,|\, h^*(b) \in \mathcal{T}^{-1}(1) = \tau\}$ by (2) in Proposition 3.2. Thus $G_2 \cong T$ and hence $\int_{\mathrm{Sets}} G_2P \cong \mathrm{Top}$.

2. G_MP classifies the category of M-fuzzifying topological spaces, i.e., $\int\limits_{\mathrm{Sets}} G_MP =$ **M-FYS**; (the notation Top-$(2, M)$ is also used in the literature). The objects of $\int G_MP$ are pairs $(X, \mathcal{T})$ with $\mathcal{T} \in G_MP(X)$ i.e., $\mathcal{T} : P(X) \longrightarrow M$ is an M-fuzzifying topology in $P(X)$. A function $f : X \longrightarrow Y$ lifts to a morphism $\bar{f} : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ in $\int G_MP$ iff $G_MPf(\mathcal{T}) \leq \mathcal{S}$ i.e., iff $G_Mf\mathcal{T}(V) \geq \mathcal{S}(V)$ for all $V \in P(Y)$. By (4) $G_Mf\mathcal{T}(V) = \mathcal{T}f^{-1}(V)$ so the lifting condition on $\bar{f}$ coincides with the definition of a continuous function $\bar{f} : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ of M-fuzzifying topological spaces in [24] page 135. M-FYS is then isomorphic to the pullback of $\int G_M$ along the power-set functor.

3. Let $P_L : \mathrm{Sets} \longrightarrow \mathrm{GeoCoP}$ be the functor defined on page 62. Then $\int\limits_{\mathrm{Sets}} G_2P_L \cong$ **L - Top**, the category of L-topological spaces. The objects of $\int G_2P_L$ are pairs $(X, \mathcal{T})$ where X is a set and $\mathcal{T} \in G_2P_L(X) = G_2(L^X) = \{\mathcal{T} : L^X \longrightarrow 2 \,|\, \mathcal{T}$ satisfies (FT1) - (FT3)$\} \cong \{\tau \subseteq L^X \,|\, \tau$ is a topology in $L^X\}$, since again as in example (1), $\mathcal{T}^{-1}(1) = \tau$ is a topology in L^X. A function $f : X \longrightarrow Y$ lifts to a morphism $\bar{f} : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ in $\int G_2P_L$ iff $G_2P_Lf(\mathcal{T}) \leq \mathcal{S}$. By the definition of P_L, for $f : X \longrightarrow Y$, $P_L(f) = (f^{\leftarrow} \dashv h_*)$, so by (4), $G_2(P_Lf)(\mathcal{T}) = \mathcal{T}f^{\leftarrow}$. The lifting condition then translates to $\mathcal{T}f^{\leftarrow} \leq \mathcal{S}$ meaning that for all $\nu \in L^Y$, $\mathcal{S}(\nu) \leq \mathcal{T} f^{\leftarrow} \nu = \mathcal{T}(\nu f)$. Let $\sigma = \mathcal{S}^{-1}(1)$ and $\tau = \mathcal{T}^{-1}(1)$ be the topologies defined by the functions $\mathcal{S} : L^Y \longrightarrow 2$ and $\mathcal{T} : L^X \longrightarrow 2$; then $\mathcal{S}(\nu) \leq \mathcal{T}(\nu f)$ means that if $\nu \in \sigma$, then $\nu f \in \tau$, as required in Definition 3.5.

4. $\int G_M P_L$ =**(L,M)-FTS**, the category of (L,M)-fuzzy topological spaces. The objects of $\int G_M P_L$ are pairs $(X, \mathcal{T})$ where $\mathcal{T} \in G_M P_L(X)$, i.e., $\mathcal{T} : L^X \longrightarrow M$ is an M-valued fuzzy topology in L^X. A function $f : X \longrightarrow Y$ lifts to a morphism $\bar{f} : (X, \mathcal{T}) \longrightarrow (Y, \mathcal{S})$ in $\int G_M P_L$ iff $G_M P_L f(\mathcal{T}) \leq \mathcal{S}$ i.e., iff for all $\nu \in L^Y$, $G_M f^{\leftarrow} \mathcal{T}(\nu) = \mathcal{T}(\nu f) \geq \mathcal{S}(\nu)$. (L,M)-FTS is then isomorphic to the pullback of $\int G_M$ along the functor P_L.

In the examples above, the plete poset M was fixed. We can however generalize the dependence on M by defining a functor $G : \text{GeoCmP} \times \text{CmP} \longrightarrow \text{CoP}$ as follows. For objects $G(A, M) = G_M(A)$ where G_M is the functor of Proposition 3.10. For a morphism $(h, r) : (A, M) \longrightarrow (B, N)$ (where $h = (h^* \dashv h_*)$) and for an M-fuzzy topology $\mathcal{T} : A \longrightarrow M$, $G(h, r)(\mathcal{T}) = r\mathcal{T}h^* : B \longrightarrow N$. To be an N-fuzzy topology on B, $r\mathcal{T}h^*$ must satisfy the properties (FT1) - (FT3) of Definition 3.7.

(FT1): $\quad r\mathcal{T}h^*(0_B) = r\mathcal{T}(0_A) = r(1_M) = r(\wedge_M \emptyset) = \wedge_N \emptyset = 1_N$

$\quad r\mathcal{T}h^*(1_B) = r\mathcal{T}h^*(\wedge_B \emptyset) = r\mathcal{T}(1_A) = r(1_M) = 1_N$

(FT2): $\quad r\mathcal{T}h^*(\mu \wedge \nu) = r\mathcal{T}(h^*(\mu) \wedge h^*(\nu)) \geq r(\mathcal{T}h^*(\mu) \wedge \mathcal{T}h^*(\nu)) =$

$\quad r\mathcal{T}h^*(\mu) \wedge r\mathcal{T}h^*(\nu)$

(FT3): $\quad r\mathcal{T}h^*(\vee_B \mu_j) = r\mathcal{T}(\vee_A h^*(\mu_j)) \geq r(\wedge_M \mathcal{T}h^*(\mu_j)) = \wedge_N r\mathcal{T}h^*(\mu_j)$

The above equalities and inequalities follow from the repeated application of the cocontinuity and left exactness of h^*, the continuity of r and the fact that $\mathcal{T}$ satisfies Definition 3.7. Since for an object b in B and $\mathcal{T}_j : A \longrightarrow M$, $G(h,r)(\vee\mathcal{T}_j)(b) = r(\vee\mathcal{T}_j)(h^*(b)) = r(\wedge\mathcal{T}_j(h^*(b))) = \wedge r(\mathcal{T}_j(h^*(b))) = (\vee(r\mathcal{T}_j h^*))(b) = \vee(G(h,r)(\mathcal{T}_j))(b)$, $G(h,r)$ is cocontinuous; the functor G is then well defined and it classifies the topological category $\int G$ over GeoCmP $\times$ CmP. The objects of $\int G$ are triples $(A, M, \mathcal{T})$ with $\mathcal{T} : A \longrightarrow M$ an M-fuzzy topology on A; a morphism $(h,r) : (A, M) \longrightarrow (B, N)$ in GeoCmP $\times$ CmP lifts to a morphism $(A, M, \mathcal{T}) \xrightarrow{(\mathcal{T},(h,r),\mathcal{S})} (B, N, \mathcal{S})$ if and only if $G(h,r)(\mathcal{T}) \leq \mathcal{S}$, i.e., iff for all b in B, $r\mathcal{T}h^*(b) \geq \mathcal{S}(b)$.

For a plete poset M, let i_M : GeoCmP $\longrightarrow$ GeoCmP $\times$ CmP be the functor that assigns (A, M) to a plete poset A and $(h, 1_M)$ to a geometric morphism $h : A \longrightarrow B$. Then $G\, i_M = G_M$ and hence $\int G_M$ is the pullback of $\int G$ along i_M:

$$\begin{tikzcd}
\int G_M \arrow[r] \arrow[d] & \int G \arrow[r] \arrow[d] & \mathrm{CoP}_* \arrow[d] \\
\mathrm{GeoCmP} \arrow[r, "i_M"] & \mathrm{GeoCmP} \times \mathrm{CmP} \arrow[r, "G"] & \mathrm{CoP}
\end{tikzcd}$$

All the topological categories discussed in this chapter can be obtained by pulling back $\int G$ along a suitable functor; $\int G$ is then a universal object for the categories considered above. Moreover, any functor $Q : C \longrightarrow$ CoP that factors through G classifies a topological category of this family; (special cases of such functors classifying familiar topological categories are listed on pages 67 and 68).

Level Topologies

Definition 3.11 (cf. Discussion 1.5 in [5]) *Given an M-valued fuzzy topology $\mathcal{T} : A \longrightarrow M$, define for all $\alpha \in M$, $\mathcal{T}_\alpha = \{\mu \in A \,|\, \mathcal{T}(\mu) \geq \alpha\}$, the α-level topology associated with $\mathcal{T}$.*

By examining the properties of the (po)set of level topologies associated to a given M-valued fuzzy topology, we'll find conditions for a given set of level topologies to define an M-fuzzy topology.

It is a direct consequence of the axioms (FT1) - (FT3) that for all $\alpha \in M$, $\mathcal{T}_\alpha$ is a topology in A, i.e., $\mathcal{T}_\alpha \in T(A) =$ the set of all toplogies in A.

The set $\{\mathcal{T}_\alpha\}_{\alpha\in M}$ of α-level topologies has the following properties:

(1) $\{\mathcal{T}_\alpha\}_{\alpha\in M} \subseteq T(A)$ is ordered by inclusion and $\alpha \leq \beta$ implies that $\mathcal{T}_\beta \subseteq \mathcal{T}_\alpha$. (The converse does not hold: for example if $\mathcal{T} : A \longrightarrow M$ is given by $\mathcal{T}(\mu) = 1_M$ for all $\mu \in A$, then $\mathcal{T}_{1_M} = \mathcal{T}_{0_M} = A$, so $\mathcal{T}_{0_M} \subseteq \mathcal{T}_{1_M}$, but $1_M \nleq 0_M$.)

(2) $(\{\mathcal{T}_\alpha\}, \leq\, = \,\subseteq^{\mathrm{op}})$ is a cocomplete subposet of $(T(A), \subseteq^{\mathrm{op}})$ with $\bigvee\limits_{j\in J} \mathcal{T}_{\beta_j} = \mathcal{T}_{\vee\beta_j} = \bigcap\limits_{j\in J} \mathcal{T}_{\beta_j}$.

Proof of (2): If $\mu \in \bigcap\limits_{j\in J} \mathcal{T}_{\beta_j}$, then $\mathcal{T}(\mu) \geq \beta_j$ for al $j \in J$, which means that $\mathcal{T}(\mu) \geq \vee\beta_j$. Thus $\mu \in \mathcal{T}_{\vee b_j}$ and consequently $\bigcap\limits_{j\in J} \mathcal{T}_{\beta_j} \subseteq \mathcal{T}_{\vee\beta_j}$. If $\mu \in \mathcal{T}_{\vee\beta_j}$, then $\mathcal{T}(\mu) \geq \vee\beta_j$, which means that $\mathcal{T}(\mu) \geq \beta_j$ for all $j \in J$ and hence $\mu \in \bigcap\limits_{j\in J} \mathcal{T}_{\beta_j}$. Thus $\mathcal{T}_{\vee\beta_j} \subseteq \bigcap\limits_{j\in J} \mathcal{T}_{\beta_j}$, which then proves that $\mathcal{T}_{\vee\beta_j} = \bigcap\limits_{j\in J} \mathcal{T}_{\beta_j}$.

Since $\bigcap\limits_{j\in J} \mathcal{T}_{\beta_j} = \bigvee\limits_{j\in J} \mathcal{T}_{\beta_j}$ in $(T(A), \subseteq^{\mathrm{op}})$, the inclusion $\{\mathcal{T}_\alpha\}_{\alpha\in M} \hookrightarrow T(A)$ is co-continuous.

Since the category PoS of posets is cartesian closed, for $K, M \in$ Pos, K^M is a poset as well and if K is a cocomplete poset, so is K^M. Now let M also be a cocomplete poset and let $F_M : CoP \longrightarrow CoP$ denote the covariant hom functor $F_M = CoP(M, \_\,)$: CoP $\longrightarrow$ CoP; then for $K \in$ CoP, $F_M(K) = CoP(M, K) = \{\Gamma : M \longrightarrow K \,|\, \Gamma$ is a cocontinuous functor$\}$, and given a morphism $h : K \longrightarrow L$ in CoP, $F_M h(\Gamma) = h\Gamma$. Since both h and Γ are cocontinuous, so is $h\Gamma$. The order on $F_M(K)$ is given by $\Gamma_1 \leq \Gamma_2$ iff $\Gamma_1(\alpha) \leq \Gamma_2(\alpha)$ for all $\alpha \in M$. Suprema then in $F_M(K)$ are also defined by image: $(\bigvee \Gamma_j)(\alpha) = \overset{K}{\bigvee}(\Gamma_j(\alpha))$. $F_M(h)$ is also cocontinuous, since h is.

Then the functor T : GeoCmP $\longrightarrow$ Cop of Proposition 3.2 and F_M can be composed: $F_M T$: GeoCmP $\longrightarrow$ CoP gives for $A \in$ GeoCmP, $F_M T(A) =$ CoP$(M, T(A))$, and for a morphism $h : A \longrightarrow B$ in GeoCoP, and $\Gamma \in$ CoP$(M, T(A))$, $F_M T(h)\Gamma = T(h)\Gamma$. $F_M T$ then classifies the topological category $\int F_M T$ over GeoCmP.

Proposition 3.12 *Given a cocomplete poset M, there is a natural transformation $\eta : G_M \Rightarrow F_M T$ defined as follows: for a plete poset A, $\eta_A : G_M(A) \longrightarrow F_M T(A) =$ $CoP(M, T(A))$ sends the M-fuzzy topology $\mathcal{T} : A \longrightarrow M$ to $\eta_A \mathcal{T} = \Gamma_{\mathcal{T}} : M \longrightarrow T(A)$ where for $\alpha \in M$, $\Gamma_{\mathcal{T}}(\alpha) = \mathcal{T}_\alpha = \{\mu \in A \,|\, \mathcal{T}(\mu) \geq \alpha\}$, the α-level topology associated with $\mathcal{T}$ (Def. 3.11).*

Proof. For each $A \in$ GeoCmP, η_A is order preserving and cocontinuous: Given $\mathcal{T}_1, \mathcal{T}_2 : A \longrightarrow M$, $\mathcal{T}_1 \leq \mathcal{T}_2$ means that $\mathcal{T}_1(a) \geq \mathcal{T}_2(a)$ for all $a \in A$ which then implies that $\Gamma_{\mathcal{T}_1}(\alpha) = \{a \in A \,|\, \mathcal{T}_1(a) \geq \alpha\} \supseteq \{a \in A \,|\, \mathcal{T}_2(a) \geq \alpha\} = \Gamma_{\mathcal{T}_2}(\alpha)$.

Hence $\Gamma_{\mathcal{T}_1}(\alpha) \leq \Gamma_{\mathcal{T}_2}(\alpha)$, since $T(A)$, the set of all topologies on A is ordered by reverse inclusion; we then have that $\eta_{_A}$ preserves order. The fact that $\eta_{_A}$ preserves arbitrary suprema, i.e., $\Gamma_{\vee\mathcal{T}j} = \vee\Gamma_{\mathcal{T}_j}$, follows, for $\alpha \in M$, from the equality of the sets $\Gamma_{\vee\mathcal{T}_j}(\alpha) = \{a \in A \,|\, (\vee\mathcal{T}_j)(a) \geq \alpha\} = \{a \in A \,|\, \wedge(\mathcal{T}_j(a) \geq \alpha\}$ and $\vee(\Gamma_{\mathcal{T}_j}(\alpha)) = \vee\{a \in A \,|\, \mathcal{T}_j(\alpha) \geq \alpha\} = \bigcap\{a \in A \,|\, \mathcal{T}_j(a) \geq \alpha\}$.

For a morphism $h : A \longrightarrow B$ in GeoCmP, the diagram

$$\begin{array}{ccc} G_{_M}(A) & \xrightarrow{\eta_A} & \mathrm{CoP}(M, T(A)) \\ {\scriptstyle G_M(h)}\downarrow & & \downarrow{\scriptstyle F_M T(h)} \\ G_{_M}(B) & \xrightarrow{\eta_B} & \mathrm{CoP}(M, T(B)) \end{array}$$

commutes, since for $(\mathcal{T} : A \longrightarrow M) \in G_M(A)$, $\eta_A(\mathcal{T}) = \Gamma_{_\mathcal{T}} : M \longrightarrow T(A)$ with

$$(*) \qquad \Gamma_{_\mathcal{T}}(\alpha) = \mathcal{T}_\alpha = \{\mu \in A \,|\, \mathcal{T}(\mu) \geq \alpha\},$$

$(F_{_M}Th)(\Gamma_{_\mathcal{T}}) = (Th)\Gamma_{_\mathcal{T}} : M \longrightarrow T(B)$ with $(Th)\Gamma_{_\mathcal{T}}(\alpha) = Th(\mathcal{T}_\alpha) =$

$\{\nu \in B \,|\, h^*(\nu) \in \mathcal{T}_\alpha\} = \{\nu \in B \,|\, \mathcal{T}(h^*(\nu)) \geq \alpha\}$ by (2) on page 54 and by $(*)$ above.

Going along the diagram the other way gives for $\mathcal{T} \in G_M(A)$ and for $\nu \in B$ that $G_M h(\mathcal{T})(\nu) := \mathcal{S}(\nu) = \mathcal{T}(h^*(\nu))$ by (4) in Proposition 3.10, and $\eta_B(\mathcal{S}) = \Gamma_{_\mathcal{S}} : \alpha \longmapsto \{\nu \in B \,|\, \mathcal{S}(\nu) \geq \alpha\} = \{\nu \in B \,|\, \mathcal{T}(h^*(\nu)) \geq \alpha\}$ as above; thus $F_M T\eta_A = \eta_B G_M h$.

The natural transformation $\eta : G_{_M} \Rightarrow F_{_M}T$ then defines a functor $\int\eta : \int G_{_M} \longrightarrow \int F_{_M}T$ the obvious way.

Definition 3.13 (Def. 1.1 in [6]) *Let L be a complete lattice. We say that x is way below y,* in symbols $x \ll y$, iff for directed subsets $D \subseteq L$ the relation $y \leq \sup D$ always implies the existence of a $d \in D$ with $x \leq d$.

Definition 3.14 (Def 1.6 in [6]) *A lattice L is called a continuous lattice* if L is complete and satisfies the axiom of approximation: $x = \bigvee_{u \ll x} u$ for all $x \in L$.

Definition 3.15 (Page 119 in [2]) *A (complete) lattice is called completely distributive* when it satisfies the dual extended distributive laws:

$$\bigwedge_C \left[\bigvee_{A_\gamma} x_{\gamma,\alpha} \right] = \bigvee_{\Phi} \left[\bigwedge_C x_{\gamma,\phi(\gamma)} \right],$$

$$\bigvee_C \left[\bigwedge_{A_\gamma} x_{\gamma,\alpha} \right] = \bigwedge_{\Phi} \left[\bigvee_C x_{\gamma,\phi(\gamma)} \right],$$

for any non-void family of index sets A_γ, one for each $\gamma \in C$, provided Φ is the set of all functions ϕ with domain C and $\phi(\gamma) \in A_\gamma$.

Proposition 3.16 (Corollary 3.5 in [6]) *Every completely distributive lattice is continuous.*

Given a fuzzy topology $\mathcal{T} : A \longrightarrow M$, the set $\{\mathcal{T}_\alpha\}_{\alpha \in M}$ of associated α-level topologies can be realized as the cocontinuos functor $\Gamma_{\mathcal{T}} : M \longrightarrow (G_2(A), \subseteq^{op})$ of cocomplete posets that sends $\alpha \in M$ to $\Gamma_{\mathcal{T}}(\alpha) = \mathcal{T}_\alpha \in G_2(A)$. Property (1) (page 70) implies that $\Gamma_{\mathcal{T}}$ is order preserving; the cocontinuity of $\Gamma_{\mathcal{T}}$ follows from property (2).

The converse of the above statement holds, if M is a completely distributive lattice:

Proposition 3.17 *If M is a completely distributive lattice (and A is a cocomplete poset), then a cocontinuous functor $\Gamma : M \longrightarrow T(A)$ defines an M-fuzzy topology $\mathcal{T}_\Gamma : A \longrightarrow M$ in A by setting*

$$\mathcal{T}_\Gamma(\mu) = \bigvee\{\beta \in M \,|\, \mu \in \Gamma(\beta)\}.$$

Proof. $\mathcal{T}_\Gamma$ satisfies (FT1) since $\Gamma(\alpha)$ is a topology in A: $0_A,\ 1_A \in \Gamma(\alpha)$ for all $\alpha \in M$, so $\mathcal{T}_\Gamma(0_A) = \mathcal{T}_\Gamma(1_A) = \vee M = 1_M$. Given $\mu, \nu \in A$, $\mathcal{T}_\Gamma(\mu) \wedge \mathcal{T}_\Gamma(\nu) = \left(\bigvee_{i\in I}\{\alpha_i \,|\, \mu \in \Gamma(\alpha_i)\}\right) \wedge \left(\bigvee_{j\in J}\{\beta_j \,|\, \nu \in \Gamma(\beta_j)\}\right) = \bigvee_{I\times J}(\alpha_i \wedge \beta_j)$ by the infinite distributive law. Since $\alpha_i\wedge\beta_j \leq \alpha_i$ and $\alpha_i\wedge\beta_j \leq \beta_j$ for all $i \in I$ and $j \in J$, $\Gamma(\alpha_i\wedge\beta_j) \supseteq \Gamma(\alpha_i)$ and $\Gamma(\alpha_i\wedge\beta_j) \supseteq \Gamma(\beta_j)$. Then $\mu,\ \nu \in \Gamma(\alpha_i\wedge\beta_j)$ for all i, j, and since $\Gamma(\alpha_i\wedge\beta_j)$ is closed under finite meets, $\mu \wedge \nu \in \Gamma(\alpha_i \wedge \beta_j)$ also. Then $\mathcal{T}_\Gamma(\mu \wedge \nu) \geq \alpha_i \wedge \beta_j$ for all i, j which implies that $\mathcal{T}_\Gamma(\mu \wedge \nu) \geq \bigvee(\alpha_i \wedge \beta_j) = \mathcal{T}_\Gamma(\mu \wedge \mathcal{T}_\Gamma(\nu)$. Thus FT2 holds.

To show that $\mathcal{T}_\Gamma(\bigvee_K \mu_k) \geq \bigwedge_K \mathcal{T}_\Gamma(\mu_k)$, the complete distributivity of M is needed: given $\{\mu_k\}_{k\in K} \subseteq A$, $\bigwedge_{k\in K} \mathcal{T}_\Gamma(\mu_k) = \bigwedge_{k\in K}\left(\bigvee_{j\in J_k}\{\beta_{k,j} \in M \,|\, \mu_k \in \Gamma(\beta_{k,j})\}\right) = \bigvee_{\phi\in\Phi}\left(\bigvee_{k\in K}\{\beta_{k,\phi(k)} \in M \,|\, \mu_k \in \Gamma(\beta_{k,\phi(k)})\}\right)$ where Φ is the set of all functions ϕ with domain K and $\phi(k) \in J_k$,(by the first extended distributive law). Since for any (fixed) ϕ, $\bigwedge_{k\in K}\{\beta_{k,\phi(k)}\} \leq \beta_{k,\phi(k)}$ for all $k \in K$, $\Gamma(\wedge\beta_{k,\phi(k)}) \supseteq \Gamma(\beta_{k,\phi(k)})$ for all k. Then since $\mu_k \in \Gamma(\beta_{k,\phi(k)})$ for all k, $\mu_k \in \Gamma(\bigwedge_K \beta_{k,\phi(k)})$ also. $\Gamma(\bigwedge_K \beta_{k,\phi(k)})$ is closed under infinite suprema, thus $\bigvee_K \mu_k \in \Gamma(\bigwedge_K \beta_{k,\phi(k)})$ also for any $\phi \in \Phi$. Then by definition of $\mathcal{T}_\Gamma$, $\mathcal{T}_\Gamma(\bigvee_K \mu_k) \geq \bigvee_\Phi \left(\bigwedge_K \beta_{k,\phi(k)}\right) = \bigwedge_K \mathcal{T}_\Gamma(\mu_k)$.

Proposition 3.18 *If M is a completely distributive lattice, then the natural transformation $\eta : G_M \Rightarrow F_M T$ of Proposition 3.12 is a natural isomorphism.*

Proof. We define a natural transformation $\lambda : F_M T \Rightarrow G_M$ as follows: for $\Gamma \in F_M T(A) = \mathrm{CoP}(M, T(A))$, $\lambda_A(\Gamma) : A \longrightarrow M$ is given by

$\lambda_A(\Gamma)(\mu) = \bigvee\{\beta \in M \mid \mu \in \Gamma(\beta)\}$ as in Proposition 3.17. For each $A \in$GeoCmP, λ_A is a morphism in CoP, since it is order-preseving and cocontinuous. The natural transformation η was defined for $A \in$GeoCmP by $\eta_A(\mathcal{T}) = \Gamma_{\mathcal{T}} : \alpha \longmapsto \mathcal{T}_\alpha = \{\mu \in A \mid \mathcal{T}(\mu) \geq \alpha\}$. $\lambda_A \eta_A(\mathcal{T})(\mu) = \mathcal{T}_{\Gamma_{\mathcal{T}}}(\mu) = \vee\{\beta \in M \mid \mu \in \Gamma_{\mathcal{T}}(\beta)\} = \vee\{\beta \in M \mid \mathcal{T}(\mu) \geq \beta\}$ by the definitions of $\Gamma_{\mathcal{T}}$ and $\mathcal{T}_\Gamma$. Let $S = \{\beta \in M \mid \mathcal{T}(\mu) \geq \beta\}$. Then $\mathcal{T}(\mu) \in S$ implies that $\mathcal{T}(\mu) \leq \vee S$ and $\mathcal{T}(\mu) \geq \beta$ for all $\beta \in S$ implies that $\mathcal{T}(\mu) \geq \vee S$. Thus $\mathcal{T}(\mu) = \vee S = \mathcal{T}_{\Gamma_{\mathcal{T}}}(\mu)$ for all $\mu \in A$, and hence $\lambda_A \eta_A = 1$.

To show that $\eta_A \lambda_A(\Gamma) = \Gamma_{\mathcal{T}_\Gamma} = \Gamma$ i.e., $\Gamma_{\mathcal{T}_\Gamma}(\alpha) = \Gamma(\alpha)$ for all $\alpha \in M$, suppose that $\mu \in \Gamma_{\mathcal{T}_\Gamma}(\alpha) = \{\mu \in A \mid \mathcal{T}_\Gamma(\mu) \geq \alpha\}$ for some α, i.e. $\mathcal{T}_\Gamma(\mu) \geq \alpha$. Again let $S = \{\beta \mid \mu \in \Gamma(\beta)\}$, $\vee S = t$ and note that S is a directed set due the cocontinuity of Γ. M being a completely distributive lattice, it is also continuous, hence $t = \bigvee\limits_{\gamma \ll t} \gamma$. Since $\gamma \ll t$ and $t = \vee S$ implies that $\gamma \leq \beta$ for some $\beta \in S$ by definition 3.4, for every $\gamma \ll t$, $\Gamma(\gamma) \supseteq \Gamma(\beta)$ for some $\beta \in S$. Thus $\mu \in \Gamma(\gamma)$ for all $\gamma \ll t$ and hence $\mu \in \bigcap\limits_{\gamma \ll t} \Gamma(\gamma) = \bigvee\limits_{\gamma \ll t} \Gamma(\gamma) = \Gamma_{\vee\gamma} = \Gamma_t$. Then $\mu \in \Gamma(\alpha)$ also, since $t = \mathcal{T}_\Gamma(\mu) \geq \alpha$. Therefore $\Gamma_{\mathcal{T}_\Gamma}(\alpha) \subseteq \Gamma(\alpha)$ for all $\alpha \in M$.

If $\mu \in \Gamma(\alpha)$, then $\alpha \in S$ and hence $\alpha \leq \vee S = \mathcal{T}_\Gamma(\mu)$. Thus $\mu \in \Gamma_{\mathcal{T}_\Gamma}$ and $\Gamma(\alpha) \subseteq \Gamma_{\mathcal{T}_\Gamma}(\alpha)$ for all $\alpha \in M$. Therefore $\Gamma_{\mathcal{T}_\Gamma} = \Gamma$, which means that $\eta_A \lambda_A = 1$, for all $A \in$ GeoCop. Hence η is a natural isomorphism and $\int \eta : \int G_M \Rightarrow \int F_M T$ is an isomorphism categories.

CHAPTER 4

Classification of General Convergence Structures

Poset properties

Definition 4.1 A poset L is said to be *directed* if every finite subset of L has an upper bound in L. Dually, L is said to be *filtered* if every finite subset of L has a lower bound in L. (The definition implies that directed and filtered posets are nonempty.)

Definition 4.2 Let L be a poset. For $\alpha \subseteq L$ and $a \in L$ we write:

(i) $\downarrow\alpha = \{x \in L \,|\, x \leq a \text{ for some } a \in \alpha\}$;

(ii) $\downarrow a = \downarrow\{a\}$;

(iii) α is called a *lower set (or down-segment)* if $\downarrow\alpha = \alpha$;

(iv) α is an *ideal* if it is a directed lower set;

(v) An ideal of the form $\downarrow a$ is called a *principal ideal* ;

(vi) An ideal α is a *prime ideal* if for any pair a, b of elements of L such that $a \wedge b$ exists, $a \wedge b \in \alpha \Rightarrow a \in \alpha$ or $b \in \alpha$.

(vii) An element p in L is *prime* if for any pair $a,\ b$ of elements of L such that $a \wedge b$ exists, $a \wedge b \leq p \Rightarrow a \leq p$ or $b \leq p$.

(Note: p is a prime element iff $\downarrow p$ is a prime ideal.)

Dual notions:

(viii) $\uparrow\alpha = \{x \in L \mid x \geq a \text{ for some } a \in \alpha\}$;

(ix) $\uparrow a = \uparrow\{a\}$;

(x) α is called an *upper set (or up-segment)* if $\uparrow\alpha = \alpha$;

(xi) α is a *filter* if it is a filtered upper set;

(xii) A filter of the form $\uparrow a$ is called a *principal filter* ;

(xiii) A filter α is a *prime filter* if for any pair a, b of elements of L such that $a \vee b$ exists, $a \vee b \in \alpha \Rightarrow a \in \alpha$ or $b \in a$.

(xiv) An element p in L is *coprime* iff it is a prime element of L^{op}.

In case the poset $(L, \leq) = (PX, \subseteq)$ for some set X, upper sets are called *stacks* on X and the notation $[\alpha]$ is also used instead of $\uparrow\alpha$. A filter in the poset $(PX, \subseteq)$ gives the usual notion of a filter on the set X. A principal filter generated by a singleton $\uparrow\{x\}$ will be denoted by $\uparrow x$; filters of this form are exactly the principal prime filters in $(PX, \subseteq)$. In the case when $P(X)$ is ordered by reverse inclusion, i.e., $(L, \leq) = P(X)^{\mathrm{op}}$, a stack (resp. filter) on the set X is a lower set (resp. ideal),the principal prime filters $\uparrow x$ on X become the principal prime ideals and the prime elements are exactly the singletons $\{x\}$.

Lemma 4.3 *If p is a prime element in the plete poset L and $h : L \longrightarrow N$ is a morphism in GeoCmP, then $h(p)$ is a prime element of N and hence $\downarrow h(p)$ is a prime ideal in N.*

Proof. Suppose p is a prime in L and and $t \wedge s \leq h(p)$ in N. Then $h^*(t \wedge s) \leq h^*(h(p))$; $h^*(h(p)) \leq p$ by the adjunction $h^* \dashv h$ and $h^*(t \wedge s) = h^*(t) \wedge h^*(s)$ by the left exactness of h^*. Hence $h^*(t) \wedge h^*(s) \leq p$, and since p is prime, $h^*(t) \leq p$ od $h^*(s) \leq p$. Suppose $h^*(t) \leq p$. Then $h(h^*(t)) \leq h(p)$ and by the adjunction, $t \leq h(h^*(t))$; hence $t \leq h(p)$ follows. Thus $h(p)$ is a prime element of N and $\downarrow h(p)$ is a prime ideal; in fact $\downarrow h(p) = \downarrow h(\downarrow p)$.

Lemma 4.4 *Let Id(L) denote the set of all ideals of the poset L. If $h : L \longrightarrow N$ is continuous and $\{\alpha_j\}_{j \in J} \subseteq$ Id(L), then $\downarrow h(\cap \alpha_j) = \cap\{\downarrow h(\alpha_j)\}$.*

Proof. The inclusion $\downarrow h(\cap \alpha_j) \subseteq \cap\{\downarrow h(\alpha_j)\}$ is obvious. To show that $\cap\{\downarrow h(\alpha_j)\} \subseteq \downarrow h(\cap \alpha_j)$, suppose that $x \in \cap\{\downarrow h(\alpha_j\}$. Then for all $j \in J$, $x \in \downarrow h(\alpha_j)$ and there exists $a_j \in \alpha_j$ such that $x \leq h(a_j)$. Thus $x \leq \wedge h(a_j) = h(\wedge a_j)$ since h is continuous, and $\wedge a_j \in \alpha_j$ for all $j \in J$ since each α_j is an ideal and therefore a lower set. Then $h(\wedge a_j) \in h(\cap \alpha_j)$ and $x \in \downarrow h(\cap \alpha_j)$.

Lemma 4.5 *If α is a prime ideal in the plete poset L and $h : L \longrightarrow N$ is a morphism in GeoCmp, then $\downarrow h(\alpha)$ is a prime ideal in N.*

Proof. Suppose $s \wedge t \in \downarrow h(\alpha)$. Then $s \wedge t \leq h(a)$ for some $a \in \alpha$. Since h defines a geometric morphism, its left adjoint h^* is left exact; therefore $h^*(s \wedge t) = h^*(s) \wedge h^*(t)$. It follows form the adjunction $h^* \dashv h$ that $h^*(h(a)) \leq a$. Thus $h^*(s) \wedge h^*(t) \leq a$ which means that $h^*(s) \wedge h^*(t) \in \alpha$; α is a prime ideal, therefore $h^*(s) \in \alpha$ or $h^*(t) \in \alpha$. If $h^*(s) \in \alpha$, then $h(h^*(s)) \in h(\alpha)$. Again by the adjunction, $h(h^*(s)) \geq s$, so $s \in \downarrow h(\alpha)$ which proves that $\downarrow h(\alpha)$ is a prime ideal in N.

Stack and Filter Convergence Structures

For a set X, let $F(X)$ denote the set of all filters and $S(X)$ the set of nonempty stacks on X.

Definition 4.6 A *Stack Convergence Structure* on a set X is an order preserving function $q : (S(X), \subseteq) \longrightarrow (P(X), \subseteq)$ such that $x \in q(\uparrow x)$ for all $x \in X$. In this case the pair (X, q) is called a *Stack Convergence Space.* Stack convergence spaces form the category SCo . A morphism $f : (X, q) \longrightarrow (Y, r)$ is a function $f : X \longrightarrow Y$ such that for all $\alpha \in S(X)$, $f(x) \in r\,(\uparrow f(\alpha))$ whenever $x \in q(\alpha)$. (cf. page 354 in [18]).

Definition 4.7 If in the above definition $S(X)$ is replaced by $F(X)$, then $q : (F(X), \subseteq) \longrightarrow (PX, \subseteq)$ is called a *Filter Convergence Structure* on X and (X, q) a *Filter Convergence Space.* Filter convergence spaces are the objects of the category FCo (cf. page 348 in [18]). Morphisms in FCo are defined the same way as in SCo.

When $x \in q(\alpha)$ for some stack/filter α, we write $\alpha \xrightarrow{q} x$ and we say that " α q-converges to x" or "x is a limit point of α".

Given a filter convergence structure $q : F(X) \longrightarrow P(X)$, consider the following stronger conditions on q:

(C1) $\alpha \xrightarrow{q} x$ implies that $(\alpha \cap \uparrow x) \xrightarrow{q} x$;

(C2) $\alpha \xrightarrow{q} x$ and $\beta \xrightarrow{q} x$ implies that $(\alpha \cap \beta) \xrightarrow{q} x$;

(C3) For any $\{\alpha_j\}_{j \in J} \subseteq F(X)$ such that $\alpha_j \xrightarrow{q} x$ for all $j \in J$, $\cap \alpha_j \xrightarrow{q} x$.

Definition 4.8 A filter convergence structure $q : F(X) \longrightarrow P(X)$ is called a *local filter convergence structure* if it satisfies (C1), a *limit structure* if it satisfies (C2), and a *pretopology* if it satisfies (C3). *Local filter convergence spaces, limit spaces and pretopological spaces* are defined the obvious way and are, respectively, objects of the topological categories LCo, Lim and PrT.

Note 4.9 Since (C3) $\Rightarrow$ (C2) $\Rightarrow$ (C1) we have that LFCo $\supset$ Lim $\supset$ PrT; also LFCo is a subcategory of FCo. Moreover Top is isomorphic to a subcategory of PrT, since given a topological space (X, τ), τ defines a pretopology $q_\tau : F(X) \longrightarrow P(X)$ as follows: for $\alpha \in F(X)$, $\alpha \xrightarrow{q_\tau} x$ iff $\alpha \supseteq \mathcal{N}_x$ where $\mathcal{N}_x$ is the neighborhood filter of x.

Note 4.10: A Stack (resp. filter) Convergence Structure can equivalently be defined as a function $K : X \longrightarrow P(S(X))$ (resp. $K : X \longrightarrow P(F(X))$) such that $\uparrow K(x) = K(x)$ and $\uparrow x \in K(x)$ $\forall x \in X$ in the poset $(P(S(X)), \subseteq)$, as in [18], on page 354, or as as a subset q of $S(X) \times X$ such that $(\uparrow x, x) \in q$ $\forall x \in X$ and for $\alpha, \beta \in S(X)$ and $\alpha \subseteq \beta$, $(\alpha, x) \in q \Longrightarrow (\beta, x) \in q$ (cf. page 18 in [17]). The equivalence of these definitions follows from the fact that the category of Sets is cartesian closed and therefore the following morphism sets are isomorphic:

$\mathrm{Sets}(S(X), 2^X) \underset{c.c.}{\cong} \mathrm{Sets}(X \times S(X), 2) \cong \mathrm{Sets}(S(X) \times X, 2) \underset{c.c.}{\cong} \mathrm{Sets}(X, 2^{S(X)})$. The function $q : S(X) \longrightarrow P(X)$ that corresponds to a function $K : X \longrightarrow P(S(X))$ under the isomorphisms above is characterized as: $x \in q(\alpha)$ iff $\alpha \in K(x)$; q assigns to a stack/filter the set of its limit points, whereas K assigns to each point x the set of stacks/filters converging to it.

The functor $Q : \text{Sets} \longrightarrow \text{CoP}$ that classifies the topological category SCo is the following. For a set X,

$$Q(X) = \{q : S(X) \longrightarrow P(X) \,|\, q \text{ is a stack convergence structure on } X\}.$$

The order on $Q(X)$ is defined by $q_1 \leq q_2$ iff $q_1(\alpha) \subseteq q_2(\alpha)$ for all $\alpha \in S(X)$. With infima defined by $\left(\bigwedge_j q_j\right)(\alpha) = \bigcap_j (q_j(\alpha))$, $Q(X)$ is a complete and therefore cocomplete poset; suprema are defined by $\left(\bigvee_j q_j\right)(\alpha) = \bigcup_j (q_j(\alpha))$. For a function $f : X \longrightarrow Y$ of sets, $Qf : Q(X) \longrightarrow Q(Y)$ is defined as the left adjoint of $(Qf)_* : Q(Y) \longrightarrow Q(X)$, where for $(r : S(Y) \longrightarrow PY) \in Q(Y)$ and $\alpha \in S(X)$ $(Qf)_*$ is defined by

$$[(Qf)_*(r)](\alpha) = \{x \in X \,|\, f(x) \in r(\uparrow f(\alpha)\}. \tag{4-1}$$

$[(Qf)_*(r)](\alpha) : S(X) \longrightarrow P(X)$ is continuous, preserves order and for all $x \in X$, $\uparrow x \xrightarrow{(Qf)_*(r)} x$, so $(Qf)_*(r)$ defines a stack convergence structure on X. The functors that classify FCo, LFCo, Lim and Prt are defined similarly. $Q_0 : \text{Sets} \longrightarrow \text{CoP}$ defined for a set X as

$$Q_0(X) = \{q : F(X) \longrightarrow P(X) \,|\, q \text{ is a filter convergence structure on } X\}$$

classifies FCo. For $i = 1, 2, 3$, let $Q_i : \text{Sets} \longrightarrow \text{CoP}$ be defined as $Q_i(X) = \{q : F(X) \longrightarrow PX \,|\, q \in Q_0(X) \text{ and } q \text{ satisfies } C_i\}$. The order, infima and suprema in $Q_i(X)$, $i = 0, 1, 2, 3$ are defined the same way as for the functor Q above, and also on the morphism level each Q_i is defined the same way as Q. Then Q_1 classifies LFCo, Q_2 classifies Lim, and Q_3 classifies Prt.

If $\alpha \in F(X)$ (and $f : X \longrightarrow Y$) then $\uparrow f(\alpha) \in F(Y)$, so the definition $[(Q_i f)_*(r)](\alpha) = \{x \in X \,|\, f(x) \in r(\uparrow f(\alpha)\}$ as in (4-1) makes sense.

Generalized Stack Convergence Structure

Definition 4.11 Given a plete poset L, let $\mathcal{L}_L$ denote the set of all lower sets of L ordered by inclusion. A function $q : \mathcal{L}_L \longrightarrow L$ is a *generalized stack convergence structure* in L if *(i)* q is order-reversing, i.e., q is an L-valued presheaf on $\mathcal{L}_L$, and *(ii)* $p \geq q(\downarrow p)$ for every principal prime ideal $\downarrow p$, or equivalently, for every prime element p of L. The pair (L, q) is then called a *generalized stack convergence space,* and is the object of the category GSCo. A morphism $h : (L, q) \longrightarrow (N, r)$ in GSCo is a geometric morphism $h : L \longrightarrow N$ such that for all $\alpha \in \mathcal{L}_L$ and prime element $p \in L$, $h(p) \geq r(\downarrow h(\alpha))$ whenever $p \geq q(\alpha)$.

Theorem 4.12 *GSCo is a topological category over GeoCmP classified by the functor* $G : GeoCmP \longrightarrow CoP$ *defined as follows: for a plete poset* L, $G(L) = \{q : \mathcal{L}_L \longrightarrow L |\, q$ *is a generalized stack convergence structure in* $L\}$*; for a geometric morphism* $h : L \longrightarrow N$, $Gh : G(L) \longrightarrow G(N)$ *is defined as the left adjoint of* $(Gh)_* : G(N) \longrightarrow G(L)$*, where for* $(r : \mathcal{L}_N \longrightarrow N) \in G(N)$ *and* $\alpha \in \mathcal{L}_L$,

$$((Gh)_*(r))(\alpha) = \wedge\{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha))\}. \tag{4-2}$$

Remark: In case $(L, \leq) = P(X)^{\mathrm{op}}$, (4-2) is equivalent to (4-1).

Proof. (i) $G(L)$ is a complete (and cocomplete) poset with the order and infima defined as follows: $q_1 \leq q_2$ iff $q_1(\alpha) \geq q_2(\alpha)$ for all $\alpha \in \mathcal{L}_L$ and $\left(\underset{j}{\wedge} q_j\right)(\alpha) = \underset{j}{\vee}(q_j(\alpha))$. (Similarly, suprema are given by $\left(\underset{j}{\vee} q_j\right)(\alpha) = \underset{j}{\wedge}(q_j(\alpha))$.) $\wedge q_j$ is clearly order reversing, and $p \geq q_j(\downarrow p)$ for all j implies that $p \geq \vee(q_j(\downarrow p)) = (\wedge q_j)(\downarrow p)$; thus $\wedge q_j$ defines a generalized stack convergence structure in L.

(ii) To see that $G(h)_*(r)$ defines a generalized stack convergence structure in L, we must first show that $(Gh)_*(r)$ is order reversing. Assume that $\alpha, \beta \in \mathcal{L}_L$ and $\alpha \subseteq \beta$. Let $A = \{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha))\}$ and $B = \{x \in L \,|\, h(x) \geq r(\downarrow h(\beta))\}$. Then $((Gh)_*(r))(\alpha) = \wedge A$ and $((Gh)_*(r))(\beta) = \wedge B$. $\alpha \subseteq \beta$ implies that $\downarrow h(\alpha) \subseteq \downarrow h(\beta)$, and since r is order reversing, $r(\downarrow h(\alpha)) \geq r(\downarrow h(\beta))$. Thus $A \subseteq B$, and hence $\wedge A \geq \wedge B$, i.e., $((Gh)_*(r))(\alpha) \geq ((G(h)_*(r))(\beta)$. Next we must show that for all principal prime ideals $\downarrow p$ of L, $p \geq ((G(h)_*(r))(\downarrow p)$. Let $S = \{x \in L \,|\, h(x) \geq r(\downarrow h(\downarrow p))\}$. Then by definition $((Gh)_*(r))\,(\downarrow p) = \wedge S$. To prove that $p \geq ((Gh)_*(r))\,(\downarrow p) = \wedge S$, we'll show that $p \in S$, i.e., $h(p) \geq r(\downarrow h(\downarrow p))$. By Lemma 4.3 $h(p)$ is a prime element of N and therefore $\downarrow h(p)$ is a prime ideal; $r : \mathcal{L}_L \longrightarrow N$ satisfies requirement *(ii)* in definition 4.11, hence $h(p) \geq r(\downarrow h(p))$. Since $\downarrow h(p) = \downarrow h(\downarrow p)$, $h(p) \geq r(\downarrow h(\downarrow p))$ also holds and therefore $p \in S$ and $p \geq \wedge S = ((Gh)_*(r)(\downarrow p)$. $(Gh)_*(r)$ then defines a generalized stack convergence structure in L.

(iii) Next we must show that $(Gh)_*$ is continuous, i.e., $[(Gh)_*(\bigwedge_j r_j)](\alpha) = [\bigwedge_j((Gh)_* r_j)](\alpha)$ for all $\alpha \in \mathcal{L}_L$. $[(Gh)_*(\bigwedge_j r_j)](\alpha) = \wedge\{x \in L \,|\, h(x) \geq (\bigwedge_j r_j)(\downarrow h(\alpha))\} = \wedge\{x \in L \,|\, h(x) \geq \bigvee_j (r_j(\downarrow h(\alpha)))\}$ and $[\bigwedge_j((Gh)_* r_j)](\alpha) = \bigvee_j[((Gh)_* r_j)(\alpha)] = \bigvee_j (\wedge\{x \in L \,|\, h(x) \geq r_j(\downarrow h(\alpha))\})$. Let $T = \{x \in L \,|\, h(x) \geq \bigvee_j (r_j(\downarrow h(\alpha))\}$ and let $S_j = \{x \in L \,|\, h(x) \geq r_j(\downarrow h(\alpha))\}$ and denote $\wedge S_j$ by s_j. Also, let $\wedge T = t$ and $\bigvee_j s_j = s$. Then we must show that $t = s$.

It follows from the definitions of the sets involved that $T \subseteq S_j$ for all j. Thus $\wedge T \geq \wedge S_j$, i.e., $t \geq s_j$ for all j, which implies that $t \geq \bigvee_j s_j$. Hence $t \geq s$.

Since h is continuous and $s_j = \bigwedge_{x \in S_j}\{x\}$, $h(s_j) = h(\bigwedge_{x \in S_j}\{x\}) = \bigwedge_{x \in S_j}\{h(x)\}$ for all j. Since $r_j(\downarrow h(\alpha)) \leq h(x)$ for all $x \in S_j$, $r_j(\downarrow h(\alpha))$ is a lower bound for the set $\{h(x)\}_{x\in S_j}$. Therefore $r_j(\downarrow h(\alpha)) \leq \bigwedge_{x\in S_j}\{h(x)\} = h(s_j)$ for all j, which then implies that $\bigvee_j\{r_j(\downarrow h(\alpha))\} \leq \bigvee_j\{h(s_j)\}$. Since $s = \bigvee_j s_j$, $s \geq s_j$ for all j; with h preserving order , we then have that $h(s) \geq h(s_j)$ for all j. Thus $h(s) \geq \bigvee_j\{h(s_j)\} \geq \bigvee_j r_j(\downarrow h(\alpha))$. This by definition of T means that $s \in T$. Hence $s \geq t = \wedge T$. Thus $s = t$, which proves the continuity of $(Gh)_*$. Then the left adjoint Gh of $(Gh)_*$ is cocontiuous.

For the functor G to be well defined, it is left to show that for morphisms $h : L \longrightarrow N$ and $g : N \longrightarrow Q$, $(G(gh))_* = (Gh)_* \circ (Gg)_*$. For $(s : \mathcal{L}_Q \longrightarrow Q) \in G(Q)$ and $\alpha \in \mathcal{L}_L$, $[(G(gh))_*(s)](\alpha) = \wedge\{x \in L \,|\, gh(x) \geq s(\downarrow (gh)(\alpha))\}$ and $[((Gh)_* \circ (Gg)_*)(s)](\alpha) = (Gh)_*[((Gg)_*(s))(\alpha)] = \wedge\{x \in L \,|\, h(x) \geq ((Gg)_*(s))(\downarrow h(\alpha))\} = \wedge\{x \in L \,|\, h(x) \geq \wedge\{y \in N \,|\, g(y) \geq s(\downarrow (g(\downarrow h(\alpha)))\}$. Then the desired equality follows directly from the fact that $\downarrow gh(\alpha) = \downarrow g(\downarrow h(\alpha))$.

GSCo is then a topological category over GeoCmP classified by the functor G, i.e., $\int_{\text{GeoCmP}} G = \text{GSCo}$ by Theorem 2.50., making GSCo canonically isomorphic to the pullback of the universal topological category along the functor G. GSCo is the family-universal category for topological categories with a stack convergence type structure: any functor $Q : A \longrightarrow \text{CoP}$ that factors through G defines a topological category in this family. In particular, by composing the functor G with the functors P and Φ of Example 2.48 we obtain the following

Proposition 4.13 $\int_{\text{Sets}} G\Phi P = \text{SCo}$.

Proof. The objects of $\int_{\text{Sets}} G\Phi P =$ are pairs (X, q) where X is a set and $q \in G\Phi P(X)$:

$$X \overset{P}{\longmapsto} P(X) \overset{\Phi}{\longmapsto} P(X)^{\text{op}} \overset{G}{\longmapsto} \{q : \mathcal{L}_{P(X)^{\text{op}}} \longrightarrow P(X)^{\text{op}}\}$$

with q as in Definition 4.11, i.e., q is order reversing and $p \geq q(\downarrow p)$ for all prime elements p of $P(X)^{\mathrm{op}}$. Since down-segments of $P(X)^{\mathrm{op}}$ are exactly the stacks on X ($\mathcal{L}_{P(X)^{\mathrm{op}}} = S(X)$), and the prime the prime elements of $P(X)^{\mathrm{op}}$ are the singletons $\{x\}$ for $x \in X$, a function q as above can equivalently be described as $q : S(X) \longrightarrow P(X)$ with q preserving order and $\{x\} \subseteq q(\uparrow x)$, i.e., $x \in q(\uparrow x)$ for all $x \in X$. Thus q gives a stack convergence structure on X, and hence $\int_{\mathrm{Sets}} G\Phi P$ and SCo have the same objects.

A function $f : X \longrightarrow Y$ lifts to a morphism $\overline{h} : (X, q) \longrightarrow (Y, r)$ in $\int_{\mathrm{Sets}} G\Phi P$ iff $[(G\Phi P)(f)](q) \leq r$ (lifting condition in Remark 2.24 (ii)). Following the actions of the functors P, Φ and G we have the following:

$$(X \xrightarrow{f} Y) \overset{P}{\longmapsto} \left(P(X) \underset{f^{-1}}{\overset{\forall_f}{\rightleftarrows}} P(Y) \right) \overset{\Phi}{\longmapsto} \left(P(X)^{\mathrm{op}} \underset{(f^{-1})^{\mathrm{op}}}{\overset{\exists_f^{\mathrm{op}}}{\rightleftarrows}} P(Y)^{\mathrm{op}} \right).$$

Using the notation $h = ((f^{-1})^{\mathrm{op}} \dashv \exists_f^{\mathrm{op}})$ for the geometric morphism

$\Phi P(f) : P(X)^{\mathrm{op}} \longrightarrow P(Y)^{\mathrm{op}}$, Gh is defined by the adjunction

$$\{r : S(Y) \longrightarrow P(Y)\} \underset{Gh}{\overset{(Gh)_*}{\rightleftarrows}} \{q : S(X) \longrightarrow P(X)\}$$

where both q and r are stack convergence structures on X and Y respectively, $Gh \dashv (Gh)_*$ and $(Gh)_*$ is defined (as in Theorem 4.12) for a stack $\alpha \in S(X)$ by $((Gh)_*(r))(\alpha) = \wedge\{S \in P(X)^{\mathrm{op}} \mid h(S) \geq r(\downarrow h(\alpha))\}$.

On one hand, due to the adjunction $Gh \dashv (Gh)_*$, the lifting condition $(Gh)(q) \leq r$ as above is equivalent to $q \leq (Gh)_*(r)$ which then means that for all stacks α on X, $q(\alpha) \geq (Gh)_*(r)(\alpha)$. The relation $\geq$ is now interpreted in $P(X)^{\mathrm{op}}$, so relative to $P(X)$ we have that h lifts to a morphism $\overline{h} : (x, q) \longrightarrow (Y, r)$ iff $q(\alpha) \subseteq (Gh)_*(r)(\alpha)$ for all $\alpha \in S(X)$.

On the other hand, in the definition of $(Gh)_*$ above, the relation $\geq$ and the operations $\wedge$ and $\downarrow$ are all interpreted again in $P(X)^{\mathrm{op}}$ and $P(Y)^{\mathrm{op}}$. Rewriting the definition relative to $P(X)$ and $P(Y)$ and using the fact that $h(S) = f(S)$ since $\exists_f = \mathrm{Im} f$ we obtain that $[(Gh)_*(r)](\alpha) = \cup\{S \subseteq X \mid f(S) \subseteq r(\uparrow f(\alpha))\} = \{x \in X \subseteq \mid f(x) \in r(\uparrow f(\alpha))\}$; then the lifting condition becomes $q(\alpha) \subseteq \{x \in X \mid f(x) \subseteq r(\uparrow f(\alpha))\}$. This is equivalent to $x \in q(\alpha) \Rightarrow f(x) \in r(\uparrow f(\alpha))$ which is exactly the condition that defines morphisms in SCo. Thus $\int\limits_{\mathrm{Sets}} G\Phi P = \mathrm{SCo}$ which means that SCo is isomorphic to the canonical pullback of the universal topological category along the functors $G\Phi P$.

Generalized Filter Convergence Structures

With a construction similar to the one in Theorem 4.12 we can generalize the notion of a filter convergence structure to posets.

Definition 4.14 Given a cocomplete poset L, let $\mathrm{Id}(L)$ denote the set of all ideals of L ordered by inclusion. A function $q : \mathrm{Id}(L) \longrightarrow L$ is a *generalized filter convergence structure* in L if *(i)* q is order-reversing and *(ii)* $p \geq q(\downarrow p)$ for every principal prime ideal $\downarrow p$. The pair (L, q) is then called a *generalized filter convergence space* and is an object of the category GFCo. Morphisms in GFCo are defined the same way as in GSCo (Definition 4.11).

The definition of morphisms in GFCo makes sense, since for an ideal α of L and $h : L \longrightarrow N$ a morphism in GeoCmP, $\downarrow h(\alpha)$ is an ideal of N: $\downarrow h(\alpha)$ is clearly a down-segment, and for $x, y \in \downarrow h(\alpha)$, $x \leq h(a)$ and $x \leq h(b)$ for some $a, b \in \alpha$; since

α is an ideal, $a \vee b \in \alpha$ and therefore $h(a \vee b) \geq h(a) \vee h(b) \geq x \vee y$, which means that $x \vee y \in \downarrow h(\alpha)$.

Remark 4.15 If in definition 4.14 $(L, \leq) = P(X)^{\mathrm{op}}$ for some set X, then $\mathrm{Id}(L) = F(X)$ and q defines a filter convergence structure on X as in Definition 4.7.

Again we'll consider the following stronger conditions on q for p a prime element of L and $\alpha, \beta \in \mathrm{Id}(L)$:

(GC1) $p \geq q(\alpha)$ implies that $p \geq q(\alpha \cap \downarrow p)$;

(GC2) $p \geq q(\alpha)$ and $p \geq q(\beta)$ implies that $p \geq q(\alpha \cap \beta)$;

(GC3) For any $\{\alpha_j\}_{j \in J} \subseteq \mathrm{Id}(L)$ such that $p \geq q(\alpha_j)$ for all $j \in J$, $p \geq q(\bigcap_{j \in J} \alpha_j)$.

If $(L, \leq) = P(X)^{\mathrm{op}}$ for some set X, then properties (GC1) - (GC3) are equivalent to properties (C1) - (C3).

Definition 4.16 A generalized filter convergence structure $q : \mathrm{Id}(L) \longrightarrow L$ is called a *generalized local filter convergence structure* if it satisfies (GC1), a *generalized limit structure* if it satisfies (GC2) and a *generalized pretopology* if it satisfies (GC3).

The pairs (L, q) where q is a generalized filter convergence structure in L satisfying (GC1), (GC2) or (GC3) are the objects of the categories GLFCo, GLim and GPrT respectively. Morphisms in these categories are defined the same way as in GSCo (Definition 4.11).

Clearly (GC3) $\Rightarrow$ (GC2) $\Rightarrow$ (GC1), therefore GPrT $\subseteq$ GLim $\subseteq$ GLFCo $\subseteq$ GFCo.

The classifying functors for the above categories are defined in the following theorems.

Theorem 4.17 *GFCo is a topological category over GeoCmP classified by the functor $G_0 : GeoCmP \longrightarrow CoP$ defined as follows: for a plete poset L, $G_0(L) = \{q : (Id)(L) \longrightarrow L | \, q$ is a generalized filter convergence structure in $L\}$; for a geometric morphism $h : L \longrightarrow N,\ G_0h : G_0(L) \longrightarrow G_0(N)$ is defined again as the left adjoint of $(G_0h)_* : G_0(N) \longrightarrow G_0(L)$, where for $(r : Id(N) \longrightarrow N) \in G_0(N)$ and $\alpha \in Id(L)$, $((G_0h)_*(r))(\alpha) = \wedge\{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha))\}$.*

The proof of the above theorem is analogous to that of Theorem 4.12.

Thus GFCo $= \int\limits_{\mathrm{GeoCmP}} G_0$ and GFCo is isomorphic to the pullback of the universal topological category along the functor G_0; moreover, it is the family universal category for topological categories with a filter convergence type structure, as any functor $Q \longrightarrow$ CoP that factors through G_0 defines a category in this family. In particular, the classifying functor of FCo factors through G_0 as shown in the following proposition.

Proposition 4.18 *FCo* $= \int\limits_{\mathrm{Sets}} G_0\Phi P.$

Proof. Given a set X, $G_0\Phi P(X) = G_0(P(X)^{\mathrm{op}}) =$

$\{q : F(X) \longrightarrow P(X) \,|\, q$ is a filter convergence structure on $X\}$ by Remark 4.15.

Thus the objects (X, q) of $= \int\limits_{\mathrm{Sets}} G_0\Phi P$ are filter convergence spaces. The rest of the proof is identical to that of Proposition 4.13.

Proposition 4.19 *GLFCo is a topological category over GeoCmP classified by the functor $G_1 : GeoCmP \longrightarrow CoP$ defined on the object level as follows: for a plete* *poset L as $G_1(L) = \{q : Id(L) \longrightarrow L | \, q \in G_0(L)$ and q satisfies (GC1)$\}$. On the morphism level G_1 is defined the same way as G_0.*

Proof. Again the proof for the most part is analogous to that of Theorem 4.12. We have to verify though that $G_1(L)$ is a complete poset. As before, order and infima are defined as $q_1 \leq q_2$ iff $q_1(\alpha) \geq q_2(\alpha)$ for all $\alpha \in \mathrm{Id}(L)$, and $(\bigwedge_j q_j)(\alpha) = \bigvee_j (q_j(\alpha))$. We have to show that $\bigwedge_j q_j$ defines a generalized filter convergence structure, i.e., that $\bigwedge_j q_j$ satisfies (GC1). Let $\{q_j\}_{j \in J}$ be a set of generalized local filter convergence funtions, and let $\alpha \in \mathrm{Id}(L)$. We have to show that for a prime element p in L, $\left(p \geq (\bigwedge_j q_j)(\alpha)\right) \Rightarrow \left(p \geq (\bigwedge_j q_j)(\alpha \cap \downarrow p)\right)$. $p \geq (\bigwedge_j q_j)(\alpha)$ means by definition that $p \geq \bigvee_j q_j(\alpha)$, and hence $p \geq q_j(\alpha)$ for all $j \in J$. Thus $p \geq q_j(\alpha \cap \downarrow p)$ for all j since q_j satisfies (GC1), and hence $p \geq \bigvee_j q_j(\alpha \cap \downarrow p) = (\bigwedge_j q_j)(\alpha \cap \downarrow p)$. We also have to verify that for a geometric morphism $h : L \longrightarrow N$ and for $r \in G_1(N)$, $(G_1 h)_*(r)$ also satisfies (GC1). Thus we must show that for a prime p in L, $p \geq [(G_1 h)_*(r)](\alpha)$ implies that $p \geq [(G_1 h)_*(r)](\alpha \cap \downarrow p)$. So assume that $p \geq [(G_1 h)_*(r)](\alpha) =$

$\wedge\{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha))\}$ (as in Theorem 4.12), let

$T = \{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha))\}$ and $\wedge T = t$. By the continuity of h, $h(\wedge T) = \wedge h(T)$, so we have that $h(t) = h(\wedge T) = \wedge\{h(x) \in N \,|\, h(x) \geq r(\downarrow h(\alpha))\}$. $r(\downarrow h(a))$ is a lower bound for $h(T)$ and hence $r(\downarrow h(\alpha)) \leq \wedge h(T) = h(t)$. This means that $t \in T$ and therefore $p \in T$ also, since $p \geq t \Rightarrow h(p) \geq h(t) \geq r(\downarrow h(\alpha))$. By Lemma 4.3 $h(p)$ is a prime element in N and r satisfies (GC1), so $h(p) \geq r(\downarrow h(\alpha))$ implies that $h(p) \geq r(\downarrow h(\alpha) \cap \downarrow h(p))$. By Lemma 4.4 $\downarrow h(\alpha) \cap \downarrow h(p) = \downarrow h(\alpha \cap \downarrow p)$, so we have that $h(p) \geq r(\downarrow h(\alpha \cap \downarrow p))$, and hence $p \in \{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha \cap \downarrow p))\}$. Then $p \geq \wedge\{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha \cap \downarrow p))\} = [(G_1 h)_*(r)](\alpha \cap \downarrow p)$, so $(G_1 h)_*(r)$

satisfies (GC1). The rest of the proof is analogous to that of Theorem 4.12. Thus GLFCo= $\int\limits_{\text{GeoCmP}} G_1$ an consequently LFCo $\int\limits_{\text{Sets}} G_1 \Phi P$.

GLFCo is then the family universal category for topological categories with a local filter convergence type structure.

Proposition 4.20 *The category GLim is a topological category over GeoCmP classified by the functor $G_2 : GeoCmP \longrightarrow Cop$ defined as follows: for a plete poset L, $G_2(L) = \{q : Id(L) \longrightarrow L |\, q \in G_0(L)$ and q satisfies (GC2)$\}$. On the morphism level G_2 is defined the same way as G_0.*

Proof. Again we have to verify that $G_2(L)$ is a complete poset. It is a direct consequence of the definition of $\bigwedge\limits_j q_j$ that for any prime element p in L, $p \geq (\bigwedge\limits_j q_j)(\alpha \cap \beta)$ whenever $p \geq (\bigwedge\limits_j q_j)(\alpha)$ and $p \geq (\bigwedge\limits_j q_j)(\beta)$.

We also have to show that for a morphism $h : Ł \longrightarrow N$ in GeoCmP and for $r \in (G_2)(N)$, $(G_2 h)_*(r)$ also satisfies (GC2): assume that $p \geq [(G_2 h)_*(r)](\alpha) = \wedge\{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha))\}$ and $p \geq [(G_2 h)_*(r)](\beta) = \wedge\{x \in L \,|\, h(x) \geq r(\downarrow h(\beta))\}$. Let $\{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha))\} = S$ and $\{x \in L \,|\, h(x) \geq r(\downarrow h(\beta))\} = T$. Then $p \geq \wedge S$ and $p \geq \wedge T$, and again by the continuity of h, $h(p) \geq h(\wedge S) = \wedge h(S)$, and $h(p) \geq h(\wedge T) = \wedge h(T)$. As before we have that $\wedge h(S) \geq r(\downarrow h(\alpha))$ and $\wedge h(S) \geq r(\downarrow h(\beta))$ and therefore $h(p) \geq r(\downarrow h(\alpha))$ and $h(p) \geq r(\downarrow h(\beta))$ also hold. Then, since r satisfies (GC2), $h(p) \geq r(\downarrow h(\alpha) \cap \downarrow h(\beta)) = r(\downarrow h(\alpha \cap \beta))$; hence $p \in \{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha \cap \beta))\}$ which implies that $p \geq \wedge\{x \in L \,|\, h(x) \geq r(\downarrow h(\alpha \cap \beta))\} = [(Gh)_*(r)](\alpha \cap \beta)$. Thus $(Gh)_*(r)$ satisfies (GC2). The rest of the proof is the same as for Theorem 4.12. We have then that GLim= $\int\limits_{\text{GeoCmP}} G_2$ and that Lim= $\int\limits_{\text{Sets}} G_2 \Phi P$. GLim is then the family universal category for topological categories with a limit structure.

Proposition 4.21 *The category GPrT is a topological category over GeoCmP classified by the functor $G_3 : GeoCmP \longrightarrow Cop$ defined as follows: for a plete poset L, $G_3(L) = \{q : Id(L) \longrightarrow L \,|\, q \in G_0(L)$ and q satisfies (GC3)$\}$. For morphisms G_3 is defined the same way as G_0.*

It can be verified essentially in the same way as in the proof of Proposition 4.20 that both $\bigwedge\limits_j q_j$ and $(Gh)_*(r)$ satisfy (GC3). So again we have that GPrT= $\int\limits_{\text{GeoCmP}} G_3$ and also that PrT= $\int\limits_{\text{Sets}} G_3 \Phi P$.

GPrT is then the family universal for topological categories with a pretopology type structure.

Whether the relationships between the classical members of the families of topological categories discussed still hold for the generalized versions would be worth investigating. Top is (isomorphic to) a bireflective subcategory of FCo. The inclusion functor $\iota : \text{Top} \hookrightarrow \text{FCo}$ is defined as follows: an object (X, τ) in Top gives a filter convergence structure $q_\tau : F(X) \longrightarrow P(X)$ on the set X by sending a filter α to $q_\tau(\alpha) = \{x \in X \,|\, \mathcal{N}_x \subseteq \alpha\}$ where $\mathcal{N}_x$ denotes the neighborhood filter of the point x associated with the topology τ. The traditional definition of a neighborhood filter, $\mathcal{N}_x = \{S \subseteq X \,|\, \exists\, U \in \tau \text{ with } x \in U \subseteq S\}$, can equivalently be defined using the notation of this chapter as $\mathcal{N}_x = \uparrow(\tau \cap \uparrow x)$ and then generalized to posets; using this approach, a generalized filter convergence structure defined as

$q_\tau(\alpha) = \vee\{a \in A \mid \uparrow(\tau \cap \uparrow a) \subseteq \alpha\}$ can be associated to an object (A, τ) of TOP, to see if $\int T =$ TOP is isomorphic to a bireflective subcateogry of $\int G_0\Phi =$ GFCo.

The family universal category could be examined for cartesian closedness in cases when the classical member of the family, e.g. the category of filter convergence spaces, is cartesian closed. Furthermore, the family universal category for other types of topological structures could be identified.

References

1. Adámek, J., Herrlich, H., and Stecker, G., *Abstract and Concrete Categories The Joy of Cats*, Wiley, 1990.

2. Birkhoff, G., *Lattice Theory*, American Mathematical Society, 1967.

3. Borceux, F., *Handbook of Categorical Algebra*, Cambridge University Press, 1994.

4. Chang, C.L., *Fuzzy Topological Spaces*, J. Math. Anal. Appl. **24** (1968), 182-190.

5. García, G., Vicente, P., Šostak, A. P., *Even Fuzzy Topologies and Related Structures*, Quaestiones Mathematicae **20** (1997), 327-350.

6. Gierz, G. et. al., *A Compendium of Continuous Lattices*, Springer-Verlag, 1980.

7. Goguen, J., *L-fuzzy Sets*, J. Math. Anal. Appl. **18** (1967), 145-174.

8. Goguen, J., *The Fuzzy Tychonoff Theorem*, J. Math. Anal. Appl. **43** (1973), 734-742.

9. Herrlich, H., *A Concept of Nearness*, General Topology and Its Applications **5** (1974), 191-212.

10. Herrlich, H., *Topological Functors*, General Topology and Its Applications **4** (1974), 125-142.

11. Johnstone, P.T., *Stone Spaces*, Cambridge Studies in Advanced Mathematics, 1982.

12. Johnstone, P.T., *Topos Theory*, Academic Press, 1977.

13. Lowen, R., *Convergence in Fuzzy Topological Spaces*, General Topology and Its Applications **10** (1979), 147-160.

14. MacLane, S., *Categories for the Working Mathematician*, Springer-Verlag, 1994.

15. MacLane, S., Moerdijk, I., *Sheaves in Geometry and Logic*, Springer-Verlag, 1994.

16. Nel, L. D., *Initially Structured Categories and Cartesian Closedness*, Can. J. Math. **XXVII** (1975), 1361-1377.

17. Preuss, G., *Theory of Topological Stuctures*, D. Reidel, 1988.

18. Schwarz, F., *Connections between Convergence and Nearness*, Lecture Notes in Mathematics **719** (1978), Springer-Verlag, 345 - 357.

19. Šostak, A.P., *On a Fuzzy Topological Structure*, Supplemento Rendiconti del Circolo Matematico di Palermo **Series 2, No 11** (1985), 89-103.

20. Thomason, R. W., *Homotopy Colimits in the Category of Small Categories*, Mathematical Proceedings of the Cambridge Philosophical Society **85** (1979), 91 - 109.

21. Wyler, O., *On the Categories of General Topology and Topological Algebra*, Archiv der Mathematik **XXII** (1971), 7-17.

22. Wyler, O., *Top Categories and Categorical Topology*, General Topology and Its Applications **1** (1971), 17-28.

23. Zadeh, L., *Fuzzy Sets*, Inform. and Control **8** (1965), 338-353.

24. Zhang, D., *L-Fuzzifying Topologies as L-Topologies*, Fuzzy Sets and Systems **125** (2002), 135-144.

25. Zhang, D., *On the Relationship between several Basic Categories in Fuzzy Topology*, Quaestiones Mathematicae **25** (2002), 289-301.