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Quasisymmetric Functions and Permutation Statistics for Coxeter Groups and Wreath Product Groups

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QUASISYMMETRIC FUNCTIONS AND PERMUTATION STATISTICS FOR COXETER GROUPS AND WREATH PRODUCT GROUPS

By

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Eulerian quasisymmetric functions were introduced by Shareshian and Wachs in order to obtain a q-analog of Euler's exponential generating function formula for the Eulerian polynomials. They are defined via the symmetric group, and applying the stable and nonstable principal specializations yields formulas for joint distributions of permutation statistics. We consider the wreath product of the cyclic group with the symmetric group, also known as the group of colored permutations. We use this group to introduce *colored Eulerian quasisymmetric functions*, which are a generalization of Eulerian quasisymmetric functions. We derive a formula for the generating function of these colored Eulerian quasisymmetric functions, which reduces to a formula of Shareshian and Wachs for the Eulerian quasisymmetric functions. We show that applying the stable and nonstable principal specializations yields formulas for joint distributions of colored permutation statistics. The family of colored permutation groups includes the family of symmetric groups and the family of hyperoctahedral groups, also called the type A Coxeter groups and type B Coxeter groups, respectively. By specializing our formulas to these cases, they reduce to the Shareshian-Wachs q-analog of Euler's

formula, formulas of Foata and Han, and a new generalization of a formula of Chow and Gessel.

This dissertation is dedicated to Kiki Moschella.

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Introduction

A permutation statistic is a map $f : S_n \to \mathbb{N}$ where S_n is the symmetric group and $\mathbb{N}$ is the set of nonnegative integers. The modern study of enumerating permutations according to statistics began in the early 20th century with the work of MacMahon [19], [20]. He studied four fundamental permutation statistics, namely the descent number, excedance number, inversion number, and major index, which we denote by des, exc, inv, and maj respectively. [1] In particular, he was the first to observe the now classic result that des and exc are equidistributed [20], that is

$$A_n(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)} = \sum_{\pi \in S_n} t^{\mathrm{exc}(\pi)}.$$

The polynomials $A_n(t)$ are well-studied and called the Eulerian polynomials. [2] They were first introduced by Euler in the 18th century in the form

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k \geq 0} (k+1)^n t^k. \tag{1}$$

Euler also established the following formula for the exponential generating function for the Eulerian polynomials

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z}. \tag{2}$$

[1] Defined in Chapter 1.

[2] It is also common to define the Eulerian polynomials to be $tA_n(t)$.

The permutation statistics inv and maj have frequently been used in the literature to obtain q-analogs of classical results. One of the earliest instances of this is Carlitz's [5] formula for a q-analog of Euler's original definition (1) of the Eulerian polynomials

$$\frac{\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)}}{(t;q)_{n+1}} = \sum_{k \geq 0} [k+1]_q^n t^k. \tag{3}$$

A more recent instance is the following formula for the bivariate distribution of maj and exc due Shareshian and Wachs [31], [32]

$$\sum_{\substack{n \geq 0 \\ \pi \in S_n}} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)} \frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(tqz) - tq\exp_q(z)}. \tag{4}$$

which is a q-analog of Euler's formula (2). To prove this formula, they introduce a family of quasisymmetric functions $Q_{n,j,k}(\mathbf{x})$, called Eulerian quasisymmetric functions, where $\mathbf{x}$ denotes the infinite set of variables $\{x_1, x_2, ...\}$. They compute the following generating function formula

$$\sum_{n,j,k \geq 0} Q_{n,j,k}(\mathbf{x}) t^j r^k z^n = \frac{(1-t)H(rz)}{H(tz) - tH(z)}, \tag{5}$$

where $H(z) := \sum_{i \geq 0} h_i(\mathbf{x}) z^i$ and $h_i(\mathbf{x})$ is the complete homogeneous symmetric function of degree i. The Eulerian quasisymmetric functions are constructed (and so named) because applying the stable principal specialization (i.e. setting $x_i = q^{i-1}$) to (5) yields the formula (4). The following formula of Foata and Han

$$\sum_{\substack{n \geq 0 \\ \pi \in S_n}} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}(\pi)} t^{\mathrm{exc}(\pi)} r^{\mathrm{fix}(\pi)} \frac{z^n}{(p;q)_{n+1}} = \sum_{k \geq 0} \frac{p^k (1-tq)(z;q)_k (tqz;q)_k}{(rz;q)_{k+1} \left[(z;q)_k - tq(tqz;q)_k\right]} \tag{6}$$

was obtained subsequently to (4), and can be obtained by applying a different specialization to (5) called the nonstable principal specialization. The Eulerian quasisymmetric functions are also quite interesting in their own right, and many other properties are investigated in [31], [32].

The symmetric group belongs to a family of groups known as Coxeter groups, and is also called the type A Coxeter group. Some of the earliest results on permutation statistics for general Coxeter groups were obtained by Reiner [22], [23], [24], [25], and Brenti [4]. The inv and des statistic have natural Coxeter group generalizations. For the hyperoctahedral group (the type B Coxeter group) we will denote the Coxeter descent statistic by $\mathrm{des_B}$. Adin and Roichman [2] introduced an analog of the major index statistic for the hyperoctahedral group, which they called the flag major index, denoted fmaj. Adin, Brenti, and Roichman [1] use fmaj and another statistic fdes to obtain a type B analog of Carlitz's identity (3). More recently Chow and Gessel [6] obtain a different type B analog of (3), which is given by

$$\frac{\sum_{\pi\in B_n} q^{\mathrm{fmaj}(\pi)} t^{\mathrm{des_B}(\pi)}}{(t;q^2)_{n+1}} = \sum_{k\geq 0} [2k+1]_q^n t^k. \tag{7}$$

A new result (Theorem 2.4.13) presented in this thesis is the following type B analog of (6), which reduces to (7) by setting $t = r = s = 1$,

$$\sum_{\substack{n\geq 0\\ \pi\in B_n}} \frac{z^n}{(p;q^2)_{n+1}} q^{\mathrm{fmaj}(\pi)} t^{\mathrm{fexc}(\pi)} p^{\mathrm{des_B}(\pi)} r^{\mathrm{fix}^+(\pi)} s^{\mathrm{neg}(\pi)}$$

$$= \sum_{k\geq 0} \frac{p^k(1-t^2q^2)(z;q^2)_k(t^2q^2z;q^2)_k}{(rz;q^2)_{k+1}\left[(1+sqt)(z;q^2)_k - (t^2q^2+sqt)(t^2q^2z;q^2)_k\right]}. \tag{8}$$

This formula includes fexc, which is a natural type B analog of exc, as well as two additional type B statistics fix^+ and neg. Foata and Han introduced fexc and showed it is equidistributed with the statistic fdes of Adin, Brenti and Roichman. Our formula (8) is similar, but not equivalent, to a formula of Foata and Han [12] which involves fdes rather than $\mathrm{des_B}$ and reduces to the type B analog of (3) due to Adin, Brenti and Roichman, while ours reduces to (7). However both (8) and the similar formula of Foata and Han reduce to the following type B analog of (4) when p approaches 1 and $r = s = 1$

$$\sum_{\substack{n\geq 0\\ \pi\in B_n}} \frac{z^n}{[n]_{q^2}!} t^{\mathrm{fexc}(\pi)} q^{\mathrm{fmaj}(\pi)} = \frac{(1-tq)\exp_{q^2}(z)}{\exp_{q^2}(t^2q^2z) - tq\exp_{q^2}(z)}. \tag{9}$$

Equation (8) is a special case of formula (10) below involving permutation statistics on a more general class of groups that contain both the symmetric group and the hyperoctahedral group. The groups in this class are wreath products $C_N \wr S_n$ of the cyclic group C_N with the symmetric group S_n, also called colored permutation groups. Some of the earliest results on permutation statistics for colored permutation groups were obtained by Reiner [22], Steingrímmsson [35], and Poirier [21]. Our formula (10) reduces to the following colored permutation generalization of Carlitz's identity (3) and the Chow-Gessel type B analog of Carlitz's identity (7), [3]

$$\frac{\sum_{\pi\in C_N\wr S_n} q^{\mathrm{flagmaj}(\pi)} t^{\mathrm{des}^*(\pi)}}{(t;q^N)_{n+1}} = \sum_{k\geq 0} [Nk+1]_q^n t^k,$$

where flagmaj and des* are generalizations of maj and des for $C_N \wr S_n$.

Our more general formula (Theorem 3.3.1) for a multivariate distribution of colored permutation statistics is

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{(p;q)_{n+1}} t^{\mathrm{exc}(\pi)} r^{\vec{\mathrm{fix}}(\pi)} s^{\vec{\mathrm{col}}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)}$$

$$= \sum_{l\geq 0} \frac{p^l(1-tq)(z;q)_l(tqz;q)_l\left(\prod_{m=1}^{N-1}(s_mz;q)_l\right)\left(\prod_{m=1}^{N-1}(r_ms_mz;q)_l\right)^{-1}}{(r_0z;q)_{l+1}\left[\left(1+\sum_{m=1}^{N-1}s_m\right)(z;q)_l - \left(tq+\sum_{m=1}^{N-1}s_m\right)(tqz;q)_l\right]}. \tag{10}$$

This formula is similar, but not equivalent, to a formula of Foata and Han [13] which

[3] This colored permutation generalization of Carlitz's identity was also independently obtained by Chow and Mansour [7].

involves a different descent statistic. We also obtain the following colored permutation generalization of (9) and (4) [4]

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{[n]_q!} t^{\mathrm{exc}(\pi)} r^{\vec{\mathrm{fix}}(\pi)} s^{\vec{\mathrm{col}}(\pi)} q^{\mathrm{maj}(\pi)}$$

$$= \frac{\exp_q(r_0 z)(1-tq)\left(\prod_{m=1}^{N-1} \mathrm{Exp}_q(-s_m z)\exp_q(r_m s_m z)\right)}{\left(1+\sum_{m=1}^{N-1} s_m\right)\exp_q(tqz) - \left(tq + \sum_{m=1}^{N-1} s_m\right)\exp_q(z)}. \tag{11}$$

Our proof of (10) and (11) involves a nontrivial generalization of techniques developed by Shareshian and Wachs in order to prove (4). We introduce a family of quasisymmetric functions which we call *colored Eulerian quasisymmetric functions*. They are a generalization of the Eulerian quasisymmetric functions of Shareshian and Wachs. We obtain the following generating function formula for our colored Eulerian quasisymmetric functions, which generalizes (5)

$$\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} Q_{n,j,\vec{\alpha},\vec{\beta}}(\mathbf{x}) z^n t^j r^{\vec{\alpha}} s^{\vec{\beta}} = \frac{H(r_0 z)(1-t)\left(\prod_{m=1}^{N-1} E(-s_m z)H(r_m s_m z)\right)}{\left(1+\sum_{m=1}^{N-1} s_m\right)H(tz) - \left(t+\sum_{m=1}^{N-1} s_m\right)H(z)}, \tag{12}$$

where $E(z) := \sum_{i\geq 0} e_i(\mathbf{x})z^i$ and $e_i(\mathbf{x})$ is the elementary symmetric function of degree i. We show that applying the stable and nonstable principal specializations to (12) yields (10) and (11), respectively.

We begin in Chapter 1 with notation and preliminaries for the symmetric group. We define the fundamental permutation statistics discussed above, and examine some classic results. Next we briefly discuss symmetric and quasisymmetric functions. This allows us to study the permutation enumeration techniques developed by Gessel and Reutenauer [15] in Sections 1.6, and those of Shareshian and Wachs [31], [32] in Section 1.7.

[4]Equation (11) was also independently obtained by Foata and Han [13].

In Chapter 2 we define general Coxeter groups and some natural statistics on them. We see how the symmetric group, or type A Coxeter group, fits into this family of groups. Next we define the type B Coxeter group, also called the hyperoctahedral group, and study its combinatorial description as the signed permutation group. We define several statistics for the signed permutation group, which are analogous to their symmetric group counterparts. We then examine type B analogs of classic type A results, including our new result (8), and its consequence (9).

In Chapter 3 we define wreath products of cyclic groups with symmetric groups, and their combinatorial interpretation as colored permutation groups. This family of groups includes the symmetric group and the hyperoctahedral group, but is in general different from the family of Coxeter groups. We discuss statistics for this group and examine several results. In particular we present our new results given in (10) and (11), and show how they imply (8) and (9).

We introduce our colored Eulerian quasisymmetric functions in Chapter 4 and prove a fundamental lemma regarding this family of quasisymmetric functions. We then show how our formulas for multivariate distributions of colored permutation statistics follow from (12).

The bulk of our remaining work is to then prove (12). In Chapter 5 we begin by obtaining a combinatorial description of the monomials appearing in each colored Eulerian quasisymmetric function. From this combinatorial description, we deduce a recurrence relation which is equivalent to the desired generating function formula. Chapter 6 is devoted to establishing this recurrence relation, which must be done separately for two different cases.

We close with Chapter 7 by presenting some recurrence and closed formulas, which are equivalent to (12). These formulas are also specialized to obtain recurrence and closed formulas for (10) and (11). In conclusion, we discuss some of our future work.

Chapter 1

The Symmetric Group

1.1 Preliminaries

For $n \geq 1$ let S_n denote the *symmetric group* on the set $[n] := \{1, 2, ..., n\}$, i.e. the group of bijections from $[n]$ to itself with multiplication given by function composition. It will also be convenient to define $S_0 := \{\theta\}$ where θ denotes the empty word. An element $\pi \in S_n$ is called a *permutation* and it can be written in two-line notation

$$\pi = \begin{bmatrix} 1 & 2 & ... & n \\ \pi(1) & \pi(2) & ... & \pi(n) \end{bmatrix},$$

and one-line notation (i.e. as a word over $[n]$)

$$\pi = \pi(1), \pi(2), ..., \pi(n).$$

We can also write a permutation as a product of *cycles*. Let $i_1, i_2, ..., i_k$ be distinct positive integers, a cycle

$$(i_1, i_2, ..., i_k)$$

denotes a bijection which maps i_j to i_{j+1} for $j = 1, 2, ..., k-1$, maps i_k to i_1, and each integer not appearing in the cycle is mapped to itself. Given any $\pi \in S_n$, it can be written as a product of disjoint cycles. For example

$$3, 2, 1, 5, 6, 4 = (1, 3)(2)(4, 5, 6).$$

We let Id denote the identity permutation which maps each integer in $[n]$ to itself.

1.2 Permutation Statistics and Eulerian Polynomials

A *permutation statistic* f is a map from the union of all symmetric groups to the set of nonnegative integers, which we denote by $\mathbb{N}$. The modern study of permutation statistics was initiated by MacMahon [20]. We will see that there are several naturally defined permutation statistics which have connections to other areas of mathematics, and lead to some beautiful results. Here we define some of these permutation statistics.

Definition 1.2.1. Let $\pi \in S_n$.

- The set of fixed points of π, denoted $\mathrm{FIX}(\pi)$, is defined by

$$\mathrm{FIX}(\pi) := \{i \in [n] : \pi(i) = i\}.$$

- The number of fixed points, denoted $\mathrm{fix}(\pi)$, is defined by

$$\mathrm{fix}(\pi) := |\mathrm{FIX}(\pi)|.$$

- The descent set of π, denoted $\mathrm{DES}(\pi)$, is defined by

$$\mathrm{DES}(\pi) := \{i \in [n-1] : \pi(i) > \pi(i+1)\}.$$

- The descent number, denoted des(π), is defined by

$$\mathrm{des}(\pi) := |\mathrm{DES}(\pi)|.$$

- The excedance set of π, denoted EXC(π), is defined by

$$\mathrm{EXC}(\pi) := \{i \in [n-1] : \pi(i) > i\}.$$

- The excedance number, denoted exc(π), is defined by

$$\mathrm{exc}(\pi) := |\mathrm{EXC}(\pi)|.$$

- An inversion of π is a pair $(\pi(i), \pi(j))$, such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. The inversion number of π, denoted inv(π), is defined to be the number of inversions of π.

- The major index of π, denoted maj(π), is defined by

$$\mathrm{maj}(\pi) := \sum_{i \in \mathrm{DES}(\pi)} i.$$

The major index is named after Major Percy Alexander MacMahon, who introduced this statistic along with descent number and excedance number. Given two permutation statistics f_1, f_2, we say they are *equidistributed* if

$$|\{\pi \in S_n : f_1(\pi) = k\}| = |\{\pi \in S_n : f_2(\pi) = k\}|$$

for all $n, k \in \mathbb{N}$. Equivalently, f_1, f_2, are equidistributed if the following polynomial identity holds for all $n \in \mathbb{N}$

$$\sum_{\pi \in S_n} t^{f_1(\pi)} = \sum_{\pi \in S_n} t^{f_2(\pi)}.$$

MacMahon [20] was the first to observe in the early 20$^{\text{th}}$ century that the descent number and excedance number are equidistributed. Here we describe a bijection called Foata's First Fundamental Transformation (see [14]), which shows that these statistics are equidistributed.

Theorem 1.2.2 (MacMahon)**.** *Descent number and excedance number are equidistributed. Equivalently,*

$$\sum_{\pi \in S_n} t^{\text{des}(\pi)} = \sum_{\pi \in S_n} t^{\text{exc}(\pi)}$$

for all $n \in \mathbb{N}$.

Proof. Given $w \in S_n$, write w in cycle notation where

1) each cycle is written with its largest element first,

2) the cycles are written in increasing order of their largest element.

Then let $\hat{w}$ be the permutation in one-line notation obtained by removing parenthesis. For example if

$$w = 5,6,8,7,2,1,9,3,4 = (1,5,2,6)(3,8)(4,7,9) = (6,1,5,2)(8,3)(9,4,7),$$

then

$$\hat{w} = 6,1,5,2,8,3,9,4,7.$$

Next we describe the inverse of the map $w \mapsto \hat{w}$. Given a permutation w, we call w_i a left-to-right maximum if $w_i > w_j$ for $j = 1,2,...,i-1$. So if $w \mapsto \hat{w}$, then placing a left parenthesis before each left-to-right maximum of $\hat{w}$, and then placing right parenthesis before each left parenthesis and at the end of $\hat{w}$, takes us back to w

written in cycle form. Therefore this map is a bijection.

This map also has the property that $i \notin \mathrm{DES}(\hat{w})$ if and only if $w(\hat{w}(i)) \geq \hat{w}(i)$. Let $\mathrm{wexc}(w) := |\{i \in [n] : w(i) \geq i\}|$, called the number of weak excedances of a permutation. Thus $n - \mathrm{des}(\hat{w}) = \mathrm{wexc}(w)$.

Let $b : S_n \to S_n$ be the involution which replaces i by $n+1-i$ and rewrites the word in reverse order. That is, if $w = w_1, w_2, ..., w_n$, then

$$b(w_1, w_2, ..., w_n) = n+1-w_n, n+1-w_{n-1}, ..., n+1-w_1.$$

It follows that $w_i \leq i$ if and only if $b(w)(n+1-i) = n+1-w_i \geq n+1-i$. Thus $\mathrm{wexc}(b(w)) = n - \mathrm{exc}(w)$.

If we apply the map b followed by the hat map, we have the desired result. That is,

$$\mathrm{exc}(w) = n - \mathrm{wexc}(b(w)) = \mathrm{des}(\widehat{b(w)}).$$

□

Note that while descent number and excedance number are equidistributed, in general they are not equal. For example if $\pi = 3, 2, 1$, then $\mathrm{des}(\pi) = 2$ and $\mathrm{exc}(\pi) = 1$.

It so happens that des and exc have an intimate connection with the Eulerian polynomials, which we denote by $A_n(t)$ for $n \in \mathbb{N}$. Euler introduced these polynomials in the 18$^{\text{th}}$ century in the form

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k\geq 0}(k+1)^n t^k,$$

in order to study the Dirichlet eta function. Euler also proved (see [17]) the following generating function formula for these polynomials

$$\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z}. \tag{1.1}$$

In the mid 20^{th} century, after MacMahon's result that des and exc are equidistributed, Riordan [27] discovered the following combinatorial interpretation of the Eulerian polynomials (see also [10]):

$$A_n(t) = \sum_{\pi \in S_n} t^{\operatorname{des}(\pi)} = \sum_{\pi \in S_n} t^{\operatorname{exc}(\pi)} \tag{1.2}$$

for all $n \in \mathbb{N}$. In fact, equation (1.2) is the modern definition of the Eulerian polynomials. [1]

Later on we will see that inversion number plays an important role in the study of the symmetric group as a Coxeter group. MacMahon [19] was the first to prove that inversion number is equidistributed with major index. Here, we choose to give a sketch of Foata's [9] bijective proof that maj and inv are equidistributed.

Theorem 1.2.3 (MacMahon). *The permutation statistics major index and inversion number are equidistributed.*

Proof. We describe Foata's second fundamental transformation $\Phi : S_n \to S_n$; we refer the reader to [9] for the proof that Φ is a bijection and has the property that $\operatorname{maj}(w) = \operatorname{inv}(\Phi(w))$ for all $w \in S_n$. Let $\mathbb{P}$ denote the set of positive integers. Given $a \in \mathbb{P}$, and a word $w = w_1, w_2, ..., w_n$ consisting of distinct positive integers different from a, the factorization of w induced by a is

$$w = \alpha_1 \cdot \alpha_2 \cdots \alpha_k, \text{ where}$$

1) if $w_n < a$, then the last letter of each α_i is less than a and all other letters of α_i are greater than a,

2) if $w_n > a$, then the last letter of each α_i is greater than a and all other letters of α_i are less than a.

[1] It is also common to define the Eulerian polynomials to be $tA_n(t)$

Given a word α, let $\tilde{\alpha}$ be the word obtained from α by moving the last letter to the front. Let

$$F_a(w) = \tilde{\alpha_1} \cdot \tilde{\alpha_2} \cdots \tilde{\alpha_k},$$

where $w = \alpha_1 \cdot \alpha_2 \cdots \alpha_k$ is the factorization of w induced by a.

Then Φ is defined recursively by

$$\Phi(w_1) = w_1,$$

$$\Phi(w_1, w_2, ..., w_n) = F_{w_n}(\Phi(w_1, w_2, ..., w_{n-1})) \cdot w_n.$$

For example, consider $w = 3, 6, 4, 1, 2, 5 \in S_6$.

$$\Phi(3) = 3$$

$$\Phi(3, 6) = F_6(3) \cdot 6 = 3, 6$$

$$\Phi(3, 6, 4) = F_4(3, 6) = \widetilde{3, 6} \cdot 4 = 6, 3, 4$$

$$\Phi(3, 6, 4, 1) = F_1(6, 3, 4) = \tilde{6} \cdot \tilde{3} \cdot \tilde{4} \cdot 1 = 6, 3, 4, 1$$

$$\Phi(3, 6, 4, 1, 2) = F_2(6, 3, 4, 1) = \widetilde{6, 3, 4, 1} \cdot 2 = 1, 6, 3, 4, 2$$

$$\Phi(3, 6, 4, 1, 2, 5) = F_5(1, 6, 3, 4, 2) \cdot 5 = \tilde{1} \cdot \widetilde{6, 3} \cdot \tilde{4} \cdot \tilde{2} \cdot 5 = 1, 3, 6, 4, 2, 5.$$

And for this example we see that $\text{maj}(w) = 2 + 3 = 5$, and $\text{inv}(\Phi(w)) = 5$.

□

As was the case with descent number and excedance number, we note that in general inversion number is not equal to major index. For example if $\pi = 1, 3, 2$, then $\text{inv}(\pi) = 1$ and $\text{maj}(\pi) = 2$.

1.3 q-analogs of Eulerian Polynomials

The permutation statistics inv and maj are frequently used to obtain q-analogs of classic results. In general, a q-analog of an object has the property that by setting $q = 1$, one recovers the original object. Of course there are many q-analogs of any given object, and there are no objective criteria for what is considered to be a good q-analog. Here we define a few commonly used q-analogs.

Definition 1.3.1. Let $n \in \mathbb{N}$.

- The *q-analog of n*, denoted $[n]_q$, is defined by

$$[n]_q := \sum_{i=0}^{n-1} q^i.$$

- The *q-analog of $n!$*, denoted $[n]_q!$, is defined by

$$[n]_q! := \prod_{i=1}^{n} [i]_q.$$

- We define *two q-analogs of the exponential function.* The first is denoted $\exp_q$ and is defined by

$$\exp_q(z) := \sum_{i \geq 0} \frac{z^i}{[i]_q!}.$$

- The second is denoted Exp_q and is defined by

$$\mathrm{Exp}_q(z) := \sum_{i \geq 0} \frac{q^{\binom{i}{2}} z^i}{[i]_q!},$$

  where $\binom{n}{k}$ is the binomial coefficient.

We note that for $n \geq 1$, $[n]_q = (1 - q^n)/(1 - q)$. Thus

$$[n]_q! = [1]_q[2]_q \cdots [n]_q = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q)\cdots(1-q)} = \frac{(q;q)_n}{(1-q)^n},$$

where for $n \in \mathbb{N}$ we define $(a;q)_n$ by

$$(a;q)_n := \prod_{i=0}^{n-1}(1-aq^i).$$

For $n=0$, we also have $[0]_q! = (q;q)_0/(1-q)^0 = 1$.

Our first example of using permutation statistics to obtain q-analogs was first proved by Rodrigues [28], it is a formula for q-counting permutations according to inversion number.

Proposition 1.3.2. *For $n \in \mathbb{N}$ we have*

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = [n]_q!.$$

Proof. Given $\pi \in S_n$, let $a_i = |\{(j,i) : (j,i) \text{ is an inversion of } \pi\}|$, and let $I(\pi)$ be the inversion table of π defined by

$$I(\pi) := (a_1, a_2, ..., a_n) \in T_n,$$

where $T_n := [0, n-1] \times [0, n-2] \times ... \times [0, 0]$. It follows that

$$\mathrm{inv}(\pi) = \sum_{i=1}^{n} a_i.$$

Next we show that the map $I : S_n \to T_n$ which maps π to $I(\pi)$ is a bijection. It suffices to show this map is surjective, so suppose we are given $(a_1, a_2, ..., a_n) \in T_n$. We construct a permutation π in one-line notation by starting with the empty word and successively inserting $n, n-1, ..., 1$. If $n, n-1, ..., i+1$ have already been inserted, then we insert i immediately after the letter in position a_i, so that there are a_i letters to the left of i that are larger than i. It follows that the resulting permutation $\pi \in S_n$

satisfies $I(\pi) = (a_1, a_2, ..., a_n)$. Then

$$\sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{(a_1,\ldots,a_n) \in T_n} q^{a_1+\ldots+a_n} = \prod_{i=1}^{n} \left(\sum_{a_i=0}^{n-i} q^{a_i} \right) = [n]_q!.$$

□

We note that MacMahon's [19] proof of the fact the inv and maj are equidistributed follows from the formula

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [n]_q!,$$

which he proved after Rodrigues had obtained the result given in Proposition 1.3.2.

We have seen that inversion number and major index both provide us with nice q-analogs of the formula for the number of permutations in S_n. We have also seen that the Eulerian polynomials can be expressed using descent number or excedance number. By pairing either des or exc, with either maj or inv (i.e. $(\mathrm{des}, \mathrm{maj}), (\mathrm{des}, \mathrm{inv}), (\mathrm{exc}, \mathrm{maj})$, or $(\mathrm{exc}, \mathrm{inv})$), one obtains a q-analog of the Eulerian polynomials. We remark that these bivariate distributions are all different for $n \geq 5$. The first result on these distributions was obtained by Carlitz, it pairs des and maj.

Theorem 1.3.3 ([5]). *For $n \in \mathbb{N}$ we have*

$$\frac{\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}(\pi)}}{(p; q)_{n+1}} = \sum_{k \geq 0} [k+1]_q^n p^k.$$

One can also consider q-analogs of the generating function formula for the Eulerian polynomials given in Equation (1.1). One such q-analog due to Stanley pairs inv with des.

Theorem 1.3.4 ([33]). *We have*

$$\sum_{\substack{n\geq 0\\ \pi\in S_n}} q^{\mathrm{inv}(\pi)}t^{\mathrm{des}(\pi)}\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1))-t}.$$

A different q-analog of Equation (1.1) was discovered by Shareshian and Wachs. It pairs maj and exc and also enumerates fixed points.

Theorem 1.3.5 ([32, Corollary 1.3]). *We have*

$$\sum_{\substack{n\geq 0\\ \pi\in S_n}} q^{\mathrm{maj}(\pi)}t^{\mathrm{exc}(\pi)}r^{\mathrm{fix}(\pi)}\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(tqz)-tq\exp_q(z)}.$$

Each of the above mentioned q-analogs of Eulerian polynomials was obtained by very different methods. Our main results are obtained by non-trivially extending the methods used by Shareshian and Wachs in obtaining Theorem 1.3.5. Their methods include nontrivial extensions of methods involving symmetric and quasisymmetric functions that were developed by Gessel and Reutenauer [15] in order to enumerate permutations by descent set and cycle type.

Foata and Han obtain an extension of Theorem 1.3.5, which can also be obtained using the above mentioned techniques of Shareshian and Wachs.

Theorem 1.3.6 ([11]). *We have*

$$\sum_{\substack{n\geq 0\\ \pi\in S_n}} q^{\mathrm{maj}(\pi)}p^{\mathrm{des}(\pi)}t^{\mathrm{exc}(\pi)}r^{\mathrm{fix}(\pi)}\frac{z^n}{(p;q)_{n+1}} = \sum_{l\geq 0}\frac{p^l(1-qt)(z;q)_l(ztq;q)_l}{(zr;q)_{l+1}\left[(z;q)_l-tq(ztq;q)_l\right]}.$$

1.4 Symmetric Functions

As mentioned above, symmetric functions will play an important part in the proofs of our results. While the study of symmetric functions is quite rich, we cover just a few of the basic results that we will need. For a more thorough treatment see [34] and [29]. We will consider symmetric functions over the ring $\mathbb{Q}$.

Definition 1.4.1. Let $\mathbf{x}$ denote the infinite set of variables $\{x_1, x_2, ...\}$. For $n \in \mathbb{N}$, we call α a *weak composition* of n if $\alpha = (\alpha_1, \alpha_2, ...)$ is an infinite sequence of nonnegative integers such that $\sum \alpha_i = n$. Let x^α denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2}....$ A formal power series is called *homogeneous of degree* n if it has the form

$$f(\mathbf{x}) = \sum_\alpha c_\alpha x^\alpha,$$

where the sum is over all weak compositions of n and $c_\alpha \in \mathbb{Q}$.

A formal power series $f(\mathbf{x})$ over the ring $\mathbb{Q}$ is called a *symmetric function* if

$$f(x_1, x_2, ...) = f(x_{\sigma(1)}, x_{\sigma(2)}, ...)$$

for every permutation σ of the positive integers.

We let Sym^n denote the set of all homogeneous symmetric function of degree n. Since the product of two symmetric functions is symmetric, we define the *algebra of symmetric functions*, denoted Sym, by

$$\mathrm{Sym} := \mathrm{Sym}^0 \oplus \mathrm{Sym}^1 \oplus \mathrm{Sym}^2 \oplus ....$$

Next we examine various bases for Sym. In each case will we require the notion of a *partition* of n. We call λ a partition of n and write $\lambda \vdash n$ if $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k)$ is a weakly decreasing sequence of positive integers such that $\sum \lambda_i = n$. We also consider the sequence consisting of a single zero to be the only partition of zero. Note that for weak compositions of $n \in \mathbb{N}$, we allowed entries to be zero and we did not require the sequence to be weakly decreasing. We let $\mathrm{Par}(n)$ denote the set of all partition of n, and $\mathrm{Par} = \bigcup_{n\geq 0} \mathrm{Par}(n)$.

Definition 1.4.2. For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \vdash n$, let $(\lambda_1, \lambda_2, ...)$ denote the weak composition obtained from λ by extending it by zeros. We define the *monomial symmetric function*, denoted m_λ, by

$$m_\lambda = m_\lambda(\mathbf{x}) := \sum_{\alpha} x^\alpha \in \mathrm{Sym}^n,$$

where the sum is over all weak compositions α which can be obtained from $(\lambda_1, \lambda_2, ...)$ by a distinct permutation of the entries of $(\lambda_1, \lambda_2, ...)$.

For example

$$m_{(0)} = 1,$$

$$m_{(1)} = \sum_{i\geq 1} x_i,$$

$$m_{(1,1)} = \sum_{i>j\geq 1} x_i x_j,$$

$$m_{(2)} = \sum_{i\geq 1} x_i^2,$$

$$m_{(2,1)} = \sum_{i,j\geq 1, i\neq j} x_i^2 x_j.$$

From these definitions, it is clear that $f \in \mathrm{Sym}^n$ if and only if $f = \sum_{\lambda\vdash n} c_\lambda m_\lambda$ where $c_\lambda \in \mathbb{Q}$. Thus we have the following immediate proposition.

Proposition 1.4.3. *The set* $\{m_\lambda : \lambda \in \mathrm{Par}\}$ *is a basis for* Sym.

The next two bases we discuss are the elementary symmetric functions, and the complete homogeneous symmetric functions. We will make frequent use of them.

Definition 1.4.4. For $n \in \mathbb{N}$, let 1^n denote the partition consisting of n ones. First we define e_n by

$$e_n = e_n(\mathbf{x}) := m_{1^n} = \sum_{i_1 > i_2 > \ldots > i_n \geq 1} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \in \mathrm{Par}$, the *elementary symmetric function*, denoted e_λ, is defined by

$$e_\lambda = e_\lambda(\mathbf{x}) := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}.$$

Next, for $n \in \mathbb{N}$ we define h_n by

$$h_n = h_n(\mathbf{x}) = \sum_{\lambda \vdash n} m_\lambda = \sum_{i_1 \geq i_2 \geq \ldots \geq i_n \geq 1} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k) \in \mathrm{Par}$, the *complete homogeneous symmetric function*, denoted h_λ, is defined by

$$h_\lambda = h_\lambda(\mathbf{x}) := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_k}.$$

While it is clear that the elementary and the complete homogeneous symmetric functions are indeed symmetric, it is less obvious that they are bases for Sym. We refer the reader to [34] for a proof of the following proposition.

Proposition 1.4.5. *The set* $\{e_\lambda : \lambda \in \mathrm{Par}\}$*, and the set* $\{h_\lambda : \lambda \in \mathrm{Par}\}$ *are both bases for* Sym.

Moreover, there is an interesting relationship between these two bases. To see this, we first note that they are multiplicative bases. Thus we may define an algebra endomorphism $\omega : \mathrm{Sym} \to \mathrm{Sym}$ by setting $\omega(e_n) = h_n$ for all $n \in \mathbb{N}$, and then extending linearly and multiplicatively. Hence for any $\lambda \in \mathrm{Par}$ we have $\omega(e_\lambda) = h_\lambda$. The interesting fact here is that $\omega^2(e_\lambda) = e_\lambda$ (see [34]).

Proposition 1.4.6. *The algebra endomorphism* ω *described above is an involution, i.e.* ω^2 *is the identity map. Equivalently* $\omega(h_n) = e_n$*.*

Proof. First consider the following formal power series

$$H(z) = \sum_{n\geq 0} h_n z^n$$

and

$$E(z) = \sum_{n\geq 0} e_n z^n.$$

In addition to their role in this proof, we will see these formal power series throughout this paper.

It is clear that

$$E(z) = \prod_{n\geq 1} (1 + x_n z).$$

It is also clear that

$$H(z) = \prod_{n\geq 1} (1 - x_n z)^{-1},$$

if we recognize that

$$(1 - x_n z)^{-1} = \sum_{i\geq 0} (x_n z)^i.$$

It follows that

$$H(z)E(-z) = \left(\sum_{n\geq 0} h_n z^n\right)\left(\sum_{n\geq 0} e_n (-z)^n\right) = 1, \tag{1.3}$$

thus for $n \geq 1$ we have

$$\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0. \tag{1.4}$$

Conversely, suppose $\sum_{i=0}^{n} (-1)^i u_i h_{n-i} = 0$ for all $n \geq 1$ for some collection of $u_i \in \mathrm{Sym}$ with $u_0 = 1$. Then by induction this implies $u_i = e_i$ for all i. Now we apply ω to Equation (1.4) to obtain

$$0 = \sum_{i=0}^{n} (-1)^i h_i \omega(h_{n-i}) = (-1)^n \sum_{i=0}^{n} (-1)^i h_{n-i} \omega(h_i),$$

where the last equality is obtained by replacing i with $n-i$. It follows that $\omega(h_i) = e_i$ as desired.

□

The interested reader should see [34] and [29] for more on symmetric functions. In particular, the Schur symmetric functions are arguably the most interesting basis for Sym. There are several different definitions of the Schur symmetric functions, and it is not obvious they are equivalent. Moreover, they play a central role in the connection between symmetric function theory and representation theory.

1.5 Quasisymmetric Functions

In addition to our use of symmetric functions, quasisymmetric functions are also prominently featured in this work (as the title suggests). The study of quasisymmetric functions began with the work of Gessel and Stanley. In the subsequent section we provide an example of how quasisymmetric functions may be used in permutation enumeration. Informally, a quasisymmetric function is a formal power series which is invariant under shifts of the indices, which is a weaker condition than that of a symmetric function.

Definition 1.5.1. A formal power series $f(\mathbf{x})$ with coefficients in $\mathbb{Q}$ is called *quasisymmetric* if for any sequence of positive integers $(\alpha_1, \alpha_2, ..., \alpha_k)$, the coefficient of

$$x_{j_1}^{\alpha_1} x_{j_2}^{\alpha_2} \cdots x_{j_k}^{\alpha_k}$$

in $f(\mathbf{x})$, equals the coefficient of

$$x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

in $f(\mathbf{x})$ whenever $j_1 > j_2 > ... > j_k$ and $i_1 > i_2 > ... > i_k$. For $n \in \mathbb{N}$, let QSym^n denote the set of all homogeneous quasisymmetric functions of degree n. Since the product of two quasisymmetric functions is also quasisymmetric, we define the *algebra of quasisymmetric functions*, denoted QSym, by

$$\mathrm{QSym} = \mathrm{QSym}^0 \oplus \mathrm{QSym}^1 \oplus \mathrm{QSym}^2 \oplus ....$$

The most obvious basis for QSym consists of the monomial quasisymmetric functions, which we define now.

Definition 1.5.2. Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ be a finite sequence of positive integers. If $\sum \alpha_i = n$ we call α a *composition* of n. For any sequence α of positive integers, the *monomial quasisymmetric function*, denoted M_α, is defined by

$$M_\alpha = M_\alpha(\mathbf{x}) := \sum_{i_1 > i_2 > ... > i_k \geq 1} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

We also define the sequence consisting of a single zero to be the only composition of zero, so that $M_{(0)} := 1$.

For example, $\alpha = (2, 1)$ is a composition of 3 and

$$M_{(2,1)} = \sum_{i > j \geq 1} x_i^2 x_j.$$

The coefficient of $x_2^2 x_1$ in $M_{(2,1)}$ is one, and the coefficient of $x_1^2 x_2$ in $M_{(2,1)}$ is zero, hence $M_{(2,1)}$ is quasisymmetric but not symmetric.

Clearly, the set $\{M_\alpha : \alpha \text{ is a composition of } n\}$ forms a basis of QSym^n. We will be more concerned with another basis, namely the fundamental quasisymmetric functions of Gessel.

Definition 1.5.3. For $n \in \mathbb{N}$ and a subset $T \subseteq [n-1]$, we define the *fundamental* *quasisymmetric function*, denoted $F_{T,n}$, by

$$F_{T,n} = F_{T,n}(\mathbf{x}) := \sum_{\substack{i_1 \geq i_2 \geq \ldots \geq i_n \geq 1 \\ i_j > i_{j+1} \text{ if } j \in T}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

We also define $F_{\emptyset,0} := 1$.

One thinks of the set T as telling us which inequalities of the indices must be strict. For example

$$F_{\{2\},3} = \sum_{i \geq j > k \geq 1} x_i x_j x_k.$$

Using the principle of inclusion-exclusion, one can prove that each monomial quasisymmetric function of degree n can be expressed as a linear combination of fundamental quasisymmetric functions of degree n, and vice versa. Hence the set $\{F_{T,n} : T \subseteq [n-1]\}$ is a basis for QSym^n (see [34]).

We conclude this section by noting the following identities

$$F_{\emptyset,n} = h_n \quad \text{and} \quad F_{[n-1],n} = e_n. \tag{1.5}$$

1.6 Necklaces and Ornaments

Quasisymmetric functions have been useful in the study of permutation enumeration. In this section, we examine how Gessel and Reutenauer used quasisymmetric functions in enumerating permutations of a given descent set and cycle type [15]. As mentioned previously, techniques introduced here will be extended and play an important role in proving further results.

The idea starts with encoding the descent set (see Definition 1.2.1) of a permutation as a fundamental quasisymmetric function. Indeed, given $\pi \in S_n$ we have

$\mathrm{DES}(\pi) \subseteq [n-1]$, and we associated a fundamental quasisymmetric function with π by defining

$$F_\pi = F_\pi(\mathbf{x}) := F_{\mathrm{DES}(\pi),n}(\mathbf{x}).$$

For example $\pi = 3, 2, 4, 1 \in S_4$, $\mathrm{DES}(\pi) = \{1, 3\}$, and

$$F_\pi = F_{\{1,3\},4} = \sum_{i_1 > i_2 \geq i_3 > i_4 \geq 1} x_{i_1} x_{i_2} x_{i_3} x_{i_4}.$$

As mentioned in the beginning of this chapter, a permutation $\pi \in S_n$ can be written as a product of disjoint cycles. The lengths of these cycles are unique, and form a partition λ of n which we call the *cycle type* of a permutation, and write $\lambda(\pi) = \lambda$. So for example if $\pi = (3, 1, 5)(2)(4, 6, 7)$, then $\lambda(\pi) = (3, 3, 1)$. One significance of cycle type is that for $\pi, \sigma \in S_n$, π is conjugate to σ (i.e. $\pi = \tau\sigma\tau^{-1}$ for some $\tau \in S_n$) if and only if $\lambda(\pi) = \lambda(\sigma)$.

To enumerate the permutations of a given descent set and cycle type, Gessel and Reutenauer [15] consider the sum

$$L_\lambda = L_\lambda(\mathbf{x}) := \sum_{\substack{\pi \in S_n \\ \lambda(\pi) = \lambda}} F_\pi(\mathbf{x}). \tag{1.6}$$

They show that L_λ is actually a symmetric function, and has a nice combinatorial description in terms of necklaces and ornaments, which we define now.

Definition 1.6.1. Consider the alphabet $\mathbb{P}$ of positive integers. Recall that a word of length n over $\mathbb{P}$ is a sequence of n letters, not necessarily distinct. The cyclic group of order n acts on the set of words of length n by cyclic rotation. So if z is a generator of this cyclic group and $v = v_1, v_2, ..., v_n$ is a word, then $z \cdot v = v_2, v_3, ..., v_n, v_1$. A *circular word*, denoted (v), is the orbit of v under this action. The length of (v) is just the length of v. A circular word (v) is called *primitive* if the size of the orbit is equal to the length of the word v. Equivalently, (v) is not primitive if v is a proper power of another word. For example the circular word $(1, 2, 1)$ is primitive, while $(1, 2, 1, 2)$ is not primitive since $1, 2, 1, 2 = u^2$ where $u = 1, 2$. We call a primitive circular word (v) over $\mathbb{P}$ a *necklace*. One can visualize (v) as a circular arrangement of letters obtained from v by attaching the first and last letters together. For each position of this necklace one can read the letters in a clockwise direction to obtain an element from the orbit of the circular action on v.

For example, consider the word $v = 2, 2, 4, 1, 3$, and corresponding necklace $(v) = (2, 2, 4, 1, 3)$. The orbit of v under the action of cyclic rotation is

$$\begin{aligned} v &= 2, 2, 4, 1, 3 \\ z \cdot v &= 2, 4, 1, 3, 2 \\ z^2 \cdot v &= 4, 1, 3, 2, 2 \\ z^3 \cdot v &= 1, 3, 2, 2, 4 \\ z^4 \cdot v &= 3, 2, 2, 4, 1 \end{aligned}$$

Alternatively, we can label a position y on (v) and consider a finite word u_y of length 5 obtained by reading the letters of (v) cyclically starting at position y, i.e.

$$\begin{matrix} (2, & 2, & 4, & 1, & 3) \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{matrix}$$

With this labeling, we see that $u_{y_i} = z^i \cdot v$ for $i = 0, 1, 2, 3, 4$.

An *ornament* R is a multiset of necklaces. Formally, R is map from the set of necklaces to $\mathbb{N}$ with finite support, i.e. $R((v))$ is the multiplicity of the necklace (v) in the multiset R. By arranging the lengths of the necklaces in R in weakly decreasing order, we obtain a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_k)$ and we say the ornament R has *cycle type* λ, and write $\lambda(R) = \lambda$. The set of all ornaments of cycle type λ is denoted $\mathcal{R}(\lambda)$.

Given a necklace (v) where $v = v_1, v_2, ..., v_n$, the *weight* of a necklace, denoted wt((v)), is the monomial

$$\mathrm{wt}((v)) := x_{v_1}x_{v_2}\cdots x_{v_n}.$$

Let η denote the set of all necklaces. The weight of an ornament R is denoted wt(R), and is defined by

$$\mathrm{wt}(R) := \prod_{(v)\in\eta} \mathrm{wt}((v))^{R((v))}.$$

For example, if

$$R = \{(1,4,1),(1,4,1),(3,2,1,6)\},$$

then

$$\mathrm{wt}(R) = x_1x_4x_1x_1x_4x_1x_3x_2x_1x_6 = x_1^5x_2x_3x_4^2x_6.$$

A key result of [15] is the following:

Theorem 1.6.2 ([15])**.** *For $n \in \mathbb{P}$ and $\lambda \vdash n$, we have*

$$L_\lambda = \sum_{R\in\mathcal{R}(\lambda)} \mathrm{wt}(R).$$

Proof. We give a sketch of the proof by describing the bijection between the monomials of the left and right hand sides above. Let $s = (s_1 \geq s_2 \geq ... \geq s_n)$ be a weakly decreasing sequence of positive integers. Given $\pi \in S_n$, we say that s is DES(π)-*compatible* if $i \in \mathrm{DES}(\pi)$ implies that $s_i > s_{i+1}$. Let Com(λ) denote the set of all pairs (π, s) such that $\lambda(\pi) = \lambda$ and s is DES(π)-compatible. Define the *weight* of the pair (π, s), denoted wt((π, s)), to be the monomial

$$\mathrm{wt}((\pi,s)) := x_{s_1}x_{s_2}\cdots x_{s_n}.$$

It follows that

$$L_\lambda = \sum_{(\pi,s)\in\mathrm{Com}(\lambda)} \mathrm{wt}((\pi,s)).$$

Next we describe a weight preserving map $f : \mathrm{Com}(\lambda) \to \mathcal{R}(\lambda)$. Given $(\pi, s) \in \mathrm{Com}(\lambda)$, first write π in cycle form. For each cycle $(i_i, i_2, ..., i_k)$ appearing in π, include the necklace $(s_{i_1}, s_{i_2}, ..., s_{i_k})$ in the multiset $f((\pi, s))$. If the necklace $(s_{i_1}, s_{i_2}, ..., s_{i_k})$ already belongs to $f((\pi, s))$, then increase its multiplicity by one. We note here that Gessel and Reutenauer prove that $(s_{i_1}, s_{i_2}, ..., s_{i_k})$ is primitive, which is not obvious.

For example if $\pi = 5, 6, 7, 3, 1, 2, 8, 4 = (1, 5)(2, 6)(3, 7, 8, 4)$, then $s = 6, 6, 6, 4, 3, 3, 3, 2$ is DES(π)-compatible since $\mathrm{DES}(\pi) = \{3, 4, 7\}$. Thus $(\pi, s) \in \mathrm{Com}(\lambda)$ where $\lambda = (4, 2, 2)$. Then

$$f((\pi, s)) = \{(6, 3), (6, 3), (6, 3, 2, 4)\} \in \mathcal{R}(\lambda). \tag{1.7}$$

Given $R \in \mathcal{R}(\lambda)$, we describe $f^{-1}(R)$. If R has repeated necklaces, then place some linear order on each set of repeated necklaces. Next we want to linearly order the positions on all necklaces in R. Given a position x, let w_x denote the infinite word obtained by reading letters in that necklace clockwise starting from that position. Given two words u, v, we say that u is *lexicographically larger* than v if for some positive integer k we have $u_i = v_i$ for $i < k$, and $u_k > v_k$. Given two positions x, y in our ornament R, we say $x > y$ if w_x is lexicographically larger than w_y, or $w_x = w_y$ and x is in a larger repeated necklace than y. This linearly orders all of the positions. A permutation π in cycle form is obtained by replacing the letter in position x with i, if x is the i^{th} largest position. We let s be the weakly decreasing rearrangement of all the letters appearing in R, and $f^{-1}(R) = (\pi, s)$.

For example consider the following ornament R with positions labeled $y_1, y_2, ..., y_8$. Note we must choose a linear order on the repeated necklaces.

$$\begin{array}{ccccccccccccc} R & = & (6, & 3) & < & (6, & 3) & , & (6, & 3, & 2, & 4) \\ & & y_1 & y_2 & & y_3 & y_4 & & y_5 & y_6 & y_7 & y_8 \end{array}$$

Then the word at each position is the following:

$$
\begin{aligned}
w_{y_1} &= 6, 3, 6, 3, 6, 3, ... \\
w_{y_2} &= 3, 6, 3, 6, 3, 6, ... \\
w_{y_3} &= 6, 3, 6, 3, 6, 3, ... \\
w_{y_4} &= 3, 6, 3, 6, 3, 6, ... \\
w_{y_5} &= 6, 3, 2, 4, 6, 3, 2, 4, ... \\
w_{y_6} &= 3, 2, 4, 6, 3, 2, 4, 6, ... \\
w_{y_7} &= 2, 4, 6, 3, 2, 4, 6, 3, ... \\
w_{y_8} &= 4, 6, 3, 2, 4, 6, 3, 2, ...
\end{aligned}
$$

The lexicographically largest word occurs at positions y_1, y_3. By the order we chose on necklaces, y_3 is the largest position, followed by y_1. So we place a 1 in position y_3, and a 2 in position y_1. The next largest word is in position y_5, so we place a 3 in position y_5. Continuing in this way, we obtain a permutation π in cycle form,

$$
\begin{array}{ccccccccccccccc}
R & = & (6, & 3) & < & (6, & 3) & , & (6, & 3, & 2, & 4,) \\
\pi & = & (2, & 6) & & (1, & 5) & & (3, & 7, & 8, & 4)
\end{array}
$$

and a sequence $s = 6, 6, 6, 4, 3, 3, 3, 2$. By comparison with (1.7), we see $f^{-1}f((\pi, s)) = (\pi, s)$ for this example. Of course a rigorous proof would involve checking that these maps are well-defined, weight preserving, and inverses. We refer the reader to [15].

□

Gessel and Reutenauer use this result to express the number of permutations of a given descent set and cycle type in terms of a certain scalar product of symmetric functions (or equivalently, a scalar product of characters of S_n representations). We will not discuss this result, instead we examine how they enumerate various subsets of S_n according to descent number and major index. The main tools used here are the stable and nonstable principal specializations.

Definition 1.6.3. The *stable principal specialization*, denoted $\mathbf{ps}$, is the ring homomorphism $\mathbf{ps} : \mathrm{QSym} \to \mathbb{Q}[q]$ (where $\mathbb{Q}[q]$ denotes the ring of formal power series in variable q over $\mathbb{Q}$) defined by

$$\mathbf{ps}(x_i) = q^{i-1} \text{ for all } i.$$

For $l \in \mathbb{N}$, the *principal specialization*, denoted $\mathbf{ps}_l$, is the ring homomorphism $\mathbf{ps}_l : \mathrm{QSym} \to \mathbb{Q}[q]$ defined by

$$\mathbf{ps}_l(x_i) = \begin{cases} q^{i-1} & \text{if } 1 \leq i \leq l \\ 0 & \text{if } i > l \end{cases}.$$

Lemma 1.6.4 ([15, Lemma 5.2])**.** *For $n \in \mathbb{N}$ we have*

$$\mathbf{ps}(F_{T,n}(\boldsymbol{x})) = \frac{q^{\sum_{i \in T} i}}{(q;q)_n},$$

and

$$\sum_{l \geq 0} \mathbf{ps}_l(F_{T,n}(\boldsymbol{x}))p^l = \frac{p^{|T|+1} q^{\sum_{i \in T} i}}{(p;q)_{n+1}}.$$

The proof of this very useful lemma is obtained by rewriting the indices of summation on the left hand side in such a way that allows one to compute the sum using geometric series. To apply this lemma, consider any subset $A \subseteq S_n$ of permutations, and define

$$L_A = L_A(\mathbf{x}) := \sum_{\pi \in A} F_\pi(\mathbf{x}) = \sum_{\pi \in A} F_{\mathrm{DES}(\pi),n}(\mathbf{x}).$$

Recall from Definition 1.2.1 that $|\mathrm{DES}(\pi)| = \mathrm{des}(\pi)$ and $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{DES}(\pi)} i$. Using Lemma 1.6.4 the following theorem is obtained.

Theorem 1.6.5 ([15, Theorem 5.3])**.** *For $n \in \mathbb{N}$ and $A \subseteq S_n$ we have*

$$\mathbf{ps}(L_A) = \frac{\sum_{\pi \in A} q^{\mathrm{maj}(\pi)}}{(q;q)_n},$$

and

$$\sum_{l \geq 0} p^l \mathbf{ps}_l(L_A) = \frac{\sum_{\pi \in A} p^{\mathrm{des}(\pi)+1} q^{\mathrm{maj}(\pi)}}{(p;q)_{n+1}}.$$

By appropriate choice of A, this theorem is used by Gessel and Reutenauer [15] to study the distribution of des and maj among involutions, derangements, and cyclic permutations.

1.7 Bicolored Necklaces and Ornaments

Recall Theorem 1.3.5 which is due to Shareshian and Wachs:

$$\sum_{\substack{n \geq 0 \\ \pi \in S_n}} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)} r^{\mathrm{fix}(\pi)} \frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(tqz) - tq\exp_q(z)},$$

where $\exp_q(z) := \sum_{i \geq 0} z^i/[i]_q!$. In this section we outline the proof of this theorem. The proof uses nontrivial extensions of the techniques outlined in the previous section. In Chapters 4,5,6, we non-trivially extend these techniques further to prove our results.

The first objective is to associate a fundamental quasisymmetric function to each permutation in such a way that applying principal specializations gets the excedance number involved. In the previous section the descent set was used, here a different set will be used.

Definition 1.7.1. For $n \geq 1$, let $[\tilde{n}]$ denote the linearly ordered alphabet

$$[\tilde{n}] := \{\tilde{1} < \tilde{2} < ... < \tilde{n} < 1 < 2 < ... < n\}.$$

For $\pi \in S_n$, define $\tilde{\pi}$ to be the word over $[\tilde{n}]$ obtained from π by replacing π_i with $\tilde{\pi_i}$ whenever $i \in \mathrm{EXC}(\pi)$ (i.e., $\pi(i) > i$, see Definition 1.2.1). Then we define

$$\mathrm{DEX}(\pi) := \mathrm{DES}(\tilde{\pi}) \subseteq [n-1],$$

where the descent set of any word over an ordered alphabet consists of all i such that $w_i > w_{i+1}$. Also define $\mathrm{DEX}(\theta) := 0$ where θ denotes the empty word.

For example if $\pi = 3, 4, 1, 7, 6, 5, 2 \in S_7$, then $\mathrm{EXC}(\pi) = \{1, 2, 4, 5\}$, $\tilde{\pi} = \tilde{3}, \tilde{4}, 1, \tilde{7}, \tilde{6}, 5, 2$, and $\mathrm{DEX}(\pi) = \{3, 4, 6\}$. The motivation for this definition is seen by the following lemma.

Lemma 1.7.2 ([32, Lemma 2.2])**.** *For all $\pi \in S_n$ we have*

$$\sum_{i \in \mathrm{DEX}(\pi)} i = \mathrm{maj}(\pi) - \mathrm{exc}(\pi),$$

and

$$|\mathrm{DEX}(\pi)| = \begin{cases} \mathrm{des}(\pi) & \text{if } \pi(1) = 1 \\ \mathrm{des}(\pi) - 1 & \text{if } \pi(1) \neq 1 \end{cases}.$$

Using DEX to associate fundamental quasisymmetric functions to permutations, we proceed with the following definition.

Definition 1.7.3. For $n, j, k \geq 0$, and $\lambda \vdash n$, define the following subsets of S_n

$$W_{n,j,k} = \{\pi \in S_n : \mathrm{exc}(\pi) = j, \mathrm{fix}(\pi) = k\},$$

$$W_{\lambda,j} = \{\pi \in S_n : \lambda(\pi) = \lambda, \mathrm{exc}(\pi) = j\}.$$

The *fixed point Eulerian quasisymmetric functions*, denoted $Q_{n,j,k}$, are defined by

$$Q_{n,j,k} = Q_{n,j,k}(\mathbf{x}) := \sum_{\pi \in W_{n,j,k}} F_{\mathrm{DEX}(\pi),n}(\mathbf{x}).$$

The *cycle type Eulerian quasisymmetric functions*, denoted $Q_{\lambda,j}$, are defined by

$$Q_{\lambda,j} = Q_{\lambda,j}(\mathbf{x}) := \sum_{\pi \in W_{\lambda,j}} F_{\mathrm{DEX}(\pi),n}(\mathbf{x}).$$

The goal now is to obtain a generating function formula for

$$\sum_{n,j,k \geq 0} Q_{n,j,k} t^j r^k z^n.$$

Once such a formula is obtained, one can apply specializations to enumerate permutations according to $\mathrm{exc}, \mathrm{maj}, \mathrm{fix}$, and des. In order to find this formula, we discuss a combinatorial description of $Q_{\lambda,j}$. This description is inspired by the necklaces and ornaments of Gessel and Reutenauer [15] (see Definition 1.6.1).

Definition 1.7.4. Define a linearly order alphabet

$$\mathcal{D} := 1 < \overline{1} < 2 < \overline{2} < 3 < \overline{3} < \ldots$$

Given a letter $a \in \mathcal{D}$, let $|a|$ denote the positive integer obtained by removing the bar from a if there is one. A *bicolored necklace* is a primitive circular word (v) over $\mathcal{D}$ such that

1. every barred letter is followed (clockwise, or cyclically) by a letter of lesser or equal absolute value,
2. every unbarred letter is followed by a letter of greater or equal absolute value,
3. necklaces of length one may not consist of a single barred letter.

For example $(1, \overline{1}, 1)$ is a bicolored necklace, and so is $(2, \overline{4}, \overline{4}, 4, \overline{4}, 1, 1)$. The following three circular words are not bicolored necklaces: $(1,\overline{2},1,\overline{2}),(1,3,\overline{2},3),(\overline{1})$.

A *bicolored ornament* is a multiset of bicolored necklaces. By arranging the lengths of the bicolored necklaces in R in weakly decreasing order, we obtain a partition $\lambda=(\lambda_1,\lambda_2,...,\lambda_k)$ and we say the bicolored ornament R has *cycle type* λ, and write $\lambda(R)=\lambda$. The set of all bicolored ornaments of cycle type λ with exactly j barred letters is denoted $\mathcal{R}(\lambda,j)$.

Given a bicolored necklace (v) of length n, the *weight* of (v), denoted $\mathrm{wt}((v))$, is defined to be the monomial

$$\mathrm{wt}((v)) := x_{|v_1|}x_{|v_2|}\cdots x_{|v_n|}.$$

Let η denote the set of all bicolored necklaces. The weight of a bicolored ornament is defined to be

$$\mathrm{wt}(R) := \prod_{(v)\in\eta} \mathrm{wt}((v))^{R((v))}.$$

Theorem 1.7.5 ([32, Corollary 3.3]). *For $n\in\mathbb{P}$, $j\in\mathbb{N}$, and $\lambda\vdash n$ we have*

$$Q_{\lambda,j} = \sum_{R\in\mathcal{R}(\lambda,j)} \mathrm{wt}(R).$$

Proof. We give a sketch of the proof by discussing how one modifies the bijection of Gessel and Reutenauer described in the proof of Theorem 1.6.2. Given $\pi\in S_n$, we call a weakly decreasing sequence $s=(s_1\geq s_2\geq ...\geq s_n)$ of positive integers $\mathrm{DEX}(\pi)$-*compatible* if $i\in\mathrm{DEX}(\pi)$ implies $s_i>s_{i+1}$. Given $\lambda\vdash n$, we define

$$\mathrm{Com}(\lambda,j) := \{(\pi,s) : \pi\in W_{\lambda,j} \text{ and } s \text{ is } \mathrm{DEX}(\pi)\text{-compatible}\}.$$

We define the *weight* of (π,s), denoted $\mathrm{wt}((\pi,s))$, to be the monomial

$$\mathrm{wt}((\pi,s)) := x_{s_1}x_{s_2}\cdots x_{s_n}.$$

Thus

$$Q_{\lambda,j} = \sum_{(\pi,s)\in \mathrm{Com}(\lambda,j)} \mathrm{wt}((\pi,s)).$$

As in the proof of Theorem 1.6.2, we construct a weight preserving map f : $\mathrm{Com}(\lambda,j) \to \mathcal{R}(\lambda,j)$. Given $(\pi,s) \in \mathrm{Com}(\lambda,j)$, let t be the sequence of letters obtained from s by replacing s_i with $\overline{s_i}$ whenever $i \in \mathrm{EXC}(\pi)$. Next, write π in cycle form. For each cycle $(i_i, i_2, ..., i_k)$ appearing in π, include the bicolored necklace $(t_{i_1}, t_{i_2}, ..., t_{i_k})$ in the multiset $f((\pi,s))$. If the necklace $(t_{i_1}, t_{i_2}, ..., t_{i_k})$ already belongs to $f((\pi,s))$, then increase its multiplicity by one. We note here that Shareshian and Wachs prove that $(t_{i_1}, t_{i_2}, ..., t_{i_k})$ is in fact a bicolored necklace.

For example, let $\pi = 4,5,1,3,2,8,6,7 = (1,4,3)(2,5)(6,8,7) \in W_{\lambda,j}$ where $\lambda = (3,3,2)$, and $j = 3$. Note that $\tilde{\pi} = \tilde{4},\tilde{5},1,3,2,\tilde{8},6,7$, and $\mathrm{DEX}(\pi) = \{4,5\}$, so $s = 5,4,4,4,3,2,2,2$ is $\mathrm{DEX}(\pi)$-compatible and $(\pi,s) \in \mathrm{Com}(\lambda,j)$. Then $t = \overline{5},\overline{4},4,4,3,\overline{2},2,2$, and

$$f((\pi,s)) = (\overline{5},4,4),(\overline{4},3),(\overline{2},2,2) \in \mathcal{R}(\lambda,j).$$

The inverse map is given by the same description appearing the proof of Theorem 1.6.2, except that the word at each position is over the alphabet $\mathcal{D}$. So we use the order on $\mathcal{D}$ to determine if the word at position x is lexicographically larger than the word at some position y. For example, if $R = (\overline{5},4,4),(\overline{4},3),(\overline{2},2,2)$ with positions $y_1, y_2, ..., y_8$ labeled consecutively, then clearly $w_{y_1} = \overline{5},4,4,\overline{5},4,4,..$ is the largest word and we place a 1 in position y_1. Since $\overline{4} > 4$, the next largest word is $w_{y_4} = \overline{4},3,\overline{4},3,...$, and we place a 2 in position y_4. If we continue this example, we see that we recover (π,s) above.

While the description of the bijection of Shareshian and Wachs is somewhat similar to that of Gessel and Reutenauer, there is much more work needed here to prove that f is in fact well-defined, and actually a bijection. For more details we refer the reader to [31], [32].

□

Shareshian and Wachs use this theorem to prove that the quasisymmetric function $Q_{\lambda,j}$ is actually symmetric. Another use that we discuss next, is to establish a recurrence for $Q_{n,j,k}$. In order to accomplish this, we introduce the bicolored banners of Shareshian and Wachs.

Definition 1.7.6. A *bicolored banner* is a word B over $\mathcal{D}$ of finite length such that

1. if $B(i)$ is barred then $|B(i)| \geq |B(i+1)|$,
2. if $B(i)$ is unbarred then $|B(i)| \leq |B(i+1)|$,
3. the last letter of B is unbarred.

The *weight* of a bicolored banner B of length n, denoted $\mathrm{wt}(B)$, is the monomial

$$\mathrm{wt}(B) = x_{|B(1)|}x_{|B(2)|}\cdots x_{|B(n)|}.$$

A *Lyndon word* over an ordered alphabet is a word that is strictly lexicographically larger that all its circular rearrangements. A *Lyndon factorization* of a word over an ordered alphabet is a factorization into a lexicographically weakly increasing sequence of Lyndon words. It is a fact that every word over an ordered alphabet has a unique Lyndon factorization. We say that a word of length n has *Lyndon type* λ, if $\lambda \vdash n$ and the parts of λ equal the lengths of the words in the Lyndon factorization (see [18, Theorem 5.1.5]). Let $\mathcal{K}(\lambda, j)$ denote the set of all bicolored banners of Lyndon type λ with exactly j barred letters.

Using Lyndon factorization, the following theorem is obtained.

Theorem 1.7.7 ([32, Theorem 3.6])**.** *There exists a weight preserving bijection from*

$\mathcal{R}(\lambda, j)$ to $K(\lambda, j)$, consequently

$$Q_{\lambda,j} = \sum_{B \in K(\lambda,j)} \mathrm{wt}(B),$$

for $n, j \in \mathbb{N}$ and $\lambda \vdash n$.

It is through this bicolored banner description that a recurrence for $Q_{n,j,k}$ is obtained. This recurrence is equivalent to the following generating function formula for $Q_{n,j,k}$.

Theorem 1.7.8 ([32, Theorem 1.2])**.** *We have*

$$\sum_{n,j,k \geq 0} Q_{n,j,k}(\mathbf{x}) t^j r^k z^n = \frac{(1-t)H(rz)}{H(tz) - tH(z)}.$$

Recall from Section 1.4 that $H(z) = \sum_{n \geq 0} h_n z^n$, from Equation (1.5) that $h_n = F_{\emptyset,n}$, and from Definition 1.3.1 that $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$. Putting all this together with Lemma 1.6.4, we have

$$\mathbf{ps}(H(z(1-q)) = \sum_{n \geq 0} \frac{z^n}{[n]_q!} = \exp_q(z).$$

We also recall Euler's exponential generating function formula for the Eulerian polynomials, Equation (1.1)

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z}.$$

So in comparing Theorem 1.7.8 with Euler's result in Equation (1.1), it is natural to call Theorem 1.7.8 a symmetric function analog of Euler's result. Moreover, by replacing $z \mapsto z(1-q)$, and $t \mapsto tq$, and applying the stable principal specialization $\mathbf{ps}$ to Theorem 1.7.8, we obtain the q-analog of Equation (1.1) due to Shareshian and

Wachs, Theorem 1.3.5

$$\sum_{\substack{n\geq 0\\ \pi\in S_n}} q^{\mathrm{maj}(\pi)}t^{\mathrm{exc}(\pi)}r^{\mathrm{fix}(\pi)}\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(rz)}{\exp_q(tqz)-tq\exp_q(z)}.$$

We remark that the formula for the distribution of $\mathrm{maj}, \mathrm{des}, \mathrm{exc}$, and fix appearing Theorem 1.3.6 also follows from Theorem 1.7.8 by applying the principal specialization $\mathbf{ps}_l$.

In the last two sections we have discussed techniques and results that will be relevant to proving our results later on. In the next two chapters we discuss well-known ways in which one can generalize the symmetric group. The first is a study of Coxeter groups, and the second is to consider certain wreath products. We will observe that there is some overlap in these two topics.

Chapter 2

The Hyperoctahedral Group

In this chapter we will see how the symmetric group fits into a larger class of groups called Coxeter groups. First we define Coxeter groups and recognize the symmetric group as part of this class of groups. We continue the chapter by considering the type B Coxeter group, also called the hyperoctahedral group.

2.1 Coxeter Groups

Informally, a Coxeter group is a group that can be presented in a simple way using generators and relations, where the generators are sometimes thought of as reflections. Coxeter groups arise in many different areas of mathematics, some examples include symmetry groups of regular polytopes, and the Weyl groups of simple Lie algebras. The finite Coxeter groups can be realized as reflection groups of finite dimensional Euclidean spaces. Moreover, the finite irreducible Coxeter groups have been completely classified: there are four infinite families, and six exceptional irreducible Coxeter groups. We refer the reader to [3] for more information on Coxeter groups.

Definition 2.1.1. A *Coxeter system* (W, S) consists of a group W and a set of generators S such that the following relations form a presentation of W:

1) $s_i^2 = \mathrm{Id}$ for all $s_i \in S$,

2) $(s_i s_j)^{m(s_i,s_j)} = \mathrm{Id}$, where $m(s_i, s_j) \in \{1, 2, ..., \infty\}$ for all $s_i, s_j \in S$.

If $m(s_i, s_j) = \infty$ for some $s_i, s_j \in S$, this means there is no relation of the form $(s_i s_j)^m$. The group W is called a *Coxeter group*. Given a Coxeter system (W, S), one can write the coefficients $m(s_i, s_j)$ such that $s_i, s_j \in S$ as a matrix called a *Coxeter matrix*.

We note that for W to be a group, $m(s_i, s_j) = m(s_j, s_i)$ for all $s_i, s_j \in S$. We also note that two generators s_i and s_j commute if and only if $m(s_i, s_j) = 2$

Two fundamental aspects of a Coxeter system that we will be particularly interested in are that of Coxeter length and right descent set. Later we will discuss the connections between certain permutation statistics, and length and right descent set.

Definition 2.1.2. Let (W, S) be any Coxeter system. We can express each $w \in W$ in terms of the generators S. If

$$w = s_{i_1} s_{i_2} \cdots s_{i_k}$$

where $s_{i_j} \in S$ for $1 \leq j \leq k$ and k is minimal among all such expressions, then k is called the *Coxeter length* of w and we write $l(w) = k$. The expression $s_{i_1} s_{i_2} \cdots s_{i_k}$ is called a *reduced word* for w.

The *right descent set* of w, denoted $D_R(w)$, is defined to be

$$D_R(w) := \{s_i \in S : l(ws_i) < l(w)\}.$$

Note that since we also consider permutations as words, we will have to distinguish between the Coxeter length of a permutation defined above, and the length of a permutation as a word, which we may denote by $\text{length}(\pi) = n$ if $\pi \in S_n$.

2.2 The Type A Coxeter Group

The most basic nontrivial example of a Coxeter group is the symmetric group, also called the type A Coxeter group. To realize S_n as a Coxeter group, we must present a list of generators S so that (S_n, S) is a Coxeter system, which we call the *type A Coxeter system*. For $i = 1, 2, ..., n-1$, define

$$\tau_i := (i, i+1) \in S_n$$

written in cycle form. The element τ_i is called an *adjacent transposition*, and we choose $S = \{\tau_1, \tau_2, ..., \tau_{n-1}\}$. Given a permutation $\pi \in S_n$ written in one line notation, we note that $\pi\tau_i$ is obtained from π by switching $\pi(i)$ and $\pi(i+1)$. For example let $\pi = 2, 5, 3, 1, 6, 4 \in S_6$, then $\pi\tau_4 = 2, 5, 3, 6, 1, 4$. From this it is clear that S generates S_n. Moreover, it is an easy exercise to verify that with these generators, S_n has the presentation

$$\begin{aligned}
&\tau_i^2 = \mathrm{Id} \quad \text{for } 1 \leq i \leq n-1, \\
&(\tau_i\tau_j)^2 = \mathrm{Id} \quad \text{for } |i-j| > 1, \\
&(\tau_i\tau_{i+1})^3 = \mathrm{Id} \quad \text{for } 1 \leq i \leq n-2,
\end{aligned}$$

where Id denotes the identity permutation of S_n (i.e. $\mathrm{Id} = 1, 2, ..., n$ written in one-line notation). Thus (S_n, S) is indeed a Coxeter system. Given $\pi \in S_n$, we let $l_{\mathrm{A}}(\pi)$ and $\mathrm{DES}_{\mathrm{A}}(\pi)$ denote the Coxeter length and right descent set of π respectively with respect to the Coxeter system (S_n, S). We also note that the corresponding Coxeter matrix, for say $n = 6$, has the form

$$m = \begin{bmatrix} 1 & 3 & 2 & 2 & 2 \\ 3 & 1 & 3 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 3 & 1 & 3 \\ 2 & 2 & 2 & 3 & 1 \end{bmatrix}$$

Recall from Definition 1.2.1 that for $1 \leq i < j \leq n$ and $\pi \in S_n$, the pair $(\pi(i), \pi(j))$

is called an inversion of π if $\pi(i) > \pi(j)$. The inversion number of π is defined by

$$\mathrm{inv}(\pi) := |\,\{(\pi(i), \pi(j)) : (\pi(i), \pi(j)) \text{ is an inversion of } \pi\}\,|.$$

This statistic gives us a combinatorial description of Coxeter length for S_n.

Proposition 2.2.1. *Let $\pi \in S_n$. Then $l_{\mathrm{A}}(\pi) = \mathrm{inv}(\pi)$.*

Proof. First we show that $\mathrm{inv}(\pi) \leq l_{\mathrm{A}}(\pi)$. Recall that multiplying π by τ_j on the right switches the letters $\pi(j)$ and $\pi(j+1)$. Thus if $\pi(j) > \pi(j+1)$, then $\mathrm{inv}(\pi\tau_j) = \mathrm{inv}(\pi) - 1$. And if $\pi(j) < \pi(j+1)$, then $\mathrm{inv}(\pi\tau_j) = \mathrm{inv}(\pi) + 1$. Since $\mathrm{inv}(\mathrm{Id}) = 0$, it follows that $\mathrm{inv}(\tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}) \leq k$, which implies $\mathrm{inv}(\pi) \leq l_{\mathrm{A}}(\pi)$.

Next we show that $l_{\mathrm{A}}(\pi) \leq \mathrm{inv}(\pi)$. Induct on $k = \mathrm{inv}(\pi)$. If $k = 0$, then $\pi = \mathrm{Id}$ and $l_{\mathrm{A}}(\mathrm{Id}) = 0$. Now let $k = \mathrm{inv}(\pi) > 0$, and assume that $l_{\mathrm{A}}(\sigma) \leq \mathrm{inv}(\sigma)$ whenever $\mathrm{inv}(\sigma) < k$. Since $\mathrm{inv}(\pi) > 0$, there exists j such that $\pi(j) > \pi(j+1)$. Thus

$$\mathrm{inv}(\pi\tau_j) = \mathrm{inv}(\pi) - 1 < k,$$

and

$$l_{\mathrm{A}}(\pi\tau_j) \leq \mathrm{inv}(\pi\tau_j) = \mathrm{inv}(\pi) - 1 = k - 1.$$

It follows that

$$l_{\mathrm{A}}(\pi) \leq k = \mathrm{inv}(\pi).$$

□

Recall that our discussion of q-analogs began in Section 1.3. Through the work of Rodrigues [28] and MacMahon [19], we saw that enumerating permutations by inversion number or major index provides a nice q-analog of $n!$. Since l_{A} is equal to inv, we update the corresponding formula now,

$$\sum_{\pi \in S_n} q^{l_{\mathrm{A}}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{inv}(\pi)} = \sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} = [n]_q!. \tag{2.1}$$

Next we show the connection between the right descent set (see Definition 2.1.2) of our Coxeter system (S_n, S) , and the descent set from Definition 1.2.1.

Proposition 2.2.2. *Let $\pi \in S_n$. Then $i \in \mathrm{DES}(\pi)$ if and only if $\tau_i \in D_R(\pi)$.*

Proof. If $i \in \mathrm{DES}(\pi)$, then multiplying by τ_i on the right switches $\pi(i)$ and $\pi(i+1)$ and $l_{\mathrm{A}}(\pi\tau_i) = \mathrm{inv}(\pi\tau_i) = \mathrm{inv}(\pi) - 1 < l_{\mathrm{A}}(\pi)$, thus $\tau_i \in D_R(\pi)$.

If $i \notin \mathrm{DES}(\pi)$, then $l_{\mathrm{A}}(\pi\tau_i) = \mathrm{inv}(\pi\tau_i) = \mathrm{inv}(\pi) + 1 > l_{\mathrm{A}}(\pi)$, and $\tau_i \notin D_R(\pi)$.

□

Since the Eulerian polynomial satisfies

$$A_n(t) = \sum_{\pi \in S_n} t^{\mathrm{des}(\pi)},$$

one could consider an Eulerian polynomial (or formal power series) for any Coxeter system (W, S) to be

$$\sum_{w \in W} t^{|D_R(w)|}.$$

2.3 The Type B Coxeter Group

Our next example of a Coxeter group is the type B Coxeter group, also called the hyperoctahedral group, and is the focus of this chapter. It is a generalization of the symmetric group, and it has a subgroup isomorphic to the symmetric group. The corresponding type B Coxeter system is similar to the type A Coxeter system, except that it contains an additional exceptional generator.

Definition 2.3.1. For $n \in \mathbb{P}$, consider a set of generators $S := \{\tau_0, \tau_1, \tau_2, ..., \tau_{n-1}\}$, and a group with presentation

$$\begin{aligned}
&\tau_i^2 = \mathrm{Id} \quad \text{for } 0 \leq i \leq n-1,\\
&(\tau_i \tau_j)^2 = \mathrm{Id} \quad \text{for } |i-j| > 1,\\
&(\tau_i \tau_{i+1})^3 = \mathrm{Id} \quad \text{for } 1 \leq i \leq n-2,\\
&(\tau_0 \tau_1)^4 = \mathrm{Id}.
\end{aligned}$$

We call the this group the *type B Coxeter group*, and we denote it by B_n. We call the corresponding Coxeter system (B_n, S) the *type B Coxeter system.* Given $w \in B_n$, we let $l_{\mathrm{B}}(w)$ denote the Coxeter length with respect to this Coxeter system.

We note that the corresponding Coxeter matrix, for say $n = 5$, has the form

$$m = \begin{bmatrix} 1 & 4 & 2 & 2 & 2 \\ 4 & 1 & 3 & 2 & 2 \\ 2 & 3 & 1 & 3 & 2 \\ 2 & 2 & 3 & 1 & 3 \\ 2 & 2 & 2 & 3 & 1 \end{bmatrix}$$

The type B Coxeter group has a nice combinatorial description as a group of *signed permutations.* This is the group of bijections on the set $\{-n, -n+1, ..., -1, 1, 2, ..., n\}$, subject to the constraint that a signed permutation π must satisfy $\pi(-i) = -\pi(i)$ for all $i \in [n] = \{1, 2, ..., n\}$. Because of this constraint, it suffices to describe a signed permutation by specifying where it maps the elements in $[n]$. A signed permutation π can be written in two-line notation

$$\pi = \begin{bmatrix} 1 & 2 & ... & n \\ \pi(1) & \pi(2) & ... & \pi(n) \end{bmatrix},$$

and one-line notation

$$\pi = \pi(1), \pi(2), ..., \pi(n),$$

where $\pi(i) \in \{-n, -n+1, ..., -1, 1, 2, ..., n\}$ and

$$|\pi| := |\pi(1)|, |\pi(2)|, ..., |\pi(n)| \in S_n.$$

Let $\text{sgn}(\pi(i)) := \pi(i)/|\pi(i)|$, that is the sign of $\pi(i)$. Clearly, S_n is a subgroup of the group of signed permutations.

We can also write a signed permutation as product of cycles. Let $i_1, i_2, ..., i_k \in \{-n, -n+1, ..., -1, 1, 2, ..., n\}$ be integers with distinct absolute values. The cycle

$$(i_1, i_2, ..., i_k)$$

denotes a bijection which maps $|i_j|$ to i_{j+1} for $j = 1, 2, ..., k-1$, maps $|i_k|$ to i_1, and if $\pm i$ does not appear in the cycle then $|i|$ is mapped to itself. Given any signed permutation π, it can be written as a product of disjoint cycles. For example

$$\pi = -3, 2, 1, -5, -6, 4, -7 = (1, -3)(2)(4, -5, -6)(-7) \in B_7,$$

and

$$|\pi| = 3, 2, 1, 5, 6, 4, 7 \in S_n.$$

We also let Id denote the identity element of the group of signed permutations (i.e Id $= 1, 2, ..., n$ written in one-line notation).

One realizes that the group of signed permutations is the type B Coxeter group B_n, by identifying the generators in S with elements of the signed permutation group. We let $\tau_0 = (-1)$, and note that multiplying a signed permutation π by τ_0 on the right has the effect of changing the sign of $\pi(1)$, and leaving the rest of π unchanged. For $1 \le i \le n-1$, we let $\tau_i = (i, i+1)$, which we also call an adjacent transposition since multiplying π on the right by $(i, i+1)$ has the effect of switching $\pi(i)$ and $\pi(i+1)$ without changing any signs. From this, it follows that the signed permutations

$\{\tau_0, \tau_1, ..., \tau_{n-1}\}$ generate the entire group of signed permutations. Moreover, we leave it to the reader to verify that they satisfy the relations in Definition 2.3.1 (see also [3]). Thus the type B Coxeter group is equal to the group of signed permutations, which we also call the hyperoctahedral group, denoted B_n. It will also be convenient to define $B_0 := \{\theta\}$ where θ denotes the empty word.

2.4 Statistics for the Hyperoctahedral Group

A signed permutation statistic, or simply permutation statistic, f is a map from the union of all signed permutation groups to $\mathbb{N}$. In Chapter 1 we found the study of permutation statistics to be a rich topic. Moreover, we found that the naturally defined statistics given in Definition 1.2.1 have a wide range of applications, and connections to the Coxeter interpretation of the symmetric group. Thus we seek so-called type B analogs of these statistics. First we define several permutation statistics for B_n, and then discuss their relevance. In this definition we use the usual order on the nonzero integers between $-n$ and n, i.e. $-n < -n+1 < ... < -1 < 1 < 2 < ... < n$.

Definition 2.4.1. Let $\pi \in B_n$.

- The set of positive fixed points (or simply fixed points) of π, is denoted $\mathrm{FIX}^+(\pi)$ and is defined by

$$\mathrm{FIX}^+(\pi) := \{i \in [n] : \pi(i) = i\}.$$

- The number of fixed points of π, denoted $\mathrm{fix}^+(\pi)$, is

$$\mathrm{fix}^+(\pi) := |\mathrm{FIX}^+(\pi)|.$$

- The set of negative fixed points of π, denoted $\mathrm{FIX}^-(\pi)$, is defined by

$$\mathrm{FIX}^-(\pi) := \{i \in [n] : \pi(i) = -i\}.$$

- The number of negative fixed points of π, denoted $\mathrm{fix}^-(\pi)$, is

$$\mathrm{fix}^-(\pi) := |\mathrm{FIX}^-(\pi)|.$$

- The negative letters of π, denoted $\mathrm{Neg}(\pi)$, are defined by

$$\mathrm{Neg}(\pi) := \{i \in [n] : \pi(i) < 0\}.$$

- The number of negative letters of π, denoted $\mathrm{neg}(\pi)$, is

$$\mathrm{neg}(\pi) := |\mathrm{Neg}(\pi)|.$$

- The descent set of π, denoted $\mathrm{DES}(\pi)$, is defined by

$$\mathrm{DES}(\pi) := \{i \in [n-1] : \pi(i) > \pi(i+1)\}.$$

- The descent number, denoted $\mathrm{des}(\pi)$, is defined by

$$\mathrm{des}(\pi) := |\mathrm{DES}(\pi)|.$$

- The type B descent set of π, denoted $\mathrm{DES_B}(\pi)$, is defined by

$$\mathrm{DES_B}(\pi) := \begin{cases} \mathrm{DES}(\pi) & \text{if } \pi(1) > 0 \\ \mathrm{DES}(\pi) \cup \{0\} & \text{if } \pi(1) < 0 \end{cases}.$$

- The type B descent number of π, denoted $\mathrm{des_B}(\pi)$, is defined by

$$\mathrm{des_B}(\pi) = |\mathrm{DES_B}(\pi)|.$$

- An inversion of π is a pair $(\pi(i), \pi(j))$, such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$. The inversion number of π, denoted $\mathrm{inv}(\pi)$, is defined to be the number of inversions of π.

- The type B inversion number of π, denoted $\mathrm{inv_B}(\pi)$, is defined by

$$\mathrm{inv_B}(\pi) := \mathrm{inv}(\pi) - \sum_{i \in \mathrm{Neg}(\pi)} \pi(i).$$

First we seek a type B analog to Proposition 2.2.1, which stated that l_{A} is equal to inv for the symmetric group. While the inv defined above is the most natural extension of inversion number, it does not have the property of being equal to l_{B} (the Coxeter length with respect to (B_n, S)), so we do not consider it to be a "good" type B analog for inversion number. We do consider $\mathrm{inv_B}$ to be a good type B analog of inversion number, because it resembles the definition of the type A inversion number, and it has the property of being equal to l_{B}. Brenti [4] introduced $\mathrm{inv_B}$ and proved the following proposition.

Proposition 2.4.2. *Let $\pi \in B_n$. Then $l_{\mathrm{B}}(\pi) = \mathrm{inv_B}(\pi)$.*

Proof. First we show $\mathrm{inv_B}(\pi) \leq l_{\mathrm{B}}(\pi)$. For $1 \leq i \leq n-1$ multiplying π on the right by τ_i switches $\pi(i+1)$ and $\pi(i)$, and does not change any signs. Consequently $i \in \mathrm{DES}(\pi)$ if and only if $\mathrm{inv_B}(\pi\tau_i) = \mathrm{inv_B}(\pi) - 1$, and $i \notin \mathrm{DES}(\pi)$ if and only if $\mathrm{inv_B}(\pi\tau_i) = \mathrm{inv_B}(\pi) + 1$. Multiplying π on the right by τ_0 changes the sign of the first letter of π. If $\pi(1) > 0$, then we lose $\pi(1) - 1$ inversions when multiplying on the right by τ_0. And if $\pi(1) < 0$ then we gain $\pi(1) - 1$ inversions when multiplying on the right by τ_0. In either case we have $\mathrm{inv}(\pi\tau_0) = \mathrm{inv}(\pi) - \pi(1) + \mathrm{sgn}(\pi(1))$. It follows that

$$\mathrm{inv}_{\mathrm{B}}(\pi\tau_0) = \mathrm{inv}(\pi\tau_0) + \chi(\pi(1) > 0)\pi(1) - \sum_{i \in \mathrm{Neg}(\pi) - \{1\}} \pi(i),$$

where we define $\chi(P) := 0$ if the statement P is is false, and $\chi(P) := 1$ otherwise. Thus

$$\begin{aligned} \mathrm{inv}_{\mathrm{B}}(\pi\tau_0) &= \mathrm{inv}(\pi) - \pi(1) + \mathrm{sgn}(\pi(1)) + \chi(\pi(1) > 0)\pi(1) - \sum_{i \in \mathrm{Neg}(\pi) - \{1\}} \pi(i) \\ &= \mathrm{inv}_{\mathrm{B}}(\pi) + \mathrm{sgn}(\pi(1)). \end{aligned}$$

It follows from these results and the fact that $\mathrm{inv}_{\mathrm{B}}(\mathrm{Id}) = 0$, that the type B inversion number of a product of k generators is less than or equal to k.

Now we show that $l_{\mathrm{B}}(\pi) \leq \mathrm{inv}_{\mathrm{B}}(\pi)$. Induct on $k = \mathrm{inv}_{\mathrm{B}}(\pi)$. If $k = 0$, then $\pi = \mathrm{Id}$ and $l_{\mathrm{B}}(\mathrm{Id}) = 0$. Now let $k = \mathrm{inv}_{\mathrm{B}}(\pi) > 0$, and assume that $l_{\mathrm{B}}(\sigma) \leq \mathrm{inv}_{\mathrm{B}}(\sigma)$ whenever $\mathrm{inv}_{\mathrm{B}}(\sigma) < k$. Since $\mathrm{inv}_{\mathrm{B}}(\pi) > 0$, either $\mathrm{DES}(\pi) \neq \emptyset$, or $\mathrm{DES}(\pi) = \emptyset$ and $\mathrm{Neg}(\pi) \neq \emptyset$. If the latter holds, then the negative letters must occur at the beginning of the word π, in particular $\pi(1) < 0$. In either case, we claim that there exists a generator τ such that

$$\mathrm{inv}_{\mathrm{B}}(\pi\tau) = \mathrm{inv}_{\mathrm{B}}(\pi) - 1 < k.$$

This follows from the fact that if $\mathrm{DES}(\pi) \neq \emptyset$,then set $\tau = \tau_i$ for some $i \in \mathrm{DES}(\pi)$. If $\mathrm{DES}(\pi) = \emptyset$ and $\pi(1) < 0$, then set $\tau = \tau_0$.

Thus

$$l_{\mathrm{B}}(\pi\tau) \leq \mathrm{inv}_{\mathrm{B}}(\pi\tau) = \mathrm{inv}_{\mathrm{B}}(\pi) - 1 = k - 1.$$

It follows that

$$l_{\mathrm{B}}(\pi) \leq k = \mathrm{inv}_{\mathrm{B}}(\pi).$$

□

As a corollary to Proposition 2.4.2, we verify that $\mathrm{DES_B}$ provides us with a type B analog of Proposition 2.2.2. Recall that we defined the right descent set for any Coxeter system in Definition 2.1.1. So in the following proposition, $D_R(\pi)$ refers to the right descent set of an element $\pi \in B_n$ with respect to the Coxeter system (B_n, S).

Proposition 2.4.3. *Let $\pi \in B_n$. Then $i \in \mathrm{DES_B}(\pi)$ if and only if $\tau_i \in D_R(\pi)$.*

Proof. In the proof of Proposition 2.4.2 we showed that if $i \in \mathrm{DES_B}(\pi)$, then $l_{\mathrm{B}}(\pi\tau_i) = \mathrm{inv_B}(\pi\tau_i) = \mathrm{inv_B}(\pi) - 1 < l_{\mathrm{B}}(\pi)$. And if $i \notin \mathrm{DES_B}(\pi)$, then $l_{\mathrm{B}}(\pi\tau_i) = \mathrm{inv_B}(\pi\tau_i) = \mathrm{inv}_B(\pi) + 1 > l_{\mathrm{B}}(\pi)$.

□

Next we define the *flag major index* statistic introduced by Adin and Roichman [2]. It is a major index like statistic that is equidistributed with l_{B} and $\mathrm{inv_B}$, thus a good type B analog to the major index for S_n (see Theorem 1.2.3).

Definition 2.4.4. For $\pi \in B_n$, the major index of π, denoted $\mathrm{maj}(\pi)$, is defined by

$$\mathrm{maj}(\pi) := \sum_{i \in \mathrm{DES}(\pi)} i.$$

The *flag major index* of π, denoted $\mathrm{fmaj}(\pi)$, is defined by

$$\mathrm{fmaj}(\pi) := 2\mathrm{maj}(\pi) + \mathrm{neg}(\pi).$$

Theorem 2.4.5 ([2, Theorem 2.2, Theorem 3.1])**.** *The statistic* fmaj *is equidistributed with l_{B}, thus also equidistributed with* $\mathrm{inv_B}$*.*

Moreover, we also have the following type B analog to Equation (2.1)

$$\sum_{\pi \in B_n} q^{l_{\mathrm{B}}(\pi)} = \sum_{\pi \in B_n} q^{\mathrm{inv_B}(\pi)} = \sum_{\pi \in B_n} q^{\mathrm{fmaj}(\pi)} = \prod_{i=1}^{n} [2i]_q. \tag{2.2}$$

A question first posed by Foata, is to extend the formula for the bivariate distribution of descent number and major index (the Carlitz identity, Theorem 1.3.3) to the hyperoctahedral group B_n. An answer to this question is given by Adin, Brenti, and Roichman [1] in Theorem 2.4.7 below. They use the flag major index and also introduce a descent like statistic called the *flag descent number*. They show that the flag descent number is equidistributed with the cardinality of a multiset they introduce called the negative descent multiset. The negative descent multiset is shown to have a natural description as a descent set in terms of the Coxeter group definition. They compute a formula for the bivariate distribution of the flag descent number and the flag major index, thus providing a type B analog to Carlitz's identity Theorem 1.3.3, and an answer to Foata's question.

Definition 2.4.6. For $\pi \in B_n$, the flag descent number of π, denoted $\mathrm{fdes}(\pi)$, is defined by

$$\mathrm{fdes}(\pi) := 2\mathrm{des}(\pi) + \chi(\pi(1) < 0).$$

Recall that $\chi(P) = 0$ if the statement P is is false, and $\chi(P) = 1$ otherwise.

Theorem 2.4.7 ([1, Theorem 4.2])**.** *For $n \in \mathbb{N}$ we have*

$$\frac{\sum_{\pi \in B_n} q^{\mathrm{fmaj}(\pi)} p^{\mathrm{fdes}(\pi)}}{(1-p)\prod_{i=1}^{n}(1-p^2q^{2i})} = \sum_{k \geq 0} [k+1]_q^n p^k.$$

As an analog to MacMahon's [20] classic result that excedance number and descent number are equidistributed (Theorem 1.2.2), we consider the following excedance like statistic of Foata and Han called the *flag excedance number*.

Definition 2.4.8. For $\pi \in B_n$, the excedance set of π, denoted $\mathrm{EXC}(\pi)$, is defined by

$$\mathrm{EXC}(\pi) := \{i \in [n-1] : \pi(i) > i\}.$$

The excedance number, denoted $\mathrm{exc}(\pi)$, is defined by

$$\mathrm{exc}(\pi) := |\mathrm{EXC}(\pi)|.$$

The flag excedance number of π, denoted $\mathrm{fexc}(\pi)$, is defined by

$$\mathrm{fexc}(\pi) := 2\mathrm{exc}(\pi) + \mathrm{neg}(\pi).$$

The following theorem of Foata and Han shows that the pair $(\mathrm{fdes}, \mathrm{fexc})$ is a nice type B analog to the pair of symmetric group statistics $(\mathrm{des}, \mathrm{exc})$.

Theorem 2.4.9 ([12])**.** *The statistics* fdes *and* fexc *are equidistributed on B_n. Equivalently,*

$$\sum_{\pi \in B_n} t^{\mathrm{fdes}(\pi)} = \sum_{\pi \in B_n} t^{\mathrm{fexc}(\pi)}.$$

This theorem is proved by specializing the following type B analog of Theorem 1.3.6 due to Foata and Han.

Theorem 2.4.10 ([12, Theorem 1.1])**.** *We have*

$$\sum_{n \geq 0} (1+t) \sum_{\pi \in B_n} s^{\mathrm{fexc}(\pi)} t^{\mathrm{fdes}(\pi)} q^{\mathrm{fmaj}(\pi)} Y_0^{\mathrm{fix}^+(\pi)} Y_1^{\mathrm{fix}^-(\pi)} Z^{\mathrm{neg}(\pi)} \frac{u^n}{(t^2; q^2)_{n+1}}$$

$$= \sum_{r \geq 0} t^r \frac{(u; q^2)_{\lfloor r/2 \rfloor + 1}}{(uY_0; q^2)_{\lfloor r/2 \rfloor + 1}} \frac{(-usqY_1Z; q^2)_{\lfloor (r+1)/2 \rfloor}}{(-usqZ; q^2)_{\lfloor (r+1)/2 \rfloor}} F_r(u; s, q, Z),$$

where

$$\begin{aligned} F_r(u; s, q, Z) &= (us^2q^2; q^2)_{\lfloor r/2 \rfloor} (1 - s^2q^2)(u; q^2)_{\lfloor (r+1)/2 \rfloor} (u; q^2)_{\lfloor r/2 \rfloor} \\ &\times \Big[(u; q^2)_{\lfloor r/2 \rfloor + 1} \big((u; q^2)_{\lfloor (r+1)/2 \rfloor} \big((u; q^2)_{\lfloor r/2 \rfloor} - s^2q^2 (us^2q^2; q^2)_{\lfloor r/2 \rfloor} \big) \\ &\qquad + sqZ (u; q^2)_{\lfloor r/2 \rfloor} \big((u; q^2)_{\lfloor (r+1)/2 \rfloor} - (us^2q^2; q^2)_{\lfloor (r+1)/2 \rfloor} \big) \big) \Big]^{-1}. \end{aligned}$$

As a corollary, Foata and Han obtain the following type B analog of Theorem 1.3.5.

Corollary 2.4.11 ([12, Corollary 1.2]). *We have*

$$\sum_{n\geq 0}\sum_{\pi\in B_n} s^{\mathrm{fexc}(\pi)}q^{\mathrm{fmaj}(\pi)}Y_0^{\mathrm{fix}^+(\pi)}Y_1^{\mathrm{fix}^-(\pi)}Z^{\mathrm{neg}(\pi)}\frac{u^n}{(q^2;q^2)_n}$$

$$= \frac{\exp_{q^2}(uY_0)\mathrm{Exp}_{q^2}(usqY_1Z)(1-s^2q^2)}{\mathrm{Exp}_{q^2}(usqZ)\left[\exp_{q^2}(us^2q^2)-s^2q^2\exp_{q^2}(u)+sqZ\left(\exp_{q^2}(us^2q^2)-\exp_{q^2}(u)\right)\right]}.$$

Clearly, Corollary 2.4.11 implies a formula for the four-variate distribution of $(\mathrm{fexc}, \mathrm{fmaj}, \mathrm{fix}^+, \mathrm{neg})$. At the end of Chapter 3 we show that this formula for the four-variate distribution of $(\mathrm{fexc}, \mathrm{fmaj}, \mathrm{fix}^+, \mathrm{neg})$ also follows from a new result of this thesis, Theorem 3.3.1.

We turn our attention now to a different and more recent solution to Foata's problem of extending Carlitz's identity, Theorem 1.3.3, to the hyperoctahedral group B_n. Since $\mathrm{DES_B}$ is the right descent set for the Coxeter system (B_n, S) (see Proposition 2.4.3), it is the most natural analog to the type A descent set. We would like to have a type B analog of Carlitz's identity that involves $\mathrm{des_B}$, and Chow and Gessel obtain just such a result.

Theorem 2.4.12 ([6, Theorem 3.7]). *For $n \in \mathbb{N}$ we have*

$$\frac{\sum_{\pi\in B_n} q^{\mathrm{fmaj}(\pi)}p^{\mathrm{des_B}(\pi)}}{(p;q^2)_{n+1}} = \sum_{k\geq 0}[2k+1]_q^n p^k.$$

Proof. Here we give a sketch of the proof from [6]. First define

$$B_{n,k}(q) := \sum_{\substack{\pi\in B_n\\ \mathrm{des_B}(\pi)=k}} q^{\mathrm{fmaj}(\pi)}.$$

Next, the following recurrence for $B_{n,k}(q)$ is obtained by analyzing how fmaj and $\mathrm{des_B}$

change when inserting $\pm n$ into a signed permutation from B_{n-1}.

$$B_{n,k}(q) = [2k+1]_q B_{n-1,k}(q) + q^{2k-1}[2n-2k+1]_q B_{n-1,k-1}(q),\ \ 1 \leq k \leq n-1.$$

The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!}.$$

Using the above recursion and a few identities involving q-binomial coefficients, the following identity is obtained by induction

$$[2r+1]_q^n = \sum_{k=0}^{n} B_{n,k}(q) \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2}.$$

This identity is used to prove that

$$\sum_{r \geq 0} [2r+1]_q^n p^r = \sum_{r \geq 0} \sum_{k=0}^{n} B_{n,k}(q) \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2} p^r$$

$$= \sum_{k=0}^{n} B_{n,k}(q) p^k \sum_{r \geq k} \begin{bmatrix} r+n-k \\ n \end{bmatrix}_{q^2} p^{r-k}$$

$$= \sum_{\pi \in B_n} q^{\mathrm{fmaj}(\pi)} p^{\mathrm{des_B}(\pi)} \sum_{s \geq 0} \begin{bmatrix} n+s \\ n \end{bmatrix}_{q^2} p^s.$$

The result now follows from the following identity

$$\sum_{k\geq 0}\begin{bmatrix} n+k \\ n \end{bmatrix}_q p^k = \frac{1}{(p;q)_{n+1}}.$$

□

A new result we present in this thesis, is a formula for the five variate distribution for $\mathrm{fmaj}, \mathrm{fexc}, \mathrm{des}_{\mathrm{B}}, \mathrm{fix}^{+}, \mathrm{neg}$. This is a type B analog to Theorem 1.3.6 and reduces to Theorem 2.4.12 by setting $t = r = s = 1$.

Theorem 2.4.13.

$$\sum_{\substack{n\geq 0\\ \pi\in B_n}} \frac{z^n}{(p;q^2)_{n+1}} q^{\mathrm{fmaj}(\pi)} t^{\mathrm{fexc}(\pi)} p^{\mathrm{des}_{\mathrm{B}}(\pi)} r^{\mathrm{fix}^{+}(\pi)} s^{\mathrm{neg}(\pi)}$$

$$= \sum_{k\geq 0} \frac{p^k(1-t^2q^2)(z;q^2)_k(t^2q^2z;q^2)_k}{(rz;q^2)_{k+1}\left[(1+sqt)(z;q^2)_k-(t^2q^2+sqt)(t^2q^2z;q^2)_k\right]}.$$

At the end of Chapter 3 we show that Theorem 2.4.13 follows from another more general new result of this thesis, Theorem 3.3.2. While Theorem 2.4.13 has similarities to Theorem 2.4.10 of Foata and Han, we note that these formulas are not equivalent. The difference is that the descent statistic in Theorem 2.4.10 is fdes, while the descent statistic in Theorem 2.4.13 is $\mathrm{des}_{\mathrm{B}}$. Thus Theorem 2.4.10 reduces to the Adin, Brenti, Roichman type B analog of Carlitz's identity (Theorem 2.4.7), while our result in Theorem 2.4.13 reduces to the Chow-Gessel type B analog of Carlitz's identity (Theorem 2.4.12). We conclude this chapter by demonstrating how Theorem 2.4.13 reduces to Theorem 2.4.12.

By setting $t = r = s = 1$, Theorem 2.4.13 reduces to

$$\sum_{\substack{n\geq 0\\ \pi\in B_n}} \frac{z^n}{(p;q^2)_{n+1}} q^{\mathrm{fmaj}(\pi)} p^{\mathrm{des}_{\mathrm{B}}(\pi)} = \sum_{k\geq 0} \frac{p^k(1-q^2)(z;q^2)_k(q^2z;q^2)_k}{(z;q^2)_{k+1}[(1+q)(z;q^2)_k-(q^2+q)(q^2z;q^2)_k]}.$$

Note that $(zq^2;q^2)_k = (z;q^2)_{k+1}/(1-z)$, thus

$$\sum_{\substack{n\geq 0\\ \pi\in B_n}} \frac{z^n}{(p;q^2)_{n+1}} q^{\mathrm{fmaj}(\pi)} p^{\mathrm{des_B}(\pi)} = \sum_{k\geq 0} \frac{p^k(1-q)(1+q)(z;q^2)_k}{(1-z)[(1+q)(z;q^2)_k - (q^2+q)(q^2z;q^2)_k]}$$

$$= \sum_{k\geq 0} \frac{p^k(1-q)(z;q^2)_k}{(1-z)(z;q^2)_k - q(z;q^2)_{k+1}} = \sum_{k\geq 0} \frac{p^k(1-q)}{(1-z)-q(1-zq^{2k})}$$

$$= \sum_{k\geq 0} \frac{p^k(1-q)}{(1-q)-z(1-q^{2k+1})} = \sum_{k\geq 0} \frac{p^k}{1-z[2k+1]_q} = \sum_{n,k\geq 0} z^n [2k+1]_q^n p^k.$$

Now extract the coefficient of z^n from both sides to obtain Theorem 2.4.12 of Chow and Gessel.

Some of the earliest results on enumerating signed permutations according to statistics are due to Reiner [23]. He derives a multivariate generating function formula involving type B descent number, major index and cycle type. His major index is different from the major index statistics we have defined in this thesis, and is not equidistributed with length for the hyperoctahedral group. We note that his proof is a type B analog of the techniques of Gessel and Reutenauer discussed in Section 1.6. To prove our results presented in this thesis, we will develop a colored permutation, or wreath product, generalization of the techniques of Shareshian and Wachs discussed in Section 1.7. In the following chapter, we will see that the family of colored permutation groups includes the family of signed permutation groups. Because of this, we are able to specialize our results to the signed permutation group. Although our techniques and results have similarities with those of Reiner, they are in fact different.

Chapter 3

The Colored Permutation Group

In the previous chapter we explored a way to generalize the symmetric group using Coxeter group theory. In this chapter we consider another generalization of the symmetric group, namely the wreath product of the cycle group with the symmetric group, also called the colored permutation group. We will observe that there are close connections between colored permutation groups and Coxeter groups.

3.1 Colored Permutations

First we define a wreath product in general, and then turn our attention to the particular class of wreath products that we will be interested in. Let H, G be groups with G acting on a set Γ. Let H^Γ be the direct product

$$H^\Gamma := \prod_{\gamma \in \Gamma} H_\gamma,$$

where each $H_\gamma = H$. Define an action of G on H^Γ by

$$g \cdot h_\gamma := h_{g^{-1}\gamma},$$

for $g \in G$, $\gamma \in \Gamma$, and $h = (h_\gamma)_{\gamma \in \Gamma} \in H^\Gamma$. The *wreath product* of H and G with respect to Γ, denoted $H \wr_\Gamma G$, is the corresponding semidirect product $H^\Gamma \rtimes G$. That is, for $h_1, h_2 \in H^\Gamma$ and $g_1, g_2 \in G$, the group multiplication in $H \wr_\Gamma G$ is given by

$$(h_1, g_1)(h_2, g_2) = (h_1(g_1 \cdot h_2), g_1 g_2).$$

We will study a particular class of wreath products called colored permutation groups.

Definition 3.1.1. For $N, n \in \mathbb{P}$, let C_N denote the cyclic group of order N and S_n the symmetric group on $[n] = \{1, 2, ..., n\}$. S_n acts on $[n]$ in the usual way, i.e. for $\pi \in S_n$ and $i \in [n]$, $\pi \cdot i = \pi(i)$. We define the *colored permutation group* to be the wreath product $C_N \wr_{[n]} S_n$, which we simply denote by $C_N \wr S_n$. For $n = 0$, it will be convenient to define $C_N \wr S_0 := \{\theta\}$ where θ denotes the empty word.

In Chapter 2 we saw that the type B Coxeter group has a nice and useful combinatorial description as a group of signed permutations. There is also a nice combinatorial description of $C_N \wr S_n$ which illuminates the reason for calling this group the colored permutation group. In this and subsequent chapters we will primarily use this combinatorial description. We consider an *N-colored permutation of length n*, or simply a *colored permutation* to be a bijection on the following (ordered) alphabet of N-colored integers from 1 to n

$$\mathcal{E} := \left\{1^{N-1} < 2^{N-1} ... < n^{N-1} < 1^{N-2} < 2^{N-2} ... < n^{N-2} < ... < 1^0 < 2^0 < ... < n^0\right\}. \tag{3.1}$$

Given such a bijection π, we require that

$$\pi(i^0) = k^{\epsilon_i} \text{ implies } \pi(i^j) = k^{(\epsilon_i + j \mod N)} \tag{3.2}$$

for $1 \leq i \leq n$ and $0 \leq j \leq N-1$. So just as with signed permutations, it suffices to describe a colored permutation by specifying where it maps $1^0, 2^0, ..., n^0$. We usually do this by writing a colored permutation π in one-line notation (i.e. a word over $\mathcal{E}$), so $\pi(i) = \pi_i \in \mathcal{E}$ denotes the i^{th} letter of the colored permutation π. We let $|\pi_i|$ denote the positive integer obtained by removing the superscript, and let $\epsilon_i \in \{0, 1, ..., N-1\}$ denote the superscript, or color, of the i^{th} letter of the word. If π is word of length n over $\mathcal{E}$, we denote by $|\pi|$ the word

$$|\pi| := |\pi_1|, |\pi_2|, ..., |\pi_n|.$$

Given any word π of length n over the alphabet $\mathcal{E}$, π defines a colored permutation if and only if $|\pi| \in S_n$.

For example, elements of this group may be written in one line or two line notation as follows:

$$\pi = 3^2, 5^0, 4^1, 1^2, 2^1 = \begin{bmatrix} 1^0 & 2^0 & 3^0 & 4^0 & 5^0 \\ 3^2 & 5^0 & 4^1 & 1^2 & 2^1 \end{bmatrix} \in C_3 \wr S_5,$$

then

$$|\pi| = 3, 5, 4, 1, 2 \in S_5.$$

And in particular, $|\pi_3| = 4$ and $\epsilon_3 = 1$ so that $\pi_3 = |\pi_3|^{\epsilon_3} = 4^1$.

We can also write a colored permutation π in cycle notation using the convention that j^{ϵ_j} follows i^{ϵ_i} means that $\pi_i = j^{\epsilon_j}$. It is easy to see that a colored permutation decomposes into a product of disjoint cycles. Continuing with the previous example, we can write it in cycle notation as

$$\pi = (1^2, 3^2, 4^1)(2^1, 5^0).$$

Proposition 3.1.2. *The group of N-colored permutations of length n is the wreath product $C_N \wr S_n$.*

Proof. Let π be a colored permutation whose letters have colors $\epsilon_1, \epsilon_2, ..., \epsilon_n$ respectively. We identify π with the element $(\epsilon, |\pi|) \in C_N \wr S_n$ where $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_n) \in (C_N)^n$. Given another colored permutation π' with corresponding element $(\epsilon', |\pi'|) \in C_N \wr S_n$, the wreath product multiplication is given by

$$(\epsilon, |\pi|)(\epsilon', |\pi'|) = (\epsilon(|\pi| \cdot \epsilon'), |\pi||\pi'|) = (\delta, \sigma),$$

where $\sigma = |\pi||\pi'|$, and if we write C_N additively then $\delta_i = \epsilon_i + \epsilon'_{|\pi|^{-1}(i)} \mod N$.

Considering the colored permutation multiplication $\pi\pi'$, clearly $|\pi\pi'| = |\pi||\pi'|$. Thus by (3.2), the color of the i^{th} letter of the word $\pi\pi'$ is $\epsilon_i + \epsilon'_{|\pi|^{-1}(i)} \mod N$, as desired.

□

It is easy to see that $C_1 \wr S_n \cong S_n$ and $C_2 \wr S_n \cong B_n$. From our study of type A and B Coxeter groups, we can see how to obtain generators for arbitrary $C_N \wr S_n$. For $1 \leq i \leq n-1$, let $\tau_i := (i^0, (i+1)^0)$ (we can think of τ_i as the usual adjacent transposition). Now the extra distinguished generator τ_0 is defined by $\tau_0(1) = 1^1$ and $\tau_0(i) = i$ for $2 \leq i \leq n$. In cycle notation this can be written as $\tau_0 := (1^1)$. Note that multiplying a colored permutation π on the right by $(i^0, (i+1)^0)$ switches the letters $\pi(i)$ and $\pi(i+1)$ without changing any colors, while multiplying on the right by τ_0 adds one (mod N) to the color of $\pi(1)$. Thus the elements $\tau_0, \tau_1, \tau_2, ..., \tau_{n-1}$ generate all of $C_N \wr S_n$. We leave for the reader to verify that this group has the presentation (see also [16])

$$\begin{aligned}
&\tau_i^2 = \mathrm{Id} \quad \text{for } i = 1, 2, ..., n-1,\\
&\tau_0^N = \mathrm{Id},\\
&(\tau_i\tau_j)^2 = \mathrm{Id} \quad \text{if } |i-j| > 1,\\
&(\tau_i\tau_{i+1})^3 = \mathrm{Id} \quad \text{for } i = 1, 2, ..., n-2,\\
&(\tau_0\tau_1)^{2N} = \mathrm{Id}.
\end{aligned}$$

We see that for $N > 2$, we almost have a Coxeter system, except that $\tau_0^2 \neq \mathrm{Id}$.

However, wreath products do provide a nice generalization of both the type A and type B Coxeter groups, in a way that is Coxeter-like.

3.2 Statistics for the Colored Permutation Group

A colored permutation statistic, or simply permutation statistic, f is a map from the union of all colored permutation groups to $\mathbb{N}$. In this section we discuss several permutation statistics that are colored analogs of previous statistics.

Definition 3.2.1. Let $\pi \in C_N \wr S_n$.

- For an integer k such that $0 \leq k \leq N-1$, we define the k^{th} color fixed point set of π, denoted $\mathrm{FIX}_k(\pi)$, by

$$\mathrm{FIX}_k(\pi) := \left\{ i \in [n] : \pi_i = i^k \right\}.$$

- The k^{th} color fixed point number of π, denoted $\mathrm{fix}_k(\pi)$, is defined by

$$\mathrm{fix}_k(\pi) := |\mathrm{FIX}_k(\pi)|.$$

- It will also be convenient to define the fixed point vector of π, denoted $\vec{\mathrm{fix}}(\pi) \in \mathbb{N}^N$, by

$$\vec{\mathrm{fix}}(\pi) := (\mathrm{fix}_0(\pi), \mathrm{fix}_1(\pi), ..., \mathrm{fix}_{N-1}(\pi)).$$

- For an integer m such that $1 \leq m \leq N-1$, we define the m^{th} color set of π, denoted $\mathrm{COL}_m(\pi)$, by

$$\mathrm{COL}_m(\pi) := \{ i \in [n] : \epsilon_i = m \}.$$

- The m^{th} color number of π, denoted $\text{col}_m(\pi)$, is defined by

$$\text{col}_m(\pi) := |\text{COL}_m(\pi)|.$$

- And we define the color vector of π, denoted $\vec{\text{col}}(\pi) \in \mathbb{N}^{N-1}$, by

$$\vec{\text{col}}(\pi) := (\text{col}_1(\pi), \text{col}_2(\pi), ..., \text{col}_{N-1}(\pi)).$$

- The descent set of π, denoted $\text{DES}(\pi)$, is defined by

$$\text{DES}(\pi) := \{i \in [n-1] : \pi_i > \pi_{i+1}\};$$

  note that this is computed with respect to the order given on $\mathcal{E}$ in (3.1).

- The descent number of π, denoted $\text{des}(\pi)$, is defined by

$$\text{des}(\pi) := |\text{DES}(\pi)|.$$

- The starred descent set of π, denoted $\text{DES}^*(\pi)$, is defined by

$$\text{DES}^*(\pi) := \begin{cases} \text{DES}(\pi) & \text{if } \epsilon_1 = 0 \\ \text{DES}(\pi) \cup \{0\} & \text{if } \epsilon_1 > 0 \end{cases}.$$

- The starred descent number of π, denoted $\text{des}^*(\pi)$, is defined by

$$\text{des}^*(\pi) := |\text{DES}^*(\pi)|.$$

For example if

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1^2 & 4^0 & 8^1 & 6^0 & 5^2 & 3^2 & 7^0 & 2^1 \end{bmatrix} \in C_3 \wr S_8,$$

then $\mathrm{DES}^*(\pi) = \{0, 2, 4, 5, 7\}$, $\mathrm{des}(\pi) = 4$, $\mathrm{des}^*(\pi) = 5$, $\vec{\mathrm{fix}}(\pi) = (1, 0, 2)$, and $\vec{\mathrm{col}}(\pi) = (2, 3)$.

Now we introduce the cv-cycle type (short for color vector cycle type) of a colored permutation $\pi \in C_N \wr S_n$. As noted above, π decomposes into a product of disjoint cycles. Let $\lambda = (\lambda_1 \geq ... \geq \lambda_k)$ be a partition of n. Let $\vec{\beta^1}, ..., \vec{\beta^k}$ be a sequence of vectors in $\mathbb{N}^{N-1}$ with each $\left|\vec{\beta^i}\right| \leq \lambda_i$, where the absolute value of a vector $\vec{\beta} \in \mathbb{N}^{N-1}$ is the sum of its components, i.e. $|\vec{\beta}| := \beta_1 + \beta_2 + ... + \beta_{N-1}$. Consider the multiset of pairs

$$\check{\lambda} = \left\{ (\lambda_1, \vec{\beta^1}), (\lambda_2, \vec{\beta^2}), ..., (\lambda_k, \vec{\beta^k}) \right\}.$$

We say that π has cv-cycle type $\check{\lambda}(\pi) = \check{\lambda}$ if each pair $(\lambda_i, \vec{\beta^i})$ corresponds to exactly one cycle of length λ_i with color vector $\vec{\beta^i}$ in the decomposition of π. Note that $\vec{\mathrm{col}}(\pi) = \vec{\beta^1} + ... + \vec{\beta^k}$ using component wise addition. Consider the following example in $C_3 \wr S_9$, let

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3^2 & 4^1 & 1^1 & 2^2 & 8^0 & 9^2 & 5^1 & 7^1 & 6^0 \end{bmatrix}$$
$$= (1^1, 3^2)(2^2, 4^1)(6^0, 9^2)(5^1, 8^0, 7^1),$$

so $\check{\lambda}(\pi) = \{(3, (2, 0)), (2, (1, 1)), (2, (1, 1)), (2, (0, 1))\}$.

The cv-cycle type is actually a refinement of the cycle type, which determines the conjugacy classes of the colored permutation group. Given $\pi \in C_N \wr S_n$, we say π has cycle type $\{(\lambda_1, j_1), (\lambda_2, j_2), ..., (\lambda_k, j_k)\}$ where

$$j_i = \sum_{m=1}^{N-1} m\beta_m^i \mod N.$$

So for example the cycle type of the colored permutation above is $\{(3,2),(2,0),(2,0),(2,2)\}$. However we will only be using the cv-cycle type in this paper.

As mentioned in the previous chapter, the flag major index was introduced by Adin and Roichman [2] as an analog to the major index (see Definition 2.4.4). In fact, they introduced a more general statistic for the colored permutation group, which we define now.

Definition 3.2.2. For $\pi \in C_N \wr S_n$, the major index of π, denoted $\text{maj}(\pi)$, is defined by

$$\text{maj}(\pi) := \sum_{i \in \text{DES}(\pi)} i.$$

The flag major index of π, denoted $\text{flagmaj}(\pi)$, is defined by

$$\text{flagmaj}(\pi) := N \cdot \text{maj}(\pi) + \sum_{m=1}^{N-1} m \cdot \text{col}_m(\pi).$$

For $N = 1$, we see immediately that flagmaj reduces to maj, which is equidistributed with l_{A} on $C_1 \wr S_n$. We also see that $\text{des}^*(\pi) = \text{des}(\pi)$ for all $\pi \in C_1 \wr S_n$. For $N = 2$ we identify the 1-colored letters with negative letters, but the order on $\mathcal{E}$ from (3.1) is not the same as the usual order on $\pm 1, \pm 2, ..., \pm n$. However, we still have that $\text{flagmaj}, \text{fmaj}$, and l_{B} are all equidistributed on $C_2 \wr S_n$. It is also a fact that des^* and des_{B} are equidistributed on $C_2 \wr S_n$. In Proposition 3.3.5 below, we describe a bijection which shows that the pair $(\text{flagmaj}, \text{des}^*)$ is jointly equidistributed with $(\text{fmaj}, \text{des}_{\text{B}})$ on $C_2 \wr S_n$. This bijection actually shows that several other statistics are also jointly equidistributed on $C_2 \wr S_n$. For $N > 2$, flagmaj is not equidistributed with length in terms of the generators described above, but this statistic does play a central role in Adin and Roichman's [2] study of $C_N \wr S_n$ actions on tensor powers of polynomial rings, and the Hilbert series of the diagonal action invariant algebra.

Haglund, Loehr, and Remmel [16] introduce another major index like statistic called *root major index.* They show this statistic is equidistributed with flagmaj, and they obtain a polynomial formula for the distribution.

Definition 3.2.3. For $\pi \in C_N \wr S_n$, the root major index of π, denoted $\mathrm{rootmaj}(\pi)$, is defined by

$$\mathrm{rootmaj}(\pi) := \mathrm{maj}(\pi) + \sum_{m=1}^{N-1} \sum_{i \in \mathrm{COL}_m(\pi)} m \cdot |\pi(i)|.$$

Theorem 3.2.4 ([16, Theorem 4.5])**.** *For $N \in \mathbb{P}$ and $n \in \mathbb{N}$ we have*

$$\sum_{\pi \in C_N \wr S_n} q^{\mathrm{flagmaj}(\pi)} = \sum_{\pi \in C_N \wr S_n} q^{\mathrm{rootmaj}(\pi)} = \prod_{i=1}^{n} [Ni]_q.$$

The flag excedance number of Foata and Han can also be extended to the colored permutation group.

Definition 3.2.5. For $\pi \in C_N \wr S_n$, the excedance set of π, denoted $\mathrm{EXC}(\pi)$, is defined by

$$\mathrm{EXC}(\pi) := \left\{ i \in [n] : \pi_i > i^0 \right\}.$$

The excedance number of π, denoted $\mathrm{exc}(\pi)$, is defined by

$$\mathrm{exc}(\pi) := |\mathrm{EXC}(\pi)|.$$

The flag excedance number of π, denoted $\mathrm{flagexc}(\pi)$, is defined by

$$\mathrm{flagexc}(\pi) := N \cdot \mathrm{exc}(\pi) + \sum_{m=1}^{N-1} m \cdot \mathrm{col}_m(\pi).$$

3.3 Multivariate Distributions of Colored Permutation Statistics

In this section we present some of the main results of this thesis, which enumerate colored permutations according to various statistics.

Theorem 3.3.1. *We have*

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{[n]_q!} t^{\mathrm{exc}(\pi)} r^{\vec{\mathrm{fix}}(\pi)} s^{\vec{\mathrm{col}}(\pi)} q^{\mathrm{maj}(\pi)}$$

$$= \frac{\exp_q(r_0 z)(1-tq)\left(\prod_{m=1}^{N-1}\mathrm{Exp}_q(-s_m z)\exp_q(r_m s_m z)\right)}{\left(1+\sum_{m=1}^{N-1} s_m\right)\exp_q(tqz)-\left(tq+\sum_{m=1}^{N-1} s_m\right)\exp_q(z)}.$$

Theorem 3.3.2. *We have*

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{(p;q)_{n+1}} t^{\mathrm{exc}(\pi)} r^{\vec{\mathrm{fix}}(\pi)} s^{\vec{\mathrm{col}}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)}$$

$$= \sum_{l\geq 0} \frac{p^l(1-tq)(z;q)_l(tqz;q)_l\left(\prod_{m=1}^{N-1}(s_m z;q)_l\right)\left(\prod_{m=1}^{N-1}(r_m s_m z;q)_l\right)^{-1}}{(r_0 z;q)_{l+1}\left[\left(1+\sum_{m=1}^{N-1} s_m\right)(z;q)_l-\left(tq+\sum_{m=1}^{N-1} s_m\right)(tqz;q)_l\right]}.$$

In Chapter 4 we show that Theorem 3.3.1 and Theorem 3.3.2 are obtained by applying the stable and nonstable principal specializations respectively to our colored permutation generalization of Theorem 1.7.8 of Shareshian and Wachs, which we state in Theorem 4.1.3.

Next we examine some special cases of Theorem 3.3.1 and Theorem 3.3.2. First, we see that by a simple variable substitution Theorem 3.3.2 reduces to a colored permutation analog of Carlitz's identity. This formula enumerates colored permutations according to fmaj and des*, and is equivalent to a formula obtained independently by Chow and Mansour [7].

Theorem 3.3.3. *We have*

$$\frac{\sum_{\pi\in C_N\wr S_n} q^{\text{flagmaj}} p^{\text{des}^*}}{(p;q^N)_{n+1}} = \sum_{l\geq 0}[Nl+1]_q^n p^l$$

Proof. In Theorem 3.3.2, replace $t\mapsto 1$, $r_k\mapsto 1$ for $0\leq k\leq N-1$, $s_m\mapsto q^m$ for $1\leq m\leq N-1$, and $q\mapsto q^N$, to obtain

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{(p;q^N)_{n+1}} q^{\text{flagmaj}(\pi)} p^{\text{des}^*(\pi)}$$

$$= \sum_{l\geq 0} \frac{p^l(1-q^N)(z;q^N)_l(q^Nz;q^N)_l}{(z;q^N)_{l+1}\left[[N]_q(z;q^N)_l + (1-q^N-[N]_q)(q^Nz;q^N)_l\right]}$$

$$= \sum_{l\geq 0} \frac{p^l(1-q^N)(z;q^N)_l(z;q^N)_{l+1}/(1-z)}{(z;q^N)_{l+1}\left[[N]_q(z;q^N)_l + (1-q^N-[N]_q)(z;q^N)_{l+1}/(1-z)\right]}$$

$$= \sum_{l\geq 0} \frac{p^l(1-q^N)(z;q^N)_l}{(1-z)[N]_q(z;q^N)_l + (1-q^N-[N]_q)(z;q^N)_{l+1}}$$

$$= \sum_{l\geq 0} \frac{p^l(1-q^N)}{(1-z)[N]_q + (1-q^N-[N]_q)(1-zq^{Nl})}$$

$$= \sum_{l\geq 0} \frac{p^l(1-q^N)}{(1-q^N) - z\left([N]_q + q^{Nl}(1-q^N-[N]_q)\right)}$$

$$= \sum_{l\geq 0} \frac{p^l(1-q^N)}{(1-q^N) - z\left((1-q^{Nl})[N]_q + q^{Nl}(1-q^N)\right)} = \sum_{l\geq 0} \frac{p^l}{1-z\left([Nl]_q + q^{Nl}\right)}$$

$$= \sum_{l\geq 0} \frac{p^l}{1-z[Nl+1]_q} = \sum_{l,n\geq 0} z^n[Nl+1]_q^n p^l.$$

Now extract the coefficient of z^n from both sides to obtain the desired result.

□

Next we notice an interesting corollary of Theorem 3.3.2.

Corollary 3.3.4. *For $N \geq 2$ and ω a primitive N^{th} root of unity, we have*

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{(p;q)_{n+1}} t^{\mathrm{exc}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)} \left(\prod_{m=1}^{N-1} \omega^{m\cdot \mathrm{col}_m(\pi)}\right) = \sum_{l\geq 0} \frac{p^l}{1-zq^l} = \sum_{n,l\geq 0} z^n q^{nl} p^l.$$

Taking the coefficient of z^n on both sides we have

$$\sum_{\pi\in C_N\wr S_n} t^{\mathrm{exc}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)} \left(\prod_{m=1}^{N-1} \omega^{m\cdot \mathrm{col}_m(\pi)}\right) = (p;q)_n.$$

Proof. In Theorem 3.3.2 set $r_k = 1$ for $0 \leq k \leq N-1$, and set $s_m = \omega^m$ for $1 \leq m \leq N-1$, noting that $\sum_{m=1}^{N-1} \omega^m = -1$, to obtain

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{(p;q)_{n+1}} t^{\mathrm{exc}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)} \left(\prod_{m=1}^{N-1} \omega^{m\cdot \mathrm{col}_m(\pi)}\right)$$

$$= \sum_{l\geq 0} \frac{p^l(1-tq)(z;q)_l(tqz;q)_l}{(z;q)_{l+1}\,(-tq+1)\,(tqz;q)_l} = \sum_{l\geq 0} \frac{p^l(z;q)_l}{(z;q)_{l+1}} = \sum_{l\geq 0} \frac{p^l}{1-zq^l} = \sum_{n,l\geq 0} z^n q^{nl} p^l.$$

Thus

$$\frac{1}{(p;q)_{n+1}} \sum_{\pi\in C_N\wr S_n} t^{\mathrm{exc}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)} \left(\prod_{m=1}^{N-1} \omega^{m\cdot \mathrm{col}_m(\pi)}\right) = \sum_{l\geq 0} (q^n p)^l = \frac{1}{1-pq^n},$$

and the result follows from this.

□

For $N = 1$, we noticed that $C_1 \wr S_n \cong S_n$. Moreover, in this case the definitions of the colored permutation statistics $\mathrm{des}^*, \mathrm{exc}, \mathrm{maj}, \mathrm{fix}_0$ are exactly equal to the permutation statistics $\mathrm{des}, \mathrm{exc}, \mathrm{maj}, \mathrm{fix}$. Theorem 3.3.1 reduces to Theorem 1.3.5 of Shareshian and Wachs, and Theorem 3.3.2 reduces to Theorem 1.3.6 of Foata and Han.

For $N = 2$, we have $C_2 \wr S_n \cong B_n$, where the 1-colored letters are identified with the negative integers. As mentioned above, the order given in (3.1) for the alphabet of colored letters $\mathcal{E}$, does not agree with the usual order on the integers. So for instance $\mathrm{des_B}(2^1, 3^1, 1^0) = 2$, while $\mathrm{des}^*(2^1, 3^1, 1^0) = 1$. However, we do have the following proposition.

Proposition 3.3.5. *There exists a bijection* $\gamma : B_n \to C_2 \wr S_n$ *such that*

$$\mathrm{des_B}(\pi) = \mathrm{des}^*(\gamma(\pi)), \qquad \mathrm{neg}(\pi) = \mathrm{col}_1(\gamma(\pi)), \qquad \mathrm{maj}(\pi) = \mathrm{maj}(\gamma(\pi)),$$

$$\mathrm{fmaj}(\pi) = \mathrm{flagmaj}(\gamma(\pi)), \qquad \mathrm{exc}(\pi) = \mathrm{exc}(\gamma(\pi))$$

$$\mathrm{fexc}(\pi) = \mathrm{flagexc}(\gamma(\pi)), \qquad \mathrm{fix}^+(\pi) = \mathrm{fix}_0(\gamma(\pi)).$$

Proof. Given $\pi \in B_n$, $\gamma(\pi)$ is obtained by rewriting each run of consecutive negative letters in reverse order, then replacing positive letters with 0-colored letters, and negative letters with 1-colored letters. For example

$$\gamma(2, -1, -5, 3, 8, -7, -4, -6) = 2^0, 5^1, 1^1, 3^0, 8^0, 6^1, 4^1, 7^1.$$

It is clear that $\mathrm{neg}(\pi) = \mathrm{col}_1(\gamma(\pi))$, $\mathrm{exc}(\pi) = \mathrm{exc}(\gamma(\pi))$, and $\mathrm{fix}^+(\pi) = \mathrm{fix}_0(\gamma(\pi))$. It follows that $\mathrm{fexc}(\pi) = \mathrm{flagexc}(\gamma(\pi))$. We note that in general $\mathrm{fix}^-(\pi) \neq \mathrm{fix}_1(\gamma(\pi))$.

Clearly, $0 \in \mathrm{DES_B}(\pi)$ if and only if $0 \in \mathrm{DES}^*(\pi)$. Now consider $i > 0$. If $\pi(i)$ and $\pi(i+1)$ have different signs or are both positive, then it is clear that $i \in \mathrm{DES_B}(\pi)$ if and only if $i \in \mathrm{DES}^*(\pi)$. If $\pi(i)$ and $\pi(i+1)$ are both negative, then $i \in \mathrm{DES_B}(\pi)$ if and only if $|\pi(i)| < |\pi(i+1)|$ if and only if $i \in \mathrm{DES}^*(\pi)$. Consequently $\mathrm{des_B}(\pi) = \mathrm{des}^*(\gamma(\pi))$, $\mathrm{maj}(\pi) = \mathrm{maj}(\gamma(\pi))$, and $\mathrm{fmaj}(\pi) = \mathrm{flagmaj}(\gamma(\pi))$.

□

Proof of Theorem 2.4.13. Consider Theorem 3.3.2 in the case $N = 2$. Since $\mathrm{flagmaj}(\pi) = 2\mathrm{maj}(\pi) + \mathrm{col}_1(\pi)$ and $\mathrm{flagexc}(\pi) = 2\mathrm{exc}(\pi) + \mathrm{col}_1(\pi)$ for all $\pi \in C_2 \wr S_n$, we replace

$q \mapsto q^2$, $t \mapsto t^2$ and $s_1 \mapsto qts$, $r_1 \mapsto 1$, and $r_0 \mapsto r$. Then apply Proposition 3.3.5 to obtain

$$\sum_{\substack{n\geq 0\\ \pi\in B_n}} \frac{z^n}{(p;q^2)_{n+1}} q^{\mathrm{fmaj}(\pi)} t^{\mathrm{fexc}(\pi)} p^{\mathrm{des_B}(\pi)} r^{\mathrm{fix}^+(\pi)} s^{\mathrm{neg}(\pi)}$$

$$= \sum_{l\geq 0} \frac{p^l (1-t^2q^2)(z;q^2)_l (t^2q^2z;q^2)_l}{(rz;q^2)_{l+1}\left[(1+sqt)(z;q^2)_l - (t^2q^2+sqt)(t^2q^2z;q^2)_l\right]},$$

as desired.

□

If we also consider Theorem 3.3.1 in the case $N = 2$, we can again replace $q \mapsto q^2$, $t \mapsto t^2$ and $s_1 \mapsto qts$, $r_1 \mapsto 1$, and $r_0 \mapsto r$, and apply Proposition 3.3.5. The resulting formula is for the four-variate distribution of $(\mathrm{fmaj}, \mathrm{fexc}, \mathrm{neg}, \mathrm{fix}^+)$. This formula is already implied by Corollary 2.4.11 of Foata and Han, which is a five-variate distribution of $(\mathrm{fmaj}, \mathrm{fexc}, \mathrm{neg}, \mathrm{fix}^+, \mathrm{fix}^-)$.

Chapter 4

Colored Eulerian Quasisymmetric Functions

In the preceding chapter we introduced the colored permutation group, and presented our formulas for multivariate distributions of colored permutation statistics. In order to prove these results, we first introduce *colored Eulerian quasisymmetric functions*, which are a colored permutation analog of the Eulerian quasisymmetric functions of Shareshian and Wachs [31], [32] (see Section 1.7). Analogous to Theorem 1.7.8, we present a generating function formula for our colored Eulerian quasisymmetric functions, the proof of which is the focus of Chapters 5 and 6. We conclude Chapter 4 with proofs of Theorem 3.3.1 and Theorem 3.3.2.

4.1 Colored Eulerian Quasisymmetric Functions

To find a colored permutation analog of the Eulerian quasisymmetric functions, we must decide how to associate a fundamental quasisymmetric function to each colored permutation. For $n \in \mathbb{N}$, we extend the definition $\mathrm{DEX}(\pi) \subseteq [n-1]$ from Definition 1.7.1. We will use this set in our definition of colored Eulerian quasisymmetric functions. First, we construct a new ordered alphabet

$$\mathcal{A} := \left\{ \widetilde{1}^0 < \widetilde{2}^0 < \ldots < \widetilde{n}^0 \right\} < \mathcal{E},$$

where $\mathcal{E}$ has the same order as defined in (3.1) above, but now the letters with a tilde are less than the letters in $\mathcal{E}$.

Definition 4.1.1. Given any colored permutation $\pi \in C_N \wr S_n$ written as a word, construct a new word $\widetilde{\pi}$ of length n over $\mathcal{A}$ as follows: if $i \in \mathrm{EXC}(\pi)$, then replace π_i by $\widetilde{\pi}_i$, otherwise leave π_i alone. For example if $\pi = 2^0, 3^2, 1^0, 6^0, 5^0, 4^3 \in C_4 \wr S_6$, then $\mathrm{EXC}(\pi) = \{1, 4\}$ and $\widetilde{\pi} = \widetilde{2^0}, 3^2, 1^0, \widetilde{6^0}, 5^0, 4^3$. Then we define the set

$$\mathrm{DEX}(\pi) := \mathrm{DES}(\widetilde{\pi}),$$

where the descent set of any word over an ordered alphabet consists of all i such that $w_i > w_{i+1}$. Also define $\mathrm{DEX}(\theta) := 0$ where θ denotes the empty word. Using the example above we have

$$\mathrm{DEX}(2^0, 3^2, 1^0, 6^0, 5^0, 4^3) = \mathrm{DES}(\widetilde{2^0}, 3^2, 1^0, \widetilde{6^0}, 5^0, 4^3) = \{3, 5\}\,.$$

This is a natural extension of the former definition, and Lemma 4.2.1 below will show that it is also a useful definition.

For $T \subseteq [n-1]$, recall that the fundamental quasisymmetric function of degree n is given by (see Definition 1.5.3)

$$F_{T,n}(\mathbf{x}) := \sum_{\substack{i_1 \geq i_2 \geq \ldots \geq i_n \geq 1 \\ i_j > i_{j+1} \text{ if } j \in T}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Definition 4.1.2. Let N be arbitrary but fixed, for $n, j \in \mathbb{N}, \vec{\alpha} \in \mathbb{N}^N, \vec{\beta} \in \mathbb{N}^{N-1}$, we define

$$W_{n,j,\vec{\alpha},\vec{\beta}} := \left\{ \pi \in C_N \wr S_n : \mathrm{exc}(\pi) = j, \vec{\mathrm{fix}}(\pi) = \vec{\alpha}, \vec{\mathrm{col}}(\pi) = \vec{\beta} \right\}.$$

We then define the *fixed point colored Eulerian quasisymmetric functions* as

$$Q_{n,j,\vec{\alpha},\vec{\beta}} = Q_{n,j,\vec{\alpha},\vec{\beta}}(\mathbf{x}) := \sum_{\pi \in W_{n,j,\vec{\alpha},\vec{\beta}}} F_{\mathrm{DEX}(\pi),n}(\mathbf{x}).$$

Given $j \in \mathbb{N}$ and a particular cv-cycle type $\check{\lambda} = \left\{(\lambda_1, \vec{\beta^1}), (\lambda_2, \vec{\beta^2}), ..., (\lambda_k, \vec{\beta^k})\right\}$, we define

$$W_{\check{\lambda},j} := \left\{\pi \in C_N \wr S_n : \check{\lambda}(\pi) = \check{\lambda}, \mathrm{exc}(\pi) = j\right\},$$

where $\lambda_1 + \lambda_2 + ... + \lambda_k = n$.

We then define the *cv-cycle type colored Eulerian quasisymmetric functions* by

$$Q_{\check{\lambda},j} = Q_{\check{\lambda},j}(\mathbf{x}) := \sum_{\pi \in W_{\check{\lambda},j}} F_{\mathrm{DEX}(\pi),n}(\mathbf{x}).$$

Recall that $C_N \wr S_0 = \{\theta\}$ where θ denotes the empty word. Then $\mathrm{DEX}(\theta) = \emptyset$, $\mathrm{des}(\theta) = \mathrm{des}^*(\theta) = \mathrm{maj}(\theta) = \mathrm{exc}(\theta) = 0, \vec{\mathrm{fix}}(\theta) = \vec{\mathrm{col}}(\theta) = \vec{0}$, and $F_{\emptyset,0} = 1$, thus $Q_{0,0,\vec{0},\vec{0}} = 1$. We also note that the definitions of $Q_{n,j,\vec{\alpha},\vec{\beta}}$ and $Q_{\check{\lambda},j}$ agree with the definitions of $Q_{n,j,k}$ and $Q_{\lambda,j}$ from Definition 1.7.3, whenever $\vec{\beta} = \vec{0}$ or $\vec{\beta^i} = \vec{0}$ for each $\vec{\beta^i}$ of $\check{\lambda}$.

Recall from Section 1.4 that $H(z) = \sum_{i \geq 0} h_i(\mathbf{x}) z^i$ and h_i is the complete homogeneous symmetric function of degree i, and $E(z) = \sum_{i \geq 0} e_i(\mathbf{x}) z^i$ and e_i is the elementary symmetric function of degree i. We now state a main theorem of this thesis, which is a colored permutation analog of Theorem 1.7.8.

Theorem 4.1.3. *Fix $N \in \mathbb{P}$ and let $r^{\vec{\alpha}} = r_0^{\alpha_0} \cdots r_{N-1}^{\alpha_{N-1}}$ and $s^{\vec{\beta}} = s_1^{\beta_1} \cdots s_{N-1}^{\beta_{N-1}}$. Then*

$$\sum_{\substack{n,j \geq 0 \\ \vec{\alpha} \in \mathbb{N}^N \\ \vec{\beta} \in \mathbb{N}^{N-1}}} Q_{n,j,\vec{\alpha},\vec{\beta}}(\mathbf{x}) z^n t^j r^{\vec{\alpha}} s^{\vec{\beta}} = \frac{H(r_0 z)(1-t)\left(\prod_{m=1}^{N-1} E(-s_m z) H(r_m s_m z)\right)}{\left(1 + \sum_{m=1}^{N-1} s_m\right) H(tz) - \left(t + \sum_{m=1}^{N-1} s_m\right) H(z)}.$$

Chapters 5 and 6 will be devoted to proving Theorem 4.1.3. The next two corollaries state immediate and interesting consequences of this theorem.

Corollary 4.1.4. *The quasisymmetric function $Q_{n,j,\vec{\alpha},\vec{\beta}}(\mathbf{x})$ is actually symmetric.*

Corollary 4.1.5. *For $N \geq 2$ and ω a primitive N^{th} root of unity, we have*

$$\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} Q_{n,j,\vec{\alpha},\vec{\beta}}(\mathbf{x}) z^n t^j \left(\prod_{m=1}^{N-1} \omega^{m\cdot\beta_m}\right) = 1.$$

Proof. Recall from Equation 1.3 in Section 1.4 that $H(z)E(-z) = 1$. Also note that $\sum_{m=1}^{N-1} \omega^m = -1$. The corollary is obtained by setting $r_k = 1$ for $0 \leq k \leq N-1$, and setting $s_m = \omega^m$ for $1 \leq m \leq N-1$. □

4.2 Specializations

In Section 3.3 we presented our formulas for multivariate distributions of colored permutation statistics. This section is devoted to proving Theorem 3.3.1 and Theorem 3.3.2, which are obtained by applying specializations to Theorem 4.1.3. Recall the principal specializations from Definition 1.6.3

$$\mathbf{ps}(x_i) = q^{i-1},$$

and

$$\mathbf{ps}_l(x_i) = \begin{cases} q^{i-1} & \text{if } 1 \leq i \leq l \\ 0 & \text{if } i > l \end{cases}.$$

Also recall from Lemma 1.6.4 that

$$\mathbf{ps}(F_{T,n}(\mathbf{x})) = \frac{q^{\sum_{i\in T} i}}{(q;q)_n},$$

and

$$\sum_{l\geq 0}\mathbf{ps}_l(F_{T,n}(\mathbf{x}))p^l = \frac{p^{|T|+1}q^{\sum_{i\in T} i}}{(p;q)_{n+1}}.$$

Therefore, we will need the following lemma, whose proof is nearly identical to the proof of Lemma 1.7.2 (see [32, Lemma 2.2]).

Lemma 4.2.1. *For every $\pi \in C_N \wr S_n$ we have*

$$|\mathrm{DEX}(\pi)| = \begin{cases} \mathrm{des}^*(\pi) - 1 & \text{if } \pi_1 \neq 1^0 \\ \mathrm{des}^*(\pi) & \text{if } \pi_1 = 1^0 \end{cases} \tag{4.1}$$

and

$$\sum_{i\in \mathrm{DEX}(\pi)} i = \mathrm{maj}(\pi) - \mathrm{exc}(\pi). \tag{4.2}$$

Proof. First define the following sets

$$J := \{i \in [n-1] : i \notin \mathrm{EXC}(\pi) \text{ and } i+1 \in \mathrm{EXC}(\pi)\},$$

$$K := \{i \in [n-1] : i \in \mathrm{EXC}(\pi) \text{ and } i+1 \notin \mathrm{EXC}(\pi)\}.$$

As in the proof of [32, Lemma 2.2] we have $K \subseteq \mathrm{DES}(\pi)$ and

$$\mathrm{DEX}(\pi) = \left(\mathrm{DES}(\pi) \biguplus J\right) - K.$$

Let $J = \{j_1 < ... < j_t\}$ and $K = \{k_1 < ... < k_s\}$ and we consider two cases.

Case 1: Suppose $1 \notin \mathrm{EXC}(\pi)$.

Since n is never an excedance position, it follows that $t = s$ and $j_1 < k_1 < j_2 < k_2 < ... < j_t < k_t$, thus

$$\sum_{i\in \mathrm{DEX}(\pi)} i = \sum_{i\in \mathrm{DES}(\pi)} i - \sum_{m=1}^{t}(k_m - j_m).$$

Since

$$\mathrm{EXC}(\pi) = \biguplus_{m=1}^{t} \{j_m+1, j_m+2, ..., k_m\},$$

it follows that

$$\mathrm{exc}(\pi) = \sum_{m=1}^{t} (k_m - j_m)$$

and (4.2) holds.

Case 2: Suppose $1 \in \mathrm{EXC}(\pi)$.

This implies that $s = t+1$ and that $k_1 < j_1 < k_2 < ... < j_t < k_{t+1}$. Again using the fact that $\mathrm{DEX}(\pi) = (\mathrm{DES}(\pi) \uplus J) - K$, we write

$$\sum_{i \in \mathrm{DEX}(\pi)} i = \left(\sum_{i \in \mathrm{DES}(\pi)} i \right) - k_1 - \sum_{m=1}^{t} (k_{m+1} - j_m).$$

Since

$$\mathrm{EXC}(\pi) = \{1, 2, ..., k_1\} \biguplus_{m=1}^{t} \{j_m+1, j_m+2, ..., k_{m+1}\},$$

we have

$$\mathrm{exc}(\pi) = k_1 + \sum_{m=1}^{t} (k_{m+1} - j_m)$$

and (4.2) holds again.

To prove (4.1), first consider the case when $\pi_1 > 1^0$. As noted above, this implies that $s = t+1$ thus $|\mathrm{DEX}(\pi)| = \mathrm{des}(\pi) - 1 = \mathrm{des}^*(\pi) - 1$. If $\pi_1 = 1^0$, then $s = t$ thus $|\mathrm{DEX}(\pi)| = \mathrm{des}(\pi) = \mathrm{des}^*(\pi)$. If $\pi_1 < 1^0$, then $s = t$ and $|\mathrm{DEX}(\pi)| = \mathrm{des}(\pi) = \mathrm{des}^*(\pi) - 1$. □

We are now ready to show how Theorem 3.3.1 follows from Theorem 4.1.3.

Proof of Theorem 3.3.1. Using Lemma 1.6.4 and Lemma 4.2.1 we have

$$\mathbf{ps}\left(Q_{n,j,\vec{\alpha},\vec{\beta}}\right) = \sum_{\pi \in W_{n,j,\vec{\alpha},\vec{\beta}}} \frac{q^{\mathrm{maj}(\pi)} q^{-j}}{(q;q)_n}. \tag{4.3}$$

Since $h_n = F_{\emptyset,n}$ and $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$, it follows that

$$\mathbf{ps}\left(H(z(1-q))\right) = \exp_q(z) = \sum_{n\geq 0} \frac{z^n}{[n]_q!}. \tag{4.4}$$

Also, since $e_n = F_{[n-1],n}$, it follows that

$$\mathbf{ps}\left(E(z(1-q))\right) = \mathrm{Exp}_q(z) = \sum_{n\geq 0} \frac{z^n q^{\binom{n}{2}}}{[n]_q!}.$$

If we then set $z \mapsto z(1-q)$ and $t \mapsto tq$ in Theorem 4.1.3 and apply **ps** to both sides, we obtain the desired result. □

Proving Theorem 3.3.2 takes more work, but is similar to the proofs of [32, Lemma 2.4 and Corollary 1.4].

Proof of Theorem 3.3.2. Again, using Lemma 1.6.4 Lemma 4.2.1 we have

$$\sum_{l\geq 0} \mathbf{ps}_l\left(Q_{n,j,\vec{\alpha},\vec{\beta}}\right) p^l = \frac{1}{(p;q)_{n+1}} \sum_{\pi\in W_{n,j,\vec{\alpha},\vec{\beta}}} p^{|\mathrm{DEX}(\pi)|+1} q^{\mathrm{maj}(\pi)-j}$$

$$= \frac{1}{(p;q)_{n+1}} \sum_{\substack{\pi\in W_{n,j,\vec{\alpha},\vec{\beta}} \\ \pi(1)\neq 1^0}} p^{\mathrm{des}^*(\pi)} q^{\mathrm{maj}(\pi)-j} + \frac{p}{(p;q)_{n+1}} \sum_{\substack{\pi\in W_{n,j,\vec{\alpha},\vec{\beta}} \\ \pi(1)= 1^0}} p^{\mathrm{des}^*(\pi)} q^{\mathrm{maj}(\pi)-j}.$$

If we define the following quantities

$$X_{n,j,\vec{\alpha},\vec{\beta}}(p,q) := \sum_{l\geq 0} \mathbf{ps}_l\left(Q_{n,j,\vec{\alpha},\vec{\beta}}\right) p^l,$$

$$Y_{n,j,\vec{\alpha},\vec{\beta}}(p,q) := \frac{1}{(p;q)_{n+1}} \sum_{\substack{\pi\in W_{n,j,\vec{\alpha},\vec{\beta}} \\ \pi(1)= 1^0}} p^{\mathrm{des}^*(\pi)} q^{\mathrm{maj}(\pi)-j},$$

$$a_{n,j,\vec{\alpha},\vec{\beta}}(p,q) := \sum_{\pi\in W_{n,j,\vec{\alpha},\vec{\beta}}} p^{\mathrm{des}^*(\pi)} q^{\mathrm{maj}(\pi)},$$

then they are related by the following equation

$$\frac{a_{n,j,\vec{\alpha},\vec{\beta}}(p,q)}{q^j(p;q)_{n+1}} = X_{n,j,\vec{\alpha},\vec{\beta}}(p,q) + (1-p)Y_{n,j,\vec{\alpha},\vec{\beta}}(p,q). \tag{4.5}$$

Define a bijection

$$\gamma : \left\{\pi \in W_{n,j,\vec{\alpha},\vec{\beta}} : \pi(1) = 1^0\right\} \to W_{n-1,j,\vec{\alpha^{(1)}},\vec{\beta}},$$

where $\vec{\alpha^{(1)}} := (\alpha_0 - 1, \alpha_1, \alpha_2, ..., \alpha_{N-1})$, by setting

$$\gamma(\pi)(i) = (|\pi(i+1)| - 1)^{\epsilon_{i+1}}$$

for $1 \leq i \leq n-1$.

For example, if $\pi = 1^0, 3^1, 2^1, 4^2$ in one-line notation, then $\gamma(\pi) = 2^1, 1^1, 3^2$. It is clear that γ is well-defined and a bijection, we would also like to know how γ changes the starred descent number and the major index. Let π be any colored permutation in the domain of γ. Since $\pi(1) = 1^0$, $0 \notin \mathrm{DES}^*(\pi)$. If $i \geq 2$, then $i \in \mathrm{DES}^*(\pi)$ if and only if $i-1 \in \mathrm{DES}^*(\gamma(\pi))$. Also, $1 \in \mathrm{DES}^*(\pi)$ if and only if $\pi(2) < 1^0$ if and only if $0 \in \mathrm{DES}^*(\gamma(\pi))$. It follows that $\mathrm{des}^*(\pi) = \mathrm{des}^*(\gamma(\pi))$ and $\mathrm{maj}(\pi) = \mathrm{maj}(\gamma(\pi)) + \mathrm{des}^*(\pi)$. Thus

$$\begin{aligned} Y_{n,j,\vec{\alpha},\vec{\beta}}(p,q) &= \frac{1}{(p;q)_{n+1}} \sum_{\pi \in W_{n-1,j,\vec{\alpha^{(1)}},\vec{\beta}}} p^{\mathrm{des}^*(\pi)} q^{\mathrm{maj}(\pi)+\mathrm{des}^*(\pi)-j} \\ &= \frac{a_{n-1,j,\vec{\alpha^{(1)}},\vec{\beta}}(qp,q)}{q^j(p;q)_{n+1}}. \end{aligned}$$

Note that $(1-p)/(p;q)_{n+1} = 1/(qp;q)_n$, so that when we substitute this expression for $Y_{n,j,\vec{\alpha},\vec{\beta}}(p,q)$ back in to (4.5) we get

$$\frac{a_{n,j,\vec{\alpha},\vec{\beta}}(p,q)}{q^j(p;q)_{n+1}} = X_{n,j,\vec{\alpha},\vec{\beta}}(p,q) + \frac{a_{n-1,j,\alpha^{\vec{(1)}},\vec{\beta}}(qp,q)}{q^j(qp;q)_n}.$$

Let $\alpha^{\vec{(h)}} := (\alpha_0 - h, \alpha_1, \alpha_2, ..., \alpha_{N-1})$, so that we can iterate this recurrence relation to obtain

$$\frac{a_{n,j,\vec{\alpha},\vec{\beta}}(p,q)}{q^j(p;q)_{n+1}} = \sum_{h=0}^{\alpha_0} X_{n-h,j,\alpha^{\vec{(h)}},\vec{\beta}}(q^h p, q).$$

Recalling the definition of $X_{n,j,\vec{\alpha},\vec{\beta}}(p,q)$, we have

$$a_{n,j,\vec{\alpha},\vec{\beta}}(p,q) = q^j(p;q)_{n+1} \sum_{h=0}^{\alpha_0} X_{n-h,j,\alpha^{\vec{(h)}},\vec{\beta}}(q^h p, q)$$

$$= (p;q)_{n+1} \sum_{l\geq 0} p^l \sum_{h=0}^{\alpha_0} \mathbf{ps}_l\left(Q_{n-h,j,\alpha^{\vec{(h)}},\vec{\beta}}\right) q^{hl+j}.$$

Lastly, we need the fact that $\mathbf{ps}_l(H(z)) = 1/(z;q)_l$ and $\mathbf{ps}_l(E(z)) = (-z;q)_l$ (see [34]) to complete the proof,

$$\sum_{\substack{n\geq 0\\ \pi\in C_N\wr S_n}} \frac{z^n}{(p;q)_{n+1}} t^{\mathrm{exc}(\pi)} r^{\vec{\mathrm{fix}}(\pi)} s^{\vec{\mathrm{col}}(\pi)} q^{\mathrm{maj}(\pi)} p^{\mathrm{des}^*(\pi)}$$

$$= \sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} \frac{z^n}{(p;q)_{n+1}} t^j r^{\vec{\alpha}} s^{\vec{\beta}} a_{n,j,\vec{\alpha},\vec{\beta}}(p,q)$$

$$= \sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} z^n t^j r^{\vec{\alpha}} s^{\vec{\beta}} \sum_{l\geq 0} p^l \sum_{h=0}^{\alpha_0} \mathbf{ps}_l\left(Q_{n-h,j,\alpha^{\vec{(h)}},\vec{\beta}}\right) q^{hl+j}$$

$$= \sum_{l\geq 0} p^l \sum_{h\geq 0} (zr_0q^l)^h \mathbf{ps}_l \left(\sum_{\substack{n,j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N,\alpha_0\geq h\\ \vec{\beta}\in\mathbb{N}^{N-1}}} Q_{n-h,j,\alpha^{\vec{(h)}},\vec{\beta}} z^{n-h}(tq)^j r^{\alpha^{\vec{(h)}}} s^{\vec{\beta}} \right).$$

By Theorem 4.1.3 this is equal to

$$= \sum_{l\geq 0} \frac{p^l}{1 - zr_0q^l} \mathbf{ps}_l \left(\frac{H(r_0z)(1-tq)\left(\prod_{m=1}^{N-1} E(-s_mz)H(r_ms_mz)\right)}{\left(1+\sum_{m=1}^{N-1} s_m\right) H(tqz) - \left(tq + \sum_{m=1}^{N-1} s_m\right) H(z)} \right)$$

$$= \sum_{l\geq 0} \frac{p^l \left(\frac{1}{(r_0z;q)_l}\right)(1-tq)\left(\prod_{m=1}^{N-1}(s_mz;q)_l \frac{1}{(r_ms_mz;q)_l}\right)}{(1-zr_0q^l)\left[\left(1+\sum_{m=1}^{N-1} s_m\right)\frac{1}{(tqz;q)_l} - \left(tq+\sum_{m=1}^{N-1} s_m\right)\frac{1}{(z;q)_l}\right]}$$

$$= \sum_{l\geq 0} \frac{p^l(1-tq)(z;q)_l(tqz;q)_l\left(\prod_{m=1}^{N-1}(s_mz;q)_l\right)\left(\prod_{m=1}^{N-1}(r_ms_mz;q)_l\right)^{-1}}{(r_0z;q)_{l+1}\left[\left(1+\sum_{m=1}^{N-1} s_m\right)(z;q)_l - \left(tq+\sum_{m=1}^{N-1} s_m\right)(tqz;q)_l\right]}.$$

□

Chapter 5

Colored Necklaces and Colored Ornaments

As mentioned earlier, Chapters 5 and 6 will be devoted to the proof of Theorem 4.1.3. In this chapter we introduce colored necklaces and colored ornaments. They are a multicolored generalization of the bicolored necklaces and bicolored ornaments of Shareshian and Wachs (see Section 1.7), which are in turn generalizations of the monochromatic necklaces and ornaments of Gessel and Reutenauer (see Section 1.6). We construct a bijection which shows that the cv-cycle type colored Eulerian quasisymmetric functions can be expressed in terms of weights of colored ornaments. This bijection and its proof are similar to the bicolored versions appearing in Theorem 1.7.5 (see also [31], [32]). We will conclude this chapter by showing that Theorem 4.1.3 is equivalent to a certain recurrence relation.

5.1 A Combinatorial Description of the Colored Eulerian Quasisymmetric Functions

Definition 5.1.1. Let $s = (s_1 \geq s_2 \geq ... \geq s_n)$ be a weakly decreasing sequence of positive integers. For $\pi \in C_N \wr S_n$, we say that s is DEX(π)-*compatible* if $i \in \mathrm{DEX}(\pi)$ implies that $s_i > s_{i+1}$. Then define the set $\mathrm{Com}(\check{\lambda}, j)$ as follows

$$\mathrm{Com}(\check{\lambda}, j) := \left\{(\pi, s) : \check{\lambda}(\pi) = \check{\lambda}, \mathrm{exc}(\pi) = j, \text{ and } s \text{ is DEX}(\pi)\text{-compatible}\right\}.$$

Define the *weight* of the pair (π, s), denoted $\mathrm{wt}((\pi, s))$, to be the monomial

$$\mathrm{wt}((\pi, s)) := x_{s_1} x_{s_2} \cdots x_{s_n}.$$

From this definition, it follows that we can express the colored Eulerian quasisymmetric functions as

$$Q_{\check{\lambda},j} = \sum_{(\pi,s)\in \mathrm{Com}(\check{\lambda},j)} \mathrm{wt}((\pi, s)).$$

Let $\mathcal{B}$ be an infinite totally ordered alphabet with letters and order given by

$$\mathcal{B} := \left\{ 1^0 < 1^1 < ... < 1^{N-1} < \overline{1^0} < 2^0 < 2^1 < ... < 2^{N-1} < \overline{2^0} < ... \right\}. \tag{5.1}$$

Let u be any positive integer. We call $\overline{u}$ a *barred* letter, while letters without a bar are called *unbarred*. For $0 \leq m \leq N - 1$ we say a letter is *m-colored* if it is of the form u^m, we also say that $\overline{u^0}$ is 0-colored. Note that only 0-colored letters may be barred. The *absolute value* of a letter is the positive integer obtained by removing any colors or bars, so $|u^m| = \left|\overline{u^0}\right| = u$.

Next we review the notion of a circular primitive word. The cyclic group of order n acts on the set of words of length n by cyclic rotation. So if z is a generator of this cyclic group and $v = v_1, v_2, ..., v_n$, then $z \cdot v = v_2, v_3, ..., v_n, v_1$. A *circular word*, denoted (v), is the orbit of v under this action. A circular word (v) is called *primitive* if the size of the orbit is equal to the length of the word v. Equivalently, a word is not primitive if it is a proper power of another word. For example the circular word $(\overline{3^0}, 3^0, 3^2)$ is primitive, while $(4^2, 3^1, 4^2, 3^1)$ is not primitive since $4^2, 3^1, 4^2, 3^1 = w^2$ where $w = 4^2, 3^1$. One can visualize (v) as a circular arrangement of letters, called a necklace, obtained from v by attaching the first and last letters together. For each position of this necklace one can read the letters in a clockwise direction to obtain an element from the orbit of the circular action (see also Sections 1.6 and 1.7, and

[15],[23],[31],[32]).

Definition 5.1.2. A *colored necklace* is a circular primitive word (v) over the alphabet $\mathcal{B}$, such that

1. Every barred letter is followed by a letter of lesser or equal absolute value,

2. Every 0-colored unbarred letter is followed by a letter of greater or equal absolute value,

3. Words of length one may not consist of a single barred letter.

Note that letters with color greater than zero may be followed by any letter from $\mathcal{B}$. Also note that for a colored necklace (v), we define its color vector in the same way we did for colored permutations. That is,

$$\vec{\text{col}}((v)) = \vec{\beta} \in \mathbb{N}^{N-1}$$

means that v has exactly β_i letters with color $i \in [N-1]$.

For $v = v_1, v_2, ..., v_n$, we define the *weight* of the colored necklace (v), denoted $\text{wt}((v))$, to be the monomial

$$\text{wt}((v)) := x_{|v_1|} x_{|v_2|} \cdots x_{|v_n|}.$$

A *colored ornament* is a multiset of colored necklaces. Formally, a colored ornament R is a map with finite support from the set η of colored necklaces to $\mathbb{N}$. We define the *weight* of a colored ornament R, denoted $\text{wt}(R)$, to be

$$\text{wt}(R) := \prod_{(v) \in \eta} \text{wt}((v))^{R((v))}.$$

Similar to the cv-cycle type of a colored permutation, the *cv-cycle type* $\check{\lambda}(R)$ of a colored ornament R is the multiset

$$\check{\lambda}(R) = \left\{(\lambda_1, \vec{\beta^1}), (\lambda_2, \vec{\beta^2}), ..., (\lambda_k, \vec{\beta^k})\right\}$$

where each colored necklace of R corresponds to precisely one pair $(\lambda_i, \vec{\beta^i})$ where this colored necklace has length λ_i and color vector $\vec{\beta^i}$.

For example let $N = 4$ and

$$R = (\overline{5^0}, \overline{5^0}, 5^2, 3^0, 3^0, 6^1, \overline{7^0}), (3^3, 3^1), (3^3, 3^1), (4^0, \overline{5^0}), (2^0), (1^3).$$

Then

$$\check{\lambda}(R) = \{(7,(1,1,0)), (2,(1,0,1)), (2,(1,0,1)), (2,(0,0,0), (1,(0,0,0)), (1,(0,0,1))\}.$$

Let $\mathcal{R}(\check{\lambda}, j)$ denote the set of all colored ornaments of cv-cycle type $\check{\lambda}$, and exactly j barred letters.

Theorem 5.1.3. *There exists a weight preserving bijection* $f : \mathrm{Com}(\check{\lambda}, j) \to \mathcal{R}(\check{\lambda}, j)$.

Proof. Let $(\pi, s) \in \mathrm{Com}(\check{\lambda}, j)$ where $s = (s_1, s_2, ..., s_n)$. First we map (π, s) to the pair (σ, α) where $\sigma \in S_n$ and α is a weakly decreasing sequence of n letters from $\mathcal{B}$. We let $\sigma = |\pi|$, and we obtain α from s by replacing each s_i with one of the following

$$s_i \mapsto \overline{s_i^0} \qquad \text{if } i \in \mathrm{EXC}(\pi),$$

$$s_i \mapsto s_i^m \qquad \text{if } \epsilon_i = m \text{ and } i \notin \mathrm{EXC}(\pi).$$

Then for each cycle $(i_1, ..., i_k)$ appearing in σ, add the colored necklace $(\alpha_{i_1}, ..., \alpha_{i_k})$ to the multiset $f((\pi, s))$.

When doing an example, it helps to write the identity permutation as word on top, below that the word for the colored permutation π, and below that the sequence

s, as follows

$$\begin{array}{rcllllllll} \mathrm{Id} & = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \pi & = & 8^1 & 3^0 & 2^2 & 5^0 & 1^2 & 6^1 & 4^0 & 7^0 \\ \widetilde{\pi} & = & 8^1 & \widetilde{3^0} & 2^2 & \widetilde{5^0} & 1^2 & 6^1 & 4^0 & 7^0 \\ s & = & 6 & 5 & 5 & 4 & 4 & 4 & 4 & 3 \end{array}.$$

One can check that $\mathrm{DEX}(\pi) = \{1, 3\}$ so that s is $\mathrm{DEX}(\pi)$-compatible (note that s has an optional decrease from s_7 to s_8). Then

$$\sigma = (1, 8, 7, 4, 5)(2, 3)(6),$$

$$\alpha = 6^1, \overline{5^0}, 5^2, \overline{4^0}, 4^2, 4^1, 4^0, 3^0,$$

and

$$f((\pi, s)) = (6^1, 3^0, 4^0, \overline{4^0}, 4^2), (\overline{5^0}, 5^2), (4^1).$$

It is clear that f preserves cv-cycle type, weight, and the number of excedances of π is equal to the number of barred letters in $f((\pi, s))$. Since f preserves cv-cycle type, and since fixed points of any color cannot be excedances, it is also clear that the colored necklaces in $f((\pi, s))$ obey rule 3 in Definition 5.1.2. To prove that rules 1 and 2 are also obeyed, we first prove the following

<u>Claim:</u> α is a weakly decreasing sequence with respect to the order on $\mathcal{B}$ given in (5.1).

Indeed, since $|\alpha_i| = s_i$, we know that $|\alpha_i| = s_i \geq s_{i+1} = |\alpha_{i+1}|$. So suppose $s_i = s_{i+1}$ and $\alpha_i = s_i^{m_1}$ while $\alpha_{i+1} = \overline{s_{i+1}^0}$. This means that $i \notin \mathrm{EXC}(\pi)$ while $i + 1 \in \mathrm{EXC}(\pi)$. Thus $i \in \mathrm{DEX}(\pi)$ but $s_i = s_{i+1}$, contradicting that s is $\mathrm{DEX}(\pi)$-compatible. Also, if $\alpha_{i+1} = s_{i+1}^{m_2}$ with $m_1 < m_2$, then i is again an element of $\mathrm{DEX}(\pi)$. This proves the claim.

To check that rule 1 is obeyed, suppose α_i is a barred letter. Then if $\sigma(i) = j$, we must have $i < j$. By the claim above, $\alpha_i \geq \alpha_j$. To check rule 2, suppose α_i is

0-colored and unbarred. Then $\sigma(i) = j$ with $i > j$ and the claim tells us that $\alpha_i \leq \alpha_j$.

To show that f is well-defined, it remains to show that each word in $f((\pi, s))$ is primitive. Suppose $(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_k})$ is a nonprimitive colored necklace in $f((\pi, s))$ obtained from the cycle $(i_1, i_2, ..., i_k)$ of σ, where i_1 is the smallest element of the cycle. Thus for some divisor d of k we have $(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_k}) = (\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_d})^{k/d}$. In particular we have

$$\alpha_{i_1} = \alpha_{i_{d+1}} \text{ and } i_1 < i_{d+1}. \tag{5.2}$$

Since the sequence α is weakly decreasing, this implies that $\alpha_i = \alpha_{i_1}$ for all $i \in B := \{i : i_1 \leq i \leq i_{d+1}\}$, and $B \cap \mathrm{DEX}(\pi) = \emptyset$. Moreover, either $B \cap \mathrm{EXC}(\pi) = B$ or $\emptyset$, and $\epsilon_i = \epsilon_{i_1}$ for all $i \in B$. So in fact $B \cap \mathrm{DES}(\pi) = \emptyset$ and

$$\sigma(i_1) < \sigma(i_1 + 1) < \sigma(i_1 + 2) < ... < \sigma(i_{d+1}). \tag{5.3}$$

From (5.3), we find that $i_2 = \sigma(i_1) < \sigma(i_{d+1}) = i_{d+2}$. Since $(\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_k}) = (\alpha_{i_1}, \alpha_{i_2}, ..., \alpha_{i_d})^{k/d}$, we now have $\alpha_{i_2} = \alpha_{i_{d+2}}$ with $i_2 < i_{d+2}$, similar to (5.2). The same argument will show $i_3 = \sigma(i_2) < \sigma(i_{d+2}) = i_{d+3}$, and we can repeat this argument until $i_{k-d+1} = \sigma(i_{k-d}) < \sigma(i_k) = i_1$, contradicting the minimality of i_1.

Thus far we have proved that $f : \mathrm{Com}(\check{\lambda}, j) \to \mathcal{R}(\check{\lambda}, j)$ is well-defined. Next, we will describe the inverse map $g : \mathcal{R}(\check{\lambda}, j) \to \mathrm{Com}(\check{\lambda}, j)$ and show that it is well-defined. Let $R \in \mathcal{R}(\check{\lambda}, j)$ and if R has any repeated colored necklaces, fix some total order on these repeated colored necklaces. For each position x of each colored necklace, let w_x denote the infinite word obtained by reading the colored necklace clockwise starting at position x. Let $w_x > w_y$ mean that w_x is lexicographically larger than w_y, using the order on $\mathcal{B}$ (see (5.1)). If $w_x = w_y$ for distinct positions x, y, then it must be that x, y are positions in distinct copies of a repeated colored necklace, since words are primitive. We can then break the tie using the total order on repeated colored necklaces.

This totally orders all the positions on all of the colored necklaces of R by letting $x > y$ if and only if

(1) $w_x > w_y$

or

(2) $w_x = w_y$ and x is in a colored necklace which is larger in the total order on these repeated colored necklaces.

If x is the i^{th} largest position of R, then we replace the letter in position x by i. After doing this for each position, we have a permutation denoted $\sigma(R) \in S_n$ written in cycle form. We then obtain a colored permutation denoted $\pi(R)$ by setting $\pi(R)(i) = (\sigma(R)(i))^{\epsilon_x}$ where ϵ_x is the color of the letter formerly occupying position x. A sequence $s(R)$ is obtained by simply taking the weakly decreasing rearrangement of the absolute values of all the letters appearing in R. We then set $g(R) = (\pi(R), s(R))$.

For example, consider the following colored ornament

$$R = \; (5^1, \;\; \overline{5^0}, \;\; 3^0) \;\; < \;\; (5^1, \;\; \overline{5^0}, \;\; 3^0), \;\; (4^1, \;\; 3^0, \;\; 3^1, \;\; 4^1), \;\; (3^1), \;\; (3^0) \; .$$

By ranking each position, we obtain $\sigma(R)$ as follows

$$\begin{array}{rccccccccccccccc} R = & (5^1, & \overline{5^0}, & 3^0) & < & (5^1, & \overline{5^0}, & 3^0), & (4^1, & 3^0, & 3^1, & 4^1), & (3^1), & (3^0) \\ \sigma(R) = & (4, & 2, & 10) & & (3, & 1, & 9) & (6, & 11, & 7, & 5) & (8) & (12) \end{array} .$$

So $g(R)$ is the pair

$$\begin{array}{rcccccccccccc} \mathrm{Id} = & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ \pi(R) = & 9^0 & 10^0 & 1^1 & 2^1 & 6^1 & 11^1 & 5^1 & 8^1 & 3^0 & 4^0 & 7^0 & 12^0 \\ s(R) = & 5 & 5 & 5 & 5 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 \end{array} .$$

It is easy to see that $\check{\lambda}(\pi(R)) = \check{\lambda}$, and that g does not depend on the ordering of repeated colored necklaces in R. Also, it follows from rules 1 and 2 in Definition 5.1.2 that if x is the i^{th} largest position, then the letter in position x is barred if and only if $i \in \text{EXC}(\pi(R))$, thus $\text{exc}(\pi(R)) = j$.

To show that g is well-defined, it remains to show that the sequence s is DEX(π)-compatible. Suppose $s_i = s_{i+1}$. Let x be the i^{th} largest position in R, and y be the $(i+1)^{\text{th}}$ largest position, in particular $x > y$. Given any word w, let $F(w)$ denote the first letter of the word. So $s_i = s_{i+1}$ means that $|F(w_x)| = |F(w_y)|$. If $F(w_x) > F(w_y)$, then one can easily check that $i \notin \text{DEX}(\pi)$ as desired.

So assume $F(w_x) = F(w_y)$. Let u denote the position immediately following x cyclically, and let v denote the position immediately following y. Since $x > y$ it follows that $u > v$. Since $\sigma(R)(i)$ is equal to the rank of position u, and $\sigma(R)(i+1)$ is equal to the rank of position v, we have $\sigma(R)(i) < \sigma(R)(i+1)$. Since $F(w_x) = F(w_y)$, then $\epsilon_x = \epsilon_y$ and this implies that $i \notin \text{DES}(\pi(R))$. Moreover, since either $i, i+1 \in \text{EXC}(\pi(R))$ or $i, i+1 \notin \text{EXC}(\pi(R))$ we have $i \notin \text{DEX}(\pi(R))$. Thus the map g is well-defined.

The proof of Theorem 5.1.3 will be complete once we show that $f \circ g = g \circ f = \text{id}$. It not hard to see that $f \circ g = \text{id}$, and that if we apply $g \circ f$ to (π, s) we will recover the sequence s. So we need to prove that applying $g \circ f$ to (π, s) will also bring us back to the colored permutation π.

Let $(\pi, s) \mapsto (\sigma, \alpha)$ in the first step of f, and let p_i be the position occupied by α_i in $f((\pi, s))$. Order the cycles of σ from largest to smallest so that the minimum elements of the cycles increase. Use this to order repeated colored necklaces in $f((\pi, s))$ so that we know how to break ties if $w_{p_i} = w_{p_j}$. We want to show the following:

(i) if $i < j$, then $w_{p_i} \geq w_{p_j}$,

and

(ii) if $i < j$ and $w_{p_i} = w_{p_j}$, then i is in a cycle of σ whose minimum element is less than the minimum element of the cycle containing j.

In order to prove both (i) and (ii), we first establish that

(iii) If $i < j$ and $w_{p_i} \leq w_{p_j}$, then $\alpha_i = \alpha_j$ and $\sigma(i) < \sigma(j)$.

Indeed, α is weakly decreasing so that $i < j$ implies $\alpha_i \geq \alpha_j$. And $w_{p_i} \leq w_{p_j}$ implies that $\alpha_i = F(w_{p_i}) \leq F(w_{p_j}) = \alpha_j$, so $\alpha_i = \alpha_j$. This implies that $\alpha_i = \alpha_{i+1} = ... = \alpha_j$, which means $s_i = s_{i+1} = ... = s_j$, $\epsilon_i = \epsilon_{i+1} = ... = \epsilon_j$, and all the letters $\alpha_i, \alpha_{i+1}, ..., \alpha_j$ are either all barred or all unbarred. Since s is DEX(π)-compatible, $k \notin \text{DEX}(\pi)$ for $i \leq k \leq j-1$. This implies that $|\pi(i)| < |\pi(i+1)| < ... < |\pi(j)|$, which means that $\sigma(i) < \sigma(i+1) < ... < \sigma(j)$. This establishes (iii).

To prove (i), suppose $i < j$ but $w_{p_i} < w_{p_j}$. Using (iii), we have $\sigma(i) < \sigma(j)$ and $\alpha_i = \alpha_j$. Since $F(w_{p_i}) = F(w_{p_j})$, we must have $w_{p_{\sigma(i)}} < w_{p_{\sigma(j)}}$. Now apply (iii) again with $\sigma(i), \sigma(j)$ taking the role of i, j. Then $\sigma^2(i) < \sigma^2(j)$ and $\alpha_{\sigma(i)} = \alpha_{\sigma(j)}$, which implies $w_{p_{\sigma^2(i)}} < w_{p_{\sigma^2(j)}}$. Apply (iii) again to obtain $\sigma^3(i) < \sigma^3(j)$, $\alpha_{\sigma^2(i)} = \alpha_{\sigma^2(j)}$ and $w_{p_{\sigma^3(i)}} < w_{p_{\sigma^3(j)}}$. By repeating this argument, we see that $\alpha_{\sigma^m(i)} = \alpha_{\sigma^m(j)}$ for all m, but this implies that $w_{p_i} = w_{p_j}$, a contradiction.

To prove (ii), suppose $i < j$ and $w_{p_i} = w_{p_j}$. Using (iii) we have $\sigma(i) < \sigma(j)$, and $w_{p_i} = w_{p_j}$ implies $w_{p_{\sigma(i)}} = w_{p_{\sigma(j)}}$. Applying (iii) again we have $\sigma^2(i) < \sigma^2(j)$ and $w_{p_{\sigma^2(i)}} = w_{p_{\sigma^2(j)}}$. Repeating this argument, we have $\sigma^m(i) < \sigma^m(j)$ for all m. Thus the cycle of σ containing i has a smaller minimum element, than the cycle containing j.

This completes the proof that $f : \text{Com}(\check{\lambda}, j) \to \mathcal{R}(\check{\lambda}, j)$ is a bijection. □

From Definition 5.1.1, we had expressed the cv-cycle type colored Eulerian quasisymmetric functions as a sum of weights of pairs (π, s). Using Theorem 5.1.3 we can now express it as a sum of weights of colored ornaments.

Corollary 5.1.4. *For all $\check{\lambda}$ and j we have*

$$Q_{\check{\lambda},j} = \sum_{(\pi,s)\in \mathrm{Com}(\check{\lambda},j)} \mathrm{wt}((\pi,s)) = \sum_{R\in \mathcal{R}(\check{\lambda},j)} \mathrm{wt}(R).$$

Remark 5.1.5. It is possible to use Corollary 5.1.4 to prove that $Q_{\check{\lambda},j}$ is also a symmetric function. One method is to use the colored ornament description of $Q_{\check{\lambda},j}$ to derive a colored analog of [32, Corollary 6.1], which involves plethysm (see [36]). Another possible method is a bijective approach as in [32, Theorem 5.8].

5.2 A Recurrence for the Fixed Point Colored Eulerian Quasisymmetric Functions

From Corollary 5.1.4, we obtain the following results concerning the fixed point colored Eulerian quasisymmetric functions.

Corollary 5.2.1. *For $n, j \in \mathbb{N}$, $\vec{\alpha} \in \mathbb{N}^N$, $\vec{\beta} \in \mathbb{N}^{N-1}$ we have*

$$Q_{n,j,\vec{\alpha},\vec{\beta}} = Q_{n-|\vec{\alpha}|,j,\vec{0},\vec{\beta}-(\alpha_1,\alpha_2,\ldots,\alpha_{N-1})} \prod_{k=0}^{N-1} h_{\alpha_k},$$

where $|\vec{\alpha}| := \sum_{k=0}^{N-1} \alpha_k$, and recall that h_{α_k} is the complete homogeneous symmetric function of degree α_k.

Proof. Corollary 5.1.4 implies that $Q_{n,j,\vec{\alpha},\vec{\beta}}$ is equal to the sum of weights of all colored ornaments with exactly α_k colored necklaces of length one consisting of a single k-colored letter, for $0 \leq k \leq N-1$. The result now follows from the fact that the weight of α_k colored necklaces of length one consisting of a single k-colored letter is h_{α_k}.

□

Corollary 5.2.2. *Theorem 4.1.3 is equivalent to*

$$\sum_{\substack{n,j\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} Q_{n,j,\vec{0},\vec{\beta}} z^n t^j s^{\vec{\beta}} = \frac{(1-t)\prod_{m=1}^{N-1} E(-s_m z)}{\left(1+\sum_{m=1}^{N-1} s_m\right) H(tz) - \left(t+\sum_{m=1}^{N-1} s_m\right) H(z)}. \tag{5.4}$$

Proof. In one direction, take the formula from Theorem 4.1.3 and simply set $r_0 = r_1 = ... = r_{N-1} = 0$. For the other direction, start with (5.4) and multiply both sides by $H(r_0 z)\prod_{m=1}^{N-1} H(r_m s_m z)$. The left hand side becomes

$$\left(\sum_{n\geq 0} r_0^n z^n h_n\right)\left(\prod_{m=1}^{N-1}\sum_{n\geq 0}(r_m s_m)^n z^n h_n\right)\left(\sum_{\substack{n,j\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} Q_{n,j,\vec{0},\vec{\beta}} z^n t^j s^{\vec{\beta}}\right)$$

$$= \sum_{n\geq 0} z^n \sum_{\substack{j\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}\\ \vec{\alpha}\in\mathbb{N}^{N}}} Q_{n-|\vec{\alpha}|,j,\vec{0},\vec{\beta}} t^j s^{\vec{\beta}+(\alpha_1,\ldots,\alpha_{N-1})} r^{\vec{\alpha}} \prod_{k=0}^{N-1} h_{\alpha_k}.$$

By Corollary 5.2.1, this is equal to the left hand side of Theorem 4.1.3. □

Corollary 5.2.3. *Equation* (5.4) *is equivalent to the recurrence relation*

$$Q_{n,j,\vec{0},\vec{\beta}} = \sum_{\substack{0\leq i\leq n-2\\ j-n+i<k<j}} Q_{i,k,\vec{0},\vec{\beta}} h_{n-i} + \sum_{m=1}^{N-1} \chi\left(\beta_m > 0\right)\left(\sum_{\substack{0\leq i\leq n-1\\ j-n+i<k\leq j}} Q_{i,k,\vec{0},\vec{\beta}(\hat{m})} h_{n-i}\right)$$

$$+\chi\left(j=0\right)\chi\left(\left|\vec{\beta}\right| = n\right)(-1)^n \prod_{m=1}^{N-1} e_{\beta_m},$$

where if $\vec{\beta} = (\beta_1, \beta_2, ..., \beta_{N-1})$ *then*

$$\vec{\beta}(\hat{m}) := (\beta_1, ..., \beta_{m-1}, \beta_m - 1, \beta_{m+1}, ..., \beta_{N-1}).$$

(Recall that $\chi(P) = 0$ *if the statement* P *is false, and* $\chi(P) = 1$ *if the statement* P *is true.)*

Proof. Let

$$I := \sum_{\substack{n,j\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} Q_{n,j,\vec{0},\vec{\beta}} z^n t^j s^{\vec{\beta}}.$$

Then the recurrence relation is equivalent to

$$I = \sum_{\substack{n,j\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} z^n \sum_{\substack{0\leq i\leq n-2\\ j-n+i<k<j}} Q_{i,k,\vec{0},\vec{\beta}} h_{n-i} t^j s^{\vec{\beta}}$$

$$+\sum_{m=1}^{N-1} \sum_{\substack{n,j\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} z^n \sum_{\substack{0\leq i\leq n-1\\ j-n+i<k\leq j}} Q_{i,k,\vec{0},\vec{\beta}(\hat{m})} h_{n-i} t^j s^{\vec{\beta}(\hat{m})} s_m + \sum_{\vec{\beta}\in\mathbb{N}^{N-1}} (-z)^{|\vec{\beta}|} s^{\vec{\beta}} \prod_{m=1}^{N-1} e_{\beta_m}$$

$$= \sum_{\substack{n,k\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} z^n \sum_{0\leq i\leq n-2} Q_{i,k,\vec{0},\vec{\beta}} s^{\vec{\beta}} h_{n-i} \sum_{j=k+1}^{k+n-i-1} t^j$$

$$+\sum_{m=1}^{N-1} \sum_{\substack{n,k\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} z^n \sum_{0\leq i\leq n-1} Q_{i,k,\vec{0},\vec{\beta}(\hat{m})} s^{\vec{\beta}(\hat{m})} h_{n-i} s_m \sum_{j=k}^{k+n-i-1} t^j + \prod_{m=1}^{N-1} E(-s_m z)$$

$$= \sum_{\substack{n,k\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} z^n \sum_{0\leq i\leq n-2} Q_{i,k,\vec{0},\vec{\beta}} t^k s^{\vec{\beta}} t h_{n-i} \left[n-i-1\right]_t$$

$$+\sum_{m=1}^{N-1} \sum_{\substack{n,k\geq 0\\ \vec{\beta}\in\mathbb{N}^{N-1}}} z^n \sum_{0\leq i\leq n-1} Q_{i,k,\vec{0},\vec{\beta}(\hat{m})} t^k s^{\vec{\beta}(\hat{m})} s_m h_{n-i} \left[n-i\right]_t + \prod_{m=1}^{N-1} E(-s_m z)$$

$$= I \sum_{n\geq 2} t \left[n-1\right]_t h_n z^n + \sum_{m=1}^{N-1} I \sum_{n\geq 1} s_m \left[n\right]_t h_n z^n + \prod_{m=1}^{N-1} E(-s_m z).$$

If we let

$$A := \sum_{n\geq 2} t \left[n-1\right]_t h_n z^n$$

and

$$B := \sum_{m=1}^{N-1} \sum_{n\geq 1} s_m \left[n\right]_t h_n z^n,$$

then

$$I = \frac{\prod_{m=1}^{N-1} E(-s_m z)}{1 - A - B}. \tag{5.5}$$

Next we compute the denominator of this expression,

$$1 - A - B = 1 + \sum_{n \geq 2} h_n z^n t \left(\frac{t^{n-1} - 1}{1 - t} \right) + \sum_{m=1}^{N-1} \sum_{n \geq 1} h_n z^n s_m \left(\frac{t^n - 1}{1 - t} \right)$$

$$= \frac{1}{1-t} \left[1 - t + H(tz) - tz - 1 - t(H(z) - z - 1) + \sum_{m=1}^{N-1} (H(tz) - H(z)) s_m \right]$$

$$= \frac{1}{1-t} \left[\left(1 + \sum_{m=1}^{N-1} s_m \right) H(tz) - \left(t + \sum_{m=1}^{N-1} s_m \right) H(z) \right].$$

Substituting this back into (5.5) gives the desired result. □

Thus Theorem 4.1.3 will be proved once we establish the recurrence relation in Corollary 5.2.3, and this will be done in Chapter 6.

Chapter 6

Colored banners

The previous chapter has shown that Theorem 4.1.3 is equivalent to the recurrence relation appearing in Corollary 5.2.3. This chapter will be devoted to establishing this recurrence relation, thus proving Theorem 4.1.3. There are two cases which will be treated separately, the case $\left|\vec{\beta}\right| = n$ and the case $\left|\vec{\beta}\right| < n$ (recall that the absolute value of a vector $\vec{\beta} \in \mathbb{N}^{N-1}$ is $\left|\vec{\beta}\right| = \sum_{m=1}^{N-1} \beta_m$).

6.1 Establishing the Recurrence, Part I

First we consider the case $\left|\vec{\beta}\right| = n$, and define

$$D_{n,\vec{\beta}}(\mathbf{x}) := Q_{n,0,\vec{0},\vec{\beta}}(\mathbf{x}).$$

(Note that $\mathbf{x} := \{x_1, x_2, ...\}$ denotes our usual set of commuting variables which we often omit, but we include here for the sake of clarity in the proof of Theorem 6.1.1). Our goal is to compute the following recurrence relation for $D_{n,\vec{\beta}}(\mathbf{x})$, and then check that it agrees with the recurrence relation appearing in Corollary 5.2.3 in the case when $\left|\vec{\beta}\right| = n$.

Theorem 6.1.1. *For $n \in \mathbb{N}$ and $\left|\vec{\beta}\right| = n$ we have*

$$D_{n,\vec{\beta}}(\mathbf{x}) = (-1)^n \left(\prod_{m=1}^{N-1} e_{\beta_m}(\mathbf{x}) \right) + \sum_{m=1}^{N-1} \chi(\beta_m > 0) D_{n-1,\vec{\beta}(\hat{m})}(\mathbf{x}) h_1(\mathbf{x}),$$

recalling that $\vec{\beta}(\hat{m}) = (\beta_1, ..., \beta_{m-1}, \beta_m - 1, \beta_{m+1}, ..., \beta_{N-1})$.

Proof. Similar to the definition of $\mathcal{R}(\check{\lambda}, j)$, we let $\mathcal{R}(n, j, \vec{\alpha}, \vec{\beta})$ denote the set of all colored ornaments of size n with j barred letters, β_i letters of color $i \in [N-1]$, and α_i colored necklaces consisting of a single i-colored letter where $0 \leq i \leq N-1$ (as usual, N is arbitrary but fixed). Hence

$$D_{n,\vec{\beta}}(\mathbf{x}) = \sum_{R \in \mathcal{R}(n,0,\vec{0},\vec{\beta})} \mathrm{wt}(R).$$

Since $\left|\vec{\beta}\right| = n$, the key fact is that the colored necklace rules of Definition 5.1.2 present no restrictions, since there are no 0-colored letters in this case. Therefore $\mathcal{R}(n, 0, \vec{0}, \vec{\beta})$ can be viewed as a set of Gessel-Reutenauer ornaments as in Section 1.6, but over the alphabet

$$\left\{1^1, 1^2, ..., 1^{N-1}, 2^1, 2^2, ..., 2^{N-1}, ...\right\}.$$

For a necklace (v) over this alphabet where $v = v_1^{\epsilon_1}, v_2^{\epsilon_2}, ..., v_n^{\epsilon_n}$ with $v_i \in \mathbb{P}$ and $\epsilon_i \in [N-1]$, we define a new weight by

$$\widetilde{\mathrm{wt}}((v)) := x_{v_1,\epsilon_1} x_{v_2,\epsilon_2} \cdots x_{v_n,\epsilon_n},$$

where

$$X := \{x_{1,1}, x_{1,2}, ..., x_{1,N-1}, x_{2,1}, x_{2,2}, ..., x_{2,N-1}, ...\}$$

is a set of commuting variables.

Then by [15, Theorem 3.6] we have

$$\sum_{\substack{\vec{\beta} \in \mathbb{N}^{N-1} \\ \left|\vec{\beta}\right| = n}} D_{n,\vec{\beta}}(\mathbf{x}) s^{\vec{\beta}} = \sum_{\substack{\vec{\beta} \in \mathbb{N}^{N-1} \\ \left|\vec{\beta}\right| = n}} \sum_{R \in \mathcal{R}(n,0,\vec{0},\vec{\beta})} \widetilde{\mathrm{wt}}(R) \Bigg|_{x_{i,j} = x_i s_j} = D_n(X) \Bigg|_{x_{i,j} = x_i s_j},$$

where D_n is the quasisymmetric generating function for derangements in S_n, as described by Gessel and Reutenauer in [15, Section 8]. By Equation (8.2) of [15], D_n satisfies the following recurrence

$$D_n(X) = h_1(X)D_{n-1}(X) + (-1)^n e_n(X).$$

Next we compute the right hand side of this equation evaluated at $x_{i,j} = x_i s_j$. First we have

$$D_{n-1}(X)\Big|_{x_{i,j}=x_i s_j} = \sum_{\substack{\vec{\beta}\in\mathbb{N}^{N-1} \\ |\vec{\beta}|=n-1}} D_{n-1,\vec{\beta}}(\mathbf{x}) s^{\vec{\beta}}.$$

Next,

$$h_1(X)\Big|_{x_{i,j}=x_i s_j} = \sum_{i\geq 1} x_i \left(\sum_{m=1}^{N-1} s_m\right) = h_1(\mathbf{x}) \sum_{m=1}^{N-1} s_m.$$

And finally

$$e_n(X)\Big|_{x_{i,j}=x_i s_j} = \sum_{\substack{\vec{\beta}\in\mathbb{N}^{N-1} \\ |\vec{\beta}|=n}} \prod_{m=1}^{N-1} s_m^{\beta_m} e_{\beta_m}(\mathbf{x}) = \sum_{\substack{\vec{\beta}\in\mathbb{N}^{N-1} \\ |\vec{\beta}|=n}} s^{\vec{\beta}} \prod_{m=1}^{N-1} e_{\beta_m}(\mathbf{x}).$$

Thus

$$\sum_{\substack{\vec{\beta}\in\mathbb{N}^{N-1} \\ |\vec{\beta}|=n}} D_{n,\vec{\beta}}(\mathbf{x}) s^{\vec{\beta}} = D_n(X)\Big|_{x_{i,j}=x_i s_j}$$

$$= h_1(\mathbf{x}) \left(\sum_{m=1}^{N-1} s_m \sum_{\substack{\vec{\beta}\in\mathbb{N}^{N-1} \\ |\vec{\beta}|=n-1}} D_{n-1,\vec{\beta}}(\mathbf{x}) s^{\vec{\beta}} \right) + (-1)^n \sum_{\substack{\vec{\beta}\in\mathbb{N}^{N-1} \\ |\vec{\beta}|=n}} s^{\vec{\beta}} \prod_{m=1}^{N-1} e_{\beta_m}(\mathbf{x}).$$

Extracting the coefficient of $s^{\vec{\beta}}$ from both sides gives the desired result.

□

Next we show that Theorem 6.1.1 agrees with the recurrence relation appearing in Corollary 5.2.3 in the case that $\left|\vec{\beta}\right| = n$.

Corollary 6.1.2. *In the case* $\left|\vec{\beta}\right| = n$*, Theorem 6.1.1 establishes the recurrence relation appearing in Corollary 5.2.3, thus proving Theorem 4.1.3 in this case.*

Proof. In Corollary 5.2.3, set $j = 0$ and $\left|\vec{\beta}\right| = n$ so that the left hand side equals $D_{n,\vec{\beta}}(\mathbf{x})$. In the first sum on the right hand side, $k < j = 0$. By definition, $Q_{i,k,\vec{0},\vec{\beta}} = 0$ if $k < 0$, thus

$$\sum_{\substack{0 \le i \le n-2 \\ j-n+i<k<j}} Q_{i,k,\vec{0},\vec{\beta}}(\mathbf{x}) h_{n-i}(\mathbf{x}) = 0.$$

In the next sum, we note that $|\vec{\beta}(\hat{m})| = n - 1$. Consequently, the only nonzero terms are when $i = n - 1$ and $k = 0$, that is

$$\sum_{m=1}^{N-1} \chi\left(\beta_m > 0\right) \left(\sum_{\substack{0 \le i \le n-1 \\ j-n+i<k\le j}} Q_{i,k,\vec{0},\vec{\beta}(\hat{m})}(\mathbf{x}) h_{n-i}(\mathbf{x}) \right) = \sum_{m=1}^{N-1} \chi(\beta_m > 0) D_{n-1,\vec{\beta}(\hat{m})}(\mathbf{x}) h_1(\mathbf{x}).$$

Lastly,

$$\chi\left(j = 0\right) \chi\left(\left|\vec{\beta}\right| = n\right) (-1)^n \prod_{m=1}^{N-1} e_{\beta_m}(\mathbf{x}) = (-1)^n \prod_{m=1}^{N-1} e_{\beta_m}(\mathbf{x}),$$

as desired.

□

6.2 Establishing the Recurrence, Part II

It now remains to consider the case $\left|\vec{\beta}\right| < n$ for establishing the recurrence relation in Corollary 5.2.3. For this we introduce colored banners, which are a generalization of the bicolored banners of Shareshian and Wachs (see Section 1.7). While the content of this section is inspired by the work of Shareshian and Wachs, our bijection appearing in Theorem 6.2.4 and the subsequent proof are considerably more complicated than the corresponding bijection and proof of [32, Theorem 3.7]. Though it is not obvious, we note that our bijection does in fact reduce to that of Shareshian and Wachs in the case $N = 1$.

Definition 6.2.1. A *colored banner* (which we simply call a banner) is a word B of finite length over the alphabet $\mathcal{B}$ such that

1. if $B(i)$ is barred then $|B(i)| \geq |B(i+1)|$,
2. if $B(i)$ is 0-colored and unbarred, then $|B(i)| \leq |B(i+1)|$ or i equals the length of B,
3. the last letter of B is unbarred.

Recall that a Lyndon word over an ordered alphabet is a word that is strictly lexicographically larger than all its circular rearrangements. And a Lyndon factorization of a word is a factorization into a lexicographically weakly increasing sequence of Lyndon words. It is a fact that every word has a unique Lyndon factorization. We say that a word of length n has Lyndon type λ (where λ is a partition of n) if parts of λ equal the lengths of the words in the Lyndon factorization (see [18, Theorem 5.1.5]).

We will apply Lyndon factorization to banners, but we will do so using a new order $<_B$ on the alphabet $\mathcal{B}$ as follows

$$\begin{gathered} 1^1 <_B 1^2 <_B ... <_B 1^{N-1} \\ <_B 2^1 <_B 2^2 <_B ... <_B 2^{N-1} \\ <_B 3^1 <_B 3^2 <_B ... <_B 3^{N-1} \\ \vdots \\ <_B 1^0 <_B \overline{1^0} <_B 2^0 <_B \overline{2^0} <_B 3^0 <_B \overline{3^0} <_B ... \end{gathered} \tag{6.1}$$

(The reason for choosing this order will become apparent in the proof of Theorem 6.2.4). We define the weight $\mathrm{wt}(B)$ of a banner $B(i), ..., B(n)$ to be the monomial

$x_{|B(1)|}...x_{|B(n)|}$. And we define the cv-cycle type of a banner B to be the multiset

$$\check{\lambda}(B) = \left\{(\lambda_1, \vec{\alpha^1}), ..., (\lambda_k, \vec{\alpha^k})\right\}$$

if B has Lyndon type λ with respect to $<_B$, and the corresponding word of length λ_i in the Lyndon factorization has color vector $\vec{\alpha^i}$. Then $K(\check{\lambda}, j)$ will denote the set of all banners of cv-cycle type $\check{\lambda}$ with exactly j barred letters.

Theorem 6.2.2. *There exists a weight preserving bijection from $\mathcal{R}(\check{\lambda}, j)$ to $K(\check{\lambda}, j)$, consequently*

$$Q_{\check{\lambda},j} = \sum_{B \in K(\check{\lambda},j)} \mathrm{wt}(B).$$

Proof. The proof uses Lyndon factorization and is identical to the proof of [32, Theorem 3.6]. □

Definition 6.2.3. A 0-*colored marked sequence*, denoted $(\omega, b, 0)$, is a weakly increasing sequence ω of positive integers, together with a positive integer b, which we call the mark, such that $1 \leq b < \mathrm{length}(\omega)$. The set of all 0-colored marked sequences with $\mathrm{length}(\omega) = n$ and mark equal to b will be denoted $M(n, b, 0)$.

For $1 \leq m \leq N - 1$, an m-*colored marked sequence*, denoted (ω, b, m), is a weakly increasing sequence ω of positive integers, together with a nonnegative integer b such that $0 \leq b < \mathrm{length}(\omega)$. The set of all m-colored marked sequences with $\mathrm{length}(\omega) = n$ and mark equal to b will be denoted $M(n, b, m)$.

We will use colored marked sequences in Theorem 6.2.4 below, where one can think of the map γ as removing a colored marked sequence (ω, b, m) from a banner. The sequence ω corresponds to the absolute values of the letters removed, b corresponds to the number of barred letters removed, and one of the letters removed has color m while the rest of the letters removed all have color 0.

Let $K_0(n, j, \vec{\beta})$ denote the set of all banners of length n, with Lyndon type having no parts of size 1, color vector equal to $\vec{\beta}$, and j bars. For $m \in [N-1]$ and $\beta_m > 0$, define

$$X_m := \biguplus_{\substack{0 \le i \le n-1 \\ j-n+i < k \le j}} K_0(i, k, \vec{\beta}(\hat{m})) \times M(n-i, j-k, m),$$

and let $X_m := 0$ if $\beta_m = 0$. We also define

$$X_0 =: \biguplus_{\substack{0 \le i \le n-2 \\ j-n+i < k < j}} K_0(i, k, \vec{\beta}) \times M(n-i, j-k, 0).$$

Theorem 6.2.4. *If $\left|\vec{\beta}\right| < n$, then there is a bijection*

$$\gamma : K_0(n, j, \vec{\beta}) \to \biguplus_{m=0}^{N-1} X_m$$

such that if $\gamma(B) = (B', (\omega, b, m))$, then $wt(B) = wt(B')wt(\omega)$ where if $\omega = \omega_1, \omega_2, ..., \omega_l$ then $wt(\omega) = x_{\omega_1} x_{\omega_2} \cdots x_{\omega_l}$.

Corollary 6.2.5. *Theorem 6.2.4 establishes the recurrence relation appearing in Corollary 5.2.3 in the case that $\left|\vec{\beta}\right| < n$, thus completing the proof of Theorem 4.1.3.*

Proof. This follows from the fact that

$$\sum_{(\omega, j-k) \in M(n-i, j-k, m)} \text{wt}(\omega) = h_{n-i}.$$

□

In order to prove Theorem 6.2.4, we will need the following lemma (see [8, Lemma 4.3]).

Lemma 6.2.6. *Let B be a banner. If the Lyndon type of B has no parts of size one, then B has a unique increasing factorization (with respect to $<_B$). By increasing factorization of B, we mean that B has the form $B = B_1 \cdot B_2 \cdot ... \cdot B_d$ where each B_i*

has the form

$$B_i = (\underbrace{a_i, ..., a_i}_{p_i \text{ times}}) \cdot u_i,$$

where $a_i \in \mathcal{B}$, $p_i > 0$, and u_i is a word of positive length over the alphabet $\mathcal{B}$ whose letters are all strictly less than a_i with respect to $<_B$ (see (6.1)), and $a_1 \leq_B a_2 \leq_B ... \leq_B a_d$. Note that the increasing factorization is a refinement of the Lyndon factorization.

For example, the Lyndon factorization of the word

$$(6^1, 1^2, 5^1, \overline{4^0}, \overline{4^0}, 4^1, 4^0, \overline{4^0}, 3^2, 5^0, 7^1)$$

is

$$(6^1, 1^2, 5^1) \cdot (\overline{4^0}, \overline{4^0}, 4^1, 4^0, \overline{4^0}, 3^2) \cdot (5^0, 7^1),$$

which has no parts of size one, so its increasing factorization is

$$(6^1, 1^2, 5^1) \cdot (\overline{4^0}, \overline{4^0}, 4^1, 4^0) \cdot (\overline{4^0}, 3^2) \cdot (5^0, 7^1).$$

Next we prove Theorem 6.2.4. As noted at the beginning of this section, the $N = 1$ case of this proof reduces to the proof of [32, Theorem 3.7]. In the general case that $N > 1$ (and $\left|\vec{\beta}\right| < n$), the proof is inspired by [32, Theorem 3.7], but significantly more complicated.

Proof of Theorem 6.2.4. Describing γ (and its inverse) requires us to consider many different cases. For convenience we will make a note of which case $\gamma(B)$ falls under when considering $\gamma^{-1}(\gamma(B))$ (and vice versa) so that one can check that γ is indeed a bijection.

First, we take the increasing factorization of B, say $B = B_1 \cdot B_2 \cdot ... \cdot B_d$. Let

$$B_d = (\underbrace{a, ..., a}_{p \text{ times}}) \cdot u,$$

where $a \in \mathcal{B}$, $p > 0$, and u is a word of positive length over the alphabet $\mathcal{B}$ whose letters are all strictly less than a with respect to the order $<_B$ from (6.1). We observe that $\left|\vec{\beta}\right| < n$ implies that a is 0-colored, since we have taken the increasing factorization with respect to $<_B$. For ease of notation, we will write

$$B_d = (\underbrace{a, ..., a}_{p \text{ times}}) \cdot u = a^p \cdot u,$$

where it is understood that a is 0-colored, and the superscript p means that the letter a is repeated p times.

Case 1, γ (Case 1.1, γ^{-1})

$B_d = a^p c$ where a is unbarred and $c \in \mathcal{B}$. Since the banner rules in Definition 6.2.1 require that $|c| \geq |a|$, this can only happen if c has positive color, say c has color $m > 0$. Then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' =: B_1 \cdot ... \cdot B_{d-1},$$

$$\omega := (\underbrace{|a|, ..., |a|}_{p \text{ times}}, |c|), \text{ and } b := 0.$$

For example if

$$B_d = (4^0, 4^0, 4^0, 4^0, 9^2),$$

then

$$(\omega, b, m) = ((4, 4, 4, 4, 9), 0, 2).$$

Case 2, γ

$B_d = a^p c, i_1, i_2, ...i_l$ where a is unbarred, $c \in \mathcal{B}$, and i_1 is unbarred. Again since a is unbarred, c must have positive color. Next, we find the index s such that $1 \leq s \leq l$

and either one of the following subcases hold:

Case 2.1, γ (Case 1.1, γ^{-1})

$i_1 \leq_B i_2 \leq_B ... \leq_B i_l$ are all 0-colored and unbarred. We then take $s = l$ and set

$$\gamma(B) := (B', (\omega, b, m)),$$

where c has color $m > 0$, and where

$$B' =: B_1 \cdot ... \cdot B_{d-1},$$

$$\omega := (|i_1|, ..., |i_l|, \underbrace{|a|, ..., |a|}_{p \text{ times}}, |c|), \text{ and } b := 0.$$

For example if

$$B_d = (4^0, 4^0, 9^1, 2^0, 2^0, 3^0),$$

then

$$(\omega, b, m) = ((2, 2, 3, 4, 4, 9), 0, 1).$$

Case 2.2, γ (Case 4.2, γ^{-1})

$i_1 \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s is barred. Then set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' =: B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p c i_{s+1} ... i_l,$$

$$\omega =: (|i_1|, ..., |i_s|), \text{ and } b := 1.$$

For example if

$$B_d = (5^0, 5^0, 8^3, 1^0, 4^0, \overline{4^0}, 2^0, 7^1),$$

then

$$\tilde{B}_d = (5^0, 5^0, 8^3, 2^0, 7^1),$$

$$(\omega, b, 0) = ((1, 4, 4), 1, 0).$$

Case 2.3, γ (Cases 1.2 and 2.1, γ^{-1})

$i_1 \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s is positively colored, say i_s has color $m > 0$. Then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' =: B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p c i_{s+1} ... i_l,$$

$$\omega := (|i_1|, ..., |i_s|), \text{ and } b := 0.$$

For example if

$$B_d = (5^0, 5^0, 8^1, 1^0, 4^0, 7^3, 6^2),$$

then

$$\tilde{B}_d = (5^0, 5^0, 8^1, 6^2),$$

$$(\omega, b, m) = ((1, 4, 7), 0, 3).$$

Case 3, γ

$B_d = a^p c, i_1, i_2, ... i_l$ where a is unbarred, $c \in \mathcal{B}$, and i_1 is barred. Again this implies c must have positive color. First, find the index r such that $i_1 \geq_B ... \geq_B i_{r-1}$

are all barred while i_r is unbarred (note $1 < r \leq l$). Then find the index s such that $r \leq s \leq l$ and either one of the following subcases hold:

Case 3.1, γ (Case 4.3, γ^{-1})

$i_r \leq_B ... \leq_B i_s$ are all 0-colored, unbarred, and $|i_s| \leq |i_{r-1}|$, while $|i_{s+1}| > |i_{r-1}|$ or $s = l$. Then set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p c i_{s+1} ... i_l,$$

$$\omega := (|i_r|, |i_{r+1}|, ..., |i_s|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|), \text{ and } b := r - 1.$$

For example if

$$B_d = (4^0, 4^0, 6^1, \overline{3^0}, \overline{3^0}, \overline{2^0}, 1^0, 2^0, 3^0),$$

then

$$\tilde{B}_d = (4^0, 4^0, 6^1, 3^0),$$

$$(\omega, b, 0) = ((1, 2, 2, 3, 3), 3, 0).$$

Case 3.2, γ (Case 4.2, γ^{-1})

$i_r \leq_B ... \leq_B i_{s-1}$ are all 0-colored, unbarred, and $|i_{s-1}| \leq |i_{r-1}|$, while i_s is barred and $|i_s| \leq |i_{r-1}|$. Then set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' =: B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p c i_{s+1} ... i_l,$$

$$\omega := (|i_r|, |i_{r+1}|, ..., |i_s|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|), \text{ and } b := r.$$

For example if

$$B_d = (6^0, 6^0, 8^2, \overline{5^0}, \overline{4^0}, \overline{4^0}, 2^0, 4^0, \overline{4^0}, 1^0, 9^1),$$

then

$$\tilde{B}_d = (6^0, 6^0, 8^2, 1^0, 9^1),$$

$$(\omega, b, 0) = ((2, 4, 4, 4, 4, 5), 4, 0).$$

Case 3.3, γ (Case 3.2, γ^{-1})

$i_r \leq_B ... \leq_B i_{s-1}$ are all 0-colored, unbarred, and $|i_{s-1}| \leq |i_{r-1}|$, while i_s is positively colored, say i_s has color $m > 0$, and $|i_s| \leq |i_{r-1}|$. Then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where c has color $m > 0$, and where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p c i_{s+1} ... i_l,$$

$$\omega := (|i_r|, |i_{r+1}|, ..., |i_s|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|), \text{ and } b := r - 1.$$

For example if

$$B_d = (5^0, 8^1, \overline{4^0}, \overline{2^0}, 1^0, 1^1, 1^3),$$

then

$$\tilde{B}_d = (5^0, 8^1, 1^3),$$

$$(\omega, b, m) = ((1, 1, 2, 4), 2, 1).$$

Case 4, γ

$B_d = a^p i_1, i_2, ...i_l$ where a is barred and i_1 is unbarred. Then find the index s such that $1 \leq s \leq l$ and either one of the following subcases hold:

Case 4.1, γ (Case 4.1, γ^{-1})

$i_1 \leq_B ... \leq_B i_l$ are all 0-colored and unbarred, so we take $s = l$ and set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1},$$

$$\omega := (|i_1|, |i_2|, ..., |i_l|, \underbrace{|a|, ..., |a|}_{p \text{ times}}), \text{ and } b := p.$$

For example if

$$B_d = (\overline{5^0}, \overline{5^0}, \overline{5^0}, 1^0, 3^0, 5^0),$$

then

$$(\omega, b, 0) = ((1, 3, 5, 5, 5, 5), 3, 0).$$

Case 4.2, γ (Case 4.4, γ^{-1})

$i_1 \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s is barred. Then set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_{s+1} ... i_l,$$

$$\omega := (|i_1|, |i_2|, ..., |i_s|), \text{ and } b := 1.$$

For example if

$$B_d = (\overline{5^0}, \overline{5^0}, \overline{5^0}, 1^0, 3^0, \overline{4^0}, 3^1, 7^2),$$

then

$$\tilde{B}_d = (\overline{5^0}, \overline{5^0}, \overline{5^0}, 3^1, 7^2),$$

$$(\omega, b, 0) = ((1, 3, 4), 1, 0).$$

Case 4.3, γ (Cases 1.3 and 2.2, γ^{-1})

$i_1 \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s is positively colored, say i_s has color $m > 0$, and $|i_{s+1}| \leq |a|$. Then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_{s+1} ... i_l,$$

$$\omega := (|i_1|, |i_2|, ..., |i_s|), \text{ and } b := 0.$$

For example if

$$B_d = (\overline{6^0}, \overline{6^0}, 2^0, 2^0, 4^0, 8^1, \overline{5^0}, 4^2),$$

then

$$\tilde{B}_d = (\overline{6^0}, \overline{6^0}, \overline{5^0}, 4^2),$$

$$(\omega, b, m) = ((2, 2, 4, 8), 0, 1).$$

Case 4.4, γ (Case 3.1, γ^{-1})

$i_1 \leq_B ... \leq_B i_{l-1}$ are all 0-colored and unbarred while i_l is positively colored, say

i_l has color $m > 0$, and $|i_l| \leq |a|$. Take $s = l$ and set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1},$$

$$\omega := (|i_1|, |i_2|, ..., |i_l|, \underbrace{|a|, ..., |a|}_{p \text{ times}}), \text{ and } b := p.$$

For example if

$$B_d = (\overline{6^0}, \overline{6^0}, 2^0, 2^0, 4^0, 5^3),$$

then

$$(\omega, b, m) = ((2, 2, 4, 5, 6, 6), 2, 3).$$

Case 4.5, γ (Case 2.3, γ^{-1})

$i_1 \leq_B ... \leq_B i_{l-1}$ are all 0-colored and unbarred while i_l is positively colored, say i_l has color $m > 0$, and $|i_l| > |a|$. Take $s = l$ and set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_1 ... i_{l-1},$$

$$\omega := (|i_l|), \text{ and } b := 0.$$

For example if

$$B_d = (\overline{6^0}, \overline{6^0}, 2^0, 2^0, 4^0, 7^3),$$

then

$$\tilde{B}_d = (\overline{6^0}, \overline{6^0}, 2^0, 2^0, 4^0),$$

$$(\omega, b, m) = ((7), 0, 3).$$

Case 4.6, γ (Case 2.3, γ^{-1})

$i_1 \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s, i_{s+1} are both positively colored, say i_{s+1} has color $m > 0$, and $|i_{s+1}| > |a|$. Then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_1 ... i_s i_{s+2} i_{s+3} ... i_l,$$

$$\omega := (|i_{s+1}|), \text{ and } b := 0.$$

For example if

$$B_d = (\overline{6^0}, \overline{6^0}, 2^0, 2^0, 4^0, 7^3, 8^2),$$

then

$$\tilde{B}_d = (\overline{6^0}, \overline{6^0}, 2^0, 2^0, 4^0, 7^3),$$

$$(\omega, b, m) = ((8), 0, 2).$$

Case 5, γ

$B_d = a^p i_1, i_2, ... i_l$ where a and i_1 are barred. First, find the index r such that $i_1 \geq_B ... \geq_B i_{r-1}$ are all barred while i_r is unbarred (note $1 < r \leq l$). Then find the index s such that $r \leq s \leq l$ and either one of the following subcases hold:

Case 5.1, γ (Case 4.1, γ^{-1})

$i_r \leq_B ... \leq_B i_l$ are all 0-colored, unbarred, and $|i_l| \leq |i_{r-1}|$. Then we take $s = l$ and set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1},$$

$$\omega := ((|i_r|, |i_{r+1}|, ..., |i_l|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|, \underbrace{|a|, ..., |a|}_{p \text{ times}}), \text{ and } b := p + r - 1.$$

For example if

$$B_d = (\overline{7^0}, \overline{7^0}, \overline{6^0}, \overline{4^0}, 1^0, 2^0, 4^0),$$

then

$$(\omega, b, 0) = ((1, 2, 4, 4, 6, 7, 7), 4, 0).$$

Case 5.2, γ (Case 4.5, γ^{-1})

$i_r \leq_B ... \leq_B i_s$ are all 0-colored, unbarred, $|i_s| \leq |i_{r-1}|$, and $|i_{r-1}| < |i_{s+1}| \leq |a|$. Then set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_{s+1} ... i_l,$$

$$\omega := ((|i_r|, |i_{r+1}|, ..., |i_s|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|), \text{ and } b := r - 1.$$

For example if

$$B_d = (\overline{7^0}, \overline{7^0}, \overline{6^0}, \overline{4^0}, 1^0, 2^0, 5^0, 8^2, 1^0),$$

then

$$\tilde{B}_d := (\overline{7^0}, \overline{7^0}, 5^0, 8^2, 1^0),$$

$$(\omega, b, 0) = ((1, 2, 4, 6), 2, 0).$$

Case 5.3, γ (Case 2.4, γ^{-1})

$i_r \leq_B ... \leq_B i_s$ are all 0-colored, unbarred, $|i_s| \leq |i_{r-1}|$, and i_{s+1} is positively colored, say i_{s+1} has color $m > 0$, with $|i_{s+1}| > |a|$. If $|i_{r-1}| \geq |i_{s+2}|$, then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_1 ... i_{r-2} i_r i_{r+1} ... i_s i_{r-1} i_{s+2} i_{s+3} ... i_l,$$

$$\omega := (|i_{s+1}|), \text{ and } b := 0.$$

For example if

$$B_d = (\overline{7^0}, \overline{7^0}, \overline{6^0}, \overline{4^0}, 1^0, 2^0, 8^1, 1^0),$$

then

$$\tilde{B}_d = (\overline{7^0}, \overline{7^0}, \overline{6^0}, 1^0, 2^0, \overline{4^0}, 1^0),$$

$$(\omega, b, m) = ((8), 0, 1).$$

Case 5.4, γ (Case 2.3, γ^{-1})

$i_r \leq_B ... \leq_B i_s$ are all 0-colored, unbarred, $|i_s| \leq |i_{r-1}|$, and i_{s+1} is positively colored, say i_{s+1} has color $m > 0$, with $|i_{s+1}| > |a|$. If $|i_{r-1}| < |i_{s+2}|$ or if $s + 1 = l$, then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_1 ... i_s i_{s+2} i_{s+3} ... i_l,$$

$$\omega := (|i_{s+1}|), \text{ and } b := 0.$$

For example if

$$B_d = (\overline{7^0}, \overline{7^0}, \overline{6^0}, \overline{4^0}, 1^0, 2^0, 8^1, 5^0),$$

then

$$\tilde{B}_d = (\overline{7^0}, \overline{7^0}, \overline{6^0}, \overline{4^0}, 1^0, 2^0, 5^0),$$

$$(\omega, b, m) = ((8), 0, 1).$$

Case 5.5, γ (Case 4.4, γ^{-1})

$i_r \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s is barred with $|i_s| \leq |i_{r-1}|$. Then set

$$\gamma(B) := (B', (\omega, b, 0)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_{s+1} ... i_l,$$

$$\omega := (|i_r|, |i_{r+1}|, ..., |i_s|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|), \text{ and } b := r.$$

For example if

$$B_d = (\overline{6^0}, \overline{6^0}, \overline{5^0}, 1^0, 2^0, 2^0, \overline{4^0}, \overline{2^0}, 2^0),$$

then

$$\tilde{B}_d = (\overline{6^0}, \overline{6^0}, \overline{2^0}, 2^0),$$

$$(\omega, b, 0) = ((1, 2, 2, 4, 5), 2, 0).$$

Case 5.6, γ (Case 3.1, γ^{-1})

$i_r \leq_B ... \leq_B i_{l-1}$ are all 0-colored and unbarred while i_l is positively colored, say i_l has color $m > 0$, with $|i_l| \leq |i_{r-1}|$. Then take $s = l$ and set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1},$$

$$\omega := (|i_r|, |i_{r+1}|, ..., |i_l|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|, \underbrace{|a|, ..., |a|}_{p \text{ times}}), \text{ and } b := p + r - 1.$$

For example if

$$B_d = (\overline{6^0}, \overline{6^0}, \overline{5^0}, 1^0, 2^0, 2^0, 4^5),$$

then

$$(\omega, b, m) = ((1, 2, 2, 4, 5, 6, 6), 3, 5).$$

Case 5.7, γ (Case 3.3, γ^{-1})

$i_r \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s is positively colored, say i_s has color $m > 0$, with $|i_s| \leq |i_{r-1}|$. If $|i_{s+1}| \leq |a|$, then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' =: B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_{s+1} ... i_l,$$

$$\omega := (|i_r|, |i_{r+1}|, ..., |i_s|, |i_{r-1}|, |i_{r-2}|, ..., |i_1|), \text{ and } b := r - 1.$$

For example if

$$B_d = (\overline{8^0}, \overline{8^0}, \overline{5^0}, 1^0, 2^0, 2^0, 4^5, 7^3),$$

then

$$\tilde{B}_d = (\overline{8^0}, \overline{8^0}, 7^3),$$

$$(\omega, b, m) = ((1, 2, 2, 4, 5), 1, 5).$$

Case 5.8, γ (Case 2.3, γ^{-1})

$i_r \leq_B ... \leq_B i_{s-1}$ are all 0-colored and unbarred while i_s is positively colored with $|i_s| \leq |i_{r-1}|$. If $|i_{s+1}| > |a|$, then i_{s+1} must be positively colored, say i_{s+1} has color $m > 0$. Then set

$$\gamma(B) := (B', (\omega, b, m)),$$

where

$$B' := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d,$$

$$\tilde{B}_d := a^p i_1 ... i_s i_{s+2} i_{s+3} ... i_l,$$

$$\omega := (|i_{s+1}|), \text{ and } b := 0.$$

For example if

$$B_d = (\overline{8^0}, \overline{8^0}, \overline{5^0}, 1^0, 2^0, 2^0, 4^5, 9^3),$$

then

$$\tilde{B}_d = (\overline{8^0}, \overline{8^0}, \overline{5^0}, 1^0, 2^0, 2^0, 4^5),$$

$$(\omega, b, m) = ((9), 0, 3).$$

This completes the description of the map γ. Next we describe γ^{-1}. Suppose we are given a banner B with increasing factorization $B = B_1 \cdot ... \cdot B_d$ where $B_d = a^p j_1 ... j_l$, and an m-colored marked sequence (ω, b, m) where $0 \leq m \leq N-1$ and $\omega = (\omega_1, ..., \omega_q)$. Here the letter a may have any color, and we do not specify its color. For this letter only we use the superscript p to denote that a is repeated p

times where $p > 0$.

Case 1, γ^{-1}

Suppose $m > 0$, $b = 0$, $q > 1$, and one of the following subcases hold:

Case 1.1, γ^{-1} (Cases 1 and 2.1, γ)

$\omega^0_{q-1} \geq_B a$. If ω_{q-1} appears $q-1$ or q times in the sequence ω, then set

$$B_{d+1} := \underbrace{\omega^0_{q-1}...\omega^0_{q-1}}_{q-1 \text{ times}} \omega^m_q.$$

Otherwise, ω_{q-1} appears r times with $r < q-1$ and we set

$$B_{d+1} := \underbrace{\omega^0_{q-1}...\omega^0_{q-1}}_{r \text{ times}} \omega^m_q \omega^0_1 \omega^0_2 ... \omega^0_{q-r-1}.$$

In either case set

$$\gamma^{-1}\left(B, (\omega, b, m)\right) := B_1 \cdot ... \cdot B_d \cdot B_{d+1}.$$

For an example of this case (and for most of the cases below), refer to corresponding case of γ.

Case 1.2, γ^{-1} (Case 2.3, γ)

$\omega^0_{q-1} <_B a$, and a is unbarred. Then set

$$\tilde{B}_d := a^p j_1 \omega^0_1 \omega^0_2 ... \omega^0_{q-1} \omega^m_q j_2 ... j_l,$$

$$\gamma^{-1}\left(B, (\omega, b, m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 1.3, γ^{-1} (Case 4.3, γ)

$\omega^0_{q-1} <_B a$, and a is barred. Then set

$$\tilde{B}_d := a^p \omega^0_1 \omega^0_2 ... \omega^0_{q-1} \omega^m_q j_1 j_2 ... j_l,$$

$$\gamma^{-1}\left(B,(\omega,b,m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 2, γ^{-1}

Suppose $m > 0$, $b = 0$, $q = 1$, and one of the following subcases hold:

Case 2.1, γ^{-1} (Case 2.3, γ)

a is unbarred. Then set

$$\tilde{B}_d := a^p j_1 \omega_1^m j_2 ... j_l,$$

$$\gamma^{-1}\left(B,(\omega,b,m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

For example if

$$B_d = (5^0, 5^0, 5^0, 8^1, 2^3, 4^1),$$

$$(\omega, b, m) = ((4), 0, 2),$$

then

$$\tilde{B}_d = (5^0, 5^0, 5^0, 8^1, 4^2, 2^3, 4^1).$$

Case 2.2, γ^{-1} (Case 4.3, γ)

a is barred and $\omega_1 \leq |a|$. Then set

$$\tilde{B}_d := a^p \omega_1^m j_1 j_2 ... j_l,$$

$$\gamma^{-1}\left(B,(\omega,b,m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

For example if

$$B_d = (\overline{6^0}, \overline{6^0}, \overline{4^0}, 2^2, 1^0, 3^1),$$

$$(\omega, b, m) = ((5), 0, 3),$$

then

$$\tilde{B}_d = (\overline{6^0}, \overline{6^0}, 5^3, \overline{4^0}, 2^2, 1^0, 3^1).$$

Case 2.3, γ^{-1} (Cases 4.5, 4.6, 5.4, 5.8, γ)

a is barred, $\omega_1 > |a|$, and we find the index s such that $1 \leq s \leq l$ and one of the following subcases hold:

Case 2.3.1 $j_1 \leq_B ... \leq_B j_l$ are all 0-colored and unbarred, so we take $s = l$.

Case 2.3.2 $j_1 \leq_B ... \leq_B j_{s-1}$ are all 0-colored and unbarred while j_s is positively colored.

Case 2.3.3 $j_1 \geq_B ... \geq_B j_{r-1}$ are all barred while $j_r \leq_B ... \leq_B j_l$ are all 0-colored, unbarred, and $|j_l| \leq |j_{r-1}|$. Then take $s = l$.

Case 2.3.4 $j_1 \geq_B ... \geq_B j_{r-1}$ are all barred while $j_r \leq_B ... \leq_B j_s$ are all 0-colored, unbarred, $|j_s| \leq |j_{r-1}|$, and $|j_{s+1}| > |j_{r-1}|$.

Case 2.3.5 $j_1 \geq_B ... \geq_B j_{r-1}$ are all barred while $j_r \leq_B ... \leq_B j_{s-1}$ are all 0-colored, unbarred, and j_s is positively colored with $|j_s| \leq |j_{r-1}|$.

Once the index s is found, we set

$$\tilde{B}_d := a^p j_1 j_2 ... j_s \omega_1^m j_{s+1} j_{s+2} ... j_l,$$

and

$$\gamma^{-1}\left(B, (\omega, b, m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Note that in Cases 2.3.1-2.3.5, j_s is an unbarred letter, thus the banner rules in Definition 6.2.1 are not violated.

Case 2.4, γ^{-1} (Case 5.3, γ)

a is barred, $\omega_1 > |a|$, and we find the index s such that $1 \leq s \leq l$ and one of the following subcases hold:

Case 2.4.1 $j_1 \leq_B ... \leq_B j_{s-1}$ are all 0-colored and unbarred while j_s is barred. Then set

$$\tilde{B}_d := a^p j_s j_1 j_2 ... j_{s-1} \omega_1^m j_{s+1} j_{s+2} ... j_l,$$

$$\gamma^{-1}\left(B,(\omega,b,m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 2.4.2 $j_1 \geq_B ... \geq_B j_{r-1}$ are all barred while $j_r \leq_B ... \leq_B j_{s-1}$ are all 0-colored, unbarred, and j_s is barred with $|j_s| \leq |j_{r-1}|$. Then set

$$\tilde{B}_d := a^p j_1 j_2 ... j_{r-1} j_s j_r j_{r+1} ... j_{s-1} \omega_1^m j_{s+1} j_{s+2} ... j_l,$$

$$\gamma^{-1}\left(B,(\omega,b,m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 3, γ^{-1}

Suppose $m > 0$, $b > 0$, and one of the following subcases hold:

Case 3.1, γ^{-1} (Cases 4.4 and 5.6, γ)

$\overline{\omega^0}_q \geq_B a$, then set

$$B_{d+1} := \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+1} \omega_1^0 \omega_2^0 ... \omega_{q-b-1}^0 \omega_{q-b}^m,$$

$$\gamma^{-1}\left(B,(\omega,b,m)\right) := B_1 \cdot ... \cdot B_d \cdot B_{d+1}.$$

Case 3.2, γ^{-1} (Case 3.3, γ)

$\overline{\omega^0}_q <_B a$, and a is unbarred. Then set

$$\tilde{B}_d := a^p j_1 \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+1} \omega_1^0 \omega_2^0 ... \omega_{q-b-1}^0 \omega_{q-b}^m j_2 ... j_l,$$

$$\gamma^{-1}\left(B,(\omega,b,m)\right) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 3.3, γ^{-1} (Case 5.7, γ)

$\overline{\omega^0}_q <_B a$, and a is barred. Then set

$$\tilde{B}_d := a^p \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+1} \omega_1^0 \omega_2^0 ... \omega_{q-b-1}^0 \omega_{q-b}^m j_1 j_2 ... j_l,$$

$$\gamma^{-1}(B,(\omega,b,m)) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 4, γ^{-1}

Suppose $m = 0$, and one of the following subcases hold:

Case 4.1, γ^{-1} (Cases 4.1 and 5.1, γ)

$\overline{\omega^0}_q \geq_B a$, then set

$$B_{d+1} := \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+1} \omega_1^0 \omega_2^0 ... \omega_{q-b}^0,$$

$$\gamma^{-1}(B,(\omega,b,m)) := B_1 \cdot ... \cdot B_d \cdot B_{d+1}.$$

Case 4.2, γ^{-1} (Cases 2.2 and 3.2, γ)

$\overline{\omega^0}_q <_B a$, a is unbarred, and $\omega_{q-b+1} \geq |j_2|$. Then set

$$\tilde{B}_d := a^p j_1 \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+2} \omega_1^0 \omega_2^0 ... \omega_{q-b}^0 \overline{\omega^0}_{q-b+1} j_2 ... j_l,$$

$$\gamma^{-1}(B,(\omega,b,m)) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 4.3, γ^{-1} (Case 3.1, γ)

$\overline{\omega^0}_q <_B a$, a is unbarred, and $\omega_{q-b+1} < |j_2|$. Then set

$$\tilde{B}_d := a^p j_1 \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+1} \omega_1^0 \omega_2^0 ... \omega_{q-b}^0 j_2 ... j_l,$$

$$\gamma^{-1}(B,(\omega,b,m)) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 4.4, γ^{-1} (Cases 4.2 and 5.5, γ)

$\overline{\omega^0}_q <_B a$, a is barred, and $\omega_{q-b+1} \geq |j_1|$. Then set

$$\tilde{B}_d := a^p \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+2} \omega_1^0 \omega_2^0 ... \omega_{q-b}^0 \overline{\omega^0}_{q-b+1} j_1 j_2 ... j_l,$$

$$\gamma^{-1}(B,(\omega,b,m)) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

Case 4.5, γ^{-1} (Case 5.2, γ)

$\overline{\omega^0}_q <_B a$, a is barred, and $\omega_{q-b+1} < |j_1|$. Then set

$$\tilde{B}_d := a^p \overline{\omega^0}_q \overline{\omega^0}_{q-1} ... \overline{\omega^0}_{q-b+1} \omega_1^0 \omega_2^0 ... \omega_{q-b}^0 j_1 j_2 ... j_l,$$

$$\gamma^{-1}(B,(\omega,b,m)) := B_1 \cdot ... \cdot B_{d-1} \cdot \tilde{B}_d.$$

This completes the description of γ^{-1}. One can check case by case that both maps are well-defined and in fact inverses of each other. □

Chapter 7

Recurrence and Closed Formulas

In this chapter, we first present some recurrence and closed form formulas which are equivalent to Theorems 4.1.3 and 3.3.1. We close with some remarks on future work.

7.1 Recurrence and Closed Formulas

Corollary 7.1.1. *Let $Q_n(t,r,s)$ denote*

$$Q_n(t,r,s) := \sum_{\substack{j\geq 0\\ \vec{\alpha}\in\mathbb{N}^N\\ \vec{\beta}\in\mathbb{N}^{N-1}}} Q_{n,j,\vec{\alpha},\vec{\beta}}\, t^j r^{\vec{\alpha}} s^{\vec{\beta}}.$$

Then for $n \geq 1$, $Q_n(t,r,s)$ satisfies the following recurrence relation

$$Q_n(t,r,s) = \left(\sum_{\substack{\vec{\mu}\in\mathbb{N}^N\\ \vec{\nu}\in\mathbb{N}^{N-1}\\ |\vec{\mu}|+|\vec{\nu}|=n}} (-1)^{|\vec{\nu}|} h_\mu e_\nu r^{\vec{\mu}} \prod_{m=1}^{N-1} s_m^{\nu_m+\mu_m} \right)$$

$$+\sum_{k=0}^{n-1} Q_k(t,r,s) h_{n-k} \left(t\,[n-k-1]_t + [n-k]_t \left(\sum_{m=1}^{N-1} s_m \right)\right).$$

Proof. From Theorem 4.1.3 and equation (5.5) in the proof of Corollary 5.2.3, we have

$$\sum_{n\geq 0} Q_n(t,r,s)z^n = \frac{-H(r_0z)\left(\prod_{m=1}^{N-1} E(-s_mz)H(r_ms_mz)\right)}{\sum_{n\geq 0}\left(t\,[n-1]_t + [n]_t\left(\sum_{m=1}^{N-1} s_m\right)\right)h_nz^n},$$

where $[-1]_t := -t^{-1}$. Thus

$$\left(\sum_{n\geq 0} Q_n(t,r,s)z^n\right)\left(\sum_{n\geq 0}\left(t\,[n-1]_t + [n]_t\left(\sum_{m=1}^{N-1} s_m\right)\right)h_nz^n\right)$$

$$= -H(r_0z)\left(\prod_{m=1}^{N-1} E(-s_mz)H(r_ms_mz)\right).$$

Next we take the coefficient of z^n on both sides,

$$\sum_{k=0}^{n} Q_k(t,r,s)h_{n-k}\left(t\,[n-k-1]_t + [n-k]_t\left(\sum_{m=1}^{N-1} s_m\right)\right)$$

$$= \sum_{\substack{\vec{\mu}\in\mathbb{N}^N\\ \vec{\nu}\in\mathbb{N}^{N-1}\\ |\vec{\mu}|+|\vec{\nu}|=n}} (-1)^{|\vec{\nu}|+1}h_\mu e_\nu r^{\vec{\mu}}\prod_{m=1}^{N-1} s_m^{\nu_m+\mu_m}.$$

Solving for $Q_n(t,r,s)$ yields the desired recurrence.

□

Corollary 7.1.2. *For $n \geq 1$ we have*

$$Q_n(t,r,s) = \sum_{l=0}^{n-1}\sum_{\substack{k_0,\ldots,k_l\geq 1\\ \sum k_i=n}} P_{k_l}\left(\prod_{j=0}^{l-1} h_{k_j}C_{k_j}\right) + \left(\prod_{i=0}^{l} h_{k_i}C_{k_i}\right),$$

where

$$P_k = P_k(r,s,\mathbf{x}) := \sum_{\substack{\vec{\mu}\in\mathbb{N}^N\\ \vec{\nu}\in\mathbb{N}^{N-1}\\ |\vec{\mu}|+|\vec{\nu}|=k}} (-1)^{|\vec{\nu}|}h_\mu e_\nu r^{\vec{\mu}}\prod_{m=1}^{N-1} s_m^{\nu_m+\mu_m}$$

and

$$C_k = C_k(t,s) := t\,[k-1]_t + [k]_t\left(\sum_{m=1}^{N-1} s_m\right).$$

Proof. Using the above notation, Corollary 7.1.1 can be written as

$$Q_n(t,r,s) = P_n + \sum_{k=0}^{n-1} Q_k(t,r,s) h_{n-k} C_{n-k}$$

$$= P_n + h_n C_n + \sum_{k=1}^{n-1} Q_k(t,r,s) h_{n-k} C_{n-k}$$

And now we show that the right hand side of Corollary 7.1.2 satisfies the same recurrence. Indeed

$$P_n + h_n C_n + \sum_{k=1}^{n-1} \left[\sum_{l=1}^{k} \sum_{\substack{k_1,\dots,k_l \geq 1 \\ \sum k_i = k}} P_{k_l} \left(\prod_{j=1}^{l-1} h_{k_j} C_{k_j} \right) + \left(\prod_{i=1}^{l} h_{k_i} C_{k_i} \right) \right] h_{n-k} C_{n-k}$$

$$= P_n + h_n C_n + \sum_{k=1}^{n-1} \left[\sum_{l=1}^{n-k} \sum_{\substack{k_1,\dots,k_l \geq 1 \\ \sum k_i = n-k}} P_{k_l} \left(\prod_{j=1}^{l-1} h_{k_j} C_{k_j} \right) + \left(\prod_{i=1}^{l} h_{k_i} C_{k_i} \right) \right] h_k C_k$$

$$= \sum_{l=0}^{n-1} \sum_{\substack{k_0,\dots,k_l \geq 1 \\ \sum k_i = n}} P_{k_l} \left(\prod_{j=0}^{l-1} h_{k_j} C_{k_j} \right) + \left(\prod_{i=0}^{l} h_{k_i} C_{k_i} \right).$$

□

Let

$$A_n^{\mathrm{maj},\mathrm{exc},\vec{\mathrm{fix}},\vec{\mathrm{col}}}(q,t,r,s) := \sum_{\pi \in C_N \wr S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{exc}(\pi)} r^{\vec{\mathrm{fix}}(\pi)} s^{\vec{\mathrm{col}}(\pi)},$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[n-k]_q![k]_q!},$$

$$\begin{bmatrix} n \\ k_0,\dots,k_l \end{bmatrix}_q := \frac{[n]_q!}{[k_0]_q![k_1]_q!\dots[k_l]_q!},$$

$$\begin{bmatrix} n \\ \vec{\mu}, \vec{\nu} \end{bmatrix}_q := \frac{[n]_q!}{[\mu_0]_q![\mu_1]_q!...[\mu_{N-1}]_q![\nu_1]_q![\nu_1]_q!...[\nu_{N-1}]_q!}$$

if $\vec{\mu} \in \mathbb{N}^N, \vec{\nu} \in \mathbb{N}^{N-1}$ and $|\vec{\mu}| + |\vec{\nu}| = n$. We now apply the stable principal specialization to Corollaries 7.1.1 and 7.1.2 to obtain a recurrence and closed form formula for $A_n^{\text{maj,exc},\vec{\text{fix}},\vec{\text{col}}}(q,t,r,s)$.

Corollary 7.1.3. *For $n \geq 1$ we have*

$$A_n^{\text{maj,exc},\vec{\text{fix}},\vec{\text{col}}}(q,t,r,s)$$

$$= \left(\sum_{\substack{\vec{\mu} \in \mathbb{N}^N \\ \vec{\nu} \in \mathbb{N}^{N-1} \\ |\vec{\mu}|+|\vec{\nu}|=n}} (-1)^{|\vec{\nu}|} \begin{bmatrix} n \\ \vec{\mu}, \vec{\nu} \end{bmatrix}_q r^{\vec{\mu}} \left(\prod_{m=1}^{N-1} q^{\binom{\nu_m}{2}} s_m^{\nu_m+\mu_m} \right) \right)$$

$$+ \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q A_k^{\text{maj,exc},\vec{\text{fix}},\vec{\text{col}}}(q,t,r,s) \left(tq[n-k-1]_{tq} + [n-k]_{tq} \sum_{m=1}^{N-1} s_m \right),$$

and

$$A_n^{\text{maj,exc},\vec{\text{fix}},\vec{\text{col}}}(q,t,r,s)$$

$$= \sum_{l=0}^{n-1} \sum_{\substack{k_0,...,k_l \geq 1 \\ \sum k_i = n}} \left(\sum_{\substack{\vec{\mu} \in \mathbb{N}^N \\ \vec{\nu} \in \mathbb{N}^{N-1} \\ |\vec{\mu}|+|\vec{\nu}|=k_l}} (-1)^{|\vec{\nu}|} \begin{bmatrix} n \\ k_0, ..., k_{l-1}, \vec{\mu}, \vec{\nu} \end{bmatrix}_q r^{\vec{\mu}} \prod_{m=1}^{N-1} q^{\binom{\nu_m}{2}} s_m^{\nu_m+\mu_m} \right)$$

$$\times \left(\prod_{j=0}^{l-1} tq[k_j - 1]_{tq} + [k_j]_{tq} \sum_{m=1}^{N-1} s_m \right)$$

$$+ \begin{bmatrix} n \\ k_0, ..., k_l \end{bmatrix}_q \left(\prod_{i=0}^{l} tq[k_i - 1]_{tq} + [k_i]_{tq} \sum_{m=1}^{N-1} s_m \right).$$

7.2 Future work

In this paper we have generalized the main results of Shareshian and Wachs in [31], [32]. In Sections 5-7 of [32], the authors investigate many other interesting properties exhibited by the Eulerian quasisymmetric functions and the relevant joint distribution formulas. We plan to present the corresponding generalizations of these properties in a forthcoming paper. This includes (as mentioned in Remark 5.1.5) a detailed proof that the cv-cycle type colored Eulerian quasisymmetric function $Q_{\check{\lambda},j}$, is in fact a symmetric function.

We expect a further study of $Q_{\check{\lambda},j}$ to be quite fruitful. In [30], Sagan, Shareshian, and Wachs show that the q-analog of the Eulerian numbers and their cycle type refinement introduced in [31], [32] provide an instance of the cyclic sieving phenomenon (see also [26]). We suspect that our colored q-analog of the Eulerian numbers and their cycle type refinement will also provide an instance of the cyclic sieving phenomenon. We plan to present such results in a future paper.

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