

2018-08-10

# Periodic Solutions of Abstract Semilinear Equations with Applications to Biological Models

Qiuyi Su

University of Miami, luckyqiuyi@gmail.com

Follow this and additional works at: [https://scholarlyrepository.miami.edu/oa\\_dissertations](https://scholarlyrepository.miami.edu/oa_dissertations)

---

## Recommended Citation

Su, Qiuyi, "Periodic Solutions of Abstract Semilinear Equations with Applications to Biological Models" (2018). *Open Access Dissertations*. 2183.

[https://scholarlyrepository.miami.edu/oa\\_dissertations/2183](https://scholarlyrepository.miami.edu/oa_dissertations/2183)

This Open access is brought to you for free and open access by the Electronic Theses and Dissertations at Scholarly Repository. It has been accepted for inclusion in Open Access Dissertations by an authorized administrator of Scholarly Repository. For more information, please contact [repository.library@miami.edu](mailto:repository.library@miami.edu).

UNIVERSITY OF MIAMI

PERIODIC SOLUTIONS OF ABSTRACT SEMILINEAR EQUATIONS WITH  
APPLICATIONS TO BIOLOGICAL MODELS

By

Qiuyi Su

A DISSERTATION

Submitted to the Faculty  
of the University of Miami  
in partial fulfillment of the requirements for  
the degree of Doctor of Philosophy

Coral Gables, Florida

August 2018

©2018  
Qiuyi Su  
All Rights Reserved

UNIVERSITY OF MIAMI

A dissertation submitted in partial fulfillment of  
the requirements for the degree of  
Doctor of Philosophy

PERIODIC SOLUTIONS OF ABSTRACT SEMILINEAR EQUATIONS WITH  
APPLICATIONS TO BIOLOGICAL MODELS

Qiuyi Su

Approved:

---

Shigui Ruan Ph.D.  
Professor of Mathematics

---

Robert Stephen Cantrell Ph.D.  
Professor of Mathematics

---

Chris Cosner Ph.D.  
Professor of Mathematics

---

Shigui Ruan Ph.D.  
Professor of Mathematics

---

Don DeAngelis Ph.D.  
Professor of Biology

---

Guillermo Prado, Ph.D.  
Dean of the Graduate School

SU, QIUYI  
Periodic Solutions of Abstract Semilinear  
Equations with Applications to Biological  
Models

(Ph.D., Mathematics)  
(August 2018)

Abstract of a dissertation at the University of Miami.

Dissertation supervised by Professor Shigui Ruan.  
No. of pages in text. (85)

I studied the existence of periodic solutions to the abstract semilinear evolution equation

$$\frac{du}{dt} = A(t)u(t) + F(t, u(t)), \quad t \geq 0$$

in a Banach space  $X$ , where  $A(t)$  is a  $T$ -periodic linear operator on  $X$  (not necessarily densely defined); and  $F : [0, \infty) \times \overline{D(A)} \rightarrow X$  is continuous and  $T$ -periodic in  $t$ . The idea is to combine Poincare map technique with fixed point theorems to derive various conditions on the operator  $A(t)$  and the map  $F(t, u)$  to ensure that the abstract evolution equation has periodic solutions. Three cases are considered: (i) If  $A(t) = A$  is time-independent and is a Hille-Yosida operator, conditions on  $F$  are given to guarantee the existence of mild periodic solutions; (ii) If  $A(t)$  is time-dependent and satisfies the hyperbolic condition, sufficient conditions on  $A(t)$  and  $F$  are presented to ensure the existence of mild periodic solutions; (iii) If  $A(t) = A$  is time-independent, is a Hille-Yosida operator and generates a compact semigroup, the existence of mild periodic solutions is also discussed. As applications, the main results are applied to establish the existence of periodic solutions in a delayed periodic red-blood cell model; age-structured models with periodic harvesting, diffusive logistic equations with periodic coefficients, and periodic diffusive Nicholson' blowflies equation with delay.

*To my family*

## **Acknowledgments**

First I would like to express my special appreciation and thanks to my advisor, Professor Shigui Ruan, not only for his academic support but also for his care in life and career. He guided me to read papers and encouraged me to do research over the years. He also provided me with research assistantship for two years, which allowed me focus more on research. He helped me come up with the topic of the thesis and guided me through the development. He also helped me a lot in finding a job. His advice on both research and career has been invaluable. I would also like to thank my committee members, Prof. Robert Stephen Cantrell, Prof. Chris Cosner and Prof. Don DeAngelis for serving as my committee members and for providing helpful comments. I would like to thank my family for their love and encouragement. For my parents and grandparents who raised me and supported me in all my pursuits. Finally, I would like to thank my friends for their concern and support and the good memories they give me.

QIUYI SU

*University of Miami*

*August 2018*

# Contents

<b>List of Figures</b>	<b>vii</b>
<b>CHAPTER 1 Introduction</b>	<b>1</b>
<b>CHAPTER 2 Preliminaries</b>	<b>6</b>
<b>CHAPTER 3 Existence of Periodic Solutions</b>	<b>11</b>
3.1 Time-independent operators . . . . .	11
3.2 Time-dependent operators . . . . .	15
3.3 Time-independent operators - Revisited . . . . .	20
<b>CHAPTER 4 Applications</b>	<b>27</b>
4.1 Age-structured population models with periodic harvesting and constant boundary value . . . . .	27
4.2 Age-structured population models with periodic harvesting and global population dependent boundary value . . . . .	34
4.3 The diffusive logistic model with periodic coefficients . . . . .	48
4.4 Retarded functional differential equations . . . . .	59
4.5 Partial functional differential equations . . . . .	68
<b>CHAPTER 5 Conclusions and Future Study</b>	<b>82</b>





## List of Figures

4.1	A $T$ -periodic solution of (4.1) starting at $u(0, a) = 0$ and with boundary condition $u(t, 0) = 0$ , where $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , $T = 1$ , $v(t, a) \equiv 0$ and $f(t, a) = 1 + 5a(1 - a) \sin(2\pi t)$ . . . . .	30
4.2	A $T$ -periodic solution of (4.1) starting at $u(0, a) = 0$ and with boundary condition $u(t, 0) = 0$ , where $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , $T = 1$ , $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$ and $f(t, a) = 1 + 5a(1 - a) \sin(2\pi t)$ . . . . .	33
4.3	A $T$ -periodic solution of (4.2) starting at $u(0, a) = 1$ and with global boundary condition $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , $T = 1$ , $v(t, a) \equiv 0$ , $\gamma(t, a) = 0.2a^2(1 - a)(1 + \sin(2\pi t))$ and $f(t, a) = 1 + 2 \sin(2\pi t)$ . . . . .	40
4.4	A solution of (4.2) starting at $u(0, a) = 1$ and with boundary condition $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , $T = 1$ , $v(t, a) \equiv 0$ , $\gamma(t, a) = 4a^2(1 - a)(1 + \sin(2\pi t))$ and $f(t, a) = 1 + 2 \sin(2\pi t)$ . . . . .	41
4.5	A $T$ -periodic solution of (4.2) starting at $u(0, a) = 1$ and with global boundary condition $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , $T = 1$ , $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$ , $\gamma(t, a) = 0.2a^2(1 - a)(1 + \sin(2\pi t))$ and $f(t, a) = 1 + 2 \sin(2\pi t)$ . . . . .	47

4.6	A solution of (4.2) starting at $u(0, a) = 1$ and with boundary condition $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , $T = 1$ , $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$ , $\gamma(t, a) = 4a^2(1 - a)(1 + \sin(2\pi t))$ and $f(t, a) = 1 + 2 \sin(2\pi t)$ . . . . .	48
4.7	A $T$ -periodic solution of the diffusive logistic equation (4.3) starting at $u(0, a) = 1$ and with boundary condition $u(t, 0) = u(t, 1) = 1$ , where $r(t) = 0.15 + 0.1 \cos(2\pi t)$ , $T = 1$ and $K(t) = 15 + \sin(2\pi t)$ . . . . .	59
4.8	A $T$ -periodic solution of the delayed periodic red-blood cell model (4.20) with $r = 1$ starting at $\varphi(\theta) = 0.2, \theta \in [-1, 0]$ , where $p(t) = 0.3 + 0.2 \sin(2\pi t)$ , $T = 1$ and $\gamma(t) = 0.15 + 0.05 \cos(2\pi t)$ . . . . .	67
4.9	An irregular solution of the delayed periodic red-blood cell model (4.20) with $r = 1$ starting at $\varphi(\theta) = 0.2, \theta \in [-1, 0]$ , where $p(t) = 3 + 2 \sin(2\pi t)$ , $T = 1$ and $\gamma(t) = 10 + 5 \cos(2\pi t)$ . . . . .	68
4.10	A $T$ -periodic solution of the delayed reaction-diffusion equation (4.26) with initial condition $\varphi(x, t) = 0.5$ for $t \in [-1, 0]$ , $x \in [0, 1]$ , where $b(t) = 0.15 + 0.15 \sin(2\pi t)$ , $T = 1$ and $a = 1$ . . . . .	77
4.11	A solution of the delayed reaction-diffusion equation (4.26) with initial condition $\varphi(x, t) = 0.5$ for $t \in [-1, 0]$ , $x \in [0, 1]$ , where $b(t) = 1.5 + 10 \sin(2\pi t)$ , $T = 1$ and $a = 1$ . . . . .	78
4.12	A $T$ -periodic solution of the diffusive Nicholson's blowflies equation (4.28) with initial condition $\varphi(x, t) = 0.1$ for $t \in [-1, 0]$ , $x \in [0, 1]$ , where $\beta(t) = 0.025 + 0.015 \cos(2\pi t)$ , $T = 1$ and $\tau = 1$ . . . . .	81

# Chapter 1

## Introduction

The aim of this work is to study the existence of periodic solutions for the following abstract semilinear equation

$$\frac{du}{dt} = Au(t) + F(t, u(t)), \quad t \geq 0 \quad (1.1)$$

and abstract semilinear evolution equation

$$\frac{du}{dt} = A(t)u(t) + F(t, u(t)), \quad t \geq 0 \quad (1.2)$$

in a Banach space  $X$ , where  $A$  is a linear operator on  $X$  (not necessarily densely defined) satisfying the Hille-Yosida condition;  $A(t)$  is a  $T$ -periodic linear operators on  $X$  (not necessarily densely defined) satisfying the hyperbolic conditions (A1)-(A3) introduced by Tanaka [1995, 1996], which will be specified later; and  $F : [0, \infty) \times \overline{D(A)} \rightarrow X$  is continuous and  $T$ -periodic in  $t$ .

One aspect of studying the existence of periodic solutions is the Massera Theorem. Massera [1950] studied the existence of  $T$ -periodic solutions for the following ordinary differential equation

$$\frac{du}{dt} = f(t, u(t)), \quad t \in \mathbb{R}, \quad (1.3)$$

where  $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and  $T$ -periodic in  $t$ . He proved that the existence of  $T$ -periodic solutions of equation (1.3) is equivalent to the existence of a bounded solution on  $\mathbb{R}_+$  of equation (1.3). Important facts, results and references on periodic solutions of ordinary differential equations can be found in Farkas [1994].

The problem on the existence of periodic solutions for differential equations in infinite dimensional spaces has been investigated in various directions. One of them is to generalize the Massera Theorem to infinite dimensional systems. In fact, Massera and Schäffer [1959] studied the relationship between the periodic solutions and bounded solutions for the linear equation

$$\frac{du}{dt} = A(t)u + f(t), \quad t \geq 0 \quad (1.4)$$

in infinite dimensional spaces. Chow [1973] and Chow and Hale [1974] established the existence of a periodic solution under the existence of a bounded solution for the nonhomogeneous linear functional differential equation

$$\frac{du}{dt} = L(t, u_t) + f(t), \quad (1.5)$$

where  $u_t \in C_r := C([-r, 0], \mathbb{R})$ ,  $L : (-\infty, +\infty) \times C_r \rightarrow \mathbb{R}$  is continuous, linear with respect to the second argument and  $T$ -periodic in  $t$ ,  $T \geq r$ , and  $f$  is continuous and  $T$ -periodic. They proved that Massera Theorem holds for equation (1.5) by showing that the Poincaré map defined by  $P : \varphi \rightarrow u_T(\cdot, \varphi, f)$ , where  $u_T(\cdot, \varphi, f)$  is the unique mild solution of equation (1.5) initiated at  $\varphi$ , has a fixed point.

Prüss [1979] studied (1.1) under the condition that  $A$  generates a  $C_0$ -semigroup  $\{U(t)\}_{t \geq 0}$  of type  $(M, \omega)$ ,  $D(A)$  is closed, bounded and convex,  $F$  is continuous and  $T$ -periodic in  $t$ . By constructing a Poincaré map and using Schauder's fixed point theorem and  $k$ -set contraction argument, he proved the existence of mild  $T$ -periodic solutions when  $U(t)$  is compact for  $t > 0$  or  $\omega < 0$  and  $F$  is compact.

By applying Horn's fixed point theorem to the Poincaré map, Liu [1998] and Ezzinbi and Liu [2002] established the existence of bounded and ultimate bounded solutions of evolution equations with or without delay, implying the existence of periodic solutions. Kato et al. [2002] studied the periodic solution of abstract linear inhomogeneous differential equations in Banach space and presented a Messera type theorem. Benkhalt and Ezzinbi [2004] and Kpoumiè et al. [2018] proved that under some conditions, the existence of a bounded solution for some nondensely defined nonautonomous partial functional differential equations implies the existence of periodic solutions. The approach was to construct a map on the space of  $T$ -periodic functions from the corresponding nonhomogeneous linear equation and use a fixed-point theorem concerning set-valued maps to prove the existence of a fixed point for this map. Li [2011] used analytic semigroup theory to discuss the existence and stability of periodic solutions in evolution equations with multiple delays. Li et al. [1999] proved several Massera-type criteria for linear periodic evolution equations with delay and applied the results to nonlinear evolution equations, functional and partial differential equations.

Nguyen and Ngo [2016b,a] investigated the abstract semilinear evolution equation (1.2) when  $A(t)$  is  $T$ -periodic,  $F$  is  $T$ -periodic in  $t$  and satisfies the  $\varphi$ -Lipschitz condition  $\|F(t, x_1) - F(t, x_2)\| \leq \varphi(t) \|x_1 - x_2\|$  for  $\varphi(t)$  being a real and positive function belonging to an admissible function space. They proved the existence of periodic solutions to (1.2) in the case that the family  $\{A(t)\}_{t \geq 0}$  generates a strongly continuous, exponentially bounded evolution family. They started with the linear equation (1.4) and used the Cesàro limit to prove the existence of periodic solutions. Then they constructed a map from periodic solutions of (1.4) and used the admissibility of function spaces combined with the Banach fixed point argument to prove the existence of a unique fixed point of the constructed map. The existence and uniqueness of a periodic

solution of (1.2) follows from the existence and uniqueness of the fixed point. Naito et al. [2000] developed a decomposition technique to prove the existence of periodic solutions to periodic evolution equations in the form of (1.4). Vrabie [1990] studied the existence of periodic mild solutions to nonlinear evolution inclusions that include equation (1.1).

In this paper, we study the existence of mild periodic solutions of the abstract semilinear equation (1.1) and abstract semilinear evolution equation (1.2) in a setting that includes several types of equations such as delay differential equations, first-order hyperbolic partial differential equations, and reaction-diffusion equations. In chapter 2, we recall some preliminary results on semigroups generated by a Hille-Yosida operator, the evolution family and the existence theorem of solutions of nonhomogeneous linear equations (2.1) and (2.5). In chapter 3, we start with the linear equations (2.1) and (2.5) to show the existence of mild periodic solutions, whose initial value is controlled by the norm of the input function  $f(t)$ . Using this result and the fixed point argument, we prove the existence of mild periodic solutions of (1.1) and (1.2) under some assumptions on  $F$ . At the end of chapter 3, we also discuss the case where the semigroup  $\{U(t)\}_{t \geq 0}$  generated by  $A$  in (1.1) is compact for  $t > 0$  and give existence theorem of mild periodic solutions of (1.1). The approach is also to start with the linear equation (2.1) to show the existence of mild periodic solutions of it and use this result combined with the Schauder's fixed point theorem to prove the existence of mild periodic solutions of (1.1). In chapter 4 we use the main results of this paper to discuss the existence of periodic solutions in several types of equations and biological models, age-structured models with periodic harvesting, diffusive logistic models with periodic coefficients, functional differential equations including red blood cell models with delay, and partial functional differential equations including diffusive Nicholson's blowflies equation

with delay. A brief discussion on the conclusions and future study is given in Chapter 5.



## Chapter 2

### Preliminaries

In this chapter, we consider the nonhomogeneous linear Cauchy problem

$$\begin{cases} \frac{du}{dt} = Au(t) + f(t), & t \geq 0, \\ u(0) = x \in \overline{D(A)}. \end{cases} \quad (2.1)$$

where the linear operator  $A$  is densely or non-densely defined in a Banach space  $X$ , the function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous and  $T$ -periodic.

First we make the following assumptions.

**Assumption 2.0.1.** (a)  $A : D(A) \subset X \rightarrow X$  is a linear operator and there exist real constants,  $M \geq 1$  and  $\omega \in \mathbb{R}$ , such that  $(\omega, \infty) \subset \rho(A)$  and  $\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}$  for  $n \geq 1$  and  $\lambda > \omega$ ;

(b)  $x \in X_0 = \overline{D(A)}$ ;

(c)  $f : [0, \infty) \rightarrow X$  is continuous.

A linear operator  $A : D(A) \subset X \rightarrow X$  satisfying Assumption 2.0.1 (a) is called a *Hille-Yosida operator*.

**Remark 2.0.1.** Note that the renorming lemma (Lemma 5.1 in Pazy [1983]) holds. By exactly the same argument as in Pazy [1983], we see that if  $\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}$  for

$n \geq 1$  and  $\lambda > \omega$ , then there exists a norm  $|\cdot|$  on  $X$  which is equivalent to the original norm  $\|\cdot\|$  on  $X$  and satisfies  $\|x\| \leq |x| \leq M \|x\|$  for  $x \in X$  and  $|(\lambda I - A)^{-n}| \leq \frac{1}{(\lambda - \omega)^n}$  for  $n \geq 1$  and  $\lambda > \omega$ . That is, without loss of generality,  $M$  can be chosen to be 1.

**Definition 2.0.1** (Magal and Ruan [2007, 2009]). A continuous function  $u : [0, \infty) \rightarrow X$  is called a *mild (or an integrated) solution* to (2.1) if

$$u(t) = U_A(t)x + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-s) \lambda (\lambda I - A)^{-1} f(s) ds. \quad (2.2)$$

for all  $t \geq 0$ .

The existence theorem for (2.1) is as follows:

**Theorem 2.0.2** (Da Prato and Sinestrari [1987]). *Under Assumption 2.0.1, there exists a unique mild solution to (2.1) with value in  $X_0 = \overline{D(A)}$ . Moreover,  $u$  satisfies the estimate*

$$\|u(t)\| \leq M e^{\omega t} \|x\| + \int_0^t M e^{\omega t} \|f(s)\| ds \quad (2.3)$$

for all  $t \geq 0$ .

If  $\overline{D(A)} \neq X$ ; that is,  $A$  is nondensely defined, let  $X_0 = \overline{D(A)}$ . If  $f(t) = 0$ , then the family of operators  $\{U_A(t)\}_{t \geq 0}$  with  $U_A(t) : X_0 \rightarrow X_0$ ,  $t \geq 0$ , defined by  $U_A(t)x = u(t)$  for all  $t \geq 0$  is the  $C_0$ -semigroup generated by  $A_0$ , the part of  $A$  in  $X_0$ . For the rest of the article, we denote by  $\{U_A(t)\}_{t \geq 0}$  the semigroup generated by  $A_0$ .

Kato [1970] initiated a study on the evolution family of solutions of the hyperbolic linear evolution Cauchy problem

$$\begin{cases} \frac{du}{dt} = A(t)u(t), & t \geq s \\ u(s) = x \in X \end{cases} \quad (2.4)$$

in a Banach space  $X$ . To recall some results about the linear evolution Cauchy problem (2.4), we make the following assumptions.

**Assumption 2.0.3.** (A1)  $D(A(t)) := D$  is independent of  $t$  and not necessarily densely defined;

(A2) The family  $\{A(t)\}_{t \geq 0}$  is stable in the sense that there are constants  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $(\omega, \infty) \subset \rho(A(t))$  for  $t \in [0, \infty)$  and

$$\left\| \prod_{j=1}^k (\lambda I - A(t_j))^{-1} \right\| \leq \frac{M}{(\lambda - \omega)^{-k}}$$

for  $\lambda > \omega$  and every finite sequence  $\{t_j\}_{j=1}^k$  with  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$  and  $k = 1, 2, \dots$ ;

(A3) The mapping  $t \rightarrow A(t)x$  is continuously differentiable in  $X$  for each  $x \in D$ .

Now we recall some classical results due to Kato [1970].

**Theorem 2.0.4** (Kato [1970]). *Let  $\{A(t), D(A(t))\}_{t \geq 0}$  be a family of linear operators on a Banach space  $X$  satisfying Assumption 2.0.3 such that  $D$  is dense in  $X$ . Then the Cauchy problem (2.4) is well-posed and the family of operators  $\{A(t)\}_{t \geq 0}$  generates an evolution family  $\{U(t, s)\}_{t \geq s \geq 0}$ . Moreover, for  $x \in D$  the map  $t \rightarrow U(t, s)x$  is the unique continuous function which solves the Cauchy problem (2.4).*

For  $\lambda > 0$ ,  $0 \leq s \leq t$  and  $x \in \overline{D}$ , set

$$U_\lambda(t, s)x = \prod_{i=\lceil \frac{s}{\lambda} \rceil + 1}^{\lceil \frac{t}{\lambda} \rceil} (I - \lambda A(i\lambda))^{-1}x.$$

**Theorem 2.0.5** (Tanaka [1996]). *Let  $\{A(t)\}_{t \geq 0}$  be a family of linear operators on a Banach space  $X$  satisfying Assumption 2.0.3. If  $x \in D$  satisfies the condition that  $A(s)x \in \overline{D}$ , then there exists an evolution family  $\{U(t, s)\}_{t \geq s \geq 0}$  defined on  $\overline{D}$  by  $U(t, s)x = \lim_{\lambda \rightarrow 0^+} U_\lambda(t, s)x$  uniformly for  $x \in \overline{D}$  and satisfying:*

(i)  $U(t, s)D(s) \subset D(t)$  for all  $0 \leq s \leq t$ , where  $D(t) := \{x \in D : A(t)x \in \overline{D}\}$ ;

(ii) for all  $x \in D(s)$  and  $t \geq s$ , the mapping  $t \mapsto U(t, s)x$  is continuous in  $D$ ;

(iii) for all  $x \in D(s)$  and  $t \geq s$ , the mapping  $t \mapsto U(t, s)x$  is continuously differentiable with

$$\partial_t U(t, s)x = A(t)U(t, s)x$$

and

$$\partial_s^+ U(t, s)x = -U(t, s)A(s)x.$$

**Theorem 2.0.6** (Oka and Tanaka [2005], Tanaka [1996]). *Assume that  $\{A(t)\}_{t \geq 0}$  satisfies Assumption 2.0.3. Then the limit*

$$U(t, s)x = \lim_{\lambda \rightarrow 0^+} U_\lambda(t, s)x$$

exists for  $x \in \overline{D}$ ,  $0 \leq s \leq t$ , where the convergence is uniform on  $\Gamma := \{(t, s) : 0 \leq s \leq t\}$ . Moreover, the family  $\{U(t, s) : (t, s) \in \Gamma\}$  satisfies the following properties:

(i) For  $x \in \overline{D}$ ,  $\lambda > 0$  and  $0 \leq s \leq r \leq t$ , one has

$$U_\lambda(t, t)x = x$$

and

$$U_\lambda(t, s)x = U_\lambda(t, r)U_\lambda(r, s)x;$$

(ii)  $U(t, s) : \overline{D} \rightarrow \overline{D}$  for  $(t, s) \in \Gamma$ ;

(iii)  $U(t, t)x = x$  and  $U(t, s)x = U(t, r)U(r, s)x$  for  $x \in \overline{D}$  and  $0 \leq s \leq r \leq t$ ;

(iv) the mapping  $(t, s) \rightarrow U(t, s)x$  is continuous on  $\Gamma$  for any  $x \in \overline{D}$ ;

(v)  $\|U(t, s)x\| \leq Me^{\omega(t-s)} \|x\|$  for  $x \in \overline{D}$  and  $(t, s) \in \Gamma$ .

In the following, we give some results on the existence of solutions for the following non-densely defined nonhomogeneous linear evolution Cauchy problem

$$\begin{cases} \frac{du}{dt} = A(t)u(t) + f(t), & t \in [0, a] \\ u(0) = x, \end{cases} \quad (2.5)$$

where  $f : [0, a] \rightarrow X$  is a function. The following theorem gives a generalized variation of constant formula for equation (2.5).

**Theorem 2.0.7** (Tanaka [1995]). *Let  $x \in \overline{D}$  and  $f \in L^1([0, a], X)$ . Then the limit*

$$u(t) := U(t, 0)x + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, r)f(r)dr \quad (2.6)$$

*exists uniformly for  $t \in [0, a]$ , and  $u$  is a continuous function on  $[0, a]$ .*

As in Tanaka [1995, 1996], for  $x \in \overline{D}$  a continuous function  $u : [0, a] \rightarrow X$  is called a *mild (or an integrated) solution* of equation (2.5) if it satisfies (2.6). Furthermore, we have the following estimate.

**Lemma 2.0.1** (Kpoumiè et al. [2018]). *Assume that  $f \in L^1([0, a], X)$ . If  $u$  is a mild solution of (2.5), then*

$$\|u(t)\| \leq Me^{\omega t} \|x\| + \int_0^t Me^{\omega(t-s)} \|f(s)\| ds. \quad (2.7)$$

## Chapter 3

# Existence of Periodic Solutions

In this chapter we will present our main results on the existence of periodic solutions in systems (1.1) and (1.2) under different conditions..

### 3.1 Time-independent operators

We first assume that the operator is time-independent and consider the nonhomogeneous linear equations

$$\frac{du}{dt} = Au(t) + f(t) \quad (3.1)$$

and the semilinear equation

$$\frac{du}{dt} = Au(t) + F(t, u), \quad (3.2)$$

where  $A : D(A) \subset X \rightarrow X$  is a linear operator,  $f \in C([0, \infty), X)$  and  $F \in C([0, \infty) \times \overline{D(A)}, X)$  are both  $T$ -periodic in  $t$ .

We have the following results for the nonhomogeneous linear equation (3.1).

**Theorem 3.1.1.** *Assume that  $A$  is a Hille-Yosida operator with  $M \geq 1$  and  $\omega \in \mathbb{R}$ ,  $f \in C([0, \infty), X)$  is  $T$ -periodic, i.e.  $f(t + T) = f(t)$  for all  $t \geq 0$ . Further, suppose that  $\omega < 0$ . Then the linear equation (3.1) has a unique mild  $T$ -periodic solution  $u_0(t)$ .*

Moreover, we have

$$\|u_0(0)\| < N \sup_{s \in [0, T]} \|f(s)\|, \quad N = \frac{T}{1 - e^{\omega T}}.$$

*Proof.* Since  $A$  is a Hille-Yosida operator, the Cauchy problem (2.1) has a unique mild solution  $u(t) : [0, \infty) \rightarrow \overline{D(A)}$  on  $t \in [0, \infty)$  for each  $x \in \overline{D(A)}$  by Theorem 2.0.2.

Now by the variation of constant formula, we have

$$u(t) = U_A(t)x + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-s) \lambda (\lambda I - A)^{-1} f(s) ds, \quad (3.3)$$

where  $\{U_A(t)\}_{t \geq 0}$  is the  $C_0$ -semigroup generated by  $A$  on  $\overline{D(A)}$ . Let  $P_T : \overline{D(A)} \rightarrow \overline{D(A)}$  be the Poincaré map, i.e.,

$$P_T(x) = u(T) = U_A(T)x + \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(T-s) \lambda (\lambda I - A)^{-1} f(s) ds. \quad (3.4)$$

Since by assumption  $\omega < 0$ ,  $\|U_A(T)\| \leq M e^{\omega T}$ . Without loss of generality (W.L.O.G.), assume that  $M = 1$  (see section 1.5 Lemma 5.1 in Pazy [1983] for the proof). Then  $\|U_A(T)\| \leq e^{\omega T} < 1$ . Thus, the operator  $I - U_A(T)$  is invertible and  $P_T(x) = x$  has a unique solution

$$x_0 = (I - U_A(T))^{-1} \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(T-s) \lambda (\lambda I - A)^{-1} f(s) ds, \quad (3.5)$$

i.e.,  $x_0$  is a unique fixed point of  $P_T$ .

Now let  $u_T(t) = u(t + T)$ , where  $u(t)$  is the unique solution of (2.1) with initial

value  $x_0$ . Then

$$\begin{aligned}
u_T(t) &= U_A(t+T)x_0 + \lim_{\lambda \rightarrow +\infty} \int_0^{t+T} U_A(t+T-s)\lambda(\lambda I - A)^{-1}f(s)ds \\
&= U_A(t)U_A(T)x_0 + \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(t)U_A(T-s)\lambda(\lambda I - A)^{-1}f(s)ds \\
&\quad + \lim_{\lambda \rightarrow +\infty} \int_T^{t+T} U_A(t+T-s)\lambda(\lambda I - A)^{-1}f(s)ds \\
&= U_A(t)u(T) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-\theta)\lambda(\lambda I - A)^{-1}f(\theta+T)d\theta \\
&= U_A(t)u_T(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-\theta)\lambda(\lambda I - A)^{-1}f(\theta)d\theta.
\end{aligned}$$

Since  $u_T(0) = u(T) = x_0$ ,  $u_T(t)$  is also a mild solution of (2.1) with initial value  $x_0$ . By the uniqueness of solutions,  $u_T(t) = u(t)$ . Thus, we have  $u(t+T) = u(t)$  for  $t \in [0, \infty)$ .

Moreover, by (3.5), we have

$$\begin{aligned}
\|x_0\| &\leq \frac{\left\| \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1}f(s)ds \right\|}{\|I - U_A(T)\|} \\
&\leq \frac{\lim_{\lambda \rightarrow +\infty} \frac{\lambda}{\|\lambda I - A\|} \sup_{s \in [0, T]} \|f(s)\| \int_0^T e^{\omega(T-s)} ds}{|1 - \|U_A(T)\||} \\
&\leq \frac{T}{1 - e^{\omega T}} \sup_{s \in [0, T]} \|f(s)\|,
\end{aligned}$$

i.e.,  $\|u_0(0)\| \leq \frac{T}{1 - e^{\omega T}} \sup_{s \in [0, T]} \|f(s)\|$ . This completes the proof.  $\blacksquare$

Now we make the following assumptions.

**Assumption 3.1.2.** (H1)  $A$  is a Hille-Yosida operator on  $X$ ; i.e., there exist  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $(\omega, \infty) \subset \rho(A)$  and  $\|(\lambda I - A)^{-n}\|_{L(X)} \leq \frac{M}{(\lambda - \omega)^n}$  for  $\lambda > \omega$ ,  $n \geq 1$ ;

(H2)  $F : [0, \infty) \times \overline{D(A)} \rightarrow X$  is continuous and Lipschitz on bounded sets; i.e., for each  $C > 0$  there exists  $K_F(C) \geq 0$  such that  $\|F(t, u) - F(t, v)\| \leq K_F(C) \|u - v\|$  for  $t \in [0, \infty)$  and  $\|u\| \leq C$  and  $\|v\| \leq C$ ;



(H3)  $F : [0, \infty) \times \overline{D(A)} \rightarrow X$  is continuous and bounded on bounded sets; i.e., there exists  $L_F(T, \rho) \geq 0$  such that  $\|F(t, u)\| \leq L_F(T, \rho)$  for  $t \leq T$  and  $\|u\| \leq \rho$ .

With these assumptions, we have the following result for equation (3.2).

**Theorem 3.1.3.** *Let Assumption 3.1.2 hold with  $\omega < 0$  and  $F$  being  $T$ -periodic in  $t$ . Suppose that there exists  $\rho > 0$  such that  $(N+T)K_F(\rho) < 1$  and  $(N+T)L_F(T, \rho) \leq \rho$ , where  $N = \frac{T}{1-e^{\omega T}}$ . Then the semilinear equation (3.2) has a mild  $T$ -periodic solution.*

*Proof.* Denote

$$B_\rho = \{v \in C(\mathbb{R}_+, \overline{D(A)}), v(t+T) = v(t), \|v\| = \sup_{s \in [0, T]} \|v(s)\| \leq \rho\}.$$

By Theorem 3.1.1, for each  $v \in B_\rho$  let  $f(t) = F(t, v(t))$ , then (3.1) has a mild  $T$ -periodic solution

$$u(t) = U_A(t)u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1}F(l, v(l))dl. \quad (3.6)$$

Define an operator  $\phi$  on  $B_\rho$  by  $\phi(v)(t) = u(t)$ . Then

$$\|\phi(v)(t)\| \leq M e^{\omega t} \|u(0)\| + \lim_{\lambda \rightarrow +\infty} \int_0^t M e^{\omega(t-l)} \frac{M\lambda}{\lambda - \omega} \|F(l, v(l))\| dl.$$

W.L.O.G., let  $M = 1$  (See Lemma 5.1 in section 1.5 of Pazy [1983]). Since  $\|u(0)\| \leq \frac{T}{1-e^{\omega T}} \sup_{s \in [0, T]} \|f(s)\|$ , we have

$$\begin{aligned} \|\phi(v)(t)\| &\leq e^{\omega t} N \sup_{s \in [0, T]} \|F(s, v(s))\| + \lim_{\lambda \rightarrow +\infty} T \frac{\lambda}{\lambda - \omega} \sup_{s \in [0, T]} \|F(s, v(s))\|, \\ \sup_{s \in [0, T]} \|\phi(v)(t)\| &\leq (N+T) \sup_{s \in [0, T]} \|F(s, v(s))\| \leq (N+T)L_F(T, \rho) \leq \rho. \end{aligned}$$

So  $\phi$  maps  $B_\rho$  to  $B_\rho$ . Furthermore, let  $v_1, v_2 \in B_\rho$ . Then

$$\begin{aligned} \phi(v_1)(t) - \phi(v_2)(t) &= u_1(t) - u_2(t) \\ &= U_A(t)(u_1(0) - u_2(0)) \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1}(F(s, v_1(s)) - F(s, v_2(s))), \end{aligned}$$

$$\begin{aligned} \|\phi(v_1(t)) - \phi(v_2(t))\| &\leq Me^{\omega t} \|u_1(0) - u_2(0)\| \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_0^t Me^{\omega(t-s)} \frac{\lambda}{\lambda - \omega} \|F(s, v_1(s)) - F(s, v_2(s))\| ds. \end{aligned}$$

Again let  $M = 1$ . Since  $\|u_1(0) - u_2(0)\| \leq N \sup_{s \in [0, T]} \|F(s, v_1(s)) - F(s, v_2(s))\|$ , by the result in Theorem 3.1.1, we have

$$\begin{aligned} \|\phi(v_1)(t) - \phi(v_2)(t)\| &\leq e^{\omega t} N \sup_{s \in [0, T]} \|F(s, v_1(s)) - F(s, v_2(s))\| \\ &\quad + \lim_{\lambda \rightarrow +\infty} T \frac{\lambda}{\lambda - \omega} \sup_{s \in [0, T]} \|F(s, v_1(s)) - F(s, v_2(s))\|, \\ \sup_{s \in [0, T]} \|\phi(v_1)(t) - \phi(v_2)(t)\| &\leq (N + T)K_F(\rho) \sup_{s \in [0, T]} \|v_1(s) - v_2(s)\|. \end{aligned}$$

So it implies that

$$\|\phi(v_1) - \phi(v_2)\| \leq (N + T)K_F(\rho) \sup_{s \in [0, T]} \|v_1(s) - v_2(s)\|.$$

Since  $(N + T)K_F(\rho) < 1$ , by Banach Fixed Point Theorem,  $\phi : B_\rho \rightarrow B_\rho$  has a fixed point; i.e., there exists  $u \in B_\rho$  such that

$$u(t) = U_A(t)u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, u(s))ds,$$

which is a  $T$ -periodic solution for (3.2). ■

## 3.2 Time-dependent operators

Now consider the linear evolution equation

$$\frac{du}{dt} = A(t)u(t) + f(t) \tag{3.7}$$

and the semilinear evolution equation

$$\frac{du}{dt} = A(t)u(t) + F(t, u), \tag{3.8}$$

where  $A(t)$  is a  $T$ -periodic linear operator on a Banach space  $X$ ,  $f : \mathbb{R}_+ \rightarrow X$  is continuous and  $T$ -periodic, and  $F : \mathbb{R}_+ \times X \rightarrow X$  is continuous and  $T$ -periodic in  $t$ .

We make the following assumptions.

**Assumption 3.2.1.** (A1)  $D(A(t)) := D$  is independent of  $t$  and not necessarily densely defined;

(A2) The family  $\{A(t)\}_{t \geq 0}$  is stable in the sense that there are constants  $M \geq 1$  and  $\omega \in \mathbb{R}$  such that  $(\omega, \infty) \subset \rho(A(t))$  for  $t \in [0, \infty)$  and

$$\left\| \prod_{j=1}^k (\lambda I - A(t_j))^{-1} \right\| \leq \frac{M}{(\lambda - \omega)^{-k}}$$

for  $\lambda > \omega$  and every finite sequence  $\{t_j\}_{j=1}^k$  with  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$  and  $k = 1, 2, \dots$ ;

(A3) The mapping  $t \rightarrow A(t)x$  is continuously differentiable in  $X$  for each  $x \in D$ .

For  $\lambda > 0$ ,  $0 \leq s \leq t$ , and  $x \in \overline{D}$ . Set

$$U_\lambda(t, s)x = \prod_{i=\lceil \frac{s}{\lambda} \rceil + 1}^{\lceil \frac{t}{\lambda} \rceil} (I - \lambda A(i\lambda))^{-1}x. \quad (3.9)$$

Then the generalized variation of constant formula of (3.7) with initial value  $u(0) = x$  is given by

$$u(t) = U(t, 0)x + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, r)f(r)dr. \quad (3.10)$$

Now we state and prove the results for the nonhomogeneous linear evolution equation (3.7).

**Theorem 3.2.2.** *Let Assumption 3.2.1 hold,  $Me^{\omega T} < 1$ ,  $f \in C([0, \infty), X)$ ,  $f(t + T) = f(t)$  for  $t \in [0, \infty)$ , and  $\omega < 0$ . Then the linear evolution equation (3.7) has a unique mild  $T$ -periodic solution  $u(t)$ . Moreover,  $\|u(0)\| \leq N \sup_{s \in [0, T]} \|f(s)\|$ , where  $N = \frac{MT}{1 - Me^{\omega T}}$ .*

*Proof.* By assumption, the variation of constant formula (3.10) holds. Let  $P_T : \overline{D} \rightarrow \overline{D}$  be the Poincaré map

$$P_T(x) = u(T) = U(T, 0)x + \lim_{\lambda \rightarrow 0^+} \int_0^T U_\lambda(T, r)f(r)dr. \quad (3.11)$$

Since  $\|U(T, 0)\| \leq Me^{\omega T} < 1$ ,  $I - U(T, 0)$  is invertible.  $P_T$  has a unique fixed point which is given by  $x = (I - U(T, 0))^{-1} \lim_{\lambda \rightarrow 0^+} \int_0^T U_\lambda(T, r)f(r)dr$ .

Now let  $u(t)$  be the unique solution with initial value  $u(0) = x$ . Let  $u_T(t) = u(t + T)$ . Then

$$\begin{aligned} u_T(t) &= U(t + T, 0)x + \lim_{\lambda \rightarrow 0^+} \int_0^{T+t} U_\lambda(T + t, r)f(r)dr \\ &= U(t + T, T)U(T, 0)x + \lim_{\lambda \rightarrow 0^+} \int_0^T U_\lambda(T + t, T)U_\lambda(T, r)f(r)dr \\ &\quad + \lim_{\lambda \rightarrow 0^+} \int_T^{T+t} U_\lambda(T + t, r)f(r)dr \\ &= U(t + T, T)U(T, 0)x + \lim_{\lambda \rightarrow 0^+} U_\lambda(T + t, T) \int_0^T U_\lambda(T, r)f(r)dr \\ &\quad + \lim_{\lambda \rightarrow 0^+} \int_T^{t+T} U_\lambda(T + t, r)f(r)dr \\ &= U(t, 0)U(T, 0)x + U(T + t, T) \lim_{\lambda \rightarrow 0^+} \int_0^T U_\lambda(T, r)f(r)dr \\ &\quad + \lim_{\lambda \rightarrow 0^+} \int_T^{t+T} U_\lambda(T + t, r)f(r)dr \\ &= U(t, 0)U(T, 0)x + U(t, 0) \lim_{\lambda \rightarrow 0^+} \int_0^T U_\lambda(T, r)f(r)dr \\ &\quad + \lim_{\lambda \rightarrow 0^+} \int_T^{t+T} U_\lambda(T + t, r)f(r)dr \\ &= U(t, 0)[U(T, 0)x + \lim_{\lambda \rightarrow 0} \int_0^T U_\lambda(T, r)f(r)dr] \\ &\quad + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(T + t, T + s)f(T + s)ds \\ &= U(t, 0)u(T) + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, s)f(s)ds \\ &= U(t, 0)x + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, s)f(s)ds. \end{aligned}$$

So  $u_T(t)$  is a solution of (3.7) with initial value  $u_T(0) = x$ . By uniqueness,  $u_T(t) = u(t)$ , i.e.  $u(t+T) = u(t)$  for  $t \in [0, \infty)$ . Furthermore,

$$\begin{aligned}
\|x\| &= \|(I - U(T, 0))^{-1}\| \left\| \lim_{\lambda \rightarrow 0^+} \int_0^T U_\lambda(T, r) f(r) dr \right\| \\
&\leq \frac{1}{\|I - U(T, 0)\|} \lim_{\lambda \rightarrow 0^+} \int_0^T \|U_\lambda(T, r)\| \|f(r)\| dr \\
&\leq \frac{1}{\|I - U(T, 0)\|} \lim_{\lambda \rightarrow 0^+} \int_0^T \left\| \prod_{i=[\frac{r}{\lambda}] + 1}^{[\frac{T}{\lambda}]} (I - \lambda A(i\lambda))^{-1} \right\| \|f(r)\| dr \\
&\leq \frac{1}{\|I - U(T, 0)\|} \lim_{\lambda \rightarrow 0^+} \int_0^T M \left( \frac{1}{1 - \lambda\omega} \right)^{[\frac{T}{\lambda}] - [\frac{r}{\lambda}]} \|f(r)\| dr \\
&\leq \frac{1}{\|I - U(T, 0)\|} \lim_{\lambda \rightarrow 0^+} \int_0^T M \left( \frac{1}{1 - \lambda\omega} \right)^{\frac{T-r}{\lambda} + 1} \|f(r)\| dr \\
&\leq \frac{1}{\|1 - \|U(T, 0)\|\|} \lim_{\lambda \rightarrow 0^+} \int_0^T \frac{M}{1 - \lambda\omega} e^{-\omega(T-r)\frac{\ln(1-\lambda\omega)}{\lambda\omega}} \|f(r)\| dr \\
&\leq \frac{MT}{1 - Me^{\omega T}} \sup_{s \in [0, T]} \|f(s)\|.
\end{aligned}$$

This completes the proof. ■

**Remark 3.2.1.** Note that since the method in the proof of Lemma 5.1 in section 1.5 of Pazy [1983] does not work for the family of operators  $A(t)$ , we cannot assume  $M$  to be 1 in this case.

In order to study the semilinear evolution equation (3.8), we introduce the following definition.

**Definition 3.2.1.** A continuous function  $u : \mathbb{R}_+ \rightarrow X$  is called a *mild (or an integrated) solution* of equation (3.8) if it satisfies the following

$$u(t) = U(t, 0)u(0) + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, \sigma) F(\sigma, u(\sigma)) d\sigma, \quad t \geq 0. \quad (3.12)$$

Next we establish the existence of periodic solutions for the semilinear evolution equation (3.8).

**Theorem 3.2.3.** *Let Assumption 3.1.2 (H2) (H3) and Assumption 3.2.1 hold,  $\omega < 0$ ,  $Me^{\omega T} < 1$ ,  $F(t + T, \cdot) = F(t, \cdot)$  for  $t \geq 0$ . Suppose that there exists  $\rho > 0$  such that  $M(N + T)K_F(\rho) < 1$  and  $M(N + T)L_F(T, \rho) \leq \rho$ , where  $N = \frac{MT}{1 - Me^{\omega T}}$ . Then the semilinear evolution equation (3.8) has a mild  $T$ -periodic solution.*

*Proof.* Let  $B_\rho = \{v \in C(\mathbb{R}_+, \overline{D}), v(t + T) = v(t), \|v\| = \sup_{s \in [0, T]} \|v(s)\| \leq \rho\}$ . By Theorem 3.2.2, for each  $v \in B_\rho$  let  $f(t) = F(t, v(t))$ , then (3.7) has a unique mild  $T$ -periodic solution given by

$$u(t) = U(t, 0)u(0) + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, r)F(r, v(r))dr, t \geq 0. \quad (3.13)$$

Let  $\phi$  be an operator on  $B_\rho$  defined by  $\phi(v)(t) = u(t)$ . Then by the argument in Theorem 3.2.2, we have

$$\begin{aligned} \|\phi(v)(t)\| &\leq Me^{\omega t} \|u(0)\| + \int_0^t Me^{\omega(t-r)} \|F(r, v(r))\| dr, \\ \sup_{t \in [0, T]} \|\phi(v)(t)\| &\leq MN \sup_{r \in [0, T]} \|F(r, v(r))\| + MT \sup_{r \in [0, T]} \|F(r, v(r))\| \\ &\leq M(N + T)L_F(T, \rho) \\ &\leq \rho. \end{aligned}$$

So  $\phi : B_\rho \rightarrow B_\rho$ . Moreover, let  $v_1, v_2 \in B_\rho$ , then

$$\begin{aligned} \phi(v_1)(t) - \phi(v_2)(t) &= u_1(t) - u_2(t) \\ &= U(t, 0)(u_1(0) - u_2(0)) \\ &\quad + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, r)[F(r, v_1(r)) - F(r, v_2(r))]dr, \end{aligned}$$

$$\begin{aligned} \|\phi(v_1)(t) - \phi(v_2)(t)\| &\leq Me^{\omega t} \|u_1(0) - u_2(0)\| \\ &\quad + \int_0^t Me^{\omega(t-r)} \|F(r, v_1(r)) - F(r, v_2(r))\| dr, \end{aligned}$$

$$\begin{aligned}
\sup_{t \in [0, T]} \|\phi(v_1)(t) - \phi(v_2)(t)\| &\leq MN \sup_{r \in [0, T]} \|F(r, v_1(r)) - F(r, v_2(r))\| \\
&\quad + MT \sup_{r \in [0, T]} \|F(r, v_1(r)) - F(r, v_2(r))\| \\
&\leq M(N + T)K_F(\rho) \sup_{t \in [0, T]} \|v_1(t) - v_2(t)\|.
\end{aligned}$$

Thus, we have

$$\|\phi(v_1) - \phi(v_2)\| \leq M(N + T)K_F(\rho) \|v_1 - v_2\|.$$

Since  $M(N + T)K_F(\rho) < 1$ , by Banach Fixed Point Theorem,  $\phi$  has a fixed point  $u \in B_\rho$ ; i.e.,

$$u(t) = U(t, 0)u(0) + \lim_{\lambda \rightarrow 0^+} \int_0^t U_\lambda(t, r)F(r, u(r))dr,$$

which is a mild  $T$ -periodic solution for (3.8). ■

### 3.3 Time-independent operators - Revisited

Now consider (3.1) and (3.2) again when  $A$  is time independent. We will investigate the case when  $A$  is compact.

**Theorem 3.3.1.** *Let Assumption 3.1.2 (H1) hold,  $f \in C([0, \infty), X)$ ,  $f(t + T) = f(t)$ . Assume that  $U_A(T)$  is compact on  $\overline{D(A)}$ . If there exists  $x \in \overline{D(A)}$  such that the Cauchy problem (2.1) has a unique bounded mild solution  $u : [0, \infty) \rightarrow \overline{D(A)}$  for  $u(0) = x \in \overline{D(A)}$ , then the nonhomogeneous linear equation (3.1) has a mild  $T$ -periodic solution.*

*Proof.* It suffices to prove that the Poincaré map  $P_T$  has a fixed point  $x_0$ , where

$$P_T(x) = U_A(T)x + \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(t - s)\lambda(\lambda I - A)^{-1}f(s)ds.$$

By the same argument as in the proof of Theorem 3.1.1, let  $u(t)$  be the solution with initial value  $x$ ,  $u(t + T) = u(t)$  for  $t \geq 0$ , which implies that  $u(t)$  is a  $T$ -periodic solution of (3.1).

Suppose  $P_T$  has no fixed point, i.e.,

$$x = U_A(T)x + \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1}f(s)ds$$

has no solution in  $\overline{D(A)}$ . Let  $P = U_A(T) : \overline{D(A)} \rightarrow \overline{D(A)}$  and

$$x_0 = \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1}f(s)ds \in \overline{D(A)}.$$

Then  $x = Px + x_0$  has no solution in  $\overline{D(A)}$ . So 1 is an eigenvalue of  $P$ . Since  $P$  is assumed to be compact on  $\overline{D(A)}$ ,  $I - P$  is Fredholm, thus  $\mathcal{R}(I - P)$  is closed in  $\overline{D(A)}$ . Then there exists  $x^* \in \overline{D(A)}$  such that  $x^*((I - P)x) = 0$  for each  $x \in \overline{D(A)}$  and  $x^*(x_0) \neq 0$ . Let

$$x_n = P_T^n(x) = P^n x + (P^{n-1} + \dots + I)x_0,$$

where  $x$  is chosen such that (3.1) has a unique bounded solution for  $u(0) = x$ . Then

$$\begin{aligned} x^*(x_n) &= x^*[P^n x + (P^{n-1} + \dots + I)x_0] \\ &= x^*(P^n x) + x^*[(P^{n-1} + \dots + I)x_0] \\ &= (P')^n x^*(x) + [(P')^{n-1} + \dots + I]x^*(x_0). \end{aligned}$$

Note that  $x^*(x) = x^*(Px)$ , so  $P'x^*(x) = x^*(x)$  for  $x \in \overline{D(A)}$ . Then we get  $x^*(x_n) = x^*(x) + nx^*(x_0)$ . Let  $n \rightarrow \infty$ , it follows that  $nx^*(x_0) \rightarrow \infty$ . Then  $x^*(x_0) \rightarrow \infty$ , which contradicts the fact that  $x_n$  is bounded, since (3.1) has a unique bounded solution for  $x \in \overline{D(A)}$ . Therefore,  $P_T$  has a fixed point in  $\overline{D(A)}$  and (3.1) has a mild T-periodic solution. ■

Finally we prove an existence theorem of periodic solutions for the semilinear equation (3.2) when the operator  $A$  is compact.

**Theorem 3.3.2.** *Let Assumption 3.1.2 (H1) (H3) hold and  $F(t + T, x) = F(t, x)$  for  $t \geq 0$ ,  $x \in \overline{D(A)}$ . Let  $U_A(t)$  be compact on  $\overline{D(A)}$  for  $t > 0$ . Suppose that there*



exists  $\rho > 0$  such that  $(N + T)L_F(T, \rho) \leq \rho$ , where  $N = \frac{T}{1-e^{\omega T}}$  for  $\omega < 0$ , and  $(N + T)e^{\omega T}L_F(T, \rho) \leq \rho$ , where  $N = \frac{Te^{\omega T}}{\|I-U_A(T)\|}$  for  $\omega \geq 0$ . If for each  $T$ -periodic  $f \in C([0, \infty), X)$ , there exists  $x \in \overline{D(A)}$  such that the Cauchy problem (2.1) has a unique bounded mild solution for  $u(0) = x \in \overline{D(A)}$ , then the semilinear equation (3.2) has a  $T$ -periodic solution.

*Proof.* Define

$$B_\rho = \{v \in C(\mathbb{R}^+, \overline{D(A)}), v(t+T) = v(t), \|v\| = \sup_{s \in [0, T]} \|v(s)\| \leq \rho\}.$$

By Theorem 3.3.1, for each  $v \in B_\rho$ , let  $f(t) = F(t, v(t))$ . Then equation (3.1) has a unique mild  $T$ -periodic solution given by

$$u(t) = U_A(t)u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1}F(l, v(l))dl. \quad (3.14)$$

Moreover,

$$u(0) = (I - U_A)^{-1} \lim_{\lambda \rightarrow +\infty} \int_0^T U_A(T-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds, \quad (3.15)$$

$$\|u(0)\| \leq \begin{cases} \frac{e^{\omega T}T}{\|I-U_A(T)\|} \sup_{s \in [0, T]} \|F(s, v(s))\|, & \omega \geq 0, \\ \frac{T}{\|I-U_A(T)\|} \sup_{s \in [0, T]} \|F(s, v(s))\|, & \omega < 0. \end{cases} \quad (3.16)$$

Since  $\frac{T}{\|I-U_A(T)\|} \leq \frac{T}{1-e^{\omega T}}$  for  $\omega < 0$ , let

$$N = \begin{cases} \frac{T}{1-e^{\omega T}}, & \omega < 0, \\ \frac{e^{\omega T}T}{\|I-U_A(T)\|}, & \omega \geq 0. \end{cases}$$

Then we have  $\|u(0)\| \leq N \sup_{s \in [0, T]} \|F(s, v(s))\|$ . Define an operator  $\phi$  on  $B_\rho$  as follows:

$$\phi(v)(t) = u(t) = U_A(t)u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1}F(l, v(l))dl.$$

Then

$$\begin{aligned} \|\phi(v)(t)\| &\leq Me^{\omega t} \|u(0)\| + \int_0^t Me^{\omega(t-l)} \|F(l, v(l))\| dl \\ &\stackrel{W.L.O.G.M=1}{=} e^{\omega t} \|u(0)\| + \int_0^t e^{\omega(t-l)} \|F(l, v(l))\| dl. \end{aligned}$$

It follows that if  $\omega < 0$ ,

$$\begin{aligned} \sup_{t \in [0, T]} \|\phi(v)(t)\| &\leq \|u(0)\| + T \sup_{t \in [0, T]} \|F(t, v(t))\| \\ &\leq (N + T)L_F(T, \rho) \\ &\leq \rho. \end{aligned}$$

If  $\omega \geq 0$ ,

$$\begin{aligned} \sup_{t \in [0, T]} \|\phi(v)(t)\| &\leq \|u(0)\| + Te^{\omega T} \sup_{t \in [0, T]} \|F(t, v(t))\| \\ &\leq e^{\omega T} (N + T)L_F(T, \rho) \\ &\leq \rho. \end{aligned}$$

So  $\phi : B_\rho \rightarrow B_\rho$ .

Next, we show that  $\phi$  is compact. Let  $t > 0$ ,  $u \in \phi(B_\rho)$ . Then there exists  $v \in B_\rho$  such that

$$u(t) = U_A(t)u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-l)\lambda(\lambda I - A)^{-1}F(l, v(l))dl.$$

Let  $0 < \varepsilon < t$ , then

$$\begin{aligned} u(t) &= U_A(t)u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^{t-\varepsilon} U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{t-\varepsilon}^t U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &= U_A(t)u(0) + U_A(\varepsilon) \lim_{\lambda \rightarrow +\infty} \int_0^{t-\varepsilon} U_A(t-\varepsilon-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_{t-\varepsilon}^t U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds. \end{aligned}$$

Since  $\|F(s, v(s))\| \leq L_F(t, \rho)$ ,  $\|\lambda(\lambda I - A)^{-1}F(s, v(s))\| \leq \frac{\lambda}{\lambda - \omega}L_F(t, \rho)$ . It then follows that  $\lim_{\lambda \rightarrow +\infty} \int_0^{t-\varepsilon} U_A(t - \varepsilon - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds$  is bounded, by the compactness of  $U_A(\varepsilon)$ , it follows that

$$\{U_A(\varepsilon) \lim_{\lambda \rightarrow +\infty} \int_0^{t-\varepsilon} U_A(t - \varepsilon - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds, v \in B_\rho\}$$

is relatively compact in  $\overline{D(A)}$ . Moreover, there exists some  $b > 0$  such that

$$\left\| \lim_{\lambda \rightarrow +\infty} \int_{t-\varepsilon}^t U_A(t - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \right\| \leq b\varepsilon$$

for  $v \in B_\rho$ . Hence,  $\{u(t), v \in \phi(B_\rho)\}$  is relatively compact in  $\overline{D(A)}$  for each  $t > 0$ .

By the periodicity,  $\{u(0) : u \in \phi(B_\rho)\}$  is relatively compact in  $\overline{D(A)}$ .

Now we show the equi-continuity of  $\{u(t), v \in \phi(B_\rho)\}$ . For  $T + \varepsilon \geq t > \tau > 0$ , we have

$$\begin{aligned} u(t) - u(\tau) &= (U_A(t) - U_A(\tau))u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &\quad - \lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(\tau - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &= (U_A(t) - U_A(\tau))u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &\quad - \lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(t - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(t - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &\quad - \lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(\tau - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &= (U_A(t) - U_A(\tau))u(0) + \lim_{\lambda \rightarrow +\infty} \int_\tau^t U_A(t - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \\ &\quad + \lim_{\lambda \rightarrow +\infty} \int_0^\tau (U_A(t - \tau) - I)U_A(\tau - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds, \end{aligned}$$

$$\begin{aligned} \|u(t) - u(\tau)\| &\leq \|U_A(t) - U_A(\tau)\| \rho + \left\| \lim_{\lambda \rightarrow +\infty} \int_\tau^t U_A(t - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \right\| \\ &\quad + \left\| (U_A(t - \tau) - I) \lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(\tau - s)\lambda(\lambda I - A)^{-1}F(s, v(s))ds \right\|. \end{aligned}$$

Since  $\{U_A(t)\}_{t>0}$  is compact on  $\overline{D(A)}$ , it is continuous in uniform topology. Then  $\lim_{t \rightarrow \tau} \|U_A(t) - U_A(\tau)\| = 0$ . Since  $\|F(s, v(s))\| \leq L_F(T + \varepsilon, \rho)$  for  $v \in B_\rho$ ,  $0 < s < T + \varepsilon$ , there exists  $C > 0$  such that

$$\left\| \lim_{\lambda \rightarrow +\infty} \int_\tau^t U_A(t-s) \lambda (\lambda I - A)^{-1} F(s, v(s)) ds \right\| \leq C(t - \tau) \text{ for } v \in B_\rho.$$

Then

$$\lim_{t \rightarrow +\tau} \left\| \lim_{\lambda \rightarrow +\infty} \int_\tau^t U_A(t-s) \lambda (\lambda I - A)^{-1} F(s, v(s)) ds \right\| \leq \lim_{t \rightarrow +\tau} C(t - \tau) = 0$$

uniformly for  $v \in B_\rho$ . Since  $\{u(t) : v \in \phi(B_\rho)\}$  is relatively compact in  $\overline{D(A)}$  for each  $t \geq 0$  as shown above,  $\{u(t) - U_A(t)u(0) : v \in B_\rho\}$  is also relatively compact in  $\overline{D(A)}$  for each  $t \geq 0$ , which implies that  $\{\lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(\tau-s) \lambda (\lambda I - A)^{-1} F(s, v(s)) ds, v \in B_\rho\}$  is relatively compact in  $\overline{D(A)}$  for each  $\tau > 0$ . So there exists a compact set  $K \subset \overline{D(A)}$  such that

$$\lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(\tau-s) \lambda (\lambda I - A)^{-1} F(s, v(s)) ds \in K$$

for all  $v \in B_\rho$ .

Since  $\lim_{h \rightarrow 0} \sup_{\alpha \in K} \|(U_A(h) - I)\alpha\| = 0$  for compact  $K$ , it follows that

$$\lim_{t \rightarrow \tau} \sup_{v \in B_\rho} \left\| (U_A(t - \tau) - I) \lim_{\lambda \rightarrow +\infty} \int_0^\tau U_A(\tau-s) \lambda (\lambda I - A)^{-1} F(s, v(s)) ds \right\| = 0.$$

Summarizing the above analysis, we have

$$\lim_{t \rightarrow \tau, t > \tau > 0} \sup_{v \in B_\rho} \|u(t) - u(\tau)\| = 0.$$

Similarly,

$$\lim_{t \rightarrow \tau, \tau > t > 0} \sup_{v \in B_\rho} \|u(t) - u(\tau)\| = 0.$$

By periodicity,  $u(t)$  is also equi-continuous at  $t = 0$ . Now by Arzelà-Ascoli theorem,  $\phi(B_\rho)$  is relatively compact in  $C = \{\varphi | \varphi \in C(\mathbb{R}_+, \overline{D(A)}), \varphi(t+T) = \varphi(t)\}$ . So  $\phi$  has

a fixed point in  $B_\rho$ ; i.e., there exists  $u \in B_\rho$  such that

$$u(t) = U_A(t)u(0) + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-s)\lambda(\lambda I - A)^{-1}F(s, u(s))ds,$$

which is a mild  $T$ -periodic solution for (3.2). ■

**Remark 3.3.1.** Note that if  $F$  is bounded, i.e.,  $\|F(t, x)\| \leq B$  for each  $t \in [0, \infty)$  and  $x \in \overline{D(A)}$ , it is a special case of Theorem 3.3.2. In this case, we choose  $\rho \geq (N+T)B$ , then  $\|\phi(v)\| \leq (N+T)B \leq \rho$ , which implies that  $\phi : B_\rho \rightarrow B_\rho$ . By the argument in Theorem 3.3.2,  $\phi$  has a fixed point in  $B_\rho$ , which is a  $T$ -periodic solution for (3.2).

## Chapter 4

### Applications

The results obtained in last chapter can be applied to study the existence of periodic solutions in several types of equations including delay differential equations, first-order hyperbolic partial differential equations, and reaction-diffusion equations, in particular some biological and physical models described by these equations. In this chapter we consider age-structured population models with periodic harvesting and the diffusive logistic equation with periodic coefficients.

#### 4.1 Age-structured population models with periodic harvesting and constant boundary value

Consider the following problem (Anița et al. [1998]):

$$\begin{cases} \partial_t u(t, a) + \partial_a u(t, a) + \mu(a)u(t, a) = f(t, a) - v(t, a)u(t, a), & (t, a) \in \mathbb{R}_+ \times [0, a^+], \\ u(t, 0) = u_0, \\ u(t, a) = u(t + T, a), \end{cases} \quad (4.1)$$

where  $t$  is the time variable,  $a$  is the age variable, and  $u(t, a)$  is the density of the population at time  $t$  with age  $a$ . This is a linear model for an age-structured population (see for instance Iannelli [1995] and Webb [1985]), where  $\mu(a)$  is the age-specific death

rate. Moreover, the population is subject to a  $T$ -periodic external flow  $f(t, a)$  and a  $T$ -periodic age-specific harvesting effort  $v(t, a)$  (see for instance Anița et al. [1998]).

**(i) No harvesting.** First, we are concerned with the case  $v(t, a) \equiv 0$ .

**Proposition 4.1.1.** *Assume that*

(i)  $f \in C([0, \infty), L^1[0, a^+))$ ,  $f(t, a) = f(t + T, a)$  for  $t \geq 0$ ,  $a \in [0, a^+)$ ;

(ii)  $\mu(a) \in L^1[0, a^+)$  and there exists  $\mu_- > 0$  such that  $\mu(a) \geq \mu_-$  for  $a \in [0, a^+)$ .

Then there exists  $u(t, a) \in C([0, \infty), L^1[0, a^+))$  such that  $u(t, a)$  is a mild  $T$ -periodic solution of problem (4.1).

*Proof.* Consider the phase space  $X := L^1[0, a^+)$ . Define the linear operator  $A : D(A) \subset X \rightarrow X$  by

$$A\varphi = -\varphi' - \mu\varphi$$

with  $D(A) = \{\varphi \in W^{1,1}[0, a^+), \varphi(0) = 0\}$ . Then  $\overline{D(A)} = X$ . Consider the map  $F : \mathbb{R}_+ \rightarrow X$  given by

$$F(t)(a) = f(t, a).$$

Then the partial differential equation (4.1) can be written as

$$\frac{\partial u}{\partial t} = -\frac{\partial u}{\partial a} - \mu(a)u + f(t) = Au + f(t),$$

which can be further written as the abstract Cauchy problem (3.1).

Notice that for  $\lambda > -\mu_-$ , we have  $\lambda \in \rho(A)$ . Now let

$$(\lambda I - A)\phi = \varphi.$$

Then by the definition of  $A$ , we have

$$\phi'(a) + (\mu(a) + \lambda)\phi(a) = \varphi(a).$$

Solving  $\phi$  in terms of  $\varphi$ , we have

$$(\lambda I - A)^{-1}\varphi(a) = \phi(a) = \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau} \varphi(s) ds.$$

W.L.O.G. we assume that  $\varphi(a) \equiv 0$  for  $a > a^+$  and extend  $\varphi(a)$  to the whole  $\mathbb{R}_+$ . If  $\mu_- \leq \mu(a)$ , then

$$\begin{aligned} \|(\lambda I - A)^{-1}\varphi\|_{L^1} &= \left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau} \varphi(s) ds \right\|_{L^1} \\ &= \int_0^{a^+} \left| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau} \varphi(s) ds \right| da \\ &\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau} |\varphi(s)| ds da \\ &\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \mu_-(a-s)} |\varphi(s)| ds da \\ &\leq \int_0^\infty \int_0^a e^{-\lambda(a-s) - \mu_-(a-s)} |\varphi(s)| ds da \\ &= \int_0^\infty \int_s^\infty e^{-\lambda(a-s) - \mu_-(a-s)} |\varphi(s)| da ds \\ &= \int_0^\infty \left( \int_s^\infty e^{-(\lambda+\mu_-)a} da \right) e^{(\lambda+\mu_-)s} |\varphi(s)| ds \\ &= \frac{1}{\mu_- + \lambda} \int_0^\infty |\varphi(s)| ds \\ &= \frac{1}{\mu_- + \lambda} \int_0^{a^+} |\varphi(s)| ds \\ &= \frac{1}{\lambda + \mu_-} \|\varphi\|_{L^1}. \end{aligned}$$

Thus, we have

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda + \mu_-}.$$

So  $A$  is a Hille-Yoshida operator with  $M = 1$  and  $\omega = -\mu_- < 0$ , which satisfies the assumptions in Theorem 3.1.1. Moreover, since  $f \in C([0, \infty), X)$ , equation (3.1) has a unique solution for each initial  $u(0) \in \overline{D(A)} = X$  by Theorem 2.0.2. Therefore, there exists a mild  $T$ -periodic solution  $u(t, a)$  of (4.1) as desired.  $\blacksquare$



In the following, we choose specific functions and parameters which satisfy the assumptions in Proposition 4.1.1 and perform some numerical simulations to show that there is a periodic solution.

Let  $T = 1$ ,  $a^+ = 1$ ,  $\mu(a) = \frac{e^{-4a}}{1.0-a}$  and  $f(t, a) = 1 + 5a(1 - a) \sin(2\pi t)$ . We can see that  $\mu_- = 0.199$ . By Theorem 2.0.2, equation (3.1) has a unique solution for each initial  $u(0) \in \overline{D(A)} = X$ . So all assumptions in Theorem 3.1.1 are satisfied and there is a unique 1-periodic solution which is shown in Figure 4.1.

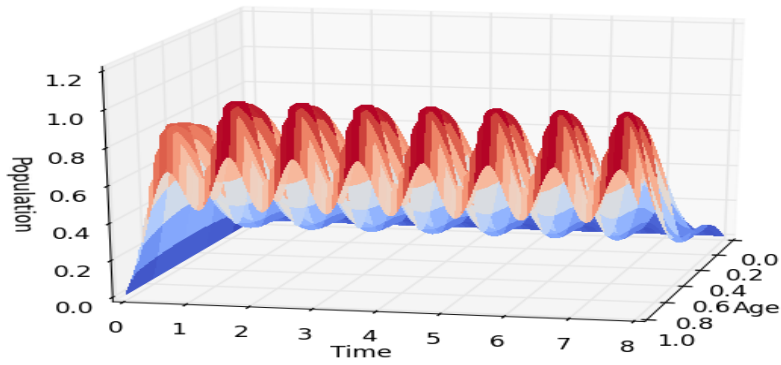


Figure 4.1: A  $T$ -periodic solution of (4.1) starting at  $u(0, a) = 0$  and with boundary condition  $u(t, 0) = 0$ , where  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ ,  $T = 1$ ,  $v(t, a) \equiv 0$  and  $f(t, a) = 1 + 5a(1 - a) \sin(2\pi t)$ .

**(ii) Periodic harvesting.** Now we consider the case when the harvest term  $v(t, a)$  is nonzero and  $T$ -periodic in  $t$ .

**Proposition 4.1.2.** *Assume that*

(i)  $f \in C([0, \infty), L^1[0, a^+))$ ,  $f(t, a) = f(t + T, a)$  for  $t \geq 0$ ,  $a \in [0, a^+)$ ;

(ii)  $\mu(a) \in L^1[0, a^+)$  and there exists  $\mu_- > 0$  such that  $\mu(a) \geq \mu_-$  for  $a \in [0, a^+)$ ;

(iii)  $v(t, a) \in C^1([0, \infty), L^1[0, a^+))$ ,  $v(t, a) = v(t + T, a)$  and there exists  $v_- > 0$  such that  $v(t, a) \geq v_-$  for  $t \geq 0$ ,  $a \in [0, a^+)$ .

Then there exists  $u(t, a) \in C([0, \infty), L^1[0, a^+))$  such that  $u(t, a)$  is a mild  $T$ -periodic solution of problem (4.1).

*Proof.* Let  $X := L^1[0, a^+]$ . Define the time-dependent  $T$ -periodic linear operator  $A(t) : D(A(t)) \subset X \rightarrow X$  by

$$A(t)\varphi = -\varphi' - \mu(a)\varphi - v(t, a)\varphi$$

with  $D(A(t)) = D = \{\varphi \in W^{1,1}[0, a^+], \varphi(0) = 0\}$ . Then  $\overline{D(A(t))} = \overline{D} = X$ . Consider the map  $f : \mathbb{R}_+ \rightarrow X$  given by

$$f(t)(a) = f(t, a)$$

Then the partial differential equation (4.1) can be written as the evolution equation (3.7).

Notice that for  $\lambda > -\mu_-$ , we have  $\lambda \in \rho(A(t))$  for  $\forall t \geq 0$ . Now let  $t \in \mathbb{R}_+$  and let

$$(\lambda I - A(t))\phi = \varphi.$$

Then by the definition of  $A(T)$  we have

$$\phi'(a) + (\mu(a) + v(t, a) + \lambda)\phi(a) = \varphi(a).$$

Solving  $\phi$  in terms of  $\varphi$ , we have

$$(\lambda I - A(t))^{-1}\varphi(a) = \phi(a) = \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau - \int_s^a v(t, \tau)d\tau} \varphi(s)ds.$$

W.L.O.G. we assume that  $\varphi(a) \equiv 0$  for  $a > a^+$  and extend  $\varphi(a)$  to the whole  $\mathbb{R}_+$ . If

$\mu_- \leq \mu(a)$  and  $v_- \leq v(t, a)$  for  $\forall t \in \mathbb{R}_+, a \in [0, a^+]$ , then

$$\begin{aligned}
\|(\lambda I - A(t))^{-1}\varphi\|_{L^1} &= \left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t, \tau) d\tau} \varphi(s) ds \right\|_{L^1} \\
&= \int_0^{a^+} \left| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t, \tau) d\tau} \varphi(s) ds \right| da \\
&\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t, \tau) d\tau} |\varphi(s)| ds da \\
&\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)| ds da \\
&\leq \int_0^\infty \int_0^a e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)| ds da \\
&= \int_0^\infty \int_s^\infty e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)| da ds \\
&= \int_0^\infty \left( \int_s^\infty e^{-(\lambda + \mu_- + v_-)a} da \right) e^{(\lambda + \mu_- + v_-)s} |\varphi(s)| ds \\
&= \frac{1}{\lambda + \mu_- + v_-} \int_0^\infty |\varphi(s)| ds \\
&= \frac{1}{\lambda + \mu_- + v_-} \int_0^{a^+} |\varphi(s)| ds \\
&= \frac{1}{\lambda + \mu_- + v_-} \|\varphi\|_{L^1}.
\end{aligned}$$

So we have

$$\|(\lambda I - A(t))^{-1}\| \leq \frac{1}{\lambda + \mu_- + v_-}.$$

Thus,

$$\left\| \prod_{j=1}^k (\lambda I - A(t_j))^{-1} \right\| \leq \frac{1}{(\lambda + (\mu_- + v_-))^{-k}}$$

for  $\lambda > -(\mu_- + v_-)$  and every finite sequence  $\{t_j\}_{j=1}^k$  with  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$  and  $k = 1, 2, \dots$ . We have checked that (A1) and (A2) of Assumption 3.2.1 hold for  $\{A(t)\}_{t \geq 0}$  with  $M = 1$  and  $\omega = -(\mu_- + v_-) < 0$ . Moreover, (iii) implies (A3) of Assumption 3.2.1. In addition, since  $f \in C([0, \infty), X)$ , (3.7) has a unique solution for each  $u(0) \in \bar{D} = X$  by Theorem 2.0.7. Therefore, there is a mild  $T$ -periodic solution for problem (4.1) by Theorem 3.2.2. ■

Proposition 4.1.2 indicates that if the external function  $f(t, u)$  is continuous and  $T$ -periodic, the mortality function  $\mu(a)$  is integrable and bounded from below, and the harvesting function  $v(t, a)$  is continuously differentiable,  $T$ -periodic in  $t$ , and bounded from below, then the model has a  $T$ -periodic mild solution.

Now we choose specific  $\mu(a)$ ,  $v(t, a)$  and  $f(t, a)$  which satisfy the assumptions in Proposition 4.1.2 and perform numerical simulations to demonstrate the existence of periodic solutions. Let  $T = 1$ ,  $a^+ = 1$ ,  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ ,  $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$  and  $f(t, a) = 1 + 5a(1 - a) \sin(2\pi t)$ . It then follows that  $\omega = -(\mu_- + v_-) = -0.299 < 0$ . Now all assumptions in Proposition 4.1.2 are satisfied, it follows that equation (4.1) has a unique mild 1-periodic solution, which is shown in Figure 4.2.

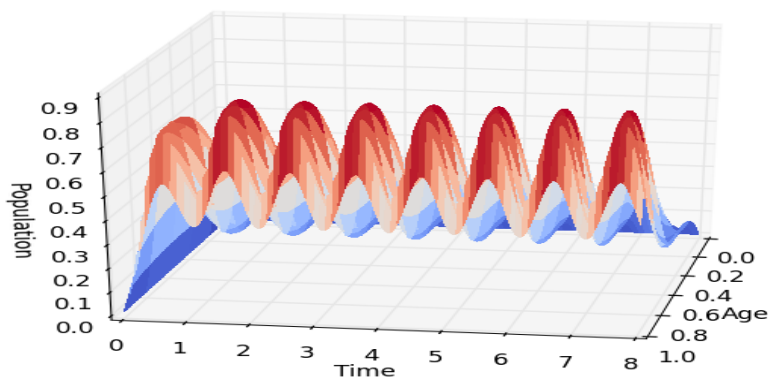


Figure 4.2: A  $T$ -periodic solution of (4.1) starting at  $u(0, a) = 0$  and with boundary condition  $u(t, 0) = 0$ , where  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ ,  $T = 1$ ,  $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$  and  $f(t, a) = 1 + 5a(1 - a) \sin(2\pi t)$ .

## 4.2 Age-structured population models with periodic harvesting and global population dependent boundary value

Consider the following problem (Anița et al. [1998]):

$$\begin{cases} \partial_t u(t, a) + \partial_a u(t, a) + \mu(a)u(t, a) = f(t, a) - v(t, a)u(t, a), & (a, t) \in [0, a^+] \times \mathbb{R}_+, \\ u(t, 0) = \int_0^{a^+} \gamma(t, a)u(t, a)da, \\ u(t, a) = u(t + T, a), \end{cases} \quad (4.2)$$

where  $t$  is the time variable,  $a$  is the age variable, and  $u(t, a)$  is the density of a population at time  $t$  with age  $a$ ,  $\mu(a)$  is the age-specified death rate, and  $\gamma(t, a)$  is the age-specified  $T$ -periodic birth rate. Moreover, there is a  $T$ -periodic external flow  $f(t, a)$  and a  $T$ -periodic age-specified harvesting effort  $v(t, a)$ .

**(i) No harvesting.** Once again, first we consider the case  $v(t, a) \equiv 0$ .

**Proposition 4.2.1.** *Assume that*

- (i)  $f \in C([0, \infty), L^1[0, a^+))$ ,  $f(t, a) = f(t + T, a)$  for  $t \geq 0$ ,  $a \in [0, a^+)$  and  $\sup_{t \in [0, T]} \int_0^{a^+} |f(t, a)| da \leq f_+(T)$ ;
- (ii)  $\mu(a) \in L^1[0, a^+)$  and there exists  $\mu_- > 0$  such that  $\mu(a) \geq \mu_-$  for  $a \in [0, a^+)$ ;
- (iii)  $\gamma(t, a) \in C([0, \infty), L^1[0, a^+))$ ,  $\gamma(t, a) = \gamma(t + T, a)$  and there exists  $\gamma_+ > 0$  such that  $0 \leq \gamma(t, a) \leq \gamma_+$  for  $t \geq 0$ ,  $a \in [0, a^+)$ ;
- (iv)  $(\frac{T}{1-e^{-\mu_-T}} + T)\gamma_+ < 1$  and the inequality  $(\frac{T}{1-e^{-\mu_-T}} + T)(\gamma_+\rho + f_+(T)) \leq \rho$  has solution.

Then problem (4.2) has a mild  $T$ -periodic solution  $u(t, a) \in C([0, \infty), L^1[0, a^+))$ .

*Proof.* Consider the space  $X := \mathbb{R} \times L^1(0, a^+)$  endowed with the product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^1(0, a^+)}.$$

Define the linear operator  $A : D(A) \subset X \rightarrow X$  by

$$A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi \end{pmatrix}$$

with  $D(A) = \{0\} \times W^{1,1}(0, a^+)$ , and  $\overline{D(A)} = \{0\} \times L^1(0, a^+) \neq X$ . Define  $F : \mathbb{R}_+ \times \overline{D(A)} \rightarrow X$  by

$$F \left( t, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) = \begin{pmatrix} \int_0^{a^+} \gamma(t, a)\phi(a)da \\ f(t, a) \end{pmatrix}.$$

Then the partial differential equation (4.2) can be written as the abstract semilinear equation (3.2). Notice that for  $\lambda > \mu_-$ , we have  $\lambda \in \rho(A)$ . Let

$$(\lambda I - A) \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \theta \\ \varphi \end{pmatrix}.$$

Then

$$(\lambda I - A)^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix}.$$

Since by definition of  $A$

$$(\lambda I - A) \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \phi(0) \\ \phi' + (\mu + \lambda)\phi \end{pmatrix},$$

we have

$$\phi(0) = \theta,$$

$$\phi'(a) + (\mu(a) + \lambda)\phi(a) = \varphi(a).$$

Hence,

$$\begin{aligned}
\phi(a) &= \theta e^{-\int_0^a (\mu(s)+\lambda)ds} + e^{-\int_0^a (\mu(s)+\lambda)ds} \int_0^a e^{\int_0^s (\mu(\tau)+\lambda)d\tau} \varphi(s) ds \\
&= \theta e^{-\lambda a - \int_0^a \mu(s)ds} + e^{-\lambda a - \int_0^a \mu(s)ds} \int_0^a e^{\lambda s + \int_0^s \mu(\tau)d\tau} \varphi(s) ds \\
&= \theta e^{-\lambda a - \int_0^a \mu(s)ds} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau} \varphi(s) ds.
\end{aligned}$$

So

$$(\lambda I - A)^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ \theta e^{-\lambda a - \int_0^a \mu(\tau)d\tau} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau} \varphi(s) ds \end{pmatrix},$$

$$\begin{aligned}
\left\| (\lambda I - A)^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \right\|_{L^1} &= \left\| \theta e^{-\lambda a - \int_0^a \mu(\tau)d\tau} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau} \varphi(s) ds \right\|_{L^1} \\
&\leq |\theta| \left\| e^{-\lambda a - \int_0^a \mu(\tau)d\tau} \right\|_{L^1} + \left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau)d\tau} \varphi(s) ds \right\|_{L^1}.
\end{aligned}$$

W.L.O.G. we assume that  $\varphi(a) \equiv 0$  for  $a \geq a^+$  and extend  $\varphi(a)$  to the whole  $\mathbb{R}_+$ . Since

$\mu(a) \geq \mu_-$ , we have

$$\begin{aligned}
\left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau} \varphi(s) ds \right\|_{L^1} &= \int_0^{a^+} \left| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau} \varphi(s) ds \right| da \\
&\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau} |\varphi(s)| ds da \\
&\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \mu_-(a-s)} |\varphi(s)| ds da \\
&\leq \int_0^\infty \int_0^a e^{-\lambda(a-s) - \mu_-(a-s)} |\varphi(s)| ds da \\
&= \int_0^\infty \int_s^\infty e^{-\lambda(a-s) - \mu_-(a-s)} |\varphi(s)| da ds \\
&= \int_0^\infty \left( \int_s^\infty e^{-(\mu_- + \lambda)a} da \right) e^{(\mu_- + \lambda)s} |\varphi(s)| ds \\
&= \frac{1}{\mu_- + \lambda} \int_0^\infty |\varphi(s)| ds \\
&= \frac{1}{\mu_- + \lambda} \int_0^{a^+} |\varphi(s)| ds \\
&= \frac{1}{\mu_- + \lambda} \|\varphi\|_{L^1}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\left\| e^{-\lambda a - \int_0^a \mu(\tau) d\tau} \right\|_{L^1} &= \int_0^{a^+} e^{-\lambda a - \int_0^a \mu(\tau) d\tau} da \\
&\leq \int_0^{a^+} e^{-(\lambda + \mu_-)a} da \\
&\leq \int_0^\infty e^{-(\lambda + \mu_-)a} da \\
&= \frac{1}{\mu_- + \lambda}.
\end{aligned}$$

So we obtain

$$\left\| (\lambda I - A)^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \right\|_{L^1} \leq \frac{1}{\lambda + \mu_-} [|\theta| + \|\varphi\|_{L^1}].$$

It follows that

$$\|(\lambda I - A)^{-1}\|_{L(X)} \leq \frac{1}{\lambda + \mu_-}$$



for  $\lambda > -\mu_-$ . Therefore,  $A$  is a Hille-Yosida operator with  $M = 1$  and  $\omega = -\mu_- < 0$ .

Moreover, since

$$F \left( t, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) = \begin{pmatrix} \int_0^{a^+} \gamma(t, a) \phi(a) da \\ f(t, a) \end{pmatrix},$$

$$F \left( t, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \right) - F \left( t, \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right) = \begin{pmatrix} \int_0^{a^+} \gamma(t, a) \phi_1(a) da \\ f(t, a) \end{pmatrix} - \begin{pmatrix} \int_0^{a^+} \gamma(t, a) \phi_2(a) da \\ f(t, a) \end{pmatrix}$$

$$= \begin{pmatrix} \int_0^{a^+} \gamma(t, a) [\phi_1(a) - \phi_2(a)] da \\ 0 \end{pmatrix}.$$

Assume  $\gamma(t, a) \leq \gamma_+$ , then it follows that

$$\left\| F \left( t, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \right) - F \left( t, \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right) \right\| = \left| \int_0^{a^+} \gamma(t, a) [\phi_1(a) - \phi_2(a)] da \right|$$

$$\leq \gamma_+ \int_0^{a^+} |\phi_1(a) - \phi_2(a)| da$$

$$= \gamma_+ \|\phi_1 - \phi_2\|_{L^1}$$

$$= \gamma_+ \left\| \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right\|.$$

So we have  $K_F(\rho) \equiv \gamma_+$ .

Assume that  $\sup_{t \in [0, T]} \int_0^{a^+} |f(t, a)| \leq f_+(T)$ , then for  $\|\phi\|_{L^1} \leq \rho$  we have

$$\begin{aligned} \left\| F \left( t, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) \right\| &= \left| \int_0^{a^+} \gamma(t, a) \phi(a) da \right| + \int_0^{a^+} |f(t, a)| da \\ &\leq \gamma_+ \int_0^{a^+} |\phi(a)| da + \int_0^{a^+} |f(t, a)| da \\ &\leq \gamma_+ \rho + f_+(T). \end{aligned}$$

So  $L_F(T, \rho) = \gamma_+ \rho + f_+(T)$ . Now we have checked conditions of Assumption 3.1.2. Moreover, since  $K_F(\rho) = \gamma_+$  and  $L_F(T, \rho) = \gamma_+ \rho + f_+(T)$ , we have  $(N + T)K_F(\rho) = (\frac{T}{1-e^{-\mu-T}} + T)\gamma_+ < 1$  and there exists  $\rho > 0$  such that  $(N + T)L_F(T, \rho) = (\frac{T}{1-e^{-\mu-T}} + T)(\gamma_+ \rho + f_+(T)) \leq \rho$ . In addition, the Cauchy problem (2.1) has a unique mild solution for each  $x \in \overline{D(A)}$  and each  $f \in C([0, \infty), X)$  by Theorem 2.0.2. Hence, all assumptions of Theorem 3.1.3 are satisfied and (4.2) has a mild  $T$ -periodic solution  $u(t, a) \in ([0, \infty), L^1(0, a^+))$ .  $\blacksquare$

Proposition 4.2.1 implies that if the external function  $f(t, a)$  is continuous and  $T$ -periodic and its integral over all ages is bounded above by  $f_+(T)$ , the death rate  $\mu(a)$  is integrable and bounded below, the birth rate  $\gamma(t, a)$  is continuous,  $T$ -periodic and bounded above by a constant  $\gamma_+$  which satisfies  $\gamma_+ < \frac{1}{\frac{T}{1-e^{-\mu-T}} + T}$  and there exists  $\rho > 0$  such that  $(\frac{T}{1-e^{-\mu-T}} + T)(\gamma_+ \rho + f_+(T)) \leq \rho$ , then the model has a  $T$ -periodic mild solution.

Next we choose specific functions and parameters for problem (4.2) which satisfy conditions in Proposition 4.2.1 and simulate the periodic solutions.

Let  $T = 1$  and  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , from the above discussion we know that  $\omega = -\mu_- = -0.199$ , then  $N = \frac{T}{1-e^{\omega T}} = \frac{1}{1-e^{-0.199}} \approx 5.5417$ . Let  $a^+ = 1$ ,  $f(t, a) = 1 + 2 \sin(2\pi t)$  and  $\gamma(t, a) = 0.2a^2(1 - a)(1 + \sin(2\pi t))$ . Then  $K_F(\rho) = \gamma_+ = 0.4 \times \frac{4}{27} \approx 0.059$

and  $(N + T) \times K_F(\rho) \approx 6.5417 \times 0.059 \approx 0.386 < 1$  for each  $\rho > 0$ . Furthermore,  $L_F(T, \rho) = 0.059\rho + 3$ , then  $(N + T)L_F(T, \rho) \leq \rho \Leftrightarrow 6.5417 \times (0.059\rho + 3) \leq \rho \Leftrightarrow 0.386\rho + 19.625 \leq \rho$ , which means that  $\rho \geq 31.96$ . Now all assumptions of Proposition 4.2.1 are satisfied. Thus, equation (4.2) has a 1-periodic solution which is shown in Figure 4.3.

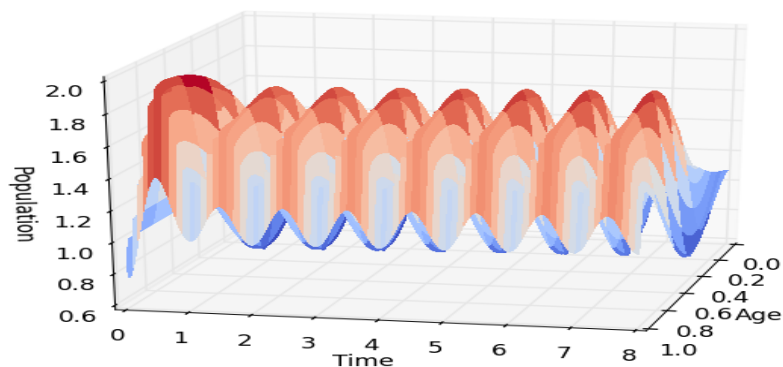


Figure 4.3: A  $T$ -periodic solution of (4.2) starting at  $u(0, a) = 1$  and with global boundary condition  $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ ,  $T = 1$ ,  $v(t, a) \equiv 0$ ,  $\gamma(t, a) = 0.2a^2(1 - a)(1 + \sin(2\pi t))$  and  $f(t, a) = 1 + 2 \sin(2\pi t)$ .

Now we change the parameters a little bit such that the assumptions of Proposition 4.2.1 are NOT satisfied. Let  $\gamma(t, a) = 4a^2(1 - a)(1 + \sin(2\pi t))$ , then  $\gamma_+ = 8 \times \frac{4}{27} = 1.18$  and  $(N + T)K_F(\rho) = 6.5417 \times 1.18 \approx 7.7192 > 1$ . So assumptions of Proposition 4.2.1 are not satisfied. Figure 4.4 shows a solution with the same initial value as the previous one in this case, which is no longer periodic.

**(ii) Periodic harvesting.** Now we let  $v(t, a)$  be nonzero and  $T$ -periodic in the time variable  $t$ .

**Proposition 4.2.2.** *Assume that*

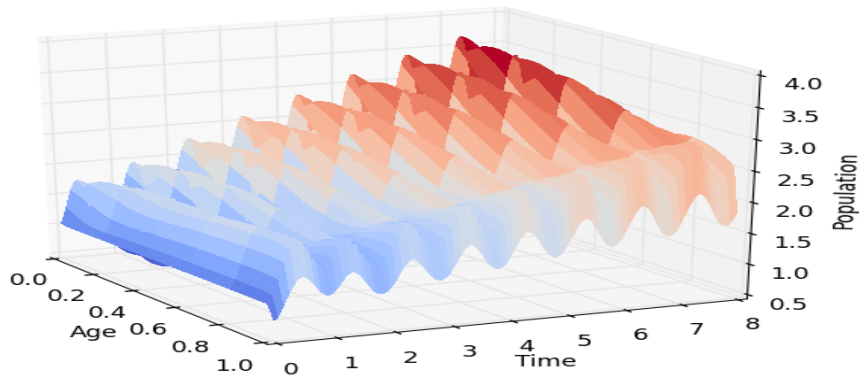


Figure 4.4: A solution of (4.2) starting at  $u(0, a) = 1$  and with boundary condition  $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ ,  $T = 1$ ,  $v(t, a) \equiv 0$ ,  $\gamma(t, a) = 4a^2(1 - a)(1 + \sin(2\pi t))$  and  $f(t, a) = 1 + 2 \sin(2\pi t)$ .

(i)  $f \in C([0, \infty), L^1[0, a^+))$ ,  $f(t, a) = f(t + T, a)$  for  $t \geq 0$ ,  $a \in [0, a^+)$  and  $\sup_{t \in [0, T]} \int_0^{a^+} |f(t, a)| da \leq f_+(T)$ ;

(ii)  $\mu(a) \in L^1[0, a^+)$  and there exists  $\mu_- > 0$  such that  $\mu(a) \geq \mu_-$  for  $a \in [0, a^+)$ ;

(iii)  $\gamma(t, a) \in C([0, \infty), L^1[0, a^+))$ ,  $\gamma(t, a) = \gamma(t + T, a)$  and there exists  $\gamma_+ > 0$  such that  $0 \leq \gamma(t, a) \leq \gamma_+$  for  $t \geq 0$ ,  $a \in [0, a^+)$ ;

(iv)  $v(t, a) \in C^1([0, \infty), L^1[0, a^+))$ ,  $v(t, a) \geq v_- > 0$  and  $v(t, a) = v(t + T, a)$  for  $t \geq 0$ ,  $a \in [0, a^+)$ ;

(v)  $(\frac{T}{1-e^{-(\mu_-+v_-)T}} + T)\gamma_+ < 1$  and the inequality  $(\frac{T}{1-e^{-(\mu_-+v_-)T}} + T)(\gamma_+\rho + f_+(T)) \leq \rho$  has solution.

Then problem (4.2) has a mild  $T$ -periodic solution  $u(t, a) \in C([0, \infty), L^1[0, a^+))$ .

*Proof.* Consider the space  $X := \mathbb{R} \times L^1(0, a^+)$  endowed with the product norm

$$\left\| \begin{pmatrix} \alpha \\ \varphi \end{pmatrix} \right\| = |\alpha| + \|\varphi\|_{L^1(0, a^+)}.$$

Define the time-dependent linear operator  $A(t) : D(A(t)) \subset X \rightarrow X$  by

$$A \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi(0) \\ -\varphi' - \mu\varphi - v(t)\varphi \end{pmatrix}$$

with  $D(A(t)) = D = \{0\} \times W^{1,1}(0, a^+)$ , and  $\overline{D(A(t))} = \overline{D} = \{0\} \times L^1(0, a^+) \neq X$ .

Define  $F : \mathbb{R}_+ \times \overline{D} \rightarrow X$  by

$$F \left( t, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) = \begin{pmatrix} \int_0^{a^+} \gamma(t, a)\phi(a)da \\ f(t, a) \end{pmatrix}.$$

Then the partial differential equation (4.2) can be written as the evolution equation (3.8).

Notice that for  $\lambda > -\mu_- - v_-$ , we have  $\lambda \in \rho(A(t))$  for  $\forall t \geq 0$ . For some  $t \in \mathbb{R}_+$ ,

let

$$\begin{aligned} (\lambda I - A(t)) \begin{pmatrix} 0 \\ \phi \end{pmatrix} &= \begin{pmatrix} \theta \\ \varphi \end{pmatrix}, \\ (\lambda I - A(t))^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} &= \begin{pmatrix} 0 \\ \phi \end{pmatrix}. \end{aligned}$$

Since by definition of  $A(t)$

$$(\lambda I - A(t)) \begin{pmatrix} 0 \\ \phi \end{pmatrix} = \begin{pmatrix} \phi(0) \\ \phi' + (\lambda + \mu + v(t))\phi \end{pmatrix},$$

we have

$$\phi(0) = \theta,$$

$$\phi'(a) + (\lambda + \mu(a) + v(t, a))\phi(a) = \varphi(a).$$

Then

$$\begin{aligned}\phi(a) &= \theta e^{-\int_0^a (\lambda + \mu(s) + v(t,s)) ds} + e^{-\int_0^a (\lambda + \mu(s) + v(t,s)) ds} \int_0^a e^{\int_0^s (\lambda + \mu(\tau) + v(t,\tau)) d\tau} \varphi(s) ds \\ &= \theta e^{-\lambda a - \int_0^a \mu(s) ds - \int_0^a v(t,s) ds} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t,\tau) d\tau} \varphi(s) ds.\end{aligned}$$

So for  $\lambda > -\mu_- - v_-$

$$\begin{aligned}(\lambda I - A(t))^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} &= \\ \begin{pmatrix} 0 \\ \theta e^{-\lambda a - \int_0^a \mu(s) ds - \int_0^a v(t,s) ds} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t,\tau) d\tau} \varphi(s) ds \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}& \left\| (\lambda I - A(t))^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \right\|_{L^1} \\ &= \left\| \theta e^{-\lambda a - \int_0^a \mu(s) ds - \int_0^a v(t,s) ds} + \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t,\tau) d\tau} \varphi(s) ds \right\|_{L^1} \\ &\leq |\theta| \left\| e^{-\lambda a - \int_0^a \mu(\tau) d\tau - \int_0^a v(t,\tau) d\tau} \right\|_{L^1} \\ &\quad + \left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t,\tau) d\tau} \varphi(s) ds \right\|_{L^1}.\end{aligned}$$

W.L.O.G. assume  $\varphi(a) \equiv 0$  for  $a > a^+$  and extend  $\varphi(a)$  to the whole  $\mathbb{R}_+$ . Since

$\mu(a) \geq \mu_-$  and  $v(t, a) \geq v_-$ , we have

$$\begin{aligned}
& \left\| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t, \tau) d\tau} \varphi(s) ds \right\|_{L^1} \\
&= \int_0^{a^+} \left| \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t, \tau) d\tau} \varphi(s) ds \right| da \\
&\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \int_s^a \mu(\tau) d\tau - \int_s^a v(t, \tau) d\tau} |\varphi(s)| ds da \\
&\leq \int_0^{a^+} \int_0^a e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)| ds da \\
&\leq \int_0^\infty \int_0^a e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)| ds da \\
&= \int_0^\infty \int_s^\infty e^{-\lambda(a-s) - \mu_-(a-s) - v_-(a-s)} |\varphi(s)| da ds \\
&= \int_0^\infty \left( \int_s^\infty e^{-(\lambda + \mu_- + v_-)a} da \right) e^{(\lambda + \mu_- + v_-)s} |\varphi(s)| ds \\
&= \frac{1}{\lambda + \mu_- + v_-} \int_0^\infty |\varphi(s)| ds \\
&= \frac{1}{\lambda + \mu_- + v_-} \int_0^{a^+} |\varphi(s)| ds \\
&= \frac{1}{\lambda + \mu_- + v_-} \|\varphi\|_{L^1}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
& \left\| e^{-\lambda a - \int_0^a \mu(\tau) d\tau - \int_0^a v(t, \tau) d\tau} \right\|_{L^1} \\
&= \int_0^{a^+} e^{-\lambda a - \int_0^a \mu(\tau) d\tau - \int_0^a v(t, \tau) d\tau} da \\
&\leq \int_0^{a^+} e^{-(\lambda + \mu_- + v_-)a} da \\
&\leq \int_0^\infty e^{-(\lambda + \mu_- + v_-)a} da \\
&= \frac{1}{\lambda + \mu_- + v_-},
\end{aligned}$$

So we obtain

$$\left\| (\lambda I - A(t))^{-1} \begin{pmatrix} \theta \\ \varphi \end{pmatrix} \right\|_{L^1} \leq \frac{1}{\lambda + \mu_- + v_-} (|\theta| + \|\varphi\|_{L^1})$$

for all  $t \in \mathbb{R}_+$  and  $\lambda > -(\mu_- + v_-)$ . It then follows that

$$\|(\lambda I - A(t))^{-1}\| \leq \frac{1}{\lambda + \mu_- + v_-}$$

for all  $t \in \mathbb{R}_+$  and  $\lambda > -(\mu_- + v_-)$  so that

$$\left\| \prod_{j=1}^k (\lambda I - A(t_j))^{-1} \right\| \leq \frac{1}{(\lambda + \mu_- + v_-)^k}$$

for  $\lambda > -(\mu_- + v_-)$  and every finite sequence  $\{t_j\}_{j=1}^k$  with  $0 \leq t_1 \leq t_2 \leq \dots \leq t_k$  and  $k = 1, 2, \dots$ . Hence, Assumption 3.2.1 holds for  $\{A(t)\}_{t \geq 0}$ .

Moreover, we have

$$F \left( t, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) = \begin{pmatrix} \int_0^{a^+} \gamma(t, a) \phi(a) da \\ f(t, a) \end{pmatrix},$$

$$\begin{aligned} F \left( t, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \right) - F \left( t, \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right) &= \begin{pmatrix} \int_0^{a^+} \gamma(t, a) \phi_1(a) da \\ f(t, a) \end{pmatrix} \\ &\quad - \begin{pmatrix} \int_0^{a^+} \gamma(t, a) \phi_2(a) da \\ f(t, a) \end{pmatrix} \\ &= \begin{pmatrix} \int_0^{a^+} \gamma(t, a) (\phi_1(a) - \phi_2(a)) da \\ 0 \end{pmatrix}. \end{aligned}$$

From the discussion of the case  $v(t, a) \equiv 0$ , we obtain

$$\left\| F \left( t, \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \right) - F \left( t, \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right) \right\| \leq \gamma_+ \left\| \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \right\|.$$



where  $\gamma(t, a) \leq \gamma_+$ . So  $K_F(\rho) = \gamma_+$ . Assume  $\sup_{t \in [0, T]} \int_0^{a^+} |f(t, a)| \leq f_+(T)$ , then from the discussion in the case  $v(t, a) \equiv 0$ , for  $\|\phi\|_{L^1} \leq \rho$ ,

$$\left\| F \left( t, \begin{pmatrix} 0 \\ \phi \end{pmatrix} \right) \right\| \leq \gamma_+ \rho + f_+(T).$$

Thus, we have  $L_F(T, \rho) = \gamma_+ \rho + f_+(T)$ . So we have checked Assumption 3.1.2 (H2)(H3) and Assumption 3.2.1 (A1)(A2), and (iv) implies Assumption 3.2.1 (A3). Moreover, we have

$$M(N + T)K_F(\rho) = \left( \frac{T}{1 - e^{-(\mu_- + v_-)T}} + T \right) \gamma_+ < 1$$

and there exists  $\rho > 0$  such that

$$M(N + T)L_F(T, \rho) = \left( \frac{T}{1 - e^{-(\mu_- + v_-)T}} + T \right) (\gamma_+ \rho + f_+(T)) \leq \rho.$$

Furthermore, Theorem 2.0.7 implies that the linear revolution Cauchy problem (2.5) has a unique mild solution for each  $u(0) \in \bar{D}$ . So all assumptions in Theorem 3.2.3 are satisfied which ensures that there is a mild  $T$ -periodic solution.  $\blacksquare$

Proposition 4.2.2 indicates that if the external function  $f(t, a)$  is continuous,  $T$ -periodic in  $t$  and its integral over all ages is bounded above by  $f_+(T)$ , the death rate  $\gamma(t, a)$  is integrable and bounded below by  $\mu_- > 0$ , the birth rate  $\gamma(t, a)$  is continuous,  $T$ -Periodic in  $t$  and bounded above by  $\gamma_+ > 0$ , the harvesting effort  $v(t, a)$  is continuously differentiable in  $t$  and integrable in  $a$ ,  $T$ -periodic and bounded below by  $v_- > 0$ , the upper bound  $\gamma_+$  of the birth rate  $\gamma(t, a)$  is bounded above by  $\gamma_+ < \frac{1}{\frac{T}{1 - e^{-(\mu_- + v_-)T}} + T}$  and there exists  $\rho > 0$  such that  $\left( \frac{T}{1 - e^{-(\mu_- + v_-)T}} + T \right) (\gamma_+ \rho + f_+(T)) \leq \rho$ , then the model has a  $T$ -periodic mild solution.

As an example, now we choose some specific functions and coefficients for problem (4.2) such that they satisfy conditions in Proposition 4.2.2.

Let  $T = 1$ ,  $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$  and  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ , then  $\omega = -\mu_- - v_- = -0.299 < 0$  and  $N = \frac{MT}{1 - Me^{\omega T}} = \frac{1}{1 - e^{-0.299}} \approx 3.87$ . Let  $a^+ = 1$ ,  $f(t, a) = 1 + 2 \sin(2\pi t)$  and  $\gamma(t, a) = 0.2a^2(1 - a)(1 + \sin(2\pi t))$ . Then  $K_F(\rho) \approx 0.059$ ,  $(N + T)K_F(\rho) \approx 4.87 \times 0.059 \approx 0.28733 < 1$  for all  $\rho > 0$ . In addition,  $L_F(T, \rho) = 0.059\rho + 3$ , then  $(N + T)L_F(T, \rho) \leq \rho \Leftrightarrow 4.87 \times (0.059\rho + 3) \leq \rho \Leftrightarrow 0.28733\rho + 14.61 \leq \rho$ , which means that  $\rho \geq 20.5$ . Then equation (4.2) has a mild 1-periodic solution by Proposition 4.2.2. A solution of equation (4.2) is shown in Figure 4.5.

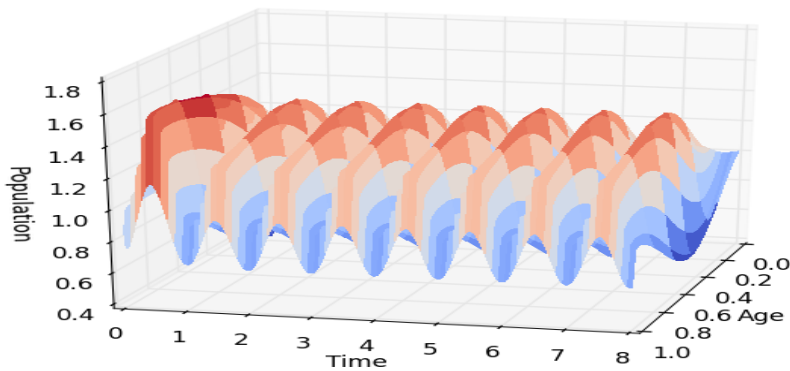


Figure 4.5: A  $T$ -periodic solution of (4.2) starting at  $u(0, a) = 1$  and with global boundary condition  $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ ,  $T = 1$ ,  $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$ ,  $\gamma(t, a) = 0.2a^2(1 - a)(1 + \sin(2\pi t))$  and  $f(t, a) = 1 + 2 \sin(2\pi t)$ .

Again, we change the parameters a little bit such that the assumptions of Proposition 4.2.2 are NOT satisfied. Let  $\gamma(t, a) = 4a^2(1 - a)(1 + \sin(2\pi t))$ , then  $\gamma_+ = 1.18$  and  $(N + T)K_F(\rho) = 4.87 \times 1.18 \approx 5.7466 > 1$ . Then assumptions of Proposition 4.2.2 are not satisfied. Figure 4.6 shows a solution with the same initial value as the previous

one, which is not periodic.

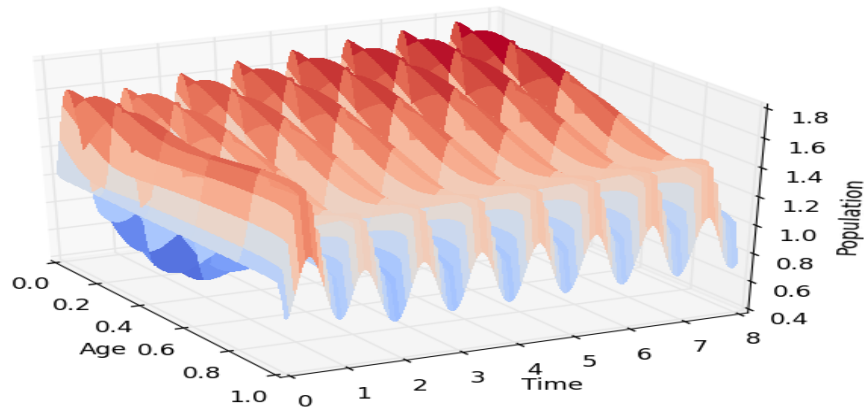


Figure 4.6: A solution of (4.2) starting at  $u(0, a) = 1$  and with boundary condition  $u(t, 0) = \int_0^1 \gamma(t, a)u(t, a)da$ , where  $\mu(a) = \frac{e^{-4a}}{1.0-a}$ ,  $T = 1$ ,  $v(t, a) = 0.5 + 0.4a(1 - a) \sin(2\pi t)$ ,  $\gamma(t, a) = 4a^2(1 - a)(1 + \sin(2\pi t))$  and  $f(t, a) = 1 + 2 \sin(2\pi t)$ .

### 4.3 The diffusive logistic model with periodic coefficients

This subsection is concerned with a diffusive logistic model in  $T$ -periodic environment.

Consider the following problem (Hess [1991], Ward Jr. [1979])

$$\begin{cases} \partial_t u(t, x) = \partial_x^2 u(t, x) + r(t)u(t, x)\left[1 - \frac{u(t, x)}{K(t)}\right], & t \in \mathbb{R}_+, x \in [0, 1], \\ u(t, 0) = u(t, 1) = 1, \\ u(t, x) = u(t + T, x), \end{cases} \quad (4.3)$$

where  $t$  is the time variable,  $x$  is the space variable, and  $u(t, x)$  is the density of a population at time  $t$  and location  $x$ . In the logistic term, we have a  $T$ -periodic intrinsic growth rate  $r(t)$  and a  $T$ -periodic carrying capacity  $K(t)$ . Moreover, we give constant boundary values.

Hess [1991] studied this kind of problem and gave existence theorems of periodic solutions under the existence of a positive supersolution and under assumptions on the eigenvalues of the linearized problem. I'll give the existence theorem of periodic solutions to this problem under another kind of assumptions by using a different method.

Let  $v(t, a) = u(t, a) - 1$ , then

$$\begin{cases} \partial_t v(t, x) = \partial_x^2 v(t, x) + r(t)[v(t, x) + 1][1 - \frac{v(t, x) + 1}{K(t)}], & t \in \mathbb{R}_+, x \in [0, 1], \\ v(t, 0) = v(t, 1) = 0, \\ v(t, x) = v(t + T, x), \end{cases} \quad (4.4)$$

where  $r(t)$  and  $K(t)$  are  $T$ -periodic. The existence of solutions for (4.3) and that for (4.4) are equivalent. From now on, we consider (4.4).

Let  $X = C[0, 1]$ . Define

$$Au = u''.$$

Then  $D(A) = \{u \in C^2[0, 1] : u(0) = u(1) = 0\}$ ,  $\overline{D(A)} = C[0, 1] = \{u \in C[0, 1] : u(0) = u(1) = 0\} \neq C[0, 1] = X$ . By separation of variable (see section 4.1 in Strauss [1992]), it follows that  $A$  generates a semigroup  $\{U_A(t)\}_{t \geq 0}$  on  $\overline{D(A)}$  given by

$$U_A(t)f(x) = \sum_{n=1}^{\infty} (2 \int_0^1 f(\xi) \sin(n\pi\xi) d\xi) \sin(n\pi x) e^{-(n\pi)^2 t}.$$

Define  $F : \mathbb{R}_+ \times \overline{D(A)} \rightarrow X$  by

$$F(t, \varphi) = r(t)(\varphi + 1)(1 - \frac{\varphi + 1}{K(t)}).$$

Then as before, we can rewrite (4.4) as abstract Cauchy problem (3.2).

**Proposition 4.3.1.** *Assume that*

- (i)  $r(t) \in C[0, \infty)$ , there exists  $r_+ > 0$  such that  $0 \leq r(t) \leq r_+$  for  $t \geq 0$ ,  $r(t) = r(t + T)$ ;

(ii)  $K(t) \in C[0, \infty)$ , there exists  $k_- > 0$  such that  $K(t) \geq k_-$  for  $t \geq 0$ ,  $K(t) = K(t + T)$ ;

(iii) There exists  $\rho > 0$  such that  $(\frac{T}{\|I - U_A(T)\|} + T)r_+(\rho + 1)(1 + \frac{1+\rho}{k_-}) \leq \rho$ .

Then problem (4.3) has a mild  $T$ -periodic solution.

*Proof.* It suffices to prove the following

(a)  $A$  is Hille-Yoshida operator with  $M = 1$  and  $\omega = 0$ ;

(b) There exists  $L_F(T, \rho) \geq 0$  such that  $\|F(t, u)\| \leq L_F(T, \rho)$  for  $t \leq T$  and  $\|u\| \leq \rho$ ;

(c)  $U_A(t)$  is compact on  $\overline{D(A)}$  for  $t > 0$ ;

(d) There exists  $\rho > 0$  such that  $(\frac{T}{\|I - U_A(T)\|} + T)r_+(\rho + 1)(1 + \frac{1+\rho}{k_-}) \leq \rho$ ;

(e) The Cauchy problem (2.1) has a unique mild solution for each  $x \in \overline{D(A)}$  and  $f \in C([0, \infty), X)$ ,  $f(t + T) = f(t)$ . Moreover there exists  $x \in \overline{D(A)}$  such that the solution  $u(t)$  with  $u(0) = x$  is bounded.

Note that if we rewrite (4.4) as abstract Cauchy problem (3.2), (a)-(e) cover all assumptions in Theorem 3.3.2. Then the existence of a mild  $T$ -periodic solution to problem (4.4) is guaranteed by Theorem 3.3.2. Thus, we get existence of a mild  $T$ -periodic solution to problem (4.3).

Now we prove (a)-(e).

(a) Let  $\psi \in X$ . Let  $\lambda > 0$ . Then

$$(\lambda I - A)\varphi = \psi \Leftrightarrow \lambda\varphi - \varphi'' = \psi$$

Set  $\hat{\varphi} = \varphi'$ . Then

$$\begin{aligned} (\lambda I - A)\varphi = \psi &\Leftrightarrow \begin{cases} \varphi' = \hat{\varphi} \\ \hat{\varphi}' = \lambda\varphi - \psi \end{cases} \\ &\Leftrightarrow \begin{cases} \sqrt{\lambda}\varphi' + \hat{\varphi}' = \sqrt{\lambda}(\sqrt{\lambda}\varphi + \hat{\varphi}) - \psi \\ \sqrt{\lambda}\varphi' - \hat{\varphi}' = -\sqrt{\lambda}(\sqrt{\lambda}\varphi - \hat{\varphi}) + \psi. \end{cases} \end{aligned}$$

Define

$$\begin{aligned} w &= (\sqrt{\lambda}\varphi + \hat{\varphi}), \\ \hat{w} &= (\sqrt{\lambda}\varphi - \hat{\varphi}). \end{aligned}$$

Then we have

$$(\lambda I - A)\varphi = \psi \Leftrightarrow \begin{cases} w' = \sqrt{\lambda}w - \psi, \\ \hat{w}' = -\sqrt{\lambda}\hat{w} + \psi. \end{cases} \quad (4.5)$$

The first equation of (4.5) is equivalent to

$$e^{-\sqrt{\lambda}x}w(x) = e^{-\sqrt{\lambda}y}w(y) - \int_y^x e^{-\sqrt{\lambda}l}\psi(l)dl, \quad \forall x \geq y. \quad (4.6)$$

In (4.6) let  $y = 0$ , then we obtain

$$w(x) = e^{\sqrt{\lambda}x}w(0) - e^{\sqrt{\lambda}x} \int_0^x e^{-\sqrt{\lambda}l}\psi(l)dl, \quad (4.7)$$

where  $w(0) = \sqrt{\lambda}\varphi(0) + \hat{\varphi}(0) = \hat{\varphi}(0)$ . In (4.6) let  $x = 1$ , we have

$$w(y) = e^{\sqrt{\lambda}y-\sqrt{\lambda}}w(1) + e^{\sqrt{\lambda}y} \int_y^1 e^{-\sqrt{\lambda}l}\psi(l)dl, \quad (4.8)$$

where  $w(1) = \sqrt{\lambda}\varphi(1) + \hat{\varphi}(1) = \hat{\varphi}(1)$ .

The second equation of (4.5) is equivalent to

$$e^{\sqrt{\lambda}x}\hat{w}(x) = e^{\sqrt{\lambda}y}\hat{w}(y) + \int_y^x e^{\sqrt{\lambda}l}\psi(l)dl, \quad \forall x \geq y. \quad (4.9)$$

In (4.9) let  $y = 0$ , then we have

$$\hat{w}(x) = e^{-\sqrt{\lambda}x}\hat{w}(0) + e^{-\sqrt{\lambda}x} \int_0^x e^{\sqrt{\lambda}l}\psi(l)dl, \quad (4.10)$$

where  $\hat{w}(0) = \sqrt{\lambda}\varphi(0) - \hat{\varphi}(0) = -\hat{\varphi}(0)$ . In (4.9) let  $x = 1$ , we have

$$\hat{w}(y) = e^{\sqrt{\lambda}-\sqrt{\lambda}y}\hat{w}(1) - e^{-\sqrt{\lambda}y} \int_y^1 e^{\sqrt{\lambda}l}\psi(l)dl, \quad (4.11)$$

where  $\hat{w}(1) = \sqrt{\lambda}\varphi(1) - \hat{\varphi}(1) = -\hat{\varphi}(1)$ .

From (4.7) and (4.10), we have

$$e^{2\sqrt{\lambda}x}\hat{w}(x) + w(x) = \int_0^x e^{\sqrt{\lambda}x}(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l})\psi(l)dl, \quad (4.12)$$

where  $x \in [0, 1]$ . Combining (4.8) and (4.11), we obtain

$$e^{2\sqrt{\lambda}(1-x)}w(x) + \hat{w}(x) = \int_x^1 e^{-\sqrt{\lambda}x}(e^{2\sqrt{\lambda}-\sqrt{\lambda}l} - e^{\sqrt{\lambda}l})\psi(l)dl. \quad (4.13)$$

Since  $\hat{w} = \sqrt{\lambda}\varphi - \hat{\varphi}$  and  $w = \sqrt{\lambda}\varphi + \hat{\varphi}$ , (4.12) and (4.13) can be written as

$$\sqrt{\lambda}(e^{2\sqrt{\lambda}x} + 1)\varphi + (1 - e^{2\sqrt{\lambda}x})\hat{\varphi} = \int_0^x e^{\sqrt{\lambda}x}(e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l})\psi(l)dl \quad (4.14)$$

and

$$(e^{2\sqrt{\lambda}(1-x)} + 1)\sqrt{\lambda}\varphi + (e^{2\sqrt{\lambda}(1-x)} - 1)\hat{\varphi} = \int_x^1 e^{-\sqrt{\lambda}x}(e^{2\sqrt{\lambda}-\sqrt{\lambda}l} - e^{\sqrt{\lambda}l})\psi(l)dl. \quad (4.15)$$

Combining (4.14) and (4.15), we have the following

$$\begin{aligned} \varphi(x) &= \frac{(e^{2\sqrt{\lambda}-\sqrt{\lambda}x} - e^{\sqrt{\lambda}x}) \int_0^x (e^{\sqrt{\lambda}l} - e^{-\sqrt{\lambda}l})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &\quad - \frac{(e^{-\sqrt{\lambda}x} - e^{\sqrt{\lambda}x}) \int_x^1 (e^{2\sqrt{\lambda}-\sqrt{\lambda}l} - e^{\sqrt{\lambda}l})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &= \frac{\int_0^x (e^{2\sqrt{\lambda}-\sqrt{\lambda}(x-l)} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}(x-l)})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &\quad - \frac{\int_x^1 (e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(l-x)} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(l-x)} + e^{\sqrt{\lambda}(x+l)})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &= \frac{\int_0^x (e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}|x-l|})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &\quad + \frac{\int_x^1 (e^{2\sqrt{\lambda}-\sqrt{\lambda}|l-x|} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}|l-x|})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &= \frac{\int_0^1 (e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}|x-l|})\psi(l)dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)}. \end{aligned}$$

Since  $\varphi \in \overline{D(A)}$ , it follows that

$$\begin{aligned} \|\varphi\| &= \sup_{x \in [0,1]} |\varphi(x)| \\ &= \sup_{x \in [0,1]} \left| \frac{\int_0^1 (e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}|x-l|}) \psi(l) dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \right|. \end{aligned}$$

Since  $e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}|x-l|} \geq 0$  for  $x \in [0, 1]$  and  $l \in [0, 1]$ ,

we have

$$\begin{aligned} \|\varphi\| &\leq \sup_{x \in [0,1]} |\psi(x)| \sup_{x \in [0,1]} \frac{\int_0^1 \left| e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}|x-l|} \right| dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &= \sup_{x \in [0,1]} |\psi(x)| \sup_{x \in [0,1]} \frac{\int_0^1 (e^{2\sqrt{\lambda}-\sqrt{\lambda}|x-l|} - e^{2\sqrt{\lambda}-\sqrt{\lambda}(x+l)} - e^{\sqrt{\lambda}(x+l)} + e^{\sqrt{\lambda}|x-l|}) dl}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &= \sup_{x \in [0,1]} |\psi(x)| \sup_{x \in [0,1]} \frac{e^{2\sqrt{\lambda}} \left( \frac{2}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}} e^{\sqrt{\lambda}(x-1)} - \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}x} + \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}(x+1)} - \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}x} \right)}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &\quad + \frac{-\frac{2}{\sqrt{\lambda}} + \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}(x-1)} + \frac{1}{\sqrt{\lambda}} e^{\sqrt{\lambda}x} - \frac{1}{\sqrt{\lambda}} e^{\sqrt{\lambda}(x+1)} + \frac{1}{\sqrt{\lambda}} e^{\sqrt{\lambda}x}}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &\leq \sup_{x \in [0,1]} |\psi(x)| \sup_{x \in [0,1]} \frac{\left( \frac{2}{\sqrt{\lambda}} - \frac{1}{\sqrt{\lambda}} e^{\sqrt{\lambda}(x-1)} - \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}x} + \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}(x+1)} - \frac{1}{\sqrt{\lambda}} e^{-\sqrt{\lambda}x} \right)}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &\quad \times (e^{2\sqrt{\lambda}} - 1) \\ &\leq \sup_{x \in [0,1]} |\psi(x)| \sup_{x \in [0,1]} \frac{(e^{2\sqrt{\lambda}} - 1) \frac{2}{\sqrt{\lambda}}}{2\sqrt{\lambda}(e^{2\sqrt{\lambda}} - 1)} \\ &= \frac{1}{\lambda} \sup_{x \in [0,1]} |\psi(x)| \\ &= \frac{1}{\lambda} \|\psi\|. \end{aligned}$$

Now we have  $\|(\lambda I - A)^{-1}\psi\| \leq \frac{1}{\lambda} \|\psi\|$ , which implies that  $\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$ . So  $A$  is Hille-Yoshida with  $M = 1$  and  $\omega = 0$ , which completes the proof of (a).



For  $\|\varphi\| \leq \rho$  and  $t \in [0, 1]$ ,

$$\left\| r(t)(\varphi + 1)\left(1 - \frac{1 + \varphi}{K(t)}\right) \right\| \leq r_+(\rho + 1)\left(1 + \frac{1 + \rho}{k_-}\right).$$

So we have  $L_F(1, \rho) = r^+(\rho + 1)\left(1 + \frac{1 + \rho}{k_-}\right)$ , which implies (b).

To prove (c), it suffices to prove uniform boundedness and equicontinuity of  $U_A(t)u(x)$  on  $\{u \in \overline{D(A)} : \|u\| \leq M_0\}$  for any  $M_0 > 0$ . Then (c) follows from Arzelà-Ascoli Theorem. For  $\|u\| \leq M_0$ ,

$$\begin{aligned} |U_A(t)u(x)| &= \left| \sum_{n=1}^{\infty} \left(2 \int_0^1 u(\xi) \sin(n\pi\xi) d\xi\right) \sin(n\pi x) e^{-(n\pi)^2 t} \right| \\ &\leq 2 \sum_{n=1}^{\infty} \left( \int_0^1 |u(\xi)| |\sin(n\pi\xi)| d\xi \right) |\sin(n\pi x)| e^{-(n\pi)^2 t} \\ &\leq 2 \sup_{\xi \in [0,1]} |u(\xi)| \sum_{n=1}^{\infty} \left( \int_0^1 |\sin(n\pi\xi)| d\xi \right) e^{-(n\pi)^2 t} \\ &= 2 \sup_{\xi \in [0,1]} |u(\xi)| \sum_{n=1}^{\infty} \frac{2}{\pi} e^{-(n\pi)^2 t} \\ &\leq \frac{4}{\pi} M_0 \sum_{n=1}^{\infty} \frac{1}{n^2 \pi^2 t} \\ &\leq \frac{8}{\pi^3 t} M_0. \end{aligned}$$

Then

$$\|U_A(t)u\| = \sup_{x \in [0,1]} |U_A(t)u(x)| \leq \frac{8}{\pi^3 t} M_0,$$

which implies that  $U_A(t)u(x)$  is uniformly bounded on  $\{u \in \overline{D(A)} : \|u\| \leq M_0\}$  for

any  $M_0 > 0$ . Now we prove equicontinuity.

$$\begin{aligned}
& |U_A(t)u(x) - U_A(t)u(y)| \\
&= \left| \sum_{n=1}^{\infty} \left( 2 \int_0^1 u(\xi) \sin(n\pi\xi) d\xi \right) (\sin(n\pi x) - \sin(n\pi y)) e^{-(n\pi)^2 t} \right| \\
&\leq \sum_{n=1}^{\infty} 2 \left( \int_0^1 |u(\xi)| |\sin(n\pi\xi)| d\xi \right) |\sin(n\pi x) - \sin(n\pi y)| e^{-(n\pi)^2 t} \\
&\leq 2 \sup_{\xi \in [0,1]} |u(\xi)| \sum_{n=1}^{\infty} \left( \int_0^1 |\sin(n\pi\xi)| d\xi \right) |\sin(n\pi x) - \sin(n\pi y)| e^{-(n\pi)^2 t} \\
&\leq \frac{4}{\pi} M_0 \sum_{n=1}^{\infty} |\sin(n\pi x) - \sin(n\pi y)| e^{-(n\pi)^2 t} \\
&\leq \frac{4}{\pi} M_0 \sum_{n=1}^{\infty} n\pi |x - y| e^{-(n\pi)^2 t} \\
&\leq \frac{4}{\pi} M_0 \sum_{n=1}^{\infty} n\pi |x - y| \frac{2}{n^4 \pi^4 t^2} \\
&= \frac{8}{\pi^4 t^2} M_0 \sum_{n=1}^{\infty} \frac{1}{n^3} |x - y| \\
&\leq \frac{12}{\pi^4 t^2} M_0 |x - y|.
\end{aligned}$$

So  $U_A(t)u(x)$  is equicontinuous on  $\{u \in \overline{D(A)} : \|u\| \leq M_0\}$  for any  $M_0 > 0$ . This completes the proof of (c).

(d) It follows directly from assumption (iii).

(e) Claim (a) together with Theorem 2.0.2 implies that the Cauchy problem (2.1) has a unique mild solution for each  $x \in \overline{D(A)}$  and  $f \in C([0, \infty), X)$  with  $f(t) = f(t+T)$ , which is the first part of (e).

Now we check that there is a bounded solution. From the variation of constant formula

$$u(t) = U_A(t)u_0 + \lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-s) \lambda (\lambda I - A)^{-1} f(s) ds,$$

we first consider the first part

$$U_A(t)u_0(x) = \sum_{n=1}^{\infty} \left( 2 \int_0^1 u_0(\xi) \sin(n\pi\xi) d\xi \right) \sin(n\pi x) e^{-(n\pi)^2 t}.$$

Then we have

$$\begin{aligned} |U_A(t)u_0(x)| &\leq \sup_{x \in [0,1]} \sum_{n=1}^{\infty} \left| \left( 2 \int_0^1 u_0(\xi) \sin(n\pi\xi) d\xi \right) \sin(n\pi x) \right| e^{-(n\pi)^2 t} \\ &\leq 2 \sum_{n=1}^{\infty} \sup_{\xi \in [0,1]} |u_0(\xi)| \frac{2}{\pi} e^{-(n\pi)^2 t} \\ &= \sup_{\xi \in [0,1]} |u_0(\xi)| \left( \sum_{n=1}^{\infty} \frac{4}{\pi} e^{-(n\pi)^2 t} \right). \end{aligned}$$

It follows that

$$\lim_{t \rightarrow +\infty} \sup_{x \in [0,1]} |U_A(t)u_0(x)| = 0.$$

So there exists an  $M > 0$  such that  $|U_A(t)u_0(x)| \leq M$  for  $t \in [0, \infty)$  and  $x \in [0, 1]$ .

Now we consider the second part  $\lim_{\lambda \rightarrow +\infty} \int_0^t U_A(t-s) \lambda(\lambda I - A)^{-1} f(s) ds$  and have

$$\begin{aligned} &|U_A(t-s) \lambda(\lambda I - A)^{-1} f(s)| \\ &= \left| \sum_{n=1}^{\infty} \left( 2 \int_0^1 \lambda(\lambda I - A)^{-1} f(s)(\xi) \sin(n\pi\xi) d\xi \right) \sin(n\pi x) e^{-(n\pi)^2(t-s)} \right| \\ &\leq 2 \sum_{n=1}^{\infty} \int_0^1 \|\lambda(\lambda I - A)^{-1}\| |f(s)(\xi)| |\sin(n\pi\xi)| d\xi e^{-(n\pi)^2(t-s)} \\ &\leq 2 \sum_{n=1}^{\infty} \sup_{\xi \in [0,1], s \in [0,1]} |f(s)(\xi)| \frac{2}{\pi} e^{-(n\pi)^2(t-s)}. \end{aligned}$$

It follows that

$$\begin{aligned}
& \left| \int_0^t U_A(t-s) \lambda (\lambda I - A)^{-1} f(s) ds \right| \\
& \leq 2 \sup_{\xi \in [0,1], \tau \in [0,1]} |f(\tau)(\xi)| \int_0^t \sum_{n=1}^{\infty} \frac{2}{\pi} e^{-(n\pi)^2(t-s)} ds \\
& = 2 \sup_{\xi \in [0,1], \tau \in [0,1]} |f(\tau)(\xi)| \sum_{n=1}^{\infty} \frac{2}{\pi} e^{-(n\pi)^2 t} \int_0^t e^{(n\pi)^2 s} ds \\
& = 2 \sup_{\xi \in [0,1], \tau \in [0,1]} |f(\tau)(\xi)| \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{1}{(n\pi)^2} (1 - e^{-(n\pi)^2 t}) \\
& < 2 \sup_{\xi \in [0,1], \tau \in [0,1]} |f(\tau)(\xi)| \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{1}{(n\pi)^2} \\
& \leq \frac{4}{\pi^3} \sup_{\xi \in [0,1], \tau \in [0,1]} |f(\tau)(\xi)| \times 2 \\
& = \frac{8}{\pi^3} \sup_{\xi \in [0,1], \tau \in [0,1]} |f(\tau)(\xi)|.
\end{aligned}$$

Hence, there exists an  $M_0 > 0$  such that

$$\lim_{\lambda \rightarrow +\infty} \left| \int_0^t U_A(t-s) \lambda (\lambda I - A)^{-1} f(s) ds \right| \leq M_0, \quad \forall t \geq 0.$$

Combining the above two parts, we have for each  $u_0 \in \overline{D(A)}$ , the solution to the Cauchy problem (2.1) is bounded for all  $t \geq 0$ , which completes the proof of the second part of (e). ■

Proposition 4.3.1 indicates that if the intrinsic growth rate  $r(t)$  is continuous,  $T$ -periodic and bounded above by a constant  $r_+$ , the carrying capacity  $K(t)$  is continuous,  $T$ -periodic and bounded below by a constant  $k_-$ , and there is  $\rho > 0$  such that all the parameters satisfy the inequality  $(\frac{T}{\|I - U_A(T)\|} + T)r_+(\rho+1)(1 + \frac{1+\rho}{k_-}) \leq \rho$ , then the model has a  $T$ -periodic mild solution.

Now we choose specific functions and parameters. Let  $T = 1$ ,  $r(t) = 0.15 + 0.1 \cos(2\pi t)$  and  $K(t) = 15 + \sin(2\pi t)$ , then  $F(t, \varphi) = (0.15 + 0.1 \cos(2\pi t))(\varphi +$

1)(1 - \frac{1+\varphi}{15+\sin(2\pi t)}). N = \frac{1}{\|I-U\_A(1)\|}, where

$$U_A(t)[f(x)] = \sum_{n=1}^{\infty} (2 \int_0^1 f(\xi) \sin(n\pi\xi) d\xi) \sin(n\pi x) e^{-(n\pi)^2 t},$$

$$\begin{aligned} \sup_{x \in [0,1]} |U_A(1)[f(x)]| &= \sup_{x \in [0,1]} \left| \sum_{n=1}^{\infty} (2 \int_0^1 f(\xi) \sin(n\pi\xi) d\xi) \sin(n\pi x) e^{-(n\pi)^2} \right| \\ &\leq \sup_{\xi \in [0,1]} |f(\xi)| \sum_{n=1}^{\infty} 2 \times \frac{2}{\pi} e^{-(n\pi)^2}, \end{aligned}$$

in which

$$\begin{aligned} e^{-(n\pi)^2} &= \frac{1}{e^{(n\pi)^2}} \\ &= \frac{1}{1 + (n\pi)^2 + \frac{(n\pi)^4}{2} + \dots} \\ &\leq \frac{2}{(n\pi)^4}. \end{aligned}$$

Thus,

$$\sum_{n=1}^{\infty} 2 \times \frac{2}{\pi} e^{-(n\pi)^2} \leq \frac{4}{\pi} \times \frac{2}{(n\pi)^4} = \frac{8}{\pi^5} \sum_{n=1}^{\infty} \frac{1}{n^4} < \frac{4}{3} \times \frac{8}{\pi^5} = \frac{32}{3\pi^5} < \frac{6}{\pi^3}.$$

So we derive

$$\sup_{x \in [0,1]} |U_A(1)[f(x)]| < \sup_{\xi \in [0,1]} |f(\xi)| \times \frac{6}{\pi^3},$$

i.e.,

$$\|U_A(1)\| < \frac{6}{\pi^3}.$$

Then

$$N = \frac{1}{\|I - U_A(1)\|} < \frac{1}{1 - \frac{6}{\pi^3}} \approx 1.24.$$

For  $\|\varphi\| \leq \rho$  and  $t \in [0, 1]$

$$\left\| r(t)(\varphi + 1) \left(1 - \frac{1 + \varphi}{K(t)}\right) \right\| \leq 0.25(\rho + 1) \left(1 + \frac{1 + \rho}{14}\right).$$

So  $r_+ = 0.25$ .

Then from  $(\frac{T}{\|I-U_A(T)\|} + T)r_+(\rho + 1)(1 + \frac{1+\rho}{k_-}) \leq \rho$ , we get  $2.24 \times 0.25(\rho + 1)(1 + \frac{1+\rho}{14}) \leq \rho$ , i.e.  $(\rho + 1)(\rho + 15) \leq 25\rho$ , which is also equivalent to  $\rho^2 - 9\rho + 15 \leq 0$ , where we get  $\frac{9-\sqrt{21}}{2} \leq \rho \leq \frac{9+\sqrt{21}}{2}$ , such  $\rho$  exists.

Now all the assumptions in Proposition 4.3.1 are satisfied, we conclude that (4.4) has a mild 1-periodic solution, i.e., (4.3) has a mild 1-periodic solution. The graph in Figure 4.7 shows the mild 1-periodic solution to the first equation and second boundary condition in (4.3) with initial value  $u \equiv 1$ , which confirms our result.

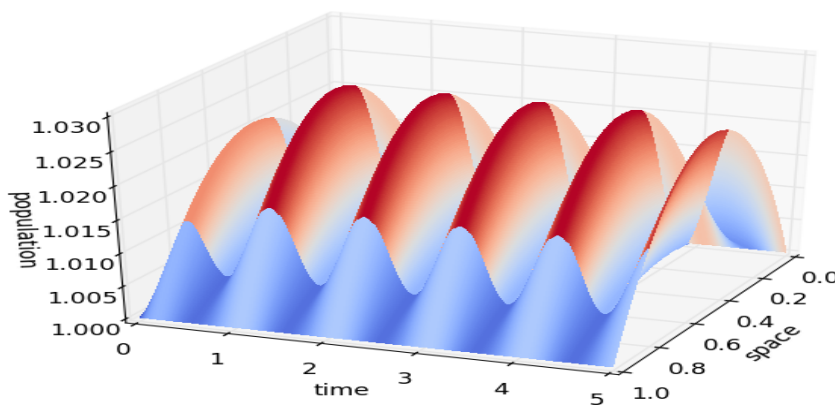


Figure 4.7: A  $T$ -periodic solution of the diffusive logistic equation (4.3) starting at  $u(0, a) = 1$  and with boundary condition  $u(t, 0) = u(t, 1) = 1$ , where  $r(t) = 0.15 + 0.1 \cos(2\pi t)$ ,  $T = 1$  and  $K(t) = 15 + \sin(2\pi t)$ .

#### 4.4 Retarded functional differential equations

The existence of periodic solutions in periodic functional differential equations has been studied by many researchers (see, for example, Chow [1973] and Chow and Hale [1974]), we refer to the classical references of Hale and Verduyn Lunel [1993] and

Burton [1983], and the references cited therein. In this subsection, we will apply the results in section 3 to obtain existence of periodic solutions in periodic functional differential equations. Namely, we will first consider a general class of retarded periodic functional differential equations, then we will consider a delayed red-blood cell model with periodic coefficients.

For  $r \geq 0$ , let  $\mathcal{C} = C([-r, 0], \mathbb{R}^n)$  be the Banach space of continuous functions from  $[-r, 0]$  to  $\mathbb{R}^n$  endowed with the supremum norm

$$\|\varphi\| = \sup_{\theta \in [-r, 0]} |\varphi(\theta)|_{\mathbb{R}^n}$$

Consider the retarded functional differential equations (RFDE) of the form

$$\begin{cases} \frac{dx(t)}{dt} = Bx(t) + \hat{L}(x_t) + f(t, x_t), \forall t \geq 0, \\ x_0 = \varphi \in \mathcal{C}, \end{cases} \quad (4.16)$$

where  $x_t \in \mathcal{C}$  is defined by  $x_t(\theta) = x(t + \theta)$  for  $\theta \in [-r, 0]$ ,  $B \in M_n(\mathbb{R})$  is an  $n \times n$  real matrix,  $\hat{L} : \mathcal{C} \rightarrow \mathbb{R}^n$  is a bounded linear operator given by

$$\hat{L}(\varphi) = \int_{-r}^0 d\eta(\theta)\varphi(\theta),$$

here  $\eta : [-r, 0] \rightarrow M_n(\mathbb{R})$  is a map of bounded variation, i.e.  $V(\eta, [-r, 0]) = \sup \sum_{i=1}^n \|\eta(\theta_{i+1}) - \eta(\theta_i)\| < +\infty$  in which the supremum is taken over all subdivisions  $-r = \theta_1 < \theta_2 < \dots < \theta_n < \theta_{n+1} = 0$ , and  $f : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$  is a continuous map.

Now following Liu et al. [2008] we rewrite (4.16) as an abstract non-densely defined Cauchy problem so that our theorems can be applied. First, we write it as a PDE. Define  $u \in C([0, \infty) \times [-r, 0], \mathbb{R}^n)$  by

$$u(t, \theta) = x(t + \theta), \quad \forall t \geq 0, \forall \theta \in [-r, 0].$$

If  $x \in C^1([-r, +\infty), \mathbb{R}^n)$ , then

$$\frac{\partial u(t, \theta)}{\partial t} = x'(t + \theta) = \frac{\partial u(t, \theta)}{\partial \theta}.$$

So we have

$$\frac{\partial u(t, \theta)}{\partial t} - \frac{\partial u(t, \theta)}{\partial \theta} = 0, \forall t \geq 0, \forall \theta \in [-r, 0].$$

Moreover, for  $\theta = 0$ , we have

$$\begin{aligned} \frac{\partial u(t, 0)}{\partial \theta} &= x'(t) = Bx(t) + \hat{L}(x_t) + f(t, x_t) \\ &= Bu(t, 0) + \hat{L}(u(t, \cdot)) + f(t, u(t, \cdot)), \forall t \geq 0. \end{aligned}$$

Thus,  $u$  satisfies the PDE

$$\begin{cases} \frac{\partial u(t, \theta)}{\partial t} - \frac{\partial u(t, \theta)}{\partial \theta} = 0, \\ \frac{\partial u(t, 0)}{\partial \theta} = Bu(t, 0) + \hat{L}(u(t, \cdot)) + f(t, u(t, \cdot)), \forall t \geq 0, \\ u(0, \cdot) = \varphi \in \mathcal{C}. \end{cases} \quad (4.17)$$

To rewrite (4.17) as an abstract non-densely defined Cauchy problem, let  $X = \mathbb{R}^n \times \mathcal{C}$  with the usual product norm

$$\left\| \begin{pmatrix} x \\ \varphi \end{pmatrix} \right\| = |x|_{\mathbb{R}^n} + \|\varphi\|.$$

Define the linear operator  $A : D(A) \subset X \rightarrow X$  by

$$A \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}, \forall \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in D(A), \quad (4.18)$$

with  $D(A) = \{0_{\mathbb{R}^n}\} \times C^1([-r, 0], \mathbb{R}^n)$ . Then  $\overline{D(A)} = \{0_{\mathbb{R}^n}\} \times \mathcal{C} \neq X$ . Define

$L : \overline{D(A)} \rightarrow X$  by

$$L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} = \begin{pmatrix} \hat{L}(\varphi) \\ 0_{\mathcal{C}} \end{pmatrix}$$



and  $F : \mathbb{R} \times \overline{D(A)} \rightarrow X$  by

$$F \left( t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \right) = \begin{pmatrix} f(t, \varphi) \\ 0_C \end{pmatrix}.$$

Set

$$v(t) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ u(t) \end{pmatrix}.$$

Then the PDE (4.17) can be written as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + L(v(t)) + F(t, v(t)), \quad t \geq 0; \quad v(0) = \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \in \overline{D(A)}. \quad (4.19)$$

Now we give an existence theorem of periodic solutions for equation (4.16).

**Assumption 4.4.1.** (B1)  $f : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$  is Lipschitz on bounded sets; i.e., for each

$C > 0$  there exists  $K_f(C) \geq 0$  such that  $\|f(t, u) - f(t, v)\| \leq K_f(C) \|u - v\|$  for  $t \in [0, \infty)$  and  $\|u\| \leq C$  and  $\|v\| \leq C$ ;

(B2)  $f : \mathbb{R} \times \mathcal{C} \rightarrow \mathbb{R}^n$  is bounded on bounded sets; i.e., there exists  $L_f(T, \rho) \geq 0$  such

that  $\|f(t, u)\| \leq L_f(T, \rho)$  for  $t \leq T$  and  $\|u\| \leq \rho$ .

With these assumptions and the notation  $\omega_0(B) := \sup_{\lambda \in \sigma(B)} \operatorname{Re}(\lambda)$ , we have the following result for equation (4.16).

**Theorem 4.4.2.** *Let Assumption 4.4.1 hold with  $\omega_0(B) < 0$  and  $f$  being  $T$ -periodic in*

*$t$ . Suppose that there exists  $\rho > 0$  such that  $(N + T)(K_f(\rho) + V(\eta, [-r, 0])) < 1$  and*

*$(N + T)(L_f(T, \rho) + V(\eta, [-r, 0]))\rho \leq \rho$ , where  $N = \frac{T}{1 - e^{\omega_0(B)T}}$ , then equation (4.16)*

*has a  $T$ -periodic solution.*

*Proof.* Since (4.16) can be written as (4.19), denote  $G(t, v(t)) = L(v(t)) + F(t, v(t))$ ,

it suffices to prove that

- (a)  $A$  satisfies Assumption 3.1.2 (H1) with  $\omega < 0$ ;
- (b)  $G : [0, \infty) \times 0_{\mathbb{R}^n} \times \mathcal{C} \rightarrow \mathbb{R}^n \times \mathcal{C}$  satisfies Assumption 3.1.2 (H1) (H2);
- (c) There exists  $\rho > 0$  such that  $(N + T)K_G(\rho) < 1$  and  $(N + T)L_G(T, \rho) \leq \rho$ ,  
 where  $N = \frac{T}{1 - e^{\omega T}}$ .

Then it follows from Theorem 3.1.3 that equation (4.19) has a  $T$ -periodic mild solution, which implies that equation (4.17) has a  $T$ -periodic mild solution with initial  $u(0, \cdot) = \varphi_0 \in \mathcal{C}$ . Meanwhile, by Theorem 2.1 in Hale and Verduyn Lunel [1993], equation (4.16) has a unique solution  $x_0(t) \in C^1([0, \infty), \mathbb{R}^n)$  with initial  $x_0(\theta) = \varphi_0(\theta)$  for  $\theta \in [-r, 0]$ . Therefore,  $x_0(t)$  is a  $T$ -periodic solution for (4.16).

From Lemma 7.1 in Magal and Ruan [2018], we know that  $A$  as defined in (4.18) is a Hille-Yoshida operator with  $\omega = \omega_0(B) < 0$  and  $M = 1$ , which proves (a).

For  $\varphi_1, \varphi_2 \in \mathcal{C}$  such that  $\|\varphi_1\| \leq C$  and  $\|\varphi_2\| \leq C$ , we have

$$\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix} \in 0_{\mathbb{R}^n} \times \mathcal{C} = \overline{D(A)}$$

and

$$\left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix} \right\| = \|\varphi_1\| \leq C, \left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix} \right\| = \|\varphi_2\| \leq C.$$

Then

$$\begin{aligned}
& \left\| G\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}\right) - G\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}\right) \right\| \\
&= \left\| L\left(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}\right) - L\left(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}\right) + F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}\right) - F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}\right) \right\| \\
&\leq \left\| L\left(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}\right) - L\left(\begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}\right) \right\| + \left\| F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}\right) - F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}\right) \right\| \\
&= \left\| \begin{pmatrix} \int_{-r}^0 d\eta(\theta)(\varphi_1(\theta) - \varphi_2(\theta)) \\ 0_{\mathcal{C}} \end{pmatrix} \right\| + \left\| \begin{pmatrix} f(t, \varphi_1) - f(t, \varphi_2) \\ 0_{\mathcal{C}} \end{pmatrix} \right\| \\
&= \left| \int_{-r}^0 d\eta(\theta)(\varphi_1(\theta) - \varphi_2(\theta)) \right|_{\mathbb{R}^n} + |f(t, \varphi_1) - f(t, \varphi_2)|_{\mathbb{R}^n} \\
&\leq K_f(C) \|\varphi_1 - \varphi_2\| + V(\eta, [-r, 0]) \|\varphi_1 - \varphi_2\| \\
&= (K_f(C) + V(\eta, [-r, 0])) \|\varphi_1 - \varphi_2\| \\
&= (K_f(C) + V(\eta, [-r, 0])) \left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix} \right\|.
\end{aligned}$$

So there exists  $K_G(C) = K_f(C) + V(\eta, [-r, 0])$  such that

$$\left\| G\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix}\right) - G\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix}\right) \right\| \leq K_G(C) \left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi_2 \end{pmatrix} \right\|.$$

Furthermore, for  $t \leq T$  and  $\left\| \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \right\| \leq \rho$ , we have

$$\begin{aligned}
\left\| G\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) \right\| &= \left\| L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} + F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) \right\| \\
&\leq \left\| L \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix} \right\| + \left\| F\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) \right\| \\
&= \left\| \begin{pmatrix} \int_{-r}^0 d\eta(\theta)\varphi(\theta) \\ 0_c \end{pmatrix} \right\| + \left\| \begin{pmatrix} f(t, \varphi) \\ 0_c \end{pmatrix} \right\| \\
&= \left| \int_{-r}^0 d\eta(\theta)\varphi(\theta) \right|_{\mathbb{R}^n} + |f(t, \varphi)|_{\mathbb{R}^n} \\
&\leq V(\eta, [-r, 0])\rho + L_f(T, \rho).
\end{aligned}$$

So there exists  $L_G(T, \rho) = V(\eta, [-r, 0])\rho + L_f(T, \rho)$  such that  $\left\| G\left(t, \begin{pmatrix} 0_{\mathbb{R}^n} \\ \varphi \end{pmatrix}\right) \right\| \leq L_G(T, \rho)$ , which completes the proof of (b).

With  $K_G(C)$  and  $L_G(T, \rho)$  given as above, (c) follows directly from the assumptions. ■

**(ii) A delayed periodic red-blood cell model.** Now as an example, we consider Consider a delayed red-blood cell model with periodic coefficients which is a modification of the model of Wazewska-Czyzewska and Lasota [1976] (see also Arino and Kimmel [1986]):

$$N'(t) = -\mu N(t) + p(t)e^{-\gamma(t)N(t-r)} \quad (4.20)$$

where  $N(t)$  denotes the number of red-blood cells at time  $t$ ,  $\mu \in (0, \infty)$  is the probability of death of a red-blood cell,  $p(t)$  and  $\gamma(t)$  are positive and  $T$ -periodic continuous

functions related to the production of red-blood cells per unit time and  $r$  is the time required to produce a red-blood cell.

**Proposition 4.4.1.** *Assume that*

(i)  $p \in C([0, \infty), \mathbb{R}^+)$ ,  $p(t+T) = p(t)$  for  $t \geq 0$  and  $p(t) \leq p_+$  for  $t \geq 0$ ;

(ii)  $\gamma \in C([0, \infty), \mathbb{R}^+)$ ,  $\gamma(t+T) = \gamma(t)$  for  $t \geq 0$  and  $\gamma(t) \leq \gamma_+$  for  $t \geq 0$ ;

(iii) *There exists  $\rho > 0$  such that  $(\frac{T}{1-e^{-\mu T}} + T)p_+\gamma_+e^{\gamma+\rho} < 1$  and  $(\frac{T}{1-e^{-\mu T}} + T)p_+e^{\gamma+\rho} \leq \rho$ .*

*Then equation (4.20) has a  $T$ -periodic solution.*

*Proof.* Equation (4.20) can be written as equation (4.16), where  $B = -\mu$ ,  $\hat{L} = 0$  and  $f(t, \varphi) = p(t)e^{-\gamma(t)\varphi(-r)}$ . Then it suffices to check assumptions of Theorem 4.4.2. First note that  $\omega_0(B) = -\mu < 0$ . Since  $\hat{L} = 0$ ,  $V(\eta, [-r, 0]) = 0$ . For  $\varphi_1, \varphi_2 \in C([-r, 0], \mathbb{R})$  and  $\|\varphi_1\| \leq \rho$ ,  $\|\varphi_2\| \leq \rho$ , by the mean value theorem we have

$$\begin{aligned} |f(t, \varphi_1) - f(t, \varphi_2)| &= |p(t)(e^{-\gamma(t)\varphi_1(-r)} - e^{-\gamma(t)\varphi_2(-r)})| \\ &\leq p(t)\gamma(t)e^{\gamma(t)\rho} \|\varphi_1 - \varphi_2\| \\ &\leq p_+\gamma_+e^{\gamma+\rho} \|\varphi_1 - \varphi_2\|. \end{aligned}$$

So we can pick  $K_f(\rho) = p_+\gamma_+e^{\gamma+\rho}$ . Moreover, for  $\varphi \in C([-r, 0], \mathbb{R})$ ,  $\|\varphi\| \leq \rho$  and  $0 \leq t \leq T$ ,

$$|f(t, \varphi)| = |p(t)e^{-\gamma(t)\varphi(-r)}| \leq p_+e^{\gamma+\rho}.$$

So we get  $L_f(T, \rho) = p_+e^{\gamma+\rho}$ . Then Assumption (iii) implies  $(N+T)(K_f(\rho) + V(\eta, [-r, 0])) < 1$  and  $(N+T)(L_f(T, \rho) + V(\eta, [-r, 0])) \leq \rho$  in the assumption of Theorem 4.4.2. The conclusion follows from Theorem 4.4.2. ■

Proposition 4.4.1 indicates that if the production related function  $p(t)$  is continuous,  $T$ -periodic and bounded above by a constant  $p_+$ , the production related function  $\gamma(t)$  is continuous,  $T$ -periodic and bounded above by a constant  $\gamma_+$ , and there exists  $\rho > 0$  such that the parameters satisfy the inequalities  $(\frac{T}{1-e^{-\mu T}} + T)p_+\gamma_+e^{\gamma+\rho} < 1$  and  $(\frac{T}{1-e^{-\mu T}} + T)p_+e^{\gamma+\rho} \leq \rho$ , then the model has a  $T$ -periodic solution.

Now we choose parameters for equation (4.20) such that assumptions in Proposition 4.4.1 are satisfied and perform numerical simulations to show the existence of a  $T$ -periodic solution. Let  $T = 1$ ,  $r = 1$ ,  $\mu = 10$ ,  $p(t) = 0.3 + 0.2 \sin(2\pi t)$  and  $\gamma(t) = 0.15 + 0.05 \cos(2\pi t)$ . It can be easily checked we have all the assumptions in Proposition 4.4.1, then there exists a 1-periodic solution, which can be seen from Figure 4.8.

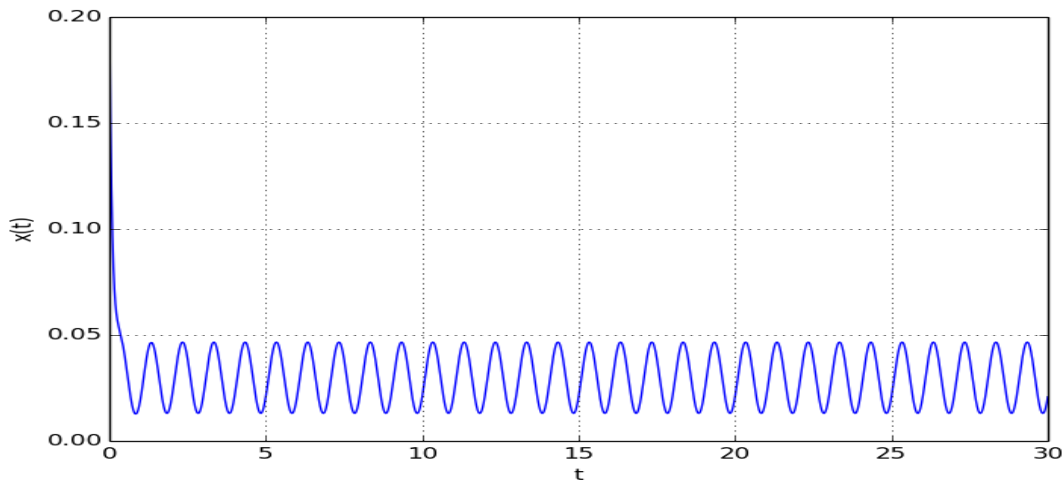


Figure 4.8: A  $T$ -periodic solution of the delayed periodic red-blood cell model (4.20) with  $r = 1$  starting at  $\varphi(\theta) = 0.2$ ,  $\theta \in [-1, 0]$ , where  $p(t) = 0.3 + 0.2 \sin(2\pi t)$ ,  $T = 1$  and  $\gamma(t) = 0.15 + 0.05 \cos(2\pi t)$ .

Now we change the parameters so that assumptions in Proposition 4.4.1 are not satisfied. Let  $T = 1$ ,  $r = 1$ ,  $\mu = 10$ ,  $p(t) = 3 + 2 \sin(2\pi t)$  and  $\gamma(t) = 10 + 5 \cos(2\pi t)$ . Figure 4.9 shows a solution in this scenario.

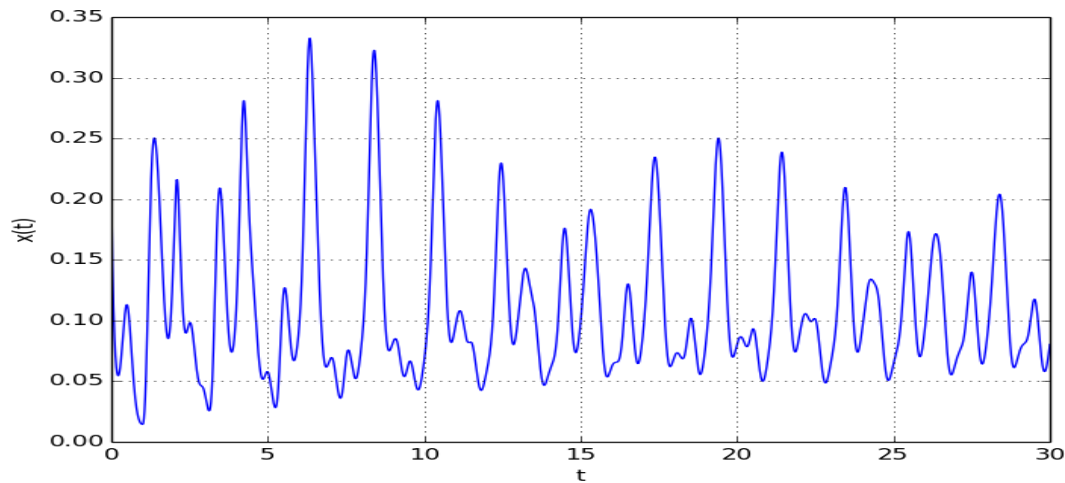


Figure 4.9: An irregular solution of the delayed periodic red-blood cell model (4.20) with  $r = 1$  starting at  $\varphi(\theta) = 0.2, \theta \in [-1, 0]$ , where  $p(t) = 3 + 2 \sin(2\pi t)$ ,  $T = 1$  and  $\gamma(t) = 10 + 5 \cos(2\pi t)$ .

## 4.5 Partial functional differential equations

Following the settings in Wu [1996] and Ducrot et al. [2013], we can also use the results in chapter 3 to study the existence of periodic solutions in abstract evolution equations with delay (Liu [1998], Ezzinbi and Liu [2002], Benkhalt and Ezzinbi [2004], Kpoumiè et al. [2018]) and partial functional differential equations with periodicity (Li [2011], Li et al. [1999]).

### (i) Periodic partial functional differential equations.

Let  $B : D(B) \subset Y \rightarrow Y$  be a linear operator on a Banach space  $(Y, \|\cdot\|_Y)$ . Assume that  $B$  is a Hille-Yosida operator; that is, there exist  $\omega_B \in \mathbb{R}$  and  $M_B > 0$  such that  $(\omega_B, +\infty) \subset \rho(B)$  and

$$\|(\lambda I - B)^{-n}\| \leq \frac{M_B}{(\lambda - \omega_B)^n}, \quad \forall \lambda > \omega_B, \quad n \geq 1.$$

Set  $Y_0 := \overline{D(B)}$ . Consider the part of  $B$  in  $Y_0$ , denoted  $B_0$ , which is defined by

$$B_0 y = B y, \quad \forall y \in D(B_0)$$

with

$$D(B_0) := \{y \in D(B) : B y \in Y_0\}.$$

Note that this construction is introduced for the existence theory.

For  $r \geq 0$ , set  $C := C([-r, 0]; Y)$  endowed with the supremum norm

$$\|\varphi\|_\infty = \sup_{\theta \in [-r, 0]} \|\varphi(\theta)\|_Y.$$

Consider the partial functional differential equations (PFDE):

$$\begin{cases} \frac{dy(t)}{dt} = B y(t) + \hat{L}(y_t) + f(t, y_t), \quad \forall t \geq 0, \\ y_0 = \varphi \in C_B. \end{cases} \quad (4.21)$$

where  $C_B := \{\varphi \in C([-r, 0]; Y) : \varphi(0) \in \overline{D(B)}\}$ ,  $y_t \in C_B$  is defined by  $y_t(\theta) = y(t + \theta)$ ,  $\theta \in [-r, 0]$ ,  $\hat{L} : C_B \rightarrow Y$  is a bounded linear operator, and  $f : \mathbb{R} \times C_B \rightarrow Y$  is a continuous map.

Now we rewrite the PFDE (4.21) as an abstract non-densely defined Cauchy problem such that our theorems can be applied. First, we regard the PFDE (4.21) as a PDE. Define  $u \in C([0, +\infty) \times [-r, 0], Y)$  by

$$u(t, \theta) = y(t + \theta), \quad \forall t \geq 0, \quad \forall \theta \in [-r, 0].$$

If  $y \in C^1([-r, +\infty), Y)$ , then

$$\frac{\partial u(t, \theta)}{\partial t} = y'(t + \theta) = \frac{\partial u(t, \theta)}{\partial \theta}.$$

Moreover, for  $\theta = 0$ , we obtain

$$\frac{\partial u(t, 0)}{\partial \theta} = y'(t) = B y(t) + \hat{L}(y_t) + f(t, y_t) = B u(t, 0) + \hat{L}(u(t, \cdot)) + f(t, u(t, \cdot)), \quad \forall t \geq 0.$$



Therefore, we deduce that  $u$  satisfies the PDE

$$\begin{cases} \frac{\partial u(t,\theta)}{\partial t} - \frac{\partial u(t,\theta)}{\partial \theta} = 0, \\ \frac{\partial u(t,0)}{\partial \theta} = Bu(t,0) + \hat{L}(u(t,\cdot)) + f(t, u(t,\cdot)), \quad \forall t \geq 0, \\ u(0,\cdot) = \varphi \in C_B. \end{cases} \quad (4.22)$$

In order to write the PDE (4.22) as an abstract non-densely defined Cauchy problem, we extend the state space to take into account the boundary conditions. Let  $X = Y \times C$  with the usual product norm

$$\left\| \begin{pmatrix} y \\ \varphi \end{pmatrix} \right\| = \|y\|_Y + \|\varphi\|_\infty.$$

Define the linear operator  $A : D(A) \subset X \rightarrow X$  by

$$A \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} = \begin{pmatrix} -\varphi'(0) + B\varphi(0) \\ \varphi' \end{pmatrix}, \quad \forall \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in D(A) \quad (4.23)$$

with

$$D(A) = \{0_Y\} \times \{\varphi \in C^1([-r,0], Y), \varphi(0) \in D(B)\}.$$

Note that  $A$  is non-densely defined because

$$X_0 := \overline{D(A)} = 0_Y \times C_B \neq X.$$

Now define  $L : X_0 \rightarrow X$  by

$$L \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} := \begin{pmatrix} \hat{L}(\varphi) \\ 0_C \end{pmatrix}$$

and  $F : \mathbb{R} \times X_0 \rightarrow X$  by

$$F(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}) := \begin{pmatrix} f(t, \varphi) \\ 0_C \end{pmatrix}.$$

Let

$$v(t) := \begin{pmatrix} 0_Y \\ u(t) \end{pmatrix}.$$

Then we can rewrite the PDE (4.22) as the following non-densely defined Cauchy problem

$$\frac{dv(t)}{dt} = Av(t) + L(v(t)) + F(t, v(t)), \quad t \geq 0; \quad v(0) = \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \in X_0. \quad (4.24)$$

To state an existence theorem of periodic solutions for equation (4.21), we make the following assumptions.

**Assumption 4.5.1.** (C1)  $f : \mathbb{R} \times C_B \rightarrow Y$  is Lipschitz on bounded sets; i.e., for each

$C > 0$  there exists  $K_f(C) \geq 0$  such that  $\|f(t, u) - f(t, v)\| \leq K_f(C) \|u - v\|$  for  $t \in [0, \infty)$  and  $\|u\| \leq C$  and  $\|v\| \leq C$ ;

(C2)  $f : \mathbb{R} \times C_B \rightarrow Y$  is bounded on bounded sets; i.e., there exists  $L_f(T, \rho) \geq 0$  such that  $\|f(t, u)\| \leq L_f(T, \rho)$  for  $t \leq T$  and  $\|u\| \leq \rho$ .

With these assumptions, we have the following result for equation (4.21).

**Theorem 4.5.2.** *Let Assumption 4.5.1 hold with  $\omega_B < 0$  and  $f$  being  $T$ -periodic in  $t$ . Suppose that there exists  $\rho > 0$  such that  $(N + T)(K_f(\rho) + \|\hat{L}\|) < 1$  and  $(N + T)(L_f(T, \rho) + \|\hat{L}\| \rho) \leq \rho$ , where  $N = \frac{T}{1 - e^{\omega_B T}}$ , then equation (4.21) has a  $T$ -periodic solution.*

*Proof.* Since (4.21) can be written as (4.24), denote  $G(t, v(t)) = L(v(t)) + F(t, v(t))$ , it suffices to prove that

- (a)  $A$  satisfies Assumption 3.1.2 (H1) with  $\omega < 0$ ;
- (b)  $G : [0, \infty) \times \{0_Y\} \times C_B \rightarrow Y \times C$  satisfies Assumption 3.1.2 (H1) (H2);

(c) There exists  $\rho > 0$  such that  $(N + T)K_G(\rho) < 1$  and  $(N + T)L_G(T, \rho) \leq \rho$ , where  $N = \frac{T}{1 - e^{\omega T}}$ .

It follows from Theorem 3.1.3 that equation (4.24) has a  $T$ -periodic mild solution, which implies that equation (4.21) has a  $T$ -periodic mild solution with initial value  $u(0, \cdot) = \varphi \in C_B$ . Meanwhile, by Theorem 2.1 in Hale and Verduyn Lunel [1993], equation (4.21) has a unique solution  $y_0(t) \in C^1([-r, \infty), Y)$  with initial condition  $y_0(\theta) = \varphi(\theta)$  for  $\theta \in [-r, 0]$ . Therefore,  $y_0(t)$  is a  $T$ -periodic solution for (4.21).

From Lemma 3.6 in Ducrot et al. [2013], we know that  $A$  as defined in (4.23) is a Hille-Yoshida operator with  $\omega = \omega_B < 0$  and  $M = 1$ , which proves (a).

For  $\varphi_1, \varphi_2 \in C_B$  such that  $\|\varphi_1\| \leq C$  and  $\|\varphi_2\| \leq C$ , we have

$$\begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}, \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix} \in 0_Y \times C_B = \overline{D(A)}$$

and

$$\left\| \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix} \right\| = \|\varphi_1\| \leq C, \left\| \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix} \right\| = \|\varphi_2\| \leq C.$$

Then

$$\begin{aligned}
& \left\| G\left(t, \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}\right) - G\left(t, \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix}\right) \right\| \\
&= \left\| L\left(\begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}\right) - L\left(\begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix}\right) + F\left(t, \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}\right) - F\left(t, \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix}\right) \right\| \\
&\leq \left\| L\left(\begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}\right) - L\left(\begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix}\right) \right\| + \left\| F\left(t, \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}\right) - F\left(t, \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix}\right) \right\| \\
&= \left\| \begin{pmatrix} \hat{L}(\varphi_1 - \varphi_2) \\ 0_C \end{pmatrix} \right\| + \left\| \begin{pmatrix} f(t, \varphi_1) - f(t, \varphi_2) \\ 0_C \end{pmatrix} \right\| \\
&= \|\hat{L}(\varphi_1 - \varphi_2)\|_Y + \|f(t, \varphi_1) - f(t, \varphi_2)\|_Y \\
&\leq K_f(C) \|\varphi_1 - \varphi_2\| + \|\hat{L}\| \|\varphi_1 - \varphi_2\| \\
&= (K_f(C) + \|\hat{L}\|) \|\varphi_1 - \varphi_2\| \\
&= (K_f(C) + \|\hat{L}\|) \left\| \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix} \right\|.
\end{aligned}$$

So there exists  $K_G(C) = K_f(C) + \|\hat{L}\|$  such that

$$\left\| G\left(t, \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix}\right) - G\left(t, \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix}\right) \right\| \leq K_G(C) \left\| \begin{pmatrix} 0_Y \\ \varphi_1 \end{pmatrix} - \begin{pmatrix} 0_Y \\ \varphi_2 \end{pmatrix} \right\|.$$

Furthermore, for  $t \leq T$  and  $\left\| \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \right\| \leq \rho$ , we have

$$\begin{aligned}
\left\| G\left(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}\right) \right\| &= \left\| L \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} + F\left(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}\right) \right\| \\
&\leq \left\| L \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix} \right\| + \left\| F\left(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}\right) \right\| \\
&= \left\| \begin{pmatrix} \hat{L}(\varphi) \\ 0_C \end{pmatrix} \right\| + \left\| \begin{pmatrix} f(t, \varphi) \\ 0_C \end{pmatrix} \right\| \\
&= \|\hat{L}(\varphi)\|_Y + \|f(t, \varphi)\|_Y \\
&\leq \|\hat{L}\| \rho + L_f(T, \rho).
\end{aligned}$$

So there exists  $L_G(T, \rho) = \|\hat{L}\| \rho + L_f(T, \rho)$  such that  $\left\| G\left(t, \begin{pmatrix} 0_Y \\ \varphi \end{pmatrix}\right) \right\| \leq L_G(T, \rho)$ ,

which completes the proof of (b).

With  $K_G(C)$  and  $L_G(T, \rho)$  given as above, (c) follows directly from the assumptions. ■

### (ii) A reaction-diffusion equation with time delay

Let us consider the following periodic reaction-diffusion equation with time delay:

$$\begin{cases} \partial_t u(x, t) = \partial_x^2 u(x, t) - au(x, t) - b(t)u(x, t - r), & 0 \leq x \leq 1, t \geq 0 \\ u(0, t) = u(1, t) = 0.5, & t \geq 0 \\ u(x, t) = \phi(t)(x), & 0 \leq x \leq 1, -r \leq t \leq 0, \end{cases} \quad (4.25)$$

where  $a \geq 0$ ,  $b \in C([0, \infty), \mathbb{R}_+)$  is  $T$ -periodic. We will study the existence of  $T$ -periodic solution of problem (4.25).

Let  $v(x, t) = u(x, t) - 0.5$ , then we have the following equation:

$$\begin{cases} \partial_t v(x, t) = \partial_x^2 v(x, t) - av(x, t) - b(t)v(x, t - r) - 0.5a - 0.5b(t), & 0 \leq x \leq 1, t \geq 0 \\ v(0, t) = v(1, t) = 0, & t \geq 0 \\ v(x, t) = \phi(t)(x) - 0.5, & 0 \leq x \leq 1, -r \leq t \leq 0. \end{cases} \quad (4.26)$$

We know that the existence of  $T$ -periodic solutions of equation (4.26) is equivalent to the existence of  $T$ -periodic solutions of equation (4.25).

Let  $X = C(0, 1)$  and  $B : X \rightarrow X$  be defined by

$$B\phi = \phi'' - a\phi$$

with

$$D(B) = \{\phi \in C^2([0, 1], \mathbb{R}), \phi(0) = \phi(1) = 0\}.$$

Let  $C_B := \{\phi \in C([-r, 0], X) : \phi(0) \in \overline{D(B)}\}$  and define  $f : [0, \infty) \times C_B \rightarrow X$  by

$$f(t, \phi) = -b(t)\phi(-r) - 0.5a - 0.5b(t).$$

Then equation (4.26) can be written as

$$\begin{cases} \frac{dy(t)}{dt} = By(t) + f(t, y_t), & \forall t \geq 0 \\ y_0 = \varphi \in C_B \end{cases} \quad (4.27)$$

**Proposition 4.5.1.** *Assume that*

(i)  $a > 0$ ,  $0 \leq b(t) \leq b_+$  and  $b(t + T) = b(t)$  for  $t \geq 0$ ;

(ii)  $(\frac{T}{1 - e^{-aT}} + T)b_+ < 1$ ;

(iii) *There exists  $\rho > 0$  such that  $(\frac{T}{1 - e^{-aT}} + T)(0.5a + 0.5b_+ + b_+\rho) \leq \rho$ .*

*Then equation (4.26) thus (4.25) has a  $T$ -periodic solution.*

*Proof.* Since equation (4.26) can be written as (4.27), it suffices to check assumptions of Theorem 4.5.2.

Let  $\psi \in X$  and let  $\lambda > -a$ . Then

$$(\lambda I - B)\varphi = \psi \Leftrightarrow (\lambda + a)\varphi - \varphi'' = \psi.$$

Following exactly the same way as in the proof of part (a) of Proposition 4.3.1, we obtain that

$$\|(\lambda I - B)^{-1}\| \leq \frac{1}{\lambda + a}, \quad \forall \lambda > -a,$$

which implies that  $\omega_B = -a < 0$ . For  $\varphi_1, \varphi_2 \in C_B$  and  $\|\varphi_1\| \leq C, \|\varphi_2\| \leq C$ , we have

$$\begin{aligned} \|f(t, \varphi_1) - f(t, \varphi_2)\| &= \|-b(t)\varphi_1(-r) - 0.5a - 0.5b(t) + b(t)\varphi_2(-r) + 0.5a + 0.5b(t)\| \\ &= \|b(t)(\varphi_2(-r) - \varphi_1(-r))\| \\ &\leq \|b(t)\| \|\varphi_2(-r) - \varphi_1(-r)\| \\ &\leq b_+ \|\varphi_1 - \varphi_2\|. \end{aligned}$$

So  $K_f(\rho) = b_+$  for  $\forall \rho > 0$ . Moreover, for  $\varphi \in C_B$  with  $\|\varphi\| \leq \rho$  and  $0 \leq t \leq T$ ,

$$\begin{aligned} \|f(t, \varphi)\| &= \|-b(t)\varphi(-r) - 0.5a - 0.5b(t)\| \\ &\leq b_+ \|\varphi\| + 0.5a + 0.5b_+ \\ &\leq b_+\rho + 0.5a + 0.5b_+. \end{aligned}$$

So we have  $L_f(T, \rho) = b_+\rho + 0.5a + 0.5b_+$ . Therefore, assumptions (ii) and (iii) imply  $(N + T)(K_f(\rho) + \|\hat{L}\|) < 1$  and  $(N + T)(L_f(T, \rho) + \|\hat{L}\| \rho) \leq \rho$  in Theorem 4.5.2, respectively. The conclusion follows from Theorem 4.5.2.  $\blacksquare$

Now we choose parameters for equation (4.26) such that assumptions in Proposition 4.5.1 are satisfied. We will perform numerical simulation to demonstrate the existence of  $T$ -periodic solutions.

Let  $T = 1$ ,  $a = 1$  and  $b(t) = 0.15 + 0.15 \sin(2\pi t)$ . We can verify that Proposition 4.5.1 holds, so there exists a  $T$ -periodic solution, which can be seen from Figure 4.10.

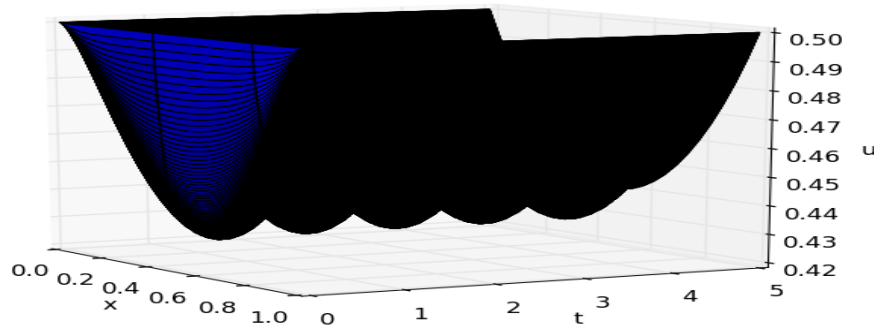


Figure 4.10: A  $T$ -periodic solution of the delayed reaction-diffusion equation (4.26) with initial condition  $\varphi(x, t) = 0.5$  for  $t \in [-1, 0]$ ,  $x \in [0, 1]$ , where  $b(t) = 0.15 + 0.15 \sin(2\pi t)$ ,  $T = 1$  and  $a = 1$ .

Now we change the parameters so that the conditions in Proposition 4.5.1 do hold. Let  $T = 1$ ,  $a = 1$  and  $b(t) = 1.5 + 10 \sin(2\pi t)$ . Figure 4.11 gives a solution in this scenario.

**(iii) The diffusive Nicholson's blowflies equation.** We consider the diffusive Nicholson's blowflies equation (So and Yang [1998], Yang and So [1998], So et al. [2000])

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - \tau u(t,x) + \beta(t)\tau u(t-1,x)e^{u(t-1,x)}, & t \geq 0, x \in [0, 1] \\ u(t,0) = u(t,1) = 0.1, & t \geq 0, \end{cases} \quad (4.28)$$

where  $\beta(t)$  is  $T$ -periodic. To study existence of  $T$ -periodic solution of equation (4.28),



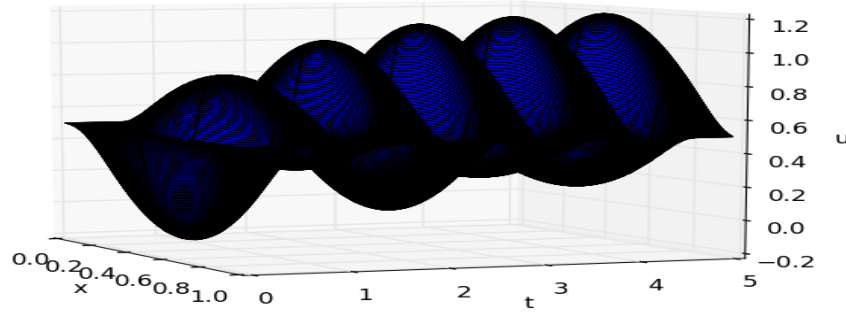


Figure 4.11: A solution of the delayed reaction-diffusion equation (4.26) with initial condition  $\varphi(x, t) = 0.5$  for  $t \in [-1, 0]$ ,  $x \in [0, 1]$ , where  $b(t) = 1.5 + 10 \sin(2\pi t)$ ,  $T = 1$  and  $a = 1$ .

let  $v(t, x) = u(t, x) - 0.1$ . Then we have

$$\begin{cases} \frac{\partial v(t, x)}{\partial t} = \frac{\partial^2 v(t, x)}{\partial x^2} - \tau v(t, x) + \beta(t)\tau v(t-1, x)e^{-[v(t-1, x)+0.1]} \\ \quad + 0.1\beta(t)\tau e^{-[v(t-1, x)+0.1]} - 0.1\tau \\ v(t, 0) = v(t, 1) = 0, \end{cases} \quad (4.29)$$

We know that existence of  $T$ -periodic solutions of equation (4.29) is equivalent to the existence of  $T$ -periodic solutions of equation (4.28).

Let  $X = C[0, 1]$  and let  $B : X \rightarrow X$  be defined by

$$B\phi = \phi'' - \tau\phi$$

with  $D(B) = \{\phi \in C^2([0, 1], \mathbb{R}), \phi(0) = \phi(1) = 0\}$ . Let  $C_B := \{\phi \in C([-1, 0], X) : \phi(0) \in \overline{D(B)}\}$  and define  $f : [0, \infty) \times C_B \rightarrow X$  by

$$f(t, \phi) = \beta(t)\tau\phi(-1)e^{-[\phi(-1)+0.1]} + 0.1\beta(t)\tau e^{-[\phi(-1)+1]} - 0.1\tau.$$

Then equation (4.29) can be written as

$$\frac{dy(t)}{dt} = By(t) + f(t, y_t), \quad \forall t \geq 0 \quad (4.30)$$

**Proposition 4.5.2.** *Assume that*

(i)  $\tau > 0$ ,  $0 \leq \beta(t) \leq \beta_+$  and  $\beta(t) = \beta(t + T)$  for  $\forall t \geq 0$ ;

(ii) *There exists  $\rho > 0$  such that  $(\frac{T}{1-e^{-\tau T}} + T)\beta_+\tau e^{-0.1}(\rho + 1.1)e^\rho < 1$  and  $(\frac{T}{1-e^{-\tau T}} + T)\tau(0.1 + \beta_+\rho e^{\rho-0.1} + 0.1\beta_+e^{\rho-0.1}) \leq \rho$ .*

*Then equation (4.29) thus (4.28) has a  $T$ -periodic solution.*

*Proof.* Since equation (4.29) can be written as (4.30), it suffices to check assumptions of Theorem 4.5.2. Let  $\psi \in X$  and let  $\lambda > -\tau$ . Then

$$(\lambda I - B)\varphi = \psi \Leftrightarrow (\lambda + \tau)\varphi - \varphi'' = \psi.$$

By following exactly the same way as in the proof of part (a) of Proposition 4.3.1, we obtain that

$$\|(\lambda I - B)^{-1}\| \leq \frac{1}{\lambda + \tau}, \quad \forall \lambda > -\tau,$$

which implies that  $\omega_B = -\tau < 0$ . For  $\varphi_1, \varphi_2 \in C_B$  and  $\|\varphi_1\| \leq \rho$ ,  $\|\varphi_2\| \leq \rho$ , we have

$$\begin{aligned} f(t, \varphi_1) - f(t, \varphi_2) &= \beta(t)\tau\varphi_1(-1)e^{-[\varphi_1(-1)+0.1]} + 0.1\beta(t)\tau e^{-[\varphi_1(-1)+0.1]} - 0.1\tau \\ &\quad - \beta(t)\tau\varphi_2(-1)e^{-[\varphi_2(-1)+0.1]} - 0.1\beta(t)\tau e^{-[\varphi_2(-1)+0.1]} + 0.1\tau \end{aligned}$$

and

$$\begin{aligned}
\|f(t, \varphi_1) - f(t, \varphi_2)\| &\leq \|\beta(t)\tau\varphi_1(-1)e^{-[\varphi_1(-1)+0.1]} - \beta(t)\tau\varphi_2(-1)e^{-[\varphi_2(-1)+0.1]}\| \\
&\quad + \|\|0.1\beta(t)\tau e^{-[\varphi_1(-1)+0.1]} - 0.1\beta(t)\tau e^{-[\varphi_2(-1)+0.1]}\| \\
&\leq \|\beta(t)\tau e^{-0.1}(\varphi_1(-1)e^{-\varphi_1(-1)} - \varphi_1(-1)e^{-\varphi_2(-1)})\| \\
&\quad + \|\beta(t)\tau e^{-0.1}(\varphi_1(-1)e^{-\varphi_2(-1)} - \varphi_2(-1)e^{-\varphi_2(-1)})\| \\
&\quad + \|\|0.1\beta(t)\tau e^{-0.1}(e^{-\varphi_1(-1)} - e^{-\varphi_2(-1)})\| \\
&\leq \beta_+\tau e^{-0.1}(\rho + 1)e^\rho \|\varphi_1 - \varphi_2\| + 0.1\beta_+\tau e^{-0.1}e^\rho \|\varphi_1 - \varphi_2\| \\
&= \beta_+\tau e^{-0.1}(\rho + 1.1)e^\rho \|\varphi_1 - \varphi_2\|.
\end{aligned}$$

So we have  $K_f(\rho) = \beta_+\tau e^{-0.1}(\rho + 1.1)e^\rho$  for  $\rho > 0$ . Moreover, for  $\varphi \in C_B$  with  $\|\varphi\| \leq \rho$  and  $0 \leq t \leq T$ ,

$$\begin{aligned}
\|f(t, \varphi)\| &= \|\beta(t)\tau\varphi(-1)e^{-[\varphi(-1)+0.1]} + 0.1\beta(t)\tau e^{-[\varphi(-1)+0.1]} - 0.1\tau\| \\
&\leq \beta_+\tau e^{-0.1} \|\varphi(-1)e^{-\varphi(-1)}\| + 0.1\beta_+\tau e^{-0.1} \|e^{-\varphi(-1)}\| + 0.1\tau \\
&\leq \tau(0.1 + \beta_+\rho e^{\rho-0.1} + 0.1\beta_+e^{\rho-0.1}).
\end{aligned}$$

Hence, we have  $L_f(T, \rho) = \tau(0.1 + \beta_+\rho e^{\rho-0.1} + 0.1\beta_+e^{\rho-0.1})$ . Therefore, assumption (ii) implies  $(N + T)(K_f(\rho) + \|\hat{L}\|) < 1$  and  $(N + T)(L_f(T, \rho) + \|\hat{L}\| \rho) \leq \rho$  in Theorem 4.5.2. The conclusion follows from Theorem 4.5.2.  $\blacksquare$

Proposition 4.5.2 indicates that if the production related function  $\beta(t)$  is  $T$ -periodic and bounded above by a constant  $\beta_+$ , and there exists  $\rho > 0$  such that  $\beta_+ < \frac{1}{\tau e^{-0.1}(\rho+1.1)e^\rho(\frac{T}{1-e^{-\tau T}}+T)}$  and  $(\frac{T}{1-e^{-\tau T}} + T)\tau(0.1 + \beta_+\rho e^{\rho-0.1} + 0.1\beta_+e^{\rho-0.1}) \leq \rho$ , then the model has a  $T$ -periodic solution.

Now we choose parameters for equation (4.28) such that assumptions in Proposition 4.5.2 are satisfied. Let  $T = 1$ ,  $\tau = 1$  and  $\beta(t) = 0.025 + 0.015 \cos 2\pi t$  in equation

(4.28), then it's easy to check that assumptions of Proposition 4.5.2 are satisfied. So there exists a  $T$ -periodic solution, which can be seen from Figure 4.12.

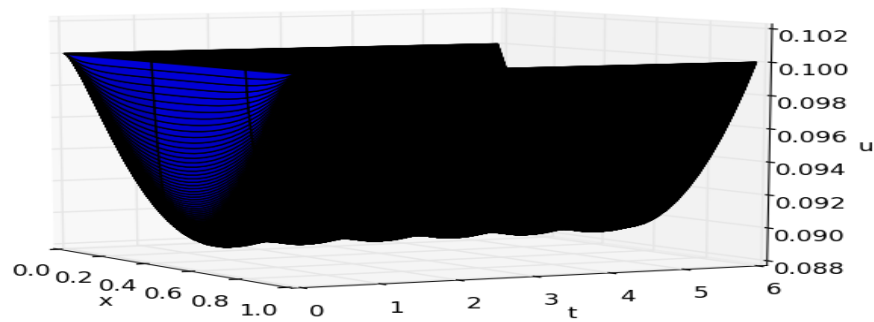


Figure 4.12: A  $T$ -periodic solution of the diffusive Nicholson's blowflies equation (4.28) with initial condition  $\varphi(x, t) = 0.1$  for  $t \in [-1, 0]$ ,  $x \in [0, 1]$ , where  $\beta(t) = 0.025 + 0.015 \cos(2\pi t)$ ,  $T = 1$  and  $\tau = 1$ .

## Chapter 5

### Conclusions and Future Study

In conclusion, I gave a few theorems on the existence of periodic solutions of abstract semilinear equations and abstract semilinear evolution equations and applied them to age-structured models with periodic harvesting, diffusive logistic equations with periodic coefficients, periodic functional differential equations including delayed red-blood cell models, and periodic partial functional differential equations including diffusive Nicholson's blowflies equation.

Further studies could go towards two directions:

One is further theoretical study. So far only the existence of periodic solutions was obtained for these equations, the next step is to study stability of the existing periodic solutions.

The other one is application. Besides the biological and medical models I mentioned in the application chapter, the theorems can also be applied to other models such as periodic wave equations and periodic epidemic models (with delay, age-structure or diffusion).

## Bibliography

- Anița, S., Iannelli, M., Kim, M.-Y., and Park, E.-J. (1998). Optimal harvesting for periodic age-dependent population dynamics. *SIAM J. Appl. Math.*, 58(5):1648–1666.
- Arino, O. and Kimmel, M. (1986). Stability analysis of models of cell production systems. *Math. Modelling*, 7:1269–1300.
- Benkhalt, R. and Ezzinbi, K. (2004). Periodic solutions for some partial functional differential equations. *J. Appl. Math. Stochastic Anal.*, 1:9–18.
- Burton, T. A. (1983). *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*. Academic Press, New York.
- Chow, S.-N. (1973). Remarks on one-dimensional delay differential equations. *J. Math. Anal. Appl.*, 41:426–429.
- Chow, S.-N. and Hale, J. K. (1974). Strongly limit-compact maps. *Funkcialaj Ekvacioj*, 17:31–38.
- Da Prato, G. and Sinestrari, E. (1987). Differential operators with non-dense domain. *Ann. Sc. Norm. Pisa Cl. Sci.*, 14:285–344.
- Ducrot, A., Magal, P., and Ruan, S. (2013). Projectors on the generalized eigenspaces for partial differential equations with time delay, in “*Infinite Dimensional Dynamical Systems*” (Eds. J. Mallet-Paret, J. Wu, Y. Yi, and H. Zhu). *Fields Institute Commun.*, 64:353–390.
- Ezzinbi, K. and Liu, J. (2002). Periodic solutions of non-densely defined delay evolution equations. *J. Appl. Math. Stochastic Anal.*, 15(2):105–114.
- Farkas, M. (1994). *Periodic Motions*, volume 104. Springer, New York.
- Hale, J. K. and Verduyn Lunel, S. M. (1993). *Introduction to Functional Differential Equations*, volume 99. Springer-Verlag, New York.

- Hess, P. (1991). *Periodic-Parabolic Boundary Value Problems and Positivity*. Longman Scientific and Technical, Harlow.
- Iannelli, M. (1995). *Mathematical Theory of Age-Structured Population Dynamics*. Giardini Editori e Stampatori, Pisa.
- Kato, J., Naito, T., and Shin, J. S. (2002). Bounded solutions and periodic solutions to linear differential equations in banach spaces. *Vietnam J. Math.*, 30:561–575.
- Kato, T. (1970). Linear evolution equations of hyperbolic type. *J. Fac. Sci. Univ. Tokyo*, 17:241–258.
- Kpoumiè, M. E., Ezzinbi, K., and Békollè, D. (2018). Periodic solutions for some non-densely nonautonomous partial functional differential equations in fading memory spaces. *Differential Equations Dynam. Systems*, 26(1-3):177–197.
- Li, Y. (2011). Existence and asymptotic stability of periodic solution for evolution equations with delays. *J. Funct. Anal.*, 261:1309–1324.
- Li, Y., Cong, F., Lin, Z., and Liu, W. (1999). Periodic solutions for evolution equations. *Nonlinear Anal.*, 36:275–293.
- Liu, J. H. (1998). Bounded and periodic solutions of finite delay evolution equations. *Nonlinear Anal.*, 34:101–111.
- Liu, Z., Magal, P., and Ruan, S. (2008). Projectors on the generalized eigenspaces for functional differential equations using integrated semigroups. *J. Differential Equations*, 224:1784–1809.
- Magal, P. and Ruan, S. (2007). On integrated semigroups and age structured models in  $l^p$  spaces. *Differential Integral Equations*, 20(2):197–239.
- Magal, P. and Ruan, S. (2009). On semilinear cauchy problems with non-dense domain. *Adv. Differential Equations*, 14:1041–1084.
- Magal, P. and Ruan, S. (2018). *Theory and Applications of Abstract Semilinear Cauchy Problems*. Springer, New York.
- Massera, J. L. (1950). The existence of periodic solutions of systems of differential equations. *Duke Math. J.*, 17:457–475.
- Massera, J. L. and Schäffer, J. J. (1959). Linear differential equations and functional analysis. ii. equations with periodic coefficients. *Ann. of Math.*, 69:88–104.
- Naito, T., Nguyen, V. M., Miyazaki, R., and Shin, J. S. (2000). A decomposition theorem for bounded solutions and the existence of periodic solutions of periodic differential equations. *J. Differential Equations*, 160:263–282.

- Nguyen, T. H. and Ngo, Q. D. (2016a). Existence, uniqueness and conditional stability of periodic solutions to evolution equations. *J. Math. Anal. Appl.*, 433:1190–1203.
- Nguyen, T. H. and Ngo, Q. D. (2016b). Periodic solutions to evolution equations: existence, conditional stability and admissibility of function spaces. *Ann. Polon. Math.*, 116:173–195.
- Oka, H. and Tanaka, N. (2005). Evolution operators generated by nondensely defined operators. *Math. Nachr.*, 278(11):1285–1296.
- Pazy, A. (1983). *Semigroups of Linear Operator and Applications to Partial Differential Equations*, volume 44. Springer, New York.
- Prüss, J. (1979). Periodic solutions of semilinear evolution equations. *Nonlinear Anal.*, 3:601–612.
- So, J. W.-H., Wu, J., and Yang, Y. (2000). Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson's blowflies equation. *Appl. Math. Comput.*, 111:33–51.
- So, J. W.-H. and Yang, Y. (1998). Dirichlet problem for the diffusive Nicholson's blowflies equation. *J. Differential Equations*, 150:317–348.
- Strauss, W. A. (1992). *Partial Differential Equations*. Wiley.
- Tanaka, N. (1995). Semilinear equations in hyperbolic case. *Nonlinear Anal.*, 24(5):773–788.
- Tanaka, N. (1996). Quasilinear evolution equations with non-densely defined operators. *Differential Integral Equations*, 5:1067–1106.
- Vrabie, I. I. (1990). Periodic solutions for nonlinear evolution equations in a Banach space. *Proc. Amer. Math. Soc.*, 109:653–661.
- Ward Jr., J. R. (1979). Boundary value problems for differential equations in Banach space. *J. Math. Anal. Appl.*, 70:589–598.
- Wazewska-Czyzewska, M. and Lasota, A. (1976). Mathematical problems of the dynamics of the red-blood cells system. *Ann. Polish Math. Soc. III Appl. Math.*, 6:23–40.
- Webb, G. F. (1985). *Theory of Nonlinear Age-Dependent Population Dynamics*. Marcel Dekker, New York.
- Wu, J. (1996). *Theory and Applications of Partial Functional Differential Equations*, volume 119. Springer-Verlag, New York.
- Yang, Y. and So, J. W.-H. (1998). Dynamics for the diffusive Nicholson's blowflies equation. "Dynamical Systems and Differential Equations, II", pages 333–352.