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Brittney Ellzey
University of Miami, brittney.ellzey@gmail.com

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Brittney Ellzey

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Brittney Ellzey

Approved:

Michelle L. Wachs, Ph.D.
Professor of Mathematics

Richard P. Stanley, Ph.D.
Professor of Mathematics

Bruno Benedetti, Ph.D.
Assistant Professor of
Mathematics

Guillermo J. Prado, Ph.D.
Dean of the Graduate School

Rafael Nepomechie, Ph.D.
Professor of Physics

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In 1912, Birkhoff introduced the chromatic polynomial of a graph, which counts the number of proper colorings of a graph. In 1995, Stanley introduced the chromatic symmetric function of a graph, a symmetric function analog of the chromatic polynomial of a graph. The Stanley-Stembridge e-positivity conjecture is a long-standing conjecture that states that the chromatic symmetric function of a certain class of graphs has nonnegative coefficients when expanded in the elementary symmetric function basis. In 2012, Shareshian and Wachs introduced the chromatic quasisymmetric function of a labeled graph, a refinement of the chromatic symmetric function. Shareshian and Wachs described their own e-positivity conjecture for chromatic quasisymmetric functions which generalizes the Stanley-Stembridge conjecture. There is ample support for these e-positivity conjectures, including weaker positivity results in other symmetric function bases.

In the first part of this thesis, we extend the work of Shareshian and Wachs from labeled graphs to a wider class of graphs, namely directed graphs. We introduce the notion of chromatic quasisymmetric function of a directed graph. For acyclic digraphs, our definition is equivalent to that of Shareshian and Wachs. We give an expansion in terms of Gessel's fundamental quasisymmetric function basis for the chromatic quasisymmetric function of all digraphs, which shows that all the coefficients are nonnegative. We use this expansion to derive a power sum symmetric function basis expansion with positive coefficients for the chromatic quasisymmetric function of all digraphs whose chromatic quasisymmetric function has symmetric function coefficients. We describe a class of digraphs, which we call circular indifference digraphs,

and show that their chromatic quasisymmetric functions are symmetric. These circular indifference digraphs include the directed cycle, for which we provide an e-basis generating function formula that shows that its chromatic quasisymmetric function is e-positive. We generalize the e-positivity conjecture of Shareshian and Wachs to the class of circular indifference digraphs. Our positivity results and computer calculations provide evidence for this conjecture.

A Smirnov word is a word over the positive integers such that consecutive letters are distinct. The descent enumerator of Smirnov words is equivalent to the chromatic quasisymmetric function of the path graph. Shareshian and Wachs found a nice e-basis generating function expansion of this descent enumerator that shows that it is e-positive. Specializing this result gave them a q-analog of Euler's exponential generating function of the classical Eulerian polynomials.

In the second part of this thesis, we consider descent enumerators for restricted Smirnov words, where we put restrictions on the relationship between the first and last letter. We also consider cyclic descent enumerators for Smirnov words. Our work on these descent enumerators refines our work on the chromatic quasisymmetric function of the directed cycle. We obtain nice e-basis generating function formulas that show that some of these descent enumerators are e-positive. We also provide expansions for the various descent enumerators in Gessel's fundamental quasisymmetric function basis. By specializing our fundamental and e-basis expansions, we obtain formulas for polynomials that are variations on the q-Eulerian polynomials studied by Shareshian and Wachs. We give a factorization of the expansion coefficients of the various descent enumerators in the power sum symmetric function basis involving the Eulerian polynomials. In addition, this work with Smirnov word descent enumerators enables us to derive an e-basis expansion formula for the chromatic quasisymmetric function of the labeled cycle, which shows that it is e-positive. This is notable, because the labeled cycle is not a graph that is covered by any of the current e-positivity conjectures.

To my Honey Paw

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This work would not have been possible without the help of many people, and I'd like to thank a few of them here.

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My family has loved and encouraged me throughout my studies, and I am so incredibly grateful for them. In addition, the professors, graduate students, and staff at the University of Miami have become like a family to me over the past few years, and I appreciate them for pushing me to be a better mathematician and for supporting me through the tough times.

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Contents

Chapter 1

Introduction

1.1 Graph colorings

A *proper coloring* of a graph $G = (V, E)$ is a map $\kappa : V \to \mathbb{P}$, where $\mathbb{P}$ denotes the positive integers, such that for all $\{u, v\} \in E$, we have that $\kappa(u) \neq \kappa(v)$. In 1912 Birkhoff [8] introduced the *chromatic polynomial* of a graph, $\chi_G(k)$, which equals the number of proper colorings of G using only the colors in $[k] \coloneqq \{1, 2, \ldots, k\}$. For example, let $P_n = ([n], E)$ be the path graph defined by $E = \{\{i, i+1\} \mid i \in [n-1]\}$. We see that

$$\chi_{P_n}(k) = k(k-1)^{n-1} \tag{1.1}$$

since there are k choices for the color on vertex $\mathbf{1}$ and $k-1$ choices for the other $n-1$ vertices. One can show that the chromatic polynomial is always a polynomial in k, justifying its name (see Section 2.3 for more information).

In 1995 Stanley [58] introduced a symmetric function analog of the chromatic polynomial. The *chromatic symmetric function* of a graph $G = (V, E)$ is defined by

$$X_G(\mathbf{x}) \coloneqq \sum_{\kappa \in C(G)} \mathbf{x}_\kappa, \tag{1.2}$$

where $\mathbf{x} = x_1, x_2, \ldots$, $C(G)$ is the set of proper colorings of G and $\mathbf{x}_\kappa = \prod_{v \in V} x_{\kappa(v)}$. One can see that this is a symmetric function, because permuting the variables is the same as permuting the colors in each proper coloring (see Section 3.1 for basic information on symmetric functions). For a symmetric function $f(\mathbf{x})$ and a positive integer k, we let $f(1^k)$ denote the result when we set $x_i = 1$ for $1 \leq i \leq k$ and $x_i = 0$ for $i > k$ in $f(\mathbf{x})$. One can see that $X_G(1^k) = \chi_G(k)$, so the chromatic symmetric function reduces to the chromatic polynomial.

Stanley showed a number of interesting results about $X_G(\mathbf{x})$. He discusses expansions in various bases for the ring of symmetric functions, including the power sum symmetric function basis (p-basis) and the elementary symmetric function basis (e-basis). See Section 3.1 for the definitions of these bases. He shows that $\omega X_G(\mathbf{x})$ is p-positive, i.e., $\omega X_G(\mathbf{x})$ has nonnegative coefficients when expanded in the p-basis, where ω is the standard involution on symmetric functions defined in Section 3.1.

A stronger property than p-positivity of $\omega X_G(\mathbf{x})$ is e-positivity of $X_G(\mathbf{x})$, but not all graphs have e-positive chromatic symmetric functions. Stanley shows that the chromatic symmetric functions of the path graph P_n and of the cycle graph $C_n = ([n], E)$, defined by $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{\{1, n\}\}$, are e-positive by obtaining the formulas

$$\sum_{n \geq 0} X_{P_n}(\mathbf{x}) z^n = \frac{\sum_{i \geq 0} e_i(\mathbf{x}) z^i}{1 - \sum_{i \geq 2} (i-1) e_i(\mathbf{x}) z^i} \tag{1.3}$$

and

$$\sum_{n \geq 2} X_{C_n}(\mathbf{x}) z^n = \frac{\sum_{i \geq 2} i(i-1) e_i(\mathbf{x}) z^i}{1 - \sum_{i \geq 2} (i-1) e_i(\mathbf{x}) z^i}, \tag{1.4}$$

where $e_i(\mathbf{x})$ is the elementary symmetric function defined in Section 3.1. Equation (1.3) is equivalent to a result of Carlitz, Scoville, and Vaughn [12] on Smirnov words, which are defined in Section 1.2.

A long-standing conjecture about chromatic symmetric functions involves the e-positivity of a particular class of graphs. We say that a poset P is $(a+b)$-free if it contains no induced subposet that consists of a disjoint union of a chain of a vertices and a chain of b vertices. The incomparability graph $\mathrm{Inc}(P)$ of a poset P is the graph whose vertex set is the elements of P with edges between incomparable elements of P.

Conjecture 1.1.1 (Stanley-Stembridge [58] [65])**.** *Let P be a $(3+1)$-free poset. Then $X_{\mathrm{Inc}(P)}(\mathbf{x})$ is e-positive.*

This conjecture is stated in terms of chromatic symmetric functions, but it originated from Stembridge's work on immanants, where it is stated in a weaker form in terms of immanants [66, Conjecture 4.4]. It is also stated in an equivalent form in terms of immanants by Stanley and Stembridge in [65, Conjecture 5.5]. We discuss Stanley's work on chromatic symmetric functions in more detail in Section 3.2.

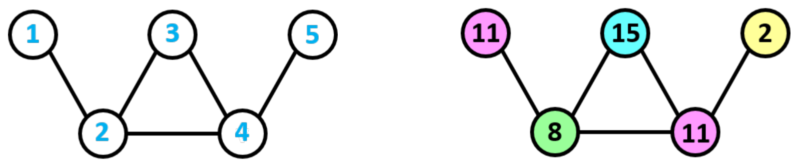
In 2012 Shareshian and Wachs [52, 53] presented a refinement of the chromatic symmetric function. We call a graph *labeled* if its vertex set is $[n] := \{1, 2, \dots, n\}$. For a labeled graph $G = ([n], E)$ and a proper coloring $\kappa \in C(G)$, define the number of *ascents* of κ as

$$\mathrm{asc}(\kappa) := |\{\{i, j\} \in E \mid i < j, \kappa(i) < \kappa(j)\}|.$$

Then the *chromatic quasisymmetric function* of a labeled graph $G = ([n], E)$ is defined by

$$X_G(\mathbf{x}, t) := \sum_{\kappa \in C(G)} t^{\mathrm{asc}(\kappa)} \mathbf{x}_\kappa. \tag{1.5}$$

Example 1.1.2. Suppose we have the following labeled graph G (pictured on the left) and a proper coloring of G (pictured on the right).



In this proper coloring the edges $\{2,3\}$ and $\{2,4\}$ are ascents. Hence this coloring corresponds to the term $t^2 x_2 x_8 x_{11}^2 x_{15}$ in $X_G(\mathbf{x},t)$.

In the chromatic quasisymmetric function of a graph, the coefficients of powers of t are not symmetric functions in general. They belong to a class of formal power series called quasisymmetric functions. See Section 4.1 for a discussion of quasisymmetric functions. Setting $t = 1$ gives back Stanley's chromatic symmetric function.

Shareshian and Wachs gave an expansion for $X_G(\mathbf{x},t)$ in a certain basis for the ring of quasisymmetric functions called Gessel's fundamental basis (F-basis) when G is the incomparability graph of a poset P on $[n]$. They also showed that when G belongs to a class of labeled graphs called natural unit interval graphs, defined in Section 4.2, $X_G(\mathbf{x},t)$ is actually symmetric, i.e., the coefficients of t in $X_G(\mathbf{x},t)$ are symmetric functions. Shareshian and Wachs conjectured and Athanasiadis [4] proved a p-positivity formula for $\omega X_G(\mathbf{x},t)$ when G is a natural unit interval graph.

Shareshian and Wachs [54, Theorem 7.2] also prove the following e-basis expansion formula for the chromatic quasisymmetric function of the labeled path.

$$\sum_{n\geq 0} X_{P_n}(\mathbf{x},t)z^n = \frac{\displaystyle\sum_{i\geq 0} e_i(\mathbf{x})z^i}{1 - \displaystyle\sum_{i\geq 2}[i-1]_t e_i(\mathbf{x})z^i}, \tag{1.6}$$

where

$$[n]_t := 1 + t + \cdots + t^{n-1}.$$

It is not hard to see that this formula establishes e-positivity of $X_{P_n}(\mathbf{x},t)$. Note that this a nice t-analog of Stanley's generating function formula for the chromatic symmetric function of the path given in (1.3). Shareshian and Wachs also give the following e-positivity conjecture for chromatic quasisymmetric functions.

Conjecture 1.1.3 (Shareshian-Wachs [53] [52]). *Let $G = ([n], E)$ be a natural unit interval graph. Then the palindromic*[1] *polynomial $X_G(\mathbf{x}, t)$ is e-positive and e-unimodal*[2]*.*

Unit interval graphs are the incomparability graphs of posets that are both $(3+1)$-free and $(2+2)$-free, so the class of graphs considered in the Shareshian-Wachs conjecture is smaller than that of the Stanley-Stembridge conjecture; however, Guay-Pacquet [35] showed that if the Stanley-Stembridge conjecture holds for incomparability graphs of posets that are both $(3+1)$-free and $(2+2)$-free, then it holds in general. Hence the Shareshian-Wachs e-positivity conjecture implies the Stanley-Stembridge e-positivity conjecture. We describe the work of Shareshian and Wachs on chromatic quasisymmetric functions of labeled graphs in more detail in Section 4.2.

There is an important connection between chromatic quasisymmetric functions of natural unit interval graphs and Hessenberg varieties, which was conjectured by Shareshian and Wachs and was proven by Brosnan and Chow [10] and later by Guay-Paquet [36]. This connection to Hessenberg varieties gives a possible approach to proving Conjecture 1.1.3. Clearman, Hyatt, Shelton, and Skandera [17] found an algebraic interpretation of chromatic quasisymmetric functions of natural unit interval graphs in terms of characters of type A Hecke algebras evaluated at Kazhdan-Lusztig basis elements. Haglund and Wilson [38] discovered a connection between chromatic quasisymmetric functions and Macdonald polynomials.

In Chapter 5 we extend the definition of chromatic quasisymmetric function to directed graphs. Let $\overrightarrow{G} = (V, E)$ be a directed graph and let $\kappa \in C(\overrightarrow{G})$, i.e., let κ be

[1] Shareshian and Wachs show that for a graph $G = ([n], E)$, if $X_G(\mathbf{x}, t)$ is symmetric, then it is palindromic, i.e., if $X_G(\mathbf{x}, t) = \sum_{j=0}^{|E|} a_j(\mathbf{x}) t^j$, then $a_j(\mathbf{x}) = a_{|E|-j}(\mathbf{x})$ for all $0 \leq j \leq \frac{|E|-1}{2}$.

[2] See Section 4.2 for the definition of e-unimodal.

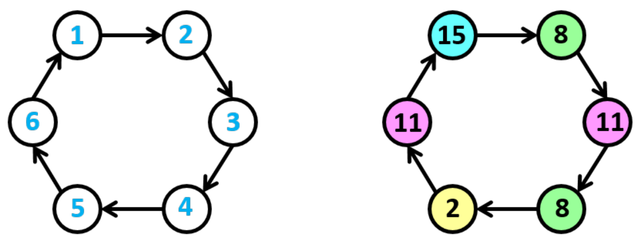
a proper coloring of $\overrightarrow{G}$. We can define the *ascents* of κ as

$$\operatorname{asc}(\kappa) := |\{(i,j) \in E \mid \kappa(i) < \kappa(j)\}|.$$

Then the *chromatic quasisymmetric function* of a directed graph $\overrightarrow{G}$ is defined as

$$X_{\overrightarrow{G}}(\mathbf{x}, t) := \sum_{\kappa \in C(\overrightarrow{G})} t^{\operatorname{asc}(\kappa)} \mathbf{x}_\kappa. \tag{1.7}$$

Example 1.1.4. Suppose we have a directed graph, which we call the directed cycle $\overrightarrow{C_6}$ (pictured on the left), and a proper coloring of $\overrightarrow{C_6}$ (pictured on the right).



In this proper coloring the edges $(2,3)$, $(5,6)$, and $(6,1)$ are ascents. Hence this coloring corresponds to the term $t^3 x_2 x_8^2 x_{11}^2 x_{15}$ in $X_{\overrightarrow{C_6}}(\mathbf{x}, t)$. Note that we put labels on the vertices of $\overrightarrow{C_6}$ so that we may refer back to them; however, the labels of a directed graph do not affect its chromatic quasisymmetric function, as the ascents are counted using only the direction of the edges.

Any labeled graph can be turned into a directed graph by orienting edges from smaller labels to larger labels. Below we show an example.

By this process, one can see that our definition agrees with the definition of Shareshian and Wachs for acyclic digraphs. Setting $t = 1$ gives us back the chromatic symmetric function of the underlying undirected graph of $\overrightarrow{G}$.

We give an expansion of the chromatic quasisymmetric function of *any* digraph in terms of Gessel's fundamental quasisymmetric function basis using a permutation statistic we call G-descents (see Section 5.2 for the definition). Our formula does not reduce to the Shareshian-Wachs formula in the case of incomparability graphs, so our formula provides a new expansion in this case. We use our F-basis expansion to prove one of our main results.

Theorem 1.1.5. *Let $\overrightarrow{G}$ be a directed graph such that $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric. Then $\omega X_{\overrightarrow{G}}(\mathbf{x},t)$ is p-positive.*

In fact, we derive a p-basis expansion for $\omega X_{\overrightarrow{G}}(\mathbf{x},t)$ when $\overrightarrow{G}$ is *any* digraph such that $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric. In comparison the Athanasiadis-Shareshian-Wachs p-positivity formula for $\omega X_G(\mathbf{x},t)$ applies only when G is a natural unit interval graph. Our formula does not reduce to theirs, so we get a new formula in the case of natural unit interval graphs. When $\overrightarrow{G} = \overrightarrow{C_n}$, the directed cycle, we show that the p-basis expansion coefficients of $\omega X_{\overrightarrow{G}}(\mathbf{x},t)$ have a nice factorization involving the classical Eulerian polynomials defined in Section 2.1.

Next we address the question of which $\overrightarrow{G}$ give us symmetric $X_{\overrightarrow{G}}(\mathbf{x},t)$. We introduce a class of directed graphs, which we call circular indifference digraphs (see Section 5.4 for the definition), and we show that if $\overrightarrow{G}$ is a circular indifference digraph, then $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric. When natural unit interval graphs are turned into digraphs by orienting each edge from smaller label to larger label, they are contained in the class of circular indifference digraphs. In fact, all acyclic circular indifference digraphs can be obtained this way. The simplest circular indifference digraph that is not acyclic is the directed cycle, $\overrightarrow{C_n} = ([n],E)$ defined by $E = \{(i,i+1) \mid i \in [n-1]\} \cup \{(n,1)\}$.

We provide the following generating function formula for $X_{\overrightarrow{C_n}}(\mathbf{x},t)$ in terms of the elementary symmetric function basis:

$$\sum_{n\geq 2} X_{\overrightarrow{C_n}}(\mathbf{x},t)z^n = \frac{t\sum\limits_{k\geq 2} k[k-1]_t e_k(\mathbf{x})z^k}{1-t\sum\limits_{k\geq 2}[k-1]_t e_k(\mathbf{x})z^k}. \tag{1.8}$$

It is not hard to see that this formula establishes e-positivity of $X_{\overrightarrow{C_n}}(\mathbf{x},t)$. Note that this is a t-analog of Stanley's formula for $X_{C_n}(\mathbf{x})$, the chromatic symmetric function of the undirected cycle as shown in (1.4). We also present and give evidence for the following generalization of the Shareshian-Wachs e-positivity conjecture for all circular indifference digraphs.

Conjecture 1.1.6. *Let $\overrightarrow{G} = (V,E)$ be a circular indifference digraph. Then the palindromic*[3] *polynomial $X_{\overrightarrow{G}}(\mathbf{x},t)$ is e-positive and e-unimodal.*

Since natural unit interval graphs are contained in the class of circular indifference digraphs, our conjecture implies the Shareshian-Wachs e-positivity conjecture, which in turn implies the Stanley-Stembridge e-positivity conjecture. In addition, Stanley [58] defines the class of circular indifference graphs, which are the underlying undirected graphs of our circular indifference digraphs, and he suggests that they may have e-positive chromatic symmetric functions. This generalized e-positivity conjecture also encompasses this speculation of Stanley.

A connection between chromatic quasisymmetric functions of directed graphs and LLT polynomials was explored by Alexandersson and Panova [2]. Chromatic quasisymmetric functions of directed graphs also appear in the work of Awan and Bernardi [5] on Tutte polynomials.

[3] Palindromicity is established in Proposition 5.1.8.

1.2 Smirnov words

A proper coloring of the path P_n can be viewed as a word over the positive integers $\mathbb{P}$ where adjacent letters are distinct. These words are sometimes called *Smirnov words* (after [33], see also [57]). The second portion of this thesis, which is joint work with Wachs [22], focuses on Smirnov words.

Let W_n denote the set of Smirnov words of length n. We can define the number of descents of a Smirnov word $w = w_1 w_2 \cdots w_n \in W_n$ as

$$\operatorname{des}(w) := |\{i \in [n-1] \mid w_i > w_{i+1}\}|.$$

The descent enumerator of Smirnov words is defined as

$$W_n(\mathbf{x}, t) := \sum_{w \in W_n} t^{\operatorname{des}(w)} \mathbf{x}_w,$$

where $\mathbf{x}_w = x_{w_1} x_{w_2} \cdots x_{w_n}$. The descent enumerator $W_n(\mathbf{x}, t)$ arose in the work of Shareshian and Wachs [54] on q-Eulerian polynomials and motivated their work on chromatic quasisymmetric functions, as

$$W_n(\mathbf{x}, t) = X_{P_n}(\mathbf{x}, t),$$

the chromatic quasisymmetric function for the labeled path. Thus (1.6) gives a nice expansion for $W_n(\mathbf{x}, t)$ in the e-basis. The $t = 1$ case of (1.6), which was given in (1.3), was originally proved by Carlitz, Scoville, and Vaughn [12] through their work on Smirnov words. The symmetric function $W_n(\mathbf{x}, 1)$ has also been studied by Stanley [58] and Dollhopf, Goulden, and Greene [19].

We can define a circular version of the descent enumerator of Smirnov words as

$$\tilde{W}_n^{\neq}(\mathbf{x},t) = \sum_{\substack{w \in W_n \\ w_1 \neq w_n}} t^{\mathrm{cdes}(w)} \mathbf{x}_w,$$

where $\mathrm{cdes}(w)$ is the number of cyclic descents of σ, defined by

$$\mathrm{cdes}(w) := |\{i \in [n] \mid w_i > w_{i+1}\}|, \tag{1.9}$$

where $w_{n+1} := w_1$. It is not difficult to see that

$$\tilde{W}_n^{\neq}(\mathbf{x},t) = X_{\overrightarrow{C_n}}(\mathbf{x},t),$$

the chromatic quasisymmetric function of the directed cycle. We obtain results on $\tilde{W}_n^{\neq}(\mathbf{x},t)$, including the e-basis expansion formula (1.8), through our work with chromatic quasisymmetric functions of directed graphs.

In Chapter 6, we refine the work on $W_n(\mathbf{x},t)$ and $\tilde{W}_n^{\neq}(\mathbf{x},t)$ by considering the descent enumerators of restricted Smirnov words, i.e., Smirnov words where we put restrictions on the relationship between the first and last letters of the word. For example we define

$$W_n^{<}(\mathbf{x},t) = \sum_{\substack{w \in W_n \\ w_1 < w_n}} t^{\mathrm{des}(w)} \mathbf{x}_w.$$

The descent enumerators $W_n^{>}(\mathbf{x},t)$ and $W_n^{=}(\mathbf{x},t)$ are defined similarly. It is an exercise of Grinberg and Reiner [34] to show that these three descent enumerators of restricted Smirnov words are symmetric. We expand upon this by providing expansions of these restricted descent enumerators in various bases.

We obtain e-basis expansions for $W_n^{<}(\mathbf{x},t), W_n^{>}(\mathbf{x},t)$ and $W_n^{=}(\mathbf{x},t)$, which show that $W_n^{<}(\mathbf{x},t)$ and $W_n^{>}(\mathbf{x},t)$ are e-positive and e-unimodal. From our e-basis expansions, one can recover the e-basis expansion formulas (1.6) and (1.8) for $W_n(\mathbf{x},t)$ and

$\tilde{W}_n^{\neq}(\mathbf{x},t)$, respectively, using the relationships

$$W_n(\mathbf{x},t) = W_n^{<}(\mathbf{x},t) + W_n^{>}(\mathbf{x},t) + W_n^{=}(\mathbf{x},t),$$

and

$$\tilde{W}_n^{\neq}(\mathbf{x},t) = tW_n^{<}(\mathbf{x},t) + W_n^{>}(\mathbf{x},t).$$

However this does not provide new proofs of (1.6) and (1.8) as our proof relies on these formulas.

In addition we obtain an e-basis expansion of another cyclic descent enumerator,

$$\tilde{W}_n(\mathbf{x},t) := \sum_{w \in W_n} t^{\mathrm{cdes}(w)} \mathbf{x}_w,$$

using

$$\tilde{W}_n(\mathbf{x},t) = tW_n^{<}(\mathbf{x},t) + W_n^{>}(\mathbf{x},t) + W_n^{=}(\mathbf{x},t).$$

The expansion is given by

$$\sum_{n\geq 1} \tilde{W}_n(\mathbf{x},t) z^n = \frac{\frac{\partial}{\partial t} \sum\limits_{i\geq 0} e_i(\mathbf{x})(tz)^i}{1 - \sum\limits_{i\geq 2} t[i-1]_t e_i(\mathbf{x})}. \tag{1.10}$$

We are also able to derive an e-basis expansion formula for the chromatic quasisymmetric function of the labeled cycle $C_n = ([n], E)$, defined by $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{\{1,n\}\}$, using

$$X_{C_n}(\mathbf{x},t) = W_n^{<}(\mathbf{x},t) + tW_n^{>}(\mathbf{x},t).$$

Note that the chromatic quasisymmetric function $X_{C_n}(\mathbf{x},t)$ of the labeled cycle is different than the chromatic quasisymmetric function $X_{\overrightarrow{C_n}}(\mathbf{x},t)$ of the directed cycle.

From our expansion, one can see that $X_{C_n}(\mathbf{x}, t)$ is e-positive. This is notable since C_n is not covered by any of the current e-positivity conjectures for $n \geq 4$.

We obtain F-basis and p-basis expansions for the various descent enumerators defined in this section. The p-basis expansions involve the classical Eulerian polynomials given by

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)},$$

where $\mathfrak{S}_n$ is the symmetric group. By specialization of our F-basis and e-basis expansions, we get formulas for variations of the q-Eulerian polynomials defined by Shareshian and Wachs, described below.

Shareshian and Wachs [54, 52] define the q-Eulerian polynomials $A_n(q,t)$ by

$$A_n(q,t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})} t^{\mathrm{des}(\sigma)}, \tag{1.11}$$

where $\mathrm{maj}_{\leq 2}$ is a permutation statistic defined in Section 6.4. We note that $A_n(1,t)$ are the classical Eulerian polynomials. By specializing their F-basis and e-basis expansions of $W_n(\mathbf{x}, t)$, they obtain the formula

$$\sum_{n \geq 1} A_n(q,t) \frac{z^n}{[n]_q!} = \frac{\displaystyle\sum_{i \geq 1} [i]_t \frac{z^i}{[i]_q!}}{\displaystyle 1 - \sum_{i \geq 2} t[i-1]_t \frac{z^i}{[i]_q!}}, \tag{1.12}$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$. Setting $q = 1$ gives a classical result of Euler on the Eulerian polynomials. See (2.1).

In Section 6.4, we study variations of the q-Eulerian polynomials. For example, let us define the cyclic q-Eulerian polynomial $\tilde{A}_n(q,t)$ by

$$\tilde{A}_n(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})} t^{\mathrm{cdes}(\sigma)}. \tag{1.13}$$

Through specialization of our F-basis and e-basis expansions of $\tilde{W}_n(\mathbf{x}, t)$, we obtain the formula

$$\sum_{n\geq 1} \tilde{A}_n(q,t)\frac{z^n}{[n]_q!} = \frac{\frac{\partial}{\partial t}\exp_q(tz)}{1 - \sum_{i\geq 2} t[i-1]_t \frac{z^i}{[i]_q!}}, \tag{1.14}$$

where $\exp_q(z) \coloneqq \sum_{i\geq 0}\frac{z^i}{[i]_q!}$. We also obtain similar results for other variations of the q-Eulerian polynomials. See Section 6.4.

This thesis is organized as follows. In Chapter 2, we review some classical results on permutation statistics, Eulerian polynomials, q-analogs of Eulerian polynomials, and the chromatic polynomial of a graph. In Chapter 3 we review some basic symmetric function theory and discuss the chromatic symmetric function. In Chapter 4 we review quasisymmetric function theory and discuss the chromatic quasisymmetric function of labeled graphs. In Chapter 5 we present our results on chromatic quasisymmetric functions of directed graphs. In Chapter 6 we present our results on descent enumerators of restricted Smirnov words. In Appendix A we discuss some relationships between various classes of graphs and directed graphs discussed in this thesis.

Chapter 2

Combinatorial polynomials

2.1 Eulerian polynomials

For $n \in \mathbb{P}$, let $[n]$ denote the set $\{1, 2, \cdots, n\}$. A *permutation* of $[n]$ is a bijection from $[n]$ to itself. Let $\mathfrak{S}_n$ denote the set of permutations of $[n]$. In this thesis, we will most commonly express a permutation in one-line notation. By this we mean that if $\sigma \in \mathfrak{S}_n$, then $\sigma : [n] \to [n]$ is a bijection, and we can write σ as $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$, where $\sigma_i \coloneqq \sigma(i)$ and $\cdot$ represents concatenation.

A *permutation statistic* is a function $f : \mathfrak{S}_n \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Here we define a few commonly studied permutation statistics, as well as a few sets associated with permutations.

Definition 2.1.1. Let $n \in \mathbb{P}$ and let $\sigma \in \mathfrak{S}_n$.

- The *descent set* of σ is defined as

$$\mathrm{DES}(\sigma) \coloneqq \{i \in [n-1] \mid \sigma_i > \sigma_{i+1}\}.$$

- The number of descents of σ is

$$\operatorname{des}(\sigma) := |\operatorname{DES}(\sigma)|.$$

- Similarly, the *ascent set* of σ is defined as

$$\operatorname{ASC}(\sigma) := \{i \in [n-1] \mid \sigma_i < \sigma_{i+1}\}.$$

- The number of ascents of σ is

$$\operatorname{asc}(\sigma) := |\operatorname{ASC}(\sigma)|.$$

- The *major index* of σ is

$$\operatorname{maj}(\sigma) := \sum_{i \in \operatorname{DES}(\sigma)} i.$$

- The number of *inversions* of σ is

$$\operatorname{inv}(\sigma) := |\{(\sigma_i, \sigma_j) \mid i < j \text{ and } \sigma_i > \sigma_j\}|.$$

- The number of *excedances* of σ is

$$\operatorname{exc}(\sigma) := |\{i \in [n-1] \mid \sigma_i > i\}|.$$

To make sure we understand these definitions fully, let us look at an example. Let $\sigma = 132794568 \in \mathfrak{S}_9$. Then $\operatorname{DES}(\sigma) = \{2,5\}$ so $\operatorname{des}(\sigma) = 2$. On the other hand, all positions that are not descents are ascents, so $\operatorname{ASC}(\sigma) = \{1,3,4,6,7,8\}$ and hence $\operatorname{asc}(\sigma) = 6$. By adding the elements of $\operatorname{DES}(\sigma)$, we see that $\operatorname{maj}(\sigma) = 2 + 5 = 7$. The inversions of σ are the pairs (i,j) that are out of order in σ, so in this case $\mathrm{inv}(\sigma) = |\{(3,2),(7,4),(9,4),(7,5),(9,5),(7,6),(9,6),(9,8)\}| = 8$. Lastly, the number of excedences of σ is $\mathrm{exc}(\sigma) = |\{2,4,5\}| = 3$.

The major index of a permutation was named after Major Percy MacMahon, who did extensive work with permutations statistics. It is clear by reversing permutations, i.e., letting $\sigma(i) = \sigma(n+1-i)$, that ascents and descents are equidistributed, i.e.,

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{asc}(\sigma)}.$$

MacMahon [42, vol. 1, p.186] was the first to observe that descents and excedances are equidistributed, i.e.,

$$\sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

This result is surprising, because for a given permutation $\sigma \in \mathfrak{S}_n$, it is not in general true that $\mathrm{des}(\sigma) = \mathrm{exc}(\sigma)$. Any permutation statistic that is equidistributed with des is called an *Eulerian statistic*. This is because this equidistribution result of MacMahon is closely related to a set of polynomials defined years before by Euler.

The Eulerian polynomials, denoted by $A_n(t)$ for each $n \in \mathbb{N}$, were first introduced in 1749 by Euler [23] in the formula

$$\sum_{k \geq 1} (k+1)^n t^k = \frac{A_n(t)}{(1-t)^{n+1}}$$

while studying the Dirichlet eta function. Euler also proved the generating function

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{(1-t)e^z}{e^{tz} - te^z}, \tag{2.1}$$

where e^z is the usual exponential function. In 1958, after the work of MacMahon, Riordan [49] discovered that

$$A_n(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{asc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}.$$

In fact, this is now how the Eulerian polynomials are usually defined.

The Eulerian polynomials have a number of interesting properties, including the fact that they are both palindromic and unimodal, as defined below.

Definition 2.1.2. Let $P(t) \in \mathbb{Q}[t]$ be a polynomial with coefficients in $\mathbb{Q}$. Then $P(t)$ can be expressed as $P(t) = \sum_{i=0}^{n} a_n t^n$, where $a_i \in \mathbb{Q}$ for all i and $a_n \neq 0$.

We say that $P(t)$ is *palindromic* if $a_i = a_{n-i}$ for each $i \in \mathbb{N}$ with $i \leq n$.

We say that $P(t)$ is *unimodal* if there exists some $j \in \mathbb{P}$ such that $a_{i-1} \leq a_i$ for all $0 < i \leq j$ and $a_i \geq a_{i+1}$ for all $j \leq i < n$. In other words, the coefficients of $P(t)$ increase from a_0 to a_j and then decrease from a_j to a_n.

It is not too difficult to see that that Eulerian polynomials are palindromic by noting that for any $\sigma \in \mathfrak{S}_n$ if $\operatorname{des}(\sigma) = k$, then $\operatorname{des}(\sigma^{rev}) = \operatorname{asc}(\sigma) = n - 1 - k$, where σ^{rev} is the reverse of σ, i.e., if $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$, then $\sigma^{rev} = \sigma_n\sigma_{n-1} \cdots \sigma_1$. Showing their unimodality is a bit trickier. See [26] for more information on this.

2.2 q-analogs

A q-analog of an object has the property that setting $q = 1$ gives back the original object. For example for $n \in \mathbb{P}$, we define the q-analog of n, denoted $[n]_q$ as

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

Clearly setting $q = 1$ in $[n]_q$ returns the number n. We can also define the q-analog of $n!$ to be

$$[n]_q! := [1]_q[2]_q \cdots [n]_q.$$

One of the most classical q-analog results is the formula

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{inv}(\sigma)} = [n]_q! = \sum_{\sigma \in \mathfrak{S}_n} q^{\operatorname{maj}(\sigma)}.$$

Rodriguez [51] proved the first inequality, and MacMahon [43] proved the second, which shows that inversions and major index are equidistributed. In fact any permutation statistic that is equidistributed with inv and maj is called a *Mahonian statistic.* Letting $q = 1$ in the formula above gives us the well-known fact that $|\mathfrak{S}_n| = n!$.

A number of q-analogs of the Eulerian polynomials have been studied over the years by looking at pairs of permutation statistics. For any two permutation statistics f_1, f_2, define

$$A_n^{(f_1,f_2)}(q,t) = \sum_{\sigma\in\mathfrak{S}_n} q^{f_1(\sigma)} t^{f_2(\sigma)}.$$

The q-analogs $A^{(\mathrm{inv},\mathrm{des})}(q,t)$, $A^{(\mathrm{maj},\mathrm{des})}(q,t)$, and $A^{(\mathrm{inv},\mathrm{exc})}(q,t)$ have been well studied. (For a few of these studies, see [6, 7, 11, 16, 24, 25, 27, 28, 32, 37, 48, 55, 64, 63, 67].) For example, Stanley [63] showed that

$$\sum_{n\geq 0} A_n^{(\mathrm{inv},\mathrm{des})}(q,t)\frac{z^n}{[n]_q!} = \frac{1-t}{\mathrm{Exp}_q(z(t-1)) - t},$$

where $\mathrm{Exp}_q(z) := \sum_{n\geq 0} q^{\binom{n}{2}} \frac{z^n}{[n]_q!}$. Substituting $q = 1$ gives the classical generating function formula of Euler (2.1). More recently, Shareshian and Wachs [54] found the formula

$$\sum_{n\geq 0} A_n^{(\mathrm{maj},\mathrm{exc})}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)}, \tag{2.2}$$

where $\exp_q(z) = \sum_{n\geq 0} \frac{z^n}{[n]_q!}$. Again setting $q = 1$ gives the classical generating function formula of Euler (2.1). In Section 6.4 we obtain expansions of a few variations of these q-analogs.

2.3 The chromatic polynomial

A graph $G = (V, E)$ is defined as a set V of vertices together with the edges, E, which is a collection of pairs of vertices. We say G is *simple* if E does not contain loops, i.e., no edge between a vertex and itself, and E does not contain multiple edges, i.e., there is at most one edge between any two distinct vertices. In this dissertation we will assume that all graphs are simple.

A proper coloring $\kappa : V \to \mathbb{P}$ of a graph $G = (V, E)$ is an assignment of positive integers, which we can think of as colors, to the vertices of G such that adjacent vertices have different colors; in other words, if $\{i, j\} \in E$, then $\kappa(i) \neq \kappa(j)$. The most famous theorem involving graph colorings is the Four-Color Theorem, which states that any planar graph, i.e., any graph that can be drawn in the plane with no intersecting edges, can be colored with at most four colors.

While attempting to prove the Four-Color Theorem, Birkhoff [8] introduced the chromatic polynomial of a planar graph in 1912. This definition was later extended to all graphs in 1933 by Whitney [68], who was a student of Birkhoff. The chromatic polynomial of a graph, G, denoted $\chi_G(k)$, gives the number of proper colorings of G using the colors of $[k]$. Though it is not obvious, the chromatic polynomial of a graph is actually a polynomial. For example if $G = P_3$, the path graph on 3 vertices, then $\chi_G(k) = k(k-1)^2$. In fact, if $G = (V, E)$ is any tree, i.e., any graph with no cycles, then $\chi_G(k) = k(k-1)^{|V|-1}$.

The chromatic polynomial has a number of interesting properties. For example Stanley [62] showed that for any graph $G = (V, E)$, the expression $(-1)^{|V|}\chi_G(-1)$ gives the number of acyclic orientations of G. The chromatic polynomial can even be defined recursively. Let $G = (V, E)$ be a graph and let $e \in E$. Then $G - e$ is the graph with the edge e deleted, i.e., $G - e = (V, E - e)$. On the other hand $G \backslash e$ is the graph G with e contracted, i.e., if $e = \{u, v\}$, then to obtain $G \backslash e$, we delete the edge e from E and identify vertex u with vertex v. Then

$$\chi_G(k) = \chi_{G-e}(k) - \chi_{G\backslash e}(k).$$

Since this graph $G = (V, E)$ with $E = \emptyset$ has $\chi_G(k) = k^{|V|}$, we can calculate the chromatic polynomial of any graph using this recursion. This recursion can also be used to show by induction that every chromatic polynomial is actually a polynomial.

We would also like to present an interesting result of Whitney, but first we need some notation. A set partition of $[n]$ is defined as $\pi = \{B_1, B_2, \ldots, B_k\}$, where

- for each i, we have $B_i \subseteq [n]$ with $B_i \neq \emptyset$,
- for each $1 \leq i < j \leq k$, we have $B_i \cap B_j = \emptyset$, and
- $\displaystyle\bigcup_{i=1}^{k} B_i = [n]$.

We call these B_i the blocks of π. For example $\{\{2,5,7\},\{1,3\},\{4\},\{6\}\}$ is a set partition of [7]. We can create a poset Π_n on set partitions of $[n]$, called the partition lattice, such that for any two set partitions $\pi, \gamma \in \Pi_n$ we have that $\pi <_{\Pi_n} \gamma$ if every block of π is contained in a block of γ, i.e., for each $B_i \in \pi$, there exists a $B_j \in \gamma$ such that $B_i \subseteq B_j$.

Given a graph $G = ([n], E)$ with vertex set $[n]$, we say that $\pi \in \Pi_n$ is *connected* if for each block $B_i \in \pi$, the induced subgraph of G on the vertices of B_i is connected. The connected set partitions form a subposet of Π_n, which we call the *bond lattice* of G, denoted L_G. The smallest element of L_G, which is the set partition where each block contains only one element, will be denoted $\hat{0}$.

Lastly, let us define the *Möbius function* of a poset P, denoted μ_P. The Möbius function is defined recursively from intervals of P into the integers. For any $s \in P$, we have that $\mu_P(s,s) = 1$. For any $s, t \in P$ with $s <_P t$, we have that $\mu_P(s,t) = -\sum_{s \leq u < t} \mu_P(s,u)$. More information on the Möbius function can be found in [59, Chapter 3]. Whitney discovered the following expansion of the chromatic polynomial of a graph.

Theorem 2.3.1 (Whitney [69]). *For any (simple) graph G,*

$$\chi_G(n) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) n^{|\pi|},$$

where $|\pi|$ denotes the number of blocks of π.

Though the chromatic polynomial did not help Birkhoff prove the Four Color Theorem as he had hoped, it has become the topic of much mathematical study. The chromatic polynomial has been generalized to the Tutte polynomial, which has applications to fields such as knot theory and computational physics. It is currently a major topic of study in algebraic graph theory with many open problems surrounding it, such as characterizing graphs with the same chromatic polynomials and determining which polynomials are chromatic polynomials. Most importantly for us, the chromatic polynomial has been generalized to the chromatic symmetric function, which has itself become a heavily studied topic in the field of algebraic combinatorics.

Chapter 3

Chromatic symmetric functions

In 1995 Stanley [58] introduced a symmetric function analog of the chromatic polynomial, called the chromatic symmetric function. Before we delve into this topic, let us review some standard theory on symmetric functions.

3.1 Symmetric functions

The theory of symmetric functions is quite broad, so here we will review only a few basic definitions and results that we need for our work. More information on symmetric functions can be found in [60, Chapter 7] and [41].

A *symmetric function* $f(\mathbf{x})$ over a commutative ring R is a formal power series in infinitely many variables, which we denote $\mathbf{x} = x_1, x_2, x_3, \ldots$, with coefficients in R so that for any permutation σ of the positive integers $\mathbb{P}$, we have that

$$f(x_1, x_2, x_3, \ldots) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \ldots).$$

In other words, permuting the variables does not change $f(\mathbf{x})$. For the purposes of this thesis, we will let $R = \mathbb{Q}$, the field of rational numbers. Let $\Lambda_{\mathbb{Q}}$ denote the $\mathbb{Q}$-algebra of symmetric functions with coefficients in $\mathbb{Q}$. In fact $\Lambda_{\mathbb{Q}}$ is a graded algebra with

$$\Lambda_{\mathbb{Q}} = \Lambda_{\mathbb{Q}}^0 \oplus \Lambda_{\mathbb{Q}}^1 \oplus \Lambda_{\mathbb{Q}}^2 \oplus \cdots ,$$

where $\Lambda_{\mathbb{Q}}^n$ is the $\mathbb{Q}$-vector space of homogeneous symmetric functions of degree n.

Now we would like to discuss a few bases for $\Lambda_{\mathbb{Q}}$, but first we must define the notion of a partition. For any $n \in \mathbb{N}$, we say λ is a partition of n, denoted $\lambda \vdash n$, if $\lambda = (\lambda_1, \lambda_2, \cdots)$ with $\lambda_1 \geq \lambda_2 \geq \cdots$, $\lambda_i \in \mathbb{N}$ for each i and $\sum_{i=1}^{k} \lambda_i = n$. Notice that using this definition, each partition ends with an infinite string of 0's. For notational convenience, we usually do not include the 0's, so for example the partition $(5, 5, 3, 2, 0, 0, \cdots)$ of 15 would be written as $(5, 5, 3, 2)$. In addition, we let $l(\lambda)$ denote the length of λ, which is the number of $\lambda_i \neq 0$. For example $l((5, 5, 3, 2)) = 4$. Let Par_n denote the set of partitions of n and let $\operatorname{Par} \coloneqq \bigcup_{i \geq 0} \operatorname{Par}_i$ denote the set of all partitions. So for example,

$$\operatorname{Par}_5 = \{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)\}.$$

One can show that $\dim(\Lambda_{\mathbb{Q}}^n) = |\operatorname{Par}_n|$, so the bases for $\Lambda_{\mathbb{Q}}$ are indexed by partitions. The first basis we would like to discuss is the elementary symmetric function basis, also known as the e-basis. For each $m \in \mathbb{P}$, define

$$e_m(\mathbf{x}) \coloneqq \sum_{i_1 < i_2 < \cdots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

So for example $e_1(\mathbf{x}) = x_1 + x_2 + x_3 + \cdots$ and $e_2(\mathbf{x}) = x_1x_2 + x_1x_3 + x_2x_3 + \cdots$. We also define $e_0(\mathbf{x}) \coloneqq 1$. Then for any partition $\lambda = (\lambda_1, \lambda_2, \cdots , \lambda_k) \vdash n$, we can define

$$e_\lambda(\mathbf{x}) \coloneqq e_{\lambda_1}(\mathbf{x}) e_{\lambda_2}(\mathbf{x}) \cdots e_{\lambda_k}(\mathbf{x}).$$

Then $\{e_\lambda(\mathbf{x}) \mid \lambda \in \mathrm{Par}_n\}$ form a $\mathbb{Q}$-basis for $\Lambda^n_{\mathbb{Q}}$ and so $\{e_\lambda(\mathbf{x}) \mid \lambda \in \mathrm{Par}\}$ form a $\mathbb{Q}$-basis for $\Lambda_{\mathbb{Q}}$. In fact, $e_1(\mathbf{x}), e_2(\mathbf{x}), \cdots$ are algebraically independent and generate $\Lambda_{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra, so $\Lambda = \mathbb{Q}[e_1(\mathbf{x}), e_2(\mathbf{x}), \cdots]$.

The other basis we will use most often in this thesis is the power sum symmetric function basis, also known as the p-basis. For each $m \in \mathbb{P}$, define

$$p_m(\mathbf{x}) \coloneqq \sum_{i \geq 1} x_i^m.$$

For example $p_1(\mathbf{x}) = x_1 + x_2 + x_3 + \cdots$ and $p_2(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + \cdots$. We define $p_0(\mathbf{x}) \coloneqq 1$. Then for any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$, we define

$$p_\lambda(\mathbf{x}) \coloneqq p_{\lambda_1}(\mathbf{x}) p_{\lambda_2}(\mathbf{x}) \cdots p_{\lambda_k}(\mathbf{x}).$$

As in the case of the elementary basis, $\{p_\lambda(\mathbf{x}) \mid \lambda \in \mathrm{Par}_n\}$ form a $\mathbb{Q}$-basis for $\Lambda^n_{\mathbb{Q}}$ and so $\{p_\lambda(\mathbf{x}) \mid \lambda \in \mathrm{Par}\}$ form a $\mathbb{Q}$-basis for $\Lambda_{\mathbb{Q}}$. In fact, $p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots$ are algebraically independent and generate $\Lambda_{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra, so $\Lambda = \mathbb{Q}[p_1(\mathbf{x}), p_2(\mathbf{x}), \cdots]$.

The last basis we would like to define is the complete homogeneous symmetric function basis, also known as the h-basis. For each $m \in \mathbb{P}$, define

$$h_m(\mathbf{x}) \coloneqq \sum_{i_1 \leq i_2 \leq \cdots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

For example $h_1(\mathbf{x}) = x_1 + x_2 + x_3 + \cdots$ and $h_2(\mathbf{x}) = x_1^2 + x_2^2 + x_1 x_2 + \cdots$. We define $h_0(\mathbf{x}) \coloneqq 1$. Then for any partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k) \vdash n$, we define

$$h_\lambda(\mathbf{x}) \coloneqq h_{\lambda_1}(\mathbf{x}) h_{\lambda_2}(\mathbf{x}) \cdots h_{\lambda_k}(\mathbf{x}).$$

As in the case of the previous two bases, $\{h_\lambda(\mathbf{x}) \mid \lambda \in \mathrm{Par}_n\}$ form a $\mathbb{Q}$-basis for $\Lambda^n_{\mathbb{Q}}$ and so $\{h_\lambda(\mathbf{x}) \mid \lambda \in \mathrm{Par}\}$ form a $\mathbb{Q}$-basis for $\Lambda_{\mathbb{Q}}$. In fact, $h_1(\mathbf{x}), h_2(\mathbf{x}), \cdots$ are algebraically independent and generate $\Lambda_{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra, so $\Lambda = \mathbb{Q}[h_1(\mathbf{x}), h_2(\mathbf{x}), \cdots]$.

One important symmetric function basis that we do not define here but may reference on occasion is the Schur basis. The Schur basis is arguably the most interesting and useful of the symmetric function bases; however, we do not use this basis in our work, and as Schur functions are a bit more difficult to define than the three bases described here, we will omit the definition. More information on Schur functions can be found in [60, Chapter 7].

Throughout this thesis, we use ω to denote the usual involution on $\Lambda_{\mathbb{Q}}$ defined by

$$\omega h_\lambda(\mathbf{x}) = e_\lambda(\mathbf{x}).$$

It can be shown that for any partition $\lambda \vdash n$, we have that $\omega p_\lambda(\mathbf{x}) = (-1)^{n-l(\lambda)} p_\lambda(\mathbf{x})$.

For any basis, $b = \{b_\lambda \mid \lambda \vdash n\}$, of $\Lambda^n_{\mathbb{Q}}$, we say that a symmetric function, $f(\mathbf{x}) \in \Lambda^n_{\mathbb{Q}}$ is *b-positive* if the expansion of the symmetric function in terms of the b_λ basis has nonnegative coefficients. It is a well-known fact that for any symmetric function $f(\mathbf{x}) \in \Lambda^n_{\mathbb{Q}}$, if $f(\mathbf{x})$ is h-positive, then it is also p-positive. In this thesis, we will use the immediate corollary that for any symmetric function $f(\mathbf{x}) \in \Lambda^n_{\mathbb{Q}}$, if $f(\mathbf{x})$ is e-positive, then $\omega f(\mathbf{x})$ is p-positive. We also reference the fact that if $f(\mathbf{x})$ is e-positive, then $f(\mathbf{x})$ is also Schur-positive.

3.2 Chromatic symmetric functions

As mentioned previously, Stanley defined a symmetric function refinement of the chromatic polynomial called the *chromatic symmetric function* of a graph. We will restate the definition here for convenience.

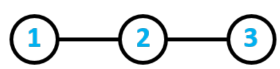
Definition 3.2.1 (Stanley [58]). For any graph $G = (V, E)$ let $C(G)$ denote the set of proper colorings of G. The chromatic symmetric function of G is defined as

$$X_G(\mathbf{x}) = \sum_{\kappa \in C(G)} \mathbf{x}_\kappa,$$

where $\mathbf{x}_\kappa = \prod_{v \in V} x_{\kappa(v)}$.

Notice that permuting the variables of $X_G(\mathbf{x})$ is equivalent to permuting the colors (which are positive integers); however, this simply gives us a different proper coloring of G, so it does not change the expansion. Hence $X_G(\mathbf{x})$ is a symmetric function. In fact, if G has n vertices, then $X_G(\mathbf{x})$ is homogeneous of degree n, so $X_G(\mathbf{x}) \in \Lambda^n_{\mathbb{Q}}$.

For a symmetric function $f(\mathbf{x}) \in \Lambda^n_{\mathbb{Q}}$ and for $k \in \mathbb{P}$, we define $f(1^k)$ as the value obtained from setting $x_i = 1$ for $i \leq k$ and $x_i = 0$ for $i > k$ in $f(\mathbf{x})$. Then $X_G(1^k)$ is the number of colorings of G that use only the colors in $[k]$, hence $X_G(\mathbf{x}) = \chi_G(k)$, where $\chi_G(k)$ is the chromatic polynomial of G evaluated at k.



As an example, let us calculate the chromatic symmetric function of the path graph, P_3, as shown above. There are $3! = 6$ possible ways to color P_3 with 3 different colors and 2 possible ways to color P_3 with two different colors (by putting one color on the outer two vertices and the other color on the middle vertex). Hence

$$X_{P_3}(\mathbf{x}) = 6 \sum_{i<j<k} x_i x_j x_k + \sum_{i<j} (x_i^2 x_j + x_i x_j^2) \tag{3.1}$$

$$= e_{21}(\mathbf{x}) + 3e_3(\mathbf{x}) \tag{3.2}$$

Stanley proved the following p-basis expansion for the chromatic symmetric function of any graph.

Theorem 3.2.2 (Stanley [58, Theorem 2.6])**.** *Let G be any graph and let L_G denote the bond lattice of G. Then*

$$X_G(\mathbf{x}) = \sum_{\pi \in L_G} \mu(\hat{0}, \pi) p_{\text{type}(\pi)}(\mathbf{x}),$$

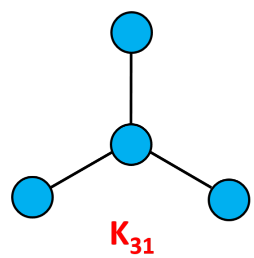
where $\mu(\hat{0}, \pi)$ is the Möbius function of L_G and type(π) *is the partition formed by ordering the sizes of the blocks of π in decreasing order.*

It is clear that $p_n(1^k) = k$ for any n, so specializing this result gives the result of Whitney for chromatic polynomials (see Theorem 2.3.1). Now let us note two useful facts. First the Möbius function of L_G alternates in sign, i.e., for all $\pi \in L_G$, we have $(-1)^{n-|\pi|}\mu(\hat{0}, \pi)$ is always positive, where n is the number of vertices of G and $|\pi|$ is the number of blocks of π. As we mentioned earlier for $\lambda \vdash n$, we have $\omega p_\lambda(\mathbf{x}) = (-1)^{n-l(\lambda)} p_\lambda(\mathbf{x})$. Combining these two facts, we get the following corollary.

Corollary 3.2.3 (Stanley [58, Corollary 2.7])**.** *For any (simple) graph G, we have that $\omega X_G(\mathbf{x})$ is p-positive. In fact*

$$\omega X_G(\mathbf{x}) = \sum_{\pi \in L_G} |\mu(\hat{0}, \pi)| p_{\text{type}(\pi)}(\mathbf{x}).$$

The natural question to ask is whether $X_G(\mathbf{x})$ has positive coefficients in other symmetric function bases. Unfortunately, that is not always the case. For the claw graph, K_{31}, we have that $X_{K_{31}}(\mathbf{x}) = e_4(\mathbf{x}) + 5e_{31}(\mathbf{x}) - 2e_{22}(\mathbf{x}) + e_{211}(\mathbf{x})$.



Then one could ask if there is a class of graphs with e-positive chromatic symmetric functions. The most well-known conjecture involving chromatic symmetric functions of graphs is the Stanley-Stembridge e-positivity conjecture. First let us say that a poset is *(a+b)-free* if it has no induced subposet that is the disjoint union of a chain with a elements and a chain b elements. The *incomparability graph* of a poset P, denoted $\mathrm{Inc}(P)$, is the graph with the elements of P as vertices and edges between incomparable elements of P.

Conjecture 3.2.4 (Stanley-Stembridge [66] [58]). *Let P be a $(3+1)$-free poset. Then $X_{\mathrm{Inc}(P)}(\mathbf{x})$ is e-positive.*

A weaker result that the chromatic symmetric functions of incomparability graphs of (3+1)-free posets are Schur-positive follows from the work of Haiman [39]. Gasharov [29] gave a combinatorial interpretation of the coefficients in the Schur basis in terms of a combinatorial object called P-tableau.

The simplest connected graph that is the incomparability graph of a $(3+1)$-free poset is the path graph, P_n. Recall that we define $P_n = ([n], E)$ to be the graph on $[n]$ with edge set $E = \{\{i, i+1\} \mid i \in [n-1]\}$. Stanley gives a nice e-basis expansion of the path, shown below.

Proposition 3.2.5 (Stanley [58, Proposition 5.3]). *Let P_n be the path graph. Then*

$$\sum_{n\geq 0} X_{P_n}(\mathbf{x})z^n = \frac{\sum_{i\geq 0} e_i(\mathbf{x})z^i}{1-\sum_{i\geq 2}(i-1)e_i(\mathbf{x})z^i}. \tag{3.3}$$

Consequently, $X_{P_n}(\mathbf{x})$ is e-positive for all $n \in \mathbb{N}$.

Stanley also defines a class of graphs, which he names circular indifference graphs, that seem to be e-positive. The simplest connected circular indifference graph that is not the incomparability graph of a poset is the cycle C_n. For $n \geq 2$, let $C_n = ([n], E)$ be the graph with edge set $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{\{1, n\}\}$. Stanley proves the following e-basis generating function formula for $X_{C_n}(\mathbf{x}, t)$.

Proposition 3.2.6 (Stanley [58, Proposition 5.4])**.** *Let C_n be the cycle graph. Then*

$$\sum_{n\geq 2} X_{C_n}(\mathbf{x})z^n = \frac{\sum_{i\geq 2} i(i-1)e_i(\mathbf{x})z^i}{1-\sum_{i\geq 2}(i-1)e_i(\mathbf{x})z^i}. \tag{3.4}$$

Consequently $X_{C_n}(\mathbf{x})$ is e-positive for all $n \in \mathbb{P}$ with $n \geq 2$.

Since their introduction, chromatic symmetric functions have been extensively studied. Some of these studies include [35], [61], [29], [30], [14], [13], [45], [40], [70], [46], [31], [47], [18].

Chapter 4

Chromatic quasisymmetric functions of labeled graphs

In 2012 Shareshian and Wachs introduced a quasisymmetric generalization of the chromatic symmetric function for labeled graphs. They did so by introducing an extra variable to record the number of ascents of each proper coloring. Before we give the formal definition of chromatic quasisymmetric functions, let us discuss some of the basic theory of quasisymmetric functions.

4.1 Quasisymmetric functions

The main work of this thesis involves quasisymmetric functions, as one may suspect by the title. As with symmetric functions, the theory of quasisymmetric functions is quite rich; however, we will address only the basic definitions and results needed for our work. More information on quasisymmetric functions can be found in [60, Chapter 7].

A *quasisymmetric function* $f(\mathbf{x})$ over a commutative ring R is a formal power series in infinitely many variables, which we denote $\mathbf{x} = x_1, x_2, x_3, \ldots$, with coefficients in R so that for positive integers $a_1, a_2, \ldots, a_k, i_1, i_2, \ldots, i_k \in \mathbb{N}$ with $i_1 < i_2 < \cdots < i_k$, we have that the coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ in $f(\mathbf{x})$ is the same as the coefficient of $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$ in $f(\mathbf{x})$. In other words, $f(\mathbf{x})$ is invariant under shifting the indices of the variables. Notice that all symmetric functions are quasisymmetric functions, but not all quasisymmetric functions are symmetric functions. For example $f(\mathbf{x}) = \sum_{i<j<k} x_i^2 x_j x_k^3$ is a quasisymmetric function but not a symmetric function.

Let $\mathrm{QSym}_{\mathbb{Q}}$ denote the $\mathbb{Q}$-vector space of quasisymmetric functions, and let $\mathrm{QSym}_{\mathbb{Q}}^n$ denote the $\mathbb{Q}$-vector space of homogeneous quasisymmetric functions of degree n. Since the product of two quasisymmetric functions is quasisymmetric, we see that $\mathrm{QSym}_{\mathbb{Q}}$ is actually a graded algebra with

$$\mathrm{QSym}_{\mathbb{Q}} = \mathrm{QSym}_{\mathbb{Q}}^0 \oplus \mathrm{QSym}_{\mathbb{Q}}^1 \oplus \mathrm{QSym}_{\mathbb{Q}}^2 \oplus \cdots .$$

Before we describe a few bases of $\mathrm{QSym}_{\mathbb{Q}}$, let us define the notion of a composition. For $n \in \mathbb{P}$, we define a *composition* of n to be an infinite sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ where each $\alpha_i \in \mathbb{N}$, there exists some $k \in \mathbb{P}$ such that $\alpha_i > 0$ for $i \leq k$ and $\alpha_i = 0$ for $i > k$ and such that $\sum_i \alpha_i = n$. For notational convenience we do not write the trailing zeros, so for example the composition $(4, 1, 6, 2, 0, 0, \dots)$ of 13 can be written as $(4, 1, 6, 2)$. Compositions are simply partitions where the parts need not be decreasing. Let $\mathrm{Comp}(n)$ denote the set of compositions of n and define $\mathrm{Comp}(0) = \{\emptyset\}$. We can also let $\mathrm{Comp} = \bigcup_{n \geq 0} \mathrm{Comp}(n)$. As an example, we have that

$$\mathrm{Comp}(4) = \{(4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1)\}.$$

There is a natural correspondence between $\mathrm{Comp}(n)$ and subsets of $[n-1]$ for $n \in \mathbb{P}$. (Note that we will let $[0] = \emptyset$.) Let $n \in \mathbb{P}$ and $(\alpha_1, \alpha_2, \dots, \alpha_k) \in \mathrm{Comp}(n)$. This corresponds to the subset $\{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$ of $[n-1]$. Similarly for a subset $\{a_1, a_2, \dots, a_m\} \subseteq [n-1]$ with $a_1 < a_2 < \cdots < a_m$, we can associate the composition $(a_1, a_2 - a_1, \dots, n - a_m)$ of n. One can easily see that this defines a bijection. From this we see that $|\mathrm{Comp}(n)| = 2^{n-1}$. It turns out that $\dim(\mathrm{QSym}_{\mathbb{Q}}^n) = 2^{n-1}$ for each $n \in \mathbb{P}$, so our bases for $\mathrm{QSym}_{\mathbb{Q}}^n$ will be indexed by either compositions of n or subsets of $n-1$.

The most natural basis for $\mathrm{QSym}_{\mathbb{Q}}$ is the monomial quasisymmetric basis. Let $n \geq 1$ and $\alpha = (a_1, a_2, \ldots, a_k) \in \mathrm{Comp}(n)$. Then

$$M_\alpha(\mathbf{x}) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}.$$

Let $M_\emptyset(\mathbf{x}) = 1$. If we let 1^n denote the composition of n that contains all 1's, we see that $M_{1^n}(\mathbf{x}) = e_n(\mathbf{x})$. Additionally, we have that $M_{(n)}(\mathbf{x}) = p_n(\mathbf{x})$. On the other hand, $M_{(2,1)} = \sum_{i<j} x_i^2 x_j$ is not a symmetric function. The set $\{M_\alpha(\mathbf{x}) \mid \alpha \in \mathrm{Comp}(n)\}$ forms a $\mathbb{Q}$-basis for $\mathrm{QSym}_{\mathbb{Q}}^n$, and hence $\{M_\alpha(\mathbf{x}) \mid \alpha \in \mathrm{Comp}\}$ forms a $\mathbb{Q}$-basis for $\mathrm{QSym}_{\mathbb{Q}}$.

The basis we will focus on in this thesis is Gessel's fundamental quasisymmetric function basis, also known as the F-basis. For $n \in \mathbb{P}$ and for each $S \subseteq [n-1]$, we define[1]

$$F_{n,S}(\mathbf{x}) = \sum_{\substack{i_1 \geq i_2 \geq \cdots \geq i_n \\ i_j > i_{j+1} \text{ if } j \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Define $F_{0,\emptyset}(\mathbf{x}) = 1$. For example $F_{n,[n-1]}(\mathbf{x}) = e_n(\mathbf{x})$, and $F_{n,\emptyset}(\mathbf{x}) = h_n(\mathbf{x})$. Again the set $\{F_{n,S}(\mathbf{x}) \mid S \subseteq [n-1]\}$ forms a $\mathbb{Q}$-basis for $\mathrm{QSym}_{\mathbb{Q}}^n$ and hence $\{F_{n,S}(\mathbf{x}) \mid n \in \mathbb{N}, S \subseteq [n-1]\}$ forms a $\mathbb{Q}$-basis for $\mathrm{QSym}_{\mathbb{Q}}$. Note that the involution ω on $\Lambda_{\mathbb{Q}}^n$ can be extended to $\mathrm{QSym}_{\mathbb{Q}}^n$ and can be described by $\omega F_{n,S}(\mathbf{x}) = F_{n,[n-1]\setminus S}(\mathbf{x})$.

4.2 Chromatic quasisymmetric functions of labeled graphs

Through their work on Eulerian quasisymmetric functions [54], Shareshian and Wachs discovered a t-analog of Stanley's e-basis expansion of the chromatic symmetric func-

[1] Note that our definition is different from the standard definition of Gessel's fundamental basis. Our $F_{n,S}(\mathbf{x})$ is equal to $L_{\alpha(S)}(\mathbf{x})$ defined in [60, Chapter 7], where $\alpha(S)$ is the reverse of the composition of n associated to S.

tion of the path graph. This led them to introduce a quasisymmetric refinement of Stanley's chromatic symmetric function called the *chromatic quasisymmetric function* of a graph. Let $G = ([n], E)$ be a graph, and let $\kappa : [n] \to \mathbb{P}$ be a proper coloring of G. We say that an edge $\{i, j\}$ of G is an *ascent* of κ if $i < j$ and $\kappa(i) < \kappa(j)$. Let $\operatorname{asc}(\kappa)$ denote the number of ascents of κ. Then the chromatic quasisymmetric function of G is given by

$$X_G(\mathbf{x}, t) = \sum_{\kappa \in C(G)} t^{\operatorname{asc}(\kappa)} \mathbf{x}_\kappa.$$

Henceforth when we use the term "labeled graph," we are referring to a graph with vertex set $[n]$. Note that the chromatic quasisymmetric function of a labeled graph depends on the labeling chosen and not just on the isomorphism class of the graph. We can easily see that setting $t = 1$ gives Stanley's chromatic symmetric function.

In the Shareshian-Wachs chromatic quasisymmetric function of a labeled graph, we can see that the coefficient of t^j for each $j \in \mathbb{N}$ is a quasisymmetric function, so $X_G(\mathbf{x}, t) \in \mathrm{QSym}_{\mathbb{Q}}[t]$, where $\mathrm{QSym}_{\mathbb{Q}}[t]$ is the ring of polynomials in t whose coefficients are in $\mathrm{QSym}_{\mathbb{Q}}$. Note that $\mathrm{QSym}_{\mathbb{Q}}[t]$ is equivalent to $\mathrm{QSym}_{\mathbb{Q}[t]}$, i.e., the ring of quasisymmetric functions with coefficients in the ring $\mathbb{Q}[t]$ of polynomials in t with coefficients in $\mathbb{Q}$. In this thesis, we tend to view these chromatic quasisymmetric functions as elements of $\mathrm{QSym}_{\mathbb{Q}}[t]$, but we may sometimes view them as elements of $\mathrm{QSym}_{\mathbb{Q}[t]}$ when convenient.

The coefficients of the chromatic quasisymmetric function of a graph do not necessarily have to be symmetric functions. The chromatic quasisymmetric function of the labeled graph, P_3, which is a path on 3 vertices labeled $1 - 2 - 3$, is symmetric[2]. In fact,

$$X_{P_3}(\mathbf{x}, t) = e_3(\mathbf{x}) + t(e_{21}(\mathbf{x}) + e_3(\mathbf{x})) + t^2 e_3(\mathbf{x}).$$

[2]Note that many times the term "symmetric" is interchangeable with the term "palindromic"; however, in this thesis we will use the term "symmetric" to imply that the coefficients of a function $f(\mathbf{x}) \in \mathrm{QSym}_{\mathbb{Q}}[t]$ are symmetric functions, i.e., that $f(\mathbf{x}) \in \Lambda_{\mathbb{Q}}[t]$.

(Compare this with the chromatic symmetric function of P_3 given in (3.2).) On the other hand, the chromatic quasisymmetric function of the graph G given by $2-1-3$ is not symmetric (see [52, Example 3.2]), since

$$X_G(\mathbf{x},t) = (F_{3,\emptyset}(\mathbf{x}) + F_{3,\{2\}}(\mathbf{x})) + 2tF_{3,\emptyset}(\mathbf{x}) + t^2(F_{3,\emptyset}(\mathbf{x}) + F_{3,\{1\}}(\mathbf{x})).$$

Then one might ask if there is a nice class of graphs with symmetric chromatic quasisymmetric functions. To answer this question, let us define the class of natural unit interval graphs. Note that there are a number of equivalent ways to define these graphs, so we choose one that is most convenient for us. See Appendix A for more information.

Definition 4.2.1. Let I be a finite set of closed unit intervals on the real line. We can write the intervals of I in the form $[a_i, a_i+1]$ for $1 \leq i \leq n$ with $a_1 < a_2 < \cdots < a_n$. Let P be the poset on $[n]$ such that $i <_P j$ if $a_i + 1 < a_j$. Posets that can be formed this way are called *natural unit interval orders.*

Natural unit interval graphs are the incomparability graphs of natural unit interval orders.

Shareshian and Wachs showed that if G is a natural unit interval graph, then $X_G(\mathbf{x},t)$ is symmetric. Notice that in our earlier example, $1-2-3$ is a natural unit interval graph, but $2-1-3$ is not.

Shareshian and Wachs used the theory of P-partitions to obtain a formula for $X_G(\mathbf{x},t)$ in terms of Gessel's fundamental quasisymmetric basis when G is the incomparability graph of a poset that uses P-descents. We discuss this in detail in Section 5.2. Note that since natural unit interval graphs are incomparability graphs of posets, this formula gives their F-basis expansion.

Using this F-basis expansion, Athanasiadis [4] was able to prove a conjecture of Shareshian and Wachs for the p-basis expansion of the chromatic quasisymmetric function of natural unit interval graphs. This is discussed in detail in Section 5.3. One of the most well-known conjectures resulting from their work is their e-positivity conjecture.

Conjecture 4.2.2 (Shareshian-Wachs [52][53]). *Let $G = ([n], E)$ be a natural unit interval graph. Then the palindromic*[3] *polynomial $X_G(\mathbf{x}, t)$ is e-positive and e-unimodal.*

In other words, if $X_G(\mathbf{x}, t) = \sum_{j=0}^{|E|} a_j(\mathbf{x})t^j$, then $a_j(\mathbf{x})$ is e-positive for all j and $a_{j+1}(\mathbf{x}) - a_j(\mathbf{x})$ is e-positive for all $j \leq \frac{|E|-1}{2}$.

The class of unit interval graphs is equivalent to the class of incomparability graphs of $(3+1)$ and $(2+2)$-free posets, so the class of graphs for the Shareshian-Wachs conjecture is smaller than the class of graphs for the Stanley-Stembridge conjecture. However, Guay-Pacquet [35] proved that if the Stanley-Stembridge conjecture holds for $(3+1)$ and $(2+2)$-free posets, then it holds for all $(3+1)$-free posets. Hence, the Shareshian-Wachs conjecture implies the Stanley-Stembridge conjecture.

Shareshian and Wachs [52] proved the weaker result that the chromatic quasisymmetric functions of natural unit interval graphs are Schur-positive. They give a combinatorial interpretation of the coefficients in the Schur basis expansion using the P-tableau described by Gasharov [29], and their result reduces to the result of Gasharov in the $t = 1$ case.

Shareshian and Wachs also obtained an e-basis generating function formula for $X_{P_n}(\mathbf{x}, t)$, which is a nice t-analog of the formula of Stanley's (see (3.3)) in the case of the unlabeled path. They showed the following:

Theorem 4.2.3 (Shareshian-Wachs [52][54]). *Let $P_n = ([n], E)$ be the labeled path graph. Then*

$$\sum_{n\geq 0} P_n(\mathbf{x}, t)z^n = \frac{\sum_{i\geq 0} e_i(\mathbf{x})z^i}{1 - \sum_{i\geq 2}[i-1]_t e_i(\mathbf{x})z^i}. \tag{4.1}$$

[3]Shareshian and Wachs established that if G is a natural unit interval graph, $X_G(\mathbf{x}, t)$ is a palindromic polynomial in t; in other words, $X_G(\mathbf{x}, t) = \sum_{i=0}^{|E|} a_i(\mathbf{x})t^i$ so that $a_i(\mathbf{x}) = a_{|E|-i}(\mathbf{x})$ for all $0 \leq i \leq |E|$.

From this, one can obtain the corollary that $X_{P_n}(\mathbf{x}, t)$ is e-positive and e-unimodal.

Much work has been done on chromatic quasisymmetric functions in the past few years. There is an important connection between chromatic quasisymmetric functions of natural unit interval graphs and Hessenberg varieties, which was conjectured by Shareshian and Wachs and was proven by Brosnan and Chow [10] and later by Guay-Paquet [36]. This connection to Hessenberg varieties gives a possible approach to proving Conjecture 4.2.2. Clearman, Hyatt, Shelton, and Skandera [17] found an algebraic interpretation of chromatic quasisymmetric functions of natural unit interval graphs in terms of characters of type A Hecke algebras evaluated at Kazhdan-Lusztig basis elements. Recently, Haglund and Wilson [38] discovered a connection between chromatic quasisymmetric functions and Macdonald polynomials.

Chapter 5

Chromatic quasisymmetric functions of directed graphs

The definition of chromatic quasisymmetric functions of labeled graphs has a natural extension to directed graphs, which we explore in this chapter. In Section 5.1, we give some basic results on chromatic quasisymmetric functions of digraphs as well as a few examples. In Section 5.2 we present our F-basis expansion for the chromatic quasisymmetric function of *all* digraphs, as well as a specialization of this expansion that refines a result of Chung and Graham. In Section 5.3 we show that for any digraph $\overrightarrow{G}$ such that $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric, we have that $\omega X_{\overrightarrow{G}}(\mathbf{x},t)$ is p-positive and we give a combinatorial interpretation of the coefficients. For the directed cycle $\overrightarrow{C_n}$, we give a factorization of the coefficients of $\omega X_{\overrightarrow{C_n}}(\mathbf{x},t)$ in the p-basis involving the Eulerian polynomials. In Section 5.4 we define the class of circular indifference digraphs and show that these digraphs have symmetric chromatic quasisymmetric functions. Lastly in Section 5.5 we present a generalized e-positivity conjecture for circular indifference digraphs and provide some support. We prove an e-basis expansion for $X_{\overrightarrow{C_n}}(\mathbf{x},t)$ showing that it is e-positive. We also provide a combinatorial interpretation for the coefficients of $X_{P_n}(\mathbf{x},t)$ and $X_{\overrightarrow{C_n}}(\mathbf{x},t)$ in the e-

basis involving acyclic orientations. Note that most of the results of this chapter can be found in the author's papers [21, 20].

5.1 Basic properties

We extend the definition of chromatic quasisymmetric function from labeled graphs to directed graphs, but before we discuss this further, let us fix some notation involving directed graphs.

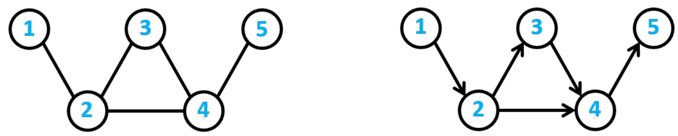
A *directed graph* (or *digraph*) $\overrightarrow{G} = (V, E)$ is a set of vertices V together with a set E of ordered pairs of vertices, called edges. The pair $(u, v) \in E$ denotes an edge directed from u to v. We say a directed graph $\overrightarrow{G} = (V, E)$ is *simple* if there are no loops, i.e., $(v, v) \notin E$ for all $v \in V$, and for any distinct vertices $u, v \in V$ there can be at most one edge directed from u to v. Note that we do allow two edges between u and v, but the edges must have opposite orientations. For notational convenience, we distinguish an undirected graph, G, from a directed graph, $\overrightarrow{G}$, with an arrow. Throughout this paper, we will refer to the *underlying undirected graph* of a digraph $\overrightarrow{G}$ by which we mean the simple undirected graph obtained by removing the orientation from the edges of $\overrightarrow{G}$ and combining any double edges into single edges. By a proper coloring of a digraph, we mean a proper coloring of the underlying undirected graph.

Definition 5.1.1. Let $\overrightarrow{G}$ be a directed graph and let $C(\overrightarrow{G})$ be the set of proper colorings of $\overrightarrow{G}$. For a proper coloring $\kappa \in C(\overrightarrow{G})$, we define the number of ascents of κ as $\operatorname{asc}(\kappa) = |\{(i, j) \in E \mid \kappa(i) < \kappa(j)\}|$, i.e., the number of ascents of κ is the number of edges of $\overrightarrow{C_n}$ directed from a smaller color to a bigger color. Then the *chromatic quasisymmetric function of a directed graph* $\overrightarrow{G}$ is

$$X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{\kappa \in C(\overrightarrow{G})} t^{\operatorname{asc}(\kappa)} \mathbf{x}_\kappa,$$

where $\mathbf{x}_\kappa = \prod_{v \in V} x_\kappa$.

We see that setting $t = 1$ in $X_{\overrightarrow{G}}(\mathbf{x}, t)$ gives Stanley's chromatic symmetric function $X_G(\mathbf{x})$ of the underlying undirected graph G of $\overrightarrow{G}$. If we take a labeled graph $G = ([n], E)$, we can create a digraph $\overrightarrow{G} = ([n], E)$ by orienting each edge from smaller label to larger label. Below we repeat our earlier example of this process.



Then we see that the Shareshian-Wachs definition of the chromatic quasisymmetric function of G is the same as our definition of the chromatic quasisymmetric function of $\overrightarrow{G}$. Every acyclic digraph can be obtained in this manner, so the Shareshian-Wachs definition is the same as our definition when we restrict ourselves to acyclic digraphs.

Both of the following propositions follow easily from the definition of the chromatic quasisymmetric function of a digraph.

Proposition 5.1.2. *For any digraph $\overrightarrow{G} = (V, E)$ with $|V| = n$, we have $X_{\overrightarrow{G}}(\mathbf{x}, t) \in$ $\mathrm{QSym}^n_{\mathbb{Q}}[t]$.*

Proposition 5.1.3. *Let $\overrightarrow{G}$ and $\overrightarrow{H}$ be digraphs on disjoint vertex sets and let $\overrightarrow{G+H}$ denote the graph formed by the disjoint union of $\overrightarrow{G}$ and $\overrightarrow{H}$. Then $X_{\overrightarrow{G+H}}(\mathbf{x}, t) =$ $X_{\overrightarrow{G}}(\mathbf{x}, t)X_{\overrightarrow{H}}(\mathbf{x}, t)$.*

Now let us look at a few examples of digraphs and their corresponding chromatic quasisymmetric functions.

Example 5.1.4. For any digraph, $\overrightarrow{G}$, on whose underlying undirected graph is the complete graph, K_n, it is easy to see that $X_{\overrightarrow{G}}(\mathbf{x}, t) = p(t)e_n(\mathbf{x})$, where $p(t) = \sum_\kappa t^{\mathrm{asc}(\kappa)}$ and κ varies over all proper colorings of $\overrightarrow{G}$ using only the colors in $[n]$. From this we can see that $X_{\overrightarrow{G}}(\mathbf{x}, t)$ is e-positive. Specifically if $\overrightarrow{G}$ is acyclic, then $p(t) = [n]_t!$, where $[n]_t = 1 + t + \cdots + t^{n-1}$ and $[n]_t! = [n]_t[n-1]_t \cdots [1]_t$ (see [52, Example 2.4]). By Proposition 5.5.6 if $\overrightarrow{G}$ contains all pairs of double edges, $p(t) = n!t^{\binom{n}{2}}$.

Example 5.1.5. Let $\overrightarrow{P_n} = (V, E)$ denote the *directed path* on n vertices with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\}$. From the work of Shareshian and Wachs [54, Theorem 7.2](see Theorem 4.2.3 in this thesis) on the labeled path graph, we know

$$\sum_{n\geq 0} X_{\overrightarrow{P_n}}(\mathbf{x}, t) z^n = \frac{\sum_{k\geq 0} e_k(\mathbf{x}) z^k}{1 - t\sum_{k\geq 2} [k-1]_t e_k(\mathbf{x}) z^k}, \tag{5.1}$$

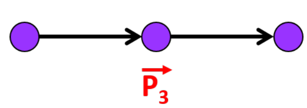
which refines Stanley's formula for the chromatic symmetric function of the undirected path, P_n [58, Proposition 5.3] (see Proposition 3.2.5 in this thesis). From this formula, we can see that $X_{\overrightarrow{P_n}}(\mathbf{x}, t)$ is symmetric, e-positive, and e-unimodal [52, Corollary C.5].

Example 5.1.6. Let us define the *directed cycle* on n vertices, denoted $\overrightarrow{C_n} = (V, E)$, as the digraph with vertex set $V = \{v_1, v_2, \cdots, v_n\}$ and edge set $E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\} \cup \{(v_n, v_1)\}$. In Theorem 5.5.2, we show that

$$\sum_{n\geq 2} X_{\overrightarrow{C_n}}(\mathbf{x}, t) z^n = \frac{t\sum_{k\geq 2} k[k-1]_t e_k(\mathbf{x}) z^k}{1 - t\sum_{k\geq 2} [k-1]_t e_k(\mathbf{x}) z^k}, \tag{5.2}$$

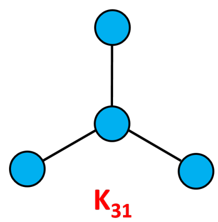
and hence $X_{\overrightarrow{C_n}}(\mathbf{x}, t)$ is symmetric. In fact, in Corollary 5.5.4, we show that the coefficients are e-positive and e-unimodal. Equation (5.2) is a t-analog of Stanley's formula for the chromatic symmetric function of the undirected cycle, C_n [58, Proposition 5.4](see Proposition 3.2.6 in this thesis).

Unfortunately, not every orientation of a given graph has a symmetric chromatic quasisymmetric function. The smallest example of this is the path on 3 vertices. The orientations of the path are given by



There is only one orientation ($\overrightarrow{P_3}$) with a symmetric chromatic quasisymmetric function. The other two orientations ($\overrightarrow{K_{12}}$ and $\overrightarrow{K_{21}}$) do not have symmetric chromatic quasisymmetric functions. See Example 5.2.3.

On the other hand, there are also graphs that do not admit any orientation whose associated chromatic quasisymmetric function is symmetric. The graph K_{31} is given by



None of the orientations of K_{31} have chromatic quasisymmetric functions that are symmetric.

Let $\rho : \mathrm{QSym}_{\mathbb{Q}} \to \mathrm{QSym}_{\mathbb{Q}}$ be the involution defined on the monomial quasisymmetric function basis, M_α, by $\rho(M_\alpha) = M_{\alpha^{rev}}$ for each composition α, where α^{rev} is the reverse of α, i.e., if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ then $\alpha^{rev} = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$. Note that every symmetric function is fixed by ρ. We can extend ρ to $\mathrm{QSym}_{\mathbb{Z}}[t]$ by linearity. Then the next propositions follow easily from [52, Proposition 2.6, Corollary 2.7, Corollary 2.8]. Note in [52], Shareshian and Wachs prove these statements for labeled graphs; however, the same proof works for digraphs.

Proposition 5.1.7. *Let $\overrightarrow{G} = (V, E)$ be a digraph on n vertices. Then*

$$\rho(X_{\overrightarrow{G}}(\mathbf{x}, t)) = \sum_{\kappa \in \kappa(\overrightarrow{G})} t^{\mathrm{des}(\kappa)} \mathbf{x}_\kappa,$$

where $\mathrm{des}(\kappa)$ *is the number of directed edges $(u, v) \in E$ such that $\kappa(u) > \kappa(v)$.*

Hence if $X_{\overrightarrow{G}}(\mathbf{x},t)$ *is symmetric, then*

$$X_{\overrightarrow{G}}(\mathbf{x},t) = \sum_{\kappa\in\kappa(\overrightarrow{G})} t^{\operatorname{des}(\kappa)}\mathbf{x}_\kappa.$$

Proposition 5.1.8. *For a digraph* $\overrightarrow{G} = (V,E)$*, if* $X_{\overrightarrow{G}}(\mathbf{x},t)$ *is symmetric, then* $X_{\overrightarrow{G}}(\mathbf{x},t)$ *is palindromic in* t *with center of symmetry* $\frac{|E|}{2}$.

5.2 Expansion in Gessel's fundamental quasisymmetric function basis

Shareshian and Wachs gave an expansion for $\omega X_G(\mathbf{x},t)$ in terms of Gessel's fundamental quasisymmetric basis when G is the incomparability graph of a poset P. The $t=1$ case of their formula for chromatic symmetric functions of incomparability graphs was proved by Chow [13, Corollary 2].

To describe their expansion, we first need a couple of definitions. Recall that $\mathfrak{S}_n$ is the group of permutations of $[n]$. Let P be a poset on $[n]$ and let $\sigma\in\mathfrak{S}_n$. We can define the set of P-descents of σ as

$$\mathrm{DES}_P(\sigma) = \{i \mid \sigma_i >_P \sigma_{i+1}\}.$$

Note that if P is the total order on $[n]$, then P-descents are just the usual descents of a permutation, as defined in Section 2.1.

Now let $G=([n],E)$ be a labeled graph and let $\sigma\in\mathfrak{S}_n$. We can define the number of G-inversions of σ as

$$\operatorname{inv}_G(\sigma) = |\{\{\sigma_i,\sigma_j\}\in E \mid i<j \text{ and } \sigma_i>\sigma_j\}|.$$

If G is the complete graph on $[n]$, i.e., the graph with all possible edges, then G-inversions are the usual inversions of a permutation, as defined in Section 2.1.

Theorem 5.2.1 (Shareshian-Wachs [52, Theorem 3.1], Chow (t=1) [13, Corollary 2])**.** *Let P be a poset on $[n]$ and let $G = ([n], E)$ be the incomparability graph of P. Then*

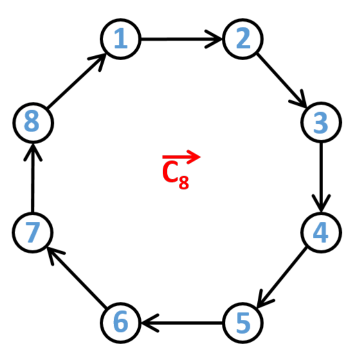
$$\omega X(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_G(\sigma)} F_{n,\mathrm{DES}_P(\sigma)}(\mathbf{x}). \tag{5.3}$$

Consequently $\omega X_G(\mathbf{x}, t)$ is F-positive.

In this section, we present an F-basis expansion of $\omega X_{\overrightarrow{G}}(\mathbf{x}, t)$ for *all* digraphs, which shows that $\omega X_{\overrightarrow{G}}(\mathbf{x}, t)$ is F-positive for *all* digraphs. In general our formula does not reduce to the formula of Shareshian and Wachs, so this gives another combinatorial description of the coefficients in the F-expansion for incomparability graphs of posets.

We may assume without loss of generality that the vertex set of a digraph $\overrightarrow{G}$ is $[n]$. The labeling chosen does not affect the chromatic quasisymmetric function of $\overrightarrow{G}$, as it would for the chromatic quasisymmetric function of a labeled graph defined by Shareshian and Wachs.

Let $\overrightarrow{G} = ([n], E)$ be a digraph and let $\sigma \in \mathfrak{S}_n$. Define a $\overrightarrow{G}$*-inversion* of σ as a directed edge (u, v) of $\overrightarrow{G}$ such that $\sigma^{-1}(u) > \sigma^{-1}(v)$, i.e., v precedes u in σ. Notice that a $\overrightarrow{G}$-inversion does not need to be a usual inversion; if $v < u$, then it will not be. Let $\mathrm{inv}_{\overrightarrow{G}}(\sigma)$ be the number of $\overrightarrow{G}$-inversions of σ.



For example, let $\overrightarrow{G}$ be $\overrightarrow{C_8}$, the directed cycle on [8] as shown above, and let $\sigma = 25413786 \in \mathfrak{S}_8$. The $\overrightarrow{C_8}$-inversions of σ are $(1,2)$, $(3,4)$, $(4,5)$, and $(6,7)$, so $\mathrm{inv}_{\overrightarrow{C_8}}(\sigma) = 4$.

The notion of a $\overrightarrow{G}$-inversion of a permutation is an easy modification of the idea of a G-inversion of a permutation defined by Shareshian and Wachs. On the other hand since we do not generally work with incomparability graphs of posets, modifying the idea of a P-descent of a permutation is a bit trickier. We do so by defining the G-descents of a permutation for any labeled graph $G = ([n], E)$.

Now let $G = ([n], E)$ be an undirected graph and let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$. For each $x \in [n]$, define the (G,σ)-*rank* of x, denoted $\mathrm{rank}_{(G,\sigma)}(x)$, as the length of the longest subword $\sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_k}$ of σ such that $\sigma_{i_k} = x$ and for each $1 \le j < k$, $\{\sigma_{i_j}, \sigma_{i_{j+1}}\} \in E$. We say σ has a G-*descent* at i with $1 \le i < n$ if either of the following conditions holds:

- $\mathrm{rank}_{(G,\sigma)}(\sigma_i) > \mathrm{rank}_{(G,\sigma)}(\sigma_{i+1})$
- $\mathrm{rank}_{(G,\sigma)}(\sigma_i) = \mathrm{rank}_{(G,\sigma)}(\sigma_{i+1})$ and $\sigma_i > \sigma_{i+1}$.

Let $\mathrm{DES}_G(\sigma)$ be the set of G-descents of σ.

For example, let $G = C_8$ be the cycle on 8 vertices labeled with [8] in cyclic order. In other words, C_8 is the underlying undirected graph of the directed cycle $\overrightarrow{C_8}$, pictured above. Let $\sigma = 25413786 \in \mathfrak{S}_8$. By attaching the (G,σ)-rank of each letter as a superscript, we get $2^1 5^1 4^2 1^2 3^3 7^1 8^3 6^2$. We can see from this that $\mathrm{DES}_G(\sigma) = \{3, 5, 7\}$.

Theorem 5.2.2. *Let $\overrightarrow{G} = ([n], E)$ be a digraph. Then*

$$\omega X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{\overrightarrow{G}}(\sigma)} F_{n,\mathrm{DES}_G(\sigma)}(\mathbf{x}), \tag{5.4}$$

where $F_{n,S}(\mathbf{x})$ is Gessel's fundamental quasisymmetric function and G is the underlying undirected graph of $\overrightarrow{G}$.

Proof. The first half of this proof closely follows the proof of [52, Theorem 3.1]. The second part of the proof is quite different. Let G be the underlying undirected graph of $\overrightarrow{G}$ and let $G_{\bar{a}}$ be an acyclic orientation of the graph G, possibly different from the given orientation on $\overrightarrow{G}$. Then define $\text{asc}(G_{\bar{a}})$ to be the number of edges of $G_{\bar{a}}$ whose orientation matches the orientation of $\overrightarrow{G}$. For each acyclic orientation $G_{\bar{a}}$, we will let $C(G_{\bar{a}})$ be the set of proper colorings, κ, of G such that if (i, j) is a directed edge of $G_{\bar{a}}$, then $\kappa(i) < \kappa(j)$. It is clear that

$$X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{G_{\bar{a}} \in AO(G)} t^{\text{asc}(G_{\bar{a}})} \sum_{\kappa \in C(G_{\bar{a}})} \mathbf{x}_{\kappa}, \tag{5.5}$$

where $AO(G)$ is the set of acyclic orientations of G.

From each acyclic orientation, $G_{\bar{a}}$, we can create a poset, $P_{\bar{a}}$, on $[n]$ by letting $i <_{P_{\bar{a}}} j$ if there is an edge from i to j in $G_{\bar{a}}$ and extending transitively. We define a labeling of $P_{\bar{a}}$ as a bijection from $P_{\bar{a}}$ to $[n]$. Now we give $P_{\bar{a}}$ a decreasing labeling $w_{\bar{a}} : P_{\bar{a}} \to [n]$, i.e., if $x <_{P_{\bar{a}}} y$, then $w_{\bar{a}}(x) > w_{\bar{a}}(y)$. Let $L(P_{\bar{a}}, w_{\bar{a}})$ be the set of linear extensions of $P_{\bar{a}}$ with the labeling $w_{\bar{a}}$. For any subset $S \subseteq [n-1]$, define $n - S = \{i \mid n - i \in S\}$. Then by the theory of P-partitions (see [60, Chapter 7] for a reference), we have

$$\sum_{\kappa \in C(G_{\bar{a}})} \mathbf{x}_{\kappa} = \sum_{\sigma \in L(P_{\bar{a}}, w_{\bar{a}})} F_{n, n-\text{DES}(\sigma)}(\mathbf{x}), \tag{5.6}$$

where $\text{DES}(\sigma)$ is the usual descent set of a permutation, i.e., $\text{DES}(\sigma) = \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}$.

Let $e : P_{\bar{a}} \to [n]$ be the identity map, i.e., the map that takes each element of $P_{\bar{a}}$ to its original label. Then $L(P_{\bar{a}}, e)$ is the set of linear extensions of $P_{\bar{a}}$ with its original labeling, e. For $\sigma \in L(P_{\bar{a}}, e)$, let $w_{\bar{a}}\sigma$ denote the product of $w_{\bar{a}}$ and σ in $\mathfrak{S}_n$.

For $\sigma \in \mathfrak{S}_n$, we have $w_{\bar{a}}\sigma \in L(P_{\bar{a}}, w_{\bar{a}})$ if and only if $\sigma \in L(P_{\bar{a}}, e)$. So (5.6) becomes

$$\sum_{\kappa \in C(G_{\bar{a}})} \mathbf{x}_\kappa = \sum_{\sigma \in L(P_{\bar{a}}, e)} F_{n, n-\mathrm{DES}(w_{\bar{a}}\sigma)}(\mathbf{x}).$$

Combining this with (5.5) gives us that

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{G_{\bar{a}} \in AO(G)} t^{\mathrm{asc}(G_{\bar{a}})} \sum_{\sigma \in L(P_{\bar{a}}, e)} F_{n, n-\mathrm{DES}(w_{\bar{a}}\sigma)}(\mathbf{x}).$$

Since each $\sigma \in \mathfrak{S}_n$ is a linear extension of a unique acyclic orientation, $G_{\bar{a}(\sigma)}$, of G, we can rewrite this as

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{asc}(G_{\bar{a}(\sigma)})} F_{n, n-\mathrm{DES}(w_{\bar{a}(\sigma)}\sigma)}(\mathbf{x}),$$

where $w_{\bar{a}(\sigma)}$ is a decreasing labeling of $P_{\bar{a}(\sigma)}$.

For $\sigma \in \mathfrak{S}_n$, let $\mathrm{ASC}(\sigma)$ denote the usual ascent set of a permutation, i.e., $\mathrm{ASC}(\sigma) = \{i \in [n-1] \mid \sigma(i) < \sigma(i+1)\}$. Also define $\sigma^{\mathrm{rev}} \in \mathfrak{S}_n$ by letting $\sigma^{\mathrm{rev}}(i) = \sigma(n+1-i)$ for all i. It is not hard to see that $\mathrm{asc}(G_{\bar{a}(\sigma)}) = \mathrm{inv}_{\vec{G}}(\sigma^{\mathrm{rev}})$ and $n - \mathrm{DES}(w_{\bar{a}(\sigma)}\sigma) = \mathrm{ASC}((w_{\bar{a}(\sigma)}\sigma)^{\mathrm{rev}})$, so we have

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{\vec{G}}(\sigma^{\mathrm{rev}})} F_{n, \mathrm{ASC}((w_{\bar{a}(\sigma)}\sigma)^{\mathrm{rev}})}(\mathbf{x}).$$

Then by reversing σ and letting $\tilde{w}_{\bar{a}(\sigma)}$ be an increasing labeling of $P_{\bar{a}(\sigma)}$, we have that

$$X_{\vec{G}}(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{\vec{G}}(\sigma)} F_{n, \mathrm{ASC}(\tilde{w}_{\bar{a}(\sigma)}\sigma)}(\mathbf{x}).$$

Finally applying the involution ω to both sides of the equation gives us

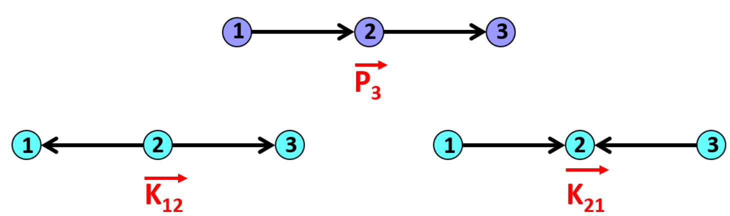
$$\omega X_{\vec{G}}(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{\vec{G}}(\sigma)} F_{n, \mathrm{DES}(\tilde{w}_{\bar{a}(\sigma)}\sigma)}(\mathbf{x}).$$

Up to this point, $\tilde{w}_{\bar{a}(\sigma)}$ has been any increasing labeling of $P_{\bar{a}(\sigma)}$, but now we will make it a specific one. Note that we will refer to the original labeling of the vertices of G as the G-labeling of the graph. For each acyclic orientation $G_{\bar{a}}$ and for each vertex v, define $\mathrm{rank}_{\bar{a}}(v)$ as the length of the longest chain of $P_{\bar{a}}$ from a minimal element of $P_{\bar{a}}$ to v. We say v is a $\mathrm{rank}_{\bar{a}}$ i element if $\mathrm{rank}_{\bar{a}}(v) = i$. To determine the labeling $\tilde{w}_{\bar{a}(\sigma)}$, first we label the $\mathrm{rank}_{\bar{a}(\sigma)}$ 0 element with the smallest G-label as 1. Then we label the $\mathrm{rank}_{\bar{a}(\sigma)}$ 0 element with the next smallest G-label as 2. We continue this process until all $\mathrm{rank}_{\bar{a}(\sigma)}$ 0 elements are labeled. Then we repeat this process with the $\mathrm{rank}_{\bar{a}(\sigma)}$ 1 elements and continue inductively until all elements are labeled.

Notice that for all $x \in [n]$, we have $\mathrm{rank}_{(G,\sigma)}(x) = \mathrm{rank}_{\bar{a}(\sigma)}(x) + 1$. So using the labeling $\tilde{w}_{\bar{a}(\sigma)}$ constructed above, if i is a descent of $\tilde{w}_{\bar{a}(\sigma)}\sigma$, then $\sigma(i+1)$ was labeled before $\sigma(i)$ in the labeling $\tilde{w}_{\bar{a}(\sigma)}$. Then either $\sigma(i+1)$ has a smaller $\bar{a}(\sigma)$-rank than $\sigma(i)$ or they have the same $\bar{a}(\sigma)$-rank and $\sigma(i) > \sigma(i+1)$. But in either case, i is also a G-descent of σ. A similar argument shows that if i is a G-descent of σ, then i is also a descent of $\tilde{w}_{\bar{a}(\sigma)}\sigma$. Hence $\mathrm{DES}(\tilde{w}_{\bar{a}(\sigma)}\sigma) = \mathrm{DES}_G(\sigma)$, and the theorem is proven. $\square$

Note that our formula requires that $\overrightarrow{G}$ be labeled with $[n]$. Each labeling of the vertices of $\overrightarrow{G}$ gives a distinct combinatorial description of the coefficients in the F-basis.

Example 5.2.3. Let us give labelings to the digraphs $\overrightarrow{P_3}$, $\overrightarrow{K_{12}}$, and $\overrightarrow{K_{21}}$ mentioned in Section 5.1 as follows:



To expand the chromatic quasisymmetric function of each of these in the F-basis, we need to calculate $\mathrm{DES}_G(\sigma)$ and $\mathrm{inv}_{\overrightarrow{G}}(\sigma)$ for each $\sigma \in \mathfrak{S}_3$. Since $\mathrm{DES}_G(\sigma)$ only depends on the underlying undirected graph, this will be the same for each of $\overrightarrow{P_3}$, $\overrightarrow{K_{12}}$, and $\overrightarrow{K_{21}}$. The calculations are shown in the chart below.

σ	$\mathrm{DES}_G(\sigma)$	$\mathrm{inv}_{\overrightarrow{P_3}}(\sigma)$	$\mathrm{inv}_{\overrightarrow{K_{12}}}(\sigma)$	$\mathrm{inv}_{\overrightarrow{K_{21}}}(\sigma)$
123	$\emptyset$	0	1	1
132	$\emptyset$	1	2	0
213	$\emptyset$	1	0	2
231	$\{2\}$	1	0	2
312	$\{1\}$	1	2	0
321	$\emptyset$	2	1	1

Using this, we have that

$$\omega X_{\overrightarrow{P_3}}(\mathbf{x},t) = (F_{3,\emptyset}(\mathbf{x})) + t(2F_{3,\emptyset}(\mathbf{x}) + F_{3,\{1\}}(\mathbf{x}) + F_{3,\{2\}}(\mathbf{x})) + t^2(F_{3,\emptyset}(\mathbf{x})) \tag{5.7}$$

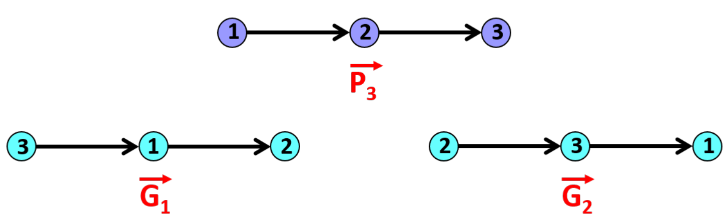
$$= h_3(\mathbf{x}) + t(h_{21}(\mathbf{x}) + h_3(\mathbf{x})) + t^2 h_3(\mathbf{x}) \tag{5.8}$$

$$\omega X_{\overrightarrow{K_{12}}}(\mathbf{x},t) = (F_{3,\emptyset}(\mathbf{x}) + F_{3,\{2\}}(\mathbf{x})) + t(2F_{3,\emptyset}(\mathbf{x})) + t^2(F_{3,\emptyset}(\mathbf{x}) + F_{3,\{1\}}(\mathbf{x})), \tag{5.9}$$

$$\omega X_{\overrightarrow{K_{21}}}(\mathbf{x},t) = (F_{3,\emptyset}(\mathbf{x}) + F_{3,\{1\}}(\mathbf{x})) + t(2F_{3,\emptyset}(\mathbf{x})) + t^2(F_{3,\emptyset}(\mathbf{x}) + F_{3,\{2\}}(\mathbf{x})). \tag{5.10}$$

From these expansions, one can see that $X_{\overrightarrow{P_3}}(\mathbf{x},t)$ is symmetric and palindromic, but $X_{\overrightarrow{K_{12}}}(\mathbf{x},t)$ and $X_{\overrightarrow{K_{21}}}(\mathbf{x},t)$ are neither.

Example 5.2.4. Now let us look at an example of a the digraph P_3 with 3 different labelings, shown below, and calculate the F-basis expansion of $\omega X_{P_3}(\mathbf{x},t)$ in 3 different ways.



The G-descents and $\overrightarrow{G}$-inversions for each of these graphs is shown in the chart below

σ	$\mathrm{DES}_{P_3}(\sigma)$	$\mathrm{DES}_{G_1}(\sigma)$	$\mathrm{DES}_{G_2}(\sigma)$	$\mathrm{inv}_{\overrightarrow{P_3}}(\sigma)$	$\mathrm{inv}_{\overrightarrow{G_1}}(\sigma)$	$\mathrm{inv}_{\overrightarrow{G_2}}(\sigma)$
123	$\emptyset$	$\emptyset$	$\emptyset$	0	1	1
132	$\emptyset$	$\{2\}$	$\emptyset$	1	1	2
213	$\emptyset$	$\emptyset$	$\{1\}$	1	2	1
231	$\{2\}$	$\emptyset$	$\emptyset$	1	1	0
312	$\{1\}$	$\emptyset$	$\emptyset$	1	0	1
321	$\emptyset$	$\{1\}$	$\{2\}$	2	1	1

In all three cases we get that

$$\omega X_{\overrightarrow{P_3}}(\mathbf{x},t) = (F_{3,\emptyset}(\mathbf{x})) + t(2F_{3,\emptyset}(\mathbf{x}) + F_{3,\{1\}}(\mathbf{x}) + F_{3,\{2\}}(\mathbf{x})) + t^2(F_{3,\emptyset}(\mathbf{x})).$$

By specializing (5.4), we obtain a t-analog of a result of Chung and Graham [15, Theorem 2] on the chromatic polynomial of a graph, which we became aware of after obtaining our results. Let us define the *t-analog of the chromatic polynomial of a digraph* $\overrightarrow{G}$ as

$$\chi_{\overrightarrow{G}}(m,t) = \sum_{\kappa \in \kappa_m(\overrightarrow{G})} t^{\mathrm{asc}(\kappa)},$$

where $\kappa_m(\overrightarrow{G})$ is the set of proper colorings of $\overrightarrow{G}$ using only colors in $[m]$. From the definition, we can see that for any $m \in \mathbb{P}$,

$$\chi_{\overrightarrow{G}}(m,t) = X_{\overrightarrow{G}}(1^m,t).$$

Also, if we set $t = 1$, we see that

$$\chi_{\overrightarrow{G}}(m,1) = \chi_G(m),$$

where G is the underlying undirected graph of $\overrightarrow{G}$ and $\chi_G(m)$ is the chromatic polynomial.

For $k, l \in \mathbb{P}$ and a digraph $\overrightarrow{G} = ([n], E)$ with underlying undirected graph, G, let $\delta_{\overrightarrow{G}}(k, l)$ denote the number of permutations $\sigma \in \mathfrak{S}_n$ such that $|\operatorname{DES}_G(\sigma)| = k$ and $\operatorname{inv}_{\overrightarrow{G}}(\sigma) = l$.

Corollary 5.2.5 (t=1 case [15, Theorem 2])**.** *Let $\overrightarrow{G} = ([n], E)$ be a digraph on n vertices. Then*

$$\chi_{\overrightarrow{G}}(m, t) = \sum_{k,l \geq 0} \delta_{\overrightarrow{G}}(k, l)\tbinom{m+k}{n} t^l.$$

Consequently[1]*, this is a polynomial in m whose coefficients are palindromic polynomials in t.*

Proof. We know that for any $S \subseteq [n-1]$ with $|S| = k$, we have $F_{n,S}(1^m) = \binom{m+n-1-k}{n}$ (see [60, Section 7.19]). Applying ω to both sides of (5.4) gives us that $X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{\sigma \in \mathfrak{S}_n} F_{n,[n-1]\setminus \operatorname{DES}_G(\sigma)}(\mathbf{x}) t^{\operatorname{inv}_{\overrightarrow{G}}(\sigma)}$. Then we have

$$\begin{aligned}
\chi_{\overrightarrow{G}}(m, t) &= X_{\overrightarrow{G}}(1^m, t) \\
&= \sum_{\sigma \in \mathfrak{S}_n} F_{n,[n-1]\setminus \operatorname{DES}_G(\sigma)}(1^m) t^{\operatorname{inv}_{\overrightarrow{G}}(\sigma)} \\
&= \sum_{\sigma \in \mathfrak{S}_n} \tbinom{m+|\operatorname{DES}_G(\sigma)|}{n} t^{\operatorname{inv}_{\overrightarrow{G}}(\sigma)} \\
&= \sum_{k,l \geq 0} \delta_{\overrightarrow{G}}(k, l)\tbinom{m+k}{n} t^l.
\end{aligned}$$

□

[1] Athanasiadis [3] observed this when $G = \operatorname{Inc}(P)$.

Example 5.2.6. Let us compute $\chi_{\overrightarrow{P_3}}(m,t)$ using our calculations of $\mathrm{DES}_{P_3}(\sigma)$ and $\mathrm{inv}_{\overrightarrow{P_3}}(\sigma)$ from Example 5.2.3.

$$\begin{aligned}\chi_{\overrightarrow{P_3}}(m,t) &= \binom{m}{3} + 2t\binom{m}{3} + 2t\binom{m+1}{3} + t^2\binom{m}{3} \\ &= \frac{m^3}{6}(1+4t+t^2) - \frac{m^2}{2}(1+2t+t^2) + \frac{m}{3}(1+t+t^2)\end{aligned}$$

We see that the coefficient of each power of m is a palindromic polynomial in t. If we set $t=1$, we get

$$\begin{aligned}\chi_{\overrightarrow{P_3}}(m,1) &= m^3 - 2m^2 + m \\ &= m(m-1)^2,\end{aligned}$$

which is the chromatic polynomial $\chi_{P_3}(m)$ of P_3.

5.3 Expansion in the power sum symmetric function basis

In [58], Stanley shows that for *any* graph G, the symmetric function $\omega X_G(\mathbf{x})$ is p-positive (see Theorem 3.2.2). Since not every graph has a symmetric chromatic quasisymmetric function, here we restrict ourselves to graphs that do. In this section, we establish p-positivity for all symmetric $\omega X_{\overrightarrow{G}}(\mathbf{x},t)$ by deriving a p-expansion formula. In Section 5.4, we introduce a class of digraphs with symmetric chromatic quasisymmetric functions, which includes natural unit interval graphs as well as the directed cycle, thereby extending the symmetry result of Shareshian and Wachs. Our p-expansion formula does not reduce to the Shareshian-Wachs-Athanasiadis formula [52] [4] for natural unit interval graphs mentioned in the introduction. It reduces to a new formula.

Before we discuss our p-basis expansion, we would like to review the p-basis expansion of Athanasiadis, Shareshian, and Wachs. In addition, we would like to call attention to an interesting and useful result of Adin and Roichman [1] that Athanasiadis [4] used to prove the p-basis expansion for chromatic quasisymmetric functions of natural unit interval graphs.

5.3.1 From the fundamental quasisymmetric function basis to the power sum symmetric function basis

Let $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, ..., \lambda_l)$. Define $s_i = \lambda_1 + \lambda_2 + ... + \lambda_i$, for $1 \leq i \leq l$ and $s_0 = 0$. A set $A \subseteq [n-1]$ is λ-*unimodal* if for $0 \leq i < l$, the intersection of A with each set of the form $\{s_i + 1, ..., s_{i+1} - 1\}$ is either the empty set or a prefix of the latter. Additionally, define $S(\lambda) = \{s_1, s_2, \cdots, s_{l-1}\}$.

Example 5.3.1. Let $\lambda = (5, 3, 3, 2, 1) \vdash 14$. Then $S(\lambda) = \{5, 8, 11, 13\}$. Let A be the set $A = \{1, 2, 3, 6, 11\}$. Let us check the following intersections to see if A is λ-unimodal:

$$A \cap \{1, 2, 3, 4\} = \{1, 2, 3\},$$

$$A \cap \{6, 7\} = \{6\},$$

$$A \cap \{9, 10\} = \emptyset,$$

$$A \cap \{12\} = \emptyset.$$

We see that A is λ-unimodal, because $\{1, 2, 3\}$ is a prefix of $\{1, 2, 3, 4\}$ and $\{6\}$ is a prefix of $\{7\}$. On the other hand if we let $B = \{1, 2, 4, 6, 11\}$ then B is not λ-unimodal, because $B \cap \{1, 2, 3, 4\} = \{1, 2, 4\}$ is not a prefix of $\{1, 2, 3, 4\}$.

For each $\lambda \vdash n$, let

$$z_\lambda := \prod_i m_i(\lambda)! i^{m_i(\lambda)},$$

where $m_i(\lambda)$ is the multiplicity of i in λ for each i, i.e., the number of parts of λ equal to i. The following result is implicit in the work of Adin and Roichman [1, Theorem 3.6], and stated explicitly and proved by Athanasiadis.

Proposition 5.3.2 (Athansiadis [4, Proposition 3.2])**.** *Let $X(\mathbf{x}) \in \Lambda_R^n$ be a homogeneous symmetric function of degree n over a commutative $\mathbb{Q}$-algebra R and suppose that*

$$X(\mathbf{x}) = \sum_{S \subseteq [n-1]} a_S F_{n,S}(\mathbf{x})$$

for some $a_S \in R$. Then

$$X(\mathbf{x}) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda(\mathbf{x}) \sum_{S \in U_\lambda} (-1)^{|S \setminus S(\lambda)|} a_S,$$

where U_λ is the set of λ-unimodal subsets of $[n-1]$.

Example 5.3.3. Recall from Example 5.2.3 that $\omega X_{\overrightarrow{P_3}}(\mathbf{x}, t)$ is symmetric and

$$\omega X_{\overrightarrow{P_3}}(\mathbf{x}, t) = (1 + 2t + t^2) F_{3,\emptyset}(\mathbf{x}) + 2t F_{3,\{1\}}(\mathbf{x}) + 2t F_{3,\{2\}}(\mathbf{x}).$$

So we can find the p-basis expansion of $\omega X_{\overrightarrow{P_3}}(\mathbf{x}, t)$ using this proposition. Let us first calculate a few useful values for each $\lambda \vdash 3$, shown in the chart below.

λ	$S(\lambda)$	U_λ	z_λ^{-1}
(3)	$\emptyset$	$\{\emptyset, \{1\}, \{1,2\}\}$	$\frac{1}{3}$
(2,1)	$\{2\}$	$\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$	$\frac{1}{2}$
(1,1,1)	$\{1,2\}$	$\{\emptyset, \{1\}, \{2\}, \{1,2\}\}$	$\frac{1}{6}$

Then applying Proposition 5.3.2 to our F-basis expansion of $\omega X_{\overrightarrow{P_3}}(\mathbf{x}, t)$ gives us

$$\begin{aligned}\omega X_{\overrightarrow{P_3}}(\mathbf{x},t) = &\frac{1}{3}p_3(\mathbf{x})[(-1)^{|\emptyset\setminus\emptyset|}(1+2t+t^2)+(-1)^{|\{1\}\setminus\emptyset|}(t)]\\ &+\frac{1}{2}p_{21}(\mathbf{x})[(-1)^{|\emptyset\setminus\{2\}|}(1+2t+t^2)+(-1)^{|\{1\}\setminus\{2\}|}(t)+(-1)^{|\{2\}\setminus\{2\}|}(t)]\\ &+\frac{1}{6}p_{111}(\mathbf{x})[(-1)^{|\emptyset\setminus\{1,2\}|}(1+2t+t^2)+(-1)^{|\{1\}\setminus\{1,2\}|}(t)+(-1)^{|\{2\}\setminus\{1,2\}|}(t)].\end{aligned}$$

Simplifying this gives us

$$\omega X_{\overrightarrow{P_3}}(\mathbf{x},t) = \frac{1}{3}p_3(\mathbf{x})(1+t+t^2)+\frac{1}{2}p_{21}(\mathbf{x})(1+2t+t^2)+\frac{1}{6}p_{111}(\mathbf{x})(1+4t+t^2).$$

Athanasiadis used the F-basis decomposition of Shareshian and Wachs [52](see Theorem 5.2.1 of this thesis) along with Proposition 5.3.2 to prove the p-basis expansion conjecture of Shareshian and Wachs. To state this theorem, we first need a bit of notation.

Let P be a poset on $[n]$ and let $\sigma \in \mathfrak{S}_n$. As defined in the previous chapter, the P-descents of σ are given by $\mathrm{DES}_P(\sigma) = \{i \in [n-1] \mid \sigma_i >_P \sigma_{i+1}\}$. A nontrivial left-to-right P-maximum of a word $w = w_1 w_2 \cdots w_k$ with distinct letters over $[n]$ is a w_j with $j \geq 1$ such that $w_j >_P w_i$ for all $i < j$. Now for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \vdash n$, define $N_{P,\lambda}$ to be the subset of permutations $\sigma \in \mathfrak{S}_n$ such that when σ is cut into contiguous segments $\alpha_1, \alpha_2, \ldots, \alpha_m$ of sizes $\lambda_1, \lambda_2, \ldots, \lambda_m$, each α_i contains no P-descents and no nontrivial left-to-right P-maxima.

Theorem 5.3.4 (Conjectured by Shareshian-Wachs [58], proved by Athanasiadis [4])**.** *Let P be a natural unit interval order on $[n]$, and let $G = ([n], E)$ be the incomparability graph of P. Then*

$$\omega X_G(\mathbf{x},t) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda(\mathbf{x}) \sum_{\sigma \in N_{P,\lambda}} t^{\mathrm{inv}_G(\sigma)},$$

where again $\mathrm{inv}_G(\sigma) = |\{\{\sigma_i, \sigma_j\} \in E \mid i > j, \sigma_i < \sigma_j\}|$. *Consequently* $\omega X_G(\mathbf{x}, t)$ *is* p*-positive when* G *is a natural unit interval graph.*

In the special case that G is the path graph P_n, Shareshian and Wachs obtained a nice factorization of the coefficients in the p-basis, namely

$$\omega X_{P_n}(\mathbf{x}, t) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda(\mathbf{x}) A_{l(\lambda)}(t) \prod_{i=1}^{l(\lambda)} [\lambda_i]_t, \tag{5.11}$$

where $l(\lambda)$ is the length of λ, i.e., the number of parts of λ, and $A_n(t)$ is the Eulerian polynomial.

5.3.2 A power sum symmetric function expansion for directed graphs

Let $G = ([n], E)$ be an undirected labeled graph and let $w = w_1 w_2 \cdots w_k$ be a word with distinct letters in $[n]$. We say w_j with $1 < j \leq k$ is a *G-isolated letter* of w if there is no w_i with $1 \leq i < j$ such that $\{w_i, w_j\} \in E$.

Now for any undirected labeled graph $G = ([n], E)$ and any partition $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, we define $N_\lambda(G)$ as the set of all $\sigma \in \mathfrak{S}_n$ such that when σ is divided up into contiguous segments $\alpha_1, \alpha_2, \cdots, \alpha_l$ of sizes $\lambda_1, \lambda_2, \cdots, \lambda_l$, each α_i has no G-isolated letters and contains no G-descents of σ. Note that the G-descents here are determined by the entire σ and cannot be determined by looking at the α_i's individually.

Let $G = C_8$, the cycle on 8 vertices labeled cyclically with $[8]$ and let $\sigma = 43587162 \in \mathfrak{S}_8$. Then attaching the (G, σ)-rank to each letter gives $4^1 3^2 5^2 8^1 7^2 1^2 6^3 2^3$ and hence $\mathrm{DES}_G(\sigma) = \{3, 5, 7\}$. If $\lambda = (3, 2, 2, 1)$, then $\alpha_1 = 435, \alpha_2 = 87, \alpha_3 = 16, \alpha_4 = 2$. None of the α_i contain any G-descents; however, in α_3, 6 is a G-isolated letter, so $\sigma \notin N_\lambda(G)$. However if $\lambda = (3, 2, 1, 1, 1)$, then $\sigma \in N_\lambda(G)$.

For each $\lambda \vdash n$, let $z_\lambda = \prod_i m_i(\lambda)! i^{m_i(\lambda)}$, where $m_i(\lambda)$ is the multiplicity of i in λ for each i, i.e., the number of parts of λ equal to i.

Theorem 5.3.5. *Let $\overrightarrow{G}$ be a digraph such that $X_{\overrightarrow{G}}(\mathbf{x}, t)$ is symmetric. Then $\omega X_{\overrightarrow{G}}(\mathbf{x}, t)$ is p-positive. In fact,*

$$\omega X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda(\mathbf{x}) \sum_{\sigma \in N_\lambda(G)} t^{\mathrm{inv}_{\overrightarrow{G}}(\sigma)}, \tag{5.12}$$

where G is the underlying undirected graph of $\overrightarrow{G}$.

We would like to point out that the proof by Athanasiadis [4, Theorem 3.1] of Theorem 5.3.4 does not generalize to the directed graph case. He uses the F-basis decomposition for natural unit interval graphs given by Shareshian and Wachs [52, Theorem 3.1] (see Theorem 5.2.1 of this thesis) involving P-descents, Proposition 5.3.2, and a formula for the coefficient of $\frac{1}{n} p_n(\mathbf{x})$ [52, Lemma 7.4]. Although we also use Proposition 5.3.2, our proof involves a sign-reversing involution as well as our F-basis decomposition given in Theorem 5.2.2.

Proof of Theorem 5.3.5. Combining Proposition 5.3.2 with our F-basis expansion (5.4), we have that

$$\omega X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda(\mathbf{x}) \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{DES}_G(\sigma) \in U_\lambda}} (-1)^{|\,\mathrm{DES}_G(\sigma) \setminus S(\lambda)|} t^{\mathrm{inv}_{\overrightarrow{G}}(\sigma)}, \tag{5.13}$$

where U_λ is the set of λ-unimodal sets on $[n-1]$.

For each $\lambda \vdash n$, let us define the set

$$D_\lambda(G) \coloneqq \{\sigma \in \mathfrak{S}_n \mid \mathrm{DES}_G(\sigma) \in U_\lambda\}.$$

Note that $N_\lambda(G) \subseteq D_\lambda(G)$. In order to prove the theorem, we will construct for each $\lambda \vdash n$ a sign-reversing, $\mathrm{inv}_G(\sigma)$-preserving involution, γ_λ, on $D_\lambda(G) \setminus N_\lambda(G)$.

That is $\gamma_\lambda : D_\lambda(G)\backslash N_\lambda(G) \to D_\lambda(G)\backslash N_\lambda(G)$ will satisfy the following for all $\sigma \in D_\lambda(G)\backslash N_\lambda(G)$:

- $\gamma_\lambda^2(\sigma) = \sigma$,
- γ_λ changes $|\operatorname{DES}_G(\sigma)\backslash S(\lambda)|$ by 1, and
- $\operatorname{inv}_{\overrightarrow{G}}(\sigma) = \operatorname{inv}_{\overrightarrow{G}}(\gamma_\lambda(\sigma))$.

Now let us fix some useful notation. Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$. We define a total order, $<_G$, on $[n]$ by $x <_G y$ if $\operatorname{rank}_{(G,\sigma)}(x) < \operatorname{rank}_{(G,\sigma)}(y)$ or if $\operatorname{rank}_{(G,\sigma)}(x) = \operatorname{rank}_{(G,\sigma)}(y)$ and $x < y$. Using this notation, there is a G-descent of σ at i if and only if $\sigma_i >_G \sigma_{i+1}$.

Fix λ and let $\sigma \in D_\lambda(G)\backslash N_\lambda(G)$. Break σ up into contiguous segments of sizes $\lambda_1, \lambda_2, \cdots \lambda_l$ called $\alpha_1, \alpha_2, \cdots \alpha_l$. Then let α_i be the first segment of σ that either has a G-isolated letter or a G-descent. Since $\operatorname{DES}_G(\sigma) \in U_\lambda$, there must exist a unique k such that $s_{i-1}+1 \leq k \leq s_i$ and α_i is of the form $\alpha_i = \sigma_{s_{i-1}+1}\,\sigma_{s_{i-1}+2}\cdots\sigma_{k-1}\,\sigma_k\,\sigma_{k+1}\cdots\sigma_{s_i}$, where $\sigma_{s_{i-1}+1} >_G \sigma_{s_{i-1}+2} >_G \cdots >_G \sigma_{k-1} >_G \sigma_k <_G \sigma_{k+1} <_G \cdots <_G \sigma_{s_i}$.

Define the involution by setting $\gamma_\lambda(\sigma) \coloneqq \alpha_1\alpha_2\cdots\tilde{\alpha}_i\cdots\alpha_l$, where $\tilde{\alpha}_i$ is obtained from α_i by considering the following cases using the k from the previous paragraph:

First define σ_m as the largest G-isolated letter in α_i such that $m > k$, i.e., σ_m is the G-isolated letter with the largest label that appears after σ_k. If there are no G-isolated letters after σ_k, then define $\sigma_m = 0$.

Case 1: $\sigma_m \neq 0$ and $\sigma_m > \sigma_{s_{i-1}+1}$.

Obtain $\tilde{\alpha}_i$ by moving σ_m before $\sigma_{s_{i-1}+1}$. Since σ_m is a G-isolated letter and thus is not connected to any letter before it in α_i, this rearrangement will not affect the number of $\overrightarrow{G}$-inversions, but it will create one new G-descent between σ_m and $\sigma_{s_{i-1}+1}$. Notice that $\gamma_\lambda(\sigma) \in D_\lambda(G)\backslash N_\lambda(G)$.

Case 2a: $\sigma_m = 0$.

Case 2b: $\sigma_m \neq 0$ and $\sigma_m < \sigma_{s_{i-1}+1}$.

In both cases, obtain $\tilde{\alpha_i}$ by moving $\sigma_{s_{i-1}+1}$ to the first spot after σ_k that will not create a new G-descent. We see that this reduces the number of G-descents by 1. Now notice that wherever we finally place $\sigma_{s_{i-1}+1}$, all the σ_j that come before this position must satisfy $\sigma_j <_G \sigma_{s_{i-1}+1}$. It follows that there is no edge between $\sigma_{s_{i-1}+1}$ and σ_j since if there were then we would have $\mathrm{rank}_{(G,\sigma)}(\sigma_j) > \mathrm{rank}_{(G,\sigma)}(\sigma_{s_{i-1}+1})$. Hence this rearrangement does not affect the number of $\overrightarrow{G}$-inversions. Notice that $\gamma_\lambda(\sigma) \in D_\lambda(G)\backslash N_\lambda(G)$.

Now notice that we have covered all cases and these cases are mutually exclusive. We leave it to the reader to check that Case 1 and Case 2 will reverse each other, so this is the involution we were looking for.

Then the only elements of $D_\lambda(G)$ that remain in (5.13) are those of $N_\lambda(G)$. Since these permutations have all their G-descents in $S(\lambda)$ by definition, the theorem is proven. $\square$

In [52, Proposition 7.8], Shareshian and Wachs showed that when G is a natural unit interval graph, the coefficient of each $z_\lambda^{-1}p_\lambda(\mathbf{x})$ in $\omega X_G(\mathbf{x},t)$ factors. Though the coefficients do not generally factor in the digraph case, we show in Theorem 5.3.7 below that the coefficient of each $z_\lambda^{-1}p_\lambda(\mathbf{x})$ in $\omega X_{\overrightarrow{G}}(\mathbf{x},t)$ does have a nice factorization involving the Eulerian polynomials when $\overrightarrow{G}$ is the directed cycle, $\overrightarrow{C_n}$, as defined in Example 5.1.6. We show in Section 5.4 that $X_{\overrightarrow{C_n}}(\mathbf{x},t)$ is symmetric.

The following lemma is a special case of [9, Theorem 3.1] but is proven here for completeness.

Lemma 5.3.6. *Let $A_k(t)$ denote the Eulerian polynomial. For $k \geq 2$, we have*

$$\sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma\ k-cycle}} t^{\mathrm{exc}(\sigma)} = tA_{k-1}(t).$$

Proof. If we write each σ in cycle form with k written as the last element of the cycle, i.e., $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_{k-1}, k)$, then we obtain $\mu = \sigma_1\sigma_2\cdots\sigma_{k-1} \in \mathfrak{S}_{k-1}$. This gives us a bijection between k-cycles $\sigma \in \mathfrak{S}_k$ and elements $\mu \in \mathfrak{S}_{k-1}$. In addition, $\mathrm{exc}(\sigma) = \mathrm{asc}(\mu)+1$ since the pair (σ_{k-1}, k) will always form an excedance, but (k, σ_1) will never form an excedance. Hence, we have the following:

$$\begin{aligned}\sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma\ k-cycle}} t^{\mathrm{exc}(\sigma)} &= \sum_{\mu \in \mathfrak{S}_{k-1}} t^{1+\mathrm{asc}(\mu)} \\ &= tA_{k-1}(t).\end{aligned}$$

$\square$

Theorem 5.3.7. *Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition of n. If $k \geq 2$, then*

$$\sum_{\sigma \in N_{C_n,\lambda}} t^{\mathrm{inv}_{\overrightarrow{C_n}}(\sigma)} = ntA_{k-1}(t) \prod_{i=1}^{k} [\lambda_i]_t, \tag{5.14}$$

where $[n]_t \coloneqq 1 + t + \cdots + t^{n-1}$. In the case that $\lambda = (n)$, we have

$$\sum_{\sigma \in N_{C_n,(n)}} t^{\mathrm{inv}_{\overrightarrow{C_n}}(\sigma)} = nt[n-1]_t. \tag{5.15}$$

Hence the coefficient of $\frac{1}{n}p_n(\mathbf{x})$ in $\omega X_{\overrightarrow{C_n}}(\mathbf{x}, t)$ is $nt[n-1]_t$ and for all other $\lambda \vdash n$, the coefficient of $z_\lambda^{-1}p_\lambda(\mathbf{x})$ in $\omega X_{\overrightarrow{C_n}}(\mathbf{x}, t)$ is $ntA_{k-1}(t)\prod_{i=1}^{k}[\lambda_i]_t$.

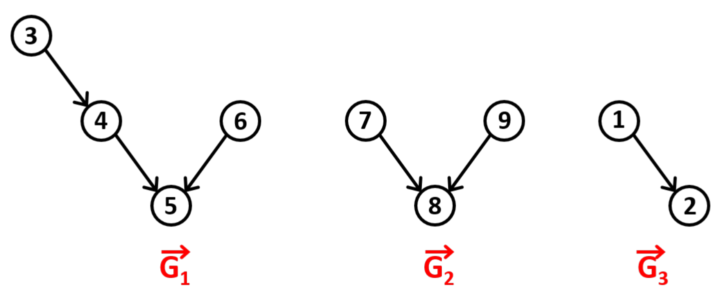
Proof. For the following proof, we will fix the labeling of $\overrightarrow{C_n}$ by $[n]$ so that $E(\overrightarrow{C_n}) = \{(i, i+1) \mid 1 \leq i < n\} \cup \{(n, 1)\}$. Note that any labeling of the vertices of $\overrightarrow{C_n}$ with $[n]$ will work the same way.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition of n and let $\sigma \in \mathfrak{S}_n$ be partitioned into pieces of size $\lambda_1, \lambda_2, \cdots, \lambda_k$ so that $\sigma = \alpha_1\alpha_2\cdots\alpha_k$, where $\cdot$ represents concatenation. Then we know $\sigma \in N_{C_n,\lambda}$ if and only if each α_i has no C_n-descents and no C_n-isolated letters.

For each α_i, we will construct a connected acyclic digraph $\overrightarrow{G_i}$ on the letters of α_i such that the underlying undirected graph, G_i, is an induced subgraph of C_n.

Let $\overrightarrow{G_i}$ be the directed graph whose vertex set is the set of letters of α_i and whose edges have the form (a, b) if b precedes a in α_i and $\{a, b\} \in E(C_n)$. Then each $\overrightarrow{G_i}$ is a connected acyclic digraph with a unique sink, which is the first letter of α_i. Indeed if there were another sink, then the second sink would be a C_n-isolated letter of α_i. Hence if $\lambda \neq (n)$, each underlying undirected graph, G_i, is a path of length λ_i in C_n. If $\lambda = (n)$, then $G_1 = C_n$.

For example, let $n = 9$, $\lambda = (4, 3, 2)$ and $\sigma = 546389721$. Then $\alpha_1 = 5463$, $\alpha_2 = 897$, and $\alpha_3 = 21$. The corresponding acyclic digraphs are as shown below:



We can uniquely recover σ from the k-tuple $(\overrightarrow{G_1}, \overrightarrow{G_2}, \cdots, \overrightarrow{G_k})$. For each vertex x in each $\overrightarrow{G_i}$, let us define rank(x) as follows. Let $x \in \overrightarrow{G_i}$ and let V_x be the set of all vertices y in $\overrightarrow{G_1}, \cdots, \overrightarrow{G_{i-1}}$ such that $\{x, y\}$ is an edge of C_n. If x is a sink of $\overrightarrow{G_i}$, then $\text{rank}(x) = \max\{1, \max_{y \in V_x}(\text{rank(y)} + 1)\}$. If x is not a sink of $\overrightarrow{G_i}$, then there exists a unique vertex z of $\overrightarrow{G_i}$ such that $(x, z) \in E(\overrightarrow{G_i})$. Then $\text{rank}(x) = \max\{\text{rank(z)} +$

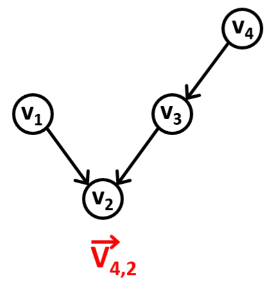
$1, \max_{y \in V_x}(\text{rank(y)}+1)\}$. Then create each α_i by starting with all vertices of $\overrightarrow{G_i}$ of rank 1 in increasing order of their label, then all vertices of $\overrightarrow{G_i}$ of rank 2 in increasing order of their label, etc. Then we have $\sigma = \alpha_1\alpha_2\cdots\alpha_k$. Notice that for all $x \in [n]$, we have that $\text{rank}(x) = \text{rank}_{(C_n,\sigma)}(x)$.

Notice that the number of $\overrightarrow{C_n}$-inversions of α_i is the number of directed edges of $\overrightarrow{G_i}$ that are oriented in the same direction as the corresponding directed edge of $\overrightarrow{C_n}$.

Case 1: $\lambda = (n)$. In this case $\overrightarrow{G_1}$ is an acyclic orientation of C_n with a unique sink. So we need to find the number of $\overrightarrow{C_n}$-inversions of the corresponding σ, i.e., the number of edges of $\overrightarrow{G_1}$ that are oriented the same direction as the corresponding edge in $\overrightarrow{C_n}$. In order to construct an acyclic orientation of C_n with a unique sink (and hence a unique source), we have n choices for a sink and then $n-1$ choices remaining for a source. There are two paths from the sink to the source. One path is oriented as in $\overrightarrow{C_n}$ and the other path is oriented opposite $\overrightarrow{C_n}$. The number of edges of the path oriented the same direction as $\overrightarrow{C_n}$ can be $1, 2, \cdots$, or $n-1$, depending on the choice of the source. So

$$\sum_{\sigma \in N_{C_n,(n)}} t^{\text{inv}_{\overrightarrow{C_n}}(\sigma)} = n(t + t^2 + \cdots + t^{n-1}) = nt[n-1]_t.$$

Case 2: $\lambda \neq (n)$. For $a, b \in \mathbb{P}$ with $b \leq a$, define a V-digraph $\overrightarrow{V}_{a,b}$ to be a digraph with vertex set $\{v_1, v_2, \cdots, v_a\}$ and edge set $\{(v_i, v_{i+1}) \mid 1 \leq i < b\} \cup \{(v_{i+1}, v_i) \mid b \leq i < a\}$. We will call v_1 the first vertex of $\overrightarrow{V}_{a,b}$ and v_a the last vertex of $\overrightarrow{V}_{a,b}$. For $1 \leq i < a$ we say the successor of v_i is v_{i+1}. Let $V_{a,b}$ denote the underlying undirected graph of $\overrightarrow{V}_{a,b}$. For all $a, b \in \mathbb{P}$ with $b \leq a$, we can see that $V_{a,b}$ is a path. For example, $\overrightarrow{V}_{4,2}$ is shown below:



Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. Then we will construct a bijection from $N_{C_n,\lambda}$ to the set M_λ of $(k+2)$-tuples $(x, \mu, \overrightarrow{V}_{\lambda_1,b_1}, \overrightarrow{V}_{\lambda_2,b_2}, \cdots, \overrightarrow{V}_{\lambda_k,b_k})$, where

- $x \in [n]$,
- $\mu \in \mathfrak{S}_k$ is a k-cycle, and
- for each i, we have $1 \leq b_i \leq \lambda_i$.

Let $\sigma \in N_{C_n,\lambda}$. Recall our earlier map from $\sigma \in N_{C_n,\lambda}$ to the k-tuples $(\overrightarrow{G_1}, \overrightarrow{G_2}, \cdots, \overrightarrow{G_k})$. For each $1 \leq i \leq k$, define b_i as one more than the number of edges of $\overrightarrow{G_i}$ that match the orientation of $\overrightarrow{C_n}$. Then $\overrightarrow{V}_{\lambda_i,b_i}$ is simply $\overrightarrow{G_i}$ without labels. To determine $\mu = (a_1, a_2, \cdots, a_k)$, we start by letting $a_1 = j_1$ where $\overrightarrow{G_{j_1}}$ contains the vertex labeled 1. From the remaining $\overrightarrow{G_i}$, let $\overrightarrow{G_{j_2}}$ be the digraph with the smallest label on its sink. Then let $a_2 = j_2$. From the remaining $\overrightarrow{G_i}$, let $\overrightarrow{G_{j_3}}$ be the digraph with the smallest label on its sink. Then let $a_3 = j_3$. We continue this process until we find a_k. Lastly, to determine x, suppose 1 is on the d^{th} vertex of $\overrightarrow{G_i}$. Then $x = \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} + d$.

In the other direction, suppose we have

$$(x, \mu, \overrightarrow{V}_{\lambda_1,b_1}, \overrightarrow{V}_{\lambda_2,b_2}, \cdots, \overrightarrow{V}_{\lambda_k,b_k}) \in M_\lambda.$$

For each $1 \leq i \leq k$, we will say that the successor of the last vertex of $\overrightarrow{V}_{\lambda_i,b_i}$ is the first vertex of $\overrightarrow{V}_{\lambda_{\mu(i)},b_{\mu(i)}}$.

There exists unique $1 \leq i \leq k$ and $1 \leq d \leq \lambda_i$ such that $x = \lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} + d$. Place a 1 on the d^{th} vertex of $\overrightarrow{V}_{\lambda_i,b_i}$. Then place a 2 on its successor, and continue labeling successors in order until all n vertices are labeled. Now the labeled $\overrightarrow{V}_{\lambda_i,b_i}$ is the same as $\overrightarrow{G_i}$, so we can recover σ as described earlier. One can check that this is a bijection.

Now suppose we have some $\sigma \in \mathfrak{S}_n$ that corresponds to

$$(x, \mu, \overrightarrow{V}_{\lambda_1,b_1}, \overrightarrow{V}_{\lambda_2,b_2}, \cdots, \overrightarrow{V}_{\lambda_k,b_k}) \in M_\lambda.$$

Notice that using the bijection, the number of $\overrightarrow{C_n}$-inversions of α_i is equal to $b_i - 1$. One can check that the number of $\overrightarrow{C_n}$-inversions between distinct α_i in σ is the same as the number of excedances of μ^{-1}, because for each $i \in [k]$, there is an edge of $\overrightarrow{C_n}$ directed from the last vertex of $\overrightarrow{V}_{\lambda_i,b_i}$ to the first vertex of $\overrightarrow{V}_{\lambda_{\mu(i)},b_{\mu(i)}}$. Then one can see that

$$\mathrm{inv}_{\overrightarrow{C_n}}(\sigma) = \mathrm{exc}(\mu^{-1}) + (b_1 - 1) + (b_2 - 1) + \cdots (b_k - 1).$$

Using Lemma 5.3.6, we see that

$$\sum_{\substack{\mu \in \mathfrak{S}_k \\ \mu\ k-cycle}} t^{\mathrm{exc}(\mu^{-1})} = \sum_{\substack{\mu \in \mathfrak{S}_k \\ \mu\ k-cycle}} t^{\mathrm{exc}(\mu)} = tA_{k-1}(t).$$

Now since for each $1 \leq i \leq k$, we have $1 \leq b_i \leq \lambda_i$, and since we have n choices for x in the bijection, we can see that (5.14) is true. $\square$

5.4 Symmetry

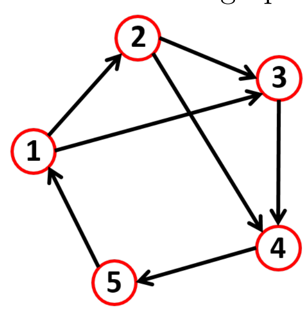
In this section, we define circular indifference digraphs and show that they have symmetric chromatic quasisymmetric functions. For $a, b \in [n]$, we define the *circular* *interval* $[a,b]$ of $[n]$ as

$$[a,b] := \begin{cases} \{a, a+1, a+2, ..., b\} & \text{if } a \leq b \\ \{a, a+1, a+2, \cdots, n, 1, 2, \cdots, b\} & \text{if } a > b \end{cases}.$$

Definition 5.4.1. We call a digraph, $\overrightarrow{G} = ([n], E)$, a *circular indifference digraph* if there exists a collection of circular intervals, I, of $[n]$ such that

$$E = \{(i,j) \mid [i,j] \text{ is contained in a circular interval of } I\}.$$

Example 5.4.2. Suppose we have the set of circular intervals $I = \{[1,3],[2,4],[4,5],[5,1]\}$. Then the corresponding circular indifference digraph is shown below.



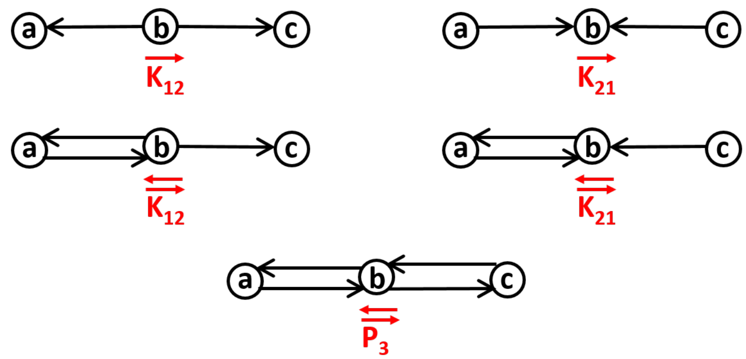
The underlying undirected graphs of these circular indifference digraphs are the *circular indifference graphs* defined by Stanley in [58]. We discuss circular indifference graphs and their relation to other well-known classes of graphs in Appendix A.

In [52, Theorem 4.5], Shareshian and Wachs show that $X_G(\mathbf{x}, t)$ is symmetric if G is a natural unit interval graph. As discussed in Appendix A, when natural unit interval graphs are viewed as digraphs, they are acyclic circular indifference digraphs. Next we extend the symmetry result of Shareshian and Wachs to all circular indifference digraphs. Our proof of symmetry is similar to that of Shareshian and Wachs. First we need the following lemmas.

Let us define five digraphs we will need for the next lemma.

- $\overrightarrow{K_{12}} = (\{a,b,c\}, \{(b,a),(b,c)\})$.
- $\overrightarrow{K_{21}} = (\{a,b,c\}, \{(a,b),(c,b)\})$.
- $\overleftrightarrow{K_{12}} = (\{a,b,c\}, \{(a,b),(b,a),(b,c)\})$.
- $\overleftrightarrow{K_{21}} = (\{a,b,c\}, \{(a,b),(b,a),(c,b)\})$.
- $\overleftrightarrow{P_3} = (\{a,b,c\}, \{(a,b),(b,a),(b,c),(c,b)\})$.

Below we see all five digraphs.



Lemma 5.4.3. *Let $\overrightarrow{G}$ be a digraph that has no induced subdigraphs isomorphic to $\overrightarrow{K_{12}}, \overrightarrow{K_{21}}, \overleftrightarrow{K_{12}}, \overleftrightarrow{K_{21}}$ or $\overleftrightarrow{P_3}$. Then the underlying undirected graph, G, is claw-free, i.e., G does not contain an induced subgraph isomorphic to K_{31}.*

Proof. Let $\overrightarrow{G}$ be a digraph whose underlying undirected graph is the claw, K_{31}. It is not difficult to see that $\overrightarrow{G}$ must have an induced subdigraph isomorphic to one of the five digraphs listed. But this means that any digraph that contains an induced claw subgraph must contain a forbidden subdigraph. $\square$

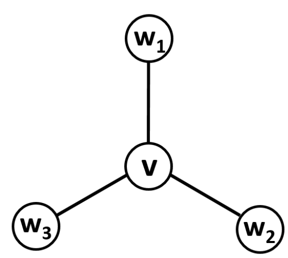
For the next lemma, we need a few definitions. We say that a digraph, $\overrightarrow{G}$, is *connected* if its underlying undirected graph, G, is connected. We say that a subdigraph, $\overrightarrow{H}$, is a *connected component* of $\overrightarrow{G}$ if H is a connected component of G. As in Example 5.1.5, we say a digraph, $\overrightarrow{G} = (V,E)$ is a *directed path* if its vertex set is

$V = \{v_1, v_2, \cdots, v_n\}$ and its edge set is $E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\}$. As defined in Example 5.1.6, we say a digraph, $\overrightarrow{G} = (V, E)$, is a *directed cycle* if its vertex set is $V = \{v_1, v_2, \cdots, v_n\}$ and its edge set is $E = \{(v_i, v_{i+1}) \mid 1 \leq i < n\} \cup \{(v_n, v_1)\}$.

Lemma 5.4.4. *Let $\overrightarrow{G}$ be a digraph that has no induced subdigraphs isomorphic to $\overrightarrow{K_{12}}, \overrightarrow{K_{21}}, \overleftrightarrow{K_{12}}, \overleftrightarrow{K_{21}}$ or $\overleftrightarrow{P_3}$. Let κ be a proper coloring of $\overrightarrow{G}$. For $a \in \mathbb{P}$, define $\overrightarrow{G_{\kappa,a}}$ as the induced subdigraph of $\overrightarrow{G}$ of all vertices colored by a or $a+1$. Then each connected component of $\overrightarrow{G_{\kappa,a}}$ is either a directed cycle with an even number of vertices or a directed path.*

Proof. Let $G_{\kappa,a}$ be the underlying undirected graph of $\overrightarrow{G_{\kappa,a}}$. First note that $G_{\kappa,a}$ cannot have any cycles of odd length, because then two vertices with the same color would be adjacent, which contradicts the fact that κ is a proper coloring.

We can also see that $G_{\kappa,a}$ cannot have any vertex adjacent to more than two other vertices. Indeed, suppose vertex v were adjacent to vertices w_1, w_2, and w_3 in $G_{\kappa,a}$, as in the following figure:



Since $G_{\kappa,a}$ has no 3-cycles, w_1, w_2, and w_3 have no edges between them. Then we see that $G_{\kappa,a}$ contains a claw as an induced subgraph, but this contradictions Lemma 5.4.3

Then since every vertex has degree at most 2, every connected component of $G_{\kappa,a}$ must be either a path or a cycle of even length. Since every induced subdigraph of $\overrightarrow{G}$ must avoid $\overrightarrow{K_{12}}$, $\overrightarrow{K_{21}}$, $\overleftrightarrow{K_{12}}$, $\overleftrightarrow{K_{21}}$, and $\overleftrightarrow{P_3}$, the only possible connected components are the ones listed in the lemma. $\square$

Theorem 5.4.5. *Let $\overrightarrow{G}$ be a digraph that has no induced subdigraphs isomorphic to $\overrightarrow{K_{12}}, \overrightarrow{K_{21}}, \overleftrightarrow{K_{12}}, \overleftrightarrow{K_{21}}$, or $\overset{\leftrightarrows}{P_3}$. Then $X_{\overrightarrow{G}}(\mathbf{x}, t)$ is symmetric.*

Proof. By Proposition 5.1.3, we can assume without loss of generality that $\overrightarrow{G}$ is connected.

We will construct an involution, ϕ_a, for each $a \in \mathbb{P}$ on the set of proper colorings of $\overrightarrow{G}$ that switches the number of occurrences of the color a and the number of occurrences the color $a+1$, leaves the number of occurrences of all other colors the same, and does not change the number of ascents of the coloring. This will then prove the theorem.

So let κ be a proper coloring of $\overrightarrow{G}$ and let $\overrightarrow{G_{\kappa,a}}$ be the induced subdigraph of $\overrightarrow{G}$ containing only the vertices colored by a and $a+1$. By Lemma 5.4.4, each component of $\overrightarrow{G_{\kappa,a}}$ is a directed path or a directed cycle of even length.

Let $\phi_a(\kappa)$ be the the coloring of $\overrightarrow{G}$ obtained from κ by replacing each occurrence of a with $a+1$ and replacing each $a+1$ with a in the components of $\overrightarrow{G_{\kappa,a}}$ that are paths with an odd number of vertices. For the other components of $\overrightarrow{G_{\kappa,a}}$ (paths and cycles with an even number of vertices), the colors of $\phi_a(\kappa)$ are the same as those of κ.

Note that in a path of with an odd number of vertices in $\overrightarrow{G_{\kappa,a}}$, exactly half of the edges are ascents of κ. Hence, if we change all a's to $a+1$'s and vice versa, we will change all ascents to descents and vice versa, but the number of ascents of κ is preserved. It is then easy to see that ϕ_a is an involution that meets the desired conditions and hence the theorem is proven.

□

Lemma 5.4.6. *Circular indifference digraphs do not have any induced subdigraphs isomorphic to $\overrightarrow{K_{12}}, \overrightarrow{K_{21}}, \overleftrightarrow{K_{12}}, \overleftrightarrow{K_{21}}$, or $\overset{\leftrightarrows}{P_3}$.*

Proof. Let $\overrightarrow{G}$ be a circular indifference digraph arising from a set of circular intervals, I, on $[n]$, and suppose $\overrightarrow{G}$ contains an induced subdigraph, $\overrightarrow{H}$, isomorphic to $\overrightarrow{K_{12}}$.

Suppose $\overrightarrow{H}$ has vertex set $\{a,b,c\}$ and edge set $\{(b,a),(b,c)\}$. Then the circular intervals $[b,a]$ and $[b,c]$ are both contained in circular intervals of I. But then either $[b,a] \subset [b,c]$ and hence $[a,c] \subset [b,c]$, which is contained in a circular interval of I, or $[b,c] \subset [b,a]$ and hence $[c,a] \subset [b,a]$, which is contained in a circular interval of I. Either way there is an edge between a and c in $\overrightarrow{G}$, which is a contradiction. Similar arguments show that $\overrightarrow{G}$ cannot contain any induced subdigraphs isomorphic to $\overrightarrow{K_{21}}$, $\overleftrightarrow{K_{12}}$, $\overleftrightarrow{K_{21}}$, or $\overleftrightarrow{P_3}$. □

Corollary 5.4.7. *Let $\overrightarrow{G}$ be a digraph such that all connected components of $\overrightarrow{G}$ are circular indifference digraphs. Then $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric.*

Proof. Combine Lemma 5.4.6 with Theorem 5.4.5. □

It turns out that the class of digraphs from Theorem 5.4.5 are not the only digraphs with symmetric chromatic quasisymmetric functions. In fact, let $C_n = ([n],E)$ be the labeled cycle with $E = \{\{i,i+1\} \mid i \in [n-1]\} \cup \{\{1,n\}\}$. If we turn C_n into a digraph by orienting each edge from smaller label to larger label, then we get the directed cycle with one edge oriented backwards. The chromatic quasisymmetric function of the labeled cycle is symmetric (see [34, Exercise 2.84]), but its associated directed graph contains induced subdigraphs isomorphic to both $\overrightarrow{K_{21}}$ and $\overrightarrow{K_{12}}$. See Section 6.6 for further results on the chromatic quasisymmetric function of the labeled cycle.

5.5 Expansion in the elementary symmetric function basis

In this section, we provide some evidence for our generalized e-positivity conjecture, which we restate here.

Conjecture 5.5.1. *Let $\overrightarrow{G} = (V, E)$ be a circular indifference digraph. Then the palindromic*[2] *polynomial $X_{\overrightarrow{G}}(\mathbf{x}, t)$ is e-positive and e-unimodal. In other words, if $X_{\overrightarrow{G}}(\mathbf{x}, t) = \sum_{j=0}^{|E|} a_j(\mathbf{x})t^j$, then $a_j(\mathbf{x})$ is e-positive for all j and $a_{j+1}(\mathbf{x}) - a_j(\mathbf{x})$ is e-positive for all $j \leq \frac{|E|-1}{2}$.*

Below we take a look at the simplest example of a circular indifference digraph that is not acyclic, namely the directed cycle, $\overrightarrow{C_n}$, and prove an e-basis generating function formula for $X_{\overrightarrow{C_n}}(\mathbf{x}, t)$.

Theorem 5.5.2. *Let $\overrightarrow{C_n}$ be the directed cycle of length n. Then*

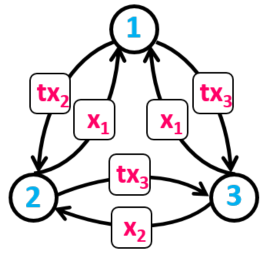
$$\sum_{n\geq 2} X_{\overrightarrow{C_n}}(\mathbf{x}, t)z^n = \frac{t\sum_{k\geq 2} k[k-1]_t e_k(\mathbf{x})z^k}{1 - t\sum_{k\geq 2}[k-1]_t e_k(\mathbf{x})z^k}, \tag{5.16}$$

where $[n]_t = 1 + t + t^2 + \cdots + t^{n-1}$.

Proof. This proof is more involved than Stanley's proof for the $t = 1$ case (see Proposition 3.2.6), but it also uses the Transfer-Matrix Method [59, Section 4.7]. So let us start with a brief review of the transfer matrix method. A *walk* of length d on a directed graph $\overrightarrow{G} = ([n], E)$ is a sequence of vertices $v_0, v_1, \ldots, v_d$ such that $(v_{i-1}, v_i) \in E$ for all $i \in [d]$. A walk is *closed* if $v_0 = v_d$. We attach weights in some commutative ring R to the edges of G. Let $\mathrm{wt} : E \to R$ be the weight function. Now define the weight $\mathrm{wt}(w)$ of a walk $w := v_0, v_1, \ldots, v_d$ to be the product $\mathrm{wt}(v_0, v_1)\,\mathrm{wt}(v_1, v_2)\ldots\mathrm{wt}(v_{d-1}, v_d)$.

Let $\overrightarrow{G} = ([n], E)$ be the digraph with $E = \{(i, j) \mid i \neq j\}$. Let us attach a weight to each edge (i, j) so that $wt((i, j)) = tx_j$ if $i < j$ and $wt((i, j)) = x_j$ if $i > j$. For example if $n = 3$ then $\overrightarrow{G}$ is shown below.

[2]See Proposition 5.1.8



We can view all proper colorings of all $\overrightarrow{C_d}$ for $d \geq 2$ using only n colors as closed walks of length d on $\overrightarrow{G}$. Each time we take a step, the colors either increase (and we need to count an ascent) or decrease. The weighted adjacency matrix of $\overrightarrow{G}$ is given by

$$A = \begin{bmatrix} 0 & tx_2 & tx_3 & \dots & tx_n \\ x_1 & 0 & tx_3 & \dots & tx_n \\ x_1 & x_2 & 0 & \dots & tx_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & x_3 & \dots & 0 \end{bmatrix}.$$

Let $Q(z) = \det(I - zA)$. By [59, Corollary 4.7.3], we know that

$$\sum_{d \geq 2} X_{\overrightarrow{C}_d}(\mathbf{x}, t)|_{x_1, x_2, \dots x_n} z^d = \frac{-zQ'(z)}{Q(z)}.$$

So we need to compute

$$Q(z) = \det(I - zA) = \det \begin{bmatrix} 1 & -tx_2 z & -tx_3 z & \dots & -tx_n z \\ -x_1 z & 1 & -tx_3 z & \dots & -tx_n z \\ -x_1 z & -x_2 z & 1 & \dots & -tx_n z \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_1 z & -x_2 z & -x_3 z & \dots & 1 \end{bmatrix}.$$

First let us describe the notation we will use. For $\sigma \in \mathfrak{S}_n$, $\mathrm{FIX}(\sigma) = \{i \mid \sigma(i) = i\}$, $\mathrm{exc}(\sigma) = |\{i \mid \sigma(i) > i\}|$, and $\mathrm{sgn}(\sigma)$ is the usual sign function on permutations. We say σ is a derangement if $\mathrm{FIX}(\sigma) = \emptyset$. Also for any $S \subseteq [n]$, define $\mathbf{x}_S = \prod_{i \in S} x_i$.

Then

$$\begin{aligned}
Q(z) &= \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma)(-1)^{n-|\mathrm{FIX}(\sigma)|} z^{n-|\mathrm{FIX}(\sigma)|} t^{\mathrm{exc}(\sigma)} \mathbf{x}_{[n]\setminus \mathrm{FIX}(\sigma)} \\
&= 1 + \sum_{k=1}^{n} (-1)^k z^k \sum_{\substack{S \subseteq [n] \\ |S|=k}} \mathbf{x}_S \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{FIX}(\sigma)=[n]\setminus S}} \mathrm{sgn}(\sigma) t^{\mathrm{exc}(\sigma)} \\
&= 1 + \sum_{k=1}^{n} (-1)^k z^k \sum_{\substack{S \subseteq [n] \\ |S|=k}} \mathbf{x}_S \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma \text{ is a derangement}}} \mathrm{sgn}(\sigma) t^{\mathrm{exc}(\sigma)} \\
&= 1 + \sum_{k=1}^{n} (-1)^k z^k e_k(x_1, x_2, ..., x_n) \sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma \text{ is a derangement}}} \mathrm{sgn}(\sigma) t^{\mathrm{exc}(\sigma)}.
\end{aligned}$$

By [44, Corollary 5.11], we have that

$$\sum_{\substack{\sigma \in \mathfrak{S}_k \\ \sigma \text{ is a derangement}}} \mathrm{sgn}(\sigma) t^{\mathrm{exc}(\sigma)} = (-1)^{k+1} t[k-1]_t.$$

Hence

$$\begin{aligned}
Q(z) &= 1 + \sum_{k=1}^{n} (-1)^k z^k e_k(x_1, x_2, ..., x_n)(-1)^{k+1} t[k-1]_t \\
&= 1 - t \sum_{k \geq 2} e_k(x_1, x_2, ..., x_n)[k-1]_t z^k.
\end{aligned}$$

Then we have

$$\begin{aligned}
-zQ'(z) &= -z(-t \sum_{k \geq 2} e_k(x_1, x_2, ..., x_n) k [k-1]_t z^{k-1}) \\
&= t \sum_{k \geq 2} e_k(x_1, x_2, ..., x_n) k [k-1]_t z^k.
\end{aligned}$$

Letting n go to infinity gives us our result. $\square$

Now let us look at some consequence of this expansion. But first we need a simple lemma.

Lemma 5.5.3. *Let* $(g_n(t))_{n\geq 0}$ *be a sequence of polynomials in* $\mathbb{Q}[t]$*, such that each* $g_n(t)$ *positive, palindromic, and unimodal with center of symmetry* $\frac{n+r}{2}$*, where* r *is some fixed constant in* $\mathbb{N}$*. If* $(G_n(\mathbf{x},t))_{n\geq 0}$ *is a sequence of polynomials in* $\Lambda_{\mathbb{Q}}[t]$ *that satisfies*

$$\sum_{n\geq 0} G_n(\mathbf{x},t)z^n = \frac{\sum\limits_{n\geq 0} g_n(t)e_n(\mathbf{x})z^n}{1-t\sum\limits_{k\geq 2}[k-1]_t e_k(\mathbf{x})z^k}$$

then each $G_n(\mathbf{x},t)$ *is palindromic, e-positive, and e-unimodal with center of symmetry* $\frac{n+r}{2}$.

Proof. We use Propositions B.1 and B.3 of [52]. Since

$$\frac{1}{1-t\sum\limits_{k\geq 2}[k-1]_t e_k(\mathbf{x})z^k} = \sum_{m\geq 0}\left(\sum_{i\geq 2} t[i-1]_t e_i(\mathbf{x})z^i\right)^m, \tag{5.17}$$

we have,

$$G_n(\mathbf{x},t) = \sum_{m\geq 1} \sum_{\substack{k_1\geq 0 \\ k_2,\ldots,k_m\geq 2 \\ \sum_{i=1}^m k_i = n}} e_{k_1}(\mathbf{x})\ldots e_{k_m}(\mathbf{x})t^{m-1}g_{k_1}(t)\prod_{i=2}^m [k_i-1]_t.$$

For each nonzero $g_{k_1}(t)$, the polynomial $t^{m-1}g_{k_1}(t)\prod_{i=2}^m[k_i-1]_t$ is a product of palindromic, positive, unimodal polynomials. Hence, the product is also palindromic, positive and unimodal with center of symmetry equal to

$$m-1+\frac{k_1+r}{2}+\sum_{i=2}^m \frac{k_i-2}{2} = \frac{n+r}{2}.$$

Since each such product has the same center of symmetry, $G_n(\mathbf{x},t)$ is palindromic, e-positive, and e-unimodal with center of symmetry $\frac{n+r}{2}$. □

Then the following corollary follows easily from Theorem 5.5.2 and Lemma 5.5.3.

Corollary 5.5.4. *$X_{\overrightarrow{C_n}}(\mathbf{x},t)$ is a palindromic, e-positive and e-unimodal polynomial in t.*

In fact, for each $n \geq 2$ we have

$$X_{\overrightarrow{C_n}}(\mathbf{x},t) = \sum_{\lambda \vdash n} e_\lambda(\mathbf{x}) \sum_{\mu:\lambda(\mu)=\lambda} \mu_1 t[\mu_1 - 1]_t t[\mu_2 - 1]_t \cdots t[\mu_{l(\lambda)} - 1]_t, \tag{5.18}$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_{l(\lambda)})$ is a composition of n, $l(\lambda)$ is the length of λ, and $\lambda(\mu) = \lambda$ means that when the parts of μ are written in decreasing order, we get the partition λ.

Now let us take a look at e-basis expansions of another class of digraphs, namely digraphs whose underlying undirected graph is the complete graph, i.e the graph with an edge between every distinct pair of vertices.

Proposition 5.5.5. *Let $\overrightarrow{G} = ([n], E)$ be a digraph whose underlying undirected graph, G, is the complete graph, K_n. Then*

$$X_{\overrightarrow{G}}(\mathbf{x},t) = p(t)e_n(\mathbf{x}),$$

where

$$p(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{inv}_{\overrightarrow{G}}(\sigma)}.$$

As a result, $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric and e-positive.

Proof. Since every vertex of $\overrightarrow{G}$ is adjacent to every other vertex, we can see that for every $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ we have $\mathrm{rank}_{(G,\sigma)}(\sigma_i) = i$ for each $i \in [n]$, so σ contains no G-descents. Taking ω of both sides of our F-basis expansion from Theorem 5.2.2 and the fact that $\omega F_{n,\emptyset}(\mathbf{x}) = e_n(\mathbf{x})$ gives us our result. $\square$

In [52], Shareshian and Wachs introduce a class of graphs, which they call $G_{n,r}$ graphs, where $n \in \mathbb{P}$ and $1 \leq r \leq n$. The vertices of $G_{n,r}$ are labeled by $[n]$ and for $1 \leq i < j \leq n$ there is an edge between i and j if $0 < j - i < r$. For example, $G_{n,1}$ is the graph on n vertices with no edges, $G_{n,2}$ is the labeled path on n elements, and $G_{n,n}$ is the complete graph on n elements. Shareshian and Wachs proved that their e-positivity conjecture holds for all $G_{n,r}$ when $r = 1, 2, n-2, n-1, n$ and they tested by computer all $G_{n,r}$ for $n \leq 8$. Hence if these graphs are turned into digraphs by orienting their edges from smaller label to larger label, our e-positivity conjecture holds for the same graphs.

We present a circular analog of these graphs, which we will call $\overrightarrow{G}^c_{n,r}$, where $n \in \mathbb{P}$ and $1 \leq r \leq n$. We define $\overrightarrow{G}^c_{n,r} = ([n], E)$, where $E = \{(i,j) \mid 0 < j - i \pmod{n} < r\}$. In other words, $\overrightarrow{G}^c_{n,r}$ is the circular indifference digraph on $[n]$ arising from the set of circular intervals

$$I = \{[i, i+r-1] \mid 1 \leq i \leq n-r+1\} \cup \{[i, i+r-1-n] \mid n-r+2 \leq i \leq n\}.$$

For example, $\overrightarrow{G}^c_{n,1} = ([n], \emptyset)$, $\overrightarrow{G}^c_{n,2}$ is the directed cycle, $\overrightarrow{C_n}$, and $\overrightarrow{G}^c_{n,n} = ([n], E)$, where $E = \{(i,j) \mid i \neq j\}$. Corollary 5.5.4 proves that our e-positivity conjecture (Conjecture 5.5.1) holds for $\overrightarrow{G}^c_{n,2}$. It is easy to see that our e-positivity conjecture holds for $\overrightarrow{G}^c_{n,1}$. Below we show that our conjecture holds for $\overrightarrow{G}^c_{n,r}$ when $r = n-1, n$. We used a computer to test our conjecture for all other $\overrightarrow{G}^c_{n,r}$ for $n \leq 8$.

Proposition 5.5.6. *For all $n \geq 1$ we have*

$$X_{\overrightarrow{G}^c_{n,n}}(\mathbf{x}, t) = n! e_n(\mathbf{x}) t^{\binom{n}{2}}, \tag{5.19}$$

and

$$X_{\overrightarrow{G}^c_{n,n-1}}(\mathbf{x},t) = ne_n(\mathbf{x})t^{\binom{n-1}{2}-n+1}A_{n-1}(t). \tag{5.20}$$

Proof. First we will prove (5.19). From Proposition 5.5.5, we see that

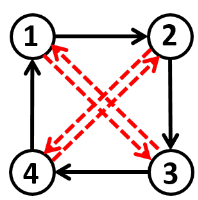
$$X_{\overrightarrow{G}^c_{n,n}}(\mathbf{x},t) = e_n(\mathbf{x}) \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\overrightarrow{G}^c_{n,n}}(\sigma)}.$$

But since every pair (i,j) is an edge, we see that for every $\sigma \in \mathfrak{S}_n$, $\mathrm{inv}_{\overrightarrow{G}^c_{n,n}}(\sigma) = \binom{n}{2}$. Combining this with the fact that $|\mathfrak{S}_n| = n!$, we have our formula for $X_{\overrightarrow{G}^c_{n,n}}(\mathbf{x},t)$.

Now let us prove (5.20). The graph $\overrightarrow{G}^c_{n,n-1} = ([n], E)$ has edge set $E = \{(i,j) \mid i-j \neq 0, 1, 1-n\}$. The set, E, can be divided into two types: the exterior edges,

$$\{(1,2),(2,3),\cdots,(n-1,n),(n,1)\},$$

which form the directed cycle, $\overrightarrow{C_n}$, and the interior edges, which are the remaining edges. Note that the interior edges are two-way edges; that is, if (a,b) is an interior edge, then so is (b,a). Below is $\overrightarrow{G}^c_{4,3}$ where the exterior edges are solid black arrows and the interior edges are dotted red arrows.



Again we will use Proposition 5.5.5, so we know

$$X_{\overrightarrow{G}^c_{n,n-1}}(\mathbf{x},t) = e_n(\mathbf{x}) \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\overrightarrow{G}^c_{n,n-1}}(\sigma)}.$$

Notice that for each $\sigma \in \mathfrak{S}_n$, we have $\binom{n}{2} - n$ $\overrightarrow{G}^c_{n,n-1}$-inversions coming from the interior edges. In order to count the $\overrightarrow{G}^c_{n,n-1}$-inversions from the exterior edges, recall that the exterior edges form the directed cycle, $\overrightarrow{C_n}$, so we need to find $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\overrightarrow{C_n}}(\sigma)}$.

Setting $\lambda = 1^n$ in (5.14) gives us that $\sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{inv}_{\overrightarrow{C_n}}(\sigma)} = ntA_{n-1}(t)$. Combining all this gives us (5.20). $\square$

The following theorem can be easily proven using the same proof technique as Stanley used to prove [58, Theorem 3.3].

Theorem 5.5.7. *Let $\overrightarrow{G}$ be a digraph on n vertices such that $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric. Suppose we have the expansion $X_{\overrightarrow{G}}(\mathbf{x},t) = \sum_{\lambda\vdash n} c_\lambda(t)e_\lambda(\mathbf{x})$. Then*

$$\sum_{\substack{\lambda\vdash n\\ l(\lambda)=k}} c_\lambda(t) = \sum_{G_{\bar{a}}\in AO_k(G)} t^{\mathrm{asc}_{\overrightarrow{G}}(G_{\bar{a}})}, \tag{5.21}$$

where G is the underlying undirected graph of $\overrightarrow{G}$, $AO_k(G)$ is the set of acyclic orientations of G with exactly k sinks and $\mathrm{asc}_{\overrightarrow{G}}(G_{\bar{a}})$ is the number of directed edges of $\overrightarrow{G}$ that are oriented as in $G_{\bar{a}}$.

Corollary 5.5.8. *Let $\overrightarrow{G}$ be a digraph on n vertices such that $X_{\overrightarrow{G}}(\mathbf{x},t)$ is symmetric. Then*

$$c_{(n)}(t) = \sum_{G_{\bar{a}}\in AO_1(G)} t^{asc_{\overrightarrow{G}}(G_{\bar{a}})}.$$

So for any symmetric $X_{\overrightarrow{G}}(\mathbf{x},t)$, the coefficient of $e_n(\mathbf{x})$ in the e-basis expansion is a polynomial in t with nonnegative coefficients.

For the directed path and the directed cycle, we can refine (5.21) by giving a combinatorial interpretation of each $c_\lambda(t)$ in terms of acyclic orientations. We already know that the $c_\lambda(t)$ have positive coefficients by the formula for the directed path given in [54, Theorem 7.2] (see Theorem 4.2.3 of this thesis) and by our formula for the directed cycle (see Theorem 5.5.2 and Corollary 5.5.4), but perhaps these interpretations can be generalized to show e-positivity for a larger class of graphs.

For the next proposition, let $\overrightarrow{C_n} = ([n], E)$ denote the directed cycle, where

$$E = \{(i, i+1) \mid 1 \leq i < n\} \cup \{(n, 1)\},$$

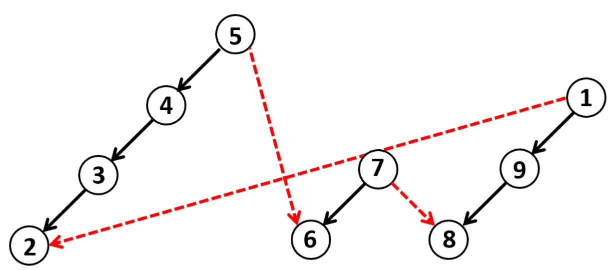
and let C_n denote its underlying undirected graph. For an acyclic orientation of C_n, denoted $G_{\bar{a}}$, we say that i and j are *consecutive sinks* of $G_{\bar{a}}$ if i and j are both sinks of $G_{\bar{a}}$ and there are no other sinks in the circular interval $[i, j]$.

Proposition 5.5.9. *Let* $X_{\overrightarrow{C_n}}(\mathbf{x}, t) = \sum_{\lambda \vdash n} c_\lambda(t) e_\lambda(\mathbf{x})$. *Then*

$$c_\lambda(t) = \sum_{G_{\bar{a}} \in AO_\lambda(C_n)} t^{\mathrm{asc}_{\overrightarrow{C_n}}(G_{\bar{a}})},$$

where $AO_\lambda(C_n)$ *is the set of all acyclic orientations* $G_{\bar{a}}$ *of* C_n *such that the number of vertices between consecutive sinks of* $G_{\bar{a}}$ *is* $\lambda_1 - 1$, $\lambda_2 - 1$,...,$\lambda_k - 1$ *in any order and* $\mathrm{asc}_{\overrightarrow{C_n}}(G_{\bar{a}})$ *is the number of directed edges of* $\overrightarrow{C_n}$ *that are oriented as in* $G_{\bar{a}}$.

Example 5.5.10. In the acyclic orientation of C_9 shown below, there are 3 vertices between sinks **2** and **6**, 1 vertex between sinks **6** and **8** and 2 vertices between sinks **8** and **2**, so this corresponds to e_{432}. There are 3 edges that match the original cyclic orientation of $\overrightarrow{C_9}$, shown by the dotted red arrows, hence this acyclic orientation corresponds to $t^3 e_{432}$.



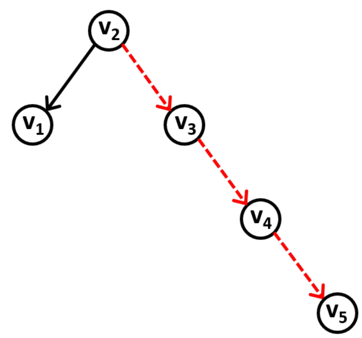
Proof. By Corollary 5.5.4

$$c_\lambda = \sum_{\mu : \lambda(\mu) = \lambda} \mu_1 t[\mu_1 - 1]_t t[\mu_2 - 1]_t \cdots t[\mu_{l(\lambda)} - 1]_t, \tag{5.22}$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_{l(\lambda)})$ is a composition of n, $l(\lambda)$ is the length of λ, and $\lambda(\mu) = \lambda$ means that when the parts of μ are written in decreasing order, this is the partition λ.

It follows from this that $c_\lambda = 0$ if any of the parts of $\lambda = 1$. We also have $AO_\lambda(C_n)$ is empty in that case, which means that the result holds in that case. We can now assume that λ has no parts of size 1.

For $a, b \in \mathbb{P}$ with $1 \le b < a$, define a *mountain*, $\overrightarrow{M}_{a,b} = (V, E)$, as a digraph on a vertices $V = \{v_1, v_2, \cdots, v_a\}$ with edge set $E = \{(v_i, v_{i-1}) \mid 1 < i \le a-b\} \cup \{(v_i, v_{i+1}) \mid a-b \le i < a\}$. We will say v_1 is the first vertex of the mountain and v_a is the last. For each $i = 1, 2, \cdots, a-1$, we say that v_{i+1} is the successor of v_i and v_i is the predecessor of v_{i+1}. Below, we show $\overrightarrow{M}_{5,3}$.



We can obtain an acyclic orientation of C_n from each term of the inner sum of (5.22) as follows. For each $1 \le i \le, l(\lambda)$, suppose we choose the t^{j_i} term from the $t[\mu_i - 1]_t$ factor. From this choice of j_i's, we can create a sequence of mountains, $\overrightarrow{M}_{\mu_1+1,j_1}, \overrightarrow{M}_{\mu_2+1,j_2}, \cdots \overrightarrow{M}_{\mu_{l(\lambda)}+1,j_{l(\lambda)}}$, on pairwise disjoint vertex sets. Then we attach the mountains by identifying the last vertex of $\overrightarrow{M}_{\mu_i+1,j_i}$ with the first vertex of $\overrightarrow{M}_{\mu_{i+1}+1,j_{i+1}}$ for $1 \le i < l(\lambda)$ and by identifying the last vertex of $\overrightarrow{M}_{\mu_{l(\lambda)}+1,j_{l(\lambda)}}$ with the first vertex of $\overrightarrow{M}_{\mu_1+1,j_1}$.

We will place the label **1** on one of the vertices, v, from $\overrightarrow{M}_{\mu_1+1,j_1}$, excluding the last vertex, so the μ_1 factor in (5.22) is for our μ_1 choices. We label the successor of v with **2** and continue labeling successors in order until we reach the predecessor of v.

It should be clear that we get a unique acyclic orientation in this manner and that every acyclic orientation can be built with this method. This proves our proposition.

□

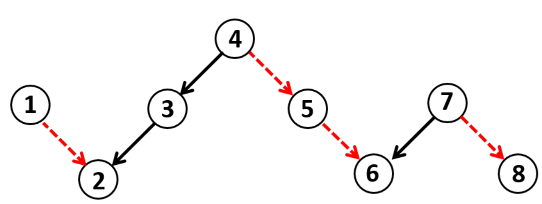
For the following proposition, let $\overrightarrow{P_n} = ([n], E)$ denote the directed path, where $E = \{(i, i+1) \mid 1 \leq i < n\}$, and let P_n denote the underlying undirected graph. For an acyclic orientation of P_n, denoted $G_{\bar{a}}$, we say that i and j are *consecutive sinks* of $G_{\bar{a}}$ if i and j are both sinks of $G_{\bar{a}}$, and there are no other sinks in the circular interval $[i, j]$. Notice that this includes the sink with the largest label and the sink with the smallest label.

Proposition 5.5.11. *Let* $X_{\overrightarrow{P_n}}(\mathbf{x}, t) = \sum_{\lambda \vdash n} c_\lambda e_\lambda$. *Then*

$$c_\lambda = \sum_{G_{\bar{a}} \in AO_\lambda(P_n)} t^{\mathrm{asc}_{\overrightarrow{P_n}}(G_{\bar{a}})},$$

where $AO_\lambda(P_n)$ *is the set of all acyclic orientations of* P_n *such that the number of vertices between consecutive sinks is* $\lambda_1 - 1$, $\lambda_2 - 1, \ldots, \lambda_k - 1$ *in any order and* $\mathrm{asc}_{\overrightarrow{P_n}}(G_{\bar{a}})$ *is the number of directed edges of* $\overrightarrow{P_n}$ *that are oriented as in* $G_{\bar{a}}$.

Example 5.5.12. In the acyclic orientation of P_8 shown below, there are 3 vertices between sinks **2** and **6**, 1 vertex between sinks **6** and **8** and 1 vertex between sinks **8** and **2**, so this corresponds to e_{422}. There are 4 edges that match the original orientation of $\overrightarrow{P_8}$, shown by the dotted red arrows, hence this acyclic orientation corresponds to $t^4 e_{422}$.



Proof. In [54, Theorem 7.2] (see Theorem 4.2.3 of this thesis), Shareshian and Wachs showed that

$$\sum_{n\geq 0} X_{P_n}(\mathbf{x},t)z^n = \frac{\sum\limits_{k\geq 0} e_k(\mathbf{x})z^k}{1 - t\sum\limits_{k\geq 2}[k-1]_t e_k(\mathbf{x})z^k}.$$

From this we can get that for $n \geq 1$,

$$X_{\overrightarrow{P_n}}(\mathbf{x},t) = \sum_{\lambda\vdash n} e_\lambda(\mathbf{x}) \sum_{\mu:\lambda(\mu)=\lambda} [\mu_1]_t t[\mu_2-1]_t t[\mu_3-1]_t \cdots t[\mu_{l(\lambda)}-1]_t. \tag{5.23}$$

(see [52, Table 1]).

We can obtain an acyclic orientation of P_n from each term of the inner sum of (5.23) as follows. For each $2 \leq i \leq l(\lambda)$, suppose we choose t^{j_i} from the $t[\mu_i - 1]_t$ factor. From this choice of j_i's, we can create a sequence of mountains, $\overrightarrow{M}_{\mu_2+1,j_2}, \cdots \overrightarrow{M}_{\mu_{l(\lambda)}+1,j_{l(\lambda)}}$, with disjoint vertex sets. Then we attach the mountains by identifying the last vertex of $\overrightarrow{M}_{\mu_i+1,j_i}$ with the first vertex of $\overrightarrow{M}_{\mu_{i+1}+1,j_{i+1}}$ for $2 \leq i < l(\lambda)$.

Now suppose we choose the t^j term from the $[\mu_1]_t$ factor. Then let $\overrightarrow{Q}_1 = (V,E)$ denote the digraph with vertex set $\{v_1, v_2, \cdots, v_{j+1}\}$ and edge set $E = \{(v_i, v_{i+1}) \mid 1 \leq i \leq j\}$. We will say v_1 is the first vertex of $\overrightarrow{Q}_1$ and v_{j+1} is the last. For each $i = 1, 2, \cdots, j$, we say that v_{i+1} is the successor of v_i and v_i is the predecessor of v_{i+1}. Let $\overrightarrow{Q}_2 = (V,E)$ denote the digraph with vertex set $\{v_1, v_2, \cdots, v_{\mu_1-j}\}$ and edge set $E = \{(v_i, v_{i-1}) \mid 1 < i \leq \mu_1 - j\}$. We will say v_1 is the first vertex of $\overrightarrow{Q}_2$ and v_{μ_1-j} is the last. For each $i = 1, 2, \cdots, \mu_1 - j - 1$, we say that v_{i+1} is the successor of v_i and v_i is the predecessor of v_{i+1}.

Then identify the last vertex of $\overrightarrow{Q}_1$ with the first vertex of $\overrightarrow{M}_{\mu_2+1,j_2}$ and identify the first vertex of $\overrightarrow{Q}_2$ with the last vertex of $\overrightarrow{M}_{\mu_{l(\lambda)+1},j_{l(\lambda)}}$.

Label the resulting digraph by placing **1** on the first vertex, v, of $\overrightarrow{Q}_1$. Label the successor of v with **2**, and continue labeling successors in order until all vertices are labeled.

It should be clear that we get a unique acyclic orientation in this manner and that every acyclic orientation can be built with this method. This proves our proposition.

□

We would like to point out that a weaker conjecture than Conjecture 5.5.1 is that the chromatic quasisymmetric functions of all circular indifference digraphs are Schur-positive. To our knowledge this is an open question. In the Schur basis expansions of Gasharov [29] for the chromatic symmetric functions of incomparability graphs of (3+1)-free posets and of Shareshian and Wachs [52] for the chromatic quasisymmetric functions of natural unit interval graphs, the coefficients are interpreted in terms of P-tableau, which use the structure of the poset associated with the graph. Since circular indifference digraphs are not incomparability graphs of posets in general, one would need a different type of tableau to describe the coefficients.

Chapter 6

Restricted Smirnov words

A proper coloring of the path P_n can be viewed as a word over the positive integers $\mathbb{P}$ where adjacent letters are distinct. These words are sometimes called *Smirnov words* (after [33], see also [57]). In fact, the chromatic quasisymmetric function $X_{P_n}(\mathbf{x}, t)$ of the path is equal to the descent enumerator of Smirnov words, defined by

$$W_n(\mathbf{x}, t) := \sum_{w \in W_n} t^{\mathrm{des}(w)} \mathbf{x}_w,$$

where W_n is the set of Smirnov words of length n and for $w \in W_n$ we let

$$\mathrm{des}(w) := |\{i \in [n-1] \mid w_i > w_{i+1}\}|.$$

In this chapter we study the descent enumerators of restricted Smirnov words, where we put restrictions on the relationship between the first and last letter of each word. We define the restricted descent enumerators

$$W_n^{<}(\mathbf{x}, t) := \sum_{\substack{w \in W_n \\ w_1 < w_n}} t^{\mathrm{des}(w)} \mathbf{x}_w,$$

$$W_n^{>}(\mathbf{x},t) \coloneqq \sum_{\substack{w \in W_n \\ w_1 > w_n}} t^{\operatorname{des}(w)} \mathbf{x}_w,$$

and

$$W_n^{=}(\mathbf{x},t) \coloneqq \sum_{\substack{w \in W_n \\ w_1 = w_n}} t^{\operatorname{des}(w)} \mathbf{x}_w.$$

It is an exercise of Grinberg and Reiner [34] that these restricted descent enumerators are symmetric. In this chapter we present our joint work with Wachs [22] where we expand these restricted descent enumerators in various bases.

In Section 6.1 we discuss some basic properties of restricted Smirnov word descent enumerators and describe their relationship with chromatic quasisymmetric functions. In Section 6.2 we present e-basis generating function formulas for the restricted Smirnov word descent enumerators $W_n^{<}(\mathbf{x},t)$, $W_n^{>}(\mathbf{x},t)$, and $W_n^{=}(\mathbf{x},t)$ and show that $W_n^{<}(\mathbf{x},t)$ and $W_n^{>}(\mathbf{x},t)$ are e-positive and e-unimodal. We use these formulas to derive an e-basis expansion of a variation of the Smirnov word descent enumerator $\tilde{W}_n(\mathbf{x},t)$ involving cyclic descents.

In Section 6.3 we provide expansions for the various descent enumerators in terms of Gessel's fundamental quasisymmetric function basis. By applying the stable principal specialization to our F-basis and e-basis expansions, we obtain variations of the q-Eulerian polynomials $A_n(q,t)$ studied by Shareshian and Wachs and defined in Section 1.2, that involve the permutation statistic $\operatorname{maj}_{\geq 2}$ paired with descents, cyclic descent, and cyclic ascents. We present these expansions in Section 6.4. In Section 6.5 we use our F-basis expansions to find p-basis expansions of these descent enumerators and give a combinatorial interpretation for their coefficients.

From the e-basis expansion of the restricted Smirnov word descent enumerators, we can derive an e-basis expansion of the chromatic quasisymmetric function of the labeled cycle, C_n, which is e-positive. Our results on this can be found in Section 6.6. Although it follows from our e-basis expansion that the coefficient of t in the restricted Smirnov word descent enumerators are symmetric functions, in Section 6.7 we present a combinatorial proof of their symmetry.

6.1 Basic definitions and properties

In this section, we define the restricted Smirnov word descent enumerators that we will be studying in the following sections and state some relationships between them, as well as their relationship to chromatic quasisymmetric functions.

We define the descent enumerator of Smirnov words of length n to be

$$W_n(\mathbf{x},t) = \sum_{w \in W_n} t^{\mathrm{des}(w)} \mathbf{x}_w,$$

where W_n is the set of Smirnov words and $\mathrm{des}(w) = |\{i \in [n-1] \mid w_i > w_{i+1}\}|$. Let $P_n = ([n], E)$ be the labeled path. By reading each proper coloring of P_n in reverse, we see that $W_n(\mathbf{x},t) = X_{P_n}(\mathbf{x},t)$. Hence the Shareshian and Wachs formula for $X_{P_n}(\mathbf{x},t)$ given in Theorem 4.2.3 is equivalent to the generating function formula,

$$\sum_{n \geq 0} W_n(\mathbf{x},t) z^n = \frac{\sum_{k \geq 0} e_k(\mathbf{x}) z^k}{1 - t \sum_{k \geq 2} [k-1]_t e_k(\mathbf{x}) z^k}. \tag{6.1}$$

.

Similarly we can define a circular version of $W_n(\mathbf{x},t)$ given by

$$\tilde{W}_n^{\neq}(\mathbf{x},t) = \sum_{\substack{w \in W_n \\ w_1 \neq w_n}} t^{\mathrm{cdes}(w)} \mathbf{x}_w,$$

where $\mathrm{cdes}(w)$ is the number of *cyclic descents* of w, that is

$$\mathrm{cdes}(w) = |\{i \in [n] \mid w_i > w_{i+1}\}|$$

with $w_{n+1} \coloneqq w_1$. Let $\overrightarrow{C_n}$ be the directed cycle described in Example 5.1.6. We see that by reading each proper coloring of $\overrightarrow{C_n}$ in the reverse order, we have $\tilde{W}_n^{\neq}(\mathbf{x},t) = X_{\overrightarrow{C_n}}(\mathbf{x},t)$. It follows that our formula given in Theorem 5.5.2 is equivalent to the generating function formula,

$$\sum_{n\geq 2} \tilde{W}_n^{\neq}(\mathbf{x},t)z^n = \frac{t\sum\limits_{k\geq 2} k[k-1]_t e_k(\mathbf{x})z^k}{1-t\sum\limits_{k\geq 2}[k-1]_t e_k(\mathbf{x})z^k}. \tag{6.2}$$

Now let us define the following descent enumerators of restricted Smirnov words that we will study in this chapter:

$$W_n^{<}(\mathbf{x},t) \coloneqq \sum_{\substack{w\in W_n \\ w_1<w_n}} t^{\mathrm{des}(w)}\mathbf{x}_w,$$

$$W_n^{>}(\mathbf{x},t) \coloneqq \sum_{\substack{w\in W_n \\ w_1>w_n}} t^{\mathrm{des}(w)}\mathbf{x}_w,$$

and

$$W_n^{=}(\mathbf{x},t) \coloneqq \sum_{\substack{w\in W_n \\ w_1=w_n}} t^{\mathrm{des}(w)}\mathbf{x}_w.$$

These refine $W_n(\mathbf{x},t)$ and $\tilde{W}_n^{\neq}(\mathbf{x},t)$, because we have that

$$W_n(\mathbf{x},t) = W_n^{<}(\mathbf{x},t) + W_n^{>}(\mathbf{x},t) + W_n^{=}(\mathbf{x},t) \tag{6.3}$$

and

$$\tilde{W}_n^{\neq}(\mathbf{x},t) = tW_n^{<}(\mathbf{x},t) + W_n^{>}(\mathbf{x},t). \tag{6.4}$$

In addition, we consider a few more descent enumerators of Smirnov words, given by

$$W_n^{\neq}(\mathbf{x},t) \coloneqq \sum_{\substack{w\in W_n \\ w_1\neq w_n}} t^{\mathrm{des}(w)}\mathbf{x}_w,$$

$$\tilde{W}_n(\mathbf{x}, t) := \sum_{w \in W_n} t^{\mathrm{cdes}(w)} \mathbf{x}_w,$$

and

$$\tilde{W}_n^a(\mathbf{x}, t) := \sum_{w \in W_n} t^{\mathrm{casc}(w)} \mathbf{x}_w,$$

where $\mathrm{casc}(w)$ is the number of *cyclic ascents* of w defined by

$$\mathrm{casc}(w) = |\{i \in [n] : w_i \leq w_{i+1}\}|$$

with $w_{n+1} := w_1$.

We will use the fact that

$$W_n^{\neq}(\mathbf{x}, t) = W_n^{<}(\mathbf{x}, t) + W_n^{>}(\mathbf{x}, t)$$

and

$$\tilde{W}_n(\mathbf{x}, t) = \tilde{W}_n^{\neq}(\mathbf{x}, t) + W_n^{=}(\mathbf{x}, t).$$

Note that there is a natural involution on Smirnov words defined by reversing each word. So if $w = w_1 w_2 \cdots w_n \in W_n$, then we can define $w^{rev} := w_n w_{n-1} \cdots w_1$ and we see that $w^{rev} \in W_n$. When we reverse a word, descents become ascents and vice versa, so we have that $\mathrm{des}(w) = \mathrm{asc}(w^{rev}) = n - 1 - \mathrm{des}(w^{rev})$ and similarly $\mathrm{cdes}(w) = \mathrm{casc}(w^{rev}) = n - \mathrm{cdes}(w^{rev})$. Hence using this involution, we get the following identities:

$$W_n^{>}(\mathbf{x}, t) = t^{n-1} W_n^{<}(\mathbf{x}, t^{-1}) \tag{6.5}$$

and

$$\tilde{W}_n^a(\mathbf{x}, t) = t^n \tilde{W}_n(\mathbf{x}, t^{-1}). \tag{6.6}$$

Using the same involution, it is easy to see that if the roles of $W^{<}(\mathbf{x}, t)$ and $W^{>}(\mathbf{x}, t)$ are switched in (6.5) and the roles of $\tilde{W}_n(\mathbf{x}, t)$ and $\tilde{W}_n^a(\mathbf{x}, t)$ are switched in (6.6), these identities still hold.

6.2 Expansion in the elementary symmetric function basis

In this section we present and prove our main result on restricted Smirnov word enumerators, giving generating function formulas for them in terms of the elementary symmetric function basis.

Theorem 6.2.1. *We have*

$$\sum_{n\geq 1} W_n^<(\mathbf{x},t)z^n = \frac{1}{D(\mathbf{x},t,z)}\sum_{i\geq 2} a_i(t)\ e_i(\mathbf{x})z^i \tag{6.7}$$
$$\sum_{n\geq 1} W_n^>(\mathbf{x},t)z^n = \frac{1}{D(\mathbf{x},t,z)}\sum_{i\geq 2} b_i(t)\ e_i(\mathbf{x})z^i \tag{6.8}$$
$$\sum_{n\geq 1} W_n^=(\mathbf{x},t)z^n = \frac{1}{D(\mathbf{x},t,z)}(e_1(\mathbf{x})z - \sum_{i\geq 2} c_i(t)\ e_i(\mathbf{x})z^i), \tag{6.9}$$

where

$$\begin{aligned} D(\mathbf{x},t,z) &:= 1-\sum_{i\geq 2} t[i-1]_t e_i(\mathbf{x})z^i, \qquad (6.10)\\ a_i(t) &:= \frac{d}{dt}[i]_t \qquad = \sum_{j=0}^{i-2}(j+1)t^j,\\ b_i(t) &:= t^{i-1}a_i(t^{-1}) \quad = \sum_{j=1}^{i-1}(i-j)t^j,\\ c_i(t) &:= it[i-2]_t, \end{aligned}$$

for all $i \geq 2$.

Before proving the theorem, we observe that

$$\begin{aligned} a_i(t)+b_i(t) &= 1+(i+1)t+(i+1)t^2+\cdots+(i+1)t^{i-2}+t^{i-1}\\ &= [i]_t + it[i-2]_t. \end{aligned}$$

Hence,

$$a_i(t) + b_i(t) - c_i(t) = [i]_t,$$

which shows that Theorem 6.2.1 refines (6.1) since $W_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t) + W_n^=(\mathbf{x}, t) = W_n(\mathbf{x}, t)$. Also

$$ta_i(t) + b_i(t) = it[i-1]_t,$$

which shows that Theorem 6.2.1 also refines (6.2) since $tW_n^<(\mathbf{x}, t) + W_n^>(\mathbf{x}, t) = \tilde{W}_n^{\neq}(\mathbf{x}, t)$.

We prove (6.9) first. Then we use (6.9), (6.1), and (6.2) to derive (6.7). Equation (6.8) follows from (6.7). Our proof of (6.9) uses the transfer-matrix method discussed in [60] and borrows ingredients from the proof of Theorem 5.5.2. See the proof of Theorem 5.5.2 for a review of the transfer matrix method.

We will need the following result from [59, Section 4.7]. Let $\overrightarrow{G} = ([k], E)$ be a digraph with edge weights $w : E \to R$, where R is some commutative ring. Let A be the weighted adjacency matrix of $\overrightarrow{G}$. For each $i, j \in [k]$, define $\mathcal{W}_{i,j,n}$ to be the set of walks of length n from i to j on $\overrightarrow{G}$ and let

$$U_{i,j,n} := \sum_{w \in \mathcal{W}_{i,j,n}} \mathrm{wt}(w).$$

Theorem 4.7.2 of [59] states that for all $i, j \in [k]$,

$$\sum_{n \geq 0} U_{i,j,n} z^n = \frac{(-1)^{i+j} \det(I - zA : j, i)}{\det(I - zA)}, \tag{6.11}$$

where $(B : j, i)$ is the matrix obtained from B by removing row j and column i.

Proof of (6.9). As in the proof of Theorem 5.5.2, we view a Smirnov word $w_1 w_2 \dots w_n$ over the alphabet $[k]$ as a walk $w_1, w_2, \dots, w_n$ of length $n-1$ on the digraph $\overrightarrow{G} = ([k], E)$, where

$$E = \{(i, j) : i, j \in [k] \text{ and } i \neq j\},$$

and we set

$$\mathrm{wt}((i,j)) := \begin{cases} x_j & \text{if } i < j \\ \\ tx_j & \text{if } i > j. \end{cases}$$

Note that if w is a Smirnov word over the alphabet $[k]$ then

$$t^{\mathrm{des}(w)} x_w = x_{w_1}\, \mathrm{wt}(w),$$

where w_1 is the first letter of w. Hence

$$W_n^{=}(x_1, \dots, x_k, t) := W_n^{=}(x_1, \dots, x_k, 0, 0, \dots, t) = \sum_{i=1}^{k} x_i U_{i,i,n-1}.$$

It follows from this and (6.11) that

$$\begin{aligned} \sum_{n \geq 1} W_n^{=}(x_1, \dots, x_k, t) z^n &= z \sum_{i=1}^{k} x_i \sum_{n \geq 0} U_{i,i,n} z^n \\ &= z \sum_{i=1}^{k} x_i \frac{\det(I - zA : i, i)}{\det(I - zA)} \\ &= \frac{z \sum_{i=1}^{k} x_i \det(I - zA : i, i)}{\det(I - zA)}, \end{aligned} \tag{6.12}$$

where A is the weighted adjacency matrix of $\overrightarrow{G}$, i.e.,

$$A = \begin{bmatrix} 0 & x_2 & x_3 & \dots & x_k \\ tx_1 & 0 & x_3 & \dots & x_k \\ tx_1 & tx_2 & 0 & \dots & x_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ tx_1 & tx_2 & tx_3 & \dots & 0 \end{bmatrix}.$$

In the proof of Theorem 5.5.2, we showed that[1]

$$\det(I - zA) = 1 - \sum_{j\geq 2} e_j(x_1, \ldots, x_k)\, t[j-1]_t z^j. \tag{6.13}$$

It follows that

$$\det(I - zA : i, i) = 1 - \sum_{j\geq 2} e_j(x_1, \ldots, \hat{x}_i, \ldots, x_k)\, t[j-1]_t z^j,$$

where $\hat{x}_i$ denotes deletion of x_i. Multiplying both sides by x_i and summing over all $i \in [k]$ yields,

$$\sum_{i=1}^{k} x_i \det(I - zA : i, i) = \sum_{i=1}^{k} x_i - \sum_{j\geq 2} \sum_{i=1}^{k} x_i e_j(x_1, \ldots, \hat{x}_i, \ldots, x_k)\, t[j-1]_t z^j.$$

One can see that

$$\sum_{i=1}^{k} x_i e_j(x_1, \ldots, \hat{x}_i, \ldots, x_k) = (j+1) e_{j+1}(x_1, \ldots, x_k),$$

since both sides enumerate $(j+1)$-subsets of $[k]$ with a distinguished element. Hence

$$\sum_{i=1}^{k} x_i \det(I - zA : i, i) = e_1(x_1, \ldots, x_k) - \sum_{j\geq 2} (j+1) e_{j+1}(x_1, \ldots, x_k) t[j-1]_t z^j.$$

Upon multiplying both sides by z, we see that the numerator of the right hand side of (6.12) is

$$e_1(x_1, \ldots, x_k) z - \sum_{j\geq 3} j\, e_j(x_1, \ldots, x_k) t[j-2]_t z^j.$$

[1] This is obtained from the formula in the proof of Theorem 5.5.2 by replacing t with t^{-1} and each x_i with tx_i

It therefore follows from (6.12) and (6.13) that

$$\sum_{n\geq 1} W_n^{=}(x_1,\ldots,x_k,t)z^n = \frac{e_1(x_1,\ldots,x_k)z - \sum_{j\geq 3} j\, e_j(x_1,\ldots,x_k)t[j-2]_t z^j}{1-\sum_{j\geq 2} e_j(x_1,\ldots,x_k)\, t[j-1]_t z^j}.$$

The desired result (6.9) follows by taking the limit as k goes to infinity. □

Proof of (6.7). It follows from (6.3), (6.1), and (6.9) that

$$\begin{aligned}\sum_{n\geq 1} W_n^{<}(\mathbf{x},t)z^n + \sum_{n\geq 1} W_n^{>}(\mathbf{x},t)z^n &= \sum_{n\geq 1} W_n(\mathbf{x},t) - \sum_{n\geq 1} W_n^{=}(\mathbf{x},t)\\ &= \frac{B(\mathbf{x},t,z)}{D(\mathbf{x},t,z)},\end{aligned} \tag{6.14}$$

where

$$\begin{aligned}B(\mathbf{x},t,z) &= \sum_{i\geq 1}[i]_t e_i(\mathbf{x})z^i - (e_1(\mathbf{x})z - \sum_{i\geq 2} it[i-2]_t\ e_i(\mathbf{x})z^i)\\ &= \sum_{i\geq 2}[i]_t e_i(\mathbf{x})z^i + \sum_{i\geq 2} it[i-2]_t\ e_i(\mathbf{x})z^i\\ &= \sum_{i\geq 2}([i]_t + it[i-2]_t)e_i(\mathbf{x})z^i\,.\end{aligned}$$

It follows from (6.4) and (6.2) that

$$\begin{aligned}t(\sum_{n\geq 1} W_n^{<}(\mathbf{x},t)z^n) + \sum_{n\geq 1} W_n^{>}(\mathbf{x},t)z^n &= \sum_{n\geq 2} \tilde{W}_n^{\neq}(\mathbf{x},t)\\ &= \frac{C(\mathbf{x},t,z)}{D(\mathbf{x},t,z)},\end{aligned} \tag{6.15}$$

where

$$C(\mathbf{x},t,z) = \sum_{i\geq 2} it[i-1]_t\ e_i(\mathbf{x})z^i.$$

By subtracting (6.14) from (6.15), we obtain

$$
\begin{aligned}
(t-1)\sum_{n\geq 1} W_n^{<}(\mathbf{x},t)z^n &= \frac{C(\mathbf{x},t,z)-B(\mathbf{x},t,z)}{D(\mathbf{x},t,z)} \\
&= \frac{\sum_{i\geq 2}(it^{i-1}-[i]_t)e_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
(t-1)(1+2t+3t^2+\cdots+(i-1)t^{i-2}) &= (i-1)t^{i-1}-[i-1]_t \\
&= it^{i-1}-[i]_t.
\end{aligned}
$$

Hence

$$
\sum_{n\geq 1} W_n^{<}(\mathbf{x},t)z^n = \frac{\sum_{i\geq 2}(1+2t+3t^2+\cdots+(i-1)t^{i-2})\, e_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)}
$$

as desired. $\square$

Proof of (6.8). Recall that $W_n^{>}(\mathbf{x},t) = t^{n-1}W_n^{<}(\mathbf{x},t^{-1})$. It follows from this and (6.7) that

$$
\begin{aligned}
\sum_{n\geq 1} W_n^{>}(\mathbf{x},t)z^n &= t^{-1}\sum_{n\geq 1} W_n^{<}(\mathbf{x},t^{-1})(tz)^n \\
&= t^{-1}\frac{\sum_{i\geq 2} a_i(t^{-1})e_i(\mathbf{x})t^i z^i}{D(\mathbf{x},t^{-1},tz)} \\
&= \frac{\sum_{i\geq 2} b_i(t)e_i(\mathbf{x})z^i}{D(\mathbf{x},t^{-1},tz)}.
\end{aligned}
$$

Since $D(\mathbf{x},t^{-1},tz) = D(\mathbf{x},t,z)$, the result holds. $\square$

We obtain equivalent formulations of (6.7) and (6.8) by multiplying the numerators and denominators of the right side of the equations by $t-1$.

Corollary 6.2.2. *We have*

$$\sum_{n\geq 1} W_n^{<}(\mathbf{x},t)z^n = \frac{\sum_{i\geq 2}(it^{i-1}-[i]_t)\ e_i(\mathbf{x})z^i}{tE(\mathbf{x},z)-E(\mathbf{x},tz)} \tag{6.16}$$

$$\sum_{n\geq 1} W_n^{>}(\mathbf{x},t)z^n = \frac{\sum_{i\geq 2}(t[i]_t-it)\ e_i(\mathbf{x})z^i}{tE(\mathbf{x},z)-E(\mathbf{x},tz)}, \tag{6.17}$$

where $E(\mathbf{x},z) := \sum_{n\geq 0} e_n(\mathbf{x})z^n$.

We have the following immediate consequence of Theorem 6.2.1 and Lemma 5.5.3.

Corollary 6.2.3 (of Theorem 6.2.1)**.** *For all $n \geq 2$, $W_n^{<}(\mathbf{x},t)$ and $W_n^{>}(\mathbf{x},t)$ are e-positive.*

Note that it follows from Theorem 6.2.1 that the coefficient of $e_n(\mathbf{x})$ in the e-basis expansion of $W_n^{=}(\mathbf{x},t)$ is $-nt[n-2]_t$ if $n \geq 2$. Hence $W_n^{=}(\mathbf{x},t)$ fails to be e-positive. However, observe that the coefficient $c_\lambda(t)$ of $e_\lambda(\mathbf{x})$ is in $\mathbb{N}[t]$ if the smallest part of λ is 1, and $-c_\lambda(t) \in \mathbb{N}[t]$ otherwise.

Recall that

$$W_n^{\neq}(\mathbf{x},t) := \sum_{\substack{w\in W_n\\ w_1\neq w_n}} x_w t^{\mathrm{des}(w)}.$$

Since $W_n^{\neq}(\mathbf{x},t) = W_n^{<}(\mathbf{x},t) + W_n^{>}(\mathbf{x},t)$, it follows from Corollary 6.2.3 that $W_n^{\neq}(\mathbf{x},t)$ is e-positive. We can say more.

Corollary 6.2.4 (of Theorem 6.2.1)**.** *We have,*

$$\sum_{n\geq 1} W_n^{\neq}(\mathbf{x},t)z^n = \frac{\sum_{i\geq 2}([i]_t + it[i-2]_t)e_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)}. \tag{6.18}$$

Consequently, $W_n^{\neq}(\mathbf{x},t)$ is palindromic, e-positive, and e-unimodal, with center of symmetry $\frac{n-1}{2}$.

Proof. Equation (6.18) is evident from Theorem 6.2.1. The consequence follows from Lemma 5.5.3 since $[i]_t + it[i-2]_t$ is palindromic, positive, and unimodal with center of symmetry $\frac{i-1}{2}$. □

We defined two other variations on the descent enumerator given by

$$\tilde{W}_n(\mathbf{x},t) \coloneqq \sum_{w\in W_n} x_w t^{\mathrm{cdes}(w)}$$

$$\tilde{W}_n^a(\mathbf{x},t) \coloneqq \sum_{w\in W_n} x_w t^{\mathrm{casc}(w)},$$

where

$$\mathrm{cdes}(w) = |\{i \in [n] : w_i > w_{i+1}\}|,$$

$$\mathrm{casc}(w) = |\{i \in [n] : w_i \leq w_{i+1}\}|,$$

and $w_{n+1} \coloneqq w_1$.

From Theorem 6.2.1, we can also obtain the following expansions.

Corollary 6.2.5 (of Theorem 6.2.1)**.** *We have,*

$$\sum_{n\geq 1} \tilde{W}_n(\mathbf{x},t)z^n = \frac{\sum_{i\geq 1} it^{i-1}e_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)} = \frac{\frac{\partial}{\partial t}\sum_{i\geq 0} e_i(\mathbf{x})(tz)^i}{D(\mathbf{x},t,z)} \tag{6.19}$$

$$\sum_{n\geq 1} \tilde{W}_n^a(\mathbf{x},t)z^n = \frac{t\sum_{i\geq 1} ie_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)} = \frac{tz\frac{\partial}{\partial z}\sum_{i\geq 0} e_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)}. \tag{6.20}$$

Consequently, $\tilde{W}_n(\mathbf{x},t)$ and $\tilde{W}_n^a(\mathbf{x},t)$ are e-positive.

Remark 6.2.6. For the sake of comparison, note that equation (6.1) can be restated as

$$1 + \sum_{n\geq 1} W_n(\mathbf{x},t)z^n = \frac{\sum_{i\geq 0} e_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)}.$$

Proof. Equation (6.19) follows from the fact that

$$\tilde{W}_n(\mathbf{x}, t) = \tilde{W}_n^{\neq}(\mathbf{x}, t) + W_n^{=}(\mathbf{x}, t)$$

and equations (6.2) and (6.9). Alternatively, one uses the fact that

$$\tilde{W}_n(\mathbf{x}, t) = tW_n^{<}(\mathbf{x}, t) + (W_n(\mathbf{x}, t) - W_n^{<}(\mathbf{x}, t)) \tag{6.21}$$

and equations (6.7) and (6.1).

Equation (6.20) follows from equation (6.19) and the fact that $\tilde{W}_n^a(\mathbf{x}, t) = t^n \tilde{W}_n(\mathbf{x}, t^{-1})$. So one can replace t by t^{-1} and z by tz in (6.19) to obtain (6.20).

The consequence follows from Lemma 5.5.3. □

Note that it follows from (6.19) that the coefficient of $e_{3,2}$ in $\tilde{W}_5(\mathbf{x}, t)$ is $(3t^2)(t) + (2t)(t[2]_t) = 5t^3 + 2t^2$, which is not palindromic. Hence $\tilde{W}_5(\mathbf{x}, t)$ fails to be palindromic. Similarly $\tilde{W}_5^a(\mathbf{x}, t)$ fails to be palindromic.

6.3 Expansion in Gessel's fundamental quasisymmetric function basis

In [52], Shareshian and Wachs provided a formula for the chromatic quasisymmetric functions of labeled incomparability graphs in Gessel's fundamental quasisymmetric function basis in terms of P-descents, which is given in Theorem 5.2.1. Note that the labeled path $P_n = ([n], E)$ with edge set $E = \{\{i, i+1\} \mid i \subseteq [n-1]\}$ is the incomparability graph of the poset P on $[n]$ defined by $i <_P j$ if $j - i \geq 2$. For $\sigma \in \mathfrak{S}_n$, define $\mathrm{DES}_{\geq 2}(\sigma) = \{i \in [n-1] \mid \sigma(i) - \sigma(i+1) \geq 2\}$. Then we can see that for any $\sigma \in \mathfrak{S}_n$, we have $\mathrm{DES}_P(\sigma) = \mathrm{DES}_{\geq 2}(\sigma)$. Also for $G = P_n$ one can check that $\mathrm{inv}_G(\sigma) = \mathrm{des}(\sigma^{-1})$. Hence by applying (5.3) to $X_{P_n}(\mathbf{x}, t) = W_n(\mathbf{x}, t)$, we obtain the following expansion:

$$\omega W_n(\mathbf{x},t) = \sum_{\sigma\in\mathfrak{S}_n} F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}(\mathbf{x})t^{\mathrm{des}(\sigma)}. \tag{6.22}$$

Note that this is a different expansion than the one obtained by applying (5.4).

In this section, we give analogous expansions for $\omega W_n^<(\mathbf{x},t)$, $\omega W_n^<(\mathbf{x},t)$, $\omega\tilde{W}_n(\mathbf{x},t)$, and $\omega\tilde{W}_n^a(\mathbf{x},t)$. These expansions immediately yield expansions for other descent enumerators such as $\tilde{W}_n^{\neq}(\mathbf{x},t) = X_{\vec{C}_n}(\mathbf{x},t)$ and for the chromatic quasisymmetric function of the directed cycle $X_{C_n}(\mathbf{x},t)$. These expansion formulas are different from the ones obtained by applying (5.4).

For $\sigma \in \mathfrak{S}_n$, define

$$\mathrm{ASC}_{\geq 2}(\sigma) := \{i \in [n-1] : \sigma(i+1) - \sigma(i) \geq 2\}.$$

Theorem 6.3.1. *For all $n \geq 1$,*

$$\omega W_n^<(\mathbf{x},t) = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(1)<\sigma(n)}} t^{\mathrm{des}(\sigma)} F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}(\mathbf{x}) \tag{6.23}$$

$$\omega W_n^>(\mathbf{x},t) = \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(1)>\sigma(n)}} t^{\mathrm{des}(\sigma)} F_{n,\mathrm{ASC}_{\geq 2}(\sigma^{-1})}(\mathbf{x}) \tag{6.24}$$

$$\omega\tilde{W}_n(\mathbf{x},t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{cdes}(\sigma)} F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}(\mathbf{x}) \tag{6.25}$$

$$\omega\tilde{W}_n^a(\mathbf{x},t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{casc}(\sigma)} F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}(\mathbf{x}). \tag{6.26}$$

Proof of (6.23). The first part of the proof is similar to that of [52, Theorem 3.1] and Theorem 5.2.2 applied to the n-cycle C_n. The second part diverges somewhat from these proofs.

Part 1: Given an acyclic orientation $\bar{a}$ of C_n, let $E_{\bar{a}}(C_n)$ be the set of directed edges of C_n under the orientation $\bar{a}$. Let $AO_n^>$ be the set of acyclic orientations $\bar{a}$ of

C_n such that $(n,1) \in E_{\bar{a}}(C_n)$. For each $\bar{a} \in AO_n^>$, let $W_{\bar{a}} \subseteq W_n$ be the set of Smirnov words $w = w_1 w_2 \cdots w_n$ such that the following hold:

- $w_n < w_1$,
- $w_i < w_{i+1}$ if $(i, i+1) \in E_{\bar{a}}(C_n)$ and $i \in [n-1]$,
- $w_i > w_{i+1}$ if $(i+1, i) \in E_{\bar{a}}(C_n)$ and $i \in [n-1]$.

Let $\mathrm{asc}(\bar{a})$ be the number of edges of $E_{\bar{a}}(C_n)$ of the form $(i, i+1)$ for $i \in [n-1]$. Then by reversing the Smirnov words, we can see that

$$W_n^<(\mathbf{x}, t) = \sum_{\substack{w \in W_n \\ w_1 > w_n}} \mathbf{x}_w t^{\mathrm{asc}(w)} = \sum_{\bar{a} \in AO_n^>} t^{\mathrm{asc}(\bar{a})} \sum_{w \in W_{\bar{a}}} \mathbf{x}_w. \tag{6.27}$$

Now for each acyclic orientation $\bar{a} \in AO_n^>$ define a poset $P_{\bar{a}}$ on $[n]$ by letting $i <_{P_{\bar{a}}} j$ if $(i,j) \in E_{\bar{a}}(C_n)$ and taking the transitive closure of these relations. Let us define a *labeling* of $P_{\bar{a}}$ to be a bijection from $P_{\bar{a}}$ to $[n]$. So a labeling is just a permutation in $\mathfrak{S}_n$. A labeling ρ is said to be decreasing if $\rho(i) > \rho(j)$ for all $i <_{P_{\bar{a}}} j$. For any labeling ρ of $P_{\bar{a}}$, let $L(P_{\bar{a}}, \rho)$ be the set of linear extensions of $P_{\bar{a}}$ with the labeling ρ.

Now fix a decreasing labeling $\rho_{\bar{a}}$ of $P_{\bar{a}}$ for each $\bar{a} \in AO_n^>$. For any subset $S \subseteq [n-1]$, define $n - S = \{i \mid n - i \in S\}$. Then by the theory of P-partitions [60, Corollary 7.19.5], we have that

$$\sum_{w \in W_{\bar{a}}} \mathbf{x}_w = \sum_{\sigma \in L(P_{\bar{a}}, \rho_{\bar{a}})} F_{n, n-\mathrm{DES}(\sigma)}, \tag{6.28}$$

where $\mathrm{DES}(\sigma)$ is the usual descent set of a permutation, i.e., $\mathrm{DES}(\sigma) = \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$.

Let $e : P_{\bar{a}} \to [n]$ be the identity labeling of $P_{\bar{a}}$, and hence $L(P_{\bar{a}}, e)$ is the set of linear extensions of $P_{\bar{a}}$ with its original labeling. Note that $\sigma \in L(P_{\bar{a}}, e)$ if and only if $\rho_{\bar{a}}\sigma \in L(P_{\bar{a}}, \rho_{\bar{a}})$, where $\rho_{\bar{a}}\sigma$ denotes the product of $\rho_{\bar{a}}$ and σ in $\mathfrak{S}_n$. Hence from

(6.28), we have

$$\sum_{w \in W_{\bar{a}}} \mathbf{x}_w = \sum_{\sigma \in L(P_{\bar{a}}, e)} F_{n, n-\mathrm{DES}(\rho_{\bar{a}}\sigma)}. \tag{6.29}$$

Note that if $\sigma \in L(P_{\bar{a}}, e)$ and $\bar{a} \in AO_n^{>}$ then $\sigma^{-1}(1) > \sigma^{-1}(n)$. Conversely, every permutation $\sigma \in \mathfrak{S}_n$ with $\sigma^{-1}(1) > \sigma^{-1}(n)$ is a linear extension in $L(P_{\bar{a}}, e)$ for a unique $\bar{a} \in AO_n^{>}$. Let $\bar{a}(\sigma)$ denote the unique acyclic orientation of C_n associated with σ. Now combining this with (6.27) and (6.29) yields,

$$W_n^{<}(\mathbf{x}, t) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^{-1}(1) > \sigma^{-1}(n)}} t^{\mathrm{asc}(\bar{a}(\sigma))} F_{n, n-\mathrm{DES}(\rho_{\bar{a}(\sigma)}\sigma)},$$

where recall $\rho_{\bar{a}(\sigma)}$ is a decreasing labeling of $P_{\bar{a}(\sigma)}$. Note that $\mathrm{asc}(\bar{a}(\sigma)) = \mathrm{des}((\sigma^R)^{-1})$, where σ^R is the reverse of σ. Hence

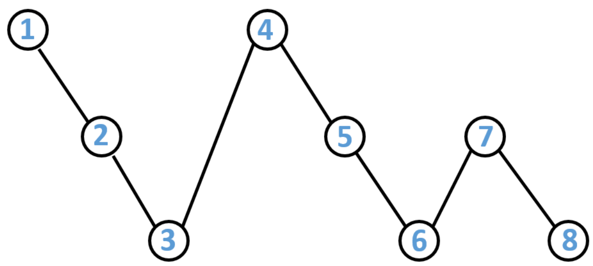
$$W_n^{<}(\mathbf{x}, t) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma^{-1}(1) > \sigma^{-1}(n)}} t^{\mathrm{des}((\sigma^R)^{-1})} F_{n, n-\mathrm{DES}((\rho_{\bar{a}(\sigma)}\sigma)}. \tag{6.30}$$

Part 2: As in the proof of [52, Theorem 3.1], our next step is to construct a particular decreasing labeling $\tilde{\rho}_{\bar{a}(\sigma)}$ of $P_{\bar{a}(\sigma)}$ for each $\sigma \in L(P_{\bar{a}}, e)$. However since C_n is not the incomparability graph of a poset, the construction used in the proof of [52, Theorem 3.1] does not work in this case. The construction used here is also quite different from that of Theorem 5.2.2. Let p be the "smallest" maximal element of $P_{\bar{a}(\sigma)}$ (that is, p is maximal in the poset $P_{\bar{a}(\sigma)}$ and is less than all the other maximal elements in the natural order on $[n]$) and let $\tilde{\rho}_{\bar{a}(\sigma)}(p) = 1$. Now remove p from the poset and let q be the smallest maximal element of the remaining poset and let $\tilde{\rho}_{\bar{a}(\sigma)}(q) = 2$. Continue this process inductively. It is clear that $\tilde{\rho}_{\bar{a}(\sigma)}$ is a decreasing labeling of $P_{\bar{a}(\sigma)}$.

Claim. If x and y are incomparable in $P := P_{\bar{a}(\sigma)}$, then $x < y$ implies $\tilde{\rho}_{\bar{a}(\sigma)}(x) < \tilde{\rho}_{\bar{a}(\sigma)}(y)$.

Proof of Claim. One can see this by drawing the Hasse diagram of P minus the edge $(n, 1)$ as a zig-zag path on $[n]$ with the elements of $[n]$ increasing as one moves from left to right. The path consists of up-segments and down-segments. An up-segment is a maximal chain of P of the form $a <_P a+1 <_P \cdots <_P a+j$, where $j \geq 1$, and a down-segment is a maximal chain with top and bottom removed unless it's 1 or n, of the form $a >_P a+1 >_P \cdots >_P a+j$, where $j \geq 0$.

Below we see an example of one such P on $[8]$.



In our example, the up-segments are $3 <_P 4$ and $6 <_P 7$, and the down segments are $1 >_P 2$, 5 and 8.

Between any two down-segments there is an up-segment. Let α_i be the ith segment from the left for each i. One can see that under the labeling $\tilde{\rho}_{\bar{a}(\sigma)}$, the segment α_1 gets the smallest labels, the segment α_2 gets the next smallest labels, and so on. Now if x and y are incomparable, they are in different segments α_i and α_j. Clearly if $x < y$ then $i < j$, which implies that x gets a smaller label then y. Hence, the claim holds.

Now we show that

$$\mathrm{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma) = [n-1] \setminus \mathrm{ASC}_{\geq 2}(\sigma), \tag{6.31}$$

for all $\sigma \in \mathfrak{S}_n$. If $i \in \mathrm{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)$ then $\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i) > \tilde{\rho}_{\bar{a}(\sigma)}\sigma(i+1)$. It thus follows from the claim that if $\sigma(i)$ and $\sigma(i+1)$ are incomparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i) > \sigma(i+1)$, which implies $i \notin \mathrm{ASC}_{\geq 2}(\sigma)$. On the other hand if $\sigma(i)$ and $\sigma(i+1)$ are comparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i+1)$ covers $\sigma(i)$ since $\sigma \in L(P_{\bar{a}(\sigma)}, e)$. This implies that either

$\sigma(i+1)=\sigma(i)+1$ or $\sigma(i+1)=\sigma(i)-1$. In either case, $i\notin \mathrm{ASC}_{\geq 2}(\sigma)$. Thus

$$\mathrm{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)\subseteq [n-1]\setminus \mathrm{ASC}_{\geq 2}(\sigma).$$

Conversely, if $i\notin \mathrm{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)$ then $\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i)<\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i+1)$. It thus follows from the claim that if $\sigma(i)$ and $\sigma(i+1)$ are incomparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i)<\sigma(i+1)$. Since j and $j+1$ are comparable in $P_{\bar{a}(\sigma)}$ for all $j\in[n-1]$, we have $\sigma(i+1)-\sigma(i)\geq 2$. Thus $i\in \mathrm{ASC}_{\geq 2}(\sigma)$. On the other hand if $\sigma(i)$ and $\sigma(i+1)$ are comparable in $P_{\bar{a}(\sigma)}$ then $\sigma(i)<_{P_{\bar{a}(\sigma)}}\sigma(i+1)$ since $\sigma\in L(P_{\bar{a}(\sigma)},e)$. But since ρ is a decreasing labeling $\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i)>\tilde{\rho}_{\bar{a}(\sigma)}\sigma(i+1)$, which contradicts our assumption that $i\notin \mathrm{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)$. Hence this case is impossible. We have shown

$$\mathrm{DES}(\tilde{\rho}_{\bar{a}(\sigma)}\sigma)\supseteq [n-1]\setminus \mathrm{ASC}_{\geq 2}(\sigma),$$

which completes the proof of (6.31).

Recall that the involution ω acts by $\omega F_{n,S}:=F_{n,[n-1]\setminus S}$. Hence by (6.31), equation (6.30) becomes

$$\begin{aligned}\omega W_n^{<}(\mathbf{x},t) &= \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^{-1}(1)>\sigma^{-1}(n)}} t^{\mathrm{des}((\sigma^R)^{-1})}F_{n,n-\mathrm{ASC}_{\geq 2}(\sigma)}\\ &= \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma^{-1}(1)<\sigma^{-1}(n)}} t^{\mathrm{des}(\sigma^{-1})}F_{n,\mathrm{DES}_{\geq 2}(\sigma)}\end{aligned}$$

□

Proof of (6.24). A similar proof can be given here. One can also use (6.23) to prove this. Indeed, by the involution on W_n which reverses Smirnov words, we obtain

$$W_n^{>}(\mathbf{x},t)=t^{n-1}W_n^{<}(\mathbf{x},t^{-1}).$$

By the involution on $\mathfrak{S}_n$, which reverses permutations,

$$\sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)>\sigma(n)}} t^{\mathrm{des}(\sigma)}F_{n,\mathrm{ASC}_{\geq 2}(\sigma^{-1})} = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)<\sigma(n)}} t^{n-1-\mathrm{des}(\sigma)}F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}.$$

The result now follows from (6.23). □

Proof of (6.25). We use the fact that $\tilde{W}_n(\mathbf{x},t) = tW_n^{<}(\mathbf{x},t) + (W_n(\mathbf{x},t) - W_n^{<}(\mathbf{x},t))$. By (6.22) and (6.23),

$$\omega W_n(\mathbf{x},t) - \omega W_n^{<}(\mathbf{x},t) = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)>\sigma(n)}} t^{\mathrm{des}(\sigma)}F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}.$$

It follows from this and (6.23) that

$$\begin{aligned}\tilde{W}_n(\mathbf{x},t) &= \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)<\sigma(n)}} t^{\mathrm{des}(\sigma)+1}F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})} + \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(1)>\sigma(n)}} t^{\mathrm{des}(\sigma)}F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}\\ &= \sum_{\sigma\in\mathfrak{S}_n} t^{\mathrm{cdes}(\sigma)}F_{n,\mathrm{DES}_{\geq 2}(\sigma^{-1})}.\end{aligned}$$

□

Proof of (6.26). This follows from (6.25) and the fact that $\tilde{W}_n^a(\mathbf{x},t) = t^n\tilde{W}_n(\mathbf{x},t^{-1})$. □

6.4 Specializations

There are various ways to specialize expansions in the fundamental quasisymmetric functions to obtain enumerative results. One way is by setting $x_i = 1$ if $i \in [m]$ and $x_i = 0$ otherwise, in a formal power series $f(\mathbf{x})$. Recall that we denote this specialization by $f(1^m)$. It is not difficult to show that (see [60, Section 7.19]),

$$F_{n,S}(1^m) = \binom{m+n-1-|S|}{n},$$

for all $S \subseteq [n-1]$. It is evident that

$$W_n(1^m,t) = \sum_{w \in W_n \cap [m]^n} t^{\mathrm{des}(w)}.$$

Hence by (6.22) and the fact that $\omega F_{n,S} = F_{n,[n-1]\setminus S}$,

$$\sum_{w \in W_n \cap [m]^n} t^{\mathrm{des}(w)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} \binom{m + |\operatorname{DES}_{\geq 2}(\sigma^{-1})|}{n},$$

for all $m, n \in \mathbb{P}$. From this we see that $\sum_{w \in W_n \cap [m]^n} t^{\mathrm{des}(w)}$ is a polynomial in m with coefficients in $\mathbb{Q}[t]$. Analogous enumerative results can by obtained by applying the same specialization to the expansions in Theorem 6.3.1. In this section we obtain different enumerative results by applying a different specialization, called the stable principal specialization, to the expansions in Theorem 6.3.1.

In [54] Shareshian and Wachs prove that by taking stable principal specialization of $W_n(\mathbf{x},t)$ one gets the q-analog of the Eulerian polynomials defined by

$$A_n(q,t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)},$$

where $\mathrm{maj}(\sigma) = \sum_{i \in \mathrm{DES}(\sigma)} i$ and $\mathrm{exc}(\sigma) = |\{i \mid \sigma(i) > i\}|$. Shareshian and Wachs obtain in [52], the following q-analog of MacMahon's classical result equating the distribution of exc and des on $\mathfrak{S}_n$:

$$A_n(q,t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})} t^{\mathrm{des}(\sigma)}, \tag{6.32}$$

where $\mathrm{maj}_{\geq 2}(\sigma) = \sum_{i \in \mathrm{DES}_{\geq 2}(\sigma)} i$.

Now let

$$[i]_q! := [i]_q[i-1]_q \dots [1]_q \quad \text{and} \quad \exp_q(z) := \sum_{i\geq 0} \frac{z^i}{[i]_q!}.$$

The following q-analog of Euler's exponential generating function for Eulerian polynomials,

$$\sum_{n\geq 1} A_n(q,t)\frac{z^n}{[n]_q!} = \frac{\displaystyle\sum_{i\geq 1}[i]_t \frac{z^i}{[i]_q!}}{1 - \displaystyle\sum_{i\geq 2} t[i-1]_t \frac{z^i}{[i]_q!}} \tag{6.33}$$

is obtained in [54] by taking stable principal specialization of both sides of (6.1).

In this section, we refine their result for the following variations of $A_n(q,t)$:

$$A_n^{<}(q,t) := \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(1)<\sigma(n)}} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})} t^{\mathrm{des}(\sigma)},$$

$$\tilde{A}_n(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})} t^{\mathrm{cdes}(\sigma)},$$

$$\tilde{A}_n^{a}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}_{\geq 2}(\sigma^{-1})} t^{\mathrm{casc}(\sigma)}.$$

Theorem 6.4.1. *We have*

$$\sum_{n\geq 2} A_n^{<}(q,t)\frac{z^n}{[n]_q!} = \frac{\dfrac{\partial}{\partial t}\displaystyle\sum_{i\geq 2}[i]_t \frac{z^i}{[i]_q!}}{1 - \displaystyle\sum_{i\geq 2} t[i-1]_t \frac{z^i}{[i]_q!}} \tag{6.34}$$

$$\sum_{n\geq 1} \tilde{A}_n(q,t)\frac{z^n}{[n]_q!} = \frac{\dfrac{\partial}{\partial t}\exp_q(tz)}{1 - \displaystyle\sum_{i\geq 2} t[i-1]_t \frac{z^i}{[i]_q!}} \tag{6.35}$$

$$\sum_{n\geq 1} \tilde{A}_n^{a}(q,t)\frac{z^n}{[n]_q!} = \frac{tz\,\dfrac{\partial}{\partial z}\exp_q(z)}{1 - \displaystyle\sum_{i\geq 2} t[i-1]_t \frac{z^i}{[i]_q!}}. \tag{6.36}$$

For the sake of comparison, note that (6.33) can be restated as

$$1+\sum_{n\geq 1} A_n(q,t)\frac{z^i}{[n]_q!} = \frac{\exp_q(z)}{1-\sum_{i\geq 2} t[i-1]_t \frac{z^i}{[i]_q!}}. \tag{6.37}$$

Proof. The *stable principal specialization* $\mathrm{ps}(G(\mathbf{x}))$ of a quasisymmetric function $G(\mathbf{x})$ is obtained from $G(\mathbf{x})$ by substituting q^{i-1} for x_i for all $i \geq 1$. It is not difficult to see that

$$\mathrm{ps}(F_{n,S}(\mathbf{x})) = \frac{\sum_{i\in S} q^i}{(1-q)(1-q^2)\dots(1-q^n)}$$

for all $S \subseteq [n-1]$ (see [60, Lemma 7.19.10]). In particular

$$\mathrm{ps}(\omega e_n(\mathbf{x})) = \mathrm{ps}(\omega F_{n,[n-1]}(\mathbf{x})) = \mathrm{ps}(F_{n,\emptyset}(\mathbf{x})) = \frac{1}{(1-q)(1-q^2)\dots(1-q^n)}.$$

Hence by (6.23),

$$\mathrm{ps}(\omega W_n^{<}(\mathbf{x},t)) = \frac{A_n^{<}(q,t)}{(1-q)(1-q^2)\dots(1-q^n)}.$$

We apply ω to both sides of (6.7), take the stable principal specialization, and replace z by $(1-q)z$ to get (6.34).

Equations (6.35) and (6.36) are obtained similarly, using Corollary 6.2.5 and Theorem 6.3.1. $\square$

Note that for $n \geq 2$, $\tilde{A}_n(1,t) = \tilde{A}_n^a(1,t)$. When q is set equal to 1, (6.36) and (6.37) reduce respectively to

$$\sum_{n\geq 1} \tilde{A}_n^a(1,t)\frac{z^n}{n!} = \frac{tz\,e^z}{1-\sum_{i\geq 2} t[i-1]_t \frac{z^i}{i!}}$$

and

$$1+\sum_{n\geq 1} A_n(1,t)\frac{z^n}{n!} = \frac{e^z}{1-\sum_{i\geq 2} t[i-1]_t \frac{z^i}{i!}}.$$

Hence for $n \geq 2$,

$$\tilde{A}_n(1,t) = \tilde{A}_n^a(1,t) = ntA_{n-1}(1,t).$$

This can easily be proved directly by observing that

$$\tilde{A}_n^a(1,t) = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma(n)=n}} \sum_{\tau\in\mathcal{C}_\sigma} t^{\mathrm{casc}(\tau)},$$

where $\mathcal{C}_\sigma$ is the set of circular rearrangements of σ. For $\sigma \in \mathfrak{S}_n$ such that $\sigma(n) = n$,

$$\sum_{\tau\in\mathcal{C}_\sigma} t^{\mathrm{casc}(\tau)} = nt^{\mathrm{casc}(\sigma)} = nt^{\mathrm{asc}(\sigma|_{n-1})+1},$$

where $\sigma|_{n-1}$ denotes the restriction of σ to $[n-1]$.

6.5 Expansion in the power sum symmetric function basis

In [52, Proposition 7.9], Shareshian and Wachs proved that for each $\lambda \vdash n$, the coefficient of $z_\lambda^{-1}p_\lambda(\mathbf{x})$ in $\omega W_n(\mathbf{x},t)$ is

$$A_{l(\lambda)}(t)\prod_{i=1}^{l(\lambda)}[\lambda_i]_t, \tag{6.38}$$

where $A_k(t)$ are the classical Eulerian polynomials defined in Section 2.1.

From Theorem 5.3.7, we know that the coefficient of $\frac{1}{n}p_n(\mathbf{x})$ in $\omega\tilde{W}_n^{\neq}(\mathbf{x},t)$ is

$$nt[n-1]_t, \tag{6.39}$$

and for $\lambda \vdash n$ with $l(\lambda) \geq 2$, the coefficient of $z_\lambda^{-1} p_\lambda(\mathbf{x})$ in $\omega \tilde{W}_n^{\neq}(\mathbf{x}, t)$ is

$$nt A_{l(\lambda)-1}(t) \prod_{i=1}^{l(\lambda)} [\lambda_i]_t. \tag{6.40}$$

Here we present some more results of this style for the restricted Smirnov word enumerators.

Theorem 6.5.1. *For all $\lambda \vdash n$, the coefficient of $z_\lambda^{-1} p_\lambda$ in $\omega W_n^{<}(\mathbf{x}, t)$ equals*

$$\frac{d}{dt}(t A_{l(\lambda)-1}(t) \prod_{i=1}^{l(\lambda)} [\lambda_i]_t),$$

where $A_0(t) = t^{-1}$.

From this theorem, we get the following corollary.

Corollary 6.5.2. *The coefficient of $z_\lambda^{-1} p_\lambda(\mathbf{x}, t)$ in $\omega \tilde{W}_n(\mathbf{x}, t)$ is*

$$A_{l(\lambda)-1}(t) \sum_{i=1}^{l(\lambda)} \lambda_i t^{\lambda_i} \prod_{j \neq i} [\lambda_j]_t, \tag{6.41}$$

where again we let $A_0(t) = t^{-1}$.

Proof. Recall that $\tilde{W}_n(\mathbf{x}, t) = t W_n^{<}(\mathbf{x}, t) + (W_n(\mathbf{x}, t) - W_n^{<}(\mathbf{x}, t))$. Fix $\lambda \vdash n$ with $l(\lambda) = k$. Then combining (6.38) with Theorem 6.5.1 gives us that the coefficient of $z_\lambda^{-1} p_\lambda(\mathbf{x})$ in $\omega \tilde{W}_n(\mathbf{x}, t)$ is

$$(t-1)\frac{d}{dt}(t A_{k-1}(t) \prod_{i=1}^{k} [\lambda_i]_t) + A_k(t) \prod_{i=1}^{k} [\lambda_i]_t. \tag{6.42}$$

When $\lambda = (n)$ and hence $k = 1$, one can easily check that the corollary holds. If $\lambda \neq (n)$ and hence $k \geq 2$, we can use the well-known identity that

$$A_k(t) = t(1-t)A'_{k-1}(t) + (1 + (k-1)t)A_{k-1}(t)$$

along with (6.42) to prove the corollary. $\square$

In this thesis, we present a combinatorial proof of Theorem 6.5.1 using our F-basis expansion of Theorem 6.3.1, but in the author's paper with Wachs [22], we give an algebraic proof that uses our e-basis expansion from Theorem 6.2.1. For our proof of Theorem 6.5.1, we will need a lemma, but first let us define some notation.

For $\sigma \in \mathfrak{S}_n$, define the number of (<2)-inversions of σ by $\mathrm{inv}_{<2}(\sigma) := \mathrm{des}(\sigma^{-1})$, i.e.,

$$\mathrm{inv}_{<2}(\sigma) := |\{i \in [n-1] \mid \sigma^{-1}(i) > \sigma^{-1}(i+1)\}|.$$

For a word $w = w_1 w_2 \cdots w_l$ with distinct letters over the alphabet $[n]$, we say that w_i with $i > 1$ is a $(\geq 2)^*$-*maximum* of w if

$$2 \leq w_i - w_j < n - 1$$

for all $1 \leq j < i$. Note that if 1 precedes n in w, then n is not considered a $(\geq 2)^*$-maximum.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \vdash n$ and define $N_\lambda^<$ as the set of all $\sigma \in \mathfrak{S}_n$ such that $\sigma(1) < \sigma(n)$ and when σ^{-1}, written in one-line notation, is broken up into contiguous segments $\alpha_1, \alpha_2, \cdots, \alpha_l$ of sizes $\lambda_1, \lambda_2, \cdots, \lambda_l$ respectively, each α_i has no (≥ 2)-descents and no $(\geq 2)^*$-maxima.

Lemma 6.5.3. *For all $n \geq 1$,*

$$\omega W_n^<(\mathbf{x}, t) = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda(\mathbf{x}) \sum_{\sigma \in N_\lambda^<} t^{\mathrm{des}(\sigma)} \tag{6.43}$$

Proof. Note that this proof is similar to that of Theorem 5.3.5. Combining Proposition 5.3.2, Equation (6.23), and the fact that for all $\sigma \in \mathfrak{S}_n$, we know $\mathrm{des}(\sigma) =$

$\mathrm{inv}_{<2}(\sigma^{-1})$, we have that

$$\omega W_n^{<}(\mathbf{x},t) = \sum_{\lambda\vdash n} z_\lambda^{-1} p_\lambda(\mathbf{x}) \sum_{\substack{\sigma\in\mathfrak{S}_n \\ \sigma(1)<\sigma(n) \\ \mathrm{DES}_{\geq 2}(\sigma^{-1})\in U_\lambda}} (-1)^{|\,\mathrm{DES}_{\geq 2}(\sigma^{-1})\backslash S(\lambda)|} t^{\mathrm{inv}_{<2}(\sigma^{-1})} \tag{6.44}$$

For each $\lambda \vdash n$, define

$$D_\lambda^{<} := \{\sigma \in \mathfrak{S}_n \mid \sigma(1) < \sigma(n) \text{ and } \mathrm{DES}_{\geq 2}(\sigma^{-1}) \in U_\lambda\}.$$

Notice that $N_\lambda^{<} \subseteq D_\lambda^{<}$. We will find a sign-reversing, $\mathrm{inv}_{<2}$-preserving involution $\phi_\lambda^{<} : D_\lambda^{<} \to D_\lambda^{<}$ that fixes all elements of $N_\lambda^{<}$ and for each $\sigma \in D_\lambda^{<}\backslash N_\lambda^{<}$, we have

- $(\phi_\lambda^{<})^2(\sigma) = \sigma$,
- $\phi_\lambda^{<}$ changes $|\,\mathrm{DES}_{\geq 2}(\sigma^{-1})\backslash S(\lambda)|$ by 1, and
- $\mathrm{inv}_{<2}(\sigma^{-1}) = \mathrm{inv}_{<2}(\phi_\lambda^{<}(\sigma^{-1}))$.

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \vdash n$ and $\sigma \in D_\lambda^{<}\backslash N_\lambda^{<}$, and let σ^{-1}, written in one-line notation, be broken into contiguous segments $\alpha_1, \alpha_2, \cdots, \alpha_l$ of sizes $\lambda_1, \lambda_2, \cdots, \lambda_l$ respectively. Let i be the smallest index such that α_i contains a (≥ 2)-descent or a (≥ 2)*-maximum. Then α_i is of the form

$$\alpha_i = \sigma_{s_{i-1}+1}\sigma_{s_{i-1}+2}\cdots\sigma_{k-1}\sigma_k\sigma_{k+1}\cdots\sigma_{s_i},$$

where $s_j - s_{j-1} \geq 2$ for $s_{i-1}+1 < j \leq k$ and $s_j - s_{j+1} < 2$ for $k \leq j < s_i$. Then $\phi_\lambda^{<}$ will change α_i according to the following cases and will fix all other α_j.

Define σ_m as the largest (≥ 2)*-maximum of α_i such that $m > k$. If there is no such m, then define $\sigma_m = 0$

Case 1:$\sigma_m \neq 0$

Then move σ_m before $\sigma_{s_{i-1}+1}$. This will increase (≥ 2)-descents by 1 since $\sigma_m - \sigma_{s_{i-1}+1} \geq 2$. If $m \neq s_i$, we should check that we do not create a (≥ 2)-descent between σ_{m-1} and σ_{m+1}. But $\sigma_m - \sigma_{m-1} \geq 2$ and since there is not a (≥ 2)-descent between σ_m and σ_{m+1}, we know $\sigma_m - \sigma_{m+1} < 2$. So $\sigma_{m+1} > \sigma_{m-1}$, meaning we do not create a new (≥ 2)-descent here. Since $\sigma_m - \sigma_j \geq 2$ for all $s_{i-1}+1 \leq j < m$, then this will not change the (< 2)-inversions of σ^{-1}. Since we do not allow n to be a (≥ 2)-maximum if it follows 1, we will still have that $\phi_\lambda^<(\sigma)(1) < \phi_\lambda^<(\sigma)(n)$. One can easily see that $\phi_\lambda^<(\sigma)$ is still in $D_\lambda^<$.

Case 2:$\sigma_m = 0$

Then move $\sigma_{s_{i-1}+1}$ to the first place after σ_k that will not create a new (≥ 2)-descent. This will decrease the (≥ 2)-descents by 1 . This will not change the (< 2)-inversions, because for all σ_j with $s_{i-1}+1 < j \leq k$ we have that $\sigma_{s_{i-1}+1} - s_j \geq 2$ and for any σ_j with $k > j$ if $|\sigma_j - \sigma_{s_{i-1}+1}| < 2$, then $\sigma_{s_{i-1}+1}$ would be placed before σ_j since this would not create a new (≥ 2)-descent. Notice that $1 \neq \sigma_{s_{i-1}+1}$ in this case, because this would imply that α_i has no (≥ 2)*-maxima and no (≥ 2)-descents. Hence we still have that $\phi_\lambda^<(\sigma)(1) < \phi_\lambda^<(\sigma)(n)$. One can easily see that $\phi_\lambda^<(\sigma)$ is still in $D_\lambda^<$.

Note that **Case 1** and **Case 2** will reverse each other and hence $\phi_\lambda^<(\sigma)$ is an involution. The only elements of $D_\lambda^<$ that remain are those of $N_\lambda^<$. Since for all $\sigma \in N_\lambda^<$, we have $\mathrm{DES}_{\geq 2}(\sigma^{-1}) \subseteq S(\lambda)$, we are done. □

Proof of Theorem 6.5.1. Note that this proof is similar to the proof of Theorem 5.3.7. We will use the fact that for each $\lambda \vdash n$, the coefficient of $z_\lambda^{-1} p_\lambda(\mathbf{x})$ in $W_n^<(\mathbf{x}, t)$ is

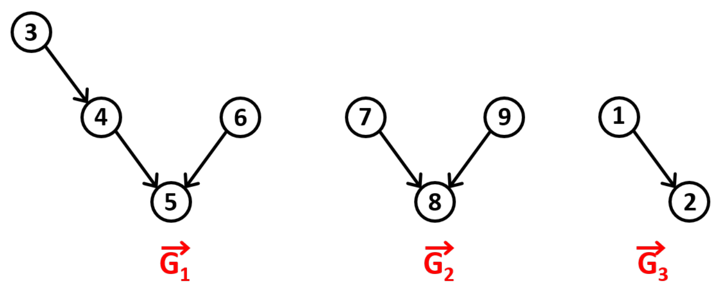
$$\sum_{\sigma \in N_\lambda^<} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in N_\lambda^<} t^{\mathrm{inv}_{<2}(\sigma^{-1})}.$$

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition of n and let $\sigma \in N_\lambda^<$ so that σ^{-1} is partitioned into pieces, $\alpha_1, \alpha_2, \cdots, \alpha_k$ of sizes $\lambda_1, \lambda_2, \cdots, \lambda_k$ so that $\sigma = \alpha_1\alpha_2 \cdots \alpha_k$, where $\cdot$ represents concatenation. Then we know $\sigma \in N_\lambda^<$ if and only if each α_i has no (≥ 2)-descents and no (≥ 2)*-maxima.

Let $C_n = ([n], E)$ be a graph defined by $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{\{1, n\}\}$. For each α_i, we will construct a connected acyclic digraph $\overrightarrow{G_i}$ on the letters of α_i such that the underlying undirected graph G_i is an induced subgraph of C_n.

Let $\overrightarrow{G_i}$ be the directed graph whose vertex set is the set of letters of α_i and whose edges have the form (a, b) if b precedes a in α_i and $\{a, b\} \in E(C_n)$.

For example, let $n = 9$, $\lambda = (4, 3, 2)$ and $\sigma = 543687921$. Then $\alpha_1 = 5436$, $\alpha_2 = 879$, and $\alpha_3 = 21$. The corresponding acyclic digraphs as as shown below:



Define a *sink* of a digraph to be a vertex with no outgoing edges.

Claim: For each $\sigma \in N_\lambda^<$ with $\sigma = \alpha_1\alpha_2 \cdots \alpha_k$, the $\overrightarrow{G_i}$ associated to each α_i is a connected acyclic digraph with a unique sink, which is the first letter of α_i.

Proof of claim: It is clear from the way $\overrightarrow{G_i}$ is defined that it must be acyclic. Then we only need to show that there is a unique sink, which would also imply that $\overrightarrow{G_i}$ is connected.

Suppose some $\overrightarrow{G_i}$ has two distinct sinks. Then let $\alpha_i = a_1a_2 \cdots a_{\lambda_i}$ be the part of σ associated to $\overrightarrow{G_i}$. It is clear that a_1 must be a sink. So let a_I be the sink with $I \neq 1$ such that I is minimal. Then the induced subgraph of C_n on the vertices of

$\{a_1, a_2, \ldots, a_{I-1}\}$ is connected; otherwise, this set would contain more than one sink of $\overrightarrow{G_i}$.

Case 1: $n \in \{a_1, a_2, \ldots, a_{I-1}\}$.

Suppose $a_j = n$. Then in order to avoid (≥ 2)-descents, we must have $a_{j+1} = n-1, a_{j+1} = n-2, \ldots, a_{I-1} = n-(I-1-j)$. But then again to avoid a (≥ 2)-descent, we must have $a_I = n-(I-j)$. But then there would be an edge of $\overrightarrow{G_i}$ oriented from a_I to a_{I-1}, contradicting the fact that a_I is a sink of $\overrightarrow{G_i}$.

Case 2: $n \notin \{a_1, a_2, \ldots, a_{I-1}\}$.

Since $\{a_1, a_2, \ldots, a_{I-1}\}$ form a connected subgraph of C_n, we must have that there exists $c, d \in \mathbb{P}$ such that $\{a_1, a_2, \ldots, a_{I-1}\} = \{c, c+1, \ldots, d\}$. Then we cannot have $a_I \leq c-2$, because this would create a (≥ 2)-descent between a_{I-1} and a_I. We cannot have $a_I \geq d+2$, because either this makes a_I a (≥ 2)*-maximum or in the case that $c = 1$ and $a_I = n$, this means there is an edge from $a_I = n$ to 1 in $\overrightarrow{G_i}$, contradicting the assumption that a_I is a sink. Then we must have that $a_I = c-1$ or $a_I = d+1$. But then there is an edge from a_I to c or d, respectively, so a_I cannot be a sink of $\overrightarrow{G_i}$.

So our claim is proven.

From the claim, we see that if $\lambda \neq (n)$, each underlying undirected graph, G_i, is a path of length λ_i in C_n. If $\lambda = (n)$, then $G_1 = C_n$.

We can uniquely recover σ from the k-tuple $(\overrightarrow{G_1}, \overrightarrow{G_2}, \cdots, \overrightarrow{G_k})$. Indeed for each $\overrightarrow{G_i}$, recreate α_i by starting with the sink of $\overrightarrow{G_i}$. Then remove this vertex from $\overrightarrow{G_i}$ and choose the sink of the remaining digraph with the smallest label to be the second letter of α_i. Then remove this sink from the remaining digraph and repeat the process until all vertices of $\overrightarrow{G_i}$ have been used.

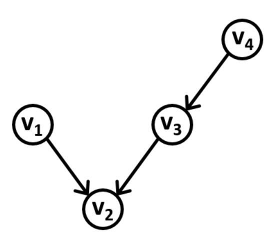
Notice that the number of (<2)-inversions of α_i is the number of directed edges of the associated $\overrightarrow{G_i}$ of the form $(j, j+1)$ with $j \in [n-1]$.

Case 1: $\lambda = (n)$.

In this case $\overrightarrow{G_1}$ is an acyclic orientation of C_n with a unique sink (and hence a unique source, i.e., a vertex with no incoming edges) and so that the edge between 1 and n is oriented from n to 1. So we need to find the number of (<2)-inversions of the corresponding σ^{-1}, i.e., the number of edges of $\overrightarrow{G_1}$ of the form $(i, i+1)$. In order to construct an acyclic orientation of C_n meeting our criteria, we can choose any $1 \leq j < n$ to be our sink. Then we must choose a k with $j < k \leq n$ to be our source. Then $\overrightarrow{G_1}$ has edges of the form $(i, i+1)$ for $1 \leq i \leq j-1$ and $k \leq i \leq n-1$. Summing over all acyclic orientations of C_n with a unique sink and with the edge $(n, 1)$, we get that the coefficient of $\frac{1}{n}p_n$ in $\omega W_n^{<}(\mathbf{x}, t)$ is $\sum_{1 \leq j < k \leq n} t^{j-1+n-k} = \frac{d}{dt}[n]_t$.

Case 2: $\lambda \neq (n)$.

For $a, b \in \mathbb{P}$ with $b \leq a$, define a V-digraph $\overrightarrow{V}_{a,b}$ to be a digraph with vertex set $\{v_1, v_2, \cdots, v_a\}$ and edge set $\{(v_i, v_{i+1}) \mid 1 \leq i < b\} \cup \{(v_{i+1}, v_i) \mid b \leq i < a\}$. We will call v_1 the first vertex of $\overrightarrow{V}_{a,b}$ and v_a the last vertex of $\overrightarrow{V}_{a,b}$. For $1 \leq i < a$ we say the successor of v_i is v_{i+1}. Let $V_{a,b}$ denote the underlying undirected graph of $\overrightarrow{V}_{a,b}$. For all $a, b \in \mathbb{P}$ with $b \leq a$, we can see that $V_{a,b}$ is a path. For example, $\overrightarrow{V}_{4,2}$ is shown below:



$\overrightarrow{\mathbf{V}}_{4,2}$

Case 2a: $\sigma \in N_{\lambda}^{<}$ with 1 and n in the same α_i

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. Then we will construct a bijection from the set

$$N_\lambda^<(s) \coloneqq \{\sigma \in N_\lambda^< \mid 1 \text{ and } n \text{ in the same } \alpha_i \text{ of } \sigma\}$$

to the set M_λ^{S} of $(k+3)$-tuples

$$(I, j, \mu, \overrightarrow{V}_{\lambda_1,b_1}, \overrightarrow{V}_{\lambda_2,b_2}, \cdots, \overrightarrow{V}_{\lambda_k,b_k}),$$

where

- $I \in [k]$ with $\lambda_I \geq 2$
- $\mu \in \mathfrak{S}_k$ is a k-cycle
- $2 \leq b_I \leq \lambda_I$
- for each $i \neq I$, we have $1 \leq b_i \leq \lambda_i$
- $1 \leq j \leq b_I - 1$

Let $\sigma \in N_\lambda^<(s)$. Define I so that α_I contains 1 and n. Recall our earlier map from $\sigma \in N_\lambda^<$ to the k-tuples $(\overrightarrow{G_1}, \overrightarrow{G_2}, \cdots, \overrightarrow{G_k})$. For each $1 \leq i \leq k$, define b_i as one more than the number of edges of $\overrightarrow{G_i}$ of the form $(i, i+1)$ for some $i \in [n-1]$. Then $\overrightarrow{V}_{\lambda_i,b_i}$ is simply $\overrightarrow{G_i}$ without labels. Since 1 and n are in $\overrightarrow{G}_I$, we will automatically have $2 \leq b_I \leq \lambda_I$. To determine $\mu = (a_1, a_2, \cdots, a_k)$, we start by letting $a_1 = I$. From the remaining $\overrightarrow{G_i}$ with $i \neq I$, let $\overrightarrow{G_{j_2}}$ be the digraph with the smallest label on its sink. Then let $a_2 = j_2$. From the remaining $\overrightarrow{G_i}$, let $\overrightarrow{G_{j_3}}$ be the digraph with the smallest label on its sink. Then let $a_3 = j_3$. We continue this process until we find a_k. Lastly, let j be the label on the sink of $\overrightarrow{G_I}$.

In the other direction, suppose we have

$$(I, j, \mu, \overrightarrow{V}_{\lambda_1,b_1}, \overrightarrow{V}_{\lambda_2,b_2}, \cdots, \overrightarrow{V}_{\lambda_k,b_k}) \in M_\lambda^{\mathrm{S}}.$$

For each $1 \leq i \leq k$, we will say that the successor of the last vertex of $\overrightarrow{V}_{\lambda_i,b_i}$ is the first vertex of $\overrightarrow{V}_{\lambda_{\mu(i)},b_{\mu(i)}}$. Place the label j on the sink of $\overrightarrow{G_I}$. Then place a $j+1$ on its successor, a $j+2$ on the successor of that vertex, etc. When we finally place the label n on a vertex, then place a 1 on its successor, a 2 on the successor of that vertex, etc. until all vertices are labeled. Now the labeled $\overrightarrow{V}_{\lambda_i,b_i}$ is the same as $\overrightarrow{G_i}$, so we can recover σ as described earlier. One can check that this is a bijection.

Now suppose we have some $\sigma \in N_\lambda^<(s)$ that corresponds to

$$(I, j, \mu, \overrightarrow{V}_{\lambda_1,b_1}, \overrightarrow{V}_{\lambda_2,b_2}, \cdots, \overrightarrow{V}_{\lambda_k,b_k}) \in M_\lambda^{\mathrm{S}}.$$

Notice that using the bijection, the number of (<2)-inversions of α_i is equal to $b_i - 1$ for $i \neq I$ and equal to $b_i - 2$ when $i = I$. One can check that the number of (<2)-inversions between distinct α_i in σ^{-1} is the same as the number of excedances of μ^{-1}. Using Lemma 5.3.6, we see that

$$\sum_{\substack{\mu \in \mathfrak{S}_k \\ \mu \ k-cycle}} t^{\mathrm{exc}(\mu^{-1})} = \sum_{\substack{\mu \in \mathfrak{S}_k \\ \mu \ k-cycle}} t^{\mathrm{exc}(\mu)} = tA_{k-1}(t).$$

Then we get the following formula

$$\begin{aligned}
\sum_{\sigma \in N_\lambda^<(s)} t^{\mathrm{inv}_{<2}(\sigma^{-1})} &= \sum_{\mu \in \mathfrak{S}_k} t^{\mathrm{exc}(\mu^{-1})} \Big(\prod_{\substack{1 \leq i \leq k \\ i \neq I}} \sum_{b_i=1}^{\lambda_i} t^{b_i - 1} \Big) \Big(\sum_{b_I=2}^{\lambda_I} \sum_{j=1}^{b_I - 1} t^{b_I - 2} \Big) \\
&= tA_{k-1}(t) \Big(\prod_{\substack{1 \leq i \leq k \\ i \neq I}} [\lambda_i]_t \Big) \Big(\sum_{b_I=2}^{\lambda_I} (b_I - 1) t^{b_I - 2} \Big) \\
&= tA_{k-1}(t) \frac{d}{dt} \prod_{i=1}^{k} [\lambda_i]_t
\end{aligned}$$

Case 2b: $\sigma \in N_\lambda^<$ with 1 and n in different α_i

Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$. Then we will construct a bijection from the set

$$N_\lambda^<(d) \coloneqq \{\sigma \in N_\lambda^< \mid 1 \text{ and } n \text{ in different } \alpha_i \text{ of } \sigma\}$$

to the set M_λ^{d} of $(k+2)$-tuples

$$(j, \mu, \overrightarrow{V}_{\lambda_1, b_1}, \overrightarrow{V}_{\lambda_2, b_2}, \cdots, \overrightarrow{V}_{\lambda_k, b_k}),$$

where

- $j \in [k]$
- $\mu \in \mathfrak{S}_k$ is a k-cycle with $\mu(j) < j$
- for each i, we have $1 \le b_i \le \lambda_i$

Let $\sigma \in N_\lambda^<(d)$. Recall our earlier map from $\sigma \in N_\lambda^<$ to the k-tuples $(\overrightarrow{G_1}, \overrightarrow{G_2}, \cdots, \overrightarrow{G_k})$. For each $1 \le i \le k$, define b_i as one more than the number of edges of $\overrightarrow{G_i}$ of the form $(i, i+1)$ for some $i \in [n-1]$. Then $\overrightarrow{V}_{\lambda_i, b_i}$ is simply $\overrightarrow{G_i}$ without labels. Define j so that α_j contains n. To determine $\mu = (a_1, a_2, \cdots, a_k)$, we start by letting $a_1 = j_1$ where $\overrightarrow{G_{j_1}}$ contains the vertex labeled 1. From the remaining $\overrightarrow{G_i}$, let $\overrightarrow{G_{j_2}}$ be the digraph with the smallest label on its sink. Then let $a_2 = j_2$. From the remaining $\overrightarrow{G_i}$, let $\overrightarrow{G_{j_3}}$ be the digraph with the smallest label on its sink. Then let $a_3 = j_3$. We continue this process until we find a_k. Notice that a_k will be j, so since 1 precedes n in σ, we have that $a_1 = \mu(j) < a_k$.

In the other direction, suppose we have

$$(j, \mu, \overrightarrow{V}_{\lambda_1, b_1}, \overrightarrow{V}_{\lambda_2, b_2}, \cdots, \overrightarrow{V}_{\lambda_k, b_k}) \in M_\lambda^{\mathrm{d}}.$$

For each $1 \le i \le k$, we will say that the successor of the last vertex of $\overrightarrow{V}_{\lambda_i, b_i}$ is the first vertex of $\overrightarrow{V}_{\lambda_{\mu(i)}, b_{\mu(i)}}$. Place the label n on the last vertex of $\overrightarrow{G_j}$. Then place a 1 on its successor, a 2 on the successor of that vertex, etc. until all vertices are labeled. Now the labeled $\overrightarrow{V}_{\lambda_i,b_i}$ is the same as $\overrightarrow{G_i}$, so we can recover σ as described earlier. One can check that this is a bijection.

Now suppose we have some $\sigma \in N_\lambda^<(d)$ that corresponds to

$$(j, \mu, \overrightarrow{V}_{\lambda_1,b_1}, \overrightarrow{V}_{\lambda_2,b_2}, \cdots, \overrightarrow{V}_{\lambda_k,b_k}) \in M_\lambda^{\mathrm{d}}.$$

Notice that using the bijection, the number of (<2)-inversions of α_i is equal to $b_i - 1$. One can check that the number of (<2)-inversions between distinct α_i in σ^{-1} is one less than the number of excedances of μ^{-1}. Then one can see that

$$\begin{aligned}
\sum_{\sigma \in N_\lambda^<(d)} t^{\mathrm{inv}_{<2}(\sigma^{-1})} &= \sum_{j=1}^{k-1} \sum_{\substack{\mu \in \mathfrak{S}_k \\ \mu \text{ is a k-cycle} \\ \mu(j)<j}} t^{\mathrm{exc}(\mu^{-1})-1} \Big(\prod_{i=1}^{k} \sum_{b_i=1}^{\lambda_i} t^{b_i-1}\Big) \\
&= \sum_{j=1}^{k-1} \sum_{\substack{\mu \in \mathfrak{S}_k \\ \mu \text{ is a k-cycle} \\ \mu^{-1}(j)>j}} t^{\mathrm{exc}(\mu^{-1})-1} \Big(\prod_{i=1}^{k} [\lambda_i]_t\Big) \\
&= \sum_{\substack{\mu \in \mathfrak{S}_k \\ \mu \text{ is a k-cycle}}} (\mathrm{exc}(\mu^{-1})) t^{\mathrm{exc}(\mu^{-1})-1} \Big(\prod_{i=1}^{k} [\lambda_i]_t\Big) \\
&= \frac{d}{dt}(tA_{k-1}(t)) \Big(\prod_{i=1}^{k} [\lambda_i]_t\Big).
\end{aligned}$$

Then **Case 1** gives us our result when $\lambda = (n)$, and putting together **Case 2a** and **Case 2b** gives us our result when $\lambda \neq (n)$.

$\square$

6.6 The labeled cycle

Let $C_n = ([n], E)$ be the labeled cycle, i.e., let $E = \{\{i, i+1\} \mid i \in [n-1]\} \cup \{\{1, n\}\}$. Recall that the chromatic quasisymmetric function $X_{C_n}(\mathbf{x}, t)$ of the labeled cycle is different than the chromatic quasisymmetric function $X_{\overrightarrow{C_n}}(\mathbf{x}, t)$ of the directed cycle. In this section, we will present a generating function for the chromatic quasisymmetric function $X_{C_n}(\mathbf{x}, t)$ of the labeled cycle in terms of the elementary basis, which follows from our work on descent enumerators of Smirnov words. Using our formula we show that $X_{C_n}(\mathbf{x}, t)$ is e-positive. This is interesting, because for $n \geq 4$, C_n is not a natural unit interval graph, nor is it a circular indifference digraph (when turned into a digraph by orienting edges from smaller label to larger label). So $X_{C_n}(\mathbf{x}, t)$ is not included in the e-positivity conjecture of Shareshian and Wachs (Conjecture 4.2.2), nor in the generalized e-positivity conjecture (Conjecture 5.5.1). In fact, it is not even included in the symmetry result of Shareshian and Wachs for labeled graphs or our symmetry result for directed graphs (see Corollary 5.4.7).

From each proper coloring $\kappa : [n] \to \mathbb{P}$ of the labeled cycle C_n, we can form a Smirnov word $w = \kappa(n)\kappa(n-1)\cdots\kappa(1)$. From this correspondence, we see that we get the relationship

$$X_{C_n}(\mathbf{x}, t) = W_n^{<}(\mathbf{x}, t) + tW_n^{>}(\mathbf{x}, t). \tag{6.45}$$

Since $W_n^{<}(\mathbf{x}, t)$ and $W_n^{>}(\mathbf{x}, t)$ are symmetric, we get that $X_{C_n}(\mathbf{x}, t)$ is symmetric as well. By combining (6.45) with the formulas given in Theorem 6.2.1, we get the following corollary.

Corollary 6.6.1 (of Theorem 6.2.1)**.** *We have,*

$$\sum_{n\geq 2} X_{C_n}(\mathbf{x}, t)z^n = \frac{\sum_{i\geq 2}([2]_t[i]_t + it^2[i-3]_t)e_i(\mathbf{x})z^i}{D(\mathbf{x}, t, z)}, \tag{6.46}$$

where $D(\mathbf{x},t,z) := 1 - \sum_{i\geq 2} t[i-1]_t e_i(\mathbf{x})z^i$ *as defined in (6.10), and* $[-n]_t := \frac{t^{-n}-1}{t-1} = -t^{-n}[n]_t$ *for* $n \geq 0$.

Note that (6.46) reduces to Stanley's formula (3.4) for the chromatic symmetric function $X_{C_n}(\mathbf{x})$ when $t=1$.

Theorem 6.6.2. *Let* $n \geq 2$.

1. *If* n *is odd,* $X_{C_n}(\mathbf{x},t)$ *is e-positive, palindromic, and e-unimodal with center of symmetry* $\frac{n}{2}$.
2. *If* n *is even,*
   (a) $X_{C_n}(\mathbf{x},t)$ *is e-positive and palindromic with center of symmetry* $\frac{n}{2}$*, but is not e-unimodal.*
   (b) $X_{C_n}(\mathbf{x},t)+t^{\frac{n}{2}}e_{2^{\frac{n}{2}}}(\mathbf{x})$ *is e-positive, palindromic, and e-unimodal with center of symmetry* $\frac{n}{2}$.

Proof. Let $U_n(\mathbf{x},t)$ and $V_n(\mathbf{x},t)$ be defined respectively by

$$\sum_{n\geq 2} U_n(\mathbf{x},t)z^n = \frac{([2]_t[2]_t + 2t^2[2-3]_t)e_2(\mathbf{x})z^2}{D(\mathbf{x},t,z)}$$

and

$$\sum_{n\geq 2} V_n(\mathbf{x},t)z^n = \frac{\sum_{i\geq 3}([2]_t[i]_t + it^2[i-3]_t)e_i(\mathbf{x})z^i}{D(\mathbf{x},t,z)}.$$

Then $X_{C_n}(\mathbf{x},t) = U_n(\mathbf{x},t) + V_n(\mathbf{x},t)$.

We have,

$$\sum_{n\geq 2} U_n(\mathbf{x},t)z^n = \frac{(1+t^2)e_2(\mathbf{x})z^2}{D(\mathbf{x},t,z)}.$$

It follows from (5.17) that

$$U_n(\mathbf{x},t) = \sum_{m\geq 1} \sum_{\substack{k_2,\ldots,k_m \geq 2 \\ \sum_{i=2}^m k_i = n-2}} e_2 e_{k_2}\ldots e_{k_m} t^{m-1}(1+t^2)\prod_{i=2}^m [k_i - 1]_t. \tag{6.47}$$

Note that for any $k \geq 3$

$$(1+t^2)[k]_t = 1 + t + 2t^2 + \cdots + 2t^{k-1} + t^k + t^{k+1},$$

and for $k = 2$,

$$(1+t^2)[k]_t = 1 + t + t^2 + t^3.$$

In either case, $(1+t^2)[k]_t$ is palindromic and unimodal with center of symmetry $\frac{k+1}{2}$.

We will now use Propositions B.1 and B.3 of [52]. Consider the term of the right side of (6.47) corresponding to the $(m-1)$-tuple $(k_2, \dots, k_m)$. If $k_j \geq 3$ for some j then since $(1+t^2)[k_j - 1]_t$ is positive, palindromic, and unimodal, $t^{m-1}(1+t^2)\prod_{i=2}^m [k_i - 1]_t$ is a product of positive, palindromic and unimodal polynomials. Hence $t^{m-1}(1+t^2)\prod_{i=2}^m [k_i - 1]_t$ is positive, palindromic and unimodal with center of symmetry

$$m - 1 + \frac{k_j}{2} + \sum_{\substack{i=2 \\ i \neq j}}^{m} \frac{k_i - 2}{2} = \frac{n}{2}.$$

It follows that the coefficient of each e_λ in $U_n(\mathbf{x}, t)$, where λ has a part of size at least 3, is positive, palindromic and unimodal with center of symmetry $\frac{n}{2}$. If λ does not have a part of size at least 3 then all the parts must be 2, which means that n is even. Hence if n is odd then $U_n(\mathbf{x}, t)$ is e-positive, palindromic, and e-unimodal with center of symmetry $\frac{n}{2}$.

Now if λ does not have a part of size at least 3 then $\lambda = 2^m$, where $n = 2m$. By (6.47), the coefficient of e_λ in $U_n(\mathbf{x}, t)$ is $t^{m-1}(1+t^2)$. It follows that if n is even, $U_n(\mathbf{x}, t) + t^m e_{2^m}$ is e-positive, palindromic, and e-unimodal, with center of symmetry $m = \frac{n}{2}$.

It follows from Lemma 5.5.3 that $V_n(\mathbf{x}, t)$ is also e-positive, palindromic, and e-unimodal, with center of symmetry $\frac{n}{2}$. Since $X_{C_n}(\mathbf{x}, t) = U_n(\mathbf{x}, t) + V_n(\mathbf{x}, t)$, Parts (1) and (2b) hold. Palindromicity of $X_{C_n}(\mathbf{x}, t)$ in the even case follows from Part (2b).

The assertion in Part (2a) that $X_{C_n}(\mathbf{x},t)$ is not e-unimodal in the even case follows from the fact the coefficient of $e_{2^m}(\mathbf{x})$ is not unimodal. □

6.7 Combinatorial proof of symmetry

It follows from our e-basis expansions of $W_n^<(\mathbf{x},t)$, $W_n^>(\mathbf{x},t)$, and $X_{C_n}(\mathbf{x},t)$ that they have symmetric function coefficients. In this section, we give a combinatorial proof of symmetry.

Theorem 6.7.1. *$W_n^<(\mathbf{x},t)$, $W_n^>(\mathbf{x},t)$, and $X_{C_n}(\mathbf{x},t)$ are in $\Lambda_{\mathbb{Z}}[t]$.*

Proof. For this proof, we will view Smirnov words $w = w_1w_2\cdots w_n$ with $w_1 < w_n$ as proper colorings $C^<(C_n)$of the labeled cycle graph $C_n = ([n], E)$ where the colors increase along the edge $\{1, n\}$, i.e., $\kappa \in C^<(C_n)$ means that $\kappa : [n] \to \mathbb{P}$ is a proper coloring of C_n and $\kappa(1) < \kappa(n)$. Then $\mathrm{des}(w)$ is the number of edges $\{i, i+1\}$ of C_n where $\kappa(i) > \kappa(i+1)$.

For each $a \in \mathbb{P}$, we will define an involution ω_a on the set $C^<(C_n)$ that exchanges the number of vertices colored with a and the number of vertices colored with $a+1$ but does not change the number of descents of the coloring. This then proves the theorem.

Notice that for any $a \in \mathbb{P}$ and any coloring $\kappa \in C^<(C_n)$, either the entire graph is colored with the colors a and $a+1$ (in which case n must be even) or the parts of the graph colored with a and $a+1$ form a set of disjoint paths (see Lemma 5.4.4). For the purpose of describing ω_a, we will define an a-chain as a maximal path in C_n such that each vertex is colored with either a or $a+1$, and define the length of an a-chain to be the number of vertices in the path. Now let $\kappa \in C^<(C_n)$ and define $\omega_a(\kappa)$ as follows:

(1) For a-chains of odd length, $\omega_a(\kappa)$ switches the colors a and $a+1$ on the vertices.

(2) If both vertex $\mathbf{1}$ and vertex $\mathbf{n}$ are contained in an a-chain of odd length, then let $\mathbf{i}$ be the end vertex of the chain with the smallest value.

(2a) If $\mathbf{i}$ is even, then after the a and $a+1$'s are switched, shift the colors down a vertex, i.e., $\omega_a(\kappa)(\mathbf{j}-\mathbf{1}) = \kappa(\mathbf{j})$ for $1 < j \leq n$, and $\omega_a(\kappa)(\mathbf{n}) = \kappa(\mathbf{1})$.

(2b) If $\mathbf{i}$ is odd, then after the a and $a+1$'s are switched, shift the colors up a vertex, i.e., $\omega_a(\kappa)(\mathbf{j}+\mathbf{1}) = \kappa(\mathbf{j})$ for $1 \leq j < n$, and $\omega_a(\kappa)(\mathbf{1}) = \kappa(\mathbf{n})$.

First let us check that ω_a is well defined, i.e., that for any $\kappa \in C_n^<$ we have that $\omega_a(\kappa) \in C_n^<$. It is easy to see that $\omega_a(\kappa)$ is a proper coloring of C_n, so we just need to check that $\omega_a(\kappa)(\mathbf{1}) < \omega_a(\kappa)(\mathbf{n})$. If at most one of $\mathbf{1}$ or $\mathbf{n}$ is colored with a or $a+1$ then ω_a should not affect the relative order. If both $\mathbf{1}$ and $\mathbf{n}$ are colored with a and $a+1$ then we must have $\kappa(\mathbf{1}) = a$ and $\kappa(\mathbf{n}) = a+1$. If these are contained in an even a-chain, then the colors do not change, so their relative order remains the same. If they are contained in an odd a-chain, then initially their colors will be switched; however, whether we fall into Case 2a or Case 2b, once the colors are rotated, we again have $\omega_a(\kappa)(\mathbf{1}) = a$ and $\omega_a(\kappa)(\mathbf{n}) = a+1$, so again their relative orders remain the same. Hence ω_a is well-defined.

Note that ω_a is an involution, which follows easily from the fact that step 2 will cause an odd a-chain that contains $\mathbf{1}$ and $\mathbf{n}$ to still be an odd a-chain that contains $\mathbf{1}$ and $\mathbf{n}$ and the label of vertex $\mathbf{i}$ mentioned in step 2 switches parity, hence, applying $\omega_a(\omega_a(\kappa)) = \kappa$. Similarly it is easy to see that ω_a switches the number of occurrences of the color a and the number of occurrences of the color $a+1$.

Now we need to check that ω_a preserves descents. Clearly, ω_a preserves the number of descents in a-chains of even length, because nothing changes. In chains of odd length that do not contain $\mathbf{1}$ and $\mathbf{n}$, notice that there is always the same number of ascents as descents, so once the colors are switched, all ascents become descents and vice versa, but the number of descents remains the same.

To see that ω_a preserves the number of descents in a-chains of odd length that contain **1** and **n**, let us look at the two cases. Let **i** be the end of the a-chain containing **1** and **n** with smallest label, and let **n-j** be the label on the other end of the a-chain. We can divide this a-chain into two parts, P_1 and P_2. Let P_1 be the part of the a-chain from **1** to **i** and let P_2 be the part of the a-chain from **n-j** to **n**.

(3a) If **i** is even, then in κ we see that P_1 contains one more ascent than descent, but P_2 contains the same number of ascents as descents. Once the colors of a and $a+1$ are switched, P_2 still contains the same number of ascents as descents, but now P_1 contains one more descent than ascent. When we rotate the colors, P_1 loses a vertex, but now it has the same number of ascents as descents, and P_2 gains a vertex, but now it has one more ascent than descent. Also notice that when we rotate the colors, it does not change the number of descents of the rest of the coloring.

(3b) If **i** is odd, then in κ we see that P_1 contains the same number of ascents as descents, but P_2 contains one more ascents than descent. Once the colors of a and $a+1$ are switched, P_1 still contains the same number of ascents as descents, but now P_2 contains one more descent than ascent. When we rotate the colors, P_2 loses a vertex, but now it has the same number of ascents as descents, and P_1 gains a vertex, but now it has one more ascent than descent. Also notice that when we rotate the colors, it does not change the number of descents of the rest of the coloring.

Hence ω_a is an involution that changes the number of occurrences of a and $a+1$ in each proper coloring in $C^{<}(C_n)$ and preserves the number of descents, so we see that $W_n^{<}(\mathbf{x}, t) \in \Lambda_Z[t]$.

Notice that if we apply ω_a to $C^{>}(C_n)$, the set of colorings, κ, of C_n with $\kappa(\mathbf{1}) > \kappa(\mathbf{n})$, then ω_a preserves the number of descents of the coloring and switches the number of occurrences of color a and color $a+1$, so this shows that $W^{>}(\mathbf{x},t)$ is symmetric as well.

We can also apply ω_a to all proper colorings of C_n. We see that ω_a would preserve the descents of each coloring and switch the number of occurrences of color a and color $a+1$, which shows that $X_{C_n}(\mathbf{x},t)$ is symmetric. $\square$

Appendix

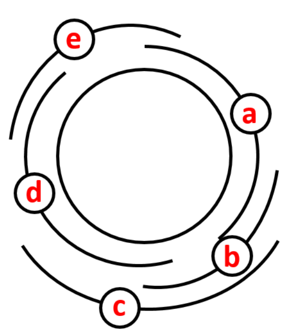
Appendix A

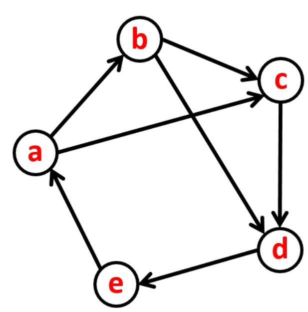
Graph classes

In this section, we will discuss a few properties of circular indifference digraphs, which we defined in Section 5.4. Then we will take a look at how these graphs relate to other graphs found in the literature.

Definition A.0.1. Suppose we have a finite collection of arcs positioned around a circle of any radius so that no arc properly contains another. We consider the starting point of an arc as the counterclockwise-most endpoint of the arc. We can construct a digraph, which we call a *proper circular arc digraph* by assigning a vertex to each arc and having an edge from arc A to arc B if the starting point of arc B is contained in arc A. The underlying undirected graph is called a *proper circular arc graph*.

Example A.0.2. Here we see a collection of proper circular arcs positioned around a circle and the corresponding proper circular arc digraph.





Theorem A.0.3. *Let $\overrightarrow{G}$ be a connected digraph. Then the following statements are equivalent:*

1. *$\overrightarrow{G}$ is a proper circular arc digraph.*

2. *$\overrightarrow{G}$ is a circular indifference digraph.*

Proof. First let's show (2) $\implies$ (1). Let $\overrightarrow{G}$ be a circular indifference digraph on $[n]$ that comes from the set of circular intervals

$$I = \{[a_1, b_1], [a_2, b_2], \cdots, [a_k, b_k]\}$$

of $[n]$. From these intervals, we can construct a set of intervals

$$\tilde{I} = \{[1, c_1], [2, c_2], ..., [n, c_n]\}$$

such that for each $i \in [n]$, we have that $[i, c_i]$ is the largest circular interval that is contained in an interval of I and that has i as its left endpoint. It is easy to see that each interval in I must be contained in an interval of $\tilde{I}$ and vice versa, so I and $\tilde{I}$ are both associated to $\overrightarrow{G}$.

We will construct n proper arcs on a circle so that the corresponding proper circular arc graph is $\overrightarrow{G}$. Draw a circle and place n points equally spaced around the circle. Label these points in cyclic order with $[n]$. For each circular interval of $\tilde{I}$, we will place an arc on the circle. Start with a circular interval $[i, c_i] \in \tilde{I}$ of maximal size and draw an arc from slightly before i to slightly after c_i. Continue this process with all the circular intervals of $\tilde{I}$ in weakly decreasing order of their sizes. Note that if the next interval has right endpoint the same c_i as a previous interval, the newest arc (coming from a circular interval of smaller size) should extend slightly past the previous arc to avoid having one interval properly contained in another.

The arc we construct from the interval $[i, c_i]$ that starts just before i on the circle is the arc that corresponds to vertex i in $\overrightarrow{G}$. Since arc i will contain the starting points of all the arcs corresponding to the vertices in $[i, c_i]$, we have that (i, j) is an edge of the proper circular arc digraph for each $i < j \leq c_i$. From here, we can see that the proper circular arc digraph associated to this set of arcs is isomorphic to the circular indifference digraph, $\overrightarrow{G}$, we started with.

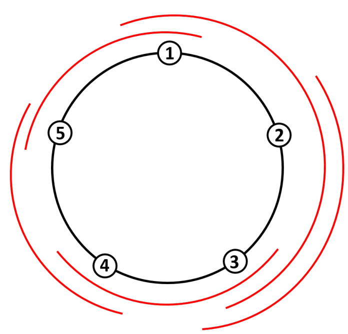
Here is an example of this process. Suppose we have the circular intervals

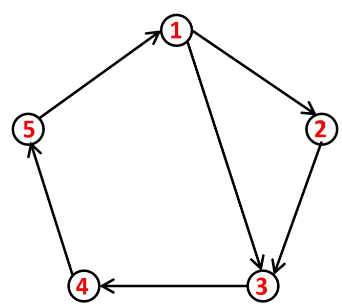
$$I = \{[1, 3], [3, 4], [4, 5], [5, 1]\}$$

on $[n]$. Then

$$\tilde{I} = \{[1, 3], [2, 3], [3, 4], [4, 5], [5, 1]\}.$$

The arcs that we would draw are shown in the figure below. We can see that both I and the arcs on this circle are associated with the digraph given below.





Now let's show $(1) \implies (2)$.

Let $\overrightarrow{G}$ be a proper circular arc digraph on n vertices that comes from some proper arc representation on a circle. Label one of the arcs 1. Now find the first arc that begins clockwise after arc 1. Label this arc 2. Then find the next arc that begins clockwise after arc 2. Label this arc 3. Continue this until all n arcs are labeled with the labels $[n]$. Now create a set of circular intervals of $[n]$, called I, as follows.

Let I be the set of all $[i, j]$ such that arc i contains the starting point of arc j. If arc i contains the starting point of arc j, then it must also contain the starting point of all arcs in $[i, j]$, hence we can see that the proper circular arc digraph associated with the set of arcs on the circle is isomorphic to the circular indifference digraph on $[n]$ associated with I. □

Another class of graphs we want to look at is the class of simple digraphs that do not have any induced subgraphs isomorphic to $\overrightarrow{K_{12}}$ and $\overrightarrow{K_{21}}$ as defined in Section 5.4 and displayed below.

For notational convenience, we will call these $\{\overrightarrow{K_{12}}, \overrightarrow{K_{21}}\}$-free digraphs.

Theorem A.0.4. *Let G be a simple connected graph. Then the following statements are equivalent:*

1. *G is isomorphic to a proper circular arc graph.*

2. *G is isomorphic to a circular indifference graph.*

3. *G admits an orientation that makes it a $\{\overrightarrow{K_{12}}, \overrightarrow{K_{21}}\}$-free digraph.*

Proof. The equivalence of (1) and (3) was shown by Skrien in [56], and the equivalence of (1) and (2) follows from Theorem A.0.3. □

Now let us look at the non-circular version of these graphs.

Definition A.0.5. Suppose that we have a collection of intervals, I, of the ordered set $[n]$. Then we can construct a graph, $G = ([n], E)$, with edge set $E = \{\{i, j\} \mid i \neq j \text{ and } i, j \text{ contained in the same interval of } I\}$. This is called an *indifference graph*.

Definition A.0.6. Suppose we have a finite collection of intervals on the real line. We can associate a graph to this interval representation by letting each interval correspond to a vertex and allowing two distinct vertices to be adjacent if their corresponding intervals overlap. This is called an *interval graph*. If no interval properly contains another, this is called a *proper interval graph*. If each interval has length 1, this is called a *unit interval graph*.

Note that unit interval graphs are just natural unit interval graphs, defined in Section 4.2, without the labels. The following theorem is the acyclic or non-circular version of Theorem A.0.4.

Theorem A.0.7. *Let G be a simple graph. Then the following statements are equivalent:*

1. *G is isomorphic to a proper interval graph.*

2. *G is isomorphic to a unit interval graph.*

3. *G is isomorphic to an indifference graph.*

4. *G admits an acyclic orientation that makes it a $\{\overrightarrow{K_{12}}, \overrightarrow{K_{21}}\}$-free digraph.*

The equivalence of (1) and (2) was shown by Roberts in [50]. The equivalence of (1) and (4) was shown by Skrien in [56]. The equivalence of (1) and (3) is well-known, but can be shown by an analogous argument to the one given in the proof of Theorem A.0.3.

Note that if we turn a natural unit interval graph of Shareshian and Wachs into a digraph by orienting edges from smaller labels to larger labels, then we get an acyclic $\{\overrightarrow{K_{12}}, \overrightarrow{K_{21}}\}$-free digraph, and in fact, every acyclic $\{\overrightarrow{K_{12}}, \overrightarrow{K_{21}}\}$-free digraph comes from a natural unit interval graph (see [52, Section 4] for more information on natural unit interval graphs).

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