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C^0-(in)extendibility of Spacetimes

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UNIVERSITY OF MIAMI

C^0-(IN)EXTENDIBILITY OF SPACETIMES

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Ever since the realization, from the Hawking-Penrose singularity theorems, that
singularities in spacetime can develop under generic circumstances, the question has
been considered as to what extent general relativity is a classically deterministic the-
ory. The essence of Penrose's strong cosmic censorship conjecture [23] is that, indeed,
general relativity is deterministic. Put in rough physical terms, under reasonable
physical conditions, spacetime should not develop naked singularities, that is to say,
no singularity (due e.g. to curvature blow-up) should ever be visible to any observer.
Such singularities would undermine the predictive ability of general relativity.

More modern statements of the strong cosmic censorship conjecture focus on the
Cauchy problem for the Einstein equations.

Strong cosmic censorship conjecture: *The maximal globally hyperbolic develop-
ment of 'generic initial data' for the Einstein equations is inextendible as a 'suitably
regular' Lorentzian manifold.*

Formulating a precise statement of the strong cosmic censorship conjecture is itself
a challenge because one needs to make precise the phrases 'generic initial data' and
'suitably regular Lorentzian manifold'. Understanding the latter is where general rel-
ativity in low regularity and in particular (in-)extendibility results become significant.
In [9], Dafermos and Luk show that the conjecture is false when 'suitably regular' is
taken to mean a Lorentzian manifold with a C^0 metric.

Prior to recent work of Sbierski [25], very little had been done to address the issue
of the extendibility (or not) of Lorentzian manifolds with metrics at lower regularity.
In [25] Sbierski develops methods for establishing the C^0-inextendibility of Lorentzian

manifolds, which he uses to prove the C^0-inextendibility of Minkowski space and the maximally analytic Schwarzschild spacetime.

In chapter one of this thesis we review the properties of C^0 spacetimes. In chapter two we establish C^0-inextendibility results applicable to the asymptotic regions of black hole spacetimes where future timelike completeness is assumed to hold. We then show how these techniques can be applied to establish the C^0-inextendibility of Minkowski, de Sitter, and anti-de Sitter spaces.

In chapter three we show that a class of $k = -1$ inflationnary FLRW spacetimes dubbed 'Milne-like' are in fact C^0-extendible (i.e. they extend through the big bang). We prove that a certain subclass of these spacetimes also do not admit curvature singularities. We also show that the cosmological constant appears as an initial condition for Milne-like spacetimes and that these spacetimes have a notion of Lorentz invariance.

In chapter four we give brief overviews of other results obtained in the C^∞ (smooth) spacetime category. We establish a theorem linking cosmological singularities to 3-manifold topology. We prove the invisibility of (weakly) trapped surfaces in asymptotically de Sitter spacetimes. Lastly, we establish an existence result for Cauchy surfaces with constant mean curvature from a spacetime curvature condition.

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Second, I would like to thank my family for supporting me through my years in academia and instilling into me a curious wonder of the world. I am forever grateful for our family dinner discussions which left a lifelong impression on me.

Finally, I would like to thank my friends for always being there for me. I cherish all the joys and struggles. We'll forever laugh and love in the universe we dreamt.

Contents

Chapter 1

C^0-Causal Theory

1.1 C^0 Spacetimes

Let $k \geq 1$. A C^k *manifold* is a topological space M endowed with a maximal C^k atlas. The *dimension* of M is the dimension of the atlas. This atlas structure allows us to define C^k curves over M. We can use C^1 curves to generate tangent vectors at a point $p \in M$. This construction yields the tangent space $T_p M$ and the tangent bundle TM which is a C^{k-1} manifold. More generally, one has C^{k-1} tensor bundles over M. By Theorem 2.9 in [18], any maximal C^1 atlas has a unique maximal smooth subatlas. Therefore we can always assume our manifold is smooth by working in this subatlas.

Let $k \geq 0$. A C^k *metric* on a C^{k+1} manifold M is a nondegenerate symmetric tensor $g \colon TM \times TM \to \mathbb{R}$ with constant index whose components in any coordinate system are C^k functions. Symmetric means $g(X, Y) = g(Y, X)$. Nondegenerate means $g(X, Y) = 0$ for all $Y \in TM$ implies $X = 0$. With constant index means there is an integer r such that at each point $p \in M$, there is a basis $e_1, \ldots, e_r, \ldots, e_n \in T_p M$ such that $g(e_i, e_i) = 1$ for $1 \leq i \leq r$ and $g(e_i, e_i) = -1$ for $r + 1 \leq i \leq n$. If $r = n$, then g

is called a *Riemannian* metric and (M, g) a Riemannian manifold. If $r = n - 1$, then g is called a *Lorentzian* metric and (M, g) a Lorentzian manifold.

If (M, g) is a Lorentzian manifold, then a nonzero vector $X \in T_pM$ is *timelike*, *null*, or *spacelike* if $g(X, X) < 0, = 0, > 0$, respectively. A nonzero vector is *causal* if it is either timelike or null. Note that our convention is that a null or causal vector is necessarily not the zero vector. A Lorentzian manifold (M, g) is *time-oriented* provided there is a C^0 timelike vector field $X \in TM$. A causal vector $Y \in T_pM$ is *future directed* if $g(X, Y) < 0$ and *past directed* if $g(X, Y) > 0$. Note that $-X$ defines an opposite time-orientation, and so any statement/theorem in a spacetime which is time-oriented by X has a time dual statement/theorem with the $-X$ time orientation.

Let $k \geq 0$. A C^k *spacetime* is a pair (M, g) where M is a connected, Hausdorff, and second-countable C^{k+1} manifold and g is a C^k Lorentzian metric such that (M, g) is time-oriented. The Hausdorff condition guarantees uniqueness of limits. The second-countable property allows us to construct partitions of unity whenever needed. For example, the following result, which has played a significant role in causal theory, relies on the existence of a partition of unity.

Proposition 1.1.1. *Let M be a C^1 connected second-countable Hausdorff manifold. Then there is a complete Riemannian metric h on M in the smooth subatlas.*

Proof. We could construct h via a partition of unity, but there's another argument using the Hopf-Rinow theorem [22]. Working in the C^∞ subatlas, we apply the Whitney embedding theorem [19] to obtain a smooth proper embedding $f \colon M \to \mathbb{R}^N$. By pulling back the Euclidean metric onto M, we have a Riemannian manifold (M, h). Let d_h be the distance function on M induced by h. Since f is proper, any closed set in M corresponds to a closed subset of $\mathbb{R}^N$. Therefore any closed and bounded subset of (M, d_h) will be a closed and bounded subset within $f(M) \subset \mathbb{R}^N$ which is compact by the Heine-Borel theorem. Therefore (M, h) is complete by Hopf-Rinow. $\qquad\square$

1.2 Timelike curves

Let (M, g) be a C^0 spacetime. A *timelike curve* is a piecewise C^1 map $\gamma \colon [a, b] \to M$ such that $\gamma'(t)$ is future-directed timelike at all its differentiable points, and if $t_0 \in [a, b]$ is a break point, then $\lim_{t \nearrow t_0} \gamma'(t)$ and $\lim_{t \searrow t_0} \gamma'(t)$ are both future-directed timelike. Note this means that $\gamma|_{[b-\varepsilon, b)}$ can be extended to a C^1 timelike curve; this will be needed to prove timelike future sets are open. Letting timelike curves be piecewise C^1 allows us to concatenate two timelike curves to form another timelike curve.
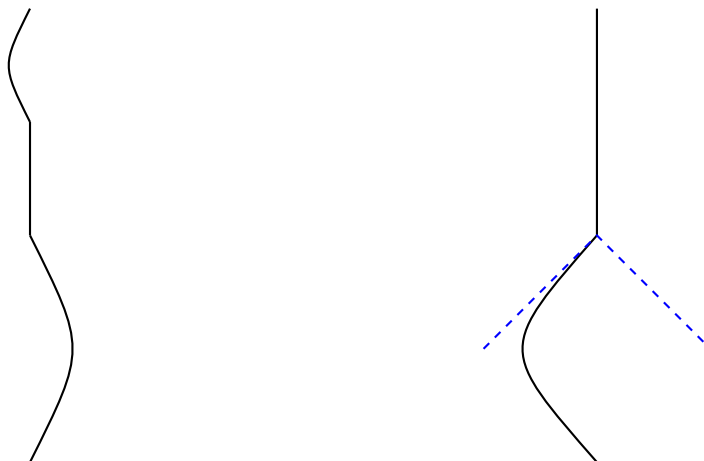
Figure 1.1: The curve on the left is a timelike curve. The curve on the right is not a timelike curve even though it is timelike at all its differentiable points. We don't count it as a timelike curve because it approaches a null vector at its break point.

A *unit* timelike curve is one such that $g(\gamma', \gamma') = -1$ at all its differentiable points. Given a set $S \subset M$ and a timelike curve γ, we will write $\gamma \subset S$ instead of $\gamma\big([a, b]\big) \subset S$. Likewise with $\gamma \cap S$. Note that 'future-directed' is implicit in the definition of a timelike curve. We will also define causal curves this way.

Definition 1.2.1. Given a set S and neighborhood U, we define the *timelike future* of S within U as the set

$$I^+(S, U) = \{p \mid \text{there is a timelike } \gamma \colon [a, b] \to U \text{ with } \gamma(a) \in S,\ \gamma(b) = p,\ \gamma \subset U\}$$

The *timelike past* $I^-(S, U)$ is defined time dually. If $U = M$, then we will write $I^+(S)$ instead. If $S = \{p\}$, then we will write $I^+(p, U)$ instead. If we wish to emphasize the Lorentzian metric g being used, we will write $I_g^+(S, U)$.

Definition 1.2.2. The *Minkowski* metric in $\mathbb{R}^{n+1}$ is $\eta = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + \delta_{ij} dx^i dx^j$. For $0 < \varepsilon < 1$, we define the *narrow* and *wide* Minkowski metrics

$$\eta^\varepsilon = -\frac{1-\varepsilon}{1+\varepsilon}(dx^0)^2 + \delta_{ij} dx^i dx^j = \eta + \frac{2\varepsilon}{1+\varepsilon}(dx^0)^2$$

$$\eta^{-\varepsilon} = -\frac{1+\varepsilon}{1-\varepsilon}(dx^0)^2 + \delta_{ij} dx^i dx^j = \eta - \frac{2\varepsilon}{1-\varepsilon}(dx^0)^2$$

For example, $\eta^{3/5}$ and $\eta^{-3/5}$ have light cones with slopes 2 and $1/2$, respectively.

Lemma 1.2.3 ([25]). *Fix $p \in M$. For any $0 < \varepsilon < 1$ there is a coordinate system $\phi \colon U_\varepsilon \to \mathbb{R}^{n+1}$ with the following properties*

(1) $\phi(p) = 0$

(2) $g_{\mu\nu}(p) = \eta_{\mu\nu}$

(3) $I_{\eta^\varepsilon}^+(p, U_\varepsilon) \subset I^+(p, U_\varepsilon) \subset I_{\eta^{-\varepsilon}}^+(p, U_\varepsilon)$.

Moreover if $\gamma \colon [a, b] \to M$ is a unit timelike curve with $\gamma(b) = p$, then we can choose the coordinate system so that $\phi \circ \gamma(t) = (t - b, 0, \ldots, 0)$.

Proof. Pick a coordinate system $\phi\colon U \to \mathbb{R}^{n+1}$ with $\phi(p) = 0$ and apply Gram-Schmidt to obtain (2). By continuity of the metric, given any $\varepsilon' > 0$, we can shrink our neighborhood so that $|g_{\mu\nu}(x) - \eta_{\mu\nu}| < \varepsilon'$. Let $X = X^\mu \partial_\mu$ be any tangent vector with $X^0 = 1$. Then

$$g(X, X) < \eta(X, X) + \varepsilon' \sum_{\mu, \nu} X^\mu X^\nu$$

$$= \eta^\varepsilon(X, X) - \frac{2\varepsilon}{1+\varepsilon} + \varepsilon' \sum_{\mu, \nu} X^\mu X^\nu$$

$$= \eta^\varepsilon(X, X) - \frac{2\varepsilon}{1+\varepsilon} + \varepsilon' \left[1 + \sum_i X^i + \sum_{i, j} X^i X^j \right]$$

If X is η^ε-timelike, then $|X^i|^2/|X^0|^2 < (1 - \varepsilon)/(1 + \varepsilon)$. Since $X^0 = 1$, we have

$$g(X, X) < \eta^\varepsilon(X, X) - \frac{2\varepsilon}{1+\varepsilon} + \varepsilon' \left[1 + n\sqrt{\frac{1-\varepsilon}{1+\varepsilon}} + n^2 \frac{1-\varepsilon}{1+\varepsilon} \right]$$

where $n + 1$ is the dimension of the spacetime. By taking $\varepsilon' > 0$ small enough, we can ensure $2\varepsilon/(1+\varepsilon)$ is larger than the bracket term. This proves the first inclusion. The proof of the second is analogous.

Now let $\gamma\colon [a, b] \to M$ be a unit timelike curve with $\gamma(b) = p$. Let (y^0, y^i) be the coordinates on U. Since $g_{\mu\nu}(p) = \eta_{\mu\nu}$, we can shrink U so that y^0 is a time function (i.e. ∇y^0 is past-directed timelike). Since we required $\lim_{t \nearrow b} \gamma'(t)$ to be future-directed timelike, the function $(y^0 \circ \gamma)'(t)$ approaches a nonzero number as $t \nearrow b$. Therefore the inverse function theorem guarantees an interval $(b - \delta, b + \delta)$ around b and a diffeomorphism $f\colon (b - \delta, b + \delta) \to (-\delta', \delta')$ such that $f = y^0 \circ \gamma$ on $(b - \delta, b]$. Let $U' \subset U$ be the preimage of $(-\delta', \delta)$ under y^0. We define new coordinates (x^0, x^i) on U' by

$$x^0(q) = y^0(q) \quad \text{and} \quad x^i(q) = y^i(q) - y^i\left(\gamma \circ f^{-1} \circ y^0(q) \right).$$

With these coordinates we have

$$x^0 \circ \gamma(t) = t - b \quad \text{and} \quad x^i \circ \gamma(t) = 0.$$

□
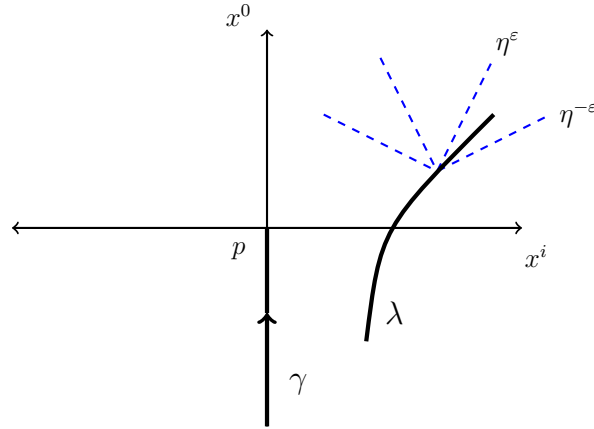
Figure 1.2: The coordinate system appearing in Lemma 1.2.3. The point p is located at the origin where the metric is exactly Minkowski (i.e. $g_{\mu\nu}(p) = \eta_{\mu\nu}$). The timelike curve γ makes up the negative x^0-axis. Any timelike curve $\lambda \subset U_\varepsilon$ will always be $\eta^{-\varepsilon}$-timelike but it may be η^ε-spacelike.

Proposition 1.2.4 ([25]). *If U is an open set, then $I^+(p, U)$ is open.*

Proof. Fix $q \in I^+(p, U)$ and let $\gamma \subset U$ be a future timelike curve with $\gamma(1) = q$. Let $\phi \colon U_\varepsilon \to \mathbb{R}^{n+1}$ be a coordinate system from Lemma 1.2.3 centered around q. Choose $\varepsilon = 3/5$ so that η^ε has lightcones with slope 2. Then for $t < 1$, we have $I_{\eta^\varepsilon}^+\big(\gamma(t), U_\varepsilon\big)$ is open since it's just the interior of a cone, and this set is contained in $I^+\big(\gamma(t), U_\varepsilon\big)$. The result follows by choosing $U_\varepsilon \subset U$. □

Corollary 1.2.5. *$I^+(S, U)$ is open.*

Proof. $I^+(S, U) = \bigcup_{p \in S} I^+(p, U)$. □

1.3   Causal curves

Fix a C^0 spacetime (M, g) with a complete Riemannian metric h. Let $I \subset \mathbb{R}$ be an interval (i.e. any connected subset of $\mathbb{R}$ with nonempty interior). A *locally Lipschitz* curve $\gamma \colon I \to M$ is a curve such that for any compact $K \subset I$ there is a constant C_K such that for any $a, b \in K$, we have

$$d_h\big(\gamma(a), \gamma(b)\big) \leq C_K |b - a|$$

where d_h is the Riemannian distance function associated with h. By Rademacher's theorem, locally Lipschitz curves are differentiable almost everywhere and locally in L^∞. A *causal curve* is a locally Lipschitz curve $\gamma \colon I \to M$ such that γ' is future-directed causal almost everywhere. Note that 'future directed' is implicit in the definition of a causal curve. We will abuse notation and write $\gamma \subset U$ instead of $\gamma(I) \subset U$. Likewise with $\gamma \cap U$.

Definition 1.3.1. Given a set S and a neighborhood U, we define the *causal future* of S within U as

$$J^+(S, U) = \{p \mid \text{there is a causal } \gamma \colon [a, b] \to U \text{ with } \gamma(a) \in S, \, \gamma(b) = p, \, \gamma \subset U\} \cup S$$

We include the union with S for J^+, because our definition of a causal curve does not include the trivial curve. The *causal past* $J^-(S, U)$ is defined time dually. If $U = M$, then we will write $J^+(S)$ instead. If $S = \{p\}$, then we will write $J^+(p, U)$ instead. Likewise with J^-.

By Proposition 2.3.1 in [4], the definition of causal curves does not depend on the choice of complete Riemannian metric h.

Proposition 1.3.2 ([4])**.** *Let h_1 and h_2 be two complete Riemannian metrics on M. Then a curve $\gamma\colon I \to M$ is locally Lipschitz with respect to h_1 if and only if it is locally Lipschitz with respect to h_2.*

One of the reasons for using locally Lipschitz curves to define causal curves is that they can be parameterized by h-arclength. Specifically we have

Proposition 1.3.3 ([4])**.** *Let $\gamma\colon I \to M$ be a causal curve and h a Riemannian metric on M. Then γ admits a reparameterization $\tilde{\gamma}$ such that $h(\tilde{\gamma}', \tilde{\gamma}') = 1$ almost everywhere and for all $a, b \in I$, we have*

$$d_h\big(\tilde{\gamma}(a), \tilde{\gamma}(b)\big) \leq |a - b|.$$

Proof. By Rademacher's theorem γ' is differentiable almost everywhere and locally in L^∞. Therefore the integral

$$s(t) = \int_{t_0}^t \sqrt{h(\gamma', \gamma')}$$

is well-defined and finite. Since γ is causal, $\gamma' \neq 0$ almost everywhere. Therefore $s(t)$ is strictly increasing; hence invertible. The reparameterization we seek is $\tilde{\gamma} = \gamma \circ s^{-1}$. Then the inequality follows from:

$$b - a = \int_a^b dt = \int_a^b \sqrt{h(\tilde{\gamma}', \tilde{\gamma}')} \geq \inf_\sigma \int \sqrt{h(\sigma', \sigma')} = d_h\big(\tilde{\gamma}(a), \tilde{\gamma}(b)\big).$$

$\square$

We set out now to prove that a causal curve must be future-directed causal at all its differentiable points. We will need two lemmas first. Recall that τ is a *time function* if its gradient $\nabla \tau$ is a past-directed timelike vector field.

Lemma 1.3.4. *Let τ be a time function and $\gamma\colon [a,b] \to M$ a causal curve. Then*

$$\gamma \subset \big\{p \mid \tau(p) > \tau \circ \gamma(a)\big\}.$$

Proof. Integrating gives

$$\tau \circ \gamma(t) - \tau \circ \gamma(a) = \int_0^t (\tau \circ \gamma)' = \int_0^t g(\nabla\tau, \gamma') \; > \; 0.$$

The last inequality holds because $g(\nabla\tau, \gamma') > 0$ almost everywhere. $\qquad\square$

Lemma 1.3.5. *Let $\gamma\colon [a,b] \to U_\varepsilon$ be a causal curve where U_ε is a coordinate neighborhood as in Lemma* 1.2.3. *Then $\gamma \subset I^+_{\eta^{-\varepsilon}}\big(\gamma(a), U_\varepsilon\big)$.*

Proof. Pick $\varepsilon = 3/5$, then $\eta^{-\varepsilon}$ has lightcones with slope $1/2$. With this ε, consider the hyperplanes in U_ε given by $x^0 - \frac{1}{2}x^1 = $ constant. Since these hyperplanes are $\eta^{-\varepsilon}$-spacelike, they are g-spacelike. Let τ be the g-time function such that $\nabla\tau$ is orthogonal to these hyperplanes. Note that $\nabla\tau$ is η^ε-timelike (lightcones with slope 2). Therefore it's g-timelike. Apply Lemma 1.3.4. Now replace $\partial/\partial x^1$ with any arbitrary unit direction orthogonal to $\partial/\partial x^0$, and apply Lemma 1.3.4 again to conclude that $\gamma \subset I^+_{\eta^{-\varepsilon}}\big(\gamma(a), U_\varepsilon\big)$. Clearly this proof does not depend on the specific choice of $\varepsilon = 3/5$. $\qquad\square$

Proposition 1.3.6. *Let $\gamma\colon I \to M$ be a causal curve. If γ is differentiable at $t_0 \in I$, then $\gamma'(t_0)$ is future-directed causal.*

Proof. Shift the parameterization so that $t_0 = 0$. First we show $\gamma'(0) \neq 0$. Construct a coordinate system $\phi\colon U_\varepsilon \to \mathbb{R}^{n+1}$ as in Lemma 1.2.3 centered around $\gamma(0)$. Choose $\varepsilon = 3/5$ so $\eta^{-\varepsilon}$ has lightcones with slope $1/2$. Then Lemma 1.3.5 implies

$$|\phi \circ \gamma(t) - \phi \circ \gamma(0)| \geq \frac{1}{2}|t|.$$

Here we are using the standard Euclidean norm on the left hand side. Therefore the difference quotient $|\phi \circ \gamma(t) - \phi \circ \gamma(0)/t|$ is bounded below. Hence $\gamma'(0) \neq 0$.

Next we show $\gamma'(0)$ cannot be spacelike. Let (x^0, x^i) be the coordinates on U_ε and put $\gamma^\mu = x^\mu \circ \gamma$. Without loss of generality suppose $\gamma'(0)$ is unit spacelike and rotate coordinates so that $\gamma'(0) = \partial/\partial x^1$. Let e_1 be the pushforward of $\partial/\partial x^1$. By definition of the derivative we have

$$\gamma^\mu(t) = te_1 + f^\mu(t)$$

where f^μ satisfies $f^\mu(t)/t \to 0$ as $t \to 0$. Therefore, by choosing $t > 0$ small enough, we can ensure $\gamma(t) \notin I^+_{\eta^{-\varepsilon}}(\gamma(0), U_\varepsilon)$ which contradicts Lemma 1.3.5. The proof that $\gamma'(0)$ cannot be past-directed causal is analogous. $\qquad\square$

A useful fact we will use frequently is that we can bound both the Lorentzian length and h-length of causal curves in small open sets. This is intuitively clear because causal curves 'can only go up.'

Proposition 1.3.7 ([15]). *Given any $p \in M$ and $\varepsilon > 0$, there is a neighborhood U such that $L(\gamma) < \varepsilon$ and $L_h(\gamma) < \varepsilon$ for all causal curves $\gamma \subset U$.*

Proof. Choose a neighborhood $\phi \colon U_\varepsilon \to \mathbb{R}^{n+1}$ as in Lemma 1.2.3 with $\varepsilon = 3/5$ so the lightcones have slope of $\eta^{-\varepsilon}$ have slope $1/2$. We first establish the bound on the Lorentzian length. Let $\gamma \subset U_\varepsilon$ be any causal curve. Put $X = \gamma'$. By Lemma 1.3.5, we know that X is $\eta^{-\varepsilon}$ timelike and since x^0 is a time function, $x^0 \circ \gamma$ is strictly increasing, and so we can reparameterize γ by x^0. Therefore $X^0 = 1$. By continuity of the metric, there is an $\varepsilon' > 0$ such that $|g_{\mu\nu}(x) - \eta_{\mu\nu}| < \varepsilon'$ for all $x \in U_\varepsilon$. Therefore

$$-g(X, X) < -\eta(X, X) + \varepsilon' \sum_{\mu, \nu} X^\mu X^\nu < -1 + \varepsilon' \left[1 + \sum_i X^i + \sum_{i, j} X^i X^j \right].$$

Since $|X^i| < 2$, we have $-g(X, X) < -1 + \varepsilon'(1 + 2n + 4n^2)$. Choosing the coordinate neighborhood so that $-\varepsilon' < x^0 < \varepsilon'$ gives $L(\gamma) < 2\varepsilon'\left[-1 + \varepsilon'(1 + 2n + 4n^2)\right]$, and we can choose $\varepsilon' > 0$ as small as we desire.

Now for the h-length. Since $\eta^{-\varepsilon}$ have lightcones with slopes $1/2$, we have $|X^i/X^0| < 2$. Since $X^0 = 1$, we have

$$h(\gamma', \gamma') = h_{\mu\nu}X^\mu X^\nu = h_{00} + 2h_{0i}X^i + h_{ij}X^i X^j$$

Set $H = \sup\left\{|h_{\mu\nu}(q)| \mid q \in U_\varepsilon, \text{ and } \mu, \nu\right\}$. Then $H < \infty$ by choosing U_ε to have compact closure. Since $|X^i| < 2$, we have $h(\gamma', \gamma') < 4(n + 1)H$. Therefore with $-\varepsilon' < x^0 < \varepsilon'$, we have $L_h(\gamma) \leq 2\varepsilon'\sqrt{4(n + 1)H}$. □

Let $\gamma\colon [a, b) \to M$ be a causal curve. Suppose there exists a $p \in M$ such that $\gamma(t_n) \to p$ for any sequence $t_n \nearrow b$. Then p is called a *future endpoint* of γ. One is tempted to define a new curve $\tilde\gamma\colon [a, b] \to M$ such that $\tilde\gamma(t) = \gamma(t)$ for $t < b$ and $\tilde\gamma(b) = p$. However it could be the case that $\tilde\gamma$ is not locally Lipschitz. For example if one extends the curve $t \mapsto \left(\sqrt{t + 1}, 0\right)$ from $(-1, 0]$ to $[-1, 0]$ in the obvious way, then the new curve defined on $[-1, 0]$ will not be locally Lipschitz because $\left(\sqrt{t + 1} - \sqrt{t' + 1}\right)/(t - t')$ diverges as t and t' approach 0. However if we reparameterize causal curves with respect to h-arclength, then these problems go away.

Let $\gamma\colon [a, b) \to M$ be a causal curve. We say γ is *future extendible* if there is a causal curve $\tilde\gamma\colon [a, b] \to M$ with $\tilde\gamma(t) = \gamma(t)$ for all $t \in [a, b)$. If no such $\tilde\gamma$ exists, then we say γ is *future inextendible*. Analogous definitions hold for *past inextendible* and *inextendible*. One of the major advantages of parameterizing causal curves with respect to a complete Riemannian metric h is that future inextendibility coincides with $b = \infty$. By Theorem 2.5.5 in [4], we have

Theorem 1.3.8. *Let h be a complete Riemannian metric on M and $\gamma\colon [a, b) \to M$ a causal curve parameterized by h-arclength.*

- *If $b = \infty$, then γ is future inextendible.*

- *If $b < \infty$, then γ can be extended to a future inextendible causal curve.*

Definition 1.3.9. Let $\gamma_n \colon I \to M$ be a sequence of causal curves. A locally Lipschitz curve $\gamma \colon I \to M$ is a *limit curve* of the γ_n if there is a subsequence γ_{n_k} that converges to γ uniformly on compact subsets of I. If γ is also a causal curve, then we say γ is a *causal limit curve* of the sequence γ_n.

From Theorem 1.6 in [5], we have the following limit curve theorem.

Theorem 1.3.10 (Limit Curve Theorem)**.** *Let $\gamma_n \colon \mathbb{R} \to M$ be a sequence of causal curves parameterized by h-arclength. If p is an accumulation point of $\{\gamma_n\}$, then there is an inextendible causal limit curve $\gamma \colon \mathbb{R} \to M$ of the γ_n which passes through p.*

1.4 Globally hyperbolic spacetimes

Definition 1.4.1. An open set $U \subset M$ is *causally convex* if $\gamma \cap U$ is connected for any causal curve γ which intersects it. A spacetime (M, g) is *strongly causal* if it has a topological basis of causally convex sets. If (M, g) is strongly causal and $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$, then (M, g) is *globally hyperbolic.*

In this section we prove a fundamental result for globally hyperbolic spacetimes. Recall a causal curve γ from p to q is a *causal maximizer* if $L(\gamma) \geq L(\lambda)$ for all causal curves λ from p to q.

Theorem 1.4.2 ([15, 24])**.** *Let (M, g) be a globally hyperbolic spacetime. Then given any $q \in J^+(p)$, there is a causal maximizer γ from p to q. Moreover $L(\gamma) < \infty$.*

Before proving Theorem 1.4.2, we have to establish some facts about strongly causal spacetimes. The first is the 'no imprisonment' property.

Proposition 1.4.3. *Suppose (M, g) is strongly causal and K is compact. Then there is no inextendible causal curve contained in K.*

Proof. By Proposition 1.3.7, for each $x \in K$, there is a neighborhood U_x such that $L_h(\gamma) \leq 1$ for all $\gamma \subset U_x$. By strong causality, there are causally convex neighborhoods $V_x \subset U_x$. Since K is compact and covered by $\{V_x\}_{x \in K}$, there is a finite subcover $V_1, \ldots, V_m$. Since these neighborhoods are causally convex, any causal curve contained in their union must have h-length bounded by m. Consequently, inextendibility in K is forbidden by Theorem 1.3.8. $\square$

The following is an important consequence of the limit curve theorem.

Proposition 1.4.4 ([15])**.** *Suppose (M, g) is strongly causal and K is compact. If $\gamma_n \colon [a_n, b_n] \to K$ is a sequence of causal curves with $\gamma_n(a_n) \to p$ and $\gamma_n(b_n) \to q$, then there is a causal limit curve γ from p to q.*

Proof. By compactness, we can cover K by finitely many causally convex neighborhoods to prove

$$\sup \{L_h(\gamma) \mid \gamma \subset K\} < \infty.$$

Assume each γ_n is parameterized by h-arclength. By Theorem 1.3.8, we can extend these to inextendible causal curves $\tilde{\gamma}_n \colon \mathbb{R} \to M$. By the limit curve theorem, there is a subsequence $\tilde{\gamma}_{n_k}$ which converges to an inextendible causal curve $\tilde{\gamma}$ which passes through p. Thus $a_{n_k} \to a > -\infty$. For the b_{n_k}, we either have (1) $b_{n_k} \to \infty$ or (2) $b_{n_k} \to b < \infty$. The first scenario is ruled out by the bound above. Therefore the second scenario must hold. Then $\tilde{\gamma}|_{[0, b]}$ is a causal limit curve from p to q. $\square$

The proof of Theorem 1.4.2 hinges on the upper semi-continuity of the Lorentzian length functional. We will first use this to prove Theorem 1.4.2, and then we will prove the upper semi-continuity of the length functional.

Proposition 1.4.5 ([15, 24]). *The Lorentzian length functional is upper semi-continuous.*

This means that if $\gamma_n \colon [a,b] \to M$ converges uniformly to $\gamma \colon [a,b] \to M$, then by choosing all n sufficiently large, the lengths of γ_n cannot be that much greater than the length of γ. More precisely, given any $\varepsilon > 0$ there is an N such that $L(\gamma_n) \le L(\gamma) + \varepsilon$ for all $n \ge N$.

Proof of Theorem 1.4.2:

Set

$$\mathcal{L} = \sup \{ \, L(\gamma) \mid \gamma \text{ is a causal curve from } p \text{ to } q \}$$

We first show $\mathcal{L} < \infty$. Let $K = J^+(p) \cap J^-(q)$. By Proposition 1.3.7 for each $x \in K$, there is a neighborhood U_x such that $L(\gamma) \le 1$ for all $\gamma \subset U_x$. By strong causality, there are causally convex neighborhoods $V_x \subset U_x$. Since K is compact and covered by $\{V_x\}_{x \in K}$, there is a finite subcover $V_1, \dots, V_m$. Therefore $\mathcal{L}$ is bounded by m.

By definition of $\mathcal{L}$ there is a sequence of causal curves γ_n from p to q satisfying $\mathcal{L} \le L(\gamma_n) + 1/n$. Proposition 1.4.4 guarantees a causal limit curve γ from p to q. By restricting to a subsequence, we can assume γ_n converges uniformly to γ. By upper semi-continuity of the length functional, given any $\varepsilon > 0$ there exists an N such that $n \ge N$ implies $L(\gamma) + \varepsilon \ge L(\gamma_n) \ge \mathcal{L} - 1/n$. Since this is true for all $n \ge N$, we have $L(\gamma) + \varepsilon \ge \mathcal{L}$. Since ε was arbitrary, γ is a causal maximizer from p to q.

$\square$

Proof of Proposition 1.4.5:

By Proposition 1.2 of [5], there is a family of smooth Lorentzian metrics $\{g_\varepsilon : \varepsilon > 0\}$ such that g_ε is wider than g (i.e., $g(X,X) \le 0$, $X \ne 0 \implies g_\varepsilon(X,X) < 0$), g_ε converges uniformly on compact subsets of M to g as $\varepsilon \to 0$, and for all $X \in TM$ with $|X|_h = 1$, we have $|g(X,X) - g_\varepsilon(X,X)| < \varepsilon$.

Hence, the curves γ and γ_n are future causal curves in (M, g_ε). By Proposition 1.3.7, there exists $C > 0$ and a partition $a = s_0 < s_1 < \cdots < s_k = b$ of $[a, b]$ such that, for $i = 0, ..., k-1$, $\gamma([s_i, s_{i+1}])$ lies in a neighborhood V_i with the property that every g-causal curve in V_i has h-length less than C.

Now let σ be a g-causal curve in V parameterized by h-arclength, then a.e.,

$$\sqrt{|g(\sigma', \sigma')|} < \sqrt{|g_\varepsilon(\sigma', \sigma')| + \varepsilon} < \sqrt{|g_\varepsilon(\sigma', \sigma')|} + \sqrt{\varepsilon}$$

and so

$$L_g(\sigma) < L_{g_\varepsilon}(\sigma) + L_h(\sigma)\sqrt{\varepsilon} < L_{g_\varepsilon}(\sigma) + C\sqrt{\varepsilon}.$$

Switching the roles of g and g_ε, we establish that

$$|L_g(\sigma) - L_{g_\varepsilon}(\sigma)| < C\sqrt{\varepsilon}. \tag{1.1}$$

It follows that,

$$L_g(\gamma) > L_{g_\varepsilon}(\gamma) - Ck\sqrt{\varepsilon}, \quad \text{and for large } n, \quad L_{g_\varepsilon}(\gamma_n) > L_g(\gamma_n) - Ck\sqrt{\varepsilon}, \tag{1.2}$$

since for large n, $\gamma_n([s_i, s_{i+1}]) \subset V_i$.

Now we use (1.2) along with the fact that the length functional is upper semicontinuous in the smooth spacetime (M, g_ε). Indeed, Corollary 2.4.11 in [4] implies that Lipschitz causal curves are continuous causal curves as defined in [20]. Upper semicontinuity for Lipschitz curves in smooth spacetimes then follows from Theorem 2.4(a) in [20].

Hence, we have,

$$L_g(\gamma) > L_{g_\varepsilon}(\gamma) - Ck\sqrt{\varepsilon}$$
$$\geq \limsup_{n\to\infty} L_{g_\varepsilon}(\gamma_n) - Ck\sqrt{\varepsilon}$$
$$\geq \limsup_{n\to\infty} \left(L_g(\gamma_n) - Ck\sqrt{\varepsilon} \right) - Ck\sqrt{\varepsilon}$$
$$= \limsup_{n\to\infty} L_g(\gamma_n) - 2Ck\sqrt{\varepsilon}\,.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. $\qquad\square$

Chapter 2

C^0-inextendibility Results

2.1 Spacetime extensions

Fix $k \geq 0$. Let (M, g) be a C^k spacetime. Let $0 \leq l \leq k$. A C^l spacetime $(M_{\text{ext}}, g_{\text{ext}})$ with the same dimension as (M, g) is a C^l-*extension* of (M, g) if there is an isometric C^{l+1}-embedding

$$(M, g) \quad \hookrightarrow \quad (M_{\text{ext}}, g_{\text{ext}})$$

such that the image of M under the embedding is a proper subset of M_{ext}. We identify M with its image under the embedding. The topological boundary of M within M_{ext} is denoted by $\partial(M, M_{\text{ext}}) = \overline{M} \setminus M$. If $(M_{\text{ext}}, g_{\text{ext}})$ is a C^l-extension for all $l \geq 0$, then we say $(M_{\text{ext}}, g_{\text{ext}})$ is a *smooth* or C^∞-extension.

For the rest of this section, we will fix a C^l-extension $(M_{\text{ext}}, g_{\text{ext}})$ of a C^k spacetime (M, g).

Proposition 2.1.1. $\partial(M, M_{\text{ext}}) \neq \emptyset$.

Proof. If this were not true, then $\overline{M} = M$, and so M_{ext} would be the disjoint union of the nonempty open sets M and $M_{\text{ext}} \setminus M$. However, this implies M_{ext} is not connected which contradicts the definition of a spacetime. $\square$

Definition 2.1.2. Let (M, g) be a C^k spacetime and $(M_{\text{ext}}, g_{\text{ext}})$ a C^l-extension. A timelike curve $\gamma \colon [a, b] \to M_{\text{ext}}$ is called *future terminating* for a point $p \in \partial(M, M_{\text{ext}})$ provided $\gamma(b) = p$ and $\gamma\big([a, b)\big) \subset M$. It is called *past terminating* if $\gamma(a) = p$ and $\gamma\big((a, b]\big) \subset M$. The *future* and *past boundaries* of M with respect to M_{ext} are

$$\partial^+(M, M_{\text{ext}}) = \{p \in \partial(M, M_{\text{ext}}) \mid \text{there is a future terminating timelike curve for } p\}$$

$$\partial^-(M, M_{\text{ext}}) = \{p \in \partial(M, M_{\text{ext}}) \mid \text{there is a past terminating timelike curve for } p\}$$

Remark. If $(M_{\text{ext}}, g_{\text{ext}})$ is clear from context, then we will simply write $\partial^+ M$ for $\partial^+(M, M_{\text{ext}})$. Likewise for $\partial^- M$ and ∂M.
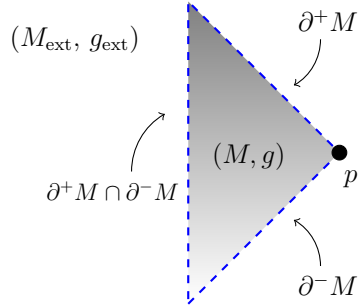
Figure 2.1: $(M_{\text{ext}}, g_{\text{ext}})$ is two-dimensional Minkowski space. (M, g) is the shaded triangle within the extension. Various points of $\partial^+ M$ and $\partial^- M$ are shown. The point $p \in \partial M$ is neither in $\partial^+ M$ nor in $\partial^- M$.

Lemma 2.1.3. *Let* $\gamma \colon [a, b] \to M_{\text{ext}}$ *be a timelike curve from* p *to* q.

(1) *If* $p \in M$ *and* $q \notin M$, *then* γ *intersects* $\partial^+ M$.

(2) *If* $p \notin M$ *and* $q \in M$, *then* γ *intersects* $\partial^- M$.

Proof. Consider case (1). Define $t_* = \sup\{t \in [a, b] \mid \gamma\big([a, t)\big) \subset M\}$. Since M is open we have $t_* > a$. Since $q \notin M$, we have $\gamma(t_*) \notin M$. On the other hand

$\gamma(t_*)$ is an accumulation point of M. Hence $\gamma(t_*) \in \partial M$. The restriction $\gamma|_{[a, t_*]}$ is a future terminating timelike curve for $\gamma(t_*)$. Hence $\gamma(t_*) \in \partial^+ M$. Case (2) follows by reversing the time orientation. $\qquad \square$

Proposition 2.1.4. $\partial^+ M \cup \partial^- M \neq \emptyset$.

Proof. Fix $p \in \partial M$. Let $U \subset M_{\text{ext}}$ be an open set around p. Fix $q \in I^-(p, U)$. Let $\gamma \subset U$ be a timelike curve from q to p. We either have $q \in M$ or $q \notin M$. First assume $q \in M$. Then Lemma 2.1.3 implies $\gamma \cap \partial^+ M \neq \emptyset$. Now assume $q \notin M$. Since $p \in \partial M$, the open set $I^+(q, U)$ of p contains a point $r \in M$. Hence there is a timelike curve $\lambda \subset U$ from q to r. Then Lemma 2.1.3 shows $\lambda \cap \partial^- M \neq \emptyset$. $\qquad \square$

Recall a subset $S \subset M_{\text{ext}}$ is *achronal* if $I^+(S, M_{\text{ext}}) \cap S = \emptyset$. The *edge* of a locally achronal set S is the set of points $p_0 \in \overline{S}$ such that every neighborhood U of p_0 contains a timelike curve $\gamma \subset U$ from a point $p \in I^-(p_0, U)$ to a point $q \in I^+(p_0, U)$ such that $\gamma \cap S = \emptyset$. We say S is *edgeless* if S is disjoint from its edge. Achronal and edgeless sets in a spacetime are topological hypersurfaces by Proposition 14.25 in [22]. Note the proof of that proposition only requires a C^0 metric.

Proposition 2.1.5 ([12]). *If $\partial^- M = \emptyset$, then $\partial^+ M$ is achronal and edgeless. Hence it's a topological hypersurface in M_{ext}.*

Proof. Suppose there is a timelike curve $\gamma \colon [a, b] \to M_{\text{ext}}$ with endpoints $p = \gamma(a)$ and $q = \gamma(b)$ on $\partial^+ M$. Then $I^+(p)$ is an open set of q. Therefore $I^+(p)$ intersects M, and so there is a point $r \in M$ and a timelike curve λ from p to r. Since $p \notin M$ and $r \in M$, Lemma 2.1.3 implies λ intersects $\partial^- M$. This contradicts the assumption $\partial^- M \neq \emptyset$.

Suppose p_0 is an edge point of $\partial^+ M$. Then given any neighborhood U of p_0, there is a timelike $\gamma \subset U$ which begins at $p \in I^-(p_0, U)$ and ends at $q \in I^+(p_0, U)$. Then $I^+(p, U)$ and $I^-(q, U)$ are open sets which intersect $\partial M \setminus \overline{\partial^- M}$. Therefore a couple

applications of Lemma 2.1.3 implies $p \notin M$ and $q \in M$. Therefore another application of Lemma 2.1.3 implies γ must intersect $\partial^- M$. Again this is a contradiction.

$\square$

2.2   Timelike completeness as an obstruction to C^0-extensions

The main theorem we prove in this section is

Theorem 2.2.1. *Suppose (M, g) is C^2, globally hyperbolic, and has a C^0-extension. If $\partial^+ M \neq \emptyset$, then there is a timelike geodesic γ in M which has a future endpoint on ∂M.*

We note that the corresponding result is known to hold in the Riemannian case: A complete Riemannian manifold (M, g) is C^0-inextendible. It s instructive to compare the two cases. In the Riemannian case, a neighborhood of a boundary point and finds a length-minimizer that connects a point in M to this boundary point. The portion of this curve in M has to be an inextendible geodesic, which by the assumption of completeness has to have infinite length. This gives the contradiction.

In the Lorentzian case one proceeds analogously, one considers a length-maximizer connecting the boundary point with a causally related point in M. The difference now is, that in order to obtain the contradiction, one has to rule out the subtle possiblility that the part of this length-maximizer that is contained in M, is a *null* geodesic (note that this length-maximizer might have non-trivial extent in the complement of M). It is here that we make use of the global hyperbolicity of M.

Before proving Theorem 2.2.1, we use it to prove the following C^0-inextendibility results.

Theorem 2.2.2. *If (M, g) is C^2, globally hyperbolic, and future timelike geodesically complete, then $\partial^+ M = \emptyset$ for any C^0-extension.*

Proof. Suppose not. Then there is an extension $(M_{\mathrm{ext}}, g_{\mathrm{ext}})$ with $\partial^+ M \neq \emptyset$. By Theorem 2.2.1, there is a future inextendible timelike geodesic γ with a future endpoint $p \in \partial M$. By Proposition 1.3.7, there is a neighborhood U of p such that $L(\lambda) < 1$ for any causal curve $\lambda \subset U$. However, the restriction $\gamma|_U$ is a future inextendible timelike geodesic in M which has infinite Lorentzian length by timelike completeness. $\square$

Thus, by Proposition 2.1.4, we have

Theorem 2.2.3 ([15])**.** *If (M, g) is C^2, globally hyperbolic, and timelike geodesically complete, then (M, g) is C^0-inextendible.*

The proof of Theorem 2.2.1 hinges on the following lemma. This lemma shows the fundamental role of global hyperbolicity in spacetime extensions.

Lemma 2.2.4. *Suppose (M, g) is globally hyperbolic. Let $p_0 \in \partial^+ M$ and U_ε be a coordinate neighborhood as in Lemma* 1.2.3. *Then for any $p, q \in M \cap U_\varepsilon$, we have*

$$I_{\eta^\varepsilon}^+(p,\, U_\varepsilon) \cap I_{\eta^\varepsilon}^-(q,\, U_\varepsilon) \,\subset\, M.$$

Proof. We can assume $q \in I_{\eta^\varepsilon}^+(p, U_\varepsilon)$ otherwise we have the empty set. By performing an η^ε Lorentz transformation, we can also assume p and q lie on a vertical line, and by shifting coordinates, we can put p and q on the x^0-axis. Seeking a contradiction, suppose there is a point $r \in I_{\eta^\varepsilon}^+(p, U_\varepsilon) \cap I_{\eta^\varepsilon}^-(q, U_\varepsilon)$ such that $r \notin M$. Consider the triangle Δ formed by the straight lines joining p, q, and r. By rotating coordinates, we can assume Δ lies in the (x^0, x^1)-plane and lies in $x^1 \geq 0$. We can foliate Δ with vertical line segments $T(x^1)$. For example $T(0)$ is just the portion of γ from p to q. Define $x_*^1 = \sup \{x^1 \mid T(x_1) \subset M\}$. Since $T(0) \subset M$, we have $x_*^1 > 0$. Since $r \notin M$,

we have $x_*^1 \notin M$. Thus there is a point $p_* \in T(x_*^1)$ such that $p_* \notin M$. We generate a sequence of points $p_n \in T(x_*^1 - 1/n)$ converging to the point p_*. Hence p_* is the only accumulation point of p_n. Since $T(x_*^1 - 1/n) \subset M$, we have generated a sequence of points $p_n \in J^+(p, M) \cap J^-(q, M)$ whose only accumulation point is $p_* \notin M$. This contradicts the global hyperbolicity of M. □

Proof of Theorem 2.2.1:

Let $p_0 \in \partial^+ M$. Let γ be a future terminating timelike curve for p_0. Introduce a coordinate $\phi \colon U_\varepsilon \to \mathbb{R}^{n+1}$ around p_0 as in Lemma 1.2.3 and let γ comprise the negative x^0-axis. Choose $\varepsilon = 3/5$ so that $\eta^{-\varepsilon}$ has light cones with slope $1/2$. Let p_1 be a point on the negative x^0-axis and p_2 on the positive x^0 axis with $x^0(p_2) = -x^0(p_1)$. Then $V = I_{\eta^{-\varepsilon}}^+(p_1, U_\varepsilon) \cap I_{\eta^{-\varepsilon}}^-(p_2, U_\varepsilon)$ is a neighborhood of p_0. Moreover V is globally hyperbolic since the x^0-plane is a Cauchy surface. Let $p \in V$ be a point on the negative x^0-axis. Then there is a causal maximizer $\lambda \colon [a, b] \to V$ from p to p_0. Redefine γ so that its the portion of the negative x^0-axis from p to p_0. Then $L(\lambda) \geq L(\gamma)$. Since λ is a maximizer and (M, g) is a C^2 spacetime, the restriction $\lambda|_M$ is either a timelike or null geodesic [3]. By shrinking V sufficiently small, we will show that $\lambda|_M$ being a null geodesic contradicts $L(\lambda) \geq L(\gamma)$. See Figure 2.2.

If $\lambda\big([a, b)\big) \subset M$, then $L(\lambda) \geq L(\gamma) > 0$ implies $\lambda|_M$ must be a timelike geodesic in which case we are done. Otherwise we reparameterize $\lambda \colon [0, 2] \to M_{\text{ext}}$ such that $\lambda_1 = \lambda|_{[0,1)}$ and $\lambda_2 = \lambda|_{[1,2]}$ where $\lambda_1 \subset M$ and $\lambda(1) \in \partial M$. Since λ is a causal maximizer, we have λ_1 is a geodesic in M. Thus it suffices to show λ_1 is a timelike geodesic.

Seeking a contradiction, suppose λ_1 is a null geodesic. Then $L(\lambda) = L(\lambda_2)$ Now we set out to put bounds on $L(\gamma)$ and $L(\lambda_2)$. Since x^0 is a time function, we can reparameterize γ and λ_2 with respect to the x^0 time coordinate. With respect to this parameterization, we have $\gamma \colon [a, 0] \to M_{\text{ext}}$ and $\lambda_2 \colon [b, 0] \to M_{\text{ext}}$ with $b > a$. Since $\eta^{3/5}$ and $\eta^{-3/5}$ have lightcones with slope 2 and $1/2$, Lemma 2.2.4 and $\lambda_2(b) \notin M$

together imply

$$\frac{b}{a} \leq \frac{4}{5}.$$

However by choosing V small enough we will show b/a can be made arbitrarily close to 1. Indeed let g_{ext} have components $g_{\mu\nu}$ in V. Then, by continuity of the metric, given any $\varepsilon > 0$, we can shrink our neighborhood V so that $|g_{\mu\nu} - \eta_{\mu\nu}| < \varepsilon$ is satisfied in V. Therefore

$$L(\gamma) = \int_a^0 \sqrt{-g_{\mu\nu}\gamma'^\mu\gamma'^\nu} = \int_a^0 \sqrt{-g_{00}} \geq a\sqrt{1-\varepsilon}.$$

Now we put a bound on $L(\lambda)$. To simplify notation, put $X = \lambda_2'$. Then

$$\begin{aligned}
L(\lambda) = L(\lambda_2) &= \int_b^0 \sqrt{-g_{\mu\nu}X^\mu X^\nu} \\
&= \int_b^0 \left[-g_{00} - 2\sum_i g_{0i}X^i - 2\sum_{i<j} g_{ij}X^i X^j - \sum_i g_{ii}|X^i|^2 \right]^{1/2} \\
&\leq \int_b^0 \left[-g_{00} - 2\sum_i g_{0i}(2) - 2\sum_{i<j} g_{ij}(2)(2) \right]^{1/2} \\
&\leq b\sqrt{(1+\varepsilon) + 4n\varepsilon + 4n(n-1)\varepsilon} \\
&= b\sqrt{1 + \varepsilon + 4n\varepsilon^2}
\end{aligned}$$

where $n + 1$ is the dimension of the spacetime. Therefore $L(\lambda) \geq L(\gamma)$ yields the following constraint on a and b

$$\frac{b}{a} \geq \frac{\sqrt{1-\varepsilon}}{\sqrt{1+\varepsilon+4n\varepsilon^2}}.$$

Thus we obtain a contradiction by choosing $\varepsilon > 0$ small enough so that the right hand side is greater than $4/5$. $\qquad\square$
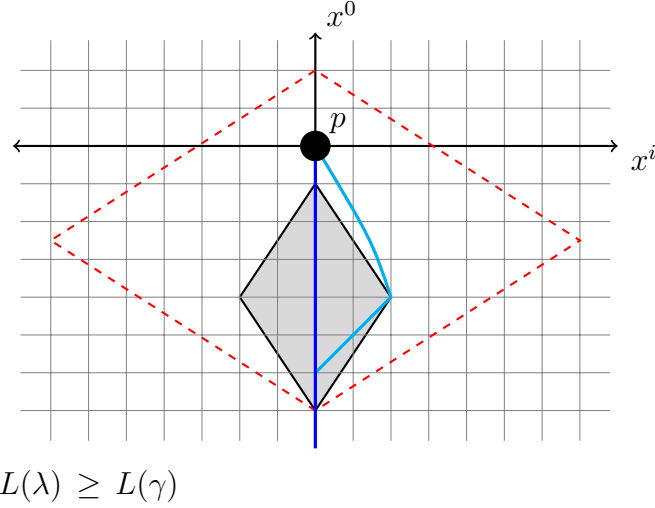
$$L(\lambda) \geq L(\gamma)$$

Figure 2.2: The dashed red diamond is the globally hyperbolic subset of p. The blue curve comprising the negative x^0-axis is γ and the teal curve is the causal maximizer λ. The shaded region represents the portion lying in M which follows from Lemma 2.2.4.

2.3 Timelike completeness as an obstruction to Lipschitz extensions

A *Lipschitz spacetime* (M, g) is one where M is $C^{1,1}$ and g is $C^{0,1}$ (i.e. the metric components $g_{\mu\nu}$ in any coordinate system satisfy a local Lipschitz condition). If (M, g) is a Lipschitz spacetime and there is no Lipschitz spacetime $(M_{\text{ext}}, g_{\text{ext}})$, then we say (M, g) is $C^{0,1}$-inextendible. In this section we prove two theorems about Lipschitz spacetimes.

Theorem 2.3.1 ([16])**.** *Suppose (M, g) is a Lipschitz spacetime. If γ is a causal maximizer from p to q and $L(\gamma) > 0$, then γ is timelike almost everywhere.*

Theorem 2.3.2 ([16])**.** *Suppose (M, g) is a smooth (at least C^2) spacetime which is timelike geodesically complete. Then (M, g) is $C^{0,1}$-inextendible.*

We will first use Theorem 2.3.1 to prove Theorem 2.3.2. Then we will prove Theorem 2.3.1.

Proof of Theorem 2.3.2:

Seeking a contradiction, suppose such a Lipschitz extension $(M_{\text{ext}}, g_{\text{ext}})$ exists. Then there is some point $p \in \partial^+ M \cup \partial^- M$. Without loss of generality, assume $p \in \partial^+ M$. Following the proof of Theorem 2.2.1, the causal maximizer γ joining the point $q \in M$ to $p \in \partial^+ M$, must be timelike almost everywhere. However, since (M, g) is C^2 and maximizers are geodesics in C^2 spacetime [3], we would have $L(\gamma|_M) = \infty$ which contradicts the fact that we can bound the length of causal curves in arbitrarily small neighborhoods. $\qquad\square$

Proof of Theorem 2.3.1:

Let $\gamma \colon [a, b] \to M$ be a causal maximizer from p to q with $L(\gamma) > 0$. If γ' is not timelike almost everywhere, then from Proposition 1.3.6, the set

$$N = \{t \mid \gamma'(t) \text{ is null}\} \subset [a, b]$$

has positive measure. We will arrive at a contradiction by showing that there is another causal curve λ from p to q such that $L(\lambda) > L(\gamma)$. See Figure 2.3.

By compactness we can cover $[a, b]$ by finitely many intervals $I_1, \ldots, I_N$ such that $\gamma \circ I_i$ is contained in some coordinate neighborhood U_i for each $i = 1, \ldots, N$. Since N has positive measure and $L(\gamma) > 0$, by inducting on i, there exists some j such that $\gamma \circ I_j$ is null on a set of positive measure and timelike on a set of positive measure.

By the previous paragraph we may assume without loss of generality that $\gamma\big([a, b]\big) \subset U_{3/5}$ where $\phi \colon U_{3/5} \to \mathbb{R}^{n+1}$ is a coordinate system from Lemma 1.2.3. Moreover we can assume γ is parameterized by the x^0 time function. By translating, we can assume $\gamma(0) = 0$ and $\gamma'(0)$ is future-directed timelike with $a < 0 < b$, and

$$N_{[a,0]} = \{t \mid \gamma'(t) \text{ is null}\} \cap [a, 0]$$

has nonzero measure, i.e. if μ denotes the Lebesgue measure on $[a, 0]$, then $\mu(N_{[a,0]}) > 0$. Since $\gamma'(0)$ is timelike, we can rotate coordinates so that $\gamma'(0) = \partial/\partial x^0$. Putting $\gamma^\mu = x^\mu \circ \gamma$, the definition of the derivative gives $\gamma^\mu(t) = \gamma'^\mu(0)t + f^\mu(t)$ where $f^\mu(t)/t \to 0$ as $t \to 0$. Since $\gamma'(0) = \partial/\partial x^0$, there exists a $t_0 > 0$ such that $\gamma\big([0, t_0]\big) \subset I^+_{\eta^{3/5}}\big(\gamma(0), U_{3/5}\big)$.

Put $\gamma_1 = \gamma|_{[a,0]}$. From the proof of Lemma 1.15 in [7] (which is where the Lipschitz assumption is used), given any $\varepsilon > 0$, there is a timelike curve $\lambda_1 : [a, 0] \to U_{3/5}$ given by such that $\lambda(a) = \gamma(a)$ and $\lambda_1^0(0) = C\varepsilon$ and $\lambda_1^i(0) = 0$ for all $i = 1, \ldots, n$, and

$$g(\lambda_1', \lambda_1') \leq g(\gamma_1', \gamma_1') - \varepsilon$$

The constant $C > 0$ in $\lambda_1^0(0) = C\varepsilon$ is the endpoint of the function f that appears in the proof of Lemma 1.15 in [7]. Therefore

$$\begin{aligned}
L(\lambda_1) &= \int_a^0 \sqrt{-g(\lambda_1', \lambda_1')} \\
&= \int_{N_{[a,0]}} \sqrt{-g(\lambda_1', \lambda_1')} + \int_{[a,0] \setminus N_{[a,0]}} \sqrt{-g(\lambda_1', \lambda_1')} \\
&\geq \mu\big(N_{[a,0]}\big)\sqrt{\varepsilon} + L(\gamma_1).
\end{aligned}$$

Since there exists a $t_0 > 0$ such that $\gamma\big([0, t_0]\big) \subset I^+_{\eta^{3/5}}\big(\gamma(0), U_{3/5}\big)$, by choosing ε sufficiently small, we can concatenate λ_1 with another timelike curve $\lambda_2 : [0, \varepsilon] \to U_{3/5}$ such that $\lambda_2(t_0) = \gamma(\varepsilon)$. Put $\gamma_2 = \gamma|_{[0,\varepsilon]}$. Put $X = \gamma_2'$. By continuity of the metric,

there is an ε_0 such that $|g_{\mu\nu} - \eta_{\mu\nu}| < \varepsilon_0$ within $U_{3/5}$. Therefore

$$
\begin{aligned}
L(\gamma_2) &= \int_0^\varepsilon \sqrt{-g_{\mu\nu} X^\mu X^\nu} \\
&= \int_0^\varepsilon \left[-g_{00} - 2\sum_i g_{0i} X^i - 2\sum_{i<j} g_{ij} X^i X^j - \sum_i g_{ii} |X^i|^2 \right]^{1/2} \\
&\leq \int_0^\varepsilon \left[-g_{00} - 2\sum_i g_{0i}(2) - 2\sum_{i<j} g_{ij}(2)(2) \right]^{1/2} \\
&\leq \varepsilon \sqrt{(1+\varepsilon_0) + 4n\varepsilon_0 + 4n(n-1)\varepsilon_0} \\
&= \varepsilon \sqrt{1 + \varepsilon_0 + 4n\varepsilon_0^2}
\end{aligned}
$$

The factors of 2 follow from the bounds on the lightcones in $U_{3/5}$. Finally, choose ε small enough so that $\mu\left(N_{[a,0]}\right)\sqrt{\varepsilon} > \varepsilon\sqrt{1 + \varepsilon_0 + 4n\varepsilon_0^2}$. Then we have

$$
\begin{aligned}
L(\lambda_1) + L(\lambda_2) &\geq L(\lambda_1) \\
&\geq \mu\left(N_{[a,0]}\right)\sqrt{\varepsilon} + L(\gamma_1) \\
&> \varepsilon\sqrt{1 + \varepsilon_0 + 4n\varepsilon_0^2} + L(\gamma_1) \\
&\geq L(\gamma_2) + L(\gamma_1).
\end{aligned}
$$

By concatenating λ_2 with $\gamma|_{[\varepsilon,b]}$, we have produced a causal curve λ from $\gamma(a)$ to $\gamma(b)$ such that $L(\lambda) > L(\gamma)$. This contradicts γ being a causal maximizer. $\qquad\square$
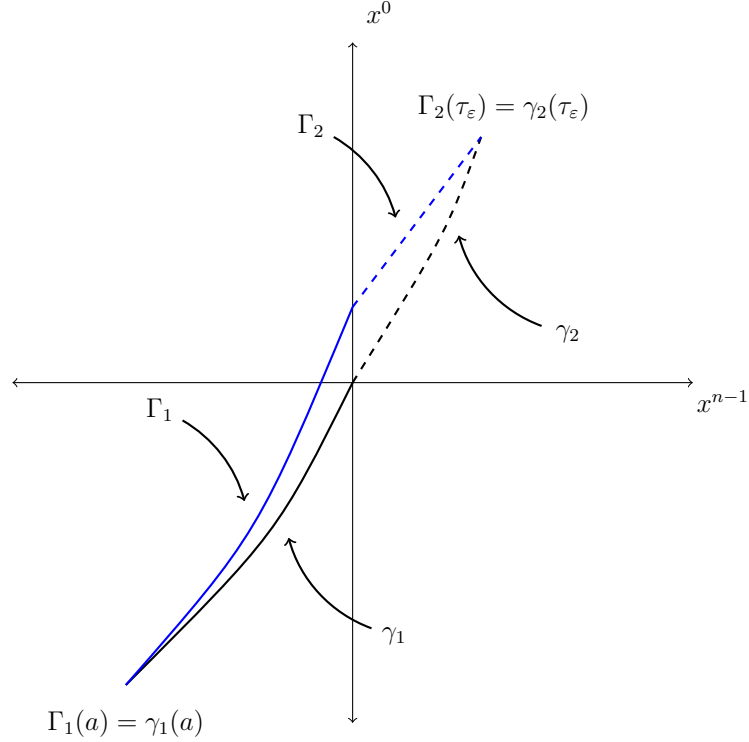
Figure 2.3: The causal curve formed by concatenating Γ_1 and Γ_2 has Lorentzian length greater than that of $\gamma|_{[a,\tau_\varepsilon]}$. Hence γ cannot be maximizing.

2.4 Future one-connectedness and future divergence

In this section we prove the following inextendibility result which will be used to prove the inextendibility of anti-de Sitter space.

Theorem 2.4.1. *If (M, g) is future one-connected and future divergent, then $\partial^+ M = \emptyset$ for any C^0-extension.*

Theorem 2.4.1 was first proved by Sbierski in [25] in the context of proving the inextendibility of Minkowski space and the exterior region of the Schwarzschild spacetime.

By Proposition 2.1.4 we have the following C^0-inextendibility theorem.

Theorem 2.4.2. *If (M, g) is future and past one-connected and future and past divergent, then (M, g) is C^0-inextendible.*

First we define future-one connectedness and future divergence. Recall the *Lorentzian distance function* is the function $d \colon M \times M \to [0, +\infty]$ where $d(p, q) = 0$ if $q \notin J^+(p)$ and $d(p, q) = \sup_\gamma L(\gamma)$ where the supremum is taken over all causal curves γ joining p to q.

Definition 2.4.3. Let (M, g) be a C^0 spacetime.

- (M, g) is *future one-connected* if given any $p, q \in M$ and any timelike curves γ and λ from p to q, there is a fixed-endpoint homotopy of timelike curves between γ and λ.

- (M, g) is *future divergent* if given any future inextendible timelike curve $\gamma \colon [a, b) \to M$, one has $\lim_{t \to b} d\big(\gamma(a), \gamma(t)\big) = \infty$ where d is the Lorentzian distance function.

Past one-connected and *past* divergence are defined by time-dualizing.

Proof of Theorem 2.4.1:

Suppose not. Then there is an extension $(M_{\text{ext}}, g_{\text{ext}})$ with a point $p_0 \in \partial^+ M$. Let γ be a future terminating timelike curve for p which comprises the negative x^0-axis of a neighborhood U_ε constructed from Lemma 1.2.3. Choose $\varepsilon = 3/5$ so that $\eta^{-\varepsilon}$ has lightcones with slope $1/2$. By Proposition 1.3.7, we can assume that the Lorentzian length of any timelike curve in U_ε is bounded by 1. Let p be a point on γ below p_0. By future divergence, there is a point q on γ and a timelike curve $\lambda \subset M$ from p to q such that $L(\lambda) > 1$. Thus a contradiction is achieved if we can show $\lambda \subset U_\varepsilon$.

Choose p and q so that $V = I^+_{\eta^{-\varepsilon}}(p, U_\varepsilon) \cap I^-_{\eta^{-\varepsilon}}(q, U_\varepsilon)$ has compact closure in U_ε. Redefine γ so that it's just the portion from p to q and reparameterize so that γ and λ both have domains $[a, b]$. By future one-connectedness, there is a homotopy $\Gamma \colon [0, 1] \times [a, b] \to M$ given by $\Gamma_s(t)$ such that Γ_s is a timelike curve from p to q and

$\Gamma_0 = \gamma$ and $\Gamma_1 = \lambda$. Let $I = \{s \in [0,1] \mid \Gamma_s \subset U_\varepsilon\}$. Since $0 \in I$ and $[0,1]$ is connected, it suffices to show I is both open and closed. I is open because U_ε is open. I is closed because $\overline{V} \subset U_\varepsilon$.

$\square$

In this subsection the spacetimes (M, g) we will be interested in are warped products. Specifically M will have manifold structure $M = I \times \Sigma$ where $I \subset \mathbb{R}$ is an open interval and Σ is a d-dimensional manifold. Let (I, dt^2) and (Σ, h) be Riemannian manifolds. If $\eta : M \to I$ and $\pi : M \to \Sigma$ denote the projection maps, then the metric g on M is given by $g = -\eta^* dt^2 + a^2 \pi^* h$ where $a : M \to (0, \infty)$ is some smooth function which depends only on t. We abuse notation and write $g = -dt^2 + a^2 h$ and $a(t, p) = a(t)$.

We will show under suitable hypotheses that these spacetimes are future one-connected and future divergent. We will apply these results to open FLRW spacetimes in Section 3.

Whether or not (M, g) is future one-connected depends only on its conformal class. By making the coordinate change $\tau = \int_c^t \frac{1}{a(s)} ds$, with $c \in I$, the metric becomes

$$g_{(\tau, p)} = a^2\big(t(\tau)\big)[-d\tau^2 + h_p].$$

Proposition 2.4.4. *Let $\gamma_i : [\tau_0, \tau_f] \to (M, -d\tau^2 + h)$, $i = 1, 2$, be two future directed timelike curves with coinciding endpoints and each parameterized by τ: $\gamma_i(\tau) = \big(\tau, \overline{\gamma}_i(\tau)\big)$. If the images of $\overline{\gamma}_1$ and $\overline{\gamma}_2$ lie completely in a common normal neighborhood U of (Σ, h) based at $\overline{\gamma}_1(\tau_0) = \overline{\gamma}_2(\tau_0)$, then γ_1 and γ_2 are timelike homotopic.*

Proof. The idea of the proof is as follows (cf. also [25]): Using the exponential map, we construct a homotopy from $\overline{\gamma}_1$ to the unique length minimizing geodesic connecting $\overline{\gamma}_1(\tau_0)$ and $\overline{\gamma}_1(\tau_f)$. If this homotopy is given by $\overline{\Gamma}_1(s, \tau)$, then we lift this homotopy to M via $\Gamma_1(s, \tau) = \big(\tau, \overline{\Gamma}_1(s, \tau)\big)$, and show that Γ_1 is a timelike homotopy. We then

repeat the same process for γ_2 and construct an analogous timelike homotopy Γ_2. The desired timelike homotopy is then the concatenation of Γ_1 and Γ_2. Since the procedure is symmetric, we only construct Γ_1 for γ_1 and omit the subscript.

Let $\gamma : [\tau_0, \tau_f] \to (M, -d\tau^2 + h)$ be a future directed timelike curve with $\overline{\gamma}$ lying in a normal neighborhood U of (Σ, h) based at $\overline{\gamma}(\tau_0)$. For each $s \in [\tau_0, \tau_f]$, let $\overline{\sigma}_s : [\tau_0, s] \to \Sigma$ be the unique length minimizing geodesic from $\overline{\gamma}(\tau_0)$ to $\overline{\gamma}(s)$ in U. The speed of $\overline{\sigma}_s$ is $|\overline{\sigma}_s'|_h = L(\overline{\sigma}_s)/(s - \tau_0)$. Now lift this curve to M via $\sigma_s : [\tau_0, s] \to M$ given by $\sigma_s(\tau) = \big(\tau, \overline{\sigma}_s(\tau)\big)$. To show that σ_s is timelike (in fact, a timelike geodesic), it suffices to show $|\overline{\sigma}_s'|_h < 1$. Since γ is a timelike curve, we must have $|\overline{\gamma}'(\tau)|_h < 1$ for all $\tau \in [\tau_0, \tau_f]$. Integrating yields $L(\overline{\gamma}|_{[\tau_0, s]}) < s - \tau_0$. Therefore

$$|\overline{\sigma}_s'|_h = \frac{L(\overline{\sigma}_s)}{s - \tau_0} \leq \frac{L(\overline{\gamma}|_{[\tau_0, s]})}{s - \tau_0} < 1. \tag{2.1}$$

Therefore σ_s is a future directed timelike curve between $\gamma(\tau_0)$ and $\gamma(s)$. Now we define the homotopy $\overline{\Gamma} : [\tau_0, \tau_f] \times [\tau_0, \tau_f] \to \Sigma$ between $\overline{\gamma}$ and $\overline{\sigma}_{\tau_f}$ via

$$\overline{\Gamma}(s, \tau) = \Big(\overline{\sigma}_s * \overline{\gamma}|_{[s, \tau_f]}\Big)(\tau) = \begin{cases} \overline{\sigma}_s(\tau) & \text{for } \tau_0 \leq \tau \leq s \\ \overline{\gamma}|_{[s, \tau_f]} & \text{for } s \leq \tau \leq \tau_f \end{cases}$$

and define $\Gamma : [\tau_0, \tau_f] \times [\tau_0, \tau_f] \to M$ by $\Gamma(s, \tau) = \big(\tau, \overline{\Gamma}(s, \tau)\big)$. We have shown that for each s, $\Gamma(s, \cdot)$ is a future directed timelike curve and $\Gamma(\tau_0, \cdot) = \gamma$ and $\Gamma(\tau_f, \cdot) = \sigma_{\tau_f}$. Thus Γ is a future directed timelike homotopy between γ and σ_{τ_f}. $\qquad \square$

Corollary 2.4.5. *Suppose at every point $p \in \Sigma$ there exists $0 \in U_p \subset T_p\Sigma$ such that the exponential map, $\exp_p : U_p \to \Sigma$, is a diffeomorphism onto Σ. Then any spacetime conformal to $(M, -d\tau^2 + h)$ is future one-connected. Hence (M, g) is future one-connected.*

Remark. We note that Corollary 2.4.5 applies in particular to the case that Σ is a *Hadamard space* (i.e. a simply connected Riemannian manifold with nonpositive sectional curvature), for which we know that the exponential map is a global diffeomorphism about every point.

For future divergence we have the following proposition and corollary. The ideas in the proofs also appear in Sbierski's paper [25].

Proposition 2.4.6. *Suppose* (Σ, h) *is a complete Riemannian manifold and* $I = (t_1, \infty)$, *such that* $\tau(t) \to \infty$ *as* $t \to \infty$. *Then* $(M, g = -d\tau^2 + h)$ *is future divergent.*

Proof. Let $\gamma \colon [\tau_0, \infty) \to M$ be a future inextendible timelike curve parameterized by τ:

$$\gamma(\tau) = \left(\tau, \overline{\gamma}(\tau)\right).$$

Fix $T \in (\tau_0, \infty)$. Let $\overline{\sigma} \colon [\tau_0, T] \to \Sigma$ be a length minimizing geodesic between $\overline{\gamma}(\tau_0)$ and $\overline{\gamma}(T)$. Since $\overline{\sigma}$ is parameterized by τ, the argument which led to (2.1) also gives

$$|\overline{\sigma}'|_h = L(\overline{\sigma})/(T - \tau_0) < 1.$$

Define $\sigma \colon [\tau_0, T] \to \Sigma$ by $\sigma(\tau) = \left(\tau, \overline{\sigma}(\tau)\right)$. Since $|\overline{\sigma}'|_h < 1$, σ is a timelike curve (in fact timelike geodesic) connecting $\gamma(\tau_0)$ to $\gamma(T)$. Thus we have

$$
\begin{aligned}
d_g\left(\gamma(\tau_0), \gamma(T)\right) &\geq L_g(\sigma) \\
&= \int_{\tau_0}^{T} \sqrt{1 - |\overline{\sigma}'|_h^2}\, d\tau \\
&= \sqrt{(T - \tau_0)^2 - (T - \tau_0)^2 |\overline{\sigma}'|_h^2} \\
&= \sqrt{(T - \tau_0)^2 - d_h^2\left(\overline{\gamma}(\tau_0), \overline{\gamma}(T)\right)}
\end{aligned}
\tag{2.2}
$$

where d_h is the Riemannian distance function on Σ.

Fix $\tau_1 \in [\tau_0, T]$. Then since γ is timelike, we have $d_h\big(\overline{\gamma}(\tau_0), \overline{\gamma}(\tau_1)\big) < \tau_1 - \tau_0$ and $d_h\big(\overline{\gamma}(\tau_1), \overline{\gamma}(T)\big) < T - \tau_1$. Therefore there exists an $\epsilon > 0$ such that $d_h\big(\overline{\gamma}(\tau_0), \overline{\gamma}(\tau_1)\big) = \tau_1 - \tau_0 - \epsilon$. By the triangle inequality we have

$$d_h\big(\overline{\gamma}(\tau_0), \overline{\gamma}(T)\big) \leq d_h\big(\overline{\gamma}(\tau_0), \overline{\gamma}(\tau_1)\big) + d_h\big(\overline{\gamma}(\tau_1), \overline{\gamma}(T)\big)$$
$$< (\tau_1 - \tau_0 - \epsilon) + (T - \tau_1)$$
$$= T - \tau_0 - \epsilon.$$

Using this in (2.2), we have

$$d\big(\gamma(\tau_0), \gamma(T)\big) \geq \sqrt{(T - \tau_0)^2 - (T - \tau_0 - \epsilon)^2}$$
$$= \sqrt{2\epsilon(T - \tau_0) - \epsilon^2}.$$

Therefore $\lim_{T \to \infty} d\big(\gamma(\tau_0), \gamma(T)\big) = \infty$. $\qquad\square$

Corollary 2.4.7. *Assume the hypotheses of Proposition 2.4.6. Then (M, g) is future divergent so long as $a(t)$ is bounded away from 0 for all large t.*

Proof. Let $\gamma \colon [\tau_0, \infty) \to M$ be a timelike curve in (M, g) parameterized by τ. Then

$$g\big(\gamma'(\tau), \gamma'(\tau)\big) = -|\gamma'(\tau)|_g^2 = a^2\big(t(\tau)\big)\Big[-1 + |\overline{\gamma}'(\tau)|_h^2\Big].$$

Since $a(t)$ is bounded away from 0 for all large t, there exist $\tau_1 \in [\tau_0, \infty)$ and $b > 0$ such that $a\big(t(\tau)\big) > b$ for all $\tau \geq \tau_1$. So for all $\tau > \tau_1$, we have $|\gamma'(\tau)|_g^2 > b\big(1 - |\overline{\gamma}'(\tau)|_h^2\big)$, from which it follows that $L_g(\gamma|_{[\tau_1, \tau]}) > b\, L_{g_0}(\gamma|_{[\tau_1, \tau]})$, where $g_0 = -d\tau^2 + h$. The result then follows from Proposition 2.4.6. $\qquad\square$

2.5 C^0-inextendibility of Minkowski, de Sitter, and anti-de Sitter spaces

In this section we use the techniques so far developed to prove the C^0-inextendibility of all simply connected spacetimes with constant sectional curvature. The C^0-inextendibility of Minkowski space and de Sitter space follow from Theorem 2.2.3 since each of these spaces are globally hyperbolic and timelike complete.

However anti-de Sitter is not globally hyperbolic. We will show anti-de Sitter space is future one-connected and future divergent. Then, since anti-de Sitter is time symmetric, Theorem 2.4.2 implies it's C^0-inextendible.

Definition 2.5.1. The $n+1$-dimensional *anti-de Sitter space (adS)* is the spacetime $(\mathbb{R} \times S^n_+, g)$, where g is given by

$$g = \frac{1}{\cos^2 \chi}\Big[-dt^2 + d\chi^2 + \sin^2 \chi d\Omega^2_{n-1}\Big],$$

and $(\chi, \omega) \in [0, \frac{\pi}{2}) \times S^{n-1}$ are spherical coordinates on the (open) hemisphere S^n_+.

Proposition 2.5.2. *Anti-de Sitter space is future one-connected and future divergent.*

Proof. Since the round hemisphere satisfies the exponential map property, Corollary 2.4.5 implies that anti-de Sitter space is future one-connected. To show that adS space is future divergent, let $\gamma : [0, t_f) \to (\mathbb{R}^{n+1}, g)$ be a future directed inextendible timelike curve parameterzed by t (by a time translation we can assume γ begins at $t = 0$). For each $t \in [0, t_f)$, we have $\gamma(t) = (t, \chi(t), \omega(t))$, where ω represents coordinates on S^{n-1}. There are essentially two cases to consider: There exists $t_k \nearrow t_f$ such that (i) $\lim_{k\to\infty} \chi(t_k) < \pi/2$ (or does not exist) or (ii) $\lim_{k\to\infty} \chi(t_k) = \pi/2$. In either case, for k sufficiently large, there exists a t-line segment σ_k from some $p_k \in \partial I^+\big(\gamma(0)\big)$ to $\gamma(t_k)$

Moreover, one has $\lim_{k\to\infty} L(\sigma_k) = \infty$. In case (i) this follows from the fact that we must have $t_f = \infty$. In case (ii) this follows from the fact that $\lim_{k\to\infty} \cos\left(\chi(t_k)\right) = 0$, so that the conformal factor becomes arbitrarily large, and that for k sufficiently large, $t_k - t(p_k)$ is uniformly positive. Since for each k there exists a null geodesic from $\gamma(0)$ to p_k, we conclude that adS space is future divergent. $\square$

Chapter 3

Spacetime Extensions of the Big Bang

3.1 Definition of coordinate singularities

In this section we give a precise definition of what we mean by a 'coordinate singularity.' Our goal is to identify when we have made a 'poor' choice of coordinates. Before giving the definition, we start with a couple of motivating examples.

Motivating examples:

(1) Consider the smooth spacetime (M, g) where $M = (0, \infty) \times \mathbb{R}$ with the metric $g = -\tau^2 d\tau^2 + dx^2$. Since the metric becomes degenerate at $\tau = 0$, we cannot extend (M, g) using the coordinates (τ, x). However, if we introduce the coordinate $t = \frac{1}{2}\tau^2$, then, with respect to these coordinates, the spacetime manifold becomes $(0, \infty) \times \mathbb{R}$ with metric $-dt^2 + dx^2$. Since the metric is nondegenerate at $t = 0$, we have no problem extending the spacetime using the coordinates (t, x). As such, we say (τ, x) were a 'poor choice' of coordinates, and $\tau = 0$ merely represents a coordinate singularity. This example demonstrates that a coordinate singularity depends on a spacetime being inextendible within one coordinate system while being extendible in another.

(2) Consider the smooth spacetime (M, g) where $M = (0, \infty) \times \mathbb{R}$ and $g = -f(t)dt^2 +$ dx^2 where $f: (0, \infty) \to \mathbb{R}$ is the smooth function given by $f(t) = 1 + \sqrt{t}$. In this case (M, g) extends through $t = 0$ via the spacetime $M_{\text{ext}} = \mathbb{R} \times \mathbb{R}$ and $g_{\text{ext}} = -dt^2 + dx^2$ for $t \leq 0$ and $g_{\text{ext}} = g$ for $t > 0$. However $(M_{\text{ext}}, g_{\text{ext}})$ is not a smooth extension of (M, g). It is only a C^0-extension. In this case we would not say that (t, x) are a 'poor' choice of coordinates for (M, g), since the coordinates (t, x) can still be used to extend the spacetime just not smoothly.

Fix $k \geq 0$. Let (M, g) be a C^k spacetime with dimension $n+1$. Recall a *coordinate system* is an element of the maximal C^{k+1}-atlas for M. Specifically, a coordinate system is a C^{k+1}-diffeomorphism $\phi: U \to \phi(U) \subset \mathbb{R}^{n+1}$ where U is an open subset of M.

Let $\phi: U \to \mathbb{R}^{n+1}$ be a coordinate system for a C^k spacetime (M, g). Let $\Omega = \phi(U) \subset \mathbb{R}^{n+1}$. Then (U, g) is C^k-isometric to $(\Omega, \phi_* g)$ where ϕ_* is the push forward. Let $0 \leq l \leq k$. Suppose there exists an open set $\Omega' \subset \mathbb{R}^{n+1}$ which properly contains Ω and a C^l-Lorentzian metric g' on Ω' such that (Ω', g') is a C^l-extension of $(\Omega, \phi_* g)$. Then we say (Ω', g') is a C^l-*coordinate extension* of $(\Omega, \phi_* g)$. If such an (Ω', g') exists, then we say ϕ is *not C^l-maximal*. If no such (Ω', g') exists, then we say ϕ is C^l-*maximal*. For example, the coordinates (τ, x) in the first example above are C^0-maximal. The coordinates (t, x) in the second example are C^1-maximal but not C^0-maximal.

Remark. Another way of saying a coordinate system ϕ is C^0-maximal is that the metric components $g_{\mu\nu}$ with respect to ϕ are maximally extended.

Definition 3.1.1 (Coordinate singularity)**.** Fix $k \geq 0$ and $0 \leq l \leq k$. Let (M, g) be a C^k spacetime and let $\phi: U \to \mathbb{R}^{n+1}$ be a coordinate system which is C^0-maximal. We say ϕ admits a C^l-*coordinate singularity* for (M, g) if there is a C^l-extension

$(M_{\text{ext}}, g_{\text{ext}})$ and a coordinate system $\psi \colon V \to \mathbb{R}^{n+1}$ for M_{ext} such that

$$V \cap M = U \quad \text{and} \quad V \cap (M_{\text{ext}} \setminus M) \neq \emptyset.$$

Remark. In the definition ϕ represents the 'poor' choice of coordinates. ψ represents the 'better' choice of coordinates.

As an illustration consider (M, g) from example (1) above. The coordinate system $\phi = (\tau, x)$ is a C^∞-coordinate singularity for (M, g). This follows because

- ϕ is C^0-maximal

- $(M_{\text{ext}}, g_{\text{ext}})$ is a C^∞-extension of (M, g) where $M_{\text{ext}} = \mathbb{R}^2$ and $g_{\text{ext}} = -dt^2 + dx^2$.

- In this example we simply take $U = M$ and $V = M_{\text{ext}}$ and $\psi = (t, x)$.

3.2 The big bang is a coordinate singularity for Milne-like spacetimes

Let $I \subset \mathbb{R}$ be an open interval. Let (Σ, h) be a three-dimensional complete Riemannian manifold with constant sectional curvature. We say (M, g) is an *FLRW spacetime* if $M = I \times \Sigma$ and $g = -d\tau^2 + a^2(\tau)h$ where $a \colon I \to (0, \infty)$ is a continuous function called the *scale factor*. The integral curves of $\partial/\partial\tau$ are called the *comoving observers*. Physically, they model the trajectories of galaxies.

Remark. We don't assume any differentiability assumption on the scale factor. Therefore the lowest regularity class for FLRW spacetimes is C^0.

Let $(\mathbb{R}^3, h)$ be hyperbolic space with sectional curvature $k = -1$. Let (M, g) be the corresponding FLRW spacetime. We use the standard coordinates $\xi = (\tau, R, \theta, \phi)$

for M where $\xi\colon U \to \mathbb{R}^4$ and $U = I \times (0, \infty) \times (0, \pi) \times (0, 2\pi)$. With respect to the coordinate system $\xi = (\tau, R, \theta, \phi)$, the metric is

$$g = -d\tau^2 + a^2(\tau)\Big[dR^2 + \sinh^2(R)(d\theta^2 + \sin^2\theta\, d\phi^2)\Big]. \tag{3.1}$$

We will first demonstrate how $\xi = (\tau, R, \theta, \phi)$ admits a C^∞-coordinate singularity for (M, g) in two familiar cases: (1) when (M, g) is the Milne universe and (2) when (M, g) is the open-slicing coordinate system for de Sitter space. Then we will show how $\xi = (\tau, R, \theta, \phi)$ admits a C^0-coordinate singularity for a class of inflationary spacetimes which we have dubbed 'Milne-like.'

3.2.1   The Milne universe

Let $(\mathbb{R}^3, h)$ be hyperbolic space with sectional curvature $k = -1$. The Milne universe is the corresponding FLRW spacetime (M, g) given by $M = (0, \infty) \times \mathbb{R}^3$ and with scale factor $a(\tau) = \tau$. With respect to the coordinate system $\xi = (\tau, R, \theta, \phi)$, the metric is

$$g = -d\tau^2 + \tau^2\Big[dR^2 + \sinh^2(R)\,(d\theta^2 + \sin^2\theta\, d\phi^2)\Big]. \tag{3.2}$$

We introduce a new coordinate system $\zeta = (t, r, \theta, \phi)$ where θ and ϕ are unchanged, but t and r are given by

$$t = \tau\cosh(R) \quad \text{and} \quad r = \tau\sinh(R). \tag{3.3}$$

Then we have $-dt^2 + dr^2 = -d\tau^2 + \tau^2 dR^2$, so that the metric in the coordinate system $\zeta = (t, r, \theta, \phi)$ is

$$g = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.4}$$

which is just the usual Minkowski metric. The coordinate system $\xi = (\tau, R, \theta, \phi)$ is C^0-maximal, but we can find a C^∞-extension via $\zeta = (t, r, \theta, \phi)$. Therefore

Proposition 3.2.1. $\xi = (\tau, R, \theta, \phi)$ *admits a C^∞-coordinate singularity for (M, g).*

The constant τ slices are hyperboloids sitting inside the future lightcone of the origin. We take the extension to be $(M_{\text{ext}}, g_{\text{ext}}) =$ Minkowski space. As $\tau \to 0$, these slices approach the lightcone at the origin $\mathcal{O}$ in Minkowski space where the extended metric g_{ext} is nondegenerate.
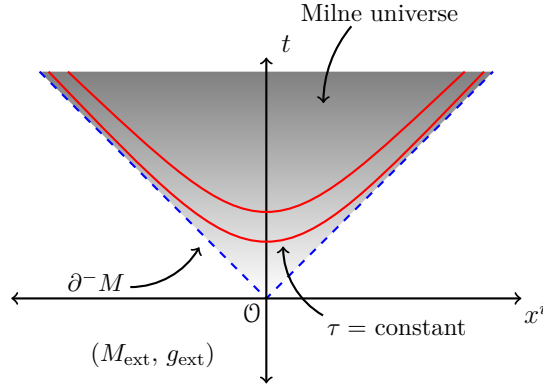

Figure 3.1: The Milne universe sits inside the future lightcone of the origin $\mathcal{O}$ in the extension which is just Minkowski space. It's foliated by constant τ slices which are hyperboloids.

3.2.2   De Sitter space

The *open slicing coordinate system* for de Sitter space is a $k = -1$ FLRW spacetime $M = (0, \infty) \times \mathbb{R}^3$ with scale factor $a(\tau) = \sinh(\tau)$. With respect to the coordinate system $\xi = (\tau, R, \theta, \phi)$, the metric is

$$g = -d\tau^2 + \sinh^2(\tau)\Big[dR^2 + \sinh^2(R)\,(d\theta^2 + \sin^2\theta\,d\phi^2)\Big]. \tag{3.5}$$

We introduce a new coordinate system $\zeta = (t, r, \theta, \phi)$ where θ and ϕ are unchanged, but t and r are given by

$$t = b(\tau)\cosh(R) \quad \text{and} \quad r = b(\tau)\sinh(R), \tag{3.6}$$

where $b(\tau) = \tanh(\tau/2) = \sinh\tau/(1+\cosh\tau)$. Then $b'(\tau) = b(\tau)/a(\tau)$, and so we have the following relationship between (t, r) and (τ, R).

$$\left(\frac{a(\tau)}{b(\tau)}\right)^2 \left(-dt^2 + dr^2\right) = -d\tau^2 + a^2(\tau)dR^2. \tag{3.7}$$

Therefore the metric is

$$g = \left(\frac{a(\tau)}{b(\tau)}\right)^2 \left[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \tag{3.8}$$

which is conformal to the Minkowski metric. Using $b(\tau) = \tanh(\tau/2)$ and $b^2(\tau) = t^2 - r^2$, we have $\tau = 2\tanh^{-1}(\sqrt{t^2 - r^2})$. Therefore $1/b'(\tau) = a(\tau)/b(\tau) = 2/(1 - t^2 + r^2)$. Thus the metric in the coordinate system $\zeta = (t, r, \theta, \phi)$ is

$$g = \left(\frac{2}{1 - t^2 + r^2}\right)^2 \left[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \tag{3.9}$$

The coordinate system $\xi = (\tau, R, \theta, \phi)$ is C^0-maximal, but we can define a C^∞-extension via $\zeta = (t, r, \theta, \phi)$. Thus

Proposition 3.2.2. $\xi = (\tau, R, \theta, \phi)$ *admits a C^∞-coordinate singularity for (M, g).*

The constant τ slices are hyperboloids sitting inside the future lightcone at the origin. We take the extension to be $(M_{\text{ext}}, g_{\text{ext}}) = $ a smooth spacetime conformal to Minkowski space. As $\tau \to 0$, these slices approach the lightcone where the extended metric g_{ext} is nondegenerate.
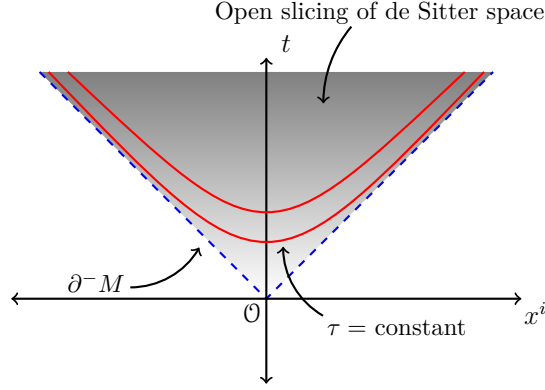
Figure 3.2: The open slicing coordinates of de Sitter space sits inside the future lightcone at the origin $\mathcal{O}$ in a spacetime conformal to Minkowski space.

3.2.3   Milne-like spacetimes

Now we wish to show that $\xi = (\tau, R, \theta, \phi)$ is a coordinate singularity for scale factors that can model the dynamics of our universe. That is, we wish to show $\tau = 0$ is a coordinate singularity for a suitably chosen scale factor $a(\tau)$ which

- begin inflationary $a(\tau) \sim \sinh(\tau)$

- then transitions to a radiation dominated era $a(\tau) \sim \sqrt{\tau}$

- then transitions to a matter dominated era $a(\tau) \sim \tau^{2/3}$

- and ends in a dark energy dominated era $a(\tau) \sim e^{\Lambda \tau}$

If we assume for small τ, the scale factor satisfies $a(\tau) \sim \tau$, then, by curve fitting, we can use $a(\tau)$ to represent each of the above eras, thus modeling the dynamics of our universe. To make this precise, we assume for small τ, the scale factor satisfies $a(\tau) = \tau + o(\tau^{1+\varepsilon})$ for some $\varepsilon > 0$ (i.e. $\left[a(\tau) - \tau\right]/\tau^{1+\varepsilon} \to 0$ as $\tau \to 0$). In particular any convergent Taylor expansion $a(\tau) = \sum_{n=1}^{\infty} c_n \tau^n$ (with $c_1 = 1$) will satisfy this condition.

Definition 3.2.3.

(1) Let (M, g) be an FLRW spacetime. We say (M, g) is *inflationary* if the scale factor for small τ satisfies $a(\tau) = \tau + o(\tau^{1+\varepsilon})$ for some $\varepsilon > 0$.

(2) We say (M, g) is *Milne-like* if it is an inflationary FLRW spacetime such that $(\Sigma, h) = (\mathbb{R}^3, h)$ where h is the hyperbolic metric with sectional curvature $k = -1$. We assume the coordinate system $\xi = (\tau, R, \theta, \phi)$ is C^0-maximal.

Remarks.

- The motivation for the word 'inflationary' comes in Section 3.4 where we show that the particle horizon is infinite for scale factors which obey $a(\tau) = \tau + o(\tau^{1+\varepsilon})$.

- A C^k Milne-like spacetime is one such that the spacetime is C^k (i.e. the scale factor $a(\tau)$ is a C^k function).

- For inflationary spacetimes we have $a(0) := \lim_{\tau \to 0} a(\tau) = 0$.

- The coordinate system $\xi = (\tau, R, \theta, \phi)$ is defined for all $\tau \in I = (0, \tau_{\max})$ where $\tau_{\max} \in (0, +\infty]$. For our universe, we expect $\tau_{\max} = +\infty$ due to dark energy.

The next theorem improves and refines Theorem 3.4 in [12].

Theorem 3.2.4. $\xi = (\tau, R, \theta, \phi)$ *admits a C^0-coordinate singularity for Milne-like spacetimes.*

Proof. Let (M, g) be a Milne-like spacetime. With respect to the coordinate system $\xi = (\tau, R, \theta, \phi)$, the metric is

$$g = -d\tau^2 + a^2(\tau)\left[dR^2 + \sinh^2(R)(d\theta^2 + \sin^2\theta\, d\phi^2)\right]. \qquad (3.10)$$

Fix any $\tau_0 \in I$. The specific choice does not matter; any τ_0 will do. Define a new coordinate system $\zeta = (t, r, \theta, \phi)$ by

$$t = b(\tau)\cosh(R) \quad \text{and} \quad r = b(\tau)\sinh(R) \tag{3.11}$$

where $b \colon I \to (0, \infty)$ is given by

$$b(\tau) = \exp\left(\int_{\tau_0}^{\tau} \frac{1}{a(s)} ds\right). \tag{3.12}$$

Note that $b(\tau)$ is an increasing C^1 function and hence it's invertible. Therefore τ as a function of t and r is

$$\tau = b^{-1}\left(\sqrt{t^2 - r^2}\right). \tag{3.13}$$

Note that t and r are defined for all points such that $t^2 - r^2 < b^2(\tau_{\max})$. With respect to the coordinate system $\zeta = (t, r, \theta, \phi)$, the metric takes the form

$$g = \Omega^2\left(\tau(t, r)\right)\left[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.14}$$

where

$$\Omega(\tau) = \frac{1}{b'(\tau)} = \frac{a(\tau)}{b(\tau)}. \tag{3.15}$$

Now we prove $\xi = (\tau, R, \theta, \phi)$ admits a C^0-coordinate singularity for (M, g). For this it suffices to show $\Omega(0) := \lim_{\tau \to 0} \Omega(\tau)$ exists and is a finite positive number. Indeed this will imply the Lorentzian metric given by equation (3.14) extends continuously through $\tau = 0$ which corresponds to the lightcone $t = r$, i.e. this will imply that (M, g) is C^0-extendible via $\zeta = (t, r, \theta, \phi)$.

To show $0 < \Omega(0) < \infty$, put $b'(0) := \lim_{\tau \to 0} b'(\tau) = \lim_{\tau \to 0} b(\tau)/a(\tau)$. By our definition of an inflationary spacetime, there is an $\varepsilon_0 > 0$ such that $a(\tau) = \tau + o(\tau^{1+\varepsilon_0})$. Therefore $\lim_{\tau \to 0} f(\tau)/\tau^{1+\varepsilon_0} = 0$ where $f(\tau)$ is given by $a(\tau) = \tau + f(\tau)$. Therefore

for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $0 < \tau < \delta$, we have $|f(\tau)| < \varepsilon\tau^{1+\varepsilon_0}$.

Hence $\tau - \varepsilon\tau^{1+\varepsilon_0} < \tau + f(\tau) < \tau + \varepsilon\tau^{1+\varepsilon_0}$. Thus $b(\tau)/a(\tau)$ is squeezed between

$$\frac{1}{a(\tau)} \exp\left(-\int_\tau^{\tau_0} \frac{1}{(\tau - \varepsilon\tau^{1+\varepsilon_0})} ds\right) < \frac{b(\tau)}{a(\tau)} < \frac{1}{a(\tau)} \exp\left(-\int_\tau^{\tau_0} \frac{1}{(\tau + \varepsilon\tau^{1+\varepsilon_0})} ds\right)$$
(3.16)

Evaluating the integrals, we find

$$\frac{1}{\tau_0} \left(\frac{\tau}{a(\tau)}\right) \left(\frac{1 - \varepsilon\tau^{\varepsilon_0}}{1 + \varepsilon\tau_0^{\varepsilon_0}}\right)^{-1/\varepsilon_0} < \frac{b(\tau)}{a(\tau)} < \frac{1}{\tau_0} \left(\frac{\tau}{a(\tau)}\right) \left(\frac{1 + \varepsilon\tau^{\varepsilon_0}}{1 + \varepsilon\tau_0^{\varepsilon_0}}\right)^{-1/\varepsilon_0}$$
(3.17)

Since this holds for all $0 < \tau < \delta$, we have $\Omega(0) = 1/b'(0) = \tau_0$. $\qquad\square$
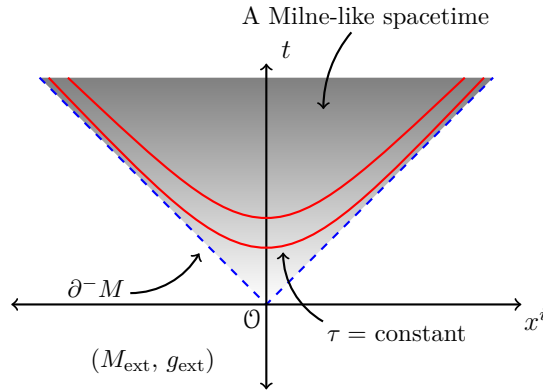

Figure 3.3: A Milne-like spacetime sits inside the future lightcone at the origin $\mathcal{O}$ in a spacetime extension $(M_{\text{ext}}, g_{\text{ext}})$ which is conformal to Minkowski space.

3.3 Curvature singularities

In this section we give a precise definition of what we mean by a 'curvature singularity.' We then establish for a certain class of Milne-like spacetimes the absence of curvature singularities. Let's fix notation by recalling the definition of curvature.

Fix $k \geq 2$. Let (M, g) be a C^k spacetime and ∇ its unique compatible affine connection. Then the *Riemann curvature tensor* is the $(3, 1)$ tensor defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

where $[X, Y]$ is the Lie derivative of Y with respect to X. Let $\{\partial_\mu\}$ be a coordinate vector basis with dual one-form basis $\{dx^\mu\}$. The *components* of the Riemann curvature tensor with respect to $\{\partial_\mu\}$ are defined by $R_{\mu\nu\alpha}{}^\beta = dx^\beta\left(R(\partial_\mu, \partial_\nu)\partial_\alpha\right)$. Using index notation $\nabla_X Y = (X^\mu \nabla_\mu Y^\nu)\partial_\nu$ and the linear and Leibniz properties of the affine connection, we have

$$R_{\mu\nu\alpha}{}^\beta Z^\alpha = (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)Z^\beta.$$

Here we see the non-commutativity of the second covariant derivatives of Z expressed in terms of the components of the curvature tensor.

Definition 3.3.1. Fix $k \geq 2$. Let (M, g) be a C^k spacetime. A *curvature invariant* on (M, g) is a scalar function which is a polynomial in the components of the metric $g_{\mu\nu}$, its inverse $g^{\mu\nu}$, and the curvature tensor $R_{\mu\nu\alpha}{}^\beta$.

Examples of curvature invariants:

(1) The scalar curvature $R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} R_{\mu\alpha\nu}{}^\alpha$.

(2) The Kretschmann scalar $R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta}$.

(3) $R_{\mu\nu} R^{\mu\nu}$.

Definition 3.3.2 (Curvature singularity)**.** Fix $k \geq 2$. Let (M, g) be a C^k spacetime. We say (M, g) admits a *future curvature singularity* if there is a future inextendible

timelike curve $\gamma \colon [a, b) \to M$ and a curvature invariant C such that $C \circ \gamma(t)$ diverges as $t \to b$. Time-dualizing the definition gives *past* curvature singularities.

Example. As an example, the interior Schwarzschild spacetime (M, g) is the smooth spacetime defined by $M = \mathbb{R} \times (0, 2m) \times S^2$ with metric $g = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2$ where $f(r) = (1 - 2m/r)$ with $m > 0$ and $(S^2, d\Omega^2)$ is the usual round metric. A calculation shows that the Kretschmann scalar $C = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is given by $C = 48m^2/r^6$. Take the timelike curve $\gamma \colon (m, 0) \to M$ given by $\gamma(r) = (t_0, r, \omega_o)$ where $t_0 \in \mathbb{R}$ and $\omega_0 \in S^2$ are fixed. Then $C \circ \gamma(r) \to \infty$ as $r \to 0$. Thus (M, g) admits a future curvature singularity.

The open slicing coordinate system for de Sitter space is a Milne-like spacetime with scale factor $a(\tau) = \sinh(\tau)$. We saw that this spacetime admits a C^∞-extension. Therefore all curvature quantities are finite-valued at $\partial^- M$, and so we have no past curvature singularities in this case. This is expected since this spacetime is just a subset of de Sitter space.

In this section we will show that Milne-like spacetimes do not admit curvature singularities provided the second derivative of the scale factor satisfies

$$a''(\tau) = \alpha\tau + C\tau^3 + o(\tau^3).$$

Here $\alpha, C \in \mathbb{R}$ are just constants. The limiting condition implies $\alpha = a'''(0)$. Note $a(\tau) = \sinh(\tau)$ satisfies this limiting condition. In fact it applies to any convergent Taylor expansion $a(\tau) = \sum_{n=1}^{\infty} c_n \tau^n$ with $c_1 = 1$ and $c_2 = 0$ and $c_4 = 0$.

An example of a Milne-like spacetime where we do have a curvature singularity is given by the scale factor $a(\tau) = \tau + \tau^2$. The scalar curvature is $R(\tau) = 6a''(\tau)/a(\tau) = 12/(\tau + \tau^2) \to \infty$ as $\tau \to 0$. This is consistent with our result because in this case we have $a''(\tau) = 2 \neq \alpha\tau + C\tau^3 + o(\tau^3)$.

Lemma 3.3.3. *Fix $k \geq 2$. Let (M, g) be a C^k Milne-like spacetime. Suppose the second derivative of the scale factor satisfies $a''(\tau) = \alpha\tau + C\tau^3 + o(\tau^3)$ where $\alpha, C \in \mathbb{R}$. Then for any $p \in \partial^- M$, the limits of*

$$\frac{\partial\Omega}{\partial t}, \quad \frac{\partial\Omega}{\partial r}, \quad \frac{\partial^2\Omega}{\partial t^2}, \quad \frac{\partial^2\Omega}{\partial r^2}$$

as $(t, r, \theta, \phi) \to p$ all exist and are finite.

Theorem 3.3.4. *Fix $k \geq 2$. Let (M, g) be a C^k Milne-like spacetime. Suppose the second derivative of the scale factor satisfies $a''(\tau) = \alpha\tau + C\tau^3 + o(\tau^3)$ where $\alpha, C \in \mathbb{R}$. Then (M, g) admits no past curvature singularities.*

Proof. Let $\gamma \colon (0, b] \to M$ be any past-inextendible timelike curve parameterized by τ (we can parameterize by τ since it's a time function). Since γ is past inextendible and timelike, Figure 3.4 shows that there exists a point $p \in \partial^- M$ such that $p = \lim_{\tau \searrow 0} \gamma(\tau)$. More rigorously, the point p can be determined by writing out γ in the $\zeta = (t, r, \theta, \phi)$ coordinate system.

$$t(p) = \lim_{\tau \to 0} t \circ \gamma(\tau) \qquad\qquad r(p) = \lim_{\tau \to 0} r \circ \gamma(\tau)$$

$$\theta(p) = \lim_{\tau \to 0} \theta \circ \gamma(\tau) \qquad\qquad \phi(p) = \lim_{\tau \to 0} \phi \circ \gamma(\tau)$$

The existence of these limits follows from γ being past-inextendible and timelike. Since any curvature invariant is constructed out of first and second derivatives of the metric coefficients (i.e. the first and second derivatives of Ω in this case), Lemma 3.3.3 implies any curvature invariant has a finite-value quantity at p. Thus there are no past curvature singularities for (M, g).
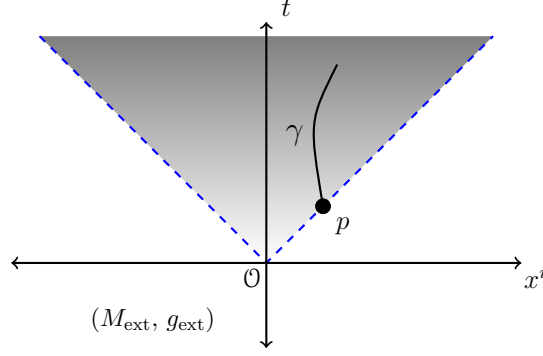
$\square$

Figure 3.4: A past-inextendible timelike curve γ inside a Milne-like spacetime (M, g) terminates at a past endpoint $p \in \partial^- M$. If $a''(\tau) = \alpha\tau + C\tau^3 + o(\tau^3)$, then any curvature invariant along γ will limit to a well-defined finite value at p.

Proof of Lemma 3.3.3:

Recall the coordinate system $\zeta = (t, r, \theta, \phi)$ from the proof of Theorem 3.2.4. Here t and r are given by

$$t(\tau, R) = b(\tau)\cosh(R) \quad \text{and} \quad r(\tau, R) = b(\tau)\sinh(R) \tag{3.18}$$

where $b\colon I \to (0, \infty)$ is given by $b(\tau) = \exp\left(\int_{\tau_0}^{\tau} \frac{1}{a(s)}ds\right)$ for some $\tau_0 > 0$. With respect to the coordinate system $\zeta = (t, r, \theta, \phi)$, the metric takes the form

$$g = \Omega^2\big(\tau(t, r)\big)\Big[-dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\Big] \tag{3.19}$$

where $\Omega = 1/b' = a/b$. We have to show for any $t_0 \geq 0$ the limits

$$\lim_{(t,r)\to(t_0,t_0)} \frac{\partial\Omega}{\partial t}(t, r) \qquad \lim_{(t,r)\to(t_0,t_0)} \frac{\partial\Omega}{\partial r}(t, r) \qquad \lim_{(t,r)\to(t_0,t_0)} \frac{\partial^2\Omega}{\partial t^2}(t, r) \qquad \lim_{(t,r)\to(t_0,t_0)} \frac{\partial^2\Omega}{\partial r^2}(t, r)$$

exist and are finite. Note that t and r appearing in the limits above are defined on the open set $U = \Big\{(t, r, \theta, \phi) \mid t^2 - r^2 < b^2(\tau_{\max}) \text{ and } t > 0 \text{ and } r \geq 0\Big\}$ and where $\tau_{\max} \in (0, +\infty]$ is given from the interval $I = (0, \tau_{\max})$ of the scale factor.

Note that b is a strictly increasing C^1 function which is never zero. Therefore it is invertible and the derivative of its inverse is $(b^{-1})'\big(b(\tau)\big) = 1/b'(\tau)$. Recall $\tau = b^{-1}\big(\sqrt{t^2 - r^2}\big)$. Therefore $\partial\tau/\partial t = t/(b'b)$. Since $\Omega = a/b = 1/b'$, the chain rule gives

$$\frac{\partial\Omega}{\partial t} = \Omega'\frac{\partial\tau}{\partial t} = \left(\frac{a' - 1}{b}\right)\left(\frac{t}{b'b}\right) \tag{3.20}$$

Let's simplify notation by letting $a(\tau) = \tau + f(\tau)$. Then we have

$$\frac{\partial\Omega}{\partial t} = \left(\frac{f'}{b^2 b'}\right)t \tag{3.21}$$

Taking another derivative, we get

$$\frac{\partial^2\Omega}{\partial t^2} = t\frac{\partial}{\partial t}\left(\frac{f'}{b^2 b'}\right) + \left(\frac{f'}{b^2 b'}\right) = t\left(\frac{f'}{b^2 b'}\right)'\frac{\partial\tau}{\partial t} + \left(\frac{f'}{b^2 b'}\right) \tag{3.22}$$

$$= t\left[\frac{f''(b^2 b') - f'\big(2b(b')^2 + b^2 b''\big)}{b^4 (b')^2}\right]\left(\frac{t}{b'b}\right) + \left(\frac{f'}{b^2 b'}\right) \tag{3.23}$$

$$= t^2\left[\frac{f''}{b^3 (b')^2} - \frac{2f'}{b^4 b'} - \frac{f'b''}{b^3 (b')^3}\right] + \left(\frac{f'}{b^2 b'}\right) \tag{3.24}$$

Plugging $b'' = (b/a)' = (b'a - a'b)/a^2 = -bf'/a^2$ into the above expression gives

$$\frac{\partial^2\Omega}{\partial t^2} = t^2\left[\frac{f''}{b^3 (b')^2} - \frac{2f'}{b^4 b'} + \frac{(f')^2}{a^2 b^2 (b')^3}\right] + \left(\frac{f'}{b^2 b'}\right) \tag{3.25}$$

Fix $\varepsilon > 0$. From the proof of Theorem 3.2.4, there exists a $\delta > 0$ such that for all $0 < \tau < \delta$, we have

$$\left(\frac{\tau}{\tau_0}\right)\left(\frac{1 - \varepsilon\tau^{\varepsilon_0}}{1 + \varepsilon\tau_0^{\varepsilon_0}}\right)^{-1/\varepsilon_0} < b(\tau) < \left(\frac{\tau}{\tau_0}\right)\left(\frac{1 + \varepsilon\tau^{\varepsilon_0}}{1 + \varepsilon\tau_0^{\varepsilon_0}}\right)^{-1/\varepsilon_0} \tag{3.26}$$

where ε_0 is given by $a(\tau) = \tau + o(\tau^{1+\varepsilon_0})$. Since $f''(\tau) = \alpha\tau + O(\tau^3)$, we have $f'(\tau) = \frac{1}{2}\alpha\tau^2 + O(\tau^4)$. Using (3.26) along with $b' = b/a$, we see that for small τ, we

have $b \sim \tau/\tau_0$ and $b' \sim 1/\tau_0$. Therefore the squeeze theorem gives

$$\lim_{\tau \to 0} \left(\frac{f''}{b^3(b')^2} - \frac{2f'}{b^4 b'} \right) = 0. \tag{3.27}$$

Note that we needed $O(\tau^3)$ in $a''(\tau) = \alpha\tau + O(\tau^3)$ to get the above equality. Similarly, we have both of the limits

$$\lim_{\tau \to 0} \frac{(f')^2}{a^2 b^2 (b')^3} \quad \text{and} \quad \lim_{\tau \to 0} \frac{f'}{b^2 b'} \tag{3.28}$$

exist and are both finite. Plugging in (3.27) and (3.28) into (3.25) and (3.21), we see that for any $t_0 \geq 0$, we have that the limits

$$\lim_{(t,r) \to (t_0, t_0)} \frac{\partial \Omega}{\partial t} \quad \lim_{(t,r) \to (t_0, t_0)} \frac{\partial^2 \Omega}{\partial t^2} \tag{3.29}$$

exist and are finite. Similarly, we obtain the same conclusion for $\frac{\partial \Omega}{\partial r}$ and $\frac{\partial^2 \Omega}{\partial r^2}$. This completes the proof of Lemma 3.3.3.

$\square$

3.4 The geometrical solution to the horizon problem

Our definition for an inflationary FLRW spacetime was one whose scale factor satisfies $a(\tau) = \tau + o(\tau^{1+\varepsilon})$ for some $\varepsilon > 0$. Our motivation is that these spacetimes solve the horizon problem, and this is true for $k = +1$, 0, or -1. However, what's unique about Milne-like spacetimes is that they extend into a larger spacetime because the big bang is just a coordinate singularity. This offers a new geometrical picture of how Milne-like spacetimes solve the horizon problem as we discuss below.

We briefly recall the horizon problem in cosmology. It is the main motivating reason for inflationary theory. The problem comes from the uniform temperature of the CMB radiation. From any direction in the sky, we observe the CMB temperature as 2.7 K. The uniformity of this temperature is puzzling: if we assume the universe exists in a radiation dominated era all the way down to the big bang (i.e. no inflation), then the points p and q on the surface of last scattering don't have intersecting past lightcones. So how can the CMB temperature be so uniform if p and q were never in causal contact in the past?
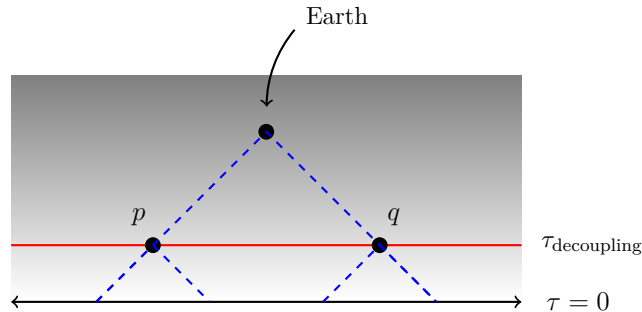

Figure 3.5: The horizon problem. Without inflation the past lightcones of p and q never intersect. But then why does the Earth measure the same 2.7 K temperature from every direction?

By using conformal time $\tilde{\tau}$ given by $d\tilde{\tau} = d\tau/a(\tau)$, it is an elementary exercise to show that there is no horizon problem provided the *particle horizon* at the moment of last scattering is infinite:

$$\int_0^{\tau_{\text{decoupling}}} \frac{1}{a(\tau)} d\tau = \infty. \tag{3.30}$$

This condition widens the past lightcones of p and q so that they intersect before $\tau = 0$.
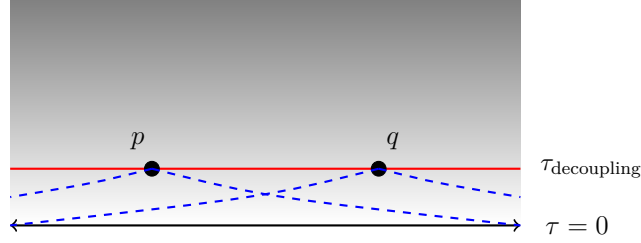
Figure 3.6: Inflation solves the horizon problem by widening the past lightcones.

Proposition 3.4.1. *The particle horizon for an inflationary spacetime is infinite.*

Proof. From the definition of an inflationary spacetime, we have

$$\lim_{\tau \to 0} \frac{a(\tau)}{\tau} = 1. \tag{3.31}$$

Therefore for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|a(\tau)/\tau - 1| < \varepsilon$ for all $0 < \tau < \delta$. Hence $1/a(\tau) > 1/(1+\varepsilon)\tau$ for all $0 < \tau < \delta$. Then the particle horizon at the moment of last scattering is

$$\int_0^{\tau_{\text{decoupling}}} \frac{1}{a(\tau)} d\tau \geq \int_0^\delta \frac{1}{a(\tau)} d\tau \geq \int_0^\delta \frac{1}{(1+\varepsilon)\tau} d\tau = \infty \tag{3.32}$$

Thus the particle horizon is infinite. □

For Milne-like spacetimes, the origin $\mathcal{O}$ plays a unique role. The lightcones of any two points must intersect above $\mathcal{O}$. This follows from the metric being conformal to Minkowski space, $g_{\mu\nu} = \Omega^2(\tau)\eta_{\mu\nu}$. As such the lightcones are given by 45 degree angles; see Figure 3.7 which clarifies the situation depicted in Figure 3.6.

Also we observe that the comoving observers all emanate from the origin $\mathcal{O}$. Indeed a comoving observer $\gamma(\tau)$ is specified by a point (R_0, θ_0, ϕ_0) on the hyperboloid.

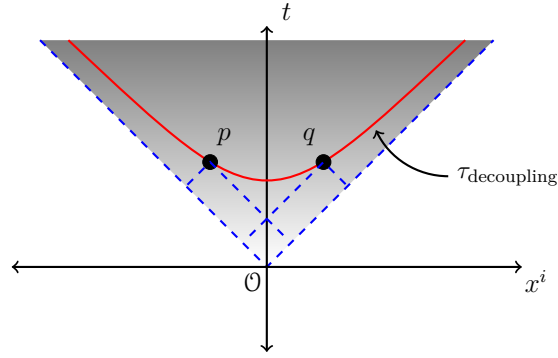$$\gamma(\tau) = (\tau, R_0, \theta_0, \phi_0). \tag{3.33}$$

Figure 3.7: A Milne-like spacetime modeling our universe. The points p and q have past lightcones which intersect at some point above $\mathcal{O}$.

In the (t, r, θ, ϕ) coordinates introduced in equation (3.11), the comoving observer is given by

$$\gamma(\tau) = \big(t(\tau), r(\tau), \theta_0, \phi_0\big) \tag{3.34}$$

where

$$t(\tau) = b(\tau)\cosh(R_0) \quad \text{and} \quad r(\tau) = b(\tau)\sinh(R_0). \tag{3.35}$$

Thus the relationship between t and r for γ is $t = \coth(R_0)r$. Therefore for any comoving observer, we have $t = Cr$ for some $C > 1$. Thus the comoving observers emanate from the origin.
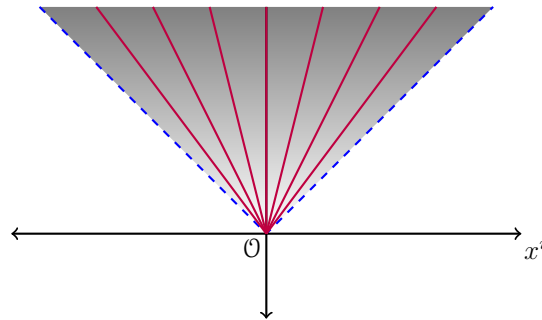

Figure 3.8: The comoving observers in a Milne-like spacetime. They all emanate from the origin $\mathcal{O}$.

3.5 The cosmological constant appears as an initial condition

In this section we show how the cosmological constant Λ can appear as an initial condition for Milne-like spacetimes. This may help explain the origin of Λ. If dark energy is really modeled by a cosmological constant and not by some other model (e.g. quintessence), then Λ would have been fixed at the big bang.

Fix $k \geq 2$. For this section let (M, g) denote a C^k Milne-like spacetime. Consider the Einstein equations with a cosmological constant

$$G_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} \;=\; 8\pi T_{\mu\nu}. \tag{3.36}$$

Let $u = \partial/\partial\tau$ denote the four-velocity of the comoving observers and let e be any unit spacelike orthogonal vector (its choice does not matter by isotropy). We define the *normal* energy density $\rho_{\text{normal}}(\tau)$ and *normal* pressure function $p_{\text{normal}}(\tau)$ in terms of the energy-momentum tensor

$$\rho_{\text{normal}} \;=\; T_{\mu\nu}u^{\mu}u^{\nu} \tag{3.37}$$

$$p_{\text{normal}} \;=\; T_{\mu\nu}e^{\mu}e^{\nu} \tag{3.38}$$

Then the energy density ρ and pressure function p in terms of ρ_{normal} and p_{normal} are given by

$$\rho \;=\; \frac{1}{8\pi}G_{\mu\nu}u^{\mu}u^{\nu} \;=\; \rho_{\text{normal}} + \frac{\Lambda}{8\pi} \tag{3.39}$$

$$p \;=\; \frac{1}{8\pi}G_{\mu\nu}e^{\mu}e^{\nu} \;=\; p_{\text{normal}} - \frac{\Lambda}{8\pi} \tag{3.40}$$

If $\rho_{\text{normal}} = p_{\text{normal}} = 0$ (e.g. de Sitter), then the equation of state for the cosmological constant is fixed for all τ.

$$\rho = -p = \frac{\Lambda}{8\pi}. \tag{3.41}$$

We show that this equation of state appears as an initial condition. For the following theorem, we define $\rho(0) := \lim_{\tau \to 0} \rho(\tau)$. Likewise with $p(0)$ and $\rho_{\text{normal}}(0)$ and $p_{\text{normal}}(0)$.

Theorem 3.5.1. *Consider a Milne-like spacetime and suppose the scale factor satisfies $a''(\tau) = \alpha\tau + o(\tau)$. Then*

$$\rho(0) = -p(0) = \frac{3}{8\pi}\alpha.$$

We prove Theorem 3.5.1 at the end of this section. First we understand its implications. If the cosmological constant Λ is the dominant energy source during the Planck era, then we have the following connection between Λ and the initial condition of the scale factor.

Proposition 3.5.2. *Suppose the scale factor satisfies $a''(\tau) = \alpha\tau + o(\tau)$ and we have $\rho_{\text{normal}}(0) = p_{\text{normal}}(0) = 0$. Then*

$$\Lambda = 3\alpha = 3a'''(0).$$

Proof. This follows from Theorem 3.5.1 and equation (3.39). Note that the limiting condition implies $\alpha = a'''(0)$. $\qquad\square$

Remark. In (3+1)-dimensional de Sitter space we have $T_{\mu\nu} = 0$ and $\Lambda = 3$. In the open slicing coordinates of de Sitter, we have $a(\tau) = \sinh(\tau)$. Hence $\alpha = a'''(0) = 1$. Therefore de Sitter space is a special example of Proposition 3.5.2.

Now we examine how an inflaton scalar field behaves in the limit $\tau \to 0$. We will demonstrate that slow-roll inflation follows if the initial condition for the potential is given by the cosmological constant: $V|_{\tau=0} = \Lambda/8\pi$. Recall the energy-momentum tensor for a scalar field ϕ is

$$T_{\mu\nu}^{\phi} = \nabla_\mu\phi\nabla_\nu\phi - \left[\frac{1}{2}\nabla^\sigma\phi\nabla_\sigma\phi + V(\phi)\right]g_{\mu\nu}. \tag{3.42}$$

And its energy density and pressure function are

$$\rho_\phi(\tau) = \frac{1}{2}\phi'(\tau)^2 + V\big(\phi(\tau)\big) \quad \text{and} \quad p_\phi(\tau) = \frac{1}{2}\phi'(\tau)^2 - V\big(\phi(\tau)\big). \tag{3.43}$$

Proposition 3.5.3. *Suppose the scale factor satisfies $a''(\tau) = \alpha\tau + o(\tau)$ and we have*

$$\rho_{\text{normal}} \to \rho_\phi \to 0 \quad \text{and} \quad p_{\text{normal}} \to p_\phi \to 0$$

as $\tau \to 0$. Then the initial condition $V\big(\phi(0)\big) = \Lambda/8\pi$ implies $\phi'(0) = 0$. Hence it yields an era of slow-roll inflation.

Proof. Since $\rho_{\text{normal}} \to \rho_\phi$ as $\tau \to 0$, Theorem 3.5.1 implies $\rho_\phi(0) = (3/8\pi)a'''(0)$. Since $\rho_\phi \to 0$ as $\tau \to 0$, Proposition 3.5.2 implies $\rho_\phi(0) = \Lambda/8\pi$. Thus the initial condition $V\big(\phi(0)\big) = \Lambda/8\pi$ implies $\phi'(0) = 0$. □

Proof of Theorem 3.5.1:

Friedmann's equations are $(8\pi/3)\rho = H^2 - 1/a^2$ and $8\pi p = -2a''/a - (8\pi/3)\rho$ where $H = a'/a$ is the Hubble parameter. Using $a(\tau) = \tau + f(\tau)$, the Friedmann equations become

$$\frac{8\pi}{3}\rho(\tau) = \left(\frac{a'(\tau)}{a(\tau)}\right)^2 - \frac{1}{a(\tau)^2} = \frac{2f'(\tau) + f'(\tau)^2}{\big[\tau + f(\tau)\big]^2} = \frac{\big(f'(\tau)/\tau\big)\big[2/\tau + f'(\tau)/\tau\big]}{\big(1 + f(\tau)/\tau\big)^2} \tag{3.44}$$

and

$$-8\pi p(\tau) = 2\frac{a''(\tau)}{a(\tau)} + \frac{8\pi}{3}\rho(\tau) = \frac{2f''(\tau)/\tau}{1 + f(\tau)/\tau} + \frac{8\pi}{3}\rho(\tau). \tag{3.45}$$

By definition of an inflationary spacetime, we have $f'(0) := \lim_{\tau\to 0} f(\tau)/\tau = 0$. Also, since $a''(\tau) = \alpha\tau + o(\tau)$, we have $0 = a''(0) = f''(0) = \lim_{\tau\to 0} f'(\tau)/\tau$ and $\alpha = \lim_{\tau\to 0} f''(\tau)/\tau$. Therefore for all $\varepsilon > 0$, there is a $\delta > 0$ such that $|f''(\tau)/\tau - \alpha| < \varepsilon$ for all $0 < \tau < \delta$. Integrating this expression gives $(\alpha - \varepsilon)\tau/2 < f'(\tau)/\tau < (\alpha + \varepsilon)\tau/2$. Plugging this into the first Friedmann equation yields $8\pi\rho(0)/3 = \alpha$. Using this for the second Friedmann equation yields $-8\pi p(0) = 3\alpha$.

$\square$

3.6   Lorentz invariance

In this section we show that the isometry group for Milne-like spacetimes contains the Lorentz group. Since Lorentz invariance plays a pivotal role in QFT (e.g. the field operators are constructed out of finite dimensional irreducible represenations of the Lorentz group, Milne-like spacetimes are a good background model if one wants to develop a quantum theory of cosmology.

Remark. In this section Λ will always denote an element of the Lorentz group (i.e. a Lorentz transformation) and not the cosmological constant.

Let $\eta_{\mu\nu}$ be the Minkowski metric. The *Lorentz group* is

$$L = O(1,3) = \{\Lambda \mid \eta_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu \eta_{\alpha\beta}\}. \tag{3.46}$$

A Lorentz transformation Λ shifts elements in Minkowski space via $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$, but it leaves the hyperboloids fixed. More generally this applies to any Milne-like spacetime by the same map.
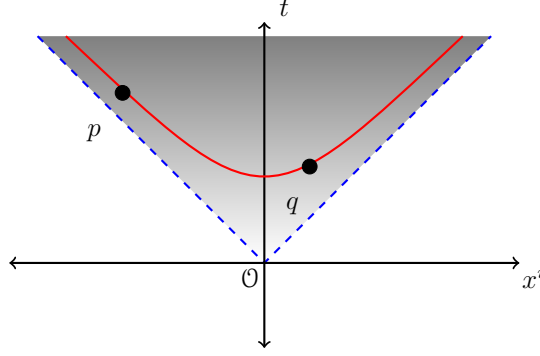
Figure 3.9: A Lorentz transformation Λ based at $\mathcal{O}$ can shifts points p to other points $q = \Lambda p$ on the same $\tau = $ constant slice. For Milne-like spacetimes, Ω is a function of τ. Therefore $\Omega(\tau) = \Omega(\tau \circ \Lambda)$, e.g. in this figure we would have $\Omega(\tau(p)) = \Omega(\tau(q))$.

For a Milne-like spacetime, we have $g_{\mu\nu} = \Omega^2(\tau)\eta_{\mu\nu}$ where $\eta_{\mu\nu}$ is the usual Minkowski metric. Since a Lorentz transformation leaves hyperboloids invariant, we have

$$\Omega(\tau) = \Omega(\tau \circ \Lambda). \tag{3.47}$$

Recall the Lorentz group $L = O(1,3)$ has four connected components $L_+^\uparrow$, $L_-^\uparrow$, $L_+^\downarrow$, $L_-^\downarrow$. The $\pm$ corresponds to $\det \Lambda = \pm 1$, the $\uparrow$ corresponds to $\Lambda^0_{\ 0} \geq 1$, and the $\downarrow$ corresponds to $\Lambda^0_{\ 0} \leq -1$.

Lorentz transformations fix the origin (i.e. $\Lambda \mathcal{O} = \mathcal{O}$) and are isometries on the spacetime manifold with boundary $(M \cup \partial^- M) \setminus \{0\}$. We will say that any map which fixes $\mathcal{O}$ and is an isometry on the spacetime manifold with boundary $(M \cup \partial^- M) \setminus \{\mathcal{O}\}$ is an *isometry on $M \cup \partial^- M$*. Note that the set of isometries on $M \cup \partial^- M$ forms a group via composition. Since Milne-like spacetimes are defined for $t > 0$, only the subgroup $L^\uparrow = L_+^\uparrow \cup L_-^\uparrow$ acts by isometries on Milne-like spacetimes. If (M, g) admits a C^2-extension, then we obtain an isomorphism.

Theorem 3.6.1. *Let (M, g) be a Milne-like spacetime. Then any $\Lambda \in L^\uparrow$ is an isometry on $M \cup \partial^- M$.*

Theorem 3.6.2. *If a Milne-like spacetime admits a C^2-extension, then* $\mathrm{L}^{\uparrow}$ *is isomorphic to the group of isometries on* $M \cup \partial^- M$.

Remark. To the best of the author's knowledge, Theorem 3.6.2 is a new result. Its proof relies on the existence of $\partial^- M$.

Proofs of Theorems 3.6.1 *and* 3.6.2:

Let Λ be an element of $\mathrm{L}^{\uparrow}$. It produces a unique map, $x \mapsto \Lambda x$ via $x^\mu \mapsto \Lambda^\mu{}_\nu x^\nu$ where $(x^0, x^1, x^2, x^3) = (t, x, y, z)$ are the conformal Minkowski coordinates introduced in the proof of Theorem 3.2.4. Since (M, g) is only defined for $t > 0$, we must restrict to Lorentz transformations $\Lambda \in \mathrm{L}^{\uparrow}$. Consider a point p in the spacetime and a tangent vector $X = X^\mu \partial_\mu$ at p. Then Λ acts on X by $d\Lambda(X) = \Lambda^\mu{}_\nu X^\nu \partial_\mu$ and sending it to the point Λp. Since our metric is $g_{\mu\nu} = \Omega^2(\tau)\eta_{\mu\nu}$ and $\Omega(\Lambda p) = \Omega(p)$, we have

$$
\begin{aligned}
g_{\mu\nu}(d\Lambda X)^\mu (d\Lambda Y)^\nu &= \Omega^2(\tau \circ \Lambda p)\, \eta_{\mu\nu}(d\Lambda X)^\mu (d\Lambda Y)^\nu \\
&= \Omega^2(\tau \circ p)\, \eta_{\mu\nu}(\Lambda^\mu{}_\alpha X^\alpha)(\Lambda^\nu{}_\beta Y^\beta) \\
&= \Omega^2(\tau \circ p)\, \eta_{\alpha\beta} X^\alpha Y^\beta \\
&= g_{\alpha\beta} X^\alpha Y^\beta.
\end{aligned}
$$

Thus Λ is an isometry. This proves Theorem 3.6.1.

Now we prove Theorem 3.6.2. By Theorem 3.6.1 we have $\mathrm{L}^{\uparrow}$ is a subgroup, so it suffices to show it's the whole group. Suppose f is an isometry on $M \cup \partial^- M$. The differential map $df_{\mathcal{O}}$ is a linear isometry on the tangent space at $\mathcal{O}$. Therefore $df_{\mathcal{O}}$ corresponds to an element of the Lorentz group, say $\Lambda^\mu{}_\nu$. It operates on vectors X at $\mathcal{O}$ via $df(X) = \Lambda^\mu{}_\nu X^\nu \partial_\mu$. Now we define the isometry $\tilde{f}$ by $\tilde{f}(x) = \Lambda^\mu{}_\nu x^\nu$. Consider the set

$$
A = \{p \in M \cup \partial^- M \mid df_p = d\tilde{f}_p\}.
$$

Note that if $df_p = d\tilde{f}_p$, then $f(p) = \tilde{f}(p)$. Hence it suffices to show $A = M \cup \partial^- M$. A is nonempty since $\mathcal{O} \in A$, and A is closed because $df - d\tilde{f}$ is continuous. So since $M \cup \partial^- M$ is connected, it suffices to show A is open in the subspace topology. Let $p \in A$. Since Ω is C^2, there is a normal neighborhood U about p. If $q \in U$, there is a vector X at p such that $\exp_p(X) = q$. Since isometries map geodesics to geodesics, they satisfy the property $f \circ \exp_p = \exp_{f(p)} \circ df_p$ for all points in U (see page 91 of [22]). Therefore

$$f(q) = f\left(\exp_p(X)\right) = \exp_{f(p)}(df_p X) = \exp_{\tilde{f}(p)}(d\tilde{f}_p X) = \tilde{f}\left(\exp_p(X)\right) = \tilde{f}(q).$$

Thus $f(q) = \tilde{f}(q)$ for all $q \in U$; hence $df_q = d\tilde{f}_q$ for all $q \in U$. Therefore A is open.

$\square$

3.7 C^0-inextendibility results within spherically symmetric spacetimes

The open FLRW spacetimes possess spherical symmetry, so we wish to describe this spherical symmetry in the class of C^0 spacetimes. The definition of spherical symmetry given in [21, Box 23.3], makes sense at the C^0 level. There, it is assumed that the group of isometries of spacetime (M^{d+1}, g) contains $SO(d)$ as a subgroup, such that the orbits of this action are spacelike $(d-1)$-spheres ($d = 3$ in their discussion). It is further assume that there exists a timelike vector field u invariant under the $SO(d)$ group action. Then, under these assumptions, their arguments lead (in the case $d = 3$; in fact any d odd would suffice) to the existence about every point of M local coordinates $(x, y, \omega \in S^{d-1})$ such that with respect to these coordinates the

metric takes the form

$$g = A(x, y)dx^2 + 2B(x, y)dxdy + C(x, y)dy^2 + R^2(x, y)d\Omega_{d-1}^2. \tag{3.48}$$

If coordinates (x, y, ω) can be introduced so that the metric takes this form, we will say that spacetime is *spherically symmetric* and will refer to the coordinates (x, y, ω) as *spherically symmetric coordinates*. The choice of radial function r is unique in the following sense: If (x, y, ω) and $(\bar{x}, \bar{y}, \omega)$ are spherically symmetric coordinates, such that x and y are solely functions of $\bar{x}$ and $\bar{y}$, then both coordinate systems induce the same radial function on the overlap. It should be noted that the usual procedure one uses to eliminate the cross term cannot be applied in the C^0 setting because this requires a Lipschitz condition on A, B, and C.

We will say that (M, g) is *strongly spherically symmetric* if about every point there are coordinates (t, r, ω) such that in this coordinate neighborhood the metric takes the form

$$g = -F(t, r)dt^2 + G(t, r)dr^2 + r^2 d\Omega_{d-1}^2, \tag{3.49}$$

and we call (t, r, ω) *strongly spherically symmetric coordinates*. To achieve the metric form (3.49) via a change of coordinates from (3.48), requires greater regularity on the metric, at least C^1, and, in addition, a C^1 genericity condition on r. We note that Milne-like spacetimes are strongly spherically symmetric.

We were able to find C^0-extensions for a Milne-like spacetime (M, g) by writing (M, g) in strongly spherically symmetric coordinates. A natural question to ask is: Can strongly spherically symmetric coordinates be used to find a C^0-extension for $k = 0$ FLRW spacetimes? What about $k = -1$ FLRW spacetimes that are not Milne-like? The results in the following subsections answer in the negative.

Theorem 3.7.1. *Let (M, g) be a $k = 0$ FLRW spacetime with metric*

$$g = -d\tau^2 + a^2(\tau)\Big[dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\Big]$$

where the scale factor $a(t)$ satisfies $a'(0) := \lim_{\tau \to 0^+} a'(\tau) \in (0, \infty]$. Then, subject to a suitable initial condition, there exists a unique transformation of the form,

$$t = t(\tau, R) \quad r = r(\tau, R) \tag{3.50}$$

such that g takes the strongly spherically symmetric form

$$g = -F(t, r)dt^2 + G(t, r)dr^2 + r^2 d\Omega_{d-1}^2,$$

where F and G are regular (away from a curve in the R-τ plane along which the Jacobian determinant $J(R, \tau) = \frac{\partial(t,r)}{\partial(\tau,R)}$ vanishes).

Now suppose that M admits a C^0-extension M_{ext}, and consider the behavior of the metric in these coordinates on approach to $\partial^- M$. Let $\gamma : [0, 1] \to M_{\text{ext}}$ be a future directed timelike curve with past end point $\gamma(0) \in \partial^- M$, and suppose R has a finite positive limit along γ as $\tau \to 0^+$. (Note, by the achronality of $\partial^- M$, $\gamma((0, 1]) \subset M$.) Then the following hold along γ.

(a) $\lim_{\tau \to 0^+} G = 0$.

(b) If F has a finite nonzero limit as $\tau \to 0^+$, then $t \to \pm\infty$ as $\tau \to 0^+$.

Remark. By a 'suitable initial condition', we mean the following: The transformation (3.50) is unique up to a function f which is determined by specifying t along a certain curve in the first quadrant of the (R, τ)-plane. This is shown in the proof below.

Proof. We begin by solving explicitly for r, t, G, and F in terms of τ, and R. Immediately, we find $r = Ra(\tau)$. To see this, consider a codimension 2 surface of constant

t and r. Since t and r are functions of τ and R only, a surface of constant t and r corresponds to a surface of constant τ and R. By restricting the metric to this surface, we have $r^2 d\Omega_{d-1}^2 = R^2 a^2(\tau) d\Omega_{d-1}^2$ and hence $r = Ra(\tau)$.

We have

$$dt^2 = (\partial_\tau t)^2 d\tau^2 + 2(\partial_\tau t)(\partial_R t) d\tau dR + (\partial_R t) dR^2$$

$$dr^2 = R^2 a'^2 d\tau^2 + 2Raa' d\tau dR + a^2 dR^2.$$

So we want

$$-d\tau^2 + a^2(\tau)\left[dR^2 + R^2 d\Omega_{d-1}^2\right] = -Fdt^2 + Gdr^2 + r^2 d\Omega_{d-1}^2 \qquad (3.51)$$

From equation (3.51), we find

$$-1 = -F(\partial_\tau t)^2 + GR^2 a'^2 \qquad \Longrightarrow \qquad F(\partial_\tau t)^2 = 1 + GR^2 a'^2 \qquad (3.52)$$

$$0 = -F(\partial_\tau t)(\partial_R t) + GRaa' \qquad \Longrightarrow \qquad F^2(\partial_\tau t)^2(\partial_R t)^2 = G^2 R^2 a^2 a'^2 \qquad (3.53)$$

$$a^2 = -F(\partial_R t)^2 + Ga^2 \qquad \Longrightarrow \qquad F(\partial_R t)^2 = a^2(G-1) \qquad (3.54)$$

By substituting (3.52) and (3.54) into (3.53), we find

$$G(R,\tau) = \frac{1}{1 - R^2 a'^2} \qquad (3.55)$$

Substituting this into (3.52) and (3.54), we find

$$F(\partial_\tau t)^2 = \frac{1}{1 - R^2 a'^2} \quad \text{and} \quad F(\partial_R t)^2 = \frac{R^2 a^2 a'^2}{1 - R^2 a'^2} \qquad (3.56)$$

Therefore $(\partial_R t / \partial_\tau t)^2 = (Raa')^2$. Since we require the metric to be Lorentzian, the leftmost equation in (3.53) implies that we must have $(\partial_R t / \partial_\tau t) = Raa'$. A solution to this PDE must be constant along the integral curves of $d\tau/dR = -Raa'$ in the (τ, R)-plane, so a general solution for t is

$$t(R, \tau) = f\left(\frac{R^2}{2} + \int \frac{1}{aa'}\right) \tag{3.57}$$

where f is some smooth function. f is uniquely determined by specifying t on a curve which is transversal to the curves $\frac{R^2}{2} + \int \frac{1}{aa'} = $ const. Thus there is a degree of freedom when choosing strongly spherically symmetric coordinates.

In summary we have

- $r = Ra(\tau)$

- $t = f\left(\frac{R^2}{2} + \int \frac{1}{aa'}\right)$

- $G = \frac{1}{1 - R^2 a'^2}$

- $F = G(\partial_\tau t)^{-2} = G\left(\frac{aa'}{f'}\right)^2$.

The Jacobian of the transformation is

$$J = (\partial_R \tau)(\partial_\tau r) - (\partial_\tau t)(\partial_R r) = f'[R^2 a' - 1/a'].$$

Therefore F and G are regular everywhere except where the Jacobian vanishes, namely along the curve $R^2 a'(\tau)^2 = 1$ (since, from (3.56) and (3.57), $f' \neq 0$). Also, note that t and r change causal character here.

We can write the metric as

$$g = -Fdt^2 + Gdr^2 + r^2 d\Omega^2$$
$$= \frac{1}{1 - R^2 a'^2}\left[-\left(\frac{aa'}{f'}\right)^2 dt^2 + dr^2\right] + r^2 d\Omega^2. \tag{3.58}$$

In equation (3.58), R and τ are smooth implicit functions of r and t away from $R^2 a'(\tau)^2 = 1$.

Now restrict to γ. Along γ we see $G \to 0$ as $\tau \to 0^+$ since $R = r/a$. This establishes (a). To prove (b), let us use s to denote the argument of f. Then

$$s(r, \tau) = \frac{r^2}{2a^2} + \int \frac{1}{aa'}$$
$$= \frac{\frac{1}{2}r^2 + a^2 \int \frac{1}{aa'}}{a^2} \tag{3.59}$$

F is given by

$$F(\tau) = \frac{1}{1 - r^2(a'/a)^2} \left(\frac{aa'}{f'\big(s(\tau)\big)} \right)^2$$
$$= \left(\frac{a^4}{(a/a')^2 - r^2} \right) \frac{1}{\big[f'\big(s(\tau)\big)\big]^2} \tag{3.60}$$

Rearranging (3.59) and (3.60) gives us

$$s^2 \big[f'(s)\big]^2 = \left(\frac{\frac{1}{2}r^2 + a^2 \int \frac{1}{aa'}}{a^2} \right)^2 \left(\frac{a^4}{F\big[(a/a')^2 - r^2\big]} \right)$$
$$= \frac{\big[\frac{1}{2}r^2 + a^2 \int \frac{1}{aa'}\big]^2}{F\big[(a/a')^2 - r^2\big]} \tag{3.61}$$

Now assume F has a finite nonzero limit as $\tau \to 0^+$. Since $(a/a') \to 0$ and $a^2 \int \frac{1}{aa'} \to 0$ (by L'Hôpital's rule) as $\tau \to 0^+$ along γ, there is a constant $0 < c < \infty$ such that

$$\lim_{\tau \to 0^+} s^2 \big[f'(s)\big]^2 = c^2$$

Note that $\tau \to 0^+$ implies $s \to \infty$ along γ. Therefore the above limit is equivalent to $\lim_{s \to \infty} s^2 \big[f'(s)\big]^2 = c^2$. As noted above, $f' \neq 0$, so it follows that $\lim_{s \to \infty} s f'(s) = \pm c$. Fix $0 < \epsilon < c/2$. Then there exists an S such that $s > S$ implies $|s f'(s) \mp c| < c/2$

so $f'(s) > \pm\frac{c}{2s}$. By integrating over all $s > S$, we find that $f(s) \to \pm\infty$ as $s \to \infty$. Hence $t \to \pm\infty$ as $\tau \to 0^+$ along γ. $\square$

Corollary 3.7.2. *Let (M, g) be a $k = 0$ FLRW spacetime where the scale factor satisfies the condition in Theorem 3.7.1. Then there is no C^0 strongly spherically symmetric extension of (M, g).*

Remark. By the conclusion we mean precisely the following: There is no point $p \in \partial^- M$ for which there exist strongly spherically symmetric coordinates (t, r, ω) defined on a neighborhood U of p such that on $U \cap M$, t and r are functions of τ and R only and $g = \psi^* g_{ext}$, where ψ is the transformation: $(\tau, R) \to (t, r)$.

Proof. Suppose there is such an extension. Then there is a point $p \in \partial^- M$ and a neighborhood U of p with strongly spherically symmetric coordinates (t, r, ω) such that t and r are as in the remark. But then the conclusions (a) and (b) of Theorem 3.7.1 apply. In particular, Theorem 3.7.1 implies that $G(p) = 0$, so the extended metric is degenerate at p. $\square$

We have analogous statements of Theorem 3.7.1 and Corollary 3.7.2 for hyperbolic FLRW spacetimes. However, we have to rule out the Milne-like spacetimes since we know these admit C^0-extensions.

Theorem 3.7.3. *Let (M, g) be a hyperbolic FLRW spacetime where the scale factor $a(\tau)$ satisfies*

$$a'(0) := \lim_{\tau \to 0^+} a'(\tau) \in [0, \infty], \quad a'(0) \neq 1.$$

Then, subject to a suitable initial condition, there exists a unique transformation of the form,

$$t = t(\tau, R) \quad r = r(\tau, R)$$

such that g takes the strongly spherically symmetric form

$$g = -F(t,r)dt^2 + G(t,r)dr^2 + r^2 d\Omega_{d-1}^2,$$

where F and G are regular (away from a curve in the R-τ plane along which the Jacobian determinant $J(R,\tau) = \frac{\partial(t,r)}{\partial(\tau,R)}$ vanishes). Suppose M admits a C^0-extension M_{ext}. Let γ be a timelike curve in M_{ext} with past end point on $\partial^- M$, such that R has a finite positive limit along γ as $\tau \to 0^+$. Then we have $\lim_{\tau \to 0^+} G = 0$ along γ.

Proof. The proof is hardly different from the proof of Theorem 3.7.1. The same analysis leads to the following expressions for r, t, G, and F

- $r = \sinh(R)a(\tau)$

- $t = f\left(\ln\left(\cosh(R) \right) + \int \frac{1}{aa'} \right)$

- $G = \left[\cosh^2(R) - \sinh^2(R)a'^2(\tau) \right]^{-1}$

- $F = \cosh^2(T)G(\partial_\tau t)^{-2} = \cosh^2(R)G \left(\frac{aa'}{f'} \right)^2.$

where f is some differentiable function which is uniquely determined by specifying t on a curve which is transversal to the curves $\ln\left(\cosh(R) \right) + \int \frac{1}{aa'} = \text{const}.$

We have

$$\begin{aligned}
G &= \left[\cosh^2\left(\sinh^{-1}(r/a) \right) - \sinh^2\left(\sinh^{-1}(r/a) \right)a'^2 \right]^{-1} \\
&= \left[(r/a)^2 + 1 - r^2 a'^2/a^2 \right]^{-1} \\
&= \frac{a^2}{r^2(1 - a'^2) + a^2}
\end{aligned} \tag{3.62}$$

Since $a'(0) \neq 1$, it follows that $G \to 0$ as $\tau \to 0^+$ along γ. $\square$

Corollary 3.7.4. *Let (M, g) be a $k = -1$ FLRW spacetime where the scale factor satisfies the condition in Theorem 3.7.3. Then there is no C^0 strongly spherically symmetric extension of (M, g).*

Proof. The remark following Corollary 3.7.2 still applies. The proof is then essentially the same as the proof of Corollary 3.7.2.

Chapter 4

Other Results in the Smooth Spacetime Category

In this chapter we give brief descriptions of results obtained with Greg Galloway and Piotr Chruściel in the smooth spacetime category [14, 6, 13].

4.1 Topology and singularities in spacetimes with compact Cauchy surfaces obeying the null energy condition

The classical Hawking cosmological singularity theorem [17, p. 272] establishes past timelike geodesic incompleteness in spatially closed spacetimes that at some stage are future expanding. This singularity theorem requires the Ricci tensor of spacetime to satisfy the strong energy condition, $\mathrm{Ric}(X, X) \geq 0$ for all timelike vectors X. In spacetimes obeying the Einstein equations with positive cosmological constant, $\Lambda > 0$, this energy condition is not in general satisfied, and the conclusion then need not hold; de Sitter space, which is geodesically complete, is an immediate example. But this is not just a feature of vacuum spacetimes; dust filled FLRW spacetimes with positive cosmological constant provide other examples. For the FLRW models discussed in [11, Section 3], the co-moving Cauchy surfaces are assumed to be compact, and, apart

from the time-dependent scale factor, have constant curvature $k = +1, 0, -1$. These three cases are topologically quite distinct. For instance, in the $k = +1$ (spherical space) case, the Cauchy surfaces have finite fundamental group, while in the $k = 0, -1$ (toroidal and hyperbolic 3-manifold) cases, the fundamental group is infinite. Moreover, it is only in the $k = +1$ case, that the past big-bang singularity can be avoided.

In [1], this topology dependent behavior was studied in a much broader context (not requiring any special symmetries) for spacetimes which are asymptotically de Sitter in the sense of admitting a regular spacelike conformal (Penrose) compactification. Originally motivated by work of Witten and Yau pertaining to the AdS/CFT correspondence, the results obtained in [1] establish connections between the bulk spacetime (e.g., its being nonsingular) and the geometry and topology of the conformal boundary. These results extend to this more general setting the behavior seen in the FLRW models.

Here we present a result of a similar nature, which explicitly relates the occurrence of singularities in spacetime to the topology of its Cauchy surfaces. By taking advantage of advances in our understanding of the topology of 3-manifolds, specfically the positive resolution of Thurston's geometrization conjecture, and subsequent consequences of it, we are able to signficantly strengthen aspects of some of the results in [1] for $3 + 1$ dimensional spacetimes.

Theorem 4.1.1 ([14])**.** *Suppose V is a smooth compact spacelike Cauchy surface in a $3 + 1$ dimensional spacetime (M, g) that satisfies the null energy condition (NEC), $\mathrm{Ric}(X, X) \geq 0$ for all null vectors X. Suppose further that V is expanding in all directions (i.e. the second fundamental form of V is positive definite). Then either*

(i) V is a spherical space, or

(ii) M is past null geodesically incomplete.

By a *spherical space*, we mean that V is a quotient of the 3-sphere S^3, $V = S^3/\Gamma$, where Γ is isomorphic to a subgroup of $SO(4)$. Typical examples are the 3-sphere itself, lens spaces and the Poincaré dodecahedral space. By taking quotients of de Sitter space, we see that there are geodesically complete spacetimes satisfying the assumptions of the theorem, having Cauchy surface topology that of any spherical space. Nevertheless, one can view Theorem 4.1.1 as a singularity theorem: Under the assumptions of the theorem, if V is not a spherical space, i.e. if V is not a 3-sphere, or a quotient thereof, then (M, g) is past null geodesically incomplete.

The proof involves several geometrically interesting elements. In addition to recent results in 3-manifold topology, the proof makes use of a fundamental existence result for minimal surfaces due to Schoen and Yau [26], in addition to a well known existence result coming from geometric measure theory. Ultimately, the proof depends on (a slight refinement of) the Penrose singularity theorem.

4.2 Weakly trapped surfaces in asymptotically de Sitter spacetimes

A classical result in the theory of black holes asserts that trapped surfaces are, in a suitable sense, 'externally invisible'. Somewhat more precisely, for spacetimes (M, g) which are asymptotically flat (in the sense of admitting a suitably regular future null infinity $\mathcal{I}^+$) and which satisfy appropriate energy and causality conditions, no (future) trapped surface ($\theta^\pm < 0$) can be contained in $I^-(\mathcal{I}^+, \widetilde{M})$, where $\widetilde{M} = M \cup \mathcal{I}^+$. In fact, this result also extends to (future) weakly trapped surfaces ($\theta^\pm \leq 0$). The aim of this work is to prove an analogue of this result in asymptotically de Sitter space-times. In [6] we establish the following theorem.
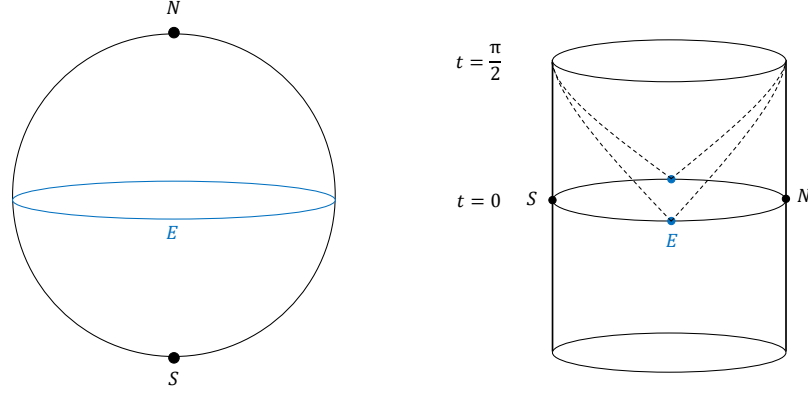
Figure 4.1: Left: An equator in the sphere S^n. Right: A spacetime diagram of de-Sitter space. Here the equator is represented by the two blue points. The causal future of E covers all of $\mathcal{I}^+ = \{t = \pi/2\}$, but the timelike future of E misses the north and south pole points on $\mathcal{I}^+$.

Theorem 4.2.1 ([6]). *Consider a future asymptotically de Sitter spacetime (M, g) which is future causally simple and satisfies the null energy condition. Let $A \subset M$ be such that $J^+(A, \widetilde{M})$ does not contain all of $\mathcal{I}^+$. Then there are no future weakly trapped surfaces contained in $J^+(A, \widetilde{M}) \cap I^-(\mathcal{I}^+, \widetilde{M})$.*

The proof is an application of the maximum principle for null hypsurfaces. Below we give some examples.

De Sitter Space. Consider de Sitter space,

$$M = (-\pi/2, \pi/2) \times S^n \,, \quad g = \cos^{-2}(t)(-dt^2 + d\omega^2) \,,$$

where $d\omega^2$ is the usual round metric on the sphere S^n. It conformally embeds into the Einstein static universe $(M', g') = (\mathbb{R} \times S^n, -dt^2 + d\omega^2)$. The future conformal completion is $\widetilde{M} = (-\pi/2, \pi/2] \times S^n$ with $\mathcal{I}^+ = \{\pi/2\} \times S^n$. Note that for any $p \in M$, $J^+(p, \widetilde{M})$ does not cover all of $\mathcal{I}^+$. Hence, by Theorem 4.2.1 there are no future weakly trapped surfaces in $J^+(p, M)$.

Now consider any t-slice $\Sigma_t = \{t\} \times S^n$. Let K be the second fundamental form of Σ_t within M. Let S be any hypersurface in Σ_t and H the mean curvature of S

within Σ_t. Then

$$\theta^\pm = \mathrm{tr}_S K \pm H \,, \tag{4.1}$$

If $0 < t < \pi/2$, then $\mathrm{tr}_S K > 0$ and so either θ^+ or θ^- is positive. Consequently there are no future weakly trapped surfaces in Σ_t for $t > 0$.[1] On the other hand, there are many future weakly trapped surfaces in Σ_0. For example take $S = E$ to be the equator. Since Σ_0 is totally geodesic within M and E is a minimal surface within Σ_0, we have $K = H = 0$. Thus, by (4.1), $\theta^\pm = 0$, and so E is future weakly (in fact, marginally) trapped. This is consistent with Theorem 4.2.1 since $J^+(E, \widetilde{M})$ contains all of $\mathfrak{I}^+$; see Figure 4.1. Let's also recognize that Theorem 4.2.1 cannot be weakened by replacing 'causal future' with 'timelike future' because $I^+(E, \widetilde{M})$ does not contain all of $\mathfrak{I}^+$, since it misses the north pole and south pole associated with the equator.

The above example shows that *any* compact embedded minimal surface in Σ_0 is a future weakly trapped surfac. But there are infinitely many such examples in the 3-sphere, of arbitrary genus. (The maximum principle implies that any such example cannot be contained in an open hemisphere; somewhat amusingly, this also follows from Theorem 4.2.1.) An interesting example in this case, where dim $M = 4$, is the Clifford torus $C \subset \Sigma_0$. Expressing S^3 as the sphere in $\mathbb{R}^4$,

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \,,$$

the Clifford torus is defined by the equations $x_1^2 + x_2^2 = 1/2 = x_3^2 + x_4^2$. The Clifford torus is a minimal surface, so one again has $\theta^\pm = 0$, and C is a weakly trapped surface in M. This is consistent with Theorem 4.2.1 since the causal future of C covers all of $\mathfrak{I}^+$. In fact, the causal future of C already covers the time slice $\{\frac{\pi}{4}\} \times S^3$; see Figure 4.2.

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[1] In fact, it can be shown that there are no future weakly trapped surfaces in the spacetime region $t > 0$.
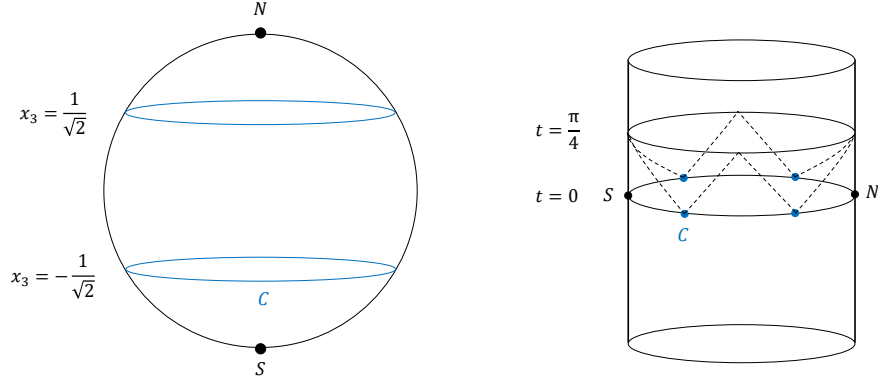
Figure 4.2: Left: A slice of the Clifford torus in the $x_4 = 0$ plane. Right: A spacetime diagram of de-Sitter space with $x_2 = x_4 = 0$. The Clifford torus is represented by the four blue points located at $(x_1, x_3) = (\pm 1/\sqrt{2}, \pm 1/\sqrt{2})$. The causal future of C already covers the time slice $\{\frac{\pi}{4}\} \times S^3$.

Schwarzschild-de Sitter space. Schwarzschild-de Sitter space is the spacetime $M = \mathbb{R} \times \mathbb{R} \times S^2$ with metric in static form,

$$g = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1}dr^2 + d\omega^2, \qquad (4.2)$$

with $m > 0$, $\Lambda > 0$, and $9\Lambda m^2 < 1$. The Penrose diagram for (M, g) is given in Figure 4.3. $\mathcal{I}^+$ has topology $\mathbb{R} \times S^2$. (Here we consider a conformal compactification consisting of a single component of future null infinity.) The g_{tt}-component of (4.2) has positive roots $r_1 < r_2$, corresponding to a black hole event horizon and a cosmological horizon, respectively. Regular Kruskal-Szekeres type coordinates can be defined near $r = r_1$ and $r = r_2$.
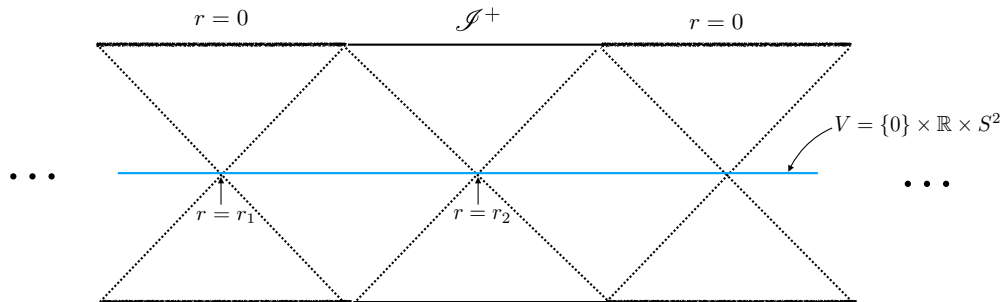
Figure 4.3: Schwarzschild-de Sitter Space.

Consider the totally geodesic time slice $V = \{0\} \times \mathbb{R} \times S^2$. Let A be the subset of V consisting of the union of all rotationally symmetric 2-spheres S_r, for $r_1 < r \leq r_0$. Provided $r_0 < r_2$, Theorem 4.2.1 implies that there are no future weakly trapped surfaces in $J^+(A, \widetilde{M}) \cap I^-(\mathscr{I}^+, \widetilde{M})$. In particular, we note that the 2-spheres S_r have positive null expansion with respect to the null normal pointing to the right. However, when we allow $r_0 = r_2$, then $J^+(A, \widetilde{M}) \cap I^-(\mathscr{I}^+, \widetilde{M})$ contains the 2-sphere at $r = r_2$, which is future weakly trapped, since it is minimal. Again this is consistent with Theorem 4.2.1, since now $J^+(A)$ contains all of $\mathscr{I}^+$.

4.3 Existence of CMC Cauchy surfaces from a space-time curvature condition

Constant mean curvature (CMC) spacelike hypersurfaces have played an important role in mathematical general relativity. In particular, as is well-known, the problem of finding solutions to the Einstein constraint equations is made much simpler by assuming CMC data. There are also many known advantages for solving the Einstein evolution equations if one works in CMC gauge. In the recent paper [10], Dilts and Holst review the issue of the existence of CMC slices in globally hyperbolic spacetimes with compact Cauchy surfaces. As discussed in [10], most such existence results ultimately rely on barrier methods. However, a well-known example of Bartnik

[2] shows that not all cosmological spacetimes have CMC Cauchy surfaces. Vacuum examples were later obtained by Chruściel, Isenberg and Pollack [8] using gluing methods. These examples share certain properties. By examining various features of Bartnik's example, Dilts and Holst formulate several conjectures concerning the existence of CMC Cauchy surfaces. We do not settle any of these conjectures here. Nevertheless, motivated by some of their considerations, we have obtained a new CMC existence result which relies on a certain spacetime curvature condition.

Theorem 4.3.1 ([13]). *Let (M, g) be a spacetime with compact Cauchy surfaces. Suppose (M, g) is future timelike geodesically complete and has everywhere nonpositive timelike sectional curvatures, i.e. $K \leq 0$ everywhere. Then (M, g) contains a CMC Cauchy surface.*

Some remarks about the curvature assumption are in order. Recall, for any timelike 2-plane, $T \subset T_pM$, the timelike sectional curvature $K(T)$ is given by

$$K(T) = -g\big(R(u, e)e, u\big) = -\langle R(u, e)e, u \rangle, \qquad (4.3)$$

where $\{u, e\}$ is any basis for T with $g(u, u) = -1$ and $g(e, e) = 1$ and R is the Riemann curvature tensor. In particular, $K(T)$ is independent of the orthonormal basis chosen. (Our sign convention for R is that of [3] and opposite that of [22].) Standard analysis of the Jacobi equation shows that $K \leq 0$ physically corresponds to attractive tidal forces; i.e. it describes gravitational attraction in the strongest sense.

The Ricci tensor evaluated on a unit timelike vector $u \in T_pM$ can be expressed as minus the sum of timelike sectional curvatures. Specifically, let $\{u, e_1, \ldots, e_n\}$ be an orthonormal basis for T_pM with $g(u, u) = -1$. Let $T_i \subset T_pM$ be the timelike plane spanned by $\{u, e_i\}$. Then

$$\mathrm{Ric}(u, u) = \sum_{i=1}^{n} \langle R(u, e_i)e_i, u \rangle = -\sum_{i=1}^{n} K(T_i). \qquad (4.4)$$

In particular the assumption of nonpositive timelike sectional curvatures implies the strong energy condition, $\text{Ric}(U, U) \geq 0$ for all timelike vectors U. As shown in [13], for FLRW spacetimes, the assumption of nonpositive timelike sectional curvatures is equivalent to the strong energy condition. In particular, sufficiently small perturbations of FLRW spacetimes which obey the strong energy condition *strictly* will have negative timelike sectional curvatures.

Since the assumption of nonpositive timelike sectional curvatures implies the strong energy condition, one is naturally led to formulate the following conjecture.

Conjecture. *Let (M, g) be a spacetime with compact Cauchy surfaces. If (M, g) is future timelike geodesically complete and satisfies the strong energy condition, i.e. $\text{Ric}(U, U) \geq 0$ for all timelike U, then (M, g) contains a CMC Cauchy surface.*

The conjecture, if correct, is not likely to be easy to prove. In particular, it would settle the Bartnik splitting conjecture [2, Conjecture 2] in the affirmative; see [2, Corollary 1, p. 621]. The conjecture above is, in a certain sense, complimentary to Conjecture 3.5 in [10]. In this context, it would be interesting to resolve the issue of the timelike completeness/incompleteness of the examples constructed in [8, Section 5.1].

Bibliography

[1] L. Andersson and G. J. Galloway. "dS/CFT and spacetime topology". In: *Adv. Theor. Math. Phys.* 6.2 (2002), pp. 307–327. ISSN: 1095-0761.

[2] R. Bartnik. "Remarks on cosmological spacetimes and constant mean curvature surfaces". In: *Comm. Math. Phys.* 117.4 (1988), pp. 615–624. ISSN: 0010-3616.

[3] J. K. Beem, P. E. Ehrlich, and K. L. Easley. *Global Lorentzian geometry.* Second. Vol. 202. Monographs and Textbooks in Pure and Applied Mathematics. New York: Marcel Dekker Inc., 1996, pp. xiv+635. ISBN: 0-8247-9324-2.

[4] P. T. Chruściel. *Elements of causality theory.* arXiv:1110.6706. 2011.

[5] P. T. Chruściel and J. D. E. Grant. "On Lorentzian causality with continuous metrics". In: *Classical Quantum Gravity* 29.14 (2012), pp. 145001, 32. ISSN: 0264-9381.

[6] Piotr T. Chruściel, Gregory J. Galloway, and Eric Ling. "Weakly trapped surfaces in asymptotically de Sitter spacetimes". In: *Classical Quantum Gravity* 35.13 (2018), pp. 135001, 9. ISSN: 0264-9381.

[7] Piotr T. Chruściel and James D. E. Grant. "On Lorentzian causality with continuous metrics". In: *Classical Quantum Gravity* 29.14 (2012), pp. 145001, 32. ISSN: 0264-9381.

[8] Piotr T. Chruściel, James Isenberg, and Daniel Pollack. "Initial Data Engineering". In: *Communications in Mathematical Physics* 257.1 (2005), pp. 29–42.

[9] M. Dafermos and J. Luk. "The interior of dynamical black holes I: The C^0-stability of the Kerr Cauchy horizon". In: (2017). arXiv:1710.01722.

[10] James Dilts and Michael Holst. "When do spacetimes have constant mean curvature slices?" In: *arXiv:1710.03209* (2017).

[11] G. J. Galloway. "Cosmological spacetimes with $\Lambda > 0$". In: *Advances in differential geometry and general relativity*. Vol. 359. Contemp. Math. Amer. Math. Soc., Providence, RI, 2004, pp. 87–101.

[12] G. J. Galloway and E. Ling. "Some Remarks on the C^0-(In)Extendibility of Spacetimes". In: *Annales Henri Poincaré* (2017). ISSN: 1424-0661.

[13] Gregory J. Galloway and Eric Ling. "Existence of CMC Cauchy surfaces from a spacetime curvature condition". In: *Gen. Relativity Gravitation* 50.9 (2018), Art. 108, 7. ISSN: 0001-7701.

[14] Gregory J. Galloway and Eric Ling. "Topology and singularities in cosmological spacetimes obeying the null energy condition". In: *Comm. Math. Phys.* 360.2 (2018), pp. 611–617. ISSN: 0010-3616.

[15] Gregory J. Galloway, Eric Ling, and Jan Sbierski. "Timelike completeness as an obstruction to C^0-extensions". In: *Comm. Math. Phys.* 359.3 (2018), pp. 937–949. ISSN: 0010-3616.

[16] Melanie Graf and Eric Ling. "Maximizers in Lipschitz spacetimes are either timelike or null". In: *Classical Quantum Gravity* 35.8 (2018), pp. 087001, 6. ISSN: 0264-9381.

[17] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time*. Cambridge Monographs on Mathematical Physics, No. 1. London: Cambridge University Press, 1973, pp. xi+391.

[18] Morris W. Hirsch. *Differential topology*. Vol. 33. Graduate Texts in Mathematics. Corrected reprint of the 1976 original. Springer-Verlag, New York, 1994, pp. x+222. ISBN: 0-387-90148-5.

[19] John M. Lee. *Introduction to smooth manifolds*. Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[20] E. Minguzzi. "Limit curve theorems in Lorentzian geometry". In: *J. Math. Phys.* 49.9 (2008), pp. 092501, 18. ISSN: 0022-2488.

[21] C. W. Misner, K. S. Thorne, and J. A. Wheeler. *Gravitation*. W. H. Freeman and Co., San Francisco, Calif., 1973, ii+xxvi+1279+iipp.

[22] B. O'Neill. *Semi-Riemannian geometry*. Vol. 103. Pure and Applied Mathematics. New York: Academic Press Inc. [Harcourt Brace Jovanovich Publishers], 1983, pp. xiii+468. ISBN: 0-12-526740-1.

[23]   R. Penrose. "Gravitational collapse: the role of general relativity". In: *Gen. Relativity Gravitation* 34.7 (2002). Reprinted from Rivista del Nuovo Cimento **1**969, Numero Speziale I, 252–276, pp. 1141–1165. ISSN: 0001-7701.

[24]   Clemens Sämann. "Global hyperbolicity for spacetimes with continuous metrics". In: *Ann. Henri Poincaré* 17.6 (2016), pp. 1429–1455. ISSN: 1424-0637.

[25]   Fer Sbierski. "The C^0-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry". In: *J. Differential Geom.* 108.2 (2018), pp. 319–378. ISSN: 0022-040X.

[26]   R. Schoen and S.-T. Yau. "Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature". In: *Ann. of Math. (2)* 110.1 (1979), pp. 127–142. ISSN: 0003-486X.