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The Topology of k–Equal Partial Decomposition Lattices

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The lattice B_n of subsets of the set $\{1, 2, ..., n\}$ ordered by inclusion and the lattice Π_n of partitions of $\{1, 2, ..., n\}$ ordered by refinement are two of the most fundamental examples in the theory of partially ordered sets (posets). A natural well-studied q–analogue of the subset lattice is the lattice $B_n(q)$ of subspaces of the n–dimensional vector space $\mathbb{F}_q^n$ over the field $\mathbb{F}_q$ with q elements ordered by inclusion. There are many justifications for viewing this as a q–analogue. One comes from the fact that the number of maximal chains of B_n is $n!$, while the number of maximal chains of $B_n(q)$ equals the q–analogue of $n!$ which is defined by

$$[n]_q! := [n]_q[n-1]_q \dots [1]_q,$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$. Another justification comes from studying the topology of a certain simplicial complex associated with the poset, called the order complex. The order complex associated with B_n is homeomorphic to a single sphere of dimension $n-2$, while the order complex associated with $B_n(q)$

has the homotopy type of a wedge of $q^{\binom{n}{2}}$ spheres of dimension $n-2$.

It is well-known that the order complex associated with Π_n has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$. Various q–analogues of the partition lattice Π_n have been proposed over the years, starting with the Dowling lattices introduced in a 1973 paper of Dowling. Posets studied by Welker and by Hanlon, Hersh, and Shareshian involve direct sum decompositions of vector spaces over $\mathbb{F}_q$. While these posets have interesting properties analogous to those of Π_n, such as having the homotopy type of a wedge of spheres, none have the desirable property that the number of spheres is a q–analogue of $(n-1)!$. The q–analogue proposed in this thesis is the poset $\Pi_n(q)$ of direct sum decompositions of subspaces of $\mathbb{F}_q^n$ whose summands all have dimension at least 2, ordered by inclusion of summands. This is actually a q–analogue of a poset that is isomorphic to Π_n, namely the poset of partitions of subsets of $\{1, 2, ..., n\}$ in which each block has size at least 2. We show that the order complex associated with $\Pi_n(q)$ has the homotopy type of a wedge of $f(q)[n-1]_q!$ spheres of dimension $n-3$ where $f(q)$ is a polynomial in q that is equal to 1 when q is set equal to 1.

In order to prove this result, we initiate a study of a much more general class of posets, which includes Π_n, $\Pi_n(q)$, and the k–equal partition lattices introduced by Björner, Lovász, and Yao in 1992. The k–equal partition lattice $\Pi_n^{=k}$ is the subposet of Π_n consisting of partitions for which each block has size at least k or 1. In this general class, the roles of B_n and $B_n(q)$ in the definitions of Π_n and $\Pi_n(q)$ are played by an arbitrary geometric lattice L. We use shellability theory to prove that the order complex associated with a general k–equal decomposition lattice $\Pi_L^{=k}$ has the homotopy type of a wedge of spheres in varying dimensions when $k > 2$ and just in dimension $n-3$ when $k = 2$. Shellability theory also enables us to derive a complicated formula for the number of spheres in each dimension. The nontrivial step of reducing the complicated formula in the case of $\Pi_{B_n(q)}^{=2} = \Pi_n(q)$ to the desired $f_n(q)[n-1]_q!$ formula uses Stanley's theory of exponential structures.

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Chapter 1

Introduction

From a combinatorics perspective, the application of topology to the theory of partially ordered sets greatly enriched the field. Beginning with the Möbius function and its connection to the reduced Euler characteristic of the order complex of a poset, poset topology has become an important tool itself, with applications to subspace arrangements and complexity theory, among other things. For further information on topology of posets, see [21].

Given any finite poset P, define a *chain* as any totally ordered subset of P, and define the *order complex* $\Delta(P)$ of P as the abstract simplicial complex whose faces are the chains of P. Using simplicial homology over the ring of integers $\mathbb{Z}$, the reduced homology associated to the order complex $\Delta(P)$ is then denoted by $\widetilde{H}_j(P)$, and the reduced cohomology is then denoted by $\widetilde{H^j}(P)$, so that by slight abuse of terminology, we can discuss (co)homology of posets. The rank of the reduced homology group $\widetilde{H}_i(P)$ is called a *reduced Betti number* of P, and is denoted by $\widetilde{\beta}_i(P)$.

Certain posets have additional structure which we make use of to analyze topological results of their order complex combinatorially. Let P be a poset; a *lower bound* of two elements $x, y \in P$ is an element $z \in P$ such that $z \leq x$ and $z \leq y$. Upper bounds are defined similarly. A *lattice* is defined as a poset such that every pair of elements has both a greatest lower bound and a least upper bound. For pairs of elements x and y, if it exists then the greatest lower bound is denoted $x \wedge y$, and if it exists, the least upper bound is denoted $x \vee y$.

A poset P is *ranked* or *pure* if for every element x, every maximal chain whose largest element is x has the same length, denoted $\rho(x)$, and called the *rank* of x. If P has unique elements $\hat{0}$ and $\hat{1}$ such that for all $x \in P$, $\hat{0} \leq x \leq \hat{1}$, then P is called *bounded*. Let P be bounded; define the *proper part* of P by $\overline{P} := P - \{\hat{0}, \hat{1}\}$.

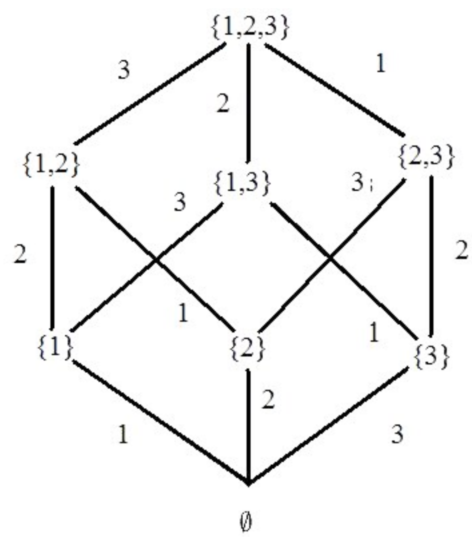
For arbitrary posets, a *Hasse diagram* is a directed graph whose vertices are elements of the poset and whose edges are the covering relations of the partial order. Let $\mathcal{E}(P)$ denote the set of coverings of P, and define an *edge labeling* as any function $\Psi : \mathcal{E}(P) \to \mathbb{Z}$. If P is bounded and admits an edge labeling that satisfies the conditions of Definition 2.2.3 below, then P is called *EL-shellable*, and the edge labeling is called an *EL-labeling*. A maximal chain

$$\hat{0} = c_0 < c_1 < \cdots < c_{m-1} < c_m = \hat{1} \subseteq P$$

is said to be *falling* if the labels of the chain weakly decrease as the chain is read in increasing order. The following theorem of Björner and Wachs connects these labelings to the topology of posets.

Theorem (see Theorem 2.2.5). *Let P be a finite EL-shellable poset under edge labeling Ψ. Then $\Delta(\overline{P})$ has the homotopy type of a wedge of spheres. Furthermore, for each i the number of spheres of dimension $i-2$, and therefore rank$(\widetilde{H_{i-2}}(\overline{P}))$, is equal to the number of falling chains of P (under Ψ) of length i.*

The *Boolean algebra* B_n is defined to be the poset consisting of all subsets of $\{1, 2, ..., n\}$ ordered by containment. Labeling each edge by the unique element in the larger subset which is not in the smaller subset makes B_n EL-shellable. There is a unique falling chain; its length is n and its label sequence is $n, n-1, ..., 1$. Therefore, $\Delta(\overline{B_n})$ has homotopy type of a single sphere of dimension $n-2$. The edge labeling of B_3 is given in the following diagram:



The Boolean algebra is a fundamental example of a well-studied type of lattice called a *geometric lattice*; Björner [2] showed all geometric lattices are EL-shellable.

There is a special class of geometric lattices called the *partition lattices*, denoted by Π_n. The elements of Π_n are the set partitions of $\{1, 2, ..., n\}$, ordered by refinement. The edges of Π_n come about by merging exactly two blocks of the finer partition. An edge labeling which assigns to each edge the largest element of the two merging blocks makes Π_n EL-shellable [22]. It can be shown that there are $(n-1)!$ falling chains of this EL-labeling of Π_n. It follows that $\Delta(\overline{\Pi_n})$ has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$.

The k–equal partition lattice $\Pi_n^{=k}$ is the induced subposet of Π_n consisting of partitions that contain no non-singleton blocks of size less than integer k. Note that when $k=2$, there are no restrictions on blocks, so $\Pi_n^{=2}=\Pi_n$. The lattice $\Pi_n^{=k}$ originated in the work of Björner, Lovász, and Yao [4] on the k–equal problem in complexity theory, where a lower bound on the complexity of the so called k–equal problem in computer science is given in terms of the Betti numbers of $\Pi_n^{=k}$. The order complex $\Delta(\overline{\Pi_n^{=k}})$ was then shown by Björner and Welker [9] to have the homotopy type of a wedge of spheres of varying dimensions. Since $\Pi_n^{=k}$ is not pure in general, Björner and Wachs [7] then extended the notion of shellability from pure to nonpure simplicial complexes, and proved the following result.

Theorem (see Proposition 2.5.3). *The lattice $\Pi_n^{=k}$ is EL-shellable for all integers $2\le k\le n$. Moreover, the order complex $\Delta(\overline{\Pi_n^{=k}})$ has the homotopy type of a wedge of spheres, where the number of spheres $\widetilde{\beta}_d(\overline{\Pi_n^{=k}})$ of dimension d is 0 unless $d=n-3-t(k-2)$ for some positive integer t. If $k=2$, then $d=n-3$ and*

$$\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}})=\sum_{t=1}^{\lfloor\frac{n}{2}\rfloor}\sum_{\substack{j_1+\cdots+j_t=n,\\ j_i\ge 2\ \forall\ i}}\binom{n-1}{j_1-1,j_2,\ldots,j_t}\prod_{i=1}^{t}(j_i-1).$$

If $k>2$, then for each possible d,

$$\widetilde{\beta}_d(\overline{\Pi_n^{=k}})=\sum_{\substack{j_1+\cdots+j_t=n,\\ j_i\ge k\ \forall\ i}}\binom{n-1}{j_1-1,j_2,\ldots,j_t}\prod_{i=1}^{t}\binom{j_i-1}{k-1}.$$

For q a prime power and n a nonnegative integer, let $B_n(q)$ be the lattice of subspaces of $\mathbb{F}_q^n$, the n–dimensional vector space over the field of order q, ordered by containment. This subspace lattice is considered to be a q–analogue of the Boolean algebra for several reasons. Define $[n]_q = \sum_{i=1}^{n} q^{i-1}$ for positive integer n, and further, define $[0]_q! = 1$, and $\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}$. Note that when $q = 1$, $[n]_q = n$, and so $[n]_q! = n!$, and $\begin{bmatrix} n \\ r \end{bmatrix}_q = \binom{n}{r}$.

One reason we consider $B_n(q)$ to be a q–analogue of B_n is that the number of elements of rank r of $B_n(q)$ (i.e., the number of r–dimensional subspaces) is $\begin{bmatrix} n \\ r \end{bmatrix}_q$, while for the Boolean algebra B_n, the number of elements of rank r is $\binom{n}{r}$. Another reason is that the number of maximal chains of $B_n(q)$ is $[n]_q!$, while the number of maximal chains of B_n is $n!$; there are many results like this for this pair of lattices. The lattice $B_n(q)$ is another example of a geometric lattice.

There have been several q–analogues of the partition lattice introduced in the literature; the Dowling lattices [11], and the f-q–order analogues of Bennett, Dempsey and Sagan [1] and Simion [15] are various examples. Other examples build on the idea of subspace lattices, including the lattice of direct sum decompositions of $\mathbb{F}_q^n$, analyzed by Welker [23], who proved that the proper part of this lattice has the homotopy type of a wedge of $\frac{1}{n}\prod_{i=1}^{n-1}(q^i - 1)f_n(q)$ spheres of dimension $n-2$, where $f_n(q)$ is a polynomial with integer coefficients.

Hanlon, Hersh, and Shareshian [13] then introduced the lattice of partial direct sum decompositions. A *partial direct sum decomposition* is a set of subspaces $\{U_1, ..., U_t\}$ of $\mathbb{F}_q^n$ whose sum is direct. These partial decompositions are ordered by $\{U_1, ..., U_t\} \leq \{W_1, ..., W_s\}$ if each summand U_i is a subspace of some summand W_j. Using discrete Morse theory, they find that the proper part of this lattice has homotopy type of a wedge of $\frac{1}{n}q^{\binom{n}{2}}\prod_{i=1}^{n-1}(q^i - 1)$ spheres of dimension $2n-3$.

Both of these examples present analogues which do not have the property that the nonvanishing Betti number of its proper part at $q = 1$ yields $(n-1)!$, which is the nonvanishing Betti number of $\overline{\Pi_n}$. Thus, one can say that they are not true q–analogues of Π_n in the purely topological sense, nor were they intended to be. The lattice of direct sum decompositions was introduced by Stanley [18] for the purpose of obtaining analogues of the compositional formula (see Section 7.1). The lattice of partial direct sum decompositions was introduced by Hanlon, Hersh, and Shareshian in order to obtain a $GL_n(\mathbb{F}_q)$–analogue of the action of the symmetric group on the homology of the proper part of the partition lattice.

We introduce then a new q–analogue $\Pi_n(q)$ of the partition lattice. For every positive integer $n > 1$ and prime power q, let $\Pi_n(q)$ denote the lattice of partial direct sum decompositions of $\mathbb{F}_q^n$ in which no summand has dimension 1. Unlike Π_n, it turns out that $\Pi_n(q)$ is not a geometric lattice. However, we show in Theorem 7.2.1 that $\Delta(\overline{\Pi_n(q)})$ has the homotopy type of a wedge of spheres of dimension $n-3$, where the number of spheres is a nice q–analogue of $(n-1)!$.

In order to prove this result, we first construct a q–analogue of $\Pi_n^{=k}$ for general $k \geq 2$. Define $\Pi_n^{=k}(q)$ as the lattice of partial direct sum decompositions of $\mathbb{F}_q^n$ in which no summand has dimension less than k. This lattice is really a q–analogue of the lattice of partial partitions of $\{1, ..., n\}$ with block sizes at least k, which is isomorphic to the k–equal partition lattice $\Pi_n^{=k}$. Note that $\Pi_n^{=2}(q) = \Pi_n(q)$.

Theorem (see Corollary 6.2.13). *The lattice* $\Pi_n^{=k}(q)$ *is EL-shellable for all integers* $2 \leq k \leq n$. *Moreover, the order complex* $\Delta(\overline{\Pi_n^{=k}(q)})$ *has the homotopy type of a wedge of spheres, where the number of spheres* $\widetilde{\beta}_d(\overline{\Pi_n^{=k}(q)})$ *of dimension* d *is 0 unless* $d = n - 3 - t(k-2)$ *for some positive integer* t. *If* $k = 2$, *then*

$d = n-3$ and

$$\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)}) = \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \geq 2 \ \forall \ i}} (t-1)! \left(\prod_{i=1}^{t} q^{\binom{\lambda_i - 1}{2}} [\lambda_i - 1]_q \right) \frac{q^{e_2(\lambda)} [n]_q!}{\prod_{i=1}^{t} [\lambda_i]_q!} \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!},$$

where $e_2(\lambda) = \sum_{1 \leq i < j \leq t} \lambda_i \lambda_j$ and $a_j(\lambda)$ is the number of parts of λ of size j. If $k > 2$, then for each possible d,

$$\widetilde{\beta}_d(\overline{\Pi_n^{=k}(q)}) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \geq k \ \forall \ i}} (t-1)! \left(\prod_{i=1}^{t} q^{\binom{\lambda_i - k + 1}{2}} \begin{bmatrix} \lambda_i - 1 \\ k-1 \end{bmatrix}_q \right) \frac{q^{e_2(\lambda)} [n]_q!}{\prod_{i=1}^{t} [\lambda_i]_q!} \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}.$$

We show that the complicated formula above reduces nicely when $k = 2$. We use the theory of *exponential structures* developed by Stanley [18] to obtain the following result.

Theorem (see Theorem 7.2.1). *The lattice $\Pi_n(q)$ is EL-shellable. The order complex $\Delta(\overline{\Pi_n(q)})$ has the homotopy type of a wedge of $\widetilde{\beta}_{n-3}(\overline{\Pi_n(q)})$ spheres of dimension $n-3$, where*

$$\widetilde{\beta}_{n-3}(\overline{\Pi_n(q)}) = \widetilde{g}_n(q) q^{\binom{n-1}{2}} [n-1]_q! \quad \text{and} \quad \widetilde{g}_n(q) = \frac{1}{nq} \left([n]_q - (1-q)^{n-1} \right).$$

Note that $\widetilde{g}_n(q) q^{\binom{n-1}{2}} = 1$ when $q = 1$, making $\Pi_n(q)$ an appealing q–analogue of Π_n. To get these results, we examine the even more general situation of decomposing a geometric lattice. Let L be a finite geometric lattice. Define an *independent set* of L to be any subset $T = \{x_1, ..., x_t\} \subseteq L - \{\hat{0}\}$ such that $\rho(x_1 \vee \cdots \vee x_t) = \sum_{i=1}^{t} \rho(x_i)$. This notion of set independence generalizes the notions of direct sum decompositions of finite vector spaces and set partitions of finite sets simultaneously.

For each positive integer k, let $\Pi_L^{=k}$ be the poset of independent subsets of L consisting of elements all of rank at least k. The order relation on independent subsets is given by $\{x_1, ..., x_t\} \leq \{y_1, ..., y_s\}$ if each x_i is less than some y_j in L. Note that when $L = B_n$, we have $\Pi_{B_n}^{=k} = \Pi_n^{=k}$, and when $L = B_n(q)$, we have $\Pi_{B_n(q)}^{=k} = \Pi_n^{=k}(q)$. We show that the poset $\Pi_L^{=k}$ is a lattice; we call $\Pi_L^{=k}$ the *k-equal partial decomposition lattice* of L. We also find an EL-labeling of $\Pi_L^{=k}$ and count its falling chains, giving us the following general theorem. Let $L_x = \{y \in L \mid y \leq x\}$ and $(L_x)_k = \{y \in L_x \mid \rho(y) \geq k\}$. For an independent set T, let $r(T) = \sum_{x \in T} \rho(x)$.

Theorem (see Theorem 6.2.8). *The lattice $\Pi_L^{=k}$ is EL-shellable for all integers $2 \leq k \leq n$. Moreover, the order complex $\Delta(\overline{\Pi_L^{=k}})$ has the homotopy type of a wedge of spheres, where the number of spheres $\widetilde{\beta}_d(\overline{\Pi_L^{=k}})$ of dimension d is 0 unless $d = n-3-t(k-2)$ for some positive integer t. If $k = 2$, then $d = n-3$ and*

$$\widetilde{\beta}_{n-3}(\overline{\Pi_L^{=2}}) = \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{T \in \Pi_L^{=2} \\ r(T)=n \\ |T|=t}} (t-1)! \prod_{x \in T} \widetilde{\beta}_{\rho(x)-3}((L_x)_2).$$

If $k > 2$, then for each possible d,

$$\widetilde{\beta}_d(\overline{\Pi_L^{=k}}) = \sum_{\substack{T \in \Pi_L^{=k} \\ r(T)=n \\ |T|=t}} (t-1)! \prod_{x \in T} \widetilde{\beta}_{\rho(x)-k-1}((L_x)_k).$$

In Chapter 2, we present preliminary information, including definitions, examples, and previous theorems, though we prove only a small selection of these theorems which turn out to be special cases of more general results presented subsequently. Chapter 3 discusses the notion of q–analogues, and presents several well-known results about q–analogues and rank selection. Chapter 4 also presents a review of several previously proven theorems and lemmas concerning geometric lattices, but these results are more fundamental to later work, and so are given in more detail. Our new results begin in Chapter 5.

In Chapter 5, we introduce the concept of independent sets, and examine the consequences of independence pertaining to $\Pi_L^{=k}$. Moreover in Chapter 6 we construct an explicit EL-labeling of the k–equal partial decomposition lattice, and exhibit a formula for determining the homotopy type of $\Delta(\overline{\Pi_L^{=k}})$, as well as the specialization of this result which applies to both $L = B_n$ and $L = B_n(q)$. Chapter 7 details results concerning the lattice $\Pi_n(q)$, which is shown to be a q–analogue of the partition lattice Π_n. Specifically, we use exponential structures to simplify the computation of the number of spheres, as well as extending other well-known results for Π_n to $\Pi_n(q)$.

Chapter 2

Background

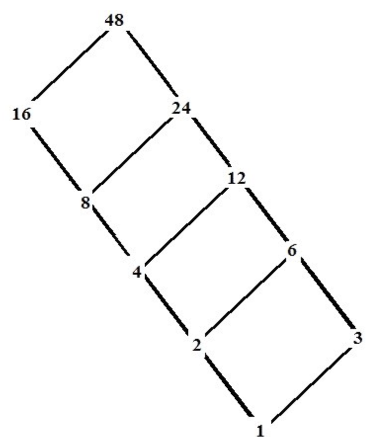
2.1 Basic Definitions

Let P be a set. A *partial ordering* $\leq_P$ is a binary relation on the elements of P satisfying the following axioms for all elements $a, b, c \in P$:

- If $a \leq_P b$ and $b \leq_P c$, then $a \leq_P c$.
- $a \leq_P a$.
- If $a \leq_P b$ and $b \leq_P a$, then $a = b$.

We call the set P a *partially ordered set*, or *poset* for short. When the poset P and the partial order $\leq_P$ are understood, we will write simply $\leq$. Further, if $x \leq_P y$ and $x \neq y$, we frequently write simply $x <_P y$. Hereafter, when we discuss posets, we shall always mean posets with finitely many elements. Given two posets P and Q, if there exists a bijection $\phi : P \to Q$ such that for all pairs $x, y \in P$, $x \leq_P y$ if and only if $\phi(x) \leq_Q \phi(y)$, then P and Q are called *isomorphic*.

Let P be a poset, and let $x, y \in P$ be such that $x < y$. If $x \leq z \leq y$ implies that $x = z$ or $y = z$ for all $z \in P$, then we say that y *covers* x, denoted by $x \lessdot y$. For any poset P, define the *Hasse diagram* of P to be the directed graph whose vertices are elements of P and whose edges are covering relations; Hasse diagrams are standardly drawn so that the smaller elements are below the larger elements. For this reason, the set of covering relations of P is also known as the *edge set* $\mathcal{E}(P)$. An example of the Hasse diagram of the positive divisors of 48 ordered by divisibility is given below:



For any poset P, we define a pair of elements $a, b \in P$ to be *comparable* if either $a \leq b$ or $b \leq a$; a pair of elements failing this condition is called *incomparable*. For example, in the Hasse diagram above, 2 and 6 are comparable, but 4 and 6 are incomparable. We then define a *totally ordered* set P as any poset such that for all pairs $a, b \in P$, then necessarily a and b are comparable.

Given $C \subseteq P$, the set C is called a *chain* if it is totally ordered; a chain C is *maximal* if for any element $a \in P - C$, then the set $\{a\} \cup C$ is not a chain. The *length* of C is one less than its cardinality. The poset P is *pure* (or also *ranked* or *graded*) if all of its maximal chains have the same length. We define the integer $\ell(P)$ as the length of the longest chain of P, and call this the length of P. If P is pure, we define a *rank function* $\rho : P \to \mathbb{N}$ by, for each $x \in P$, setting

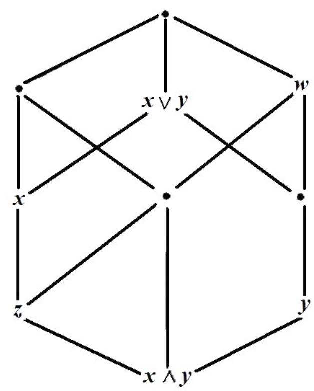
$\rho(x)$ equal to the length of the longest chain whose largest element is x.

In the poset of divisors of 48 illustrated above, the subset $\{2, 6, 24\}$ is a chain of length 2, as it is totally ordered. This chain is not maximal, however, since the inclusion of the number 12 leaves the new set totally ordered. On the other hand, the subset $\{1, 3, 6, 12, 24, 48\}$ is a maximal chain, since no other divisor of 48 is also divisible by 3, and thus would be incomparable to the prime number 3. The poset is pure since each maximal chain has length 5.

If there exists an element $\hat{0} \in P$ such that $\hat{0} \leq x$ for all $x \in P$, then $\hat{0}$ is unique and is called the minimum element of P, or the *bottom* element. Similarly, if there exists $\hat{1} \in P$ such that $x \leq \hat{1}$ for all $x \in P$, then $\hat{1}$ is also unique and is called the maximum or *top* element. If P has both a top and a bottom, we say that it is *bounded*. For example, the divisors of 48 are bounded by 1 and 48. If P is bounded, define its *proper part* $\overline{P}$ by $\overline{P} = P - \{\hat{0}, \hat{1}\}$; that is, the proper part of P is obtained by removing the top and bottom of P. We can also artificially adjoin to P a top and bottom; that is, we define $\widehat{P} := P \cup \{\hat{0}, \hat{1}\}$, where these elements satisfy the definition of top and bottom.

A *subposet* of P is a set $T \subseteq P$ with a partial order $\leq_T$ such that if $a \leq_T b$, then $a \leq_P b$. The subposet T is called *induced* if for all pairs $a, b \in T$, we have $a \leq_P b$ if and only if $a \leq_T b$. For example, chains are induced subposets. Given $x, y \in P$, define the *open interval* (x, y) as the induced subposet on the set $\{z \in P \mid x < z < y\}$, and the *closed interval* $[x, y]$ as the induced subposet on the set $\{z \in P \mid x \leq z \leq y\}$. Note that if $x \lessdot y$, then $(x, y) = \emptyset$. If P is bounded, a *lower interval* is a closed interval of the form $[\hat{0}, x]$, and an *upper interval* has the form $[x, \hat{1}]$, where $x \in P$.

Given $x, y \in P$, define a *lower bound* as an element $a \in P$ such that $a \leq x$ and $a \leq y$; an *upper bound* is an element $b \in P$ such that $x \leq b$ and $y \leq b$. If the set of lower bounds of x and y has a top element then we call this element the *meet* of x and y, denoted by $x \wedge y$. Similarly, a bottom element of the set of upper bounds is called the *join* of x and y, denoted $x \vee y$. We illustrate this with a diagram:



In this poset, the elements x and y as well as their meet and join are all labeled. On the other hand, the pair z and y do not have a join, since both $x \vee y$ and the element w are incomparable upper bounds of z and y, with no upper bounds less than either of these, so that the set of upper bounds has no minimum.

If every pair of elements of a poset has both a meet and a join, then the poset is called a *lattice*. If L is a lattice, we can extend the definition of meets and joins to collections of more than two elements; for example, the meet of the set $\{x_1, ..., x_k\} \subseteq L$ is defined as the unique maximum element of the set $\{z \in L \mid z \leq x_i \; \forall \, i\}$. In particular, meets and joins in lattices are associative, commutative, and idempotent binary operations (see [16]). Furthermore, all lattices are bounded.

The set of divisors of 48 forms a lattice, where the meet of two integers is their greatest common divisor, and their join is their least common multiple. More generally, if every pair of elements of a poset has a join, then the poset is called a *join semilattice*; similarly, if every pair of elements has a meet, then the poset is called a *meet semilattice*. Therefore, a lattice is both a join semilattice and a meet semilattice. Further, if a join semilattice has a bottom, then it is a lattice; similarly, if a meet semilattice has a top, then it is a lattice (see [16]). Suppose P is bounded, and define its *atoms* as those elements that cover $\hat{0}$.

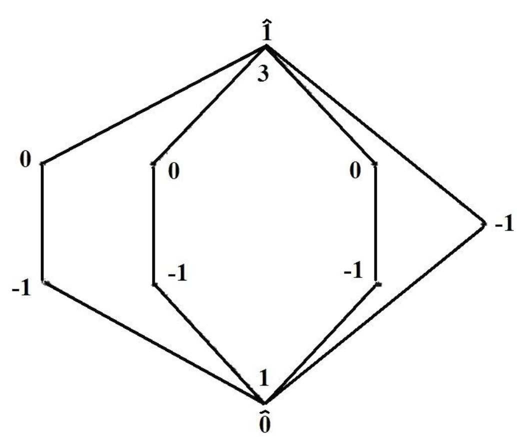
Given poset P, let $I_C(P)$ be the set of closed intervals of P. Define the *Möbius function* $\mu : I_C(P) \to \mathbb{Z}$ recursively as follows:

- If $x \in P$, then $\mu([x,x]) = 1$.

- If $x < y$, then
$$\mu([x,y]) = -\left(\sum_{z \ \mid \ x \leq z < y} \mu([x,z]) \right).$$

As this function is defined on intervals, it is customary to eliminate the interval brackets and simply indicate the bounds of the interval in question. However, we will avoid this notation to make it more clear that this is a function on intervals of P. Further, if P is bounded, then we call the value $\mu(P) = \mu([\hat{0},\hat{1}])$ the *Möbius invariant* of P. For various interpretations and uses of this Möbius function, see for example [16].

As an example, we can compute the Möbius function of the indicated interval $[x,y]$:



Here, we write the value of the $\mu([x,z])$ next to each vertex z.

Given two posets P and Q with their respective partial orders, define a new poset called the *(direct) product* $P \times Q$ with partial order $\leq_{P\times Q}$ such that $(p_1, q_1) \leq_{P\times Q} (p_2, q_2)$ if $p_1 \leq_P p_2$ and $q_1 \leq_Q q_2$. With this definition, it can be shown that

$$\mu([(p_1, q_1), (p_2, q_2)]) = \mu([p_1, p_2]) \cdot \mu([q_1, q_2]) \tag{2.1.1}$$

for every closed interval of $P \times Q$ (see [16]). Note that the definition is symmetric, so that $P \times Q \cong Q \times P$. Further, if P is pure of length n and Q is pure of length m, then $P \times Q$ is pure of length $n+m$.

2.2 Shellability and EL-labelings

In this section, we relate posets to topology.

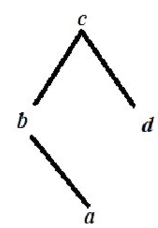
Definition 2.2.1. A family of sets Δ on vertex set V is an (abstract) simplicial complex if Δ satisfies the following properties:

- If $Y \subseteq X \in \Delta$, then $Y \in \Delta$.
- Every singleton subset of V is an element of Δ.

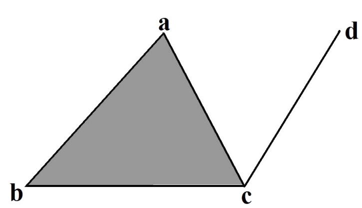
The elements of Δ are called its *faces*; a face is *maximal* if it is not contained in any other face. Define a *simplex* as a simplicial complex with only one maximal face. Define the *dimension* of a face F as $\dim(F) = |F| - 1$. The complex consisting of only the empty set is called the *empty simplicial complex*, and has dimension -1. We refer to the empty set as the *degenerate empty complex*, and say that it has dimension -2, even though it is not a proper simplicial complex.

The complex Δ is *pure* if all of its maximal faces have the same dimension; this dimension is referred to as the dimension of Δ. For any poset P, its *order complex* $\Delta(P)$ is the simplicial complex whose vertex set is P and whose faces are the chains of P. Accordingly, maximal faces of $\Delta(P)$ correspond to maximal chains of P. Note that P is pure if and only if $\Delta(P)$ is pure.

For example, consider the poset P, with the following Hasse diagram:



Then $\Delta(P) = \Big\{\{a,b,c\},\{a,b\},\{a,c\},\{b,c\},\{d,c\},\{a\},\{b\},\{c\},\{d\},\emptyset\Big\}$. We see here that the maximal faces of this simplicial complex are $\{a,b,c\}$ and $\{d,c\}$, which correspond to the two maximal chains of the poset. Geometrically, using standard depictions of simplices, we can represent this complex as:



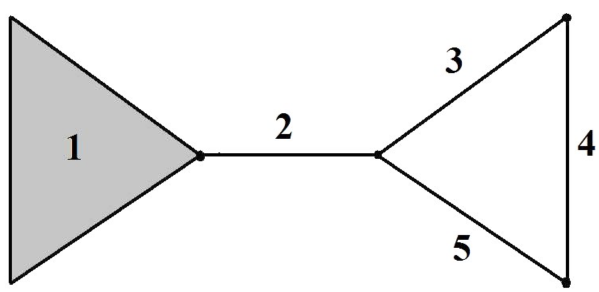
Here, the shaded triangle represents the 2-simplex $\{a,b,c\}$.

A natural question is what topological properties does the order complex of a poset (or equivalently, its geometric realization) have? In the small example above, it is evident that the geometric realization of the order complex is contractible. In general, we can see that if there is a bottom or top element in the poset, then the order complex will be contractible, being a cone. Thus, we will only consider the proper part of a bounded poset.

It is not in general a simple task to determine the topology of the order complex of an arbitrary poset. On the other hand, some techniques do exist which make these computations more tractable. For instance, shellability theory gives a standard method of obtaining topological results. Originally, shellability was defined only for pure simplicial complexes [10]. This definition was extended to include nonpure complexes by Björner and Wachs [7, 8]; this is the definition we present below.

Given a face F of a simplicial complex Δ, denote by $\langle F\rangle$ the simplicial complex generated by F. That is, $\langle F\rangle = \{G \in \Delta \mid G \subseteq F\}$. A simplicial complex is *shellable* if there exists a linear ordering of its maximal faces, $F_1, F_2, ..., F_n$ such that for $1 < k \leq n$, $\langle F_k\rangle \cap \left(\bigcup_{i=1}^{k-1} \langle F_i\rangle\right)$ is a pure simplicial complex of dimension $\dim(F_k) - 1$.

Below, we give an example of a shelling order for a geometric nonpure simplicial complex, where the numbers indicate the linear order of the maximal faces. Here, there are 5 maximal faces; four 1-faces, and one 2-face (which is shaded):



The use of shellability as a topological tool comes from the following theorem:

Theorem 2.2.2 ([7, Theorem 4.1]). *A shellable simplicial complex has the homotopy type of a wedge of spheres, where for each i, the number of i–spheres is* *the number* r_i *of* i*–dimensional maximal faces whose entire boundary is contained in the union of earlier maximal faces.*

Since a shellable complex Δ has the homotopy type of a wedge of spheres, in each dimension the reduced integral (co)homology groups are

$$\widetilde{H}_i(\Delta) \cong \widetilde{H}^i(\Delta) \cong \mathbb{Z}^{r_i}.$$

For instance in the shellable geometric complex above, the only face whose entire boundary is contained in the union of earlier maximal faces is labeled 5, and this face has dimension 1, so that the complex has the homotopy type of a single 1–sphere.

For a pure bounded poset P, Björner also introduced a method to establish shellability of its order complex without resorting to the abstract definition [3]. This method is to construct what is called an *edge-lexicographic labeling* (or *EL-labeling* for short) on the poset. Such a labeling also gives a very convenient combinatorial way of computing the reduced (co)homology. The original definition of this technique applied only to pure posets; Björner and Wachs extended the definition to nonpure posets [7], and this is the definition we will use subsequently.

Definition 2.2.3. Let P be a bounded poset, with edge set $\mathcal{E}(P)$. An EL-labeling is a function $\Psi : \mathcal{E}(P) \to \Lambda$, where Λ is a fixed totally ordered set, such that Ψ satisfies the following conditions for any comparable pair $x < y$:

1. There exists a maximal chain $c = \{c_0, c_1, ..., c_k\}$ in $[x, y]$ with $x = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_k = y$, such that $\Psi(c_{i-1} \lessdot c_i) <_\Lambda \Psi(c_i \lessdot c_{i+1})$ for all integers $0 < i < k$. That is, the labels of the chain strictly increase when read upwards in the Hasse diagram of P. Such a maximal chain is called *rising*.

2. There can be only one rising chain in $[x, y]$. That is, a rising chain must exist and be unique.

3. The unique rising chain of $[x, y]$ has a sequence of labels that, when read from the bottom to the top, lexicographically precedes the label sequence of every other maximal chain of $[x, y]$ (also read upwards) in the lexicographic order induced from Λ.

Notice that a poset may admit distinct EL-labelings. There is also a more general version of this technique called *chain-lexicographic labeling*, or *CL-labeling* [7]; in fact, Theorem 2.2.4 below applies to this more general case. However, an EL-labeling is a special type of CL-labeling, and since we can exhibit EL-labelings for all the posets we consider subsequently, we will not define the more general CL-labelings.

We now give a nonexample to illustrate the requirements that an EL-labeling must possess. Consider the poset of Figure 2.1, given by its Hasse diagram. Here, the edges are labeled by integers under their usual total order. We can verify that the only rising maximal chain of the entire poset is the bold (leftmost) chain. Now, we consider the set of maximal chains. They carry label sequences (reading from bottom to top) of $(1, 2, 3)$, $(1, 3, 2)$, $(2, 3, 1)$, and $(3, 2)$, of which the bold chain is lexicographically the smallest.

The conditions of Definition 2.2.3 are easy to verify for the maximal chains of the poset of Figure 2.1, but we observe that this is still not an EL-labeling since all proper intervals of this poset are single chains, most of which are not rising. For instance, the interval $[a, \hat{1}]$ consists of a single length 2 chain whose labels decrease. Similarly, consider the alternative labeling of Figure 2.2.

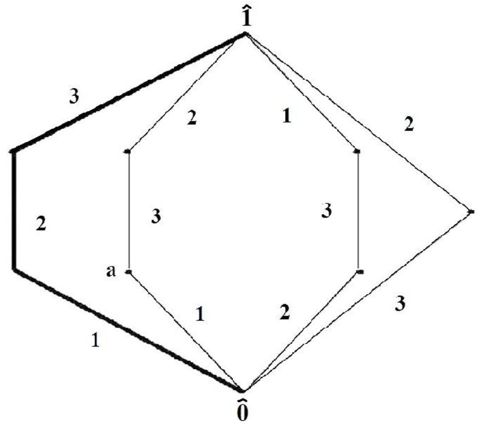


Figure 2.1: An edge labeling.

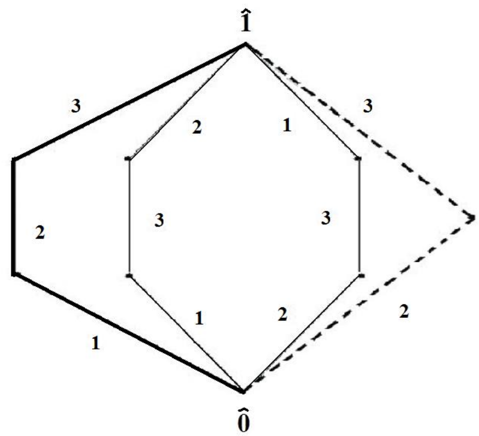
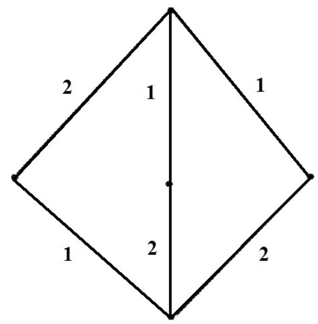


Figure 2.2: An alternative edge labeling.

Now, the dashed chain is also rising, and so the uniqueness of the rising chain fails. In this second labeling, we could then try to permute the labels of bold chain to be for instance $(2, 1, 3)$; this would again give us a unique rising chain (the dashed chain), but now the unique rising maximal chain is not lexicographically first among all maximal chains, as the chain labeled $(1, 3, 2)$ is now first.

In actuality, the poset of Figures 2.1 and 2.2 does not admit an EL-labeling. As an example of a poset that does admit an EL-labeling, consider the following:



Since any proper interval consists of only a single labeled edge, we only need to verify that there is a single rising maximal chain which is lexicographically first, and this is easily seen to be the leftmost chain. Notice that the non-rising chains don't have unique label sequences, but that this is not required in any case.

A poset P for which an EL-labeling exists is said to be *EL-shellable*. This is because of the following theorem which connects these lexicographic labelings to the earlier notions of shellability:

Theorem 2.2.4 ([7, Theorem 5.8]). *If a poset P is EL-shellable, then $\Delta(\overline{P})$ is a shellable complex.*

If poset P is EL-shellable, we define its *falling chains* as the maximal chains whose label sequences weakly decrease when read upwards in the Hasse diagram; that is, we are allowed to repeat labels, but we cannot increase at any step from the previous label. Now we can compute the homotopy type of $\Delta(\overline{P})$ in the following way:

Theorem 2.2.5 ([7, Theorem 5.9]). *Let P be a finite EL-shellable poset under edge labeling Ψ. Then $\Delta(\overline{P})$ has the homotopy type of a wedge of spheres. Furthermore, for each i, the number of spheres of dimension $i-2$ is equal to the number r_i of falling chains of P of length i. Consequently,*

$$\operatorname{rank}\left(\widetilde{H^{i-2}}(\Delta(\overline{P}))\right) = \operatorname{rank}\left(\widetilde{H_{i-2}}(\Delta(\overline{P}))\right) = r_i.$$

Given a poset P, the rank of the reduced homology group $\widetilde{H}_i(\Delta(P))$ is called a *reduced Betti number* of P, and is denoted by $\widetilde{\beta}_i(P)$. According to this theorem, if we can construct an EL-labeling on a bounded poset, then we can compute its proper part's reduced (co)homology by simply counting how many falling chains it contains of each given length. This transforms a topological problem into a combinatorial problem, for which techniques may exist to simplify the computations if the labels are chosen in special ways.

Notice that because the dimension of (co)homology is two less than the length of the chain, we have negatively indexed (co)homology for a poset whose length is less than 2. For instance, the two element chain c_1, which has length 1, has proper part equal to the empty set. We then have $\widetilde{\beta_{-1}}(\overline{c_1}) = 1$. We will not consider posets with length less than 1.

The connection to topology is then further strengthened by the following important fundamental theorem of P. Hall. First, given a simplicial complex Δ, define the *reduced Euler characteristic*

$$\widetilde{\chi}(\Delta) = \sum_{i \geq -1} (-1)^i f_i,$$

where for each i, we let f_i equal the number of faces of Δ of dimension i.

Theorem 2.2.6 (see [16]). *Let P be a bounded poset. Then $\mu(P) = \widetilde{\chi}(\Delta(\overline{P}))$.*

Further, by the Euler-Poincaré formula we obtain

$$\mu(P) = \widetilde{\chi}(\Delta(\overline{P})) = \sum_{i \geq -1} (-1)^i \widetilde{\beta}_i(\overline{P}),$$

so that we may obtain the Möbius invariant of an EL-shellable poset by counting its falling chains. If $\widetilde{\beta}_i(\overline{P}) = 0$ for all $i < \ell(P) - 2$ (as is the case when P is pure and EL-shellable), then

$$\mu(P) = (-1)^{\ell(P)}\widetilde{\beta}_{\ell(P)-2}(\overline{P}). \tag{2.2.1}$$

Let P be bounded and ranked with $\ell(P) = n$. For any subset $S \subseteq \{1, 2, ..., n-1\}$, define the *rank-selected subposet* to be the induced subposet

$$P_S := \{x \in P \mid \rho(x) \in S\}.$$

Given an EL-labeling Ψ of P, for a maximal chain

$$c := \hat{0} = c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n = \hat{1},$$

define the *descent set* $\mathrm{Des}(c)$ by

$$\mathrm{Des}(c) := \{i \in \{1, 2, ..., n-1\} \mid \Psi(c_{i-1} \lessdot c_i) \geq \Psi(c_i \lessdot c_{i+1})\}.$$

The next theorem of Björner and Wachs connects shellability of P to shellability of its rank-selected subposets:

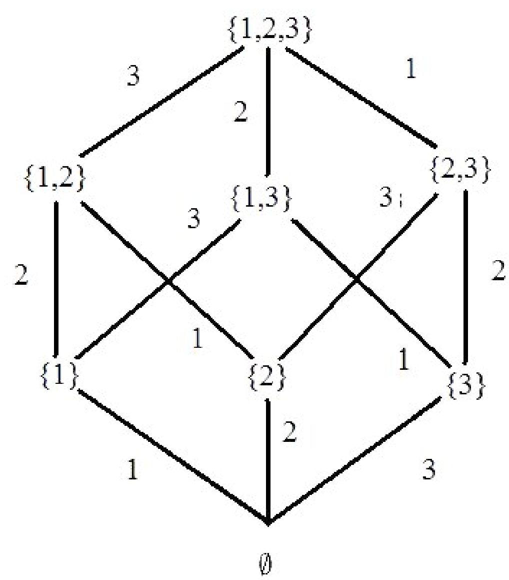
Theorem 2.2.7 ([6, Theorem 8.1]). *Let P be a pure EL-shellable poset of length n, and let $S \subseteq \{1, 2, ..., n-1\}$. Then $\Delta(P_S)$ has the homotopy type of a wedge of $(|S|-1)$–spheres. The number of spheres is the number of maximal chains of P with descent set S.*

2.3 The Boolean Algebra

Now, we consider some examples of posets for which we can construct an EL-labeling. Let c_n be the chain of length $n \geq 1$. This poset is EL-shellable using any increasing labeling of its edges; further, c_n has zero falling chains unless $n = 1$, so that the unique edge of the chain is both rising and falling. For $n > 1$, $\overline{c_n}$ has an order complex consisting of a single simplex of dimension $n-2$, which is contractible. Alternatively, we can compute the Möbius function directly for a chain, and observe that $\mu(c_n) = 0$ if $n > 1$, and that $\mu(c_1) = -1$.

The second poset we consider is the *Boolean algebra* B_n on n elements. The Boolean algebra B_n consists of all subsets of $[n] := \{1, 2, ..., n\}$, ordered by containment. For example, $\{1, 3, 5\} \lessdot \{1, 2, 3, 5\}$ in B_5. Note that if $x \lessdot y$ in B_n, there is a unique element in the set $y - x$, the complement of x in y. Further, we have $\hat{0} = \emptyset$, and $\hat{1} = [n]$. We construct an edge labeling $\Psi : \mathcal{E}(B_n) \to [n]$ by defining $\Psi(x \lessdot y)$ to be the unique element of $y - x$.

As an example, we illustrate the Hasse diagram of B_3, with the labeling Ψ indicated above:



This labeling of B_n is an EL-labeling, as the rising chain in any interval $[x, y]$ of length at least 1 will be the chain in which we add the elements of $y - x$ to x in order from smallest to largest. Further, maximal chains of $[x, y]$ correspond bijectively to permutations of the set $y - x$. This observation allows us to see that there is a unique rising chain, and that this chain is lexicographically first, as the set of permutations of any set S has a single increasing permutation, and it is lexicographically first.

Further, as there is only one permutation of $[n]$ that can be written in weakly decreasing order, namely $n, (n-1), (n-2), \cdots, 1$, there can be only one falling chain of B_n. Now, since B_n has only one falling chain, we have that $\Delta(\overline{B_n})$ has the homotopy type of a single sphere of dimension $n-2$ by Theorem 2.2.5. This is also easy to see directly since $\Delta(\overline{B_n})$ is the barycentric subdivision of the boundary of the $(n-1)$-simplex, and so is homeomorphic to the $(n-2)$-sphere (see [21]).

It is clear that B_n is a lattice with $x \wedge y = x \cap y$ and $x \vee y = x \cup y$, for all $x, y \in B_n$. We discuss in Section 4.2 a generalization of the EL-labeling Ψ of B_n above which applies to a wider class of lattices.

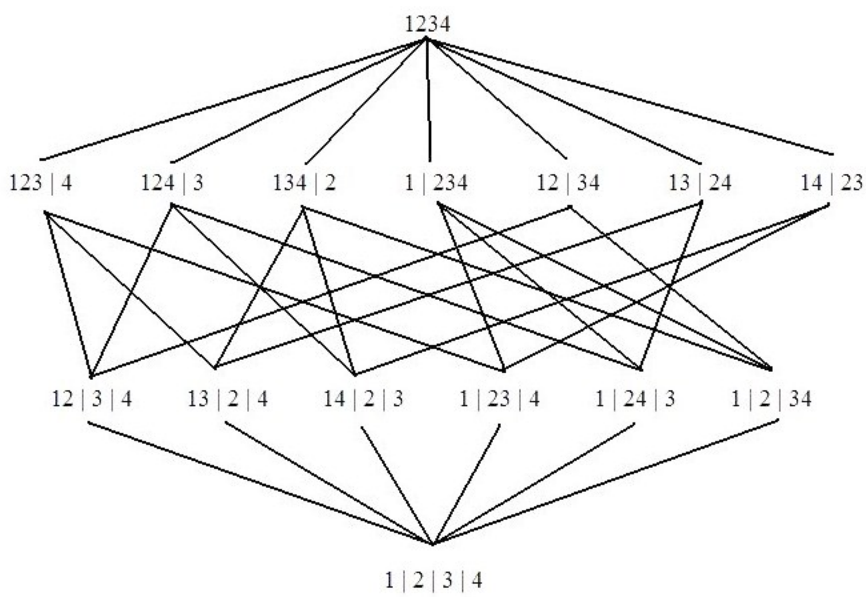
2.4 Partition Lattices

Now we describe another important family of posets. Let n be a positive integer, and define the *partition lattice* Π_n to be the poset of set partitions of $[n]$, ordered by refinement. That is, $x = \{B_1, B_2, ..., B_t\} \in \Pi_n$ if:

- $B_i \neq \emptyset$ for all i.
- $\displaystyle\bigcup_{i=1}^{t} B_i = [n]$.
- $B_i \cap B_j = \emptyset$ for all $i \neq j$.

The elements B_i of x are called *blocks*. Further, we say $x \leq y$ in Π_n if and only if each block of x is contained in a block of y.

To denote the elements of Π_n, we list the elements of its blocks without set brackets, and separate the blocks by vertical bars. For instance, in Π_n, the bottom element $\hat{0}$ has only blocks of size 1, so we write $\hat{0} = 1|2|3|\cdots|n$. The top element $\hat{1}$ is the partition consisting of a single block, and so we write $\hat{1} = 123\cdots n$. Further, the atoms of Π_n have exactly one block of size 2 while each other block is of size 1. Note that Π_n is pure, with rank function ρ satisfying $\rho(x) = n - k$, where k is the number of blocks of $x \in \Pi_n$. The partition lattice Π_4 is illustrated below:



We now describe an EL-labeling of Π_n due to Wachs [22].

Proposition 2.4.1. *Let $x \lessdot y$ in Π_n, with $x = \{B_1, B_2, ...B_t\}$. Then there is a unique pair of distinct indices $i, j \in [t]$ such that $B_k \in y$ for all $k \notin \{i, j\}$, and $(B_i \cup B_j) \in y$. The edge labeling which assigns to $x \lessdot y$ the label $\max(B_i \cup B_j)$ is an EL-labeling of Π_n.*

Proof. Consider first the upper interval $[a, \hat{1}]$, for $a = \{A_1, A_2, ..., A_t\} \in \Pi_n$. Label the blocks of a so that $\max(A_i) < \max(A_{i+1})$ for each i. We construct the first edge of a maximal chain c by forming block $A_1 \cup A_2$, which receives the label $\max(A_2)$. Further, no other possible merger will receive this label, since any other pair of blocks has a larger maximum.

Notice that $\max(A_1 \cup A_2) = \max(A_2)$. Therefore, after this first step, the blocks $A_1 \cup A_2$ and A_3 have the smallest maxima remaining. We merge in the second step $A_1 \cup A_2$ and A_3, which receives the label $\max(A_3)$, which is the smallest label possible at this step. Continuing in this fashion by merging $\bigcup_{i=1}^{j} A_i$ with A_{j+1} at each step in the chain, we generate the labels $\max(A_2), \max(A_3), ..., \max(A_t)$ in sequence, and so c is a rising chain in this interval.

Notice that the set of possible labels for any maximal chain in this interval is $\Lambda = \{\max(A_i) \mid i = 2, 3, ..., t\}$, since $\max(A_1)$ can never be a label. Now there is a unique way to arrange the labels of Λ into a strictly rising sequence; since $|\Lambda| = t-1$, and c is a chain with strictly increasing label sequence of length $t-1$, it must be uniquely rising. It is also lexicographically minimal among maximal chains, since it clearly takes on the smallest possible values at each step.

Now that upper intervals have the uniquely rising chain, we consider the arbitrary interval $[a, b]$, for $a = \{A_1, A_2, ..., A_t\}$, and $b = \{B_1, B_2, ...B_s\}$, where we arrange both sets in order of their increasing maxima, as before. Observe that for any distinct blocks A_i and A_j, that we may only merge these blocks if there exists B_k such that $A_i \cup A_j \subseteq B_k$. Consider a fixed B_k, and let $\{A_{k,1}, A_{k,2}, ..., A_{k,j_k}\}$ be the set of blocks of a which are subsets of B_k.

Since we must merge these j_k blocks together in some order, from the above discussion of upper intervals, we see that there is a unique way to do this which generates strictly increasing labels. Further, notice that since b is a partition, that no label arising in the formation of B_k can be used in the formation of any other block distinct from B_k.

We combine these observations to see that there is a unique way to generate the smallest possible label above any element in a maximal chain of $[a, b]$, by shuffling together the unique rising chains formed by each block B_k. Therefore, each interval has the requisite rising chain, and so this is an EL-labeling of Π_n. $\square$

We claim that for the labeling of Proposition 2.4.1, the falling chains will all have constant label n. Indeed, joining any block with the block containing n will always produce this label, and any edge below $\hat{1} \in \Pi_n$ is always of this type. Therefore, to be falling, we must have labels which are weakly larger than n at each edge. Since no label is larger than n, this implies that all the labels can only be exactly n.

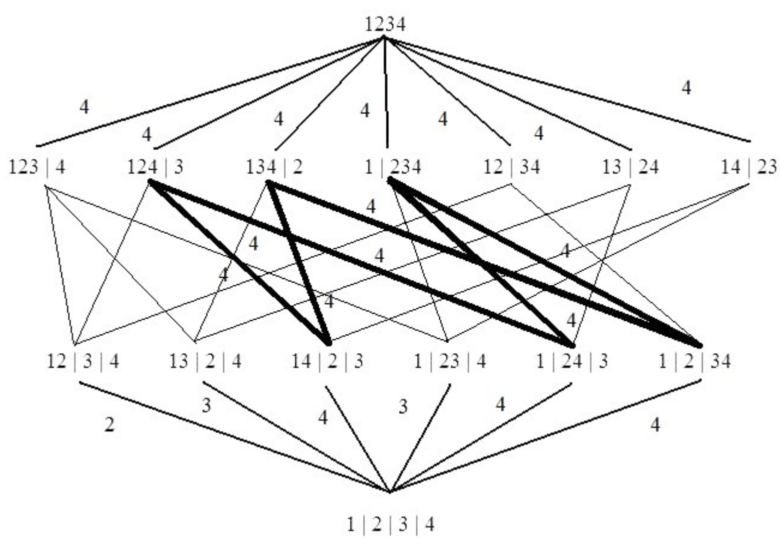


Figure 2.3: An EL-labeling of Π_4.

We illustrate this with Π_4 in Figure 2.3. We have indicated the edges in the middle of the diagram that are part of falling chains in bold; further, notice all these edges share the same label of 4, as do all edges below $\hat{1}$.

Proposition 2.4.2. *The order complex* $\Delta(\overline{\Pi_n})$ *has the homotopy type of a wedge of* $(n-1)!$ *spheres of dimension* $n-3$.

Proof. Observe first that Π_n is pure of length $n-1$. It follows from Theorem 2.2.5 and Proposition 2.4.1 that the homotopy type of $\Delta(\overline{\Pi_n})$ is that of a wedge of spheres, where the number of spheres equals the number of falling chains of the EL-labeling. Now we count the falling chains. Since falling chains have constant label n along their whole length, we need only count at each rank how many elements will generate such a label.

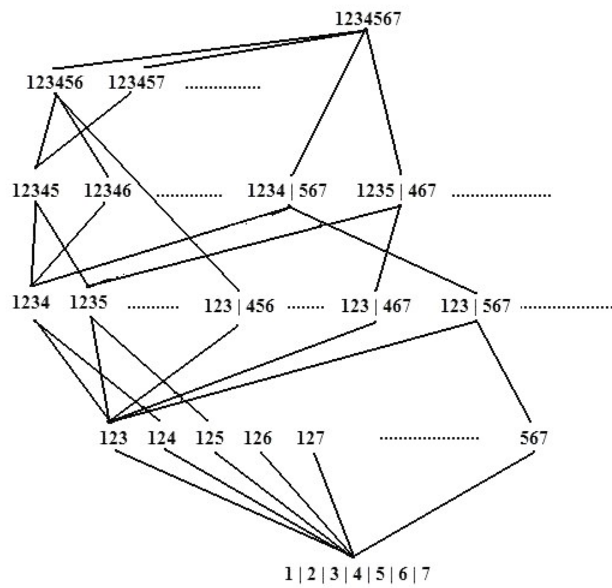
There will be $n-1$ atoms whose label above $\hat{0}$ is n, corresponding to choosing any of the other $n-1$ elements to pair with n in the atom. Thus, for an edge above an atom, since the block containing n now has two elements, we have $n-2$ singleton blocks available to choose from to merge with the block containing n, and any of these generates label n. Similarly, at rank i, the block containing n will have $i+1$ elements, leaving $n-i-1$ singletons to choose from. Thus, the number of falling chains is $\displaystyle\prod_{i=0}^{n-2}(n-i-1) = (n-1)!$. $\square$

2.5 k–Equal Partition Lattices

In this section, we discuss a subposet of the partition lattice introduced by Björner, Lovász, and Yao [4]. Let $2 \leq k \leq n$ be an integer, and define $\Pi_n^{=k}$

as the induced subposet of Π_n consisting of partitions in which every block has either size 1 or size at least k. Björner, Lovász, and Yao show that $\Pi_n^{=k}$ is also a lattice and call it the *k–equal partition lattice* [4].

Below we illustrate a portion of $\Pi_7^{=3}$.



For brevity, we have omitted all the parts of size 1. It is easy to see that even in this small example there are several distinct types of covering relations. It is also easy to see that not every maximal chain has the same length, hence $\Pi_7^{=3}$ is not pure.

When $k = 2$, there are no disallowed block sizes; therefore, $\Pi_n^{=2} = \Pi_n$. It is further evident that $\Pi_n^{=k}$ is also pure in the case when $k > \frac{n}{2}$, since in this event, each element other than the minimum can consist of only one nonsingleton block of size at least k. This case is related to posets called *truncated Boolean algebras*; these will be discussed further in Section 3.3. On the other hand, for all other values of k with $2 < k \leq \frac{n}{2}$, the lattices are not pure.

Björner, Lovász, and Yao used the topology of $\Pi_n^{=k}$ to find a lower bound on the k–equal problem of complexity theory [4]. Later, Björner and Welker showed that $\Delta(\overline{\Pi_n^{=k}})$ has the homotopy type of a wedge of spheres of varying dimensions [9]. Björner and Wachs then showed that $\Delta(\overline{\Pi_n^{=k}})$ is shellable [7]. In fact, this family of lattices was the motivating example for extending shellability theory to simplicial complexes that are not pure [7].

We now describe the EL-labeling of $\Pi_n^{=k}$ that Björner and Wachs used to establish shellability of $\Delta(\overline{\Pi_n^{=k}})$. We first create the linearly ordered product set $\Lambda = [2] \times [n]$ as our label set, where the order is lexicographic. That is, $(1,1) < (1,2) < \cdots < (1,n) < (2,1) < \cdots < (2,n)$. Next, observe there are only three distinct types of covering relations. Let $x = x_1|x_2|\cdots|x_m$, where we omit the singleton blocks of x and the nonsingleton blocks are ordered arbitrarily:

Type I - Creation:

Let $y = x_1|\cdots|x_m|z$, where z is a subset of $[n]$ of cardinality k. In this case, we label the edge $x \lessdot y$ with $(2,a)$, where $a = \max(z)$.

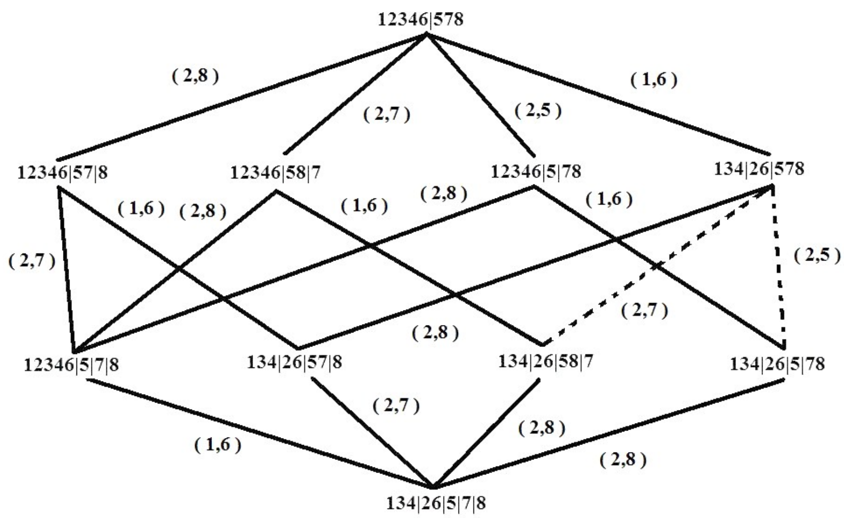
Type II - Expansion:

Let $y = x_1|\cdots|x_{m-1}|z$, where $z = x_m \cup \{a\}$ for $a \in [n]$. In this case, we label the edge $x \lessdot y$ with $(2,a)$.

Type III - Merger:

Let $y = x_1|\cdots|x_{m-2}|z$, where $z = x_{m-1} \cup x_m$. That is, we replace two blocks of x by the union of the two blocks. In this case, we label the edge $x \lessdot y$ with $(1,a)$, where $a = \max(z)$.

As an example, consider the following interval of $\Pi_8^{=2}$, complete with edge labeling described by the above label rules:



It can be checked that this labeling is an EL-labeling of this interval. We can see in this example that there are only two chains which are falling in the interval (the dashed edges are those which belong only to falling chains). Denote this edge labeling by Ψ.

Proposition 2.5.1 ([7, Theorem 6.1]). *The edge labeling* Ψ *is an EL-labeling of* $\Pi_n^{=k}$.

From the definition of Ψ, we see that a falling chain of $\Pi_n^{=k}$ will be one in which we create and/or expand some t blocks, followed by $t-1$ mergers. To count the number of falling chains of $\Pi_n^{=k}$ of length d, we can use the following proposition:

Proposition 2.5.2 ([9, Theorem 4.5]). *Let* $1 < k \leq n$. *Define* $B^{n,k}(t)$ *by*

$$B^{n,k}(t) = (t-1)! \cdot \sum_{0=i_0 \leq \cdots \leq i_t = n-tk} \prod_{j=0}^{t-1} \binom{n - jk - i_j - 1}{k-1} (j+1)^{i_{j+1}-i_j}.$$

The order complex $\Delta(\overline{\Pi_n^{=k}})$ *has the homotopy type of a wedge of spheres, where* *the number of spheres* $\widetilde{\beta}_{d-2}(\overline{\Pi_n^{=k}})$ *of dimension* $d-2$ *is given by*

$$\widetilde{\beta}_{d-2}(\overline{\Pi_n^{=k}}) = \begin{cases} \sum_{t=1}^{\lfloor \frac{n}{k} \rfloor} B^{n,k}(t), & k=2 \text{ and } d=n-1 \\ B^{n,k}(\frac{n-d-1}{k-2}), & k>2 \text{ and } \frac{n-d-1}{k-2} \in \mathbb{P} \\ 0, & otherwise. \end{cases}$$

The original proof of Proposition 2.5.2 did not make use of the EL-shellability of $\Pi_n^{=k}$, as the notion of EL-shellability was not extended to nonpure posets until after this formula was found. However, the EL-labeling Ψ gives a proof of this formula which shows how the falling chains must be constructed.

Proof idea. Since a merger carries a label which is strictly less than the label of either a creation or an expansion, to be falling a chain cannot have a merger precede a creation or an expansion. Given a falling chain c there is a unique $x \in c$ such that below x there are only creations and expansions, and above x there are only mergers. We call such an element the *pivot* of c. Let $x = \{B_1, ..., B_t\}$ be the pivot of c. Then $|B_i| \geq k$ for each i. We will count how many falling chains pass through the pivot x.

The length of such a chain will be $d = n-1-t(k-2)$, as we must have t creation steps, creating t blocks with a total of tk elements; since the pivot has n total elements in nonsingleton blocks, the remaining $n-tk$ elements must be part of some expansion. Finally, if there are t blocks in the pivot, it will take $t-1$ mergers to join them together, giving a total number of edges as $t+(n-tk)+(t-1) = n-1-t(k-2)$. Notice that if $k=2$, then $d=n-1$ for all values of t, while if $k>2$, there is a unique value of d for each value of t. This accounts for the formula for $\widetilde{\beta}_{d-2}(\overline{\Pi_n^{=k}})$, provided we can show that $B^{n,k}(t)$

equals the number of falling chains whose pivot x has t blocks.

One can show that there are $(t-1)!$ falling chains above x, and that there are

$$\sum_{0=i_0\leq\cdots\leq i_t=n-tk}\prod_{j=0}^{t-1}\binom{n-jk-i_j-1}{k-1}(j+1)^{i_{j+1}-i_j},$$

falling chains below x. Hence, $B^{n,k}(t)$ is indeed the number of falling chains whose pivot has t blocks. □

An alternative formula due to Björner and Wachs [7] is given below:

Proposition 2.5.3 ([7, Corollary 6.3]). *Let $1<k\leq n$. Define $B^{n,k}(t)$ by*

$$B^{n,k}(t)=\sum_{\substack{j_1+\cdots+j_t=n,\\ j_i\geq k\ \forall\ i}}\binom{n-1}{j_1-1,j_2,...,j_t}\prod_{i=1}^{t}\binom{j_i-1}{k-1}.$$

The order complex $\Delta(\overline{\Pi_n^{=k}})$ has the homotopy type of a wedge of spheres, where the number of spheres $\widetilde{\beta}_{d-2}(\overline{\Pi_n^{=k}})$ of dimension $d-2$ is given by

$$\widetilde{\beta}_{d-2}(\overline{\Pi_n^{=k}})=\begin{cases}\displaystyle\sum_{t=1}^{\lfloor\frac{n}{k}\rfloor}B^{n,k}(t), & k=2 \textit{ and } d=n-1\\ B^{n,k}(\frac{n-d-1}{k-2}), & k>2 \textit{ and } \frac{n-d-1}{k-2}\in\mathbb{P}\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. Again we count falling chains with pivot x. Let $x=\{B_1,...,B_t\}\in\Pi_n^{=k}$ be a pivot. Index the blocks so that $n\in B_1$, and let $j_i=|B_i|$ for each i. Since $n\in B_1$, there are $n-1$ elements remaining to be distributed among the t labeled blocks; further, of these, only j_1-1 may be chosen for B_1. Therefore, the number of ways to distribute the elements of $[n]$ among the t blocks while keeping $n\in B_1$ is $\binom{n-1}{j_1-1,j_2,...,j_t}$.

Then for each block B_i, we need to choose k elements of the block to form the initial creation, and the remaining elements will form expansions. However, the creation step which will be expanded to form B_i must contain $\max(B_i)$, as previously observed. Therefore, we can only choose $k-1$ other elements for each creation; since there are j_i elements in block B_i, this leaves us $\binom{j_i-1}{k-1}$ possible choices for the initial creation. As the blocks are disjoint, there is a unique falling sequence of expansions of B_i given a fixed creation.

Now we arrange our mergers so that we merge B_1 first with B_2, then merge B_3 into $B_1 \cup B_2$, and continue by merging B_i into $\bigcup_{j=1}^{i-1} B_j$. Notice that this gives a unique falling merger sequence among this chosen ordered collection of blocks. $\square$

Notice that the formula of Proposition 2.5.2 presumes that expansions can be performed on any existing block, and depend only on their relative order in the sequence of steps. Meanwhile, the formula of Proposition 2.5.3 presumes that once there are no more singleton blocks in the chain, the remaining merger labels of blocks are not affected by the order such mergers are performed in, and so can be specified beforehand. These two properties may not hold in a more general setting; in Section 6.2 we will give a modification of these formulas which takes these observations into account.

Chapter 3

q–Analogues

3.1 Permutation Enumeration

A *q–analogue* is a loosely defined term in general, best illustrated by example. The simplest example is the q–analogue of the positive integer n, denoted by $[n]_q$, and defined by

$$[n]_q = \sum_{i=1}^{n} q^{i-1} = \frac{q^n - 1}{q - 1}.$$

Observe that by setting $q = 1$ in the sum, we get $[n]_1 = n$, while in the fractional expression, we must take the limit as q approaches 1. To avoid taking limits, we will use the sum definition, though the fraction equivalent may be used whenever we assume $q \neq 1$. Now that we have defined a q–analogue of an integer, we can extend this to other integer-related ideas.

For instance, the q–analogue of $n!$ is given by

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$

and it is easy to see that $[n]_1! = n!$. Further, we can define $[0]_q = 0$, so that

$[0]_q! = 1$. We define the *q–binomial coefficient*,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

for all nonnegative integers $0 \leq k \leq n$. Notice that the q–binomial coefficients are also symmetric; that is, $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$. Further, they also satisfy an analogue of Pascal's identity:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

Define the *q–multinomial coefficients*

$$\begin{bmatrix} n \\ k_1, k_2, ..., k_j \end{bmatrix}_q = \frac{[n]_q!}{[k_1]_q![k_2]_q! \cdots [k_j]_q!},$$

subject to the condition that $k_1 + k_2 + \cdots + k_j = n$, completely analogous to the usual multinomial coefficients. Given these definitions alone, it is a surprising fact that q–multinomial coefficients are always in fact polynomials in q; this is in part because for positive integers m and n, with m a divisor of n, we have $\frac{[n]_q}{[m]_q} = \sum_{i=1}^{\frac{n}{m}} q^{m(i-1)}$.

Now we examine some of the connections between q–analogues and permutation enumeration. Denote by $\mathfrak{S}_n$ the set of all permutations (written as words) of $[n]$. For the word $\sigma \in \mathfrak{S}_n$, denote by $\sigma(i)$ the i^{th} entry of the word σ. Given $\sigma \in \mathfrak{S}_n$, define an *inversion* of σ to be a pair (i, j) such that $i < j$ and $\sigma(i) > \sigma(j)$; for instance, the only inversions of the word $\sigma = 123645$ are the pairs $(4, 5)$ and $(4, 6)$. Define

$$\text{inv}(\sigma) := |\{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}|,$$

and define

$$\text{Des}(\sigma) := \{i \in [n-1] \mid \sigma(i) > \sigma(i+1)\}.$$

Proposition 3.1.1. *For all* $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} q^{inv(\sigma)} = [n]_q!.$$

More generally, given any word $\omega := \omega(1)\omega(2)\cdots\omega(n)$ over alphabet $[m]$, define

$$\mathrm{inv}(\omega) := |\{(i,j) \mid i < j, \omega(i) > \omega(j)\}|.$$

Let $\mathcal{M}$ be a multiset with a_i copies of i for each $i \in [m]$. Let the cardinality of $\mathcal{M}$ be $n = a_1 + \cdots + a_m$, and let $\mathfrak{S}_{\mathcal{M}}$ be the set of arrangements of $\mathcal{M}$.

Proposition 3.1.2 (see [16]).

$$\sum_{\omega \in \mathfrak{S}_{\mathcal{M}}} q^{inv(\omega)} = \begin{bmatrix} n \\ a_1, a_2, ..., a_m \end{bmatrix}_q$$

3.2 q–Analogues of Posets

We can also form q–analogues of more elaborate structures; for instance, a q–analogue of the Boolean algebra, denoted $B_n(q)$, is defined to be the poset of all subspaces of the n–dimensional vector space $\mathbb{F}_q^n$ ordered by containment, where $\mathbb{F}_q$ denotes the finite field of order q. In particular, we must restrict the value of q here to be a prime power for the definition to make sense.

We call $B_n(q)$ the *subspace lattice*, as it is clear that any two subspaces have a meet, which is their intersection, and that they also have a join, which is their sum. Notice that $B_n(q)$ is bounded, with bottom the trivial subspace (0), and top $\mathbb{F}_q^n$. Note that just as B_n is pure with rank function satisfying $\rho(x) = |x|$

for each $x \in B_n$, the lattice $B_n(q)$ is also pure, with rank function satisfying $\rho(x) = \dim(x)$ for each $x \in B_n(q)$. That $B_n(q)$ can be viewed as a q–analogue of B_n comes from the following well-known properties:

- The total number of maximal chains of B_n is $n!$; the total number of maximal chains of $B_n(q)$ is $[n]_q!$.
- The total number of elements of rank r of B_n is $\binom{n}{r}$; the total number of elements of rank r of $B_n(q)$ is $\begin{bmatrix} n \\ r \end{bmatrix}_q$.
- The Möbius invariant of B_n is $\mu(B_n) = (-1)^n$; the Möbius invariant of $B_n(q)$ is $\mu(B_n(q)) = (-1)^n q^{\binom{n}{2}}$.
- Let P be a ranked bounded poset, and define the polynomial $\chi(P,t) = \sum_{x \in P} \mu([\hat{0}, x]) t^{\ell(P) - \rho(x)}$. This is called the *characteristic polynomial* of P. Then $\chi(B_n, t) = (t-1)^n$, and $\chi(B_n(q), t) = \prod_{i=1}^{n} (t - q^{i-1})$

This is not an exhaustive list of properties for which these two lattices have similar formulas, but we can easily see that for each of them, setting $q = 1$ for $B_n(q)$ gives the same result that B_n has.

The second property above for $B_n(q)$ states that the number of r–dimensional subspaces of $\mathbb{F}_q^n$ is $\begin{bmatrix} n \\ r \end{bmatrix}_q$. This is the $m = 0$ case of the following result, which we will need later in Section 6.2.

Lemma 3.2.1. *Let V be an m–dimensional subspace of $\mathbb{F}_q^n$. The number of distinct p–dimensional subspaces $W \subset \mathbb{F}_q^n$ such that $V \cap W = (0)$ is*

$$q^{mp} \begin{bmatrix} n-m \\ p \end{bmatrix}_q .$$

Proof. Suppose that V has ordered basis $(v_1, ..., v_m)$. We build W by choosing an ordered basis $(w_1, w_2, ..., w_p)$ in sequence. To choose w_1, we select a vector w_1 such that $w_1 \notin V$. There are $q^n - q^m$ such vectors we may choose for w_1. By our choice of w_1, we observe that $(v_1, ..., v_m, w_1)$ is also a linearly independent set, and therefore $(v_1, ..., v_m, w_1)$ is an ordered basis for the subspace we will denote by U_1.

Note that $\dim(U_1) = m + 1$. Now given subspace U_i with $\dim(U_i) = m + i$ and ordered basis $(v_1, ..., v_m, w_1, ..., w_i)$ for some fixed integer $1 \leq i < p$, we observe that there are q^{m+i} vectors in U_i, and thus we choose a vector w_{i+1} outside of U_i, for which there are $q^n - q^{m+i}$ choices. We then observe that the set $(v_1, ..., v_m, w_1, ..., w_i, w_{i+1})$ is linearly independent, and so we define U_{i+1} as the subspace with ordered basis $(v_1, ..., v_m, w_1, ..., w_{i+1})$.

Thus, we have recursively defined the sequence of subspaces $U_1, U_2, ..., U_p$, where U_i has ordered basis $(v_1, ..., v_m, w_1, ..., w_i)$ for each i. Notice that there are $\displaystyle\prod_{i=1}^{p}(q^n - q^{m+i-1})$ possible ways to choose the sequence of vectors $w_1, ..., w_p$. Now the sequence $(w_1, ..., w_p)$ is an ordered basis of a subspace of $\mathbb{F}_q^n$; denote this space by W. Further, since $(v_1, ..., v_m, w_1, ..., w_p)$ is linearly independent, $V \cap W = (0)$; we can also observe that $U_p = V \oplus W$.

On the other hand, any nontrivial subspace W has multiple ordered bases in general. In particular, if we let $V = (0)$, then $U_p = W$, and so $\displaystyle\prod_{i=1}^{p}(q^p - q^{i-1})$ counts the number of ways to select an ordered basis for W, since $m = 0$ in this case. Therefore, of the $\displaystyle\prod_{i=1}^{p}(q^n - q^{m+i-1})$ possible ordered bases, exactly $\displaystyle\prod_{i=1}^{p}(q^p - q^{i-1})$ will correspond to the same subspace W. Hence, there are $\displaystyle\prod_{i=1}^{p}\frac{q^n - q^{m+i-1}}{q^p - q^{i-1}}$ distinct subspaces W such that $\dim(W) = p$, $\dim(V) = m$, and $V \cap W = (0)$.

To simplify this product, recall that $[n]_q = \dfrac{q^n-1}{q-1}$. Thus, we have

$$\begin{aligned}\prod_{i=1}^{p}\frac{q^n-q^{m+i-1}}{q^p-q^{i-1}} &= (q^m)^p\prod_{i=1}^{p}\frac{q^{n-m}-q^{i-1}}{q^p-q^{i-1}}\\ &= q^{mp}\prod_{i=1}^{p}\frac{q^{(n-m)-i+1}-1}{q^{p-i+1}-1}\\ &= q^{mp}\prod_{i=1}^{p}\frac{[(n-m)-i+1]_q}{[p-i+1]_q}\\ &= q^{mp}\begin{bmatrix}n-m\\ p\end{bmatrix}_q\end{aligned}$$

□

We now describe a labeling of the edges of $B_n(q)$ which can be used to derive the first and third properties listed above. Each subspace U of $\mathbb{F}_q^n$ can be represented by a unique $n \times n$ reduced row echelon matrix $\mathcal{M}(U)$ whose row vectors span U. Label the columns of $\mathcal{M}(U)$ from right to left, and let

$$\mathcal{C}(U) = \{j \in [n] \mid \text{column } j \text{ of } \mathcal{M}(U) \text{ contains a leading } 1\}.$$

It is not difficult to verify that if $U \lessdot_{B_n(q)} V$, then $\mathcal{C}(U) \lessdot_{B_n} \mathcal{C}(V)$.

We construct an edge labeling $\Psi : \mathcal{E}(B_n(q)) \to [n]$ by letting $\Psi(U \lessdot_{B_n(q)} V)$ be the unique element of $\mathcal{C}(V) - \mathcal{C}(U)$. We state the next two results without proof.

Proposition 3.2.2 (see [15]). *The labeling Ψ is an EL-labeling of $B_n(q)$.*

Proposition 3.2.3. *For each maximal chain $c := (0) = U_0 \lessdot U_1 \lessdot \cdots \lessdot U_n = \mathbb{F}_q^n$ the label sequence*

$$(\Psi(U_0 \lessdot U_1), \Psi(U_1 \lessdot U_2), ..., \Psi(U_{n-1} \lessdot U_n))$$

is a permutation in $\mathfrak{S}_n$.

Moreover for each $\sigma \in \mathfrak{S}_n$*, the number of maximal chains of* $B_n(q)$ *with label sequence* σ *is* $q^{inv(\sigma)}$*.*

Corollary 3.2.4. *The number of maximal chains of* $B_n(q)$ *is* $[n]_q!$*.*

Proof. We use Proposition 3.1.1. ☐

Corollary 3.2.5. *The order complex* $\Delta(\overline{B_n(q)})$ *has the homotopy type of a wedge of* $q^{\binom{n}{2}}$ $(n-2)$*–spheres.*

There have been several q–analogues of the partition lattice introduced in the literature, starting with the Dowling lattices [11], and a class of lattices studied by Bennett, Dempsey and Sagan [1] and Simion [15]. One example which builds on the idea of the subspace lattice is the lattice $DS_n(q) \cup \{\hat{0}\}$ of direct sum decompositions of $\mathbb{F}_q^n$, as was studied by Welker [23], who proved that $\Delta(DS_n(q) - \{\hat{1}\})$ has the homotopy type of a wedge of $\frac{1}{n}\prod_{i=1}^{n-1}(q^i-1)f_n(q)$ spheres of dimension $n-2$, where $f_n(q)$ is a polynomial with integer coefficients. We will discuss $DS_n(q)$ in more detail in Section 7.1.

Hanlon, Hersh, and Shareshian [13] then introduced the lattice $PD_n(q)$ of partial direct sum decompositions. A *partial direct sum decomposition* is a set of subspaces $\{U_1, ..., U_t\}$ of $\mathbb{F}_q^n$ whose sum is direct. That is, given the set $\{U_1, ..., U_t\}$, there exists some subspace $V \subseteq \mathbb{F}_q^n$ such that every vector $v \in V$ can be expressed uniquely as $v = u_1 + u_2 + \cdots + u_t$, where $u_i \in U_i$ for each i.

These partial decompositions are ordered by $\{U_1, ..., U_t\} \leq \{W_1, ..., W_s\}$ if each U_i is contained in some W_j. Using discrete Morse theory, they find that this lattice has homotopy type of a wedge of $\frac{1}{n}q^{\binom{n}{2}}\prod_{i=1}^{n-1}(q^i-1)$ spheres of dimension $2n-3$. Note that both this formula and the formula of Welker reduce to 0 when $q = 1$, therefore are not q–analogues of $(n-1)!$.

In Chapter 5 we introduce a new q–analogue of the k–equal partition lattice which is an induced subposet of $PD_n(q)$, which we then study throughout the remaining chapters. Our new q–analogue has the homotopy type of a wedge of $(n-3)$–spheres. The number of spheres in the $k=2$ case is a nice q–analogue of $(n-1)!$, namely $[n-1]_q!$ times a polynomial in q which immediately reduces to 1 when q is set equal to 1.

3.3 Rank Selection

Recall that the rank-selected subposet P_S of P is the induced subposet consisting of elements whose ranks belong to a fixed set S, with $S \subseteq [\ell(P)-1]$.

Theorem 3.3.1. *For* $S \subseteq [n-1]$,

(1) The order complex $\Delta((B_n)_S)$ *has the homotopy type of a wedge of* $(|S|-1)$*–spheres, where the number of spheres is* $\Big|\{\sigma \in \mathfrak{S}_n \mid Des(\sigma) = S\}\Big|$.

(2) The order complex $\Delta((B_n(q))_S)$ *has the homotopy type of a wedge of* $(|S|-1)$*–spheres, where the number of spheres is* $\displaystyle\sum_{\substack{\sigma\in\mathfrak{S}_n \\ Des(\sigma)=S}} q^{inv(\sigma)}$.

Proof. (1) This follows from Theorem 2.2.7 and the EL-labeling $\Psi : B_n \to [n]$ given in Section 2.3, which induces a bijection from maximal chains to permutations in $\mathfrak{S}_n$.

(2) This follows from Theorem 2.2.7 and Propositions 3.2.2 and 3.2.3.

□

We consider now a certain type of rank selected subposet.

Definition 3.3.2. Let P be a bounded ranked poset, with length $\ell(P) = n$, and rank function ρ. For a fixed integer k such that $1 < k < n$, the *(lower) truncation* P_k of P is the rank-selected subposet corresponding to $S = \{k, k+1, ..., n-1\}$; that is,

$$P_k = \{x \in P \mid k \leq \rho(x) \leq n-1\}.$$

Note that we may define *upper truncations* similarly; however, we shall mean lower truncation whenever we say truncation. Further, since a truncation P_k is not usually bounded, we shall work with $\widehat{P_k}$ whenever we need bounded posets, such as for EL-shellability. Note that $\widehat{P_k}$ is a pure lattice.

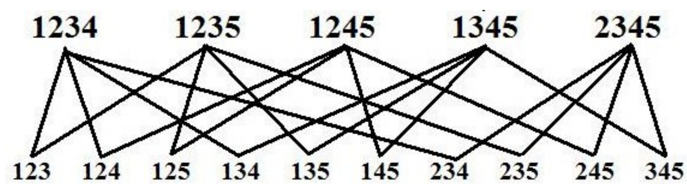


Figure 3.1: The truncation $(B_5)_3$.

Proposition 3.3.3. *The order complex* $\Delta((B_n)_k)$ *has the homotopy type of a wedge of spheres of dimension* $n-k-1$*, where the number of spheres is* $\binom{n-1}{k-1}$.

Proof. Note that we can apply Theorem 3.3.1 with $S = \{k, k+1, ..., n-1\}$. For a permutation to have descent set S, we must have that n is in position k, and all elements in positions $i < k$ are arranged in increasing order, while those in positions $i > k$ are arranged in decreasing order. The number of ways to distribute the numbers $1, ..., n-1$ into these two groups is $\binom{n-1}{k-1}$, and each such choice has a unique chain with descent set S. $\square$

Proposition 3.3.4. *The order complex* $\Delta((B_n(q))_k)$ *has the homotopy type of a wedge of spheres of dimension* $n-k-1$*, where the number of spheres is* $q^{\binom{n-k+1}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$.

Proof. From Theorem 3.3.1, we know that $S = \{k, k+1, ..., n-1\}$ allows us to find the number of spheres as $\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{Des}(\sigma)=S}} q^{\mathrm{inv}(\sigma)}$. Now, we establish the equivalence of this sum to $q^{\binom{n-k+1}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$. If $\mathrm{Des}(\sigma) = S$, then $\sigma = a_1 \cdots a_{k-1} n b_{k+1} \cdots b_n$, where $a_i < a_{i+1}$ for $1 \leq i < k-1$ and $b_j > b_{j+1}$ for $k+1 \leq j < n$. Each pair (j_1, j_2) with $k \leq j_1 < j_2 \leq n$ is an inversion, and thus there are $\binom{n-k+1}{2}$ such inversions. Therefore, since every σ with this descent set contains at least this many inversions, we factor out $q^{\binom{n-k+1}{2}}$ from the sum and consider the remaining possible inversions.

Now let W be the set of words consisting of $(k-1)$ 1s and $(n-k)$ 2s. By Proposition 3.1.2 we have for W that $\sum_{\omega\in W} q^{\mathrm{inv}(\omega)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$. Now the set of permutations with descent set $\{k, k+1, ..., n-1\}$ is in bijection with words ω consisting of $(k-1)$ 1s and $(n-k)$ 2s. The bijection takes the positions of the 1s of ω to form the increasing subword, followed by n, followed by the positions of the 2s to form the decreasing subword. For example, if $n = 9$ and $k = 6$, then $\omega = 11212112$ is identified uniquely with 124679853.

The only inversions of ω come from having a 2 precede a 1. Therefore, all of the inversions of ω correspond to inversions of type $a_i > b_j$ in σ, and so the number of inversions of ω equals the number of inversions of σ corresponding to a pair of indices (j_1, j_2) with $j_1 < k < j_2$. Since each σ had $q^{\binom{n-k+1}{2}}$ more inversions not counted by ω, we take the product to arrive at the expression $\sum_{\substack{\sigma\in\mathfrak{S}_n \\ \mathrm{Des}(\sigma)=S}} q^{\mathrm{inv}(\sigma)} = q^{\binom{n-k+1}{2}}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$, as claimed. □

Chapter 4

Geometric Lattices

4.1 Preliminaries

Recall that if a poset P is bounded, we call the elements which cover $\hat{0}$ its atoms. If every $x \in P$ is the join of some set of atoms (where the join of the empty set is $\hat{0}$), then P is called *atomic*. We have the following simple lemma concerning the atomicity of lattices:

Lemma 4.1.1. *Let L be an atomic lattice. For distinct elements $x, y \in L$, if $x \lessdot y$, then there exists an atom a such that $a < y$ and $a \nleq x$.*

Proof. Suppose that L is atomic and $x \lessdot y$, but no such atom exists. This implies that every atom $b < y$ is such that $b \leq x$. By the definition of joins, we must have that $\left(\bigvee_{\substack{b<y \\ \rho(b)=1}} b \right) \leq x$. Thus, we cannot express y as the join of any set of atoms, a contradiction. $\square$

The following proposition establishes an important collection of equivalent conditions about lattices (see [16]):

Proposition 4.1.2. *If L is a finite lattice, then the following are equivalent:*

- *L is pure and its rank function ρ satisfies for each pair $x, y \in L$*
$$\rho(x) + \rho(y) \geq \rho(x \wedge y) + \rho(x \vee y).$$
- *For each pair of elements $x, y \in L$, if $(x \wedge y) \lessdot x$, then $y \lessdot (x \vee y)$.*
- *For each pair of elements $x, y \in L$, if x and y both cover $x \wedge y$, then $x \vee y$ covers both x and y.*

A lattice L which satisfies any of the conditions of Proposition 4.1.2 is called *(upper) semimodular*. This is usually shortened to simply *semimodular* when there is no confusion, although there is a dual notion of lower semimodularity. We have the following lemma concerning semimodularity.

Lemma 4.1.3. *Let L be a semimodular lattice, and $S \subseteq L$. Then*
$$\rho\left(\bigvee_{y \in S} y\right) \leq \sum_{y \in S} \rho(y).$$

Proof. Using Proposition 4.1.2, we proceed by induction on the cardinality of S. If $|S| = 2$, then since both $\rho(x \vee z) + \rho(x \wedge z) \leq \rho(x) + \rho(z)$ and $\rho(x \wedge z) \geq 0$ by the nonnegativity of ρ, we can conclude that $\rho(x \vee z) \leq \rho(x) + \rho(z)$. Consider now $T = S \cup \{a\}$ for some element $a \notin S$. Then by the associativity of the join,
$$\begin{aligned}
\rho\left(\bigvee_{y \in T} y\right) &= \rho\left(a \vee \bigvee_{y \in S} y\right) \\
&\leq \rho(a) + \rho\left(\bigvee_{y \in S} y\right) \\
&\leq \rho(a) + \sum_{y \in S} \rho(y) \\
&= \sum_{y \in T} \rho(y).
\end{aligned}$$

$\square$

We now define a lattice to be *geometric* if it is both atomic and semimodular. It follows from Proposition 4.1.2 that every geometric lattice is pure. As a consequence of atomicity and Lemma 4.1.3, we have the following:

Corollary 4.1.4. *Let L be a geometric lattice, and fix $x \in L$. If $x = a_1 \vee \cdots \vee a_s$ for distinct atoms $a_1, ..., a_s$, then $s \geq \rho(x)$. Moreover, there exists some collection of atoms $\{a_1, ..., a_{\rho(x)}\}$ with $x = a_1 \vee \cdots \vee a_{\rho(x)}$.*

Proof. Since an atom has rank 1, observe that $\sum_{i=1}^{s} \rho(a_i) = s$. By Lemma 4.1.3, we have $\rho(x) = \rho\left(\bigvee_{i=1}^{s} a_i\right) \leq s$, as claimed. To show the existence of a set of atoms of cardinality exactly $\rho(x)$, assume that x cannot be expressed as the join of exactly $\rho(x)$ atoms, and that x is minimal among all elements with this property. Since $x \neq \hat{0}$, which is the join of $\rho(\hat{0}) = 0$ atoms, there exists $y \in L$ such that $y \lessdot x$.

Then by assumption, we have that y is the join of $\rho(y)$ atoms. But notice that for any atom $a < x$ such that $a \nleq y$ (which exists by Lemma 4.1.1), we have by semimodularity that $y \lessdot (a \vee y)$, and $a \vee y = x$, since we assumed that x covers y. Thus, we can express x as the join of $\rho(x)$ atoms, contradicting our assumption. Therefore, every x can be expressed as the join of $\rho(x)$ atoms. $\square$

Example 4.1.5. The Boolean algebra B_n is geometric. Recall that for $X, Y \in B_n$, we have that $X \wedge Y = X \cap Y$ and $X \vee Y = X \cup Y$; further, recall that $\rho(X) = |X|$. Clearly, B_n is atomic since $X = \bigvee_{a \in X} \{a\}$ for every $X \in B_n$. Moreover, it is clear that B_n is semimodular, since in fact $|X| + |Y| = |X \cap Y| + |X \cup Y|$ for all finite subsets X, Y of a set.

Example 4.1.6. The subspace lattice $B_n(q)$ is also geometric. Recall that for $X, Y \in B_n(q)$, we have that $X \wedge Y = X \cap Y$ and $X \vee Y = X + Y$; further, recall that $\rho(X) = \dim(X)$. Clearly, $B_n(q)$ is atomic since $X = \bigvee_{u \in X-(0)} (u)$, where by (u) we mean the linear subspace spanned by the vector u. Moreover, it is clear that $B_n(q)$ is semimodular, since in fact $\dim(X) + \dim(Y) = \dim(X \cap Y) + \dim(X + Y)$ for all finite dimensional subspaces X, Y of a vector space.

Example 4.1.7. The partition lattice Π_n is also geometric. Recall that atoms of Π_n are partitions with a single block of size 2 while all other blocks have size 1. For $\pi = \{B_1, B_2, ..., B_t\} \in \Pi_n$, observe that for each $B_i = \{b_1^i, b_2^i, ..., b_m^i\}$ such that $m \geq 2$, we may write $B_i = \bigcup_{j=1}^{m-1} \{b_j^i, b_{j+1}^i\}$. Therefore, we can construct each block of π by joining atoms, and thus Π_n is atomic.

To see that Π_n is semimodular, consider three partitions γ, α, and β such that $\gamma \lessdot \alpha$ and $\gamma \lessdot \beta$. Clearly $\gamma = \alpha \wedge \beta$. Let $\gamma = \{\gamma_1, \gamma_2, ..., \gamma_m\}$. We obtain α from γ by merging $\gamma_i \cup \gamma_j$, and we obtain β by merging $\gamma_k \cup \gamma_\ell$. We have two cases.

In the first case, suppose $\{i, j\} \cap \{k, \ell\} = \emptyset$. Then we clearly have that $\alpha \vee \beta$ is obtained from γ by performing both mergers $\gamma_i \cup \gamma_j$ and $\gamma_k \cup \gamma_\ell$, and so covers both α and β. In the second case, assume that $i = k$. Then we can obtain $\alpha \vee \beta$ from γ by merging $\gamma_i \cup \gamma_j \cup \gamma_\ell$, which also covers both α and β. Thus, we have that Π_n is semimodular.

4.2 Shellability of Geometric Lattices

Björner [2] proved that all geometric lattices are EL-shellable. We present the labeling in this section and a proof that it is an EL-labeling.

Definition 4.2.1. Let L be a geometric lattice with a totally ordered atom set A. The edge labeling $\Psi : \mathcal{E}(L) \to A$ is given by defining

$$\Psi(x \lessdot_L y) = \min_A\{a \in A \mid a \leq_L y, a \not\leq_L x\},$$

for each edge $x \lessdot_L y$ in L.

Notice that the label $\lambda = \Psi(x \lessdot_L y)$ is an atom of L. Further, by $\min_A$ we mean minimum with respect to $<_A$. Also, since $x \lessdot y$, and $\lambda \not\leq x$, we see that $x \vee \lambda = y$. Thus, we can equivalently say that λ is the smallest atom with respect to $<_A$ such that $x \vee \lambda = y$. This set of atoms whose minimum we seek cannot be empty by Lemma 4.1.1. If $L = B_n$, then Ψ is precisely the EL-labeling of Section 2.3.

Theorem 4.2.2. *Let L be a geometric lattice. Then the labeling Ψ of Definition 4.2.1 is an EL-labeling.*

Proof. To prove this, we must show that in any interval $[x, y] \subseteq L$, there is a unique rising chain which lexicographically precedes all other maximal chains in the interval. We begin by showing that we can construct a rising chain in the interval:

Claim 4.2.3. *Given a geometric lattice L with linearly ordered atom set A and an interval $[x, y] \subseteq L$ of length $n \geq 1$, the chain $c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n$ generated by setting $c_0 = x$ and for each $0 < i < n$ defining $c_i := c_{i-1} \vee z_i$, where $z_i = \min_A\{z \in A \mid z \leq y, z \not\leq c_{i-1}\}$, is rising.*

Proof of Claim 4.2.3. We first show that $\Psi(c_{i-1} \lessdot c_i) = z_i$. Since $z_i \leq_L c_i$ and $z_i \nleq_L c_{i-1}$, we have that

$$\begin{aligned}\Psi(c_{i-1} \lessdot c_i) &= \min_A\{a \in A \mid a \leq_L c_i, a \nleq_L c_{i-1}\} \\ &\leq_A z_i\end{aligned}$$

Now since $\{a \in A \mid a \leq_L c_i, a \nleq_L c_{i-1}\} \subseteq \{a \in A \mid a \leq_L y, a \nleq_L c_{i-1}\}$, we also have that $\Psi(c_{i-1} \lessdot c_i) \geq_A z_i$, and so $\Psi(c_{i-1} \lessdot c_i) = z_i$.

Now we need to show that $z_i <_A z_{i+1}$ for each i. Notice that $\{a \in A \mid a \leq y, a \nleq c_i\} \subsetneq \{a \in A \mid a \leq y, a \nleq c_{i-1}\}$. Therefore,

$$\begin{aligned}z_i &= \min_A\{a \in A \mid a \leq y, a \nleq c_{i-1}\} \\ &\leq_A \min_A\{a \in A \mid a \leq y, a \nleq c_i\} \\ &= z_{i+1}\end{aligned}$$

Since $z_i \notin \{a \in A \mid a \leq y, a \nleq c_i\}$, we see that $z_i \neq z_{i+1}$. Thus, we have that $\Psi(c_{i-1} \lessdot c_i) <_A \Psi(c_i \lessdot c_{i+1})$ for each i, and the chain $c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n$ is rising. $\square$

Now given that there is a rising chain in the interval, we must show it is unique.

Claim 4.2.4. *The chain constructed in Claim 4.2.3 is the only rising chain in the interval* $[x, y]$.

Proof of Claim 4.2.4. Let $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$ be a rising chain of the interval different from the chain $c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n$ of Claim 4.2.3. Since the chains are different, there exists some index i for which $x_j = c_j$ for all $j < i$, and $x_i \neq c_i$, since $c_0 = x_0 = x$. Let $\lambda_{i_1} = \Psi(c_{i-1} \lessdot c_i)$, and $\lambda_{i_2} = \Psi(c_{i-1} \lessdot x_i)$.

Since $\lambda_{i_1} = \min_A\{a \in A \mid a \leq_L y, a \nleq_L c_{i-1}\}$, we must have that $\lambda_{i_1} <_A \lambda_{i_2}$, since if $\lambda_{i_1} = \lambda_{i_2}$, then $c_i = c_{i-1} \vee \lambda_{i_1} = c_{i-1} \vee \lambda_{i_2} = x_i$. Further, this also implies that $\lambda_{i_1} \not\leq_L x_i$, since a covering must take as its label the smallest available less than the larger element of the edge. Since $\lambda_{i_1} \leq_L y$ and is smaller in the atom order than all remaining atoms also less than y, there will be some step k in the chain at which $\lambda_{i_1} \not\leq_L x_{k-1}$ but $\lambda_{i_1} \leq_L x_k$.

Hence $\Psi(x_{k-1} \lessdot x_k) = \lambda_{i_1}$. Therefore the chain $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$ has a descent, since it has label λ_{i_2} precede λ_{i_1}. This contradicts the assumption that the chain is rising; thus, $c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n$ is uniquely rising. □

Claim 4.2.5. *The chain constructed in Claim 4.2.3 lexicographically precedes all other chains of the interval* $[x, y]$.

Proof of Claim 4.2.5. To see that $c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n$ lexicographically precedes all other chains, we begin at its first label. Now since $c_0 = x$, the first label of the chain is $z_1 = \min_A\{a \in A \mid a \not\leq_L x, a \leq_L y\}$. Any other chain which does not use z_1 as its first label will therefore succeed this constructed chain in the lexicographic order since its first label is strictly larger.

On the other hand, if the first label of any other chain is also z_1, then the second element in the chain is by definition $x \vee z_1 = c_1$. Thus, the chains coincide along this first edge. We now repeat this argument for z_2, but only on chains whose first label was z_1, as we have already established that only these may equal or precede the constructed chain. By repeating this argument for each step in the chain, we see that the chain $c_0 \lessdot c_1 \lessdot \cdots \lessdot c_n$ lexicographically precedes every other chain. □

Combining these three claims, the edge labeling Ψ is an EL-labeling of L. □

As an example of this edge labeling, we illustrate Π_4 labeled in this fashion in Figure 4.1; notice that here we use an arbitrary index for the atoms, and label edges with this index for brevity. The labeled bold edges in the middle of Figure 4.1 are part of a falling chain. Notice that the falling chains of this labeling are different from the falling chains of the labeling used in Figure 2.3, but that there are still exactly six falling chains in both cases.

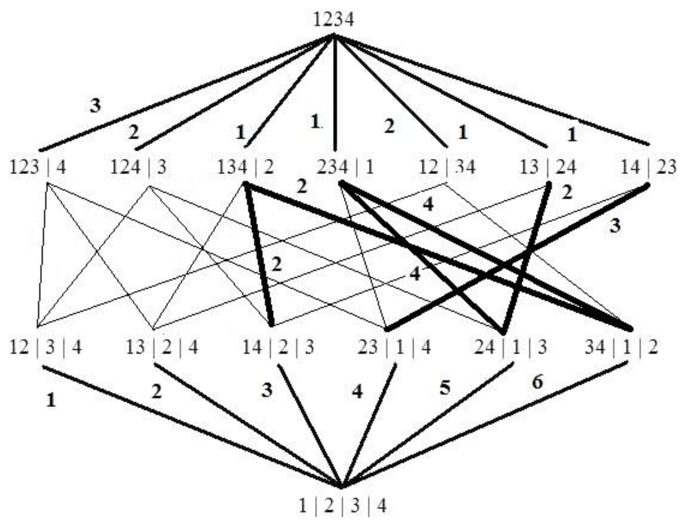


Figure 4.1: A geometric labeling of Π_4.

Chapter 5

Partial Decomposition Posets

5.1 Definition and Examples

Recall that if every pair of elements in a poset P has a join, we call P a join semilattice. Let P be a ranked join semilattice, with rank function ρ. A set $T = \{x_1, ..., x_m\} \subseteq P$ is called *independent* if $\sum_{i=1}^{m} \rho(x_i) = \rho\left(\bigvee_{i=1}^{m} x_i\right)$ and $\rho(x_i) \neq 0$ for each i. Define the *partial decomposition poset* Π_P of P as the set of independent sets T of P, where we define the order relation $\leq_{\Pi_P}$ for any pair $T, S \in \Pi_P$ by saying that $T \leq_{\Pi_P} S$ if and only if $\forall\ x \in T$ there exists $y \in S$ such that $x \leq_P y$.

It is easy to verify that the order relation $\leq_{\Pi_P}$ is a partial order. Note that Π_P has a bottom element, namely the empty set; to distinguish this from $\hat{0} \in P$ when P is bounded, we denote the minimum $T_0 := \emptyset$. Also, observe that since $\hat{1} \in P$ exists, the element $T_1 := \{\hat{1}\}$ is maximum in Π_P. Consider the following example— let P be the join semilattice given by the following Hasse diagram:

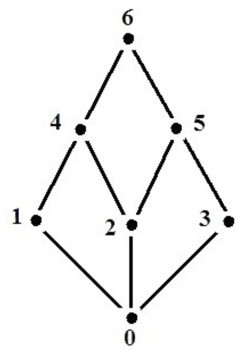
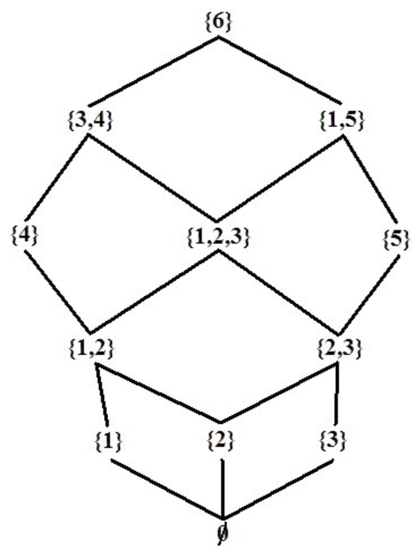


Figure 5.1: The poset P.

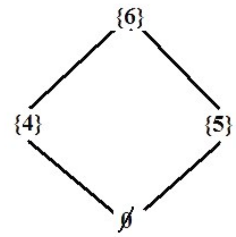
Applying the definition, we have that Π_P is given by



Define now for positive integer k the *k–equal partial decomposition poset* $\Pi_P^{=k}$ as the induced subposet of Π_P given by

$$\Pi_P^{=k} = \{T \in \Pi_P \mid \rho(x) \geq k \ \forall \ x \in T\}.$$

Further, note that if $k = \ell(P)$, then $\Pi_P^{=k} \cong c_1$, the length one chain, while if $k = 1$, then $\Pi_P^{=1} = \Pi_P$. We will only consider $k > 1$ in general. For instance, for the poset P of Figure 5.1, $\Pi_P^{=2}$ is given by



Consider the following two fundamental examples of partial decomposition posets.

Example 5.1.1. Let P be the Boolean algebra B_n. Let $T = \{A_1, A_2, ..., A_t\} \subset B_n$, with $A_i \neq \emptyset$ for each i. Since $\rho(A_i) = |A_i|$ in B_n and also $A_i \vee A_j = A_i \cup A_j$, we have that T is independent if and only if $\sum_{i=1}^{t} |A_i| = \left|\bigcup_{i=1}^{t} A_i\right|$. Clearly, we have that T is independent if and only if the sets $A_1, ..., A_t$ are mutually disjoint.

Observe that Π_{B_n} is the set of partial partitions of $[n]$. This lattice was introduced and denoted $\Pi_{\leq n}$ by Hanlon, Hersh, and Shareshian [13, Definition 3.1]. They give the definition in terms of partitions of subsets of $[n]$ rather than in terms of independent sets.

We can identify $T \in \Pi_{B_n}^{=k}$ with the element T' in the k–equal partition lattice $\Pi_n^{=k}$ by requiring that each set $A_i \in T$ is a block of T' and if $a \notin \bigcup_{i=1}^{t} A_i$, then $\{a\}$ is a singleton block of T'. This map between $\Pi_{B_n}^{=k}$ and $\Pi_n^{=k}$ is a bijection, since its inverse simply forgets the singleton blocks of T'.

Further, it preserves the order relation, since given $T \leq_{\Pi_{B_n}^{=k}} S$, any singleton of S' must also be a singleton of T'. Therefore, since each block of T is contained in some block of S and any singleton block of T' is contained in a block of S', we must have that $T' \leq_{\Pi_n^{=k}} S'$. Therefore, we have that $\Pi_{B_n}^{=k} \cong \Pi_n^{=k}$ for each positive integer $2 \leq k \leq n$. Notice that since $\Pi_n^{=2} = \Pi_n$, we have that

$$\Pi_{B_n}^{=2} \cong \Pi_n.$$

Example 5.1.2. Let $P = B_n(q)$ for prime power q. Let $T = \{U_1, ..., U_t\} \subset B_n(q)$, with $U_i \neq (0)$ for each i. To be independent, since $\rho(U_i) = \dim(U_i)$ and also $U_i \vee U_j = U_i + U_j$, we have that T is independent if and only if $\sum_{i=1}^{t} \dim(U_i) = \dim(U_1 + \cdots + U_t)$.

Let V be a finite dimensional vector space over the field $\mathbb{F}_q$, and let $U_1, U_2, ..., U_t$ be nontrivial subspaces of V. Recall that the *sum* $U_1 + \cdots + U_t$ is defined as the smallest subspace of V so that for any vector $v \in U_1 + \cdots + U_t$, we may write $v = u_1 + \cdots + u_t$ for some collection of vectors $u_i \in U_i$ for each i. The sum is called *direct* and denoted by $U_1 \oplus \cdots \oplus U_t$ if one of the following equivalent conditions hold:

- Every vector $v \in U_1 + \cdots + U_t$ has a unique expression as a sum, $v = \sum_{i=1}^{t} u_i$, where $u_i \in U_i$ for each i.

- For each i, we have that $U_i \cap (U_1 + \cdots + U_{i-1} + U_{i+1} + \cdots + U_t) = (0)$, the trivial subspace.

- $\dim(U_1 + \cdots + U_t) = \sum_{i=1}^{t} \dim(U_i)$.

We call $U_1 \oplus \cdots \oplus U_t$ a *partial direct sum decomposition* of V. Therefore, the set $T = \{U_1, ..., U_t\}$ is independent if and only if $U_1 \oplus \cdots \oplus U_t$ is a partial direct sum decomposition of $V = \mathbb{F}_q^n$. Thus, we have that $\Pi_{B_n(q)}$ is the poset $PD_n(q)$ of all partial direct sum decompositions of $\mathbb{F}_q^n$, discussed in Section 3.2. On the other hand, the poset $\Pi^{=k}_{B_n(q)}$ is the poset of partial direct sum decompositions whose summands all have dimension at least k, which has so far not been studied.

From the observation that $B_n(q)$ can be viewed as a q–analogue of B_n, and $\Pi^{=2}_{B_n} \cong \Pi_n$, we can hope that $\Pi^{=2}_{B_n(q)}$ can also be viewed as a new q–analogue of Π_n; we will provide further justification in Chapters 6 and 7. Because of these results, we will later use the alternative notation $\Pi_n^{=k}(q)$ for $\Pi_{B_n(q)}^{=k}$.

5.2 Preliminary Lemmas About Independent Sets

Before proving more substantial results concerning partial decomposition posets, we prove some results concerning the nature of Π_L for L a geometric lattice. For these lemmas, we always assume that L is a geometric lattice with atom set A.

Lemma 5.2.1. *A set T is independent in L if and only if every subset of T is independent in L.*

Proof. In one direction, the proof is completely trivial. To show the other direction, assume that T is independent and some subset S of T is not independent. Denote $T - S$ by S'.

Since T is independent, we have that $S \neq T$. By the assumption that S is not independent and by Lemma 4.1.3, $\rho\left(\bigvee_{y \in S} y\right) < \sum_{y \in S} \rho(y)$. Now observe that by join associativity we have that $\rho\left(\bigvee_{y \in T} y\right) = \rho\left(\left(\bigvee_{y \in S'} y\right) \vee \left(\bigvee_{y \in S} y\right)\right)$. Applying Lemma 4.1.3 to this, we obtain

$$\begin{aligned}
\rho\left(\left(\bigvee_{y \in S'} y\right) \vee \left(\bigvee_{y \in S} y\right)\right) &\leq \rho\left(\bigvee_{y \in S'} y\right) + \rho\left(\bigvee_{y \in S} y\right) \\
&< \sum_{y \in S'} \rho(y) + \sum_{y \in S} \rho(y) \\
&= \sum_{y \in T} \rho(y).
\end{aligned}$$

Thus, T could not be independent, a contradiction. $\square$

Lemma 5.2.2. *Let $T = \{x_1, x_2, ..., x_m\}$ be an independent set of L. If $b \in A$ is such that $b \not\leq (x_1 \vee \cdots \vee x_m)$, then both $\{x_1, ..., (x_m \vee b)\}$ and $\{x_1, ..., x_m, b\}$ are also independent sets of L.*

Proof. Notice that $b \not\leq x_m$, so by semimodularity, since $b \wedge x_m = \hat{0}$, we have that $\rho(b \vee x_m) = \rho(x_m) + 1 = \rho(x_m) + \rho(b)$. By the associativity of the join, observe that $\rho\left(b \vee \bigvee_{i=1}^{m} x_i\right) = \rho\left((b \vee x_m) \vee \bigvee_{i=1}^{m-1} x_i\right)$, so that the independence of $\{x_1, ..., x_m, b\}$ implies the independence of $\{x_1, ..., (x_m \vee b)\}$.

Let $z = \bigvee_{i=1}^{m} x_i$. Since $b \not\leq z$ and b is an atom, we have that $b \wedge z = \hat{0}$, and so $b \wedge z \lessdot b$. Therefore by semimodularity we have that $z \lessdot b \vee z$, so that $\rho(b \vee z) = \rho(z) + 1 = \rho(z) + \rho(b)$. Now since T is independent, we have that $\rho(z) = \sum_{i=1}^{m} \rho(x_i)$. Combining these results gives that $\rho\left(b \vee \bigvee_{i=1}^{m} x_i\right) = \rho(b) + \sum_{i=1}^{m} \rho(x_i)$, and so $T \cup \{b\}$ is independent. $\square$

Lemma 5.2.3. *Let T be an independent set of L, and suppose $x \in T$ and $S \subseteq T - \{x\}$. Then $x \wedge \bigvee_{y \in S} y = \hat{0}$.*

Proof. Since S and $S \cup \{x\}$ are both subsets of T, by Lemma 5.2.1 they are independent. By the semimodularity of L and the independence of both S and $S \cup \{x\}$ we then have

$$\begin{aligned}
\rho(x) + \rho\left(\bigvee_{y \in S} y\right) &\geq \rho\left(x \vee \left(\bigvee_{y \in S} y\right)\right) + \rho\left(x \wedge \left(\bigvee_{y \in S} y\right)\right) \\
&= \rho(x) + \sum_{y \in S} \rho(y) + \rho\left(x \wedge \left(\bigvee_{y \in S} y\right)\right) \\
&= \rho(x) + \rho\left(\bigvee_{y \in S} y\right) + \rho\left(x \wedge \left(\bigvee_{y \in S} y\right)\right)
\end{aligned}$$

This gives that $0 \geq \rho\left(x \wedge \left(\bigvee_{y \in S} y\right)\right)$. However, since ρ is a nonnegative function, we conclude that $\rho\left(x \wedge \left(\bigvee_{y \in S} y\right)\right) = 0$; moreover, since $\hat{0}$ is the only element of L of rank 0, we have $x \wedge \left(\bigvee_{y \in S} y\right) = \hat{0}$, as claimed. $\square$

Corollary 5.2.4. *If $T \leq_{\Pi_L} S$, then each element $x \in T$ is less than a unique element $y \in S$ in L.*

Proof. Since the elements of any independent set must have meet $\hat{0}$, there can be no distinct elements $y_1, y_2 \in S$ such that $x \leq_L y_1$ and $x \leq_L y_2$ for any $x \in T$, as $(y_1 \wedge y_2)$ would not be $\hat{0}$, a contradiction. $\square$

Lemma 5.2.5. *Let T be an independent set of L, and let $x \in T$. If $w <_L x$, then the set $(T - \{x\}) \cup \{w\}$ is independent.*

Proof. Without loss of generality, assume that $w \lessdot x$. Since L is geometric, there exists an atom a such that $w \vee a = x$. Let $z = \bigvee_{y \in T - \{x\}} y$. By associativity of the join, $z \vee x = z \vee (w \vee a) = (z \vee w) \vee a$. By semimodularity, we have that $\rho(z \vee x) = \rho((z \vee w) \vee a) \leq \rho(z \vee w) + \rho(a) = \rho(z \vee w) + 1$. By the independence of T and Lemma 5.2.1, we also have that $\rho(z \vee x) = \rho(z) + \rho(x)$. Combining this inequality and equation gives that $\rho(z) + \rho(x) - 1 \leq \rho(z \vee w)$. Since $\rho(w) = \rho(x) - 1$, we have that $\rho(z) + \rho(w) \leq \rho(z \vee w)$. But semimodularity also requires that $\rho(z) + \rho(w) \geq \rho(z \vee w)$; therefore, we must have that $\rho(z \vee w) = \rho(z) + \rho(w)$, and so $(T - \{x\}) \cup \{w\}$ is independent. $\square$

Lemma 5.2.6. *Let R be an independent subset of L. Let $T \subseteq L$ be such that $\bigvee_{y \in R} y \in T$ and $R \cap T = \emptyset$. Then T is independent if and only if $S = \left(T - \left\{\bigvee_{y \in R} y\right\}\right) \cup R$ is independent.*

Proof. Let $z = \bigvee_{y \in R} y$. We have that

$$\begin{aligned}
\rho\left(\bigvee_{y\in S} y\right) &= \rho\left(\left(\bigvee_{y\in T-\{z\}} y\right) \vee \left(\bigvee_{y\in R} y\right)\right) \\
&= \rho\left(\left(\bigvee_{y\in T-\{z\}} y\right) \vee z\right) \\
&= \rho\left(\bigvee_{y\in T} y\right).
\end{aligned}$$

We also have that

$$\begin{aligned}
\sum_{y\in S} \rho(y) &= \left(\sum_{y\in T-\{z\}} \rho(y)\right) + \rho\left(\bigvee_{y\in R} y\right) \\
&= \left(\sum_{y\in T-\{z\}} \rho(y)\right) + \rho(z) \\
&= \sum_{y\in T} \rho(y).
\end{aligned}$$

Therefore S is independent if and only if T is independent. □

Note that by combining Lemmas 5.2.5 and 5.2.6, if $T \subseteq L$ is an independent set, then we can replace any $x \in T$ by any independent subset $R \subseteq [\hat{0}, x] \subset L$ and maintain independence.

5.3 Establishing That $\Pi_L^{=k}$ Is a Lattice

As an application of the results of Section 5.2, we next show that $\Pi_L^{=k}$ is also a lattice when L is geometric. Let L be a geometric lattice, and $0 < k < \ell(L)$

a fixed integer. For the pair of distinct elements $T, S \in \Pi_L^{=k}$, define

$$T \diamond S := \Big\{(x \wedge y) \in L \mid x \in T,\ y \in S,\ \rho(x \wedge y) \geq k\Big\}. \tag{5.3.1}$$

Lemma 5.3.1. *Let L be a geometric lattice, and $0 < k < \ell(L)$ a fixed integer. Fix $T, S \in \Pi_L^{=k}$. If $T \diamond S$ is independent, then $T \diamond S = T \wedge S$.*

Proof. Let $U \in \Pi_L^{=k}$ be an arbitrary lower bound of T and S. Then for each element $z \in U$, by Corollary 5.2.4 and the definition of the order relation, there exists a unique $x \in T$ and a unique $y \in S$ such that $z \leq_L x$ and $z \leq_L y$; thus z is a lower bound of x and y in L.

Now since $x, y, z \in L$, by the definition of meets we have $z \leq_L (x \wedge y)$. Thus, each element in U is less than or equal to the meet of some unique pair of elements, one from T and one from S. Since we define $T \diamond S$ to be the set of all such meets, we conclude $U \leq_{\Pi_L^{=k}} T \diamond S$. Thus, $T \diamond S$ is the meet. □

Theorem 5.3.2. *Let L be a geometric lattice, and $0 < k \leq \ell(L)$ a fixed integer. Then $\Pi_L^{=k}$ is a lattice.*

Proof. When $k = \ell(L)$, since $\Pi_L^{=k} \cong c_1$, it is trivially a lattice; assume then that $k < \ell(L)$. We have already defined $T \diamond S$ given any two elements $T, S \in \Pi_L^{=k}$, and shown that if it is independent, then it is the meet. We now show that $T \diamond S$ is independent. To show this, we use the results of the Section 5.2. These give us three processes which allow us to transform a given independent set into a "smaller" independent set in $\Pi_L^{=k}$.

Consider T and $T \diamond S$. We begin by deleting all elements of T which are not greater than any element of $T \diamond S$, using Lemma 5.2.1. Next, we can look for any element of T which is greater than a single element of $T \diamond S$; let $x \in T$ and $y \in T \diamond S$ be such a pair, and assume $x \neq y$, or else there is nothing more to do. Since $y <_L x$, by Lemma 5.2.5 the set $(T - \{x\}) \cup \{y\}$ is independent.

Last, assume there are multiple elements of $T \diamond S$ less than some fixed element of $x \in T$. That is, let $\{y_1, ..., y_j\} \subseteq T \diamond S$ be such that each $y_i \leq_L x$. Since each $y_i \leq_L x$, we must also have that $(y_1 \vee \cdots \vee y_j) \leq_L x$. If $x \neq (y_1 \vee \cdots \vee y_j)$, then we use Lemma 5.2.5 to replace x by this join. Notice by the definition of $T \diamond S$, that there must exist distinct $z_1, ..., z_j \in S$ such that $y_i \leq_L z_i$ for each i, since they are all less than the same $x \in T$.

Therefore, since each y_i can be obtained by application of Lemma 5.2.5 to S, the set $\{y_1, ..., y_j\}$ is also independent. Applying Lemma 5.2.6 to the join $y_1 \vee \cdots \vee y_j$ allows us to preserve independence as we replace $y_1 \vee \cdots \vee y_j$ by the subset $\{y_1, ..., y_j\}$.

Combining these three processes shows that $T \diamond S$ is independent, since each of its elements can be obtained from T in a way which preserves independence; thus, $T \diamond S = T \wedge S$, so $\Pi_L^{=k}$ is a meet semilattice. Recall that $T_1 = \{\hat{1}\}$ is maximum in $\Pi_L^{=k}$. Since any meet semilattice with a top element is a lattice, we have that $\Pi_L^{=k}$ is a lattice. $\square$

For example, consider $L = B_9$ and $k = 2$. Using traditional block notation, let $T = 123|4567|89$ and $S = 128|3459|67$. Now $T \wedge S = 12|45|67$. We can obtain $T \wedge S$ from T by deleting block 89, replacing block 123 by 12, and then splitting block 4567 to form the set of blocks 45|67.

Because $\Pi_L^{=k}$ is a lattice when L is geometric, we shall call it the $k-$ *equal partial decomposition lattice*. It can easily be verified that these lattices are only pure either when $k = 2$, or when $k > \frac{n}{2}$. Specifically, we have that $\ell(\Pi_L^{=2}) = \ell(L) - 1$, while for $k > 2$, the lattice has maximal chains of length $(\ell(L) - 1) - t(k - 2)$ for each positive integer $t \leq \frac{\ell(L)}{k}$.

Now being a lattice, we may ask whether $\Pi_L^{=k}$ is also semimodular. We previously observed that $\Pi_{B_n}^{=k} \cong \Pi_n^{=k}$ was not pure whenever $k > 2$; since semimodularity implies purity, this lattice is also not semimodular. When $k = 2$, we have the following result:

Proposition 5.3.3. *For $k = 2$, there exists a subspace lattice $B_n(q)$ such that $\Pi_{B_n(q)}^{=2}$ is not semimodular.*

The implication of this statement is that it is not true in general that $\Pi_L^{=k}$ is geometric, even if L itself is geometric. This contrasts with the result that $\Pi_{B_n}^{=2} \cong \Pi_n$, which is geometric; the proposition implies this is an exceptional case.

Proof. Consider $L = B_{10}(2)$, the lattice of subspaces of the vector space $\mathbb{F}_2^{10}$. We will construct a pair of elements $\mathcal{T}, \mathcal{S} \in \Pi_{B_{10}(2)}^{=2}$ which will fail the condition for semimodularity; namely, we will show that $\mathcal{T} \wedge \mathcal{S} \lessdot \mathcal{T}$ but $\mathcal{S}$ is not covered by $\mathcal{T} \vee \mathcal{S}$.

We use $m \times 10$ reduced row echelon matrices with nonzero rows to represent m–dimensional subspaces of $\mathbb{F}_2^{10}$. Consider the following four subspaces of $\mathbb{F}_2^{10}$:

$$T = \begin{pmatrix} 0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&1&1&0&0 \end{pmatrix}, \; S_1 = \begin{pmatrix} 1&0&1&0&1&0&1&0&1&0 \\ 0&1&0&1&0&1&0&1&0&1 \\ 0&0&0&0&0&0&0&0&1&1 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 1&1&1&0&0&0&1&1&1&0 \\ 0&0&0&1&1&1&0&0&0&0 \end{pmatrix}, \; S_3 = \begin{pmatrix} 1&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&1&1 \\ 0&0&0&0&0&0&1&1&0&0 \end{pmatrix}.$$

We now define the sets $\mathcal{T} = \{T\}$ and $\mathcal{S} = \{S_1, S_2, S_3\}$. It is not difficult to check both that $\mathcal{S}$ is independent and that from the characterization of the meet of independent sets given in (5.3.1), $\mathcal{T} \wedge \mathcal{S} = \{V\}$, where

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

It is clear that $V \lessdot T$ in $B_{10}(2)$; hence $\mathcal{T} \wedge \mathcal{S} \lessdot \mathcal{T}$. Suppose now that $S \lessdot \mathcal{T} \vee \mathcal{S}$. Then we must have that $\mathcal{T} \vee \mathcal{S}$ has one of the following forms: (1) $\{S_i \vee S_j, S_k\}$ for some permutation ijk of the indices $1, 2, 3$ such that $T \subseteq S_i \vee S_j$; or (2) $\{S_i, S_j, S'_k\}$ for some permutation ijk of the indices $1, 2, 3$ where $S_k \lessdot_{B_n(q)} S'_k$ and $T \subseteq S'_k$.

We can see now that (1) is not possible by considering the vector $a = (0, 0, 0, 1, 0, 0, 0, 0, 0, 0)$. Note that $a \in T$, but $a \notin S_i \vee S_j$ for any pair i, j, as we can see from the following pairwise joins:

$$S_1 \vee S_2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$S_1 \vee S_3 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$S_2 \vee S_3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

We now show that (2) is also not possible. Since $a \notin S_k$ but $a \in T \subseteq S'_k$, we see that $S'_k = S_k \vee (a)$. Notice however that a can be expressed as the sum of the vectors $(0,0,0,0,0,0,0,0,1,1) \in S_1$ and $(0,0,0,1,0,0,0,0,1,1) \in S_2 \vee S_3$. Therefore, we have that $a \in S_i \vee S_j \vee S_k$, and so $S_i \vee S_j \vee S_k = S_i \vee S_j \vee S'_k$. This contradicts the independence of $\{S_i, S_j, S'_k\}$, hence $\mathcal{S}$ is not covered by $\mathcal{T} \vee \mathcal{S}$. $\square$

Chapter 6

The Topology of $\Pi_L^{=k}$

6.1 The EL-Shellability of $\Pi_L^{=k}$

We now will show that $\Pi_L^{=k}$ is EL-shellable if L is a geometric lattice and $k \geq 2$ is an integer. Denote by A the atom set of L, and fix a linear order $<_A$ on A. Since A is linearly ordered, given any subset $S \subseteq A$, define $w(S)$ to be the strictly increasing word on the letters of S. That is, if $S = \{a_1, ..., a_s\}$ is such that $a_i <_A a_{i+1}$ for each $i \in [s-1]$, then $w(S) = a_1 a_2 \cdots a_s$.

Since L is geometric, any element $x \in L$ can be written as a join of some collection of atoms of L. We say that a set of atoms $\mathrm{B}_x = \{a_1, ..., a_m\}$ is a *basis* of x if $x = \bigvee_{i=1}^{m} a_i$ and $\rho(x) = |\mathrm{B}_x|$. We then use the order $<_A$ to induce a lexicographic order on the set of words of the form $w(B_x)$ for a fixed $x \in L$. That is, given two words $w_1 = a_1 a_2 \cdots a_m$ and $w_2 = b_1 b_2 \cdots b_m$, then $w_1 < w_2$ if there exists unique index $1 \leq j \leq m$ such that $a_i = b_i$ whenever $i < j$ and $a_j <_A b_j$. Since there is a bijective correspondence between increasing words and subsets of A, this induces a lexicographic order on bases of x. Now we define a special basis in the following way:

Definition 6.1.1. Given a linear order $<_A$ of the atoms of geometric lattice L, a basis $\mathrm{U}_x = \{a_1, ..., a_m\}$ of $x \in L$ is said to be the *unique minimal basis* of x if $w(\mathrm{U}_x) \leq w(B_x)$ for any basis B_x of x.

We will always denote the unique minimal basis of the element x by U_x. Further, we can make the following observation concerning this basis:

Lemma 6.1.2. *Fix $x \in L$ for a geometric lattice L with a given linear atom order $<_A$. Suppose $U_x = \{a_1, ..., a_m\}$ where $a_1 <_A a_2 <_A \cdots <_A a_m$. Then the label sequence of the unique rising chain of $[\hat{0}, x]$ under the EL-labeling Ψ_L of L induced by the atom order (defined in Definition 4.2.1) is $(a_1, a_2, ..., a_m)$.*

Proof. Recall that the edge $y \lessdot z$ of L is labeled by finding the smallest atom a in A such that $y \vee a = z$, and assigning $y \lessdot z$ the label a. We then can construct a maximal chain $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_m$ of $[\hat{0}, x]$ by defining for each $0 \leq j \leq m$ the element $x_j = \bigvee_{i \leq j} a_i$. Note that $x_0 = \hat{0}$, while $x_m = x$ by the definition of a basis. That these elements form a maximal chain is a consequence of semimodularity, as we see that $x_j \lessdot x_{j+1}$ for each j. Consider the edge $x_j \lessdot x_{j+1}$ for a fixed $0 \leq j < m$. Suppose that the label of the edge $x_i \lessdot x_{i+1}$ is a_{i+1} for each $0 \leq i < j$, and that there exists atom $b <_A a_{j+1}$ which is minimal among all atoms such that $x_j \vee b = x_{j+1}$; the edge $x_j \lessdot x_{j+1}$ must receive label b.

Furthermore, if such a b existed, there would exist a basis B_x such that the first $j+1$ letters of $w(B_x)$ are $a_1 \cdots a_j b$. Regardless of the remaining atoms in this basis, $w(B_x)$ would then lexicographically precede $w(\mathrm{U}_x)$. Therefore, no such atom b can exist, and so the edge $x_j \lessdot x_{j+1}$ must receive label a_{j+1} for each j. Since $a_1 <_A a_2 <_A \cdots <_A a_m$, the maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_m = x$

has label sequence $(a_1, ..., a_m)$, and thus is the rising chain of $[\hat{0}, x]$. $\square$

Once we have U_x for each $x \in L$, we can use this basis to generate covering labels in $\Pi_L^{=k}$. First, we observe that there are only three types of coverings in $\Pi_L^{=k}$, generalizing the three observed for the k–equal partition lattice. We summarize these coverings as follows.

Proposition 6.1.3. *Given a geometric lattice L and a fixed integer $2 \leq k \leq \ell(L)$, then $T \lessdot S$ for $T, S \in \Pi_L^{=k}$ if and only if one of the following hold:*

Type I - Creation:

There exists a unique $y \notin T$ such that $S = T \cup \{y\}$ and $\rho(y) = k$.

Type II - Expansion:

There exists a unique $x \in T$ and $y \in S$ such that $x \lessdot_L y$ and $S = (T - \{x\}) \cup \{y\}$.

Type III - Merger:

There exist unique and distinct $x, y \in T$ such that $S = (T - \{x, y\}) \cup \{x \vee y\}$.

We can now assign labels to each type of edge as follows:

Definition 6.1.4. Let L be a geometric lattice with linearly ordered atom set A, and fix an integer $2 \leq k \leq \ell(L)$. Define the edge labeling $\Psi : \mathcal{E}(\Pi_L^{=k}) \to [2] \times A$ by labeling each type of covering step described in Proposition 6.1.3 as follows:

Type I - Creation:

$\Psi(T \lessdot S) = (2, b)$, where $b = \max_A(\mathrm{U}_y)$.

Type II - Expansion:

$\Psi(T \lessdot S) = (2, b)$, where $b = \min_A\{a \in A \mid a \vee x = y\}$.

Type III - Merger:

$\Psi(T \lessdot S) = (1, b)$, where $b = \max_A(\mathrm{U}_x \cup \mathrm{U}_y)$.

We now order the label set lexicographically, so that $(1, *) < (2, *)$ for any arbitrary entries in the second component, while for fixed m, we use the atom ordering of L to say that $(m, a) < (m, b)$ if and only if $a <_A b$. This is well-defined, since the second component is always an atom of L, which we call the *labeling atom* of the covering relation.

Further, we have the following lemma concerning mergers and their labels:

Lemma 6.1.5. *Let T be an independent set of geometric lattice L, and let $x, y \in T$ be distinct. Then*

$$\max_A(U_{x \vee y}) \leq_A \max_A(U_x \cup U_y).$$

Proof. We have $x \vee y = \left(\bigvee_{b \in \mathrm{U}_x} b\right) \vee \left(\bigvee_{b \in \mathrm{U}_y} b\right) = \left(\bigvee_{b \in \mathrm{U}_x \cup \mathrm{U}_y} b\right)$. Since $\{x, y\} \subseteq T$ is independent, $\mathrm{U}_x \cap \mathrm{U}_y = \emptyset$ by Lemma 5.2.3. Therefore $\mathrm{U}_x \cup \mathrm{U}_y$ is a basis for $x \vee y$.

Let $\hat{0} = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_{m-1} \lessdot z_m = x \vee y$ be the rising chain of $[\hat{0}, x \vee y]$ under the labeling Ψ_L of L induced by the linear order of A given in Definition 4.2.1. By Lemma 6.1.2, we have that $a := \max_A(\mathrm{U}_{x \vee y}) = \Psi_L(z_{m-1} \lessdot x \vee y)$. This implies that $a = \min_A\{b \in A \mid b \leq_L x \vee y, b \not\leq_L z_{m-1}\}$.

Assume that $\max_A(\mathrm{U}_x \cup \mathrm{U}_y) <_A a$; thus, for all $b \in \mathrm{U}_x \cup \mathrm{U}_y$, we have $b <_A a$. By the minimality of a, we have that $b \in \mathrm{U}_x \cup \mathrm{U}_y$ implies $b \leq_L z_{m-1}$. Thus,

$\bigvee_{b \in \mathrm{U}_x \cup \mathrm{U}_y} b \leq_L z_{m-1} \lneq_L x \vee y$. This contradicts that $\mathrm{U}_x \cup \mathrm{U}_y$ is a basis of $x \vee y$; thus, we must have that $\max_A(\mathrm{U}_{x \vee y}) \leq_A \max_A(\mathrm{U}_x \cup \mathrm{U}_y)$. □

With this labeling, we now state and prove the following main theorem:

Theorem 6.1.6. *Given a geometric lattice L and a positive integer $1 < k \leq \ell(L)$, the labeling Ψ of Definition 6.1.4 for the partial decomposition lattice $\Pi_L^{=k}$ is an EL-labeling.*

Proof. The proof of this theorem generalizes the proof of Proposition 2.5.1 for the k–equal partition lattice [7]. Clearly, if $k = \ell(L)$, the lattice is EL-shellable, being isomorphic to the chain c_1; assume then that $k < \ell(L)$. We begin by analyzing upper intervals $[S, T_1]$, of which there are two types.

The first type of upper interval is of the form $[S, T_1]$, for $S \neq T_0 \in \Pi_L^{=k}$. We start by indexing the elements of $S = \{x_1, ..., x_p\}$ so that for all pairs of indices $i < j$,

$$\max_A(\mathrm{U}_{x_i}) <_A \max_A(\mathrm{U}_{x_j}). \tag{6.1.1}$$

This is well-defined, since no atom is less than two elements of S simultaneously by Lemma 5.2.3. For each $i \in \{1, ..., p\}$, define $w_i = \bigvee_{j=1}^{i} x_j$. Now define $S_i = \{w_i, x_{i+1}, x_{i+2}, ..., x_p\}$ for each i, with $1 \leq i \leq p$. In particular, observe that $S_1 = S$, and $S_p = \{w_p\}$; also, observe that $w_i = w_{i-1} \vee x_i$, and that $S_i = (S_{i-1} - \{w_{i-1}, x_i\}) \cup \{w_i\}$.

Let $\rho(w_p) = r$. Define (recursively) for each i with $p+1 \leq i \leq \ell(L) - r$ the following:

- $b_i = \min_A\{a \in A \mid a \not\leq w_{i-1}\}$
- $w_i = w_{i-1} \vee b_i$

Further, define $S_i = \{w_i\}$ for each i with $p+1 \leq i \leq \ell(L)-r$.

Claim 6.1.7. *The maximal chain* $c := S_1 \lessdot S_2 \lessdot \cdots \lessdot S_p \lessdot S_{p+1} \lessdot \cdots \lessdot S_{\ell(L)-r}$ *is rising in the upper interval* $[S, T_1]$.

Proof of Claim 6.1.7. First, observe that $S_{i-1} \lessdot S_i$ for each i, since when $i \leq p$, each w_i is the merger of the two elements w_{i-1} and x_i of S_{i-1}, while when $i > p$, each w_i is an expansion of S_{i-1}. For the mergers, we need to show that for each i, we have $\max_A(\mathrm{U}_{w_i}) <_A \max_A(\mathrm{U}_{x_{i+1}})$, which will imply that the labeling atom for each merge is $\max_A(\mathrm{U}_{x_{i+1}})$. For $i = 1$ it is clear that $\max_A(\mathrm{U}_{w_1}) = \max_A(\mathrm{U}_{x_1}) <_A \max_A(\mathrm{U}_{x_2})$ by (6.1.1). Assume now that the inequality holds for some $i \geq 1$. By Lemma 6.1.5, the definition of w_{i+1}, and (6.1.1), we have that

$$\begin{aligned}
\max_A(\mathrm{U}_{w_{i+1}}) &= \max_A(\mathrm{U}_{w_i \vee x_{i+1}}) \\
&\leq_A \max_A\{\max_A(\mathrm{U}_{w_i}), \max_A(\mathrm{U}_{x_{i+1}})\} \\
&= \max_A(\mathrm{U}_{x_{i+1}}) \\
&<_A \max_A(\mathrm{U}_{x_{i+2}}).
\end{aligned}$$

Therefore, by induction on i, we have $\max_A(\mathrm{U}_{w_i}) <_A \max_A(\mathrm{U}_{x_{i+1}})$ for all $1 \leq i \leq p-1$.

Thus, for each $i \in [p-1]$,

$$\Psi(S_i \lessdot S_{i+1}) = (1, \max_A(\mathrm{U}_{x_{i+1}})). \tag{6.1.2}$$

Now we can see that the sequence of labels for this chain is precisely $(1, \max_A(\mathrm{U}_{x_2})), ..., (1, \max_A(\mathrm{U}_{x_p})), (2, b_{p+1}), ..., (2, b_{\ell(L)-r})$. By the choice of indices for the elements of S from (6.1.1), the merger subsequence is strictly increasing. That the expansion subsequence is strictly increasing is a consequence of the fact that since each b_i is the smallest atom available at the $(i-p)^{th}$ expansion step, these labels must increase, since the minimum of the set of available atoms must get strictly larger after every expansion. Thus, the chain c is rising. $\square$

(Proof of Theorem 6.1.6 continued) Note that if $|S| = 1$, then $p = 1$, so there are no merger labels in this chain. Similarly, if $w_p = \hat{1}$, there will be no expansion labels. Further, observe that in this interval, any creation or expansion which precedes a merger automatically generates a descent. Since only mergers can reduce the cardinality of an independent set, and we must finish with a singleton independent set, we must have that any rising chain must consist of a sequence of mergers followed by a sequence of expansions, where one of these sequences may be empty.

Now since we must merge all the elements of S first, and we defined $w_p = \bigvee_{i=1}^{p} x_i$, every rising chain must pass through the element $S_p = \{w_p\}$. Therefore, we may consider the two separate intervals, $[S, S_p]$ and $[S_p, T_1]$; either one of these intervals may be singleton, in which case there are no chains, and no further analysis needed. In the interval $[S, S_p]$, we seek a rising chain c_m, which must consist of only mergers, while in $[S_p, T_1]$, we seek a rising chain c_e, which must consist of only expansions. Since these intervals intersect at only one element, then the set $c_m \cup c_e$ is a maximal chain of $[S, T_1]$, and will be rising.

Notice that by our construction of c in Claim 6.1.7, if we define $c_m = S_1 \lessdot S_2 \lessdot \cdots \lessdot S_p$ and $c_e = S_p \lessdot S_{p+1} \lessdot \cdots \lessdot S_{\ell(L)-r}$, then these two chains are each rising in their respective intervals. Now to show that $c = c_m \cup c_e$ is the only rising chain in $[S, T_1]$, we can show that c_m and c_e are uniquely rising in their respective intervals.

Claim 6.1.8. *The chain c_m is the only rising chain in the interval $[S, S_p]$.*

Proof of Claim 6.1.8. Notice that since $w_p = \bigvee_{i=1}^{p} x_i$, we cannot perform any type of covering step in this interval except mergers, since both expansions and creations require an increase in the sum of the ranks of the elements of an independent set. Let d_m be a rising chain of $[S, S_p]$. Since each covering step is a merger in this interval, consider the covering step which arises from merging together two distinct elements $u, v \in R_1$ for some $R_1 \in d_m$. By definition, the label this step receives must be $(1, b)$, where $b = \max_A\{\max_A(\mathrm{U}_u), \max_A(\mathrm{U}_v)\}$.

Assume that $\max_A(\mathrm{U}_u) <_A \max_A(\mathrm{U}_v)$, so that the label of the merger $u \vee v$ is $(1, \max_A(\mathrm{U}_v))$. If $v = v_1 \vee v_2$ was the result of the previous merger of the two elements $v_1, v_2 \in R_2$ for some $R_2 \in d_m$ with $R_2 <_L R_1$, then assume that $\max_A(\mathrm{U}_{v_1}) <_A \max_A(\mathrm{U}_{v_2})$, so that the merger $v_1 \vee v_2$ generates the label $(1, \max_A(\mathrm{U}_{v_2}))$. Now by Lemma 6.1.5 we have that $\max_A(\mathrm{U}_v) \leq_A \max_A(\mathrm{U}_{v_1} \cup \mathrm{U}_{v_2}) = \max_A(\mathrm{U}_{v_2})$, so we have that $(1, \max_A(\mathrm{U}_v)) \leq (1, \max_A(\mathrm{U}_{v_2})$. But this implies that d_m is not rising, since the label $(1, \max_A(\mathrm{U}_{v_2})$ precedes the label $(1, \max_A(\mathrm{U}_v))$.

Hence we have shown that each covering of d_m is a merger $u \vee v$ where $v \in S$ and the label of the merger is $(1, \max_A(\mathrm{U}_v))$. It follows that there is a permutation $\sigma \in \mathfrak{S}_p$ such that $d_m := S'_1 \lessdot S'_2 \lessdot \cdots \lessdot S'_p$, where for each i we have $w'_i := \bigvee_{j=1}^{i} x_{\sigma(j)}$, $S'_i = \{w'_i, x_{\sigma(i+1)}, x_{\sigma(i+2)}, ..., x_{\sigma(p)}\}$, and $\Psi(S'_{i-1} \lessdot S'_i) = (1, \max_A(\mathrm{U}_{x_{\sigma(i)}}))$. Since d_m is rising, by (6.1.1) we must have that σ is the unique increasing word, $\sigma = 123 \cdots p$. Therefore $d_m = c_m$, and the rising chain is unique. $\square$

Claim 6.1.9. *The chain c_e is the only rising chain in the interval $[S_p, T_1]$.*

Proof of Claim 6.1.9. Notice that since both S_p and T_1 are singleton sets, if $R \in [S_p, T_1]$ is such that $|R| > 1$, then any maximal chain containing R must contain a creation followed by a merger, and so is not rising. Therefore, any rising chain in this interval consists entirely of expansions only. Let f_e be a maximal chain consisting of only expansions in the interval with label sequence $(2, d_{p+1}), ..., (2, d_{\ell(L)-r})$, and let i be the first index at which $b_i \neq d_i$, with b_i the i^{th} label of the chain c_e.

Since by definition b_i is the smallest atom available at this step not less than w_{i-1}, we must have that $b_i <_A d_i$. Further, we have that $b_i \nleq_L (w_{i-1} \vee d_i)$, by the definition of the edge labeling of an expansion. But since every atom is less than $\hat{1}$, we have that b_i will become an expansion label of f_e eventually, being the smallest atom available. Thus, we have $b_i = d_{i+k}$ for some positive integer k; thus, f_e is not rising, as $(2, d_i)$ precedes $(2, b_i)$ in the label sequence. □

Therefore, the chain c has a uniquely rising label sequence in $[S, T_1]$.

Claim 6.1.10. *The chain c is lexicographically first among all maximal chains of $[S, T_1]$.*

Proof of Claim 6.1.10. To show that c is lexicographically first among all maximal chains, we make use of the fact given in (6.1.2) that the rising chain we have constructed uses at each step the smallest possible label available. Since no label can be smaller at each step than the label of the rising chain, it cannot be preceded by any other maximal chain in the interval. Thus, these upper intervals of this type have the rising maximal chain we seek. □

(Proof of Theorem 6.1.6 continued) Now consider the upper interval $[T_0, T_1] = \Pi_L^{=k}$. We must clearly begin our sequence with a creation, since we cannot arrive at T_1 unless we have at least one element. However, if we have multiple creations,

we will have to have a merger which follows them, creating a descent. Since we can have only one creation, the remaining steps are expansions only.

Claim 6.1.11. *Denote the rising chain of* L *by* $C_L := \hat{0} \lessdot c_1 \lessdot \cdots \lessdot c_{\ell(L)-1} \lessdot \hat{1}$*, under the EL-labeling* Ψ_L *given in Definition 4.2.1. Then the chain* $C_{\Pi_L^{=k}} := T_0 \lessdot \{c_k\} \lessdot \{c_{k+1}\} \lessdot \cdots \lessdot \{c_{\ell(L)-1}\} \lessdot T_1$ *is a rising chain of* $\Pi_L^{=k}$*.*

Proof of Claim 6.1.11. By Lemma 6.1.2, the labeling atom of the creation $T_0 \lessdot \{c_k\}$ is the same as the largest label in the unique rising chain of $[\hat{0}, c_k]_L$. However, this chain is a subset of C_L. Thus, we have $\Psi(T_0 \lessdot \{c_k\}) = (2, \Psi_L(c_{k-1} \lessdot c_k))$, while $\Psi(\{c_i\} \lessdot \{c_{i+1}\}) = (2, \Psi_L(c_i \lessdot c_{i+1}))$ for each $i \geq k$. The label sequence of $C_{\Pi_L^{=k}}$ is therefore

$$(2, \Psi_L(c_{k-1} \lessdot c_k)), (2, \Psi_L(c_k \lessdot c_{k+1})), ..., (2, \Psi_L(c_{\ell(L)-1} \lessdot \hat{1})),$$

and so since C_L is rising, we have that $C_{\Pi_L^{=k}}$ is also rising. □

Claim 6.1.12. *The chain* $C_{\Pi_L^{=k}}$ *is uniquely rising among all maximal chains of* $\Pi_L^{=k}$*.*

Proof of Claim 6.1.12. Since we cannot have mergers and maintain a rising chain, assume $D := T_0 \lessdot \{d_k\} \lessdot \{d_{k+1}\} \lessdot \cdots \lessdot \{d_{\ell(L)-1}\} \lessdot T_1$ is a rising chain. By Lemma 6.1.2, $\Psi(T_0 \lessdot \{d_k\}) = (2, a)$ where a is the largest label in the uniquely rising chain D_k of the interval $[\hat{0}, d_k]_L$ under the EL-labeling Ψ_L of Definition 4.2.1. The labels given to the expansions above $\{d_k\}$ are directly taken from a maximal chain of the interval $[d_k, \hat{1}]_L$, as in Claim 6.1.11.

Since D is rising and $\Psi(\{d_i\} \lessdot \{d_{i+1}\}) = (2, \Psi_L(d_i \lessdot_L d_{i+1}))$ for each i, we have $E := d_k \lessdot d_{k+1} \lessdot \cdots \lessdot d_{\ell(L)-1} \lessdot d_{\ell(L)} = \hat{1}$ is rising in L. Further since $\Psi(T_0 \lessdot \{d_k\}) < \Psi(\{d_k\} \lessdot \{d_{k+1}\})$, we must have that $a <_A \Psi_L(d_k \lessdot d_{k+1})$. But notice that since $D_k \cap E = \{d_k\}$, the chain $D_k \cup E$ is a rising maximal chain of

L, and so we must have that $D = C_{\Pi_L^{=k}}$ by the uniqueness of the rising chain of L. $\square$

Claim 6.1.13. *The chain* $C_{\Pi_L^{=k}}$ *is lexicographically first among all maximal chains of* $\Pi_L^{=k}$.

Proof of Claim 6.1.13. Notice by Theorem 4.2.2 and Lemma 6.1.2 that every atom $a <_A \max_A(\mathrm{U}_{c_k})$ is such that $a \leq_L c_{k-1}$. Therefore, if $\Psi(T_0 \lessdot \{d_k\}) = (2, a)$ is such that $a <_A \max_A(\mathrm{U}_{c_k})$ for some atom $\{d_k\} \in \Pi_L^{=k}$, then $b \in \mathrm{U}_{d_k}$ implies that $b \leq_L c_{k-1}$. By the definitions of bases and joins, we then have that $d_k \leq_L c_{k-1}$, which contradicts the fact that $\rho(d_k) = k$ by the definition of $\Pi_L^{=k}$.

Now suppose $\Psi(T_0 \lessdot \{d_k\}) = (2, \max_A(\mathrm{U}_{c_k}))$; that is, we have that $a := \max_A(\mathrm{U}_{c_k}) = \max_A(\mathrm{U}_{d_k})$. Then since $\bigvee_{b \in \mathrm{U}_{d_k} - \{a\}} b \leq_L c_{k-1}$ from the previous observations, and $c_{k-1} \vee a = c_k$, we have that $d_k = \bigvee_{b \in \mathrm{U}_{d_k}} b \leq_L c_k$. Thus, if $\rho(d_k) = k$, then we must have that $d_k = c_k$. Therefore, only one creation has a label as small as the creation label of $C_{\Pi_L^{=k}}$. Since the chains above this creation are maximal chains of the upper interval $[\{c_k\}, T_1]$, Claim 6.1.10 gives that the unique rising chain is lexicographically first in this interval. Therefore, the chain $C_{\Pi_L^{=k}}$ is lexicographically first among all chains of $\Pi_L^{=k}$. $\square$

(Proof of Theorem 6.1.6 continued) Thus, any upper interval of the form $[S, T_1]$ contains the maximal chain we seek. Now consider an arbitrary interval $[S, T]$, with $S = \{s_1, ..., s_m\}$, and $T = \{t_1, ..., t_u\}$. First consider two distinct elements $t_i, t_h \in T$. By Lemma 5.2.3 $t_i \wedge t_h = \hat{0}$ for any such pair. By the definition of labeling atoms, we see that any label in any chain generated by t_i cannot also be a label in a chain generated by t_h since it would imply an atom simultaneously less than both.

For a fixed $t_i \in T$, denote by S^i the element of $[S,T]$ such that $S^i = \{t_i\} \cup V^i$, where $V^i = \{s \in S \mid s \not\leq_L t_i\}$. Observe that since $V^i \subset S$, by applying Lemma 5.2.6 to T, we have that S^i is independent. In particular, every element $Z \in [S, S^i]$ must satisfy $V^i \subseteq Z$. We now observe that the interval $[S, S^i]$ is isomorphic to the upper interval $[S - V^i, \{t_i\}]$ of $\Pi^{=k}_{L_{t_i}}$, where $L_{t_i} = [\hat{0}, t_i] \subseteq L$; here, the isomorphism is given by the map which forgets the subset V^i common to every element of $[S, S^i]$. Note that lower intervals of geometric lattices are also geometric.

Since $\Pi^{=k}_{L_{t_i}} = [T_0, \{t_i\}]$ is a lower interval of $\Pi^{=k}_L$, the edge labeling Ψ of $\Pi^{=k}_L$ restricts to the edge labeling of Definition 6.1.4 with $L = L_{t_i}$ for $\Pi^{=k}_{L_{t_i}}$. Using Claims 6.1.7 and 6.1.11, we know that every upper interval of $\Pi^{=k}_{L_{t_i}}$ has a rising chain; we have also shown that such chains are unique in Claims 6.1.8, 6.1.9, and 6.1.12. Let $S - V^i = R^i_1 \lessdot R^i_2 \lessdot R^i_3 \lessdot \cdots \lessdot R^i_{m_i} = \{t_i\}$ be the rising chain of the upper interval $[S - V^i, \{t_i\}] \subset \Pi^{=k}_{L_{t_i}}$ for each i, and denote $\Psi(R^i_j \lessdot R^i_{j+1})$ by λ^i_j.

Denote the set of all λ^i_j by Λ. Note that since $t_i \wedge t_h = \hat{0}$ and $\lambda^i_j \leq_L t_i$ for each i and each j, we have that $\lambda^{i_1}_{j_1} \neq \lambda^{i_2}_{j_2}$ if $i_1 \neq i_2$. Since the label set $[2] \times A$ is totally ordered, Λ is also totally ordered. Observe that $S = \bigcup_{i=1}^{t} R^i_1$, while $T = \bigcup_{i=1}^{t} R^i_{m_i}$. Further, any union of the form $\bigcup_{i=1}^{t} R^i_{j_i}$, where j_i is arbitrary for each i, is independent by Lemmas 5.2.5 and 5.2.6.

We now define the chain C to be $S = Z_0 \lessdot Z_1 \lessdot Z_2 \lessdot \cdots \lessdot Z_v = T$, where $Z_1 := (S - R^i_1) \cup R^i_2$ and we choose i such that $\lambda^i_1 = \min(\Lambda)$. To recursively define the remaining elements, let Λ_p be the set of labels of Λ which have been used to form Z_p for each integer p. We then define $Z_{p+1} := (Z_p - R^i_j) \cup R^i_{j+1}$, where $\lambda^i_j = \min(\Lambda - \Lambda_p)$. Notice that each Z_p is independent, being a union of the form $\bigcup_{i=1}^{t} R^i_{j_i}$. Further, we have $\Psi(Z_p \lessdot Z_{p+1}) = \lambda^i_j$. Moreover, as we select λ^i_j to be minimum at each step, we must have that C is rising.

Now we show that C is the only rising chain of $[S,T]$. Note that every maximal chain of $[S,T]$ has a similar description to that of C. Indeed, let D be the maximal chain $S = D_0 \lessdot D_1 \lessdot D_2 \lessdot \cdots \lessdot D_v = T$. Then for each $i \in [u]$, there is a unique maximal chain $M_i(D) := S - V^i = Y^i_1 \lessdot Y^i_2 \lessdot \cdots \lessdot Y^i_{m_i} = \{t_i\}$ of $[S - V^i, \{t_i\}]$ such that for all $h \geq 0$ we have that $D_{h+1} = (D_h - Y^i_j) \cup Y^i_{j+1}$ for some i, j.

We have that $\Psi(D_h \lessdot D_{h+1}) = \Psi(Y^i_j \lessdot Y^i_{j+1})$. Hence, the label set of D is $\Omega := \bigcup_{i=1}^{u} \Upsilon_i(D)$, where $\Upsilon_i(D)$ is the label set of $M_i(D)$ for each i. Assume D is rising. Then each $M_i(D)$ is rising. Since each $[S - V^i, \{t_i\}]$ has a unique rising chain, we have that $M_i(D) = M_i(C)$ for each i, where C is the rising chain constructed above. Therefore, we have $\Upsilon_i(D) = \Upsilon_i(C)$; consequently, we have that $\Omega = \Lambda$, the label set of C. Since Λ is totally ordered, there is a unique way to arrange its labels in increasing order; therefore, $D = C$.

We can see that C is also lexicographically first since at each step we choose the smallest available label of all types in the interval. That is, since each $M_i(C) := S - V^i = R^i_1 \lessdot R^i_2 \lessdot R^i_3 \lessdot \cdots \lessdot R^i_{m_i} = \{t_i\}$ is lexicographically minimal, the labels it generates are all as small as possible at each step. Further, since we construct C by choosing the smallest label of Λ at each step, we must have that C is also lexicographically minimal.

Thus in any interval, there is a unique rising maximal chain which is lexicographically first among all maximal chains. Therefore, Ψ is an EL-labeling, so that $\Pi^{=k}_L$ is EL-shellable. $\square$

6.2 The Falling Chains of $\Pi_L^{=k}$

The EL-labeling of $\Pi_L^{=k}$ is a generalization of the EL-labeling of Björner and Wachs for the k–equal partition lattice described in Section 2.5. Recall that when $L = B_n$, all the labels of a falling chain which were of the merger type had to be exactly $(1, n)$; this constancy does not hold in general.

Fix a positive integer n. We say that the multiset $\lambda = \{\lambda_1, \lambda_2, ..., \lambda_t\}$ is an *integer partition* of n if $\lambda_i \in \mathbb{P}$ for each i and $\lambda_1 + \lambda_2 + \cdots + \lambda_t = n$. Each $\lambda_i \in \lambda$ is called a *part* of λ; the *length* of λ is defined as the number of parts of λ, and is denoted by $\ell(\lambda)$. We will always index the parts of λ so that $\lambda_i \geq \lambda_{i+1}$ for each i. We frequently call λ simply a partition when the context is clear. If λ is a partition of n, then we indicate this relationship by $\lambda \vdash n$. For example, $\lambda = \{3, 3, 2, 1, 1\}$ is a partition of 10, so we write $\lambda \vdash 10$, and $\ell(\lambda) = 5$.

Given any element $T = \{x_1, ..., x_t\} \in \Pi_L$, define the *type* of T, denoted type(T), to be the multiset of integers $\{\rho(x_1), \rho(x_2), ..., \rho(x_t)\}$. Define the *total rank* $r(T)$ of T to be

$$r(T) := \sum_{x \in T} \rho(x).$$

For a fixed partition (i.e., multiset) $\lambda = \{\lambda_1, ..., \lambda_t\}$, define

$$\Pi_{L,\lambda} := \{T \in \Pi_L \mid \text{type}(T) = \lambda\}. \qquad (6.2.1)$$

For a general geometric lattice L and a fixed element $x \in L$, define

$$L_x = [\hat{0}, x].$$

Let $\rho(x) \geq k$ and define

$$\Pi_L^{=k}(x) = \{T \in [T_0, \{x\}] \mid |T| \leq 1\}.$$

Recall from Definition 3.3.2 that a truncation P_k of a ranked bounded poset P is the induced subposet consisting of all elements of ranks in the set $S =$

$\{k, k+1, ..., \ell(P)-1\}$. Also, recall that by $\widehat{P}$ we mean to adjoin both a top and bottom element to the poset P.

Lemma 6.2.1. *Let L be a geometric lattice, and fix $x \in L$ with $\rho(x) \geq k$. Then $\Pi_L^{=k}(x) \cong \widehat{(L_x)_k}$.*

Proof. We show these two posets are isomorphic by constructing a bijection between them. Define $\phi : \widehat{(L_x)_k} \rightarrow \Pi_L^{=k}(x)$ by

$$\phi(z) = \begin{cases} \{x\}, & z = \hat{1} \\ \{z\}, & z \in (L_x)_k \\ \emptyset, & z = \hat{0} \end{cases}$$

This map is well-defined since if $z \in (L_x)_k$, then both $z \leq_L x$ and $\rho(z) \geq k$ hold, so that $\{z\} \in \Pi_L^{=k}(x)$. It is clearly bijective, and moreover, it is clear that $\phi(z_1) \leq_{\Pi_L^{=k}(x)} \phi(z_2)$ if and only if $z_1 \leq_{\widehat{(L_x)_k}} z_2$. Thus, the posets are isomorphic. □

We now turn attention to counting falling chains of $\Pi_L^{=k}$; we begin with a few lemmas.

Lemma 6.2.2. *Let L be a geometric lattice, and $T = \{x_1, x_2, ..., x_t\} \in \Pi_L^{=k}$. If $r(T) = \ell(L)$, then $[T, T_1] \cong \Pi_t$.*

Proof. Notice that since the total rank of T is $\ell(L)$, all covering relations of $[T, T_1]$ must be mergers, since it is not possible to raise the total rank. Arrange the indices of the $x_i \in T$ so that $\max_A(\mathrm{U}_{x_i}) <_A \max_A(\mathrm{U}_{x_{i+1}})$ for $1 \leq i < t$.

Define the function $\phi : \Pi_t \rightarrow [T, T_1]$ by

$$\phi(\{B_1, B_2, ..., B_m\}) = \left\{ \bigvee_{i \in B_1} x_i, \bigvee_{i \in B_2} x_i, ..., \bigvee_{i \in B_m} x_i \right\},$$

For example, let $t = 9$; we have

$$\phi\left(\{\{1,3,6\},\{2\},\{4,5,7\},\{8,9\}\}\right) = \{x_1 \vee x_3 \vee x_6,\ x_2,\ x_4 \vee x_5 \vee x_7,\ x_8 \vee x_9\}.$$

Similarly, it is easy to see that $\phi(\hat{0}) = T$ and $\phi(\hat{1}) = T_1$. To show that ϕ is well-defined, we must show that $\phi(\{B_1, B_2, ..., B_m\})$ is an independent set. By Lemma 5.2.1, the set $\{x_i \in T \mid i \in B_j\}$ is independent for any block B_j. Hence, we have $\rho\left(\bigvee_{i \in B_j} x_i\right) = \sum_{i \in B_j} \rho(x_i)$. Therefore,

$$\begin{aligned}
\sum_{j=1}^{m} \rho\left(\bigvee_{i\in B_j} x_i\right) &= \sum_{j=1}^{m} \sum_{i \in B_j} \rho(x_i) \\
&= \sum_{x \in T} x \\
&= \rho\left(\bigvee_{x \in T} x\right) \\
&= \rho\left(\bigvee_{j=1}^{m}\left(\bigvee_{i \in B_j} x_i\right)\right).
\end{aligned}$$

Thus, ϕ is well-defined. This map is clearly order-preserving, so we need only to show that it is a bijection.

To show ϕ is a bijection, we show first that it is surjective. Note that if $S = \{y_1, ..., y_m\} \in [T, T_1]$, then each y_i is the join of some subset of T, as mergers are the only covering steps in this interval. Define $B_j := \{i \in [t] \mid x_i <_L y_j\}$ for each j; we can see that $\phi(\{B_1, ..., B_m\}) = \{y_1, ..., y_m\}$, and so ϕ is surjective. For injectivity, it is enough to show that for subsets $A, B \subset [t]$ we have that $\bigvee_{i \in A} x_i = \bigvee_{i \in B} x_i$ implies $A = B$.

Suppose $\bigvee_{i \in A} x_i = \bigvee_{i \in B} x_i$ and $A \neq B$. Without loss of generality assume that there exists $j \in A$ such that $j \notin B$. Now

$$x_j \vee \bigvee_{i \in B} x_i = x_j \vee \bigvee_{i \in A} x_i = \bigvee_{i \in A} x_i = \bigvee_{i \in B} x_i,$$

since $j \in A$. Since $\{x_i \in T \mid i \in B\} \cup \{x_j\} \subset T$ is independent, we cannot have that $x_j \vee \bigvee_{i \in B} x_i = \bigvee_{i \in B} x_i$. Therefore, there is no $j \in A$ such that $j \notin B$, and thus $A = B$. $\square$

Corollary 6.2.3. *Let L be a geometric lattice, and $T \in \Pi_L^{=k}$ an independent set such that $r(T) = \ell(L)$. The number of falling chains of $[T, T_1]$ is $(t-1)!$ under any EL-labeling of $\Pi_L^{=k}$.*

Proof. Recall from Proposition 2.4.2 that Π_t has the homotopy type of a wedge of $(t-1)!$ spheres of dimension $t-3$. By Theorem 2.2.5, the number of falling chains of $[T, T_1]$ under any EL-labeling must therefore be $(t-1)!$. $\square$

Lemma 6.2.4. *Let L be a geometric lattice, and $T = \{x_1, ..., x_t\} \in \Pi_L^{=k}$. Then*

$$[T_0, T] \cong \Pi_{L_{x_1}}^{=k} \times \Pi_{L_{x_2}}^{=k} \times \cdots \times \Pi_{L_{x_t}}^{=k}.$$

Proof. By Corollary 5.2.4, if $S = \{y_1, ..., y_s\} \in [T_0, T]$, then each $y_j \in S$ is less than a unique $x_i \in T$. Therefore, define $S_i = \{z \in S \mid z \leq_L x_i\}$ for each i; notice that some of the S_i may be empty, but that $\bigcup_{i=1}^{t} S_i = S$ and $S_i \cap S_j = \emptyset$ whenever $i \neq j$. By the definition of L_{x_i} and Lemma 5.2.1, we must have that $S^i \in \Pi_{L_{x_i}}^{=k}$. We define the map $\varphi : [T_0, T] \to \Pi_{L_{x_1}}^{=k} \times \Pi_{L_{x_2}}^{=k} \times \cdots \times \Pi_{L_{x_t}}^{=k}$ by

$$\varphi(S) = (S_1, S_2, ..., S_t).$$

To show that φ is a bijection, we now define a map $\psi : \Pi_{L_{x_1}}^{=k} \times \Pi_{L_{x_2}}^{=k} \times \cdots \times \Pi_{L_{x_t}}^{=k} \to [T_0, T]$ by

$$\psi(S_1, S_2, ..., S_t) = \bigcup_{i=1}^{t} S_i.$$

Observe here that $S_i \cap S_j = \emptyset$ as a consequence of Lemma 5.2.3. To see that $\bigcup_{i=1}^{t} S_i$ is independent, we apply Lemma 5.2.6, which implies that we can replace each $x_i \in T$ by the elements of S_i and preserve independence; thus, ψ is well-defined.

Now consider the composition $\psi \circ \varphi$. It is clear that $\psi \circ \varphi = id$, since φ takes S to a collection of nonintersecting subsets of S whose union is S, and then ψ forms the union of these sets. For the composition $\varphi \circ \psi$, since if $z \in \bigcup_{i=1}^{t} S_i$ then there exists a unique S_j such that $z \in S_j$, and therefore $z \leq_L x_j$ for a unique $x_j \in T$, we also have that $\varphi \circ \psi = id$, and so this pair of maps are bijections. They clearly preserve order; thus, $[T_0, T] \cong \Pi^{=k}_{L_{x_1}} \times \Pi^{=k}_{L_{x_2}} \times \cdots \times \Pi^{=k}_{L_{x_t}}$. □

Now given geometric lattice L and $T \in \Pi^{=k}_L$, define $[T_0, T]_{CE}$ to be the subposet of $[T_0, T]$ which consists of only creation and expansion coverings. Note that $\Pi^{=k}_L(x) = [T_0, \{x\}]_{CE}$; moreover, $[T_0, T]_{CE}$ is not necessarily equal as a set to $[T_0, T]$, nor is it an induced subposet.

Lemma 6.2.5. *The restriction of* Ψ *to* $[T_0, T]_{CE}$ *is an EL-labeling.*

Proof. This is because for all $R <_{[T_0,T]_{CE}} S$, the rising chain of $[R, S]$ has no mergers. □

Lemma 6.2.6. *Let* L *be a geometric lattice, and* $T = \{x_1, ..., x_t\} \in \Pi^{=k}_L$*. Then*

$$[T_0, T]_{CE} \cong (L_{x_1})_k \times (L_{x_2})_k \times \cdots \times (L_{x_t})_k.$$

Consequently, $[T_0, T]_{CE}$ *is pure.*

Proof. Since we do not allow mergers in $[T_0, T]_{CE}$, we observe that for each $S \in [T_0, T]_{CE}$, we have $|S_i| \leq 1$ for each i, where S_i is defined as in the proof of Lemma 6.2.4. Therefore, we have $S_i \in \Pi^{=k}_L(x_i)$ for each i. We then have that the map $\varsigma : [T_0, T]_{CE} \to \Pi^{=k}_L(x_1) \times \Pi^{=k}_L(x_2) \times \cdots \times \Pi^{=k}_L(x_t)$, given by

$$\varsigma(S) = (S_1, S_2, ..., S_t),$$

is order-preserving.

Similarly, define $\vartheta : \Pi_L^{=k}(x_1) \times \Pi_L^{=k}(x_2) \times \cdots \times \Pi_L^{=k}(x_t) \to [T_0, T]_{CE}$ by

$$\vartheta((S_1, ..., S_t)) = \bigcup_{i=1}^{t} S_i.$$

By the definition of $\Pi_L^{=k}(x_i)$, we have that each S_i is a set of cardinality at most 1, where if $S_i = \{s\}$, then $s \leq_L x_i$. Thus by the independence of T and Lemma 5.2.5, we have that $\left(\bigcup_{i=1}^{t} S_i\right) \leq_{\Pi_L^{=k}} T$, so ϑ is well-defined. Moreover, it is clear that $\varsigma \circ \vartheta = id$, since by Lemma 5.2.3 we must have that $S_i \cap S_j = \emptyset$ whenever $i \neq j$. Similarly, we have that $\vartheta \circ \varsigma = id$, as the S_i are defined to have union S. Thus, we have $[T, T_0]_{CE} \cong \Pi_L^{=k}(x_1) \times \Pi_L^{=k}(x_2) \times \cdots \times \Pi_L^{=k}(x_t)$.

To complete the proof of the isomorphism, we apply Lemma 6.2.1. The consequence follows from the fact that products of pure posets are pure. $\square$

Corollary 6.2.7. *Let L be a geometric lattice and $T \in \Pi_L^{=k}$. Then*

$$\mu([T_0, T]_{CE}) = \prod_{x \in T} \mu(\widehat{(L_x)_k}).$$

Consequently, we have

$$\widetilde{\beta}_{r(T)-(k-1)|T|-2}((T_0, T)_{CE}) = \prod_{x \in T} \widetilde{\beta}_{\rho(x)-k-1}((L_x)_k).$$

Proof. The first formula is a clear consequence of Lemma 6.2.6 and (2.1.1). Since L_x is pure and EL-shellable, by Theorem 2.2.7 we have that $\widetilde{\beta}_i((L_x)_k) = 0$ for all $i < \rho(x) - k - 1$. Now by (2.2.1) we can write

$$\widetilde{\beta}_{\rho(x)-k-1}((L_x)_k) = |\mu(\widehat{(L_x)_k})|.$$

By Lemma 6.2.6, $[T_0, T]_{CE}$ is pure of length $\sum_{x \in T} \ell(\Pi_L^{=k}(x)) = \sum_{x \in T} (\rho(x) - k + 1) = r(T) - (k-1)t$. Since by Lemma 6.2.5 $[T_0, T]_{CE}$ is also EL-shellable, we have

$$\widetilde{\beta}_{r(T)-(k-1)t-2}((T_0, T)_{CE}) = |\mu([T_0, T]_{CE})|,$$

so the consequence holds. $\square$

Using these results, we can now give the following general counting formula for $\Pi_L^{=k}$.

Theorem 6.2.8. *Let* L *be a geometric lattice, with* $\ell(L) = n$*, and fix an integer* $2 \leq k \leq n$*. For integer* $1 \leq t \leq \frac{n}{k}$*, define* $B^{L,k}(t)$ *by*

$$B^{L,k}(t) = \sum_{\substack{T \in \Pi_L^{=k} \\ r(T)=n \\ |T|=t}} (t-1)! \prod_{x \in T} \widetilde{\beta}_{\rho(x)-k-1}((L_x)_k).$$

The order complex $\Delta(\overline{\Pi_L^{=k}})$ *has the homotopy type of a wedge of spheres, where the number of spheres* $\widetilde{\beta}_{d-2}(\overline{\Pi_L^{=k}})$ *of dimension* $d-2$ *is given by*

$$\widetilde{\beta}_{d-2}(\overline{\Pi_L^{=k}}) = \begin{cases} \displaystyle\sum_{t=1}^{\lfloor \frac{n}{k} \rfloor} B^{L,k}(t), & k = 2 \text{ and } d = n-1 \\ B^{L,k}(\frac{n-d-1}{k-2}), & k > 2 \text{ and } \frac{n-d-1}{k-2} \in \mathbb{P} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. To compute the Betti number $\widetilde{\beta}_{d-2}(\overline{\Pi_L^{=k}})$, we make use of Theorems 2.2.5 and 6.1.6, and count the number of falling chains of $\Pi_L^{=k}$ of length d. Recall that by Definition 6.1.4, the label $(1, b)$ a merger receives in a chain is strictly less than the label given to any creation or expansion, as these are of the form $(2, b)$. Therefore, for the chain c to be falling, there must exist a unique element $T \in c$ such that the labels of coverings below T cannot be mergers, and coverings above T (if any exist) must all be mergers. Define such an element $T \in c$ as the *pivot* of c.

The number of falling chains with pivot T equals the number f_1 of falling chains of $[T_0, T]_{CE}$ times the number f_2 of falling chains of $[T, T_1]$ consisting only of mergers. To compute f_1, we recall from Lemma 6.2.5 that the restriction of Ψ to $[T_0, T]_{CE}$ is an EL-labeling. It follows from Theorem 2.2.5 and Corollary 6.2.7 that

$$f_1 = \prod_{x \in T} \widetilde{\beta}_{\rho(x)-k-1}((L_x)_k).$$

To compute f_2 we note that the independence of the pivot T guarantees that $r(T) = \ell(T) = n$ because mergers do not increase the total rank. Since creations and expansions do increase the total rank we also see that all maximal chains of $[T, T_1]$ consist only of mergers. Thus by Corollary 6.2.3,

$$f_2 = (|T| - 1)!.$$

We can now conclude that the number of falling chains with pivot T is

$$(|T| - 1)! \prod_{x \in T} \widetilde{\beta}_{\rho(x)-k-1}((L_x)_k).$$

Now consider the length d of a falling chain whose pivot is T. If $|T| = t$ for some positive integer t, then observe that $d = t + (n - kt) + (t - 1) = (n - 1) - t(k - 2)$. We see that this is so because firstly each of the t elements of T corresponds to exactly one creation, which contributes a total of tk to the total rank. Since we know that $r(T) = n$, and each expansion adds 1 to $r(T)$, there must be $n - tk$ expansions. Finally, a total of $t - 1$ mergers are necessary to combine t elements to form a singleton set, as a merger reduces cardinality by one.

Note further that if $k = 2$, then all the chains must be the same length $d = n - 1$, so that we must sum over all possible values of t to obtain all falling chains of this length. On the other hand, if $k > 2$, then each value of t determines a unique length d, and thus the value of $\widetilde{\beta}_{d-2}(\overline{\Pi_L^{=k}})$. $\square$

While Theorem 6.2.8 gives a clean expression for finding the number of falling chains of each length, we observe that in practice the number of elements of $\Pi_L^{=k}$ can be almost prohibitively large for usefulness. For instance, we can compute that $\mu(\Pi_{\Pi_6}^{=2}) = 6600$, for which it is necessary to sum over 2101 different elements of $\Pi_{\Pi_6}^{=2}$. We now give a simplification of Theorem 6.2.8 which applies to a large class of interesting posets.

Define a lattice L to be *lower interval isomorphic* if $L_x \cong L_y$ whenever $\rho(x) = \rho(y)$. Note that both B_n and $B_n(q)$ are lower interval isomorphic. For a lower interval isomorphic geometric lattice L and integer $1 < k < \ell(L)$, define the function $f_L : \mathbb{P} \to \mathbb{N}$ by $f_L(m) = \widetilde{\beta}_{m-k-1}((L_z)_k)$, for some $z \in L$ such that $\rho(z) = m$. This function is well-defined given lower interval isomorphism.

Theorem 6.2.9. *Let L be a lower interval isomorphic geometric lattice with $\ell(L) = n$, and fix an integer $2 \le k \le n$. For $1 \le t \le \frac{n}{k}$, define $B^{L,k}(t)$ by*

$$B^{L,k}(t) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \ge k\ \forall\ i}} (t-1)! \left(\prod_{i=1}^{t} f_L(\lambda_i) \right) \left| \Pi_{L,\lambda} \right|.$$

The order complex $\Delta(\overline{\Pi_L^{=k}})$ has the homotopy type of a wedge of spheres, where the number of spheres $\widetilde{\beta}_{d-2}(\overline{\Pi_L^{=k}})$ of dimension $d-2$ is given by

$$\widetilde{\beta}_{d-2}(\overline{\Pi_L^{=k}}) = \begin{cases} \displaystyle\sum_{t=1}^{\lfloor \frac{n}{k} \rfloor} B^{L,k}(t), & k = 2 \text{ and } d = n-1 \\ B^{L,k}(\frac{n-d-1}{k-2}), & k > 2 \text{ and } \frac{n-d-1}{k-2} \in \mathbb{P} \\ 0, & \text{otherwise.} \end{cases} \tag{6.2.2}$$

Proof. We apply Theorem 6.2.8 to L. We obtain the following, by removing the dependence on T and summing over all types:

$$\sum_{\substack{T \in \Pi_L^{=k} \\ r(T)=n \\ |T|=t}} (t-1)! \prod_{x \in T} \widetilde{\beta}_{\rho(x)-k-1}((L_x)_k) = \sum_{\substack{T \in \Pi_L^{=k} \\ r(T)=n \\ |T|=t}} (t-1)! \prod_{x \in T} f_L(\rho(x))$$

$$= \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \geq k \ \forall \ i}} \sum_{T \in \Pi_{L,\lambda}} (t-1)! \prod_{i=1}^{t} f_L(\lambda_i)$$

$$= \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \geq k \ \forall \ i}} (t-1)! \left(\prod_{i=1}^{t} f_L(\lambda_i) \right) \left| \Pi_{L,\lambda} \right|$$

□

Recall not only that B_n is lower interval isomorphic, but that $\Pi_{B_n}^{=k} \cong \Pi_n^{=k}$, for which similar results were given in Propositions 2.5.2 and 2.5.3.

Lemma 6.2.10. *If* $L = B_n$*, then* $f_{B_n}(p) = \binom{p-1}{k-1}$ *and for* $\lambda \vdash m \leq n$ $|\Pi_{B_n,\lambda}| = \binom{n}{\lambda_1, ..., \lambda_t, (n-m)} \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}$*, where* $a_j(\lambda)$ *equals the number of parts of* λ *of size* j*.*

Proof. To prove the lemma, we observe that B_n is lower interval isomorphic, since we have specifically that the lower interval $[\hat{0}, x]$ is isomorphic to the smaller Boolean algebra $B_{\rho(x)}$. Since $L_x = B_{\rho(x)}$, we have that $(L_x)_k = (B_{\rho(x)})_k$. By Proposition 3.3.3, this gives that $f_{B_n}(p) = \binom{p-1}{k-1}$.

We will now show that $|\Pi_{B_n,\lambda}| = \binom{n}{\lambda_1, ..., \lambda_t, (n-m)} \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}$. We manipulate the multinomial coefficient to obtain

$$\begin{aligned}
\binom{n}{\lambda_1, ..., \lambda_t, (n-m)} &= \frac{n!}{\lambda_1!\lambda_2! \cdots \lambda_t!(n-m)!} \\
&= \frac{n!}{\lambda_1!(n-\lambda_1)!} \frac{(n-\lambda_1)!}{\lambda_2!(n-\lambda_1-\lambda_2)!} \cdots \frac{(n-\lambda_1-\cdots-\lambda_{t-1})!}{\lambda_t!(n-m)!} \\
&= \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-\lambda_1-\cdots-\lambda_{t-1}}{\lambda_t} \\
&= \binom{n}{\lambda_1}\binom{n-\lambda_1}{\lambda_2} \cdots \binom{n-m+\lambda_t}{\lambda_t}.
\end{aligned}$$

From this, we can see that the product of binomial coefficients counts the number of ways to choose a sequence of subsets of $[n]$ such that the size of the i^{th} set is λ_i, and the subsets are mutually disjoint; thus, they form a set partition of $A \subset [n]$ where $|A| = m$.

Now we observe that this expression overcounts the number of set partitions of A of type λ, since without loss of generality the sequences $B_1, B_2, ..., B_t$ and $B_2, B_1, ..., B_t$ are counted as being distinct in this multinomial coefficient if $|B_1| = |B_2|$, but correspond to the same set partition $\{B_1, ..., B_t\}$. Thus, we need to determine how many sequences counted by $\binom{n}{\lambda_1, ..., \lambda_t, (n-m)}$ correspond to the same set partition S. Since there are $a_j(\lambda)$ parts of λ which equal j, we observe that there are $a_j(\lambda)$ blocks of S which have cardinality j.

We choose an arrangement of the blocks of S in which the cardinalities are weakly increasing. Therefore, we must arrange the $a_1(\lambda)$ blocks of size 1 in any order first, for which there are $a_1(\lambda)!$ possible arrangements. Following this, we can see that for each $j > 1$, we must choose one of the $a_j(\lambda)!$ arrangements consisting solely of the $a_j(\lambda)$ blocks of size j, and this arrangement must follow the arrangement consisting of blocks of size $j-1$ in the overall sequence.

Therefore, there are $\prod_{j=1}^{n} a_j(\lambda)!$ distinct arrangements of the blocks of S whose block sizes weakly decrease. We conclude that the number of distinct set partitions whose block sizes correspond to the integer partition λ is

$$|\Pi_{B_n,\lambda}| = \binom{n}{\lambda_1, ..., \lambda_t, (n-m)} \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}.$$

□

Corollary 6.2.11. *Let* $L = B_n$. *Then* $B^{B_n,k}(t)$ *is given by*

$$B^{B_n,k}(t) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \geq k \ \forall \ i}} (t-1)! \left(\prod_{i=1}^{t} \binom{\lambda_i - 1}{k-1} \right) \binom{n}{\lambda_1, ..., \lambda_t} \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}.$$

Lemma 6.2.12. *Let* $L = B_n(q)$. *Then* $f_{B_n(q)}(p) = q^{\binom{p-k+1}{2}} \begin{bmatrix} p-1 \\ k-1 \end{bmatrix}_q$ *and for* $\lambda \vdash m$,

$$|\Pi_{B_n(q),\lambda}| = q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t, (n-m) \end{bmatrix}_q \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}, \tag{6.2.3}$$

where $a_j(\lambda)$ *equals the number of parts of* λ *of size* j *and* $e_2(\lambda) = \sum_{1 \le i < j \le t} \lambda_i \lambda_j$.

Proof. To prove the lemma, we observe that $B_n(q)$ is lower interval isomorphic, since we have specifically that the lower interval $[\hat{0}, x]$ is isomorphic to the smaller subspace lattice $B_{\rho(x)}(q)$. Since we have that $L_x \cong B_{\rho(x)}(q)$, then we have that $(L_x)_k \cong (B_{\rho(x)}(q))_k$. By Proposition 3.3.4, this gives that $f_{B_n(q)}(p) = q^{\binom{p-k+1}{2}} \begin{bmatrix} p-1 \\ k-1 \end{bmatrix}_q$.

Now we need to show that (6.2.3) holds. To prove this, we recall from Lemma 3.2.1 that given the r–dimensional subspace V, the number of s–dimensional subspaces W such that $V \cap W = (0)$ is $q^{rs} \begin{bmatrix} n-r \\ s \end{bmatrix}_q$. We now choose a sequence of subspaces $W_1, ..., W_t$ such that $\dim(W_i) = \lambda_i$ for each $1 \le i \le t$ given the partition λ, and $V = W_1 \oplus \cdots \oplus W_t$ for $V \subset \mathbb{F}_q^n$ an m–dimensional subspace. The number of choices for each W_i after choosing $W_1, ..., W_{i-1}$ is $q^{(\lambda_1 + \cdots + \lambda_{i-1})\lambda_i} \begin{bmatrix} n - \lambda_1 - \cdots - \lambda_{i-1} \\ \lambda_i \end{bmatrix}_q$. Therefore, the number of sequences of subspaces $W_1, ..., W_t$ is

$$\left(\begin{bmatrix} n \\ \lambda_1 \end{bmatrix}_q \right) \left(q^{\lambda_1 \lambda_2} \begin{bmatrix} n-\lambda_1 \\ \lambda_2 \end{bmatrix}_q \right) \cdots \left(q^{(\lambda_1 + \cdots + \lambda_{t-1})\lambda_t} \begin{bmatrix} n-m+\lambda_t \\ \lambda_t \end{bmatrix}_q \right)$$

$$= q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t, (n-m) \end{bmatrix}_q.$$

Now we observe that this expression overcounts the number of independent sets of subspaces of $B_n(q)$ of type λ, since without loss of generality the sequences $W_1, W_2, ..., W_t$ and $W_2, W_1, ..., W_t$ are counted as being distinct in the product if $\dim(W_1) = \dim(W_2)$, but correspond to the same set of subspaces $\{W_1, ..., W_t\}$. Thus, we need to determine how many sequences counted by $q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t, (n-m) \end{bmatrix}_q$ correspond to the same independent set S of subspaces. Since there are $a_j(\lambda)$ parts of λ which equal j, we observe that there are $a_j(\lambda)$ elements of S which have dimension j.

We choose an arrangement of the elements of S in which the dimensions are weakly increasing. Therefore, we must arrange the $a_1(\lambda)$ subspaces of dimension 1 in any order first, for which there are $a_1(\lambda)!$ possible arrangements. Following this, we can see that for each $j > 1$, we must choose one of the $a_j(\lambda)!$ arrangements consisting solely of the $a_j(\lambda)$ subspaces of dimension j, and this arrangement must follow the arrangement consisting of subspaces of dimension $j-1$ in the overall sequence,

Therefore, there are $\prod_{j=1}^{n} a_j(\lambda)!$ distinct arrangements of S in which the dimensions weakly decrease. We conclude that the number of distinct independent sets of subspaces whose dimensions correspond to the partition λ is

$$|\Pi_{B_n(q),\lambda}| = q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t, (n-m) \end{bmatrix}_q \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}.$$

□

Corollary 6.2.13. *Let* $L = B_n(q)$. *Then* $B^{B_n(q),k}(t)$ *is given by*

$$B^{B_n(q),k}(t) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \geq k \ \forall \ i}} (t-1)! \left(\prod_{i=1}^{t} q^{\binom{\lambda_i - k + 1}{2}} \begin{bmatrix} \lambda_i - 1 \\ k-1 \end{bmatrix}_q \right) q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t \end{bmatrix}_q \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}.$$

Now we can recognize Corollary 6.2.13 as a direct q–analogue of Corollary 6.2.11. The only new terms we have obtained that do not directly correspond to terms in this original formula are all powers of q, and thus this q–analogue is clear.

Chapter 7

The Case of $\Pi_n^{=2}(q)$

Now that we have shown $\Pi_L^{=k}$ is EL-shellable for the geometric lattice L and $k > 1$, we will examine more closely the case of $\Pi_{B_n(q)}^{=2}$. Since $B_n(q)$ is lower interval isomorphic, Theorem 6.2.9 will apply, and we will get several nice results concerning this lattice.

We will begin to use the notation $\Pi_n^{=k}(q)$ rather than $\Pi_{B_n(q)}^{=k}$, following the observation from Corollaries 6.2.11 and 6.2.13 that this is a q–analogue of $\Pi_{B_n}^{=k} \cong \Pi_n^{=k}$. In this section, we will describe various results concerning $\Pi_n^{=2}(q)$. Specifically, the main ideas concern counting the falling chains of each possible length, and showing further justification that this lattice is a q–analogue of $\Pi_n^{=2} \cong \Pi_n$.

7.1 Exponential Structures

As a preliminary before proving the main results of this section, we first define an exponential structure, following [17] and [18]:

Definition 7.1.1. An *exponential structure* is a sequence of posets $Q_1, Q_2, ...,$ satisfying each of the following three axioms:

- For each positive integer n, Q_n is finite with a unique top element and is pure of length $n-1$.
- Every upper interval of Q_n is isomorphic to a partition lattice Π_t for some integer t.
- Suppose we have element π of Q_n and some minimal element ρ with $\rho \leq \pi$. Then the interval $[\rho, \pi]$ is isomorphic to $\Pi_1^{a_1} \times \Pi_2^{a_2} \times \cdots \times \Pi_n^{a_n}$ for some unique collection of nonnegative integers $a_1, ..., a_n$ satisfying $\sum ia_i = n$. We call $(a_1, ..., .a_n)$ the *shape* of π. It is then also required that the subposet $\Lambda_\pi = \{\sigma \in Q_n | \sigma \leq \pi\}$ be isomorphic to $Q_1^{a_1} \times Q_2^{a_2} \times \cdots \times Q_n^{a_n}$. In particular, if ρ' is another minimal element with $\rho' \leq \pi$, then $[\rho, \pi] \cong [\rho', \pi]$.

Further, if $Q_1, Q_2, ...$ satisfies these conditions, we denote by $M(n)$ for each positive integer n the number of minimal elements of Q_n, and call this the *denominator sequence* of the exponential structure.

Note that we use here *shape* rather than the usual nomenclature of *type*, as in [18]; this is to avoid confusion later with our previous use of *type* in terms of independent sets. The prototypical exponential structure is the sequence of partition lattices, given by $Q_n = \Pi_n$; here, we have that $M(n) = 1$.

Define the poset $DS_n(q)$ to consist of all collections $\{W_1, W_2, ..., W_k\}$ of subspaces of $\mathbb{F}_q^n$ such that $\dim(W_i) > 0$ for all i and $\mathbb{F}_q^n = W_1 \oplus \cdots \oplus W_k$, and partially ordered such that $\{W_1, ..., W_k\} \leq_{DS_n(q)} \{V_1, ..., V_m\}$ whenever each W_i is contained some V_j. Note that $DS_n(q)$ is the subposet of $\Pi_{B_n(q)}$

consisting exactly of independent sets of total rank n. Stanley [18] observes that $DS_1(q), DS_2(q), \ldots$ is an exponential structure with denominator sequence $M(n) = \dfrac{q^{\binom{n}{2}}[n]_q!}{n!}$. This follows from Lemmas 6.2.2 and 6.2.4; we can also establish the value of $M(n)$ using Lemma 6.2.12.

Specifically, recall from Lemma 6.2.12 that $q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, \ldots, \lambda_t \end{bmatrix}_q \prod_{j=1}^{t} \dfrac{1}{a_j(\lambda)}$ is the number of distinct direct sum decompositions of $\mathbb{F}_q^n$ into subspaces whose dimensions form the partition λ. We now show that to be minimum in $DS_n(q)$, the direct sum must have exactly n summands of dimension 1 each. Clearly, there can be no direct sum strictly smaller than such a decomposition in the partial order, so that such elements are minimal.

On the other hand, any summand with dimension larger than 1 can be decomposed into a direct sum of smaller subspaces. Thus, the minimal elements of $DS_n(q)$ are characterized by the unique partition $\lambda = \{1, 1, \ldots, 1\} \vdash n$; by Lemma 6.2.12, since $e_2(\{1, 1, \ldots, 1\}) = \sum_{1 \leq i < j \leq t} 1 = \binom{n}{2}$, $[1]_q! = 1$, and $a_j(\lambda) = 0$ unless $j = n$, in which case $a_n(\lambda) = n$, we get that $M(n) = \dfrac{q^{\binom{n}{2}}[n]_q!}{n!}$.

This is the structure analyzed by Welker in [23], and is another typical example of an exponential structure; note however that Welker adds a minimum element to this poset to make it a lattice. Observe that $\Pi_n^{=2}(q)$ fails the condition that its upper intervals are isomorphic to some partition lattice Π_t, so the sequence $\Pi_1^{=2}(q), \Pi_2^{=2}(q), \ldots$ is not an exponential structure.

Now any exponential structure $Q_1, Q_2, \ldots$ allows us to make use of the *compositional formula*, in which we are given functions $f : \mathbb{P} \to \mathbb{Q}$ and $g : \mathbb{N} \to \mathbb{Q}$ with $g(0) = 1$, and we define a new function $h : \mathbb{N} \to \mathbb{Q}$ that satisfies the following conditions:

- $h(0) = 1$
- $h(n) = \sum_{\pi \in Q_n} f(1)^{a_1} f(2)^{a_2} \cdots f(n)^{a_n} g(|\pi|)$ for $n \geq 1$,

where $(a_1, ..., a_n)$ is the shape of π, and $|\pi| = a_1 + \cdots + a_n$. The compositional formula says that if we define three formal power series in $\mathbb{Q}[[x]]$ by

$$F(x) = \sum_{n \geq 1} f(n) \frac{x^n}{n! M(n)},$$
$$G(x) = \sum_{n \geq 0} g(n) \frac{x^n}{n!},$$
$$H(x) = \sum_{n \geq 0} h(n) \frac{x^n}{n! M(n)},$$

then we have the relation $H(x) = G(F(x))$, the composition of $G(x)$ and $F(x)$.

7.2 Applying Exponential Structures to $\Pi_n^{=2}(q)$

Recalling Theorem 6.2.9 and Corollary 6.2.13, we have the following formula for the number $\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})$ of falling chains of $\Pi_n^{=2}(q)$ under the EL-labeling Ψ of Definition 6.1.4:

$$\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)}) = \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{\lambda \vdash n \\ \ell(\lambda)=t \\ \lambda_i \geq 2 \ \forall \ i}} (t-1)! \left(\prod_{i=1}^{t} q^{\binom{\lambda_i - 1}{2}} [\lambda_i - 1]_q \right) q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t \end{bmatrix}_q \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!}.$$

While this formula is decent, we obtain the following more convenient form:

Theorem 7.2.1. *For any integer $n > 2$, the number $\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})$ of falling chains of $\Pi_n^{=2}(q)$ under the EL-labeling Ψ of Definition 6.1.4 is*

$$\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)}) = \widetilde{g}_n(q) \cdot q^{\binom{n-1}{2}} [n-1]_q!,$$

where we define the polynomial

$$\widetilde{g_n}(q) = \frac{1}{nq}\bigg([n]_q - (1-q)^{n-1}\bigg).$$

Before we prove this result, we observe several facts. First, we have that $\mu(\Pi_n^{=2}(q)) = (-1)^{n-1}\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})$, since the lattice is pure of length $n-1$. Second, since $\widetilde{g_n}(1) = 1$ for all $n > 1$, $\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})$ forms a direct q–analogue of $(n-1)!$. Last, there are also other ways of writing the polynomial $\widetilde{g_n}$; some alternate formulations are given below in Proposition 7.3.8.

To prove Theorem 7.2.1, we apply the theory of exponential structures from Section 7.1. To this end, we will use the exponential structure $DS_1(q), DS_2(q), ...,$ and begin with the following definition.

Definition 7.2.2. Let q be a fixed prime power. Define:

- $f : \mathbb{P} \to \mathbb{Q}$ by

$$f(n) = q^{\binom{n-1}{2}}[n-1]_q$$

- $g : \mathbb{N} \to \mathbb{Q}$ by

$$g(0) = 1, \text{ and } g(n) = (n-1)! \text{ for } n \geq 1$$

Theorem 7.2.3. *For the positive integer $n \geq 2$,*

$$\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)}) = \sum_{\pi \in DS_n(q)} f(1)^{a_1} f(2)^{a_2} \cdots f(n)^{a_n} g(|\pi|),$$

with $f(n)$ and $g(n)$ defined in Definition 7.2.2.

Proof. For each $\pi \in DS_n(q)$, we can write $\pi = \{V_1, V_2, ..., V_t\}$ for some positive integer t, and such that $\dim(V_1) \geq \dim(V_2) \geq \cdots \geq \dim(V_t)$. Let $\lambda_i = \dim(V_i)$ for each $1 \leq i \leq t$, and denote by λ the partition $\{\lambda_1, \lambda_2, ..., \lambda_t\}$; note that λ

is a partition of n, since π is a direct sum decomposition of $\mathbb{F}_q^n$. We define the *type* of π to be the partition λ, and denote this by $\text{type}(\pi) = \lambda$. Let $a_i(\lambda)$ denote the number of parts of λ which equal i, for each $1 \leq i \leq n$.

Claim 7.2.4. *Given* $\pi \in DS_n(q)$ *with* $type(\pi) = \lambda$ *for* $n \geq 1$, *the sequence* $(a_1(\lambda), ..., a_n(\lambda))$ *is the shape of* π.

Proof of Claim 7.2.4. Recall that a minimal element $\rho \in DS_n(q)$ is a direct sum decomposition of $\mathbb{F}_q^n$ into n one-dimensional summands. If $\rho \leq \pi$, then each summand of ρ is contained in a single summand of π. Arrange the summands of ρ so that for $\pi = \{V_1, V_2, ..., V_t\}$, with $\lambda_i = \dim(V_i)$ for each $1 \leq i \leq t$, we have that $\rho = \{W_{1,1}, ..., W_{1,\lambda_1}, W_{2,1}, ..., W_{2,\lambda_2}, ..., W_{t,\lambda_t}\}$, where each of the λ_i summands of the form $W_{i,j}$ is contained in V_i.

Now since we can only place W_{i_1,j_1} and W_{i_2,j_2} together in a single subspace when $i_1 = i_2$, we can see that there is an isomorphism from $[\rho, \pi]$ to the poset $\Pi_{\lambda_1} \times \Pi_{\lambda_2} \times \cdots \times \Pi_{\lambda_t} = \Pi_1^{a_1(\lambda)} \times \Pi_2^{a_2(\lambda)} \times \cdots \times \Pi_n^{a_n(\lambda)}$, given by identifying $W_{i,j}$ with the number j in Π_{λ_i}. Since the shape of π is unique, we have that $(a_1(\lambda), ..., a_n(\lambda))$ is the shape of π. □

(Proof of Theorem 7.2.3 continued) Since we have that the shape of π is $(a_1(\lambda), ..., a_n(\lambda))$, we must also have that $|\pi| = \ell(\lambda)$, the number of parts of λ. Since the shape of π depends only on the type of π, we observe now that we have

$$\sum_{\pi \in DS_n(q)} f(1)^{a_1} \cdots f(n)^{a_n} g(|\pi|) = \sum_{\lambda \vdash n} \sum_{\substack{\pi \in DS_n(q) \\ \text{type}(\pi) = \lambda}} f(1)^{a_1(\lambda)} \cdots f(n)^{a_n(\lambda)} g(\ell(\lambda)) \quad (7.2.1)$$

$$= \sum_{\lambda \vdash n} f(1)^{a_1(\lambda)} \cdots f(n)^{a_n(\lambda)} g(\ell(\lambda)) \Big| \{\pi \in DS_n(q) \mid \text{type}(\pi) = \lambda\} \Big| \qquad (7.2.2)$$

$$= \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{\lambda \vdash n \\ \lambda_i \geq 2 \ \forall \ i \\ \ell(\lambda) = t}} g(t) \left(\prod_{i=1}^{t} f(\lambda_i) \right) \left| \Pi_{B_n(q),\lambda} \right| \tag{7.2.3}$$

$$= \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{\lambda \vdash n \\ \lambda_i \geq 2 \ \forall \ i \\ \ell(\lambda) = t}} (t-1)! \left(\prod_{i=1}^{t} q^{\binom{\lambda_i - 1}{2}} [\lambda_i - 1]_q \right) q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t \end{bmatrix}_q \prod_{j=1}^{n} \frac{1}{a_j(\lambda)!} \tag{7.2.4}$$

Here, we can see that (7.2.2) follows from (7.2.1) by noticing that the terms in the summation depend only on the type of π, so that we may collect all elements $\pi \in DS_n(q)$ with type λ. Then (7.2.3) follows from (7.2.2) by collecting all partitions λ with the same number of parts t, since $g(\ell(\lambda))$ depends only on this value, and restricting each part of λ to be at least 2 since $f(1) = 0$. Further, to see that $\{\pi \in DS_n(q) \mid \text{type}(\pi) = \lambda\} = \Pi_{B_n(q),\lambda}$, where $\Pi_{B_n(q),\lambda}$ is as in (6.2.1), we note that $T \in DS_n(q)$ if and only if $T \in \Pi_{B_n(q)}$ and $\text{type}(T)$ is a partition of n. Lastly, we use Lemma 6.2.12 and the definitions of $f(n)$ and $g(n)$ from Definition 7.2.2 to obtain (7.2.4), which is exactly the value of $\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})$ given by (6.2.2) and Corollary 6.2.13. $\square$

We will now apply the compositional formula to the exponential structure $DS_1(q), DS_2(q), ...$ and the functions f and g of Definition 7.2.2. Let

$$h(n) = \begin{cases} \displaystyle\sum_{\pi \in DS_n(q)} f(1)^{a_1} f(2)^{a_2} \cdots f(n)^{a_n} g(|\pi|), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

Now we see that for $n \geq 2$, we can write $h(n) = \widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})$ by Theorem 7.2.3. Further, it is clear that $h(1) = 0$, since there is only one direct sum decomposition of $\mathbb{F}_q^1$, which has only one summand of dimension 1, and $f(1) = 0$.

Thus, using Definition 7.2.2 and Theorem 7.2.3, we have that

$$H(x) = \sum_{n\geq 0} h(n)\frac{x^n}{n!M(n)} = 1 + \sum_{n\geq 2} \widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})\frac{x^n}{q^{\binom{n}{2}}[n]_q!}.$$

By the compositional formula, we have $H(x) = G(F(x))$, where

$$\begin{aligned} G(x) &= \sum_{n\geq 0} g(n)\frac{x^n}{n!} \\ &= 1 + \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} \\ &= 1 + \sum_{n\geq 1}\frac{x^n}{n} \\ &= 1 - \log(1-x), \end{aligned}$$

and

$$\begin{aligned} F(x) &= \sum_{n\geq 1} f(n)\frac{x^n}{n!M(n)} \\ &= \sum_{n\geq 2} q^{\binom{n-1}{2}}[n-1]_q\frac{x^n}{q^{\binom{n}{2}}[n]_q!} \qquad (7.2.5) \\ &= \sum_{n\geq 2}\frac{x^n}{q^n[n-1]_q!} - \sum_{n\geq 2}\frac{x^n}{q^n[n]_q!}, \qquad (7.2.6) \end{aligned}$$

since for all integers $n \geq 1$ we have the identities $[n-1]_q = \frac{1}{q}([n]_q - 1)$ and $\frac{q^{\binom{n-1}{2}}}{q^{\binom{n}{2}}} = \frac{1}{q^{n-1}}$, so that we can rewrite the coefficient of x^n in (7.2.5) as

$$\frac{q^{\binom{n-1}{2}}[n-1]_q}{q^{\binom{n}{2}}[n]_q!} = \frac{\frac{1}{q}([n]_q - 1)}{q^{n-1}[n]_q!} = \frac{1}{q^n[n-1]_q!} - \frac{1}{q^n[n]_q!},$$

from which (7.2.6) follows.

Now using the q–exponential $e_q(x) := \sum_{n\geq 0}\frac{x^n}{[n]_q!}$, we can rewrite $F(x)$ as

$$\begin{aligned} F(qz) &= z\left(e_q\left(z\right) - 1\right) - \left(e_q\left(z\right) - 1 - z\right) \\ &= 1 + e_q\left(z\right)\left(z - 1\right), \end{aligned}$$

where $z = \frac{x}{q}$ for notational simplicity. Composing this with $G(x)$ gives

$$G\big(F(qz)\big) = 1 - \log\big(1 - F(qz)\big) = 1 - \log\big((1-z)e_q(z)\big),$$

which implies

$$G\big(F(x)\big) = 1 - \log\left(1 - \frac{x}{q}\right) - \log\left(e_q\left(\frac{x}{q}\right)\right).$$

Now we wish to find a power series representation of $G(F(x))$. To accomplish this, we find power series for each of the logarithms in the formula above, then add them term by term. For the first, standard calculation gives $-\log(1-\frac{x}{q}) = \sum_{n\geq 1} \frac{x^n}{nq^n}$. To evaluate the logarithm of the q–exponential, we use the following lemma:

Lemma 7.2.5. *The following formal power series identity holds:*

$$\log(e_q(z)) = \sum_{n\geq 1}\left(\frac{(1-q)^{n-1}}{[n]_q}\frac{z^n}{n}\right).$$

Before proving this lemma, we use its result to find that

$$\begin{aligned} G(F(x)) &= 1 + \sum_{n\geq 1}\frac{x^n}{nq^n} - \sum_{n\geq 1}\left(\frac{(1-q)^{n-1}}{[n]_q}\frac{x^n}{nq^n}\right) \\ &= 1 + \sum_{n\geq 1}\frac{x^n}{n}\left(\frac{1}{q^n} - \frac{(1-q)^{n-1}}{[n]_q q^n}\right) \\ &= 1 + \sum_{n\geq 1} x^n \left(\frac{\frac{1}{qn}([n]_q - (1-q)^{n-1})q^{\binom{n-1}{2}}[n-1]_q!}{q^{\binom{n}{2}}[n]_q!}\right). \end{aligned}$$

Setting this now equal to $H(x) = 1 + \sum_{n\geq 2}\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)})\frac{x^n}{q^{\binom{n}{2}}[n]_q!}$, we conclude that $\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)}) = \frac{1}{qn}([n]_q - (1-q)^{n-1})q^{\binom{n-1}{2}}[n-1]_q!$, which is what was claimed in Theorem 7.2.1. We also note that this formula agrees with our result that

$h(1) = 0$. Thus, once we prove Lemma 7.2.5 concerning the logarithm of the q–exponential, we will have proven Theorem 7.2.1.

Proof of Lemma 7.2.5. To prove the lemma, we use several standard techniques involving formal power series (see for instance [14]). Recall first that as a power series, we have that $\log(1+z) = \sum_{n\geq 1} \frac{(-1)^{n-1}z^n}{n}$. Further, to compose two power series $E(z)$ and $L(z)$ to form $L(E(z))$, we must have that $E(z)$ has constant term 0, in which case we replace each z in the series expansion of $L(z)$ by the series $E(z)$. Consider the series $E(z) = e_q(z) - 1 = \sum_{n\geq 1} \frac{z^n}{[n]_q!}$; this series has constant term 0, so we may form the composition

$$\log(e_q(z)) = \log(1 + (e_q(z) - 1)) = \sum_{n\geq 1} (-1)^{n-1} \frac{(e_q(z) - 1)^n}{n}.$$

Notice that since $(e_q(z) - 1)$ has no constant term, then every power $(e_q(z) - 1)^n$ also has no constant term. Thus, the formal power series $\log(e_q(z))$ has no constant term as well. Further, the constant term of the claimed series $\sum_{n\geq 1} \frac{(1-q)^{n-1}}{[n]_q} \frac{z^n}{n}$ is also zero, and so the two series both have no constant term. Therefore, the two series are equal if and only if they have the same formal power series derivative with respect to z. We use this fact along with the derivative of logarithm series to claim that if we can show that $\frac{\frac{\partial}{\partial z} e_q(z)}{e_q(z)} = \sum_{n\geq 1} \left(\frac{(1-q)^{n-1}}{[n]_q} z^{n-1} \right)$, then the original power series formula will logically follow.

In the ring of formal power series, a fraction of two series is defined as the series whose product with the denominator equals the numerator; thus, we need to show that $\frac{\partial}{\partial z} e_q(z) = e_q(z) \cdot \sum_{n\geq 1} \left(\frac{(1-q)^{n-1}}{[n]_q} z^{n-1} \right)$. Writing this now in series notation, and reindexing to start all series at $n = 0$, we can write this as

$$\sum_{n\geq 0} z^n \frac{n+1}{[n+1]_q!} = \left(\sum_{n\geq 0} \frac{z^n}{[n]_q!} \right) \left(\sum_{n\geq 0} \frac{(1-q)^n}{[n+1]_q} z^n \right).$$

Multiplying the two series on the right and collecting powers of z gives that we wish to show

$$\sum_{n\geq 0} z^n \frac{n+1}{[n+1]_q!} = \sum_{n\geq 0} z^n \left(\sum_{i=0}^{n} \frac{1}{[i]_q!} \cdot \frac{(1-q)^{n-i}}{[n+1-i]_q} \right).$$

Equating the coefficients of z on both sides of this gives us that we wish to prove for all nonnegative integers n that

$$\frac{n+1}{[n+1]_q!} = \sum_{i=0}^{n} \frac{1}{[i]_q!} \cdot \frac{(1-q)^{n-i}}{[n+1-i]_q}.$$

We can show that this equation holds for various values of n; for instance, for $n = 0$, we have $\frac{1}{[0]_q!} \cdot \frac{(1-q)^0}{[1]_q} = \frac{1}{[1]_q}$, while for $n = 1$,

$$\frac{1}{[0]_q!} \cdot \frac{(1-q)^1}{[2]_q} + \frac{1}{[1]_q} \cdot \frac{(1-q)^0}{[1]_q} = \frac{1-q}{[2]_q} + 1 = \frac{2}{[2]_q}.$$

To prove this for general n, we first rewrite this equation as

$$n+1 = \sum_{i=0}^{n} \left(\frac{[n+1]_q!}{[i]_q! \cdot [n-i]_q!} \cdot \frac{[n-i]_q!(1-q)^{n-i}}{[n+1-i]_q} \right).$$

Now inside the summation on the right, we have that

$$\begin{aligned} \frac{[n+1]_q!}{[i]_q! \cdot [n-i]_q!} \cdot \frac{[n-i]_q!(1-q)^{n-i}}{[n+1-i]_q} &= \begin{bmatrix} n+1 \\ i \end{bmatrix}_q \cdot [n-i]_q! \cdot (1-q)^{n-i} \\ &= \begin{bmatrix} n+1 \\ i \end{bmatrix}_q \prod_{j=1}^{n-i} (1-q^j). \end{aligned}$$

We use this simplification now and rewrite the statement once more as

$$n+1 = \sum_{i=0}^{n} \left(\begin{bmatrix} n+1 \\ i \end{bmatrix}_q \cdot \prod_{j=1}^{n-i} (1-q^j) \right)$$

Define $P(n) = \sum_{i=0}^{n} \left(\begin{bmatrix} n+1 \\ i \end{bmatrix}_q \cdot \prod_{j=1}^{n-i} (1-q^j) \right)$ for each integer $n \geq 0$. Then we can say equivalently that we wish to show that for all integers $n \geq 0$, that $P(n) = n+1$. We have already observed that $P(0) = 1$ and $P(1) = 2$ (after suitably rearranging the case above with $n = 1$). To prove the result in general, we will prove that $R(n) := P(n+1) - P(n)$ is such that $R(n) \equiv 1$ for all $n \geq 0$. We can see that, for instance, $R(0) = P(1) - P(0) = 2 - 1 = 1$. We begin by writing

$$R(n) = \sum_{i=0}^{n+1} \left(\begin{bmatrix} n+2 \\ i \end{bmatrix}_q \cdot \prod_{j=1}^{n+1-i} (1 - q^j) \right) - \sum_{i=0}^{n} \left(\begin{bmatrix} n+1 \\ i \end{bmatrix}_q \cdot \prod_{j=1}^{n-i} (1 - q^j) \right).$$

In the first term (the $P(n+1)$ sum), we observe that if we fix a particular product $\prod_{j=1}^{k}(1-q^j)$ for some fixed integer $k \leq n$, then its coefficient is $\begin{bmatrix} n+2 \\ n+1-k \end{bmatrix}_q$. Similarly, the same product has coefficient $\begin{bmatrix} n+1 \\ n-k \end{bmatrix}_q$ in the $P(n)$ sum. Further, in $P(n+1)$, in the term with $i = 0$, for which the product is $\prod_{j=1}^{n+1}(1 - q^j)$, we observe that this product does not appear in $P(n)$, but its coefficient is exactly 1 in $P(n+1)$. Now we use both the q–analogue of Pascal's identity as well as the symmetry of the q–binomial coefficients to observe that

$$\begin{bmatrix} n+2 \\ n+1-k \end{bmatrix}_q - \begin{bmatrix} n+1 \\ n-k \end{bmatrix}_q = q^{n+1-k} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q .$$

We combine this with the previous results to see that

$$R(n) = \sum_{k=0}^{n+1} \left(q^{n+1-k} \cdot \prod_{j=0}^{k-1} (1 - q^{n+1-j}) \right),$$

since when $k = n+1$, the term simplifies to $\prod_{j=0}^{n}(1 - q^{n+1-j}) = \prod_{j=1}^{n+1}(1 - q^j)$, the unique term outlined above, and for the remaining terms (using the convention that a void product is 1), $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \cdot \prod_{j=1}^{k}(1 - q^j) = \prod_{j=0}^{k-1}(1 - q^{n+1-j})$.

For example, just to verify, we can see that

$$R(0) = \sum_{k=0}^{1} \left(q^{1-k} \cdot \prod_{j=0}^{k-1}(1 - q^{1-j}) \right) = q^1 + q^0 \cdot (1 - q) = 1,$$

and

$$R(1) = \sum_{k=0}^{2} \left(q^{2-k} \cdot \prod_{j=0}^{k-1} (1 - q^{2-j}) \right) = q^2 + q^1 \cdot (1 - q^2) + q^0 \cdot (1 - q^2)(1 - q) = 1.$$

We now show that $R(n) = 1$ for all n by induction on n. Assume that $R(n) = 1$ for some integer $n \geq 0$. By the formula we have derived above for $R(n)$, we see that

$$R(n+1) = \sum_{k=0}^{n+2} \left(q^{n+2-k} \cdot \prod_{j=0}^{k-1} (1 - q^{n+2-j}) \right).$$

When $k = 0$, the void product gives that this term is exactly q^{n+2}. We remove this term from the sum and write then

$$R(n+1) = q^{n+2} + \sum_{k=1}^{n+2} \left(q^{n+2-k} \cdot \prod_{j=0}^{k-1} (1 - q^{n+2-j}) \right).$$

If we now reindex this sum by setting $i = k - 1$, we can write

$$R(n+1) = q^{n+2} + \sum_{i=0}^{n+1} \left(q^{n+1-i} \cdot \prod_{j=0}^{i} (1 - q^{n+2-j}) \right).$$

As the product inside this sum is now never void, the factor corresponding to $j = 0$ is common to all terms in the sum; we factor this out to obtain

$$R(n+1) = q^{n+2} + (1 - q^{n+2}) \cdot \sum_{i=0}^{n+1} \left(q^{n+1-i} \cdot \prod_{j=1}^{i} (1 - q^{n+2-j}) \right).$$

Reindexing this product now by setting $\ell = j - 1$, we have

$$R(n+1) = q^{n+2} + (1 - q^{n+2}) \cdot \sum_{i=0}^{n+1} \left(q^{n+1-i} \cdot \prod_{\ell=0}^{i-1} (1 - q^{n+1-\ell}) \right).$$

But now we can observe that this summation is exactly equal to the formula we defined for $R(n)$, so we can rewrite this one last time as

$$R(n+1) = q^{n+2} + (1 - q^{n+2}) \cdot R(n),$$

and now from our inductive hypothesis that $R(n) = 1$, this reduces so that $R(n+1) = q^{n+2} + (1 - q^{n+2}) = 1$. Thus $R(n) = 1$ for all positive integers n, since $R(0) = 1$.

This then gives that $P(n+1) = P(n)+1$ for all positive integers n; combining this with $P(0) = 1$ and $P(1) = 2$, we conclude that $P(n) = n + 1$ for all nonnegative integers n. Using this, we conclude further that since the derivatives of the two series coincide term for term and they share the same constant term, we have that $\log(e_q(z)) = \sum_{n\geq 1} \left(\frac{(1-q)^{n-1}}{[n]_q} \frac{z^n}{n} \right)$, as claimed. $\square$

7.3 Further Results for $\Pi_n^{=2}(q)$

Recall from Proposition 5.3.3 that it is natural to use matrices to represent subspaces of $B_n(q)$. As we have shown that $\Pi_n^{=k}(q)$ is a q–analogue of $\Pi_n^{=k}$, we would like to include the Boolean algebra B_n as the case $B_n(1)$. To make this compatibility possible, we can use the following definition.

Definition 7.3.1. Define $\mathbb{F}_1^n$ to be the set of n coordinate vectors each of length n, and each composed of a single entry which is a 1 and the remaining $(n-1)$ entries all 0. We do not define either addition or scalar multiplication over this object, so it is not an actual vector space. For example, $\mathbb{F}_1^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$. A *subspace* of $\mathbb{F}_1^n$ is then a subset of these coordinate vectors, and a *direct sum* is a set partition of these coordinate vectors.

If we now label the columns of these coordinate vectors of $\mathbb{F}_1^n$ from right to left, i.e., $(n, n-1, ..., 3, 2, 1)$ as the column labels, then there is a bijection from subspaces of $\mathbb{F}_1^n$ to subsets of $[n]$, and another bijection from direct sums to set partitions of $[n]$, both of which preserve the partial order on both posets. Notice that if we use this definition to describe the lattice $B_n(1)$, then $B_n(1) \cong B_n$ for each positive integer n. Thus, from now on, we will include B_n as the member of the family $B_n(q)$ when $q = 1$.

Now we can exhibit further results; for instance, consider the following definition:

Definition 7.3.2. Given any ranked bounded poset P, the *Whitney numbers of the first kind* are the sequence of integers denoted by $w_0, w_1, w_2, ..., w_k, ...$, such that each w_k is the sum of the Möbius invariants of the lower intervals of length k in P. Similarly, the *Whitney numbers of the second kind* are the sequence of integers denoted by $W_0, W_1, W_2, ..., W_k, ...$ such that each W_k is the number of elements of rank k in P. In other words,

$$w_k = \sum_{\substack{t \in P \\ \rho(t)=k}} \mu([\hat{0}, t]), \qquad \text{and} \qquad W_k = \sum_{\substack{t \in P \\ \rho(t)=k}} 1$$

A well-known result concerning the partition lattice Π_n is that $w_k = s(n, n-k)$, where $s(n, i)$ is a Stirling number of the first kind, and similarly, $W_k = S(n, n-k)$, where $S(n, i)$ is a Stirling number of the second kind (see [16]). The next proposition demonstrates that these Whitney numbers of the second kind extend naturally to $\Pi_n^{=2}(q)$:

Proposition 7.3.3. *For* $j > 0$*, the Whitney number of the second kind* W_j *for* $\Pi_n^{=2}(q)$ *is*

$$W_j = \sum_{\substack{\lambda \vdash m \leq n \\ \ell(\lambda)=t \\ \lambda_i \geq 2 \ \forall \ i \\ j=m-t}} q^{e_2(\lambda)} \begin{bmatrix} n \\ \lambda_1, ..., \lambda_t, (n-m) \end{bmatrix}_q \prod_{i=1}^{n} \frac{1}{a_i(\lambda)!},$$

where $a_i(\lambda)$ *equals the number of parts of* λ *of size* i*, and* $e_2(\lambda) = \sum_{1 \leq i_1 < i_2 \leq t} \lambda_{i_1} \lambda_{i_2}$.

Here, both m and t are variables which are allowed to change, provided their difference $j = m - t$ is fixed. Note that if $t = 1$, then $e_2(\lambda) = 0$. Also, since $\Pi_n^{=2}(q)$ is bounded, we must have that $W_0 = 1$. The proof of this proposition relies on (6.2.3) and the following simple lemma:

Lemma 7.3.4. *Let* $T \in \Pi_n^{=2}(q)$*, and let* $t = |T|$ *and* $m = r(T)$*. Then* $\rho(T) = m - t$ *in* $\Pi_n^{=2}(q)$*.*

Proof. Since all maximal chains of $\Pi_n^{=2}(q)$ are of the same length, we need only compute the ranks of a single maximal chain passing through T. For this chain, we can choose the portion below T to consist only of creations and expansions, since such chains exist by construction. To generate T, we need exactly t creations; since each creation adds exactly 2 to the total rank $r(T)$ of T, we have that $m \geq 2t$.

Clearly, the number of expansions necessary is $m - 2t$, since each expansion adds 1 to the sum of the ranks. There are then a total of $t + m - 2t = m - t$ covering steps below T, so the rank in $\Pi_n^{=2}(q)$ is as claimed. □

Notice that setting $q = 1$ gives an alternative formula for the Stirling numbers of the second kind in terms of binomial coefficients and partitions.

Example 7.3.5. Consider $n = 5$. Since $m \leq 5$ and $t \leq \frac{5}{2}$, we have the following possibilities for each value of j:

j	1	2	2	3	3	4
m	2	3	4	4	5	5
t	1	1	2	1	2	1

Notice that $j \ngtr 4$. We can use this table and Proposition 7.3.3 to calculate the following with Maple:

i	W_i
0	1
1	$\begin{bmatrix} 5 \\ 2,3 \end{bmatrix}_q = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$
2	$\begin{bmatrix} 5 \\ 3,2 \end{bmatrix}_q + \frac{q^4}{2} \begin{bmatrix} 5 \\ 2,2,1 \end{bmatrix}_q =$ $1 + q + 2q^2 + 2q^3 + \frac{5}{2}q^4 + 2q^5 + 3q^6 + \frac{5}{2}q^7 + 3q^8 + \frac{5}{2}q^9 + 2q^{10} + q^{11} + \frac{1}{2}q^{12}$
3	$\begin{bmatrix} 5 \\ 4,1 \end{bmatrix}_q + q^6 \begin{bmatrix} 5 \\ 3,2 \end{bmatrix}_q =$ $1 + q + q^2 + q^3 + q^4 + q^6 + q^7 + 2q^8 + 2q^9 + 2q^{10} + q^{11} + q^{12}$
4	$\begin{bmatrix} 5 \\ 5 \end{bmatrix}_q = 1$
> 4	0

Observe that when $q = 1$, we get the sequence $1, 10, 25, 15, 1, 0, 0...,$ which is the sequence of Stirling numbers of the second kind with $n = 5$, as expected.

Now using the result of Theorem 7.2.1 and (2.1.1), we can also demonstrate a formula for the Whitney numbers of the first kind, combining our knowledge of Möbius invariants and the number of elements of each rank type in the lattice.

Proposition 7.3.6. *For* $j>0$*, the Whitney number of the first kind* w_j *for* $\Pi_n^{=2}(q)$ *is*

$$w_j=(-1)^j\sum_{\substack{\lambda\vdash m\leq n\\ \ell(\lambda)=t\\ \lambda_i\geq 2\ \forall\ i\\ j=m-t}} q^{e_2(\lambda)}\begin{bmatrix} n\\ \lambda_1,\ldots,\lambda_t,(n-m)\end{bmatrix}_q\prod_{i=1}^{n}\frac{1}{a_i(\lambda)!}\prod_{i=1}^{t}\widetilde{g}_{\lambda_i}(q)q^{\binom{\lambda_i-1}{2}}[\lambda_i-1]_q!,$$

where

$$\widetilde{g}_{\lambda_i}(q)=\frac{1}{\lambda_i q}\Big([\lambda_i]_q-(1-q)^{\lambda_i-1}\Big),$$

$a_i(\lambda)$ *is the number of parts of* λ *of size* i*, and* $e_2(\lambda)=\sum_{1\leq i_1<i_2\leq t}\lambda_{i_1}\lambda_{i_2}$.

We examine the case when $n=5$ as an example. Recalling from Example 7.3.5 the allowable values of m and t for each j, we can compute the following table using Maple:

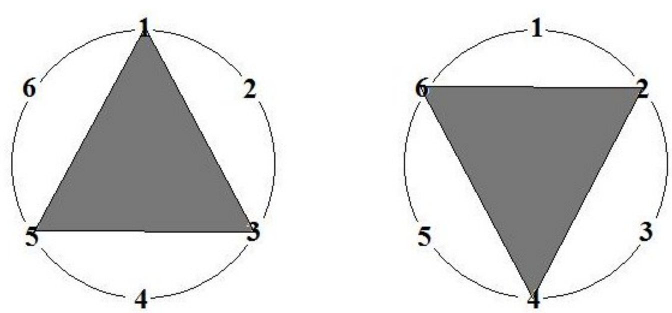
i	w_i
0	1
1	$-\begin{bmatrix}5\\2,3\end{bmatrix}_q=-(1+q+2q^2+2q^3+2q^4+q^5+q^6)$
2	$(q^2+q)\begin{bmatrix}5\\3,2\end{bmatrix}_q+\frac{q^4}{2}\begin{bmatrix}5\\2,2,1\end{bmatrix}_q=$ $q+2q^2+3q^3+\frac{9}{2}q^4+5q^5+5q^6+\frac{9}{2}q^7+4q^8+\frac{5}{2}q^9+2q^{10}+q^{11}+\frac{1}{2}q^{12}$
3	$-(1-\frac{1}{2}q+\frac{1}{2}q^2)q^3[3]_q!\begin{bmatrix}5\\4,1\end{bmatrix}_q-(q^8+q^7)\begin{bmatrix}5\\3,2\end{bmatrix}_q=$ $-\left(q^3+\frac{5}{2}q^4+4q^5+5q^6+\frac{13}{2}q^7+7q^8+\frac{13}{2}q^9+6q^{10}+5q^{11}+\frac{7}{2}q^{12}+2q^{13}+q^{14}\right)$
4	$(1-q+q^2)q^6[4]_q!\begin{bmatrix}5\\5\end{bmatrix}_q=$ $q^6+2q^7+3q^8+4q^9+4q^{10}+4q^{11}+3q^{12}+2q^{13}+q^{14}$
>4	0

Now if we evaluate these polynomials when $q=1$, we obtain the following sequence: $1,-10,35,-50,24,0,0,\ldots$, which is exactly the sequence of Stirling numbers of the first kind for $n = 5$, as desired. Notice that because they are defined in terms of sums over partitions, neither of these formulas appears to have a simple recurrence relation, as do the Stirling numbers.

We also have alternate formulations for the polynomial $\widetilde{g_n}(q)$. We begin with a definition.

Definition 7.3.7. Fix positive integers n and h such that $0 < h < n$, and fix a circular arrangement of the letters of $[n]$. Let $a_{n,h}$ be the number of distinct subsets of $[n]$ of size h such that no two letters in the subset are adjacent in the arrangement. Define the *Lucas polynomial* $L_n^*(x)$ by $L_n^*(x) := \sum_{h \geq 1} a_{n,h} x^{h-1}$.

Note that the Lucas polynomials have varying definitions in the literature, usually by taking $a_{n,h}$ as coefficients of different powers of x. Further, these polynomials have both a closed form and a recursive form. For example, we have $a_{6,3} = 2$, which we can see directly in the following diagram, which illustrates the only two possible subsets of size 3:



Proposition 7.3.8. *The following polynomials are all equal for all positive integers* n*:*

- $g_n(x) = \frac{1}{n} L_n^*(x)$
- $\displaystyle\sum_{i \geq 0} \binom{n-i-2}{i} \frac{x^i}{i+1}$

- $\displaystyle\sum_{t\geq 1}\binom{n-t}{t}\frac{x^{t-1}}{n-t}$

The following polynomials are all equal for all positive integers n:

- $\widetilde{g_n}(q) = \frac{1}{nq}\Big([n]_q - (1-q)^{n-1}\Big)$
- $\displaystyle\sum_{m\geq 0}\left((-q)^m\sum_{i\geq 0}\frac{\binom{n-i-2}{i}\binom{i}{i-m}}{i+1}\right)$
- $\displaystyle\sum_{m\geq 0}\left((-q)^m\sum_{t\geq 1}\frac{\binom{n-t}{t}\binom{t-1}{m-t+1}}{n-t}\right)$
- $\displaystyle\frac{1}{n}\sum_{i=0}^{n-2}\big(q^i + (1-q)^i\big)$
- $\displaystyle\frac{1}{n}\sum_{i=0}^{n-2}\left(1 + (-1)^i\binom{n-1}{i+1}\right)q^i$

Further, $g_n(q^2-q) = \widetilde{g_n}(q)$ for all positive integers n.

We could use the different expressions for the polynomial $\widetilde{g_n}$ and appropriate algebraic manipulation to express the number of falling chains, and thus the Möbius invariant; for instance, we could write

$$\widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)}) = \left(\frac{1}{n}L_n^*(q^2-q)\right) q^{\binom{n-1}{2}}[n-1]_q!.$$

Proof of Proposition 7.3.8. The proof is relatively simple, and relies entirely on the fact that each of the polynomials above is a solution of the following equivalent recurrence relations on these polynomials, with initial conditions $g_1(x) = \widetilde{g_1}(q) = 0$ and $g_2(x) = \widetilde{g_2}(q) = 1$:

$$g_n(x) = \frac{1}{n}\cdot\Big(1 + (n-1)\cdot g_{n-1}(x) + (n-2)\cdot x\cdot g_{n-2}(x)\Big)$$

$$\widetilde{g_n}(q) = \frac{1}{n}\cdot\Big(1 + (n-1)\cdot\widetilde{g_{n-1}}(q) + (n-2)\cdot(q^2-q)\cdot\widetilde{g_{n-2}}(q)\Big)$$

From these recurrence relations, we see that $g_n(q^2-q) = \widetilde{g_n}(q)$. □

Note that these recurrences coincide with recurrence satisfied by the Lucas polynomials. Now these recurrence relations induce the following recurrence relation on the Möbius invariants of these lattices directly:

Proposition 7.3.9. *For the sequence of Möbius invariants* $\left(\mu(\Pi_n^{=2}(q))\right)_{n>2}$*, the following recurrence relation holds, with the initial conditions* $\mu(\Pi_1^{=2}(q)) = 0$ *and* $\mu(\Pi_2^{=2}(q)) = -1$*:*

$$n \cdot \mu(\Pi_n^{=2}(q)) = (-1)^{n-1} q^{\binom{n-1}{2}} [n-1]_q! - (n-1)(q^{n-2}[n-1]_q)\mu(\Pi_{n-1}^{=2}(q))$$
$$+(n-2)(q^2-q)(q^{n-2}[n-1]_q)(q^{n-3}[n-2]_q)\mu(\Pi_{n-2}^{=2}(q))$$

Since the initial terms alternate in sign, it can be easily shown that the Möbius invariants alternate in sign as well, as each of the terms in the recurrence have the same sign for all n. As the Betti numbers of $\Delta(\overline{\Pi_n^{=2}(q)})$ are the absolute values of the Möbius invariants, the related recurrence of the Betti numbers is:

$$n \cdot \widetilde{\beta}_{n-3}(\overline{\Pi_n^{=2}(q)}) = q^{\binom{n-1}{2}}[n-1]_q! + (n-1)(q^{n-2}[n-1]_q) \cdot \widetilde{\beta}_{n-4}(\overline{\Pi_{n-1}^{=2}(q)})$$
$$+(n-2)(q^2-q)(q^{n-2}[n-1]_q)(q^{n-3}[n-2]_q) \cdot \widetilde{\beta}_{n-5}(\overline{\Pi_{n-2}^{=2}(q)})$$

We conclude with a few observations. Recall the characteristic polynomial $\chi(P,t)$ of a bounded ranked poset P with $\ell(P) = n$ is given by $\chi(P,t) = \sum_{z \in P} \mu([\hat{0}, z]) t^{n-\rho(z)} = \sum_{i=0}^{n} w_i t^{n-i}$, for w_i the Whitney numbers of the first kind. In the case that $P = \Pi_n^{=2}(q)$, the characteristic polynomials can be computed rather straightforwardly, and thus these Möbius invariants can be computed recursively in this manner as well. This can be achieved by recalling that for $T \in \Pi_n^{=2}(q)$ such that $T = \{B_1, B_2, ..., B_t\} \neq T_1$, and $\dim(B_i) = a_i$ for each i, we have by Lemma 6.2.4 and (2.1.1) that $\mu([T_0, T]) = \prod_{i=1}^{t} \mu(\Pi_{a_i}^{=2}(q))$.

We can then find all elements S in $\Pi_n^{=2}(q)$ with the same rank. In so doing, we can compute all of the coefficients of the characteristic polynomial other than the coefficient of t. Now since $0 = \sum_{\hat{0} \leq z \leq \hat{1}} \mu([\hat{0}, z])$ in any bounded poset, the remaining coefficient, which is the invariant $\mu(\Pi_n^{=2}(q))$, is completely determined.

For example, in the case when $n = 3$, we need only find the number of atoms of the lattice, since each will have Möbius invariant -1, and these are the only elements which are neither bottom nor top elements. The number of atoms is simply the number of two-dimensional subspaces of the three-dimensional vector space over the field of order q, which is $1 + q + q^2$. Thus, observing that there is a bottom element (a fact which is true for all $n > 1$, so that the coefficient of t^n is always 1 for this family of lattices), the first two terms of the characteristic polynomial are $t^3 - (1 + q + q^2)t^2$.

In order to make the coefficients sum to zero, this then uniquely determines the coefficient of t as $q + q^2$, which direct examination shows to be the Möbius invariant $\mu(\Pi_3^{=2}(q))$ in this case. Now the polynomial

$$\chi(\Pi_3^{=2}(q), t) = t^3 - (1 + q + q^2)t^2 + (q + q^2)t \tag{7.3.1}$$

factors as $t(t - 1)(t - (q^2 + q))$, similar to the case of Π_n, whose characteristic polynomial factors as $\chi(\Pi_n, t) = \prod_{i=1}^{n-1}(t - i)$ (see [16]).

However, a repetition of these procedures for $n = 4$ using (7.3.1) yields a characteristic polynomial of $t^4 - (q^4 + q^3 + 2q^2 + q + 1)t^3 + \frac{1}{2}(q^8 + q^7 + 2q^6 + 3q^5 + 5q^4 + 4q^3 + 4q^2 + 2q)t^2 - \frac{1}{2}(q^2 - q + 2)q^3(q + 1)(q^2 + q + 1)t$, which has roots $t = 0$, $t = 1$, and $t = \frac{1}{2}(q^3 + q^2 + 2q + 1 \pm \sqrt{-q^6 + q^4 + 1})$. This technique is not likely to be fruitful for proving results, as it is computationally intensive and does not appear to yield factorizable polynomials following a general pattern.

On the other hand, now that we have formulas for the Whitney numbers, we can compute both the characteristic polynomial as well as the rank generating function of $\Pi_n^{=2}(q)$. The *rank generating function* $F(P,t)$ is defined for any ranked poset P by $F(P,t) = \sum_{z \in P} t^{\rho(z)}$, and in the case of bounded posets of length n, it can be expressed as the polynomial $F(P,t) = \sum_{i=0}^{n} W_i t^i$.

Finally, it would be insightful to provide a bijective proof that $\widetilde{g_n}(q) = \frac{1}{n} L_n^*(q^2 - q)$, especially in connection with the recurrence relation satisfied by the Möbius invariants. However, no such bijections are known currently.

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