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Attractors in Dynamics with Choice

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ATTRACTORS IN DYNAMICS WITH CHOICE

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Dynamics with choice is a generalization of discrete-time dynamics where instead of the same evolution operator at every time step there is a choice of operators to transform the current state of the system. Many real life processes studied in chemical physics, engineering, biology and medicine, from autocatalytic reaction systems to switched systems to cellular biochemical processes to malaria transmission in urban environments, exhibit the properties described by dynamics with choice. We study the long-term behavior in dynamics with choice. We prove very general results on the existence and properties of global compact attractors in dynamics with choice. In addition, we study the dynamics with restricted choice when the allowed sequences of operators correspond to subshifts of the full shift. One of practical consequences of our results is that when the parameters of a discrete-time system are not known exactly and/or are subject to change due to internal instability, or a strategy, or Nature's intervention, the long term behavior of the system may not be correctly described by a system with "averaged" values for the parameters. There may be a Gestalt effect.

I dedicate this thesis to my parents, Danilo and Darinka, and my sisters Olivera and Danijela. Without their love and support completing this work would be impossible.

Posvecujem ovaj rad mojim dragim roditeljima, Danilu i Darinki, kao i mojim sestrama Danijeli i Oliveri. Bez njihove ljubavi i konstante podrske ovo dostignuce bi bilo nemoguce.

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Chapter 1

INTRODUCTION

The mathematical setting for discrete dynamics is a space X and a map $S : X \to X$. The space X is the state space, the space of all possible states of the system. The map S, the evolution operator, defines the change of states over one time step: $x \in X$ at time $t = 0$ evolves into $S(x)$ at $t = 1$, $S(S(x))$ at $t = 2$, $\ldots$, $S^n(x)$ at $t = n$, etc. If instead of one operator, S, we have a choice of evolution operators, $\{S_j\}_{j \in \mathcal{J}}$, and at every time step we choose one of them, then we have a *dynamics with choice*. One way to visualize the multitude of choice through time is to generate the infinite tree of choices. Suppose we have N operators, S_0, S_1, $\ldots$, S_{N-1}, then the root of an infinite rooted tree has N children, every child has N children, and so on. The root corresponds to $t = 0$, its children correspond to $t = 1$, the children of the children correspond to $t = 2$, etc. At every step, the children of each node are labeled 0 through $N - 1$. Beginning at the root, infinite branches (paths, strategies) represent the possible choices: for example, in Figure 1.1 we choose the path w that starts with 011... (bold edges). For this choice, the first few points in the trajectory of a point $x_0 \in X$ are $x_1 = S_0(x_0)$, $x_2 = S_1(x_1) = S_1(S_0(x_0))$, $x_3 = S_1(x_2) = S_1(S_1(S_0(x_0)))$, etc. It is natural to encode the infinite paths (beginning at the root) by one-sided infinite words (strings, sequences) on N symbols. If w is such sequence, it is convenient to align it with the set of non-negative integers $\mathbb{Z}_{\geq 0}$ and denote by $w(k)$ the $(k+1)$-st letter of w, i.e., $w = w(0)w(1)w(2)\ldots$. Thus, $w(0) = 0$, $w(1) = 1$, $w(2) = 1$, are the first three symbols of the path $w = 011\ldots$.

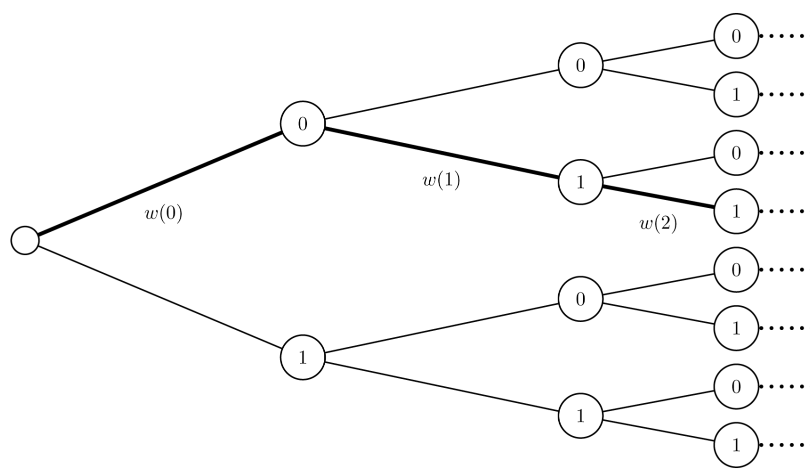

Figure 1.1: The tree of choices in the case of two operators

We study dynamics with choice, i.e., the dynamics of points and subsets of X along all possible paths simultaneously. We will explain what this means momentarily. Here we would like to emphasize that, from the point of view of long-term behavior, dynamics with choice, in general, is not the same as the union of trajectories along different infinite paths. We will return to this point later when we talk about the Gestalt effect.

Denote by Σ the space of all one-sided infinite strings (words) of symbols from the alphabet $\mathcal{J}$. We call the elements of Σ *strategies* or *plans*, because the symbols and their order in a word $w \in \Sigma$ will tell us which maps S_j and in what order are applied. We identify the strings from Σ with maps from the (semigroup of) non-negative integers, $\mathbb{Z}_{\geq 0}$, into $\mathcal{J}$; thus, for $w : \mathbb{Z}_{\geq 0} \to \mathcal{J}$ in Σ, we write it as an infinite word $w = w(0)w(1)w(2)\ldots$. The shift operator $\sigma : \Sigma \to \Sigma$ maps w to $\sigma(w)$ so that $\sigma(w)(n) = w(n+1)$; in other words, $\sigma(w) = w(1)w(2)\ldots$. Given the state space X and the family of maps S_j, $j \in \mathcal{J}$, we define the corresponding *dynamics with choice* as the dynamics on the product $\mathfrak{X} = X \times \Sigma$ generated by the iterations of the map

$$\mathfrak{S} : \ (x, w) \mapsto (S_{w(0)}(x), \sigma(w)) \, . \tag{1.1}$$

In other words, we view the dynamics $x_{n+1} = S_{w(n)}(x_n)$ as a non-autonomous system and use the skew-product (semi)flow approach (see [34]) to describe it.

Dynamics with choice is a language to describe processes where different strategies could be applied or happen. Most of mathematical models in natural sciences and engineering are expressed in terms of differential equations. Those equations are often continuous limits of discrete equations. Continuous case is easier for qualitative analysis. However, there are situations where discrete equations describe the processes better. Every realistic model comes with parameters. We are interested in situations where parameters may change due to, e.g., internal instability or outside intervention. In an illustrative example in section 6.1, the coefficients a and b are proportional to the biting rate of mosquitoes which depends, for example, on temperature and humidity which may change from day to day and during the day.

We study long-term regimes in dynamics with choice. More specifically, we define and study global compact attractors in dynamics with choice. By a global compact attractor we mean **the minimal compact set that attracts all bounded sets**, see section 2.1 for definitions and references. Thinking in terms of a model with parameters, assume we know that for each admissible fixed (in time) set of parameters the system possesses a global compact attractor. What happens when the parameters switch between admissible values? Is there an attractor? How is it related to attractors corresponding to fixed parameters? Is there a Gestalt effect? These are the questions we address here.

There are many real life and engineered systems that switch between different modes of operation (the so-called hybrid systems). When the behavior in each mode is modeled using continuous dynamics and the transitions are viewed as discrete-time events, such systems are called switching or switched. Analysis and especially control of switching systems is an area of intensive research, see, e.g., Liberzon's book [28] and the survey by Margaliot [30]. There is a natural affinity between switching systems and dynamics with choice (see, e.g., [23]).

Readers familiar with iterated function systems, [17, 8], may wonder if there is a connection between iterated function systems and dynamics with choice. Indeed there is, but we have to establish it. For the connection between the attractor in dynamics with choice and the attractor of the corresponding IFS (fractal) see section 3.2.1.

Our interest in dynamics with choice has not been motivated by fractals. We would like to understand the long-term behavior in dynamics with choice. We assume that X is a complete metric space (with metric d), the operators S_j, for $j \in \mathcal{J}$ are continuous, and each of the (semi)dynamical systems (X, d, S_j) possesses a global compact attractor. Consider the corresponding dynamics with choice as the dynamics on the product metric space [1] $\mathfrak{X} = X \times \Sigma$ generated by the operator $\mathfrak{S}$ acting according to the rule (1.1). From general theory (see section 2.1) we know that a system ought

[1] Σ can be equipped with a metric making it a compact metric space, see Section 3.1 for a specific choice. We denote here by *dist* the corresponding product-metric on $X \times \Sigma$.

to enjoy certain compactness and dissipativity properties in order for it to possess the global compact attractor.

In general, even when the individual systems (X, d, S_j) do have attractors, the system $(\mathfrak{X}, \mathit{dist}, \mathfrak{S})$ will not have a global compact attractor. There are several reasons why. One counter-example we borrow from [3] (where it is used in the context of IFS). Take $X = \mathbb{R}$ with standard metric d and define two maps, S_0 and S_1, as follows:

$$S_0(x) = \begin{cases} 0, & \text{if } x \le 0, \\ -2x, & \text{if } x > 0 \end{cases} \qquad S_1(x) = \begin{cases} -2x, & \text{if } x \le 0, \\ 0, & \text{if } x > 0 \end{cases}$$

Each of the systems (X, d, S_j) has the global compact attractor, a singleton $\{0\}$. At the same time, the trajectory $x_n = S_{w(n-1)} \circ S_{w(n-2)} \circ \cdots \circ S_{w(0)}(x_0)$ corresponding to the periodic string $w = 010101\ldots$ is unbounded for any initial point $x_0 > 0$. Hence, there is no compact attractor attracting (x_0, w).

The second example is infinite-dimensional. Let $B_0 = B_0(p_0)$ and $B_1 = B_1(p_1)$ be two disjoint closed unit balls centered at p_0 and p_1 in an infinite-dimensional Banach space. Let $X = B_0 \cup B_1$. Define the maps S_0 and S_1 as follows: on B_0 the map S_0 is a contraction and it maps B_1 to B_0; the map S_1 is a contraction on B_1 and maps B_0 to B_1:

$$S_0(x) = \begin{cases} p_0 + \frac{1}{2}(x - p_0), & \text{if } x \in B_0, \\ p_0 + (x - p_1), & \text{if } x \in B_1 \end{cases} \qquad S_1(x) = \begin{cases} p_1 + \frac{1}{2}(x - p_1), & \text{if } x \in B_1, \\ p_1 + (x - p_0), & \text{if } x \in B_0 \end{cases}$$

The system (X, d, S_0) does have the global compact attractor, $\{p_0\}$, and (X, d, S_1) does have the global compact attractor, $\{p_1\}$. The corresponding dynamics with choice, $(\mathfrak{X}, dist, \mathfrak{S})$, does have the global *closed* attractor, namely, $\mathfrak{X}$, but does not have the global *compact* attractor.

In the first example, the maps are compact (which is good), but they do not have a joint bounded absorbing set (lack of dissipativity in $(\mathfrak{X}, dist, \mathfrak{S})$). In the second example, there is a joint bounded absorbing set, $B_0 \cup B_1$, but there is not enough compactness (the maps S_j are not compact, not contracting, and, more generally, not condensing).

These examples show what kind of situations do not allow global compact attractors in the dynamics with choice. Thus, we make additional assumptions. First, we assume that there exists a bounded absorbing set that absorbs every bounded set regardless of the strategy. In applications, an absorbing set is usually a ball of the radius that depends on the parameters of the model. Our "dissipativity" assumption means that there is a common estimate on the radius for different values of the parameters.

Our second, "compactness" assumption is that each of the operators S_j is condensing with respect to a common measure of noncompactness. This assumption covers practically all situations encountered in applications: contractions, compact operators, and compact plus contractions. As their name suggests, measures of noncompactness measure how far a set is from being compact. There are several different measures of noncompactness in use, [1]. For example, the Hausdorff measure of noncompactness of a set A is the infimum of $\epsilon > 0$ such that A has a finite ϵ-net. In what follows we use only very general properties shared by all popular measures of noncompactness, see Definition 4 in section 2.2 below. Let ψ be a measure of noncompactness (as in Definition 4). An operator $S : X \to X$ is condensing with respect to ψ iff $\psi(S(A)) < \psi(A)$ for any non-compact set A, and $\psi(S(A)) = \psi(A) = 0$ if A is compact.

These two assumptions are enough if $\mathcal{J}$ is a finite set. In general, if $\mathcal{J}$ is an infinite set, possibly uncountable, we need an additional assumption concerning their dependence on the parameter j. We assume that there exists a finite partition of the set $\mathcal{J}$ such that on each of the partition sets the operators depend uniformly continuously on the parameter j.

Our assumptions on the state space and the operators guarantee that, for every fixed $j \in \mathcal{J}$, the discrete dynamics generated on X by S_j does possess a global compact attractor (in X). More generally, as we show in sections 3.3 and 3.3.1, it makes sense to define *individual attractors*, $\mathcal{A}_w$, corresponding to every string (infinite path in the tree of choices) $w \in \Sigma$. The attractors generated by each S_j correspond to "constant" strings, $w = jjj\ldots$. It is not hard to see that such attractors do not exhaust the projection onto X of the attractor in dynamics with choice (we call the projection K). There are situations when the union of all $\mathcal{A}_w$ is K (this happens, in particular, when S_j's are strict contractions). However, in general, the union $\bigcup_{w\in\Sigma} \mathcal{A}_w$ is *strictly smaller* than K. We give an example of this in section 3.3.1. In the cases when $\bigcup_{w\in\Sigma} \mathcal{A}_w$ is strictly smaller than K we say that there is **a Gestalt effect**, i.e., "the whole is greater than the sum of its parts." This is a new phenomenon. It has not been observed in the framework of Iterated Function Systems because, as we show in Lemma 24, the Gestalt effect cannot occur when operators S_j are contractions.

An important generalization of dynamics with choice is *dynamics with restricted choice*. The name should indicate that not all strategies (sequences $w = w(0)w(1)\cdots \in \Sigma$) are allowed. In particular, we consider the sets in Σ that are closed and shift invariant, i.e., *subshifts*, see [29, 20]. Given a subshift $\Lambda \subset \Sigma$, we consider the dynamics on the product-space $\mathfrak{X}_\Lambda = X \times \Lambda$ generated by the map $\mathfrak{G}$ as in (3.5).

Restricted dynamics of a sort has been considered previously, see [32, 31]. For example, the graph directed Markov systems of [31] describe iterations of uniformly contracting maps indexed by the edges of a directed (possibly infinite) graph. In this case there is a correspondence between the points of the limit set and the infinite walks through the graph (the coding space). Similarly, the directed IFSs discussed in [9] are defined with the help of the aforementioned correspondence, and the fractal (or attractor) K_Λ (the projection onto X of the attractor in dynamics with restricted choice) is understood in terms of the map from the code space to K as the image of Λ, [9, Theorem 4.16.3]. The correspondence between the points of the code space, Σ, and the points of K is possible because the maps are contractions (right away, or eventually). Our approach gives a new and more general view on restricted dynamics. We justify the name – attractor – and unveil attractors' more subtle structure. This new approach allows us to work in a much more general setting and with transformations that are not contractions. We do not have and do not use a map from the code space into the attractor.

We should mention the paper of Andres and Fišer, [2]. They use their result of [3] on the existence of the fractal (the set K in our notation) for an IFS with compact operators S_j to conclude that fixed time solution operators of systems of ordinary differential equations could play the role of maps generating the IFSs. As an illustration they use five two-dimensional systems of ODEs to produce five operators (incidentally, contractions, as noted in [2]) and plot the corresponding dragon-tail-like fractal set. Although their message is that IFSs and fractals can be generated by solution operators of ODEs, their examples can serve as an illustration for our dynamics with choice attractors.

Our definitions of dynamics with choice and dynamics with restricted choice as skew-product semi-flows fit in with the theory of non-autonomous semi-dynamical systems, see [34, 14, 21, 22] and references therein. In chapter 5 we explain how our attractors are related to the forward and pullback attractors in that theory. We should mention the paper of Cheban and Mammana, [13], on discrete inclusions $u_{i+1} \in F(u_i)$ where $F(u) = \bigcup_j f_j(u)$ and $\{f_j\}$ is a collection of maps. In [13], the authors view such an inclusion as a non-autonomous system and arrive at essentially the same skew-product flow as our dynamics with choice. Motivated by iterated function systems, they consider only contracting (exponentially, after a finite number of iterations) maps f_j, prove the existence of a global attractor, and discuss some of its properties.

Numerics for dynamics with choice is a very interesting but difficult subject. In chapter 6 section 6.1, we apply the theory to a specific example of a discrete Ross-Macdonald type model of malaria transmission. The model can be viewed as a time discretization of the ODE model, or as a pre-ODE form of the model. The reason we have chosen this model is because it is simple and we can visualize all the attractors.

More general compact and condensing operators will be needed in the study of dynamics with choice related to nonlinear dissipative partial differential equations, which I plan to address at a later time.

Structure of the thesis. In the following chapter, we give a brief introduction on notions of a global compact attractor, an iterated function system, and a pullback and a forward attractors in nonautonomous systems. The mathematical setting for the dynamics with choice is explained in chapter 3. We prove the existence of the global compact attractor in dynamics with choice and obtain its properties by looking at the connection with IFS. Dynamics with restricted choice is explained in chapter 4 and in chapter 5 we show that under additional assumption the attractor of the IFS is the pullback and forward attractor of the corresponding nonautonomous system. In chapter 6, we give an illustration for the theory of dynamics with choice, and explain in details how numerical computations were performed. In what follows, we use $\square$ to denote the end of the proof.

Chapter 2

THEORY

2.1 Attractors

In applications, dynamical systems aspire to describe the real life systems. Continuous dynamical systems are usually represented by differential equations while difference equations represent discrete dynamical systems. Here we deal with discrete dynamics. Let Y be a complete metric space with distance d_Y, and let $\Phi : Y \to Y$ be a continuous, bounded map ('bounded' means the image of a bounded set is bounded). Iterations of Φ, Φ^n, generate dynamics on (Y, d_Y). We will denote the corresponding discrete semidynamical system by (Y, d_Y, Φ), or simply (Y, Φ). It is useful to consider not only the dynamics of individual points under the action of Φ, but, more generally, the dynamics of bounded sets. The collection of all bounded subsets of Y are denote by $\mathcal{B}(Y)$. We say that the set $A \in \mathcal{B}(Y)$ attracts the set $B \in \mathcal{B}(Y)$ if

$$\vec{dist}\,(\Phi^n(B),\, A) \underset{n\to\infty}{\longrightarrow} 0\,,$$

where the one-sided distance between two sets is understood as follows:

$$\vec{dist}\,(C,\,A) = \sup_{y\in C} d_Y\,(y,\,A).$$

Definition 1. *We call a set* $\mathfrak{M} \subset Y$ **the global compact attractor** *of the system* (Y, d_Y, Φ) *if*

- $\mathfrak{M}$ *is compact,*
- $\mathfrak{M}$ *attracts every bounded subset of* Y*,*
- $\mathfrak{M}$ *is the minimal set with these two properties.*

There are several books devoted to the subject of global attractors. Our presentation given here is closer to [25]. We give results on the existence of global compact attractors and certain properties that they exhibit. We refer the reader to [5, 16, 25, 26, 35] for the proofs of the results presented here.

For a system to possess a global compact attractor, it should enjoy certain properties, namely, some form of compactness and some dissipativity. Here is the basic existence (and uniqueness) result:

Theorem 2. *The semidynamical system* (Y, dist, Φ) *has a global compact attractor if and only if it enjoys the following two properties:*

1. *("compactness") For every bounded sequence* (y_k) *in* Y *and every increasing sequence of integers* $n_k \to +\infty$*, the sequence* $\Phi^{n_k}(y_k)$ *has a convergent subsequence.*
2. *("dissipativity") There exists a bounded set* $\mathbf{B} \subset Y$ *which absorbs every bounded set in the sense that for every* $A \in \mathcal{B}(Y)$ *there exists* $m(A) > 0$ *such that*

$\Phi^n(A) \subset \mathbf{B}$ *for all* $n \geq m(A)$.

Some of the basic properties of a global compact attractor are collected in the following theorem:

Theorem 3. *Assume that* $\mathfrak{M}$ *is the global compact attractor of the semidynamical system* (Y, dist, Φ). *Then*

1. $\mathfrak{M}$ *is the union of all possible limits of sequences of the form* $\Phi^{n_k}(y_k)$, *where* y_k *is a bounded sequence in* Y *and* $n_k \to \infty$.

2. $\mathfrak{M}$ *is (strictly) invariant:* $\Phi(\mathfrak{M}) = \mathfrak{M}$.

3. $\mathfrak{M}$ *is the union of all closed bounded sets* A *with the property* $A \subset \Phi(A)$.

4. $\mathfrak{M}$ *is the maximal closed set with the property* $A \subset \Phi(A)$; *in particular,* $\mathfrak{M}$ *is the maximal (strictly) invariant closed set.*

5. *Through every point* $y \in \mathfrak{M}$ *passes a complete trajectory, i.e., there exists a two-sided sequence* $\ldots, y_{-2}, y_{-1}, y_0, y_1, \ldots$ *of points in* $\mathfrak{M}$ *such that* $y_0 = y$ *and* $y_{m+1} = \Phi(y_m)$ *for all integers* m.

6. $\mathfrak{M}$ *is the union of all complete, bounded trajectories in* Y.

In applications, people do not verify the "compactness" property of Theorem 2 directly. Instead, they use one of the known sufficient conditions that imply it. Three of the most useful sufficient conditions are:

- Φ is a compact map: $\Phi : Y \to Y$ is continuous and maps bounded sets into relatively compact sets;

- S is a contraction: $d_X(S(x), S(y)) < \gamma d_X(x, y)$ for some positive $\gamma < 1$ and for all $x, y \in X$;

- in the case Y is a Banach space, Φ is a sum of a compact operator and a strict contraction.

Compact Φ arise, e.g., in the finite-dimensional dynamics described by differential or difference equations, or, in the infinite dimensional case, in dynamics described by parabolic equations. The "compact + contraction" Φ appear, e.g., in hyperbolic problems with damping. Each of the three sufficient conditions implies that Φ is condensing with respect to some measure(s) of noncompactness. Below we give a brief account of the facts we need and refer to [1] for more details on the measures of noncompactness.

2.2 Measures of noncompactness

Measures of noncompactness assign real non-negative numbers to bounded sets with value 0 assigned exclusively to relatively compact sets. The basic examples are the Kuratowski measure of noncompactness α and the Hausdorff measure of noncompactness χ. By definition, $\alpha(A)$ is the infimum of numbers $\epsilon > 0$ such that A admits a finite cover by sets of diameter less than ϵ. The number $\chi(A)$ is the infimum of those $\epsilon > 0$ for which A possesses a finite ϵ-net in Y. In this paper we adopt the following definition of a general measure of noncompactness (our definition differs from that in [1]).

Definition 4. *A function ψ assigning non-negative real numbers to bounded subsets of (a complete metric) space Y will be called a measure of noncompactness iff it has* *the following properties:*

(i) $\psi(A) = 0$ *if and only if* A *is relatively compact;*

(ii) If $A_1 \subset A_2$, *then* $\psi(A_1) \leq \psi(A_2)$ *;*

(iii) $\psi(A_1 \cup A_2) = \max\{\psi(A_1),\, \psi(A_2)\}$ *;*

(iv) There exists a constant $c(\psi) \geq 0$ *such that* $|\psi(A_1) - \psi(A_2)| \leq c(\psi)\, d_H(A_1, A_2)$, *where* d_H *is the Hausdorff distance,* $d_H(A_1, A_2) = \max\{\vec{\text{dist}}\,(A_1, A_2),\, \vec{\text{dist}}\,(A_2, A_1)\}$.

Note that property (iv) implies that the measures of noncompactness of a bounded set and its closure are equal:

(v) $\psi(\overline{A}) = \psi(A)$.

Both α and χ enjoy all these properties. An example of a set which has a non-zero, finite Kuratowski and Hausdorff measures of noncompactness is a unit ball (B) in an infinite dimensional Banach space. In this case, $\alpha(B) = 2$ and $\chi(B) = 1$, see [1, Theorem 1.1.6]. For applications of measures of noncompactness in spaces of continuous functions, differentiable functions, integrable functions, etc., see [6] and references therein.

Definition 5. *A continuous bounded map* $\Phi : Y \to Y$ *is called condensing with respect to the measure of noncompactness* ψ *(we also say* Φ *is* ψ*-condensing) iff* $\psi(\Phi(A)) \leq \psi(A)$ *for any bounded* A, *and* $\psi(\Phi(A)) < \psi(A)$ *if* $\psi(A) > 0$ *(i.e., if* $\overline{A}$ *is not compact).*

Theorem 6. *Consider the system* (Y, d_Y, Φ). *Assume that* Φ *is condensing with respect to some measure of noncompactness* ψ *and that there exists a bounded set* $\mathbf{B}$

which absorbs every bounded set (this means the set $\mathbf{B}$ *will eventually contain the image of any bounded set). Then* (Y, d_Y, Φ) *possesses a global compact attractor.*

This is a corollary of the general Theorem 2, because a ψ-condensing map possesses the "compactness" property of Theorem 2. In the case ψ is the Kuratowski measure of noncompactness, the theorem above is proved in [33, Theorem 32].

Proof. Since $\mathbf{B}$ is an absorbing set, it is left to show "compactness" property of theorem 2. To do this, we have to show that every sequence $\Phi^{n_k}(x_k)$, for (x_k) bounded and $n_k \to \infty$, has a convergent subsequence. Let C be the set of all sequences $\{\Phi^{n_k}(x_k)\}$, with $n_k \to \infty$, and let $s = \sup\{\psi(p) \,|\, p \in C\}$. The claim is that there exists an element, $p^* \in C$, such that $\psi(p^*) = s$. To show this, pick a sequence $(p_j) \in C$ such that $\psi(p_j) \nearrow s$. We can write $p_j = \bigcup_{k=1}^{\infty} \Phi^{n_k^j}(x_k^j)$. Now, let $\tilde{p}_j = \bigcup_{k=j+1}^{\infty} \Phi^{n_k^j}(x_k^j)$ and notice that

$$\begin{aligned} \psi(p_j) &= \psi\left(\textstyle\bigcup_{k=1}^{\infty} \Phi^{n_k^j}(x_k^j)\right) = \max\left\{\psi\left(\textstyle\bigcup_{k=j+1}^{\infty} \Phi^{n_k^j}(x_k^j)\right), \psi\left(\textstyle\bigcup_{k=1}^{j} \Phi^{n_k^j}(x_k^j)\right)\right\} \\ &= \psi\left(\textstyle\bigcup_{k=j+1}^{\infty} \Phi^{n_k^j}(x_k^j)\right) = \psi(\tilde{p}_j) \end{aligned} \tag{2.1}$$

Let $p^* = \bigcup_{j=1}^{\infty} \tilde{p}_j$. Then,

$$p^* = \bigcup_{j=1}^{\infty} \bigcup_{k=j+1}^{\infty} \Phi^{n_k^j}(x_k^j) = \bigcup_{k=2}^{\infty} \bigcup_{j=1}^{k-1} \Phi^{n_k^j}(x_k^j).$$

The second union in the last term is finite, and therefore $p^* \in C$ and $\psi(p^*) \leq s$. Since,

$$\psi(p^*) = \max\left\{\psi(p_1), \ldots, \psi(p_m), \psi\left(\bigcup_{j=m+1}^{\infty} \tilde{p}_j\right)\right\},$$

it follows that $\psi(p^*) \geq \psi(p_j)$, for every j. Therefore, $\psi(p^*) = s$. This proves the claim. Now, let $\tilde{p^*} = \{\Phi^{n_k}(x_k) \,|\, \Phi^{n_k+1}(x_k) \in p^*\}$. We have that $\tilde{p^*} \in C$ and $\psi(\tilde{p^*}) \leq s$. Also, we have $\Phi(\tilde{p^*}) = p^*$ and $s = \psi(p^*) = \psi\left(\Phi(\tilde{p^*})\right) \geq \psi(\tilde{p^*})$. Since Φ is condensing map, it follows that $\psi(\tilde{p^*}) = 0$, i.e., $s = 0$. Thus, the theorem is proved. $\square$

If Y is a product of two complete metric spaces, (Y_1, d_1) and (Y_2, d_2), then we choose

$$d_Y((y_1', y_2'), (y_1'', y_2'')) \;=\; d_1((y_1', y_1'')) + d_2((y_2', y_2''))$$

as a metric on Y. If ψ_1 and ψ_2 are the measures of noncompactness on Y_1 and Y_2 respectively, then we define

$$\psi(A) = \max\{\psi_1(\mathrm{pr}_1 A),\, \psi_2(\mathrm{pr}_2 A)\},$$

to be the measure of noncompactness on the product space, where pr_k is a projection on Y_k, defines a measure of noncompactness on $Y = Y_1 \times Y_2$. To see that ψ is well-defined, we will show that all four properties in definition 4 are satisfied. Let A be a subset of Y. If A is relatively compact, then both projections, $\mathrm{pr}_1 A$ and $\mathrm{pr}_2 A$, are relatively compact and we have $\psi(A) = \max\{\psi_1(\mathrm{pr}_1 A),\, \psi_2(\mathrm{pr}_2 A)\} = 0$. On the other hand, if $\psi(A) = 0$, then both, $\psi_1(\mathrm{pr}_1 A) = 0$ and $\psi_2(\mathrm{pr}_2 A)$=0, which implies that $\mathrm{pr}_1 A$ and $\mathrm{pr}_2 A$ are relatively compact, and therefore A is relatively compact. To prove property (ii), let A_1 and A_2, be such that $A_1 \subset A_2$. Obviously, $\mathrm{pr}_i A_1 \subset \mathrm{pr}_i A_2$, and therefore $\psi_i(\mathrm{pr}_i A_1) \leq \psi_i(\mathrm{pr}_i A_2)$ for $i = 1, 2$. Since

$$\psi(A_i) = \max\{\psi_1(\mathrm{pr}_1 A_i),\, \psi_2(\mathrm{pr}_2 A_i)\},$$

then if, for example, $\psi(A_1) = \psi_2(\mathrm{pr}_2 A_1)$ and $\psi(A_2) = \psi_1(\mathrm{pr}_1 A_2)$, we know that $\psi(A_1) \leq \psi(A_2)$ because we have $\psi_2(\mathrm{pr}_2 A_1) \leq \psi_2(\mathrm{pr}_2 A_2) \leq \psi_1(\mathrm{pr}_1 A_2)$. All other possibilities are either obvious or similar to this one. This proves property (ii) of definition 4. For the remaining of the proof, A_1 and A_2 are arbitrary subsets of Y. Property (iii) is very easy to show

$$\begin{aligned}
&\psi(A_1 \cup A_2) = \psi((\mathrm{pr}_1 A_1 \cup \mathrm{pr}_1 A_2) \times (\mathrm{pr}_2 A_1 \cup \mathrm{pr}_2 A_2)) = \\
&\max\{\max\{\psi_1(\mathrm{pr}_1 A_1), \psi_1(\mathrm{pr}_1 A_2)\}, \max\{\psi_2(\mathrm{pr}_2 A_1), \psi_2(\mathrm{pr}_2 A_2)\}\} = \\
&\max\{\max\{\psi_1(\mathrm{pr}_1 A_1), \psi_2(\mathrm{pr}_2 A_1)\}, \max\{\psi_1(\mathrm{pr}_1 A_2), \psi_2(\mathrm{pr}_2 A_2)\}\} = \\
&\max\{\psi(A_1), \psi(A_2)\}.
\end{aligned} \tag{2.2}$$

Property (iv) follows immediately if $\psi(A_1) = \psi_1(\mathrm{pr}_1 A_1)$ and $\psi(A_2) = \psi_1(\mathrm{pr}_1 A_2)$, or $\psi(A_1) = \psi_2(\mathrm{pr}_2 A_1)$ and $\psi(A_2) = \psi_2(\mathrm{pr}_2 A_2)$. It takes an extra step to prove property (iv) in the case $\psi(A_1) = \psi_1(\mathrm{pr}_1 A_1)$ and $\psi(A_2) = \psi_2(\mathrm{pr}_2 A_2)$ (or A_1 and A_2 changing places). Assume, wlog, $\psi_1(\mathrm{pr}_1 A_1) \geq \psi_2(\mathrm{pr}_2 A_2)$. Then we have:

$$\begin{aligned}
\psi_1(\mathrm{pr}_1 A_1) - \psi_2(\mathrm{pr}_2 A_2) &\leq \psi_1(\mathrm{pr}_1 A_1) - \psi_1(\mathrm{pr}_1 A_2) + \psi_1(\mathrm{pr}_1 A_2) - \psi_2(\mathrm{pr}_2 A_2) \\
&\leq c(\psi_1)\, d_H(\mathrm{pr}_1 A_1, \mathrm{pr}_1 A_2) + \psi_1(\mathrm{pr}_1 A_2) - \psi_2(\mathrm{pr}_2 A_2) \\
&\leq c(\psi)\, d_H(A_1, A_2),
\end{aligned} \tag{2.3}$$

where $c(\psi) = \max\{c(\psi_1), c(\psi_2)\}$. The last inequality follows since $\psi_1(\mathrm{pr}_1 A_2) - \psi_2(\mathrm{pr}_2 A_2) \leq 0$ (assumed above). $\square$

2.3 Iterated Function Systems

Traditionally, an Iterated Function System (IFS)[2] is associated with a space X (usually $X = \mathbb{R}^n$), and a finite number of maps S_0, S_1, $\ldots$, $S_{N-1} : X \to X$ (usually S_j's are linear contractions, see [8]). The IFS $(X; S_0, S_1, \ldots, S_{N-1})$ can be viewed as a discrete dynamics on the space 2^X (of subsets of X). The evolution is generated by means of the Hutchinson-Barnsley operator:

$$\bar{F} \,:\, A \mapsto \bar{F}(A) := \overline{S_0(A) \cup S_1(A) \cup \cdots \cup S_{N-1}(A)}\,. \tag{2.4}$$

Following a long-standing tradition, people studying dynamics are first of all interested in fixed points. In the case of an IFS, those are the fixed points of the Hutchinson-Barnsley operator. As has been well illustrated by Barnsley, for many simple IFSs on the plane one can use a computer to plot their compact fixed points (sets) and obtain fascinating fractals. An example of a fractal is given in figure 2.1 which is generated by the map $\bar{F}$ starting from a point $(0, 0)$

$$\begin{aligned} S_0(x, y) &= (0, 0.16y) \\ S_1(x, y) &= (0.2x - 0.26y, 0.23x + 0.22y + 1.6) \\ S_2(x, y) &= (-0.15x + 0.28y, 0.26x + 0.24y + 0.44) \\ S_3(x, y) &= (0.85x + 0.04y, -0.04x + 0.85y + 1.6) \end{aligned} \tag{2.5}$$

For more examples on fractals, see [8, 9]. Generating fractals is one of the main motivations in the study of IFSs. In some papers a fractal is defined as the compact invariant set of an IFS, see [9] and references therein.

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[2] We use the abbreviation IFS for single and IFSs for plural forms.

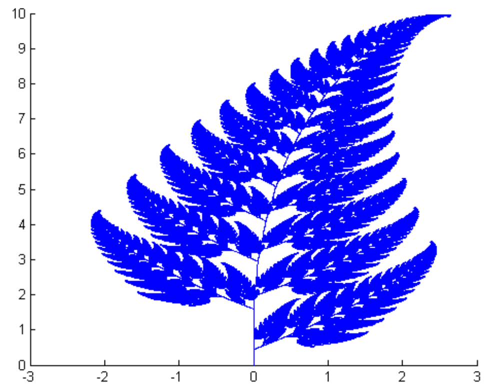

Figure 2.1: Fractal Fern

To prove that an IFS does have a fixed point, the general definition should be made more specific. One needs to specify the properties of the space X; the space 2^X should be narrowed to an appropriate class of subsets; assumptions should be made on the operators $S_0, S_1, \ldots, S_{N-1}$. As an example we state the original result of Hutchinson, [17, Section 3].

Theorem (Hutchinson). *Let X be a complete metric space (with metric d). Denote by $\bar{\mathcal{B}}(X)$ the space of all non-empty closed bounded subsets of X. Assume that each operator $S_0, S_1, \ldots, S_{N-1}$ is a strict contraction (i.e., there is a number $\gamma \in (0,1)$ such that $d(S_j(x), S_j(y)) \leq \gamma\, d(x, y)$ for every pair $x, y \in X$ and for all j). Define the evolution operator $\bar{F} : \bar{\mathcal{B}}(X) \to \bar{\mathcal{B}}(X)$ by the formula*

$$\bar{F} : A \mapsto \bar{F}(A) = \overline{S_0(A) \cup S_1(A) \cup \cdots \cup S_{N-1}(A)}\,. \tag{2.6}$$

Then there exists a unique fixed point $K \in \bar{\mathcal{B}}(X)$ of $\bar{F}$. Viewed as a subset of X, the set K is compact. Also, K attracts every closed bounded subset of X in the sense *that, for any* $C \in \bar{\mathcal{B}}(X)$*,*

$$d_H(\bar{F}^n(C),\, K) \to 0 \qquad as \quad n \to \infty\,,$$

where d_H *is the Hausdorff distance.*

The IFS with contractive operators S_j are called hyperbolic. Over the years this result has been generalized in many different directions (different assumptions on X and/or S_j), see [4] for references. When the system has infinitely many operators $\{S_j\}_{j\in\mathcal{I}}$, it is called an infinite iterated function system. In this case, the Hutchinson-Barnsley operator is defined as a closure of an infinite union:

$$\bar{F}(A) = \overline{\bigcup_{j\in\mathcal{I}} S_j(A)}.$$

For an example of an IFS with infinitely many contractions S_j, see [27].

2.4 Nonautonomous Systems

It has been known for a long time that non-autonomous dynamical systems can be viewed as autonomous dynamical systems in a larger (state) space, and there are many ways of achieving this. However, for the purposes of the analysis of the long-term behavior of solutions, the most beneficial approach was suggested by G. Sell, [34]. The modern abstract definition of a discrete non-autonomous semi-dynamical system consists of a state space, X, a base (or parameter) space, P, a map $\theta : P \to P$ that defines a dynamics on P, and a cocycle map $\varphi : \mathbb{Z}_{\geq 0} \times X \times P \to X$. The cocycle map φ has the properties:

1. $\varphi(0, x, p) = x$

2. $\varphi(n+1, x, p) = \varphi(n, \varphi(1, x, p), \theta(p))$.

The skew-product dynamics is then understood as an autonomous dynamics on the product $X \times P$ generated by the map

$$\pi(x, p) = (\varphi(1, x, p), \theta(p)) \,.$$

In what follows, X and P are assumed to be complete metric spaces, and the maps θ and $\varphi(1, \cdot, \cdot)$ are assumed to be continuous and bounded. Also, we assume that P is compact and $\theta(P) = P$.

There is a considerable literature devoted to attractors of non-autonomous systems, see [14, 11, 21, 22, 12] and references therein. Several authors (e.g., [12]) view the product $X \times P$ as a fiber bundle over the base P. Thus, it makes sense to define the attractors fibered over P. The following definitions are compiled from [12, 22].

Let $\hat{M}$ be a collection of compact subsets $M(p) \subset X$ parametrized by the points of P.

Definition 7. *$\hat{M} = \{M(p)\}_{p \in P}$ is a uniform forward attractor of the non-autonomous system $\langle X, \varphi, (P, \mathbb{Z}_{\geq 0}, \theta) \rangle$ iff*

1) $\bigcup_{q:\, \theta(q)=p} \varphi(1, M(q), q) = M(p)$;

2) The set $\bigcup_{p \in P} M(p)$ is compact;

3) $\lim\limits_{n \to +\infty} \sup_{p \in P} dist_X \left(\varphi(n, B, p), M(\theta^n(p)\right) = 0$, for every bounded $B \subset X$ and $p \in P$.

The following result follows from [22, Sec. 5]

Lemma 8. *Assume that the skew-product semi-dynamical system* $(X \times P, \pi)$ *corresponding to the non-autonomous system* $\langle X, \varphi, (P, \mathbb{Z}_{\geq 0}, \theta)\rangle$ *possesses a global compact attractor* M. *Then the slices* $M(p) = \{x \in X : (x,p) \in M\}$ *form the forward attractor.*

In the theory of non-autonomous systems, serious attention is paid to the notion of a pullback attractor because of its role in random dynamical systems, [15]. Usually this notion is introduced under the assumption that the map θ is a homeomorphism on P. If θ is not invertible, the usual definition is not applicable. However, the definition can be extended to the case of non-invertible θ, as shown in [22].

Definition 9. $\hat{M} = \{M(p)\}_{p\in P}$ *is a uniform pullback attractor of the non-autonomous system* $\langle X, \varphi, (P, \mathbb{Z}_{\geq 0}, \theta)\rangle$ *iff*

1) $\bigcup_{q:\, \theta(q)=p} \varphi(1, M(q), q) = M(p)$*;*

2) The set $\bigcup_{p\in P} M(p)$ *is compact;*

3) $\lim\limits_{n\to+\infty} \sup_{p\in P}\ dist_X\left(\varphi(n, B, \theta^{-n}(p)), M(p)\right) = 0$, *for every bounded* $B \subset X$ *and* $p \in P$.

In this definition, $\theta^{-n}(p)$ is the set of all $q \in P$ such that $\theta^n(q) = p$.

Lemma 10. *Assume that the skew-product semi-dynamical system* $(X \times P, \pi)$ *corresponding to the non-autonomous system* $\langle X, \varphi, (P, \mathbb{Z}_{\geq 0}, \theta)\rangle$ *possesses a global compact attractor* M. *Then the slices* $M(p) = \{x \in X : (x,p) \in M\}$ *form the pullback attractor.*

The lemma and its proof can be found in [22].

Chapter 3

DYNAMICS WITH CHOICE

In this chapter we define dynamics with choice and give very general sufficient conditions for the existence of global compact attractor in dynamics with choice. Also, we give an example of a gestalt effect by showing that, in our example, the union of all individual attractors does not fill up the whole attractor in dynamics with choice.

3.1 Mathematical setting

Given a complete metric space (X, d) and a set of continuous and bounded operators, $\{S_j\}_{j\in\mathcal{J}}$, we can define dynamics with choice. We assume that j is an element of a compact metric space, $(\mathcal{J}, d_{\mathcal{J}})$, which we call the *alphabet*. We define $\Sigma_{\mathcal{J}}$ to be the set of all one-sided infinite sequences whose elements are symbols in the alphabet $\mathcal{J}$. Every $w \in \Sigma_{\mathcal{J}}$ can be viewed as a map $w : \mathbb{Z}_{\geq 0} \to \mathcal{J}$. Denote $w = w(0)w(1)w(2)...$, where $w(i-1)$ is the i^{th} symbol in the sequence w. On the space of one-sided infinite sequences $\Sigma_{\mathcal{J}}$, we act with the shift operator σ, which maps w to $\sigma(w) = w(1)w(2)w(3)...$ (erasing the first symbol in the sequence).

We consider strings (words) of finite length and one-sided strings of infinite length. Denote by Σ^* the set of all finite length strings (words), and denote by $\Sigma_{\mathcal{J}}$ the set of all (one-sided) infinite strings. The word of length 0 is the empty word. Given a string $w \in \Sigma^* \cup \Sigma_{\mathcal{J}}$, $w(0)$ is the *first* letter of w, and $w(k)$ is the $(k+1)$-st letter of w. The length of w is denoted by $|w|$. If w is a finite string and $u \in \Sigma^* \cup \Sigma_{\mathcal{J}}$, their concatenation is denoted by $w.u$; if $|w| = m$, then $(w.u)(m+k) = u(k)$ for $k = 0, 1, \ldots$. For a $w \in \Sigma^*$ and $s \in \Sigma^* \cup \Sigma_{\mathcal{J}}$, we write $w \sqsubset s$ if w is the beginning of the string s, i.e., if there exists $u \in \Sigma^* \cup \Sigma_{\mathcal{J}}$ such that $s = w.u$. For an infinite string s, its first n letters form a word denoted by $s[n]$, i.e., $s[n] = s(0)s(1)\ldots s(n-1)$. The set of all words of length m will be denoted by Σ^*_m.

Define a metric $d_{\Sigma_{\mathcal{J}}}$ on $\Sigma_{\mathcal{J}}$ as follows. If $w, s \in \Sigma_{\mathcal{J}}$ then

$$d_{\Sigma_{\mathcal{J}}}(w,s) = \sum_{j=0}^{\infty} 2^{-j}\, d_{\mathcal{J}}(s(j), w(j))\,. \tag{3.1}$$

Lemma 11. *$d_{\Sigma_{\mathcal{J}}}$ is a metric on $\Sigma_{\mathcal{J}}$. Moreover, $(\Sigma_{\mathcal{J}}, d_{\Sigma_{\mathcal{J}}})$ is a compact space.*

Proof. Two sequences $s, w \in \Sigma_{\mathcal{J}}$ are the same if $s(j) = w(j)$ for all j, which then implies that $d_{\Sigma_{\mathcal{J}}}(s,w) = 0$. Similarly, if $d_{\Sigma_{\mathcal{J}}}(s,w) = 0$ then $d_{\mathcal{J}}(s(j), w(j))=0$ for all j, and $s = w$. Also, it is obvious that $d_{\Sigma_{\mathcal{J}}}(s,w) = d_{\Sigma_{\mathcal{J}}}(w,s)$ and $d_{\Sigma_{\mathcal{J}}}(s,w) \geq 0$. It is left to show the triangle inequality. Let $s, u, w \in \Sigma_{\mathcal{J}}$, then for every $j \in \mathbb{N}_{\geq 0}$ we have

$$d_{\mathcal{J}}(s(j), w(j)) \leq d_{\mathcal{J}}(s(j), u(j)) + d_{\mathcal{J}}(u(j), w(j)).$$

If we multiply the inequality above by 2^{-j} and add over all j, we get

$$\sum_{j=0}^{\infty} 2^{-j}\, d_{\mathcal{J}}(s(j), w(j)) \leq \sum_{j=0}^{\infty} 2^{-j}\, d_{\mathcal{J}}(s(j), u(j)) + \sum_{j=0}^{\infty} 2^{-j}\, d_{\mathcal{J}}(u(j), w(j)), \quad (3.2)$$

i.e., the triangle inequality

$$d_{\Sigma_{\mathcal{J}}}(s, w) \leq d_{\Sigma_{\mathcal{J}}}(s, u) + d_{\Sigma_{\mathcal{J}}}(u, w).$$

To show that $(\Sigma_{\mathcal{J}}, d_{\Sigma_{\mathcal{J}}})$ is a compact space, pick a sequence $\{w_n\} \subset \Sigma_{\mathcal{J}}$. Notice that for each j, $w_n(j)$ is a sequence of points in the compact set $\mathcal{J}$, and therefore, has a convergent subsequence. Thus, $\{w_n(0)\}$ has a subsequence converging to some point, say $w(0)$. Denote by $\{w_{n_k^1}\}$ the subsequence of $\{w_n\}$ such that $w_{n_k^1}(0) \to w(0)$. Restricted to this subsequence, there exists a subsequence $\{w_{n_k^2}\}$ such that $\{w_{n_k^2}(1)\}$ converges to some point in $\mathcal{J}$, call it $w(1)$. Notice that $w_{n_k^2}(0) \to w(0)$, since $\{w_{n_k^2}(0)\}$ is a subsequence of the convergent sequence $\{w_{n_k^1}(0)\}$. Proceeding like this, we construct sequences $\{w_{n_k^1}\} \supset \{w_{n_k^2}\} \supset \{w_{n_k^3}\}$... with the property that $w_{n_k^l}(i) \to w(i)$ for all $i = 0, ..., l-1$. The claim is that the subsequence $\{w_{n_l^l}\}$ converges to $w = w(0)w(1)w(2)...$ in the given metric. For every $\epsilon > 0$ there exists an L such that $\sum_{j=L}^{\infty} 2^{-j} < \frac{\epsilon}{2M}$ where $M = \sup_{x,y \in \mathcal{J}} d_{\mathcal{J}}(x, y)$, and there exists an $N \geq L$ such that $\forall l \geq N$, we have $\sum_{j=0}^{L-1} 2^{-j}\, d_{\mathcal{J}}(w_{n_l^l}(j), w(j)) < \frac{\epsilon}{2}$. Then, we have

$$d_{\Sigma_{\mathcal{J}}}(w_{n_l^l}, w) = \sum_{j=0}^{\infty} 2^{-j}\, d_{\mathcal{J}}(w_{n_l^l}(j), w(j))$$

$$= \sum_{j=0}^{L-1} 2^{-j}\, d_{\mathcal{J}}(w_{n_l^l}(j), w(j)) + \sum_{j=L}^{\infty} 2^{-j}\, d_{\mathcal{J}}(w_{n_l^l}(j), w(j)) \quad (3.3)$$

$$\leq \frac{\epsilon}{2} + M * \sum_{j=L}^{\infty} 2^{-j} \leq \frac{\epsilon}{2} + M * \frac{\epsilon}{2M} \leq \epsilon.$$

The claim and the lemma are proved. □

We can extend the metric $d_{\Sigma_{\mathcal{J}}}$ to the space $\Sigma_{\mathcal{J}} \cup \Sigma^*$ as follows. First, add an auxiliary point, say Q, to the set $\mathcal{J}$ and extend the metric $d_{\mathcal{J}}$ to $\tilde{\mathcal{J}} = \mathcal{J} \times \{Q\}$ by postulating $d_{\mathcal{J}}(j, Q) = diam\, \mathcal{J}$, $\forall j \in \mathcal{J}$. Next, embed $\Sigma_{\mathcal{J}} \cup \Sigma^*$ into $\Sigma_{\tilde{\mathcal{J}}}$ by leaving the strings from Σ as they are and attaching an infinite string of $Q's$ to the finite words from Σ^*:

$$\begin{aligned} &u \to u \qquad\qquad if\ u \in \Sigma \\ &u \to u.QQQQ... \quad if\ u \in \Sigma^* \end{aligned} \tag{3.4}$$

Define a metric on $\Sigma_{\tilde{\mathcal{J}}}$ as in 3.1.

Lemma 12. *$\Sigma_{\mathcal{J}} \cup \Sigma^*$ is a compact subset of $\Sigma_{\tilde{\mathcal{J}}}$.*

Proof. Since $\Sigma_{\tilde{\mathcal{J}}}$ is compact by Lemma 11, it remains to show that the set $\Sigma_{\mathcal{J}} \cup \Sigma^*$ is closed. Let w_n be a sequence in $\Sigma_{\mathcal{J}} \cup \Sigma^*$ which converges to some point w. If w_n is a sequence of infinite strings then w is an infinite string, and by Lemma 11, $w \in \Sigma_{\mathcal{J}}$. If w_n has a subsequence of finite strings then, there are two possible cases. w is a finite string or w is an infinite string. If w is a finite string, then there must exist a subsequence w_{n_k} whose lengths are $|w|$. Hence, the corresponding strings $\tilde{w}_{n_k} \in \Sigma_{\tilde{\mathcal{J}}}$ have the form $w_{n_k}.QQ\bar{Q}$. Since $w_{n_k} \to w$,

$$\tilde{w}_{n_k} \to \tilde{w} = w.QQ\bar{Q}.$$

$\tilde{w} \in \Sigma_{\tilde{\mathcal{J}}}$ because $\Sigma_{\tilde{\mathcal{J}}}$ is compact, and therefore $w \in \Sigma^*$. On the other hand, if w is an infinite string, $w \in \Sigma_{\mathcal{J}}$ if only if $w(i) \neq Q$ for all integers $i \geq 0$. Suppose there exists an N such that $w(N) = Q$. Then, there exists an infinite subsequence $\tilde{w}_{n_k} \to w$ such that either $w_{n_k}(N) = Q$ or $w_{n_k}(N) \to Q$ with $w_{n_k}(N) \neq Q$ for all k. In the first case, we get that w_{n_k} are strings whose lengths are at most N. This is impossible because the lengths must increase to infinity. The second case is impossible because $d_{\mathcal{J}}(j, Q) = \textit{diam}\, \mathcal{J}$ for all $j \in J$. A contradiction. Hence, $w \in \Sigma_{\mathcal{J}}$. $\square$

When $\mathcal{J}$ has finitely many elements, the metric $d_{\Sigma_{\mathcal{J}}}$ on $\Sigma_{\mathcal{J}}$ can be reduced to a metric $d_{\Sigma^{\diamond}_{\mathcal{J}}}(w, s) = \sum\limits_{j=0}^{\infty} 2^{-j}\, d_{\mathcal{J}}(s(j), w(j))$, with $d_{\mathcal{J}}(s(j), w(j)) = 1$ if $s(j) = w(j)$, and $d_{\mathcal{J}}(s(j), w(j)) = 0$ if $s(j) \neq w(j)$. This metric is equivalent to the metric used in [18], where the distance between two sequences is 2^{-N}, if N is the smallest integer j such that $s(j) \neq w(j)$. The shift operator, $\sigma : \Sigma_{\mathcal{J}} \to \Sigma_{\mathcal{J}}$, acts on $w \in \Sigma_{\mathcal{J}}$ by deleting its first symbol. It is easy to show that this action is continuous. Actually, we have $d_{\Sigma_{\mathcal{J}}}(\sigma(s), \sigma(w)) \leq 2\, d_{\Sigma_{\mathcal{J}}}(s, w)$.

In the following chapters, we denote the space $\Sigma_{\mathcal{J}}$ by Σ and the metric on it by d_{Σ}.

3.2 Attractors for Dynamics with Choice

Definition 13. *Dynamics with choice*[3] *is a discrete-time dynamics on the product space* $\mathfrak{X} = X \times \Sigma$ *generated by the evolution operator* $\mathfrak{S} : \mathfrak{X} \to \mathfrak{X}$*:*

$$\mathfrak{S}(x, w) = (S_{w(0)}, \sigma(w)). \tag{3.5}$$

This map is obviously continuous and bounded. Because we will consider iterations of $\mathfrak{S}$,

[3]Originally, we distinguished the cases when there are finitely many operators (which we called dynamics with choice [18]), and infinitely many operators (which we called dynamics with a range of choice [19]). We do not distinguish the two cases unless otherwise specified and we refer to it as dynamics with choice.

$$\mathfrak{S}^n(x,u) = (S_{u(n-1)} \circ \cdots \circ S_{u(1)} \circ S_{u(0)}(x), \, \sigma^n(u)),$$

we introduce the notation

$$S_w = S_{w(n-1)} \circ \cdots \circ S_{w(1)} \circ S_{w(0)},$$

if w is a word of length n. Thus, we can write $\mathfrak{S}^n(x,u) = (S_{u[n]}(x), \sigma^n(u))$.

We have the following additional assumptions on the operators S_j.

Assumption 1. *Assume there is a closed, bounded set $\mathbf{B} \subset X$ such that for every bounded $A \subset X$ there exists $m(A) > 0$ such that $S_w(A) \subset \mathbf{B}$ for every word w of length $n \geq m(A)$.*

[In applications **B** is usually a closed ball of radius that depends on the parameters of the model. Showing that for different values of the parameters there is a common estimate on the radius is enough to verify Assumption 1.]

Let ψ be a measure of noncompactness as in Definition 4.

Assumption 2. *Assume that each operator S_j is ψ-condensing.*

Note, that Assumptions 1 and 2 are enough to prove the existence of a global compact attractor in dynamics with choice when $\mathcal{J}$ is finite. Since, in general, we allow an infinite number of maps S_j, we need an additional assumption concerning their dependence on the parameter j.

Assumption 3. *There exists a finite partition $\{\mathcal{J}_p\}_{p=1}^{M}$ of the set $\mathcal{J}$, with $\mathcal{J}_p$ compact, such that for any closed, bounded $A \subset X$, the maps S_j, restricted to A, depend uniformly continuously on j when j changes within each of the sets $\mathcal{J}_p$. More precisely,* *given a closed, bounded A, for every p and every $\epsilon > 0$ there is a $\delta > 0$ such that* $\sup_{x \in A} d_X(S_i(x), S_j(x)) \leq \epsilon$ *provided* $i, j \in \mathcal{J}_p$ *and* $d_{\mathcal{J}}(i,j) \leq \delta$.

We are going to apply Theorem 2 to prove the existence of the attractor in the dynamics with choice $(\mathfrak{X}, \mathfrak{S})$. Our Assumption 1 gives the absorbing set $\mathcal{B} = \mathbf{B} \times \mathcal{J}$ in $\mathfrak{X}$. It remains to show that the map $\mathfrak{S}$ is condensing with respect to some measure of noncompactness $\psi_{\mathfrak{X}}$. Because the parameter space $\mathcal{J}$ is compact, there is a natural choice for $\psi_{\mathfrak{X}}$, namely,

$$\psi_{\mathfrak{X}}(\mathfrak{C}) = \psi_X(\mathrm{pr}_X(\mathfrak{C}))\,,$$

where $\mathrm{pr}_X(\mathfrak{C}) = \{x \in X \,:\, (x,u) \in \mathfrak{C}, \text{ for some } u \in \Sigma_{\mathcal{J}}\}$. We showed in 2.2 that $\psi_{\mathfrak{X}}$ enjoys the properties (i), (ii), (iii), and (iv) of the measures of noncompactness. With this choice of $\psi_{\mathfrak{X}}$ we prove the following fact.

Lemma 14. *The map $\mathfrak{S}$ is $\psi_{\mathfrak{X}}$-condensing.*

Proof. Let $\mathfrak{C}$ be a closed, bounded subset of $\mathfrak{X}$. Its projection on X, $C = \mathrm{pr}_X(\mathfrak{C})$, is closed and bounded in X. Pick an $\epsilon > 0$. By Assumption 3, there is $\delta > 0$ such that $\sup_{x \in C} d_X(S_i(x), S_j(x)) \leq \epsilon$, and hence $d_H(S_i(C), S_j(C)) \leq \epsilon$, provided $d_{\mathcal{J}}(i,j) \leq \delta$ and i and j lie within the same set $\mathcal{J}_p$. Let $\mathcal{I}_\delta = \{i_1, \ldots, i_R\}$ be a finite δ-net in $\mathcal{J}$. We have

$$\psi_{\mathfrak{X}}(\mathfrak{S}(\mathfrak{C})) \leq \psi_X\left(\bigcup_{i \in \mathcal{J}} S_i(C)\right) = \max_{1 \leq p \leq M} \psi_X\left(\bigcup_{i \in \mathcal{J}_p} S_i(C)\right) = \psi_X\left(\bigcup_{i \in \mathcal{J}_{p_0}} S_i(C)\right),$$

for some p_0. Now, since

$$\left|\psi_X\left(\bigcup_{i \in \mathcal{J}_{p_0}} S_i(C)\right) - \psi_X\left(\bigcup_{j \in \mathcal{J}_{p_0} \cap \mathcal{I}_\delta} S_j(C)\right)\right| \leq c(\psi_X)\, d_H\left(\bigcup_{i \in \mathcal{J}_{p_0}} S_i(C), \bigcup_{j \in \mathcal{J}_{p_0} \cap \mathcal{I}_\delta} S_j(C)\right)$$

and $d_H\left(\bigcup_{i\in\mathcal{J}_{p_0}} S_i(C), \bigcup_{j\in\mathcal{J}_{p_0}\cap\mathcal{I}_\delta} S_j(C)\right) \le \epsilon$, we obtain

$$\psi_{\mathfrak{X}}\left(\mathfrak{S}(\mathfrak{C})\right) \le \psi_X\left(\bigcup_{j\in\mathcal{J}_{p_0}\cap\mathcal{I}_\delta} S_j(C)\right) + c(\psi_X)\,\epsilon = \psi_X(S_i(C)) + c(\psi_X)\,\epsilon\,,$$

for some $i \in \mathcal{J}_{p_0} \cap \mathcal{I}_\delta$. Hence,

$$\psi_{\mathfrak{X}}\left(\mathfrak{S}(\mathfrak{C})\right) \le \psi_X(C) + c(\psi_X)\,\epsilon\,,$$

and the inequality is strict if C is not relatively compact. Since $\psi_{\mathfrak{X}}(\mathfrak{C}) = \psi_X(C)$ and ϵ was arbitrary, lemma is proved. □

Corollary 15. *For every bounded set $A \subset X$, there exists a $j_0 \in \mathcal{J}$ such that $\psi_X(\bigcup_{j\in\mathcal{J}} S_j(A)) \le \psi_X(S_{j_0}(A))$, and $\psi_X(\bigcup_{j\in\mathcal{J}} S_j(A)) < \psi_X(S_{j_0}(A))$ iff $\psi_X(S_{j_0}(A)) > 0$.*

Applying Theorem 2 we immediately obtain the following result.

Theorem 16. *Let X be a complete metric space and let S_j, for $j \in \mathcal{J}$ be a compact metric set, be continuous, bounded (i.e., take bounded sets to bounded sets) maps $X \to X$. In addition, let assumptions 1, 2, and 3 be satisfied. Then the system $(\mathfrak{X}, \mathrm{dist}, \mathfrak{S})$ has a global compact attractor, $\mathfrak{M}$.*

The attractor $\mathfrak{M}$ has the following properties.

(1) $\mathfrak{M}$ is (strictly) invariant: $\mathfrak{S}(\mathfrak{M}) = \mathfrak{M}$.

(2) $\mathfrak{M}$ is the union of all closed bounded sets $A \subset \mathfrak{X}$ with the property $A \subset \mathfrak{S}(A)$.

(3) $\mathfrak{M}$ *is the maximal closed set with the property* $A \subset \mathfrak{S}(A)$*; in particular,* $\mathfrak{M}$ *is the maximal (strictly) invariant closed set.*

(4) Through every point $(x, w) \in \mathfrak{M}$ *passes a complete trajectory. This means there exists a two-sided sequence* $\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$ *of points in* X *and a two-sided infinite string* $\ldots s(-2)s(-1)s(0)s(1)s(2)\ldots$ *such that* $x(0) = x$ *and* $s(0)s(1)s(2)\cdots = w(0)w(1)w(2)\ldots$ *and such that* $S_{s(n)}(x_n) = x_{n+1}$ *for every integer* n*.*

(5) $\mathfrak{M}$ *is the union of all complete, bounded trajectories in* $\mathfrak{X}$*.*

3.2.1 Connection with IFS

Given the state space X and operators S_j, there are two ways of describing dynamics generated by the corresponding IFS. First, one can follow the trajectories of bounded subsets of X under the iterations of the Hutchinson-Barnsley map $\bar{F}$ (here we consider the map for the infinite union, i.e.,

$$\bar{F}(A) = \overline{\bigcup_{j \in \mathcal{J}} S_j(A)}).$$

We denote such system by $(X, d, \bar{F})$. The notion of the global compact attractor as the minimal compact set that attracts all bounded sets, is well-defined for $(X, d, \bar{F})$. The second possibility is to choose the space of closed bounded sets, $\bar{\mathcal{B}}(X)$, as the state space of the system and study the dynamics of its points under the iterations of $\bar{F}$. As a rule, $\bar{\mathcal{B}}(X)$ is equipped with the Hausdorff distance d_H. Thus we obtain the second system, $(\bar{\mathcal{B}}(X), d_H, \bar{F})$. It turns out that from the point of view of global compact attractors the dynamical system $(\bar{\mathcal{B}}(X), d_H, \bar{F})$ is not very interesting (because convergence in the Hausdorff metric is too strong). It possesses an attractor (in the sense we use here) essentially only if the maps S_j are contractions, so then the attractor is just one point in $\bar{\mathcal{B}}(X)$. For more general S_j, it makes more sense to study the fixed points of $\bar{F}$.

Theorem 17. *Make the same assumptions on the space X and operators S_j as in Theorem 16. Then*

(1) The IFS $(X, d, \bar{F})$ does have a global compact attractor, K.

(2) The set K is the largest compact set in X which is invariant under the Hutchinson-Barnsley map $\bar{F}$, $K = \bar{F}(K)$.

(3) The attractor $\mathfrak{M}$ of the dynamics with choice has the following product structure:

$$\mathfrak{M} = K \times \Sigma\,.$$

In the extensive literature on IFSs the main question is the existence of "the fractal," i.e., the maximal compact set invariant under the Hutchinson-Barnsley operator $\bar{F}$. This corresponds to the second assertion of our Theorem 17. We believe that viewing "the fractal" of an IFS as the attractor of the dynamical system $(X, d, \bar{F})$ is beneficial to the theory of IFSs. This approach, in particular, points to the "right" assumptions on the space X and the operators S_j.

The following lemma is needed for the proof of the theorem. A version of its proof can be found in the proof of [16, Lemma 2.3.5].

Lemma 18. *For any bounded sequence x_n, and any sequence w_n of finite lengths increasing to infinity, under assumptions 1, 2, and 3, the sequence $S_{w_n}(x_n)$ has a convergent subsequence.*

Proof. Let $\{x_n\}$ be any bounded sequence in X and $\{w_n\} \subset \Sigma^*$ be a sequence of finite strings whose lengths are increasing to infinity. There exists a subsequence of w_n which converges to $w \in \Sigma$. Wlog denote this subsequence by w_n. Then by assumption 1 there exists $m(\{x_n\})$ such that $S_{w_n}(x_n) \subset \mathbf{B}$ for all n with $|w_n| > m(\{x_n\})$. Now, find $m(\mathbf{B})$ and define

$$\tilde{\mathbf{B}} = \bigcup_{v \in \Sigma^*_{m(\mathbf{B})}} S_v(\mathbf{B})$$

Clearly, any positive trajectory of the set $\tilde{\mathbf{B}}$ is in B, i.e., for a word v of any length we have $S_v(\tilde{\mathbf{B}}) \subset \mathbf{B}$. Let $m = m(\{x_n\}) + m(\mathbf{B})$, and let s_n be such that $w_n = w_n[m].s_n$. If we let $y_n = S_{w_n[m]}(x_n)$, then $y_n \subset \tilde{\mathbf{B}}$. It is left to prove that the sequence $S_{s_n}(y_n)$ is precompact.

Consider the union of positive trajectories of $\tilde{\mathbf{B}}$ for all possible choices of $w \in \Sigma^*_n$:

$$C_0 = \tilde{\mathbf{B}} \cup \bigcup_{n \geq 1} \bigcup_{v \in \Sigma^*_n} S_v(\tilde{\mathbf{B}})\,.$$

Define inductively

$$C_{n+1} = \bigcup_{j \in \mathcal{J}} S_j\left(C_n\right).$$

We have $C_0 \supset C_1 \supset C_2 \supset \ldots$.

Let $\mathfrak{H}$ denote a collection of all sets $A \subset \mathbf{B}$ that can be written in the following way

$$A = \bigcup_{n \geq 0} A_n \qquad \text{where } A_n \text{ is a } \textit{finite} \text{ (or empty) subset of } C_n\,.$$

Next we will show that every $A \in \mathfrak{H}$ is relatively compact. Since the sequence $S_{s_n}(y_n)$ is in $\mathfrak{H}$ then our lemma will be proved.

Let $A^* \in \mathfrak{H}$ be such that $\psi_X(A^*) = \sup_{A\in\mathfrak{H}} \psi_X(A)$. This set exists, see the proof of theorem 6 (also, see lemma 1.6.10 [1]). Denote $A^* = \bigcup_{n=0}^{\infty} A_n^*$ such that each $A_n^* \subset C_n$. Then for every $p \in A_n^*$, there exist a $q \in C_{n-1}$ and a $j \in \mathcal{J}$ such that $p = S_j(q)$. Denote by A_{n-1} the resulting subset of C_{n-1} and by $A = \bigcup_{n=0}^{\infty} A_n$. Obviously, $A \in \mathfrak{H}$ and therefore we have $\psi_X(A) \leq \psi_X(A^*)$. Consider the set $\bar{F}(A) = \bigcup_{j\in\mathcal{J}} S_j(A)$. This set is in $\mathfrak{H}$ and it contains $\bigcup_{n=1}^{\infty} A_n^*$. Since it follows from Corollary 15 that $\psi_X(\bigcup_{j\in\mathcal{J}} S_j(A)) \leq \psi_X(S_{j_0}(A))$, for some $j_0 \in \mathcal{J}$, then we have

$$\psi_X(A^*) = \psi_X(\bigcup_{n=1}^{\infty} A_n^*) \leq \psi_X(\bar{F}(A)) \leq \psi_X(S_{j_0}(A)) \leq \psi_X(A)$$

Above we used properties (i) and (iii) of ψ_X to show first equality, property (ii) for first inequality, and last inequality follows since each operator S_j is ψ_X-condensing. If A is not relatively compact, then $\psi_X(S_{j_0}(A)) < \psi_X(A)$, which implies $\psi_X(A^*) < \psi_X(A)$. This is a contradiction to $\psi_X(A^*) \geq \psi_X(A)$. Therefore, we must have $\psi_X(A^*) = \psi_X(A) = 0$. This concludes the proof of the lemma. □

Proof of Theorem 17. Consider the IFS dynamics $(X, d, \bar{F})$. This means that we follow the dynamics of bounded sets under the iterations of $\bar{F}$. Since the map $\bar{F}$ is inherently multi-valued, we cannot apply theorem 2 to show the existence of a global compact attractor. In order to do that, we need to define the ω-limit set of a bounded set ($B \subset X$), and prove that it is a nonempty, invariant, compact set which attracts B. Notice that

$$\bar{F}^n(x) = \overline{\bigcup_{w^*\in\Sigma_n^*} S_{w^*}(x)}$$

Therefore, we define ω-limit set to be

$$\omega(B) = \left\{y \in X \,|\, y = \lim S_{w_{n_k}}(x_k),\ for\,(x_k) \subset B,\, and\,(w_{n_k}) \subset \Sigma^*\right\}$$

Lemma 19. *For any bounded set $B \subset X$,*

(i) *$\omega(B)$ is nonempty,*

(ii) *if $A \subset B$, then $\omega(A) \subset \omega(B)$,*

(iii) *$\omega(B)$ is compact,*

(iv) *$\omega(B)$ is invariant in the sense that $\omega(B) = \bar{F}(\omega(B))$,*

(v) *$\omega(\omega(B)) = \omega(B)$*

(vi) *$\omega(B)$ attracts B.*

Proof. For any bounded $B \subset X$, $\omega(B)$ is a nonempty set by Lemma 18. Let $y \in \omega(A)$, where $A \subset B$, then $y = \lim S_{w_{n_k}}(x_k)$, for $(x_k) \in A$ and any sequence of finite strings w_{n_k}, whose lengths are increasing to infinity. Since, $A \subset B$, then $(x_k) \in B$, and therefore $y \in \omega(B)$. This proves property (ii).

Note that $\omega(B)$ can be characterized as follows. $\omega(B)$ is the set of all $y \in X$ such that for every $\epsilon > 0$ and every integer $k \geq 0$ there exist an $x \in B$, an $n > k$, and a $w^* \in \Sigma_n^*$ so that $S_{w^*}(x) \in \mathcal{O}_\epsilon(y)$ (where $\mathcal{O}_\epsilon(y)$ is the ϵ-neighborhood of y). Yet another way to describe $\omega(B)$ is to consider the trajectory of B and its tails:

$$D_0 = B \cup \left(\cup_{w\in\Sigma_1^*} S_w(B)\right) \cup \left(\cup_{w\in\Sigma_2^*} S_w(B)\right) \cup \dots, \qquad D_n = \bigcup_{m\geq n} \left(\cup_{w\in\Sigma_m^*} S_w(B)\right).$$

Clearly, $D_0 \supset D_1 \supset D_2 \supset \ldots$. It turns out that

$$\omega(B) = \bigcap_{n \geq 0} \overline{D_n}\,. \tag{3.6}$$

Indeed, inclusion $\subset$ is obvious. To prove the "$\supset$" part, pick a y in the intersection of the tails and set $\epsilon_n = 2^{-n}$. In D_1 there is a point $y_1 = S_{w_{m_1}}(x_1)$, $x_1 \in B$, and $w_{m_1} \in \Sigma^*_{m_1}$ such that $d\,(y_1,\, y) \leq \epsilon_1$. In D_{m_1+1} there is a point $y_2 = S_{w_{m_2}}(x_2)$, $m_2 > m_1$, such that $d\,(y_2,\, y) \leq \epsilon_2$, and so on. The limit of y_n belongs to $\omega(B)$, i.e., $y \in \omega(B)$. Therefore, $\omega(B)$ is closed.

$\omega(B)$ is invariant in the sense that $\omega(B) = \bar{F}(\omega(B)) = \overline{\bigcup_{j \in \mathcal{J}} S_j(\omega(B))}$. To show this, pick an $x \in \omega(B)$. Then, there exist a sequence $(x_k) \in B$, and a sequence $w_k \in \Sigma^*$ with lengths n_k increasing to infinity such that $x = \lim S_{w_k}(x_k)$. The last symbol in the words (w_k) give a sequence $w_k(n_k - 1) \subset \mathcal{J}$. Infinitely many elements of this sequence belong to some $\mathcal{J}_p$. This sequence has a convergent subsequence with limit, say, $j_0 \in \mathcal{J}_p$. Wlog, denote the sequence again by $(w_k(n_k - 1))$ and the corresponding sequence of words again by (w_k). By Lemma 18, there is a subsequence of $S_{w_k[n_k-1]}(x_k)$ which converges to some y. In fact, $y \in \omega(B)$. Now, for every $\epsilon > 0$ there exists k (large) such that $d\,(x, S_{w_k}(x_k)) \leq \frac{\epsilon}{3}$, $d\left(S_{w_k(n_k-1)}\left(S_{w_k[n_k-1]}(x_k)\right), S_{w_k(n_k-1)}(y)\right) \leq \frac{\epsilon}{3}$. and $d\left(S_{w_k(n_k-1)}(y), S_{j_0}(y)\right) \leq \frac{\epsilon}{3}$ (follows form assumption 3). Then, we have

$$\begin{aligned} d(x, S_j(y)) \leq d\,(x, S_{w_k}(x_k)) + d\left(S_{w_k(n_k-1)}\left(S_{w_k[n_k-1]}(x_k)\right), S_{w_k(n_k-1)}(y)\right) \\ + d\left(S_{w_k(n_k-1)}(y), S_{j_0}(y)\right) < \epsilon \end{aligned} \tag{3.7}$$

Since for every $x \in \omega(B)$ and every $\epsilon > 0$ there exists $j \in \mathcal{J}$ such that $d(x, \bigcup_{j\in\mathcal{J}} S_j(\omega(B))) < \epsilon$, it follows that $x \in \overline{\bigcup_{j\in\mathcal{J}} S_j(\omega(B))}$. To show the other side of the inclusion, pick $x \in \bigcup_{j\in\mathcal{J}} S_j(\omega(B))$, then

$$x = S_j(\lim S_{w_{n_k}}(x_k)) = \lim S_j\left(S_{w_{n_k}}(x_k)\right) = \lim S_{w_{n_k}.j}(x_k) \in \omega(B)$$

Since, $\omega(B)$ is closed and $\bigcup_{j\in\mathcal{J}} S_j(\omega(B)) \subset \omega(B)$ it follows that $\overline{\bigcup_{j\in\mathcal{J}} S_j(\omega(B))} \subset \omega(B)$. This proves property (iv).

To show compactness, note that

$$\psi_X(\omega(B)) = \psi_X\left(\bigcup_{j\in\mathcal{J}} S_j(\omega(B))\right) \leq \psi_X\left(S_{j_0}(\omega(B))\right) \leq \psi_X(\omega(B)).$$

The first equality follows from the invariance of $\omega(B)$ and Property (v) of Definition 4. The first inequality follows from Corollary 15, and the second inequality follows since maps S_j are condensing. The second inequality is strict only if $\omega(B)$ is not relatively compact, which would lead to a contradiction. Since $\omega(B)$ is closed, it is compact. This proves property (iii).

Let $y \in \omega(\omega(B))$, then there exists a sequence $(x_k) \in \omega(B)$ and a sequence of finite strings w_{n_k} of lengths increasing to infinity such that $y = \lim S_{w_{n_k}}(x_k)$. Note that for every k, $S_{w_{n_k}}(x_k) \in \bar{F}^{n_k}(\omega(B)) = \omega(B)$. The equality follows from property (iv). Now, for every $\epsilon > 0$, there exists k, such that $d(y, S_{w_{n_k}}(x_k)) \leq \epsilon$. Since, $\omega(B)$ is compact, it follows that $y \in \omega(B)$. Therefore, we have $\omega(\omega(B)) \subset \omega(B)$. Now, pick $y \in \omega(B)$. Then, $y \in \bar{F}^{n_k}(\omega(B))$ for every $n_k \geq 0$. Therefore, we can find a sequence $(x_k) \in \omega(B)$, and w_{n_k} with lengths $|w_{n_k}| = n_k$ such that for every k we have $y = S_{w_{n_k}}(x_k)$. Hence, $y = \lim S_{w_{n_k}}(x_k)$. This proves that $y \in \omega(\omega(B))$ and the property (v) is proven.

To show that for every $\epsilon > 0$ there exists an m such that $S_{w_n}(B) \subset \mathcal{O}_\epsilon(\omega(B))$ for all $n \geq m$, we argue by contradiction. Assume there exists an $\epsilon_0 > 0$ such that $S_{w_n}(B)$ does not lie inside $\mathcal{O}_{\epsilon_0}(\omega(B))$ for infinitely many n. This means that there is a sequence (x_k) in B and a sequence $n_k \nearrow +\infty$ such that $S_{w_{n_k}}(x_k) \notin \mathcal{O}_{\epsilon_0}(\omega(B))$. But we already know that $S_{w_{n_k}}(x_k)$ must have a convergent subsequence whose limit must be in $\omega(B)$. A contradiction. Therefore, $\omega(B)$ attracts B. We showed property (vi) which concludes the proof of the lemma. □

Now we return to the proof of theorem 17.

Claim: $K = \omega(\mathbf{B})$ is the global compact attractor.

Let P be any compact subset of X. For every $x \in P$, there exist an open ball $\mathcal{O}_{r_x}(x) \subset P$, and a positive number n, such that, for every $w \in \Sigma$, $S_{w[n]}(\mathcal{O}_{r_x}(x)) \subset \mathbf{B}$. From property (ii) of Lemma 19, it follows that $\omega(\mathcal{O}_{r_x}(x)) \subset \omega(\mathbf{B})$. Now, let $\{\mathcal{O}_{r_x}(x)\}_{x \in P}$ be an open cover of P. Then, since P is compact, there exists a finite subcover. Now, $P \subset \bigcup_{j=1}^{N} \mathcal{O}_{r_j}(x_j)$ gives us

$$\omega(P) \subset \omega\left(\bigcup_{j=1}^{N} \mathcal{O}_{r_j}(x_j)\right) \subset \bigcup_{j=1}^{N} \omega\left(\mathcal{O}_{r_j}(x_j)\right) \subset \omega(\mathbf{B})$$

Since $\omega(P)$ attracts P, a bigger set, K, attracts P. For any bounded set B, $\omega(B)$ is compact, and $\omega(\omega(B)) = \omega(B)$ (follows from property (v) in the above lemma). Also, $\omega(B) \subset K$. Hence, K attracts all bounded sets. K is the maximal compact set invariant under $\bar{F}$. Suppose this is not true. Then, there exists a set $A \supsetneq K$ which attracts all bounded sets and is invariant under $\bar{F}$. Both A and K are compact, therefore we can find $\epsilon > 0$ such that $A \setminus \mathcal{O}_\epsilon(K) \neq \emptyset$. Since A is invariant, and K attracts all bounded sets we have

$$A = \bar{F}^n(A) \subset \mathcal{O}_\epsilon(K).$$

A contradiction. Therefore, K is the global compact attractor of the system $(X, d, \bar{F})$ which comes with all the properties listed in Theorem 3.

To prove that $\mathfrak{M} = K \times \Sigma$ we start by showing that the slices of the attractor corresponding to different strings are all the same, i.e., the set $\{x \in X : (x, s) \in \mathfrak{M}\}$ does not depend on s.

All slices are equal. Recall that every point (x, s) in $\mathfrak{M}$ is a limit of some sequence $\mathfrak{S}^{n_k}(x_k, s_k)$ with bounded $(x_k) \subset X$ and $\sigma^{n_k}(s_k)$ converging to s. As we argued above, we can write

$$\mathfrak{S}^{n_k}(x_k, s_k) = \left(S_{w_k}(x_k), \sigma^{n_k}(s_k)\right),$$

where w_k is a prefix of length $|w_k| = n_k$ of the string s_k, i.e., $s_k = w_k.\sigma^{n_k}(s_k)$. The sequence $S_{w_k}(x_k)$ converges to x and $\sigma^{n_k}(s_k)$ converges to s. The limit of the pair will not change if we replace s_k by $w_k.s$. Clearly, for any string $u \in \Sigma$, we have

$$\lim \left(S_{w_k}(x_k), \sigma^{n_k}(w_k.u)\right) = (x, u).$$

This proves that $\mathfrak{M} = A \times \Sigma$. The set $A \subset X$ is compact because $\mathfrak{M}$ is compact.

Since $\Sigma = \bigcup_{j \in \mathcal{J}} j.\Sigma$ and since $\mathfrak{S}(\mathfrak{M}) = \mathfrak{M}$, we get $\mathfrak{S}(A \times \Sigma) = (\bigcup_{j \in \mathcal{J}} S_j(A)) \times \Sigma = A \times \Sigma$. In other words, $A = \bigcup_{j \in \mathcal{J}} S_j(A)$. Because K is the maximal compact in X with this property, we have $A \subset K$. On the other hand, $\mathfrak{S}(K \times \Sigma) = K \times \Sigma$. Since $A \times \Sigma$ is the maximal compact in $\mathfrak{X}$ with this property, we have $K \subset A$, and hence, $A = K$. This completes the proof of Theorem 17. $\square$

3.3 Individual Attractors

Every fixed strategy also generates a dynamics on X: if $w \in \Sigma$ is the (fixed) strategy, then an $x \in X$ moves to $S_{w(0)}(x)$, then to $S_{w(1)}\left(S_{w(0)}(x)\right)$, then to $S_{w(2)}\left(S_{w(1)}\left(S_{w(0)}(x)\right)\right)$, etc. Denote this dynamics by (X, d, w). This is not a (semi)dynamical system, but we should not worry about names. Certain important notions related to the long-term behavior with natural adjustments still make sense. For example, the individual, i.e., corresponding to an individual strategy w, trajectory of a set B is the union

$$B \cup S_{w[1]}(B) \cup S_{w[2]}(B) \cup \dots\ .$$

We define the individual ω-limit set of a bounded set B as

$$\omega(B,\, w) = \{y \in X \,:\, y = \lim S_{w[n_k]}(y_k) \quad \text{for some sequence } (y_k) \text{ in } B\}\,.$$

By analogy with Definition 1, we say that a set A is the global compact attractor of system (X, d, w) if it is the minimal set with the following two properties: A is compact and A attracts every bounded set under the strategy w, i.e., for any bounded B, we have $\lim_{n\to\infty} \vec{dist}\,(S_{w[n]}(B),\, A) = 0$.

Next theorem establishes the existence of individual compact attractors, $\mathcal{A}_w$, of systems (X, d, w). Along the way we establish various properties of the ω-limiting sets $\omega(B,\, w)$.

Theorem 20. *Under the Assumptions 1, 2, and 3, every system (X, d, w) has the global compact attractor, which we denote by $\mathcal{A}_w$. This attractor is the intersection* *of the closures of the tails of the trajectory of the absorbing set* $\mathbf{B}$*,*

$$\mathcal{A}_w = \bigcap_{n\geq 1} \overline{\bigcup_{k\geq n} S_{w[k]}(\mathbf{B})}\,.$$

The attractor, $\mathcal{A}_w$*, is the union of all* $\omega(B, w)$ *with bounded* B*.*

Proof. We use some notation and keep in mind the argument from the proof of Lemma 18. Due to Assumption 1, every bounded set eventually finds itself in the set $\tilde{\mathbf{B}}$ and after that stays there.

Step 1. The ω-limit sets of bounded sets are not empty.

Pick a point $x_0 \in X$ and follow its trajectory, $x_n = S_{w[n]}(x_0)$. There will be a time n such that $x_n \in \tilde{\mathbf{B}} \subset C_0$, and then inevitably $x_{n+1} \in C_1$, $x_{n+2} \in C_2$, and so on. By Lemma 18, the sequence (x_n) is relatively compact. Thus, $\omega(\{x_0\}, w) \neq \emptyset$. Because $\omega(\{x_0\}, w) \subset \omega(B, w)$ if $x_0 \in B$, we have $\omega(B, w) \neq \emptyset$.

Step 2. $\omega(B, w)$ is the intersection of the closures of the tails of its trajectory, hence $\omega(B, w)$ is closed.

Note that $\omega(B, w)$ can be characterized as follows. $\omega(B,\, w)$ is the set of all $y \in X$ such that for every $\epsilon > 0$ and every integer $k \geq 0$ there exist and $x \in B$ and $n > k$ so that $S_{w[n]}(x) \in \mathcal{O}_\epsilon(y)$ (where $\mathcal{O}_\epsilon(y)$ is the ϵ-neighborhood of y). Yet another way to describe $\omega(B,\, w)$ is to consider the trajectory of B and its tails:

$$D_0 = B \cup S_{w[1]}(B) \cup S_{w[2]}(B) \cup \dots\,, \qquad D_n = \bigcup_{m\geq n} S_{w[m]}(B)\,.$$

Clearly, $D_0 \supset D_1 \supset D_2 \supset \dots$. It turns out that

$$\omega(B,\, w) = \bigcap_{n\geq 0} \overline{D_n}\,. \tag{3.8}$$

Indeed, inclusion $\subset$ is obvious. To prove the "$\supset$" part, pick a y in the intersection of the tails and set $\epsilon_n = 2^{-n}$. In D_1 there is a point $y_1 = S_{w[m_1]}(x_1)$, $x_1 \in B$, such that $d\,(y_1,\, y) \leq \epsilon_1$. In D_{m_1+1} there is a point $y_2 = S_{w[m_2]}(x_2)$, $m_2 > m_1$, such that $d\,(y_2,\, y) \leq \epsilon_2$, and so on. The limit of y_n belongs to $\omega(B,\, w)$, i.e., $y \in \omega(B,\, w)$.

Step 3. $\omega(B, w)$ **is compact.**

Compactness of $\omega(B,\, w)$ will follow from the fact that the intersection of the closures of the sets C_n in the proof of Lemma 18 is compact, because, thanks to Assumption 1, $\bigcap\limits_{n\geq 0} \overline{D_n} \subset \bigcap\limits_{n\geq 0} \overline{C_n}$. Denote $C_* = \bigcap\limits_{n\geq 0} \overline{C_n}$. Since $\omega(B,\, w)$ is not empty, C_* is not empty as well. And it is closed. If the set C_* is not compact, then there exist $\epsilon_0 > 0$ and an infinite sequence $(y_n) \subset C_*$ such that $d\,(y_n,\, y_m) \geq \epsilon_0$ for all n and $m \neq n$. Since $y_n \in \overline{C_n}$ (in fact, the whole sequence lies in every set $\overline{C_n}$), there exists a sequence $y_{nk} \in C_n$ that converges to y_n as $k \to \infty$. For every $\epsilon > 0$ there are numbers k_n such that $d\,(y_{nk_n},\, y_n) \leq \epsilon$ for all n. When $\epsilon < \epsilon_0/2$, the Hausdorff distance between the sets $\{y_{nk_n}\}$ and $\{y_n\}$ is not greater than ϵ. Using property (iv) of the measure of noncompactness ψ, we obtain $|\psi\,(\{y_n\}) - \psi\,(\{y_{nk_n}\})\,| \leq\ c(\psi)\,\epsilon$. Now, $\psi\,(\{y_{nk_n}\}) = 0$ by Lemma 18. Then $\psi\,(\{y_n\}) \leq\ c(\psi)\,\epsilon$. Since this is true for any ϵ, we obtain $\psi\,(\{y_n\}) = 0$, a contradiction. This proves that C_* is compact.

Step 4. $\omega(B,\, w)$ **attracts** B**.**

To show that for every $\epsilon > 0$ there exists an m such that $S_{w[n]}(B) \subset \mathcal{O}_\epsilon\,(\omega(B,\, w))$ for all $n \geq m$, we argue by contradiction. Assume there exists an $\epsilon_0 > 0$ such that $S_{w[n]}(B)$ does not lie inside $\mathcal{O}_{\epsilon_0}\,(\omega(B,\, w))$ for infinitely many n. This means that there is a sequence x_k in B and a sequence $n_k \to +\infty$ such that $S_{w[n_k]}(x_k) \notin \mathcal{O}_{\epsilon_0}\,(\omega(B,\, w))$. But we already know that $S_{w[n_k]}(x_k)$ must have a convergent subsequence whose limit must be in $\omega(B,\, w)$. A contradiction.

Step 5. $\mathcal{A}_w = \omega(\tilde{\mathbf{B}},\, w)$**.**

Because every bounded set is eventually absorbed by the set $\tilde{\mathbf{B}}$, we have $\omega(B,\, w) \subset \omega(\tilde{\mathbf{B}},\, w)$. We claim that $A = \omega(\tilde{\mathbf{B}},\, w)$ is the global compact attractor.

It is clear that such A attracts every bounded set. As an ω-limit set A is compact. It only remains to show that A is the minimal compact set that attracts every bounded set. Assume it is not and there is another compact set, P, that attracts every bounded set. Then there exists $\epsilon > 0$ such that $A \setminus \mathcal{O}_\epsilon(P) \neq \emptyset$. Let $x \in A \setminus \mathcal{O}_\epsilon(P)$, then there exists $(x_k) \in \tilde{\mathbf{B}}$, and sequence (n_k) of integers increasing to infinity such that $x = \lim S_{w[n_k]}(x_k)$. This implies that P does not attract bounded set (x_k). A contradiction. In other words, $A = \mathcal{A}_w$.

Step 6. It is not hard to see that $\mathcal{A}_w = \bigcup_{\text{bounded } B} \omega(B,\, w)$. The theorem is proved. $\square$

3.3.1 Interplay between individual attractors

Recall, that (with Assumptions 1, 2, and 3) the global attractor $\mathfrak{M}$ of $(\mathfrak{X}, \mathit{dist}, \Sigma)$ is a product $\mathfrak{M} = K \times \Sigma$.

We start with a few simple observations.

Lemma 21. *$\mathcal{A}_w \subset \bar{F}\,(\mathcal{A}_w) \subset K$, where $\bar{F}$ is the Hutchinson-Barnsley operator.*

Proof. Pick a point, x, in $\mathcal{A}_w$. Then $x = \lim S_{w[n_k]}(x_k)$ for some bounded sequence (x_k) in X and $n_k \to \infty$. The last symbol in the words $w[n_k]$ give a sequence $w(n_k) \subset \mathcal{J}$. Infinitely many elements of this sequence belong to some $\mathcal{J}_p$. This sequence has a convergent subsequence with limit, say, $j_0 \in \mathcal{J}_p$. Wlog, denote the sequence again by $(w(n_k))$ and the corresponding sequence of words by $(w[n_k])$. Now, out of the sequence $(w[n_k])$ pick a subsequence (call it again $(w[n_k])$) such that $S_{w[n_k-1]}(x_k) \to y$, for some $y \in \mathcal{A}_w$ (convergence on a subsequence is ensured by Lemma 18). Now, for every $\epsilon > 0$ there exists k (large) such that for $z = S_{j_0}(y)$ we have

$$\begin{aligned} d(x, z) \leq d\left(x, S_{w(n_k)} S_{w[n_k-1]}(x_k)\right) + \\ d\left(S_{w(n_k)} S_{w[n_k-1]}(x_k), S_{w(n_k)}(y)\right) + d\left(S_{w(n_k)}(y), S_{j_0}(y)\right) < \epsilon \end{aligned} \tag{3.9}$$

Obviously, first two distances can be made smaller then $\frac{\epsilon}{3}$ for large k. The third distance can be made smaller then $\frac{\epsilon}{3}$ because of assumption 3. Since for every $x \in \mathcal{A}_w$ and every $\epsilon > 0$ we can find $z \in \bigcup_{j \in \mathcal{J}} S_j(\mathcal{A}_w)$ such that $dist(x, z) < \epsilon$, it follows that $x \in \overline{\bigcup_{j \in \mathcal{J}} S_j(\mathcal{A}_w)} = \bar{F}(\mathcal{A}_w)$. The lemma is proved. $\square$

Lemma 22. $\mathcal{A}_w \subset \mathcal{A}_{\sigma(w)}$.

Proof. Again, if $x \in \mathcal{A}_w$, then $x = \lim S_{w[n_k]}(x_k)$. Clearly,

$$S_{w[n_k]}(x_k) = S_{\sigma(w)[n_k-1]}\left(S_{w(0)}(x_k)\right) .$$

The sequence $(S_{w(0)}(x_k))$ is bounded and $\sigma(w)[n_k - 1] \to \sigma(w)$. Lemma is proved. $\square$

Corollary 23. *If the string w is periodic, then $\mathcal{A}_w = \mathcal{A}_{\sigma(w)}$.*

The union of individual attractors $\mathcal{A}_w$ lies inside of K,

$$\bigcup_{w \in \Sigma} \mathcal{A}_w \subseteq K . \tag{3.10}$$

There are many important cases when this union equals K.

Lemma 24. *We have $\bigcup\limits_{w \in \Sigma} \mathcal{A}_w = K$ in each of the following cases:*

a) The operators $\{S_j\}$ are strict contractions with the contraction factors $0 < \gamma_j \leq \gamma < 1$.

b) The operators $\{S_j\}$ are eventually strict contractions, i.e., there exist a $0 < \gamma < 1$ and an integer $M \geq 1$ such that for any finite word w^* of length $\geq M$ the operator S_{w^*} is a contraction with the factor γ.

c) $S_j^{-1}(K) \supseteq K$ for $j \in \mathcal{J}$.

d) Each operator S_j is invertible on K.

Proof. The inclusion (3.10) is obvious. To prove the equality in the special cases a), b), and c) pick an $x \in K$. There exists a sequence of points $\{x_k\} \subset K$, and a sequence w_{n_k} of lengths n_k increasing to infinity such that $x = \lim_{k\to\infty} S_{w_{n_k}}(x_k)$. We claim that $x \in \mathcal{A}_u$, where $u = w_{n_1}.w_{n_2}\dots w_{n_k}\dots$. Denote $u[m_k] = w_{n_1}.w_{n_2}\dots w_{n_k}$. The lengths of the words $u[m_k]$ go to infinity.

In the cases *a)* and *b)*, for every k and any $y \in K$ we have

$$\begin{aligned} &d(S_{w_{n_k}}(x_k), S_{u[m_k]}(y)) = d(S_{w_{n_k}}(x_k), S_{w_{n_k}} S_{u[m_{k-1}]}(y)) \\ &= d(S_{w_{n_k}}(x_k), S_{w_{n_k}}(z_k)) \end{aligned}$$

where $z_k = S_{u[m_{k-1}]}(y)$. Then, in the case $a)$, $d(S_{w_{n_k}}(x_k), S_{w_{n_k}}(z_k)) \leq \gamma^{n_k} d(x_k, z_k) \leq \gamma^{n_k} \operatorname{diam}(K)$, and in the case $b)$, $d(S_{w_{n_k}}(x_k), S_{w_{n_k}}(z_k)) \leq \gamma^{l_k}\, d(x_k, z_k) \leq \gamma^{l_k} \operatorname{diam}(K)$, where l_k is the round down of n_k/M. Therefore, $d(S_{w_{n_k}}(x_k), S_{u[m_k]}(y)) \to 0$, as $k \to \infty$. Since, $\lim_{k\to\infty} S_{u[m_k]}(y) \in \mathcal{A}_u$, and $\lim_{k\to\infty} S_{u[m_k]}(y) = \lim_{k\to\infty} S_{w_{n_k}}(x_k) = x$, it follows that $x \in \mathcal{A}_u$ and the inclusion $K \subset \bigcup_{w\in\Sigma} \mathcal{A}_w$ is proved.

In the third case, since $S_j^{-1}(K) \supseteq K$, for every $y \in K$ there exist $z_j \in K$ with $y = S_j(z_j)$, $j \in \mathcal{J}$. Therefore, for every k, we can find $y_k \in K$ such that $S_{u[m_{k-1}]}(y_k) = x_k$. Then, $S_{u[m_k]}(y_k) = S_{w_{n_k}} S_{u[m_{k-1}]}(y_k) = S_{w_{n_k}}(x_k)$. It follows that $x \in \mathcal{A}_u$.

Finally, *d)* is a special case of *c)*. This concludes the proof. $\square$

Remark 25. *The case d) may seem too restrictive. However, there are many situations where the operators S_j are not invertible on X but are invertible on the attractor K. This was first observed by Ladyzhenskaya in the case of Navier-Stokes equations, [24]. The fact is due to the invariance of K and, what is called, backward uniqueness property of certain parabolic-like equations.*

Although K equals the union of individual attractors in many cases, there are situations when K is strictly larger than that union. This is what we call a Gestalt effect. This is a new phenomenon. As we have shown in Lemma 24, the Gestalt effect cannot occur when operators S_j are contractions.

Example of a Gestalt effect.

In this example the state space X will be the space Σ_2 of one-sided infinite strings of 0's and 1's. There will be two operators, S_0 and S_1, defined as follows:

$$S_0(v) = v(2).v\,, \quad S_1(v) = v(1).v$$

for all $v = v(0)v(1)v(2)v(3)\cdots \in X$. The conditions of Theorem 16 are satisfied, so let $\mathfrak{M} = K \times \Sigma_2$ be the global compact attractor of the corresponding dynamics with choice. [Note that the global compact attractor of the system generated by S_0 is the set of all strings with period 3, and the attractor of the system generated by S_1 is the set of all strings with period 2.]

We claim that the sequence $u = 000\overline{100}$ is in K but not in $\mathcal{A}_w$ for any $w \in \Sigma_2$. Let $v = 001.\sigma^3(v)$, i.e., the first three symbols of v are 001, and let $w_k = 000...0001$ with $3k$ zeros before 1. Then, for every k, $S_{w_k}(v) = 0.001001...001$ with 001 repeating k times. Therefore, $S_{w_k}(v) \to u$ as $k \to \infty$, i.e., $u \in K$. To show that u does not belong to the union $\bigcup_{w\in\Sigma} \mathcal{A}_w$, we argue by contradiction. If $u \in \mathcal{A}_s$, then there exists a sequence $v_k \in \Sigma_2$ such that $\lim_{k\to\infty} S_{s[n_k]}(v_k) = u$, where $n_k \nearrow \infty$. Therefore, we can find l, such that $S_{s[n_l]}(v_l)$, $S_{s[n_{l+1}]}(v_{l+1})$, $\ldots$, $S_{s[n_{l+8}]}(v_{l+8})$, all begin with 000100100.... Since $S_{s[n_{l+1}]}(v_{l+1}) = S_{s(n_{l+1})}...S_{s[n_l]}(v_{l+1}) = 0001001001...$, and the action of operators S_0 and S_1 depends only on the first three symbols in the strings, it follows that $v_l[3] \neq v_{l+1}[3]$, because if $v_l[3] = v_{l+1}[3]$, then $S_{s[n_{l+1}]}(v_{l+1})$ starts with at least 4 zeros, i.e., 0000100100..., which is impossible. Similarly, $v_{l+k}[3] \neq v_{l+j}[3]$ for $j, k = 0, \ldots, 8$, $j \neq k$. But there can be only 8 different three-letter words in 2 symbols. A contradiction. Hence, u does not belong to the $\bigcup_{w\in\Sigma} \mathcal{A}_w$. $\square$

Chapter 4

DYNAMICS WITH RESTRICTED CHOICE

An interesting class of systems arises when the choice of the parameters $j \in \mathcal{J}$ at every time step is not arbitrary but is restricted by some rules. For example, consider an oriented, finite or infinite, connected graph such that each vertex has an outgoing edge. Label every edge by a symbol from $\mathcal{J}$ and consider all infinite paths in the graph. The infinite strings of symbols corresponding to the infinite paths form a shift invariant subset of $\Sigma_{\mathcal{J}}$ – the set of allowed (admissible) plans. The operators S_j acting on the states in the order allowed by those plans generate a graph directed dynamics on X, see, e.g., [31] for examples of such systems. More generally, let Λ be a closed, shift invariant subset of $\Sigma_{\mathcal{J}}$. We associate with Λ a discrete time dynamics on the space $\mathfrak{X}_\Lambda = X \times \Lambda$ generated by the iterations of the map $\mathfrak{S}$ defined in (3.5). This is what we mean by *dynamics with restricted choice.*

Theorem 26. *With Assumptions 1,2, and 3, the discrete semidynamical system* $(\mathfrak{X}_\Lambda, \mathfrak{S})$ *possesses a global compact attractor* $\mathfrak{M}_\Lambda$*.*

(1) The attractor $\mathfrak{M}_\Lambda$ *is the maximal invariant compact subset of* $\mathfrak{X}_\Lambda$ *such that* $\mathfrak{S}(\mathfrak{M}_\Lambda) = \mathfrak{M}_\Lambda$*. Clearly,* $\mathfrak{M}_\Lambda$ *is a subset of the global compact attractor* $\mathfrak{M}$ *of*

the full system $(\mathfrak{X}, \mathfrak{S})$*.*

(2) Through every point $(x(0), w)$ *passes a complete trajectory, i.e., there exists a two-sided infinite sequence of points* $..., x(-1), x(0), x(1), ...$ *and a two-sided infinite sequence* $..., w(-1), w(0), w(1), ...$ *(extending* w *in* Λ*) such that* $S_{w(n)}(x(n)) = x(n+1)$ *for all integers* n*.*

(3) Let K_Λ *denote the projection of the attractor* $\mathfrak{M}_\Lambda$ *onto the* X *component. The set* K_Λ *is a compact subset of the set* K *of Theorem 2.3. There exist compact sets* A_j*,* $j \in \mathcal{J}$*, such that* $K_\Lambda = \bigcup\limits_{j \in \mathcal{J}} A_j$ *and* $K_\Lambda = \bigcup\limits_{j \in \mathcal{J}} A_j = \bigcup\limits_{j \in \mathcal{J}} S_j(A_j)$*.*

(4) If $\Lambda = \Sigma_{\mathcal{J}'}$*, where* $\mathcal{J}'$ *is a closed subset of* $\mathcal{J}$*, then* $\mathfrak{M}_\Lambda = K_\Lambda \times \Lambda$*. In general,* $\mathfrak{M}_\Lambda$ *is not a product, the slices of* $\mathfrak{M}_\Lambda$ *corresponding to different* $w \in \Lambda$ *may be different.*

Proof. The existence of the global compact attractor, $\mathfrak{M}_\Lambda$, follows from the abstract result, Theorem 2. The assertions 1 and 2 of Theorem 26 are among the general properties of global compact attractors, see Theorem 3. Denote by K_Λ the projection of $\mathfrak{M}_\Lambda$ onto the X component. Clearly, K_Λ is compact. Also, K_Λ is a subset of the slice K corresponding to the full shift Σ, as in Theorem 17. Because of the invariance property of $\mathfrak{M}_\Lambda$, for every point $y \in K_\Lambda$ there is a j, one of the symbols of $\mathcal{J}$, and a point $x \in K_\Lambda$ such that $y = S_j(x)$. Define the sets $A_j = \{x \in K_\Lambda \,:\, S_j(x) \in K_\Lambda\}$. It is easy to see that each A_j is compact and $K_\Lambda = \bigcup_{j \in \mathcal{J}} A_j$. By construction, we have $\bigcup_{j \in \mathcal{J}} A_j = \bigcup_{j \in \mathcal{J}} S_j(A_j)$. This proves assertion 3. When $\mathcal{J}' \subset \mathcal{J}$ and $\Lambda = \Sigma_{\mathcal{J}'}$, then dynamics with restricted choice is actually full dynamics with choice, and therefore $\mathfrak{M}_\Lambda$ is a product set. Below we give a couple of different scenarios when $\mathfrak{M}_\Lambda$ is not a product set. $\square$

To analyze the slices $K_\Lambda(s) = \{x \in X : (x,s) \in \mathfrak{M}_\Lambda\}$, we follow the argument of the corresponding part of section 3.2.1.

Every point $(x,s) \in \mathfrak{M}_\Lambda$ is the limit of the form

$$(x,s) = \lim_{n_k \to \infty} \left(S_{w_k}(x_{n_k}), \sigma^{n_k}(s_{n_k})\right) ,$$

where (x_n) is a bounded sequence in X, (s_n) is a bounded sequence in Λ, and w_k is the prefix of s_{n_k}, $s_{n_k} = w_k.\sigma^{n_k}(s_{n_k})$. Because $\mathfrak{M}_\Lambda$ is invariant under $\mathfrak{S}$ and we know that the unrestricted dynamics has the global compact attractor $\mathfrak{M} = K \times \Sigma$, the sequence (x_n) can be taken from the compact K, and we may assume that $x_{n_k} \to x_* \in K$. Also, we may assume that the words w_k converge (to some infinite string $w_* \in \Lambda$). The strings $\sigma^{n_k}(s_{n_k})$ converge to s. Consider all strings $u \in \Lambda$ such that $w_k.u$ is a string in Λ for infinitely many k. For every such u we will have $x \in M_\Lambda(u)$.

We see that the number of different slices of the attractor $\mathfrak{M}_\Lambda$ that may depend on the sequence x_{n_k}, but more importantly, it depends on what strings can be attached to convergent sequences of finite words in Λ.

With every sequence (w_k) of finite words in Λ we associate the set $\mathfrak{s}((w_k))$ of one-sided infinite strings $u \in \Lambda$ such that $w_{k_\ell}.u \in \Lambda$ for some subsequence w_{k_ℓ}. In order to show that $\mathfrak{M}_\Lambda$ is not necessarily a product set, we will first show that, if Λ is a sofic shift, the number of different sets among all $\mathfrak{s}((w_k))$ is finite. The argument will be similar to the proof of Theorem 3.2.10 in [29].

Recall that Λ is a sofic shift if it has a presentation by a finite labeled graph, see [29]. This means that there is a directed graph, $G = (V, E)$, with a finite number of vertices, V, and edges, E; the edges are labeled by the symbols $0, 1, \ldots, N-1$; from every vertex begins at least one infinite directed path; the labels of the edges in the infinite directed paths form infinite one-sided strings that exhaust exactly all strings in Λ.

Lemma 27. *If Λ is a one-sided sofic subshift of Σ, then the number of different sets among all $\mathfrak{s}((w_k))$ is finite.*

Proof. Let $G = (V, E)$ be a labeled graph presenting Λ. Let (w_k) be a sequence of finite words allowed in Λ. For each word w_k pick a finite directed path in G presenting it. We can find a subsequence, (w_{k_ℓ}), such that all the words w_{k_ℓ} have the same terminal vertex in their presentation. If T is such vertex, then $w_{k_\ell}.u \in \Lambda$ for all infinite paths u starting at T. Because the number of vertices is finite, we are done. $\square$

Lemma 28. *If Λ is a one-sided sofic subshift of Σ, then the number of different slices is finite, and it is at most the number of vertices in a finite labeled graph that represents the sofic shift.*

Proof. Claim: If $u, v \in \Lambda$ and $u(0) = v(0)$, i.e., they have the same starting vertex, then $K_\Lambda(u) = K_\Lambda(v)$. Since both u and v are arbitrary chosen with $u(0) = v(0)$, it is enough to show inclusion in one direction only, e.g., $K_\Lambda(u) \subset K_\Lambda(v)$. Pick $x \in K_\Lambda(u)$, then there exists a bounded sequence $\{x_k, u_k\}$, and a sequence $n_k \nearrow \infty$ such that $S_{u_k[n_k]}(x_k) \to x$ and $\sigma^{n_k}(u_k) \to u$ as $k \to \infty$. Pick a subsequence (calling it again $\sigma^{n_k}(u_k)$), such that $\sigma^{n_k}(u_k)$ start with $u(0)$ for all k. Denote by $v_k = u_k[n_k].v \in \Lambda$, then

$$x = \lim_{k\to\infty} S_{u_k}[n_k](x_k) \ \text{and} \ \sigma^{n_k}(v_k) \to v$$

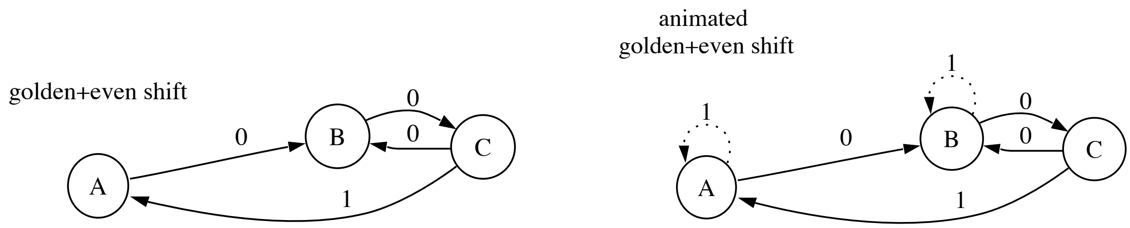

Figure 4.1: The golden+even shift and its animation

and therefore $x \in K_\Lambda(v)$. Since u and v were arbitrary, it follows that we can only get different slices from paths that start from different vertices. This proves the lemma. $\square$

Remark 29. *Even if the number of different sets among all* $\mathfrak{s}((w_k))$ *is* > 1, *the attractor* $\mathfrak{M}_\Lambda$ *may be a product,* $\mathfrak{M}_\Lambda = K_\Lambda \times \Lambda$, *with the same slice for every string in* Λ.

Indeed, let $N = 2$ and let Λ consist of the periodic string $u = 100100\ldots$ and its shifts $\sigma(u) = 00100\ldots$ and $\sigma^2(u) = 0100\ldots$. If (w_k) consists of words ending in 00, then the only string that can be attached to w_k is u. If (w_k) consists of words ending in 1, then the only string is $\sigma(u)$, and for words ending in 10 the only string is $\sigma^2(u)$. Thus, we have three different sets of the form $\mathfrak{s}((w_k))$. At the same time, the individual attractors $\mathcal{A}_u$, $\mathcal{A}_{\sigma(u)}$, and $\mathcal{A}_{\sigma^2(u)}$, are all equal, as we argue in Corollary 23.

One may ask whether $\mathfrak{M}_\Lambda$ is always a product. The answer is no, as the following example shows. Let Λ be the intersection of the one-sided golden mean shift with the even shift. In other words, Λ consists of all sequences of 0s and 1s such that between any two 1s there are two or a larger even number of 0s. A graph presenting Λ is given on Figure 4.1. We will animate this graph to define the dynamics. First, identify the nodes with three distinct points A, B, and C in $\mathbb{R}^2$, see Figure 4.1 left, and define

$X = \{A, B, C\}$. Second, define the maps S_0 and S_1 acting on points as shown by the directed edges labeled correspondingly; for example, $S_0(A) = B$, $S_1(A) = A$, and $S_0(C) = B$.

Now consider the set Λ^+ of non-empty finite words (blocks) of Λ. We divide Λ^+ in three classes and correspondingly divide the strings in Λ into three classes. The first class of words in Λ^+ consists of the words ending in 1. Such words can serve as prefixes of strings starting with an even (or infinite) number of 0s. Denote these classes by Λ^+_A and ${}_A\Lambda$. The second class of finite words consists of the words ending in odd number of 0s. The strings for which such words can serve as prefixes are the strings starting with an odd number of 0s. These classes are denoted by Λ^+_B and ${}_B\Lambda$. The last class in Λ^+ consists of words ending in even number of 0s. The corresponding strings are those starting with 1 or with an even number of 0s. These are denoted by Λ^+_C and ${}_C\Lambda$. By looking at the picture of the animated shift, it is easy to identify the possible limits of sequences $S_{w_k}(x_k)$ when w_k belong to a particular class, while $x_k \in \{A, B, C\}$. We see that if $w_k \in \Lambda^+_A$, then the limit set is $\{A, B\}$. If $w_k \in \Lambda^+_B$, then the limit set is $\{B, C\}$. Finally, if $w_k \in \Lambda^+_C$, then the limit set is again $\{B, C\}$. Thus, there are two different slices in the attractor $\mathfrak{M}_\Lambda$. One slice is $\{A, B\}$, and the other is $\{B, C\}$. We have $K_\Lambda(u) = \{A, B\}$ if $u \in {}_A\Lambda$, and $K_\Lambda(u) = \{B, C\}$ if $u \in {}_B\Lambda \cup {}_C\Lambda$. The global attractor $\mathfrak{M}_\Lambda$ is a union of the sets $\{A, B\} \times_A \Lambda$, $\{B, C\} \times_B \Lambda$, and $\{B, C\} \times_C \Lambda$.

The above example shows that the number of slices can be strictly less then the number of vertices in the finite graph. Now, we will give an example when this two numbers are equal.

Again, let $X = \{A, B, C\}$, and let Λ be given by the finite graph in figure 4.2 (left). As in the example above, let Λ^+ be the set of non-empty finite words (blocks) of Λ. There are three classes of words which end in 1, 10 and 100. The words ending

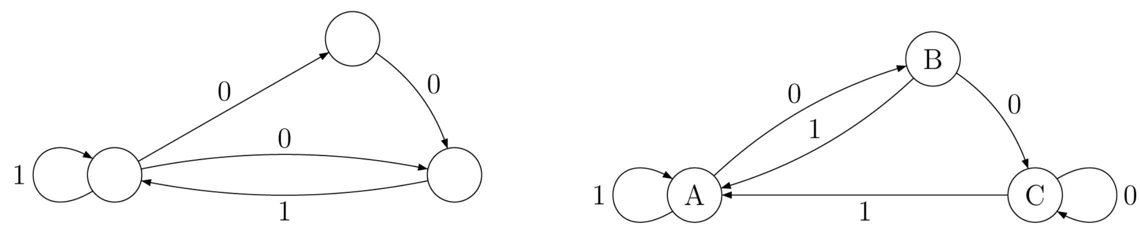

Figure 4.2: The graph representing Λ(left) and X with action of operators S_0 and S_1 (right)

in 1 are the prefixes to words starting with 01 or 001. The words ending in 10 serve as prefixes to words starting with 1 or 01, and the words ending in 100 are prefixes to words starting with 1. From figure 4.2 (right), it is not hard to see that if the sequence of words ends in 1, the limits of sequences $S_{w_{n_k}}(x_k)$ converge to A, for all bounded subsets $\{x_k\} \subset X$. Now, if the sequence of words ends in 10, then the limits of $S_{w_{n_k}}(x_k)$ will end up in B. Finally, if the sequence of words ends in 100, the limits of $S_{w_{n_k}}(x_k)$ will end up in C. Denote by ${}_A\Lambda$ the words starting with 001, ${}_B\Lambda$ the words starting with 01, ${}_C\Lambda$ the words starting with 1. Then we have three different slices in the attractor $\mathfrak{M}_\Lambda$, which are

$$\begin{aligned} &\{A\} \times_A \Lambda \\ &\{A, B\} \times_B \Lambda \\ &\{B, C\} \times_C \Lambda. \end{aligned} \tag{4.1}$$

Another example of different slices appears in numerical results reported in the next chapter.

The subshifts over a finite alphabet have been studied extensively, see [29, 20] and references therein. The subshifts over an infinite, possibly uncountable, alphabet have been studied much, much less.

Chapter 5

PULLBACK AND FORWARD ATTRACTORS

In this chapter we consider dynamics with choice as a nonautonomous semi-dynamical system in the setting of section 2.4. Let X and Σ be as before. The space Σ with the shift σ will be the parameter space. Now, define $\phi : \mathbb{Z}_{\geq 0} \times X \times \Sigma \to X$ to be the map $\phi(n, x, w) = S_{w[n]}(x)$ and $\phi(0, x, w) = x$. It is clear that ϕ is a cocycle. In the following theorem we prove lemmas 8 and 10 for dynamics with choice, i.e., for the nonautonomous system $(X, \phi, (\Sigma, \mathbb{Z}_{\geq 0}, \sigma))$. In fact, we prove a more general result by allowing any closed, shift invariant subset as the parameter space. This setup corresponds to dynamics with restricted choice presented in the previous chapter. Let $\Lambda \subseteq \Sigma$ be a subshift. Consider the nonautonomous semi-dynamical system $(X, \phi, (\Lambda, \mathbb{Z}_{\geq 0}, \sigma))$. Under the assumptions of theorem 26 the restricted dynamics with choice has a global compact attractor, $\mathfrak{M}_\Lambda \subset X \times \Lambda$. As we know, in general, $\mathfrak{M}_\Lambda$ is not a product set. For every $w \in \Lambda$ denote by $K_\Lambda(w)$ the set

$$K_\Lambda(w) = \{x \in X \mid (x, w) \in \mathfrak{M}_\Lambda\}.$$

Since $\mathfrak{M}_\Lambda$ is compact, each $K_\Lambda(w) \subset X$ is compact as well.

Theorem 30. $\bigcup_{w\in\Lambda} K_\Lambda(w)$ *is the uniform pullback and the uniform forward attractor of the nonautonomous dynamical system* $(X, \phi, (\Lambda, \mathbb{Z}_{\geq 0}, \sigma))$ *with respect to* $\bar{\mathcal{B}}(X)$ *(closed and bounded subsets of* X*).*

Proof. First, let us prove the property 1) in definitions 7 and 9, i.e.,

$$\overline{\bigcup_{u\colon \sigma(u)=w} S_{u(0)}(K_\Lambda(u))} = K_\Lambda(w). \tag{5.1}$$

Here, $u(0)$ represents the symbols which are extensions to the left of w, i.e., $u = u(0).w \in \Lambda$. Notice that only when $\Lambda = \Sigma$ the union of these symbols is the whole alphabet $\mathcal{J}$.

Let $y \in S_{u(0)}(K_\Lambda(u))$. There exists a bounded sequence $\{x_k\} \in X$, and a sequence $\{s_k\} \in \Lambda$, together with a sequence $n_k \nearrow \infty$, such that $y = S_{u(0)}(\lim_{k\to\infty} S_{u_{n_k}}(x_k))$, where $s_k = u_{n_k}.\sigma^{n_k}(s_k)$, and $\sigma^{n_k}(s_k) \to u$. Since, $\sigma^{n_k+1}(s_k) \to \sigma(u) = w$ and

$$y = S_{u(0)}(\lim_{k\to\infty} S_{u_{n_k}}(x_k)) = \lim_{k\to\infty} S_{u(0)} S_{u_{n_k}}(x_k)$$

we get that $y \in K_\Lambda(w)$. Therefore, we have that $\bigcup_{u\colon \sigma(u)=w} S_{u(0)}(K_\Lambda(u)) \subset K_\Lambda(w)$. Since $K_\Lambda(w)$ is compact, we have $\overline{\bigcup_{u\colon \sigma(u)=w} S_{u(0)}(K_\Lambda(u))} \subset K_\Lambda(w)$. Now, pick $y \in K_\Lambda(w)$. There exists a bounded sequence $\{x_k\}$, and a sequence $\{s_k\}$, such that $y = \lim_{k\to\infty} S_{s_k[n_k]}(x_k)$, and $\sigma^{n_k}(s_k) \to w$. The last symbols of the words $s_k[n_k]$, the symbols $j_k = s_k(n_k - 1)$, have a subsequence j_{k_l}, which lies in some $\mathcal{J}_p$ and converges to some element of $\mathcal{J}_p$, say j_0. Then, for every $\epsilon > 0$, there exist l (large) such that if $x = \lim S_{s_k[n_k-1]}(x_k)$ and $z = S_{j_0}(x)$, we have

$$
\begin{aligned}
&d(z,y) \leq d(S_{j_0}(x), S_{j_0}(S_{s_{k_l}[n_{k_l}-1]}(x_{k_l})))+ \\
&d(S_{j_0}(S_{s_{k_l}[n_{k_l}-1]}(x_{k_l})), S_{j_{k_l}}(S_{s_{k_l}[n_{k_l}-1]}(x_{k_l}))) + d(S_{j_{k_l}}(S_{s_{k_l}[n_{k_l}-1]}(x_{k_l})), y)) < \epsilon
\end{aligned}
\tag{5.2}
$$

It is easy to see that we can pick l large enough to ensure that all three distances above are less then $\frac{\epsilon}{3}$. Making the second distance less then $\frac{\epsilon}{3}$ is possible because of the assumption 3. Note that $x \in K_\Lambda(j_0.w)$ and $z \in S_{j_0}(K_\Lambda(j_0.w))$ by construction. Since for every $\epsilon > 0$, $d(y, \bigcup_{u\colon\sigma(u)=w} S_{u(0)}(K_\Lambda(u))) < \epsilon$, it follows that $y \in \overline{\bigcup_{u\colon\sigma(u)=w} S_{u(0)}(K_\Lambda(u))}$. The proof of (5.1) is complete.

In order to show that $\bigcup_{w\in\Lambda} K_\Lambda(w)$ is a uniform pullback attractor we need to show that

$$\lim_{n\to\infty} \sup_{w\in\Lambda} \vec{dist}(\bigcup_{u\colon\sigma^n(u)=w} \phi(n,B,u), K_\Lambda(w)) = 0.$$

for any $B \subset \bar{\mathcal{B}}(X)$. Let $(B\times\{u\}) \subset X \times \Lambda$, and consider the iterations $\mathfrak{S}^n(B\times\{u\})$. For every n, we have

$$\mathfrak{S}^n(B\times\{u\}) = (S_{u[n]}(B), \sigma^n(u)) = (\phi(n,B,u), w).$$

Because $\mathfrak{M}_\Lambda$ attracts all bounded sets, $\vec{dist}(\mathfrak{S}^n(B\times\{u\}), \mathfrak{M}_\Lambda) \to 0$. On the other hand, since $\sigma^n(u) = w$ for all n, we obtain

$$\vec{dist}(\phi(n,B,u), K_\Lambda(w)) \to 0.$$

Similarly, we have

$$\mathfrak{S}^n(B \times w) = (S_{w[n]}(B), \sigma^n(w)) = (\phi(n, B, w), \sigma^n(w))$$

and $d\vec{ist}(\mathfrak{S}^n(B \times w), \mathfrak{M}_\Lambda) \to 0$. Therefore,

$$\lim_{n\to\infty} \sup_{w\in\Lambda} d\vec{ist}(\phi(n, B, w), K_\Lambda(\sigma^n(w))) = 0.$$

This shows that $\bigcup_{w\in\Lambda} K_\Lambda(w)$ is also a uniform forward attractor. The theorem is proved. $\square$

Chapter 6

NUMERICS IN DYNAMICS WITH CHOICE

Finding attractors numerically is a difficult task. We will consider only the case when X is a bounded region in $\mathbb{R}^2$. The assumption of boundedness is natural because the attractor lies inside the bounded absorbing set. Moreover, we will assume that X is a rectangle and the maps S_j map X into itself. In the case of a hyperbolic IFS, to find its attractor, it is sufficient to look at tails of the trajectory of any one point. This makes it easy to plot the fractals. In dynamics with choice when the maps S_j are not contractions, this approach does not work, even for IFSs. In what follows, we describe a special algorithm designed to plot the attractor in dynamics with choice.

The idea is to divide X into small squares and replace the action of the maps S_j on X by their action on the squares. There are many ways of defining this new discrete action. We choose the following. Assign a number to each square for example as shown on figure 6.1. Denote by $\square_a$ the square number a and denote by c_a its center. Let the number of squares be M. For a map $S : X \to X$, we define its discrete version $\tilde{S}$ as a map from the ordered set $[1, 2, ..., M]$ into itself as follows: $\tilde{S}(a) = b$ if $S(c_a) \in \square_b$.

9	10	11	12
5	6	7	8
1	2	3	4

Figure 6.1: Numbering X

Suppose we have N maps $S_0, ..., S_{N-1}$. We discretize each map and define an $M \times M$ matrix A as follows: $A(i,j) = 1$ if $\tilde{S}_k(i) = j$ for at least one $k \in \{0, 1, ..., N-1\}$, and $A(i,j) = 0$ otherwise. The matrix A contains all the information about the discretized IFS. To find the attractor of this IFS, consider the iterations of A^n acting on the column M-vector $\mathbf{v} = [1, 1, ..., 1]^T$. After a finite number of iterations the zero components of the vector $A^n\mathbf{v}$ will stabilize. The nonzero components will give the location of the squares that make up the attractor.

$\square_a$	$\tilde{S}_0(\square_a)$	$\tilde{S}_1(\square_a)$
1	1	1
2	3	6
3	7	7
4	8	8
5	6	5
6	7	5
7	7	6
8	7	7
9	10	5
10	6	9
11	7	10
12	11	8

Table 6.1: Action of the operators S_0 and S_1

For example, let X and its division be as in figure 6.1. Let S_0 and S_1 be two operators acting from X to itself. Denote by $\tilde{S}_0$ and $\tilde{S}_1$ the discretizations of S_0

1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	1	0	0	1	0	0	0
0	1	0	0	1	0	1	0	0	1	0	0
0	0	1	0	0	1	1	1	0	0	1	0
0	0	0	1	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	1	0	1	0
0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	0	0	0	0	0	0	0

Table 6.2: Matrix A

and S_1, and let their action be as in table 6.1. Then, the matrix A is given in table 6.2. After only three iterations, the zero components of the vector $A^n\mathbf{v}$ stabilize and $A^3\mathbf{v} = [1,0,0,0,20,22,22,0,2,2,0,0]^T$. The attractor in dynamics with choice for this system is given in figure 6.2.

This is just one step of the procedure. In order to get a better approximation to the real attractor we have to refine the partition and repeat the procedure. We stop when we see that the pictures of the attractor stop changing. The final size of the squares in the partition of X is dictated by the convergence considerations. In applications, the numbers of squares was of the order 10^6. In the case when we have finitely many maps, preferably not too many maps, the matrix A is sparse, and in

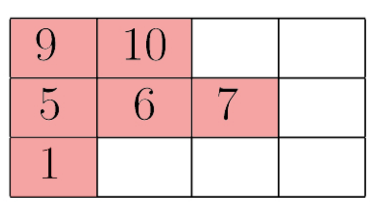

Figure 6.2: Attractor in dynamics with Choice

most cases, we are able to work with it. However, when there are large number of maps, possibly infinite, the number of nonzero components in the matrix A increases significantly. The memory required for the matrix is huge, and most likely will cause ”out of the memory” error.

Here, I am not giving justification why the picture is close to the real attractor because this is beyond the scope of this work and I plan to address it later.

6.1 Example

The simplest mathematical model of malaria transmission goes back to Ross and Macdonald. The state of the human-mosquito interaction system is described by the portion of infected humans, x, and the portion of infected mosquitoes, y. The change in time is described by the following simple system of ordinary differential equations:

$$\begin{aligned} \dot{x} &= a\,y\,(1-x) - \,r\,x \\ \dot{y} &= b\,x\,(1-y) - m\,y \end{aligned} \tag{6.1}$$

The nature of the positive coefficients a, b, r, and m is discussed in [36]. In particular, the coefficients a and b are proportional to the biting rate and the transmission efficiencies (infected human to mosquito and infected mosquito to human), r is the recovery rate (in humans), and $1/m$ is the average mosquito life-span. In practice, it is hard to measure these parameters. Also, there are many factors that affect their values, see [36], page 8, and the values may change in time.

The state space for the model (6.1) is the closed square $X = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$. For initial conditions in X the solution stays in X for all t. If the quantity $R_0 = \frac{ab}{rm}$ is < 1, all trajectories starting in X converge to the origin, and the global compact attractor consists of a single point, $P_1 = (0,0)$. If $R_0 > 1$, the equilibrium P_1 becomes unstable and there emerges the second fixed point, $P_2 = (x_*, y_*)$, inside the square X,

$$x_* = \frac{ab - rm}{b\,(a+r)}\,, \quad y_* = \frac{ab - rm}{a\,(b+m)}\,. \tag{6.2}$$

This second equilibrium is stable, and the global compact attractor of the system consists of the two equilibria, P_1 and P_2, and of the heteroclinic trajectory connecting them (and staying entirely inside X). The number R_0, known as the basic reproductive number, detects the emergence of epidemics: when $R_0 > 1$ there is a stable portion of infected population.

We consider a discrete version of equations (6.1):

$$\begin{aligned} x(t+\Delta t) &= x(t) + \Delta t\ (a\,y(t)\,(1-x(t)) - \ r\,x(t)) \\ y(t+\Delta t) &= y(t) + \Delta t\ (b\,x(t)\,(1-y(t)) - m\,y(t))\ . \end{aligned} \tag{6.3}$$

The time step map $(x(t), y(t)) \mapsto (x(t+\Delta t), y(t+\Delta t))$ maps X into itself provided

$$\Delta t\ <\ \min\{\frac{1}{a+r}\,,\ \frac{1}{b+m}\}\,. \tag{6.4}$$

The fixed points for (6.3) are the same as for (6.1). As in the continuous case, if $ab > rm$ and the time step satisfies (6.4), the global attractor for (6.3) consists of the two fixed points, P_1 and P_2, and the heteroclinic trajectory connecting them.

We choose two sets of parameters, $\mathrm{pset}_0 = \{a = 4,\ b = 6,\ r = 1,\ m = 2\}$ and $\mathrm{pset}_1 = \{a = 2,\ b = 10,\ r = 3,\ m = 2\}$, and denote the corresponding time step maps by S_0 and S_1. These sets of parameters are not related to any real-life situation

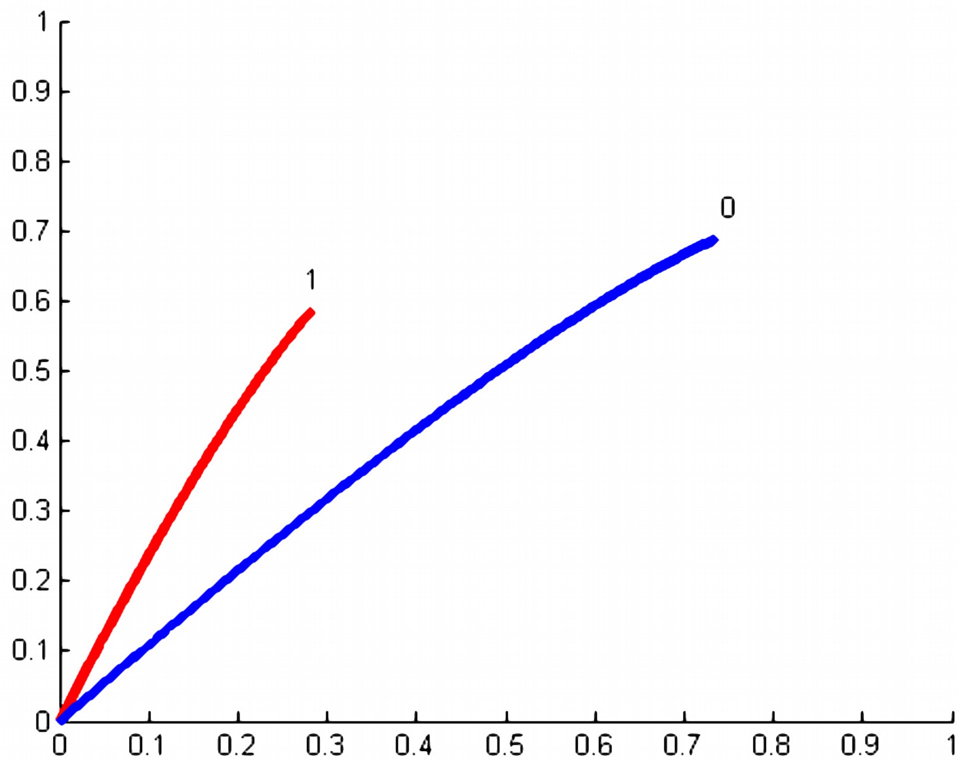

Figure 6.3: Attractors for $(X, d,\ S_0)$ (right) and $(X, d,\ S_1)$ (left).

but rather chosen to better visualize the attractors. The fixed point P_2 for pset_0 is (11/15, 11/16) and for pset_1 it is (7/25, 7/12). Figures 6.3 through 6.11 show the results of numerical computation. The results depend on the size of the time step Δt. On figures 6.3, 6.4, and 6.5, the left line (the heteroclinic trajectory) is the (global compact) attractor for the discrete system (X, S_1), and the right line is the attractor of (X, S_0). On figures 6.4 and 6.5, the two lines between the attractors of the discrete systems (X, S_0) and (X, S_1) form the individual attractor $\mathcal{A}_w$ corresponding to the periodic string $w = 1010...$ (on figure 6.5 the two lines are very close). For our example of dynamics with choice, Σ is the space of one-sided infinite strings of symbols 0 and

1. According to Theorem 17, the global compact attractor for $(\mathfrak{X}, \Sigma)$ has one slice, i.e., $\mathfrak{M} = K \times \Sigma$. The set K for $\Delta t = .05$ and for $\Delta t = .005$ are depicted on figures 6.6 and 6.7, respectively.

One may wonder if there is a Gestalt effect in this example. The answer is no. The reason for this is that the maps S_0 and S_1 are invertible on X and therefore on the attractor slice K (see Lemma 24 (iii)). In fact, for any set of positive parameters $\mathrm{pset}_j = \{a,\, b,\, r,\, m\}$ satisfying the inequalities $ab > rm$ and (6.4), the corresponding operator S_j is invertible on X. To prove this, we need to show that the map $(x(t), y(t)) \to (x(t+\Delta t), y(t+\Delta t))$ defined by the formulas (6.3) is injective. Assume it is not and $(x, y) \neq (x_1, y_1)$ are two points in X such that

$$\begin{aligned} x_1 + \Delta t\ (a\, y_1\, (1 - x_1) -\ r\, x_1) &= x + \Delta t\ (a\, y\, (1 - x) -\ r\, x) \\ y_1 + \Delta t\ (b\, x_1\, (1 - y_1) - m\, y_1) &= y + \Delta t\ (b\, x\, (1 - y) - m\, y)\,. \end{aligned} \tag{6.5}$$

This can be rewritten in a slightly different form:

$$(1 - \Delta t\, r - \Delta t\, a)\ (x_1 - x) + \Delta t\, a\, (1 - y_1)\, (x_1 - x) + \Delta t\, a\, (1 - x)(y_1 - y) = 0 \tag{6.6}$$

$$(1 - \Delta t\, m - \Delta t\, b)\ (y_1 - y) + \Delta t\, b\, (1 - x)\, (y_1 - y) + \Delta t\, b\, (x_1 - x)(1 - y_1) = 0. \tag{6.7}$$

We multiply the first equation by b, the second equation by a and subtract them to get

$$b\ (1 - \Delta t\, r - \Delta t\, a)\ (x_1 - x) = a\ (1 - \Delta t\, m - \Delta t\, b)\ (y_1 - y).$$

This implies that $x_1 - x$ and $y_1 - y$ have the same sign. Hence, all terms in equations (6.6) and (6.7) are either positive or negative simultaneously. A contradiction. $\square$

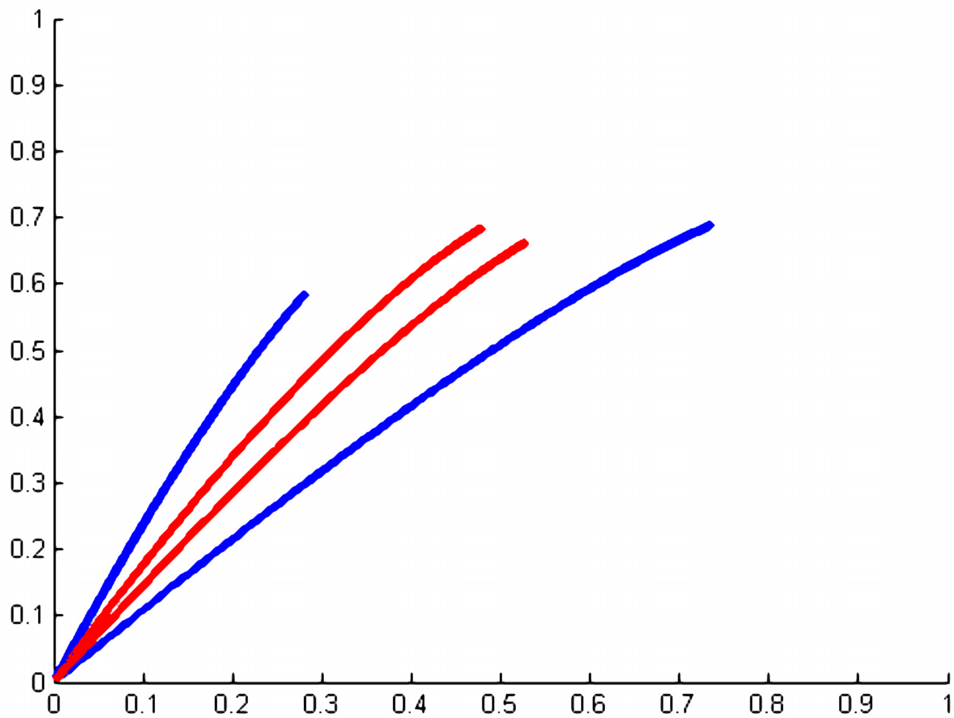

Figure 6.4: Three individual attractors $\mathcal{A}_w$ for $\Delta t = 0.05$: left: $w = 111...$; middle two: $w = 1010...$; right: $w = 000....$

We computed the attractor slice K by implementing the method explained in the beginning of this chapter in MATLAB. The invariant region for the system is the unit square, which made computations easier. The first step was to divide the unit square into smaller squares. When $\Delta t = .05$ figure (6.6), we subdivided the unit square into $10^3 \times 10^3$ squares. We labeled each square by a number between 1 and 10^6 (in the order described in the previous section), and computed the discretized maps $\tilde{S}_j$ and the corresponding $10^6 \times 10^6$ matrix A. After sufficiently many iterations of the matrix

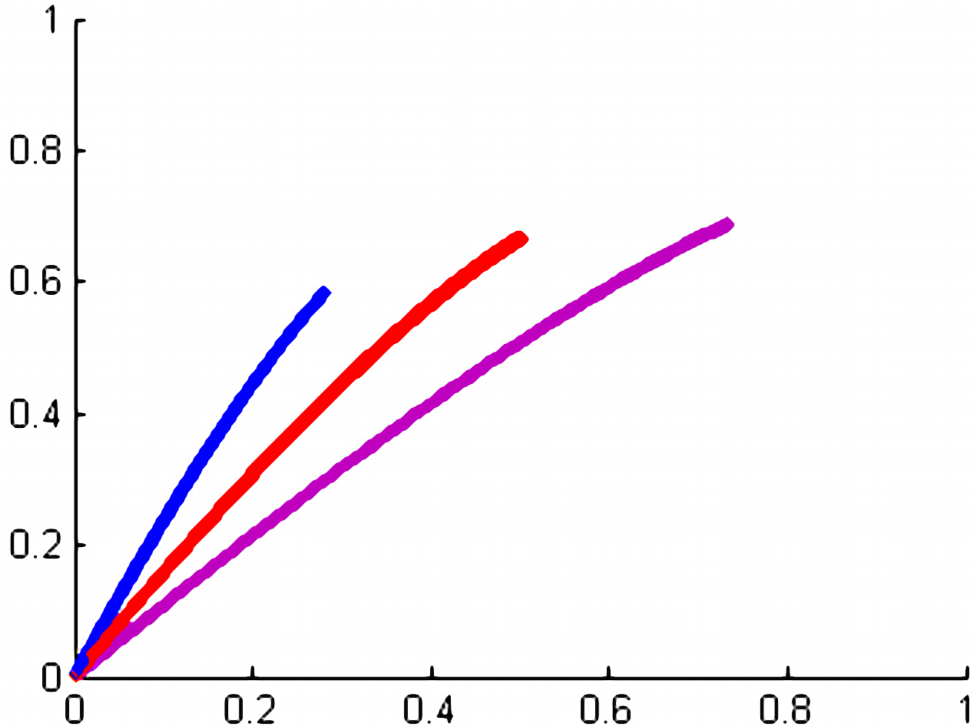

Figure 6.5: Three individual attractors $\mathcal{A}_w$ for $\Delta t = 0.005$: left: $w = 111...$; middle two (very close together): $w = 1010...$; right: $w = 000....$

A, we computed the vector $A^n\mathbf{v}$ ($\mathbf{v}$ as in previous section) whose nonzero components reveled the locations of the points of the attractor K.

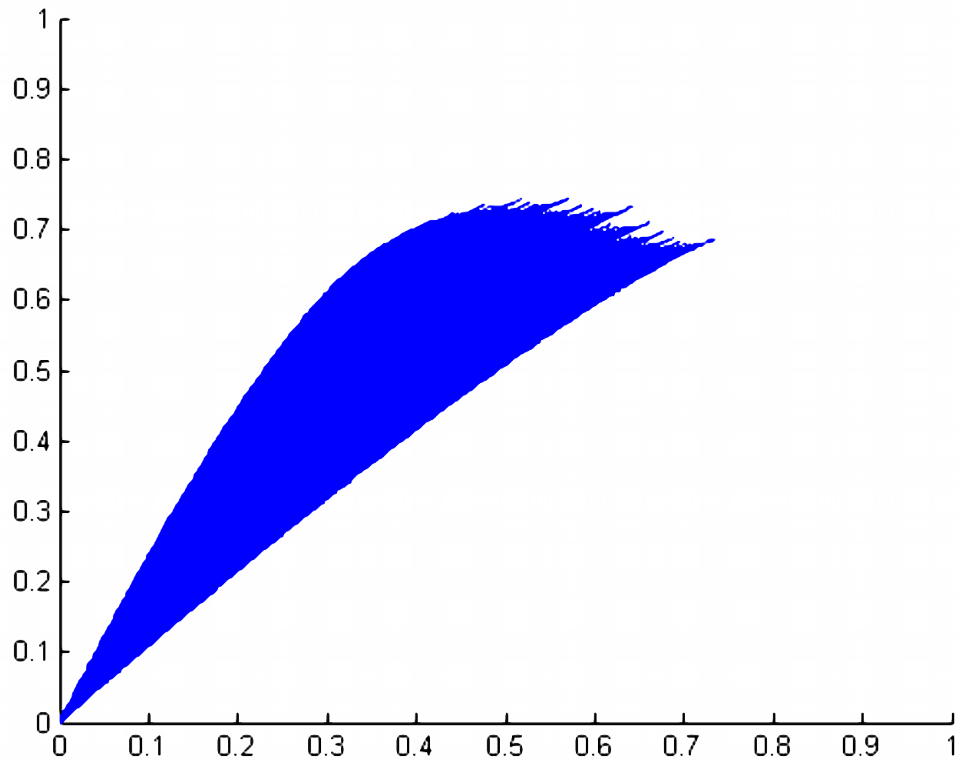

Figure 6.6: The attractor slice K; $\Delta t = 0.05$.

For the case when $\Delta t = .005$, we subdivided the unit square into $2 * 10^3 \times 2 * 10^3$ smaller squares. The matrix needed in this case was of the size $4 * 10^6 \times 4 * 10^6$. Even with such fine subdivision of the unit square, we were not able to get an accurate picture of the attractor K (as it was shown on the figure 6.7). In order to obtain the attractor in figure 6.7, we had to divide the unite square into several regions (squares) first, and then subdivide each of the regions into a large number of small squares. We computed the pieces of the attractor for each separate region in the same way as

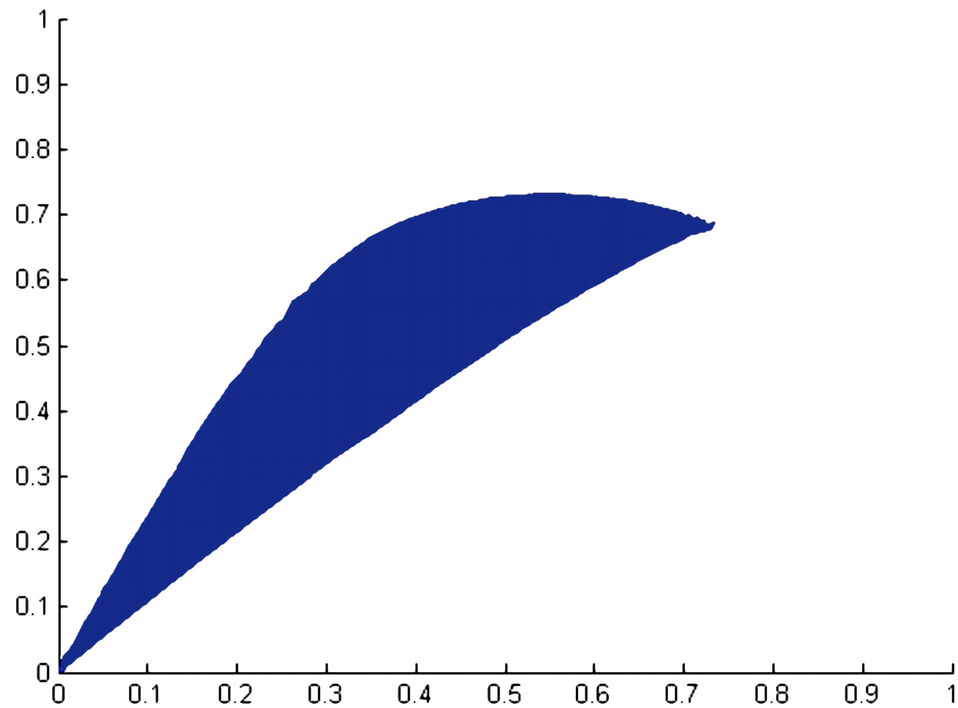

Figure 6.7: The attractor slice K; $\Delta t = 0.005$.

explained above.

In addition, we tried to compute the attractor (for this model) when we had infinitely many operators allowing the parameters to take on the values of the intervals. If we allow all four parameters to be intervals (e.g., each operator corresponds to a point between pset_0 and pset_1 - coordinate-wise), then, regardless of the subdivision, the matrix had a lot of nonzero components which made iterations of A impossible (using MATLAB). However, we were able to compute the attractor if three of the parameters were fixed and only one of them was an interval. We fixed the values for the parameters $b = 6$, $r = 3$, $m = 2$, and we let $a = [2, 7]$. In this case, Σ is the space of one-sided infinite sequences whose symbols are in $\mathcal{J} = [0, 1]$. The correspondence

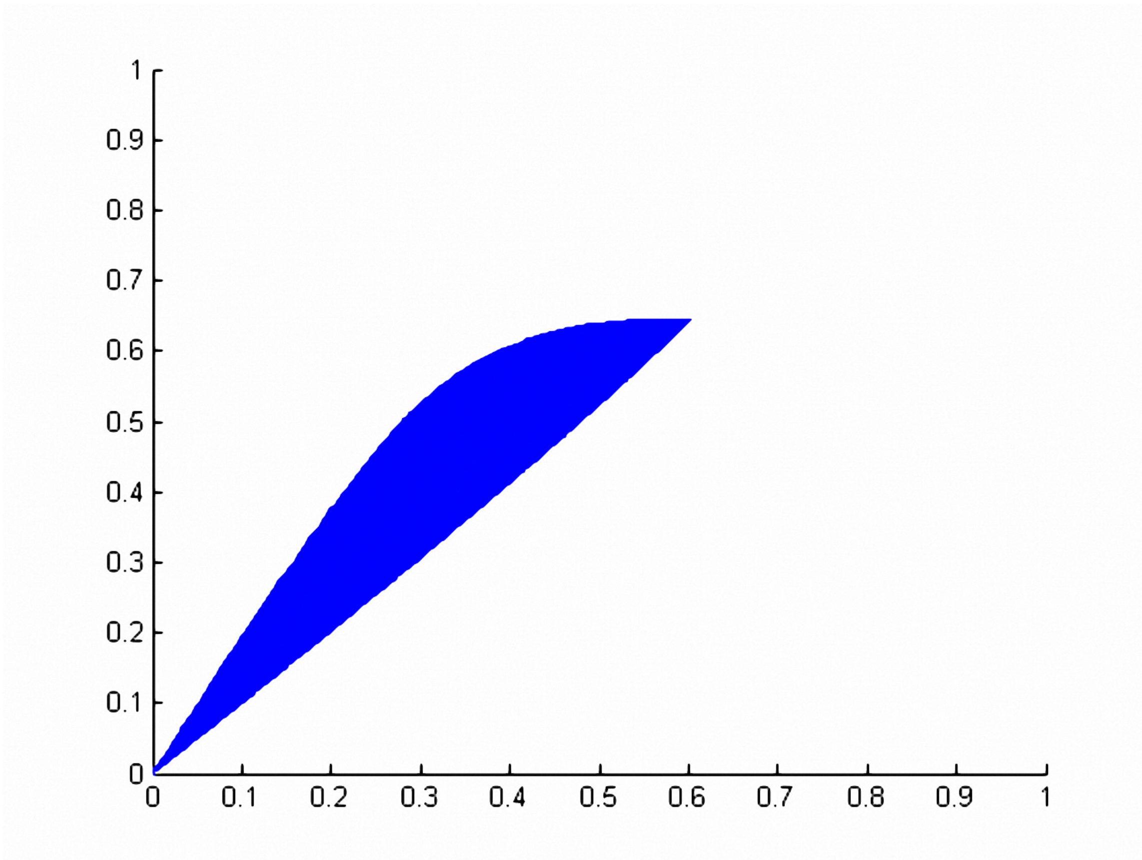

Figure 6.8: The attractor slice K. Infinitely many operators.

between an operator and a point in $\mathcal{J}$ can be seen as $S_j \leftrightarrow \{7 * j + 2 * (1 - j)\}$. The attractor slice K is given in figure 6.8. Despite the fact that we had infinitely many operators, we were able to compute the attractor of the system because the system is monotone and linear with respect to the chosen parameter a.

We have also looked at the dynamical systems corresponding to convex combinations of the parameter sets pset_0 and pset_1 and plotted their global attractors. The result is different from K, see figure 6.11 where the "convex combination" is superimposed onto the set K. We distinguish three parts of the boundary of the "convex combination." The left and right sides are the two attractors of the discrete dynamical

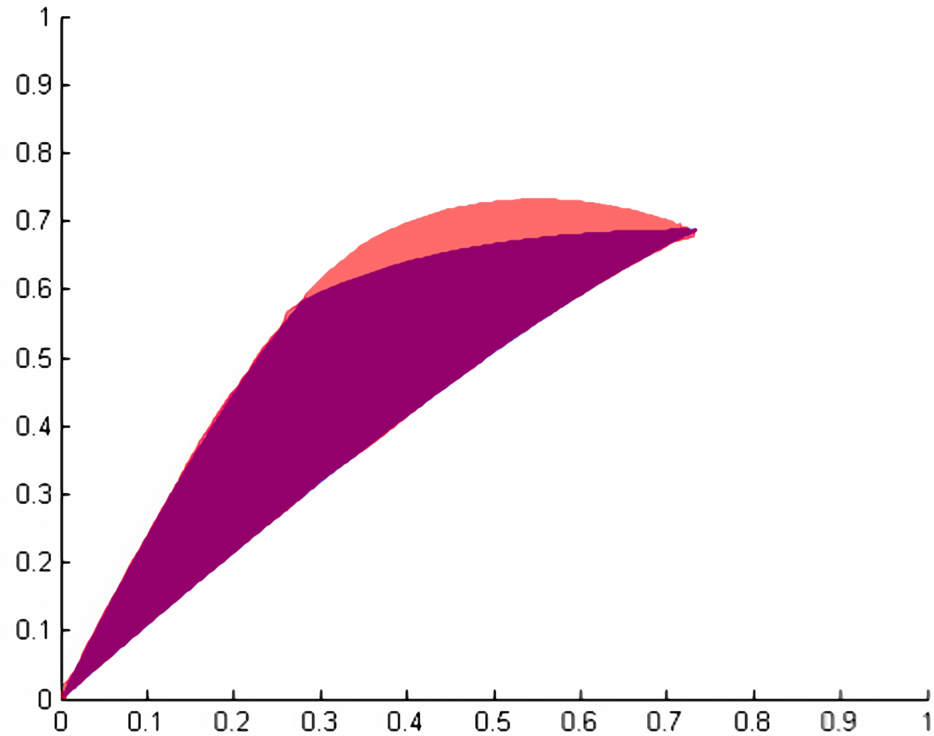

Figure 6.9: “Convex combination” superimposed over K; $\Delta t = 0.005$.

systems with parameter sets pset_0 and pset_1. The third upper part, refers to the fixed points of the systems corresponding to the convex combination of the parameter sets pset_0 and pset_1, and is given by the parametric equation

$$x_* = \frac{-4j^2 + 8j + 7}{5\,(5 - 2j)}\,, \quad y_* = \frac{-4j^2 + 8j + 7}{4(3 - j)(j + 1)}\,,$$

for $j \in \mathcal{J} = [0, 1]$.

For the attractor slice K, we cannot represent “the top of the boundary” by an equation. We only notice that when $\Delta t \to 0$, “the top of the boundary” of K becomes smooth. Note that the limit set is not an attractor of any system (6.3) with a

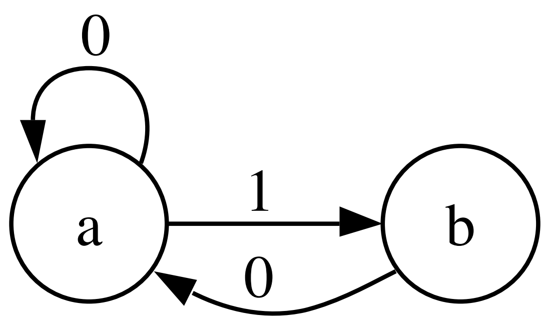

Figure 6.10: The golden mean shift.

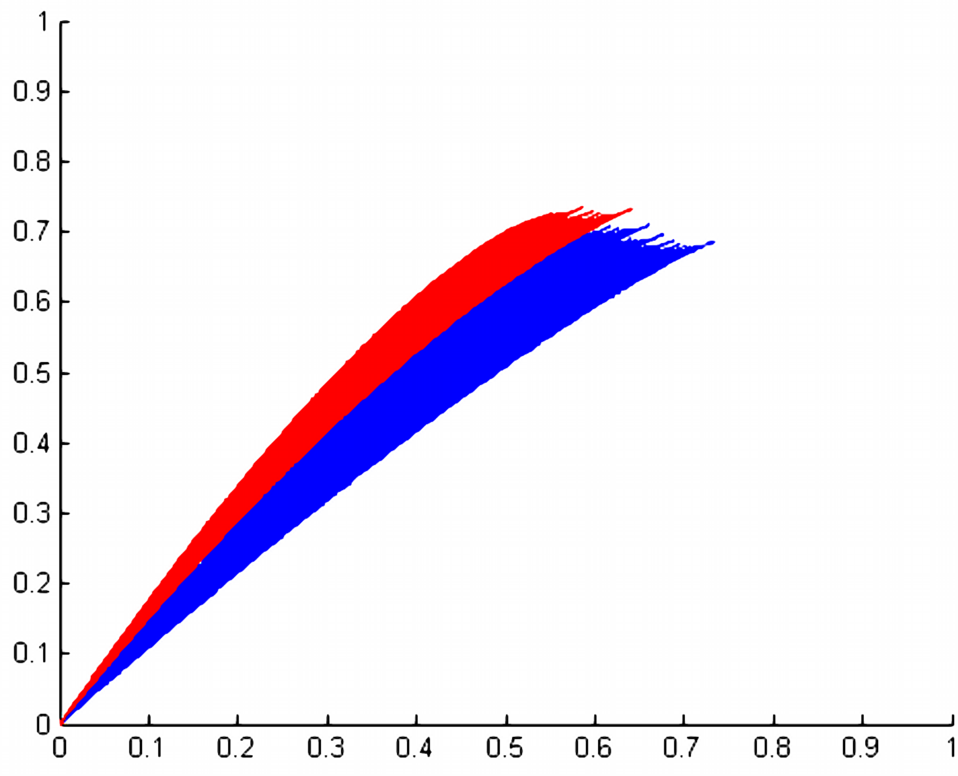

Figure 6.11: Golden mean, full; $\Delta t = 0.05$.

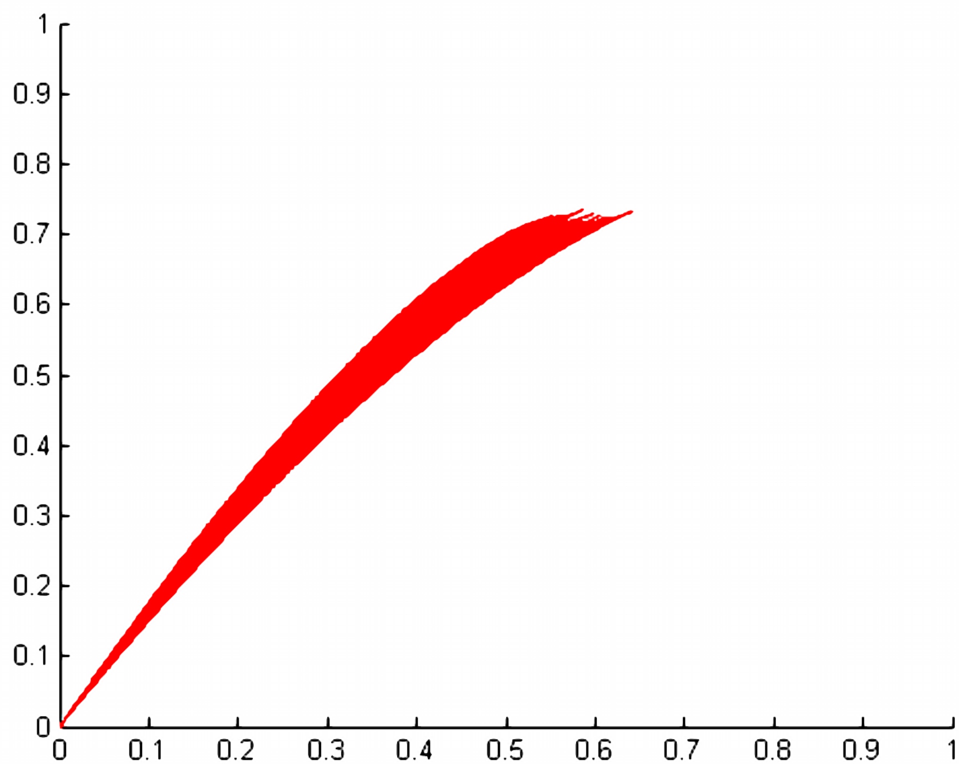

Figure 6.12: The red slice; $\Delta t = 0.05$.

fixed, averaged set of parameters a, b, r and m. It would be interesting to understand whether the limit set can be obtained as a union of the attractors of the systems (X, S_t), where the operator S_t corresponds to a certain parameter set pset_t for some curve connecting pset_0 with pset_1 in the space of parameters.

Next, we consider restricted dynamics associated with the golden mean subshift Λ (made of one-sided strings of 0s and 1s such that each 1 is necessarily followed by 0). The graph representing the golden mean shift is shown on figure 6.10.

Our analysis in chapter 4 shows that the global attractor of the restricted dynamics, $(\mathfrak{X}, \Lambda)$ may have at most two different slices: one corresponding to sequences of

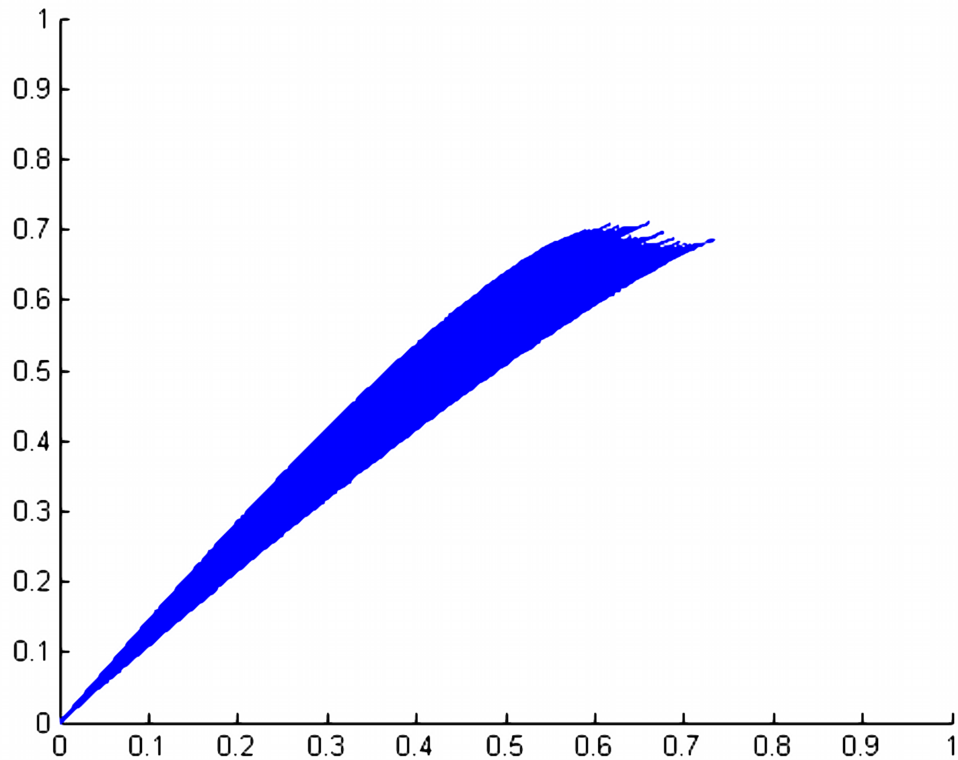

Figure 6.13: The blue slice; $\Delta t = 0.05$.

words ending in 1 (the red slice), and the other one corresponding to sequences of words ending in 0 (the blue slice). We computed the red slice by the same method as we computed the attractor slice K (explained above), except that the operators we took for the dynamics with choice were S_0 and the composition operator $S_0 \circ S_1$. The operators for the blue slice were S_0 and $S_1 \circ S_0$. In both cases, time step is $\Delta t = .05$, and the unit square was divided into $10^3 \times 10^3$ squares. The computations of the slices shows that the attractor of the restricted dynamics $(\mathfrak{X}, \Lambda)$ indeed has two slices. The slices are shown on figures 6.12 and 6.13. As point sets on the plane, the slices overlap. Their union is plotted on figure 6.11.

Appendix I

```
%dynamics with choice for ross mcdonald malaria equation
%t denotes discrete step, p is the length of a square in a partition of X
%example: attractorDWC(0.05,.001)
function attractorDWC(t,p)
n=1/p+1; %the number of squares in one row
z=n*n; %total number of squares
%two sets of parameters (pset_0 and pset_1)
a=[4 2];
b=[6 10];
r=[1 3];
m=[2 2];
%preallocation
s=ones(1,2*z);
ai=zeros(2*n,n);
%partition of the unit square;
%X and Y contain the centers of 1001 x 1001 squares
[X,Y]=meshgrid(0:p:1,0:p:1);
%calculating one-step time iteration by both maps
 for k=1:2
    nx = X + t*(a(k)*Y.*(1-X)-r(k)*X);
    ny = Y + t*(b(k)*X.*(1-Y)-m(k)*Y);
    re = round(nx*(n-1))+1;
    q = round(ny*(n-1));
```

```
    ai((k-1)*n+1:k*n,1:n)=n*q+re;
 end
%i is a row vector containing all one-step time iterations
i=(ai');
i=i(:);
j=repmat(1:z,1,2);
%the matrix A contains all one-step time iterations, i.e.,
%A(i(k),j(k))=s(k)=1 for all k=1..z
A=sparse(i,j,s,z,z);
clear i j s ai q re
%calculating the vector A^n * B
B=ones(z,1);
for k=1:2000
    B1=A*B;
    S=find(B);
    S1=find(B1);
    if size(S)==size(S1)
        if S==S1
           break;
        end
    else
        B=B1;
    end
end
figure(3),set(gcf,'doublebuffer','on'),hold on
```

```
axis([0 1 0 1])
o=size(S);
Z=zeros(2,o(1));
%retrieving the centers of the squares that belong to the attractor
for k=1:o(1)
   w=S(k)/n;
   re=floor(w);
   if re~=w
       re=re+1;
   end
   q=mod(S(k),n);
   if q==0
       q=n;
   end
   x(1)=X(re,q);
   x(2)=Y(re,q);
   Z(:,k)=x;
end
%plotting the attractor
figure(3),plot(Z(1,:),Z(2,:),'b.','markersize',1)
hold off
```

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