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For Jenn, Ellie, and Max.

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ABSTRACT

Kaschner, Scott R. Ph.D., Purdue University, May 2013. Superstable Manifolds of Invariant Circles. Major Professor: Roland K. W. Roeder.

Let $f : X \dashrightarrow X$ be a dominant meromorphic self-map, where X is a compact, connected complex manifold of dimension $n > 1$. Suppose there is an embedded copy of $\mathbb{P}^1$ that is invariant under f, with f holomorphic and transversally superattracting with degree a in some neighborhood. Suppose also that f restricted to this line is given by $z \mapsto z^b$, with resulting invariant circle S. We prove that if $a \geq b$, then the local stable manifold $\mathcal{W}^s_{\mathrm{loc}}(S)$ is real analytic. In fact, we state and prove a suitable localized version that can be useful in wider contexts. We then show that the condition $a \geq b$ cannot be relaxed without adding additional hypotheses by presenting two examples with $a < b$ for which $\mathcal{W}^s_{\mathrm{loc}}(S)$ is not real analytic in the neighborhood of any point.

1. INTRODUCTION

Let $f : X \dashrightarrow X$ be a dominant meromorphic self-map of a compact, connected complex manifold X of dimension $n > 1$. Here, the focus is on the situation in which there is $L \subset X$, an embedded copy of $\mathbb{P}^1$, with f holomorphic in a neighborhood of L, L is invariant, and $f|L$ is conjugate to $z \mapsto z^b$. We also assume L is transversally superattracting of degree a, that is, the local coordinates of f transverse to L vanish with order a. Although this is a rather special situation, it has appeared in examples from [1–4].

For such maps, the Julia set of $f|L$ is an invariant circle S, which is a hyperbolic set for f. The local stable manifold $\mathcal{W}^s_{\text{loc}}(S)$ is a real $2n-1$ dimensional manifold. We will prove:

Theorem A. *If $a \geq b$, then $\mathcal{W}^s_{\text{loc}}(S)$ has real analytic regularity.*

To prove the theorem, we will localize to the situation to a tubular neighborhood of L which is forward invariant under f. Theorem A is a direct consequence of the following:

Theorem A'. *Let N be a complex manifold with $\dim(N) \geq 2$, containing an embedded projective line L. Suppose $f \colon N \to N$ a dominant holomorphic map, L is invariant and transversally superattracting with degree a, and $f|L$ is conjugate to $z \mapsto z^b$, having invariant circle S. If $a \geq b$, then $\mathcal{W}^s_{\text{loc}}(S)$ has real analytic regularity.*

In Section 2 we prove Theorem A' by constructing a semi-conjugacy between f and $z \mapsto z^b$ on a forward invariant neighborhood of S.

The proof of Theorem A' is followed by Section 3, where we provide applications to certain specific examples, including those from [2, Sec. 6.2] and [1]. These examples are followed by Section 4, where an alternative proof of Theorem A for a specific family of maps is provided using holomorphic folations.

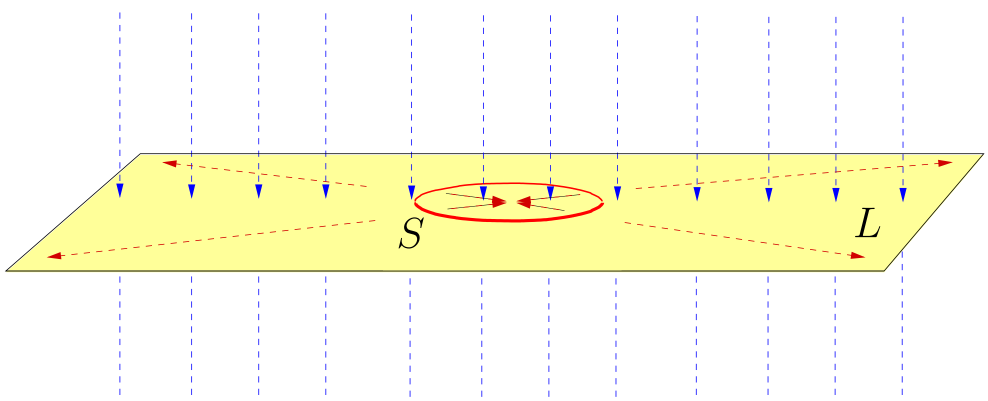

Figure 1.1. Contraction to L and repulsion from S within L

In Section 5, we show that the condition that $a \geq b$ cannot be improved without adding additional hypotheses. We'll consider two maps for which $a < b$ and $\mathcal{W}^s_{\mathrm{loc}}(S)$ is not analytic. One of them is the Migdal-Kadanoff renormalization map R for the Ising model on the Diamond Hierarchical Lattice (DHL) that was studied extensively in [3, 4]. It has $a = 2$ and $b = 4$. The other is a polynomial skew product with $a = 2$ and $b = 3$.

Let us comment a bit more on the map R. For this map, the invariant circle S has the physical context of being related to the bottom of the Lee-Yang cylinder, so it is denoted B. In [4, Lemma 3.2], the authors proved that $\mathcal{W}^s_{\mathrm{loc}}(B)$ is a C^∞ manifold. We prove:

Theorem B. *The stable manifold $\mathcal{W}^s_{\mathrm{loc}}(B)$ is not real analytic at any point.*

Proof of this theorem divides into four main parts. First we construct a co-dimension 1 Böttcher function φ defined in a neighborhood of B under the assumption that $\mathcal{W}^s_{\mathrm{loc}}(B)$ is real analytic. Next we extend the domain of φ to a neighborhood of the set obtained from L by removing the two superattracting fixed points. After that, we develop local properties of R near one of these superattracting fixed points. Lastly, we examine the behavior of φ and R in the extension, from which we derive a contradiction.

This theorem is of physical interest, since $\mathcal{W}^s_{\mathrm{loc}}(B)$ is related to phase transitions of the Ising model on the DHL at low temperatures; see [3, 4]. In §6, we'll explain how Theorem B relates to the limiting distribution of Lee-Yang and Lee-Yang-Fisher zeros at low temperatures.

1.1 Examples Illustrating Hypotheses

Consider a map $f\colon \mathbb{P}^2 \to \mathbb{P}^2$ given locally by

$$f(z, w) = (z^2 + wz, w^4 + w^3 z).$$

This map satisfies the hypotheses of Theorem A with $L = \{w = 0\}$ and $a = 3 > 2 = b$. Figure 1.1 illustrates several different slices of $\mathcal{W}^s_{\mathrm{loc}}(S)$ in the plane $\{w = c\}$ parallel

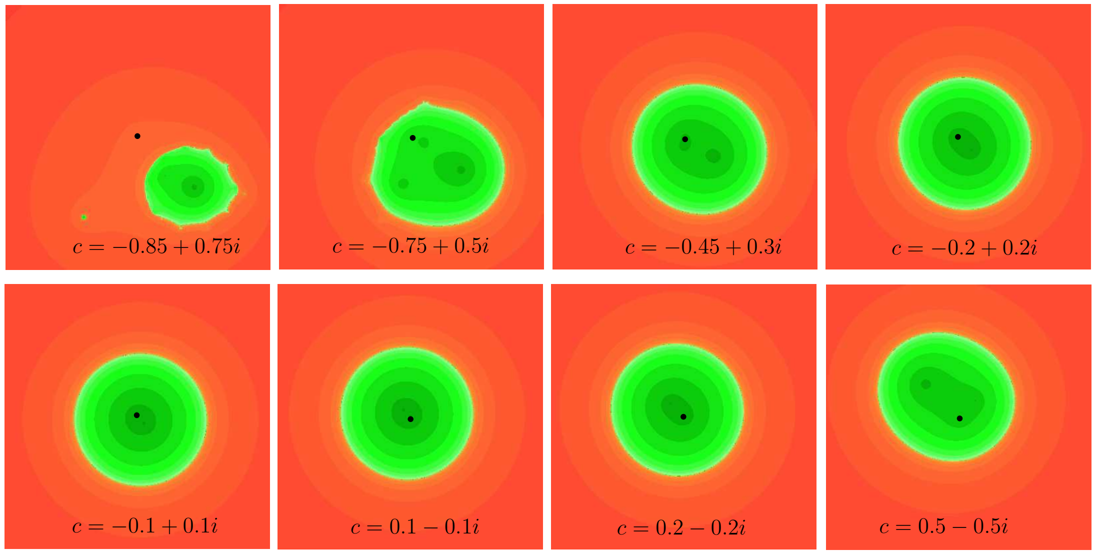

Figure 1.2. $\mathcal{W}^s(S) \cap \{w = c\}$ for f for different c values

to L. In each picture, the visible part of $\mathcal{W}^s_{\text{loc}}(S)$ is the boundary between the two colors, and the gradation of color indicates strength of repulsion from $\mathcal{W}^s_{\text{loc}}(S)$ within $\{w = c\}$.

Now consider $g\colon \mathbb{P}^2 \to \mathbb{P}^2$ given locally by

$$g(z, w) = (z^3 + wz^2, w^2).$$

Again, this map satisfies the hypotheses of Theorem A with $L = \{w = 0\}$, with the exception in this case that $a = 2 < 3 = b$. The darker coloring for small $|w|$ in Figure 1.1 illustrates the domination of the repelling direction over the attracting direction.

Both of these illustrate that examination of stable manifolds in this paper is necessarily in a very small neighborhood of S. While the slices of $\mathcal{W}^s_{\text{loc}}(S)$ appear to be at least C^1 for small $|w|$, for $|w|$ even near 1, the behavior of the stable manifold becomes far more complicated.

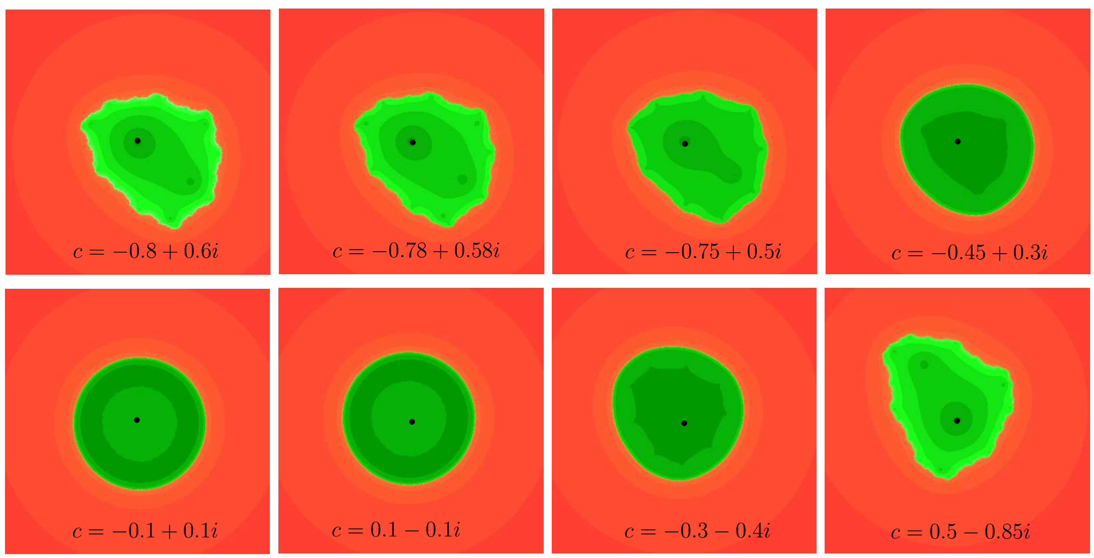

Figure 1.3. $\mathcal{W}^s(S) \cap \{w = c\}$ for g with different c values

1.2 Historical Notes

For a diffeomorphism, the existence and regularity of the local stable manifold for a hyperbolic invariant manifold N has been studied extensively Hirsch-Pugh-Shub in [5]. A strong form of hyperbolicity known as *normal hyperbolicity* is assumed in order to guarantee a C^1 local stable manifold. Specifically, N is called normally hyperbolic for f if the expansion of Df in the unstable direction transverse to N dominates the maximal expansion of Df tangent to N and the contraction of Df in the stable direction transverse to N dominates the maximal contraction of Df tangent to N; see [5, Theorem 1.1]. For C^r regularity, there is an analogous condition in terms of the r-th power of the maximal expansion/contraction tangent to N.

Although the maps considered in this paper are many-to-one, they also do not fit in the context of [5] since $f|L$ is conformal, forcing that the rates of expansion tangent to S and transverse to S are equal. Thus, S is not normally hyperbolic.

The construction of the semi-conjugacy in the proof of Theorem A' is similar to the proof of the well-known Böttcher's Theorem from one-dimensional complex dynamics [6]; see also [7, Ch. 9].

Theorem 1.2.1 (Böttcher) *If S is a Riemann surface and $f\colon S \to S$ is given by $f(z) = a_n z^n + a_{n-1} + z^{n+1} + \cdots$ with $n \geq 2$ and $a_n \neq 0$, then there exists a local holomorphic change of coordinate $w = \phi(z)$, with $\phi(0) = 0$, which conjugates f to the nth power map $w \mapsto w^n$ throughout some neighborhood of 0. Furthermore, ϕ is unique up to multiplication by an $(n-1)$st root of unity.*

In fact, many of the techniques used in this paper are similar to this classical theorem in spirit. While Böttcher's Theorem refers to a holomorphic change of coordinate (often called a Böttcher coordinate) defined in the neighborhood of a superattracting fixed point, the function we construct here is neither a coordinate, nor is it defined in a full neighborhood of a superattracting fixed point. However, by analogy, we call it a "co-dimension 1 Böttcher function."

Those interested in the mathematical legacy of Böttcher should see [8]. We will now briefly describe variants of Böttcher's Theorem in higher dimensions. It was shown by Hubbard and Papadopol in [9] that a Böttcher coordinate in higher dimension cannot exist in general. With additional hypotheses, their existence has been proved in [10, Theorem 3.2] and [11]. A more detailed criterion for existence of a Böttcher coordinate is presented in [12]. The related problem of conjugating a polynomial endomorphism to its highest degree terms in a neighborhood of the hyperplane at infinity is studied in [9, Theorem 9.3], [13, Theorem 7.4], [14], and [15, Theorem 1]. These authors prove that such a conjugacy exists on the stable set of the Julia set at infinity, so long as it satisfies suitable hyperbolicity. More recent studies of superattracting behavior appear in [16, 17].

2. PROOF OF THEOREM A'

The $\mathbb{C}^{n-1}$ bundle over $\mathbb{P}^1$ can be described by two systems of locally trivializing coordinates $(z, \boldsymbol{w}) \in \mathbb{C} \times \mathbb{C}^{n-1}$ and $(\zeta, \boldsymbol{\omega}) \in \mathbb{C} \times \mathbb{C}^{n-1}$. For $z \neq 0$, the are related by $\zeta = 1/z$ and $\boldsymbol{\omega} = A_z \boldsymbol{w}$, with $A_z : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ a linear isomorphism depending holomorphically on z. Let us choose these trivializations so that the dynamics on the zero section is $z \mapsto z^b$.

We will make use of standard multi-index notation. Given $\boldsymbol{c} \in \mathbb{Z}_+^{n-1}$ and $\boldsymbol{w} \in \mathbb{C}^{n-1}$, $\boldsymbol{w}^{\boldsymbol{c}} = w_1^{c_1} w_2^{c_2} \cdots w_{n-1}^{c_{n-1}}$ and $|\boldsymbol{c}| = c_1 + \cdots + c_{n-1}$. We will always use the standard Hermitian norm $|\boldsymbol{w}| = (|w_1|^2 + \cdots + |w_{n-1}|^2)^{1/2}$ on $\mathbb{C}^{n-1}$.

Lemma 2.0.2 *There are holomorphic functions $\boldsymbol{g}_1$ and $\boldsymbol{g}_{\boldsymbol{c}}$ for each $|c| = a$ such that in the $(z, \boldsymbol{w})$ coordinates*

$$f(z, \boldsymbol{w}) \; = \; \left(z^b + \boldsymbol{w} \cdot \boldsymbol{g}_1(z, \boldsymbol{w}), \sum_{|\boldsymbol{c}|=a} \boldsymbol{w}^{\boldsymbol{c}} \boldsymbol{g}_{\boldsymbol{c}}(z, \boldsymbol{w}) \right).$$

Similarly, there are holomorphic functions $\boldsymbol{h}_1$ and $\boldsymbol{h}_{\boldsymbol{c}}$ for each $|c| = a$ such that in the $(\zeta, \boldsymbol{\omega})$ coordinates

$$f(\zeta, \boldsymbol{\omega}) \; = \; \left(\zeta^b + \boldsymbol{\omega} \cdot \boldsymbol{h}_1(\zeta, \boldsymbol{\omega}), \sum_{|\boldsymbol{c}|=a} \boldsymbol{\omega}^{\boldsymbol{c}} \boldsymbol{h}_{\boldsymbol{c}}(\zeta, \boldsymbol{\omega}) \right).$$

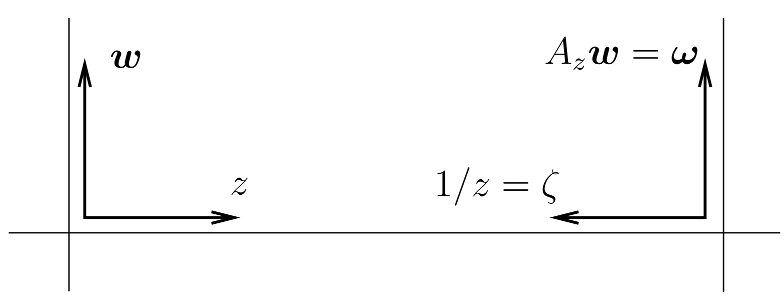

Figure 2.1. Local coordinates centered on the two fixed points in L

Proof The proof is the same in both coordinate systems, so we'll work in the $(z, \boldsymbol{w})$ system. Since $f \mid L$ is the map $z \mapsto z^b$, the first coordinate of f minus z^b vanishes on L. Since L is given by $\boldsymbol{w} = \boldsymbol{0}$, we have that the first coordinate of f is $z^b + \boldsymbol{w} \cdot \boldsymbol{g}_1(z, \boldsymbol{w})$ for some holomorphic function $\boldsymbol{g}_1$. Meanwhile, the expression for the second coordinate follows from the fact that L is transversally superattracting of degree a. ■

2.1 Hyperbolic theory

We'll now verify that the local stable manifold $\mathcal{W}^s_{\mathrm{loc}}(S)$ is a $2n-1$ real-dimensional topological manifold that is foliated by local stable manifolds of each point of S.

The hyperbolic theory for endomorphisms is somewhat less standard than for diffeomorphisms. Suitable references from the context of complex dynamics include [13, 18, 19]. For consistency, we will use definitions and results from [13, Appendix B]. Let us consider the natural extension

$$\hat{S} \;:=\; \{(x_i)_{i \le 0} \,:\, x_i \in S \text{ and } f(x_i) = x_{i+1}\}.$$

We'll denote such histories by $\hat{x} = (x_i)_{i \le 0} \in \hat{S}$. Notice that the action of f naturally lifts to an action $\hat{f} : \hat{S} \to \hat{S}$.

Lemma 2.1.1 *S is a hyperbolic set for the map f.*

Proof Note that for $x \in S$, we have

$$Df_x \;=\; \begin{bmatrix} bz^{b-1} & \dfrac{\partial}{\partial \boldsymbol{w}} \boldsymbol{g}_1(z, \boldsymbol{0}) \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix}.$$

Thus, we have $E^s(x) = \ker(Df)$ and $E^u(\hat{x}) \subset L$, so $T_x\mathbb{C}^n = E^s(x) \oplus E^u(\hat{x})$. Invariance of $E^s(x)$ follows from the fact any point in the kernel is collapsed to $(0, \boldsymbol{0})$ under Df, and invariance of $E^u(\hat{x})$ follows from the invariance of L. Also, for any $v^s \in E^s(x)$ and $v^u \in E^u(\hat{x})$ with $n \ge 0$,

$$\|Df_x^n v^s\| = 0 \le C\lambda^n \|v^s\| \text{ and } \|Df_x^n v^u\| \le C\lambda^{-n} \|v^u\|,$$

for $C = 1$ and $\lambda = 1/2$. Thus, we have that S is hyperbolic. ■

Therefore, by the stable manifold theorem (see, for example, [20, Theorem 5.2]) each point $x \in S$ will have local stable manifold $\mathcal{W}^s_{\rm loc}(x)$ that is a complex $n-1$ ball holomorphically embedded into N and each prehistory $\hat{x}$ will have a local unstable manifold $\mathcal{W}^u_{\rm loc}(\hat{x})$, which is a holomorphic disc. They depend continuously on x and $\hat{x}$. (In this case, the unstable manifolds all lie in L.)

Existence of such stable laminations has also been proved in the holomorphic context by Ushiki [21]. It can be proved in the following simple way as well, which is a direct generalization of what was done in [1, Proposition 4.2] and [3, Proposition 9.2].

By the stable manifold theorem for a point (see, for example, [22, Sec. 2.6] or [11], which hold even if Df has an eigenvalue of 0), there exists a local stable manifold, $\mathcal{W}^s_{\rm loc}((1,\mathbf{0}))$, which is the graph of a holomorphic function $z = \eta_1(\boldsymbol{w})$ defined on some $(n-1)$-dimensional open ball, Λ, in the $\boldsymbol{w}$ axis. Let $\Sigma \subset S$ to be the set of iterated preimages of $(1,\mathbf{0})$. Using a suitable invariant cone field and a well-chosen neighborhood of S, one can take iterated preimages of $\mathcal{W}^s_{\rm loc}((1,\mathbf{0}))$ so that the preimage through each $x \in \Sigma$ is expressed as the graph of a holomorphic function $\eta_x(\boldsymbol{w})$ defined on Λ, making Λ smaller if necessary. In this way, we can construct local stable manifolds over Σ, which is dense in S. The function $\eta\colon \Lambda \times \Sigma \to \mathbb{C}$ given by $\eta(\boldsymbol{w}, x) = \eta_x(\boldsymbol{w})$ defines a holomorphic motion of $\Sigma \subset \mathbb{C}$, parameterized by $\boldsymbol{w} \in \Lambda \subset \mathbb{C}^{n-1}$. We may use the λ-lemma [23, 24] to extend η continuously to a holomorphic motion of $\overline{\Sigma} = S$, obtaining stable manifolds for every point of S.

Definition 2.1.1 *A hyperbolic set $\hat{\Lambda}$ has a local product structure, if $\delta > 0$ can be chosen small enough so that for any $p \in \Lambda$ and $\hat{q} \in \hat{\Lambda}$, either $\mathcal{W}^s_\delta(p) \cap \mathcal{W}^u_\delta(\hat{q})$ is empty or it is a single point $x \in \Lambda$ so the unique history $\hat{x}$ of x satisfying $x_j \in \mathcal{W}^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \leq 0$ is completely contained in $\hat{\Lambda}$.*

Lemma 2.1.2 *S has local product structure for the map f.*

Proof By Lemma 2.1.1, S is hyperbolic. Recall that for any $\hat{q} \in \hat{S}$, we have that $\mathcal{W}^u_\delta(\hat{q}) = \mathbb{D}_\delta(q_0) \subset L$, which is the disc of radius $\delta > 0$ centered at the point q

contained in L. Since $\mathcal{W}^u_\delta(\hat{q})$ depends only on q_0, existence of a local product structure for $\hat{S}$ is very simple.

By the Stable Manifold Theorem, we may choose $\delta > 0$ small enough so that for any $p \in S$, we have $\mathcal{W}^s_\delta(p) \cap L = \{p\}$. Thus, for any two points $p, q \in S$, the intersection $\mathcal{W}^s_\delta(p) \cap \mathcal{W}^u_\delta(\hat{q}) = \{p\}$, with $p \in S$. Moreover, p has a unique prehistory $\hat{p} = (p_i)_{i\le 0}$ with $p_j \in \mathcal{W}^u_\delta(\hat{f}^j(\hat{q}))$ for all $j \le 0$, and it is completely contained in $\hat{S}$ as well. ∎

Given a neighborhood Ω of S, let

$$\mathcal{W}^s_{\text{loc}}(S) := \{x \in N \colon f^n x \in \Omega \text{ and } f^n x \to S \text{ as } n \to \infty\} \tag{2.1}$$

(where Ω is implicit in the notation, and an assertion involving $\mathcal{W}^s_{\text{loc}}(S)$ means that it holds for any sufficiently small neighborhood of S).

Since S has a local product structure $\mathcal{W}^s_{\text{loc}}(S)$ is the union of the local stable manifolds $\mathcal{W}^s_{\text{loc}}(x)$ of points $x \in \mathcal{B}$; see [13, Proposition B.6]. The local stable manifolds of points are pairwise disjoint and depend continuously on the base point, therefore we have:

Corollary 2.1.3 $\mathcal{W}^s_{\text{loc}}(S)$ *is a topological manifold of real dimension* $2n-1$.

2.2 Co-dimension 1 Böttcher function

Let $(z_n, \boldsymbol{w}_n) := f^n(z, \boldsymbol{w})$. Motivated by Böttcher's theorem [6], [7, p. 86], we consider a sequence of functions

$$\varphi_n(z, \boldsymbol{w}) = z_n^{1/b^n}.$$

We will show that the φ_n converge uniformly on compact subsets of some forward invariant neighborhood Ω of S to a holomorphic function φ that semi-conjugates f to $z \mapsto z^b$:

$$\varphi(f(z, \boldsymbol{w})) = \varphi(z, \boldsymbol{w})^b. \tag{2.2}$$

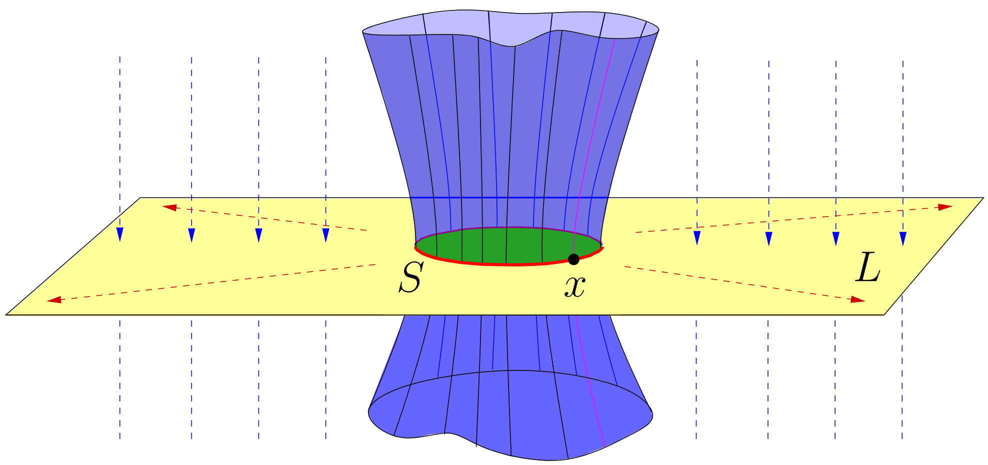

Figure 2.2. $\mathcal{W}^s_{\text{loc}}(S)$, contraction to L, and repulsion from S within L

To make sense of the b^n-th roots and the limit, we'll rewrite each φ_n as telescoping product:

$$\varphi \;=\; \lim_{n\to\infty}\varphi_n \;=\; z_0\cdot\frac{{z_1}^{1/b}}{z_0}\cdot\frac{{z_2}^{1/b^2}}{{z_1}^{1/b}}\cdot\frac{{z_3}^{1/b^3}}{{z_2}^{1/b^2}}\cdots \;=\; z_0\prod_{n=0}^{\infty}\left(\frac{z_{n+1}}{{z_n}^b}\right)^{\frac{1}{b^{n+1}}}, \tag{2.3}$$

where it follows from Lemma 2.0.2 that

$$\frac{z_{n+1}}{{z_n}^b}=\frac{z_n^b+\boldsymbol{w}_n\cdot\boldsymbol{g}_1(z_n,\boldsymbol{w}_n)}{{z_n}^b}=1+\frac{\boldsymbol{w}_n}{{z_n}^b}\cdot\boldsymbol{g}_1(z_n,\boldsymbol{w}_n). \tag{2.4}$$

In the $(\zeta,\boldsymbol{\omega})$ coordinates we have:

$$\frac{z_{n+1}}{{z_n}^b}=\frac{\zeta_n^b}{\zeta_{n+1}}=\frac{1}{1+\frac{\boldsymbol{\omega}_n}{\zeta_n^b}\cdot\boldsymbol{h}_1(\zeta_n,\boldsymbol{\omega}_n)}. \tag{2.5}$$

When working in $\mathcal{W}^s(\eta_1)$ we'll use expression (2.4), when working in $\mathcal{W}^s(\eta_2)$ we'll use expression (2.5), and when working on $\mathcal{W}^s_{\mathrm{loc}}(S)$, we'll use either.

We'll construct a forward invariant neighborhood Ω of S so that if $(z,\boldsymbol{w})\in\Omega\cap(\mathcal{W}^s(\eta_1)\cup\mathcal{W}^s_{\mathrm{loc}}(S))$, then

$$\left|\frac{\boldsymbol{w}_n}{{z_n}^b}\cdot\boldsymbol{g}_1(z_n,\boldsymbol{w}_n)\right| \;<\; \frac{1}{2}, \tag{2.6}$$

and if $(\zeta,\boldsymbol{\omega})\in\Omega\cap(\mathcal{W}^s(\eta_2)\cup\mathcal{W}^s_{\mathrm{loc}}(S))$, then

$$\left|\frac{\boldsymbol{\omega}_n}{\zeta_n^b}\cdot\boldsymbol{h}_1(\zeta_n,\boldsymbol{\omega}_n)\right| \;<\; \frac{1}{2}. \tag{2.7}$$

Then, for points in Ω, the b^n-th root is defined by taking the branch cut along the negative real axis. Moreover, this condition will also imply convergence of the infinite product (2.3) on Ω, since the corresponding sum of logarithms converges:

$$\sum_{n=1}^{\infty}\log\left|\frac{z_{n+1}}{{z_n}^b}\right|^{\frac{1}{b^{n+1}}}\le\sum_{n=1}^{\infty}\frac{1}{b^{n+1}}\log 2.$$

To construct Ω, first note that for any $K_1>0$ sufficiently small, $\{|\boldsymbol{w}|\le K_1\}\cap(\mathcal{W}^s(\eta_1)\cup\mathcal{W}^s_{\mathrm{loc}}(S))$ is a compact subset of $\mathbb{C}^n$. Since $\boldsymbol{g}_1$ is holomorphic on $\mathbb{C}^n$, there is a bound $|\boldsymbol{g}_1(z,\boldsymbol{w})|\le K_2$ on any such compact set. A similar bound holds in the other coordinate system. Therefore, it suffices to show:

Lemma 2.2.1 *Given any $K > 0$, there exists a forward invariant neighborhood of S in which*

$$\frac{|\boldsymbol{w}|}{|z|^b} < K \qquad \textit{and} \qquad \frac{|\boldsymbol{\omega}|}{|\zeta|^b} < K. \tag{2.8}$$

Proof of this lemma relies on point blow-ups, so let us provide a breif description of this technique using the definitions from [25]. Let M be a complex manifold of dimension n, and let $z = (z_1, \ldots, z_n)$ holomorphic coordinates in an open st $U \subset M$ centered around the point $p \in M$. The *blow-up* $\tilde{M}$ of M at p is the complex manifold obtained by adjoining to $M \setminus \{p\}$ the manifold

$$\tilde{U} = \{(z, l) \colon z \in l\} \subset U \times \mathbb{P}^{n-1}$$

via the isomorphism

$$\tilde{U} \setminus (z = 0) \cong U \setminus \{p\}$$

given by $(z, l) \mapsto z$. One may be tempted to think of this simply as a holomorphic change of coordinates, and while that is true, what is happening additionally is that the point p is being replaced by a projective hypersurface. There is a natural projection map $\pi \colon \tilde{M} \to M$ extending the identity on $M \setminus \{p\}$. By construction, $E = \pi^{-1}(p)$ is isomorphic to $\mathbb{C}P^{n-1}$ and is called the *exceptional divisor* of the blow-up $\tilde{M} \to M$.

Proof [Proof of Lemma 2.2.1] We will take an inductive sequence of b point blow-ups at each of the two fixed points η_1 and η_2. Using the forms of f given by Lemma 2.0.2, the calculation will be the same at each of these two points, so we'll focus on η_1, which is given by $(z, \boldsymbol{w}) = (0, \boldsymbol{0})$.

We first do a point blow-up at η_1, producing an exceptional divisor $E_{\eta_1,1}$. Let $\tilde{L}_1$ be the proper transform of L. We then blow-up the point intersection point between $E_{\eta_1,1}$ and $\tilde{L}_1$, producing a new exceptional divisor $E_{\eta_1,2}$ and proper transform $\tilde{L}_2$. We inductively do this $b-2$ additional times, each time blowing up the intersection point between the previous exceptional divisor and proper transform of L.

Consider the system of coordinates $z, \boldsymbol{\lambda} = \frac{\boldsymbol{w}}{z^b}$ centered at the intersection point of $E_{\eta_1,b}$ with $\tilde{L}_b$. Let us denote $(z', \boldsymbol{\lambda}') = \tilde{f}(z, \boldsymbol{\lambda})$, where $\tilde{f}$ is the extension of f to the final blow-up. We have

$$\begin{aligned} z' &= z^b + z^b \boldsymbol{\lambda} \cdot \boldsymbol{g}(z, z^b\boldsymbol{\lambda}) \\ \boldsymbol{\lambda}' &= \frac{\boldsymbol{w}'}{(z')^b} = \frac{\sum_{|\boldsymbol{c}|=a}(z^b\boldsymbol{\lambda})^{\boldsymbol{c}} \boldsymbol{g}_{\boldsymbol{c}}(z, z^b\boldsymbol{\lambda})}{(z^b + z^b\boldsymbol{\lambda}\cdot\boldsymbol{g}(z, z^b\boldsymbol{\lambda}))^b} = \frac{z^{b(a-b)}\sum_{|\boldsymbol{c}|=a}\boldsymbol{\lambda}^{\boldsymbol{c}}\boldsymbol{g}_{\boldsymbol{c}}(z, z^b\boldsymbol{\lambda})}{(1+\boldsymbol{\lambda}\cdot\boldsymbol{g}(z, z^b\boldsymbol{\lambda}))^b}. \end{aligned}$$

Notice that this extension $\tilde{f}$ is holomorphic in a neighborhood of $(z, \boldsymbol{\lambda}) = (0, \mathbf{0})$ and that this point is superattracting for $\tilde{f}$.

Therefore, for any $\varepsilon_1 > 0$ and $K \geq \delta_1 > 0$, sufficiently small, $\widetilde{U}_1 := \{|z| < \varepsilon_1, |\boldsymbol{\lambda}| < \delta_1\}$ will be forward invariant under $\tilde{f}$. Hence,

$$U_1(\varepsilon_1, \delta_1) := \pi\left(\widetilde{U}_1(\varepsilon_1, \delta_1)\right) = \left\{|z| < \varepsilon_1, \frac{|\boldsymbol{w}|}{|z|^b} < \delta_1\right\}$$

will be a forward invariant set for f.

As stated before, the same calculation can be done at η_2, with analogous results. In particular, for any $\varepsilon_2 > 0$ and $K \geq \delta_2 > 0$ sufficiently small we will have a forward invariant set for f of the form

$$U_2(\varepsilon_2, \delta_2) = \left\{|\zeta| < \varepsilon_2, \frac{|\boldsymbol{\omega}|}{|\zeta|^b} < \delta_2\right\}.$$

Let $V \subset N$ be a forward invariant tubular neighborhood of L and let

$$V(\varepsilon_1, \varepsilon_2) = V \setminus (\{|z| < \varepsilon_1\} \cup \{|\zeta| < \varepsilon_2\}).$$

Note that if V sufficiently small, then all points of $V(\varepsilon_1, \varepsilon_2)$ satisfy (2.8). We will show that V can be made even smaller, if necessary, in order to make

$$\Omega := V(\varepsilon_1, \varepsilon_2) \cup U_1(\varepsilon_1, \delta_1) \cup U_2(\varepsilon_2, \delta_2)$$

forward invariant.

Since $U_1(\varepsilon_1, \delta_1)$ and $U_2(\varepsilon_2, \delta_2)$ are forward invariant, we need only check that if $x \in V(\varepsilon_1, \varepsilon_2)$ and $f(x) \notin V(\varepsilon_1, \varepsilon_2)$, then $f(x) \in U_1(\varepsilon_1, \delta_1) \cup U_2(\varepsilon_2, \delta_2)$. Let us focus on $x \in \mathcal{W}^s(\eta_1)$, since the proof will be the same for $x \in \mathcal{W}^s(\eta_2)$.

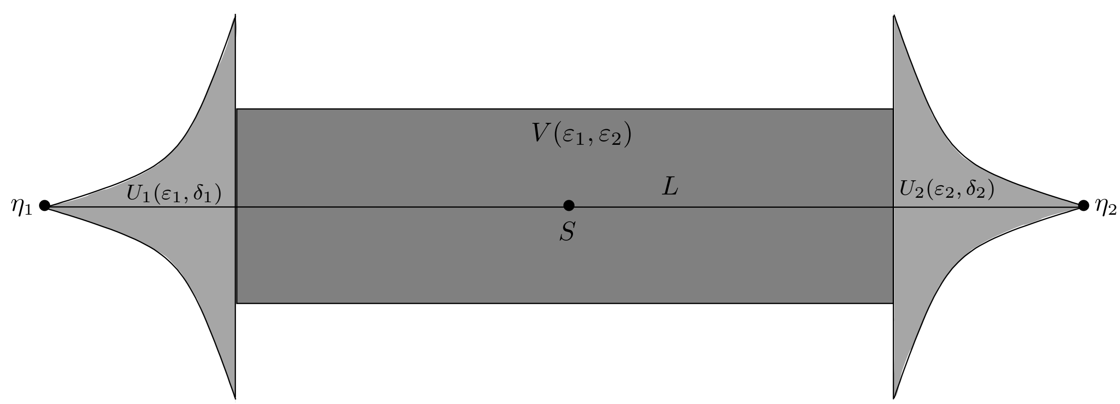

Figure 2.3. The forward invariant neighborhood Ω

Let $x = (z, \boldsymbol{w}) \in V(\varepsilon_1, \varepsilon_2) \cap \mathcal{W}^s(\eta_1)$ and let $(z_1, \boldsymbol{w}_1) = f(z, \boldsymbol{w})$. Since $(z, \boldsymbol{w}) \in V(\varepsilon_1, \varepsilon_2)$, $|\boldsymbol{w}|/|z|^b < K$, so that (2.6) and (2.4) imply that the $|z_1| \geq |z|^b/2 \geq \varepsilon_1^b/2$. Thus, we need only choose the (forward invariant) tubular neighborhood V sufficiently small so that

$$V \cap \left\{ \frac{\varepsilon_1^b}{2} \leq |z| \leq \varepsilon_1 \right\} \subset U_1(\varepsilon_1, \delta_1).$$

Doing the same thing near η_2, we construct a forward invariant neighborhood Ω satisfying (2.8). ■

2.3 Completing the proof of Theorem A'

Using the invariance (2.2), for any $(z, \boldsymbol{w}) \in \mathcal{W}^s_{\mathrm{loc}}(S)$ we have $|\varphi(z, \boldsymbol{w})| = 1$ so that $\psi := \log|\varphi|$ will be a real analytic function that vanishes on $\mathcal{W}^s_{\mathrm{loc}}(S)$. Notice on that L, we have $\varphi(z, \mathbf{0}) = z$ and hence $\psi(z, \mathbf{0}) = \log|z|$. Since the derivative $D\psi$ is non-zero on S, we have that $\{\psi = 0\}$ is a real analytic $2n - 1$ real-dimensional manifold in some neighborhood of S.

By Corollary 2.1.3, $\mathcal{W}^s(S) \subset \{\psi = 0\}$ is also a real $2n - 1$ dimensional manifold. Thus, by invariance of domain, $\mathcal{W}^s(S) = \{\psi = 0\}$ in this neighborhood.

3. EXAMPLES ILLUSTRATING THEOREM A

3.1 Regular Polynomial Endomorphisms in Two Dimensions

Suppose $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ is a degree d regular polynomial endomorphism. Then f has the form $f(x,y) = (p(x,y), q(x,y))$ for polynomials p and q. Moreover, if $p_d(x,y)$ and $q_d(x,y)$ are the degree d homogeneous terms of p and q, then in homogeneous coordinates in the line at infinity, $L_\infty = \{Z = 0\}$, f has the form

$$f\mid_{Z=0}[X:Y:Z] = [p_d(X,Y):q_d(X,Y):0]. \tag{3.1}$$

Since f is regular, p_d and q_d have no common zeros, so there is no indeterminacy on L_∞. The coordinate on L_∞ is $z = Y/X$, so if we assume $f \mid L_\infty$ is conjugate to $z \mapsto z^d$, then there are coordinates such that $p_d(X,Y) = X^d$ and $q_d(X,Y) = Y^d$. Then

$$f_\infty(z) \;=\; \frac{p_d(1,z)}{q_d(1,z)} \;=\; z^d, \tag{3.2}$$

so J_∞, the Julia set on L_∞, is a geometric circle. Thus, this situation satisfies the hypotheses of Theorem A with $a = b = d$, so we have the following:

Corollary 3.1.1 *If f is a regular polynomial endomorphism of $\mathbb{C}^2$ for which $f \mid L_\infty$ is conjugate to $z \mapsto z^d$, then $\mathcal{W}^s_{\text{loc}}(J_\infty)$ has real analytic regularity.*

Real analyticity of the stable manifold considered in [2, Section 6.2] is a direct application of Corollary 3.1.1.

3.2 Degenerate Newton Mappings

Newton mappings used to find the common roots of $P(x,y) = x(1-x)$ and $Q(x,y) = y^2 + Bxy - y$ were considered dynamically in [1]. They have the form

$$N(x,y) = \left(\frac{x^2}{2x-1}, \frac{y(Bx^2 + 2xy - Bx - y)}{(2x-1)(Bx+2y-1)}\right). \tag{3.3}$$

We will consider their extension as rational maps of $\mathbb{P}^1 \times \mathbb{P}^1$. They are skew products with the first coordinate having superattracting fixed points of degree 2 at $x = 0$ and $x = 1$, so the vertical lines $\{x = 0\} \times \mathbb{P}^1$ and $\{x = 1\} \times \mathbb{P}^1$ are transversally superattracting for N with the same degree. Using the formula, one can check that N has no indeterminate points in some neighborhood of these two lines.

Restricted to $\{x = 0\} \times \mathbb{P}^1$, N is the one-dimensional Newton map for the quadratic polynomial with roots at $y = 0$ and $y = 1$. It is therefore conjugate to $z \mapsto z^2$, having an invariant circle S_0 corresponding to the points of equal distance from $y = 0$ and $y = 1$ in $\mathbb{P}^1$. (S_0 is the closure of $\mathrm{Im}(y) = \frac{1}{2}$ in $\mathbb{P}^1$.)

Similarly, the restriction of N to $\{x = 1\} \times \mathbb{P}^1$ is the one-dimensional Newton map for the quadratic polynomial with roots at $y = 0$ and $y = 1 - B$. Thus, it is conjugate to $z \mapsto z^2$, with an invariant circle S_1 corresponding to the points of equal distance from $y = 0$ and $y = 1 - B$ within $\mathbb{P}^1$.

Both of the lines $\{0\} \times \mathbb{P}^1$ and $\{1\} \times \mathbb{P}^1$ is transversally superattracting with degree 2, with the restriction of N to each of them conjugate to $z \mapsto z^2$. Therefore, it follows immediately from Theorem A that the local stable manifolds $\mathcal{W}^s_{\mathrm{loc}}(S_0)$ and $\mathcal{W}^s_{\mathrm{loc}}(S_1)$ are real analytic. This was proven previously in [1] using more specific details of the mapping.

3.3 Example with indeterminacy

Consider the polynomial mapping $g : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$g(x,y) = \left(x^2 + y(1+xy), y^3(1+xy)\right). \tag{3.4}$$

Within $\mathbb{C}^2$, the line $L := \{y = 0\}$ is invariant and transversally superattracting with degree 3 and $g|L$ is given by $x \mapsto x^2$. Let $S := \{|x| = 1, y = 0\}$ be the invariant circle. Although there is the needed domination between the degrees $(3 > 2)$, to apply Theorem A we need to check how g extends to a neighborhood of infinity on L. The extension of g to $\mathbb{P}^2$ is given in homogeneous coordinates by

$$g[X : Y : Z] = [X^2Z^3 + YZ^2(Z^2 + XY) : Y^3(Z^2 + XY) : Z^5].$$

There is a point of indeterminacy for g at $[1 : 0 : 0]$ on the projective line $Y = 0$, which we'll also denote by L. Therefore, Theorem A does not immediately apply.

Let us perform two blowups. We first blow-up the point $[1 : 0 : 0]$ and we then blow-up the point where the proper transform of L intersects the exceptional divisor over $[1 : 0 : 0]$. We'll denote the space obtained after doing these two blow-ups by $\widetilde{\mathbb{P}^2}$, the projection by $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$, the proper transform of L after these two blow-ups by $\widetilde{L}$, the invariant circle within $\widetilde{L}$ by $\widetilde{S}$, and the lift of g to the blown-up space by $\widetilde{g} : \widetilde{\mathbb{P}^2} \to \widetilde{\mathbb{P}^2}$.

A neighborhood of $\widetilde{L}$ can be described by two systems of coordinates (x, y) and (ζ, τ), where $x = X/Z, y = Y/Z$ are the original affine coordinates on $\mathbb{C}^2$ and $\zeta = Z/X, \tau = XY/Z^2$. In the first system of coordinates, $\widetilde{g}$ is given by (3.4). In the second system of coordinates, $\widetilde{g}$ is given by

$$\widetilde{g}(\zeta, \tau) = \left(\frac{\zeta^2}{1 + \tau\,\zeta^3(1 + \tau)}, \tau^3\zeta\,(1 + \tau)\left(1 + \tau\zeta^3 + \tau^2\zeta^3\right)\right).$$

In the second system of coordinates, $\widetilde{L}$ is given by $\tau = 0$, so we see that $\widetilde{g}$ is holomorphic in a neighborhood of $\widetilde{L}$. Moreover, $\widetilde{L}$ invariant and transversally superattracting with degree 3 and $\widetilde{g}|\widetilde{L}$ still given by $x \mapsto x^2$. Therefore, Theorem A applies to give that the local stable manifold $\mathcal{W}^s_{\text{loc}}\left(\widetilde{S}\right)$ for $\widetilde{S}$ under $\widetilde{g}$ is real analytic.

Notice that $\widetilde{g}$ and g are birationally conjugate by means of π. Moreover, restricted to small neighborhoods of $\widetilde{S}$ and S, this birational conjugacy becomes an honest holomorphic conjugacy. Since the local stable manifolds $\mathcal{W}^s_{\text{loc}}\left(\widetilde{S}\right)$ and $\mathcal{W}^s_{\text{loc}}(S)$ are defined in terms of the action of iterates of $\widetilde{g}$ and g, respectively, on these small neighborhoods, we conclude that $\mathcal{W}^s_{\text{loc}}(S)$ is also real analytic.

This third example illustrates two important considerations about Theorem A. First, it illustrates that one sometimes needs to do some blow-ups in order to obtain a map without indeterminacy in a neighborhood of L.

Second, it illustrates the reason why we need to consider arbitrary $\mathbb{C}^{n-1}$ bundles over L, since blowing up points of L will change it's normal bundle. (In this example, the normal bundle of L is the hyperplane bundle, while the normal bundle of $\widetilde{L}$ is the tautological bundle.)

4. EXAMPLE OF THEOREM A PROVED USING THE FOLIATION

Consider the family of skew product maps $f_\lambda\colon \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$f_\lambda(z,w) \;=\; (z^2 + \lambda w, w^2),$$

for $\lambda \in \mathbb{C}$. The lines $L_0 := \{w = 0\}$ and $L_1 := \{z = 0\}$ are both invariant. Moreover, L_0 is transversally super-attracting, and $f_\lambda|L_0$ is the map $z \mapsto z^2$. Again, $S := \{|z| = 1, w = 0\} \subset L_0$ is an invariant circle contained in L_0.

Thus, this specific example satisfies the hypotheses of Theorem A with $a = b = 2$, so

Corollary 4.0.1 *For any $\lambda \in \mathbb{C}$, the map f_λ generates the stable manifold $\mathcal{W}^s_{\mathrm{loc}}(S)$ with real analytic regularity.*

In this situation, however, we present a proof of this result using a different technique. The corollary will be proved by constructing a backward invariant holomorphic foliation in a neighborhood of S. This foliation will have the property that each leaf intersects L_0 in a unique point, and the projection to L_0 along leaves is a holomorphic function. Since $\mathcal{W}^s_{\mathrm{loc}}(S)$ is the preimage under the projection of the real analytic set S, $\mathcal{W}^s_{\mathrm{loc}}(S)$ must also be real analytic.

4.1 Backward Invariant Neighborhoods and Cone Fields

To iteratively pull back a holomorphic foliation of L_0 in a neighborhood of S, we will need a backward invariant set that avoids critical values of f_λ (besides L_0) in order to maintain a proper holomorphic foliation. It follows from $\det(Df) = 4zw$ that L_0 and L_1 are the only critical curves, so let $B := \{|w| \geq c|z|^2\}$ for some $c > 0$.

Lemma 4.1.1 *There is a $c > 0$ such that B is backward invariant.*

Proof Suppose $f_\lambda(z,w) = (z^{'}, w^{'}) \in B$, so we have

$$c \;\le\; \frac{|w^{'}|}{|z^{'}|^2} \;=\; \frac{|w|^2}{|z^2 - aw|^2} \;=\; \frac{|w/z^2|^2}{|1 - a(w/z^2)|^2}.$$

Let $x = w/z^2$, so

$$\sqrt{c} \;\le\; \frac{|x|}{|1 + ax|} \;\le\; \frac{|x|}{|1 - |ax||},$$

which yields two cases.

Case one: If $1 > |ax|$, then $\sqrt{c} \le |x|/(1 - |ax|)$, which implies $\sqrt{c} \le \sqrt{|}x| + c|ax|$, or

$$\frac{\sqrt{c}}{1 + \sqrt{c}|a|} \;\le\; |x|.$$

Then

$$c \;\le\; \frac{\sqrt{c}}{1 + \sqrt{c}|a|} \;\le\; \left|\frac{w}{z^2}\right|$$

precisely when $c|a| + \sqrt{c} - 1 \le 0$, or

$$c \;\le\; \left(\frac{-1 + \sqrt{1 + 4|a|}}{2|a|}\right)^2. \tag{4.1}$$

Case two: If $1 < |ax|$, then $\sqrt{c} \le |x|/(|ax| - 1)$, which implies $(\sqrt{c}|a| - 1)|x| = \sqrt{c}|ax| - |x| \le \sqrt{c}$. Then if $c \le 1/|a|^2$,

$$\sqrt{c} \;\le\; \frac{\sqrt{c}}{1 - \sqrt{c}|a|} \;\le\; |x|.$$

in which case, we again have $c \le |w/z^2|$.

It follows that for

$$c \;\le\; \min\left\{\frac{1}{|a|^2}, \left(\frac{-1 + \sqrt{1 + 4|a|}}{2|a|}\right)^2\right\} \tag{4.2}$$

that B is backward invariant. ∎

The behavior of f_λ near infinity is also relevant, so extend f_λ to a map $F_\lambda \colon \mathbb{P}^2 \to \mathbb{P}^2$ by

$$F_\lambda[Z : W : T] = [Z^2 +_\lambda WT : W^2 : T^2],$$

and use local coordinates $\omega = W/Z$ and $\tau = T/Z$, so

$$\omega' = \frac{\omega^2}{1 + a\omega\tau} \quad \text{and} \quad \tau' = \frac{\tau^2}{1 + a\omega\tau}$$

Then we have $|DF_\lambda| = \frac{4\omega\tau}{(1+a\omega\tau)^3} = 0$ only on $\{\omega = 0\}$ and $\{\tau = 0\}$, and both of these lines are totally invariant. Let $B^{'} := \{|\tau| \geq c|\omega|^2\}$, where c is the constant from Lemma 4.1.1. Observe that in these coordinates,

$$c \;\leq\; \frac{|\tau'|}{|\omega'|^2} \;=\; \frac{|\tau|^2}{|\omega|^4}|1 + a\omega\tau|,$$

so

$$c \;\leq\; \sqrt{c} \;\leq\; \left|\frac{\tau}{\omega^2}\right| \sqrt{|1 + a\omega\tau|}.$$

Then in a small enough neighborhood of 0 and for small enough c, we have

$$c \;\leq\; \left|\frac{\tau}{\omega^2}\right|.$$

Thus, $B^{'}$ is backward invariant in a small enough neighborhood of 0.

Having confined the critical set to a backward invariant set $B \cup B^{'}$, we may, for some $\varepsilon > 0$, define a neighborhood of $L_0 \setminus \{0, [1:0:0]\}$,

$$\Omega_\varepsilon := \{|w| < \varepsilon\} \setminus B \cup B^{'},$$

on which we can define a foliation.

Lemma 4.1.2 *For $x \in \Omega_\varepsilon$, the vertical cone field $K^v_x = \{(u, v) \in T_x\mathbb{C}^2 \colon |v| \geq \alpha(x)|u|\}$, where $\alpha(x) = |z/a|$, is backward invariant.*

Let $\mathcal{F}$ be a proper vertical foliation of $L_0 \setminus \{0, [1 : 0 : 0]\}$ with respect to some cone field. See Figure 4.1.

Proof We will prove the horizontal cone $K^h_x = \{(u, v) \in T_x\mathbb{C}^2 \colon |v| \leq \alpha(x)|u|\}$ is forward invariant. For $x, f(x) \in \Omega_\varepsilon$, we have

$$D(f_\lambda)_x = \begin{bmatrix} 2z & -a \\ 0 & 2w \end{bmatrix}, \text{ so} \qquad D(f_\lambda)_x \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2zu - av \\ 2wv \end{bmatrix}. \tag{4.3}$$

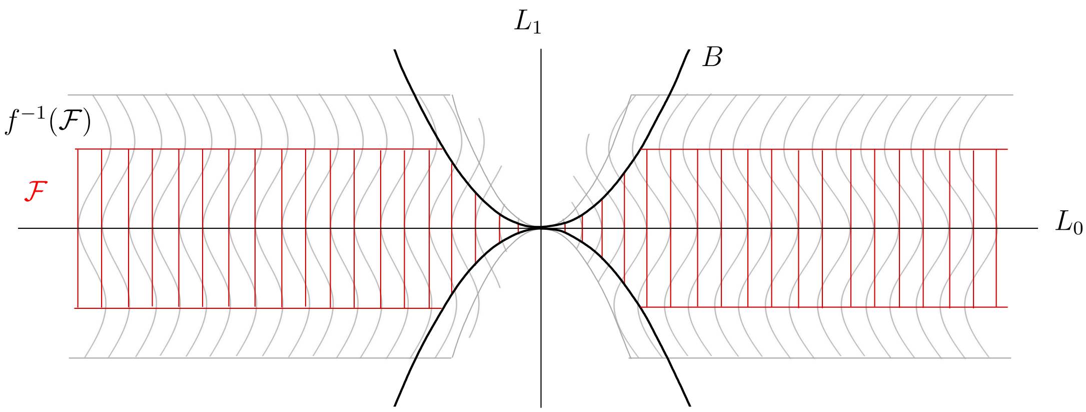

Figure 4.1. Ω_ε in a neighborhood of 0

Since $x \in \Omega_\varepsilon$, we may assume $|w| \le c|z|^2$, where c is the constant defined in Lemma 4.1.1. Moreover, since the cone K_x is horizontal, we have $|v| \le \alpha(x)|u|$. Thus, it suffices to show $|wv| \le \alpha(f(x))|zu - v|$. Since $\alpha(x) = |z/a|$,

$$1 \;\le\; \left|\frac{1}{|a|c} - 1\right| \;\le\; \left|\left|\frac{z^2}{aw}\right| - 1\right| \;\le\; \left|\frac{z^2}{aw} - 1\right| \;\le\; \frac{|z^2 - aw|/|a|}{|w|} \;=\; \frac{\alpha(f(x))}{|w|}. \quad (4.4)$$

Using this, we have

$$1 \;\le\; \frac{\alpha(f_\lambda(x))}{|w|} \;=\; \frac{\alpha(f_\lambda(x))}{|w|}\left|\frac{|z|}{\alpha(x)} - 1\right| \;\le\; \alpha(f_\lambda(x))\left|\frac{zu}{wv} - \frac{1}{w}\right| \;=\; \frac{\alpha(f_\lambda(x))|zu - v|}{|wv|}. \quad (4.5)$$

■

4.2 Pulling Back the Foliation

Lemma 4.2.1 *The intersected preimage $f_\lambda^{-1}(\mathcal{F}) \cap \Omega_\varepsilon$ is a proper vertical holomorphic foliation.*

Proof Let $\gamma_z \in \mathcal{F}$, so γ_z is a holomorphic disc through some point $z \in L_0 \setminus \{0, [1 : 0 : 0]\}$, that is γ_z is a submanifold of $\mathbb{C}^2$ of complex codimension 1. Note that for any

$x \in \Omega_\varepsilon \setminus L_0$, $D(f_\lambda)_x$ has full rank since all critical curves (besides L_0) are confined outside Ω_ε. Considering the points in L_0, recall that $f_\lambda \mid L_0$ is the map $z \mapsto z^2$, so if $x \in L_0 \setminus \{0, [1:0:0]\}$, then $\mathrm{Im}(Df_x) = L_0$. Thus, for any $x \in \gamma_z$, we have

$$\mathrm{Im}(Df_{f^{-1}(x)}) \ + \ T_x(\gamma_z) = \mathbb{C}^2,$$

where the terms of the direct sum are linearly independent since $\mathrm{Im}(Df_x) = L_0$ and $\mathcal{F}$ is a vertical foliation of L_0. Then f is transversal to L_0, so by the preimage theorem, $f_\lambda^{-1}(\gamma_z)$ is a codimension 1 complex manifold through $f_\lambda^{-1}(z) \in L_0$.

Recall that by Lemma 4.1.1 vertical cones must be backward invariant. Thus, $f_\lambda^{-1}(\gamma_z)$, the codimension 1 complex manifold through $f_\lambda^{-1}(z) \in L_0$, is a vertical holomorphic disc provided it is bounded away from the critical set. This is achieved by intersecting $f_\lambda^{-1}(\gamma_z)$ with Ω_ε. ■

At this point, we have shown that the foliation $\mathcal{F}$ the property that backward iterates are still vertical foliations, but we still require backward invariance of this foliation. For this, we consider the limit of backward iterates of $\mathcal{F}$, intersecting with Ω_ε after each iterate. That is, we define recursively

$$\mathcal{F}_n := f_\lambda^{-1}(\mathcal{F}_{n-1}) \cap \Omega_\varepsilon.$$

Lemma 4.2.2 *The sequence of backward iterates $\mathcal{F}_n$ converge to a proper vertical holomorphic foliation, $\widetilde{\mathcal{F}}$.*

Proof Let $\varphi_n \colon \Omega_\varepsilon \to L_0 \setminus \{0, [1:0:0]\}$ be the projection along each leaf $\gamma_z \in \mathcal{F}_n$ onto $z \in L_0$. Since vertical cone fields are backwards invariant, derivatives of φ_n are uniformly bounded. Then φ_n are locally bounded, so by Montel's theorem, $\{\varphi_n\}$ is a normal family. Thus, there is a subsequence φ_{n_k} that converges to a holomorphic map $\widetilde{\varphi}$.

Let $\widetilde{\mathcal{F}}$ be the foliation whose leaves are defined by $\widetilde{\gamma}_z := \widetilde{\varphi}^{-1}(z)$ for any $z \in L_0 \setminus \{0, [1:0:0]\}$. Since $\widetilde{\varphi} \mid L_0 \equiv id$, any $z \in L_0 \setminus \{0, [1:0:0]\}$ is a regular value for $\widetilde{\varphi}$, so by the preimage theorem, each $\widetilde{\gamma}_z$ is a vertical holomorphic disc. Since $\widetilde{\mathcal{F}}$

must agree with the stable foliation of S, which is backward invariant, it follows from uniqueness of holomorphic functions that $\widetilde{\mathcal{F}}$ is backward invariant. $\blacksquare$

Then $\{\widetilde{\gamma}_z \mid \widetilde{\gamma}_z \cap L_0 \in S\} = W^s(S)$ must be real analytic since it is a three real dimensional topological manifold that is a restriction to leaves of a holomorphic foliation intersecting S.

5. PROOF OF THEOREM B

We'll now show that the hypothesis in Theorem A that L is transversally superattracting with degree greater than or equal to the degree of $f|L$ cannot be eliminated without adding additional hypotheses.

The Migdal-Kadanoff Renormalization map $R : \mathbb{P}^2 \to \mathbb{P}^2$ for the Ising model on the DHL is given in homogeneous coordinates by

$$R[U : V : W] = [(U^2 + V^2)^2 : V^2(U + W)^2 : (W^2 + V^2)^2].$$

For this map, the projective line $L_0 = \{V = 0\}$ is transversally superattracting with degree 2 with R holomorphic on a forward invariant neighborhood of L_0. Restricted to L_0, R is given by $u \mapsto u^4$, where $u = U/W$, so $a = 2$ and $b = 4$. The invariant circle is denoted $B := \{V = 0, |u| = 1\}$. Below, we will show that $\mathcal{W}^s_{\text{loc}}(B)$ is not real analytic in the neighborhood of any point of B, thus proving Theorem B.

The second example for which $a < b$ and $W^s(S)$ is not real analytic is the following polynomial skew product of $f : \mathbb{P}^2 \to \mathbb{P}^2$ given in affine coordinates by

$$f(z, w) = (z^3 + 2wz^2, w^2). \tag{5.1}$$

One can check that this map is holomorphic on a forward invariant neighborhood in $\mathbb{P}^2$ of the invariant line $L = \{w = 0\}$. Moreover, L is transversally superattracting with degree 2, and $f|L$ is given by $z \mapsto z^3$. Thus, $a = 2 < 3 = b$. For this map, $\mathcal{W}^s_{\text{loc}}(S)$ is not real analytic in the neighborhood of any point of S.

In this chapter, we'll provide a detailed proof of Theorem B, showing that $\mathcal{W}^s_{\text{loc}}(B)$ is not real analytic. An adaptation of the same techniques can be used to show the analogous result for the skew product f. We leave details of this adaptation to the reader.

5.1 The Migdal-Kadanoff Renormalization

In the remainder of this section, we will adopt the notation from the recent preprints [3, 4] by Bleher, Lyubich, and Roeder. Although $R : \mathbb{P}^2 \to \mathbb{P}^2$ is more convenient for illustrating Theorem A, in the proof of Theorem B it will be more convenient to work the expression of the Migdal-Kadanoff renormalization $\mathcal{R} : \mathbb{P}^2 \to \mathbb{P}^2$ in the physical coordinates (z, t). In these coordinates, it is given by

$$(z_{n+1}, t_{n+1}) \;=\; \left(\frac{z_n^2 + t_n^2}{z_n^{-2} + t_n^2}, \frac{z_n^2 + z_n^{-2} + 2}{z_n^2 + z_n^{-2} + t_n^2 + t_n^{-2}}\right) \;:=\; \mathcal{R}(z_n, t_n). \tag{5.2}$$

We consider (z, t) as affine coordinates on $\mathbb{P}^2$ with $z = Z/Y, t = T/Y$ for some system of homogeneous coordinates $[Z : T : Y]$. The map $\mathcal{R}$ has an invariant projective line $\mathcal{L}_0 = \{T = 0\}$ that is transversally superattracting, except for an indeterminate point at $\mathbf{0} := [0 : 0 : 1]$, and $\mathcal{R}|\mathcal{L}_0$ is given by $z \mapsto z^4$. The invariant circle is given by $\mathcal{B} = \{|z| = 1, t = 0\}$.

The map R is semi-conjugate to $\mathcal{R}$ by means of a rational map $\Psi : \mathbb{P}^2 \to \mathbb{P}^2$:

$$\begin{array}{ccc} \mathbb{P}^2 & \xrightarrow{\mathcal{R}} & \mathbb{P}^2 \\ \Big\downarrow{\scriptstyle \Psi} & & \Big\downarrow{\scriptstyle \Psi} \\ \mathbb{P}^2 & \xrightarrow{R} & \mathbb{P}^2 \end{array} \tag{5.3}$$

with $[U : V : W] = \Psi([Z : T : Y]) = [Y^2 : ZT : Z^2]$. The map Ψ sends $\mathcal{L}_0$ to L_0, $\mathcal{B}$ to B, and is holomorphic in a neighborhood of $\mathcal{B}$. Therefore, $\mathcal{W}^s_{\text{loc}}(\mathcal{B}) = \Psi^{-1}(\mathcal{W}^s_{\text{loc}}(B))$. In particular, if $\mathcal{W}^s_{\text{loc}}(B)$ were real analytic in the neighborhood of any point of B, then $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ would be real analytic in the neighborhood of the preimage of that point under Ψ. So, Theorem B will follow from:

Theorem B' *The stable manifold $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ is not real analytic at any point.*

The reason we originally stated Theorem B for R rather than $\mathcal{R}$ is that R is holomorphic in a full neighborhood of L_0, so that it illustrates why the hypothesis on a and b can't be eliminated in Theorem A. One can also resolve the indeterminacy $\mathbf{0} \in \mathcal{L}_0$ for $\mathcal{R}$, placing it in the context of Theorem A, via a suitable birational modification (two blow-ups and one blow-down), but that is somewhat more complicated.

We will begin by proving the following proposition, and Theorem B' will follow shortly thereafter.

Proposition 5.1.1 $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ *is not real analytic in any full neighborhood of* $\mathcal{B}$*.*

This proposition will be proven by contradiction, so for the remainder of this section, we assume $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ is real analytic in a full neighborhood of $\mathcal{B}$. We will begin by describing the dynamics of $\mathcal{R}$ near $\mathcal{L}_0$, and after that, with the construction of a co-dimension 1 Böttcher function φ. This is followed by the extension of the domain of φ and an exploration of the behavior of φ and $\mathcal{R}$ in the extension. The section concludes with a proof of Proposition 5.1.1.

5.2 Dynamics in a Neighborhood of the Invariant Line

We will now briefly summarize basic properties of the dynamics for $\mathcal{R}$ in a neighborhood of $\mathcal{L}_0$ from [3, Sec. 4].

Let $\mathbb{D}_0 := \{|z| < 1, t = 0\} \subset \mathcal{L}_0$. The orbit of any $z \in \mathbb{D}_0$ will converge to an indeterminate point $\mathbf{0} := \{(0,0)\}$. (Informally, we will denote these points by $\mathcal{W}^s(\mathbf{0})$.) Meanwhile, points near $\mathbf{0}$ but not on $\mathcal{L}_0$ will converge to a superattracting fixed point $\eta := \{(0,1)\}$.

To see what happens for large $|z|$, we write $\mathcal{R}$ in homogeneous coordinates, obtaining

$$\mathcal{R}\colon [Z : T : Y] \mapsto [Z^2(Z^2+T^2)^2 : T^2(Z^2+Y^2)^2 : (Z^2+T^2)(T^2Z^2+Y^4)]. \tag{5.4}$$

There is another superattracting fixed point $\eta' := [1 : 0 : 0]$, which attracts all points of $\mathcal{L}_0$ with $|z| > 1$.

Lemma 5.2.1 $\mathcal{W}^s(\mathbf{0}) \cup \mathcal{W}^s_{\text{loc}}(\eta) \cup \mathcal{W}^s_{\text{loc}}(\mathcal{B}) \cup \mathcal{W}^s_{\text{loc}}(\eta')$ *fills some neighborhood of* $\mathcal{L}_0 \setminus \{\mathbf{0}\}$*.*

See [3, Lemma 4.2].

There is another invariant line $\mathcal{L}_1 := \{t = 1\}$ passing through η and η'. We have $\mathcal{R}|\mathcal{L}_1 : z \to z^2$.

For the remainder of this section, it will convenient to use a system of affine coordinates centered at η'. We will use $(\lambda = Y/Z - T/Z, \tau = T/Z)$, so that $\mathcal{L}_0 = \{\tau = 0\}$ and $\mathcal{L}_1 = \{\lambda = 0\}$. In these coordinates,

$$(\lambda_{n+1}, \tau_{n+1}) = \left(\lambda_n^2 \left(\frac{\lambda_n + 2\tau_n}{1 + \tau_n^2}\right)^2, \tau_n^2 \left(\frac{1 + (\tau_n + \lambda_n)^2}{1 + \tau_n^2}\right)^2\right) := \mathcal{R}(\lambda_n, \tau_n). \tag{5.5}$$

As before, $\mathcal{R}|\mathcal{L}_0 : \lambda \to \lambda^4$ and $\mathcal{R}|\mathcal{L}_1 : \tau \to \tau^2$.

We continue by exploring the some preliminary consequences of the hypothesis that $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is real analytic in such a full neighborhood of $\mathcal{B}$.

Proposition 5.2.1 *If $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is real analytic in a full neighborhood of $\mathcal{B}$, then there is another neighborhood Ω_0 of $\mathcal{B}$ and a holomorphic function $\varphi\colon \Omega_0 \to \mathbb{C}$ such that*

(i) if $(\lambda, \tau) \in \Omega_0$ and $\mathcal{R}(\lambda, \tau) \in \Omega_0$, then $\varphi(\mathcal{R}(\lambda, \tau)) = \varphi(\lambda, \tau)^4$,

(ii) $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$, and

(iii) $\varphi(\lambda, 0) = \lambda$.

The function φ is analogous to the one constructed in the Proof of Theorem A'. However, Proposition 5.2.1 only gives that φ is defined on a small neighborhood of $\mathcal{B}$, which may not be forward invariant under $\mathcal{R}$.

We will exploit the fact that each $x \in B$ is hyperbolic, emitting a stable manifold $\mathcal{W}^s_{\mathrm{loc}}(x)$ that is a one-dimensional holomorphic curve transverse to $\mathcal{L}_0$. Together, the union of stable manifolds of each $x \in B$ forms a foliation of $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$; see [3, Proposition 9.2].

The notion of Levi-flat real-codimension 1 hypersurfaces $\Sigma \subset \mathbb{C}^n$ will be useful; for background see [26, 27]. A C^2 hypersurface Σ is Levi flat if though each point of Σ there is a complex codimension 1, holomorphic hypersurface. The union of these hypersurfaces is called the *Levi foliation of* Σ. Thus, the preceding paragraph shows that $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is Levi flat. Note that there is another, more common but equivalent, definition of Levi-flat given in terms of vanishing an appropriate Levi $(1,1)$-form [26, page 126].

Rea's Theorem [28] holds in any codimension, but here we need only

Theorem 5.2.2 (Rea) *Suppose Σ is a Levi-flat, real analytic hypersurface defined on some open $\Omega_0 \subset \mathbb{C}^n$. Then there is a neighborhood $\Omega \subset \Omega_0$ of Σ to which the Levi foliation extends uniquely and holomorphically.*

We include a sketch of the proof, as it is rather simple in this case.

In a neighborhood of any $x \in \Sigma$, we can choose a holomorphic coordinate system $(u_1, \ldots, u_n)$ such that $\Sigma = \{\mathrm{Im} u_n = 0\}$. See [27, Remark 4.3]. In these coordinates, the Levi foliation has leaves given by $u_n = a \in \mathbb{R}$. Thus, a holomorphic extension of the foliation is obtained by letting a be complex (with small imaginary part).

To see that the extension is unique, suppose $(v_1, \ldots, v_n)$ is another holomorphic coordinate system defined in a neighborhood of x also with $\Sigma = \{\mathrm{Im} v_n = 0\}$. In these coordinates, the Levi foliation and extension are given analogously. To see that the resulting extension is the same as that obtained using the u-coordinates, it sufficed to show that the change of coordinates $(v_1, \ldots, v_n) = \psi(u_1, \ldots, u_n)$ maps vertical hyperplanes $u_n = a$ to vertical hyperplanes $v_n = b$. Since $\psi(\{\mathrm{Im} u_n = 0\}) = \{\mathrm{Im} v_n = 0\}$, and biholomorphisms send holomorphic hypersurfaces to holomorphic hypersurfaces, this is true for any real a. Hence, it is true for any complex a (where ψ is defined).

Proof [Proof of Proposition 5.2.1] As stated above, $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is foliated by a family $\mathcal{F}$ of holomorphic stable curves at each point in $\mathcal{B}$, so it's Levi flat. Since $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is assumed to be real analytic, Rea's Theorem implies that this Levi foliation extends to be a complex analytic foliation in a neighborhood of $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$. Since the foliation $\mathcal{F}$ is transverse to $\mathcal{L}_0$ at points of $\mathcal{B}$, in a small enough neighborhood $\tilde{\Omega}$, each curve γ_x of the foliation is transverse to $\mathcal{L}_0$. Then we may assume Ω is the union of connected components in $\tilde{\Omega}$ of any leaf that intersects $\tilde{\Omega} \cap \{\lambda = 0\}$. Let $\varphi \colon \Omega \to \mathbb{C}$ be the map assigning to each $(\lambda, \tau) \in \Omega$ the point where $\gamma_{(\lambda,\tau)}$ intersects $\tau = 0$. From this, (ii) and (iii) follow immediately. Note that it follows from a change of coordinates and the Implicit Function Theorem that φ is holomorphic.

Define Ω_0 to be the connected component of $\mathcal{R}^{-1}(\Omega) \cap \Omega$ containing $\mathcal{B}$. For each τ_0 with $|\tau_0|$ sufficiently small, let $\mathcal{L}_{\tau_0} := \{\tau = \tau_0\}$. Observe that $\mathcal{B}_{\tau_0} := \mathcal{W}^s_{\text{loc}}(\mathcal{B}) \cap \mathcal{L}_{\tau_0}$ is a topological circle. Since $\mathcal{B}_{\tau_0} \subset \mathcal{W}^s_{\text{loc}}(\mathcal{B})$, (i) holds on $\mathcal{B}_{\tau_0}$ and, by uniqueness properties of holomorphic functions, it holds in some open neighborhood of $\mathcal{B}_{\tau_0}$ within $\mathcal{L}_{\tau_0}$. Varying τ_0, these neighborhoods form an open neighborhood of $\mathcal{B}$ contained in Ω_0 on which (i) holds. This property then extends to all of Ω_0, since Ω_0 is connected.

■

We can suppose that the domain Ω_0 on which φ is defined, given by Proposition 5.2.1, is sufficiently small, so that it is contained in $\mathcal{W}^s(\mathbf{0}) \cup \mathcal{W}^s_{\text{loc}}(\eta) \cup \mathcal{W}^s_{\text{loc}}(\mathcal{B}) \cup \mathcal{W}^s_{\text{loc}}(\eta')$. Since $\mathcal{B}$ has a local product structure, it is isolated in the recurrent set. Proof of this is similar to [29, Proposition 4.4]. Thus, we can choose Ω_0 smaller if necessary so that each orbit enters and leaves Ω_0 at most once.

Proposition 5.2.2 *The domain Ω_0 may be extended to Ω, a neighborhood of $\mathcal{L}_0 \setminus \{\eta', \eta\}$, such that $\varphi \colon \Omega \to \mathbb{C}$ is holomorphic,*

(i) If $(\lambda, \tau) \in \Omega$ and $\mathcal{R}(\lambda, \tau) \in \Omega$, then $\varphi(\mathcal{R}(\lambda, \tau)) = \varphi(\lambda, \tau)^4$,

(ii) $\mathcal{W}^s_{\text{loc}}(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$, and

(iii) $\varphi(\lambda, 0) = \lambda$ for $x \in \mathcal{L}_0 \setminus \{\eta', \eta\}$.

In general, the push-forward of a function by a mapping is not well-defined. However, if the mapping is proper, then it is well-defined by averaging over the fibers. It was shown in [3, Sec. 4.5] that $\mathcal{R}$ has topological degree eight. In the proposition below, we mimic this push forward under a proper mapping.

Proof Let $\Omega_n := \{x \colon \mathcal{R}^{-n}\{x\} \subseteq \Omega_0\}$ and C_n be the critical value set for $\mathcal{R}^n$. For $x \in \Omega_n \setminus C_n$, we may define

$$\varphi(x) = \frac{1}{8^n} \sum_{i=1}^{8^n} \varphi(y_i)^4, \tag{5.6}$$

where $\{y_i\}_{i=1}^{8^n} = \mathcal{R}^{-n}\{x\}$. Then locally about each $x \in \Omega_n \backslash C_n$, φ is holomorphic since each branch of $\mathcal{R}^{-n}$ is holomorphic by the Inverse Function Theorem. If x follows a nontrivial loop around C_n, then $\varphi(x)$ has no monodromy since we are averaging over all of the fibers in (5.6). Moreover, since $|\varphi|$ is bounded on Ω_0, (5.6) implies $|\varphi|$ is also bounded on $\Omega_n \setminus C_n$. Therefore, by the Riemann Extension Theorem, φ can be extended through the critical value curves to be holomorphic on all of Ω_n.

If $x \in \Omega_n \cap \Omega_m$ with $n \geq m \geq 0$, then $\mathcal{R}^{-n}\{x\}, \mathcal{R}^{-m}\{x\} \subset \Omega_0$. Since any orbit enters and leaves Ω_0 at most once, for any $y_i \in \mathcal{R}^{-m}\{x\}$ and each $z_j \in \mathcal{R}^{m-n}\{y_i\}$, we have that $z_j, \mathcal{R}(z_j), \ldots, \mathcal{R}^{n-m}(z_j) = y_i \in \Omega$. Thus, $\varphi(y_i) = \varphi(\mathcal{R}^{n-m}(z_j)) = \varphi(z_j)^{4^{n-m}}$ since (i) holds on Ω_0. This implies

$$\frac{1}{8^m} \sum_{y_i \in \mathcal{R}^{-m}(x)} \varphi(y_i)^{4^m} = \frac{1}{8^n} \sum_{z_j \in \mathcal{R}^{-n}(x)} \varphi(z_j)^{4^n},$$

so that the two definition of φ agree in $\Omega_n \cap \Omega_m$.

We obtain a well-defined holomorphic function φ on

$$\Omega_\infty := \bigcup_{n=0}^{\infty} \Omega_n. \tag{5.7}$$

Then we define Ω to be the connected component of $\mathcal{R}^{-1}(\Omega_\infty) \cap \Omega_\infty$ containing $\mathcal{B}$. Now (i) holds on all of Ω using the exactly the same proof as in Proposition 5.2.1.i.

Since $\mathcal{L}_0$ is forward invariant, Ω_0 intersects $\mathcal{L}_0$, and $\mathcal{R}|\mathcal{L}_0$ is $\lambda \mapsto \lambda^4$, it follows that Ω contains $\mathcal{L}_0 \setminus \{\eta', \eta\}$. The fact that $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B}) = \{|\varphi(\lambda, \tau)| = 1\}$ also follows from the fact that $\Omega_0 \subset \Omega$. ∎

5.3 Local Properties Near the Fixed Point

In order to study the geometry of the extended domain Ω and the properties of φ, several technical results about the dynamics near η' will be required. We may choose $\varepsilon > 0$ sufficiently small so that the bidisk

$$X_\varepsilon := \{|\lambda| < \varepsilon, |\tau| < \varepsilon\}, \tag{5.8}$$

is forward invariant, and $\mathcal{R}$ strictly decreases each component in modulus. We continue by describing the trajectory of orbits as they converge to η'.

Proposition 5.3.1 *If $\varepsilon > 0$ is sufficiently small, then for any $\gamma \in \mathbb{Z}_+$, if $(\lambda_0, \tau_0) \in X_\varepsilon \setminus \mathcal{L}_0$, then $|\lambda_n|/|\tau_n|^\gamma \to 0$.*

This proposition implies that any point near η' and not on $\mathcal{L}_0$ converges to η' with an arbitrarily high degree of tangency to $\mathcal{L}_1$.

Proof We first proof the proposition when $|\lambda_0| \le |\tau_0|^\gamma$. Let $w_n := \lambda_n/\tau_n^\gamma$, so that

$$w_{n+1} = \frac{\lambda_{n+1}}{\tau_{n+1}^\gamma} = \frac{\lambda_n^2}{\tau_n^{2\gamma}} \left(\frac{(1+\tau_n^2)^{\gamma-1}(\lambda_n + 2\tau_n)}{(1+(\tau_n+\lambda_n)^2)^\gamma} \right)^2 = w_n^2 \tau_n^2 \left(\frac{(1+\tau_n^2)^{\gamma-1}(2 + w_n \tau_n^{\gamma-1})}{(1+\tau_n(1+w_n\tau_n^\gamma)^2)^\gamma} \right)^2. \tag{5.9}$$

In the (τ, w) coordinates, $(0,0)$ is a superattracting fixed point for $\mathcal{R}$. Then there is a $\delta > 0$ such that any point with $|\tau|, |w| < \delta$ is in $\mathcal{W}^s((0,0))$. The closed disk $\{\tau = 0, |w| \le 1\}$ collapses to $(0,0)$. By continuity, there exists $\varepsilon > 0$ such that

$$\mathcal{R}(\{|\tau| < \varepsilon, |w| \le 1 + \varepsilon\}) \subset \{|\tau|, |w| < \delta\} \subset \mathcal{W}^s((0,0)). \tag{5.10}$$

Thus, for $(\lambda_0, \tau_0) \in X_\varepsilon$ with $\varepsilon > 0$ sufficiently small, if $|\lambda_0| \le |\tau_0|^\gamma$, then the result follows.

Now it suffices to show that if $\tau_0 \ne 0$, then there is some $N \ge 0$ so that $|\lambda_n| \le |\tau_n|^\gamma$ for any $n \ge N$. Let

$$M_1 = \min_{(\lambda,\tau)\in \overline{X}_\varepsilon} \left| \frac{1+(\tau+\lambda)^2}{1+\tau^2} \right|^2 \quad \text{and} \quad M_2 = \max_{(\lambda,\tau)\in \overline{X}_\varepsilon} 9 \left| \frac{1}{1+\tau^2} \right|^2. \tag{5.11}$$

As long as $|\lambda_n| \ge |\tau_n|^\gamma$, we have

$$|\tau_{n+1}| \ge M_1 |\tau_n|^2 \text{ and } |\lambda_{n+1}| \le M_2 |\lambda_n|^{2+2/\gamma}. \tag{5.12}$$

This implies that

$$|\tau_n| \ge A_1 \rho_1^{2^n} \quad \text{and} \quad |\lambda_n| \le A_2 \rho_2^{(2+2/\gamma)^n} \tag{5.13}$$

for some $A_i > 0$ and $0 < \rho_i < 1$. Then

$$\frac{|\lambda_n|}{|\tau_n|^\gamma} \le \frac{A_2}{A_1} \frac{\rho_2^{(2+2/\gamma)^n}}{\rho_1^{\gamma 2^n}} = A \rho_2^{(2+2/\gamma)^n - a\gamma 2^n} \to 0, \tag{5.14}$$

where $\rho_1 = \rho_2^a$ and $A = A_2/A_1$. Thus, for some iterate m, we have $|\lambda_m| \le |\tau_m|^\gamma$. ∎

Consider the "bullet-shaped" regions $B_{\gamma,c} := \{(\lambda,\tau)\colon |\lambda| \geq c|\tau|^\gamma\}$, and let $B_\gamma \equiv B_{\gamma,1}$. We will use the following horizontal and vertical cones:

$$C^h := \{|\tau| \leq |\lambda|\} \text{ and } C^v := \{|\tau| \geq |\lambda|\}, \tag{5.15}$$

noting that $C^h = B_1$.

Corollary 5.3.1 *If $\varepsilon > 0$ is sufficiently small, then for any $\gamma \in \mathbb{Z}_+$,*

$$\mathcal{R}^{-1}(B_\gamma) \cap X_\varepsilon \subset B_\gamma.$$

Corollary 5.3.2 *For any $\gamma \in \mathbb{Z}_+$, $\displaystyle\bigcap_{n=0}^{\infty} \mathcal{R}^{-n}(B_\gamma) \cap X_\varepsilon = \mathcal{L}_0 \cap X_\varepsilon$*

Lemma 5.3.3 *For any sufficiently small $\varepsilon > \sigma > 0$ and any $\gamma \in \mathbb{Z}_+$, there exist $m \in \mathbb{Z}_+$ such that $\mathcal{R}^{-m}(B_\gamma) \cap \left(\overline{X}_\varepsilon \setminus X_\sigma\right) \subset C^h$.*

Proof Consider the compact set $K := \left(\overline{X}_\varepsilon \setminus X_\sigma\right) \cap C^v$. It suffices to prove that there exists $m \in \mathbb{Z}_+$ such that $\mathcal{R}^m(K) \subset X_\varepsilon \setminus B_\gamma$. By the proof of Proposition 1.6, for each $x \in K$, there exists m_x such that for any $m \geq m_x$, $\mathcal{R}^m x \in X_\varepsilon \setminus B_\gamma$, which is open. Then there is an open neighborhood U_x of x such that $\mathcal{R}^{m_x}(U_x) \subset X_\varepsilon \setminus B_\gamma$. Since K is compact, there exists m such that for any $x \in K$, $\mathcal{R}^m(x) \in X_\varepsilon \setminus B_\gamma$. ■

Recall that $\mathcal{R}|\mathcal{L}_0$ is $\lambda \mapsto \lambda^4$ and $\mathcal{R}|\mathcal{L}_1$ is $\tau \mapsto \tau^2$. The following distortion estimates allow local approximation of these properties near η'. Also, recall the notation $(\lambda_n, \tau_n) = \mathcal{R}^n(\lambda_0, \tau_0)$. Lastly, given two sequences x_n and y_n, we will use $x_n \asymp y_n$ to mean that $a \leq |x_n/y_n| \leq A$ for some constants $0 < a < A$.

Proposition 5.3.2 *For $\varepsilon > 0$ sufficiently small and any $\gamma \geq 1$,*

(i) If $(\lambda_i, \tau_i) \in B_\gamma \cap X_\varepsilon$ for $i = 0, \ldots, n$, then $|\lambda_n| \asymp |\lambda_0|^{4^n}$.

(ii) If $(\lambda_i, \tau_i) \in X_\varepsilon \setminus B_\gamma$ for $i = 0, \ldots, n$, then $|\tau_n| \asymp |\tau_0|^{2^n}$.

Proof Let

$$A_i \;=\; \frac{1}{|\lambda_{n-i}|^2}\left|\frac{\lambda_{n-i}+2\tau_{n-i}}{1+\tau_{n-i}^2}\right|^2 \;\le\; 1+5\left|\frac{\tau_{n-i}}{\lambda_{n-i}}\right|, \tag{5.16}$$

so that $|\lambda_{n-i+1}| = A_i|\lambda_{n-i}|^4$. Inductively, we have

$$|\lambda_n| \;=\; \left(\prod_{i=1}^{n} A_i^{4^{i-1}}\right)|\lambda_0|^{4^n}. \tag{5.17}$$

Recall the constants $M_1 \le 1 \le M_2$ from the proof of Proposition 5.3.1, which are independent of γ. We have $|\tau_n| \ge (M_1|\tau_{n-i}|)^{2^i}$ and $|\lambda_n| \le (M_2|\tau_{n-i}|)^{(2+2/\gamma)^i}$, so it follows that

$$\left|\frac{\tau_n}{\lambda_n}\right| \ge \frac{M_1^{2^i}}{M_2^{(2+2/\gamma)^i}}\left|\frac{\tau_{n-i}}{\lambda_{n-i}^{(1+1/\gamma)^i}}\right|^{2^i}. \tag{5.18}$$

This implies there is a $0<\delta<1$ such that

$$5\left|\frac{\tau_{n-i}}{\lambda_{n-i}}\right| \;\le\; 5(M_2|\lambda_{n-i}|)^{(1+1/\gamma)^i-1}\frac{M_2}{M_1}\left|\frac{\tau_n}{\lambda_n}\right|^{1/2^i} \;\le\; \delta^{(1+1/\gamma)^i}, \tag{5.19}$$

since M_2 is a fixed constant, $|\lambda_{n-i}|<\varepsilon$, and we can choose ε as small as we like.

It suffices to find uniform constants to estimate the product $\prod_{i=1}^{n} A_i^{4^{i-1}}$ independent n. Observe

$$\prod_{i=1}^{n} A_i^{4^{i-1}} \;\le\; \prod_{i=1}^{n}\left(1+5\left|\frac{\tau_{n-i}}{\lambda_{n-i}}\right|\right)^{4^{i-1}} \;\le\; \prod_{i=1}^{\infty}\left(1+\delta^{(1+1/\gamma)^i}\right)^{4^{i-1}}, \tag{5.20}$$

where the last product converges since

$$\sum_{i=1}^{\infty} 4^{i-1}\log\left(1+\delta^{(1+1/\gamma)^i}\right) \tag{5.21}$$

converges. Thus, there is a constant A such that for any n, $\prod_{i=1}^{n} A_i^{4^{i-1}} \le A$.

A similar calculation can be done to find a uniform lower bound for the product. Moreover, the proof for the vertical distortion control is similar (and easier). ■

Consider $\mathcal{R}^*(\mathcal{L}_1)$, the pullback of the curve $\mathcal{L}_1 = \{T=Y\}$, given by

$$-\,Z^2(T-Y)^2(T+Y)^2 = 0. \tag{5.22}$$

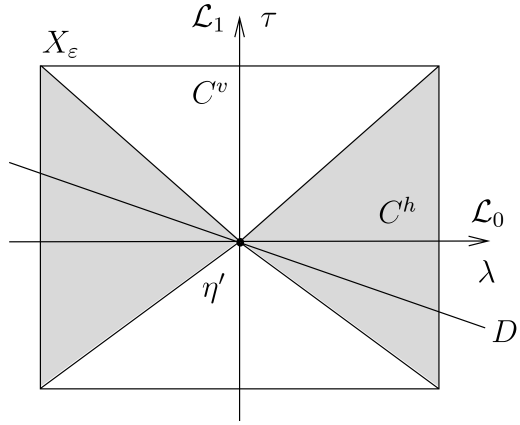

Figure 5.1. Bidisk neighborhood of η'

The pullback of $\mathcal{L}_1$ contains $\mathcal{L}_1$, $\{Z = 0\}$, and $\{T + Y = 0\}$ (each counted with multiplicity two). Call this last curve D, so in (λ, τ) coordinates,

$$D \;:=\; \{\lambda + 2\tau = 0\}. \tag{5.23}$$

Lemma 5.3.4 *If $x \in X_\varepsilon \setminus B_3$ and ε is sufficiently small, then $\mathcal{R}^{-1}\{x\} \cap C^h \neq \emptyset$ and $\mathcal{R}^{-1}\{x\} \cap C^v \neq \emptyset$.*

Proof Let $N := \{|\lambda| < \frac{1}{2}|\tau|^2\}$, and note that if $x \in X_\varepsilon \setminus B_3$, then $x \in N \cap X_\varepsilon$. Suppose $x \in N \cap X_\varepsilon$ and let $(\lambda, \tau) \in \mathcal{R}^{-1}\{x\}$. Recall that the line $D := \{\lambda + 2\tau = 0\}$ has $\mathcal{R}(D) = \mathcal{L}_1$. Also, note that N is the union over $|c| \leq 1/2$ of the curves $P_c := \{\lambda = c\tau^2\}$, and the preimage of any of these curves, $\mathcal{R}^{-1}(P_c)$, is the set of points satisfying

$$\lambda^2 \left(\frac{\lambda + 2\tau}{1 + \tau^2}\right)^2 \;=\; c\tau^4 \left(\frac{1 + (\lambda + \tau)^2}{1 + \tau^2}\right)^4. \tag{5.24}$$

It follows that if $\varepsilon > 0$ is small enough that $\left|\sqrt{c}\frac{(1+(\lambda+\tau)^2)^2}{1+\tau^2}\right| \leq 1$, then $\mathcal{R}^{-1}(P_c)$ is a set of points that satisfies

$$\left|\frac{\lambda}{\tau}\right| \frac{|\lambda + 2\tau|}{|\tau|} \;\leq\; 1. \tag{5.25}$$

Since the curve P_c is tangent to $\mathcal{L}_1$ and $\mathcal{R}(D\cup\mathcal{L}_1) = \mathcal{L}_1$, $\mathcal{R}^{-1}(P_c)$ must have a branch tangent to $\mathcal{L}_1$ and another branch tangent to D. Moreover, by (5.25), these preimage curves must be contained in C^v and C^h respectively. Thus, there is a preimage in C^h and another in C^v. ■

With a small amount of additional work, one can show that any point $x \in X_\varepsilon$ with ε sufficiently small has a preimage under the second iterate of $\mathcal{R}$ contained in $C^h \cap X_\varepsilon$.

Lemma 5.3.5 *For any sufficiently small $\varepsilon > 0$ and any $k \in \mathbb{Z}_+$, there exist $\sigma > 0$ and $\gamma \in \mathbb{Z}_+$ such that if $x \in X_\sigma \setminus B_\gamma$, then x has a preorbit $\{x^v_{k,i}\}_{i=1}^k$ of length at least k contained in $C^v \cap X_\varepsilon$.*

Proof Let $\mathcal{R}(\lambda,\tau) = (\lambda',\tau') \in X_\sigma \setminus B_\gamma$, so there is a $\delta_1 > 0$ such that

$$1 \;\geq\; \frac{|\lambda'|}{|\tau'|^\gamma} \;\geq\; \frac{|\lambda|^2}{|\tau|^{2\gamma}}|\lambda + 2\tau|^2(1-\delta_1)^{2(\gamma-1)}. \tag{5.26}$$

For large enough γ and small enough σ, Lemma 5.3.4 implies there is some preimage $(\lambda,\tau) \in C^v$. Then $|\tau| \leq |2\tau + \lambda|$, so

$$1 \;\geq\; \frac{|\lambda|}{|\tau|^{\gamma-1}}(1-\delta_1)^{\gamma-1}. \tag{5.27}$$

There are δ_i for $i = 2,\ldots,\gamma-2$ so that after repeating this process, we have $\mathcal{R}^{\gamma-2}(\lambda_0,\tau_0) \in X_\varepsilon \setminus B_\gamma$ with

$$1 \;\geq\; \frac{|\lambda_0|}{|\tau_0|^3}(1-\delta_1)^{\frac{\gamma-1}{2\gamma-4}}(1-\delta_2)^{\frac{\gamma-2}{2\gamma-3}}\cdots(1-\delta_{\gamma-2})^{\frac{4}{2}}(1-\delta_{\gamma-3})^3. \tag{5.28}$$

Pick σ small enough and $\gamma \geq k+3$ so that (5.28) implies $(\lambda_0,\tau_0) \subset C^v \cap X_\sigma$ and $\mathcal{R}^{-k}\{x\} \subset X_\varepsilon$. ■

Lemma 5.3.6 *For any $\gamma \in \mathbb{Z}_+$, there exists $\sigma > 0$ such that $B_\gamma \cap X_\sigma \subset \Omega$.*

Proof By Proposition 5.2.2, Ω contains some neighborhood of $\mathcal{L}_0 \setminus \{\eta',\eta\}$. By Lemma 5.3.2, there exists $\varepsilon > 0$ sufficiently small so that for any $\gamma \in \mathbb{Z}_+$, the

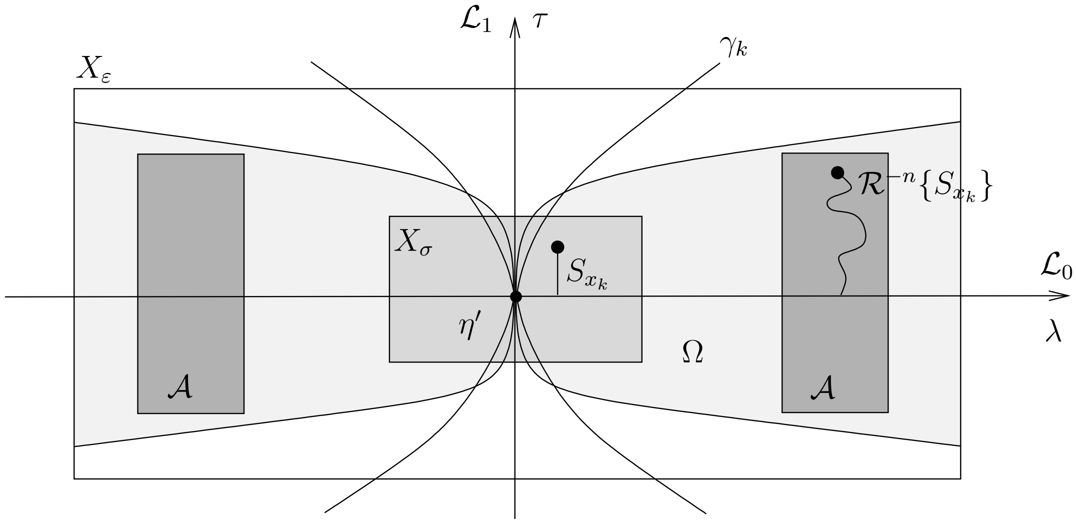

Figure 5.2. X_σ (medium gray), $\mathcal{A}$ (dark gray), and Ω (light gray)

horizontal distortion estimates can be applied in $B_\gamma \cap X_\varepsilon$. Let $\mathcal{A} := \{a\varepsilon^{4^{j+2}} < |\lambda| < 2A\varepsilon^{4^j}, |\tau| < \delta\}$, where a and A are the constants from the distortion estimate, $j \in \mathbb{Z}_+$ is chosen so that $\mathcal{A} \subset X_\varepsilon$, and $\delta < 0$ is chosen small enough so that $\mathcal{A} \subset \Omega$. See Figure 5.2.

Let $x = (\lambda_0, \tau_0) \in B_\gamma \cap X_\sigma$ and S_x be the real straight line path connecting x to $(\lambda_0, 0) \in \mathcal{L}_0$. If $\sigma < \varepsilon$ is sufficiently small, then by Corollary 5.3.2 and the horizontal distortion estimates, there is an integer n such that both $\mathcal{R}^{-n}\{S_x\}, \mathcal{R}^{-n+1}\{S_x\} \subset \mathcal{A}$. Then $S_x \subset \Omega_\infty$ and $S_x \subset \mathcal{R}^{-1}(\Omega_\infty)$, and since S_x is connected and intersects $(\mathcal{L}_0 \setminus \{\eta', \eta\}) \subset \Omega$, we have that $x \in S_x \subset \Omega$. ■

Proposition 5.3.3 *For any sequence $\{x_m\} \subset \Omega$, if $x_m \to \eta'$, then $\varphi(x_m) \to 0$.*

Proof By Lemma 5.3.6, there exists $\sigma > 0$ such that $B_3 \cap X_\sigma \subset \Omega$. By the uniformity of φ on compact sets and the fact that $\varphi|\mathcal{L}_0 = id$, if $\delta > 0$ small enough, then $\mathcal{A} := \{\sigma^{4^2} < |\lambda| < \sigma, |\tau| < \delta\} \subset B_3$, and $|\varphi(x)| < 2\sigma$ for $x \in \mathcal{A}$. By Lemma 5.3.4, there is a point in the preimage of each $x_m \in X_\sigma \setminus B_3$ contained in B_3, and Corollary 5.3.1, B_3 is backward invariant. Thus, there is a backward orbit of each x_m that remains in $B_3 \subset \Omega$. Let $\{x_{m,n}\}$ be this preorbit. If x_m sufficiently close to η', then by Corollary 5.3.2 there is an $N(m)$ such that $x_{m,N(m)} \in \mathcal{A}$. Using the invariance $\varphi(\mathcal{R}^n(x)) = \varphi(x)^{4^n}$, we have

$$|\varphi(x_m)| \;=\; |\varphi(x_{m,N})^{4^N}| \;<\; (2\sigma)^{4^N}. \tag{5.29}$$

As m goes to infinity, we need N to go to infinity as well in order for $x_{m,N}$ to remain in $\mathcal{A}$. This implies that the $\lim_{m\to\infty} |\varphi(x_m)| = 0$. ■

5.4 Proof of Non-analyticity

Proposition 5.4.1 *For any $\varepsilon > 0$ sufficiently small, there is a sequence $\{x_k\}$ converging to η' such that for each k, x_k has a preorbit of length k contained in $C^v \cap X_\varepsilon$ and a preorbit of length k contained in $C^h \cap X_\varepsilon$. Moreover, any preimage of x_k that is in X_ε is in Ω.*

Proof By Lemma 5.3.6, there exists $\varepsilon > 0$ sufficiently small so that $X_\varepsilon \cap C^h \subset \Omega$. For each $k \in \mathbb{Z}_+$, we do the following. Using Lemma 5.3.5, there exists $\gamma \in \mathbb{Z}_+$ and $\sigma > 0$ such that $x_k \in X_\sigma \setminus B_\gamma$ has a preorbit $x^v_{k,i} \subset C^v$ of length at least k. Supposing that σ is smaller if necessary, we can assure that $\mathcal{R}^{-k}\{x_k\} \subset X_\varepsilon$. Requiring that $\gamma \geq 3$, Lemma 5.3.4 implies that x_k has a first preimage, $x^h_{k,1}$, in C^h. Since C^h is backward invariant by Corollary 5.3.1, x_k has a preorbit $x^h_{k,i} \subset C^h$ of length at least k.

It remains to show that any preimage of x_k that is in X_ε is in Ω. First note that by Lemma 5.3.6, we can choose σ smaller if necessary so that $(B_{\gamma+1} \cap X_\sigma) \subset \Omega$. By Lemma 5.3.3, there is an $m \in \mathbb{Z}_+$ such that $\mathcal{R}^{-m}(B_{\gamma+1}) \cap (X_\varepsilon \setminus X_\sigma) \subset C^h$. Let $0 < \tilde{\sigma} < \sigma$ be sufficiently small that if $x \in X_{\tilde{\sigma}}$, then $\mathcal{R}^{-m}\{x\} \subset X_\sigma$. Let $x_k \in (B_{\gamma+1} \setminus B_\gamma) \cap X_{\tilde{\sigma}}$. Using that $B_{\gamma+1}$ is backward invariant, any preimage of x_k that is in X_σ will be in $(B_{\gamma+1} \cap X_\sigma) \subset \Omega$. Meanwhile, by the choice of $\tilde{\sigma}$, any preimage that is in $X_\varepsilon \setminus X_\sigma$ will be in $X_\varepsilon \cap C^h \subset \Omega$. ■

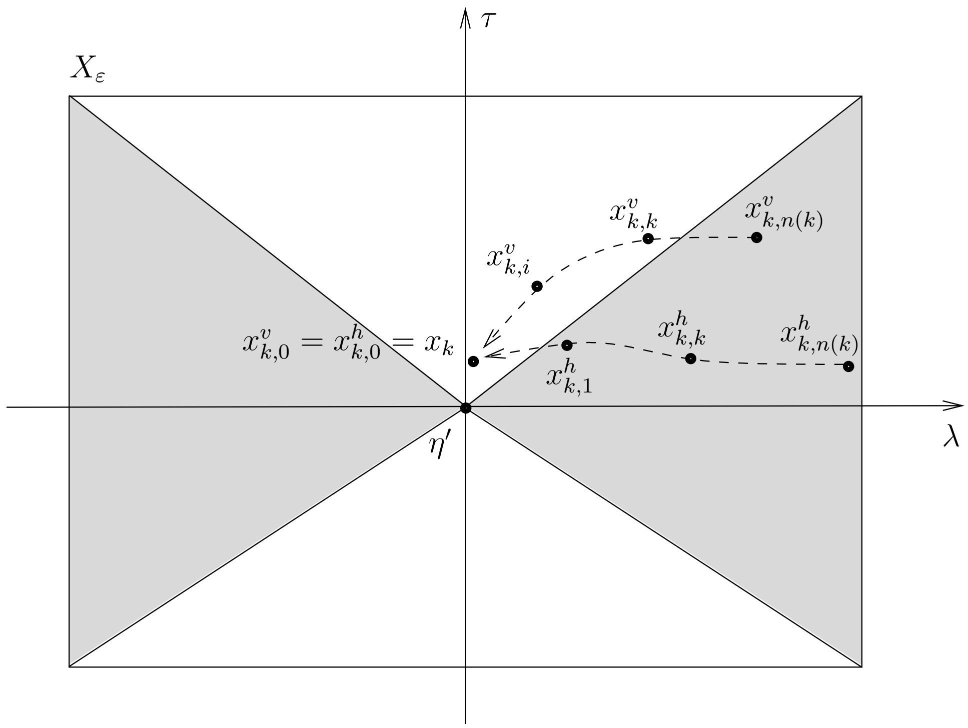

Figure 5.3. The preorbits $\{x^v_{k,i}\}$ and $\{x^h_{k,i}\}$

Proof [Proof of Proposition 5.1.1] Let $\{x_k\} \subset \Omega$ be a sequence as described in Proposition 5.4.1, and for each k, let $\{x^v_{k,i}\}_{i=1}^k \subset C^v$ and $\{x^h_{k,i}\}_{i=1}^k \subset C^h$ be preorbits of length k such that $x^v_{k,0} = x^h_{k,0} = x_k$. Each preorbit $\{x^h_{k,i}\}_{i=1}^k$ can be extended to a preorbit $\{x^h_{k,i}\}_{i=1}^{n(k)}$ with the element $x^h_{k,n(k)}$ being the last preimage remaining in X_ε. See Figure 5.3. Note that by Proposition 5.4.1 for any $0 \le i \le n(k)$, we have both $x^v_{k,i}, x^h_{k,i} \in \Omega$.

We first show there is a subsequence of $\{x^h_{k,n(k)}\}$, that converges to a point in $\mathcal{L}_0 \setminus \{\eta', \eta\}$. By construction, $x^h_{k,n(k)}$ is a preimage of $x^h_{k,1} \in C^h$, so

$$x^h_{k,n(k)} \in \bigcap_{i=0}^{n(k)-1} \mathcal{R}^{-i}(C^h) \cap X_\varepsilon. \tag{5.30}$$

Also by construction, $x^h_{k,n(k)} \in X_\varepsilon \setminus \mathcal{R}(X_\varepsilon)$, which has compact closure. Thus, there is some subsequence such that $x^h_{k_j,n(k_j)} \to x_*$ with

$$x_* \;\in\; \bigcap_{i=0}^{\infty} \mathcal{R}^{-i}(B_{\gamma_k}) \cap X_\varepsilon \;=\; \mathcal{L}_0 \cap X_\varepsilon. \tag{5.31}$$

However, since each $x^h_{k,n(k)} \in X_\varepsilon \setminus \mathcal{R}(X_\varepsilon)$, we must have $|x_*| \geq \varepsilon^4$.

By the vertical and horizontal distortion distortion estimates in Proposition 5.3.2, preimages of x_k are escaping X_ε faster along $x^h_{k,i}$ than $x^v_{k,i}$, so we also have $x^v_{k,n(k)} \subset X_\varepsilon$. Note that $x^v_{k,i}$ may be in C^h for $k \leq i \leq n(k)$. Then using both vertical and horizontal distortion, there is a constant A so that

$$\operatorname{dist}(x^v_{k,n(k)}, \eta') \;\leq\; A \operatorname{dist}(x_k, \eta')^{\frac{1}{2^k 4^{n-k}}} \;\asymp\; A \operatorname{dist}(x^h_{k,n(k)}, \eta')^{\frac{4^n}{2^k 4^{n-k}}} \;\leq\; A\varepsilon^{2^k}, \tag{5.32}$$

which converges to 0 as $k \to \infty$. Thus, the sequence $x^v_{k,n(k)}$ converges to η'.

By Proposition 5.3.3, $\varphi(x^v_{k,n(k)}) \to 0$ as $k \to \infty$. We also have that $|\varphi(x^h_{k_j,n(k_j)})| \to |\varphi(x_*)| \geq \varepsilon^4$ as $k \to \infty$. However, $x^h_{k_j,n(k_j)}$ and $x^v_{k_j,n(k_j)}$ are both nth preimages of x_{k_j}, and using the invariance $\varphi(\mathcal{R}^n(x)) = \varphi(x)^{4^n}$, this implies $|\varphi(x^v_{k_j,n(k)})| = |\varphi(x^h_{k_j,n(k)})|$ for every $n(k)$. Then $0 = |\varphi(x_*)| \geq \varepsilon^4$, a contradiction. ■

Lemma 5.4.1 *If $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is real analytic at $x \in \mathcal{B} \setminus \{(\pm i, 0)\}$, then $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is real analytic at $\mathcal{R}(x)$.*

Proof Images of real analytic hypersurfaces under holomorphic maps were considered by Baouendi and Rothschild [30]. Suppose that M is a germ of a real analytic hypersurface in $\mathbb{C}^N$ and H is the germ of a holomorphic map from $\mathbb{C}^N$ to $\mathbb{C}^N$ with $H(0) = 0$. The germ H is called *finite* if every point in some neighborhood of 0 has finitely many preimages. It is shown in [30, Theorem 4] that if H is finite and $M' := H(M)$ is smooth in some neighborhood of 0, then M' is actually real analytic.

We are in the position to apply this result, since $\mathcal{R}$ sends $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ from the neighborhood of any $x \in \mathcal{B}$ to $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ within a smaller neighborhood of $\mathcal{R}(x)$. However, we must avoid the vertical lines $z = \pm i$, which are collapsed by $\mathcal{R}$ to the fixed point $(1, 0) \in B$. Away from these lines, $\mathcal{R}$ is finite. ■

Proof [Proof of Theorem B'] By Proposition 5.1.1, there is some point $x \in \mathcal{B}$ at which $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is not real analytic. We will now use the fact that $\mathcal{R}$ is expanding on $\mathcal{B}$ to show that $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is not real analytic in the neighborhood of any point of $\mathcal{B}$.

Since $\mathcal{R}|\mathcal{B}$ is $z \mapsto z^4$, it is expanding on $\mathcal{B}$, so there is some iterate n such that $\mathcal{R}^n(U \cap \mathcal{B}) = \mathcal{B}$. Because we assumed $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is real analytic at every point of $U \cap \mathcal{B}$, we can use Lemma 5.4.1 iteratively to see that $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is real analytic at every point of $\mathcal{B}$, except perhaps at the iterated images of $(\pm i, 0)$. However, these consist of just the fixed point $(1, 0)$. To see that $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is real analytic at $(1, 0)$ note that $(1, 0)$ is also the image of $(-1, 0)$ under $\mathcal{R}$, where $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is real analytic. Thus, $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ must be real analytic at every point of $\mathcal{B}$, which is impossible by Proposition 5.1.1.

We now know that $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is not real analytic in the neighborhood of any point of $\mathcal{B}$. However, it could still be real analytic in the neighborhood of some other point. We now show that this is also impossible.

Each stable manifold $\mathcal{W}^s_{\rm loc}(x_0)$ can be expressed as the graph of a convergent power series:

$$z = h(t, z_0) = \sum_{j=0}^{\infty} a_j(z_0) t^j \qquad \text{where} \qquad x_0 = (z_0, 0). \tag{5.33}$$

Since each $\mathcal{W}^s_{\rm loc}(x_0)$ depends continuously on $z_0 \in \mathcal{B}$, the coefficients $a_j(z_0)$ are continuous functions of z_0. Therefore, there is a uniform radius of convergence $\delta > 0$. For the remainder of the proof, we suppose that the neighborhood in which $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is defined is contained in $|t| < \delta/3$.

Suppose $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is real analytic in a neighborhood of some x_1. Then one can express leaves of the stable foliation near x_1 as graphs of some convergent power series

$$z = k(t, z_1) = \sum_{j=0}^{\infty} b_j(z_1)(t - t_1)^j. \tag{5.34}$$

The function $(z_1, t) \mapsto (z, t)$, with z given by (5.34), gives a parametrization of $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ near x_1 with z_1 varying over the real analytic arc $\mathcal{W}^s_{\rm loc}(\mathcal{B}) \cap \{t = t_1\}$ and t varying over some complex disc centered at t_0. Since we have assumed $\mathcal{W}^s_{\rm loc}(\mathcal{B})$ is real analytic near x_1, the parameterization is an analytic function. In particular, $\frac{\partial^j}{\partial t^j} z$ is real analytic for each $j \geq 0$. Restricting to $t = t_1$ we see that each of the coefficients $b_j(z_1)$ is a real analytic function of z_1.

We now use this to show that $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is also real analytic in a neighborhood of the unique point x_0 for which $x_1 \in \mathcal{W}^s_{\mathrm{loc}}(x_0)$. Since $\mathcal{W}^s_{\mathrm{loc}}(x_0)$ is the graph of a holomorphic function over $|t| < \delta$, $|t_1| < \delta/3$ implies that (5.34) converges on the disc $|t - t_0| < \delta/2$. In particular, each of the holomorphic discs defined by (5.34) crosses all the way through $\mathcal{B}$. As they depend real analytically on z_1, this implies that $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$ is real analytic in a neighborhood of $x_0 \in \mathcal{B}$, which is not possible. ■

6. PHYSICAL INTERPRETATION

In this chapter we will relate Theorems B' to the Ising Model on the DHL. We refer the reader to [3,4] for physical background. The DHL is a sequence of graphs Γ_n obtained in a self-similar way. Associated to each graph is a partition function $\mathsf{Z}_n(z,t)$ whose zeros

$$\mathcal{S}_n^c := \{(z,t) \in \mathbb{C}^2 \,:\, \mathsf{Z}_n(z,t) = 0\}$$

describe the singularities of the Ising model associated to Γ_n. They are called the *Lee-Yang-Fisher zeros*. The actual physics is described by the limit $n \to \infty$. It is proved in [4] that the limiting distribution of zeros exists as a closed, positive $(1,1)$-current $\mathcal{S}^c$ on $\mathbb{P}^2$. In fact, $\mathcal{S}^c = \frac{1}{2}\Psi^* S$, where S is the Green current for R. The support of $\mathcal{S}^c$ describes locus where phase transitions occur in $\mathbb{C}^2$.

It is shown in [4] that at low complex temperatures supp $\mathcal{S}^c$ coincides with $\mathcal{W}^s_{\mathrm{loc}}(\mathcal{B})$. Combining Theorem B' with the work from [4] gives the following:

Corollary 6.0.2 *At low complex temperatures ($|t|$ small), the locus of phase transitions for the Ising model on the DHL is a 3 real-dimensional manifold that is C^∞ but not real analytic.*

A preferred subset of the Lee-Yang-Fisher zeros is obtained by requiring that $t \in [0,1]$, which correspond to "physical" temperatures. The Lee-Yang Circle Theorem [31,32] asserts that for each n and fixed $t_0 \in [0,1]$, zeros of partition function $\mathsf{Z}_n(z,t_0)$ corresponding to Γ_n lie on the unit circle $\mathbb{T}_{t_0} := \{|z| = 1,\, t = t_0\}$. Let

$$\mathcal{C} = \{|z| = 1, t \in [0,1]\}.$$

The *Lee-Yang zeros* are defined by

$$\mathcal{S}_n := \{(z,t) \in \mathcal{C} \,:\, \mathsf{Z}_n(z,t) = 0\}.$$

Isakov [33] proved for any $t_0 > 0$ sufficiently small the free energy for the Ising model on the $\mathbb{Z}^d$ lattice with $d > 1$ does not have analytic continuation through any point of the circle $\mathbb{T}_{t_0}$. This implies that the limiting distribution of Lee-Yang zeros for the $\mathbb{Z}^d$ lattice with $d > 1$ does not have real analytic density in the neighborhood of any point of the circle $t = t_0$. In the remainder of this chapter, we discuss how Corollary 6.0.2 can be related to Isakov's result.

One can check that $\mathcal{R}$ maps the Lee-Yang cylinder $\mathcal{C}$ into itself, with the Lee-Yang zeros corresponding to Γ_{n+1} obtained by pulling back the Lee-Yang zeros corresponding to Γ_n under $\mathcal{R}|\mathcal{C}$. The map $\mathcal{R}\colon \mathcal{C} \to \mathcal{C}$ was also studied previously by Bleher and Žalys [34].

In [3], Bleher, Lyubich, and Roeder describe the limiting distribution of Lee-Yang zeros for the DHL; let us provide a very brief summary. Let $\mathcal{C}_1 := \mathcal{C} \setminus \{t = 1\}$. It was shown that $\mathcal{R}\colon \mathcal{C}_1 \to \mathcal{C}_1$ is partially hyperbolic, with a unique central foliation $\mathcal{F}^c$ which is vertical (with respect to a suitable cone field) on $\mathcal{C}_1$. In particular, one can define the $\mathcal{F}^c$ holonomy map $g_t : \mathbb{T}_0 \to \mathbb{T}_t$. The limiting distribution of Lee-Yang zeros at temperature $t_0 \in [0, 1)$ is obtained as the pushforward $\mu_t = g_{t_0 *}\text{Leb}$, where Leb is the normalized Lebesgue measure on $\mathbb{T}_0$.

In a neighborhood of $\mathcal{B}$, $\mathcal{F}^c$ coincides with the stable foliation of $\mathcal{B}$, which is a union of the real analytic curves $\mathcal{W}^s_{\text{loc}}(x) \cap \mathcal{C}$, taken over $x \in \mathcal{B}$. It is shown in [4, Lemma 3.2] that the stable foliation of $\mathcal{B}$ within $\mathcal{C}$ has the same regularity that the stable manifold $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ does as a submanifold of $\mathbb{C}^2$. (In fact, $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ was shown to be a C^∞ manifold in [4] by first showing that the stable foliation of $\mathcal{B}$ within $\mathcal{C}$ is C^∞.)

Therefore, Theorem B' implies that the central foliation is not real analytic at low temperatures. Moreover, by [3], an open dense set of points from $\mathcal{C}$ have orbits converging to $\mathcal{B}$. Since $\mathcal{F}^c$ is invariant, this implies the following:

Theorem 6.0.3 *$\mathcal{F}^c$ is not real analytic in the neighborhood of any point of $\mathcal{C}$.*

Using the holonomy description of the limiting distribution of Lee-Yang zeros, we find the following modest analog of Isakov's Theorem for the DHL:

Corollary 6.0.4 *For any $z = e^{i\phi} \in \mathcal{B}$, there is a dense set of $t_0 \in [0,1]$ so that the limiting distribution of Lee-Yang zeros within $\mathbb{T}_{t_0}$ does not have real analytic density at (t_0, ϕ).*

7. OPEN PROBLEMS

7.1 Regularity of Superstable Manifolds When b Is Larger Than a

Theorem A naturally leads one to question whether there are necessary conditions for $\mathcal{W}^s_{\text{loc}}(S)$ to be real analytic. If $a < b$, the superattracting direction is not as strong as the expansion within L; we believe there should be a way to exploit this to answer the following question:

Question 1 *Is there a "generic" class of mappings f with $a < b$ for which $\mathcal{W}^s_{\text{loc}}(S)$ is not real analytic?*

Here we are attempting to generalize of the technique in Section 5 for the Migdal-Kadanoff renormalization that proves $\mathcal{W}^s_{\text{loc}}(\mathcal{B})$ is not real analytic. If f happens to be a product, the stable manifold will be real analytic, so the class of functions not producing a real analytic stable manifold is at best generic in some sense. The technique used for $\mathcal{R}$ (given by(5.2)) and f (given by (5.1)) relies on a second invariant line L_1 such that $f \mid L_1$ is the map $w \mapsto w^a$. We suspect one may use the degree a transversal superattraction of L to generate an invariant cone field to serve the same purpose in the general case.

Question 2 *For any a and b, is $\mathcal{W}^s_{\text{loc}}(S)$ a C^∞ manifold?*

Following the method in [3, Proposition 9.12], define the sequence $B_n(x) := \frac{1}{b^n} Df^n(x)$. It is not difficult to show B_n converges uniformly on compact subsets of $\mathcal{W}^s_{\text{loc}}(S)$ at super-exponential rate to a matrix-valued function $B(x)$. The goal is to prove this function B is C^∞ in any neighborhood of S, since one can use the invariance

$$bB_n(x) = B_{n-1}(f(x))Df(x) \text{ and } bB(x) = B(f(x))Df(x), \tag{7.1}$$

to extend the result to any compact subset of $\mathcal{W}^s(S)$. $B(x)$ and all of its derivatives are converging so fast that we believe Whitney's extension theorem could be used to extend B to a C^∞ function in a neighborhood of S. In this case, since L is transversally superattracting, $\ker B(x)$ would be a C^∞ holomorphic line field, so that one could integrate it to get a C^∞ foliation by holomorphic discs of S. Within $\mathcal{W}^s_{\text{loc}}(S)$, $\ker(B)$ is the field of tangent planes to the Levi foliation of $\mathcal{W}^s_{\text{loc}}(S)$. Therefore, $\mathcal{W}^s_{\text{loc}}(S)$ is formed as a union of the holomorphic discs from the C^∞ foliation.

The only missing piece is the application of Whitney's Extension Theorem [35] which provides a partial converse to Taylor's Theorem. Let U be an open subset of $\mathbb{R}^n$, and X a closed subset of U. As described in [36], Whitney's theorem asserts that a function f defined in X is the restriction of F^0, a C^m function in U ($m \in \mathbb{N}$ or $m = +\infty$) provided there exists a sequence $(F^k)_{|k|\leq m}$ of functions defined in X which satisfies for each $|k| \leq m$,

$$(R^m_x F)^k(y) \;:=\; F^k(y) - \sum_{|j|\leq m-|k|} \frac{F^{k+j}(x)}{j!} \cdot (y-x)^j \;=\; o(|x-y|^{m-|k|}). \tag{7.2}$$

Roughly speaking, one must control the tails of the Taylor expansion uniformly.

7.2 Lee-Yang Density

Recall Corollary 6.0.4. Unfortunately, the fact that $\mathcal{F}^s$ not real analytic at any point does not imply that none of the non-trivial holomomies are real analytic. Isakov [33] proved a similar result for Ising models on the $\mathbb{Z}^d$ lattice. However, Isakov's result required a great deal of difficult and complicated analysis. We would like to prove the analogus result:

Conjecture 1 *For temperature $0 < t < t_c$, the limiting density of Lee-Yang zeros for the DHL $\rho_t(\phi)$ is C^∞, but not real analytic.*

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