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ASYMPTOTICS OF THE FREDHOLM DETERMINANT CORRESPONDING TO THE FIRST BULK CRITICAL UNIVERSALITY CLASS IN RANDOM MATRIX MODELS

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To Nona, love you.

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ABSTRACT

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We study the one-parameter family of determinants $\det(I-\gamma K_{\mathrm{PII}}), \gamma\in\mathbb{R}$ of an integrable Fredholm operator K_{PII} acting on the interval $(-s,s)$ whose kernel is constructed out of the Ψ-function associated with the Hastings-McLeod solution of the second Painlevé equation. In case $\gamma=1$, this Fredholm determinant describes the critical behavior of the eigenvalue gap probabilities of a random Hermitian matrix chosen from the Unitary Ensemble in the bulk double scaling limit near a quadratic zero of the limiting mean eigenvalue density. Using the Riemann-Hilbert method, we evaluate the large s-asymptotics of $\det(I-\gamma K_{\mathrm{PII}})$ for all values of the real parameter γ.

1. INTRODUCTION

1.1 Objective

This dissertation is devoted to the asymptotical analysis of certain Fredholm determinants which appear in random matrix theory. Let $\mathcal{M}(n)$ be the unitary ensemble of random $n \times n$ Hermitian matrices $M = (M_{ij}) = \overline{M}^t$ equipped with the probability measure,

$$P^{(n,N)}(M)dM = ce^{-N\mathrm{tr}V(M)}dM, \qquad c\int\limits_{\mathcal{M}(n)} e^{-N\mathrm{tr}V(M)}dM = 1. \tag{1.1}$$

Here dM denotes the Haar measure on $\mathcal{M}(n) \simeq \mathbb{R}^{n^2}$, N is a fixed integer and the potential $V : \mathbb{R} \to \mathbb{R}$ is assumed to be real analytic satisfying the growth condition

$$\frac{V(x)}{\ln(x^2+1)} \to \infty \qquad \text{as } |x| \to \infty. \tag{1.2}$$

The principal object of the analysis of the model is the statistics of eigenvalues of the matrices from $\mathcal{M}(n)$. A classical fact [17, 42] is that the eigenvalues form a determinantal random point process with the kernel

$$K_{n,N}(x,y) = e^{-\frac{N}{2}V(x)}e^{-\frac{N}{2}V(y)}\sum_{i=0}^{n-1} p_i(x)p_i(y), \tag{1.3}$$

where $p_j(x)$ are polynomials orthonormal with respect to the weight $e^{-NV(x)}$,

$$\int\limits_{\mathbb{R}} p_i(x)p_j(x)e^{-NV(x)}dx = \delta_{ij}, \quad p_j(x) = \kappa_j x^j + \ldots \tag{1.4}$$

In particular, one of the basic statistical characteristics, the *gap probability*,

$$E_{n,N}(s) = \mathrm{Prob}\big(M \in \mathcal{M}(n) \text{ has no eigenvalues in the interval } (-s,s),\ s > 0\big)$$

is given by the formula,

$$\begin{aligned} E_{n,N}(s) &= \sum_{j=0}^{n} \frac{(-1)^j}{j!} \int_{-s}^{s} \cdots \int_{-s}^{s} \det\big(K_{n,N}(x_k, x_l)_{k,l=1}^{j}\big) dx_1 \cdots dx_j \\ &\equiv \det(I - K_{n,N}), \end{aligned}$$

where $K_{n.N}$ is the trace class operator acting on $L^2\big((-s,s), dx\big)$ with kernel $K_{n.N}(x,y)$.

Assumptions (1.2) on the potential $V(x)$ ensure [23] (see also [17] for more on the history of the subject) that the mean eigenvalue density $\frac{1}{n}K_{n,N}(x,x)$ has a limit,

$$\lim_{\substack{n,N\to\infty \\ \frac{n}{N}\to 1}} \frac{1}{n} K_{n,N}(x,x) = \rho_V(x) \geq 0, \tag{1.5}$$

whose support, $\Sigma_V \equiv \overline{\{x \in \mathbb{R} : \rho_V(x) > 0\}}$, is a finite union of intervals (simultaneously, $\rho_V(x)$ defines the density of the equilibrium measure for the logarithmic potentials in the presence of the external potential V). The limiting density $\rho_V(x)$ is determined by the potential $V(x)$. At the same time, the local statistics of eigenvalues in the large n, N limit satisfies the so-called *universality property*, i.e. it is determined only by the local characteristics of the eigenvalue density ρ_V (compare [9,24,47]). For instance, let us choose a regular point $x^* \in \Sigma_V$, i.e. $\rho_V(x^*) > 0$. Then the *bulk universality* states that

$$\lim_{n\to\infty} \frac{1}{n\rho_V(x^*)} K_{n,n}\left(x^* + \frac{\lambda}{n\rho_V(x^*)}, x^* + \frac{\mu}{n\rho_V(x^*)}\right) = K_{\sin}(\lambda,\mu) \equiv \frac{\sin\pi(\lambda-\mu)}{\pi(\lambda-\mu)} \tag{1.6}$$

uniformly on compact subsets of $\mathbb{R}$, which in turn implies [24] that for a regular point x^*,

$$\begin{aligned} \lim_{\substack{n,N\to\infty \\ \frac{n}{N}\to 1}} \mathrm{Prob}\left(M \in \mathcal{M}(n) \text{ has no eigenvalues} \in \left(x^* - \frac{s}{n\rho_V(x^*)}, x^* + \frac{s}{n\rho_V(x^*)}\right)\right) \\ = \det(I - K_{\sin}), \end{aligned} \tag{1.7}$$

where $K_{\sin}$ is the trace class operator on $L^2\big((-s,s); d\lambda\big)$ with kernel $K_{\sin}(\lambda,\mu)$ given in (1.6). (This result was first obtained for the Gaussian unitary ensemble with quadratic polynomial potential $V(x)$ in the classical works of Gaudin and Dyson.) The

Fredholm determinant in the right hand side of (1.7) admits the following asymptotic representation [26],

$$\ln\det(I - K_{\sin}) = -\frac{(\pi s)^2}{2} - \frac{1}{4}\ln(\pi s) + \frac{1}{12}\ln 2 + 3\zeta'(-1) + O\big(s^{-1}\big), \quad s \to \infty, \tag{1.8}$$

where $\zeta'(z)$ is the derivative of the Riemann zeta-function (a rigorous proof for this expansion without the constant term was obtained independently by Widom and Suleimanov - see [22] for more historical details; a rigorous proof including the constant terms was obtained independently in [27, 41] - see also [18]). This remarkable formula yields one of the most important results in random matrix theory, i.e. an explicit evaluation of *the large gap probability.*

Equation (1.7) shows that in double scaling limits the basic statistical properties of hermitian random matrices are still expressible in terms of Fredholm determinants. This is also true for the first critical case, when $\rho_V(x)$ vanishes quadratically at an interior point $x^* \in \Sigma_V$. However, in this situation the scaling limit is more complicated [10, 15]. Let $\rho_V(x^*) = \rho_V'(x^*) = 0, \rho_V''(x^*) > 0$ and $n, N \to \infty$ such that

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) = C$$

exists with $C \in \mathbb{R}$. Then the *critical bulk universality* guarantees existence of positive constants c and c_1 such that

$$\lim_{n,N\to\infty} \frac{1}{cn^{1/3}} K_{n,N}\left(x^* + \frac{\lambda}{cn^{1/3}}, x^* + \frac{\mu}{cn^{1/3}}\right) = K_{\mathrm{PII}}(\lambda, \mu; x) \tag{1.9}$$

uniformly on compact subsets of $\mathbb{R}$ where the variable x is the scaling parameter defined by the relation

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) = xc_1.$$

Here the limiting kernel $K_{\mathrm{PII}}(\lambda, \mu; x)$ is constructed out of the Ψ-function associated with a special solution of the second Painlevé equation. The precise description of the kernel $K_{\mathrm{PII}}(\lambda, \mu; x)$ is as follows.

Let $u(x)$ be the Hastings-McLeod solution of the Painlevé II equation [33], i.e. the unique real-valued solution to the boundary value problem

$$u_{xx} = xu + 2u^3, \qquad u(x) \sim \begin{cases} \mathrm{Ai}(x), & x \to +\infty; \\ \sqrt{-\frac{x}{2}}, & x \to -\infty, \end{cases}$$

where $\mathrm{Ai}(x)$ is the Airy-function (the solution $u(x)$ is in fact uniquely determined by its Airy-asymptotics at $x = +\infty$). Viewing x, $u \equiv u(x)$ and $u_x \equiv \frac{du(x)}{dx}$ as real parameters, consider the 2×2 system of linear ordinary differential equations,

$$\frac{\partial \Psi}{\partial \lambda} = A(\lambda, x)\Psi, \; A(\lambda, x) = -4i\lambda^2\sigma_3 + 4i\lambda \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix} + \begin{pmatrix} -ix - 2iu^2 & -2u_x \\ -2u_x & ix + 2iu^2 \end{pmatrix}. \tag{1.10}$$

Let $\Psi(\lambda) \equiv \Psi(\lambda, x)$ be the fundamental solution of system (1.10) which is uniquely fixed by the asymptotic condition,

$$\Psi(\lambda, x) = \Big(I + O\big(\lambda^{-1}\big)\Big) e^{-i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3}, \qquad \lambda \to \infty, \; 0 < \arg \lambda < \pi.$$

Then, the kernel $K_{\mathrm{PII}}(\lambda, \mu; x)$ is given by the formula,

$$K_{\mathrm{PII}}(\lambda, \mu; x) \equiv K_{\mathrm{PII}}(\lambda, \mu) = \frac{1}{2\pi}\left(\frac{\psi_{21}(\lambda, x)\psi_{11}(\mu, x) - \psi_{21}(\mu, x)\psi_{11}(\lambda, x)}{\lambda - \mu}\right), \tag{1.11}$$

where $\psi_{11}(\lambda, x)$ and $\psi_{21}(\lambda, x)$ are the entries of the matrix valued function $\Psi(\lambda, x) \equiv \big(\psi_{jk}(\lambda, x)\big)_{j,k=1,2}$.

Remark 1 *The function $\Psi(\lambda, x)$ can be alternatively defined as a solution of a certain matrix oscillatory Riemann-Hilbert problem. The exact formulation of this Riemann-Hilbert problem is given in chapter 2.*

One object of this thesis is the study of the Fredholm determinant

$$\det(I - K_{\mathrm{PII}}), \tag{1.12}$$

where K_{PII} is the trace class operator on $L^2\big((-s, s); d\lambda\big)$ with kernel (1.11). In virtue of (1.9), this determinant replaces the sine - kernel determinant in the description of the gap-probability near the critical point x^*, i.e. instead of (1.7) one has that

$$\lim \mathrm{Prob}\left(M \in \mathcal{M}(n) \text{ has no eigenvalues} \in \left(x^* - \frac{s}{cn^{1/3}}, x^* + \frac{s}{cn^{1/3}}\right)\right)$$

$$= \det(I - K_{\mathrm{PII}}), \tag{1.13}$$

as $n, N \to \infty$ and

$$\lim_{n,N\to\infty} n^{2/3}\left(\frac{n}{N} - 1\right) = xc_1.$$

(A proof can be obtained in a same manner as the proof of the similar equation (21) in [19] with the help of the proper estimates from [10].)

1.2 Statement of results

Our main result is the following analogue of the Dyson formula (1.8) for the Painlevé II - kernel determinant (1.12).

Theorem 1.2.1 *Let* K_{PII} *denote the trace class operator on* $L^2\big((-s,s);d\lambda\big)$ *with kernel* (1.11)*. Then as* $s \to \infty$ *the Fredholm determinant* $\det(I - K_{\mathrm{PII}})$ *behaves as*

$$\ln\det(I - K_{\mathrm{PII}}) = -\frac{2}{3}s^6 - s^4 x - \frac{1}{2}(sx)^2 - \frac{3}{4}\ln s + \int_x^\infty (y-x)u^2(y)dy$$

$$-\frac{1}{6}\ln 2 + 3\zeta'(-1) + O\big(s^{-1}\big), \tag{1.14}$$

and the error term in (1.14) *is uniform on any compact subset of the set*

$$\big\{x \in \mathbb{R}:\ -\infty < x < \infty\big\}. \tag{1.15}$$

The proof of Theorem 1.2.1 is based on a Riemann-Hilbert approach which is reviewed in chapter 2. This approach (compare [22, 34]) uses the integrable form of the Fredholm operator (1.12), allowing us to connect the resolvent kernel to the solution of a Riemann-Hilbert problem. The latter can be analysed rigorously via the Deift-Zhou nonlinear steepest descent method.

In order to describe other spectral properties of large Hermitian matrices we need to study the Fredholm determinant

$$\det\left(I - \gamma K_{\mathrm{PII}}\right) \tag{1.16}$$

for the values of γ which are different from $\gamma = 1$. Similar one-parameter families of determinants already appear in connection with the sine - kernel determinant, for instance in the famous Montgomery-Odlyzko conjecture [43, 46] concerning the zeros of the Riemann zeta-function, in the description of the emptiness formation probability and other correlation functions in one-dimensional impenetrable Bose gas [35–37] as well as in a number of other important mathematical and theoretical physics applications.

The analytical challenge of the determinants (1.16) is once again the large s asymptotics. In the case of the sine - kernel determinants, the result is well known (see e.g. [2, 4, 44, 45, 50] and [18] for more on the history of the question)

1. As $s \to \infty$

$$\ln \det \left(I - \gamma K_{\sin}\right) = 4i\nu\pi s + 2(i\nu)^2 \ln\left(\pi s\right) + \chi_{\sin} + O\left(s^{-1}\right)$$

uniformly on any compact subset of the set $\{\gamma \in \mathbb{R} : \ -\infty < \gamma < 1\}$, where

$$i\nu \equiv i\nu(\gamma) = \frac{1}{2\pi} \ln\left(1 - \gamma\right)$$

and the constant $\chi_{\sin} \equiv \chi_{\sin}(\gamma)$ is given by the equation

$$\chi_{\sin} = 2(i\nu)^2 + 4(i\nu)^2 \ln 2 + 2\int_0^{\gamma} \nu(t)\left(\ln\frac{\Gamma(\nu(t))}{\Gamma(-\nu(t))}\right)' dt. \tag{1.17}$$

The latter constant was obtained by A. Budylin and V. Buslaev as a corollary to their main result in [4], namely the asymptotics of the resolvent of the kernel $\gamma K_{\sin}(\lambda, \mu)$. Formula (1.17) also follows from the general theorem of E. Basor and H. Widom concerning the determinants of Toeplitz integral operators with piecewise continuous symbols [3].

2. For γ chosen from any compact subset of the set $\{\gamma \in \mathbb{R} : \ 1 < \gamma < \infty\}$, the Fredholm determinant $\det\left(I - \gamma K_{\sin}\right)$ has infinitely many zeros $\{s_n\}$ which accumulate at infinity, see [44, 45, 50].

In the given situation (1.16), we have the following analogues for the Painlevé II - kernel determinants (1.16).

Theorem 1.2.2 *Let* K_{PII} *denote the trace class operator on* $L^2\big((-s,s);d\lambda\big)$ *with kernel* (1.11). *As* $s \to \infty$

$$\ln\det\left(I - \gamma K_{\text{PII}}\right) = i\nu\left(\frac{16}{3}s^3 + 4xs\right) + 6(i\nu)^2\ln s + \chi_{\text{PII}} + O\left(s^{-1}\right) \tag{1.18}$$

uniformly on any compact subset of the set

$$\left\{(\gamma, x) \in \mathbb{R}^2 :\ -\infty < \gamma < 1,\ -\infty < x < \infty\right\}, \tag{1.19}$$

where

$$i\nu \equiv i\nu(\gamma) = \frac{1}{2\pi}\ln(1-\gamma)$$

and

$$\chi_{\text{PII}} = 2\left(i\nu\right)^2 + 8\left(i\nu\right)^2\ln 2 + 2\int_0^\gamma \nu(t)\left(\ln\frac{\Gamma\left(\nu(t)\right)}{\Gamma\left(-\nu(t)\right)}\right)' dt \tag{1.20}$$

with the Euler gamma-function $\Gamma(z)$.

Theorem 1.2.3 *For* (γ, x) *chosen from any compact subset of the set*

$$\left\{(\gamma, x) \in \mathbb{R}^2 :\ 1 < \gamma < \infty,\ -\infty < x < \infty\right\} \tag{1.21}$$

the Fredholm determinant $\det\left(I - \gamma K_{\text{PII}}\right)$ *has infinitely many zeros* $\{s_n\}$ *with asymptotic distribution*

$$\frac{8}{3}s_n^3 + 2xs_n + \frac{1}{\pi}\ln(\gamma-1)\ln\left(16s_n^3 + 4xs_n\right) - \arg\frac{\Gamma(1-\nu)}{\Gamma(\nu)} \sim \frac{\pi}{2} + n\pi, \quad n \to \infty. \tag{1.22}$$

The asymptotic expansions given in (1.14), (1.18) and (1.22) contain several interesting characteristica which we want to discuss in the next section.

1.3 Discussion and outline of thesis

We bring the reader's attention to the following two interesting aspects of formula (1.14). One is related to the Forrester-Chen-Eriksen-Tracy conjecture ([14, 31]; see also [8]) concerning the behavior of the large gap probabilities. The conjecture states that the probability $E(s)$ of emptiness of the (properly scaled) interval $(x^* - s, x^* + s)$ around the point x^* satisfies the estimate,

$$E(s) \sim \exp\Big(-Cs^{2\kappa+2}\Big), \tag{1.23}$$

if the mean density $\rho(x)$ behaves as $\rho \sim (x - x^*)^\kappa$. This conjecture is supported by the classical results concerning the regular bulk point ($\kappa = 0$, the sine - kernel determinant - equation (1.8)) and regular edge point ($\kappa = 1/2$, the Airy - kernel determinant - the Tracy-Widom formula, see (1.24)). For higher order critical edge points ($\kappa = 2l + 1/2$, the higher Painlevé I - kernel determinants), estimate (1.23) follows from the asymptotic results of [16]. Our asymptotic equation (1.14) supports the Forrester-Chen-Eriksen-Tracy conjecture for the first critical case in the bulk, when $\kappa = 2$.

The second important feature of the estimate (1.14) is related to the constant (with respect to s) term in this formula. Starting from the seminal works of Onsager and Kaufman on the Ising model whose mathematical needs led to the birth of the Strong Szegö Theorem in the theory of Toeplitz matrices (see e.g. [20] for more on the history of the matter), the evaluation of the constant terms in the asymptotics of different correlation and distribution functions of random matrix theory and of the theory of solvable statistical mechanics models has always been a great challenge in the field[1]. In addition to the Strong Szegö Theorem and the already mentioned works [27, 41] and [18] devoted to the rigorous derivation of Dyson's constant in (1.8), different "constant" problems were considered (and solved) in the works [2, 4, 19, 51], and [1]. More questions in the area are still open, notably the generalization of the Dyson formula to the large gap probabilities in the general β-ensembles. A comprehensive account of the state of the art in this field, with formulation of the precise conjectures

[1] As soon as the leading term in the asymptotics of a correlation function is known, the small corrections can be usually (but not always !) relatively easy determined via a relevant system of differential equations. This, however, is not true for constant terms which always need an independent derivation.

concerning the Dyson constants for general β-ensembles, can be found in the recent survey of Forrester and Sorrell [32].

Formula (1.14) provides, in particular, another generalization of the Dyson constant formula, namely, it gives the constant term in the asymptotics of the gap probability in the bulk of the $\beta = 2$ ensemble for the first critical case when the mean density is having a quadratic zero. An important new feature of the constant term in formula (1.14) is the involvement of a Painlevé transcendent which describes the dependence of this term on the scaling parameter x. This fact explains the failure of the authors of [16] to find a closed expression for the similar constant in the case of the higher universality classes corresponding to the edge behavior of the gap probability (see Section "Constant Problem" in [16]). Indeed, our result shows that for higher universality classes one has to expect that the relevant constant terms are functions of the corresponding double scaling parameters which in turn are described via the solutions of certain nonlinear systems of a generalized Painlevé type (the generalized Schlesinger equations of isomonodromy deformations). These solutions, similar to the Hastings-McLeod solution of PII participating in (1.14), are supposed to be the "new" transcendents, i.e. not expressible in terms of the known special functions (i.e. in terms of a finite number of contour integrals of elementary, elliptic or finite genus algebraic functions).

It is also interesting to notice, that the constant term c_0 in the asymptotics (1.14) can be written as

$$c_0 \equiv c_0(x) = -\ln F_{TW}(x) - \frac{1}{6}\ln 2 + 3\zeta'(-1).$$

where $F_{TW}(x)$ is the celebrated Tracy-Widom distribution function,

$$F_{TW}(x) = e^{-\int_x^\infty (y-x)u^2(y)dy}. \tag{1.24}$$

Let us finish this introductory chapter with a brief outline for the rest of the dissertation. Chapter 2 gives a short review of the Riemann-Hilbert approach for the asymptotics of integrable Fredholm operators. We then apply the general framework to the Fredholm determinant $\det(I-\gamma K_{\mathrm{PII}})$ and formulate the associated "master" Riemann-Hilbert problem (RHP). We will also evaluate logarithmic s and x derivatives of the determinant $\det(I-\gamma K_{\mathrm{PII}})$ in terms of the solution of the underlying RHP and outline a derivation of an integrable system whose tau-function is represented by $\det(I-\gamma K_{\mathrm{PII}})$. In chapter 3, following the Deift-Zhou scheme, we construct the asymptotic solution of the master RHP. Comparing to the more usual cases, an extra "undressing" step is needed to overcome the transcendentality of the kernel $K_{\mathrm{PII}}(\lambda,\mu;x)$. Here, a crucial role is played by the aforementioned alternative Riemann-Hilbert definition of the function $\Psi(\lambda,x)$. Also, the situation $\gamma>1$ requires additional steps since we have to deal with a singular or solitonic type of Riemann-Hilbert problem. The calculations of chapter 3 and 4 provide us with the asymptotics of $\ln\det(I-\gamma K_{\mathrm{PII}})$ given in (1.14), (1.18) up to the constant terms as well as the distribution of zeros as stated in (1.22). In order to determine the constant terms, we will, in chapter 5, go back to equation (1.11) and look at the behavior of the kernel $K_{\mathrm{PII}}(\lambda,\mu;x)$ as $x\to+\infty$. We will see that in the large x limit, the kernel $K_{\mathrm{PII}}(\lambda,\mu;x)$ is replaced by the following cubic generalization of the sine kernel

$$K_{\mathrm{PII}}(\lambda,\mu)\mapsto K_{\mathrm{csin}}(\lambda,\mu)=\frac{\sin\left(\frac{4}{3}(\lambda^3-\mu^3)+x(\lambda-\mu)\right)}{\pi(\lambda-\mu)}. \tag{1.25}$$

Introducing a parameter $t\in[0,1]$

$$K_{\mathrm{csin}}(\lambda,\mu)\mapsto \check{K}_{\mathrm{csin}}(\lambda,\mu)=\frac{\sin\left(\frac{4}{3}t(\lambda^3-\mu^3)+x(\lambda-\mu)\right)}{\pi(\lambda-\mu)}$$

we compute the large s behavior of $\det\left(I-\gamma\check{K}_{\mathrm{csin}}\right)$ using again the Riemann-Hilbert approach. This will be done in chapter 6. This analysis will indeed produce the constant term in (1.14), since $\det\left(I-\check{K}_{\mathrm{csin}}\right)\big|_{t=0}$ reduces to the sine kernel with known asymptotics, see (1.8)

$$\ln\det(I-\check{K}_{\mathrm{csin}})\big|_{t=0}=-\frac{(sx)^2}{2}-\frac{1}{4}\ln(sx)+\frac{1}{12}\ln 2+3\zeta'(-1)+O\big(s^{-1}\big),\quad s\to\infty$$

uniformly on any compact subset of (1.15).

On the other hand for $\gamma < 1$, we use the logarithmic γ derivative of the determinant $\det(I - \gamma K_{\text{csin}})$ combined with the estimates from chapter 6 to derive the constant given in (1.18).

Remark 2 *We do not address in this dissertation the question of the higher corrections to* (1.14) *and* (1.18). *After the leading and constant terms are determined, the higher corrections can be in principal obtained by iterating the final ratio-Riemann-Hilbert problems (see chapter 4 and 6). Alternatively, one can use the differential system related to the determinant* $\det(I - \gamma K_{\text{PII}})$, *which we have mentioned above, and which we intend to discuss in detail in a future publication.*

The analysis of the Fredholm determinants corresponding to (1.25) is of interest on its own: The cubic sine - kernel determinant $\det(I - K_{\text{csin}})$ appears in condensed matter physics [7], namely in the description of the Fermi distribution of semiclassical non-equilibrium Fermi states. In order to understand perturbations to a degenerate Fermi gas one studies the one parameter extension of determinants corresponding to the cubic sine - kernel, that is

$$\det(I - \gamma K_{\text{csin}}), \qquad \gamma \in \mathbb{R}.$$

Although our interest in the cubic sine - kernel arises through the study of the Painlevé II - kernel determinants, the analysis given in chapters 6 and 7 of the present thesis, leads to the following asymptotic results.

Theorem 1.3.1 *Let* K_{csin} *denote the trace class operator on* $L^2((-s,s);d\lambda)$ *with kernel* (1.25). *Then as* $s \to \infty$

$$\begin{aligned} \ln\det(I - \gamma K_{\text{csin}}) &= i\nu\left(\frac{16}{3}s^3 + 4xs\right) + 6(i\nu)^2\ln s - \int_x^\infty (y-x)u^2(u,\gamma)dy \\ &\quad + \chi_{\text{PII}} + O\left(s^{-1}\right) \end{aligned} \tag{1.26}$$

uniformly on any compact subset of the set (1.19), *where* χ_{PII} *is given in* (1.20),

$$i\nu \equiv i\nu(\gamma) = \frac{1}{2\pi}\ln(1-\gamma)$$

and $u = u(x, y)$ denotes the real-valued Ablowitz-Segur solution of the second Painlevé equation $u_{xx} = xu + 2u^3$ corresponding to the monodromy surface

$$\mathbb{M} = \left\{(s_1, \ldots, s_6 \middle| \; s_1 = -i\gamma, s_2 = 0, s_3 = \bar{s}_1, s_{n+3} = -s_n\right\}.$$

We should mention that a large class of the generalized sine-kernel determinants has already been considered in [40] (see eq. (1.6) there). In the case $\gamma < 1$ and after a proper re-scaling, the determinant $\det(I - \gamma K_{\text{csin}})$ can be put in the form which is very close to the one treated in [40]. However, an essential difference occurs: the fast phase function, the function $p(\lambda)$ in the notations of [40] (see eq. (1.7)), which appear as a result of the re-scaling, does not satisfy one of the key conditions of [40]; moreover, it becomes depended on the large parameter. This means that the results of [40] are not directly applicable to our case. In fact, if one formally applies the main asymptotic formula of [40] to our case, then the first two terms of our asymptotic equation (1.26) are reproduced while the constant (in s) term is not. Most significantly, the integral term with the Painlevé function does not show up. Also it is not possible to extend the techniques in [40] beyond the situation $\gamma < 1$. In fact, for $\gamma = 1$, the relevant asymptotics is given by a formula ignoring the Tracy-Widom term in (1.14).

Theorem 1.3.2 *Let K_{csin} denote the trace class operator on $L^2\big((-s, s); d\lambda\big)$ with kernel* (1.25)*. Then as $s \to \infty$ the Fredholm determinant* $\det(I - K_{\text{csin}})$ *behaves as*

$$\ln\det(I - K_{\text{csin}}) = -\frac{2}{3}s^6 - s^4 x - \frac{1}{2}(sx)^2 - \frac{3}{4}\ln s - \frac{1}{6}\ln 2 + 3\zeta'(-1) + O\big(s^{-1}\big). \quad (1.27)$$

and the error term in (1.27) *is uniform on any compact subset of the set* (1.15)*.*

The Ablowitz-Segur solution [48] to the second Painlevé equation is given by the unique solution of the boundary value problem

$$u_{xx} = xu + 2u^3, \quad u(x) \sim \gamma \mathrm{Ai}(x), \quad x \to \infty, \; \gamma \neq 1. \quad (1.28)$$

Such solutions are smooth in case $\gamma < 1$, with exponentially fast decay as $x \to +\infty$ and oscillatory behavior as $x \to -\infty$. On the other hand in case $\gamma > 1$, the solution has poles on the real axis, but is still pole-free for sufficiently large positive x, in fact (cf. [6]) for (γ, x) chosen from any compact subset of the set

$$\left\{ (\gamma, x) \in \mathbb{R}^2 : \ 1 < \gamma < \infty, \, x > \left(\frac{3}{2} \ln \gamma \right)^{2/3} \right\} \tag{1.29}$$

the solution $u = u(x, \gamma)$ to (1.28) is pole-free. This in turn implies

Theorem 1.3.3 *For* (γ, x) *chosen from any compact subset of the set* (1.29)*, the Fredholm determinant* $\det(I - \gamma K_{\mathrm{csin}})$ *has infinitely many zeros* $\{s_n\}$ *with asymptotic distribution given in* (1.22)*.*

2. RIEMANN-HILBERT PROBLEM FOR INTEGRABLE FREDHOLM OPERATORS

We define the integral kernel (1.11) in terms of the solution of a Riemann-Hilbert problem, locate its structure within the algebra of integrable Fredholm operators, set up the master RHP and perform certain preliminary steps within the Deift-Zhou nonlinear steepest descent roadmap. Also the logarithmic s and x derivatives are expressed in terms of "local" quantities associated to the master RHP and we briefly discuss the underlying differential equations.

2.1 Riemann-Hilbert approach - setup and review

The classical theory of ordinary differential equations in the complex plane implies that system (1.10) has precisely one irregular singular point of Poincaré rank 3 at infinity. This observation leads to the existence of seven canonical solutions $\Psi_n(\lambda)$ which are fixed uniquely by their asymptotics (for more detail see e.g. [29])

$$\Psi_n(\lambda) \sim \Big(I + O\big(\lambda^{-1}\big)\Big)e^{-i(\frac{4}{3}\lambda^3+x\lambda)\sigma_3}, \quad \lambda \to \infty, \ \lambda \in \Omega_n$$

where the canonical sectors Ω_n (compare Figure 2.1) are defined by

$$\Omega_n = \Big\{\lambda \in \mathbb{C} \mid \arg\lambda \in \Big(\frac{\pi}{3}(n-2), \frac{\pi}{3}n\Big), n = 1, \dots, 7\Big\}.$$

Moreover the presence of an irregular singularity gives us a non-trivial Stokes phenomenon described by the Stokes matrices S_n:

$$S_n = \big(\Psi_n(\lambda)\big)^{-1}\Psi_{n+1}(\lambda).$$

In the given situation (1.11) (see again [29]) these multipliers are

$$S_1 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_4 = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \ S_3 = \bar{S}_1, S_5 = \bar{S}_2, S_6 = \bar{S}_4, \tag{2.1}$$

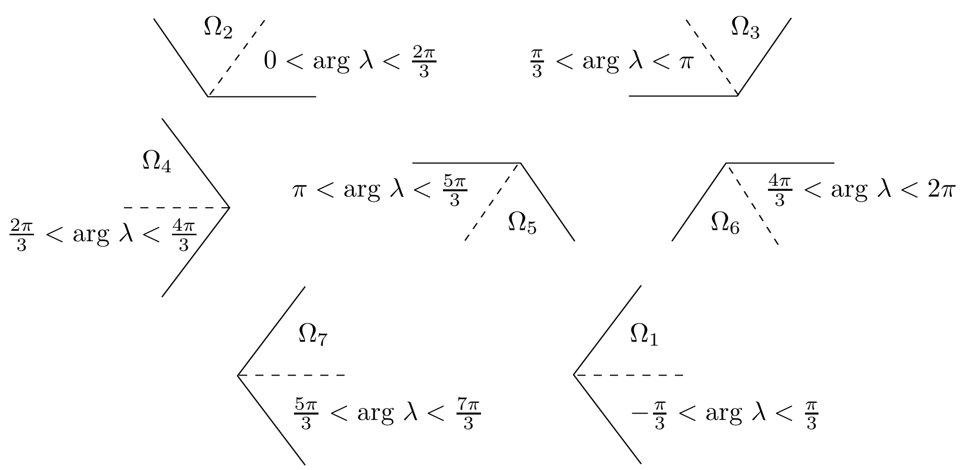

Figure 2.1. Canonical sectors of system (1.10) with the dashed lines indicating where Re $\lambda^3 = 0$

hence the required solution in (1.10) is the second and third canonical solution $\Psi(\lambda, x) \equiv \Psi_2(\lambda, x) = \Psi_3(\lambda, x)$ with asymptotics

$$\Psi(\lambda, x) \sim \Big(I + O\big(\lambda^{-1}\big)\Big) e^{-i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3}, \qquad \lambda \to \infty,\ 0 < \arg\lambda < \pi \tag{2.2}$$

and Stokes matrices as in (2.1). Now that we have defined the integral kernel (1.11) let us connect it to a Riemann-Hilbert problem: The given kernel belongs to an algebra of integrable operators first introduced in [34], see also [22]: Let Σ be an oriented contour in the complex plane $\mathbb{C}$ such as a Jordan curve. We are interested in operators of the form $\lambda I + K$ on $L^2(\Sigma)$, where K denotes an integral operator with kernel

$$K(\lambda, \mu) = \frac{\sum_{i=1}^{M} f_i(\lambda) h_i(\mu)}{\lambda - \mu}, \qquad \sum_{i=1}^{M} f_i(\lambda) h_i(\lambda) = 0, \quad M \in \mathbb{Z}_{\geq 1} \tag{2.3}$$

with functions f_i, h_i which are smooth up to the boundary of Σ. Given two operators $\lambda I + K, \check{\lambda} I + \check{K}$ of this type, the composition $(\lambda I + K)(\check{\lambda} I + \check{K})$ is again of the same form, hence we have a ring. Moreover let K^t denote the real adjoint of K, i.e.

$$K^t(\lambda, \mu) = -\frac{\sum_{i=1}^{M} h_i(\lambda) f_i(\mu)}{\lambda - \mu}.$$

Our results are based on the following facts of the theory of integrable operators (see e.g. [22]). First an algebraic Lemma, showing that the resolvent of $I-K$ is again integrable.

Lemma 1 *Given an operator $I-K$ on $L^2(\Sigma)$ in the previous ring with kernel* (2.3)*. Suppose the inverse $(I-K)^{-1}$ exists, then $I+R=(I-K)^{-1}$ lies again in the same ring with*

$$R(\lambda,\mu)=\frac{\sum_{i=1}^{M}F_i(\lambda)H_i(\mu)}{\lambda-\mu},\qquad \sum_{i=1}^{M}F_i(\lambda)H_i(\lambda)=0 \tag{2.4}$$

and the functions F_i, H_i are given by

$$F_i(\lambda)=\Big((I-K)^{-1}f_i\Big)(\lambda),\qquad H_i(\lambda)=\Big(\big(I-K^t\big)^{-1}h_i\Big)(\lambda). \tag{2.5}$$

Secondly an analytical Lemma, which connects integrable operators to a Riemann-Hilbert problem.

Lemma 2 *Let K be of integrable type such that $(I-K)^{-1}$ exists and let $Y=Y(z)$ denote the unique solution of the following $M\times M$ Riemann-Hilbert problem (RHP)*

- *$Y(z)$ is analytic for $z\in\mathbb{C}\backslash\Sigma$*
- *On the contour Σ, the boundary values of the function $Y(z)$ satisfy the jump relation*

  $$Y_+(z)=Y_-(z)\left(I-2\pi i f(z)h^t(z)\right),\qquad z\in\Sigma$$

  where $f(z)=(f_1(z),\dots,f_M(z))^t$ and similarly $h(z)=(h_1(z),\dots,h_M(z))^t$
- *At an endpoint of the contour Σ, $Y(z)$ has no more than a logarithmic singularity*
- *As $z\to\infty$*

  $$Y(z)=I+O\big(z^{-1}\big)$$

Then $Y(z)$ determines the resolvent kernel via

$$F(z)=Y(z)f(z),\qquad H(z)=\big(Y^t(z)\big)^{-1}h(z) \tag{2.6}$$

and conversely the solution of the above RHP is expressible in terms of the function $F(z)$ *using the Cauchy integral*

$$Y(z) = I - \int_{\Sigma} F(w)h^t(w)\frac{dw}{w-z}. \tag{2.7}$$

Let us use this general setup in the given situation (1.11). We have

$$\gamma K_{\mathrm{PII}}(\lambda,\mu) = \frac{f^t(\lambda)h(\mu)}{\lambda-\mu},\ f(\lambda) = i\sqrt{\frac{\gamma}{2\pi}}\begin{pmatrix}\psi_{11}(\lambda)\\ \psi_{21}(\lambda)\end{pmatrix},\ h(\mu) = i\sqrt{\frac{\gamma}{2\pi}}\begin{pmatrix}\psi_{21}(\mu)\\ -\psi_{11}(\mu)\end{pmatrix} \tag{2.8}$$

where we suppressed the x dependency in $\psi_{jk}(\lambda) \equiv \psi_{jk}(\lambda, x)$ and $\sqrt{z}$ is defined on $\mathbb{C}\backslash(-\infty, 0]$ with its branch fixed by the condition $\sqrt{z} > 0$ as $z > 0$. Lemma 2 leads us therefore to the following Y-RHP

- $Y(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[-s, s]$
- Orienting the line segment $[-s, s]$ from left to right, the following jump holds

$$Y_+(\lambda) = Y_-(\lambda)\begin{pmatrix}1+i\gamma\psi_{11}(\lambda)\psi_{21}(\lambda) & -i\gamma\psi_{11}^2(\lambda)\\ i\gamma\psi_{21}^2(\lambda) & 1-i\gamma\psi_{11}(\lambda)\psi_{21}(\lambda)\end{pmatrix},\quad \lambda\in[-s,s]$$

- At the endpoints $\lambda = \pm s$, $Y(\lambda)$ has logarithmic singularities, i.e.

$$Y(\lambda) = O\big(\ln(\lambda \mp s)\big),\quad \lambda\to\pm s$$

- As $\lambda \to \infty$ we have

$$Y(\lambda) = I + \frac{m_1}{\lambda} + O\left(\lambda^{-2}\right).$$

The given jump matrix on the segment $[-s, s]$ can be factorized using the unimodular fundamental solution $\Psi(\lambda)$ of (1.10) corresponding to the choices (2.1) and (2.2)

$$\begin{aligned} G(\lambda) &= \begin{pmatrix}1+i\gamma\psi_{11}(\lambda)\psi_{21}(\lambda) & -i\gamma\psi_{11}^2(\lambda)\\ i\gamma\psi_{21}^2(\lambda) & 1-i\gamma\psi_{11}(\lambda)\psi_{21}(\lambda)\end{pmatrix}\\ &= \begin{pmatrix}\psi_{11}(\lambda) & \psi_{12}(\lambda)\\ \psi_{21}(\lambda) & \psi_{22}(\lambda)\end{pmatrix}\begin{pmatrix}1 & -i\gamma\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi_{22}(\lambda) & -\psi_{12}(\lambda)\\ -\psi_{21}(\lambda) & \psi_{11}(\lambda)\end{pmatrix}\\ &= \Psi(\lambda)\begin{pmatrix}1 & -i\gamma\\ 0 & 1\end{pmatrix}\big(\Psi(\lambda)\big)^{-1}. \end{aligned}$$

This motivates a first series of transformations of the initial Y-RHP.

2.2 First transformations of the RHP - uniformization

We make the following substitution in the original Y-RHP

$$\widetilde{X}(\lambda) = Y(\lambda)\Psi(\lambda), \quad \lambda \in \mathbb{C}\backslash[-s, s] \tag{2.9}$$

which leads to a RHP for the function $\widetilde{X}(\lambda)$:

- $\widetilde{X}(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[-s, s]$
- The following jump holds

$$\widetilde{X}_+(\lambda) = \widetilde{X}_-(\lambda) \begin{pmatrix} 1 & -i\gamma \\ 0 & 1 \end{pmatrix}, \quad \lambda \in [-s, s] \tag{2.10}$$

- As $\lambda \to \pm s$, we have

$$\widetilde{X}(\lambda) = O\big(\ln(\lambda \mp s)\big)$$

- At infinity,

$$\widetilde{X}(\lambda) = \Big(I + O\big(\lambda^{-1}\big)\Big)\Psi(\lambda), \quad \lambda \to \infty$$

In order to uniformize the behavior of $\widetilde{X}(\lambda)$ at infinity, we will now use the Stokes phenomenon (2.1) of $\Psi(\lambda)$ and introduce more cuts to the Riemann-Hilbert problem. Let

$$X(\lambda) = \widetilde{X}(\lambda) \begin{cases} I, & \lambda \in \hat{\Omega}_1, \\ S_3, & \lambda \in \hat{\Omega}_2, \\ S_3S_4, & \lambda \in \hat{\Omega}_3, \\ S_3S_4S_6, & \lambda \in \hat{\Omega}_4, \end{cases} \tag{2.11}$$

with

$$\Gamma_1 = \Big\{\lambda \in \mathbb{C}: \ \arg(\lambda - s) = \frac{\pi}{6}\Big\}, \qquad \Gamma_3 = \Big\{\lambda \in \mathbb{C}: \ \arg(\lambda + s) = \frac{5\pi}{6}\Big\},$$
$$\Gamma_4 = \Big\{\lambda \in \mathbb{C}: \ \arg(\lambda + s) = -\frac{5\pi}{6}\Big\}, \qquad \Gamma_6 = \Big\{\lambda \in \mathbb{C}: \ \arg(\lambda - s) = -\frac{\pi}{6}\Big\},$$

then $X(\lambda)$ satisfies the following "master" RHP, depicted in Figure 2.2

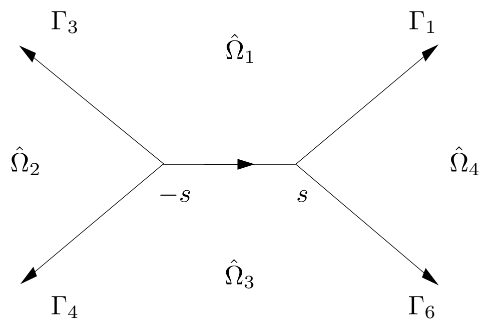

Figure 2.2. Jump contours of the master RHP

- $X(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash\big([-s,s]\cup\bigcup_k \Gamma_k\big)$
- Along the infinite rays Γ_k, $X(\lambda)$ has jumps described by the Stokes matrices

$$X_+(\lambda) = X_-(\lambda) S_k, \quad \lambda \in \Gamma_k,$$

whereas on the line segment $[-s,s]$ we have the following jump

$$X_+(\lambda) = X_-(\lambda)\begin{pmatrix} 0 & -i \\ -i & 1-\gamma \end{pmatrix}, \quad \lambda \in [-s,s]. \tag{2.12}$$

- In a neighborhood of the endpoints $\lambda = \pm s$,

$$X(\lambda) = \check{X}(\lambda)\begin{pmatrix} 1 & -\frac{\gamma}{2\pi}\ln\frac{\lambda-s}{\lambda+s} \\ 0 & 1 \end{pmatrix}\begin{cases} I, & \lambda \in \hat\Omega_1, \\ S_3, & \lambda \in \hat\Omega_2, \\ S_3S_4, & \lambda \in \hat\Omega_3, \\ S_3S_4S_6, & \lambda \in \hat\Omega_4, \end{cases} \tag{2.13}$$

where $\check{X}(\lambda)$ is analytic at $\lambda = \pm s$ and the branch of the logarithm is fixed by the condition $-\pi < \arg\frac{\lambda-s}{\lambda+s} < \pi$.[1]

- As $\lambda \to \infty$ the following asymptotical behavior holds

$$X(\lambda) = \Big(I + \frac{m_1}{\lambda} + O\big(\lambda^{-2}\big)\Big)\Big(I + \frac{m_1^{HM}}{\lambda} + O\big(\lambda^{-2}\big)\Big)e^{-i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3} \tag{2.14}$$

[1] The local behavior (2.13) of $X(\lambda)$ at the endpoints $\pm s$ can be derived directly from the a-priori information $\tilde{X}(\lambda) = O(\ln(\lambda \mp s)), \lambda \to \pm s$ and the jump condition (2.10).

with

$$m_1^{HM} = \frac{1}{2}\begin{pmatrix} -iv & u \\ u & iv \end{pmatrix}, \qquad v = (u_x)^2 - xu^2 - u^4, \quad v_x = -u^2.$$

As we are going to see in chapter 3, the latter master RHP can be solved asymptotically by approximating its solution with local model functions, however this analysis is essentially different in the regimes $\gamma = 1$ and $\gamma \neq 1$. Before we start this analysis in detail, we first connect the solution of the master RHP to the Fredholm determinant $\det(I - \gamma K_{\mathrm{PII}})$.

2.3 Logarithmic derivatives - connection to X-RHP

We wish to express certain logarithmic derivatives of the Fredholm determinants $\det(I - \gamma K_{\mathrm{PII}})$ in terms of the solution of the X-RHP. To this end recall the following classical identity, valid for any differentiable family of trace class operators [49]

$$\frac{\partial}{\partial s} \ln\det(I - \gamma K_{\mathrm{PII}}) = -\mathrm{trace}\Big((I - \gamma K_{\mathrm{PII}})^{-1} \frac{\partial}{\partial s}(\gamma K_{\mathrm{PII}}) \Big). \tag{2.15}$$

In our situation

$$\frac{\partial K_{\mathrm{PII}}}{\partial s}(\lambda, \mu) = K_{\mathrm{PII}}(\lambda, \mu)\big(\delta(\mu - s) + \delta(\mu + s)\big),$$

where, by definition

$$\int_{-s}^{s} \delta(w \mp s) f(w) dw = f(\pm s),$$

and therefore

$$-\mathrm{trace}\left((I - \gamma K_{\mathrm{PII}})^{-1} \frac{\partial}{\partial s}(\gamma K_{\mathrm{PII}}) \right) = -R(s,s) - R(-s,-s)$$

with $R(\lambda, \mu)$ denoting the kernel (see (2.4)) of the resolvent $R = (I - \gamma K_{\mathrm{PII}})^{-1}\gamma K_{\mathrm{PII}}$. The latter derivative can be simplified using the equations (see (2.8))

$$f_1(\lambda) = -h_2(\lambda), \quad f_2(\lambda) = h_1(\lambda)$$

as well as the identity $\det Y(\lambda) \equiv 1$, which is a direct consequence of the unimodularity of the jump matrix $G(\lambda)$ and Liouville's theorem. We have,

$$R(\lambda,\mu) = \frac{F_1(\lambda)H_1(\mu)+F_2(\lambda)H_2(\mu)}{\lambda-\mu} = \frac{F_1(\lambda)F_2(\mu)-F_2(\lambda)F_1(\mu)}{\lambda-\mu}.$$

Since $R(\lambda,\mu)$ is continuous along the diagonal $\lambda=\mu$ we obtain further

$$R(s,s) = F_1'(s)F_2(s)-F_2'(s)F_1(s), \quad R(-s,-s) = F_1'(-s)F_2(-s)-F_2'(-s)F_1(-s) \tag{2.16}$$

provided F_i is analytic at $\lambda=\pm s$. One way to see this is a follows. Use the connection

$$X(\lambda) = Y(\lambda)\Psi(\lambda)\begin{cases} I, & \lambda\in\hat{\Omega}_1,\\ S_3, & \lambda\in\hat{\Omega}_2,\\ S_3S_4, & \lambda\in\hat{\Omega}_3,\\ S_3S_4S_6, & \lambda\in\hat{\Omega}_4,\end{cases} \equiv Y(\lambda)\Psi(\lambda)\widehat{S}(\lambda)$$

and (2.6)

$$F(\lambda) = X(\lambda)\big(\widehat{S}(\lambda)\big)^{-1}\big(\Psi(\lambda)\big)^{-1}f(\lambda) = X(\lambda)\big(\widehat{S}(\lambda)\big)^{-1}i\sqrt{\frac{\gamma}{2\pi}}\begin{pmatrix}1\\0\end{pmatrix}$$

as well as (2.13) to derive the following local identity

$$F(\lambda) = \check{X}(\lambda)\begin{pmatrix}1 & -\frac{\gamma}{2\pi}\ln\frac{\lambda-s}{\lambda+s}\\ 0 & 1\end{pmatrix}\widehat{S}(\lambda)\big(\widehat{S}(\lambda)\big)^{-1}\frac{i}{\sqrt{2\pi}}\begin{pmatrix}1\\0\end{pmatrix} = \check{X}(\lambda)i\sqrt{\frac{\gamma}{2\pi}}\begin{pmatrix}1\\0\end{pmatrix}, \tag{2.17}$$

valid in a vicinity of $\lambda=\pm s$. But this proves analyticity of $F(\lambda)$ at the endpoints and as we shall see later on, (2.17) is all we need to connect (2.15) via (2.16) to the solution of the X-RHP. We summarize

Proposition 2.3.1 *The logarithmic s-derivative of the Fredholm determinant* (1.12) *can be expressed as*

$$\begin{aligned}\frac{\partial}{\partial s}\ln\det(I-\gamma K_{\mathrm{PII}}) &= -R(s,s)-R(-s,-s), \qquad (2.18)\\ &\qquad R(\pm s,\pm s) = F_1'(\pm s)F_2(\pm s)-F_2'(\pm s)F_1(\pm s)\end{aligned}$$

and the connection to the X-RHP is established through

$$F(\lambda) = \check{X}(\lambda)i\sqrt{\frac{\gamma}{2\pi}}\begin{pmatrix}1\\0\end{pmatrix},$$

where $\check{X}(\lambda)$ is analytic in a neighborhood of $\lambda=\pm s$, see (2.13).

Besides the logarithmic s-derivative we also differentiate with respect to x

$$\frac{\partial}{\partial x}\ln\det\left(I-\gamma K_{\mathrm{PII}}\right)=-\mathrm{trace}\left(\left(I-\gamma K_{\mathrm{PII}}\right)^{-1}\frac{\partial}{\partial x}\left(\gamma K_{\mathrm{PII}}\right)\right).$$

In our situation the kernel itself depends on x, since (see e.g. [29])

$$\frac{\partial\Psi}{\partial x}=U(\lambda,x)\Psi,\qquad U(\lambda,x)=-i\lambda\sigma_3+i\begin{pmatrix}0 & u\\ -u & 0\end{pmatrix},$$

and we have

$$\begin{aligned}\frac{\partial}{\partial x}\big(\gamma K_{\mathrm{PII}}(\lambda,\mu)\big) &= \frac{i\gamma}{2\pi}\big(\psi_{21}(\lambda,x)\psi_{11}(\mu,x)+\psi_{21}(\mu,x)\psi_{11}(\lambda,x)\big)\\ &= i\big(f_2(\lambda)h_2(\mu)-f_1(\lambda)h_1(\mu)\big)\end{aligned}$$

and with (2.5)

$$-\mathrm{trace}\left(\left(I-\gamma K_{\mathrm{PII}}\right)^{-1}\frac{\partial}{\partial x}\left(\gamma K_{\mathrm{PII}}\right)\right)=-i\int_{-s}^{s}\big(F_2(\lambda)h_2(\lambda)-F_1(\lambda)h_1(\lambda)\big)\,d\lambda.$$

On the other hand the Cauchy integral (2.7) implies

$$Y(\lambda)=I+\frac{m_1}{\lambda}+O\big(\lambda^{-2}\big),\quad \lambda\to\infty;\qquad m_1=\int_{-s}^{s}F(w)h^t(w)dw$$

so

$$\frac{\partial}{\partial x}\ln\det(I-\gamma K_{\mathrm{PII}})=i\left(m_1^{11}-m_1^{22}\right),\quad m_1=\left(m_1^{ij}\right)$$

and the connection to the X-RHP is established via (2.14). Again we summarize

Proposition 2.3.2 *The logarithmic x-derivative of the given Fredholm determinant can be expressed as*

$$\frac{\partial}{\partial x}\ln\det\left(I-\gamma K_{\mathrm{PII}}\right)=i\left(X_1^{11}-X_1^{22}\right)-v \tag{2.19}$$

with

$$X(\lambda)=\left(I+\frac{X_1}{\lambda}+O\big(\lambda^{-2}\big)\right)e^{-i(\frac{4}{3}\lambda^3+x\lambda)\sigma_3},\quad \lambda\to\infty;\qquad X_1=\left(X_1^{ij}\right).$$

Proposition 2.3.1 and 2.3.2 are sufficient to determine the large s-asymptotics of $\ln\det(I-\gamma K_{\mathrm{PII}})$ up to the constant term. As indicated in chapter 1, those constant terms will be determined through the asymptotical analysis of the cubic sine - kernel determinant (1.25) in chapters 6 and 7.

2.4 Differential equations associated with det $(I-\gamma K_{\mathrm{PII}})$

Our considerations rely only on the underlying Riemann-Hilbert problems. Nevertheless, before we move further ahead in the asymptotical analysis, we would like to take a short look into the differential equations associated with the master X-RHP.

To this end we notice that the X-RHP has unimodular constant jump matrices, thus the well-defined logarithmic derivatives $X_\lambda X^{-1}(\lambda), X_s X^{-1}(\lambda)$ and $X_x X^{-1}(\lambda)$ are rational functions. Indeed using (2.13) as well as (2.14) we have

$$\frac{\partial X}{\partial \lambda}=\left[-4i\lambda^2\sigma_3+4i\lambda\begin{pmatrix}0 & n_1\\ -n_2 & 0\end{pmatrix}+\begin{pmatrix}n_3 & n_4\\ n_5 & -n_3\end{pmatrix}+\frac{N_1}{\lambda-s}-\frac{N_2}{\lambda+s}\right]X\equiv\mathcal{A}(\lambda,s,x)X \tag{2.20}$$

where

$$N_1=-\frac{\gamma}{2\pi}\check{X}(s)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\big(\check{X}(s)\big)^{-1};\quad N_2=-\frac{\gamma}{2\pi}\check{X}(-s)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\big(\check{X}(-s)\big)^{-1}$$

and with parameters n_i which can be expressed in terms of the entries of m_1 and m_1^{HM} (see (2.14)). Moreover

$$\frac{\partial X}{\partial s}=\left[-\frac{N_1}{\lambda-s}-\frac{N_2}{\lambda+s}\right]X\equiv\mathcal{B}(\lambda,s,x)X$$

and also

$$\frac{\partial X}{\partial x}=\left[-i\lambda\sigma_3+i\begin{pmatrix}0 & n_1\\ -n_2 & 0\end{pmatrix}\right]X\equiv\mathcal{C}(\lambda,s,x)X.$$

Hence we arrive at the Lax-system for the function X,

$$\begin{cases}\frac{\partial X}{\partial\lambda}=\mathcal{A}(\lambda,s,x)X\\ \frac{\partial X}{\partial s}=\mathcal{B}(\lambda,s,x)X,\\ \frac{\partial X}{\partial x}=\mathcal{C}(\lambda,s,x)X.\end{cases}$$

Considering the compatibility conditions of the system,

$$\mathcal{A}_s - \mathcal{B}_\lambda = [\mathcal{B}, \mathcal{A}], \qquad \mathcal{A}_x - \mathcal{C}_\lambda = [\mathcal{C}, \mathcal{A}], \qquad \mathcal{B}_x - \mathcal{C}_s = [\mathcal{C}, \mathcal{B}] \tag{2.21}$$

we are lead to a system of eighteen nonlinear ordinary differential equations for the unknown quantities n_i and the entries of N_1 and N_2. Since it is possible to express the previous derivatives of $\ln\det(I - \gamma K_{\mathrm{PII}})$ solely in terms of the unknowns n_i, N_1 and N_2, one could then try to derive a differential equation for the Fredholm determinant (1.12) using (2.21). We shall devote to these issues a future publication.

3. ASYMPTOTIC SOLUTION OF THE MASTER RIEMANN-HILBERT PROBLEM

The integrable form of the Painlevé II - kernel (1.11) allowed us to connect certain logarithmic derivatives to the solution of the X-RHP, the master RHP. We will now solve the latter problem asymptotically according to the Deift-Zhou nonlinear steepest descent roadmap [24, 25]. Various special functions of Painlevé, hypergeometric and Bessel-type will be used to approximate the global solution $X(\lambda)$ by local model functions, *parametrices* and the iterative solution of a singular integral equation. We present this asymptotical resolution first for $\gamma \neq 1$, followed then by the regime $\gamma = 1$.

3.1 Rescaling, normalization and opening of lenses, $\gamma \neq 1$

We scale the variables in (2.11) as $\lambda = zs$ and normalize the asymptotics in (2.14) by introducing

$$T(z) = X(zs)e^{s^3\vartheta(z)\sigma_3}, \quad z \in \mathbb{C}\backslash\Big([-1,1] \cup \bigcup_k \Gamma_k\Big), \qquad \vartheta(z) = i\left(\frac{4}{3}z^3 + \frac{xz}{s^2}\right). \tag{3.1}$$

This leads to the following RHP

- $T(z)$ is analytic for $z \in \mathbb{C}\backslash\Big([-1,1] \cup \bigcup_k \Gamma_k\Big)$
- The jump properties of $T(z)$ are given by the equations

$$\begin{aligned} T_+(z) &= T_-(z)e^{-s^3\vartheta(z)\sigma_3}\begin{pmatrix} 0 & -i \\ -i & 1-\gamma \end{pmatrix}e^{s^3\vartheta(z)\sigma_3}, \qquad z \in [-1,1] \\ T_+(z) &= T_-(z)e^{-s^3\vartheta(z)\sigma_3}S_k e^{s^3\vartheta(z)\sigma_3}, \qquad z \in \Gamma_k \end{aligned}$$

- In a neighborhood of the endpoints $z = \pm 1$

$$T(z)e^{-s^3\vartheta(z)\sigma_3} = \check{X}(zs)\begin{pmatrix} 1 & -\frac{\gamma}{2\pi}\ln\frac{z-1}{z+1} \\ 0 & 1 \end{pmatrix}\begin{cases} I, & z \in \hat{\Omega}_1, \\ S_3, & z \in \hat{\Omega}_2, \\ S_3S_4, & z \in \hat{\Omega}_3, \\ S_3S_4S_6, & z \in \hat{\Omega}_4, \end{cases}$$

- As $z \to \infty$, we have $T(z) = I + O\big(z^{-1}\big)$

Our next move will deform the latter T-RHP to a RHP formulated according to the sign-diagram of the function Re $\vartheta(z)$, depicted in Figure 3.1. In this Figure we choose x from a compact subset of the real line, $s > 0$ is sufficiently large and

$$z_\pm = \pm i\sqrt{\frac{3x}{4s^2}}$$

denote the two vertices of the depicted curves.

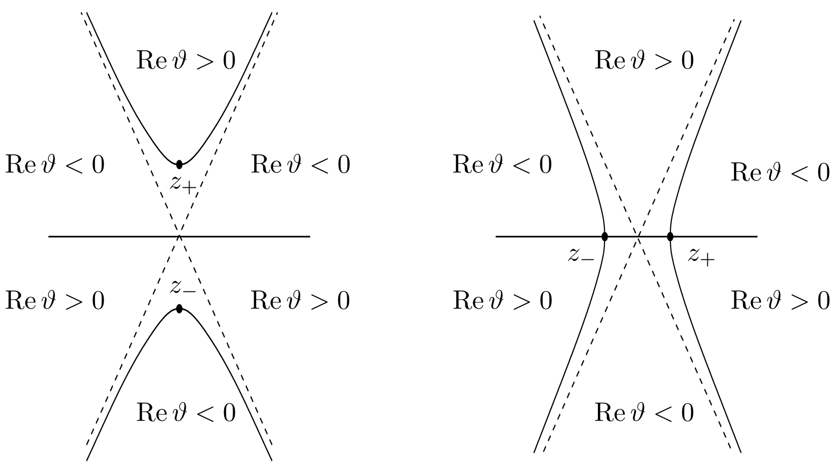

Figure 3.1. Sign-diagram for the function Re $\vartheta(z)$. In the left picture we indicate the location of $z_\pm$ as $x > 0$ and in the right picture for a particular choice of $x < 0$. Along the solid lines Re $\vartheta(z) = 0$ and the dashed lines resemble arg $z = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$

With the matrix factorization

$$\begin{pmatrix} 0 & -i \\ -i & 1-\gamma \end{pmatrix} = \begin{pmatrix} 1 & -\frac{i}{1-\gamma} \\ 0 & 1 \end{pmatrix}(1-\gamma)^{-\sigma_3}\begin{pmatrix} 1 & 0 \\ -\frac{i}{1-\gamma} & 1 \end{pmatrix} \equiv S_US_DS_L,$$

valid as long as $\gamma \neq 1$, we perform *opening of lenses* as follows. Let $\mathcal{L}_j^{\pm}$ and $\mathcal{L}_k$ denote the *upper (lower) lens*, shown in Figure 3.2, which is bounded by the contours $\gamma_{jk}^{\pm}$ and Γ_k, where

$$\gamma_{12}^{+} = \left\{ z \in \mathbb{C} : \arg z = \frac{\pi}{6} \right\}, \qquad \gamma_{21}^{+} = \left\{ z \in \mathbb{C} : \arg z = \frac{5\pi}{6} \right\},$$
$$\gamma_{32}^{-} = \left\{ z \in \mathbb{C} : \arg z = -\frac{5\pi}{6} \right\}, \qquad \gamma_{41}^{-} = \left\{ z \in \mathbb{C} : \arg z = -\frac{\pi}{6} \right\}.$$

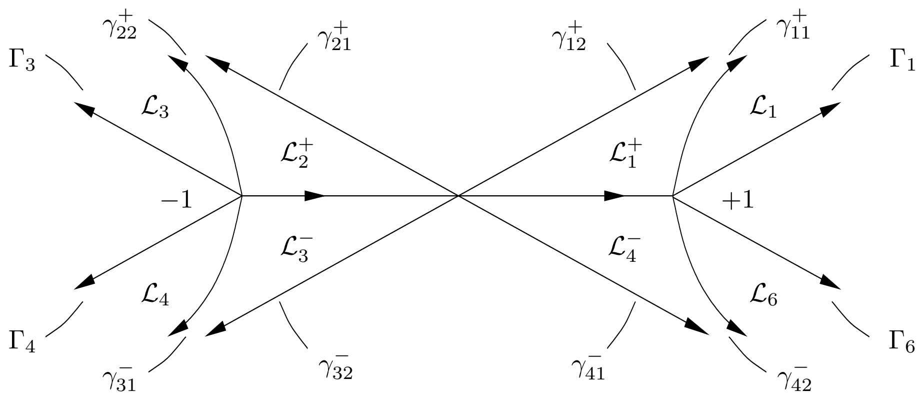

Figure 3.2. Opening of lenses – $T(z) \mapsto S(z)$

Define

$$S(z) = T(z) e^{-s^3\vartheta(z)\sigma_3} \begin{cases} S_1^{-1} e^{s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_1, \\ S_L^{-1} e^{s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_1^{+} \cup \mathcal{L}_2^{+}, \\ S_3 e^{s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_3, \\ S_4^{-1} e^{s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_4, \\ S_U e^{s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_3^{-} \cup \mathcal{L}_4^{-}, \\ S_6 e^{s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_6, \\ e^{s^3\vartheta(z)\sigma_3}, & \text{otherwise}, \end{cases} \equiv T(z)L(z) \tag{3.2}$$

then $S(z)$ solves the following RHP

- $S(z)$ is analytic for $z \in \mathbb{C} \setminus ([-1,1] \cup \mathcal{D})$ with $\mathcal{D} = \bigcup_{j,k} \left(\gamma_{jk}^{+} \cup \gamma_{jk}^{-} \right)$

- With orientation fixed as in Figure 3.2, $S(z)$ has jumps given by

$$\begin{aligned} S_+(z) &= S_-(z)e^{-s^3\vartheta(z)\sigma_3}\widehat{G}_S e^{s^3\vartheta(z)\sigma_3}, \quad z \in \mathbb{C}\setminus([-1,1]\cup\mathcal{D}) \\ &\equiv S_-(z)G_S(z) \end{aligned} \tag{3.3}$$

where the piecewise constant matrix $\widehat{G}_S$ can be read from Figure 3.3.

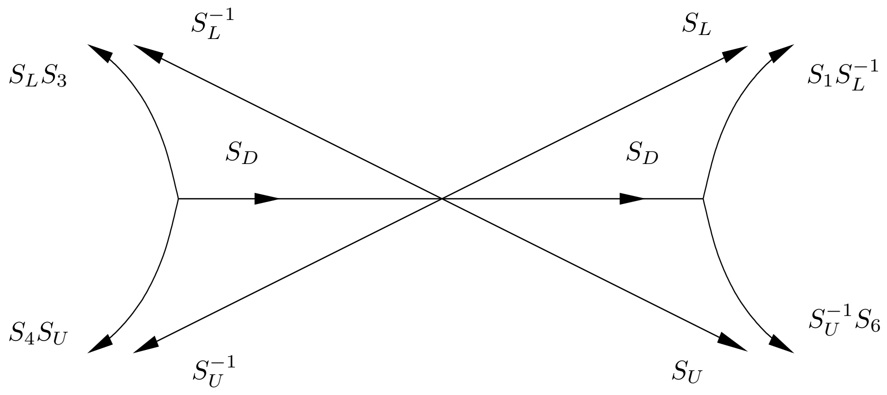

Figure 3.3. The piecewise constant matrix $\widehat{G}_S$

- As $z \to \pm 1$, we have

$$S(z)L^{-1}(z)e^{-s^3\vartheta(z)\sigma_3} = \check{X}(zs)\begin{pmatrix} 1 & -\frac{\gamma}{2\pi}\ln\frac{z-1}{z+1} \\ 0 & 1 \end{pmatrix}\begin{cases} I, & z \in \hat{\Omega}_1, \\ S_3, & z \in \hat{\Omega}_2, \\ S_3S_4, & z \in \hat{\Omega}_3, \\ S_3S_4S_6, & z \in \hat{\Omega}_4, \end{cases} \tag{3.4}$$

- At infinity, $S(z) = I + O\big(z^{-1}\big), z \to \infty$

Let us analyse the behavior of $G_S(z)$ along the infinite branches as $s \to \infty$. To this end recall the sign-diagram of the function Re $\vartheta(z)$, depicted in Figure 3.1. We have in the upper half-plane

$$G_S(z) = e^{-s^3\vartheta(z)\sigma_3}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}e^{s^3\vartheta(z)\sigma_3}, \quad z \in \gamma_{jk}^+, \quad j,k = 1,2$$

with a constant $a \in \mathbb{C}$ which can be read from Figure 3.3. Since we choose x from a compact subset of the real line and $s > 0$ is sufficiently large, $\operatorname{Re} \vartheta(z)$ is always negative on $\gamma_{jk}^{+}, j, k = 1, 2$ outside a small neighborhood around the origin and the endpoints $z = \pm 1$, hence for such z

$$G_S(z) \longrightarrow I, \quad s \to \infty \tag{3.5}$$

uniformly on any compact subset of the set (1.15) and the stated convergence is in fact exponentially fast. A similar statement holds on the infinite branches in the lower half-plane. There

$$G_S(z) = e^{-s^3\vartheta(z)\sigma_3} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} e^{s^3\vartheta(z)\sigma_3}, \quad z \in \gamma_{jk}^{-}, \ \ j, k = 1, 2$$

again with some constant $b \in \mathbb{C}$ which is given in Figure 3.3. In this situation $\operatorname{Re} \vartheta(z) > 0$ outside a small neighborhood of the origin as well as the endpoints $z = \pm 1$ and therefore

$$G_S(z) \longrightarrow I, \quad s \to \infty \tag{3.6}$$

also uniformly on any compact subset of the set (1.15). From (3.5) and (3.6) we expect, and this will be justified rigorously, that as $s \to \infty$, $S(z)$ converges to a solution of the model RHP, in which we only have to deal with the diagonal jump matrix S_D on the line segment $[-1, 1]$. Also this convergence is expected to be uniform with respect to z outside some small neighborhood of the origin as well as outside some vicinities of the endpoints $z = \pm 1$. Let us now move on to the underlying model RHP as well as the construction of the relevant parametrices.

3.2 The model RHP and parametrices for $\gamma \neq 1$

The model RHP consists in finding the piecewise analytic 2×2 matrix valued function $M(z)$ such that

- $M(z)$ is analytic for $z \in \mathbb{C} \backslash [-1, 1]$

- Along $[-1,1]$, the following jump condition holds

$$M_+(z) = M_-(z)S_D, \qquad z \in [-1,1]$$

where

$$S_D = (1-\gamma)^{-\sigma_3}$$

- $M(z) = I + O\big(z^{-1}\big), \ z \to \infty$

Only assuming that $\gamma \neq 1$, we can always solve this diagonal and thus quasi-scalar RHP (cf. [29])

$$M(z) = \exp\left[\frac{1}{2\pi i}\int_{-1}^{1}\frac{\ln(1-\gamma)^{-\sigma_3}}{\mu - z}\,d\mu\right] = \left(\frac{z+1}{z-1}\right)^{\nu\sigma_3} \tag{3.7}$$

with

$$\nu = \frac{1}{2\pi i}\ln(1-\gamma), \qquad \arg(1-\gamma) \in (-\pi,\pi] \tag{3.8}$$

and $\left(\frac{z+1}{z-1}\right)^{\nu}$ is defined on $\mathbb{C}\backslash[-1,1]$ with its branch fixed by the condition $\left(\frac{z+1}{z-1}\right)^{\nu} \to 1$ as $z \to \infty$. The function $M(z)$ as introduced in (3.7) is not the unique solution to the model RHP. But, we will see that the one we choose here will properly match with the parametrices at $z = \pm 1$.

Remark 3 *We bring the reader's attention to the important fact, that in case* $\gamma < 1$, *we have* $\arg(1-\gamma) = 0$ *and* ν *is therefore purely imaginary. However if* $\gamma > 1$, *then* $\arg(1-\gamma) = \pi$ *and* ν *equals*

$$\nu = \frac{1}{2\pi i}\ln(\gamma - 1) + \frac{1}{2} \equiv \nu_0 + \frac{1}{2}, \qquad \nu_0 \in i\mathbb{R}. \tag{3.9}$$

Later on we will see that this difference will have a very substantial impact on the whole steepest descent analysis.

We now construct a parametrix at the origin $z = 0$. The idea isto use a Ψ-function associated to the system (1.10) for our construction. More precisely let

$$P_{II}(\zeta) = \Psi_1(\zeta) \equiv \Psi_1(\zeta, x), \qquad \zeta \in \mathbb{C} \tag{3.10}$$

be the first canonical solution of system (1.10) which is uniquely fixed by the asymptotics

$$P_{II}(\zeta) = \Big(I + O\big(\zeta^{-1}\big)\Big)e^{-i(\frac{4}{3}\zeta^3 + x\zeta)\sigma_3}, \quad \zeta \to \infty, \quad \zeta \in \Omega_1.$$

Secondly, using the Stokes matrices in (2.1), we introduce

$$\widetilde{P}_{II}^{RH}(\zeta) = \begin{cases} P_{II}(\zeta), & \arg\zeta \in (-\frac{\pi}{6}, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, \frac{7\pi}{6}), \\ P_{II}(\zeta)S_1, & \arg\zeta \in (\frac{\pi}{6}, \frac{5\pi}{6}), \\ P_{II}(\zeta)S_4, & \arg\zeta \in (\frac{7\pi}{6}, \frac{11\pi}{6}), \end{cases} \tag{3.11}$$

which solves the RHP depicted in Figure 3.4.

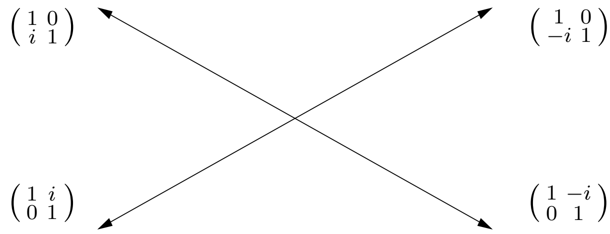

Figure 3.4. The model RHP near $z = 0$ which can be solved explicitly using the Hastings-Mcleod solution of the second Painlevé equation

More precisely, the function $\widetilde{P}_{II}^{RH}(\zeta)$ possesses the following analytic properties.

- $\widetilde{P}_{II}^{RH}(\zeta)$ is analytic for $\zeta \in \mathbb{C}\setminus \left\{\arg\zeta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$
- The following jumps hold

$$\begin{aligned} \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_+ &= \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_- S_1, & \arg\zeta &= \frac{\pi}{6} \\ \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_+ &= \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_- S_3, & \arg\zeta &= \frac{5\pi}{6} \\ \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_+ &= \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_- S_4, & \arg\zeta &= \frac{7\pi}{6} \\ \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_+ &= \Big(\widetilde{P}_{II}^{RH}(\zeta)\Big)_- S_6, & \arg\zeta &= \frac{11\pi}{6} \end{aligned}$$

- Recalling the discussion in chapter 2, the following uniform asymptotics holds, valid in a full neighborhood of infinity (cf. [29])

$$\widetilde{P}_{II}^{RH}(\zeta) = \left(I + \frac{m_1^{HM}}{\zeta} + \frac{m_2^{HM}}{\zeta} + O\big(\zeta^{-3}\big)\right) e^{-i(\frac{4}{3}\zeta^3 + x\zeta)\sigma_3}, \quad \zeta \to \infty. \tag{3.12}$$

In (3.12) (compare (1.10) and the discussion in section 2.1) we have

$$m_1^{HM} = \frac{1}{2}\begin{pmatrix} -iv & u \\ u & iv \end{pmatrix}, \quad m_2^{HM} = \frac{1}{8}\begin{pmatrix} u^2-v^2 & 2i(u_x+uv) \\ -2i(u_x+uv) & u^2-v^2 \end{pmatrix},$$

where $u = u(x)$ is the Hastings-McLeod solution of the second Painlevé equation and we put $v = (u_x)^2 - xu^2 - u^4$. Next we assemble the piecewise analytic matrix-valued function $P_{II}^{RH}(\zeta)$

$$P_{II}^{RH}(\zeta) = \begin{cases} e^{\pi i\nu\sigma_3}\widetilde{P}_{II}^{RH}(\zeta)e^{-\pi i\nu\sigma_3}, & \operatorname{Im}\zeta > 0, \\ e^{\pi i\nu\sigma_3}\widetilde{P}_{II}^{RH}(\zeta)e^{\pi i\nu\sigma_3}, & \operatorname{Im}\zeta < 0. \end{cases} \tag{3.13}$$

Together with the RHP for $\widetilde{P}_{II}^{RH}(\zeta)$, we see at once that $P_{II}^{RH}(\zeta)$ in addition to the jumps on the rays depicted in Figure 3.4, also has a jump on the real line

$$\left(P_{II}^{RH}(\zeta)\right)_+ = \left(P_{II}^{RH}(\zeta)\right)_- e^{-2\pi i\nu\sigma_3} \equiv \left(P_{II}^{RH}(\zeta)\right)_- S_D, \quad \zeta \in \mathbb{R}$$

where we orient the real line from left to right. Also on the rays, by construction,

$$\begin{aligned} \left(P_{II}^{RH}(\zeta)\right)_+ &= \left(P_{II}^{RH}(\zeta)\right)_- S_L, & \arg\zeta &= \frac{\pi}{6} \\ \left(P_{II}^{RH}(\zeta)\right)_+ &= \left(P_{II}^{RH}(\zeta)\right)_- S_L^{-1}, & \arg\zeta &= \frac{5\pi}{6} \\ \left(P_{II}^{RH}(\zeta)\right)_+ &= \left(P_{II}^{RH}(\zeta)\right)_- S_U^{-1}, & \arg\zeta &= \frac{7\pi}{6} \\ \left(P_{II}^{RH}(\zeta)\right)_+ &= \left(P_{II}^{RH}(\zeta)\right)_- S_U, & \arg\zeta &= \frac{11\pi}{6}. \end{aligned}$$

The model function $P_{II}^{RH}(\zeta)$ will now be used to construct the parametrix to the solution of the original S-RHP in a neighborhood of $z = 0$. We proceed in two steps. First define

$$\zeta(z) = sz, \quad |z| < r. \tag{3.14}$$

This change of variables is locally conformal and it enables us to define the origin parametrix $U(z)$ near $z = 0$ by the formula

$$U(z) = B_0(z)P_{II}^{RH}\big(\zeta(z)\big)e^{i(\frac{4}{3}\zeta(z)+x\zeta(z))\sigma_3}, \quad |z| < r \tag{3.15}$$

with $\zeta(z)$ as in (3.14) and the matrix multiplier

$$B_0(z)=\left(\frac{z+1}{z-1}\right)^{\nu\sigma_3}\begin{cases} I, & \operatorname{Im} z>0,\\ e^{-2\pi i\nu\sigma_3}, & \operatorname{Im} z<0,\end{cases}\qquad B_0(0)=e^{-\pi i\nu\sigma_3}. \tag{3.16}$$

By construction, in particular since $B_0(z)$ is analytic in a neighborhood of $z=0$, the parametrix $U(z)$ has jumps along the curves depicted in Figure 3.5, which are locally identical to the jump curves in the original RHP. Also these jumps are described by the same jump matrices as in the S-RHP (see (3.3)), hence the ratio of $S(z)$ with $U(z)$ is locally analytic, i.e.

$$S(z)=N_0(z)U(z),\qquad |z|<r<\frac{1}{2}. \tag{3.17}$$

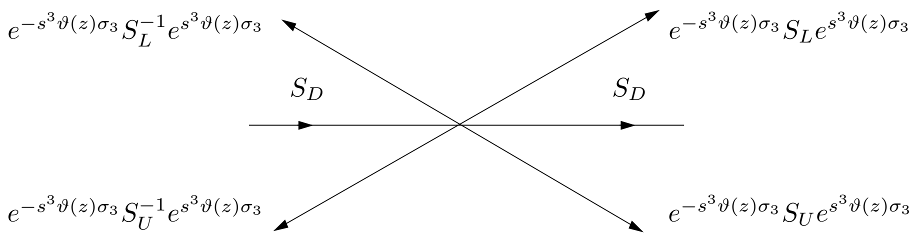

Figure 3.5. Jump graph of the parametrix $U(z)$

Let us explain the role of the left multiplier $B_0(z)$ in the definition (3.15). Observe that

$$M(z)=B_0(z)\begin{cases} I, & \operatorname{Im} z>0,\\ e^{2\pi i\nu\sigma_3}, & \operatorname{Im} z<0.\end{cases}$$

This relation together with the asymptotic equation (3.12) implies that,

$$\begin{aligned} U(z) &= B_0(z)e^{\pi i\nu\sigma_3}\Bigg[I+\frac{1}{2\zeta}\begin{pmatrix} -iv & u \\ u & iv \end{pmatrix} \\ &\quad +\frac{1}{8\zeta^2}\begin{pmatrix} u^2-v^2 & 2i(u_x+uv) \\ -2i(u_x+uv) & u^2-v^2 \end{pmatrix}+O\big(\zeta^{-3}\big)\Bigg]e^{-\pi i\nu\sigma_3}B_0^{-1}(z)M(z) \\ &= \Bigg[I+\frac{B_0(z)}{2\zeta}\begin{pmatrix} -iv & ue^{2\pi i\nu} \\ ue^{-2\pi i\nu} & iv \end{pmatrix}B_0^{-1}(z)+\frac{B_0(z)}{8\zeta^2} \\ &\quad\times\begin{pmatrix} u^2-v^2 & 2i(u_x+uv)e^{2\pi i\nu} \\ -2i(u_x+uv)e^{-2\pi i\nu} & u^2-v^2 \end{pmatrix}B_0^{-1}(z)+O\big(\zeta^{-3}\big)\Bigg]M(z) \end{aligned} \tag{3.18}$$

as $s\to\infty$ and $0<r_1\leq|z|\leq r_2<\frac{1}{2}$ (so $|\zeta|\to\infty$). Since the function $\zeta(z)$ is of order $O(s)$ on the latter annulus and $B_0(z)$ is bounded, equation (3.18) yields the matching relation between the model functions $U(z)$ and $M(z)$,

$$U(z)=\big(I+o(1)\big)M(z),\qquad s\to\infty,\ \ 0<r_1\leq|z|\leq r_2<\frac{1}{2},$$

which is crucial for the successful implementation of the nonlinear steepest descent method as we shall see in the next section. This is the reason for choosing the left multiplier $B_0(z)$ in (3.15) in the form (3.16).

For the parametrix at the right endpoint $z=+1$, we recall the Taylor expansion

$$\vartheta(z)=\vartheta(1)+i\Big(4+\frac{x}{s^2}\Big)(z-1)+O\big((z-1)^2\big),\qquad z\to 1$$

and the singular endpoint behavior

$$S(z)=O\big(\ln(z-1)\big),\qquad z\to 1. \tag{3.19}$$

Both observations suggest to use the confluent hypergeometric function $U(a,b;\zeta)$ for our construction. We will justify this idea as follows. Recall that the listed confluent hypergeometric function is defined as unique solution to Kummer's equation

$$zw''+(b-z)w'-aw=0$$

satisfying the asymptotic condition as $\zeta \to \infty$ and $-\frac{3\pi}{2} < \arg \zeta < \frac{3\pi}{2}$ (see [5])

$$U(a,b;\zeta) = \zeta^{-a}\bigg(1 - \frac{a(1+a-b)}{\zeta} + \frac{a(a+1)(1+a-b)(2+a-b)}{2\zeta^2} + O\big(\zeta^{-3}\big)\bigg).$$

Also, using the notation $U(a,\zeta) \equiv U(a,1;\zeta)$, the following monodromy relation holds on the entire universal covering of the punctured plane

$$U(1-a, e^{i\pi}\zeta) = e^{2\pi i a}U(1-a, e^{-i\pi}\zeta) - e^{i\pi a}\frac{2\pi i}{\Gamma^2(1-a)}U(a,\zeta)e^{-\zeta} \tag{3.20}$$

and moreover we have an expansion at the origin (compare to (3.19))

$$U(a,\zeta) = c_0 + c_1 \ln\zeta + c_2\zeta + c_3\zeta\ln\zeta + O\big(\zeta^2\ln\zeta\big), \quad \zeta \to 0 \tag{3.21}$$

with coefficients c_i given as

$$c_0 = -\frac{1}{\Gamma(a)}\big(\psi(a) + 2\gamma_E\big),\ c_1 = -\frac{1}{\Gamma(a)},\ c_2 = -\frac{a}{\Gamma(a)}\big(\psi(a+1) + 2\gamma_E - 2\big),\ c_3 = -\frac{a}{\Gamma(a)}$$

where γ_E is Euler's constant and we introduced the Digamma function $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Keeping the latter properties in mind, we introduce on the punctured plane (cf. [38])

$$\begin{aligned} P_{CH}(\zeta) &= \begin{pmatrix} U(\nu, e^{i\frac{\pi}{2}}\zeta)e^{2\pi i\nu}e^{-i\frac{\zeta}{2}} & -U(1-\nu, e^{-i\frac{\pi}{2}}\zeta)e^{\pi i\nu}e^{i\frac{\zeta}{2}}\frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ -U(1+\nu, e^{i\frac{\pi}{2}}\zeta)e^{\pi i\nu}e^{-i\frac{\zeta}{2}}\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & U(-\nu, e^{-i\frac{\pi}{2}}\zeta)e^{i\frac{\zeta}{2}} \end{pmatrix} \\ &\quad \times e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}, \quad \zeta \in \mathbb{C}\backslash\{0\}, \quad -\pi < \arg\zeta \leq \pi \end{aligned} \tag{3.22}$$

with ν given in (3.8). Let us collect the following asymptotic expansions. First in the sector $-\frac{\pi}{2} < \arg\zeta < \frac{\pi}{2}$

$$\begin{aligned} P_{CH}(\zeta) &= \Bigg[I + \frac{i}{\zeta}\begin{pmatrix} \nu^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{\pi i\nu} \\ \frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{-\pi i\nu} & -\nu^2 \end{pmatrix} \\ &\quad + \frac{1}{\zeta^2}\begin{pmatrix} -\frac{\nu^2}{2}(1+\nu)^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2e^{\pi i\nu} \\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2e^{-\pi i\nu} & -\frac{\nu^2}{2}(1-\nu)^2 \end{pmatrix} + O\big(\zeta^{-3}\big)\Bigg]\zeta^{-\nu\sigma_3} \\ &\quad \times e^{-i\frac{\zeta}{2}\sigma_3}e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}\begin{pmatrix} e^{i\frac{3\pi}{2}\nu} & 0 \\ 0 & e^{-i\frac{\pi}{2}\nu} \end{pmatrix}, \quad \zeta \to \infty. \end{aligned}$$

For another sector, say $\frac{\pi}{4} < \arg\zeta < \frac{5\pi}{4}$, we use (3.20) in the first column of (3.22) and obtain instead

$$\begin{aligned} P_{CH}(\zeta) &= \Bigg[I + \frac{i}{\zeta} \begin{pmatrix} \nu^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)} e^{\pi i \nu} \\ \frac{\Gamma(1+\nu)}{\Gamma(-\nu)} e^{-\pi i \nu} & -\nu^2 \end{pmatrix} \\ &\quad + \frac{1}{\zeta^2} \begin{pmatrix} -\frac{\nu^2}{2}(1+\nu)^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2 e^{\pi i \nu} \\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2 e^{-\pi i \nu} & -\frac{\nu^2}{2}(1-\nu)^2 \end{pmatrix} + O\big(\zeta^{-3}\big) \Bigg] \\ &\quad \times \zeta^{-\nu\sigma_3} e^{-i\frac{\zeta}{2}\sigma_3} e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3} \begin{pmatrix} e^{i\frac{3\pi}{2}\nu} & 0 \\ 0 & e^{-i\frac{\pi}{2}\nu} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{2\pi e^{i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} & 1 \end{pmatrix}, \quad \zeta \to \infty, \end{aligned}$$

as well as for $-\frac{5\pi}{4} < \arg\zeta < -\frac{\pi}{4}$ with a similar argument in the second column of (3.22)

$$\begin{aligned} P_{CH}(\zeta) &= \Bigg[I + \frac{i}{\zeta} \begin{pmatrix} \nu^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)} e^{\pi i \nu} \\ \frac{\Gamma(1+\nu)}{\Gamma(-\nu)} e^{-\pi i \nu} & -\nu^2 \end{pmatrix} \\ &\quad + \frac{1}{\zeta^2} \begin{pmatrix} -\frac{\nu^2}{2}(1+\nu)^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2 e^{\pi i \nu} \\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2 e^{-\pi i \nu} & -\frac{\nu^2}{2}(1-\nu)^2 \end{pmatrix} + O\big(\zeta^{-3}\big) \Bigg] \\ &\quad \times \zeta^{-\nu\sigma_3} e^{-i\frac{\zeta}{2}\sigma_3} e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3} \begin{pmatrix} e^{i\frac{3\pi}{2}\nu} & 0 \\ 0 & e^{-i\frac{\pi}{2}\nu} \end{pmatrix} \begin{pmatrix} 1 & \frac{2\pi e^{-3\pi i\nu}}{\Gamma(1-\nu)\Gamma(\nu)} \\ 0 & 1 \end{pmatrix}, \quad \zeta \to \infty. \end{aligned}$$

Also of interest for future purposes is the following identity

$$P_{CH}(\zeta) = P_{CH}(e^{-2\pi i}\zeta) \begin{pmatrix} e^{-2\pi i\nu} + \left(\frac{2\pi}{\Gamma(1-\nu)\Gamma(\nu)}\right)^2 & -\frac{2\pi e^{-i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} \\ -\frac{2\pi e^{3\pi i\nu}}{\Gamma(1-\nu)\Gamma(\nu)} & e^{2\pi i\nu} \end{pmatrix}. \tag{3.23}$$

Let us now assemble the model function

$$P_{CH}^{RH}(\zeta) = \begin{cases} P_{CH}(\zeta)\begin{pmatrix} 1 & 0 \\ \frac{2\pi e^{i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} & 1 \end{pmatrix}\begin{pmatrix} e^{-\frac{3\pi}{2}i\nu} & 0 \\ 0 & e^{\frac{\pi}{2}i\nu} \end{pmatrix}, & \arg\zeta \in (\frac{\pi}{3}, \pi), \\ P_{CH}(\zeta)\begin{pmatrix} 1 & -\frac{2\pi e^{-3\pi i\nu}}{\Gamma(1-\nu)\Gamma(\nu)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-\frac{3\pi}{2}i\nu} & 0 \\ 0 & e^{\frac{\pi}{2}i\nu} \end{pmatrix}, & \arg\zeta \in (-\pi, -\frac{\pi}{3}), \\ P_{CH}(\zeta)\begin{pmatrix} e^{-\frac{3\pi}{2}i\nu} & 0 \\ 0 & e^{\frac{\pi}{2}i\nu} \end{pmatrix}, & \arg\zeta \in (-\frac{\pi}{3}, \frac{\pi}{3}). \end{cases} \tag{3.24}$$

which solves the RHP depicted in Figure 3.6

In more detail, $P_{CH}^{RH}(\zeta)$ possesses the following analytic properties

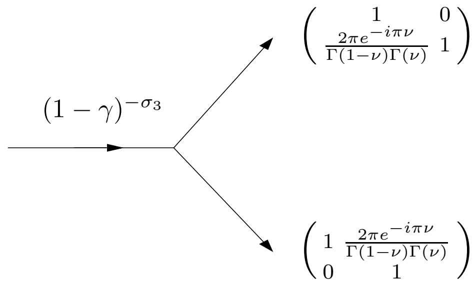

Figure 3.6. The model RHP near $z = +1$ which can be solved explicitly using confluent hypergeometric functions

- $P_{CH}^{RH}(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash\{\arg\zeta = -\pi, -\frac{\pi}{3}, \frac{\pi}{3}\}$
- The following jumps are valid, the jump contours being oriented as shown in Figure 3.6

$$\begin{aligned}
\left(P_{CH}^{RH}(\zeta)\right)_+ &= \left(P_{CH}^{RH}(\zeta)\right)_-(1-\gamma)^{-\sigma_3}, \quad \arg\zeta = -\pi \\
\left(P_{CH}^{RH}(\zeta)\right)_+ &= \left(P_{CH}^{RH}(\zeta)\right)_-\begin{pmatrix} 1 & 0 \\ \frac{2\pi e^{-i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{\pi}{3} \\
\left(P_{CH}^{RH}(\zeta)\right)_+ &= \left(P_{CH}^{RH}(\zeta)\right)_-\begin{pmatrix} 1 & \frac{2\pi e^{-i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} \\ 0 & 1 \end{pmatrix}, \quad \arg\zeta = -\frac{\pi}{3}
\end{aligned}$$

and in virtue of the classical identity

$$\Gamma(1-\nu)\Gamma(\nu) = \frac{\pi}{\sin\pi\nu}$$

the entries of the latter triangular matrices equal

$$\frac{2\pi e^{-i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} = i\gamma(1-\gamma)^{-1}.$$

- As $\zeta \to \infty$, the model function $P_{CH}^{RH}(\zeta)$ shows the following asymptotic behavior, which is valid in a full neighborhood of infinity

$$\begin{aligned} P_{CH}^{RH}(\zeta) &= \Bigg[I + \frac{i}{\zeta}\begin{pmatrix} \nu^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{\pi i\nu} \\ \frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{-\pi i\nu} & -\nu^2 \end{pmatrix} \\ &\quad + \frac{1}{\zeta^2}\begin{pmatrix} -\frac{\nu^2}{2}(1+\nu)^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2 e^{\pi i\nu} \\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2 e^{-\pi i\nu} & -\frac{\nu^2}{2}(1-\nu)^2 \end{pmatrix} + O\big(\zeta^{-3}\big)\Bigg] \\ &\quad \times \zeta^{-\nu\sigma_3}e^{-i\frac{\zeta}{2}\sigma_3}e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}, \quad \zeta \to \infty. \end{aligned}$$

In order to construct the relevant parametrix near $z = +1$, define the locally conformal change of variables

$$\zeta(z) = -2is^3\big(\vartheta(z) - \vartheta(1)\big) = \big(8s^3 + 2xs\big)\,(z-1)\big(1 + O(z-1)\big), \quad |z-1| < r. \tag{3.25}$$

The parametrix is now given by

$$V(z) = B_r(z)e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(1)\sigma_3}P_{CH}^{RH}\big(\zeta(z)\big)e^{(\frac{i}{2}\zeta(z)+s^3\vartheta(1))\sigma_3}, \quad |z-1| < r \tag{3.26}$$

with $\zeta(z)$ as in (3.25) and the matrix-valued function $B_r(z)$ equals

$$B_r(z) = \left(\zeta(z)\frac{z+1}{z-1}\right)^{\nu\sigma_3}, \quad B_r(1) = \big(16s^3 + 4xs\big)^{\nu\sigma_3}. \tag{3.27}$$

Also here, following from analyticity of $B_r(z)$, parametrix jumps match original jumps in the S-RHP. Moreover the parametrix $V(z)$ has jumps along the curves depicted in Figure 3.7, and we can always locally match the latter curves with the jump curves in the S-RHP. Furthermore, and we shall elaborate this in full detail very soon, the singular endpoint behavior of the parametrix $V(z)$ matches (3.19), i.e.

$$V(z) = O\big(\ln(z-1)\big), \quad z \to +1. \tag{3.28}$$

Hence the ratio of $S(z)$ with $V(z)$ is locally analytic, i.e.

$$S(z) = N_r(z)V(z), \quad |z-1| < \frac{1}{2}$$

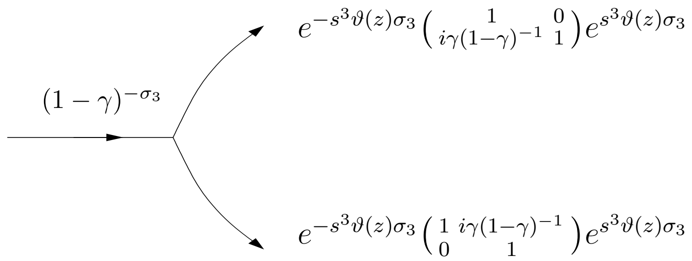

Figure 3.7. Transformation of parametrix jumps to original jumps

In the end the role of the multiplier $B_r(z)$ follows again from the following asymptotical matching relation

$$\begin{aligned}
V(z) &= B_r(z)e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(1)\sigma_3}\Bigg[I+\frac{i}{\zeta}\begin{pmatrix}\nu^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{\pi i\nu}\\ \frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{-\pi i\nu} & -\nu^2\end{pmatrix}\\
&\quad+\frac{1}{\zeta^2}\begin{pmatrix}-\frac{\nu^2}{2}(1+\nu)^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2e^{\pi i\nu}\\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2e^{-\pi i\nu} & -\frac{\nu^2}{2}(1-\nu)^2\end{pmatrix}+O\big(\zeta^{-3}\big)\Bigg]\zeta^{-\nu\sigma_3}\\
&\quad\times e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{s^3\vartheta(1)\sigma_3} \qquad (3.29)\\
&= \Bigg[I+\frac{i}{\zeta}\begin{pmatrix}\nu^2 & -i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\beta_r^2(z)\\ -i\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{2s^3\vartheta(1)}\beta_r^{-2}(z) & -\nu^2\end{pmatrix}\\
&\quad+\frac{1}{\zeta^2}\begin{pmatrix}-\frac{\nu^2}{2}(1+\nu)^2 & -i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2e^{-2s^3\vartheta(1)}\beta_r^2(z)\\ i\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2e^{2s^3\vartheta(1)}\beta_r^{-2}(z) & -\frac{\nu^2}{2}(1-\nu)^2\end{pmatrix}\\
&\quad+O\big(\zeta^{-3}\big)\Bigg]M(z) \qquad (3.30)
\end{aligned}$$

as $s\to\infty$ valid on the annulus $0<r_2\leq|z-1|\leq r_2<\frac{1}{2}$ (hence $|\zeta|\to\infty$) where we use the notation

$$\beta_r(z)=\left(\zeta(z)\frac{z+1}{z-1}\right)^{\nu}.$$

If we are dealing with the case $\gamma<1$, then

$$\beta_r^{\pm 2}(z)\frac{1}{\zeta}=O\left(s^{-3\pm 6\operatorname{Re}\nu}\right)=o(1),\qquad s\to\infty.$$

This would mean that equation (3.30) yields the matching relation between the model functions $V(z)$ and $M(z)$,

$$V(z)=\big(I+o(1)\big)M(z),\qquad s\to\infty,\ 0<r_1\leq|z-1|\leq r_2<\frac{1}{2} \qquad (3.31)$$

which is again crucial for the successful implementation of the nonlinear steepest descent method. However, if $\gamma > 1$, then

$$\nu = \frac{1}{2\pi i}\ln(\gamma-1)+\frac{1}{2} \equiv \nu_0 + \frac{1}{2}$$

and hence

$$\beta_r^2(z)\frac{1}{\zeta} = \hat{\beta}_r^2(z)\frac{z+1}{z-1} = O(1), \quad s\to\infty; \quad \hat{\beta}_r(z) = \left(\zeta(z)\frac{z+1}{z-1}\right)^{\nu_0}.$$

With this notation, we have to replace (3.31) in case $\gamma > 1$ by

$$V(z) = E_r(z)\big(I+o(1)\big)M(z), \qquad s\to\infty,\ 0<r_1\leq |z-1| \leq r_2 < \frac{1}{2} \tag{3.32}$$

where

$$E_r(z) = \begin{pmatrix} 1 & \frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(z)\frac{z+1}{z-1} \\ 0 & 1 \end{pmatrix}. \tag{3.33}$$

The appearance of the nontrivial matrix term $E_r(z)$ instead of the unit matrix in estimate (3.32) yields a very serious change in the further asymptotic analysis comparing with the matching case (3.31). We will proceed with this analysis in section 3.4.

For now, we introduce the model RHP near the other endpoint $z=-1$. Opposed to (3.22) consider

$$\begin{aligned}\widetilde{P}_{CH}(\zeta) \;&=\; \begin{pmatrix} U(-\nu, e^{-i\frac{3\pi}{2}}\zeta)e^{-i\frac{\zeta}{2}} & U(1+\nu, e^{-i\frac{\pi}{2}}\zeta)e^{\pi i\nu}e^{i\frac{\zeta}{2}}\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} \\ U(1-\nu, e^{-i\frac{3\pi}{2}}\zeta)e^{\pi i\nu}e^{-i\frac{\zeta}{2}}\frac{\Gamma(1-\nu)}{\Gamma(\nu)} & U(\nu, e^{-i\frac{\pi}{2}}\zeta)e^{2\pi i\nu}e^{i\frac{\zeta}{2}} \end{pmatrix} \\ &\times e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3} = \sigma_2 P_{CH}(e^{-i\pi}\zeta)\sigma_2, \quad 0<\arg\zeta\leq 2\pi.\end{aligned}$$

and

$$\widetilde{P}_{CH}^{RH}(\zeta) = \begin{cases} \widetilde{P}_{CH}(\zeta)\begin{pmatrix} 1 & 0 \\ \frac{2\pi e^{-3i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} & 1 \end{pmatrix}\begin{pmatrix} e^{i\frac{\pi}{2}\nu} & 0 \\ 0 & e^{-i\frac{3\pi}{2}\nu} \end{pmatrix}, & \arg\zeta\in(0,\frac{2\pi}{3}), \\ \widetilde{P}_{CH}(\zeta)\begin{pmatrix} 1 & -\frac{2\pi e^{i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^{i\frac{\pi}{2}\nu} & 0 \\ 0 & e^{-i\frac{3\pi}{2}\nu} \end{pmatrix}, & \arg\zeta\in(\frac{4\pi}{3},2\pi), \\ \widetilde{P}_{CH}(\zeta)\begin{pmatrix} e^{i\frac{\pi}{2}\nu} & 0 \\ 0 & e^{-i\frac{3\pi}{2}\nu} \end{pmatrix}, & \arg\zeta\in(\frac{2\pi}{3},\frac{4\pi}{3}). \end{cases} \tag{3.34}$$

The model function $\widetilde{P}_{CH}^{RH}(\zeta)$ solves the RHP of Figure 3.8

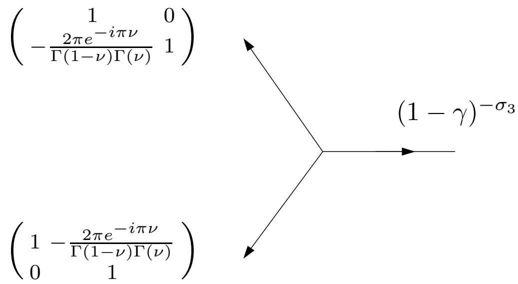

Figure 3.8. The model RHP near $z=-1$ which can be solved explicitly using confluent hypergeometric functions

- $\widetilde{P}_{CH}^{RH}(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash\{\arg\zeta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$
- Along the contour in Figure 3.8, the following jumps are valid (recall (3.23) and the symmetry relation $\widetilde{P}_{CH}^{RH}(\zeta) = \sigma_2 P_{CH}(e^{-i\pi}\zeta)\sigma_2$)

$$\begin{aligned}
\big(\widetilde{P}_{CH}^{RH}(\zeta)\big)_+ &= \big(\widetilde{P}_{CH}^{RH}(\zeta)\big)_- e^{-2\pi i\nu\sigma_3}, \quad \arg\zeta = 0 \\
\big(\widetilde{P}_{CH}^{RH}(\zeta)\big)_+ &= \big(\widetilde{P}_{CH}^{RH}(\zeta)\big)_- \begin{pmatrix} 1 & 0 \\ -\frac{2\pi e^{-i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{2\pi}{3} \\
\big(\widetilde{P}_{CH}^{RH}(\zeta)\big)_+ &= \big(\widetilde{P}_{CH}^{RH}(\zeta)\big)_- \begin{pmatrix} 1 & -\frac{2\pi e^{-i\pi\nu}}{\Gamma(1-\nu)\Gamma(\nu)} \\ 0 & 1 \end{pmatrix}, \quad \arg\zeta = \frac{4\pi}{3}
\end{aligned}$$

- From symmetry $\widetilde{P}_{CH}^{RH}(\zeta) = \sigma_2 P_{CH}(e^{-i\pi}\zeta)\sigma_2$ and the asymptotic information derived earlier for $P_{CH}(\zeta)$ in the different sectors, we deduce the following behavior, valid in a full neighborhood of infinity

$$\begin{aligned}
\widetilde{P}_{CH}^{RH}(\zeta) = & \Bigg[I + \frac{i}{\zeta}\begin{pmatrix} \nu^2 & \frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{-\pi i\nu} \\ -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{\pi i\nu} & -\nu^2 \end{pmatrix} \\
& + \frac{1}{\zeta^2}\begin{pmatrix} -\frac{\nu^2}{2}(1-\nu)^2 & \frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2 e^{-\pi i\nu} \\ \frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2 e^{\pi i\nu} & -\frac{\nu^2}{2}(1+\nu)^2 \end{pmatrix} + O\big(\zeta^{-3}\big)\Bigg] \\
& \times \big(e^{-i\pi}\zeta\big)^{\nu\sigma_3} e^{-i\frac{\zeta}{2}\sigma_3} e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}, \quad \zeta \to \infty
\end{aligned}$$

The next steps are very similar to the construction of $V(z)$. First define

$$\zeta(z) = -2is^3\big(\vartheta(z) - \vartheta(-1)\big) = \big(8s^3 + 2xs\big)(z+1)\Big(1 + O(z+1)\Big), \quad |z+1| < r \quad (3.35)$$

and secondly the parametrix $W(z)$ near the left endpoint $z=-1$ via

$$W(z)=B_l(z)e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(-1)\sigma_3}\widetilde{P}_{CH}^{RH}\big(\zeta(z)\big)e^{(\frac{i}{2}\zeta(z)+s^3\vartheta(-1))\sigma_3},\quad |z+1|<r. \quad (3.36)$$

with $\zeta(z)$ as in (3.35) and

$$B_l(z)=\left(e^{-i\pi}\zeta(z)\frac{z-1}{z+1}\right)^{-\nu\sigma_3},\qquad B_l(-1)=\big(16s^3+4xs\big)^{-\nu\sigma_3}.$$

Also here parametrix jumps match with original jumps locally on the original jump contour, see Figure 3.9,

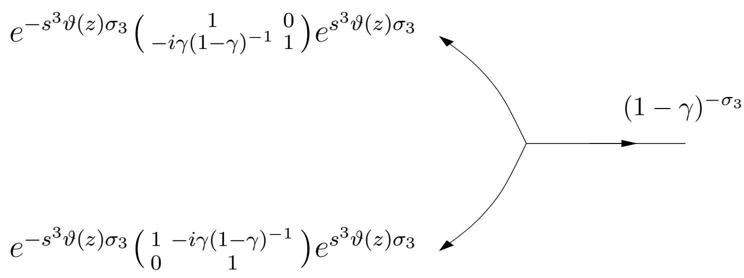

Figure 3.9. Transformation of parametrix jumps to original jumps

and with the singular endpoint behavior (see section 4.1 for a rigorous derivation)

$$W(z)=O\left(\ln(z+1)\right),\qquad z\to -1 \quad (3.37)$$

we have a locally analytic ratio of $S(z)$ and $W(z)$

$$S(z)=N_l(z)W(z),\qquad |z+1|<\frac{1}{2}.$$

The role of the left multiplier follows once more from the asymptotical matchup between the model functions

$$\begin{aligned}W(z) \;=\; &\Bigg[I+\frac{i}{\zeta}\begin{pmatrix}\nu^2 & -i\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{-2s^3\vartheta(-1)}\beta_l^2(z)\\ -i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{2s^3\vartheta(-1)}\beta_l^{-2}(z) & -\nu^2\end{pmatrix}\\ &+\frac{1}{\zeta^2}\begin{pmatrix}-\frac{\nu^2}{2}(1-\nu)^2 & -i\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2e^{-2s^3\vartheta(-1)}\beta_l^2(z)\\ i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2e^{2s^3\vartheta(-1)}\beta_l^{-2}(z) & -\frac{\nu^2}{2}(1+\nu)^2\end{pmatrix}\\ &+O\big(\zeta^{-3}\big)\Bigg]M(z),\end{aligned} \quad (3.38)$$

valid as $s \to \infty$ on the annulus $0 < r_1 \le |z+1| \le r_2 < \frac{1}{2}$ (thus $|\zeta| \to \infty$) and we introduced

$$\beta_l(z) = \left(e^{-i\pi}\zeta(z)\frac{z-1}{z+1}\right)^{-\nu}.$$

Similar to the situation at the right endpoint, estimate (3.38) implies on the annulus for $\gamma < 1$

$$W(z) = \big(I + o(1)\big)M(z), \quad s \to \infty,$$

whereas in case $\gamma > 1$, we have

$$W(z) = E_l(z)\big(I + o(1)\big)M(z), \quad s \to \infty \tag{3.39}$$

with

$$E_l(z) = \begin{pmatrix} 1 & 0 \\ -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{2s^3\vartheta(-1)}\hat{\beta}_l^{-2}(z)\frac{z-1}{z+1} & 1 \end{pmatrix}, \quad \hat{\beta}_l(z) = \left(e^{-i\pi}\zeta(z)\frac{z-1}{z+1}\right)^{-\nu_0}.$$

At this point we can use the model functions $M(z), U(z), V(z)$ and $W(z)$ to employ our next transformation.

3.3 The ratio problem – iterative solution for $\gamma < 1$

We put in this transformation

$$R(z) = S(z)\begin{cases} \big(V(z)\big)^{-1}, & |z-1| < r_1, \\ \big(U(z)\big)^{-1}, & |z| < r_2, \\ \big(W(z)\big)^{-1}, & |z+1| < r_1, \\ \big(M(z)\big)^{-1}, & |z-1| > r_1, |z+1| > r_1, |z| > r_2, \end{cases} \tag{3.40}$$

where $0 < r_1, r_2 < \frac{1}{2}$ is fixed. With $C_{0,r,l}$ denoting the clockwise oriented circles shown in Figure 3.10, the ratio-function $R(z)$ solves the following RHP

- $R(z)$ is analytic for $z \in \mathbb{C}\backslash\Sigma_R$ with $\Sigma_R = C_{0,r,l} \cup \bigcup_{i=1}^{8} \gamma_i$
- For the jumps, along the infinite branches γ_i

$$R_+(z) = R_-(z)M(z)G_S(z)\big(M(z)\big)^{-1},$$

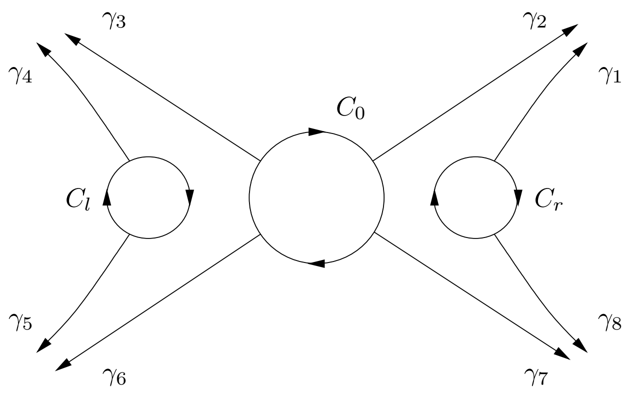

Figure 3.10. The jump graph for the ratio-function $R(z)$

with $G_S(z)$ denoting the corresponding jump matrix from (3.3). On the clockwise orientied circles C_0 and $C_{r,l}$, the jumps are described by the equations

$$
\begin{aligned}
R_+(z) &= R_-(z)U(z)\big(M(z)\big)^{-1}, \ z \in C_0,\\
R_+(z) &= R_-(z)V(z)\big(M(z)\big)^{-1}, \ z \in C_r,\\
R_+(z) &= R_-(z)W(z)\big(M(z)\big)^{-1}, \ z \in C_l.
\end{aligned}
$$

- $R(z)$ is analytic at $z = \pm 1$. This observation will follow directly from (3.28) and (3.37), which shall be proved in section 4.1
- In a neighborhood of infinity, we have $R(z) \to I$.

We emphasize that, by construction, $R(z)$ has no jumps inside of the circles $C_{0,r,l}$ and across the line segment in between. In order to apply the Deift-Zhou nonlinear steepest descent method for the ratio-RHP, all jump matrices have to be close to the unit matrix, as $s \to \infty$, compare [25]. Hence it is now important to recall the previosuly stated behavior of the jump matrices as $s \to \infty$: As mentioned before, due to the "correct" triangularity of $\widetilde{S}_i$ combined with the sign-diagram of $\operatorname{Re}\vartheta(z)$, the jump matrices corresponding to the infinite parts $\bigcup_{i=1}^{8}\gamma_i$ of the R-jump contour are in fact exponentially close to the unit matrix

$$\|MG_S\big(M\big)^{-1} - I\|_{L^2\cap L^\infty(\gamma_i)} \leq c_1 \begin{cases} e^{-c_2 s^3|z|}, & \text{emanating from } C_0; \\ e^{-c_3 s^3|z\mp 1|}, & \text{emanating from } C_{r,l}, \end{cases} \tag{3.41}$$

as $s \to \infty$ with constants $c_i > 0$ whose values are not important. Also by virtue of (3.18), $U(z)\big(M(z)\big)^{-1}$ approaches the unit matrix as $s \to \infty$,

$$\|U\big(M\big)^{-1} - I\|_{L^2\cap L^\infty(C_0)} \leq c_4 s^{-1} \tag{3.42}$$

with a constant $c_4 > 0$. However, as we have already seen, compare (3.32), (3.39), the jumps on $C_{r,l}$ have to be treated more carefully. In case $\gamma < 1$, estimates (3.30) and (3.38) yield

$$\|V\big(M\big)^{-1} - I\|_{L^2\cap L^\infty(C_r)} \leq c_5 s^{-3}, \quad \|W\big(M\big)^{-1} - I\|_{L^2\cap L^\infty(C_l)} \leq c_6 s^{-3} \tag{3.43}$$

as $s \to \infty$. The estimations (3.41), (3.42) and (3.43), which are uniform on any compact subset of the set (1.19)

$$\{(\gamma, x) \in \mathbb{R}^2 : \ -\infty < \gamma < 1, \ -\infty < x < \infty\},$$

enable us to solve the ratio-RHP for $\gamma < 1$ iteratively. Indeed the stated ratio-RHP for the function $R(z)$

- $R(z)$ is analytic for $z \in \mathbb{C}\backslash\Sigma_R$
- Along the contour depicted in Figure 3.10

$$R_+(z) = R_-(z)G_R(z), \qquad z \in \Sigma_R.$$

- As $z \to \infty$, we have $R(z) = I + O\big(z^{-1}\big)$.

is equivalent to the singular integral equation

$$R_-(z) = I + \frac{1}{2\pi i}\int_{\Sigma_R} R_-(w)\big(G_R(w) - I\big)\frac{dw}{w - z_-} \tag{3.44}$$

and by the previous estimates (3.41), (3.42) and (3.43), we have

$$\|G_R - I\|_{L^2\cap L^\infty(\Sigma_R)} \leq c_6 s^{-1} \tag{3.45}$$

uniformly on any compact subset of the set (1.15). By standard arguments (see [25]), we know that for sufficiently large s, the relevant integral operator is contracting and equation (3.44) can be solved iteratively in $L^2(\Sigma_R)$. Moreover, its unique solution satisfies

$$\|R_- - I\|_{L^2(\Sigma_R)} \leq cs^{-1}, \quad s\to\infty. \tag{3.46}$$

The latter information is all we need to compute the asymptotic expansion for the Fredholm determinant $\det(I-\gamma K_{\mathrm{PII}})$ in case $\gamma<1$ up to the constant term. Before we derive the relevant asymptotics let us first discuss the situation $\gamma>1$. In this case

$$\|V\big(M\big)^{-1} - I\|_{L^2\cap L^\infty(C_r)} \nrightarrow 0, \quad \|W\big(M\big)^{-1} - I\|_{L^2\cap L^\infty(C_l)} \nrightarrow 0 \tag{3.47}$$

and we need to employ further transformations.

3.4 Undressing and dressing – iterative solution for $\gamma > 1$

The presence of the multipliers $E_r(z)$ and $E_l(z)$ in (3.32) and (3.39) requires further transformations leading to a singular or solitonic type of Riemann-Hilbert problem. Following [11, 28], we will show how to deal with the singular structure: A key observation for our first move is that the jump matrices $G_r(z) = V(z)\big(M(z)\big)^{-1}$ and $G_l(z) = W(z)\big(M(z)\big)^{-1}$ admit the following algebraic factorizations

$$\begin{aligned} G_r(z) &= E_r(z)\widehat{G}_r(z) = \begin{pmatrix} 1 & \frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(z)\frac{z+1}{z-1} \\ 0 & 1 \end{pmatrix} \\ &\times\left[I + \frac{i}{\zeta}\begin{pmatrix} \nu^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(z)\frac{z+1}{z-1}(1-2\nu) \\ 0 & -\nu^2 \end{pmatrix} + O\big(\zeta^{-2}\big)\right], \end{aligned} \tag{3.48}$$

$$\begin{aligned} G_l(z) &= E_l(z)\widehat{G}_l(z) = \begin{pmatrix} 1 & 0 \\ -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{2s^3\vartheta(-1)}\hat{\beta}_l^{-2}(z)\frac{z-1}{z+1} & 1 \end{pmatrix} \\ &\times\left[I + \frac{i}{\zeta}\begin{pmatrix} \nu^2 & 0 \\ \frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{2s^3\vartheta(-1)}\hat{\beta}_l^{-2}(z)\frac{z-1}{z+1}(2\nu-1) & -\nu^2 \end{pmatrix} + O\big(\zeta^{-2}\big)\right] \end{aligned} \tag{3.49}$$

as $s \to \infty$ and $0 < r_1 \le |z \mp 1| \le r_2 < \frac{1}{2}$. We observe that $\|\widehat{G}_{r,l} - I\| \to 0$ as $s \to \infty$; in fact, since $|\zeta(z)| \ge cs^3$ on $C_r \cup C_l$, we have that

$$\|\widehat{G}_{r,l} - I\|_{L^2 \cap L^\infty(C_r,l)} \le c_7 s^{-3}, \qquad s \to \infty.$$

Hence, the natural idea is to pass from the function $R(z)$ to the function $P(z)$ defined by the equations

$$P(z) = \begin{cases} R(z)E_r(z), & |z-1| < r_1, \\ R(z)E_l(z), & |z+1| < r_1, \\ R(z), & |z \mp 1| > r_1, \end{cases} \tag{3.50}$$

with $0 < r_1 < \frac{1}{2}$ chosen as in (3.40). By definition, the function $P(z)$ solves the following RHP:

- $P(z)$ is analytic for $z \in \mathbb{C} \backslash \big(\Sigma_R \cup \{\pm 1\}\big)$
- $P_+(z) = P_-(z) G_P(z)$, where

$$G_P(z) = \begin{cases} \widehat{G}_{r,l}(z), & z \in C_{r,l}, \\ U(z)\big(M(z)\big)^{-1}, & z \in C_0, \\ M(z)\widetilde{S}_k\big(M(z)\big)^{-1}, & z \in \gamma_k, k = 1, \dots, 8. \end{cases}$$

- $P(z)$ has first-order pole singularities at $z = \pm 1$. More precisely let $P(z) = \big(P^{(1)}(z), P^{(2)}(z)\big)$ with $P^{(i)}(z)$ denoting the columns of the corresponding 2×2 matrix valued function. We obtain from (3.48), (3.49) and (3.50)

$$\operatorname{res}_{z=+1} P^{(2)}(z) = P^{(1)}(1) \left(\frac{2\Gamma(1-\nu)}{\Gamma(\nu)} e^{-2s^3\vartheta(1)} \hat{\beta}_r^2(1) \right) \tag{3.51}$$

$$\operatorname{res}_{z=-1} P^{(1)}(z) = P^{(2)}(-1) \left(\frac{2\Gamma(1-\nu)}{\Gamma(\nu)} e^{2s^3\vartheta(-1)} \hat{\beta}_l^{-2}(-1) \right). \tag{3.52}$$

- As $z \to \infty$, we have $P(z) \to I$.

At this point it is important to notice that the latter four properties determine $P(z)$ uniquely.

Proposition 3.4.1 *The stated singular Riemann-Hilbert problem for $P(z)$ has a unique solution.*

Proof The residue relations (3.51) and (3.52) imply

$$P(z) = \begin{cases} \hat{P}^{(+)}(z)\begin{pmatrix} 1 & \frac{2p}{z-1} \\ 0 & 1 \end{pmatrix}, & |z-1| < r; \\ \hat{P}^{(-)}(z)\begin{pmatrix} 1 & 0 \\ \frac{2p}{z+1} & 1 \end{pmatrix}, & |z+1| < r, \end{cases} \qquad p = \frac{\Gamma(1-\nu)}{\Gamma(\nu)} e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(1) \quad (3.53)$$

with

$$\vartheta(1) = i\left(\frac{4}{3} + \frac{x}{s^2}\right) = -\vartheta(-1), \qquad \hat{\beta}_r^2(1) = \left(16s^3 + 4xs\right)^{2\nu_0} = \hat{\beta}_l^{-2}(-1)$$

and where $\hat{P}^{(\pm)}(z)$ are analytic at $z = \pm 1$. Hence one establishes $\det P(z) \equiv 1$ via Liouville theorem using the normalization at infinity und unimodularity of the jump matrices. From this and representation (3.53), the ratio of any two solutions $P_1(z)$ and $P_2(z)$ of the given P-RHP, i.e.

$$P_1(z)\big(P_2(z)\big)^{-1},$$

is an entire function approaching identity at infinity; hence $P_1 = P_2$, showing uniqueness. ■

Next, all jump matrices in the P-RHP approach the identity matrix as $s \to \infty$; however $P(z)$ has singularities at $z = \pm 1$ whose structure is described by the residue relations (3.51) and (3.52). This type of Riemann-Hilbert problem is a known one in the theory of integrable systems. The way to deal with such RHPs is to use a certain "dressing" procedure which reduces the problem to the one without the pole singularities. We put

$$P(z) = (zI + B)Q(z)\begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix}, \qquad (3.54)$$

where $B \in \mathbb{C}^{2\times 2}$ is constant and see immediately that $Q(z)$ solves the following RHP:

- $Q(z)$ is analytic for $z \in \mathbb{C}\backslash\Sigma_R$

- $Q_+(z) = Q_-(z)G_Q(z)$, where

$$G_Q(z) = \begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} \widehat{G}_{r,l}(z) \begin{pmatrix} z+1 & 0 \\ 0 & z-1 \end{pmatrix}, \qquad z \in C_{r,l}$$

and

$$G_Q(z) = \begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} U(z)\big(M(z)\big)^{-1} \begin{pmatrix} z+1 & 0 \\ 0 & z-1 \end{pmatrix}, \qquad z \in C_0$$

as well as

$$G_Q(z) = \begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} M(z)\widetilde{S}_k\big(M(z)\big)^{-1} \begin{pmatrix} z+1 & 0 \\ 0 & z-1 \end{pmatrix}, \qquad z \in \gamma_k.$$

- $Q(z) \to I$, as $z \to \infty$

The Q-jump matrix $G_Q(z)$ is uniformly close to the unit matrix: therefore the Q-RHP admits direct asymptotic analysis, which can be performed after we have determined the unknown matrix B. Using the conditions (3.51) and (3.52)

$$\begin{aligned} \mathrm{res}_{z=+1}P^{(2)}(z) &= (I+B)Q^{(2)}(1) = (I+B)pQ^{(1)}(1), \\ \mathrm{res}_{z=-1}P^{(1)}(z) &= (-I+B)Q^{(1)}(-1) = (-I+B)\big(-p\big)Q^{(2)}(-1), \end{aligned}$$

so

$$B = \left(Q(-1)\begin{pmatrix} 1 \\ p \end{pmatrix}, Q(1)\begin{pmatrix} -p \\ 1 \end{pmatrix}\right)\sigma_3\left(Q(-1)\begin{pmatrix} 1 \\ p \end{pmatrix}, Q(1)\begin{pmatrix} -p \\ 1 \end{pmatrix}\right)^{-1}. \tag{3.55}$$

Let us now see for which values of s the latter matrix inverse is well-defined: Since

$$\|G_Q - I\|_{L^2\cap L^\infty(\Sigma_R)} \le c_8 s^{-1}, \quad s \to \infty \tag{3.56}$$

we can solve the Q-RHP via iteration. Indeed, it is equivalent to the singular integral equation

$$Q_-(z) = I + \frac{1}{2\pi i}\int_{\Sigma_R} Q_-(w)\big(G_Q(w) - I\big)\frac{dw}{w - z_-} \tag{3.57}$$

which can be solved iteratively in $L^2(\Sigma_R)$, its unique solution satisfies

$$\|Q_- - I\|_{L^2(\Sigma_R)} \leq \tilde{c}s^{-1}, \quad s \to \infty. \tag{3.58}$$

Combining the integral representation

$$Q(z) = I + \frac{1}{2\pi i}\int_{\Sigma_R} Q_-(w)\big(G_Q(w) - I\big)\frac{dw}{w-z}, \qquad z \notin \Sigma_R \tag{3.59}$$

with (3.56) and (3.58), we conclude

$$Q(\pm 1) = I + O\big(s^{-1}\big), \qquad s \to \infty.$$

Hence the matrix inverse in the right hand side of (3.55) exists for all sufficiently large s lying outside of the zero set of the function

$$1 + p^2$$

which consists of the points $\{s_n\}$ defined by the equation

$$\frac{8}{3}s_n^3 + 2xs_n + \frac{1}{\pi}\ln(\gamma - 1)\ln\big(16s^3 + 4xs\big) - \arg\frac{\Gamma(1-\nu)}{\Gamma(\nu)} = \frac{\pi}{2} + n\pi, \qquad n = 1, 2, \ldots$$

and which will eventually form the zeros of the Fredholm determinant as written in Theorem 1.22. Henceforth, when dealing with the situation $\gamma > 1$, we shall always assume that s stays away from the small neighborhood of the points s_n.

At this point we have gathered enough information to derive the asymptotics of $\det(I - \gamma K_{\mathrm{PII}})$ for $\gamma \neq 1$ as stated in Theorem 1.18 and 1.22 up to the constant term. However we will postpone these derivations until chapter 4, right now we will focus on the asymptotic resolution of the X-RHP in case $\gamma = 1$.

3.5 Rescaling and g-function transformation, $\gamma = 1$

We go back to section 2.2, equation (2.12) and recall that on the line segment $[-s, s]$

$$X_+(\lambda) = X_-(\lambda)\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \lambda \in [-s, s],$$

i.e. we face a permutation jump matrix. This behavior (cf. [22]) motivates the introduction of the following g-function,

$$g(z) = \frac{4i}{3}\sqrt{z^2-1}\left(z^2+\frac{1}{2}+\frac{3x}{4s^2}\right), \quad \sqrt{z^2-1}\sim z,\ z\to\infty. \tag{3.60}$$

This function is analytic outside the segment $[-1,1]$ and as $z\to\infty$

$$g(z) = \vartheta(z) + O\big(z^{-1}\big), \quad \vartheta(z) = i\left(\frac{4}{3}z^3+\frac{xz}{s^2}\right).$$

Also,

$$g_+(z)+g_-(z) = 0, \qquad z\in[-1,1]. \tag{3.61}$$

We put

$$A(z) = X(zs)e^{s^3 g(z)\sigma_3}, \quad z\in\mathbb{C}\backslash\Big([-1,1]\cup\bigcup_k \Gamma_k\Big) \tag{3.62}$$

and, taking into account (3.61), are lead to the following RHP

- $A(z)$ is analytic for $z\in\mathbb{C}\backslash\big([-1,1]\cup\bigcup_k\Gamma_k\big)$
- The jump properties of $T(z)$ are given by the equations

$$A_+(z) = A_-(z)\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad z\in[-1,1]$$

$$A_+(z) = A_-(z)e^{-s^3 g(z)\sigma_3}S_k e^{s^3 g(z)\sigma_3}, \ z\in\Gamma_k.$$

- In a neighborhood of the endpoints $z=\pm 1$

$$A(z)e^{-s^3 g(z)\sigma_3} = \check{X}(zs)\begin{pmatrix} 1 & -\frac{1}{2\pi}\ln\frac{z-1}{z+1} \\ 0 & 1 \end{pmatrix}\begin{cases} I, & z\in\hat{\Omega}_1, \\ S_3, & z\in\hat{\Omega}_2, \\ S_3S_4, & z\in\hat{\Omega}_3, \\ S_3S_4S_6, & z\in\hat{\Omega}_4, \end{cases} \tag{3.63}$$

- As $z\to\infty$, we have $A(z) = I + O(z^{-1})$

Let us analyse the behavior of the jumps along the infinite branches Γ_k as $s\to\infty$. To this end consider the sign-diagram of the function Re $g(z)$, depicted in Figure 3.11,

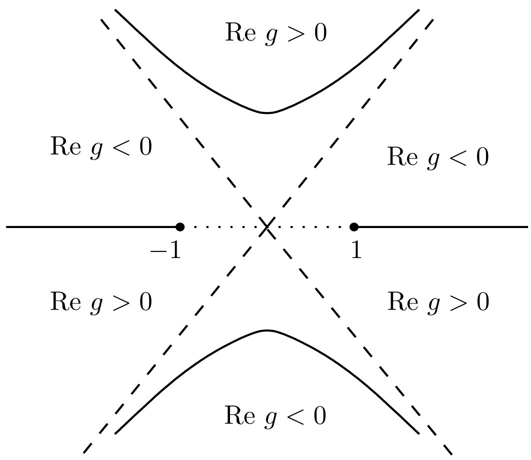

Figure 3.11. Sign-diagram for the function Re $g(z)$. Along the solid lines Re $g(z) = 0$, the dashed lines resemble arg $z = \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}$ and the dotted line indicates the branch cut of $g(z)$

where x is chosen from a compact subset of the real line and $s > 0$ is sufficiently large.

Since Re $g(z)$ is negative resp. positive along the rays Γ_1, Γ_3 resp. Γ_4, Γ_6,

$$e^{-s^3 g(z)\sigma_3} S_k e^{s^3 g(z)\sigma_3} \longrightarrow I, \quad s \to \infty \tag{3.64}$$

uniformly on any compact subset of the set (1.15) and the stated convergence is exponentially fast. Therefore, similar to our discussion in section 3.1 for $\gamma \neq 1$, we expect, and again this will be justified rigorously, that as $s \to \infty$, $A(z)$ converges to a solution of a model RHP, in which we only have to deal with the constant jump matrix on the line segment $[-1, 1]$.

3.6 The model RHP and parametrices for $\gamma = 1$

The problem is as follows: Find the piecewise analytic 2×2 matrix valued function $D(z)$ such that

- $D(z)$ is analytic for $z \in \mathbb{C}\backslash[-1, 1]$

- Along $[-1,1]$ the following jump condition holds

$$D_+(z) = D_-(z)\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad z \in [-1,1]$$

- $D(z) = I + O\big(z^{-1}\big), \quad z \to \infty$

A solution to this problem can be obtained explicitly via diagonalization (cf. [22])

$$D(z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \beta(z)^{\sigma_3}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \beta+\beta^{-1} & \beta-\beta^{-1} \\ \beta-\beta^{-1} & \beta+\beta^{-1} \end{pmatrix}, \tag{3.65}$$

with

$$\beta(z) = \left(\frac{z+1}{z-1}\right)^{1/4}$$

and $\left(\frac{z+1}{z-1}\right)^{1/4}$ is defined on $\mathbb{C}\backslash[-1,1]$ with its branch fixed by the condition $\left(\frac{z+1}{z-1}\right)^{1/4} \to 1$ as $z \to \infty$. Following the same strategy as presented in previous sections, we continue with the construction of parametrices.

For the right endpoint $z = +1$ use the local expansion

$$g(z) = \frac{4\sqrt{2}}{3}i\left(\frac{3}{2} + \frac{3x}{4s^2}\right)\sqrt{z-1}\Big(1 + O\big((z-1)^{1/2}\big)\Big), \ z \to 1, \ -\pi < \arg(z-1) \leq \pi$$

and the singular endpoint behavior (2.13)

$$A(z) = O\big(\ln(z-1)\big), \quad z \to 1.$$

Both observations suggest (cf. [22]) to use the Bessel functions $H_0^{(1)}(\zeta)$ and $H_0^{(2)}(\zeta)$ for our construction. The latter Hankel functions of first and second kind are unique independent solutions to Bessel's equation

$$zw'' + w' + w = 0$$

satisfying the following asymptotic conditions as $\zeta \to \infty$ and $-\pi < \arg\zeta < \pi$ (see [5])

$$\begin{aligned} H_0^{(1)}(\zeta) &= \sqrt{\frac{2}{\pi\zeta}}e^{i(\zeta-\frac{\pi}{4})}\left(1 - \frac{i}{8\zeta} - \frac{9}{128\zeta^2} + \frac{75i}{1024\zeta^3} + O\big(\zeta^{-4}\big)\right) \\ H_0^{(2)}(\zeta) &= \sqrt{\frac{2}{\pi\zeta}}e^{-i(\zeta-\frac{\pi}{4})}\left(1 + \frac{i}{8\zeta} - \frac{9}{128\zeta^2} - \frac{75i}{1024\zeta^3} + O\big(\zeta^{-4}\big)\right). \end{aligned}$$

Secondly $H_0^{(1)}(\zeta), H_0^{(2)}(\zeta)$ satisfy monodromy relations, valid on the entire universal covering of the punctured plane

$$H_0^{(1)}\big(\zeta e^{\pi i}\big) = -H_0^{(2)}(\zeta),\ H_0^{(2)}\big(\zeta e^{\pi i}\big) = H_0^{(1)}(\zeta) + 2H_0^{(2)}(\zeta),\ H_0^{(2)}\big(\zeta e^{-\pi i}\big) = -H_0^{(1)}(\zeta) \tag{3.66}$$

and finally the following expansions at the origin are valid (compare to (2.13))

$$H_0^{(1)}(\zeta) = a_0 + a_1 \ln\zeta + a_2\zeta^2 + a_3\zeta^2\ln\zeta + O\big(\zeta^4\ln\zeta\big),\ \zeta\to 0 \tag{3.67}$$

with coefficients a_i given as

$$a_0 = 1 + \frac{2i\gamma_E}{\pi} - \frac{2i}{\pi}\ln 2,\quad a_1 = \frac{2i}{\pi},\quad a_2 = \frac{i}{2\pi}(1-\gamma_E) - \frac{1}{4} + \frac{i}{2\pi}\ln 2,\quad a_3 = -\frac{i}{2\pi}$$

where γ_E is Euler's constant and the expansion for $H_0^{(2)}(\zeta)$ is up to the replacement $a_i \mapsto \bar{a}_i$ identical to (3.67). The latter properties in mind, define on the punctured plane $\zeta\in\mathbb{C}\backslash\{0\}$

$$P_{BE}(\zeta) = e^{i\frac{\pi}{4}\sigma_3}\begin{pmatrix} H_0^{(2)}(\sqrt{\zeta}) & H_0^{(1)}(\sqrt{\zeta}) \\ \sqrt{\zeta}\big(H_0^{(2)}\big)'(\sqrt{\zeta}) & \sqrt{\zeta}\big(H_0^{(1)}\big)'(\sqrt{\zeta}) \end{pmatrix} e^{-i\frac{\pi}{4}\sigma_3},\quad -\pi<\arg\zeta\leq\pi. \tag{3.68}$$

From the behavior of $H_0^{(1)}(\zeta)$ and $H_0^{(2)}(\zeta)$ at infinity we deduce

$$\begin{aligned} P_{BE}(\zeta) \;=\; & \sqrt{\frac{2}{\pi}}\zeta^{-\sigma_3/4}e^{i\frac{\pi}{4}}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix}\left[I + \frac{i}{8\sqrt{\zeta}}\begin{pmatrix} -1 & -2 \\ 2 & 1\end{pmatrix} + \frac{3}{128\zeta}\begin{pmatrix} 1 & -4 \\ -4 & 1\end{pmatrix}\right. \\ & \left. + \frac{15i}{1024\zeta^{3/2}}\begin{pmatrix} 1 & 6 \\ -6 & -1\end{pmatrix} + O\big(\zeta^{-2}\big)\right]e^{-i\sqrt{\zeta}\sigma_3}, \end{aligned}$$

as $\zeta\to\infty$ and $-\pi<\arg\zeta\leq\pi$. We now assemble the following model function

$$P_{BE}^{RH}(\zeta) = \begin{cases} P_{BE}(\zeta)\begin{pmatrix} 1 & 0 \\ -i & 1\end{pmatrix}, & \arg\zeta\in(\frac{\pi}{6},\pi), \\ P_{BE}(\zeta)\begin{pmatrix} 1 & i \\ 0 & 1\end{pmatrix}, & \arg\zeta\in(-\pi,-\frac{\pi}{6}), \\ P_{BE}(\zeta), & \arg\zeta\in(-\frac{\pi}{6},\frac{\pi}{6}). \end{cases} \tag{3.69}$$

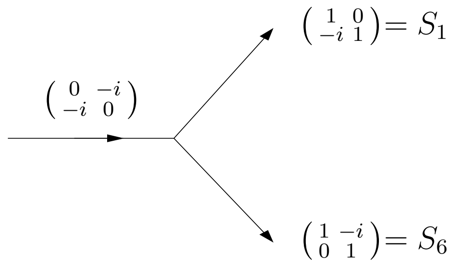

Figure 3.12. The model RHP near $z = +1$ which can be solved explicitly using Hankel functions

which solves the RHP depicted in Figure 3.12.

More precisely, the function $P_{BE}^{RH}(\zeta)$ possesses the following analytic properties.

- $P_{BE}^{RH}(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash\{\arg\zeta = -\pi, -\frac{\pi}{6}, \frac{\pi}{6}\}$
- The following jumps hold

$$\begin{aligned}\left(P_{BE}^{RH}(\zeta)\right)_+ &= \left(P_{BE}^{RH}(\zeta)\right)_- \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \qquad \arg\zeta = \frac{\pi}{6} \\ \left(P_{BE}^{RH}(\zeta)\right)_+ &= \left(P_{BE}^{RH}(\zeta)\right)_- \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \qquad \arg\zeta = -\frac{\pi}{6}\end{aligned}$$

And for the jump on the line $\arg\zeta = \pi$ we notice that the monodromy relations imply

$$\begin{aligned}H_0^{(2)}\left(\sqrt{\zeta}_+\right) &= H_0^{(2)}\left(\sqrt{\zeta}_- e^{\pi i}\right) = H_0^{(1)}\left(\sqrt{\zeta}_-\right) + 2H_0^{(2)}\left(\sqrt{\zeta}_-\right) \\ \left(H_0^{(2)}\right)'\left(\sqrt{\zeta}_+\right) &= e^{-i\pi}\left(H_0^{(1)}\right)'\left(\sqrt{\zeta}_-\right) + 2e^{-i\pi}\left(H_0^{(2)}\right)'\left(\sqrt{\zeta}_-\right)\end{aligned}$$

and

$$\begin{aligned}H_0^{(1)}\left(\sqrt{\zeta}_+\right) &= H_0^{(1)}\left(\sqrt{\zeta}_- e^{\pi i}\right) = -H_0^{(2)}\left(\sqrt{\zeta}_-\right) \\ \left(H_0^{(1)}\right)'\left(\sqrt{\zeta}_+\right) &= \left(H_0^{(2)}\right)'\left(\sqrt{\zeta}_-\right).\end{aligned}$$

Therefore

$$\left(P_{BE}(\zeta)\right)_+ = \left(P_{BE}(\zeta)\right)_- e^{i\frac{\pi}{4}\sigma_3} \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} e^{-i\frac{\pi}{4}\sigma_3} = \left(P_{BE}(\zeta)\right)_- \begin{pmatrix} 2 & -i \\ -i & 0 \end{pmatrix}$$

and hence

$$\big(P_{BE}^{RH}(\zeta)\big)_+ = \big(P_{BE}^{RH}(\zeta)\big)_- \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \arg\zeta = \pi,$$

- In order to determine the behavior of $P_{BE}^{RH}(\zeta)$ at infinity we make the following observations. First let $\arg\zeta \in (\frac{\pi}{6}, \pi)$ and consider

$$e^{-i\sqrt{\zeta}\sigma_3} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} e^{i\sqrt{\zeta}\sigma_3} = \begin{pmatrix} 1 & 0 \\ -ie^{2i\sqrt{\zeta}} & 1 \end{pmatrix}.$$

  Observe that $\mathrm{Re}\big(i\sqrt{\zeta}\big) < 0$, hence the given product approaches the identity exponentially fast as $\zeta \to \infty$. Secondly for $\arg\zeta \in (-\pi, -\frac{\pi}{6})$

$$e^{-i\sqrt{\zeta}\sigma_3} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} e^{i\sqrt{\zeta}\sigma_3} = \begin{pmatrix} 1 & ie^{-2i\sqrt{\zeta}} \\ 0 & 1 \end{pmatrix}$$

  and in this situation $\mathrm{Re}\big(-i\sqrt{\zeta}\big) < 0$, so again the product approaches the identity exponentially fast as $\zeta \to \infty$. Both cases together with the previously stated asymptotics for $P_{BE}(\zeta)$ imply therefore

$$\begin{aligned} P_{BE}^{RH}(\zeta) &= \sqrt{\frac{2}{\pi}}\zeta^{-\sigma_3/4} e^{i\frac{\pi}{4}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \left[I + \frac{i}{8\sqrt{\zeta}} \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix} \right. \\ &\quad \left. + \frac{3}{128\zeta} \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} + \frac{15i}{1024\zeta^{3/2}} \begin{pmatrix} 1 & 6 \\ -6 & -1 \end{pmatrix} + O\big(\zeta^{-2}\big) \right] e^{-i\sqrt{\zeta}\sigma_3}, \end{aligned} \tag{3.70}$$

  as $\zeta \to \infty$ in a whole neighborhood of infinity.

The model function $P_{BE}^{RH}(\zeta)$ will now be used to construct the parametrix to the solution of the original A-RHP in a neighborhood of $z = +1$. We proceed in two steps. First define

$$\zeta(z) = -s^6 g^2(z), \quad |z-1| < r, \ -\pi < \arg\zeta \le \pi \tag{3.71}$$

or respectively

$$\sqrt{\zeta(z)} = -is^3 g(z) = \frac{4s^3}{3}\sqrt{z^2-1}\left(z^2 + \frac{1}{2} + \frac{3x}{4s^2}\right).$$

This change of variables is indeed locally conformal, since

$$\zeta(z) = \frac{32s^6}{9}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^2 (z-1)\big(1 + O(z-1)\big), \quad |z-1| < r$$

and it enables us to define the right parametrix $I(z)$ near $z = +1$ by the formula:

$$I(z) = C_r(z)\frac{\sigma_3}{2}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{4}}P_{BE}^{RH}\big(\zeta(z)\big)e^{s^3 g(z)\sigma_3}, \quad |z-1| < r \tag{3.72}$$

with $\zeta(z)$ as in (3.71) and the matrix multiplier

$$C_r(z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\zeta(z)\frac{z+1}{z-1}\right)^{\sigma_3/4}, \quad C_r(1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{8s^3}{3}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)\right)^{\sigma_3/2}. \tag{3.73}$$

By construction, in particular since $C_r(z)$ is analytic in a neighborhood of $z = +1$, the parametrix $I(z)$ has jumps along the curves depicted in Figure 3.13, and we can always locally match the latter curves with the jump curves of the original RHP. Also these jumps are described by the same Stokes matrices as in the original A-RHP. Furthermore, and we will elaborate this in full detail very soon, the singular behavior of $I(z)$ at the endpoint $z = +1$ matches the singular behavior of $A(z)$:

$$I(z) = O\big(\ln(z-1)\big), \quad |z-1| < r. \tag{3.74}$$

Hence the ratio of $A(z)$ with $I(z)$ is locally analytic, i.e.

$$A(z) = N_r(z)I(z), \quad |z-1| < r < \frac{1}{2}. \tag{3.75}$$

Let us once more explain the role of the left multiplier $C_r(z)$ in the definition (3.72). Observe that

$$C_r(z)\zeta(z)^{-\sigma_3/4}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = D(z).$$

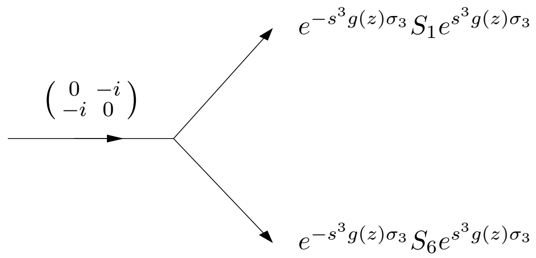

Figure 3.13. Transformation of parametrix jumps to original jumps

This relation together with the asymptotic equation (3.70) implies that,

$$\begin{aligned} I(z) &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \beta(z)^{\sigma_3} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left[I + \frac{i}{8\sqrt{\zeta}} \begin{pmatrix} -1 & -2 \\ 2 & 1 \end{pmatrix} + \frac{3}{128\zeta} \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} \right. \\ &\left. + \frac{15i}{1024\zeta^{3/2}} \begin{pmatrix} 1 & 6 \\ -6 & -1 \end{pmatrix} + O\big(\zeta^{-2}\big) \right] \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \beta(z)^{-\sigma_3} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} D(z) \\ &= \left[I + \frac{i}{16\sqrt{\zeta}} \begin{pmatrix} \beta^2 - 3\beta^{-2} & -(\beta^2 + 3\beta^{-2}) \\ \beta^2 + 3\beta^{-2} & -(\beta^2 - 3\beta^{-2}) \end{pmatrix} + \frac{3}{128\zeta} \begin{pmatrix} 1 & -4 \\ -4 & 1 \end{pmatrix} \right. \\ &\left. + \frac{15i}{2048\zeta^{3/2}} \begin{pmatrix} -(5\beta^2 - 7\beta^{-2}) & 5\beta^2 + 7\beta^{-2} \\ -(5\beta^2 + 7\beta^{-2}) & 5\beta^2 - 7\beta^{-2} \end{pmatrix} + O\big(\zeta^{-2}\big) \right] D(z) \end{aligned} \tag{3.76}$$

as $s \to \infty$ and $0 < r_1 \le |z-1| \le r_2 < 1$ (so $|\zeta| \to \infty$). Since the function $\zeta(z)$ is of order $O\big(s^6\big)$ on the latter annulus and $\beta(z)$ is bounded, equation (3.76) yields the desired matching relation between the model functions $I(z)$ and $D(z)$,

$$I(z) = \big(I + o(1)\big) D(z), \quad s \to \infty, \quad 0 < r_1 \le |z-1| \le r_2 < 1,$$

which in turn explains the choice of the left multiplier $C_r(z)$ in (3.72) in the form (3.73). We continue with the model problem near the other endpoint $z = -1$.

Consider on the punctured plane $\zeta \in \mathbb{C} \backslash \{0\}$

$$\widetilde{P}_{BE}(\zeta) = \begin{pmatrix} e^{-i\frac{3\pi}{2}} \sqrt{\zeta} \Big(H_0^{(1)}\Big)' \big(e^{-i\frac{\pi}{2}}\sqrt{\zeta}\big) & -\sqrt{\zeta} \Big(H_0^{(2)}\Big)' \big(e^{-i\frac{\pi}{2}}\sqrt{\zeta}\big) \\ -e^{i\frac{\pi}{2}} H_0^{(1)} \big(e^{-i\frac{\pi}{2}}\sqrt{\zeta}\big) & H_0^{(2)} \big(e^{-i\frac{\pi}{2}}\sqrt{\zeta}\big) \end{pmatrix}, \quad 0 < \arg\zeta \le 2\pi$$

which satisfies

$$\widetilde{P}_{BE}(\zeta) = \sqrt{\frac{2}{\pi}}\zeta^{\sigma_3/4}\begin{pmatrix}-1 & -1\\ -i & i\end{pmatrix}\left[I + \frac{1}{8\sqrt{\zeta}}\begin{pmatrix}-1 & 2\\ -2 & 1\end{pmatrix} + \frac{3}{128\zeta}\begin{pmatrix}-1 & -4\\ -4 & -1\end{pmatrix}\right.$$
$$\left. + \frac{15}{1024\zeta^{3/2}}\begin{pmatrix}-1 & 6\\ -6 & 1\end{pmatrix} + O(\zeta^{-2})\right]e^{\sqrt{\zeta}\sigma_3}$$

as $\zeta \to \infty$ and $0 < \arg\zeta \le 2\pi$. Next, instead of (3.69), define

$$\widetilde{P}^{RH}_{BE}(\zeta) = \begin{cases} \widetilde{P}_{BE}(\zeta)\begin{pmatrix}1 & 0\\ -i & 1\end{pmatrix}, & \arg\zeta \in (0, \frac{5\pi}{6}),\\ \widetilde{P}_{BE}(\zeta)\begin{pmatrix}1 & i\\ 0 & 1\end{pmatrix}, & \arg\zeta \in (\frac{7\pi}{6}, 2\pi),\\ \widetilde{P}_{BE}(\zeta), & \arg\zeta \in (\frac{5\pi}{6}, \frac{7\pi}{6}). \end{cases} \tag{3.77}$$

which solves the model RHP of Figure 3.14. More precisely, the function $\widetilde{P}^{RH}_{BE}(\zeta)$ has the following analytic properties

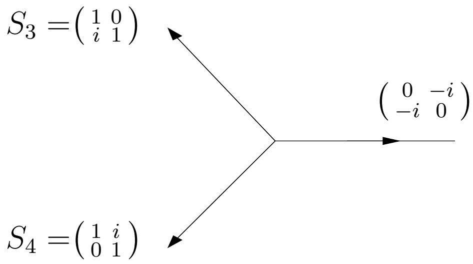

Figure 3.14. The model RHP near $z = -1$ which can be solved explicitly using Hankel functions

- $\widetilde{P}^{RH}_{BE}(\zeta)$ is analytic for $\zeta \in \mathbb{C}\backslash\{\arg\zeta = \frac{5\pi}{6}, \frac{7\pi}{6}, 2\pi\}$

- We have the following jumps on the contour depicted in Figure 3.14

$$\begin{aligned}\left(\widetilde{P}^{RH}_{BE}(\zeta)\right)_+ &= \left(\widetilde{P}^{RH}_{BE}(\zeta)\right)_- \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \quad \arg\zeta = \frac{5\pi}{6} \\ \left(\widetilde{P}^{RH}_{BE}(\zeta)\right)_+ &= \left(\widetilde{P}^{RH}_{BE}(\zeta)\right)_- \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \quad \arg\zeta = \frac{7\pi}{6}\end{aligned}$$

and on the line segment $\arg\zeta = 2\pi$

$$\begin{aligned}H_0^{(1)}\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_+}\right) &= H_0^{(1)}\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}e^{-i\pi}\right) \\ &= H_0^{(2)}\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}\right) + 2H_0^{(1)}\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}\right) \\ \left(H_0^{(1)}\right)'\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_+}\right) &= e^{i\pi}\left(H_0^{(2)}\right)'\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}\right) + 2e^{i\pi}\left(H_0^{(1)}\right)'\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}\right)\end{aligned}$$

as well as

$$\begin{aligned}H_0^{(2)}\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_+}\right) &= H_0^{(2)}\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}e^{-i\pi}\right) = -H_0^{(1)}\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}\right) \\ \left(H_0^{(2)}\right)'\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_+}\right) &= \left(H_0^{(1)}\right)'\left(e^{-i\frac{\pi}{2}}\sqrt{\zeta_-}\right)\end{aligned}$$

hence

$$\left(\widetilde{P}^{RH}_{BE}(\zeta)\right)_+ = \left(\widetilde{P}^{RH}_{BE}(\zeta)\right)_- \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \arg\zeta = 2\pi.$$

- A similar argument as given in the construction of $P^{RH}_{BE}(\zeta)$ implies

$$\begin{aligned}\widetilde{P}^{RH}_{BE}(\zeta) &= \sqrt{\frac{2}{\pi}}\zeta^{\sigma_3/4}\begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix}\left[I + \frac{1}{8\sqrt{\zeta}}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix} + \frac{3}{128\zeta}\begin{pmatrix} -1 & -4 \\ -4 & -1 \end{pmatrix}\right. \\ &\quad \left. + \frac{15}{1024\zeta^{3/2}}\begin{pmatrix} -1 & 6 \\ -6 & 1 \end{pmatrix} + O\left(\zeta^{-2}\right)\right]e^{\sqrt{\zeta}\sigma_3} \qquad (3.78)\end{aligned}$$

as $\zeta \to \infty$, valid in a full neighborhood of infinity.

Again we use the model function $\widetilde{P}^{RH}_{BE}(\zeta)$ in the construction of the parametrix to the solution of the original A-RHP near $z = -1$. Instead of (3.71)

$$\zeta(z) = s^6 g^2(z), \quad |z+1| < r, \ 0 < \arg\zeta \leq 2\pi \qquad (3.79)$$

or equivalently

$$\sqrt{\zeta(z)} = -s^3 g(z) = -\frac{4is^3}{3}\sqrt{z^2-1}\left(z^2 + \frac{1}{2} + \frac{3x}{4s^2}\right).$$

This change of the independent variable is locally conformal

$$\zeta(z) = \frac{32s^6}{9}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^2 (z+1)\big(1 + O(z+1)\big), \quad |z+1| < r$$

and allows us to define the left parametrix $J(z)$ near $z = -1$ by the formula:

$$J(z) = C_l(z)\frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix}\sqrt{\frac{\pi}{2}}\widetilde{P}_{BE}^{RH}\big(\zeta(z)\big)e^{s^3 g(z)\sigma_3}, \quad |z+1| < r \tag{3.80}$$

with the matrix multiplier

$$C_l(z) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\zeta(z)\frac{z-1}{z+1}\right)^{-\sigma_3/4}, \quad C_l(-1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\left(\frac{8is^3}{3}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)\right)^{-\sigma_3/2}. \tag{3.81}$$

Similar to the previous situation, $J(z)$ has jumps on the contour depicted in Figure 3.15 which are described by the same Stokes matrices as in the original A-RHP.

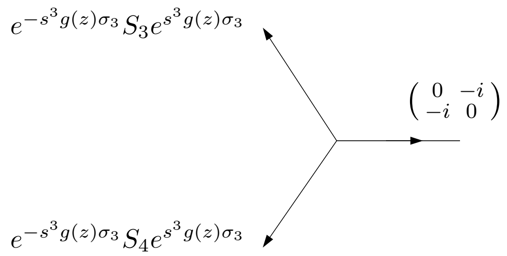

Figure 3.15. Transformation of parametrix jumps to original jumps

Also here, as we shall see in detail in the section 4.4, the singular behavior at $z = -1$ matches:

$$J(z) = O\big(\ln(z+1)\big), \quad |z+1| < r \tag{3.82}$$

Hence the ratio of parametrix $J(z)$ with $A(z)$ is locally analytic

$$A(z) = N_l(z)J(z), \quad |z+1| < r < \frac{1}{2}$$

and the left multiplier (3.81) in (3.80) provides us with the following asymptotic matchup between $J(z)$ and $D(z)$:

$$\begin{aligned} J(z) &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \beta(z)^{\sigma_3}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\Bigg[I+\frac{1}{8\sqrt{\zeta}}\begin{pmatrix} -1 & 2 \\ -2 & 1 \end{pmatrix}+\frac{3}{128\zeta}\begin{pmatrix} -1 & -4 \\ -4 & -1 \end{pmatrix} \\ &\quad +\frac{15}{1024\zeta^{3/2}}\begin{pmatrix} -1 & 6 \\ -6 & 1 \end{pmatrix}+O\big(\zeta^{-2}\big)\Bigg]\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\beta(z)^{-\sigma_3}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} D(z) \\ &= \Bigg[I+\frac{1}{16\sqrt{\zeta}}\begin{pmatrix} \beta^{-2}-3\beta^2 & \beta^{-2}+3\beta^2 \\ -(\beta^{-2}+3\beta^2) & -(\beta^{-2}-3\beta^2) \end{pmatrix}+\frac{3}{128\zeta}\begin{pmatrix} -1 & -4 \\ -4 & -1 \end{pmatrix} \\ &\quad +\frac{15}{2048\zeta^{3/2}}\begin{pmatrix} 5\beta^{-2}-7\beta^2 & 5\beta^{-2}+7\beta^2 \\ -(5\beta^{-2}+7\beta^2) & -(5\beta^{-2}-7\beta^2) \end{pmatrix}+O\big(\zeta^{-2}\big)\Bigg]D(z) \end{aligned} \tag{3.83}$$

as $s\to\infty$ and $0<r_1\leq|z+1|\leq r_2<1$, thus

$$J(z)=\big(I+o(1)\big)D(z),\qquad s\to\infty,\quad 0<r_1\leq|z+1|\leq r_2<1.$$

At this point we can use the model functions $D(z), F(z)$ and $H(z)$ to employ the final transformation.

3.7 The ratio problem – iterative solution for $\gamma=1$

In this final transformation we put

$$K(z)=A(z)\begin{cases} \big(I(z)\big)^{-1}, & |z-1|<r, \\ \big(J(z)\big)^{-1}, & |z+1|<r, \\ \big(D(z)\big)^{-1}, & |z\mp 1|>r \end{cases} \tag{3.84}$$

where $0<r<\frac{1}{4}$ remains fixed. With C_r and C_l denoting the clockwise oriented circles shown in Figure 3.16, the ratio-function $K(z)$ solves the following RHP

- $K(z)$ is anlytic for $z\in\mathbb{C}\backslash\big\{C_r\cup C_l\cup\bigcup_k\gamma_k\big\}$
- Along the infinite branches γ_k

$$K_+(z)=K_-(z)D(z)e^{-s^3g(z)\sigma_3}S_ke^{s^3g(z)\sigma_3}\big(D(z)\big)^{-1},\quad z\in\gamma_k,$$

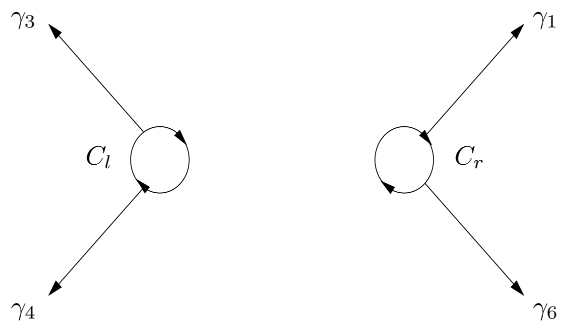

Figure 3.16. The jump graph for the ratio-function $K(z)$

and on the clockwise oriented circles $C_{r,l}$

$$\begin{aligned} K_+(z) &= K_-(z)I(z)\big(D(z)\big)^{-1}, \ z \in C_r \\ K_+(z) &= K_-(z)J(z)\big(D(z)\big)^{-1}, \ z \in C_l \end{aligned}$$

- $K(z)$ is analytic at $z = \pm 1$. This observation follows from (3.74) and (3.82), which will be proved in section 4.4.
- In a neigborhood of infinity, we have $K(z) \to I$

Due to the triangularity of all Stokes matrices S_k, the jump matrices corresponding to the infinite parts $\bigcup_k \gamma_k$ of the K-jump contour are exponentially close to the unit matrix

$$\|Me^{-s^3 g(\cdot)\sigma_3} S_k e^{s^3 g(\cdot)\sigma_3} \big(M\big)^{-1} - I\|_{L^2 \cap L^\infty(\gamma_k)} \leq c_1 e^{-c_2 s^3 |z \mp 1|} \tag{3.85}$$

emanating from $C_{r,l}$ as $s \to \infty$ with constants $c_i > 0$ whose values are not important. Moreover, by virtue of (3.76), $I(z)\big(D(z)\big)^{-1}$ approaches the unit matrix as $s \to \infty$

$$\|I\big(D\big)^{-1} - I\|_{L^2 \cap L^\infty(C_r)} \leq c_3 s^{-3} \tag{3.86}$$

and from (3.83), also $J(z)\big(D(z)\big)^{-1}$

$$\|J\big(D\big)^{-1} - I\|_{L^2 \cap L^\infty(C_l)} \leq c_4 s^{-3}, \ s \to \infty. \tag{3.87}$$

All together, with G_K denoting the jump matrix in the latter ratio-RHP and Σ_K the underlying contour

$$\|G_K - I\|_{L^2\cap L^\infty(\Sigma_K)} \leq cs^{-3}, \quad s\to\infty \tag{3.88}$$

uniformly on any compact subset of the set (1.15). The latter estimation enables us to solve the ratio-RHP iteratively, its unique solution satisfies

$$\|K_- - I\|_{L^2(\Sigma_K)} \leq cs^{-3}, \quad s\to\infty. \tag{3.89}$$

4. ASYMPTOTICS OF $\ln\det(I-\gamma K_{\mathrm{PII}})$ UP TO CONSTANT TERMS

Using the Deift-Zhou nonlinear steepest descent method, we were able to solve the master RHP asymptotically for all values of γ. Using the latter information, we will now derive the asymptotic expansions given in Theorem 1.2.1 and 1.2.2 up to the contant terms and in addition prove the large zero distribution of $\det(I-\gamma K_{\mathrm{PII}})$ as stated in Theorem 1.2.3. Our proofs rely on the logarithmic derivative identities obtained in Proposition 2.3.1 and 2.3.2.

4.1 The situation $\gamma\neq 1$ – preliminary steps

Let us recall the common part of the series of transformations, which has been used in the asymptotical solution of the original Y-RHP in case $\gamma\neq 1$

$$Y(\lambda)\mapsto \widetilde{X}(\lambda)\mapsto X(\lambda)\mapsto T(z)\mapsto S(z)\mapsto R(z).$$

In order to determine $\ln\det(I-\gamma K_{\mathrm{PII}})$ via Proposition 2.3.1, we need to connect $\check{X}(\pm s)$ and $\check{X}'(\pm s)$ to the values of $R(\pm 1)$ and $R'(\pm 1)$ of the ratio-function. This can be done as follows: From (3.40) and (3.4) for $|z-1|<r$

$$R(z)V(z)L^{-1}(z)e^{-s^3\vartheta(z)\sigma_3}=\check{X}(zs)\begin{pmatrix}1 & -\frac{\gamma}{2\pi}\ln\frac{z-1}{z+1}\\ 0 & 1\end{pmatrix}\widehat{S}(z), \tag{4.1}$$

and for $|z+1|<r$

$$R(z)W(z)L^{-1}(z)e^{-s^3\vartheta(z)\sigma_3}=\check{X}(zs)\begin{pmatrix}1 & -\frac{\gamma}{2\pi}\ln\frac{z-1}{z+1}\\ 0 & 1\end{pmatrix}\widehat{S}(z). \tag{4.2}$$

This shows that the required values of $\check{X}(\pm s)$ and $\check{X}'(\pm s)$ can be determined via comparison in (4.1) and (4.2), once we know the local expansions of $V(z)$, respectively $W(z)$ at $z=\pm 1$. Our starting point is (3.21)

$$\begin{aligned}
P_{CH}(\zeta) \;=\; & \Bigg[\begin{pmatrix} d_1(\zeta,\nu)e^{2\pi i\nu} & -d_2(\zeta,1-\nu)e^{\pi i\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ -d_1(\zeta,1+\nu)e^{\pi i\nu}\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & d_2(\zeta,-\nu)\end{pmatrix} \\
& +\zeta\begin{pmatrix} d_3(\zeta,\nu)e^{2\pi i\nu} & -d_4(\zeta,1-\nu)e^{\pi i\nu}\frac{\Gamma(1-\nu)}{\Gamma(\nu)} \\ -d_3(\zeta,1+\nu)e^{\pi i\nu}\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} & d_4(\zeta,-\nu)\end{pmatrix} + O\big(\zeta^2\ln\zeta\big)\Bigg] \\
& \times e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}, \quad \zeta\to 0 \qquad (4.3)
\end{aligned}$$

with

$$d_1(\zeta,\nu)=c_0(\nu)+c_1(\nu)\Big(\ln\zeta+i\frac{\pi}{2}\Big),\qquad d_2(\zeta,\nu)=c_0(\nu)+c_1(\nu)\Big(\ln\zeta-i\frac{\pi}{2}\Big)$$

and

$$d_3(\zeta,\nu)=-\frac{i}{2}d_1(\zeta,\nu)+i\bigg(c_2(\nu)+c_3(\nu)\Big(\ln\zeta+i\frac{\pi}{2}\Big)\bigg)$$

as well as

$$d_4(\zeta,\nu)=\frac{i}{2}d_2(\zeta,\nu)-i\bigg(c_2(\nu)+c_3(\nu)\Big(\ln\zeta-i\frac{\pi}{2}\Big)\bigg).$$

Now trace back the changes of variables

$$\zeta=\zeta(z)=-2is^3\big(\vartheta(z)-\vartheta(1)\big),\quad |z-1|<r_1,\qquad \lambda=zs$$

and deduce from (4.3) and (3.24)

$$\begin{aligned}
P_{CH}^{RH}\Big(\zeta\Big(\frac{\lambda}{s}\Big)\Big) \;=\; & \Big[P_1\big(\ln(\lambda-s)\big)+(\lambda-s)P_2\big(\ln(\lambda-s)\big) \\
& +O\big((\lambda-s)^2\ln(\lambda-s)\big)\Big]\begin{pmatrix} e^{-i\frac{3\pi}{2}\nu} & 0 \\ 0 & e^{i\frac{\pi}{2}\nu}\end{pmatrix},\qquad \lambda\to s,
\end{aligned}$$

valid in the sector $-\frac{\pi}{3}<\arg(\lambda-s)<\frac{\pi}{3}$. Here the matrix functions $P_1(\lambda)=\big(P_1^{ij}(\lambda)\big)$ and $P_2(\lambda)=\big(P_2^{ij}(\lambda)\big)$ can be determined from (4.3) and for the remaining sectors $-\pi<\arg(\lambda-s)<-\frac{\pi}{3}$ and $\frac{\pi}{3}<\arg(\lambda-s)<\pi$ we can derive similar expansions, they differ from the latter only by multiplication with a triangular matrix, see (3.24). Combining now (3.26) with the latter expansion, the left hand side of (4.1) satisfies

$$
\begin{aligned}
& R(z)V(z)L^{-1}(z)e^{-s^3\vartheta(z)\sigma_3}\Big|_{z=\frac{\lambda}{s}} = R(z)B_r(z)e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(1)\sigma_3}P_{CH}^{RH}\big(\zeta(z)\big)\Big|_{z=\frac{\lambda}{s}} \\
= \;& R(1)B_r(1)e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(1)\sigma_3}P_1\big(\ln(\lambda-s)\big)\begin{pmatrix} e^{-i\frac{3\pi}{2}\nu} & 0 \\ 0 & e^{i\frac{\pi}{2}\nu}\end{pmatrix} \\
& +(\lambda-s)\Bigg[R(1)B_r(1)e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(1)\sigma_3}P_2\big(\ln(\lambda-s)\big)+\frac{1}{s}\big(R'(1)B_r(1) \\
& +R(1)B_r'(1)\big)e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(1)\sigma_3}P_1\big(\ln(\lambda-s)\big)\Bigg]\begin{pmatrix} e^{-i\frac{3\pi}{2}\nu} & 0 \\ 0 & e^{i\frac{\pi}{2}\nu}\end{pmatrix} \\
& +O\big((\lambda-s)^2\ln(\lambda-s)\big), \quad \lambda\to s, \ -\frac{\pi}{3}<\arg(\lambda-s)<\frac{\pi}{3}.
\end{aligned}
$$

On the other hand, the right hand side in (4.1) can be expanded in the latter sector as well:

$$
\begin{aligned}
& R(z)V(z)L^{-1}(z)e^{-s^3\vartheta(z)\sigma_3}\Big|_{z=\frac{\lambda}{s}} = \Big(\check{X}(s)+(\lambda-s)\check{X}'(s)+O\big((\lambda-s)^2\big)\Big) \\
& \times\begin{pmatrix} 1 & -\frac{\gamma}{2\pi}\ln\frac{\lambda-s}{\lambda+s} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ i & 1\end{pmatrix} = \Bigg[\begin{pmatrix}\check{X}_{11}(s) & \check{X}_{12}(s) \\ \check{X}_{21}(s) & \check{X}_{22}(s)\end{pmatrix} \\
& +(\lambda-s)\begin{pmatrix}\check{X}_{11}'(s) & \check{X}_{12}'(s) \\ \check{X}_{21}'(s) & \check{X}_{22}'(s)\end{pmatrix}+O\big((\lambda-s)^2\big)\Bigg]\begin{pmatrix} 1-\frac{i\gamma}{2\pi}\ln\frac{\lambda-s}{\lambda+s} & -\frac{\gamma}{2\pi}\ln\frac{\lambda-s}{\lambda+s} \\ i & 1\end{pmatrix}
\end{aligned}
$$

which implies after comparison

$$
\begin{aligned}
\check{X}_{11}(s) &= -\frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\left(\Big(R(1)B_r(1)\Big)_{11}\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}+i\Big(R(1)B_r(1)\Big)_{12}\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)}\right) \\
\check{X}_{21}(s) &= -\frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\left(\Big(R(1)B_r(1)\Big)_{21}\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}+i\Big(R(1)B_r(1)\Big)_{22}\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)}\right).
\end{aligned}
$$

Although we currently derived the last two identities from a comparison in the sector $-\frac{\pi}{3}<\arg(\lambda-s)<\frac{\pi}{3}$, the same identities follow from a comparison in the other two sectors as well. There one uses the correct triangular matrices in (3.24) on the left hand side as well as a careful trace back of the contour deformations, and on the right hand side the corresponding matrices from (2.13). Also by comparison

$$\check{X}_{11}'(s) = \frac{2\pi i}{\gamma s} e^{i\frac{\pi}{2}\nu}\Bigg(\Big(R(1)B_r(1)\Big)_{11}\big(-2is^3\vartheta'(1)\big)\left(\frac{i}{2}-i\nu\right)\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}$$
$$+\Big(R(1)B_r(1)\Big)_{12}\big(-2is^3\vartheta'(1)\big)\left(\frac{1}{2}+\nu\right)\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)}$$
$$-\Big(R(1)B_r'(1)+R'(1)B_r(1)\Big)_{11}\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)} - \Big(R(1)B_r'(1)+R'(1)B_r(1)\Big)_{12}\frac{ie^{s^3\vartheta(1)}}{\Gamma(-\nu)}\Bigg)$$

and

$$\check{X}_{21}'(s) = \frac{2\pi i}{\gamma s} e^{i\frac{\pi}{2}\nu}\Bigg(\Big(R(1)B_r(1)\Big)_{21}\big(-2is^3\vartheta'(1)\big)\left(\frac{i}{2}-i\nu\right)\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}$$
$$+\Big(R(1)B_r(1)\Big)_{22}\big(-2is^3\vartheta'(1)\big)\left(\frac{1}{2}+\nu\right)\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)}$$
$$-\Big(R(1)B_r'(1)+R'(1)B_r(1)\Big)_{21}\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)} - \Big(R(1)B_r'(1)+R'(1)B_r(1)\Big)_{22}\frac{ie^{s^3\vartheta(1)}}{\Gamma(-\nu)}\Bigg).$$

In order to obtain the corresponding identities for $\check{X}(-s)$ and $\check{X}'(-s)$ we would use the same strategy as sketched above with the only difference that we have to work now with (4.2) rather than (4.1). We choose to skip the details and simply state the results: First

$$\begin{aligned}
\check{X}_{11}(-s) &= \frac{2\pi i}{\gamma} e^{i\frac{\pi}{2}\nu}\left(\Big(R(-1)B_l(-1)\Big)_{11}\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)} + i\Big(R(-1)B_l(-1)\Big)_{12}\frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)}\right)\\
\check{X}_{21}(-s) &= \frac{2\pi i}{\gamma} e^{i\frac{\pi}{2}\nu}\left(\Big(R(-1)B_l(-1)\Big)_{21}\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)} + i\Big(R(-1)B_l(-1)\Big)_{22}\frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)}\right),
\end{aligned}$$

followed by

$$\begin{aligned}
\check{X}_{11}'(-s) &= -\frac{2\pi i}{\gamma s} e^{i\frac{\pi}{2}\nu}\Bigg(\Big(R(-1)B_l(-1)\Big)_{11}\big(-2is^3\vartheta'(-1)\big)\left(\frac{i}{2}+i\nu\right)\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)}\\
&\quad +\Big(R(-1)B_l(-1)\Big)_{12}\big(-2is^3\vartheta'(-1)\big)\left(\frac{1}{2}-\nu\right)\frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)}\\
&\quad -\big(R(-1)B_l'(-1)+R'(-1)B_l(-1)\big)_{11}\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)}\\
&\quad -\Big(R(-1)B_l'(-1)+R'(-1)B_l(-1)\big)_{12}\frac{ie^{s^3\vartheta(-1)}}{\Gamma(\nu)}\Bigg)
\end{aligned}$$

and moreover

$$
\begin{aligned}
\check{X}'_{21}(-s) \quad = \quad & -\frac{2\pi i}{\gamma s} e^{i\frac{\pi}{2}\nu} \Bigg(\Big(R(-1)B_l(-1) \Big)_{21} \big(-2is^3\vartheta'(-1) \big) \left(\frac{i}{2} + i\nu \right) \frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)} \\
& + \Big(R(-1)B_l(-1) \Big)_{22} \big(-2is^3\vartheta'(-1) \big) \left(\frac{1}{2} - \nu \right) \frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)} \\
& - \big(R(-1)B_l'(-1) + R'(-1)B_l(-1) \big)_{21} \frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)} \\
& - \Big(R(-1)B_l'(-1) + R'(-1)B_l(-1) \big)_{22} \frac{ie^{s^3\vartheta(-1)}}{\Gamma(\nu)} \Bigg).
\end{aligned}
$$

We finish the currect section by evaluating the resolvent kernel $R(\lambda, \mu)$ at $\lambda = \mu = \pm s$. Recall (2.17)

$$
\begin{aligned}
F_1(\pm s) = i\sqrt{\frac{\gamma}{2\pi}} \check{X}_{11}(\pm s), \qquad F_2(\pm s) = i\sqrt{\frac{\gamma}{2\pi}} \check{X}_{21}(\pm s) \\
F_1'(\pm s) = i\sqrt{\frac{\gamma}{2\pi}} \check{X}'_{11}(\pm s), \qquad F_2'(\pm s) = i\sqrt{\frac{\gamma}{2\pi}} \check{X}'_{21}(\pm s)
\end{aligned}
$$

and (2.18)

$$
R(s,s) = F_1'(s)F_2(s) - F_2'(s)F_1(s).
$$

The last two identities combined with the formulae for $\check{X}_{jk}(s)$ and $\check{X}'_{jk}(s)$, we obtain

$$
\begin{aligned}
R(s,s) = & \frac{2\pi}{\gamma s} e^{i\pi\nu} \Bigg[\Big(R_{11}(1)R_{22}(1) - R_{12}(1)R_{21}(1) \Big) \frac{\big(-2is^3\vartheta'(1) \big)}{\Gamma(\nu)\Gamma(-\nu)} \\
& + \Big(R_{11}(1)R_{22}(1) - R_{12}(1)R_{21}(1) \Big) \left(1 + \frac{\vartheta''(1)}{\vartheta'(1)} \right) \frac{i\nu}{\Gamma(\nu)\Gamma(-\nu)} \\
& + \Big(R'_{11}(1)R_{22}(1) - R'_{22}(1)R_{11}(1) + R'_{12}(1)R_{21}(1) - R'_{21}(1)R_{12}(1) \Big) \frac{i}{\Gamma(\nu)\Gamma(-\nu)} \\
& + \Big(R'_{11}(1)R_{21}(1) - R'_{21}(1)R_{11}(1) \Big) (16s^3 + 4xs)^{2\nu} \frac{e^{-2s^3\vartheta(1)}}{\Gamma^2(\nu)} \\
& - \Big(R'_{12}(1)R_{22}(1) - R'_{22}(1)R_{12}(1) \Big) (16s^3 + 4xs)^{-2\nu} \frac{e^{2s^3\vartheta(1)}}{\Gamma^2(-\nu)} \Bigg].
\end{aligned}
$$

In order to simplify this identity, we use

Proposition 4.1.1 *$R(z)$ is unimodular for any $x, \gamma \in \mathbb{R}$, i.e.* $\det R(z) \equiv 1$.

Proof It is easy to verify that $\det P_{II}^{RH}(\zeta) \equiv 1$ as well as $\det P_{CH}^{RH}(\zeta) = \det \widetilde{P}_{CH}^{RH}(\zeta) \equiv 1$. Therefore one establishes from (3.15),(3.26) and (3.36)

$$\det U(z) = \det V(z) = \det W(z) \equiv 1.$$

Moreover the model function $M(z)$ is unimodular, hence the ratio function $R(z)$ has a unimodular jump matrix $G_R(z)$. But this shows that $\det R(z)$ is in fact an entire function, normalized at infinity, so by Liouville theorem

$$\det R(z) = R_{11}(z)R_{22}(z) - R_{12}(z)R_{21}(z) \equiv 1, \quad z \in \mathbb{C}.$$

■

In light of the last proposition

$$\begin{aligned} R(s,s) = \frac{2\pi}{\gamma s} e^{i\pi\nu} \Bigg[& \frac{8s^3+2xs}{\Gamma(\nu)\Gamma(-\nu)} + \frac{i\nu}{\Gamma(\nu)\Gamma(-\nu)} \frac{12+\frac{x}{s^2}}{4+\frac{x}{s^2}} + \frac{i}{\Gamma(\nu)\Gamma(-\nu)} \\ & \times \Big(R_{11}'(1)R_{22}(1) - R_{22}'(1)R_{11}(1) + R_{12}'(1)R_{21}(1) - R_{21}'(1)R_{12}(1) \Big) \\ & + \Big(R_{11}'(1)R_{21}(1) - R_{21}'(1)R_{11}(1) \Big)(16s^3+4xs)^{2\nu} \frac{e^{-2s^3\vartheta(1)}}{\Gamma^2(\nu)} \\ & - \Big(R_{12}'(1)R_{22}(1) - R_{22}'(1)R_{12}(1) \Big)(16s^3+4xs)^{-2\nu} \frac{e^{2s^3\vartheta(1)}}{\Gamma^2(-\nu)} \Bigg] \end{aligned} \tag{4.4}$$

and we notice that the current derivation did not distinguish between the cases $\gamma < 1$ and $\gamma > 1$, hence the latter identity holds as long as $\gamma \neq 1$. Similarly, using Proposition 4.1.1 once more, we also have

$$\begin{aligned} R(-s,-s) = \frac{2\pi}{\gamma s} e^{i\pi\nu} \Bigg[& \frac{8s^3+2xs}{\Gamma(\nu)\Gamma(-\nu)} + \frac{i\nu}{\Gamma(\nu)\Gamma(-\nu)} \frac{12+\frac{x}{s^2}}{4+\frac{x}{s^2}} + \frac{i}{\Gamma(\nu)\Gamma(-\nu)} \\ & \times \Big(R_{11}'(-1)R_{22}(-1) - R_{22}'(-1)R_{11}(-1) + R_{12}'(-1)R_{21}(-1) - R_{21}'(-1)R_{12}(-1) \Big) \\ & + \Big(R_{11}'(-1)R_{21}(-1) - R_{21}'(-1)R_{11}(-1) \Big)(16s^3+4xs)^{-2\nu} \frac{e^{-2s^3\vartheta(-1)}}{\Gamma^2(-\nu)} \\ & - \Big(R_{12}'(-1)R_{22}(-1) - R_{22}'(-1)R_{12}(-1) \Big)(16s^3+4xs)^{2\nu} \frac{e^{2s^3\vartheta(-1)}}{\Gamma^2(\nu)} \Bigg]. \end{aligned} \tag{4.5}$$

We will now derive (1.18) up to the constant term.

4.2 Proof of Theorem 1.2.2 up to constant terms

Using estimations (3.45) and (3.46) in (3.44), we see that as $s \to \infty$

$$R(\pm 1) = I + O\big(s^{-1}\big), \quad R'(\pm 1) = O\big(s^{-1}\big)$$

uniformly on any compact subset of the set (1.19). Also in case $\gamma < 1$, the functions

$$(16s^3 + 4xs)^{\pm 2\nu} e^{\mp 2\vartheta(1)\sigma_3}$$

are bounded as $s \to \infty$. From Proposition 2.3.1 and (4.4), (4.5), we obtain therefore

$$\begin{aligned} \frac{\partial}{\partial s} \ln\det\left(I - \gamma K_{\mathrm{PII}}\right) &= -R(s,s) - R(-s,-s) \\ &= -\frac{4\pi}{\gamma s} e^{i\pi\nu} \left[\frac{8s^3 + 2xs}{\Gamma(\nu)\Gamma(-\nu)} + \frac{i\nu}{\Gamma(\nu)\Gamma(-\nu)} \frac{12 + \frac{x}{s^2}}{4 + \frac{x}{s^2}} \right] + O\big(s^{-2}\big) \\ &= i\nu\left(16s^2 + 4x\right) + \frac{6(i\nu)^2}{s} + O\big(s^{-2}\big), \quad s \to \infty \qquad (4.6) \end{aligned}$$

uniformly on any compact subset of the set (1.19). Integrating with respect to s, we obtain the leading terms in (1.18) up to a term which still might depend on x and γ. In order to show that this term is in fact x-independent, we use Proposition 4.1.1: Trace back the transformations

$$X(\lambda) \mapsto T(z) \mapsto S(z) \mapsto R(z)$$

and obtain with (3.7) and (3.44)

$$X_1 = \lim_{\lambda\to\infty} \left(\lambda\Big(X(\lambda)e^{i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3} - I\Big)\right) = 2s\nu\sigma_3 + \frac{is}{2\pi}\int_{\Sigma_R} R_-(w)\big(G_R(w) - I\big)dw.$$

From (3.18), (3.43) and (3.46)

$$\frac{i}{2\pi}\int_{\Sigma_R} R_-(w)\big(G_R(w) - I\big)dw = \frac{i}{2\pi}\int_{C_0} \big(G_R(w) - I\big)dw + O\big(s^{-2}\big)$$

$$= \frac{i}{2\pi}\int_{C_0} B_0(w) \begin{pmatrix} -iv & ue^{2\pi i\nu} \\ ue^{-2\pi i\nu} & iv \end{pmatrix} B_0^{-1}(w)\frac{dw}{2\zeta(w)} + O\big(s^{-2}\big)$$

where the last integral can be computed by residue theorem. We obtain

$$X_1 = 2s\nu\sigma_3 + \frac{1}{2}\begin{pmatrix} -iv & u \\ u & iv \end{pmatrix} + O\big(s^{-1}\big), \qquad s\to\infty$$

uniformly on any compact subset of the set (1.19). Back to (2.19)

$$\begin{aligned}\frac{\partial}{\partial x}\ln\det\left(I-\gamma K_{\mathrm{PII}}\right) &= i\left(X_1^{11}-X_1^{22}\right)-v \\ &= 4si\nu + O\big(s^{-1}\big), \qquad s\to\infty\end{aligned}$$

which, combined with (4.6), implies

$$\ln\det\left(I-\gamma K_{\mathrm{PII}}\right) = i\nu\left(\frac{16}{3}s^3+4xs\right)+6(i\nu)^2\ln s+\chi_{\mathrm{PII}}+O\big(s^{-1}\big), \tag{4.7}$$

that is Theorem 1.18 up to a γ-dependent term χ_{PII}.

4.3 Proof of Theorem 1.2.3

In order to verify the large zero distribution of $\det\left(I-\gamma K_{\mathrm{PII}}\right)$ in case $\gamma>1$, we will again use Propositions 2.3.1 and 2.3.2. This time however we need to trace back the full series of transformations,

$$Y(\lambda)\mapsto \widetilde{X}(\lambda)\mapsto X(\lambda)\mapsto T(z)\mapsto S(z)\mapsto R(z)\mapsto P(z)\mapsto Q(z)$$

and recall (see section 3.4), that in case $\gamma>1$, all large values of s stay away from the small neighborhoods of the points $\{s_n\}$ defined by the equation

$$\frac{8}{3}s_n^3+2xs_n+\frac{1}{\pi}\ln(\gamma-1)\ln\left(16s^3+4xs\right)-\arg\frac{\Gamma(1-\nu)}{\Gamma(\nu)}=\frac{\pi}{2}+n\pi, \qquad n=1,2,\ldots$$

Still we make use of (4.4) and (4.5), however we now need to connect the required values of $R(\pm1)$ and $R'(\pm1)$ to $Q(\pm1)$ and $Q'(\pm1)$. To this end recall (3.50), (3.54) and the residue relations (3.51),(3.52). This gives

$$R(1)=(I+B)\,Q(1)\begin{pmatrix}\frac{1}{2} & -p\nu_0\frac{12+\frac{x}{s^2}}{4+\frac{x}{s^2}} \\ 0 & 0\end{pmatrix}+\big((I+B)\,Q'(1)+Q(1)\big)\begin{pmatrix}0 & -p \\ 0 & 1\end{pmatrix}$$

where (compare (3.53))

$$p = \frac{\Gamma(1-\nu)}{\Gamma(\nu)} e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(1) = e^{-i\sigma}, \qquad \nu_0 = \frac{1}{2\pi i}\ln(\gamma - 1)$$

with

$$\sigma \equiv \sigma(s,x,\gamma) = \frac{8}{3}s^3 + 2xs + \frac{1}{\pi}\ln(\gamma-1)\ln\big(16s^3+4xs\big) - \arg\frac{\Gamma(1-\nu)}{\Gamma(\nu)}. \tag{4.8}$$

Also

$$\begin{aligned} R'(1) &= (I+B)Q(1)\begin{pmatrix} -\frac{1}{4} & -p\nu_0\kappa(s,x) \\ 0 & 0 \end{pmatrix} + \Big((I+B)\frac{Q''(1)}{2} + Q'(1)\Big)\begin{pmatrix} 0 & -p \\ 0 & 1 \end{pmatrix} \\ &+\Big((I+B)Q'(1)+Q(1)\Big)\begin{pmatrix} \frac{1}{2} & -p\nu_0\frac{12+\frac{x}{s^2}}{4+\frac{x}{s^2}} \\ 0 & 0 \end{pmatrix} \end{aligned}$$

where we introduced

$$\kappa(s,x) = \frac{3(2\nu_0-1)(12+\frac{x}{s^2})^2 + 80(4+\frac{x}{s^2})}{12(4+\frac{x}{s^2})^2}.$$

Furthermore

$$R(-1) = (-I+B)Q(-1)\begin{pmatrix} 0 & 0 \\ -p\nu_0\frac{12+\frac{x}{s^2}}{4+\frac{x}{s^2}} & -\frac{1}{2} \end{pmatrix} + \Big((-I+B)Q'(-1)+Q(-1)\Big)\begin{pmatrix} 1 & 0 \\ p & 0 \end{pmatrix}$$

and

$$\begin{aligned} R'(-1) = (-I+B)Q(-1)\begin{pmatrix} 0 & 0 \\ p\nu_0\kappa(s,x) & -\frac{1}{4} \end{pmatrix} + \Big((-I+B)\frac{Q''(-1)}{2} \\ +Q'(-1)\Big)\begin{pmatrix} 1 & 0 \\ p & 0 \end{pmatrix} + \Big((-I+B)Q'(-1)+Q(-1)\Big)\begin{pmatrix} 0 & 0 \\ -p\nu_0\frac{12+\frac{x}{s^2}}{4+\frac{x}{s^2}} & -\frac{1}{2} \end{pmatrix}. \end{aligned}$$

Next we compute the values of $Q(\pm 1)$ and $Q'(\pm 1)$ via (3.59) as $s \to \infty$. Since we are going to need terms of $O(s^{-2})$, we have to iterate (3.57). First for any $z \in \Sigma_R$

$$\begin{aligned} Q_-(z) - I = \frac{1}{2\pi i}\int_{\Sigma_R} Q_-(w)\big(G_Q(w)-I\big)\frac{dw}{w-z_-} \\ = \frac{1}{2\pi i}\int_{\Sigma_R}\big(G_Q(w)-I\big)\frac{dw}{w-z_-} + O\big(s^{-2}\big) = \frac{1}{2sz}\Bigg[\begin{pmatrix} -iv & -u \\ -u & iv \end{pmatrix} \\ -\begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} B_0(z)\begin{pmatrix} -iv & ue^{2\pi i\nu} \\ ue^{-2\pi i\nu} & iv \end{pmatrix} B_0^{-1}(z)\begin{pmatrix} z+1 & 0 \\ 0 & z-1 \end{pmatrix}\Bigg] + O\big(s^{-2}\big) \end{aligned}$$

where we used that

$$\begin{pmatrix} \frac{1}{z+1} & 0 \\ 0 & \frac{1}{z-1} \end{pmatrix} B_0(z) \begin{pmatrix} -iv & ue^{2\pi i\nu} \\ ue^{-2\pi i\nu} & iv \end{pmatrix} B_0^{-1}(z) \begin{pmatrix} z+1 & 0 \\ 0 & z-1 \end{pmatrix}$$

$$= \begin{pmatrix} -iv & -u \\ -u & iv \end{pmatrix} + 4\nu_0 \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix} z + O\big(z^2\big), \quad z \to 0.$$

This leads to

$$\begin{aligned} Q(\pm 1) &= I \mp \frac{1}{2s}\begin{pmatrix} iv & u \\ u & -iv \end{pmatrix} \mp \frac{\nu_0}{s^2}\begin{pmatrix} u^2 & iu_x \\ iu_x & -u^2 \end{pmatrix} \\ &\quad + \frac{1}{8s^2}\begin{pmatrix} u^2 - v^2 & -2i(u_x + uv) \\ 2i(u_x + uv) & u^2 - v^2 \end{pmatrix} + O\big(s^{-3}\big), \quad s \to \infty \end{aligned} \tag{4.9}$$

as well as

$$\begin{aligned} Q'(\pm 1) &= \frac{1}{2s}\begin{pmatrix} iv & u \\ u & -iv \end{pmatrix} + \frac{\nu_0}{s^2}\begin{pmatrix} u^2 & iu_x \\ iu_x & -u^2 \end{pmatrix} \\ &\quad \mp \frac{1}{4s^2}\begin{pmatrix} u^2 - v^2 & -2i(u_x + uv) \\ 2i(u_x + uv) & u^2 - v^2 \end{pmatrix} + O\big(s^{-3}\big), \quad s \to \infty. \end{aligned} \tag{4.10}$$

At this point we consider the matrix

$$N = \left(Q(-1)\begin{pmatrix} 1 \\ p \end{pmatrix}, Q(1)\begin{pmatrix} -p \\ 1 \end{pmatrix} \right)$$

which appears in (3.55). In order to find the large s-asymptotics of B, we first compute an expansion for the determinant of N. From (4.9)

$$\begin{aligned} \det N &= 2p\bigg(\cos\sigma + \frac{1}{s}\big(u - v\sin\sigma\big) + \frac{\cos\sigma}{2s^2}\big(u^2 - v^2\big) + \frac{2i\nu_0}{s^2}\big(u_x + u^2\sin\sigma\big) \\ &\quad + O\big(s^{-3}\big)\bigg), \quad s \to \infty \end{aligned}$$

and since we agreeded to stay away from the small neighborhoods of the zeros of $\cos\sigma$, the latter determinant is non-zero and we can asymptotically compute the matrix B:

$$\begin{aligned} B_{11} &= \frac{2ip}{\det N}\Big(\sin\sigma + \frac{v}{s}\cos\sigma - \frac{v^2}{2s^2}\sin\sigma - \frac{2i\nu_0}{s^2}u^2\cos\sigma - \frac{u_x}{2s^2} + O\big(s^{-3}\big)\Big) \\ B_{12} &= \frac{2p}{\det N}\Big(1 + \frac{u}{s}\cos\sigma + \frac{2i\nu_0}{s^2}u_x\cos\sigma + \frac{u^2}{2s^2} - \frac{u_x}{2s^2}\sin\sigma - \frac{uv}{s^2}\sin\sigma + O\big(s^{-3}\big)\Big) \\ B_{21} &= B_{12} + O\big(s^{-3}\big), \quad B_{22} = -B_{11} + O\big(s^{-3}\big) \end{aligned}$$

and with

$$\frac{2p}{\det N} = \frac{1}{\cos\sigma}\left[1+\frac{1}{s}\left(v\tan\sigma-\frac{u}{\cos\sigma}\right)+\frac{1}{s^2}\left(v^2\tan^2\sigma-\frac{2uv\sin\sigma}{\cos^2\sigma}\right.\right.$$
$$\left.\left.+\frac{u^2}{\cos^2\sigma}-\frac{u^2-v^2}{2}-2i\nu_0u^2\tan\sigma-\frac{2i\nu_0u_x}{\cos\sigma}\right)+O\big(s^{-3}\big)\right]$$

we obtain

$$B_{11} = i\left[\tan\sigma+\frac{1}{s}\left(\frac{v\sin\sigma-u}{\cos^3\sigma}\sin\sigma+\frac{v}{\cos\sigma}\right)+\frac{1}{s^2}\left(\frac{2v^2\sin\sigma-u^2\sin\sigma-u_x}{2\cos^2\sigma}\right.\right.$$
$$-\frac{2i\nu_0(u^2\sin\sigma+u_x)\sin\sigma}{\cos^3\sigma}+\frac{(v\sin\sigma-u)^2\sin\sigma}{\cos^4\sigma}-\frac{uv\cos\sigma+2i\nu_0u^2}{\cos\sigma}$$
$$\left.+O\big(s^{-3}\big)\right]$$

and similarly

$$B_{12} = \frac{1}{\cos\sigma}+\frac{1}{s}\left(\frac{v\sin\sigma-u}{\cos^2\sigma}+u\right)+\frac{1}{s^2}\left(\frac{v^2-u_x\sin\sigma}{2\cos\sigma}\right.$$
$$\left.-\frac{2i\nu_0(u^2\sin\sigma+u_x)}{\cos^2\sigma}+\frac{(v\sin\sigma-u)^2}{\cos^3\sigma}-u^2\cos\sigma+2i\nu_0u_x\right)+O\big(s^{-3}\big).$$

At this point we have gathered enough information to go back to (4.4) and (4.5). Since $\nu=\nu_0+\frac{1}{2}$, notice that

$$R(s,s) = -i\nu_0(8s^2+2x)-i(4s^2+x)$$
$$+ip\left(16s^2+4x\right)\left(R'_{11}(1)R_{21}(1)-R'_{21}(1)R_{11}(1)\right)+O\big(s^{-1}\big)$$

and similarly

$$R(-s,-s) = -i\nu_0(8s^2+2x)-i(4s^2+x)$$
$$-ip\left(16s^2+4x\right)\left(R'_{12}(1)R_{22}(1)-R'_{22}(1)R_{12}(1)\right)+O\big(s^{-1}\big).$$

Next

$$R'_{11}(1) = -\frac{1}{4}\big((I+B)Q(1)\big)_{11}+\frac{1}{2}\big((I+B)Q'(1)+Q(1)\big)_{11}$$
$$R_{21}(1) = \frac{1}{2}\big((I+B)Q(1)\big)_{21}$$
$$R'_{21}(1) = -\frac{1}{4}\big((I+B)Q(1)\big)_{21}+\frac{1}{2}\big((I+B)Q'(1)+Q(1)\big)_{21}$$
$$R_{11}(1) = \frac{1}{2}\big((I+B)Q(1)\big)_{11}$$

and we can now combine the previously derived information on $Q(1), Q'(1)$ as well as B to derive

$$ip\left(16s^2+4x\right)\left(R'_{11}(1)R_{21}(1)-R'_{21}(1)R_{11}(1)\right)=(4s^2+x)\big(i+\tan\sigma\big)+\alpha_++O\big(s^{-1}\big)$$

with a function $\alpha_+=\alpha_+(s,x,\gamma)$ such that

$$\int\alpha_+(s,x,\gamma)\,ds=O\big(\ln s\big),\qquad s\to\infty.$$

Following the same computations for $R(-s,-s)$, we end with

$$R(-s,-s)=-i\nu_0(8s^2+2x)+(4s^2+x)\tan\sigma+\alpha_-+O\big(s^{-1}\big)$$

where $\alpha_-=\alpha_-(s,x,\gamma)$ is such that

$$\int\alpha_-(s,x,\gamma)\,ds=O\big(\ln s\big),\qquad s\to\infty.$$

By Proposition 2.3.1, we obtain

$$\begin{aligned}\frac{\partial}{\partial s}\ln\det\left(I-\gamma K_{\mathrm{PII}}\right)&=-R(s,s)-R(-s,-s)\\&=i\nu_0\left(16s^2+4x\right)-\left(8s^2+2x\right)\tan\sigma(s,x,\gamma)\\&\quad-(\alpha_++\alpha_-)+O\big(s^{-1}\big),\qquad s\to\infty\end{aligned}$$

uniformly on any compact subset of the set (1.21), outside a small neighborhood of the points $\{s_n\}$ defined by

$$\sigma(s_n,x,\gamma)=\frac{8}{3}s_n^3+2xs_n+\frac{1}{\pi}\ln(\gamma-1)\ln\left(16s_n^3+4xs_n\right)-\arg\frac{\Gamma(1-\nu)}{\Gamma(\nu)}=\frac{\pi}{2}+n\pi.$$

as $n\to\infty$. In order to finish the proof of Theorem 1.2.3, we use again Proposition 2.3.2. Tracing back all transformations, one obtains

$$\begin{aligned}X_1&=2s\nu\sigma_3+s\left(B-\sigma_3\right)+\frac{is}{2\pi}\int_{\Sigma_R}Q_-(w)\big(G_Q(w)-I\big)dw\\&=2s\nu\sigma_3+s\left(B-\sigma_3\right)+\frac{1}{2}\begin{pmatrix}-iv&-u\\-u&iv\end{pmatrix}+O\big(s^{-1}\big)\end{aligned}$$

and therefore with (2.19) and previously derived expansions

$$\frac{\partial}{\partial x}\ln\det\left(I-\gamma K_{\mathrm{PII}}\right)=4i\nu_0 s-2s\tan\sigma+O\big(s^{-1}\big),\qquad s\to\infty$$

which proves Theorem 1.2.3.

Remark 4 *We want to emphasize that our strategy in fact produced an asymptotic series for* $\ln\det\left(I-\gamma K_{\mathrm{PII}}\right)$ *of the form*

$$\begin{aligned}\ln\det\left(I-\gamma K_{\mathrm{PII}}\right) &= i\nu_0\left(\frac{16}{3}s^3+4xs\right)+\ln\left|\cos\sigma(s,x,\gamma)\right|\\ &\quad +c_0\ln s+c_1(\gamma)+O\big(s^{-1}\big),\qquad s\to\infty,\end{aligned}\tag{4.11}$$

uniformly on any compact subset of the set (1.21) *and outside a neighborhood of the points* $\{s_n\}$. *Here, the universal constant* c_0*, can be computed by a direct, although tedious, refinement of our approach.*

4.4 The situation $\gamma = 1$ – preliminary steps

We will use the same strategy as presented in sections 4.1 and 4.2 only this time customized to the series of transformations

$$Y(\lambda)\mapsto\widetilde{X}(\lambda)\mapsto X(\lambda)\mapsto A(z)\mapsto K(z).$$

In the current situation $\gamma=1$, we need to connect the values of $X(\pm s)$ and $X'(\pm s)$ to the corresponding ones of $K(\pm 1)$ and $K'(\pm 1)$. With

$$K(z)I(z)e^{-s^3g(z)\sigma_3}=\check{X}(zs)\begin{pmatrix}1 & -\frac{1}{2\pi}\ln\frac{z-1}{z+1}\\ 0 & 1\end{pmatrix}\begin{cases} I, & \lambda\in\hat{\Omega}_1,\\ S_3S_4, & \lambda\in\hat{\Omega}_3,\\ S_3S_4S_6, & \lambda\in\hat{\Omega}_4,\end{cases}\tag{4.12}$$

valid for $|z-1|<r$, and

$$K(z)J(z)e^{-s^3g(z)\sigma_3}=\check{X}(zs)\begin{pmatrix}1 & -\frac{1}{2\pi}\ln\frac{z-1}{z+1}\\ 0 & 1\end{pmatrix}\begin{cases} I, & \lambda\in\hat{\Omega}_1,\\ S_3, & \lambda\in\hat{\Omega}_2,\\ S_3S_4, & \lambda\in\hat{\Omega}_3,\end{cases}\tag{4.13}$$

which is valid for $|z+1| < r$, we can again obtain the required values via comparison in (4.12) and (4.13), once we know the local expansions of $I(z)$ and $J(z)$ at $z = \pm 1$. This time our starting point is (3.67)

$$\begin{aligned} P_{BE}^{RH}(\zeta) &= \begin{pmatrix} \bar{a}_0 + \frac{\bar{a}_1}{2}\ln\zeta & i(a_0 + \frac{a_1}{2}\ln\zeta) \\ \bar{a}_1 & a_1 \end{pmatrix} \\ &+\zeta \begin{pmatrix} \bar{a}_2 + \frac{\bar{a}_3}{2}\ln\zeta & i(a_2 + \frac{a_3}{2}\ln\zeta) \\ 2\bar{a}_2 + \bar{a}_3 + \bar{a}_3\ln\zeta & 2a_2 + a_3 + a_3\ln\zeta \end{pmatrix} + O\big(\zeta^2\ln\zeta\big), \quad (4.14) \end{aligned}$$

as $\zeta \to 0$ and $-\frac{\pi}{6} < \arg\,\zeta < \frac{\pi}{6}$. The latter expansion together with the changes of variables $\zeta = \zeta(z) = -s^6 g^2(z)$ and $\lambda = zs$ implies for $-\frac{\pi}{6} < \arg\,(\lambda - s) < \frac{\pi}{6}$

$$P_{BE}^{RH}\big(\zeta(z)\big) = P_3\big(\ln(\lambda - s)\big) + (\lambda - s)P_4\big(\ln(\lambda - s)\big) + O\big((\lambda - s)^2\ln(\lambda - s)\big), \quad \lambda \to s$$

with the matrix functions $P_3 = (P_3^{ij})$ and $P_4 = (P_4^{ij})$ being determined from (4.14). Now we combine the latter expansion with (3.72) and (3.84)

$$\begin{aligned} & K(z)I(z)e^{-s^3 g(z)\sigma_3}\Big|_{z=\frac{\lambda}{s}} = K(z)C_r(z)\frac{\sigma_3}{2}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{4}}P_{BE}^{RH}\big(\zeta(z)\big)\Big|_{z=\frac{\lambda}{s}} \\ = \; & K(1)C_r(1)\frac{\sigma_3}{2}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{4}}P_3\big(\ln(\lambda - s)\big) + (\lambda - s)\Big\{K(1)C_r(1)\frac{\sigma_3}{2}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{4}} \\ & \times P_4\big(\ln(\lambda - s)\big) + \big(K'(1)C_r(1) + K(1)C_r'(1)\big)\frac{\sigma_3}{2s}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{4}}P_3\big(\ln(\lambda - s)\big)\Big\} \\ & + O\big((\lambda - s)^2\ln(\lambda - s)\big), \quad \lambda \to s, \quad -\frac{\pi}{6} < \arg\,(\lambda - s) < \frac{\pi}{6} \end{aligned}$$

and similar identities hold for $-\pi < \arg(\lambda - s) < -\frac{\pi}{6}$ and $\frac{\pi}{6} < \arg(\lambda - s) < \pi$, they differ from (4.14) only by right multiplication with a Stokes matrix (see (3.69)). On the other hand the right hand side in (4.12) implies for $-\frac{\pi}{6} < \arg(\lambda - s) < \frac{\pi}{6}$

$$
\begin{aligned}
&K\left(\frac{\lambda}{s}\right) I\Big(\frac{\lambda}{s}\Big) e^{-s^3 g(\frac{\lambda}{s})\sigma_3} = \Big(\check{X}(s) + (\lambda - s)\check{X}'(s) + O\big((\lambda-s)^2\big)\Big) \\
&\times \begin{pmatrix} 1 & -\frac{1}{2\pi}\ln\frac{\lambda-s}{\lambda+s} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} = \left[\begin{pmatrix} \check{X}_{11}(s) & \frac{1}{2\pi}\ln(2s)\check{X}_{11}(s) + \check{X}_{12}(s) \\ \check{X}_{21}(s) & \frac{1}{2\pi}\ln(2s)\check{X}_{21}(s) + \check{X}_{22}(s) \end{pmatrix} \right. \\
&\left. +(\lambda - s) \begin{pmatrix} \check{X}'_{11}(s) & \frac{1}{2\pi}(\ln(2s)\check{X}'_{11}(s) + \frac{1}{2s}\check{X}_{11}(s)) + \check{X}'_{12}(s) \\ \check{X}'_{21}(s) & \frac{1}{2\pi}(\ln(2s)\check{X}'_{21}(s) + \frac{1}{2s}\check{X}_{21}(s)) + \check{X}'_{22}(s) \end{pmatrix} \right] \\
&\times \begin{pmatrix} 1 - \frac{i}{2\pi}\ln(\lambda - s) & -\frac{1}{2\pi}\ln(\lambda - s) \\ i & 1 \end{pmatrix} + O\big((\lambda - s)^2\ln(\lambda - s)\big), \quad \lambda \to s
\end{aligned}
$$

where we used the same notations for $\check{X}(\lambda)$ as in section 4.1, hoping this ambiguity won't lead to any confusion in the following. From a comparison of the left and right hand side in (4.12)

$$
\check{X}_{11}(s) = \sqrt{\frac{\pi}{2}} e^{-i\frac{\pi}{4}} \big(K(1)C_r(1)\big)_{11}, \qquad \check{X}_{21}(s) = \sqrt{\frac{\pi}{2}} e^{-i\frac{\pi}{4}} \big(K(1)C_r(1)\big)_{21}, \tag{4.15}
$$

and these identities have been derived in the sector $-\frac{\pi}{6} < \arg(\lambda - s) < \frac{\pi}{6}$. However by multiplying in the other sectors with the right Stokes matrices from (4.14) as well as using the appropriate Stokes matrices in (2.13), we can easily show that (4.15) follows in fact from comparison in a full neighborhood of $\lambda = +s$. Comparing now terms of $O\big((\lambda - s)\ln(\lambda - s)\big)$ we also derive

$$
\begin{aligned}
\check{X}'_{11}(s) &= -\sqrt{\frac{\pi}{2}} e^{-i\frac{\pi}{4}} \left[\frac{8s^5}{9}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^2 \Big(\big(K(1)C_r(1)\big)_{11} + 2i\big(K(1)C_r(1)\big)_{12} \Big) \right. \\
&\quad \left. -\frac{1}{s}\big(K'(1)C_r(1) + K(1)C'_r(1)\big)_{11} \right]
\end{aligned} \tag{4.16}
$$

and

$$
\begin{aligned}
\check{X}'_{21}(s) &= -\sqrt{\frac{\pi}{2}} e^{-i\frac{\pi}{4}} \left[\frac{8s^5}{9}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^2 \Big(\big(K(1)C_r(1)\big)_{21} + 2i\big(K(1)C_r(1)\big)_{22} \Big) \right. \\
&\quad \left. -\frac{1}{s}\big(K'(1)C_r(1) + K(1)C'_r(1)\big)_{21} \right].
\end{aligned} \tag{4.17}
$$

This implies the identities

$$F_1(s) = \frac{i}{\sqrt{2\pi}}\check{X}_{11}(s),\ F_2(s) = \frac{i}{\sqrt{2\pi}}\check{X}_{21}(s),\ F_1'(s) = \frac{i}{\sqrt{2\pi}}\check{X}_{11}'(s),\ F_2'(s) = \frac{i}{\sqrt{2\pi}}\check{X}_{21}'(s)$$

related to the solution of the K-RHP via (4.15), (4.16) and (4.17). A similar analysis for the left endpoint $\lambda = -s$ provides us with

$$F_1(-s) = \frac{i}{2}\big(K(-1)C_l(-1)\big)_{12}, \quad F_2(-s) = \frac{i}{2}\big(K(-1)C_l(-1)\big)_{22} \tag{4.18}$$

and

$$\begin{aligned} F_1'(-s) \; &= \; \frac{i}{2}\left[\frac{8s^5}{9}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)^2\Big(\big(K(-1)C_l(-1)\big)_{12} + 2\big(K(-1)C_l(-1)\big)_{11}\Big)\right. \\ &\quad \left. +\frac{1}{s}\big(K(-1)C_l'(-1) + K'(-1)C_l(-1)\big)_{12}\right] \end{aligned} \tag{4.19}$$

as well as

$$\begin{aligned} F_2'(-s) \; &= \; \frac{i}{2}\left[\frac{8s^5}{9}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)^2\Big(\big(K(-1)C_l(-1)\big)_{22} + 2\big(K(-1)C_l(-1)\big)_{21}\Big)\right. \\ &\quad \left. +\frac{1}{s}\big(K(-1)C_l'(-1) + K'(-1)C_l(-1)\big)_{22}\right]. \end{aligned} \tag{4.20}$$

We can now derive

$$\begin{aligned} R(s,s) \; &= \; F_1'(s)F_2(s) - F_2'(s)F_1(s) \\ &= \; -\frac{4s^5}{9}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)^2\Big[\big(K(1)C_r(1)\big)_{11}\big(K(1)C_r(1)\big)_{22} - \big(K(1)C_r(1)\big)_{21} \\ &\quad \times\big(K(1)C_r(1)\big)_{12}\Big] + \frac{i}{4s}\Big[\big(K'(1)C_r(1) + K(1)C_r'(1)\big)_{11} \\ &\quad \times\big(K(1)C_r(1)\big)_{21} - \big(K'(1)C_r(1) + K(1)C_r'(1)\big)_{21}\big(K(1)C_r(1)\big)_{11}\Big] \end{aligned}$$

as well as

$$\begin{aligned} &R(-s,-s) = F_1'(-s)F_2(-s) - F_2'(-s)F_1(-s) \\ = \; &-\frac{4s^5}{9}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)^2\Big[\big(K(-1)C_l(-1)\big)_{11}\big(K(-1)C_l(-1)\big)_{22} - \big(K(-1)C_l(-1)\big)_{21} \\ &\times\big(K(-1)C_l(-1)\big)_{12}\Big] - \frac{1}{4s}\Big[\big(K(-1)C_l'(-1) + K'(-1)C_l(-1)\big)_{12} \\ &\times\big(K(-1)C_l(-1)\big)_{22} - \big(K(-1)C_l'(-1) + K'(-1)C_l(-1)\big)_{22}\big(K(-1)C_l(-1)\big)_{12}\Big]. \end{aligned}$$

The following analogue of Proposition 4.1.1 will allow us to simplify the identities for $R(s,s)$ and $R(-s,-s)$

Proposition 4.4.1 *$R(z)$ is unimodular for any $x \in \mathbb{R}$, i.e.* $\det R(z) \equiv 1$.

Proof From (3.69) we obtain that $\det P_{BE}^{RH}(\zeta) = \frac{4i}{\pi}$, hence $\det I(z) = 1$. Similarly $\det \widetilde{P}_{BE}^{RH}(\zeta) = -\frac{4i}{\pi}$ leading to $\det J(z) = 1$. Since the model function $D(z)$ is unimodular as well, the ratio function $K(z)$ has a unimodular jump matrix $G_K(z)$. This shows that $\det K(z)$ is entire, and by normalization at infinity therefore

$$\det K(z) = K_{11}(z)K_{22}(z) - K_{21}(z)K_{12}(z) \equiv 1, \quad z \in \mathbb{C}.$$

■

Applying the latter Proposition, one checks readily

$$\big(K(1)C_r(1)\big)_{11}\big(K(1)C_r(1)\big)_{22} - \big(K(1)C_r(1)\big)_{21}\big(K(1)C_r(1)\big)_{12} = -2$$

and

$$\big(K(-1)C_l(-1)\big)_{11}\big(K(-1)C_l(-1)\big)_{22} - \big(R(-1)C_l(-1)\big)_{21}\big(K(-1)C_l(-1)\big)_{12} = -2.$$

We combine these two identities with the values of $C_r'(1)$ and $C_l'(-1)$ to deduce

$$\begin{aligned} R(s,s) &= \frac{8s^5}{9}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)^2 + \frac{2is^2}{3}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)\Big[\big(R_{11}'(1)+R_{12}'(1)\big) \\ &\quad \times\big(R_{21}(1)+R_{22}(1)\big) - \big(R_{21}'(1)+R_{22}'(1)\big)\big(R_{11}(1)+R_{12}(1)\big)\Big] \end{aligned}$$

as well as

$$R(-s,-s) = \frac{8s^5}{9}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)^2 - \frac{2is^2}{3}\left(\frac{3}{2}+\frac{3x}{4s^2}\right)\Big[\big(R_{11}'(-1)-R_{12}'(-1)\big)$$
$$\times\big(R_{21}(-1)-R_{22}(-1)\big) - \big(R_{21}'(-1)-R_{22}'(-1)\big)\big(R_{11}(-1)-R_{12}(-1)\big)\Big].$$

At this point we can derive (1.14) up to the constant term.

4.5 Proof of Theorem 1.2.1 up to constant terms

We first estimate $K(\pm 1)$. From the integral representation and (3.88), (3.89)

$$\begin{aligned} K(\pm 1) &= I + \frac{1}{2\pi i}\int_{\Sigma_K} K_-(w)\big(G_K(w) - I\big)\frac{dw}{w \mp 1} \\ &= I + \frac{1}{2\pi i}\int_{C_{r,l}} \big(G_K(w) - I\big)\frac{dw}{w \mp 1} + O\big(s^{-6}\big) = I + O\big(s^{-3}\big), \quad s \to \infty \end{aligned}$$

so

$$\begin{aligned} R(s,s) &= \frac{8s^5}{9}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^2 + \frac{2is^2}{3}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)\Big[K'_{11}(1) - K'_{22}(1) \\ &\quad + K'_{12}(1) - K'_{21}(1)\Big] + O\big(s^{-4}\big) \end{aligned}$$

and

$$\begin{aligned} R(-s,-s) &= \frac{8s^5}{9}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^2 + \frac{2is^2}{3}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)\Big[K'_{11}(-1) - K'_{22}(-1) \\ &\quad - K'_{12}(-1) + K'_{21}(-1)\Big] + O\big(s^{-4}\big), \quad s \to \infty. \end{aligned}$$

In order to compute the values $K'(\pm 1)$ one uses (3.76) and (3.83)

$$\begin{aligned} K'(\pm 1) &= \frac{1}{2\pi i}\int_{C_{r,l}} \big(G_K(w) - I\big)\frac{dw}{(w \mp 1)^2} + O\big(s^{-6}\big) \\ &= \frac{1}{2\pi i}\int_{C_r} \frac{i}{16\sqrt{\zeta(w)}}\begin{pmatrix} \beta^2 - 3\beta^{-2} & -(\beta^2 + 3\beta^{-2}) \\ \beta^2 + 3\beta^{-2} & -(\beta^2 - 3\beta^{-2}) \end{pmatrix}\frac{dw}{(w \mp 1)^2} \\ &\quad + \frac{1}{2\pi i}\int_{C_l} \frac{1}{16\sqrt{\zeta(w)}}\begin{pmatrix} \beta^{-2} - 3\beta^2 & \beta^{-2} + 3\beta^2 \\ -(\beta^{-2} + 3\beta^2) & -(\beta^{-2} - 3\beta^2) \end{pmatrix}\frac{dw}{(w \mp 1)^2} \\ &\quad + O\big(s^{-6}\big), \quad s \to \infty \end{aligned}$$

with the local variables given in (3.71), (3.79):

$$\begin{aligned} w \in C_r: \quad \frac{\beta^2(w)}{\sqrt{\zeta(w)}} &= \frac{3}{4s^3}\left(w^2 + \frac{1}{2} + \frac{3x}{4s^2}\right)^{-1}\frac{1}{w - 1}, \\ \frac{\beta^{-2}(w)}{\sqrt{\zeta(w)}} &= \frac{3}{4s^3}\left(w^2 + \frac{1}{2} + \frac{3x}{4s^2}\right)^{-1}\frac{1}{w + 1} \end{aligned}$$

and

$$w \in C_l: \quad \frac{\beta^2(w)}{\sqrt{\zeta(w)}} = \frac{3i}{4s^3}\left(w^2 + \frac{1}{2} + \frac{3x}{4s^2}\right)^{-1}\frac{1}{w-1},$$
$$\frac{\beta^{-2}(w)}{\sqrt{\zeta(w)}} = \frac{3i}{4s^3}\left(w^2 + \frac{1}{2} + \frac{3x}{4s^2}\right)^{-1}\frac{1}{w+1}.$$

By residue theorem

$$\begin{aligned} K'(\pm 1)\left(\frac{3}{2} + \frac{3x}{4s^2}\right) &= \frac{3i}{256 s^3}\begin{pmatrix} -1 & \mp 1 \\ \pm 1 & 1 \end{pmatrix} \\ &+ \frac{3i}{64s^3}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^{-1}\begin{pmatrix} -\frac{25}{8} - \frac{9x}{16s^2} & \mp(\frac{41}{8} + \frac{9x}{16s^2}) \\ \pm(\frac{41}{8} + \frac{9x}{16s^2}) & \frac{25}{8} + \frac{9x}{16s^2} \end{pmatrix} \\ &+ \frac{3i}{16s^3}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^{-2}\begin{pmatrix} -1 & \pm 1 \\ \mp 1 & 1 \end{pmatrix} + O\big(s^{-6}\big), \end{aligned}$$

and we obtain

$$R(s,s) = R(-s,-s) = \frac{8s^5}{9}\left(\frac{3}{2} + \frac{3x}{4s^2}\right)^2 + \frac{3}{8s} + O\big(s^{-3}\big), \quad s \to \infty. \tag{4.21}$$

Combining (4.21) with (2.18) we have thus derived the following asymptotics

$$\frac{\partial}{\partial s}\ln\det\left(I - K_{\mathrm{PII}}\right) = -4s^5 - 4xs^3 - x^2 s - \frac{3}{4s} + O\big(s^{-3}\big), \quad s \to \infty \tag{4.22}$$

uniformly on any compact subset of the set (1.15). Integrating with respect to s, we have verified (1.14) up to an s-independent term. In order to determine the x-dependency of this term we are now going to determine $\det\left(I - K_{\mathrm{PII}}\right)$ via Proposition 2.3.2:

$$\frac{\partial}{\partial x}\ln\det\left(I - K_{\mathrm{PII}}\right) = i\big(X_1^{11} - X_1^{22}\big) - v$$

where

$$X_1 = \lim_{\lambda\to\infty}\Big(\lambda\big(X(\lambda)e^{i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3} - I\big)\Big).$$

We first recall the definition of $\beta(z)$ and $g(z)$, hence as $z \to \infty$

$$M(z) = I + \frac{1}{2z}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + O\big(z^{-2}\big), \; e^{s^3(\vartheta(z) - g(z))\sigma_3} = I + \frac{is^3}{2z}\Big(1 + \frac{x}{s^2}\Big)\sigma_3 + O\big(z^{-2}\big),$$

which gives

$$X_1 = \frac{is^4}{2}\Big(1+\frac{x}{s^2}\Big)\sigma_3 + \frac{s}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{is}{2\pi}\int\limits_{C_{r,l}} K_-(w)\big(G_K(w)-I\big)\,dw,$$

already neglecting exponentially small contributions in the last equality. The integral can be evaluated in a similar way as we did it during the computation of (4.22), we end up with

$$\begin{aligned} \frac{i}{2\pi}\int\limits_{C_{r,l}} \Big(K_-(w)\big(G_K(w)-I\big)\Big)_{11} dw &= \frac{3i}{32s^3}\Big(\frac{3}{2}+\frac{3x}{4s^2}\Big)^{-1} + O\big(s^{-6}\big) \\ = -\frac{i}{2\pi}\int\limits_{C_{r,l}} \Big(K_-(w)\big(G_K(w)-I\big)\Big)_{22} dw, & \end{aligned}$$

i.e. together

$$\begin{aligned} \frac{\partial}{\partial x}\ln\det\left(I-K_{\mathrm{PII}}\right) &= 2is\Bigg[\frac{it}{2}\Big(1+\frac{x}{s^2}\Big)+\frac{3i}{32t}\Big(\frac{3}{2}+\frac{3x}{4s^2}\Big)^{-1}\Bigg] - v + O\big(s^{-5}\big) \\ &= -s^4 - s^2x - v - \frac{1}{8s^2} + O\big(s^{-4}\big), \quad s\to\infty, \end{aligned} \tag{4.23}$$

again uniformly on any compact subset of the set (1.15). Given the asymptotic expansions (4.22) and (4.23) we can now determine the large s-asymptotics of $\det\left(I-K_{\mathrm{PII}}\right)$ via integration

$$\ln\det\left(I-K_{\mathrm{PII}}\right) = -\frac{2}{3}s^6 - xs^4 - x^2s^2 - \frac{3}{4}\ln s + \int\limits_x^\infty (y-x)u^2(y)dy + \omega + O\big(s^{-1}\big), \tag{4.24}$$

recalling that $u(x) \sim \mathrm{Ai}(x)$ as $x\to+\infty$. As we see, (4.24) matches (1.14) up to a universal constant ω.

5. KERNEL APPROXIMATION: FROM K_{PII} TO K_{csin}

We were able to prove Theorem 1.2.3 on the zero distribution of $\det(I-\gamma K_{\text{PII}})$ in case $\gamma>1$ as well as Theorems 1.2.1 and 1.2.2 up to constant terms. In this section we will calculate the remaining constant terms. To this end we make use of an approximation argument which replaces the initial kernel $K_{\text{PII}}(\lambda,\mu;x)$ in the large positive x-limit by a cubic generalization of the sine - kernel. The latter is of integrable type and its asymptotics can be computed via an auxiliary Riemann-Hilbert problem. We prove the necessary estimates which allow us to compute the constant terms in Theorem 1.2.1 and 1.2.2 through the asymptotical solution of the auxiliary RHP, set up the auxiliary RHP and derive another set of logarithmic derivatives.

5.1 Large positive x-limit in $K_{\text{PII}}(\lambda,\mu;x)$

Within the asymptotical analysis of the master RHP, the X-RHP in chapter 3, one of the first steps allowed us to transform jumps on the infinite branches Γ_k to exponentially small contributions. This was established for $\gamma\neq 1$ via the set of transformations

$$X(\lambda)\mapsto T(z)\mapsto S(z)$$

and for $\gamma=1$ via

$$X(\lambda)\mapsto A(z).$$

Both transformations heavily rely on the underlying sign diagrams: For $\gamma\neq 1$, we pictured $\text{Re}\,\vartheta(z)$ in Figure 3.1, respectively for $\gamma=1$, $\text{Re}\,g(z)$ in Figure 3.11. In both cases it was important that for x chosen from a compact subset of the real line and s sufficiently large, one always has that the corresponding real parts are negative in the upper half-plane on the infinite parts Γ_1,Γ_3 and positive on the infinite contours

Γ_4, Γ_6 in the lower half-plane. This fact however also holds in the limit $x \to +\infty$, on the other hand it fails for $x \to -\infty$: Let

$$z_{\pm} = \pm i\sqrt{\frac{3x}{4s^2}}, \quad \gamma \neq 1 \qquad\qquad \widehat{z}_{\pm} = \pm i\sqrt{\frac{1}{2} + \frac{3x}{4s^2}}, \quad \gamma = 1$$

denote the intersection points of the algebraic curves

$$\text{Re } \vartheta(z) = 0, \quad \gamma \neq 1 \qquad\qquad \text{Re } g(z) = 0, \quad \gamma = 1$$

with the coordinate axes Re $z = 0 =$ Im z. In case $x, s > 0$, they are (independently of the distinction in γ) purely imaginary, hence the statement on the sign of Re $\vartheta(z)$ respectively Re $g(z)$ on Γ_i follows. This implies the following important Proposition, where an analogue for the cubic sine determinants (1.3.1) and (1.26) also holds, see chapter 6.

Proposition 5.1.1 *The asymptotic expansions* (4.7) *for* $\gamma < 1$ *and* (4.24) *for* $\gamma = 1$ *are uniform in the parameter* x *chosen from the set*

$$\{x \in \mathbb{R} : \ x \geq \alpha, \ \alpha < 0\}.$$

Our approach henceforth will be to study the large positive x-limit of (1.11), i.e. the large positive x-limit of the associated function $\Psi(\lambda, x)$. We begin with the following Riemann-Hilbert problem depicted in Figure 5.1, compare [29]

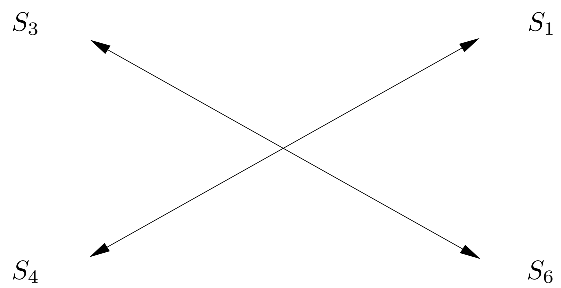

Figure 5.1. The RHP jump graph associated with the Hastings-McLeod transcendent

- $\Psi^\infty(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash\big(\bigcup_k R_k\big)$ where R_k denote the rays

$$R_k = \{\lambda \in \mathbb{C}|\ \arg\lambda = \frac{\pi}{6} + \frac{\pi}{3}(k-1)\},\ k = 1,3,4,6$$

- On the rays R_k, the boundary values of the function Ψ^∞ satisfy the jump relation

$$\Psi^\infty_+(\lambda) = \Psi^\infty_-(\lambda)S_k, \quad \lambda \in R_k,\ k = 1,3,4,6$$

- At $\lambda = \infty$ the following asymptotic behavior takes place

$$\Psi^\infty(\lambda)e^{i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3} = I + O\big(\lambda^{-1}\big)$$

which is connected to the given Ψ-function of (1.11) by

$$\Psi(\lambda, x) = \Psi^\infty(\lambda, x)S_1.$$

As we see, determining the large positive x behavior of $\Psi(\lambda, x)$ therefore reduces to an analysis of the oscillatory Ψ^∞-RHP. However the latter RHP is very well known since it is used to determine the large x-asymptotics of the Hastings-McLeod solution of the second Painlevé transcendent given in the introduction (cf. [29]). We have in fact for $\lambda \in (-s, s)$

$$\Psi^\infty(\lambda, x)e^{i(\frac{4}{3}\lambda^3 + x\lambda)\sigma_3} - I = O\left(\frac{x^{-1/4}e^{-\frac{2}{3}x^{3/2}}}{\sqrt{4\lambda^2 + x}}\right), \qquad x \to +\infty$$

hence

$$\begin{aligned} \psi_{11}(\lambda, x) &= e^{-i(\frac{4}{3}\lambda^3 + x\lambda)} + O\left(\frac{x^{-1/4}e^{-\frac{2}{3}x^{3/2}}}{\sqrt{4\lambda^2 + x}}\right) \\ \psi_{21}(\lambda, x) &= -ie^{i(\frac{4}{3}\lambda^3 + x\lambda)} + O\left(\frac{x^{-1/4}e^{-\frac{2}{3}x^{3/2}}}{\sqrt{4\lambda^2 + x}}\right) \end{aligned}$$

as $x \to +\infty$ and $\lambda \in (-s, s)$. Going back to (1.11) we obtain

$$K_{\mathrm{PII}}(\lambda, \mu) = K_{\mathrm{csin}}(\lambda, \mu) + O\left(x^{1/4}e^{-\frac{2}{3}x^{3/2}}\right), \quad x \to +\infty, \quad \lambda, \mu \in (-s, s) \tag{5.1}$$

where

$$K_{\mathrm{csin}}(\lambda, \mu) = \frac{\sin\left(\frac{4}{3}(\lambda^3 - \mu^3) + x(\lambda - \mu)\right)}{\pi(\lambda - \mu)}. \tag{5.2}$$

The latter integral kernel is a cubic generalization of the sine - kernel, see (1.6)

$$\frac{\sin x(\lambda-\mu)}{\pi(\lambda-\mu)}$$

acting on $L^2\big((-s,s);d\lambda\big)$. In order to compute the constant term in (1.14) we will introduce a parameter $t\in[0,1]$ and pass from (5.2) to

$$K_{\text{csin}}(\lambda,\mu)\mapsto \check{K}_{\text{csin}}(\lambda,\mu)=\frac{\sin\left(\frac{4}{3}t(\lambda^3-\mu^3)+x(\lambda-\mu)\right)}{\pi(\lambda-\mu)} \tag{5.3}$$

and compute the large s-asymptotics of

$$\frac{\partial}{\partial t}\ln\det\left(I-\check{K}_{\text{csin}}\right) \tag{5.4}$$

with the Riemann-Hilbert approach of chapter 2. Afterwards, using uniformity of the asymptotic expansion with respect to $t\in[0,1]$ we shall integrate

$$\int_0^1\frac{\partial}{\partial t}\ln\det(I-\check{K}_{\text{csin}})\;dt=\ln\det(I-K_{\text{csin}})-\ln\det(I-K_{\sin});$$

but since the asymptotic expansion of the sine kernel as $s\to\infty$ is known including the constant term, we know the large s-asymptotics of

$$\ln\det(I-K_{\text{csin}})$$

also up to order $O(s^{-1})$, in fact

$$\ln\det(I-K_{\text{csin}})=A_1(s,x)+\omega_0+O\big(s^{-1}\big),\quad s\to\infty \tag{5.5}$$

uniformly on any compact subset of the set (1.15) with

$$A_1(s,x)=-\frac{2}{3}s^6-s^4x-\frac{1}{2}(sx)^2-\frac{3}{4}\ln s,\qquad \omega_0=-\frac{1}{6}\ln 2+3\zeta'(-1),$$

which is the statement of Theorem 1.3.2. But we already know from (4.24)

$$\ln\det(I-K_{\text{PII}})=A_1(s,x)+\int_x^\infty(y-x)u^2(y)dy+\omega+O\big(s^{-1}\big),\quad s\to\infty \tag{5.6}$$

hence considering (5.1), Proposition 5.1.1 as well as

$$\lim_{x\to\infty}\int\limits_x^\infty (y-x)u^2(y)=0 \tag{5.7}$$

we might conjecture that $\omega=\omega_0$, which is proven in Proposition 5.1.2 below.

For the constant term χ_{PII} in Theorem 1.2.2 use a similar strategy: We compute the large s-asymptotics of

$$\frac{\partial}{\partial\gamma}\ln\det\left(I-\gamma K_{\mathrm{csin}}\right)$$

within the approach of chapter 2. After that integrate

$$\int_0^\gamma \frac{\partial}{\partial\gamma'}\ln\det\left(I-\gamma' K_{\mathrm{csin}}\right)d\gamma' = \ln\det\left(I-\gamma K_{\mathrm{csin}}\right), \qquad \gamma<1$$

and obtain

$$\ln\det\left(I-\gamma K_{\mathrm{csin}}\right)=A_2(s,x)+\chi_0+O\big(s^{-1}\big) \tag{5.8}$$

uniformly on any compact subset of the set (1.19) with

$$A_2(s,x)=i\nu\left(\frac{16}{3}s^3+4xs\right)+6(i\nu)^2\ln s-\int\limits_x^\infty (y-x)u^2(y,\gamma)dy$$

and

$$\chi_0=2\left(i\nu\right)^2+8\left(i\nu\right)^2\ln 2+2\int\limits_0^\gamma \nu(t)\left(\ln\frac{\Gamma\left(\nu(t)\right)}{\Gamma\left(-\nu(t)\right)}\right)'dt,$$

which is the statement of Theorem 1.3.1. On the other hand we know from (4.7)

$$\ln\det\left(I-\gamma K_{\mathrm{PII}}\right)=A_2(s,x)+\chi_{\mathrm{PII}}+O\big(s^{-1}\big), \tag{5.9}$$

hence again with (5.1), Proposition 5.1.1 as well as (compare (1.28))

$$\lim_{x\to\infty}\int\limits_x^\infty (y-x)u^2(y,\gamma)dy=0 \tag{5.10}$$

we conjecture $\chi_{\mathrm{PII}}=\chi_0$.

Proposition 5.1.2 *With the latter notations*

$$\omega_0=\omega \qquad \text{and} \qquad \chi_0=\chi_{\mathrm{PII}}.$$

Proof We start from the following identity for trace class operators (cf. [49])

$$\det(I-A)(I-B) = \det(I-A)\det(I-B)$$

which gives in our situation

$$\det(I-\gamma K_{\mathrm{PII}}) - \det(I-\gamma K_{\mathrm{csin}}) = -\det(I-\gamma K_{\mathrm{csin}}) \times\Big[1-\det\Big(I-(I-\gamma K_{\mathrm{csin}})^{-1}(\gamma K_{\mathrm{PII}}-\gamma K_{\mathrm{csin}})\Big)\Big], \quad \gamma \leq 1$$

provided

$$(I-\gamma K_{\mathrm{csin}})^{-1} = I + R_{\mathrm{csin}} \tag{5.11}$$

exists as a bounded operator. The latter statement will follow from the Riemann-Hilbert analysis of the auxiliary RHP given in chapter 6. Since from (5.1)

$$(K_{\mathrm{PII}}f)(\lambda) = \int_{-s}^{s} K_{\mathrm{PII}}(\lambda,\mu)f(\mu)d\mu = (K_{\mathrm{csin}}f)(\lambda) + (Ef)(\lambda)$$

where the trace class operator E has a kernel satisfying

$$E(\lambda,\mu) = O\Big(x^{1/4}e^{-\frac{2}{3}x^{3/2}}\Big), \quad x\to\infty, \quad (\lambda,\mu)\in[-s,s]\times[-s,s] \tag{5.12}$$

we obtain

$$\det(I-\gamma K_{\mathrm{PII}}) - \det(I-\gamma K_{\mathrm{csin}}) = -\det(I-\gamma K_{\mathrm{csin}})\Big[1-\det\Big(I-(I+R_{\mathrm{csin}})\gamma E\Big)\Big]$$

and therefore

$$\frac{\det(I-\gamma K_{\mathrm{PII}})}{\det(I-\gamma K_{\mathrm{csin}})} = \det\Big(I-(I+R_{\mathrm{csin}})\gamma E\Big), \quad \gamma\leq 1.$$

Now from the boundedness of $I+R_{\mathrm{csin}}$ as well as (5.12), we see that the convolution kernel of the operator

$$(I+R_{\mathrm{csin}})\gamma E$$

approaches zero exponentially fast as $x\to\infty$, thus via Hadamard's inequality in the same limit

$$\begin{aligned} &\det\Big(I-(I+\check{R}_{\mathrm{csin}})\gamma E\Big) \\ = \;& 1+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\int_{-s}^{s}\cdots\int_{-s}^{s}\det\big[\big(I+\check{R}_{\mathrm{csin}}\big)\gamma E\big](x_i,x_j)dx_1\cdots dx_n = 1+o_s(1)\end{aligned}$$

or similarly

$$\ln\det(I-\gamma K_{\mathrm{PII}}) = \ln\det(I-\gamma K_{\mathrm{csin}}) + o_s(1), \quad x\to\infty \tag{5.13}$$

We now combine (5.5),(5.6), (5.13) and obtain

$$|\omega_0-\omega| \le \frac{\alpha_1}{s} + \frac{\beta_1(s)}{x} + \int_x^\infty (y-x)u^2(y)dy$$

for all $x \ge x_0$ and $s \ge s_0$, with a universal constant α_1 and a positive function $\beta_1 = \beta_1(s)$. Recalling (5.7) we first take the limit $x\to\infty$ and afterwards $s\to\infty$ to conclude $\omega_0=\omega$. Secondly from (5.8), (5.9) and (5.13)

$$|\chi_0-\chi_{\mathrm{PII}}| \le \frac{\alpha_2}{s} + \frac{\beta_2(s)}{x} + \int_x^\infty (y-x)u^2(y,\gamma)dy$$

which by (5.10) and the same reasoning as before yields $\chi_0 = \chi_{\mathrm{PII}}$. ■

5.2 Riemann-Hilbert problem associated with $\det(I-\gamma K_{\mathbf{csin}})$

We introduce the auxiliary RHP related to the cubic sine - kernel (5.2) or (5.3). The underlying kernel is of integrable type with

$$\gamma\check{K}_{\mathrm{csin}}(\lambda,\mu) = \frac{d^t(\lambda)e(\mu)}{\lambda-\mu}, \; d(\lambda) = \sqrt{\frac{\gamma}{2\pi i}}\begin{pmatrix} e^{i(\frac{4}{3}t\lambda^3+x\lambda)} \\ e^{-i(\frac{4}{3}t\lambda^3+x\lambda)}\end{pmatrix}, \; e(\lambda) = \sqrt{\frac{1}{2\pi i}}\begin{pmatrix} e^{-i(\frac{4}{3}t\lambda^3+x\lambda)} \\ -e^{i(\frac{4}{3}t\lambda^3+x\lambda)}\end{pmatrix}$$

where we slightly abuse notation, since the appearance of t will only be used in case $\gamma=1$, for $\gamma<1$ we will analyse the problem without the parameter t. Lemma 2 implies the following Θ-RHP

- $\Theta(\lambda)$ is analytic for $\lambda\in\mathbb{C}\backslash[-s,s]$
- On the line segment $[-s,s]$ oriented from left to right, the following jump holds

$$\Theta_+(\lambda) = \Theta_-(\lambda)\begin{pmatrix} 1-\gamma & \gamma e^{2i(\frac{4}{3}t\lambda^3+x\lambda)} \\ -\gamma e^{-2i(\frac{4}{3}t\lambda^3+x\lambda)} & 1+\gamma \end{pmatrix}, \quad \lambda\in[-s,s]$$

- $\Theta(\lambda)$ has at most logarithmic endpoint singularities at $\lambda = \pm s$

$$\Theta(\lambda) = O\big(\ln(\lambda \mp s)\big), \quad \lambda \to \pm s \tag{5.14}$$

- $\Theta(\lambda) \to I$ as $\lambda \to \infty$.

We can factorize the jump matrix

$$\begin{pmatrix} 1-\gamma & \gamma e^{2i(\frac{4}{3}t\lambda^3+x\lambda)} \\ -\gamma e^{-2i(\frac{4}{3}t\lambda^3+x\lambda)} & 1+\gamma \end{pmatrix} = e^{i(\frac{4}{3}t\lambda^3+x\lambda)\sigma_3} \begin{pmatrix} 1-\gamma & \gamma \\ -\gamma & 1+\gamma \end{pmatrix} e^{-i(\frac{4}{3}t\lambda^3+x\lambda)\sigma_3}$$

and employ the following transformation

$$\Phi(\lambda) = \Theta(\lambda) e^{i(\frac{4}{3}t\lambda^3+x\lambda)\sigma_3}, \qquad \lambda \in [-s, s] \tag{5.15}$$

which leads to a RHP for the function $\Phi(\lambda)$, the auxiliary RHP:

- $\Phi(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[-s, s]$
- The following jump holds

$$\Phi_+(\lambda) = \Phi_-(\lambda) \begin{pmatrix} 1-\gamma & \gamma \\ -\gamma & 1+\gamma \end{pmatrix}, \qquad \lambda \in [-s, s] \tag{5.16}$$

- From (5.14), we deduce the following refined endpoint behavior

$$\Phi(\lambda) = \check{\Phi}(\lambda)\left[I + \frac{\gamma}{2\pi i}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \ln\left(\frac{\lambda - s}{\lambda + s}\right)\right] \tag{5.17}$$

  where $\check{\Phi}(\lambda)$ is analytic at $\lambda = \pm s$ and the branch of the logarithm is fixed by the condition $-\pi < \arg \frac{\lambda - s}{\lambda + s} < \pi$.

- At infinity, $\Phi(\lambda)$ is normalized as follows

$$\Phi(\lambda) = \Big(I + O\big(\lambda^{-1}\big)\Big) e^{i(\frac{4}{3}t\lambda^3+x\lambda)\sigma_3}, \qquad \lambda \to \infty. \tag{5.18}$$

As we are going to see in chapter 6 the latter RHP admits direct asymptotical analysis via the nonlinear steepest descent method. This analysis shows a lot of similarities to the analysis of the Painlevé II - kernel presented in chapter 3 of the current dissertation. However one major difference to (1.11) is the absence of infinite jump contours in the given Φ-RHP, hence we should not start our analysis from the X-RHP in chapter 2 and use the previously discussed large x-approximation $\Psi^{\infty}(\lambda, x)$. Again, before we start this asymptotical analysis, we first connect the relevant logarithmic derivatives to the solution of the auxiliary RHP, the Φ-RHP.

5.3 Logarithmic derivatives – connection to Φ-RHP

We will derive four identities for logarithmic derivatives. The first two are with respect to s and x and will be used to determine the expansion given in Theorem 1.3.1 up to the constant term and the zero distribution of Theorem 1.3.3. Since their derivation is almost identical to the identities given in Proposition 2.3.1 and 2.3.2, we limit ourselves to a statement of results:

Proposition 5.3.1 *The logarithmic s-derivative of the cubic sine - kernel determinant* (5.2) *can be expressed as*

$$\begin{aligned}\frac{\partial}{\partial s}\ln\det(I-\gamma K_{\text{csin}}) &= -R(s,s)-R(-s,-s),\\ &\qquad R(\pm s,\pm s)=\Pi_1'(\pm s)\Pi_2(\pm s)-\Pi_2'(\pm s)\Pi_1(\pm s)\end{aligned}$$

with $R(\lambda,\mu)$ denoting the kernel of the resolvent $R=(I-\gamma K_{\text{csin}})^{-1}\gamma K_{\text{csin}}$, that is

$$R(\lambda,\mu)=\frac{\Pi^t(\lambda)E(\mu)}{\lambda-\mu},\qquad \Pi(\lambda)=\Theta(\lambda)d(\lambda),$$

and where the connection to the Φ-RHP is established through

$$\Pi(\lambda)=\check{\Phi}(\lambda)\sqrt{\frac{\gamma}{2\pi i}}\begin{pmatrix}1\\1\end{pmatrix},\quad \lambda\to\pm s.$$

Next

Proposition 5.3.2 *The logarithmic x-derivative of the cubic sine - kernel determinant* (5.2) *can be expressed as*

$$\frac{\partial}{\partial x}\ln\det(I-\gamma K_{\mathrm{csin}}) = i\big(\Phi_1^{22}-\Phi_1^{11}\big)$$

with

$$\Phi(\lambda)=\Big(I+\frac{\Phi_1}{\lambda}+\frac{\Phi_2}{\lambda^2}+\frac{\Phi_3}{\lambda^3}+O\big(\lambda^{-4}\big)\Big)e^{i(\frac{4}{3}\lambda^3+x\lambda)\sigma_3},\ \lambda\to\infty;\ \Phi_1=\big(\Phi_1^{ij}\big).$$

Much more interesting is the derivation of an identity for the logarithmic γ-derivative, which will be used in the end to determine the constant term in Theorem 1.2.2 respectively Theorem 1.3.1. This identity is derived for $\gamma<1$ and inspired by a similar approach which was used in [21] in the asymptotics of Toeplitz determinants.

We start with

$$\frac{\partial}{\partial\gamma}\ln\det\left(I-\gamma K_{\mathrm{csin}}\right) = -\mathrm{trace}\Big(\big(I-\gamma K_{\mathrm{csin}}\big)^{-1}K_{\mathrm{csin}}\Big) = -\frac{1}{\gamma}\int_{-s}^{s}R(\lambda,\lambda)d\lambda.$$

and now wish to express the latter integral over the resolvent kernel in terms of the solution of the auxiliary RHP. Recall to this end the definition of the functions $d(\lambda),e(\lambda)$ and unimodularity of $\Theta(\lambda)$

$$\begin{aligned} R(\lambda,\lambda) &= \Pi_1'(\lambda)\Pi_2(\lambda)-\Pi_2'(\lambda)\Pi_1(\lambda)=\frac{\gamma}{\pi}\big(4\lambda^2+x\big)+\frac{\gamma}{2\pi i}\big(\Theta_{11}'(\lambda)\Theta_{22}(\lambda)\\ &\quad -\Theta_{11}(\lambda)\Theta_{22}'(\lambda)+\Theta_{12}'(\lambda)\Theta_{21}(\lambda)-\Theta_{21}'(\lambda)\Theta_{12}(\lambda)\big)+\big(\Theta_{11}'(\lambda)\Theta_{21}(\lambda)\\ &\quad -\Theta_{11}(\lambda)\Theta_{21}'(\lambda)\big)d_1^2(\lambda)+\big(\Theta_{12}'(\lambda)\Theta_{22}(\lambda)-\Theta_{12}(\lambda)\Theta_{22}'(\lambda)\big)d_2^2(\lambda)\end{aligned}$$

where ($'$) indicates differentiation with respect to λ. In terms of $\Phi(\lambda)$

$$\begin{aligned} R(\lambda,\lambda) &= \frac{\gamma}{2\pi i}\Big[\big(\Phi_{11}'(\lambda)+\Phi_{12}'(\lambda)\big)\big(\Phi_{21}(\lambda)+\Phi_{22}(\lambda)\big)\\ &\quad -\big(\Phi_{11}(\lambda)+\Phi_{12}(\lambda)\big)\big(\Phi_{21}'(\lambda)+\Phi_{22}'(\lambda)\big)\Big] \end{aligned}\tag{5.19}$$

Our next move will replace all terms involving derivatives with respect to λ. To this end we consider the differential equations associated with the Φ-RHP (compare section 2.4 where we studied the differential equations associated with the X-RHP):

All jump matrices in the Φ-RHP are unimodular and constant with respect to λ, s and x, thus the well-defined logarithmic derivatives $\Phi_\lambda\Phi^{-1}(\lambda), \Phi_s\Phi^{-1}(\lambda)$ and $\Phi_x\Phi^{-1}(\lambda)$ are rational functions. Indeed using (5.18) and (5.14)

$$\frac{\partial\Phi}{\partial\lambda} = \left[4i\lambda^2\sigma_3 - 4i\lambda\begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} + \begin{pmatrix} d & e \\ f & -d \end{pmatrix} + \frac{A}{\lambda - s} - \frac{B}{\lambda + s}\right]\Phi \tag{5.20}$$

where

$$A = \frac{\gamma}{2\pi i}\check{\Phi}(s)\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\big(\check{\Phi}(s)\big)^{-1}; \qquad B = \frac{\gamma}{2\pi i}\check{\Phi}(-s)\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\big(\check{\Phi}(-s)\big)^{-1} \tag{5.21}$$

and with parameters b, c, d, e, f which can be expressed in terms of the entries of Φ_1 and Φ_2

$$b = 2\Phi_1^{12}, \quad c = 2\Phi_1^{21}, \quad d = ix + 8i\Phi_1^{12}\Phi_1^{21} \tag{5.22}$$

$$e = 8i\Big(\Phi_1^{12}\Phi_1^{22} - \Phi_2^{12}\Big), \quad f = -8i\Big(\Phi_1^{21}\Phi_1^{11} - \Phi_2^{21}\Big). \tag{5.23}$$

Substituting (5.20) into (5.19) and recalling (5.21) we obtain with $A = (A_{ij}), B = (B_{ij})$

$$\begin{aligned} R(\lambda,\lambda) \;=\; & \frac{\gamma}{2\pi i}\Bigg[\left(8i\lambda^2 + 2d + \frac{A_{11} - A_{22}}{\lambda - s} - \frac{B_{11} - B_{22}}{\lambda + s}\right)\big(\Phi_{11}(\lambda) + \Phi_{12}(\lambda)\big) \\ & \times\big(\Phi_{21}(\lambda) + \Phi_{22}(\lambda)\big) \\ & \quad +\left(-4i\lambda b + e + \frac{A_{12}}{\lambda - s} - \frac{B_{12}}{\lambda + s}\right)\big(\Phi_{21}(\lambda) + \Phi_{22}(\lambda)\big)^2 \\ & \quad +\left(-4i\lambda c - f - \frac{A_{21}}{\lambda - s} + \frac{B_{21}}{\lambda + s}\right)\big(\Phi_{11}(\lambda) + \Phi_{12}(\lambda)\big)^2\Bigg]. \end{aligned} \tag{5.24}$$

Next, we γ-differentiate the Φ-RHP in (5.16) to obtain the following additive RHP for the function $\phi(\lambda) = \frac{\partial\Phi}{\partial\gamma}(\lambda)\big(\Phi(\lambda)\big)^{-1}$

- $\phi(\lambda)$ is analytic for $\lambda \in \mathbb{C}\backslash[-s, s]$
- Along the line segment $[-s, s]$, oriented from left to right

$$\phi_+(\lambda) = \phi_-(\lambda) + \Phi_-(\lambda)\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\big(\Phi_-(\lambda)\big)^{-1}, \quad \lambda \in [-s, s]$$

- $\phi(\lambda)$ has at most logarithmic singularities at the endpoints $\lambda = \pm s$

$$\phi(\lambda) = \frac{\partial \check{\Phi}}{\partial \gamma}(\lambda)\big(\check{\Phi}(\lambda)\big)^{-1} + \check{\Phi}(\lambda)\frac{1}{2\pi i}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\big(\check{\Phi}(\lambda)\big)^{-1}\ln\frac{\lambda - s}{\lambda + s}, \quad \lambda \to \pm s \tag{5.25}$$

- As $\lambda \to \infty$, we have $\phi(\lambda) \to 0$

If we let

$$\gamma(\lambda) = \Phi(\lambda)\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\big(\Phi(\lambda)\big)^{-1}, \quad \lambda \in \mathbb{C}\backslash[-s,s],$$

then $\gamma_+(\lambda) = \gamma_-(\lambda), \lambda \in [-s,s]$ and $\gamma(\lambda)$ is bounded as $\lambda \to \pm s$. Hence $\gamma(\lambda)$ is entire and we have a solution to the ϕ-RHP

$$\phi(\lambda) = \frac{1}{2\pi i}\int_{s}^{s}\frac{\gamma_-(w)}{w-\lambda}dw = \frac{1}{2\pi i}\int_{-s}^{s}\frac{\gamma(w)}{w-\lambda}dw$$

$$= \frac{1}{2\pi i}\int_{-s}^{s}\begin{pmatrix} -(\Phi_{11}(w)+\Phi_{12}(w))(\Phi_{21}(w)+\Phi_{22}(w)) & (\Phi_{11}(w)+\Phi_{12}(w))^2 \\ -(\Phi_{21}(w)+\Phi_{22}(w))^2 & (\Phi_{11}(w)+\Phi_{12}(w))(\Phi_{21}(w)+\Phi_{22}(w)) \end{pmatrix}\frac{dw}{w-\lambda}.$$

This solution enables us to rewrite $\int_{-s}^{s} R(\lambda,\lambda)d\lambda$ with the help of (5.24), for instance

$$\int_{-s}^{s}\lambda^n\big(\Phi_{11}(\lambda)+\Phi_{12}(\lambda)\big)\big(\Phi_{21}(\lambda)+\Phi_{22}(\lambda)\big)d\lambda = \int_{\Sigma} w^n\phi_{11}(w)dw, \quad n \in \mathbb{Z}_{\geq 0}$$

with Σ denoting a closed Jordan curve around the interval $[-s,s]$ and where we used

$$\lambda^n = \frac{1}{2\pi i}\int_{\Sigma}\frac{w^n}{w-\lambda}dw, \quad \lambda \in [-s,s].$$

Similarly

$$\begin{aligned} \int_{-s}^{s}\lambda^n\big(\Phi_{21}(\lambda)+\Phi_{22}(\lambda)\big)^2 d\lambda &= \int_{\Sigma} w^n\phi_{21}(w)dw, \\ \int_{-s}^{s}\lambda^n\big(\Phi_{11}(\lambda)+\Phi_{12}(\lambda)\big)^2 d\lambda &= -\int_{\Sigma} w^n\phi_{12}(w)dw \end{aligned}$$

and we obtain

$$\frac{\partial}{\partial\gamma}\ln\det\left(I-\gamma K_{\mathrm{csin}}\right)=-\frac{1}{\gamma}\int_{-s}^{s}R(\lambda,\lambda)d\lambda=-\frac{1}{2\pi i}\Bigg[8i\int_{\Sigma}w^2\phi_{11}(w)dw$$
$$+\int_{\Sigma}\big(2d\phi_{11}(w)+e\phi_{21}(w)+f\phi_{12}(w)\big)dw-4i\int_{\Sigma}w\big(b\phi_{21}(w)-c\phi_{12}(w)\big)dw$$
$$-\int_{-s}^{s}\Big((A_{11}-A_{22})\gamma_{11}(\lambda)+A_{12}\gamma_{21}(\lambda)+A_{21}\gamma_{12}(\lambda)\Big)\frac{d\lambda}{\lambda-s}$$
$$+\int_{-s}^{s}\Big((B_{11}-B_{22})\gamma_{11}(\lambda)+B_{12}\gamma_{21}(\lambda)+B_{21}\gamma_{12}(\lambda)\Big)\frac{d\lambda}{\lambda+s}\Bigg]. \tag{5.26}$$

Since

$$\gamma(\lambda)=\frac{2\pi i}{\gamma}A+O(\lambda-s),\ \lambda\to s,\qquad \gamma(\lambda)=\frac{2\pi i}{\gamma}B+O(\lambda+s),\ \lambda\to -s$$

and

$$(A_{11}-A_{22})A_{11}+2A_{12}A_{21}=0=(B_{11}-B_{22})B_{11}+2B_{12}B_{21},$$

we deduce that the last two integrals in (5.26) are indeed well-defined. To evaluate them, let

$$\widehat{\phi}(\lambda)\ =\ \phi(\lambda)-\check{\Phi}(s)\begin{pmatrix}-1&1\\-1&1\end{pmatrix}\big(\check{\Phi}(s)\big)^{-1}\frac{1}{2\pi i}\ln\frac{\lambda-s}{\lambda+s},\quad \lambda\in\mathbb{C}\backslash[-s,s]$$
$$\widetilde{\phi}(\lambda)\ =\ \phi(\lambda)-\check{\Phi}(-s)\begin{pmatrix}-1&1\\-1&1\end{pmatrix}\big(\check{\Phi}(-s)\big)^{-1}\frac{1}{2\pi i}\ln\frac{\lambda-s}{\lambda+s},\quad \lambda\in\mathbb{C}\backslash[-s,s].$$

From (5.25) we see that $\widehat{\phi}(\lambda)$ is bounded as $\lambda\to s$ and $\widetilde{\phi}(\lambda)$ is bounded as $\lambda\to -s$, more precisely

$$\widehat{\phi}(s)=\frac{\partial\check{\Phi}}{\partial\gamma}(s)\big(\check{\Phi}(s)\big)^{-1},\quad \widetilde{\phi}(-s)=\frac{\partial\check{\Phi}}{\partial\gamma}(-s)\big(\check{\Phi}(-s)\big)^{-1},$$

also

$$\widehat{\phi}_+(\lambda)=\widehat{\phi}_-(\lambda)+\gamma(\lambda)-\gamma(s),\quad \widetilde{\phi}_+(\lambda)=\widetilde{\phi}_-(\lambda)+\gamma(\lambda)-\gamma(-s),\ \lambda\in[-s,s],$$

hence

$$\widehat{\phi}(\lambda) = \frac{1}{2\pi i}\int_{-s}^{s}\frac{\gamma(w)-\gamma(s)}{w-\lambda}dw, \quad \widetilde{\phi}(\lambda) = \frac{1}{2\pi i}\int_{-s}^{s}\frac{\gamma(w)-\gamma(-s)}{w-\lambda}dw \tag{5.27}$$

and we conclude

$$\begin{aligned}
&\int_{-s}^{s}\Big((A_{11}-A_{22})\gamma_{11}(\lambda)+A_{12}\gamma_{21}(\lambda)+A_{21}\gamma_{12}(\lambda)\Big)\frac{d\lambda}{\lambda-s} \\
= \; & (A_{11}-A_{22})\int_{-s}^{s}\frac{\gamma_{11}(\lambda)-\gamma_{11}(s)}{\lambda-s}d\lambda + A_{12}\int_{-s}^{s}\frac{\gamma_{21}(\lambda)-\gamma_{21}(s)}{\lambda-s}d\lambda \\
& + A_{21}\int_{-s}^{s}\frac{\gamma_{12}(\lambda)-\gamma_{12}(s)}{\lambda-s}d\lambda = 2\pi i\Big((A_{11}-A_{22})\widehat{\phi}_{11}(s)+A_{12}\widehat{\phi}_{21}(s)+A_{21}\widehat{\phi}_{12}(s)\Big).
\end{aligned}$$

Similarly

$$\begin{aligned}
&\int_{-s}^{s}\Big((B_{11}-B_{22})\gamma_{11}(\lambda)+B_{12}\gamma_{21}(\lambda)+B_{21}\gamma_{12}(\lambda)\Big)\frac{d\lambda}{\lambda+s} \\
= \; & 2\pi i\Big((B_{11}-B_{22})\widetilde{\phi}_{11}(-s)+B_{12}\widetilde{\phi}_{21}(-s)+B_{21}\widetilde{\phi}_{12}(-s)\Big).
\end{aligned}$$

In order to evaluate the remaining integrals in (5.26), we recall

$$\begin{aligned}
\phi(\lambda) \; = \; & \frac{1}{\lambda}(\Phi_1)_\gamma + \frac{1}{\lambda^2}\big((\Phi_2)_\gamma - (\Phi_1)_\gamma\Phi_1\big) \\
& + \frac{1}{\lambda^3}\big((\Phi_3)_\gamma + (\Phi_1)_\gamma(\Phi_1^2-\Phi_2) - (\Phi_2)_\gamma\Phi_1\big) + O\big(\lambda^{-4}\big), \quad \lambda\to\infty
\end{aligned}$$

and apply residue theorem

$$\begin{aligned}
&\int_\Sigma \phi(w)dw = 2\pi i(\Phi_1)_\gamma, \qquad \int_\Sigma w\phi(w)dw = 2\pi i\big((\Phi_2)_\gamma - (\Phi_1)_\gamma\Phi_1\big) \\
&\int_\Sigma w^2\phi(w)dw = 2\pi i\big((\Phi_3)_\gamma + (\Phi_1)_\gamma(\Phi_1^2-\Phi_2) - (\Phi_2)_\gamma\Phi_1\big).
\end{aligned}$$

At this point we can summarize our computations.

Proposition 5.3.3 *The logarithmic γ-derivative of the cubic sine - kernel determinant* (5.2) *can be expressed as*

$$\frac{\partial}{\partial\gamma}\ln\det\left(I-\gamma K_{\mathrm{csin}}\right)=-8i\Big((\Phi_3)_\gamma+(\Phi_1)_\gamma(\Phi_1^2-\Phi_2)-(\Phi_2)_\gamma\Phi_1\Big)^{11}$$
$$-2d\Big((\Phi_1)_\gamma\Big)^{11}+4ib\Big((\Phi_2)_\gamma-(\Phi_1)_\gamma\Phi_1\Big)^{21}-4ic\Big((\Phi_2)_\gamma-(\Phi_1)_\gamma\Phi_1\Big)^{12}$$
$$-e\Big((\Phi_1)_\gamma\Big)^{21}-f\Big((\Phi_1)_\gamma\Big)^{12}+\Big((A_{11}-A_{22})\widehat{\phi}_{11}(s)+A_{12}\widehat{\phi}_{21}(s)+A_{21}\widehat{\phi}_{12}(s)\Big)$$
$$-\Big((B_{11}-B_{22})\widetilde{\phi}_{11}(-s)+B_{12}\widetilde{\phi}_{21}(-s)+B_{21}\widetilde{\phi}_{12}(-s)\Big) \tag{5.28}$$

where

$$\Phi(\lambda)\sim\Big(I+\frac{\Phi_1}{\lambda}+\frac{\Phi_2}{\lambda^2}+\frac{\Phi_3}{\lambda^3}+O\big(\lambda^{-4}\big)\Big)e^{i(\frac{4}{3}\lambda^3+x\lambda)\sigma_3},\quad \lambda\to\infty, \tag{5.29}$$

with

$$\widehat{\phi}(s)=\frac{\partial\check{\Phi}}{\partial\gamma}(s)\big(\check{\Phi}(s)\big)^{-1},\quad \widetilde{\phi}(-s)=\frac{\partial\check{\Phi}}{\partial\gamma}(-s)\big(\check{\Phi}(-s)\big)^{-1},$$

and the functions b,c,d,e,f,A,B are defined in (5.22),(5.23) *and* (5.21).

The last proposition will allow us to compute the constant term in Theorems 1.3.1 and 1.18 using Proposition 5.1.2. For Theorem 1.14, we use the Φ-RHP with $\gamma=1$ and the parameter $t\in[0,1]$ involved. Hence the following identity, which was derived in much more generality in [40], will be useful:

$$\frac{\partial}{\partial t}\ln\det\big(I-\check{K}_{\mathrm{csin}}\big)=\frac{1}{2\pi}\int_{\Sigma}\mathrm{trace}\Big[\Theta'(w)\sigma_3\Theta^{-1}(w)\Big]\frac{4}{3}w^3dw$$

where Σ denotes a closed Jordan curve around the line segment $[-s,s]$. Applying residue theorem, we have

Proposition 5.3.4 *The logarithmic t-derivative of the Fredholm determinant* $\det(I-\check{K}_{\mathrm{csin}})$ *can be expressed as*

$$\frac{\partial}{\partial t}\ln\det\big(I-\check{K}_{\mathrm{csin}}\big)=\frac{4i}{3}\mathrm{trace}\Big(-\Theta_1\sigma_3\big(\Theta_1^2-\Theta_2\big)+2\Theta_2\sigma_3\Theta_1-3\Theta_3\sigma_3\Big) \tag{5.30}$$

with

$$\Theta(\lambda)\sim I+\frac{\Theta_1}{\lambda}+\frac{\Theta_2}{\lambda^2}+\frac{\Theta_3}{\lambda^3}+O\big(\lambda^{-4}\big),\quad \lambda\to\infty$$

and the connection to the auxiliary RHP is established through (5.15).

At this point we set up all necessary steps to start the asymptotical analysis of the auxiliary RHP. Similar to the situation in chapter 3, our approach is based on the Riemann-Hilbert method and its underlying integral equations, not on an interplay of the RHP with differential equations connected to $\det(I-\gamma K_{\text{csin}})$ - we face for the cubic sine - kernel a similar situation as the one described in section 2.4. The analysis this time will be more involved, see (5.28) and (5.30): both equations involve contributions arising from terms of order $O(\lambda^{-3})$, hence we will have to iterate the underlying integral equations.

6. ASYMPTOTIC SOLUTION OF THE AUXILIARY RIEMANN-HILBERT PROBLEM

We solve the Φ-RHP according to the Deift-Zhou nonlinear steepest descent method. Most parts in the chapter below have their counterpart in the asymptotical solution of the X-RHP presented in chapter 3, the differences are only of technical nature, but not of conceptual. Again this resolution is first presented for $\gamma \neq 1$, followed then by $\gamma = 1$ which involves the Φ-RHP with a parameter t.

6.1 Rescaling and opening of lenses, $\gamma \neq 1$

We rescale $\Theta(\lambda)$ and introduce

$$\Upsilon(z) = \Phi(zs)e^{-s^3\vartheta(z)\sigma_3} \equiv \Theta(zs), \quad z \in \mathbb{C}\backslash[-1,1] \tag{6.1}$$

whose underlying RHP, up to the rescaling $\lambda = zs$, is identical to the initial Θ-RHP. To move to a RHP posed according to the sign of $\operatorname{Re}\vartheta(z)$, we recall Figure 3.1 and the LDU-factorization

$$\begin{aligned}\begin{pmatrix} 1-\gamma & \gamma \\ -\gamma & 1+\gamma \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -\gamma(1-\gamma)^{-1} & 1 \end{pmatrix} (1-\gamma)^{\sigma_3} \begin{pmatrix} 1 & \gamma(1-\gamma)^{-1} \\ 0 & 1 \end{pmatrix} \\ &\equiv \widehat{S}_L S_D^{-1} \widehat{S}_U, \end{aligned} \tag{6.2}$$

valid whenever $\gamma \neq 1$. With the same notations as in section 3.1, see also Figure 6.1 below, we define

$$\Delta(z) = \begin{cases} \Upsilon(z)e^{s^3\vartheta(z)\sigma_3}\widehat{S}_U^{-1}e^{-s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_1^+ \cup \mathcal{L}_2^+, \\ \Upsilon(z)e^{s^3\vartheta(z)\sigma_3}\widehat{S}_L e^{-s^3\vartheta(z)\sigma_3}, & z \in \mathcal{L}_3^- \cup \mathcal{L}_4^-, \\ \Upsilon(z), & \text{otherwise}, \end{cases} \equiv \Upsilon(z)\widehat{L}(z) \tag{6.3}$$

and are lead to the following RHP

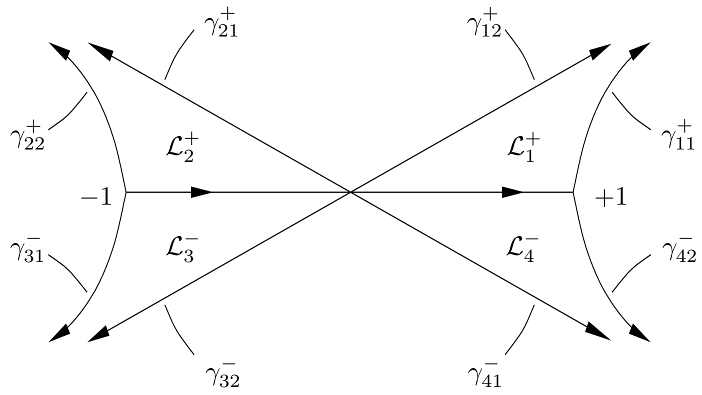

Figure 6.1. Opening of lenses – $\Upsilon(z) \mapsto \Delta(z)$

- $\Delta(z)$ is analytic for $z \in \mathbb{C}\backslash([-1,1]\cup\mathcal{D})$, with $\mathcal{D} = \bigcup_{i,j}\left(\gamma_{ji}^{+}\cup\gamma_{ji}^{-}\right)$
- The following jumps hold, with orientation fixed as in Figure 6.1

$$\Delta_+(z) = \Delta_-(z)e^{s^3\vartheta(z)\sigma_3}\widehat{G}_\Delta(z)e^{-s^3\vartheta(z)\sigma_3}, \quad z \in [-1,1]\cup\mathcal{D} \tag{6.4}$$

where the piecewise constant function $\widehat{G}_\Delta(z)$ is given by

$$\widehat{G}_\Delta(z) = \begin{cases} \widehat{S}_U^{-1}, & z \in \gamma_{11}^{+}\cup\gamma_{21}^{+}, \\ \widehat{S}_U, & z \in \gamma_{12}^{+}\cup\gamma_{22}^{+}, \\ (1-\gamma)^{\sigma_3}, & z \in [-1,1], \\ \widehat{S}_L, & z \in \gamma_{31}^{-}\cup\gamma_{41}^{-}, \\ \widehat{S}_L^{-1}, & z \in \gamma_{32}^{-}\cup\gamma_{42}^{-}. \end{cases}$$

- As $z \to \pm 1$, we have

$$\Delta(z)\widehat{L}^{-1}(z)e^{s^3\vartheta(z)\sigma_3} = \check{\Phi}(zs)\left[I + \frac{\gamma}{2\pi i}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\ln\left(\frac{z-1}{z+1}\right)\right] \tag{6.5}$$

- At infinity, $\Delta(z) = I + O(z^{-1}), z \to \infty$.

On the infinite branches in the upper half-plane, we have

$$G_\Delta(z) = e^{s^3\vartheta(z)\sigma_3}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}e^{-s^3\vartheta(z)\sigma_3}, \quad z \in \gamma_{jk}^{+}, \quad j,k = 1,2$$

and since also here x is chosen from a compact subset of the real line, the sign-diagram implies

$$G_{\Delta}(z) \longrightarrow I, \quad s \to \infty, \;\; z \in \gamma_{jk}^{+},\;\; |z| > \varepsilon > 0$$

uniformly on any compact subset of the set (1.19). With a similar statement valid in the lower half-plane we are thus lead to a model problem in which we have to solve a RHP with S_D^{-1} as jump matrix on the line segment $[-1,1]$.

6.2 The model RHP and parametrices for $\gamma \neq 1$

Find the piecewise analytic 2×2 matrix valued function $\Xi(z)$ such that

- $\Xi(z)$ is analytic for $z \in \mathbb{C}\backslash[-1,1]$
- Along the line segment $[-1,1]$, we have

$$\Xi_+(z) = \Xi_-(z) S_D^{-1}, \quad S_D = (1-\gamma)^{-\sigma_3}$$

- At infinity, $\Xi(z) = I + O\big(z^{-1}\big), z \to \infty$

This diagonal problem can always be solved (compare section 3.2) and we obtain

$$\Xi(z) = \left(\frac{z+1}{z-1}\right)^{-\nu\sigma_3}, \quad \nu = \frac{1}{2\pi i}\ln(1-\gamma)$$

where ν was introduced and its branch fixed in (3.7). We also note that ν is purely imaginary if and only if $\gamma < 1$.

In the construction of an origin parametrix, we won't use the Hastings-Mcleod solution in the given situation. Instead, start with

$$P_{II}(\zeta) = \Psi_1(\zeta), \quad \arg\zeta \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$$

as the first canonical solution of (1.10), where $u = u(x,\gamma)$ is chosen from the Ablowitz-Segur family of solutions to the second Painlevé equation, that is u solves the boundary value problem

$$u_{xx} = xu + 2u^3, \quad u(x) \sim \gamma \mathrm{Ai}(x), \quad x \to +\infty, \;\; \gamma \neq 1$$

Remark 5 *Recall that* $u = u(x, \gamma)$, *might have poles on the real line. However from our discussion in section 1.3 we know that* u *is pole free on the entire real line in case* $\gamma < 1$ *and for* $\gamma > 1$ *we restrict ourselves to values of* x *chosen from the set* (1.29). *Thus in either case,* $u = u(x, \gamma)$ *is smooth in* x *and therefore also* $P_{II}(\zeta) \equiv P_{II}(\zeta; x)$.

Now, opposed to (3.11), define

$$\widehat{P}_{II}^{RH}(\zeta) = \begin{cases} P_{II}(\zeta), & \arg\zeta \in (-\frac{\pi}{6}, \frac{\pi}{6}) \cup (\frac{5\pi}{6}, \frac{7\pi}{6}), \\ P_{II}(\zeta)M_1, & \arg\zeta \in (\frac{\pi}{6}, \frac{5\pi}{6}), \\ P_{II}(\zeta)M_2, & \arg\zeta \in (\frac{7\pi}{6}, \frac{11\pi}{6}), \end{cases} \tag{6.6}$$

with

$$M_1 = \begin{pmatrix} 1 & 0 \\ -i\gamma & 1 \end{pmatrix}, \quad M_2 = \sigma_2 M_1 \sigma_2 = \begin{pmatrix} 1 & i\gamma \\ 0 & 1 \end{pmatrix}.$$

One checks directly that $\widehat{P}_{II}^{RH}(\zeta)$ solves the model RHP shown in Figure 6.2

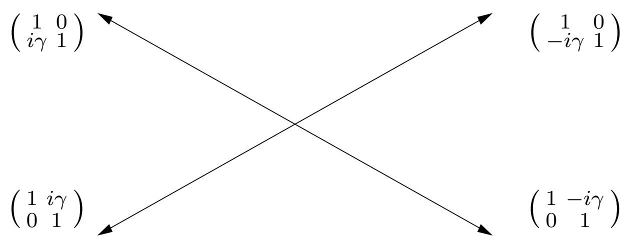

Figure 6.2. The model RHP near $z = 0$ which can be solved explicitly using the real-valued Ablowitz-Segur solution of the second Painlevé equation

- $\widehat{P}_{II}^{RH}(\zeta)$ is analytic for $\zeta \in \mathbb{C} \setminus \{\arg\zeta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}\}$.

- On the infinite rays, the following jumps hold

$$\begin{aligned}
\big(\widehat{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\widehat{P}_{II}^{RH}(\zeta)\big)_- M_1, && \arg\zeta = \frac{\pi}{6}\\
\big(\widehat{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\widehat{P}_{II}^{RH}(\zeta)\big)_- M_1^{-1}, && \arg\zeta = \frac{5\pi}{6}\\
\big(\widehat{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\widehat{P}_{II}^{RH}(\zeta)\big)_- M_2, && \arg\zeta = \frac{7\pi}{6}\\
\big(\widehat{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\widehat{P}_{II}^{RH}(\zeta)\big)_- M_2^{-1}, && \arg\zeta = \frac{11\pi}{6}
\end{aligned}$$

- In (6.6) we chose a specific Stokes phenomenon described by the following Stokes matrices

$$S_1 = M_1, S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, S_4 = M_2, S_3 = \bar{S}_1, S_5 = \bar{S}_2, S_6 = \bar{S}_4. \tag{6.7}$$

  This leads now [29] to the following uniform asymptotics, valid in a full neighborhood of infinity

$$\widehat{P}_{II}^{RH}(\zeta) \sim \Big(I + \frac{m_1}{\zeta} + \frac{m_2}{\zeta^2} + \frac{m_3}{\zeta^3} + O\big(\zeta^{-4}\big)\Big) e^{-i(\frac{4}{3}\zeta^3 + x\zeta)\sigma_3}, \quad \zeta \to \infty \tag{6.8}$$

  with

$$m_1 = \frac{1}{2}\begin{pmatrix} -iv & u \\ u & iv \end{pmatrix}, \quad m_2 = \frac{1}{8}\begin{pmatrix} u^2 - v^2 & 2i(u_x + uv) \\ -2i(u_x + uv) & u^2 - v^2 \end{pmatrix},$$

  and

$$m_3 = \frac{1}{48}\begin{pmatrix} i(v^3 - 3vu^2 + 2(xv - uu_x)) & -3(u(u^2+v^2) + 2(vu_x + xu)) \\ -3(u(u^2+v^2) + 2(vu_x + xu)) & -i(v^3 - 3vu^2 + 2(xv - uu_x)) \end{pmatrix}.$$

  where we use the abbreviation $v = (u_x)^2 - xu^2 - u^4$.

The model function (6.6) will now be used to construct the parametrix to the solution of the original Δ-RHP in a neighborhood of $z = 0$. First set

$$\mathcal{P}_{II}^{RH}(\zeta) = \begin{cases} e^{\pi i\nu\sigma_3}\widehat{P}_{II}^{RH}(\zeta)e^{-\pi i\nu\sigma_3}, & \mathrm{Im}\,\zeta > 0; \\ e^{\pi i\nu\sigma_3}\widehat{P}_{II}^{RH}(\zeta)e^{\pi i\nu\sigma_3}, & \mathrm{Im}\,\zeta < 0; \end{cases}$$

leading to a model RHP with jumps on the positive oriented real line

$$\big(\mathcal{P}_{II}^{RH}(\zeta)\big)_+ = \big(\mathcal{P}_{II}^{RH}(\zeta)\big)_-(1-\gamma)^{-\sigma_3}$$

as well as on the infinite rays $\arg\zeta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$$\begin{aligned}
\big(\mathcal{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\mathcal{P}_{II}^{RH}(\zeta)\big)_- \begin{pmatrix} 1 & 0 \\ -i\gamma(1-\gamma)^{-1} & 0 \end{pmatrix}, \quad \arg\zeta = \frac{\pi}{6} \\
\big(\mathcal{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\mathcal{P}_{II}^{RH}(\zeta)\big)_- \begin{pmatrix} 1 & 0 \\ i\gamma(1-\gamma)^{-1} & 1 \end{pmatrix}, \quad \arg\zeta = \frac{5\pi}{6} \\
\big(\mathcal{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\mathcal{P}_{II}^{RH}(\zeta)\big)_- \begin{pmatrix} 1 & i\gamma(1-\gamma)^{-1} \\ 0 & 1 \end{pmatrix}, \quad \arg\zeta = \frac{7\pi}{6} \\
\big(\mathcal{P}_{II}^{RH}(\zeta)\big)_+ &= \big(\mathcal{P}_{II}^{RH}(\zeta)\big)_- \begin{pmatrix} 1 & -i\gamma(1-\gamma)^{-1} \\ 0 & 1 \end{pmatrix}, \quad \arg\zeta = \frac{11\pi}{6}
\end{aligned}$$

and with behavior at infinity

$$\mathcal{P}_{II}^{RH}(\zeta) = \Big(I + \frac{\widetilde{m}_1}{\zeta} + \frac{\widetilde{m}_2}{\zeta^2} + \frac{\widetilde{m}_3}{\zeta^3} + O\big(\zeta^{-4}\big)\Big)e^{-i(\frac{4}{3}\zeta^3+x\zeta)\sigma_3} \begin{cases} I, & \operatorname{Im}\zeta > 0; \\ e^{2\pi i\nu\sigma_3}, & \operatorname{Im}\zeta < 0; \end{cases}$$

where

$$\widetilde{m}_1 = \frac{1}{2}\begin{pmatrix} -iv & ue^{2\pi i\nu} \\ ue^{-2\pi i\nu} & iv \end{pmatrix}, \quad \widetilde{m}_2 = \frac{1}{8}\begin{pmatrix} u^2-v^2 & 2i(u_x+uv)e^{2\pi i\nu} \\ -2i(u_x+uv)e^{-2\pi i\nu} & u^2-v^2 \end{pmatrix}.$$

and

$$\widetilde{m}_3 = \frac{1}{48}\begin{pmatrix} i(v^3-3vu^2+2(xv-uu_x)) & -3(u(u^2+v^2)+2(vu_x+xu))e^{2\pi i\nu} \\ -3(u(u^2+v^2)+2(vu_x+xu))e^{-2\pi i\nu} & -i(v^3-3vu^2+2(xv-uu_x)) \end{pmatrix}.$$

We use the same change of variables as in (3.14)

$$\zeta(z) = sz, \quad |z| < r$$

but a slightly different form for the parametrix. Instead of (3.15), define

$$\mathcal{U}(z) = \sigma_1 B_0(z) e^{-i\frac{\pi}{4}\sigma_3}\mathcal{P}_{II}^{RH}\big(\zeta(z)\big)e^{i(\frac{4}{3}\zeta(z)+x\zeta(z))\sigma_3}e^{i\frac{\pi}{4}\sigma_3}\sigma_1, \quad |z| < r \qquad (6.9)$$

with (compare (3.16))

$$B_0(z) = \left(\frac{z+1}{z-1}\right)^{\nu\sigma_3} \begin{cases} I, & \operatorname{Im} z > 0, \\ e^{-2\pi i\nu\sigma_3}, & \operatorname{Im} z < 0, \end{cases} \qquad B_0(0) = e^{-\pi i\nu\sigma_3}.$$

By construction, the parametrix $\mathcal{U}(z)$ has jumps along the curves depicted in Figure 6.3 and these jumps are described by the same matrices as in the original Δ-RHP. Thus, the ratio of $\Delta(z)$ with $\mathcal{U}(z)$ is locally analytic, i.e.

$$\Delta(z) = N_0(z)\mathcal{U}(z), \qquad |z| < r < \frac{1}{2}.$$

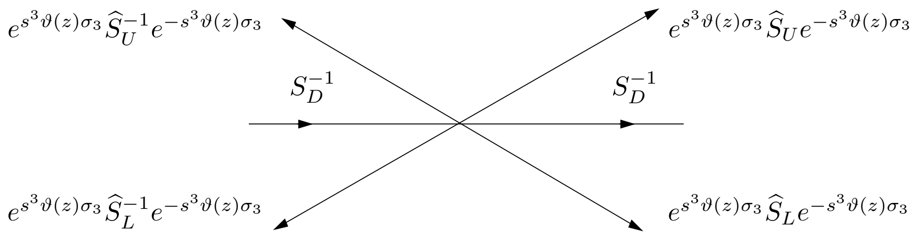

Figure 6.3. Jump graph of the parametrix $\mathcal{U}(z)$

The role of the multiplier $B_0(z)$ follows also here from the asymptotic matching condition

$$\begin{aligned}
\mathcal{U}(z) &= \sigma_1 B_0(z) e^{-i\frac{\pi}{4}\sigma_3}\Big(I + \frac{\widetilde{m}_1}{\zeta} + \frac{\widetilde{m}_2}{\zeta^2} + \frac{\widetilde{m}_3}{\zeta^3} + O\big(\zeta^{-4}\big)\Big) e^{i\frac{\pi}{4}\sigma_3} B_0^{-1}(z)\sigma_1 \Xi(z) \\
&= \Bigg[I + \frac{i}{2\zeta} B_0(z)^{-1} \begin{pmatrix} v & u e^{-2\pi i\nu} \\ -u e^{2\pi i\nu} & -v \end{pmatrix} B_0(z) \\
&\quad + \frac{1}{8\zeta^2} B_0(z)^{-1} \begin{pmatrix} u^2 - v^2 & 2(u_x + uv)e^{-2\pi i\nu} \\ 2(u_x + uv)e^{2\pi i\nu} & u^2 - v^2 \end{pmatrix} B_0(z) + \frac{i}{48\zeta^3} B_0(z)^{-1} \\
&\quad \times \begin{pmatrix} -(v^3 - 3vu^2 + 2(xv - uu_x)) & -3(u(u^2+v^2) + 2(vu_x + xu))e^{-2\pi i\nu} \\ 3(u(u^2+v^2) + 2(vu_x + xu))e^{2\pi i\nu} & v^3 - 3vu^2 + 2(xv - uu_x) \end{pmatrix} \\
&\quad \times B_0(z) + O\big(\zeta^{-3}\big) \Bigg] \Xi(z) \qquad (6.10)
\end{aligned}$$

as $s \to \infty$ and $0 < r_1 \leq |z| \leq r_2 < 1$ (so $|\zeta| \to \infty$). Since the function $\zeta(z)$ is of order $O(s)$ on the latter annulus and $B_0(z)$ is bounded, equation (6.10) yields the desired matching relation between the model functions $\mathcal{U}(z)$ and $\Xi(z)$

$$\mathcal{U}(z) = \big(I + o(1)\big)\Xi(z), \quad s \to \infty, \ \ 0 < r_1 \leq |z| \leq r_2 < 1.$$

The parametrices for the endpoints $z = \pm 1$ are almost identical to the ones constructed in section 3.2. For the right endpoint we use $P_{CH}(\zeta)$ as introduced in (3.22) and $P_{CH}^{RH}(\zeta)$ as in (3.24). With the same change of variables

$$\zeta(z) = -2is^3\big(\vartheta(z) - \vartheta(1)\big), \quad |z-1| < r$$

define for $|z-1| < r$

$$\mathcal{V}(z) = \sigma_1 e^{-i\frac{\pi}{4}\sigma_3} B_r(z) e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3} e^{-s^3\vartheta(1)\sigma_3} P_{CH}^{RH}\big(\zeta(z)\big) e^{(\frac{i}{2}\zeta(z)+s^3\vartheta(1))\sigma_3} e^{i\frac{\pi}{4}\sigma_3}\sigma_1, \quad (6.11)$$

where (compare (3.27))

$$B_r(z) = \left(\zeta(z)\frac{z+1}{z-1}\right)^{\nu\sigma_3}, \quad B_r(1) = \big(16s^3 + 4xs\big)^{\nu\sigma_3}.$$

The latter model function solves the RHP depicted in Figure 6.4 below and we expect

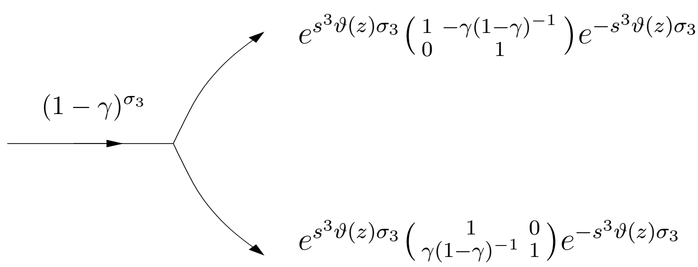

Figure 6.4. Transformation of parametrix jumps to original jumps

the singular endpoint behavior to match (6.5) (see section 4.1 or section 7.1)

$$\mathcal{V}(z) = O\big(\ln(z-1)\big), \quad z \to +1.$$

Hence the ratio of $\Delta(z)$ with $\mathcal{V}(z)$ is locally analytic, i.e.

$$\Delta(z) = N_r(z)\mathcal{V}(z), \quad |z-1| < \frac{1}{2}$$

and we also have an asymptotical matchup

$$\begin{aligned}\mathcal{V}(z) &= \sigma_1 e^{-i\frac{\pi}{4}\sigma_3} B_r(z) e^{i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3} e^{-s^3\vartheta(1)\sigma_3}\Bigg[I+\frac{i}{\zeta}\begin{pmatrix}\nu^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{\pi i\nu}\\ \frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{-\pi i\nu} & -\nu^2\end{pmatrix}\\ &\quad+\frac{1}{\zeta^2}\begin{pmatrix}-\frac{\nu^2}{2}(1+\nu)^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2e^{\pi i\nu}\\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2e^{-\pi i\nu} & -\frac{\nu^2}{2}(1-\nu)^2\end{pmatrix}+O\big(\zeta^{-3}\big)\Bigg]\\ &\quad\times\zeta^{-\nu\sigma_3}e^{s^3\vartheta(1)\sigma_3}e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{i\frac{\pi}{4}\sigma_3}\sigma_1\\ &= \Bigg[I+\frac{i}{\zeta}\begin{pmatrix}-\nu^2 & \frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{2s^3\vartheta(1)}\beta_r^{-2}(z)\\ -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\beta_r^2(z) & \nu^2\end{pmatrix}\\ &\quad+\frac{1}{\zeta^2}\begin{pmatrix}-\frac{\nu^2}{2}(1-\nu)^2 & -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2e^{2s^3\vartheta(1)}\beta_r^{-2}(z)\\ -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2e^{-2s^3\vartheta(1)}\beta_r^2(z) & -\frac{\nu^2}{2}(1+\nu)^2\end{pmatrix}\\ &\quad+O\big(\zeta^{-3}\big)\Bigg]\Xi(z) \qquad (6.12)\end{aligned}$$

as $s\to\infty$, valid on the annulus $0<r_1\leq|z-1|\leq r_2<1$ (hence $|\zeta|\to\infty$) with the abbreviation

$$\beta_r(z)=\left(\zeta(z)\frac{z+1}{z-1}\right)^{\nu}.$$

Also here, similar to section 3.3, the estimate (6.12), yields for $\gamma<1$

$$\mathcal{V}(z)=\big(I+o(1)\big)\Xi(z),\quad s\to\infty,\ 0<r_1\leq|z-1|\leq r_2<1$$

but for $\gamma>1$ this needs to be replaced by

$$\mathcal{V}(z)=\sigma_1e^{-i\frac{\pi}{4}\sigma_3}E_r(z)e^{i\frac{\pi}{4}\sigma_3}\sigma_1\big(I+o(1)\big)\Xi(z),\quad s\to\infty$$

with

$$\sigma_1e^{-i\frac{\pi}{4}\sigma_3}E_r(z)e^{i\frac{\pi}{4}\sigma_3}\sigma_1=\begin{pmatrix}1 & 0\\ -i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(z)\frac{z+1}{z-1} & 1\end{pmatrix},\quad \hat{\beta}_r(z)=\left(\zeta(z)\frac{z+1}{z-1}\right)^{\nu_0}.$$

Hence, also in the given situation, we will have to account for the nontrivial matrix $E_r(z)$ as long as $\gamma>1$.

For the remaining left endpoint, we use $\widetilde{P}_{CH}(\zeta)$ and $\widetilde{P}_{CH}^{RH}(\zeta)$ as introduced in (3.34) and the change of variables

$$\zeta(z)=-2is^3\big(\vartheta(z)-\vartheta(-1)\big),\quad |z+1|<r.$$

Introduce for $|z+1|<r$

$$\mathcal{W}(z)=\sigma_1 e^{-i\frac{\pi}{4}\sigma_3}B_l(z)e^{-i\frac{\pi}{2}(\frac{1}{2}-\nu)\sigma_3}e^{-s^3\vartheta(-1)\sigma_3}\widetilde{P}_{CH}^{RH}\big(\zeta(z)\big)e^{(\frac{i}{2}\zeta(z)+s^3\vartheta(-1))\sigma_3}e^{i\frac{\pi}{4}\sigma_3}\sigma_1, \tag{6.13}$$

with

$$B_l(z)=\left(e^{-i\pi}\zeta(z)\frac{z-1}{z+1}\right)^{-\nu\sigma_3},\quad B_l(-1)=\big(16s^3+4xs\big)^{-\nu\sigma_3}.$$

Using the stated conjugation with $\sigma_1 e^{-i\frac{\pi}{4}\sigma_3}$, we again match parametrix jumps with original jumps locally on the original jump contour, see Figure 6.5, and the singular

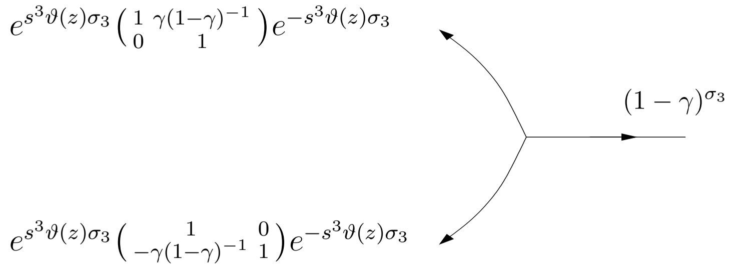

Figure 6.5. Transformation of parametrix jumps to original jumps

endpoint behavior matches (6.5):

$$\mathcal{W}(z)=O\big(\ln(z+1)\big),\quad z\to-1.$$

Thus the ratio of $\Delta(z)$ with $\mathcal{W}(z)$ is locally analytic,

$$\Delta(z)=N_l(z)\mathcal{W}(z),\quad |z+1|<\frac{1}{2}$$

and the model functions $\Delta(z),\Xi(z)$ are related through the following asymptotical matchup

$$\begin{aligned}\mathcal{W}(z) &= \Bigg[I+\frac{i}{\zeta}\begin{pmatrix}-\nu^2 & \frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{2s^3\vartheta(-1)}\beta_l^2(z)\\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}e^{-2s^3\vartheta(-1)}\beta_l^{-2}(z) & \nu^2\end{pmatrix}\\ &\quad+\frac{1}{\zeta^2}\begin{pmatrix}-\frac{\nu^2}{2}(1+\nu)^2 & -\frac{\Gamma(1-\nu)}{\Gamma(\nu)}(1-\nu)^2e^{2s^3\vartheta(-1)}\beta_l^2(z)\\ -\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}(1+\nu)^2e^{-2s^3\vartheta(-1)}\beta_l^{-2}(z) & -\frac{\nu^2}{2}(1-\nu)^2\end{pmatrix}\\ &\quad+O\big(\zeta^{-3}\big)\Bigg]\Xi(z),\end{aligned} \tag{6.14}$$

valid as $s\to\infty$ on the annulus $0<r_1\leq|z+1|\leq r_2<1$ (thus $|\zeta|\to\infty$) and we put

$$\beta_l(z)=\left(e^{-i\pi}\zeta(z)\frac{z-1}{z+1}\right)^{-\nu}.$$

For $\gamma<1$, (6.14) implies on the annulus

$$\mathcal{W}(z)=\big(I+o(1)\big)\Xi(z),\qquad s\to\infty,$$

whereas for $\gamma>1$, it needs to be replaced by

$$\mathcal{W}(z)=\sigma_1e^{-i\frac{\pi}{4}\sigma_3}E_l(z)e^{i\frac{\pi}{4}\sigma_3}\sigma_1\big(I+o(1)\big)\Xi(z),\qquad s\to\infty$$

with

$$\sigma_1e^{-i\frac{\pi}{4}\sigma_3}E_l(z)e^{i\frac{\pi}{4}\sigma_3}\sigma_1=\begin{pmatrix}1 & -i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{2s^3\vartheta(-1)}\hat{\beta}_l^{-2}(z)\frac{z-1}{z+1}\\ 0 & 1\end{pmatrix},$$

where

$$\hat{\beta}_l(z)=\left(e^{-i\pi}\zeta(z)\frac{z-1}{z+1}\right)^{-\nu_0}.$$

We now summarize the model functions $\Xi(z),\mathcal{U}(z),\mathcal{V}(z)$ and $\mathcal{W}(z)$ in order to employ our next transformation.

6.3 The ratio problem – iterative solution for $\gamma<1$

Similarly to (3.40), put

$$\mathcal{R}(z)=\Delta(z)\begin{cases}\big(\mathcal{U}(z)\big)^{-1}, & |z-1|<r_1,\\ \big(\mathcal{V}(z)\big)^{-1}, & |z|<r_2,\\ \big(\mathcal{W}(z)\big)^{-1}, & |z+1|<r_1,\\ \big(\Xi(z)\big)^{-1}, & |z-1|>r_1,|z+1|>r_1,|z|>r_2,\end{cases}$$

with $0<r_1,r_2<\frac{1}{2}$ fixed. This implies a RHP for the ratio-function $\mathcal{R}(z)$ as depicted in Figure 6.6.

- $\mathcal{R}(z)$ is analytic for $z\in\mathbb{C}\backslash\big\{C_{0,r,l}\cup\bigcup_{i=1}^{8}\gamma_i\big\}$

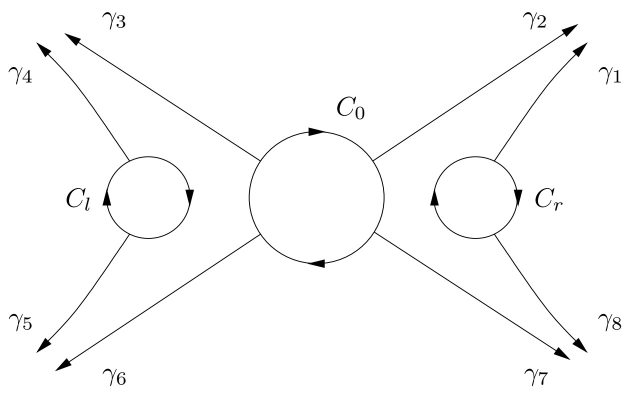

Figure 6.6. The jump graph for the ratio-function $\mathcal{R}(z)$

- For the jumps, along the infinite branches γ_i

$$\mathcal{R}_+(z) = \mathcal{R}_-(z)\Xi(z)e^{s^3\vartheta(z)\sigma_3}\widehat{G}_\Delta(z)e^{-s^3\vartheta(z)\sigma_3}\big(\Xi(z)\big)^{-1}.$$

  On the clockwise oriented circles C_0 and $C_{r,l}$, the jumps are described by the equations

$$\begin{aligned}\mathcal{R}_+(z) &= \mathcal{R}_-(z)\mathcal{U}(z)\big(\Xi(z)\big)^{-1},\ z \in C_0,\\ \mathcal{R}_+(z) &= \mathcal{R}_-(z)\mathcal{V}(z)\big(\Xi(z)\big)^{-1},\ z \in C_r,\\ \mathcal{R}_+(z) &= \mathcal{R}_-(z)\mathcal{W}(z)\big(\Xi(z)\big)^{-1},\ z \in C_l.\end{aligned}$$

- $\mathcal{R}(z)$ is analytic at $z = \pm 1$. This observation will be proven in the same way, as we verified the same statement for the function $R(z)$ in section 4.1

- In a neighborhood of infinity, we have $\mathcal{R}(z) \to I$.

Without recalling all underlying estimates (see section 3.3 for an almost identical situation), the latter ratio-RHP can be solved iteratively in case $\gamma < 1$. Indeed its underlying jump matrix $G_{\mathcal{R}}(z)$ satifies on the contour $\Sigma_{\mathcal{R}}$ as shown in Figure 6.6

$$\|G_{\mathcal{R}} - I\|_{L^2\cap L^\infty(\Sigma_{\mathcal{R}})} \leq cs^{-1}, \quad s \to \infty \tag{6.15}$$

uniformly on any compact subset of the set (1.19)

$$\big\{(\gamma, x) \in \mathbb{R}^2 :\ -\infty < \gamma < 1,\ -\infty < x < \infty\big\}.$$

Since the ratio problem is equivalent to the singular integral equation

$$\mathcal{R}_-(z) = I + \frac{1}{2\pi i}\int\limits_{\Sigma_{\mathcal{R}}} \mathcal{R}_-(w)\big(G_{\mathcal{R}}(w) - I\big)\frac{dw}{w - z_-},$$

we know that for sufficiently large s, the relevant integral operator is contracting and we can solve the latter equation iteratively in $L^2(\Sigma_{\mathcal{R}})$, its unique solution satifies

$$\|\mathcal{R}_- - I\|_{L^2(\Sigma_{\mathcal{R}})} \leq cs^{-1}, \quad s \to \infty. \tag{6.16}$$

Estimations (6.15) and (6.16) allow us to derive the asymptotics of $\det(I - \gamma K_{\text{csin}})$ as stated in Theorem 1.3.1 up to the constant term for $\gamma < 1$. Also, tracing back the transformations

$$\Theta(\lambda) \mapsto \Phi(\lambda) \mapsto \Delta(z) \mapsto \mathcal{R}(z)$$

we obtain existence and boundedness of $\Theta(\lambda), \lambda \in [-s, s]$ and hence existence and boundedness of the resolvent $I + R_{\text{csin}}$ for sufficiently large s which is needed in (5.11).

In case $\gamma > 1$, the jump matrices on C_r and C_l are not uniformly close to the unit matrix as $s \to \infty$

$$\|\mathcal{V}(\Xi)^{-1} - I\|_{L^2\cap L^\infty(C_r)} \nrightarrow 0, \quad \|\mathcal{W}(\Xi)^{-1} - I\|_{L^2\cap L^\infty(C_l)} \nrightarrow 0.$$

Again, we will use the undressing-dressing transformations of section 3.4.

6.4 Undressing and dressing – iterative solution for $\gamma > 1$

We use the notation of section 3.4 and recall that the jump matrices $\mathcal{G}_r(z) = \mathcal{V}(z)\left(\Xi(z)\right)^{-1}$ and $\mathcal{G}_l(z) = \mathcal{W}(z)\big(\Xi(z)\big)^{-1}$ can be written as

$$\mathcal{G}_r(z) = \sigma_1 e^{-i\frac{\pi}{4}\sigma_3} G_r(z) e^{i\frac{\pi}{4}\sigma_3}\sigma_1, \quad \mathcal{G}_l(z) = \sigma_1 e^{-i\frac{\pi}{4}\sigma_3} G_l(z) e^{i\frac{\pi}{4}\sigma_3}\sigma_1. \tag{6.17}$$

Hence the following steps are completely analogous to those described in section 3.4. Put

$$\mathcal{P}(z) = \begin{cases} \mathcal{R}(z)\sigma_1 e^{-i\frac{\pi}{4}\sigma_3} E_r(z) e^{i\frac{\pi}{4}\sigma_3}\sigma_1, & |z - 1| < r_1, \\ \mathcal{R}(z)\sigma_1 e^{-i\frac{\pi}{4}\sigma_3} E_l(z) e^{i\frac{\pi}{4}\sigma_3}\sigma_1, & |z + 1| < r_1, \\ \mathcal{R}(z), & |z \mp 1| > r_1. \end{cases}$$

which leads to a singular RHP posed on the contour depicted in Figure 6.6 with pole singularities at the points $z = \pm 1$. The jump matrix $G_{\mathcal{P}}(z)$ is uniformly close to the unit matrix as $s \to \infty$,

$$\|G_{\mathcal{P}} - I\|_{L^2 \cap L^\infty(\Sigma_{\mathcal{R}})} \leq cs^{-1}, \quad s \to \infty$$

and with $\mathcal{P}(z) = \big(\mathcal{P}^{(1)}(z), \mathcal{P}^{(2)}(z)\big)$ written in terms of its columns, the residue relations (3.51) and (3.52) need to be replaced by

$$\text{res}_{z=+1}\mathcal{P}^{(1)}(z) = \mathcal{P}^{(2)}(1)\bigg(-2i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(1)\bigg) \tag{6.18}$$

$$\text{res}_{z=-1}\mathcal{P}^{(2)}(z) = \mathcal{P}^{(1)}(-1)\bigg(2i\frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{2s^3\vartheta(-1)}\hat{\beta}_l^{-2}(-1)\bigg). \tag{6.19}$$

Similarly to Proposition 3.4.1, we have

Proposition 6.4.1 *The Riemann-Hilbert problem for $\mathcal{P}(z)$ has a unique solution*

Proof The residue relations (6.18),(6.19) imply

$$\mathcal{P}(z) = \begin{cases} \hat{\mathcal{P}}^{(+)}(z)\begin{pmatrix} 1 & 0 \\ -\frac{2ip}{z-1} & 1 \end{pmatrix}, & |z-1| < r; \\ \hat{\mathcal{P}}^{(-)}(z)\begin{pmatrix} 1 & \frac{2ip}{z+1} \\ 0 & 1 \end{pmatrix}, & |z+1| < r, \end{cases} \qquad p = \frac{\Gamma(1-\nu)}{\Gamma(\nu)}e^{-2s^3\vartheta(1)}\hat{\beta}_r^2(1)$$

where $\mathcal{P}^{(\pm)}(z)$ are analytic at $z = \pm 1$. Using the same arguments as in Proposition 3.4.1, we establish uniqueness. ■

Our last transformation reduces the $\mathcal{P}$-RHP to one without pole singularities. Introduce

$$\mathcal{P}(z) = \Big(zI + \widehat{B}\Big)\,\mathcal{Q}(z)\begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix}$$

where $\widehat{B} \in \mathbb{C}^{2\times 2}$ is constant and obtain the following RHP

- $\mathcal{Q}(z)$ is analytic for $z \in \mathbb{C}\backslash\big\{C_{0,r,l} \cup \bigcup_{i=1}^{8}\gamma_i\big\}$

- $\mathcal{Q}_+(z) = \mathcal{Q}_-(z) G_{\mathcal{Q}}(z)$, where

$$G_{\mathcal{Q}}(z) = \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix} \sigma_1 e^{-i\frac{\pi}{4}\sigma_3} \widehat{G}_{r,l}(z) e^{i\frac{\pi}{4}\sigma_3} \sigma_1 \begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix}, \qquad z \in C_{r,l}$$

and

$$G_Q(z) = \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix} \mathcal{U}(z) \big(\Xi(z)\big)^{-1} \begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix}, \qquad z \in C_0$$

as well as on the infinite branches γ_i

$$G_{\mathcal{Q}}(z) = \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix} \Xi(z) e^{s^3\vartheta(z)\sigma_3} \widehat{G}_{\Delta}(z) e^{-s^3\vartheta(z)\sigma_3} \big(\Xi(z)\big)^{-1} \begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix}.$$

- $\mathcal{Q}(z) \to I$, as $z \to \infty$

As we already found out, the $\mathcal{Q}$-jump matrix $G_{\mathcal{Q}}(z)$ is uniformly close to the unit matrix and therefore the $\mathcal{Q}$-RHP can be solved iteratively. We will do that once we have determined the unknown matrix $\widehat{B}$. Using the conditions (6.18) and (6.19), we have

$$\widehat{B} = \left(Q(1) \begin{pmatrix} 1 \\ ip \end{pmatrix}, Q(-1) \begin{pmatrix} ip \\ 1 \end{pmatrix} \right) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \left(Q(1) \begin{pmatrix} 1 \\ ip \end{pmatrix}, Q(-1) \begin{pmatrix} ip \\ 1 \end{pmatrix} \right)^{-1}. \tag{6.20}$$

Now we check for which values of s the latter matrix inverse is well-defined. Since

$$\|G_{\mathcal{Q}} - I\|_{L^2\cap L^\infty(\Sigma_{\mathcal{R}})} \leq cs^{-1}, \qquad s \to \infty$$

we can solve the singular integral equation

$$\mathcal{Q}_-(z) = I + \frac{1}{2\pi i} \int\limits_{\Sigma_{\mathcal{R}}} \mathcal{Q}_-(w) \big(G_{\mathcal{Q}}(w) - I\big) \frac{dw}{w - z_-}$$

iteratively in $L^2(\Sigma_{\mathcal{R}})$, its unique solution satisfies

$$\|\mathcal{Q}_- - I\|_{L^2(\Sigma_{\mathcal{R}})} \leq cs^{-1}, \qquad s \to \infty.$$

Since this implies

$$Q(\pm 1) = I + O\big(s^{-1}\big), \quad s \to \infty,$$

we see that the matrix inverse in (6.20) exists for all sufficiently large s lying outside of the zero set of the function

$$1 + p^2$$

and these are precisely all points $\{s_n\}$ defined by the equation

$$\cos\sigma(s_n, x, \gamma) = 0, \quad n = 1, 2, ...$$

where $\sigma = \sigma(s, x, \gamma)$ was already introduced in (4.8). As we did it in section 3.4 we will henceforth, when dealing with the case $\gamma > 1$, stay away from the small neigbhorhood of the points s_n. Let us now move on to the asymptotic resolution of the Φ-RHP in case $\gamma = 1$.

6.5 Rescaling and g-function transformation, $\gamma = 1$

Let us go back to (5.16) and notice that in the given situation the jump contour of the Φ-RHP consists only of the line segment $[-s, s]$ oriented from left to right with

$$\Phi_+(\lambda) = \Phi_-(\lambda)\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \lambda \in [-s, s].$$

This jump equals the jump one faces during the asymptotical analysis of the sine kernel determinant (cf. [22]). Here and there we use a g-function together with the scaling $z = \frac{\lambda}{s}$. Introduce

$$\hat{g}(z) = \frac{4i}{3}\sqrt{z^2-1}\left(z^2 t + \frac{t}{2} + \frac{3x}{4s^2}\right), \quad \sqrt{z^2-1} \sim z, \quad z \to \infty \tag{6.21}$$

being analytic outside the segment $[-1, 1]$ and as $z \to \infty$

$$\hat{g}(z) = i\left(\frac{4}{3}tz^3 + \frac{xz}{s^2}\right) + O\big(z^{-1}\big).$$

In the situation $t = 1$, (6.21) reduces to the previously used g-function (3.60), whereas for $t = 0$, we obtain the g-function used in the analysis of the sine kernel (see [22]). We put

$$\Lambda(z) = \Phi(zs)e^{-s^3\hat{g}(z)\sigma_3}, \quad z \in \mathbb{C}\backslash[-1,1] \tag{6.22}$$

and are lead to the following RHP

- $\Lambda(z)$ is analytic for $z \in \mathbb{C}\backslash[-1,1]$
- The following jump holds

$$\Lambda_+(z) = \Lambda_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 2e^{2s^3\hat{g}_+(z)} \end{pmatrix}, \quad z \in (-1,1)$$

  since

$$\hat{g}_+(z) + \hat{g}_-(z) = 0, z \in [-1,1].$$

- $\Lambda(z)$ has at most logarithmic singularities at the endpoints $z = \pm 1$
- As $z \to \infty$, $\Lambda(z) = I + O\big(z^{-1}\big)$.

Since Im $\sqrt{z^2-1}_+ > 0$ for $z \in (-1,1)$, we have Re $\hat{g}_+(z) < 0$ for $z \in (-1,1)$ showing that

$$\begin{pmatrix} 0 & 1 \\ -1 & 2e^{2s^3\hat{g}_+(z)} \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s \to \infty, \quad z \in (-1,1)$$

exponentially fast. Thus, also here, we expect, that as $s \to \infty$, $\Lambda(z)$ converges to a solution of a model RHP posed on the line segment $[-1,1]$.

6.6 The model RHP and parametrices for $\gamma = 1$

The model problem on $[-1,1]$ consists in finding the piecewise analytic 2×2 matrix valued function $\mathcal{N}(z)$ such that

- $\mathcal{N}(z)$ is analytic for $z \in [-1,1]$

- On the line segment $[-1,1]$ the following jump holds

$$\mathcal{N}_+(z) = \mathcal{N}_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad z \in [-1,1]$$

- $\mathcal{N}(z)$ has at most logarithmic singularities at the endpoints $z = \pm 1$

- $\mathcal{N}(z) = I + O\big(z^{-1}\big),\ z \to \infty$

This problem has an explicit solution

$$\mathcal{N}(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \beta(z)^{-\sigma_3}\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}, \qquad \beta(z) = \left(\frac{z+1}{z-1}\right)^{1/4} \tag{6.23}$$

and $\big(\frac{z+1}{z-1}\big)^{1/4}$ is defined on $\mathbb{C}\backslash[-1,1]$ with its branch fixed by the condition $\big(\frac{z+1}{z-1}\big)^{1/4} \to 1$ as $z \to \infty$, compare section 3.5.

The construction of endpoint parametrices is very similar to the constructions given in section 3.6. We use again Bessel functions. First for the right endpoint $z = +1$, define on the punctured plane $\zeta \in \mathbb{C}\backslash\{0\}$

$$Q_{BE}^{RH}(\zeta) = \begin{pmatrix} \sqrt{\zeta}\big(H_0^{(1)}\big)'(\sqrt{\zeta}) & \sqrt{\zeta}\big(H_0^{(2)}\big)'(\sqrt{\zeta}) \\ H_0^{(1)}(\zeta) & H_0^{(2)}(\zeta) \end{pmatrix}, \qquad -\pi < \arg\zeta \le \pi. \tag{6.24}$$

Since $Q_{BE}^{RH}(\zeta) = \sigma_1 e^{-i\frac{\pi}{4}\sigma_3}P_{BE}(\zeta)e^{i\frac{\pi}{4}\sigma_3}\sigma_1$ we can use (3.70) and deduce

$$\begin{aligned} Q_{BE}^{RH}(\zeta) \;=\; & \sqrt{\frac{2}{\pi}}\zeta^{\sigma_3/4}e^{i\frac{\pi}{4}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\left[I + \frac{i}{8\sqrt{\zeta}}\begin{pmatrix} 1 & 2i \\ 2i & -1 \end{pmatrix} + \frac{3}{128\zeta}\begin{pmatrix} 1 & -4i \\ 4i & 1 \end{pmatrix}\right. \\ & \left. + \frac{15i}{1024\zeta^{3/2}}\begin{pmatrix} -1 & -6i \\ -6i & 1 \end{pmatrix} + O\big(\zeta^{-2}\big)\right]e^{i\sqrt{\zeta}\sigma_3} \end{aligned} \tag{6.25}$$

as $\zeta \to \infty$, valid in a full neighborhood of infinity. Also on the line $\arg\zeta = \pi$ we obtain

$$\big(Q_{BE}^{RH}(\zeta)\big)_+ = \big(Q_{BE}^{RH}(\zeta)\big)_-\begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

thus (6.24) solves the RHP depicted in Figure 6.7. We use the model function $Q_{BE}^{RH}(\zeta)$

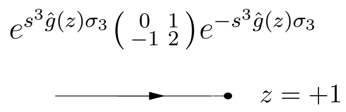

Figure 6.7. The model RHP near $z = +1$ which can be solved in terms of Hankel functions

in the construction of the parametrix to the solution of the original Φ-RHP in a neighborhood of $z = +1$. First (compare (3.71))

$$\zeta(z) = -s^6\hat{g}^2(z), \quad |z-1| < r, \quad -\pi < \arg\zeta \leq \pi \tag{6.26}$$

with

$$\sqrt{\zeta(z)} = -is^3\hat{g}(z) = \frac{4s^3}{3}\sqrt{z^2-1}\left(z^2t + \frac{t}{2} + \frac{3x}{4s^2}\right)$$

which gives a locally conformal change of variables

$$\zeta(z) = \frac{32s^6}{9}\left(\frac{3t}{2} + \frac{3x}{4s^2}\right)^2(z-1)\big(1 + O(z-1)\big), \quad |z-1| < r.$$

Secondly define the right parametrix $\mathcal{I}(z)$ near $z = +1$ by the formula

$$\mathcal{I}(z) = \mathcal{C}_r(z)\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\sqrt{\frac{\pi}{2}}e^{-i\frac{\pi}{4}}Q_{BE}^{RH}\big(\zeta(z)\big)e^{-s^3\hat{g}(z)\sigma_3}, \quad |z-1| < r \tag{6.27}$$

with $\zeta(z)$ as in (6.26) and

$$\mathcal{C}_r(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\left(\zeta(z)\frac{z+1}{z-1}\right)^{-\sigma_3/4}, \quad \mathcal{C}_r(1) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\left(\frac{8s^3}{3}\left(\frac{3t}{2} + \frac{3x}{4s^2}\right)\right)^{-\sigma_3/2}.$$

As a result of our construction the parametrix has jumps only on the line segment depicted in Figure 6.7, described by the same jump matrix as in the original Φ-RHP. Also, since $Q_{BE}^{RH}(\zeta) = \sigma_1 e^{-i\frac{\pi}{4}\sigma_3}P_{BE}(\zeta)e^{i\frac{\pi}{4}\sigma_3}\sigma_1$, the singular endpoint behavior matches. Therefore the ratio of $\Lambda(z)$ with $\mathcal{I}(z)$ is locally analytic, i.e.

$$\Lambda(z) = M_r(z)\mathcal{I}(z), \quad |z-1| < r < \frac{1}{2}$$

and moreover from (6.25)

$$\begin{aligned}\mathcal{I}(z) &= \begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\beta(z)^{-\sigma_3}\frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\Bigg[I+\frac{i}{8\sqrt{\zeta}}\begin{pmatrix}1 & 2i\\ 2i & -1\end{pmatrix}+\frac{3}{128\zeta}\begin{pmatrix}1 & -4i\\ 4i & 1\end{pmatrix}\\ &\quad +\frac{15i}{1024\zeta^{3/2}}\begin{pmatrix}-1 & -6i\\ -6i & 1\end{pmatrix}+O\big(\zeta^{-2}\big)\Bigg]\frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\beta(z)^{\sigma_3}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\mathcal{N}(z)\\ &= \Bigg[I+\frac{i}{16\sqrt{\zeta}}\begin{pmatrix}3\beta^{-2}-\beta^2 & i(3\beta^{-2}+\beta^2)\\ i(3\beta^{-2}+\beta^2) & -(3\beta^{-2}-\beta^2)\end{pmatrix}+\frac{3}{128\zeta}\begin{pmatrix}1 & -4i\\ 4i & 1\end{pmatrix}\\ &\quad +\frac{15i}{2048\zeta^{3/2}}\begin{pmatrix}5\beta^2-7\beta^{-2} & -i(5\beta^2+7\beta^{-2})\\ -i(5\beta^2+7\beta^{-2}) & -(5\beta^2-7\beta^{-2})\end{pmatrix}+O\big(\zeta^{-2}\big)\Bigg]\mathcal{N}(z) \qquad (6.28)\end{aligned}$$

as $s\to\infty$ and $0<r_1\leq|z-1|\leq r_2<1$ (so $|\zeta|\to\infty$). It is very important that the function $\zeta(z)$ is or order $O\big(s^2\big)$ on the latter annulus for all $t\in[0,1]$. Hence, since $\beta(z)$ is bounded, equation (6.28) yields the desired matching relation between the model functions $\mathcal{I}(z)$ and $\mathcal{N}(z)$,

$$\mathcal{I}(z)=\big(I+o(1)\big)\mathcal{N}(z),\qquad s\to\infty,\ 0<r_1\leq|z-1|\leq r_2<1$$

uniformly on any compact subset of the set

$$\big\{(t,x)\in\mathbb{R}^2:\ 0\leq t\leq 1, -\infty<x<\infty\big\}. \qquad (6.29)$$

For the left endpoint $z=-1$ define for $\zeta\in\mathbb{C}\backslash\{0\}$ with $0<\arg\zeta\leq 2\pi$

$$\widetilde{Q}^{RH}_{BE}(\zeta)=\begin{pmatrix}H_0^{(2)}(e^{-i\frac{\pi}{2}}\sqrt{\zeta}) & H_0^{(1)}(e^{-i\frac{\pi}{2}}\sqrt{\zeta})\\ -e^{-i\frac{\pi}{2}}\sqrt{\zeta}\big(H_0^{(2)}\big)'(e^{-i\frac{\pi}{2}}\sqrt{\zeta}) & e^{-i\frac{3\pi}{2}}\sqrt{\zeta}\big(H_0^{(1)}\big)'(e^{-i\frac{\pi}{2}}\sqrt{\zeta})\end{pmatrix}, \qquad (6.30)$$

hence, since $\widetilde{Q}^{RH}_{BE}(\zeta)=\sigma_1 e^{-i\frac{\pi}{4}\sigma_3}\widetilde{P}_{BE}(\zeta)e^{i\frac{\pi}{4}\sigma_3}\sigma_1$ we obtain from (3.78)

$$\begin{aligned}\widetilde{Q}^{RH}_{BE}(\zeta) &= \sqrt{\frac{2}{\pi}}\zeta^{-\sigma_3/4}\begin{pmatrix}i & 1\\ i & -1\end{pmatrix}\Bigg[I+\frac{1}{8\sqrt{\zeta}}\begin{pmatrix}1 & -2i\\ -2i & -1\end{pmatrix}+\frac{3}{128\zeta}\begin{pmatrix}-1 & -4i\\ 4i & -1\end{pmatrix}\\ &\quad +\frac{15}{1024\zeta^{3/2}}\begin{pmatrix}1 & -6i\\ -6i & -1\end{pmatrix}+O\big(\zeta^{-2}\big)\Bigg]e^{-\sqrt{\zeta}\sigma_3} \qquad (6.31)\end{aligned}$$

as $\zeta \to \infty$ and on the line $\arg \zeta = 2\pi$

$$\left(\widetilde{Q}_{BE}^{RH}(\zeta)\right)_+ = \left(\widetilde{Q}_{BE}^{RH}(\zeta)\right)_- \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$$

which shows that (6.30) solves the model problem of Figure 6.8.

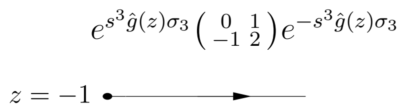

Figure 6.8. The model RHP near $z = -1$ which can be solved in terms of Hankel functions

This model problem enables us to introduce the parametrix $\mathcal{J}(z)$ in a neighborhood of $z = -1$. Define

$$\zeta(z) = s^6 \hat{g}^2(z), \qquad |z+1| < r, \quad 0 < \arg \zeta \leq 2\pi \tag{6.32}$$

with

$$\sqrt{\zeta(z)} = -s^3\hat{g}(z) = -\frac{4is^3}{3}\sqrt{z^2-1}\left(z^2 t + \frac{t}{2} + \frac{3x}{4s^2}\right),$$

a locally conformal change of variables

$$\zeta(z) = \frac{32 s^6}{9}\left(\frac{3t}{2} + \frac{3x}{4s^2}\right)^2 (z+1)\big(1 + O(z+1)\big), \qquad |z+1| < r.$$

Given the left parametrix $\mathcal{J}(z)$ near $z = -1$ by the formula

$$\mathcal{J}(z) = \mathcal{C}_l(z)\left(-\frac{i}{2}\right)\sqrt{\frac{\pi}{2}}\widetilde{Q}_{BE}^{RH}\big(\zeta(z)\big)e^{-s^3\hat{g}(z)\sigma_3}, \qquad |z+1| < r \tag{6.33}$$

where

$$\mathcal{C}_l(z) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\left(\zeta(z)\frac{z-1}{z+1}\right)^{\sigma_3/4}, \quad C_l(-1) = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\left(\frac{8is^3}{3}\left(\frac{3t}{2} + \frac{3x}{4s^2}\right)\right)^{\sigma_3/2}$$

and $\zeta = \zeta(z)$ as in (6.32), we see that the model jump matches the jump in the original S-RHP and by the symmetry relation $\widetilde{Q}_{BE}^{RH}(\zeta) = \sigma_1 e^{-i\frac{\pi}{4}\sigma_3}\widetilde{P}_{BE}(\zeta)e^{i\frac{\pi}{4}\sigma_3}\sigma_1$ also

$$\mathcal{J}(z) = O\big(\ln(z+1)\big), \quad z \to -1.$$

Hence the ratio of $\Lambda(z)$ with $\mathcal{J}(z)$ is locally analytic

$$\Lambda(z) = M_l(z)\mathcal{J}(z), \quad |z+1| < r < \frac{1}{2}$$

and via (6.31)

$$\begin{aligned}\mathcal{J}(z) \;=\; & \Bigg[I + \frac{1}{16\sqrt{\zeta}}\begin{pmatrix} 3\beta^2 - \beta^{-2} & -i(3\beta^2+\beta^{-2}) \\ -i(3\beta^2+\beta^{-2}) & -(3\beta^2-\beta^{-2})\end{pmatrix} + \frac{3}{128\zeta}\begin{pmatrix} -1 & -4i \\ 4i & -1\end{pmatrix} \\ & + \frac{15}{2048\zeta^{3/2}}\begin{pmatrix} 7\beta^2 - 5\beta^{-2} & -i(7\beta^2+5\beta^{-2}) \\ -i(7\beta^2+5\beta^{-2}) & -(7\beta^2-5\beta^{-2})\end{pmatrix} + O\big(\zeta^{-2}\big)\Bigg]\mathcal{N}(z)\end{aligned}$$

as $s \to \infty$ and $0 < r_1 \leq |z+1| \leq r_2 < 1$ (hence $|\zeta| \to \infty$). Also here the function $\zeta(z)$ in (6.32) is of order $O\big(s^2\big)$ on the latter annulus for all $t \in [0,1]$. Therefore

$$\mathcal{J}(z) = \big(I + o(1)\big)\mathcal{N}(z), \quad s \to \infty,\ 0 < r_1 \leq |z+1| \leq r_2 < 1$$

again uniformly on any compact subset of the set (6.29).

6.7 The ratio problem – iterative solution for $\gamma = 1$

Similar to (3.84) we define

$$\mathcal{K}(z) = \Lambda(z)\begin{cases} \big(\mathcal{I}(z)\big)^{-1}, & |z-1| < \varepsilon, \\ \big(\mathcal{J}(z)\big)^{-1}, & |z+1| < \varepsilon, \\ \big(\mathcal{N}(z)\big)^{-1}, & |z \mp 1| > \varepsilon \end{cases} \tag{6.34}$$

with $0 < \varepsilon < \frac{1}{4}$ fixed and are lead to the ratio-RHP depicted in Figure 6.9

More precisely, the function $\mathcal{K}(z)$ has the following analytic properties.

- $\mathcal{K}(z)$ is analytic for $z \in \mathbb{C}\backslash\{(-1+\varepsilon, 1-\varepsilon) \cup \hat{C}_r \cup \hat{C}_l\}$

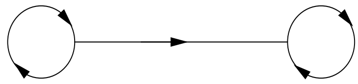

Figure 6.9. The jump graph for the ratio-function $\mathcal{K}(z)$

- The following jumps are valid on the clockwise oriented circles

$$\mathcal{K}_+(z) = \mathcal{K}_-(z)\mathcal{I}(z)\big(\mathcal{N}(z)\big)^{-1}, \quad |z-1| = \varepsilon$$
$$\mathcal{K}_+(z) = \mathcal{K}_-(z)\mathcal{J}(z)\big(\mathcal{N}(z)\big)^{-1}, \quad |z+1| = \varepsilon$$

whereas on the line segment $(-1+\varepsilon, 1-\varepsilon)$

$$\mathcal{K}_+(z) = \mathcal{K}_-(z)\mathcal{N}_+(z)\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\big(\mathcal{N}_+(z)\big)^{-1}.$$

- $\mathcal{K}(z)$ is analytic at $z = \pm 1$
- As $z \to \infty$ we have $\mathcal{K}(z) \to I$.

As a result of our construction (6.34), $\mathcal{K}(z)$ has no jumps in the parts of the original jump contour which lie inside the circles $\hat{C}_{r,l}$ and as we shall see now, the latter $\mathcal{K}$-RHP admits direct asymptotical analysis. To this end recall the matching relations and deduce

$$\|\mathcal{I}\big(\mathcal{N}\big)^{-1} - I\|_{L^2\cap L^\infty(\hat{C}_r)} \leq c_1 s^{-1}, \ \|\mathcal{J}\big(\mathcal{N}\big)^{-1} - I\|_{L^2\cap L^\infty(\hat{C}_l)} \leq c_2 s^{-1} \quad s \to \infty \quad (6.35)$$

which holds uniformly on any compact subset of the set (6.29) with some constants $c_i > 0$. Secondly recall (6.23)

$$\mathcal{N}_+(z)\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\big(\mathcal{N}_+(z)\big)^{-1} = \begin{pmatrix} \beta_+ + \beta_+^{-1} & i(\beta_+ - \beta_+^{-1}) \\ -i(\beta_+ - \beta_+^{-1}) & \beta_+ + \beta_+^{-1} \end{pmatrix}\begin{pmatrix} 1 & -2e^{2s^3\hat{g}_+(z)} \\ 0 & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} \beta_+ + \beta_+^{-1} & i(\beta_+ - \beta_+^{-1}) \\ -i(\beta_+ - \beta_+^{-1}) & \beta_+ + \beta_+^{-1} \end{pmatrix}^{-1}$$

however, as we mentioned previously, for $z \in (-1+\varepsilon, 1-\varepsilon)$

$$\mathrm{Re}\ \hat{g}_+(z) < 0,$$

i.e.

$$\|\mathcal{N}_+ \left(\begin{smallmatrix} 1 & -2 \\ 0 & 1 \end{smallmatrix}\right) \left(\mathcal{N}_+\right)^{-1} - I\|_{L^2\cap L^\infty(-1+\varepsilon,1-\varepsilon)} \leq c_3 e^{-c_4 \varepsilon s}, \quad s \to \infty \tag{6.36}$$

also here uniformly on any compact subset of the set (6.29). Thus together in the limit $s \to \infty$, with $G_{\mathcal{K}}$ denoting the jump matrix in the $\mathcal{K}$-RHP and $\Sigma_{\mathcal{K}}$ the underlying contour,

$$\|G_{\mathcal{K}} - I\|_{L^2\cap L^\infty(\Sigma_{\mathcal{K}})} \leq c_5 s^{-1}, \quad s \to \infty \tag{6.37}$$

uniformly on any compact subset of (6.29). The last estimate provides us with the unique solvability of the $\mathcal{K}$-RHP, its unique solution satifies

$$\|\mathcal{K}_- - I\|_{L^2(\Sigma_{\mathcal{K}})} \leq \hat{c} s^{-1}. \tag{6.38}$$

uniformly on any compact subset of (6.29). Tracing back the transformations

$$\Theta(\lambda) \mapsto \Phi(\lambda) \mapsto \Delta(z) \mapsto \mathcal{K}(z)$$

we also obtain existence and boundedness of $\Theta(\lambda), \lambda \in [-s, s]$ and hence existence and boundedness of the resolvent $I + R_{\mathrm{csin}}$ for sufficiently large s which is needed in (5.11).

The information derived in this chapter enables us to determine the large s-asymptotics of $\det(I - \gamma K_{\mathrm{csin}})$ and complete the proofs of Theorems 1.3.1, 1.27 and 1.3.3. The following computations however will be more involved than in chapter 4.

7. ASYMPTOTICS OF $\ln\det(I-\gamma K_{\text{csin}})$

We solved the auxiliary RHP according to the Deift-Zhou nonlinear steepest descent method in the last chapter. Using the four logarithmic derivative identities derived in section 5.3 we will now compute the relevant expansions. The major techniqual obstacle in the current section arises from the fact that we need to obtain all expanions including the constant terms. Hence we will have to iterate the relevant integral equations.

7.1 The situation $\gamma \neq 1$ – preliminary steps

We recall the transformations which have been used in the asymptotical solution of the Θ-RHP in case $\gamma < 1$

$$\Theta(\lambda) \mapsto \Phi(\lambda) \mapsto \Upsilon(z) \mapsto \Delta(z) \mapsto \mathcal{R}(z).$$

In order to determine $\det(I-\gamma K_{\text{csin}})$ from Proposition 5.3.1, we will again connect $\check{\Phi}(\pm s)$ and $\check{\Phi}'(\pm s)$ to the values of $\mathcal{R}(\pm 1)$ and $\mathcal{R}'(\pm 1)$ using the same strategy as in section 4.1. Since the relevant parametrices $\mathcal{V}(z)$ and $\mathcal{W}(z)$ are up to conjugation with $\sigma_1 e^{-i\frac{\pi}{4}\sigma_3}$ identical with $V(z)$ and $W(z)$ of section 3.2, we skip various steps in the relevant comparison and simply state the results. By proposition 5.3.1, locally

$$\Pi(\lambda) = \check{\Phi}(\lambda)\sqrt{\frac{\gamma}{2\pi i}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \check{\Phi}(\lambda) = \check{\Phi}(\pm s) + \check{\Phi}'(\pm s)\big(\lambda \mp s\big) + O\big((\lambda \mp s)\big), \quad \lambda \to \pm s$$

and from comparison

$$\begin{aligned}\check{\Phi}_{11}(s) + \check{\Phi}_{12}(s) \quad &= \quad \frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{11}\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)} \\ &\qquad -\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{12}\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}\Bigg)\end{aligned}$$

as well as

$$\check{\Phi}_{21}(s)+\check{\Phi}_{22}(s) = \frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{21}\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)} -\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{22}\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}\Bigg).$$

Moreover

$$\check{\Phi}_{11}'(s)+\check{\Phi}_{12}'(s) = \frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Bigg(\Big(\mathcal{R}'(1)-\mathcal{R}(1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Big)\big(B_r(1)\big)^{-1}\Bigg)_{11}\frac{e^{s^3\vartheta(1)}}{s\Gamma(-\nu)} -\Bigg(\Big(\mathcal{R}'(1)-\mathcal{R}(1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Big)\big(B_r(1)\big)^{-1}\Bigg)_{12}\frac{e^{-s^3\vartheta(1)}}{s\Gamma(\nu)}+(8s^2+2x) \times\Bigg\{\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{11}\Big(\frac{i}{2}+i\nu\Big)\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)}-\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{12} \times\Big(-\frac{i}{2}+i\nu\Big)\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}\Bigg\}\Bigg)$$

and

$$\check{\Phi}_{21}'(s)+\check{\Phi}_{22}'(s) = \frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Bigg(\Big(\mathcal{R}'(1)-\mathcal{R}(1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Big)\big(B_r(1)\big)^{-1}\Bigg)_{21}\frac{e^{s^3\vartheta(1)}}{s\Gamma(-\nu)} -\Bigg(\Big(\mathcal{R}'(1)-\mathcal{R}(1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Big)\big(B_r(1)\big)^{-1}\Bigg)_{22}\frac{e^{-s^3\vartheta(1)}}{s\Gamma(\nu)}+(8s^2+2x) \times\Bigg\{\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{21}\Big(\frac{i}{2}+i\nu\Big)\frac{e^{s^3\vartheta(1)}}{\Gamma(-\nu)}-\Big(\mathcal{R}(1)\big(B_r(1)\big)^{-1}\Big)_{22} \times\Big(-\frac{i}{2}+i\nu\Big)\frac{e^{-s^3\vartheta(1)}}{\Gamma(\nu)}\Bigg\}\Bigg).$$

Next

$$\check{\Phi}_{11}(-s)+\check{\Phi}_{12}(-s) = \frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big]_{12}\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)} -\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big)_{11}\frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)}\Bigg)$$

and

$$\check{\Phi}_{21}(-s)+\check{\Phi}_{22}(-s) = \frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big)_{22}\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)} -\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big)_{21}\frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)}\Bigg).$$

And finally

$$\check{\Phi}'_{11}(-s)+\check{\Phi}'_{12}(-s)=\frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Bigg(\Bigg(\mathcal{R}'(-1)-\mathcal{R}(-1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Bigg)\big(B_l(-1)\big)^{-1}\Bigg)_{12}$$
$$\times\frac{e^{-s^3\vartheta(-1)}}{s\Gamma(-\nu)}-\Bigg(\Bigg(\mathcal{R}'(-1)-\mathcal{R}(-1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Bigg)\big(B_l(-1)\big)^{-1}\Bigg)_{11}\frac{e^{s^3\vartheta(-1)}}{s\Gamma(\nu)}$$
$$+(8s^2+2x)\Bigg\{\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big)_{11}\Big(\frac{i}{2}+i\nu\Big)\frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)}$$
$$-\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big)_{12}\Big(\frac{3i}{2}+i\nu\Big)\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)}\Bigg\}\Bigg)$$

as well as

$$\check{\Phi}'_{21}(-s)+\check{\Phi}'_{22}(-s)=\frac{2\pi i}{\gamma}e^{i\frac{\pi}{2}\nu}\Bigg(\Bigg(\Bigg(\mathcal{R}'(-1)-\mathcal{R}(-1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Bigg)\big(B_l(-1)\big)^{-1}\Bigg)_{22}$$
$$\times\frac{e^{-s^3\vartheta(-1)}}{s\Gamma(-\nu)}-\Bigg(\Bigg(\mathcal{R}'(-1)-\mathcal{R}(-1)\frac{\nu\sigma_3}{2}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Bigg)\big(B_l(-1)\big)^{-1}\Bigg)_{21}\frac{e^{s^3\vartheta(-1)}}{s\Gamma(\nu)}$$
$$+(8s^2+2x)\Bigg\{\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big)_{21}\Big(\frac{i}{2}+i\nu\Big)\frac{e^{s^3\vartheta(-1)}}{\Gamma(\nu)}$$
$$-\Big(\mathcal{R}(-1)\big(B_l(-1)\big)^{-1}\Big)_{22}\Big(\frac{3i}{2}+i\nu\Big)\frac{e^{-s^3\vartheta(-1)}}{\Gamma(-\nu)}\Bigg\}\Bigg).$$

By Proposition 5.3.1, the connection to the resolvent kernel $R(\lambda,\mu)$ is established via

$$\Pi_1(\pm s)=\sqrt{\frac{\gamma}{2\pi i}}\big(\check{\Phi}_{11}(\pm s)+\check{\Phi}_{12}(\pm s)\big),\qquad \Pi_2(\pm s)=\sqrt{\frac{\gamma}{2\pi i}}\big(\check{\Phi}_{21}(\pm s)+\check{\Phi}_{22}(\pm s)\big)$$
$$\Pi'_1(\pm s)=\sqrt{\frac{\gamma}{2\pi i}}\big(\check{\Phi}'_{11}(\pm s)+\check{\Phi}'_{12}(\pm s)\big),\qquad \Pi'_2(\pm s)=\sqrt{\frac{\gamma}{2\pi i}}\big(\check{\Phi}'_{21}(\pm s)+\check{\Phi}'_{22}(\pm s)\big)$$

which, in terms of the previous identities, leads to

$$R(s,s)=\frac{2\pi i}{\gamma}e^{i\pi\nu}\Bigg(\Big[\mathcal{R}'_{11}(1)\mathcal{R}_{21}(1)-\mathcal{R}'_{21}(1)\mathcal{R}_{11}(1)\Big](16s^3+4xs)^{-2\nu}\frac{e^{2s^3\vartheta(1)}}{s\Gamma^2(-\nu)}$$
$$+\Big(\mathcal{R}'_{12}(1)\mathcal{R}_{22}(1)-\mathcal{R}'_{22}(1)\mathcal{R}_{12}(1)\Big)(16s^3+4xs)^{2\nu}\frac{e^{-2s^3\vartheta(1)}}{s\Gamma^2(\nu)}$$
$$-\Bigg(\mathcal{R}'_{11}(1)\mathcal{R}(1)_{22}-\mathcal{R}'_{22}(1)\mathcal{R}_{11}(1)+\mathcal{R}'_{12}(1)\mathcal{R}_{21}(1)-\mathcal{R}'_{21}(1)\mathcal{R}_{12}(1)$$
$$-\Big(\mathcal{R}_{11}(1)\mathcal{R}_{22}(1)-\mathcal{R}_{21}(1)\mathcal{R}_{12}(1)\Big)\nu\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\Bigg)\frac{1}{s\Gamma(\nu)\Gamma(-\nu)}$$
$$-\Big(\mathcal{R}_{11}(1)\mathcal{R}_{22}(1)-\mathcal{R}_{21}(1)\mathcal{R}_{12}(1)\Big)i\frac{8s^2+2x}{\Gamma(\nu)\Gamma(-\nu)}\Bigg).$$

In order to simplify the latter identity for $R(s,s)$, we use again unimodularity of $\mathcal{R}(z)$.

Proposition 7.1.1 *The ratio function $\mathcal{R}(z)$ is unimodular for any $x,\gamma\in\mathbb{R}$, i.e.* $\det\mathcal{R}(z)\equiv 1$.

Proof The parametrices $\mathcal{V}(z)$ and $\mathcal{W}(z)$ are up to conjugation with the factor $\sigma_1 e^{-i\frac{\pi}{4}\sigma_3}$ identical to the parametrices $V(z)$ and $W(z)$ of section 3.2, thus $\det\mathcal{V}(z)=\det\mathcal{W}(z)=1$. Also, the parametrix $\mathcal{U}(z)$ was constructed in terms of a unimodular canonical solution of system (1.10), i.e. $\det\mathcal{U}(z)=1$ as well. Thus the ratio function $\mathcal{R}(z)$ has a unimodular jump matrix $G_{\mathcal{R}}(z)$, and we obtain as in Proposition 4.1.1 the statement. ■

This implies

$$\begin{aligned}
R(s,s) &= -ie^{i\pi\nu}\frac{2\pi i}{\gamma}\frac{8s^2+2x}{\Gamma(\nu)\Gamma(-\nu)}+\frac{2\pi i}{\gamma}\frac{\nu e^{i\pi\nu}}{s\Gamma(\nu)\Gamma(-\nu)}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}-\frac{2\pi i}{\gamma}\frac{e^{i\pi\nu}}{s\Gamma(\nu)\Gamma(-\nu)}\\
&\times\Big(\mathcal{R}_{11}'(1)\mathcal{R}_{22}(1)-\mathcal{R}_{22}'(1)\mathcal{R}_{11}(1)+\mathcal{R}_{12}'(1)\mathcal{R}_{21}(1)-\mathcal{R}_{21}'(1)\mathcal{R}_{12}(1)\Big)\\
&+\frac{2\pi i}{\gamma}e^{i\pi\nu}\bigg(\Big(\mathcal{R}_{11}'(1)\mathcal{R}_{21}(1)-\mathcal{R}_{21}'(1)\mathcal{R}_{11}(1)\Big)(16s^3+4xs)^{-2\nu}\frac{e^{2s^3\vartheta(1)}}{s\Gamma^2(-\nu)}\\
&+\Big(\mathcal{R}_{12}'(1)\mathcal{R}_{22}(1)-\mathcal{R}_{22}'(1)\mathcal{R}_{12}(1)\Big](16s^3+4xs)^{2\nu}\frac{e^{-2s^3\vartheta(1)}}{s\Gamma^2(\nu)}\bigg).
\end{aligned}\tag{7.1}$$

With the same reasoning

$$\begin{aligned}
R(-s,-s) &= -ie^{i\pi\nu}\frac{2\pi i}{\gamma}\frac{8s^2+2x}{\Gamma(\nu)\Gamma(-\nu)}+\frac{2\pi i}{\gamma}\frac{\nu e^{i\pi\nu}}{s\Gamma(\nu)\Gamma(-\nu)}\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}-\frac{2\pi i}{\gamma}\frac{e^{i\pi\nu}}{s\Gamma(\nu)\Gamma(-\nu)}\\
&\Big(\mathcal{R}_{11}'(-1)\mathcal{R}_{22}(-1)-\mathcal{R}_{22}'(-1)\mathcal{R}_{11}(-1)+\mathcal{R}_{12}'(-1)\mathcal{R}_{21}(-1)-\mathcal{R}_{21}'(-1)\mathcal{R}_{12}(-1)\Big)\\
&+\frac{2\pi i}{\gamma}e^{i\pi\nu}\bigg(\Big(\mathcal{R}_{11}'(-1)\mathcal{R}_{21}(-1)-\mathcal{R}_{21}'(-1)\mathcal{R}_{11}(-1)\Big)(16s^3+4xs)^{2\nu}\frac{e^{-2s^3\vartheta(1)}}{s\Gamma^2(\nu)}\\
&+\Big(\mathcal{R}_{12}'(-1)\mathcal{R}_{22}(-1)-\mathcal{R}_{22}'(-1)\mathcal{R}_{12}(-1)\Big)(16s^3+4xs)^{-2\nu}\frac{e^{2s^3\vartheta(1)}}{s\Gamma^2(-\nu)}\bigg)
\end{aligned}\tag{7.2}$$

and we notice that our derivation of the latter two identities did not distinguish between the cases $\gamma<1$ and $\gamma>1$. At this point we will prove Theorem 1.3.1 including the constant term. Our proof uses primarily the γ-derivative of Proposition

5.3.3, the s-derivative of Proposition 5.3.1 will only be used in certain intermediate steps and to verify the stated error term.

7.2 Proof of Theorem 1.3.1

We have to determine the large s-asymptotics of the matrix coefficients Φ_1, Φ_2 and Φ_3 in (5.29). Tracing back the relevant transformations

$$\begin{aligned}\Phi_1 &= \lim_{\lambda\to\infty}\big(\lambda\big(\Phi(\lambda)e^{-i(\frac{4}{3}\lambda^3+x\lambda)\sigma_3}-I\big)\big) = -2\nu\sigma_3 s+\frac{is}{2\pi}\int\limits_{\Sigma_{\mathcal{R}}}\mathcal{R}_-(w)\big(G_{\mathcal{R}}(w)-I\big)dw\\ &= -2\nu\sigma_3 s+\frac{is}{2\pi}\int\limits_{\Sigma_{\mathcal{R}}}\big(\mathcal{R}_-(w)-I\big)\big(G_{\mathcal{R}}(w)-I\big)dw+\frac{is}{2\pi}\int\limits_{\Sigma_{\mathcal{R}}}\big(G_{\mathcal{R}}(w)-I\big)dw\end{aligned}$$

and from the standard integral representation of $\mathcal{R}_-(z), z\in\Sigma_{\mathcal{R}}$ as well as (6.16) and (6.10)

$$\begin{aligned}\mathcal{R}_-(z)-I &= \frac{1}{2\pi i}\int\limits_{C_0}\big(G_{\mathcal{R}}(w)-I\big)\frac{dw}{w-z_-}+O\big(s^{-2}\big)\\ &= \frac{1}{2\pi i}\int\limits_{C_0}\big(B_0(w)\big)^{-1}\begin{pmatrix} v & ue^{-2\pi i\nu}\\ -ue^{2\pi i\nu} & -v\end{pmatrix}B_0(w)\frac{idw}{2sw(w-z_-)}+O\big(s^{-2}\big)\\ &= \frac{i}{2sz}\left[\begin{pmatrix} v & u\\ -u & -v\end{pmatrix}-\big(B_0(z)\big)^{-1}\begin{pmatrix} v & ue^{-2\pi i\nu}\\ -ue^{2\pi i\nu} & -v\end{pmatrix}B_0(z)\right]+O\big(s^{-2}\big).\end{aligned}$$

We now improve the last estimation via iteration:

$$\mathcal{R}_-(z)-I = \mathcal{R}_1(z)+\mathcal{R}_2(z)+O\left(s^{-3}\right),\quad z\in\Sigma_{\mathcal{R}}$$

where

$$\begin{aligned}\mathcal{R}_1(z) &= \frac{1}{2\pi i}\int\limits_{C_0}\big(G_{\mathcal{R}}(w)-I\big)\frac{dw}{w-z_-}\\ \mathcal{R}_2(z) &= \frac{1}{2\pi i}\int\limits_{C_0}\mathcal{R}_1(w)\big(G_{\mathcal{R}}(w)-I\big)\frac{dw}{w-z_-}.\end{aligned}$$

Carrying out the computations we are lead to

$$\mathcal{R}_-(z)-I=\frac{i}{2sz}\Bigg[\begin{pmatrix} v & u\\ -u & -v\end{pmatrix}-\big(B_0(z)\big)^{-1}\begin{pmatrix} v & ue^{-2\pi i\nu}\\ -ue^{2\pi i\nu} & -v\end{pmatrix}B_0(z)\Bigg]$$
$$-\frac{1}{4s^2z^2}\begin{pmatrix} v & u\\ -u & -v\end{pmatrix}\Bigg[\begin{pmatrix} v & u\\ -u & -v\end{pmatrix}-\big(B_0(z)\big)^{-1}\begin{pmatrix} v & ue^{-2\pi i\nu}\\ -ue^{2\pi i\nu} & -v\end{pmatrix}B_0(z)\Bigg]$$
$$+\frac{1}{8s^2z^2}\Bigg[\begin{pmatrix} u^2-v^2 & 2(u_x+uv)\\ 2(u_x+uv) & u^2-v^2\end{pmatrix}-\big(B_0(z)\big)^{-1}$$
$$\times\begin{pmatrix} u^2-v^2 & 2(u_x+uv)\\ 2(u_x+uv) & u^2-v^2\end{pmatrix}B_0(z)\Bigg]+\frac{\nu}{s^2z}\begin{pmatrix} u^2 & -u_x\\ u_x & -u^2\end{pmatrix}+O\big(s^{-3}\big),\quad s\to\infty,$$

valid for any $z\in\Sigma_{\mathcal{R}}$. Back to our first identity for Φ_1, one starts with

$$\int_{\Sigma_{\mathcal{R}}}\big(\mathcal{R}_-(w)-I\big)\big(G_{\mathcal{R}}(w)-I\big)dw=\frac{2\pi i\nu}{s^2}\begin{pmatrix} -u^2 & -uv\\ uv & u^2\end{pmatrix}+\frac{i(-2\pi i)\nu^2}{s^3}$$
$$\times\begin{pmatrix} 2uu_x & -2u^3+v(u_x+uv)+\frac{u}{2}(u^2-v^2)\\ 2u^3-v(u_x+uv)-\frac{u}{2}(u^2-v^2) & -2uu_x\end{pmatrix}$$
$$+O\big(s^{-4}\big),\quad s\to\infty$$

and moves on to

$$\int_{\Sigma_{\mathcal{R}}}\big(G_{\mathcal{R}}(w)-I\big)dw=\frac{(-2\pi i)i}{2s}\begin{pmatrix} v & u\\ -u & -v\end{pmatrix}+\frac{(-2\pi i)\nu}{s^2}\begin{pmatrix} 0 & -(u_x+uv)\\ u_x+uv & 0\end{pmatrix}$$
$$+\frac{i(-2\pi i)\nu^2}{s^3}\begin{pmatrix} 0 & -(\frac{u}{2}(u^2+v^2)+vu_x+xu)\\ \frac{u}{2}(u^2+v^2)+vu_x+xu & 0\end{pmatrix}$$
$$+\frac{(-2\pi i)i}{4s^3+xs}\begin{pmatrix} -\nu^2 & \sqrt{-\nu^2}\cos\sigma\\ -\sqrt{-\nu^2}\cos\sigma & \nu^2\end{pmatrix}+O\big(s^{-4}\big),\qquad s\to\infty$$

with

$$\sigma=\sigma(s,x,\gamma)=\frac{8}{3}s^3+2xs+\frac{\ln|1-\gamma|}{\pi}\ln\big(16s^3+4xs\big)-\arg\frac{\Gamma(1-\nu)}{\Gamma(\nu)}.$$

Adding up

$$\int_{\Sigma_{\mathcal{R}}} \mathcal{R}_-(w)\big(G_{\mathcal{R}}(w)-I\big)dw = \frac{(-2\pi i)i}{2s}\begin{pmatrix} v & u \\ -u & -v \end{pmatrix} + \frac{(-2\pi i)\nu}{s^2}\begin{pmatrix} u^2 & -u_x \\ u_x & -u^2 \end{pmatrix}$$

$$+\frac{(-2\pi i)i\nu^2}{s^3}\begin{pmatrix} 2uu_x & -(xu+2u^3) \\ xu+2u^3 & -2uu_x \end{pmatrix}$$

$$+\frac{(-2\pi i)i}{4s^3+xs}\begin{pmatrix} -\nu^2 & \sqrt{-\nu^2}\cos\sigma \\ -\sqrt{-\nu^2}\cos\sigma & \nu^2 \end{pmatrix} + O\big(s^{-4}\big).$$

This leads to the following expansion for Φ_1

$$\Phi_1 = -2\nu\sigma_3 s + \frac{i}{2}\begin{pmatrix} v & u \\ -u & -v \end{pmatrix} + \frac{\nu}{s}\begin{pmatrix} u^2 & -u_x \\ u_x & -u^2 \end{pmatrix} + \frac{i\nu^2}{s^2}\begin{pmatrix} 2uu_x & -u_{xx} \\ u_{xx} & -2uu_x \end{pmatrix}$$

$$+\frac{i}{4s^2+x}\begin{pmatrix} -\nu^2 & \sqrt{-\nu^2}\cos\sigma \\ -\sqrt{-\nu^2}\cos\sigma & \nu^2 \end{pmatrix} + O\big(s^{-3}\big), \quad s\to\infty, \tag{7.3}$$

uniformly on any compact subset of the set (1.19) and where we used that $u = u(x,\gamma)$ solves the second Painlevé equation. Secondly

$$\Phi_2 = 2\nu^2 s^2 I - \frac{i\nu s^2}{\pi}\big(I_1+I_2\big)\sigma_3 + \frac{is^2}{2\pi}\int_{\Sigma_{\mathcal{R}}} \mathcal{R}_-(w)\big(G_{\mathcal{R}}(w)-I\big)w\,dw. \tag{7.4}$$

We need to compute

$$\begin{aligned} I_1 &= \int_{\Sigma_{\mathcal{R}}} \big(\mathcal{R}_-(w)-I\big)\big(G_{\mathcal{R}}(w)-I\big)w\,dw \\ &= \frac{(-2\pi i)i\nu}{2s^3}\begin{pmatrix} u^2v+uu_x & \frac{u}{2}(u^2-v^2) \\ \frac{u}{2}(u^2-v^2) & u^2v+uu_x \end{pmatrix} + O\big(s^{-4}\big) \end{aligned}$$

as well as

$$\begin{aligned} I_2 &= \int_{\Sigma_{\mathcal{R}}} \big(G_{\mathcal{R}}(w)-I\big)w\,dw = \frac{(-2\pi i)}{8s^2}\begin{pmatrix} u^2-v^2 & 2(u_x+uv) \\ 2(u_x+uv) & u^2-v^2 \end{pmatrix} \\ &+\frac{(-2\pi i)i\nu}{4s^3}\begin{pmatrix} 0 & u(u^2+v^2)+2(vu_x+xu) \\ u(u^2+v^2)+2(vu_x+xu) & 0 \end{pmatrix} \\ &+\frac{2\pi i}{4s^3+xs}\begin{pmatrix} 0 & \sqrt{-\nu^2}\sin\sigma \\ -\sqrt{-\nu^2}\sin\sigma & 0 \end{pmatrix} + O\big(s^{-4}\big), \end{aligned}$$

in order to obtain

$$\Phi_2 = 2\nu^2 s^2 I - i\nu s \begin{pmatrix} v & -u \\ -u & v \end{pmatrix} - 2\nu^2 \begin{pmatrix} u^2 & u_x \\ u_x & u^2 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} u^2 - v^2 & 2(u_x + uv) \\ 2(u_x + uv) & u^2 - v^2 \end{pmatrix}$$
$$-\frac{2i\nu^3}{s}\begin{pmatrix} 2uu_x & xu + 2u^3 \\ xu + 2u^3 & 2uu_x \end{pmatrix} + \frac{i\nu}{2s}\begin{pmatrix} u^2 v + uu_x & u^3 + vu_x + xu \\ u^3 + vu_x + xu & u^2 v + uu_x \end{pmatrix}$$
$$+\frac{2i\nu s}{4s^2 + x}\begin{pmatrix} \nu^2 & \sqrt{-\nu^2}\cos\sigma \\ \sqrt{-\nu^2}\cos\sigma & \nu^2 \end{pmatrix} - \frac{s}{4s^2 + x}\begin{pmatrix} 0 & \sqrt{-\nu^2}\sin\sigma \\ \sqrt{-\nu^2}\sin\sigma & 0 \end{pmatrix}$$
$$+O\big(s^{-2}\big), \quad s \to \infty. \tag{7.5}$$

Finally the computation of Φ_3

$$\Phi_3 = -\frac{2\nu}{3}s^3(1 + 2\nu^2)\sigma_3 + \frac{i\nu^2 s^3}{\pi}\int_{\Sigma_{\mathcal{R}}} \mathcal{R}_-(w)\big(G_{\mathcal{R}}(w) - I\big)dw - \frac{i\nu s^3}{\pi}\big(I_3 + I_4\big)\sigma_3$$
$$+\frac{is^3}{2\pi}\int_{\Sigma_{\mathcal{R}}} \mathcal{R}_-(w)\big(G_{\mathcal{R}}(w) - I\big)w^2\, dw. \tag{7.6}$$

Since

$$\int_{\Sigma_{\mathcal{R}}} \big(\mathcal{R}_-(w) - I\big)\big(G_{\mathcal{R}}(w) - I\big)w^2\, dw = O\big(s^{-4}\big)$$

and

$$\int_{\Sigma_{\mathcal{R}}} \mathcal{R}_-(w)\big(G_{\mathcal{R}}(w) - I\big)w^2\, dw = \frac{(-2\pi i)i}{4s^3 + xs}\begin{pmatrix} -\nu^2 & \sqrt{-\nu^2}\cos\sigma \\ -\sqrt{-\nu^2}\cos\sigma & \nu^2 \end{pmatrix}$$
$$+\frac{(-2\pi i)i}{48s^3}\begin{pmatrix} -(v^3 - 3vu^2 + 2(xv - uu_x)) & -3(u(u^2 + v^2) + 2(vu_x + xu)) \\ 3(u(u^2 + v^2) + 2(vu_x + xu)) & v^3 - 3vu^2 + 2(xv - uu_x) \end{pmatrix}$$
$$+O\big(s^{-4}\big)$$

we obtain

$$\Phi_3 = -\frac{2\nu}{3}s^3(1+2\nu^2)\sigma_3 + i\nu^2 s^2\begin{pmatrix} v & u \\ -u & -v\end{pmatrix} + 2\nu^3 s\begin{pmatrix} u^2 & -u_x \\ u_x & -u^2\end{pmatrix}$$
$$-\frac{\nu s}{4}\begin{pmatrix} u^2-v^2 & -2(u_x+uv) \\ 2(u_x+uv) & -(u^2-v^2)\end{pmatrix} + 2i\nu^4\begin{pmatrix} 2uu_x & -u_{xx} \\ u_{xx} & -2uu_x\end{pmatrix}$$
$$+\frac{2i\nu^2 s^2 + is^2}{4s^2+x}\begin{pmatrix} -\nu^2 & \sqrt{-\nu^2}\cos\sigma \\ -\sqrt{-\nu^2}\cos\sigma & \nu^2\end{pmatrix} - \frac{2i\nu s^2}{4s^2+x}\sqrt{-\nu^2}\sin\sigma\,\sigma_2$$
$$-i\nu^2\begin{pmatrix} u^2 v + uu_x & -(u^3+vu_x+xu) \\ u^3+vu_x+xu & -(u^2 v+uu_x)\end{pmatrix}$$
$$+\frac{i}{48}\begin{pmatrix} -(v^3-3vu^2+2(xv-uu_x)) & -3(u(u^2+v^2)+2(vu_x+xu)) \\ 3(u(u^2+v^2)+2(vu_x+xu)) & v^3-3vu^2+2(xv-uu_x)\end{pmatrix} + O\big(s^{-1}\big).$$

With the given information at hand, (5.22) and (5.23) lead to

$$b = 2\Phi_1^{12} = iu - \frac{2\nu}{s}u_x - \frac{2i\nu^2}{s^2}(xu+2u^3) + \frac{2i\sqrt{-\nu^2}}{4s^2+x}\cos\sigma + O\big(s^{-3}\big) \tag{7.7}$$

as well as

$$c = 2\Phi_1^{21} = -b + O\big(s^{-3}\big),\quad s\to\infty \tag{7.8}$$

and

$$\begin{aligned} d &= ix + 8i\Phi_1^{12}\Phi_1^{21} \qquad (7.9)\\ &= ix + 2iu^2 - \frac{8\nu}{s}uu_x - \frac{8i\nu^2}{s^2}\big((u_x)^2 + xu^2 + 2u^4\big) + \frac{8iu\sqrt{-\nu^2}}{4s^2+x}\cos\sigma + O\big(s^{-3}\big). \end{aligned}$$

Furthermore

$$e = 8i\Big(\Phi_1^{12}\Phi_1^{22} - \Phi_2^{12}\Big) = -2iu_x + \frac{4\nu}{s}(xu+2u^3) + \frac{8is\sqrt{-\nu^2}}{4s^2+x}\sin\sigma + O\big(s^{-2}\big)$$

and

$$f = -e + O\big(s^{-2}\big) \tag{7.10}$$

where we made use of the following identities, see (7.3) and (7.5)

$$\Phi_1^{21} = -\Phi_1^{12} + O\big(s^{-3}\big),\ \ \Phi_1^{11} = -\Phi_1^{22} + O\big(s^{-3}\big),\ \ \Phi_2^{21} = \Phi_2^{12} + O\big(s^{-2}\big),\ \ s\to\infty. \tag{7.11}$$

We have now derived enough information to evaluate the first terms listed in Proposition 5.3.3. Put

$$\begin{aligned}\mathcal{P}_1(s,x,\gamma) &= -8i\Big((\Phi_3)_\gamma + (\Phi_1)_\gamma(\Phi_1^2-\Phi_2) - (X_2)_\gamma\Phi_1\Big)^{11} - 2d\Big((\Phi_1)_\gamma\Big)^{11}\\ &\quad +4ib\Big((\Phi_2)_\gamma - (\Phi_1)_\gamma\Phi_1\Big)^{21} - 4ic\Big((\Phi_2)_\gamma - (\Phi_1)_\gamma\Phi_1\Big)^{12}\\ &\quad -e\Big((\Phi_1)_\gamma\Big)^{21} - f\Big((\Phi_1)_\gamma\Big)^{12}\end{aligned}$$

and notice that

$$\left(\frac{\partial}{\partial\gamma}\Phi_k\right)^{ij} = \frac{\partial}{\partial\gamma}\Big(\Phi_k^{ij}\Big),\quad i,j,k=1,2.$$

From (7.10) and (7.11) we obtain therefore

$$-e\Big((\Phi_1)_\gamma\Big)^{21} = -f\Big((\Phi_1)_\gamma\Big)^{12} + O\big(s^{-2}\big)$$

and via (7.7) and (7.11)

$$4ib\Big((\Phi_2)_\gamma\Big)^{21} = -4ic\Big((\Phi_2)_\gamma\Big)^{12} + O\left(\frac{\ln s}{s^2}\right).$$

Since also

$$\begin{aligned}-4ib\Big(\big(\Phi_1\big)_\gamma\Phi_1\Big)^{21} &= -4ib\Big(\big(\Phi_1^{21}\big)_\gamma\Phi_1^{11} + \big(\Phi_1^{22}\big)_\gamma\Phi_1^{21}\Big)\\ &= 4ic\Big(\big(\Phi_1^{12}\big)_\gamma\Phi_1^{22} + \big(\Phi_1^{11}\big)_\gamma\Phi_1^{12}\Big) + O\left(\frac{\ln s}{s^2}\right)\\ &= 4ic\Big(\big(\Phi_1\big)_\gamma\Phi_1\Big)^{12} + O\left(\frac{\ln s}{s^2}\right)\end{aligned}$$

as $s\to\infty$ uniformly on any compact subset of the set (1.19), we can simplify the expression for $\mathcal{P}_1(s,x,\gamma)$ asymptotically

$$\begin{aligned}\mathcal{P}_1(s,x,\gamma) &= -8i\Big(\big(\Phi_3\big)_\gamma + \big(\Phi_1\big)_\gamma(\Phi_1^2-\Phi_2) - \big(\Phi_2\big)_\gamma\Phi_1\Big)^{11} - 2d\Big(\big(\Phi_1\big)_\gamma\Big)^{11}\\ &\quad +8ib\Big(\big(\Phi_2\big)_\gamma - \big(\Phi_1\big)_\gamma\Phi_1\Big)^{21} - 2e\Big(\big(\Phi_1\big)_\gamma\Big)^{21} + O\left(\frac{\ln s}{s^2}\right).\end{aligned}\qquad(7.12)$$

Next from (7.3)

$$\Phi_1^2 = \left(4\nu^2s^2 - 2i\nu vs - 4\nu^2u^2 + \frac{u^2-v^2}{4} - \frac{8i\nu^3}{s}uu_x + \frac{4i\nu^3 s}{4s^2+x} + \frac{i\nu}{s}(vu^2+uu_x)\right)I$$
$$+O\big(s^{-2}\big)$$

with I denoting the 2×2 identity matrix. Thus

$$
\begin{aligned}
\Big(\big(X_1\big)_\gamma X_1^2\Big)^{11} &= \Big(-2\nu_\gamma s + \frac{i}{2}v_\gamma + \frac{(\nu u^2)_\gamma}{s} + \frac{i}{s^2}\big(2\nu^2 u u_x\big)_\gamma - \frac{i}{4s^2+x}\big(\nu^2\big)_\gamma \\
&+O\Big(\frac{\ln s}{s^3}\Big)\Big)\Big(4\nu^2 s^2 - 2i\nu v s - 4\nu^2 u^2 + \frac{u^2-v^2}{4} - \frac{8i\nu^3}{s}uu_x + \frac{4i\nu^3 s}{4s^2+x} \\
&+\frac{i\nu}{s}(vu^2+uu_x) + O\big(s^{-2}\big)\Big) \\
= &\ -8\nu^2\nu_\gamma s^3 + 4i\nu\nu_\gamma v s^2 + 2i\nu^2 s^2 v_\gamma + 8\nu^2\nu_\gamma u^2 s - \nu_\gamma s\frac{u^2-v^2}{2} + \nu s v v_\gamma \\
&+4\nu^2 s\big(\nu u^2\big)_\gamma + 16 i\nu_\gamma \nu^3 uu_x - \frac{8i\nu^3\nu_\gamma s^2}{4s^2+x} - 2i\nu\nu_\gamma\big(vu^2+uu_x\big) - 2i\nu^2u^2v_\gamma \\
&+\frac{i}{2}v_\gamma\frac{u^2-v^2}{4} - 2i\nu v\big(\nu u^2\big)_\gamma + 4i\nu^2\big(2\nu^2 uu_x\big)_\gamma - \frac{4i\nu^2 s^2}{4s^2+x}\big(\nu^2\big)_\gamma + O\Big(\frac{\ln s}{s}\Big).
\end{aligned}
$$

Moving on, we use

$$
\Big(\big(\Phi_1\big)_\gamma\Phi_2 + \big(\Phi_2\big)_\gamma\Phi_1\Big)^{11} = \big(\Phi_1^{11}\Phi_2^{11}\big)_\gamma + \big(\Phi_1^{12}\big)_\gamma\Phi_2^{21} + \big(\Phi_2^{12}\big)_\gamma\Phi_1^{21}
$$

and obtain

$$
\begin{aligned}
\Big(\big(\Phi_1\big)_\gamma\Phi_2 + \big(\Phi_2\big)_\gamma\Phi_1\Big)^{11} &= \Big(-4\nu^3s^3 + 3i\nu^2s^2v + 6\nu^3u^2s - 3i\nu^2u^2v + 12i\nu^4uu_x \\
-i\nu^2uu_x - \frac{6i\nu^4s^2}{4s^2+x} - \nu s\frac{u^2-3v^2}{4} &+ \frac{iv}{2}\frac{u^2-v^2}{8} + O\big(s^{-1}\big)\Big)_\gamma - \frac{\nu s}{2}uu_\gamma + \frac{s}{2}u\big(\nu u\big)_\gamma \\
-i\nu^2u_xu_\gamma + \frac{i}{2}u_\gamma\frac{u_x+uv}{4} - i\nu u\big(\nu u_x\big)_\gamma &+ i\nu u_x\big(\nu u\big)_\gamma + iu\big(\nu^2u_x\big)_\gamma - \frac{i}{2}u\frac{(u_x+uv)_\gamma}{4} \\
+O\big(s^{-1}\big).
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
\Big(\big(\Phi_2\big)_\gamma - \big(\Phi_1\big)_\gamma\Phi_1\Big)^{21} &= is(\nu u)_\gamma + i\nu_\gamma su - i\nu su_\gamma - 2(\nu^2u_x)_\gamma + \frac{(u_x+uv)_\gamma}{4} \\
-2\nu\nu_\gamma u_x + \frac{1}{4}(uv_\gamma - vu_\gamma) &+ 2\nu(\nu u_x)_\gamma + O\big(s^{-1}\big),
\end{aligned}
$$

which implies

$$
\begin{aligned}
b\Big(\big(\Phi_2\big)_\gamma - \big(\Phi_1\big)_\gamma\Phi_1\Big)^{21} &= -su(\nu u)_\gamma - su^2\nu_\gamma + s\nu uu_\gamma - 2iu(\nu^2u_x)_\gamma \\
+iu\frac{(u_x+uv)_\gamma}{4} - 2i\nu\nu_\gamma uu_x &+ iu\frac{uv_\gamma - vu_\gamma}{4} + 2i\nu u(\nu u_x)_\gamma - 2i\nu u_x(\nu u)_\gamma \\
-2i\nu\nu_\gamma uu_x + 2i\nu^2u_xu_\gamma &+ O\big(s^{-1}\big),
\end{aligned}
$$

and finally

$$d\Big(\big(\Phi_1\big)_\gamma\Big)^{11} = -2i\nu_\gamma sx - 4i\nu_\gamma u^2 s - \frac{xv_\gamma}{2} - u^2 v_\gamma + 16\nu\nu_\gamma uu_x + O\big(s^{-1}\big)$$

as well as

$$e\Big(\big(X_1\big)_\gamma\Big)^{21} = -u_x u_\gamma + O\big(s^{-1}\big).$$

At this point, we write

$$\mathcal{P}_1(s,x,\gamma) = s^3\mathcal{P}_1^{(3)}(x,\gamma) + s^2\mathcal{P}_1^{(2)}(x,\gamma) + s\mathcal{P}_1^{(1)}(x,\gamma) + \mathcal{P}_1^{(0)}(x,\gamma) + O\left(\frac{\ln s}{s}\right)$$

where $\mathcal{P}_1^{(i)}(x,\gamma)$ are independent of s. Since

$$\nu\big|_{\gamma=0} = 0$$

we get from (7.12) and the previous computations

$$\begin{aligned}\int_0^\gamma \mathcal{P}_1^{(3)}(x,t)dt &= \int_0^\gamma \left[-8i\Big(-\frac{2\nu}{3}\big(1+2\nu^2\big)\Big)_t - 8i\big(-8\nu^2\nu_t\big) + 8i\big(-4\nu^3\big)_t\right]dt \\ &= \frac{16}{3}i\nu.\end{aligned}$$

Next

$$\begin{aligned}\int_0^\gamma \mathcal{P}_1^{(2)}(x,t)dt &= \int_0^\gamma \left[-8i\big(i\nu^2 v\big)_t - 8i\big(4i\nu\nu_t v + 2i\nu^2 v_t\big) + 8i\big(3i\nu^2 v\big)_t\right]dt \\ &= 0.\end{aligned}$$

and

$$\begin{aligned}\int_0^\gamma \mathcal{P}_1^{(1)}(x,t)dt = \int_0^\gamma \Bigg[&-8i\Big(2t^3u^2 - \frac{t}{4}(u^2-v^2)\Big)_t - 8i\Big(8\nu^2\nu_t u^2 - \nu_t\frac{u^2-v^2}{2} \\ &+\nu v v_t + 4\nu^2(\nu u^2)_t\Big) + 8i\Big(6\nu^3u^2 - \nu\frac{u^2-3v^2}{4}\Big)_t + 8i\Big(-\frac{\nu}{2}uu_t + \frac{u}{2}(\nu u)_t\Big) \\ &+8i\Big(iu\big(i(\nu u)_t + i\nu_t u - i\nu u_t\big)\Big) - 2\Big(-2i\nu_t x - 4i\nu_t u^2\Big)\Bigg]dt \\ = \;& 4i\nu x.\end{aligned}$$

We complete the computations for $\mathcal{P}_1(s,x,\gamma)$ by evaluating

$$\int_0^\gamma \mathcal{P}_1^{(0)}(x,t)dt = \int_0^\gamma \Bigg[-8i\Big(4i\nu^4 uu_x - \frac{2i\nu^4 s^2}{4s^2+x} - \frac{is^2\nu^2}{4s^2+x} - i\nu^2(u^2v+uu_x)$$

$$-\frac{i}{48}(v^3-3vu^2+2(xv-uu_x))\Big)_t - 8i\Big(16i\nu^3\nu_t uu_x - \frac{8i\nu^3\nu_t s^2}{4s^2+x} - 2i\nu\nu_t(vu^2$$

$$+uu_x) - 2i\nu^2u^2v_t + iv_t\frac{u^2-v^2}{8} - 2i\nu v(\nu u^2)_t + 8i\nu^2(\nu^2 uu_x)_t - \frac{4i\nu^2 s^2}{4s^2+x}(\nu^2)_t\Big)$$

$$+8i\Big(-3i\nu^2u^2v + 12i\nu^4 uu_x - i\nu^2 uu_x - \frac{6i\nu^4 s^2}{4s^2+x} + iv\frac{u^2-v^2}{16}\Big)_t$$

$$+8i\Big(-i\nu^2 u_x u_t + iu_t\frac{u_x+uv}{8} - i\nu u(\nu u_x)_t + i\nu u_x(\nu u)_t + iu(\nu^2 u_x)_t$$

$$-iu\frac{(u_x+uv)_t}{8}\Big) - 2\Big(-\frac{xv_t}{2} - u^2v_t + 16\nu\nu_t uu_x\Big) + 8i\Big(-2iu(\nu^2u_x)_t$$

$$+iu\frac{(u_x+uv)_t}{4} - 2i\nu\nu_t uu_x + iu\frac{uv_t - vu_t}{4} + 2i\nu u(\nu u_x)_t - 2i\nu u_x(\nu u)_t$$

$$-2i\nu\nu_t uu_x + 2i\nu^2 u_x u_t\Big) - 2\big(-u_x u_t\big)\Bigg]dt$$

$$= \quad -2\nu^2 + \frac{2}{3}(xv-uu_x) + 2\int_0^\gamma u_x u_t d\gamma.$$

The next Proposition will be useful

Proposition 7.2.1 *Let $u=u(x,\gamma), \gamma<1$ denote the Ablowitz-Segur solution of the boundary value problem*

$$u_{xx} = xu + 2u^3, \quad u(x) \sim \gamma \mathrm{Ai}(x), \quad x \to +\infty.$$

Then

$$\frac{2}{3}\big(xv(x,\gamma) - u(x,\gamma)u_x(x,\gamma)\big) + 2\int_0^\gamma u_x(x,t)u_t(x,t)dt = -\int_x^\infty (y-x)u^2(y,\gamma)dy \quad (7.13)$$

where $v=(u_x)^2 - xu^2 - u^4$.

Proof Let $F(x,\gamma)$ denote the left hand side in (7.13). Using the differential equation for u as well as integration by parts, we have

$$\frac{\partial}{\partial x}F(x,\gamma) = v(x,\gamma) = \big(u_x(x,\gamma)\big)^2 - xu^2(x,\gamma) - u^4(x,\gamma)$$

and hence after integration

$$F(x,\gamma)=-\int_x^{\infty}(y-x)u^2(y,\gamma)dy+C(\gamma) \tag{7.14}$$

with a constant C, only depending on γ. Since u decays exponentially fast as $x\to\infty$, the same limit on both sides of (7.14) gives us the stated identity. ■

Let us summarize our previous computations. As $s\to\infty$, uniformly on any compact subset of the set (1.19)

$$\int_0^{\gamma}\mathcal{P}_1(s,x,t)dt=i\nu\Big(\frac{16}{3}s^3+4sx\Big)-\int_x^{\infty}(y-x)u^2(y)dy+2(i\nu)^2+O\Big(\frac{\ln s}{s}\Big) \tag{7.15}$$

To move further ahead in the equation of the γ-derivative, let us define

$$\mathcal{P}_2(s,x,\gamma)=\big(A_{11}-A_{22}\big)\widehat{\phi}_{11}(s)+A_{12}\widehat{\phi}_{21}(s)+A_{21}\widehat{\phi}_{12}(s),\ \ \widehat{\phi}(s)=\frac{\partial\check{\Phi}}{\partial\gamma}(s)\big(\check{\Phi}(s)\big)^{-1}$$

with (compare (5.21))

$$A=\frac{\gamma}{2\pi i}\check{\Phi}(s)\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\big(\check{\Phi}(s)\big)^{-1}.$$

Since

$$A_{11}=-\frac{\gamma}{2\pi i}\big(\check{\Phi}_{11}(s)+\check{\Phi}_{12}(s)\big)\big(\check{\Phi}_{21}(s)+\check{\Phi}_{22}(s)\big)$$

we can now use the identities derived in section 7.1 for $\check{\Phi}_{ij}(\pm s)$. With

$$\begin{aligned}\mathcal{R}(\pm1) &= I+\frac{1}{2\pi i}\int_{\Sigma_{\mathcal{R}}}\mathcal{R}_-(w)\big(G_{\mathcal{R}}(w)-I\big)\frac{dw}{w\mp1}\\ &= I+\frac{1}{2\pi i}\int_{C_0}\big(G_{\mathcal{R}}(w)-I\big)\frac{dw}{w\mp1}+O\big(s^{-2}\big)\\ &= I\pm\frac{i}{2s}\begin{pmatrix}v & u\\ -u & -v\end{pmatrix}+O\big(s^{-2}\big)\end{aligned} \tag{7.16}$$

and the classical identity

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z},\quad z\in\mathbb{C}\backslash\mathbb{Z}$$

one concludes

$$A_{11} = \nu + O\big(s^{-1}\big), \quad A_{22} = -\nu + O\big(s^{-1}\big)$$

uniformly on any compact subset of the set (1.19). Also

$$A_{12} = \nu(16s^3 + 4xs)^{-2\nu} e^{2s^3\vartheta(1)} \frac{\Gamma(\nu)}{\Gamma(-\nu)} + O\big(s^{-1}\big) \tag{7.17}$$

and

$$A_{21} = -\nu(16s^3 + 4xs)^{2\nu} e^{-2s^3\vartheta(1)} \frac{\Gamma(-\nu)}{\Gamma(\nu)} + O\big(s^{-1}\big). \tag{7.18}$$

Next we use (7.16) to evaluate asymptotically the identities for $\check{\Phi}_{ij}(s)$ obtained in section 7.1

$$\begin{aligned}
\check{\Phi}_{11}(s) &= (16s^3 + 4xs)^{-\nu} e^{i\frac{\pi}{2}\nu} e^{s^3\vartheta(1)} \bigg(c_0(-\nu) + c_1(-\nu)\bigg(\ln(16s^3 + 4xs) - i\frac{\pi}{2}\bigg)\bigg) \\
&\quad + O\bigg(\frac{\ln s}{s}\bigg) \\
\check{\Phi}_{12}(s) &= -(16s^3 + 4xs)^{-\nu} e^{i\frac{\pi}{2}\nu} e^{s^3\vartheta(1)} \bigg(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} + c_1(-\nu)\bigg(\ln(16s^3 + 4xs) \\
&\quad + i\frac{\pi}{2}\bigg)\bigg) + O\bigg(\frac{\ln s}{s}\bigg) \\
\check{\Phi}_{21}(s) &= -(16s^3 + 4xs)^{\nu} e^{i\frac{\pi}{2}\nu} e^{-s^3\vartheta(1)} \bigg(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)} + c_1(\nu)\bigg(\ln(16s^3 + 4xs) \\
&\quad - i\frac{\pi}{2}\bigg)\bigg) + O\bigg(\frac{\ln s}{s}\bigg) \\
\check{\Phi}_{22}(s) &= (16s^3 + 4xs)^{\nu} e^{i\frac{\pi}{2}\nu} e^{-s^3\vartheta(1)} \bigg(c_0(\nu) + c_1(\nu)\bigg(\ln(16s^3 + 4xs) + i\frac{\pi}{2}\bigg)\bigg) \\
&\quad + O\bigg(\frac{\ln s}{s}\bigg).
\end{aligned}$$

Combined with (7.17) and (7.18), we deduce the following asymptotics for $\mathcal{P}_2(s,x,\gamma)$

$$\mathcal{P}_2(s,x,\gamma) = 2\nu e^{i\pi\nu}\Big(c_0(\nu)+c_1(\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)\Big[\nu_\gamma\Big(i\frac{\pi}{2} - \ln(16s^3+4xs)\Big)\Big(c_0(-\nu)+c_1(-\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big) + \Big(c_0(-\nu) + c_1(-\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)_\gamma\Big] - 2\nu e^{i\pi\nu}\Big(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)}+c_1(\nu) \times\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)\Big[\nu_\gamma\Big(i\frac{\pi}{2}-\ln(16s^3+4xs)\Big)\Big(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} +c_1(-\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big) + \Big(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}+c_1(-\nu) \times\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)_\gamma\Big] - \nu e^{i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)}\Big(c_0(\nu)+c_1(\nu)\Big(\ln(16s^3+4xs) +i\frac{\pi}{2}\Big)\Big)\Big(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)}+c_1(\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)_\gamma + \nu e^{i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)} \times\Big(c_0(\nu)+c_1(\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)_\gamma\Big(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)}+c_1(\nu) \times\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big) + \nu e^{i\pi\nu}\frac{\Gamma(-\nu)}{\Gamma(\nu)}\Big(c_0(-\nu)+c_1(-\nu)\Big(\ln(16s^3+4xs) -i\frac{\pi}{2}\Big)\Big)\Big(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}+c_1(-\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)_\gamma - \nu e^{i\pi\nu}\frac{\Gamma(-\nu)}{\Gamma(\nu)} \times\Big(c_0(-\nu)+c_1(-\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)_\gamma\Big(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)} +c_1(-\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big) + O\Big(\frac{(\ln s)^3}{s}\Big), \quad s\to\infty.$$

What is left in the identitiy stated in Proposition 5.3.3 is the term

$$\mathcal{P}_3(s,x,\gamma) = \big(B_{11}-B_{22}\big)\widetilde{\phi}_{11}(-s) + B_{12}\widetilde{\phi}_{21}(-s) + B_{21}\widetilde{\phi}_{12}(-s),$$

with

$$B_{11} = \nu + O\big(s^{-1}\big), \qquad B_{12} = \nu(16s^3+4xs)^{2\nu}e^{-2s^3\vartheta(1)}\frac{\Gamma(-\nu)}{\Gamma(\nu)} + O\big(s^{-1}\big)$$

and

$$B_{21} = -\nu(16s^3+4xs)^{-2\nu}e^{2s^3\vartheta(1)}\frac{\Gamma(\nu)}{\Gamma(-\nu)} + O\big(s^{-1}\big), \quad s\to\infty$$

which, also here, holds uniformly on any compact subset of (1.19). Again (7.16) allows us to simplify the idenitites for $\check{\Phi}_{ij}(-s)$ obtained in section 7.1 and we are lead to the following asymptotics for $\mathcal{P}_3(s,x,\gamma)$

$$\mathcal{P}_3(s,x,\gamma) = 2\nu e^{i\pi\nu}\Big(c_0(-\nu)+c_1(-\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)\Big[\nu_\gamma\Big(i\frac{\pi}{2}$$
$$+\ln(16s^3+4xs)\Big)\Big(c_0(\nu)+c_1(\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)+\Big(c_0(\nu)+c_1(\nu)$$
$$\times\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)_\gamma\Big] - 2\nu e^{i\pi\nu}\Big(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}+c_1(-\nu)$$
$$\times\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)\Big[\nu_\gamma\Big(i\frac{\pi}{2}+\ln(16s^3+4xs)\Big)\Big(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)}$$
$$+c_1(\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)+\Big(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)}$$
$$+c_1(\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)_\gamma\Big] - \nu e^{i\pi\nu}\frac{\Gamma(-\nu)}{\Gamma(\nu)}\Big(c_0(-\nu)+c_1(-\nu)$$
$$\times\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)\Big(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}+c_1(-\nu)\Big(\ln(16s^3+4xs)$$
$$+i\frac{\pi}{2}\Big)\Big)_\gamma + \nu e^{i\pi\nu}\frac{\Gamma(-\nu)}{\Gamma(\nu)}\Big(c_0(-\nu)+c_1(-\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)_\gamma$$
$$\times\Big(c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}+c_1(-\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)+\nu e^{i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)}\Big(c_0(\nu)$$
$$+c_1(\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)\Big(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)}+c_1(\nu)\Big(\ln(16s^3+4xs)$$
$$-i\frac{\pi}{2}\Big)\Big)_\gamma - \nu e^{i\pi\nu}\frac{\Gamma(\nu)}{\Gamma(-\nu)}\Big(c_0(\nu)+c_1(\nu)\Big(\ln(16s^3+4xs)+i\frac{\pi}{2}\Big)\Big)_\gamma$$
$$\times\Big(c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)}+c_1(\nu)\Big(\ln(16s^3+4xs)-i\frac{\pi}{2}\Big)\Big)+O\Big(\frac{(\ln s)^3}{s}\Big),\ s\to\infty.$$

The two expansions for $\mathcal{P}_2(s,x,\gamma)$ and $\mathcal{P}_3(s,x,\gamma)$ combined together allow us to evaluate

$$\int_0^\gamma \big(\mathcal{P}_2(s,x,t)-\mathcal{P}_3(s,x,t)\big)dt. \tag{7.19}$$

In this evaluation it is important to recall the definitions of $c_0(\nu)$ and $c_1(\nu)$

$$c_0(\nu) = -\frac{1}{\Gamma(\nu)}\big(\psi(\nu)+2\gamma_E\big),\quad c_1(\nu) = -\frac{1}{\Gamma(\nu)}$$

as well as the functional equation of the Digamma function (see e.g. [5])

$$\psi(z) = \psi(z+1) - \frac{1}{z} = \psi(1-z) - \pi\cot\pi z, \quad z \in \mathbb{C}\backslash\{0,-1,-2,\ldots\}.$$

It implies

$$c_0(\nu)c_1(-\nu) + c_1(\nu)c_0(-\nu) - c_0(1+\nu)\frac{\Gamma(1+\nu)}{\Gamma(-\nu)}c_1(\nu) - c_1(-\nu)c_0(1-\nu)\frac{\Gamma(1-\nu)}{\Gamma(\nu)} = 0$$

and shows therefore that all terms of $O\big((\ln s)^2\big)$ in (7.19) vanish. The remaining terms of $O(\ln s)$ and $O(1)$ can be computed in a similar way, we obtain

$$\int_0^\gamma \big(\mathcal{P}_2(s,x,t) - \mathcal{P}_3(s,x,t)\big)dt = 6(i\nu)^2\ln s + 8(i\nu)^2\ln 2 \tag{7.20}$$

$$+2\int_0^\gamma \nu(t)\big(\ln\Gamma(\nu(t)) - \ln\Gamma(-\nu(t))\big)_t dt + O\left(\frac{(\ln s)^3}{s}\right)$$

as $s\to\infty$ uniformly on any compact subset of the set (1.19). The latter statement combined with (7.15) implies Theorem 1.3.1 with the "constant" term

$$\chi_0 = 2(i\nu)^2 + 8(i\nu)^2\ln 2 + 2\int_0^\gamma \nu(t)\left(\ln\frac{\Gamma(\nu(t))}{\Gamma(-\nu(t))}\right)_t dt$$

and in terms of Proposition 5.1.2 therefore completes the proof of Theorem 1.18. Up to this point, we have verified Theorem 1.3.1 with an error term of

$$O\left(\frac{(\ln s)^3}{s}\right).$$

We can use Proposition 5.3.1 to improve this error estimate. With (7.16) after simplification

$$R(s,s) = -i\nu(8s^2+2x) - \frac{3(i\nu)^2}{s} + O\big(s^{-2}\big)$$

and also

$$R(-s,-s) = -i\nu(8s^2+2x) - \frac{3(i\nu)^2}{s} + O\big(s^{-2}\big)$$

which implies via (2.18)

$$\frac{\partial}{\partial s}\ln\det(I-\gamma K_{\mathrm{csin}}) = i\nu(16s^3+4x) + \frac{6(i\nu)^2}{s} + O\big(s^{-2}\big), \quad s\to\infty$$

uniformly on any compact subset of the set (1.19). Integrating the latter equation with respect to s and comparing with (7.20), we have completed the proof of Theorem 1.3.1.

7.3 Proof of Theorem 1.3.3

In order to derive the large zero distribution of $\det(I-\gamma K_{\mathrm{csin}})$ for $\gamma>1$, we trace back all relevant transformations and use (7.1),(7.2). First

$$\theta(\lambda)\mapsto\Phi(\lambda)\mapsto\Upsilon(z)\mapsto\Delta(z)\mapsto\mathcal{R}(z)\mapsto\mathcal{P}(z)\mapsto\mathcal{Q}(z)$$

and we connect the values of $\mathcal{R}(\pm1)$ and $\mathcal{R}'(\pm1)$ to $\mathcal{Q}(\pm1)$ and its derivative:

$$\mathcal{R}(1)=\Big(\mathcal{Q}(1)+(I+\widehat{B})\mathcal{Q}'(1)\Big)\begin{pmatrix}1 & 0\\ ip & 0\end{pmatrix}+(I+\widehat{B})\mathcal{Q}(1)\begin{pmatrix}0 & 0\\ \nu_0 ip\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}} & \frac{1}{2}\end{pmatrix}\tag{7.21}$$

and

$$\mathcal{R}'(1)=\big(\mathcal{Q}(1)+(I+\widehat{B})\mathcal{Q}'(1)\big)\begin{pmatrix}0 & 0\\ \nu_0 ip\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}} & \frac{1}{2}\end{pmatrix}+(I+\widehat{B})\mathcal{Q}(1)\tag{7.22}$$
$$\times\begin{pmatrix}0 & 0\\ \nu_0 ip\widehat{\kappa}(s,x) & -\frac{1}{4}\end{pmatrix}+\left(\mathcal{Q}'(1)+(I+\widehat{B})\frac{\mathcal{Q}''(1)}{2}\right)\begin{pmatrix}1 & 0\\ ip & 0\end{pmatrix}$$

where

$$\widehat{\kappa}(s,x)=\frac{1}{2}\left(\frac{10}{3}\Big(1+\frac{x}{4s^2}\Big)+\Big(\nu_0-\frac{1}{2}\Big)\Big(3+\frac{x}{4s^2}\Big)^2\right)\Big(1+\frac{x}{4s^2}\Big)^{-2}.$$

Also

$$\mathcal{R}(-1)=\Big(\mathcal{Q}(-1)-(I-\widehat{B})\mathcal{Q}'(-1)\Big)\begin{pmatrix}0 & ip\\ 0 & 1\end{pmatrix}-(I-\widehat{B})\mathcal{Q}(-1)\begin{pmatrix}-\frac{1}{2} & -\nu_0 ip\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\\ 0 & 0\end{pmatrix}$$

and

$$\mathcal{R}'(-1)=\Big(\mathcal{Q}(-1)-(I-\widehat{B})\mathcal{Q}'(-1)\Big)\begin{pmatrix}-\frac{1}{2} & -\nu_0 ip\frac{3+\frac{x}{4s^2}}{1+\frac{x}{4s^2}}\\ 0 & 0\end{pmatrix}\tag{7.23}$$
$$+\left(\mathcal{Q}'(-1)-(I-\widehat{B})\frac{\mathcal{Q}''(-1)}{2}\right)\begin{pmatrix}0 & ip\\ 0 & 1\end{pmatrix}-(I-\widehat{B})\mathcal{Q}(-1)\begin{pmatrix}-\frac{1}{4} & \nu_0 ip\widehat{\kappa}(s,x)\\ 0 & 0\end{pmatrix}.$$

Next we evaluate $\mathcal{Q}(\pm 1)$: For any $z \in \Sigma_{\mathcal{Q}}$

$$\begin{aligned}
\mathcal{Q}_-(z) &= I + \frac{1}{2\pi i}\int\limits_{\Sigma_{\mathcal{R}}} \mathcal{Q}_-(w)\big(G_{\mathcal{Q}}(w) - I\big)\frac{dw}{w - z_-} \\
&= I + \frac{i}{2sz}\left(\begin{pmatrix} v & -u \\ u & -v \end{pmatrix} - \begin{pmatrix} \frac{1}{z-1} & 0 \\ 0 & \frac{1}{z+1} \end{pmatrix}\big(B_0(z)\big)^{-1}\right. \\
&\quad \left.\times \begin{pmatrix} v & ue^{-2\pi i\nu} \\ -ue^{2\pi i\nu} & -v \end{pmatrix} B_0(z)\begin{pmatrix} z-1 & 0 \\ 0 & z+1 \end{pmatrix}\right) + O\big(s^{-2}\big),
\end{aligned}$$

and thus

$$\begin{aligned}
\mathcal{Q}(\pm 1) &= I \pm \frac{i}{2s}\begin{pmatrix} v & -u \\ u & -v \end{pmatrix} \mp \frac{\nu_0}{s^2}\begin{pmatrix} -u^2 & -u_x \\ u_x & u^2 \end{pmatrix} \\
&\quad + \frac{1}{8s^2}\begin{pmatrix} u^2 - v^2 & -2(u_x + uv) \\ -2(u_x + uv) & u^2 - v^2 \end{pmatrix} + O\big(s^{-3}\big), \quad s \to \infty.
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{Q}'(\pm 1) &= -\frac{i}{2s}\begin{pmatrix} v & -u \\ u & -v \end{pmatrix} + \frac{\nu_0}{s^2}\begin{pmatrix} -u^2 & -u_x \\ u_x & u^2 \end{pmatrix} \\
&\quad \mp \frac{1}{4s^2}\begin{pmatrix} u^2 - v^2 & -2(u_x + uv) \\ -2(u_x + uv) & u^2 - v^2 \end{pmatrix} + O\big(s^{-3}\big).
\end{aligned}$$

These computations imply for the matrix

$$\mathcal{N} = \left(Q(1)\begin{pmatrix} 1 \\ ip \end{pmatrix}, Q(-1)\begin{pmatrix} ip \\ 1 \end{pmatrix}\right),$$

which appears in (6.20), that

$$\begin{aligned}
\det \mathcal{N} &= 2p\Big(\cos\sigma - \frac{v}{s}\sin\sigma + \frac{u}{s} \\
&\quad + \frac{2i\nu_0}{s^2}\big(u_x + u^2\sin\sigma\big) + \frac{u^2 - v^2}{2s^2}\cos\sigma + O\big(s^{-3}\big)\Big). \qquad (7.24)
\end{aligned}$$

We agreed that s stays away from the small neighborhood of the points $\{s_n\}$ defined by $\cos\sigma(s_n, x, \gamma) = 0$ and therefore, for sufficiently large s lying outside of the zero set of the latter transcendental equation, the stated determinant is non-zero. Back to (6.20), this implies

$$\widehat{B} = -\frac{2ip}{\det\mathcal{N}}\Bigg[\begin{pmatrix}\sin\sigma & -1\\ 1 & -\sin\sigma\end{pmatrix} + \frac{\cos\sigma}{s}\begin{pmatrix}v & -u\\ u & -v\end{pmatrix} + \frac{1}{s^2}\Bigg\{2i\nu_0\cos\sigma$$
$$\times\begin{pmatrix}-u^2 & -u_x\\ u_x & u^2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-v^2\sin\sigma & -u^2\\ u^2 & v^2\sin\sigma\end{pmatrix} + \frac{u_x}{2}\begin{pmatrix}-1 & \sin\sigma\\ -\sin\sigma & 1\end{pmatrix}$$
$$+uv\sin\sigma\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\Bigg\} + O\big(s^{-3}\big)\Bigg], \qquad s\to\infty$$

and with

$$-\frac{2ip}{\det\mathcal{N}} = \frac{1}{\cos\sigma}\Bigg[1 + \frac{1}{s}\Big(v\tan\sigma - \frac{u}{\cos\sigma}\Big) + \frac{1}{s^2}\Big(v^2\tan^2\sigma - \frac{2uv\sin\sigma}{\cos^2\sigma} + \frac{u^2}{\cos^2\sigma}$$
$$-\frac{u^2-v^2}{2} - 2i\nu_0 u^2\tan\sigma - \frac{2i\nu_0 u_x}{\cos\sigma}\Big) + O\big(s^{-3}\big)\Bigg]$$

we obtain in turn

$$\widehat{B} = \frac{1}{\cos\sigma}\begin{pmatrix}\sin\sigma & -1\\ 1 & -\sin\sigma\end{pmatrix} + \frac{1}{s}\Bigg\{\frac{v\sin\sigma - u}{\cos^2\sigma}\begin{pmatrix}\sin\sigma & -1\\ 1 & -\sin\sigma\end{pmatrix} + \begin{pmatrix}v & -u\\ u & -v\end{pmatrix}\Bigg\}$$
$$+\frac{1}{s^2}\Bigg\{\frac{1}{2\cos\sigma}\begin{pmatrix}2v^2\sin\sigma - u^2\sin\sigma - u_x & -v^2 + u_x\sin\sigma\\ v^2 - u_x\sin\sigma & -2v^2\sin\sigma + u^2\sin\sigma + u_x\end{pmatrix}$$
$$-\frac{2i\nu_0(u^2\sin\sigma + u_x)}{\cos^2\sigma}\begin{pmatrix}\sin\sigma & -1\\ 1 & -\sin\sigma\end{pmatrix} + \frac{(v\sin\sigma - u)^2}{\cos^3\sigma}\begin{pmatrix}\sin\sigma & -1\\ 1 & -\sin\sigma\end{pmatrix}$$
$$+\begin{pmatrix}-uv\cos\sigma - 2i\nu_0 u^2 & u^2\cos\sigma - 2i\nu_0 u_x\\ -u^2\cos\sigma + 2i\nu_0 u_x & uv\cos\sigma + 2i\nu_0 u^2\end{pmatrix}\Bigg\} + O\big(s^{-3}\big), \qquad s\to\infty$$

where all expansions are uniformly on any compact subset of the set (1.29). Let us go back to (7.1) and (7.2). Since $\nu = \nu_0 + \frac{1}{2}$, we notice

$$R(s,s) = -i\nu_0(8s^2+2x) - i(4s^2+x)$$
$$-(16s^2+4x)\Big[\mathcal{R}_{12}'(1)\mathcal{R}_{22}(1) - \mathcal{R}_{22}'(1)\mathcal{R}_{12}(1)\Big] + O\big(s^{-1}\big)$$

and similarly

$$\begin{aligned} R(-s,-s) &= -i\nu_0(8s^2+2x) - i(4s^2+x) - (16s^2+4x) \\ &\quad \times\Big[\mathcal{R}'_{11}(-1)\mathcal{R}_{21}(-1) - \mathcal{R}'_{21}(-1)\mathcal{R}_{11}(-1)\Big] + O\big(s^{-1}\big). \end{aligned}$$

Next

$$\begin{aligned} \mathcal{R}'_{12}(1) &= \frac{1}{2}\big(\mathcal{Q}(1) + (I+\widehat{B})\mathcal{Q}'(1)\big)_{12} - \frac{1}{4}\big((I+\widehat{B})\mathcal{Q}(1)\big)_{12} \\ \mathcal{R}_{22}(1) &= \frac{1}{2}\big((I+\widehat{B})\mathcal{Q}(1)\big)_{22} \\ \mathcal{R}'_{22}(1) &= \frac{1}{2}\big(\mathcal{Q}(1) + (I+\widehat{B})\mathcal{Q}'(1)\big)_{22} - \frac{1}{4}\big((I+\widehat{B})\mathcal{Q}(1)\big)_{22} \\ \mathcal{R}_{12}(1) &= \frac{1}{2}\big((I+\widehat{B})\mathcal{Q}(1)\big)_{12} \end{aligned}$$

and therefore

$$\begin{aligned} \mathcal{R}'_{12}(1)\mathcal{R}_{22}(1) - \mathcal{R}'_{22}(1)\mathcal{R}_{12}(1) &= \frac{1}{4}\Big[\big(\mathcal{Q}(1) + (I+\widehat{B})\mathcal{Q}'(1)\big)_{12}\big((I+\widehat{B})\mathcal{Q}(1)\big)_{22} \\ &\quad -\big(\mathcal{Q}(1) + (I+\widehat{B})\mathcal{Q}'(1)\big)_{22}\big((I+\widehat{B})\mathcal{Q}(1)\big)_{12}\Big]. \end{aligned}$$

We combine the previously derived information on $\mathcal{Q}(1), \mathcal{Q}'(1)$ and $\widehat{B}$ to derive

$$-(16s^2+4x)\Big[\mathcal{R}'_{12}(1)\mathcal{R}_{22}(1) - \mathcal{R}'_{22}(1)\mathcal{R}_{12}(1)\Big] = (4s^2+x)\big(i+\tan\sigma\big) + \widehat{\alpha}_+ + O\big(s^{-1}\big)$$

with a function $\widehat{\alpha}_+ = \widehat{\alpha}_+(s,x,\gamma)$ such that

$$\int \widehat{\alpha}_+(s,x,\gamma)ds = O(\ln s), \quad s\to\infty.$$

Also for $R(-s,-s)$

$$R(-s,-s) = -i\nu_0(8s^2+2x) + (4s^2+x)\tan\sigma + \widehat{\alpha}_- + O\big(s^{-1}\big). \tag{7.25}$$

where

$$\int \widehat{\alpha}_-(s,x,\gamma)ds = O(\ln s).$$

All together from Proposition 5.3.1

$$\frac{\partial}{\partial s}\ln\det\left(I-\gamma K_{\text{csin}}\right) = i\nu_0(16s^2+4x) - (8s^2+2x)\tan\sigma - \big(\widehat{\alpha}_+ + \widehat{\alpha}_-\big) + O\big(s^{-1}\big). \tag{7.26}$$

Opposed to the latter equation, we now recall Proposition 5.3.2 and evaluate the logarithmic x-derivative. For $\gamma > 1$,

$$\begin{aligned}\Phi_1 &= \lim_{\lambda\to\infty}\Big(\lambda\big(\Phi(\lambda)e^{-i(\frac{4}{3}\lambda^3+x\lambda)\sigma_3}-I\big)\Big)\\ &= -2\nu s\sigma_3 + s(\sigma_3+\widehat{B}) + \frac{is}{2\pi}\int_{\Sigma_{\mathcal{R}}}\mathcal{Q}_-(w)(w)\big(G_{\mathcal{Q}}(w)-I\big)dw\end{aligned}$$

where the expansion for $\widehat{B}$ has already been computed. From this and residue theorem

$$\begin{aligned}\Phi_1 &= -2\nu s\sigma_3 + s\sigma_3 - \frac{is}{\cos\sigma}\begin{pmatrix}\sin\sigma & -1\\ 1 & -\sin\sigma\end{pmatrix} - \frac{i}{2}\begin{pmatrix}v & -u\\ u & -v\end{pmatrix}\\ &\quad -\frac{i(v\sin\sigma-u)}{\cos^2\sigma}\begin{pmatrix}\sin\sigma & -1\\ 1 & -\sin\sigma\end{pmatrix} + O\big(s^{-1}\big),\end{aligned}$$

hence

$$\frac{\partial}{\partial x}\ln\det\left(I-\gamma K_{\mathrm{csin}}\right) = 4i\nu_0 s + v - 2s\tan\sigma - \frac{2v}{\cos^2\sigma} + \frac{2u\sin\sigma}{\cos^2\sigma} + O\big(s^{-1}\big). \quad (7.27)$$

Integrating both identities (7.26), (7.27) and comparing the result, we conclude for $s\to\infty$ away from the zeros of $\cos\sigma = 0$

$$\begin{aligned}\ln\det(I-\gamma K_{\mathrm{csin}}) &= i\nu_0\left(\frac{16}{3}s^3+4sx\right) + \ln|\cos\sigma(s,x,\gamma)| + c_2\ln s\\ &\quad -\int_x^{\infty}(y-x)u^2(y,\gamma)dy + c_3(\gamma) + O\big(s^{-1}\big), \quad (7.28)\end{aligned}$$

with real-valued constants c_i, solely depending on γ and the error term is uniform on any compact subset of the set (1.29). The given expansion (7.28) verifies the claim on the asymptotic distribution of the zeros of the Fredholm determinant as given in Theorem 1.3.3.

7.4 Proof of Theorem 1.3.2

The final two sections of this dissertation complete the proofs of Theorems 1.2.1 and 1.3.2. We use the logarithmic t-derivative as prepared in Proposition 5.3.4, i.e.

we need to derive the large s-asymptotics of the coefficients Θ_1, Θ_2 and Θ_3 in the asymptotic series

$$\Theta(\lambda) = I + \frac{\Theta_1}{\lambda} + \frac{\Theta_2}{\lambda^2} + \frac{\Theta_3}{\lambda^3} + O\big(\lambda^{-4}\big), \qquad \lambda \to \infty.$$

First trace back the relevant transformations

$$\Theta(\lambda) \mapsto \Phi(\lambda) \mapsto \Lambda(z) \mapsto \mathcal{K}(z)$$

and recall the important estimations (6.37) and (6.38)

$$\|G_{\mathcal{K}} - I\|_{L^2\cap L^\infty(\Sigma_{\mathcal{K}})} \leq cs^{-1}, \quad \|\mathcal{K}_- - I\|_{L^2(\Sigma_{\mathcal{K}})} \leq cs^{-1}, \quad s \to \infty$$

which are uniform in the parameters (x,t) chosen from any compact subset of the set

$$\big\{(x,t) \in \mathbb{R}^2: \ -\infty < x < \infty, \ 0 \leq t \leq 1\big\}. \tag{7.29}$$

We compute

$$\begin{aligned}\Theta_1 &= \lim_{\lambda\to\infty}\Big(\lambda\big(\Theta(\lambda) - I\big)\Big), \qquad \Theta_2 = \lim_{\lambda\to\infty}\Big(\lambda^2\Big(\Theta(\lambda) - I - \frac{\Theta_1}{\lambda}\Big)\Big),\\ \Theta_3 &= \lim_{\lambda\to\infty}\Big(\lambda^3\Big(\Theta(\lambda) - I - \frac{\Theta_1}{\lambda} - \frac{\Theta_2}{\lambda^2}\Big)\Big).\end{aligned}$$

and therefore need the following expansions:

$$\mathcal{N}(z) = I + \frac{i}{2z}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} + \frac{1}{8z^2}\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} + \frac{3i}{16z^3}\begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix} + O\big(z^{-4}\big),$$

as well as

$$\hat{g}(z) = i\Big(\frac{4}{3}tz^3 + \frac{xz}{s^2}\Big) - \frac{i}{2z}\Big(t + \frac{x}{s^2}\Big) - \frac{i}{2z^3}\Big(\frac{t}{3} + \frac{x}{4s^2}\Big) + O\big(z^{-5}\big).$$

and

$$\begin{aligned}\mathcal{K}(z) &= I + \frac{i}{2\pi z}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w) - I\big)dw + \frac{i}{2\pi z^2}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w) - I\big)w\,dw\\ &\quad + \frac{i}{2\pi z^3}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w) - I\big)w^2\,dw + O\big(z^{-4}\big),\end{aligned}$$

and also

$$e^{(s^3\hat g(z)-is^3(\frac{4}{3}tz^3+\frac{xz}{s^2}))\sigma_3} = I-\frac{i}{2z}(ts^3+xs)\sigma_3-\frac{1}{8z^2}(ts^3+xs)^2I$$
$$-\frac{i}{2z^3}\left(\frac{ts^3}{3}+\frac{xs}{4}\right)\sigma_3+\frac{i}{48z^3}(ts^3+xs)^3\sigma_3+O\big(z^{-4}\big).$$

They imply

$$\Theta_1 = s\lim_{z\to\infty}\Big(z\big(\mathcal{R}(z)\mathcal{N}(z)e^{(s^3\hat g(z)-is^3(\frac{4}{3}tz^3+\frac{xz}{s^2}))\sigma_3}-I\big)\Big)$$
$$= s\bigg(-\frac{i}{2}\big(ts^3+xs\big)\sigma_3-\frac{\sigma_2}{2}+\frac{i}{2\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)dw\bigg)$$

and

$$\Theta_2 = s^2\bigg(-\frac{1}{8}\big(ts^3+xs\big)^2I-\frac{1}{4}\big(ts^3+xs\big)\sigma_1+\frac{I}{8}$$
$$+\frac{1}{4\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)dw\,\big(ts^3+xs\big)\sigma_3$$
$$-\frac{i}{4\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)dw\sigma_2+\frac{i}{2\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)w\,dw\bigg).$$

Moreover

$$\Theta_3 = s^3\bigg(\frac{i}{48}\big(ts^3+xs\big)^3\sigma_3+\frac{1}{16}\big(ts^3+xs\big)^2\sigma_2-\frac{i}{16}\big(ts^3+xs\big)\sigma_3$$
$$-\frac{i}{2}\left(\frac{ts^3}{3}+\frac{xs}{4}\right)\sigma_3-\frac{3i}{16}\sigma_2-\frac{i}{16\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)dw\,\big(ts^3+xs\big)^2$$
$$-\frac{i}{8\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)dw\,(ts^3+xs)\sigma_1+\frac{i}{16\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)dw$$
$$+\frac{1}{4\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)w\,dw\,(ts^3+xs)\sigma_3$$
$$-\frac{i}{4\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)w\,dw\sigma_2+\frac{i}{2\pi}\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)w^2dw\bigg).$$

Our next move focuses on the computation of $J_n=\int_{\Sigma_{\mathcal{K}}}\mathcal{K}_-(w)\big(G_{\mathcal{K}}(w)-I\big)w^n\,dw, n=0,1,2$. As we see from (6.37) and (6.38)

$$J_n=O\big(s^{-3}\big),\quad s\to\infty,\ t>0\qquad J_n=O\big(s^{-1}\big),\quad s\to\infty,\ t=0$$

on the other hand (5.30) has to be evaluated up to $O(s^{-1})$ in order to determine the constant term in Theorem 1.27. Hence we need to iterate the underlying integral equation. First in case $t > 0$ for $z \in \Sigma_{\mathcal{K}}$

$$\mathcal{K}_-(z) - I = \frac{1}{2\pi i}\int_{\hat{C}_r}\big(G_{\mathcal{K}}(w) - I\big)\frac{dw}{w - z_-} + \frac{1}{2\pi i}\int_{\hat{C}_l}\big(G_{\mathcal{K}}(w) - I\big)\frac{dw}{w - z_-} + O\big(s^{-6}\big)$$

and if $t = 0$ the latter error term is of order $O\big(s^{-2}\big)$. Thus as $s \to \infty$

$$\begin{aligned}\mathcal{K}_-(z) - I &= \frac{1}{2\pi i}\int_{\hat{C}_r}\frac{i}{16\sqrt{\zeta}}\begin{pmatrix} 3\beta^{-2} - \beta^2 & i(3\beta^{-2} + \beta^2) \\ i(3\beta^{-2} + \beta^2) & -(3\beta^{-2} - \beta^2)\end{pmatrix}\frac{dw}{w - z_-} \\ &+ \frac{1}{2\pi i}\int_{\hat{C}_l}\frac{1}{16\sqrt{\zeta}}\begin{pmatrix} 3\beta^{2} - \beta^{-2} & -i(3\beta^{2} + \beta^{-2}) \\ -i(3\beta^{2} + \beta^{-2}) & -(3\beta^{2} - \beta^{-2})\end{pmatrix}\frac{dw}{w - z_-}\end{aligned}$$

modulo a correction term. Since

$$\begin{aligned}\int_{\hat{C}_r}\frac{\beta^{-2}(w)}{\sqrt{\zeta(w)}(w - z_-)}dw &= -\frac{3}{4s^3}\left(z^2t + \frac{t}{2} + \frac{3x}{4s^2}\right)^{-1}\frac{2\pi i}{z+1} \\ \int_{\hat{C}_r}\frac{\beta^{2}(w)}{\sqrt{\zeta(w)}(w - z_-)}dw &= \frac{3}{4s^3}\left(\left(\frac{3t}{2} + \frac{3x}{4s^2}\right)^{-1} - \left(z^2t + \frac{t}{2} + \frac{3x}{4s^2}\right)^{-1}\right)\frac{2\pi i}{z-1} \\ \int_{\hat{C}_l}\frac{\beta^{2}(w)}{\sqrt{\zeta(w)}(w - z_-)}dw &= -\frac{3i}{4s^3}\left(z^2t + \frac{t}{2} + \frac{3x}{4s^2}\right)^{-1}\frac{2\pi i}{z-1} \\ \int_{\hat{C}_l}\frac{\beta^{-2}(w)}{\sqrt{\zeta(w)}(w - z_-)}dw &= \frac{3i}{4s^3}\left(\left(\frac{3t}{2} + \frac{3x}{4s^2}\right)^{-1} - \left(z^2t + \frac{t}{2} + \frac{3x}{4s^2}\right)^{-1}\right)\frac{2\pi i}{z+1}\end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{K}_-(z) - I &= \frac{3i}{64s^3}\left(-2\left(z^2t + \frac{t}{2} + \frac{3x}{4s^2}\right)^{-1} - \left(\frac{3t}{2} + \frac{3x}{4s^2}\right)^{-1}\right) \\ &\quad\times\left[\frac{1}{z-1}\begin{pmatrix}1 & -i \\ -i & -1\end{pmatrix} + \frac{1}{z+1}\begin{pmatrix}1 & i \\ i & -1\end{pmatrix}\right] + O\big(s^{-6}\big) \qquad (7.30) \\ &\equiv \frac{3i}{64s^3}f(z,t)\left[\frac{1}{z-1}\begin{pmatrix}1 & -i \\ -i & -1\end{pmatrix} + \frac{1}{z+1}\begin{pmatrix}1 & i \\ i & -1\end{pmatrix}\right] + O\big(s^{-6}\big),\end{aligned}$$

for $t>0$ respectively with a correction term of order $O\big(s^{-2}\big)$ in case $t=0$. We are going to improve the latter estimation via iteration

$$\mathcal{K}_-(z)-I=\int\limits_{\Sigma_{\mathcal{K}}}\big(\mathcal{K}_-(w)-I\big)\big(G_{\mathcal{K}}(w)-I\big)\frac{dw}{w-z_-}+\int\limits_{\Sigma_{\mathcal{K}}}\big(G_{\mathcal{K}}(w)-I\big)\frac{dw}{w-z_-}$$

and the first integral $\mathcal{K}_-(z)-I$ is given by (7.30). By residue theorem

$$\mathcal{K}_-(z)-I=\frac{3i}{64s^3}f(z,t)\left[\frac{1}{z-1}\begin{pmatrix}1 & -i\\ -i & -1\end{pmatrix}+\frac{1}{z+1}\begin{pmatrix}1 & i\\ i & -1\end{pmatrix}\right]$$
$$-\frac{27i}{128s^6}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-2}\left[\frac{1}{z-1}\begin{pmatrix}1+\frac{i}{32} & 2i+\frac{1}{8}\\ -2i-\frac{1}{8} & 1+\frac{i}{32}\end{pmatrix}-\frac{1}{z+1}\begin{pmatrix}1+\frac{i}{32} & -2i-\frac{1}{8}\\ 2i+\frac{1}{8} & 1+\frac{i}{32}\end{pmatrix}\right]$$
$$+\frac{9i}{16s^6}h(z,t)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+O\big(s^{-9}\big),\quad s\to\infty$$

with

$$h(z,t)=\frac{\big(z^2t+\frac{t}{2}+\frac{3x}{4s^2}\big)^{-1}}{(z+1)(z-1)}\left[\left(z^2t+\frac{t}{2}+\frac{3x}{4s^2}\right)^{-1}\left(1+\frac{3i}{64}\right)+\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\right]$$

for $t>0$ and with an error term of order $O\big(s^{-3}\big)$ in case $t=0$. Having the latter information we first compute J_0

$$J_0=\int\limits_{\Sigma_{\mathcal{K}}}\big(\mathcal{K}_-(w)-I\big)\big(G_{\mathcal{K}}(w)-I\big)dw+\int\limits_{\Sigma_{\mathcal{K}}}\big(G_{\mathcal{K}}(w)-I\big)dw.$$

All integrals can be evaluated via residue theorem, we summarize the results

$$\frac{1}{2\pi i}\int\limits_{\Sigma_{\mathcal{K}}}\big(G_{\mathcal{K}}(w)-I\big)dw=\frac{3i}{32s^3}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\sigma_3-\frac{27}{512s^6}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-2}\sigma_2$$
$$+\frac{405i}{65536s^9}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\left[\frac{5}{4}\frac{\frac{27t}{2}+\frac{3x}{4s^2}}{\frac{3t}{2}+\frac{3x}{4s^2}}+\frac{7}{4}\right]\sigma_3+O\big(s^{-12}\big),$$

as $s\to\infty$ for $t>0$ and with an error term of order $O\big(s^{-4}\big)$ in case $t=0$. Similarly

$$\frac{1}{2\pi i}\int\limits_{\hat{C}_{r,l}}\big(\mathcal{K}_-(w)-I\big)\big(G_{\mathcal{K}}(w)-I\big)dw=\frac{27}{512s^6}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-2}\sigma_2$$
$$+\frac{81}{1024s^9}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\left(1-\frac{3i}{256}\right)\sigma_3+\frac{81}{8192s^9}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\left(1-\frac{5i}{32}\right)\sigma_3$$
$$+\frac{27}{32768s^9}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-4}\left[49t+3it+\left(\frac{211}{2}+3i\right)\left(\frac{19t}{2}+\frac{3x}{4s^2}\right)\right]\sigma_3$$
$$-\frac{81}{8192s^9}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\left[\left(1+\frac{i}{32}\right)\left(3i-4it\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\right)\right.$$
$$\left.-\left(2i+\frac{1}{8}\right)\left(3+4t\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\right)\right]\sigma_1+O\big(s^{-12}\big),\quad s\to\infty$$

and we summarize

$$\frac{J_0}{2\pi i}=\frac{3i}{32s^3}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\sigma_3+\frac{27}{64s^9}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\big(a(s,t)\sigma_3+b(s,t)\sigma_1\big)+O\big(s^{-12}\big) \tag{7.31}$$

as $s\to\infty$ for $t>0$ respectively with an error term of order $O\big(s^{-4}\big)$ for $t=0$. Here the functions $a=a(s,t)$ and $b=b(s,t)$ can be read of from the previous lines, we state J_0 in this form since as we will see, only the structure of the term of order $O\big(s^{-9}\big)$ matters. Moving on to J_1 and J_2 similar computations imply

$$\frac{J_1}{2\pi i}=\frac{3}{32s^3}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\sigma_1-\frac{81}{2048s^6}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-2}I+O\big(s^{-9}\big) \tag{7.32}$$

and

$$\frac{J_2}{2\pi i}=\frac{3i}{32s^3}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\sigma_3+O\big(s^{-9}\big) \tag{7.33}$$

in the limit $s\to\infty$ for $t>0$ or in case $t=0$ with adjusted error terms. It is now straight forward to use the given information (7.31), (7.32) and (7.33) to obtain the large s-asymptotics for Θ_1,Θ_2 and Θ_3. Once we have the latter expansions we go back to (5.30)

$$\frac{4i}{3}\mathrm{trace}\,\big(-3\Theta_3\sigma_3\big)=\frac{s^3}{6}\big(ts^3+xs\big)^3-\frac{s^3}{2}\big(ts^3+xs\big)-4s^3\left(\frac{ts^3}{3}+\frac{xs}{4}\right)$$
$$+\frac{3}{32}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\big(ts^3+xs\big)^2-\frac{27i}{64s^6}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\big(ts^3+xs\big)^2a(s,t)$$
$$-\frac{15}{32}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}-\frac{81}{512s^3}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-2}\big(ts^3+xs\big)+O\big(s^{-3}\big),\quad s\to\infty$$

and

$$\frac{4i}{3}\mathrm{trace}\,\big(2\Theta_2\sigma_3\Theta_1\big) = -\frac{3}{16}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\big(ts^3+xs\big)^2$$
$$+\frac{27i}{32s^6}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\big(ts^3+xs\big)^2 a(s,t)-\frac{s^3}{3}\big(ts^3+xs\big)^3+s^3\big(ts^3+xs\big)$$
$$+\frac{3}{16}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}+\frac{21}{256s^3}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-2}\big(ts^3+xs\big)+O\big(s^{-3}\big)$$

as well as

$$\frac{4i}{3}\mathrm{trace}\,\big(-\Theta_1\sigma_3(\Theta_1^2-\Theta_2)\big) = \frac{39}{512s^3}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-2}\big(ts^3+xs\big)$$
$$+\frac{3}{32}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}\big(ts^3+xs\big)^2-\frac{27i}{64s^6}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-3}\big(ts^3+xs\big)^2 a(s,t)$$
$$-\frac{s^3}{2}\big(ts^3+xs\big)+\frac{s^6}{6}\big(ts^3+xs\big)^3-\frac{3}{32}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}+O\big(s^{-3}\big)$$

in all cases as $s\to\infty$ uniformly on any compact subset of the set (7.29). Now use Proposition 5.3.4 and add up the latter three identities

$$\frac{\partial}{\partial t}\ln\det\big(I-\check{K}_{\mathrm{csin}}\big) = -\frac{4}{3}ts^6-xs^4-\frac{3}{8}\left(\frac{3t}{2}+\frac{3x}{4s^2}\right)^{-1}+O\big(s^{-3}\big),\quad s\to\infty$$

uniformly on any compact subset of the set (7.29). Now integrate and obtain

$$\begin{aligned}\int_0^1\frac{\partial}{\partial t}\ln\det(I-\check{K}_{\mathrm{csin}})\,dt &= -\frac{2}{3}s^6-s^4x-\frac{1}{4}\ln\left(\frac{3}{2}+\frac{3x}{4s^2}\right)+\frac{1}{4}\ln\left(\frac{3x}{4s^2}\right)+O\big(s^{-3}\big)\\ &= -\frac{2}{3}s^6-s^4x-\frac{1}{2}\ln s+\frac{1}{4}\ln x-\frac{1}{4}\ln 2+O\big(s^{-2}\big).\end{aligned}$$

On the other hand

$$\int_0^1\frac{\partial}{\partial t}\ln\det(I-\check{K}_{\mathrm{csin}})\,dt = \ln\det(I-K_{\mathrm{csin}})-\ln\det(I-K_{\mathrm{sin}})$$

and we know (see [26])

$$\ln\det(I-K_{\mathrm{sin}}) = -\frac{(xs)^2}{2}-\frac{1}{4}\ln(sx)+\frac{1}{12}\ln 2+3\zeta'(-1)+O\big(s^{-1}\big),\ s\to\infty$$

hence together as $s\to\infty$

$$\ln\det(I-K_{\mathrm{csin}}) = -\frac{2}{3}s^6-xs^4-\frac{(xs)^2}{2}-\frac{3}{4}\ln s-\frac{1}{6}\ln 2+3\zeta'(-1)+O\big(s^{-1}\big)\quad (7.34)$$

and the error term is uniform on any compact subset of the set (1.15). This proves Theorem 1.3.2.

7.5 Proof of Theorem 1.2.1 with constant term

From Proposition 5.1.2 and equations (4.24), (7.34) we obtain immediately

$$\ln\det(I - K_{\mathrm{PII}}) = -\frac{2}{3}s^6 - s^4 x - \frac{3}{4}\ln s + \int_x^\infty (y-x)u^2(y)dy + \omega + O\big(s^{-1}\big)$$

uniformly on any compact subset of the set (1.15) with $\omega = -\frac{1}{6}\ln 2 + 3\zeta'(-1)$. This proves Theorem 1.2.1.

8. SUMMARY

The current thesis focused on the asymptotical analysis of two one-parameter families of Fredholm determinants $\det(I-\gamma K), \gamma \in \mathbb{R}$ with the trace class operators $K = K_{\mathrm{PII}}$ and $K = K_{\mathrm{csin}}$ acting on $L^2\big((-s,s);d\lambda\big)$. We were able to derive the large s-asymptotics in both cases for $\gamma \leq 1$ including the "constant" terms and stated the large zero distributions for $\gamma > 1$. We want to discuss some possibilities for future projects related to the determinants studied in this thesis.

- The stated expansions show, that for both kernels, the point $\gamma = 1$ is a critical point, i.e. at this value of the parameter the large s-behavior of all determinants undergoes a qualitative change. Hence it is a natural question to ask for the relevant double-scale asymptotics as $s \to \infty, \gamma \to 1$. So far the only attempt to describe such transitional behavior was done by Dyson in case of the sine - kernel determinant. He uses a Coulomb gas interpretation and derives a heuristic formula for the double-scale asymptotics which involves Jacobi theta-functions associated with a certain elliptic curve. It is desirable to turn Dyson's analysis into a rigorous approach and to extend the strategy to the kernels $K = K_{\mathrm{PII}}$ and $K = K_{\mathrm{csin}}$.

- We mentioned in section 2.4 the possibility to derive a differential equation associated with $\det(I-\gamma K_{\mathrm{PII}})$. If available, this equation considerably reduces the computational effort in the asymptotical analysis. In case of the sine - kernel Jimbo, Miwa, Mori and Sato derived an integrable system whose tau-function is represented by $\det(I-K_{\sin})$. This result connects the latter determinant to the fifth Painlevé equation and gives hope that similar systems can be derived for $K = K_{\mathrm{PII}}$ as well as $K = K_{\mathrm{csin}}$.

- In case $\gamma > 1$ we mainly focused on the large zero distribution of $\det(I - \gamma K)$, although our analysis produces asymptotic series up to certain "constant" terms, see (4.11) and (7.34). On one hand it would be nice to compute those constants, on the other hand the appearance of the Ablowitz-Segur solution in (7.34) leads to the interesting question of what happens to $\det(I - \gamma K_{\text{csin}})$, the underlying Riemann-Hilbert problem and expansion (7.34) in case we choose x to coincide with one of the poles of u?

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