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D-Bar and Dirac Type Operators on Classical and Quantum Domains

For the degree of Doctor of Philosophy

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D-BAR AND DIRAC TYPE OPERATORS ON CLASSICAL AND QUANTUM DOMAINS

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of

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For my parents, Mary Scheffer and Scott McBride, who have supported my journey since the beginning. Also for my sweetheart, Nicole Ashpole, who has kept my spirits high during the dry spells of research.

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ABSTRACT

McBride, Matthew Scott Ph.D., Purdue University, August 2012. D-bar and Dirac Type Operators on Classical and Quantum Domains. Major Professor: Slawomir Klimek.

I study d-bar and Dirac operators on classical and quantum domains subject to the APS boundary conditions, APS like boundary conditions, and other types of global boundary conditions. Moreover, the inverse or inverse modulo compact operators to these operators are computed. These inverses/parametrices are also shown to be bounded and are also shown to be compact, if possible. Also the index of some of the d-bar operators are computed when it doesn't have trivial index. Finally a certain type of limit statement can be said between the classical and quantum d-bar operators on specialized complex domains.

1. INTRODUCTION

Analysis of operators, especially unbounded operators is rich in theory and has diverse applications to other fields of mathematics. For example if one was studying differential operators, one particular thing that can be studied is the spectrum of that operator. This can lead to expansion theorems given a certain potential which allows solutions to a vast amount of differential and partial differential equations to be computed and studied. Also with the theory of partial differential equations, knowing how a differential operator behaves will allow people to esitmate solutions to certain types of partial differential equations through integral estimates of the differential operator. A priori the operator is of course unbounded however when restricted to a proper domain, usually some kind of initial value or boudary value such as Dirichlet or Von Neumann is assumed to be satisfied, the operator on that domain will then usually be bounded. However different types of initial or boundary values do lead to of course different types of restrictions to the operator.

Another useful application of analysis of unbounded operators is in noncommutative geometry. In this theory one tries to describe what a noncommutative space is and the types of operators that act there. When one tries to describe the space's structure, as one would describe geometrical and differential-geometrical properties of say, $L^2(\mathbb{R})$, differential-difference operators arise naturally in this description. Knowing how differential operators behave on known commutative spaces, one expects there to be a very close analogy and similar theory in the quantum case. One particular operator that comes up is the Dirac operator, a first order differential operator whose square is a second order elliptic operator, such as the Laplacian. Knowing how Dirac operators behave, one expects their quantum analogs to be similar and to be some type of commutator. One differential-geometrical concept that comes to play is the index of the operator.

The Atiyah-Patodi-Singer Index Theorem (APS) establishs an index formlua for Dirac operators on a closed manifold with boundary which depends only on the differential-geometric properties of the manifold. The original proof of the Atiyah-Singer Index Theorem, also only had closed manifolds without boundary, was quite complicated and in [2], Aityah, Patodi, and Singer devised a new proof using heat kernels. If D is a Dirac or d-bar operator, meaning it is the square root of a second order elliptic operator, and D^* is its adjoint, then DD^* and D^*D are self adjoint and their non-zero eigenvalues have the same multiplicities, however their zero eigenspaces may have different multplicities. Then it was shown in [2] that for $t \geq 0$

$$\text{Index}(D) := \dim \text{Ker}(D) - \dim \text{Ker}(D^*) = \text{tr}\left(e^{-tD^*D}\right) - \text{tr}\left(e^{-tDD^*}\right).$$

For this to be established, the authors first had to show that D was a Fredholm operator, meaning, that D was closed; and has closed range since D is unbounded, and that D also had finite dimensional kernel and cokernel. They were able to establish this by showing D was invertible modulo compact operators. This is when the APS boundary condition had to be initially setup. If we let M be a closed manifold with boundary, then the main idea the authors exploited was to attach an infinite cylinder on the collar of the boundary and required that the Dirac operator decomposes into a certain form. In otherwords if Y is the boundary of M and $Y \times \mathbb{R}_{\geq 0}$ is the attachment of the infinite (half-infinite) cylinder to the boundary and D is the Dirac operator on M, then the decomposition of D that was required is

$$D = \frac{\partial}{\partial t} + B$$

over $Y \times \mathbb{R}_{\geq 0}$, where B is a first order self adjoint elliptic operator acting on $C^\infty(Y, E)$ and E is a vector bundle over Y. The APS boundary condition is that for sections $f(y,t)$ of E lifted to $Y \times \mathbb{R}_{\geq 0}$, then one requires that

$$Pf(\cdot\ , 0) = 0 \tag{1.1}$$

where P is the spectral projection of B. This is a non-local boundary condition that aleviates problems that classical boundary conditions, such as Dirichlet, have when trying to do global analysis. Since most of this thesis is spent finding inverses or inverses modulo compact operators to classical and quantum Dirac operators over different closed manifolds with boundary, time will be taken now to state the main technical theorem that was established in [2]. First more notation must be added. The space of all C^∞ functions satisfying (1.1) will be denoted by $C^\infty(Y \times \mathbb{R}_{\geq 0}, E; P)$ and C^∞_{comp} will denote functions vanishing for $t \geq C$ for some C. Also H^k will denote the Sobolev space of sections with derivatives up to order k in L^2. Now the theorem can be stated and it is

Theorem 1.0.1 *There is a linear operator*

$$Q : C^\infty_{comp}(Y \times \mathbb{R}_{\geq 0}, E) \to C^\infty(Y \times \mathbb{R}_{\geq 0}, E; P)$$

such that

(i) $DQg = g$ *for all* $g \in C^\infty_{comp}(Y \times \mathbb{R}_{\geq 0}, E)$

(ii) $QDf = f$ *for all* $f \in C^\infty(Y \times \mathbb{R}_{\geq 0}, E; P)$

(iii) *The kernel* $Q(y, t; z, v)$ *of* Q *is* C^∞ *for* $t \neq v; y, z \in Y$ *and* $t, v \in \mathbb{R}_{\geq 0}$

(iv) Q *extends to a continuous map* $H^{k-1} \to H^k_{loc}$ *for all integers* $k \geq 1$.

The main technique the authors used to prove this was expanding the solutions in terms of the eigenfunctions of B, in otherwords they did a spectral/Fourier decomposition and then used some functional analysis techniques in estimations. In their paper, [2], the authors only considered Dirac operators on closed manifolds with boundary where the functions on the manifold commuted.

Over the years, applications to quantum mechanics have given the rise for a non-commutative analog to the APS theorem. As of now, there is no known generic index formula, or even a standard technique, like the above theorem, that generalizes the index theorem to non-commutative spaces. The goal of this thesis is to find an inverse or inverse modulo compact operators, sometimes called parametrices (parametrix s.) of classical Dirac operators and quantum Dirac operators on simple domains subject to APS-like boundary conditions, the important fact being that the boundary condition is a non-local condition. These domains will be the disk, the annulus, sometimes called the finite cylinder, the punctured disk and the solid torus. It's important to note that all these domains with the exception of the finite cylinder, do not have the cylindrical structure on the boundary that the APS theory requires, thus the need for a slightly different but similar global boundary condition when compared to the exact APS boundary condition is necessary. Also another goal is showing that under mild conditions these parametrices are compact operators in their own right and if the indices of the classical and quantum Dirac operators are not trivial, then they must be the same. The last goal is showing that on two of the domains, the disk and annulus, the quantum parametrix converges to the classical one in some sense described later. The overall point of these goals is that through out this thesis, a generic technique emerges that works for all the cases considered here and may be a technique that is needed to discovering the general method that would yield a quantum analog of an APS-like Theorem. Another goal is that there is no generic framework when dealing with non-commutative spaces, no standards, etc. In order to understand these spaces better, non-trivial examples must be developed and understood.

This thesis is divided into five chapters not including this chapter or the summary chapter. The first chapter discusses classical d-bar operators on the disk and annulus. It also discusses a natural choice for a quantum d-bar operator on the quantum versions of the disk and annulus through C^*-algebras. The second chapter is a shorter chapter that discusses classical Dirac type operators on the punctured disk and it also discusses the non-commutative analongs. In third chapter, a quantization-deformation of the d-bar operator on the disk and annulus is discussed. It shows that through continuous fields of Hilbert spaces and under suitable conditions, the quantum parametrix converges to the classical one. In the fourth chapter, Dirac type operators on the classical solid torus are discussed. Moreover a different type of non-local boundary condition is discussed that is similar in spirit to the APS boundary condition and similar to the APS-like boundary condition used in the first three chapters. Finally in the last chapter quantum Dirac type operators are discussed on the quanutm solid torus subject to the non-local boundary condition that was discussed in chapter four but of course in the noncoummtative sense. Throughout this entire thesis, there are infinite products that arise and a convention needs to be made. Let $\{a_n\}$ be a sequence of complex numbers for $n \in \mathbb{N}$ or $n \in \mathbb{Z}$. For $n \in \mathbb{N}$, we say the infinite product

$$\prod_{n=0}^{\infty} a_n$$

exists and will be denoted

$$\prod_{n=0}^{\infty} a_n < \infty$$

if for $N > 0$

$$\lim_{N\to\infty} \prod_{n=0}^{N} a_n$$

exists and is nonzero. For $n \in \mathbb{Z}$, we say the infinite product

$$\prod_{n\in\mathbb{Z}} a_n$$

exists and will be denoted

$$\prod_{n\in\mathbb{Z}} a_n < \infty$$

if for $M, N > 0$

$$\lim_{M,N\to\infty} \prod_{n=-M}^{N} a_n$$

exists and is nonzero. Let $\{A_n\}$ be a sequence of complex invertible matices for $n \in \mathbb{N}$. We say the infinite product, here we multiply from the left,

$$\prod_{n=0}^{\infty} A_n$$

exists if for $N > 0$

$$\lim_{N\to\infty} \prod_{n=0}^{N} A_n$$

exists and the resulting limit is an invertible matrix.

2. SUMMARY

The sections of this chapter summarize each chapter by describing the main problem in each chapter and gives a short explaination of the rest of the sections in that chapter.

2.1 Summary of Chapter 3

In this chapter one considers noncommutative analogs of the d-bar operator on simple complex plane domains with boundary: disk and annulus. In both cases the corresponding quantum domain, its boundary, a d-bar operator, and an analog of the L^2 Hilbert space of functions on the domain is constructed using a weighted shift, subject to suitable assumptions. The weighted shift plays the role of the complex coordinate z.

For such d-bar operators one also considers boundary conditions of Atiyah, Patodi, Singer (APS) type [2]. This can be done so that both the commutative and the noncommutative setup appear in close analogy. The main result of the chapter is that of the quantum d-bar operators subject to APS conditions are unbounded Fredholm operators. Additionally their index is computed.

Recall that an unbounded operator D is called a Fredholm operator if D is closed, has closed range, and finite dimensional kernel and cokernel. Equivalently, see [27], a closed opearotor D is Fredholm if it has a bounded parametrix Q such that both $QD - I$ and $DQ - I$ are compact. The technical part of the paper consist of finding such a parametrix.

The celebrated APS boundary condition was introduced in [2] to handle the index theory for geometrical operators on manifolds with boundary when usual local boundary conditions were not available. Because it is non-local, the APS condition seems to be naturally suited to consider in noncommutative geometry. A more general class of APS-type boundary conditions was described in [5]. Here only simple APS-type boundary conditions given by spectral projections are considered.

This chapter is a continuation and an extension of [6], which considered APS theory on the noncommutative unit disk. Here the chapter is presented in a somewhat different and more detailed treatment of the disk case as well as a similar theory on the cylinder. In particular the modifications that is considered here yield a compact parametrix for the d-bar operators, which was not the case in [6].

Noncommutative domains considered in this chapter were previously discussed in [15, 16]. Other papers that studied d-bar operator in similar situations (but not the APS boundary conditions) are: [4], [14], [26], [28]- [32]. A related study of an example of APS boundary conditions in the context of noncommutative geometry is contained in [7], another one is in [23].

The ideas in this chapter can be further extended in several directions. The present setup fits into deformation-quantization scheme and so it will be desirable to consider classical limit of the quantum d-bar operators. Other, different, possibly higher dimensional examples should also be constructed. Because of the compact parametrix, the d-bar operators of this paper can be used to define Fredholm modules over quantum domains (with boundary), which will be interesting to explore. While the computation of the index in the present work is fairly straightforward, it is a challenging question to find a noncommutative framework for such calculations in general.

The chapter is organized as follows. In the preliminary section 3.1 we describe the classical d-bar operators on domains in complex plane subject to APS-type boundary conditions and compute their index. Section 3.2 contains the main constructions of the paper: quantum disk, quantum annulus, Hilbert spaces, d-bar operators, APS-type boundary conditions. The main results are also stated in this section. Section 3.3 is the longest of the chapter. It contains detailed analysis of some finite difference operators in weighted ℓ^2 spaces. The operators are essentially unbounded Jacobi operators, see [34]. That analysis constitutes the technical backbone of the paper. Section 3.4 introduces noncommutative Fourier transform on our quantum domains. The Fourier transform essentially diagonalizes the d-bar operators and thus reduces their analysis to the analysis of the difference operators of the previous section. Finally, section 3.5 describes proofs of the main results.

2.2 Summary of Chapter 4

The main technical and computational part of the Atiyah, Patodi, Singer paper [2] is the initial section containing a study of a nonlocal boundary value problem for the first order differential operators of the form $\Gamma(\frac{\partial}{\partial t} + B)$ on the semi-infinite cylinder $\mathbb{R}_+ \times Y$, where $t \in \mathbb{R}_+$ and B, Γ live on the boundary Y. The novelty of the chapter was the boundary condition, now called the APS boundary condition, that involved a spectral projection of B. The authors explicitly compute and estimate the fundamental solutions on the cylinder. This is later used to construct a parametrix for the analogical boundary value problem on a manifold with boundary by gluing it with a contribution from the interior, see also [5].

This chapter aims, in a special case, to reproduce such results in the noncommutative setup of [8]. A similar but different study of an example of APS boundary conditions in the context of noncommutative geometry is contained in [7].

This chapter is a continuation of the analysis started in [6] and chapter 3. The goal of the article and chapter was to provide simple examples of Dirac type operators on noncommutative compact manifolds with boundary and then study Atiyah-Patodi-Singer type boundary conditions and the corresponding index problem. This was done for the noncommutative disk and the noncommutative annulus and for two somewhat different types of operators constructed by taking commutators with weighted shifts.

In this chapter one considers such non-commutative analogs of the Dirac type operator $\frac{\partial}{\partial t} + \frac{1}{i}\frac{\partial}{\partial \varphi}$ on the cylinder $\mathbb{R}_+ \times S^1$, which is viewed as a punctured disk. Using a weighted shift, which plays the role of the complex coordinate z on the disk, quantum Dirac operators, and analogs of the L^2 Hilbert space of functions in which they act are constructed. One then consider the boundary condition of Atiyah, Patodi, Singer. This is done in close analogy with the commutative case. The main result of this chapter is that a quantum Dirac operator has an inverse which, minus the zero mode, is bounded just like in Proposition 2.5 of [2]. In contrast with the previous chapter the analysis here is more subtle because of the noncompactness of the cylinder. In particular the components of a parametrix are not compact operators and we use the Schur-Young inequality to estimate their norms. It is hoped that in the future such results will be needed to contruct spectral triples and a noncommutative index theory of quantum manifolds with boundary.

The chapter is organized as follows. In section 4.1 the classical APS result for the operator $-2\overline{z}\frac{\partial}{\partial\overline{z}}$ on the cylinder is stated and re-proved using the Schur-Young inequality. Section 4.2 contains the construction of the non-commutative punctured disk and the first type of noncommutative analogs of the operator from the previous section. The operators here are similar to those of [6]. Also in this section a non-commutative Fourier decomposition of the Hilbert spaces and the operators is discussed. Section 4.3 contains the construction and the analysis of the Fourier components of the parametrix and the proof of the main result. Finally in section 4.4 one considers the "balanced" versions of the quantum Dirac operators in the spirit of chapter 3 and it is shown how to modify the previous arguments to estimate the parametrix.

2.3 Summary of Chapter 5

According to the broadest and the most flexible definition, a quantum space is simply a noncommutative algebra. Noncommutative geometry [8] studies what could be considered "geometric properties" of such quantum spaces.

One of the most basic examples of a quantum space is the quantum unit disk $C(\mathbb{D}_t)$ of [15]. It is defined as the universal unital C^*-algebra with the generators z_t

and $\overline{z_t}$ which are adjoint to each other, and satisfy the following commutation relation: $[\overline{z_t}, z_t] = t(I - z_t\overline{z_t})(I - \overline{z_t}z_t)$, for a continuous parameter $0 < t < 1$.

It was proved in [15] that $C(\mathbb{D}_t)$ has a more concrete representation as the C^*-algebra generated by the unilateral weighted shift with the weights given by the formula:

$$w_t(k) = \sqrt{\frac{(k+1)t}{1+(k+1)t}}. \tag{2.1}$$

In fact, as a C^*-algebra, $C(\mathbb{D}_t)$ is isomorphic to the Toeplitz algebra. Moreover the family $C(\mathbb{D}_t)$ is a deformation, and even deformation - quantization of the algebra of continuous functions on the disk $C(\mathbb{D})$ obtained in the limit as $t \to 0$, called the classical limit.

The quantum unit disk is one of the simplest examples of a quantum manifold with boundary. It is also an example of a quantum complex domain, with z_t playing the role of a quantum complex coordinate. Additionally, biholomorphisms of the unit disk naturally lift to automorphisms of $C(\mathbb{D}_t)$, see [15].

In view of this complex analytic interpretation of the quantum unit disk, there is a natural need to define analogs of complex partial derivatives as some kind of unbounded operators on $C(\mathbb{D}_t)$ and its various Hilbert space completions. Such constructions have been described in several places in the literature, see for example [4], [6], [14], [17], [18], [36]. In this chapter one is primarily concerned with one such choice, the so-called balanced d and d-bar operators of [17] which is describe below.

One notices that $S_t := [\overline{z_t}, z_t]$ is an invertible trace class operator (with an unbounded inverse) and defines

$$D_t a = S_t^{-1/2}[a, z_t]S_t^{-1/2}$$

and

$$\overline{D_t} a = S_t^{-1/2}[\overline{z_t}, a]S_t^{-1/2},$$

for appropriate $a \in C(\mathbb{D}_t)$. These two operators have the following easily seen properties

$$D_t(1) = 0, \quad D_t(z_t) = 0, \quad D_t(\overline{z_t}) = 1$$
$$\overline{D_t}(1) = 0, \quad \overline{D_t}(z_t) = 1, \quad \overline{D_t}(\overline{z_t}) = 0$$

which makes them plausible candidates for quantum complex partial derivatives. To make an even better case of their suitability, one would like to know that in some kind of interpretation of the limit as $t \to 0$, they indeed become the classical partial derivatives. This problem was posed at the end of [14] and it is the subject of the present chapter.

In fact one considers here a broader classical limit problem by studying quite general families of unilateral weights $w_t(k)$, and not just those given by (2.1). Like in [17] such unilateral shifts are still considered coordinates of quantum disks. Additionally one also considers bilateral shifts and the C^*-algebras they generate. They are quantum analogs of annuli and can be analyzed very similarly to the quantum disks.

One starts with giving a concrete meaning to the classical limit $t \to 0$, which involves two important steps. The first step is to consider certain bounded functions of the quantum d and d-bar operators to properly manage their unboundedness. In this chapter one chooses to work with the inverses of the operators D_t subject to APS boundary conditions [2] since they are easy to describe and the results of [6], [17] can be utilized.

The second step of this chapter's approach to the classical limit is the choice of framework for studying limits of objects living in different spaces. Such a natural framework is provided by the language of continuous fields, in this case of continuous fields of Hilbert spaces, see [11]. Following [6] and [17] one defines, using operators S_t, weighted Hilbert space completions $\mathcal{H}_t$, $0 < t < 1$, of the above quantum domains, while $\mathcal{H}_0$ is the classical L^2 space. One then equips that family of Hilbert spaces with a natural structure of continuous field, namely the structure generated by the polynomials in complex quantum and classical coordinates.

In this setup the study of the classical limit becomes a question of continuity, a property embedded in the definition of the continuous field. Consequently, inverses of the operators D_t subject to APS boundary conditions, are considered as morphisms of the continuous fields of Hilbert spaces. The main result of this chapter is that in such a sense the limit of D_t is indeed $\frac{\partial}{\partial \bar{z}}$.

The chapter is organized as follows. In section 5.1 a review of the definitions and properties of continuous fields of Hilbert spaces and their morphisms is presented. In section 5.2 one describes the constructions of the quantum disk, the quantum annulus, Hilbert spaces of L^2 "functions" on those quantum spaces, d-bar operators and their inverses subject to APS conditions. One states the conditions on weights $w_t(k)$ and provide example of such weights. A construction of the generating subspace Λ needed for the construction of the continuous field of Hilbert spaces is done. The main results of this chapter are also formulated at the end of that section. Finally, section 5.3 contains the proofs of the results.

2.4 Summary of Chapter 6

The celebrated Atiyah-Patodi-Singer boundary condition [2] for Dirac operators on closed manifolds with boundary was introduced as a key ingredient in the generalization of the Atiyah-Singer index theorem. It is a non-local boundary condition which makes the Dirac operator Fredholm. An advantage of the APS condition is that further detailed analysis can be carried out where local conditions such as Dirichlet and Neumann may not be well behaved.

That theory works, however, under the assumption that both the manifold and the operator have a cylindrical structure near the boundary. A semi-infinite cylinder can then be smoothly attached to the manifold and the Dirac operator can be naturally extended over to the cylinder. The APS boundary condition can be then described in the following geometrical terms: a sufficiently regular section is in the domain of the operator if it extends to a square integrable solution on the cylinder. The issue though is that many concrete natural operators do not have a cylindrical structure near the boundary. Even a simple d-bar operator $\partial/\partial\bar{z}$ on a disk in the complex plane does not have this structure. The solid torus example studied here is not cylindrical near boundary either.

In this chapter one considers a Dirac operator on the solid torus considered geometrically as the product of the unit disk and the unit circle. A construction of another non-local boundary condition similar in spirit to the APS boundary condition is done. It was inspired by the non-local boundary conditions discussed in [25] and [12] to get around the necessity of having a cylindrical structure near the boundary. The boundary condition that is proposed in this chapter has the same geometrical interpretation as the APS condition. Namely, one cam consider the solid torus as a subset of the bigger noncompact space of the plane cross the unit circle. The domain of the Dirac operator is defined in full analogy with APS as consisting of those sufficiently regular sections which extend to square integrable solutions on the complement of the solid torus.

The motivation for studying this particular example comes from the larger project of developing a concept of a noncommutative manifold with boundary and noncommutative elliptic boundary conditions. This is done by studying examples, starting with two-dimensional domains and continuing with more complex cases. In particular the efforts of the whole project were concentrated on studying quantum analogs of Dirac operators subject to APS like boundary conditions, see [6], [17], [18], and [19] as well as chapters 3, 4, and 5. The solid torus studied in this chapter is possibly the simplest three dimensional example, yet significantly more difficult then the two-dimensional examples studied in the previous chapters. While the standard APS theory does not apply, the example however seems to have an attractive noncommutative version. This noncommutative version is the topic of the next chapter. It is shown in this chapter that the Dirac operator on the solid torus subject to the non-local boundary condition is self-adjoint and has no kernel. Using a partial Fourier transform one obtains an explicit formula for its inverse. One then shows that the inverse is a compact operator, which is the main feature of elliptic boundary conditions. It is proven that the inverse is a p-th Schatten class operator for $p > 3$. This is obtained by direct analysis of the formula for the inverse using subtle estimates involving modified Bessel functions.

The chapter is organized as follows. In section 6.1 the Hilbert space and the Dirac operator D are defined, and the boundary condition is stated. The section also contains the computation of the kernel of the Dirac operator, and the computation of its inverse Q. The proof of the main theorem, the compactness of Q, and the Schatten class computation is contained in section 6.2. In the last section, section 6.3, a collection of numerous facts about the modified Bessel functions are stated. Some of these facts are classical and some are more recent.

2.5 Summary of Chapter 7

Finding quantum analogs of Dirac type operators on manifolds with boundary and global boundary conditions has been a hot topic ever since Atiyah, Patodi, and Singer had the break-through index theorem for in their paper [2]. Their theorem didn't consider the case for non-commutative spaces. One of the aspects is to find an appropriate analog to Dirac type operators on some domain and finding an inverse or inverse modulo compact operators. Some examples in simple domains, such as the disk, annulus, and punctured disk have been made in [17] and [18]. Moreover in these papers, the authors showed how similar the setup and results are between the commutative and quantum cases. Also in those papers the global boundary condition imposed was the classical APS boundary condition. In chapter 6, we will discuss Dirac type operators on the solid torus, in the commutative sense, with a different type of nonlocal boundary condition that was inspired by [25]. This chapter will follow up the same type of analysis done with commutative solid torus only now the quantum analogs will be considered.

In this chapter we use the non-local boundary condition that will be used in chapter 6 and construct the quantum analog of it. Of course the first issue is too describe what the quantum solid torus even is. The boundary condition can be thought of extending functions beyond the boundary of the solid torus which makes sense geometrically in the classical case. However in the quantum analog, there are obvious obstructions that need to be addressed. For example what does one mean for the "outside" of the boundary to the quantum solid torus. The idea is to translate the boundary condition into an equivalent scaling requirement. With this established the same exact scaling condition can be easily translated to the quantum case.

Also in this chapter we consider the non-commutative analog of a Dirac type operator D on the solid torus with boundary. The quantum domain, its boundary, and a quantum analog of the Dirac type operator are all constructed. Also an analog of the L^2 Hilbert space of functions on the domain is constructed using two weighted shifts subject to suitable assumptions. The analysis done here is to compute the kernel of the Dirac type operator, calculate its parametrix and analyze it. It will also be shown that the parametrix is a compact operator. It was shown in chapter 6 that the parametrix to the classical Dirac type operator was also compact. In [17], the balanced d-bar operator was introduced to produce a compact parametrix.

The chapter is organized as follows. In section 2 the quantum solid torus is discussed. The Hilbert space and the Dirac operator, D, used are also defined in the noncommutative sense. As in [17], the Hilbert space is formed through C^*-algebras. The boundary condition is defined and the domain of the Dirac operator is also stated. Finally at the end of this section the statement of the main theorem is stated. Section 3 contains the computation of the kernel of the Dirac operator. It also houses the special solutions that are in the kernel as well as a few properties of special solutions. The last part of the section contains the analysis of the kernel of D subject to its domain. In section 4 the computation of the parametrix, Q, to the Dirac type operator D is made. Properties of the special solutions are also discussed here as these properties are relavent for the analysis done on the parametrix which is also done in the section. Finally at the end of the section the proof of the main theorem is shown.

3. D-BAR OPERATORS ON THE CLASSICAL AND QUANTUM DISK AND ANNULUS

3.1 The d-bar operator on domains in the complex plane

In this section a review of the basic aspects of the APS theory for the d-bar operator on simple domains in the complex plane $\mathbb{C}$ is done. First some notation is made. The first domain is the disk:

$$\mathbb{D} = \{z \in \mathbb{C} : \ |z| \leq \rho_+\}$$
$$\partial\mathbb{D} = \{z \in \mathbb{C} : \ |z| = \rho_+\} \simeq S^1.$$

The second domain is an annulus in the complex plane $\mathbb{C}$:

$$\mathbb{A}_{\rho_-,\rho_+} = \{z \in \mathbb{C} : \ 0 < \rho_- \leq |z| \leq \rho_+\}$$
$$\partial\mathbb{A}_{\rho_-,\rho_+} = \{z \in \mathbb{C} : \ |z| = \rho_\pm\} \simeq S^1 \cup S^1,$$

which can also be viewed as a finite cylinder.

For each of those domains we will consider the d-bar operator:

$$D = \frac{\partial}{\partial \overline{z}}$$

defined on the space of smooth functions.

Concentration on the unit disk will be done first. In this case one has the short exact sequence:

$$0 \longrightarrow C_0^\infty(\mathbb{D}) \longrightarrow C^\infty(\mathbb{D}) \xrightarrow{r} C^\infty(\partial\mathbb{D}) \longrightarrow 0 \tag{3.1}$$

where $r : C^\infty(\mathbb{D}) \to C^\infty(\partial\mathbb{D})$ is the restriction map to the boundary, $rf(\varphi) = f(\rho_+ e^{i\varphi})$. Here $C_0^\infty(\mathbb{D})$ is the space of smooth functions on $\mathbb{D}$ vanishing at the boundary and $z \in \mathbb{D}$ has polar representation $z = \rho e^{i\varphi}$.

Next one considers the APS-like boundary conditions on D. Notice that the APS theory cannot be applied directly in this case since the operator D does not quite decompose into tangential (boundary) and transverse parts near boundary. However this is only a minor technical annoyance, and it is clear that $-i\partial/\partial\varphi$ is the correct boundary operator. The APS-type boundary conditions considered in this chapter are given in terms of the spectral projections of the boundary operator $-i\partial/\partial\varphi$ as follows. Let $\pi_A(I)$ be the spectral projection of a self-adjoint operator A onto interval I. For an integer N one introduces P_N:

$$P_N = \pi_{\frac{1}{i}\frac{\partial}{\partial\varphi}}(-\infty, N]. \tag{3.2}$$

In other words P_N is the orthogonal projection in $L^2(S^1)$ onto $\mathrm{span}\{e^{in\varphi}\}_{n \le N}$.

The main object of the APS theory is the operator D_N defined to be the operator D with the domain:

$$\mathrm{dom}(D_N) = \{f \in C^\infty(\mathbb{D}) \subset L^2(\mathbb{D}) \ : \ rf \in \mathrm{Ran}\ P_N\}.$$

The following theorem is stated, see [6] for details.

Theorem 3.1.1 *The closure of the operator D_N is an unbounded Fredholm operator in $L^2(\mathbb{D})$ and it has the following index: $Index(D_N) = N + 1$.*

Next a discussion of the annulus is presented. Some functional analytic details are skipped, however the index calculation shown is done in a similar fashion to what was done in [6] in the disk case.

If one lets $r_\pm$ be the restriction to the boundary map i.e. $r_\pm f(\varphi) = f(\rho_\pm e^{i\varphi})$, then one has the short exact sequence:

$$0 \longrightarrow C_0^\infty(\mathbb{A}_{\rho_-,\rho_+}) \longrightarrow C^\infty(\mathbb{A}_{\rho_-,\rho_+}) \overset{r=r_+\oplus r_-}{\longrightarrow} C^\infty(S^1) \oplus C^\infty(S^1) \longrightarrow 0 \tag{3.3}$$

where $C_0^\infty(\mathbb{A}_{\rho_-,\rho_+})$ is the space of smooth functions on $\mathbb{A}_{\rho_-,\rho_+}$ which are zero on the boundary.

The key to index calculation of the d-bar operator is the following proposition. In what follows we use the usual inner product on $L^2(\mathbb{A}_{\rho_-,\rho_+})$:

$$\langle f, g\rangle = \int_{\mathbb{A}_{\rho_-,\rho_+}} \overline{f(z)}g(z)\frac{dz\wedge d\overline{z}}{-2i\pi}.$$

Proposition 3.1.2 *Let D be the operator*

$$D = \frac{\partial}{\partial\overline{z}}$$

on $C^\infty(\mathbb{A}_{\rho_-,\rho_+})$. Then the kernel of D is the set of bounded holomorphic functions on $\mathbb{A}_{\rho_-,\rho_+}$. Moreover

$$\langle Df, g\rangle = \langle f, \overline{D}g\rangle + \int_0^{2\pi} \overline{r_+ f(\varphi)}r_+ g(\varphi)\rho_+ e^{-i\varphi}\frac{d\varphi}{2\pi} - \int_0^{2\pi} \overline{r_- f(\varphi)}r_- g(\varphi)\rho_- e^{-i\varphi}\frac{d\varphi}{2\pi}$$

where $f, g \in C^\infty(\mathbb{A}_{\rho_-,\rho_+})$ and

$$\overline{D} = -\frac{\partial}{\partial z}.$$

Proof The first conclusion is clear. The integration by parts formula follows immediately from Stokes' Theorem. ■

In order to define APS-type boundary conditions here we take extra caution since the boundary has two components. Let $P_N^\pm$ be the spectral projections in $L^2(S^1)$ of the boundary operators $\pm\frac{1}{i}\frac{\partial}{\partial\varphi}$ onto interval $(-\infty, N]$ i.e.:

$$P_N^\pm = \pi_{\pm\frac{1}{i}\frac{\partial}{\partial\varphi}}(-\infty, N] \tag{3.4}$$

where $\pm$ is introduced due to the boundary orientations of the inner circle and outer circle. Then, for integers M, N, we define the operator $D_{M,N}$ to be equal to D with domain

$$\mathrm{dom}(D_{M,N}) = \{f \in C^\infty(\mathbb{A}_{\rho_-,\rho_+}) \ : \ r_+ f \in \mathrm{Ran}\ P_M^+,\ r_- f \in \mathrm{Ran}\ P_N^-\}.$$

An immediate corollary of this definition is the description of the kernel of $D_{M,N}$.

Corollary 3.1.3 *Let $D_{M,N}$ be as defined above, then*

$$Ker(D_{M,N}) = \begin{cases} \left\{f \ : \ f(z) = \sum_{n=-N}^{M} c_n z^n\right\} & \textit{if } N+M \geq 0 \\ 0 & \textit{otherwise.} \end{cases}$$

It follows from Proposition 3.1.2 that the adjoint of $D_{M,N}$, is (the closure of) the operator $\overline{D}_{M,N}$ which is equal to $\overline{D}$ but with the following domain

$$\mathrm{dom}(\overline{D}_{M,N}) = \{f \in C^\infty(\mathbb{A}_{\rho_-,\rho_+}) \ : \ e^{-i\varphi} r_+ f \in \mathrm{Ker}\ P_M^+,\ e^{-i\varphi} r_- f \in \mathrm{Ker}\ P_N^-\}.$$

Moreover, one has the following description of the kernel of $\overline{D}_{M,N}$

$$\mathrm{Ker}(\overline{D}_{M,N}) = \begin{cases} \left\{f \ : \ f(z) = \sum_{n=N}^{-(M+2)} c_n z^n\right\} & \text{if } N+M < 0 \\ 0 & \text{otherwise.} \end{cases}$$

The following theorem is the corresponding index theorem for the commutative cylinder.

Theorem 3.1.4 *The closure of the operator $D_{M,N}$ is an unbounded Fredholm operator. Its index is given by: $Index(D_{M,N}) = M + N + 1$.*

Proof To show the Fredholm property one follows [2]. If $f \in C^\infty(\mathbb{A}_{\rho_-,\rho_+})$ then $f(z)$ has the following Fourier representation:

$$f(z) = \sum_{n\in\mathbb{Z}} f_n(\rho) e^{in\varphi}.$$

This Fourier representation is exactly the spectral decomposition of [2] using the eigenvectors of the boundary operators $\pm i\partial/\partial\varphi$. In the Fourier transform the operator

D decomposes into sum of ordinary differential operators which allows for explicit calculation of a parametrix just like in [2].

The index computation is as follows. Indeed one has:

$$\begin{aligned}\dim \operatorname{Ker}(D_{M,N}) &= \#\{n \mid -N \leq n \leq M\} \\ &= \begin{cases} 0 & \text{if } M+N<0 \\ M+N+1 & \text{if } M+N \geq 0. \end{cases}\end{aligned}$$

In a similar fashion

$$\begin{aligned}\dim \operatorname{Ker}(D^*_{M,N}) &= \#\{n \mid N \leq n \leq -(M+2)\} \\ &= \begin{cases} -(M+N+1) & \text{if } N<0 \\ 0 & \text{if } N \geq 0. \end{cases}\end{aligned}$$

Consequently

$$\operatorname{Index}(D_{M,N}) = \dim \operatorname{Ker}(D_{M,N}) - \dim \operatorname{Ker}(D^*_{M,N}) = M+N+1.$$

■

Attention is now turned to the d-bar operator in the quantum domains.

3.2 The d-bar operator on the non-commutative domains

In this section one defines the main objects of this chapter: quantum disk, quantum annulus, Hilbert spaces of L^2 "functions", and d-bar operators. The main results are also stated at the end of this section.

In the following definitions we let $\mathbb{S}$ be either $\mathbb{N}$ or $\mathbb{Z}$. The main input of the theory is a weighted shift U_W in $\ell^2(\mathbb{S})$. Conceptually, U_W is a noncommutative complex coordinate on the corresponding noncommutative domain.

Definition: Let $\{e_k\}$, $k \in \mathbb{S}$ be the canonical basis for $\ell^2(\mathbb{S})$. Given a bounded sequence of numbers $\{w_k\}$, called weights, the weighted shift U_W is an operator in $\ell^2(\mathbb{S})$ defined by:

$$U_W e_k = w_k e_{k+1}.$$

Also one will need the usual shift operator U which is defined by

$$U e_k = e_{k+1}$$

and the diagonal operator W defined by

$$W e_k = w_k e_k. \tag{3.5}$$

Note that U_W decomposes to $U_W = UW$ and $W = (U_W^* U_W)^{1/2}$ as in the polar decomposition. If $\mathbb{S} = \mathbb{N}$ then the shift U_W is called unilateral and it will be used to define a quantum disk. If $\mathbb{S} = \mathbb{Z}$ then the shift U_W is called bilateral and it will be used to define a quantum annulus (also called a quantum cylinder).

The following conditions on U_W are required:

Condition 1. The weights are uniformly positive $w_k \geq \epsilon > 0$, for every $k \in \mathbb{S}$.

Condition 2. The shift U_W is hyponormal, i.e.

$$S = [U_W^*, U_W] \geq 0.$$

Condition 3. The operator S defined in *condition 2* is injective.

These conditions have some implications that need a remark. First note how S acts on the basis $\{e_k\}$

$$\begin{aligned} Se_k &= (U_W^* U_W - U_W U_W^*)\, e_k \\ &= (w_k^2 - w_{k-1}^2)\, e_k = s_k e_k, \end{aligned} \tag{3.6}$$

where $s_k := w_k^2 - w_{k-1}^2$. It follows that the conditions 2 and 3 mean that the weights w_k form a strictly increasing sequence. Hence the following limits exist and are positive numbers:

$$w^{\pm} := \lim_{k \to \pm\infty} w_k.$$

Secondly, observe that S is a trace class operator with easily computable trace: $\mathrm{tr}(S) = (w^+)^2$ in the unilateral case and $\mathrm{tr}(S) = (w^+)^2 - (w^-)^2$ in the bilateral case. Moreover S is invertible with unbounded inverse.

Let $C^*(W)$ be the $C^* -$ algebra generated by U_W. Then it is known that there are short exact sequences analogous to (3.1) and (3.3). Let $\mathcal{K}$ be the ideal of compact operators. Then in the unilateral case the $C^* -$ algebra generated by U_W is the Non-Commutative Disk of [15] with the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(W) \xrightarrow{r} C(S^1) \longrightarrow 0.$$

Similarly, in the bilateral case the $C^* -$ algebra generated by U_W is the Non-Commutative Cylinder, see [16], with the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(W) \overset{r=r_+\oplus r_-}{\longrightarrow} C(S^1) \oplus C(S^1) \longrightarrow 0.$$

In the above we let again, abusing notation, r be the restriction map in the disk case and $r_\pm$ in the cylinder case. These two sequences are described in [10].

Now the definitions of the quantum d-bar operators are next to discuss. With slight abuse, we will use the same notation for both classical and quantum operators.

Define the Hilbert space $\mathcal{H}$ as the completion of $C^*(W)$ with respect to the inner product $\langle\ ,\ \rangle_S$ defined as follows:

$$\langle a, b\rangle_S = \mathrm{tr}(S^{1/2} b S^{1/2} a^*)$$

where $a, b \in C^*(W)$. It is easy to verify that $\langle a, a\rangle_S$ is well-defined and positive. Note that the inner product $\langle\ ,\ \rangle_S$ is slightly different than the one defined in [6]. This is done (among other reasons) to make definitions more symmetric.

The basic idea of the definition of a quantum d-bar operator, explained in [6], is to replace derivatives with commutators and so to consider operators of the form $a \mapsto P[Q, a]R$, where P, Q, R are possibly unbounded operators affiliated with $C^*(W)$. The choices are made so that it is possible to impose APS like boundary conditions, prove the Fredholm property and compute the index. Additionally it is advantageous for the operator D to have algebraic relations with U_W and $\overline{U_W}$ similar to the relations of the complex partial derivative with z and $\bar{z}$. With that in mind, the following definition of a quantum d-bar operator is made D in $\mathcal{H}$:

$$Da = S^{-1/2}\,[a, U_W]\,S^{-1/2}$$

where the domain of D is the set of those $a \in \mathcal{H}$ for which $S^{1/2}DaS^{1/2}(Da)^*$ is trace class. It will be verified later that $\mathrm{Dom}(D)$ is dense and that for $a \in \mathrm{Dom}(D)$, $r(a)$ is a square integrable function on the boundary of the domain. This definition is again somewhat different than the one considered in [6]: it is symmetric with respect to left/right multiplication, and the operator D has better functional-analytic properties.

A straightforward computation shows the following identities:

$$\begin{aligned}
D(U_W^n) &= 0\\
D(U_W^*) &= 1\\
D((U_W^*)^n) &= S^{-1/2}\,[(U_W^*)^n, U_W]\,S^{-1/2}\\
&= S^{-1/2}(U_W^*)^{n-1}S^{1/2}\\
&\qquad - S^{-1/2}(U_W^*)^{n-2}SU_W^*S^{-1/2} - \cdots - S^{1/2}(U_W^*)^{n-1}S^{-1/2}.
\end{aligned}$$

The first two computations show that D looks like $\frac{\partial}{\partial \bar{z}}$ if U_W was z and the third computation illustrates the non-commutativity of the situation.

We proceed to the definitions of the APS-type boundary conditions on D. Let again P_N be the orthogonal projection in $L^2(S^1)$ defined in equation (3.2), and let $P_N^{\pm}$ be the orthogonal projections defined in equation (3.4). Now we can define D_N, $D_{M,N}$ in full analogy with the previous section. The operator D_N equals the unilateral operator D with domain

$$\mathrm{dom}(D_N) = \{a \in \mathrm{Dom}(D) :\ r(a) \in \mathrm{Ran}\ P_N\}\,.$$

Similarly, the operator $D_{M,N}$ equals the bilateral operator D with domain

$$\operatorname{dom}(D_{M,N}) = \left\{a \in \operatorname{Dom}(D) :\ r_+(a) \in \operatorname{Ran}\, P_N^+,\ r_-(a) \in \operatorname{Ran}\, P_M^-\right\}.$$

The main results of this chapter can now be stated.

Theorem 3.2.1 *For the non-commutative disk case, the operator D_N is an unbounded Fredholm operator. Moreover ind$(D_N) = N + 1$.*

This is a slight modification from [6], where a somewhat different version of D_N was considered. Additionally one has:

Theorem 3.2.2 *For the non-commutative cylinder case, the operator $D_{M,N}$ is an unbounded Fredholm operator. Moreover ind$(D_{M,N}) = M + N + 1$.*

The proofs are contained in the last section of this chapter.

3.3 Analysis of finite difference operators

In this section a detailed analysis of certain finite difference operators related to Jacobi matrices is presented. As indicated in the introduction, these operators come up as components of D and its adjoint in Fourier transforms. This will be fully explained in the following section.

As before $\mathbb{S}$ is either $\mathbb{Z}$ or $\mathbb{N}$. Given a sequences of positive numbers $a = \{a_n\}_{n\in\mathbb{S}}$ called weights, the Hilbert Space $\ell^2_a(\mathbb{S})$ is defined by

$$\ell^2_a(\mathbb{S}) = \left\{ f = \{f_n\}_{n\in\mathbb{S}} \ :\ \sum_{n\in\mathbb{S}} \frac{1}{a_n}|f_n|^2 < \infty \right\}$$

with inner product given by $\langle f, g\rangle = \sum_{n\in\mathbb{S}} \frac{1}{a_n}\overline{f_n} g_n$. If a sequence $\{f_n\} \in \ell^2_a(\mathbb{S})$ has limits, $\lim_{n\to\pm\infty} f_n$, they will be denoted $f_{\pm\infty}$.

Given two weight sequences a and a' we will be studying throughout this section the following unbounded Jacobi type difference operators between $\ell^2_a(\mathbb{S})$ and $\ell^2_{a'}(\mathbb{S})$:

$$
\begin{gathered}
Af_n = a_n(f_n - c_{n-1}f_{n-1}) \quad \text{where} \\
\mathrm{dom}(A) = \left\{ f \in \ell^2_{a'}(\mathbb{S}) \ : \ \|Af\|_{\ell^2_a(\mathbb{S})} < \infty \right\} \\
\text{and} \\
\overline{A}f_n = a'_n(f_n - \overline{c_n}f_{n+1}) \quad \text{where} \\
\mathrm{dom}(\overline{A}) = \left\{ f \in \ell^2_a(\mathbb{S}) \ : \ \|\overline{A}f\|_{\ell^2_{a'}(\mathbb{S})} < \infty \right\}
\end{gathered}
$$

for $n \in \mathbb{S}$. If $\mathbb{S} = \mathbb{N}$ it is assumed in the above that $f_{-1} = 0$.

The coefficients a_n, a'_n, and $c_n \in \mathbb{C}$ are assumed to satisfy:

$$
0 < |c_n| \leq 1 \ , \ \sum_{n \in \mathbb{S}} \frac{1}{a'_n} = C' < \infty \ , \ \sum_{n \in \mathbb{S}} \frac{1}{a_n} = C < \infty \ , \prod_{n \in \mathbb{S}} \frac{1}{c_n} < \infty. \tag{3.7}
$$

Also define:

$$
K = \prod_{n \in \mathbb{S}} \frac{1}{|c_n|}.
$$

For the product involving the complex c_n, when one says the product is finite it is meant that the limit of a finite product exists. The goal of this section is to establish the Fredholm properties of the operators A, $\overline{A}$ and related operators obtained by imposing conditions at infinities. This is done by constructing a parametrix for each operator. The discussion will be split into two separate but similar cases: unilateral and bilateral.

3.3.1 Unilateral case

First one needs to study the kernels of A and $\overline{A}$, in order to see if these operators have inverses or not.

Proposition 3.3.1 *Given A and $\overline{A}$ above one has*

$$
\begin{gathered}
Ker\, A = \{0\} \\
dim\ Ker\, \overline{A} = 1.
\end{gathered}
$$

Proof First consider the equation $Af_n = 0$ which is $a_n(f_n - c_{n-1}f_{n-1}) = 0$ for $n = 0, 1, 2\dots$ Then solving recursively one can see that the only solution to the equation is $f_0 = f_1 = \cdots = f_n = 0$ for all n. This shows that Ker A is trivial and thus A is an invertible operator.

Secondly consider the equation $\overline{A}f_n = 0$ which is $a'_n(f_n - \overline{c_n}f_{n+1}) = 0$ for $n = 0, 1, 2\dots$ Then solving recursively one has

$$\begin{aligned} n = 0 \Rightarrow &\quad f_1 = \tfrac{1}{\overline{c_0}} f_0 \\ n = 1 \Rightarrow &\quad f_2 = \tfrac{1}{\overline{c_0 c_1}} f_0 \\ \vdots \quad & \quad\quad \vdots \end{aligned}$$

which in general gives

$$f_n = \frac{1}{\overline{c_0 c_1 \cdots c_{n-1}}} f_0,$$

thus showing that $\overline{A}$ has a one dimensional kernel provided that $f_n \in \ell^2_a(\mathbb{N})$. Notice the following

$$|f_n| = \frac{1}{|c_0 \cdots c_{n-1}|}|f_0| \leq \prod_{i=0}^{\infty} \frac{1}{|c_i|}|f_0| = K|f_0|$$

since $|c_i| \leq 1$ for all $i = 0, 1, \dots$ From this it follows that

$$\|f\|^2 \leq \sum_{n=0}^{\infty} \frac{1}{a_n} K^2 |f_0|^2 = CK^2|f_0|^2 < \infty$$

with the constants defined at the beginning of the section. Thus this completes the proof. ■

Next it's shown how to find the inverse T of A and one studies its properties.

Proposition 3.3.2 *There exists an operator $T \in B(\ell^2_a(\mathbb{N}), \ell^2_{a'}(\mathbb{N}))$ such that $TA = I_{\ell^2_a(\mathbb{N})}$ and $AT = I_{\ell^2_{a'}(\mathbb{N})}$. Indeed it is given by the formula 3.8 below. In particular A is an unbounded Fredholm operator with zero index.*

Proof From Proposition 3.3.1 we know that A is invertible so let $\{g_n\} \in \ell^2_a(\mathbb{N})$ and $\{f_n\} \in \text{dom}(A)$ and consider the equation $Af_n = g_n$ which is $a_n(f_n - c_{n-1}f_{n-1}) = g_n$ for $n = 0, 1, 2 \ldots$ As above, solving for each n recursively one arrives at the following formula

$$(Tg)_n = \sum_{i=0}^{n} \frac{1}{a_i} \left(\prod_{j=i}^{n-1} c_j \right) g_i, \tag{3.8}$$

where in the above one sets, for convenience:

$$\prod_{j=n}^{n-1} c_j = 1.$$

The next item shown is that $T \in B(\ell^2_a(\mathbb{N}), \ell^2_{a'}(\mathbb{N}))$. Divide and multiply each term as follows

$$(Tg)_n = \frac{1}{a_n} g_n + \frac{c_{n-1}}{a_{n-1}} g_{n-1} + \cdots + \frac{c_{n-1} \cdots c_0}{a_0} g_0 =$$

$$= \frac{\sqrt{a_n}}{a_n} \frac{g_n}{\sqrt{a_n}} + \frac{c_{n-1}\sqrt{a_{n-1}}}{a_{n-1}} \frac{g_{n-1}}{\sqrt{a_{n-1}}} + \cdots + \frac{c_{n-1} \cdots c_0 \sqrt{a_0}}{a_0} \frac{g_0}{\sqrt{a_0}}.$$

Since $\|Tg\|^2 = \sum_{n=0}^{\infty} \frac{1}{a'_n} |Tg_n|^2$ and since $|c_n| \leq 1$ for every n, using the Cauchy - Schwarz inequality one has

$$\begin{aligned} |(Tg)_n|^2 &\leq \left(\left(\frac{\sqrt{a_n}}{a_n} \right)^2 + \cdots + \left(\frac{\sqrt{a_0}}{a_0} \right)^2 \right) \left(\frac{1}{a_n} |g_n|^2 + \cdots + \frac{1}{a_0} |g_0|^2 \right) \leq \\ &\leq \left(\sum_{n=0}^{\infty} \frac{1}{a_n} \right) \|g\|^2 = C\|g\|^2. \end{aligned}$$

Consequently:

$$\begin{aligned} \|Tg\|^2 &\leq \sum_{n=0}^{\infty} \frac{1}{a'_n} C \|g\|^2 = \\ &= C'C\|g\|^2, \end{aligned}$$

which implies that $\|T\| \leq \sqrt{C'C}$, thus one has $T \in B(\ell^2_a(\mathbb{N}), \ell^2_{a'}(\mathbb{N}))$. A straightforward calculation shows that $TA = I_{\ell^2_a(\mathbb{N})}$ and $AT = I_{\ell^2_{a'}(\mathbb{N})}$. ■

An important corollary from this proposition is the existence of limits at infinity for sequences which are in the domain of A.

Corollary 3.3.3 *Let $f = \{f_n\} \in dom(A)$, then $\lim_{n\to\infty} f_n = f_\infty$ exists and is given by the following formula*

$$f_\infty = \sum_{i=0}^{\infty} \frac{1}{a_i} \left(\prod_{j=i}^{\infty} c_j \right) Af_i. \tag{3.9}$$

Proof If $f \in \text{dom}(A)$, then write f as, $f = T(Af)$, then one has the following

$$f_n = \sum_{i=0}^{n} \frac{1}{a_i} \left(\prod_{j=i}^{n-1} c_j \right) Af_i.$$

Using assumptions 3.7 and estimating as above, we see that the formula 3.9 is well defined. Now the fact that $\lim_{n\to\infty} f_n = f_\infty$ follows from a simple $\varepsilon/2$ argument. ■

We now wish to consider the operator $\overline{A}$ and determine if it has bounded right inverse since Proposition 3.3.1 tells us that $\overline{A}$ has a one dimensional kernel. The next proposition will show this. The following notation will be used: if V be a closed subspace of a Hilbert space H, then we denote Proj_V, to be the orthogonal projection onto V.

Proposition 3.3.4 *Given $\overline{A}$ from above then there exists a $\overline{T} \in B(\ell^2_{a'}(\mathbb{N}), \ell^2_a(\mathbb{N}))$ such that $\overline{AT} = I_{\ell^2_{a'}(\mathbb{N})}$ and $\overline{TA} = I_{\ell^2_a(\mathbb{N})} - Proj_{Ker\,\overline{A}}$. In particular $\overline{A}$ is an unbounded Fredholm operator with index equal to one.*

Proof From Proposition 3.3.1 we know that $\overline{A}$ has a one dimensional kernel spanned by the following vector $\Omega \in \text{Ker}(\overline{A})$:

$$\Omega_n = \prod_{i=n}^{\infty} \overline{c_i} = \left(\prod_{i=0}^{n-1} \frac{1}{\overline{c_i}} \right) \left(\prod_{i=0}^{\infty} \overline{c_i} \right).$$

Next consider the equation $\overline{A}g_n = a'_n(g_n - \overline{c_n}g_{n+1}) = f_n$ for $n = 0, 1, 2, \ldots$ As before solve the equation recursively and one will arrive at the formula

$$g_{n+1} = \prod_{i=0}^{n} \frac{1}{\overline{c_i}} g_0 - \sum_{i=0}^{n} \frac{1}{a'_i} \left(\prod_{j=i}^{n} \frac{1}{\overline{c_j}} \right) f_i.$$

where g_0 is arbitrary. To finish the construction of $\overline{T}$ one needs to choose g_0 so that $\overline{TA} = I_{\ell^2_a(\mathbb{Z})} - \mathrm{Proj}_{\mathrm{Ker}\,\overline{A}}$ as it's clear that $\overline{AT} = I_{\ell^2_{a'}(\mathbb{Z})}$.

The disadvantage of the above formula for $\overline{T}$ is that it does not translate easily to the bilateral case. Anticipating it, we rewrite the above solution in an equivalent but different looking form:

$$\begin{aligned} (\overline{T}f)_n = g_n &= \sum_{i=n}^{\infty} \frac{1}{a'_i} \left(\prod_{j=n}^{i-1} \overline{c_j} \right) f_i - \left(\prod_{i=n}^{\infty} \overline{c_i} \right) L(f) \\ &= (\overline{T_0}f)_n - \Omega_n \, L(f), \end{aligned} \tag{3.10}$$

where we set $\prod_{j=n}^{n-1} \overline{c_j} = 1$ and $L(f)$ is an arbitrary constant. This form of solution is also explained conceptually when considering bilateral case.

For $\overline{T}f$ to be orthogonal to Ker $\overline{A}$, one needs $\langle \Omega, \overline{T}f \rangle = 0$ for the above $\Omega \in$ Ker $\overline{A}$. From this one can deduce that $L(f)$ is the following following linear functional of f:

$$L(f) := \frac{\langle \Omega, \overline{T_0}f \rangle}{||\Omega||^2} = \frac{\sum_{n=0}^{\infty} \sum_{i=n}^{\infty} \frac{1}{a'_n} \frac{1}{a'_i} \left(\prod_{j=n}^{i-1} c_j \right) \left(\prod_{k=n}^{\infty} \overline{c_k} \right) f_i}{\sum_{n=0}^{\infty} \frac{1}{a'_n} \left(\prod_{i=n}^{\infty} |c_i|^2 \right)}.$$

It is straightforward to verify now that $\overline{TA} = I_{\ell^2_a(\mathbb{Z})} - \mathrm{Proj}_{\mathrm{Ker}\,\overline{A}}$ and that $\overline{AT} = I_{\ell^2_{a'}(\mathbb{Z})}$. All that remains is to show the boundedness of $\overline{T}$. The operator $\overline{T_0}$ is bounded by $\sqrt{CC'}$ in exactly the same way as the operator T is Proposition 3.3.2. To estimate $L(f)$ notice that

$$C' \geq ||\Omega||^2 = \sum_{n=0}^{\infty} \frac{1}{a'_n} \left(\prod_{i=n}^{\infty} |c_i|^2 \right) \geq \sum_{n=0}^{\infty} \frac{1}{a'_n} \left(\prod_{i=0}^{\infty} |c_i|^2 \right) = \frac{C'}{K^2},$$

which implies that $|L(f)| \leq K\sqrt{C}||f||$ and $||\overline{T}|| \leq \sqrt{CC'} + K\sqrt{CC'}$. This completes the proof. ■

Again one gets a corollary on the existence of limits at infinity for sequences which are in the domain of $\overline{A}$.

Corollary 3.3.5 *Let $f \in dom(\overline{A})$, then f_∞ exists and is given by the following formula*

$$f_\infty = -L(Af).$$

Proof The proof for the $\overline{T_0}$ term is identical to the proof of the Corollary 3.3.3. To compute the limit of the other term notice that:

$$\Omega_n = \prod_{i=n}^{\infty} \overline{c_i} = \frac{\prod_{i=0}^{\infty} \overline{c_i}}{\prod_{i=0}^{n-1} \overline{c_i}} \to 1$$

as $n \to \infty$. ■

The above corollaries allow us to consider "boundary" conditions on A and $\overline{A}$. Define the operators A_0 and $\overline{A_0}$ as follows: A_0 is the operator A but with domain

$$\mathrm{dom}(A_0) = \{f \in \mathrm{dom}(A) : f_\infty = 0\},$$

and $\overline{A_0}$ is the operator $\overline{A}$ with domain

$$\mathrm{dom}(\overline{A_0}) = \{f \in \mathrm{dom}(\overline{A}) : f_\infty = 0\}.$$

The four operators are closely related as shown by the following computation of the adjoint of A.

Proposition 3.3.6 *The adjoint of A has the following formula*

$$A^* = \overline{A_0}.$$

Moreover the adjoint of $\overline{A}$ has the following formula

$$\overline{A}^* = A_0.$$

Proof Computing the inner product one has:

$$\langle Af, g\rangle = \sum_{n=0}^{\infty} \frac{1}{a_n} \overline{a_n(f_n - c_{n-1}f_{n-1})} g_n = \sum_{n=0}^{\infty} \overline{(f_n - c_{n-1}f_{n-1})} g_n =$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \overline{(f_n - c_{n-1}f_{n-1})} g_n = \lim_{N\to\infty} \left(\sum_{n=0}^{N} \overline{f_n} g_n - \sum_{n=0}^{N} \overline{c_{n-1}f_{n-1}} g_n \right).$$

Then, setting $n-1 \mapsto n$ one arrives at

$$\begin{aligned} \langle Af, g\rangle &= \lim_{N\to\infty} \left(\sum_{n=0}^{N} \overline{f_n}(g_n - \overline{c_n} g_{n+1}) - c_N \overline{f_N} g_{N+1} \right) = \\ &= \sum_{n=0}^{\infty} \frac{1}{a'_n} \overline{f_n} a'_n (g_n - \overline{c_n} g_{n+1}) - \overline{f_\infty} g_\infty = \\ &= \langle f, \overline{A} g\rangle - \overline{f_\infty} g_\infty. \end{aligned}$$

Here note that $\prod c_n^{-1} < \infty$ and $|c_n| \le 1$ implies that the c_n converge to 1.

The functional $f \to f_\infty$ is not continuous thus implying that if $f \in \mathrm{dom}(A^*)$, then $f_\infty = 0$ and if $g \in \mathrm{dom}(\overline{A}^*)$, then $g_\infty = 0$. This completes the proof. ■

It follows that all four operators are Fredholm operators where the parametrix in each case is T, $\overline{T}$, or their adjoints. For completeness we compute the adjoint of T and of $\overline{T}$: this is not necessary for the main argument but may possibly be useful in future applications.

Proposition 3.3.7 *The adjoint of T is equal to $\overline{T_0}$ of 3.10, i.e. it has the following formula:*

$$(T^* f)_n = \overline{T_0} f_n = \sum_{k=n}^{\infty} \frac{1}{a'_k} \left(\prod_{j=n}^{k-1} \overline{c_j} \right) f_k.$$

Similarly:

$$\overline{T}^* f = Tf - \frac{\langle \Omega, f\rangle}{||\Omega||} T\Omega.$$

Proof Looking at the inner product one has

$$\langle Tg, f\rangle = \sum_{n=0}^{\infty} \frac{1}{a_n}\overline{(Tg)_n} f_n = \sum_{n=0}^{\infty} \frac{1}{a_n}\left(\frac{1}{a_n}\overline{g_n} + \frac{\overline{c_{n-1}}}{a_{n-1}}\overline{g_{n-1}} + \cdots + \frac{\overline{c_{n-1}\cdots c_0}}{a_0}\overline{g_0}\right) f_n =$$

$$= \sum_{n=0}^{\infty} \frac{1}{a_n}\left(\frac{1}{a_n}\overline{g_n} f_n\right) + \sum_{n=0}^{\infty} \frac{1}{a_n}\left(\frac{\overline{c_{n-1}}}{a_{n-1}}\overline{g_{n-1}} f_n\right) + \cdots + \sum_{n=0}^{\infty} \frac{1}{a_n}\left(\frac{\overline{c_{n-1}\cdots c_0}}{a_0}\overline{g_0} f_n\right).$$

Then using $n \mapsto j+1$ in the second sum, $n \mapsto j+2$ in the third sum and so on and relabeling the indices, one has

$$\begin{aligned} \langle Tg, f\rangle &= \sum_{n=0}^{\infty} \frac{1}{a_n}\overline{g_n}\left(\frac{1}{a_n} f_n\right) + \sum_{n=0}^{\infty} \frac{1}{a_n}\overline{g_n}\left(\frac{\overline{c_n}}{a_{n+1}} f_{n+1}\right) + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{a'_n}\overline{g_n}\left(\frac{1}{a'_n} f_n\right) + \sum_{n=0}^{\infty} \frac{1}{a'_n}\overline{g_n}\left(\frac{\overline{c_n}}{a'_{n+1}} f_{n+1}\right) + \cdots \\ &\quad + \sum_{n=0}^{\infty} \frac{1}{a'_n}\overline{g_n}\left(\frac{\overline{c_n\cdots c_{n+k}}}{a'_{n+(k+1)}} f_{n+(k+1)}\right) + \cdots \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{1}{a'_n}\overline{g_n}\left(\frac{1}{a'_n} f_n + \frac{\overline{c_n}}{a'_{n+1}} f_{n+1} + \cdots + \frac{\overline{c_n\cdots c_{n+k}}}{a'_{n+(k+1)}} f_{n+(k+1)} + \cdots\right) = \langle g, T^* f\rangle.$$

This then shows the first result. For the second formula we notice that we just showed that $\overline{T_0}^* = T$ and the second term comes from an easy computation of the adjoint of the projection $f \to L(f)\Omega$. ∎

Combining Propositions 3.3.2, 3.3.4, and 3.3.6 we get the following results about A_0 and $\overline{A_0}$.

Corollary 3.3.8 *A_0 is an unbounded Fredholm operator with index equal to minus one. One has*

$$A_0 T_0 = I_{\ell^2_{a'}(\mathbb{N})} - Proj_{Coker(A_0)}$$

$$T_0 A_0 = I_{\ell^2_a(\mathbb{N})}$$

where $T_0 := \overline{T}^*$

One also has:

Corollary 3.3.9 $\overline{A_0}$ *is an unbounded Fredholm operator with index zero, and*

$$\overline{A_0 T_0} = I_{\ell^2_a(\mathbb{N})}$$
$$\overline{T_0 A_0} = I_{\ell^2_{a'}(\mathbb{N})}.$$

It turns out that more can be said about the parametrices introduced above.

Proposition 3.3.10 *Each of the parametrix operators: T, T_0, $\overline{T}$, $\overline{T_0}$ is a Hilbert-Schmidt operator.*

Proof Only the details for the operator T are presented, as other cases are similar. In fact the proposition already follows from the way we estimated the norm of T since T is an integral operator. An alternative proof is given here. First note that $\|T\|^2_{HS} = tr(T^*T) = \sum_{i=0}^{\infty} \|Te_i\|^2$ where $\{e_i\}$ is the canonical basis for $\ell^2_a(\mathbb{N})$. So

$$(Te_i)_n = \frac{1}{a_n}(e_i)_n + \frac{c_{n-1}}{a_{n-1}}(e_i)_{n-1} + \cdots + \frac{c_{n-1}\cdots c_0}{a_0}(e_i)_0.$$

It follows that $(Te_i)_n = 0 \ \forall n < i$, and

$$(Te_i)_i = \frac{\sqrt{a_i}}{a_i}$$
$$(Te_i)_{i+1} = \frac{c_i}{a_i}\sqrt{a_i}$$
$$(Te_i)_{i+2} = \frac{c_{i+1}c_i}{a_i}\sqrt{a_i}$$
$$\vdots$$

Then we estimate

$$\|Te_i\|^2 = \frac{1}{a_i}\sum_{k=0}^{\infty} \frac{1}{a'_{i+k}}|c_i c_{i+1}\cdots c_{i+k}|^2 \leq \frac{1}{a_i}C',$$

and consequently

$$\|T\|^2_{HS} = \sum_{i=0}^{\infty} \|Te_i\|^2 \leq C' \sum_{i=0}^{\infty} \frac{1}{a_i} \leq CC' \quad \Rightarrow \|T\|_{HS} \leq \sqrt{CC'}.$$

■

Now the attention will be shifted to the bilateral case and one would like to study the same type of properties as considered in the unilateral case. It turns out that both A and $\overline{A}$ have one dimensional kernels in that case, one has to use infinite products for some expressions, and there are more options of imposing conditions at infinities. However the analytic aspects of the theory are no different than the unilateral case and so we provide less detail in some estimates to avoid repetitiveness.

3.3.2 Bilateral case

As in the unilateral case oen starts with the study of the kernels of A and $\overline{A}$. It turns out that both A and $\overline{A}$ have one dimensional kernels. First recall the constants defined at the beginning of this section

$$C = \sum_{n\in\mathbb{Z}} \frac{1}{a_n} < \infty \ , \ C' = \sum_{n\in\mathbb{Z}} \frac{1}{a'_n} < \infty \quad \text{and} \quad K = \prod_{n\in\mathbb{Z}} \frac{1}{|c_n|} < \infty.$$

Proposition 3.3.11 *Given A and $\overline{A}$ above we have:*

$$dim\ KerA = 1$$

$$dim\ Ker\overline{A} = 1.$$

Proof First the investigation of the kernel of A is done. To this end one needs to solve the equation $Af_n = a_n(f_n - c_{n-1}f_{n-1}) = 0$ for $n \in \mathbb{Z}$. This is done recursively and, for $n \geq 0$, one arrives at the following

$$f_n = \left(\prod_{i=-1}^{n-1} c_i\right) f_{-1}, \quad n \geq 0.$$

Next, in a similar fashion, solve the equation for $n < 0$ to get the following

$$f_{-n} = \left(\prod_{i=-2}^{-n} \frac{1}{c_i}\right) f_{-1}, \quad n \geq 1.$$

The two formulas above can be written compactly in the following semi-infinite product

$$f_n = \left(\prod_{i=-\infty}^{n-1} c_i \right) \alpha$$

for any constant α. To see that the kernel of A is indeed one dimensional, we need to verify that $\{f_n\} \in \ell^2_{a'}(\mathbb{Z})$. Using the fact that $|c_i| \leq 1$ for all i one has that

$$\|f\|^2_{\ell^2_{a'}(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \frac{1}{a'_n} \left| \prod_{i=-\infty}^{n-1} c_i \right|^2 |\alpha|^2 \leq |\alpha|^2 \sum_{n\in\mathbb{Z}} \frac{1}{a'_n} = |\alpha|^2 C' < \infty,$$

thus $\{f_n\} \in \ell^2_{a'}(\mathbb{Z})$.

Next the equation $\overline{A} f_n = a'_n(f_n - \overline{c_n} f_{n+1}) = 0$ for $n \in \mathbb{Z}$ needs to be studied. One gets

$$f_n = \left(\prod_{i=0}^{n-1} \frac{1}{\overline{c_i}} \right) f_0 \quad \text{for} \quad n \geq 0$$

and the similar formula for $n < 0$

$$f_{-n} = \left(\prod_{i=1}^{-n} \overline{c_i} \right) f_0 \quad \text{for} \quad n \geq 1.$$

Also one has the same type of semi-infinite product for $\overline{A}$:

$$f_n = \left(\prod_{i=n}^{\infty} \overline{c_i} \right) \beta$$

for any constant β. As with A, to guarantee that the kernel of $\overline{A}$ is one dimensional, we need to verify that $\{f_n\} \in \ell^2_a(\mathbb{Z})$. Using the fact that $|c_i| \leq 1$ for all i one has that

$$\|f\|^2_{\ell^2_a(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} \frac{1}{a_n} \left| \prod_{i=n}^{\infty} \overline{c_i} \right|^2 |\beta|^2 \leq |\beta|^2 \sum_{n\in\mathbb{Z}} \frac{1}{a_n} = |\beta|^2 C < \infty.$$

This completes the proof. ∎

Next we construct a parametrix for A.

Proposition 3.3.12 *There exists a $T \in B(\ell^2_a(\mathbb{Z}), \ell^2_{a'}(\mathbb{Z}))$ such that $AT = I_{\ell^2_a(\mathbb{Z})}$ and $TA = I_{\ell^2_{a'}(\mathbb{Z})} - Proj_{KerA}$. In particular A is an unbounded Fredholm operator with index equal to one.*

Proof We start by looking at the equation $Af_n = a_n(f_n - c_{n-1}f_{n-1}) = g_n$ which can be written as:

$$f_n - c_{n-1}f_{n-1} = \frac{g_n}{a_n}. \tag{3.11}$$

The variation of constants method is usded to solve (3.11). First observe that the homogeneous equation $f_n - c_{n-1}f_{n-1} = 0$ has the following solution by the kernel calculation in Proposition 3.3.11: $f_n = \left(\prod_{i=-\infty}^{n-1} c_i\right)\alpha$ for some constant α. Consequently set

$$f_n = \left(\prod_{j=-\infty}^{n-1} c_j\right)\alpha_n$$

and substitute this into equation (3.11). This leads to the following equation for α_n:

$$\alpha_n - \alpha_{n-1} = \left(\prod_{j=-\infty}^{n-1} \frac{1}{c_j}\right)\frac{g_n}{a_n}$$

which has a solution given by:

$$\alpha_n = \sum_{i=-\infty}^{n} \frac{1}{a_i}\left(\prod_{j=-\infty}^{i-1} \frac{1}{c_j}\right) g_i.$$

Therefore one has a particular solution of equation (3.11):

$$f_n = \sum_{i=-\infty}^{n} \frac{1}{a_i}\left(\prod_{j=i}^{n-1} c_j\right) g_i,$$

and the general solution is

$$f_n = \sum_{i=-\infty}^{n} \frac{1}{a_i}\left(\prod_{j=i}^{n-1} c_j\right) g_i - \left(\prod_{i=-\infty}^{n-1} c_i\right)\alpha.$$

The above expression gives the formula for T:

$$(Tg)_n = (T_1 g)_n - \alpha(g)\Omega_n^-, \tag{3.12}$$

where

$$(T_1 g)_n := \sum_{i=-\infty}^{n} \frac{1}{a_i}\left(\prod_{j=i}^{n-1} c_j\right) g_i \tag{3.13}$$

and $\Omega_n^- := \prod_{i=-\infty}^{n-1} c_i$, and $\alpha(g)$ arbitrary.

It's is clear from the construction that $AT = I_{\ell_a^2(\mathbb{Z})}$. To make sure that one gets $TA = I_{\ell_{a'}^2(\mathbb{Z})} - \text{Proj}_{\text{Ker } A}$, a choice on $\alpha(g)$ must be made just as in the unilateral case:

$$\alpha(g) := \frac{\langle \Omega^-, T_1 g\rangle}{||\Omega^-||^2} = \frac{\sum_{n\in\mathbb{Z}} \sum_{i=-\infty}^{n} \frac{1}{a_n}\frac{1}{a_i} \left(\prod_{k=-\infty}^{n-1} \overline{c_k}\right) \left(\prod_{j=i}^{n-1} c_j\right) g_i}{\sum_{n\in\mathbb{Z}} \frac{1}{a_n} \left(\prod_{i=-\infty}^{n-1} |c_i|^2\right)}.$$

Convergence of the sums and products and the boundedness of T is established just as in the unilateral case. The operator $\overline{T_1}$ is bounded by $\sqrt{CC'}$ in essentially the same way as the operator T is Proposition 3.3.2. To see that we write

$$(T_1 g)_n = \frac{\sqrt{a_n}}{a_n} \frac{1}{\sqrt{a_n}} g_n + \frac{c_{n-1}\sqrt{a_{n-1}}}{a_{n-1}} \frac{1}{\sqrt{a_{n-1}}} g_{n-1} + \cdots$$

and estimate using the Cauchy-Schwarz inequality and the fact that the $|c_i| \leq 1$ for all i:

$$\begin{aligned} |(T_1 g)_n|^2 &\leq \left[\left(\frac{\sqrt{a_n}}{a_n}\right)^2 + \left(\frac{\sqrt{a_{n-1}}}{a_{n-1}}\right)^2 + \cdots\right] \left(\frac{1}{a_n}|g_n|^2 + \frac{1}{a_{n-1}}|g_{n-1}|^2 + \cdots\right) \\ &\leq \left(\sum_{i=-\infty}^{n} \frac{1}{a_i}\right) \|g\|^2. \end{aligned}$$

Consequently

$$\|T_1 g\|^2 = \sum_{n\in\mathbb{Z}} \frac{1}{a'_n} |T_1 g_n|^2 \leq \sum_{n\in\mathbb{Z}} \frac{1}{a'_n} C\|g\|^2 = (C'C)\|g\|^2.$$

To estimate $\alpha(g)$ we notice that

$$C' \geq ||\Omega||^2 = \sum_{n\in\mathbb{Z}} \frac{1}{a'_n} \left(\prod_{i=n}^{\infty} |c_i|^2\right) \geq \sum_{n\in\mathbb{Z}} \frac{1}{a'_n} \left(\prod_{i\in\mathbb{Z}} |c_i|^2\right) = \frac{C'}{K^2},$$

which implies that $|\alpha(g)| \leq K\sqrt{C}||g||$ and $||T|| \leq \sqrt{CC'} + K\sqrt{CC'}$. This completes the proof. ∎

An important corollary from this proposition is the existence of limits at infinities for the sequences which are in the domain of A.

Corollary 3.3.13 *Let $f \in dom(A)$, then $f_{\pm\infty}$ exist and are given by the following formulas:*

$$f_{\infty} = \sum_{i=-\infty}^{\infty} \frac{1}{a_i} \left(\prod_{j=i}^{\infty} c_j \right) Af_i - \left(\prod_{i=-\infty}^{\infty} c_i \right) \alpha(Af)$$
$$f_{-\infty} = \alpha(Af).$$

Proof Using the previous proposition and the methods invoked in Corollaries 3.3.3 and 3.3.5 yields the desired result. ∎

Next analogous results about the $\overline{A}$ are stated.

Proposition 3.3.14 *There exists a $\overline{T} \in B(\ell^2_{a'}(\mathbb{Z}), \ell^2_a(\mathbb{Z}))$ such that $\overline{AT} = I_{\ell^2_{a'}(\mathbb{Z})}$ and $\overline{TA} = I_{\ell^2_{a'}(\mathbb{Z})} - Proj_{Ker\overline{A}}$. In particular $\overline{A}$ is an unbounded Fredholm operator with index equal to one.*

Proof The solution of the equation

$$a'_n(f_n - \overline{c_n} f_{n+1}) = g_n \quad \text{for } n \in \mathbb{Z}$$

is given the following formula

$$(\overline{T}g)_n = \sum_{i=n}^{\infty} \frac{1}{a'_i} \left(\prod_{j=n}^{i-1} \overline{c_j} \right) g_i - \left(\prod_{i=n}^{\infty} \overline{c_i} \right) \beta(g) = \overline{T_0} g_n - \beta(g) \Omega_n^+, \tag{3.14}$$

where we set $\prod_{j=n}^{n-1} \overline{c_j} = 1$ and $\beta(g)$ is an arbitrary constant. Here

$$(\overline{T_0} g)_n := \sum_{i=n}^{\infty} \frac{1}{a'_i} \left(\prod_{j=n}^{i-1} \overline{c_j} \right) g_i, \tag{3.15}$$

and $\Omega_n^+ := \prod_{i=n}^{\infty} \overline{c_i}$.

One has the relation $\overline{AT} = I_{\ell^2_{a'}(\mathbb{Z})}$, however to make sure one has $\overline{TA} = I_{\ell^2_a(\mathbb{Z})} - \mathrm{Proj}_{\mathrm{Ker}\ \overline{A}}$, one needs to make the following choice of $\beta(g)$:

$$\beta(g) = \frac{\langle \Omega^+, \overline{T_0} g \rangle}{||\Omega^+||^2} = \frac{\sum_{n\in\mathbb{Z}} \sum_{i=n}^{\infty} \frac{1}{a'_n} \frac{1}{a'_i} \left(\prod_{j=n}^{i-1} c_j \right) \left(\prod_{k=n}^{\infty} \overline{c_k} \right) g_i}{\sum_{n\in\mathbb{Z}} \frac{1}{a'_n} \left(\prod_{i=n}^{\infty} |c_i|^2 \right)}.$$

The previous methods yield

$$\|\overline{T}\| \leq \sqrt{CC'} + K\sqrt{CC'} < \infty,$$

and the statement of the proposition follows. ■

An immediate corollary is the following:

Corollary 3.3.15 *Let $f \in dom(\overline{T})$, then $f_{\pm\infty}$ exist and*

$$\begin{aligned} f_{\infty} &= \beta(Af) \\ f_{-\infty} &= \sum_{i=-\infty}^{\infty} \frac{1}{a_i'} \left(\prod_{j=-\infty}^{i-1} \overline{c_j} \right) Af_i - \left(\prod_{i=-\infty}^{\infty} \overline{c_i} \right) \beta(Af). \end{aligned}$$

Imposing vanishing conditions at infinities one can construct the following six operators. A_0 is the operator A but with domain

$$\mathrm{dom}(A_0) = \{f \in \mathrm{dom}(A) : f_{\infty} = 0\}$$

and $\overline{A_0}$ is the operator $\overline{A}$ with domain

$$\mathrm{dom}(\overline{A_0}) = \{f \in \mathrm{dom}(\overline{A}) : f_{\infty} = 0\}.$$

A_1 is the operator A with domain

$$\mathrm{dom}(A_1) = \{f \in \mathrm{dom}(A) : f_{-\infty} = 0\}$$

and $\overline{A_1}$ is the operator $\overline{A}$ with domain

$$\mathrm{dom}(\overline{A_1}) = \{f \in \mathrm{dom}(\overline{A}) : f_{-\infty} = 0\}.$$

Finally A_2 is the operator A with domain

$$\mathrm{dom}(A_2) = \{f \in \mathrm{dom}(A) : f_{\pm\infty} = 0\}$$

and $\overline{A_2}$ is the operator $\overline{A}$ with domain

$$\mathrm{dom}(\overline{A_2}) = \{f \in \mathrm{dom}(\overline{A}) : f_{\pm\infty} = 0\}.$$

The above operators are related by the calculation of adjoints of A and $\overline{A}$.

Proposition 3.3.16 *With the above definitions we have:*

$$A^* = \overline{A_2},\ A_0^* = \overline{A_1},\ A_1^* = \overline{A_0},\ A_2^* = \overline{A},\ \overline{A}^* = A_2,\ \overline{A_0}^* = A_1,\ \overline{A_1}^* = A_0,\ \overline{A_2}^* = A.$$

Proof This easily follows from the integration by parts formula:

$$\langle Af, g\rangle = \langle f, \overline{A}g\rangle - \overline{f_\infty} g_\infty + \overline{f_{-\infty}} g_{-\infty}.$$

∎

It follows from the definitions and the kernel calculations for A and $\overline{A}$ that the just introduced six operators $A_0, A_1, A_2, \overline{A_0}, \overline{A_1}, \overline{A_2}$ have no kernel, while the adjoint calculation shows that only $A_2, \overline{A_2}$ have cokernel (of dimension one).

Next a parametrix for each of the above operators is built. So far we have constructed T, formula 3.12, and $\overline{T}$, formula (3.14). In view of the above proposition we set $T_2 := \overline{T}^*$ and $\overline{T_2} := T^*$. We have also introduced T_1, formula (3.13), and $\overline{T_0}$, formula (3.15) and one can verify like in Proposition 3.3.7 that $T_1^* = \overline{T_0}$. The following similar looking operators are introduced:

$$(\overline{T_1}g)_n := \sum_{i=-\infty}^{n} \frac{1}{a_i} \left(\prod_{j=i}^{n-1} \overline{c_j} \right) g_i$$

and

$$(T_0 g)_n := \sum_{i=n}^{\infty} \frac{1}{a'_i} \left(\prod_{j=n}^{i-1} c_j \right) g_i,$$

for which one has $T_0^* = \overline{T_1}$. Then we get the following summary of the Fredholm properties of our operators.

Proposition 3.3.17 *With the above definitions one has*

$$A_0T_0 = I_{\ell^2_a(\mathbb{Z})} \text{ and } T_0A_0 = I_{\ell^2_{a'}(\mathbb{Z})}$$
$$A_1T_1 = I_{\ell^2_a(\mathbb{Z})} \text{ and } T_1A_1 = I_{\ell^2_{a'}(\mathbb{Z})}$$
$$A_2T_2 = I_{\ell^2_a(\mathbb{Z})} - Proj_{Coker(A_2)} \text{ and } T_2A_2 = I_{\ell^2_{a'}(\mathbb{Z})}$$
$$\overline{T_0A_0} = I_{\ell^2_{a'}(\mathbb{Z})} \text{ and } \overline{A_0T_0} = I_{\ell^2_a(\mathbb{Z})}$$
$$\overline{T_1A_1} = I_{\ell^2_{a'}(\mathbb{Z})} \text{ and } \overline{A_1T_1} = I_{\ell^2_a(\mathbb{Z})}$$
$$\overline{T_2A_2} = I_{\ell^2_{a'}(\mathbb{Z})} \text{ and } \overline{A_2T_2} = I_{\ell^2_a(\mathbb{Z})} - Proj_{Coker(\overline{A_2})}.$$

In particular all six operators are unbounded Fredholm operators with index zero for A_0, A_1, $\overline{A_0}$, $\overline{A_1}$ *and index minus one for* A_2, $\overline{A_2}$.

We conclude this section with a simple observation on functional-analytic properties of the parametrices.

Proposition 3.3.18 *Each of the 8 parametrix operators:* T, T_0, T_1, T_2, $\overline{T}$, $\overline{T_0}$, $\overline{T_1}$, $\overline{T_2}$ *is a Hilbert-Schmidt operator.*

3.4 Fourier Transform in quantum domains

In this section the Fourier Transform in the quantum domains is consided, and one will get decomposition theorems for the Hilbert space $\mathcal{H}$ and the operator D, defined in section 3.2. The following discussion covers both cases $\mathbb{S} = \mathbb{N}$ and $\mathbb{S} = \mathbb{Z}$ in a fairly uniform manner: there are only a few places where the difference between the unilateral and the bilateral cases needs to be covered separately. An extensive use of the label operator will be made and it is defined as:

$$Ke_k = ke_k,$$

where $\{e_k\}$, $k \in \mathbb{S}$ is the canonical basis for $\ell^2_a(\mathbb{S})$. The label operator lets one write different diagonal operators as its functions. For example two previously introduced operators can be expressed, with some notational abuse, as $W = W(K)$, and $S = S(K)$, see 3.5 and 3.6, with $W(k) = w_k$, and $S(k) = s_k = w_k^2 - w_{k-1}^2$. Additionally, the elements of $\ell_a^2(\mathbb{S})$ will also be written using the function notation i.e. $\{f_k\} = \{f(k)\}$. If $\{f(k)\}$ has a limit at $\pm\infty$ it is denoted by $f(\pm\infty)$.

For the purpose of the following discussion define

$$a^{(n)}(k) = S^{-1/2}(k)S^{-1/2}(k+n). \tag{3.16}$$

Then one has the following lemma which is essentially a Fourier decomposition of the Hilbert space $\mathcal{H}$.

Lemma 3.4.1 *Let $a^{(n)} = \{a^{(n)}(k)\}$ be the sequence of positive numbers defined above. The map $I : \bigoplus_{m=0}^{\infty} \ell^2_{a^{(m)}}(\mathbb{S}) \oplus \bigoplus_{n=1}^{\infty} \ell^2_{a^{(n)}}(\mathbb{S}) \to \mathcal{H}$ given by*

$$\bigoplus_{m=0}^{\infty}\{f_m(k)\}_{k\in\mathbb{S}} \oplus \bigoplus_{n=1}^{\infty}\{g_n(k)\}_{k\in\mathbb{S}} \xrightarrow{I} \sum_{m=0}^{\infty} U^m f_m(K) + \sum_{n=1}^{\infty} g_n(K)(U^*)^n$$

is well-defined and is an isomorphism of Hilbert spaces.

Proof First we need to show that I is an isometry. This will only be done for the $g_n(K)$ terms as the calculation for the $f_n(K)$ terms is essentially identical. It follows that

$$\begin{aligned}
\left\| \sum_{n=1}^{\infty} g_n(K)(U^*)^n \right\|_{\mathcal{H}}^2 &= \operatorname{tr}\left(S^{1/2}(K) \sum_{n=1}^{\infty} g_n(K)(U^*)^n S^{1/2}(K) \sum_{l=1}^{\infty} U^n \overline{g_l(K)} \right) \\
&= \operatorname{tr}\left(S^{1/2}(K) S^{1/2}(K+n) \sum_{n=1}^{\infty} |g_n(K)|^2 \right) \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \frac{1}{a^{(n)}(k)} |g_n(k)|^2 = \sum_{n=1}^{\infty} \|\{g_n(k)\}\|^2_{\ell^2_{a^{(n)}}} \\
&= \|\{g_n(k)\}\|^2_{\bigoplus_{n=1}^{\infty} \ell^2_{a^{(n)}}}
\end{aligned}$$

and thus the norms are the same and I is an isometry on its range. To show that Ran $I = \mathcal{H}$ it must be demonstrated that Ran I is dense in $\mathcal{H}$.

First note that $C^*(W)$ is dense in $\mathcal{H}$ by construction. Define $\delta_l(k)$ to be the following function:

$$\delta_l(k) = \begin{cases} 1 & k = l \\ 0 & k \neq l. \end{cases}$$

Then the (not normalized) canonical basis in $\ell^2_{a^{(m)}}(\mathbb{S})$ corresponds through the map I to $U^m\delta_l(K)$ and similarly the canonical basis in $\ell^2_{a^{(n)}}(\mathbb{S})$ corresponds to $\delta_l(K)(U^*)^n$. Note that $U^m\delta_l(K)$ and $\delta_l(K)(U^*)^n$ sit inside $C^*(W)$, so all that is required is to show they generate a dense set in $C^*(W)$ in the topology induced by $\mathcal{H}$ (they do not in the usual topology of $C^*(W)$). However this is clear since

$$\sum_{l\leq L} \delta_l(K) \underset{L\to\infty}{\to} I \quad \text{in } \mathcal{H}$$

because the operator S is trace class. It follows that U, U^* are in Ran I, and thus Ran I is a dense subspace of $\mathcal{H}$. ■

In what follows it will be convenient sometimes to write the Fourier series for $a \in \mathcal{H}$ in one of two ways:

$$a = \sum_{m=0}^{\infty} U^m f_m(K) + \sum_{n=1}^{\infty} g_n(K)(U^*)^n = \sum_{m=1}^{\infty} U^m f_m(K) + \sum_{n=0}^{\infty} g_n(K)(U^*)^n$$

where we always set $f_0(k) = g_0(k)$.

The Fourier transform described in the above lemma will now be used to find a decomposition of D in terms of the operators A and $\overline{A}$ defined in the previous section. Recall that those operators depend on sequences of weights a, a' and coefficients c subject to conditions 3.7. Since in the following the parameters vary, one will need appropriate decorations on A and $\overline{A}$. To do that, in addition to sequences (3.16), the following is introduced:

$$c^{(n)}(k) := W(k)W^{-1}(k+n+1). \tag{3.17}$$

Next define the operators $A^{(n)}$ as follows:

$$\begin{aligned}
&A^{(n)} : \mathrm{dom}(A^{(n)}) \subset \ell^2_{a^{(n+1)}}(\mathbb{S}) \to \ell^2_{a^{(n)}}(\mathbb{S})\\
&\text{where } \mathrm{dom}(A^{(n)}) = \{f \in \ell^2_{a^{(n+1)}}(\mathbb{S}) \ : \ \|A^{(n)} f\|_{\ell^2_{a^{(n)}}(\mathbb{S})} < \infty\}\\
&A^{(n)} f(k) = a^{(n)}(k) \left(f(k) - c^{(n)}(k-1)\, f(k-1)\right).
\end{aligned}$$

The corresponding formal adjoints $\overline{A}^{(n)}$ are defined in the same way as in the previous section i.e.

$$\begin{aligned}
&\overline{A}^{(n)} : \mathrm{dom}(\overline{A}^{(n)}) \subset \ell^2_{a^{(n)}}(\mathbb{S}) \to \ell^2_{a^{(n+1)}}(\mathbb{S})\\
&\text{where } \mathrm{dom}(\overline{A}^{(n)}) = \{f \in \ell^2_{a^{(n)}}(\mathbb{S}) \ : \ \|\overline{A}^{(n)} f\|_{\ell^2_{a^{(n+1)}}(\mathbb{S})} < \infty\}\\
&\overline{A}^{(n)} f(k) = a^{(n+1)}(k)(f(k) - \overline{c}^{(n)}(k) f(k+1)).
\end{aligned}$$

Additionally one will need the following diagonal operator $W^{(m)}(K) := W(K+m)$ i.e.

$$W^{(m)} f(k) := W(k+m) f(k)$$

for $f \in \ell^2_{a^{(n)}}(\mathbb{S})$. Clearly $W^{(m)}$ is a bounded, invertible, self-adjoint operator with a bounded inverse.

Now the main decomposition theorem can be stated. A minor difficulty here is that D is not diagonal with respect to the Fourier decomposition of the Hilbert space but rather shifts the components by one.

Theorem 3.4.2 *With the above notation the operator D has the following decomposition: $Da = \sum_{m=1}^{\infty} U^m f'_m(K) + \sum_{n=0}^{\infty} g'_n(K)(U^*)^n$, where $a = \sum_{m=0}^{\infty} U^m f_m(K) + \sum_{n=1}^{\infty} g_n(K)(U^*)^n$ and $f'_{m+1} = -\overline{A}^{(m)} W^{(m)} f_m$ and $g'_{n-1} = W^{(n-1)} A^{(n-1)} g_n$. We write symbolically:*

$$D \cong \left((-\overline{A}^{(m)} W^{(m)})_{m=0}^{\infty}, (W^{(n-1)} A^{(n-1)})_{n=1}^{\infty} \right).$$

Proof We compute the expression $Da = S^{-1/2}(K)\,[a, UW(K)]\, S^{-1/2}(K)$ using the Fourier decomposition: $a = \sum_{m=0}^{\infty} U^m f_m(K) + \sum_{n=1}^{\infty} g_n(K)(U^*)^n$. We use the following commutation relation

$$f(K)U = Uf(K+1).$$

Then one obtains, setting in the unilateral case $W(-1) = f_n(-1) = g_n(-1) = 0$,

$$\begin{aligned}
Da &= S^{-1/2}(K)\,[a, UW(K)]\,S^{-1/2}(K) \\
&= \sum_{m=0}^{\infty} S^{-1/2}(K)\,(U^m f_m(K)UW(K) - UW(K)U^m f_m(K))\,S^{-1/2}(K) \\
&+ \sum_{n=1}^{\infty} S^{-1/2}(K)\left(g_n(K)(U^*)^{n-1}W(K) - UW(K)g_n(K)(U^*)^n\right)S^{-1/2}(K).
\end{aligned}$$

The above expression is equal to

$$\begin{aligned}
&-\sum_{m=0}^{\infty} U^{m+1}S^{-1/2}(K)S^{-1/2}(K+m+1)\,(W(K+m)f_m(K) - W(K)f_m(K+1)) \\
&+\sum_{n=1}^{\infty} S^{-1/2}(K)S^{-1/2}(K+n-1) \\
&\qquad \times (W(K+n-1)g_n(K) - W(K-1)g_n(K-1))\,(U^*)^{n-1},
\end{aligned}$$

which can be written as:

$$\begin{aligned}
&-\sum_{m=0}^{\infty} U^{m+1}a^{(m+1)}(K) \\
&\qquad \times \left(W(K+m)f_m(K) - \frac{W(K)}{W(K+m+1)}W(K+m+1)f_m(K+1)\right) \\
&+\sum_{n=1}^{\infty} W(K+n-1)a^{(n-1)}(K)\left(g_n(K) - \frac{W(K-1)}{W^(K+n-1)}g_n(K-1)\right)(U^*)^{n-1}.
\end{aligned}$$

This is equal to:

$$\begin{aligned}
&-\sum_{m=0}^{\infty} U^{m+1}a^{(m+1)}(K)\left(W^{(m)}(K)f_m(K) - c^{(m)}(K)W^{(m)}(K+1)f_m(K+1)\right) \\
&+\sum_{n=1}^{\infty} W^{(n-1)}(K)a^{(n-1)}(K)\left(g_n(K) - c^{(n-1)}(K-1)g_n(K-1)\right)(U^*)^{n-1}.
\end{aligned}$$

Consequently

$$Da = -\sum_{m=0}^{\infty} U^{m+1}\overline{A}^{(m)}W^{(m)}f_m(K) + \sum_{n=1}^{\infty} W^{(n-1)}A^{(n-1)}g_n(K)(U^*)^{n-1}.$$

Next we need to verify that the $a^{(n)}$, see (3.16), and the $c^{(n)}$, see (3.17), satisfy the conditions 3.7. Note that since w_k is an increasing sequence converging to $w^+ > 0$ one has $|c^{(n)}(k)| = \left|\frac{w_k}{w_{k+n+1}}\right| \leq 1$.

In the unilateral case, $\mathbb{S} = \mathbb{N}$, we compute

$$K^{(n)} := \prod_{k=0}^{\infty} \frac{1}{c^{(n)}(k)} = \frac{(w^+)^{n+1}}{w_0 \cdots w_n} < \infty.$$

Next note that

$$\begin{aligned} C^{(n)} &:= \sum_{k=0}^{\infty} \frac{1}{a^{(n)}(k)} = \sum_{k=0}^{\infty} \sqrt{s_k s_{k+n}} \leq \sqrt{\sum_{k=0}^{\infty} s_k} \sqrt{\sum_{k=0}^{\infty} s_{k+n}} \\ &= \sqrt{w^+} \sqrt{\sum_{k=n}^{\infty} s_k} < \infty, \end{aligned}$$

with the constant $C^{(n)}$ going to zero as $n \to \infty$.

In the bilateral case ($k \in \mathbb{Z}$) we have

$$K^{(n)} := \prod_{k=-\infty}^{\infty} \frac{1}{c^{(n)}(k)} = \frac{(w^+)^{n+1}}{(w^-)^{n+1}} < \infty.$$

Next we estimate

$$\begin{aligned} C^{(n)} &:= \sum_{k=-\infty}^{\infty} \frac{1}{a^{(n)}(k)} = \sum_{k \leq -n/2} \sqrt{s_k s_{k+n}} + \sum_{k > -n/2} \sqrt{s_k s_{k+n}} \\ &\leq \sqrt{w^+ - w^-} \sqrt{\sum_{k \leq -n/2} s_k} + \sqrt{w^+ - w^-} \sqrt{\sum_{k > n/2} s_k} < \infty, \end{aligned}$$

and again the constant $C^{(n)}$ goes to zero as $n \to \infty$. ■

As we will see later on, the significance of $\lim_{n \to \infty} C^{(n)} = 0$ is that it implies compactness of a parametrix of D, subject to APS boundary conditions.

It is stated here without a proof the analogous result for the formal adjoint $\overline{D}$ of D. We define

$$\overline{D}b := S^{-1/2}(K)[b, W(K)U^*]S^{-1/2}(K).$$

on the maximal domain, like the operator D. One has the following decomposition.

Theorem 3.4.3 *With the above notation the operator $\overline{D}$ can be written as*

$$\overline{D} \cong \left((-W^{(m)}A^{(m)})_{m=0}^{\infty}, (\overline{A}^{(n-1)}W^{(n-1)})_{n=1}^{\infty} \right).$$

3.5 Proofs of the index theorems in the quantum disk and annulus

One is now in a position to consider the proofs of the main results of this chapter. A rephrasement of the statements of the theorems from section 3.2 is made here adding more detail. The operator D_N equals the unilateral operator D with domain

$$\mathrm{dom}(D_N) = \{a \in \mathrm{Dom}(D) : \ r(a) \in \mathrm{Ran}\ P_N\}.$$

The first of the main results of this chapter will now be proven.

Theorem 3.5.1 *The operator D_N defined above is an unbounded Fredholm operator with index $ind(D_N) = N + 1$. In fact, there is a bounded operator Q_N such that $Ker(Q_N) = Coker(D_N)$, $D_N Q_N = I - Proj_{Coker(D_N)}$, and $Q_N D_N = I - Proj_{Ker(D_N)}$. Moreover the parametrix Q_N is a compact operator.*

Proof All the hard work has been done. It's now just a matter of piecing together appropriate results from the previous sections. First we analyze the APS boundary conditions. Let $a = \sum_{n=0}^{\infty} U^n f_n(K) + \sum_{n=1}^{\infty} g_n(K)(U^*)^n$ be in $\mathrm{dom}(D_N)$. Then the restriction $r(a)$ from section 3.2 is well defined. We note that r acts on U, U^*, and $f(K)$ in the following way

$$\begin{aligned} r(U) &= e^{i\varphi} \\ r(U^*) &= e^{-i\varphi} \\ r(f(K)) &= f(\infty) \cdot I := \lim_{k\to\infty} f(k) \cdot I. \end{aligned}$$

The third equation holds because the difference $f(K) - f(\infty) \cdot I$ is a compact operator, and r vanishes on compact operators. Consequently one sees that r acts on $a \in \mathrm{Dom}(D)$ in the following way:

$$r(a) = \sum_{n=0}^{\infty} e^{in\varphi} f_n(\infty) + \sum_{n=1}^{\infty} g_n(\infty) e^{-in\varphi}.$$

This means that for $r(a)$ to be in the range of P_N, where Ran $P_N = \operatorname{span}_{n\leq N}\{e^{in\varphi}\}$, one has the following: if $N \geq 0$, then $f_n(\infty) = 0$ for $n > N$, and if $N < 0$, then $f_n(\infty) = 0$ for all n and $g_n(\infty) = 0$ for $n < -N$. Thus from Theorem 3.4.2 and from Proposition 3.3.6 one can represent D_N subject to the APS boundary conditions as follows

$$D_N = \begin{cases} \left((-\overline{A}^{(m)} W^{(m)})_{m=0}^{N}, (-\overline{A_0}^{(m)} W^{(m)})_{m=N+1}^{\infty}, (W^{(n-1)} A^{(n-1)})_{n=1}^{\infty}\right) \\ \text{for } N \geq 0 \\ \left((-\overline{A_0}^{(m)} W^{(m)})_{m=0}^{\infty}, (W^{(n-1)} A_0^{(n-1)})_{n=1}^{-N-1}, (W^{(n-1)} A^{(n-1)})_{n=-N}^{\infty}\right) \\ \text{for } N < 0 \end{cases}$$

Also note from Theorem 3.4.2, Proposition 3.3.6 and the above analysis of the APS conditions, one can represent D_N^* as follows

$$D_N{}^* = \begin{cases} \left((-W^{(m)} A_0^{(m)})_{m=0}^{N}, (-W^{(m)} A^{(m)})_{m=N+1}^{\infty}, (\overline{A_0}^{(n-1)} W^{(n-1)})_{n=1}^{\infty}\right) \\ \text{for } N \geq 0 \\ \left((-W^{(m)} A^{(m)})_{m=0}^{\infty}, (\overline{A}^{(n-1)} W^{(n-1)})_{n=1}^{-N-1}, (\overline{A_0}^{(n-1)} W^{(n-1)})_{n=-N}^{\infty}\right) \\ \text{for } N < 0 \end{cases}$$

From these representations and from Proposition 3.3.1, one gets the following

$$\dim \operatorname{Ker} D_N = \begin{cases} N+1 & \text{for } N \geq 0 \\ 0 & \text{for } N < 0 \end{cases}$$

and

$$\dim \operatorname{Ker} D_N{}^* = \begin{cases} 0 & \text{for } N \geq 0 \\ -(N+1) & \text{for } N < 0 \end{cases}$$

and thus the index calculation follows. To conclude that D_N is a Fredholm operator we need to construct a parametrix. One builds Q_N in the following fashion:

$$Q_N = \begin{cases} \left((-V^{(m)}\overline{T}^{(m)})_{m=0}^{N}, (-V^{(m)}\overline{T_0}^{(m)})_{m=N+1}^{\infty}, (T^{(n-1)}V^{(n-1)})_{n=1}^{\infty}\right) \\ \text{for } N \geq 0 \\ \left((-V^{(m)}\overline{T_0}^{(m)})_{m=0}^{\infty}, ({T_0}^{(n-1)}V^{(n-1)})_{n=1}^{-N-1}, (T^{(n-1)}V^{(n-1)})_{n=-N}^{\infty}\right) \\ \text{for } N < 0 \end{cases}$$

where $T^{(n)}$, $\overline{T}^{(n)}$, $T_0^{(n)}$, and $\overline{T_0}^{(n)}$ are, correspondingly, the parametrices for $A^{(n)}$, $\overline{A}^{(n)}$, $A_0^{(n)}$ and $\overline{A_0}^{(n)}$, as defined in section 3.3, and

$$V^{(m)} := \left(W^{(m)}\right)^{-1}.$$

From Corollary 3.3.8 and Propositions 3.3.4 and 3.3.2, it follows that

$$Q_N D_N = \begin{cases} I - \text{Proj}_{\text{Ker } D_N} & \text{for } N \geq 0 \\ I & \text{for } N < 0 \end{cases}$$

and

$$D_N Q_N = \begin{cases} I & \text{for } N \geq 0 \\ I - \text{Proj}_{\text{Ker } D_N{}^*} & \text{for } N < 0. \end{cases}$$

From the construction, the kernel of each T operator is the cokernel of the corresponding A operator, which implies that $Ker(Q_N) = Coker(D_N)$.

Finally all that remains is to show that Q_N is a bounded, and in fact, a compact operator. Notice that $T^{(n-1)}V^{(n-1)}$ and $-V^{(m)}\overline{T_0}^{(m)}$ are compact operators (in fact Hilbert-Schmidt operators) with norms that can be estimated as follows:

$$||T^{(n-1)}V^{(n-1)}|| \leq \frac{1}{w_0}\sqrt{C^{(n-1)}C^{(n)}}$$

and similarly

$$||V^{(m)}\overline{T_0}^{(m)}|| \leq \frac{1}{w_0}\sqrt{C^{(m)}C^{(m+1)}}.$$

Since $C^{(n)} \to 0$ as $n \to \infty$, it follows from the decomposition that Q_N is compact as a uniform limit of compact operators. Thus this completes the proof. ■

Now one considers the non-commutative cylinder case. The operator $D_{M,N}$ equals the bilateral operator D with domain

$$\mathrm{dom}(D_{M,N}) = \left\{a \in \mathrm{Dom}(D): \ r_+(a) \in \mathrm{Ran}\ P_N^+,\ r_-(a) \in \mathrm{Ran}\ P_M^-\right\}.$$

Theorem 3.5.2 *The operator $D_{M,N}$ above is an unbounded Fredholm operator with index $ind(D_{M,N}) = M+N+1$. In fact, there is a bounded operator $Q_{M,N}$ such that that $Ker(Q_{M,N}) = Coker(D_{M,N})$, $D_{M,N}Q_{M,N} = I - Proj_{Coker(D_{M,N})}$, and $Q_{M,N}D_{M,N} = I - Proj_{Ker(D_{M,N})}$. Moreover the parametrix $Q_{M,N}$ is a compact operator.*

Proof The proof is analogous to the previous proof, however there are more cases to consider. This is due to the way how this chapter treated both the disk and the cylinder in complete parallel so far. A different Fourier transform of the Hilbert space could also have been considered leading to an easier index calculation. However that would have made the corresponding decompositions of D different and more complicated to analyze.

Let $a = \sum_{n=0}^{\infty} U^n f_n(K) + \sum_{n=1}^{\infty} g_n(K)(U^*)^n$ be in $\mathrm{dom}(D_{M,N})$. Then we have

$$r_\pm(a) = \sum_{n=0}^{\infty} e^{in\varphi} f_n(\pm\infty) + \sum_{n=1}^{\infty} g_n(\pm\infty)e^{-in\varphi}.$$

We need $r_+(a)$ to be in Ran $P_N^+ = \underset{n\leq N}{\mathrm{span}}\{e^{in\varphi}\}$, and for $r_-(a)$ to be in Ran $P_M^- = \underset{-M\leq n}{\mathrm{span}}\{e^{in\varphi}\}$, so one is led to consider the following six cases. In each case we list the decomposition of the operator $D_{M,N}$ (in the first line), its adjoint ${D_{M,N}}^*$ (in the second line), and the parametrix $Q_{M,N}$ (in the third line).

<u>*Case 1*</u> *:* $M + N \geq 0$

Case 1(a) : $N \geq 0,\ M > 0$

$$\Big((-\overline{A}^{(m)} W^{(m)})_{m=0}^{N}, (-\overline{A_0}^{(m)} W^{(m)})_{m=N+1}^{\infty},$$
$$(W^{(n-1)} A^{(n-1)})_{n=1}^{M}, (W^{(n-1)} A_1^{(n-1)})_{n=M+1}^{\infty}\Big)$$

$$\Big((-W^{(m)} {A_2}^{(m)})_{m=0}^{N}, (-W^{(m)} {A_1}^{(m)})_{m=N+1}^{\infty},$$
$$(\overline{A_2}^{(n-1)} W^{(n-1)})_{n=1}^{M}, (\overline{A_0}^{(n-1)} W^{(n-1)})_{n=M+1}^{\infty}\Big)$$

$$\Big((-V^{(m)}\overline{T}^{(m)})_{m=0}^{N}, (-V^{(m)}\overline{T_0}^{(m)})_{m=N+1}^{\infty},$$
$$(T^{(n-1)}V^{(n-1)})_{n=1}^{M}, (T_1^{(n-1)}V^{(n-1)})_{n=M+1}^{\infty}\Big)$$

Case 1(b) : N < 0, M > 0

$$\Big((-\overline{A_0}^{(m)}W^{(m)})_{m=0}^{\infty}, (W^{(n-1)}A_0^{(n-1)})_{n=1}^{-N-1},$$
$$(W^{(n-1)}A^{(n-1)})_{n=-N}^{M}, (W^{(n-1)}A_1^{(n-1)})_{n=M+1}^{\infty}\Big)$$

$$\Big((-W^{(m)}A_1{}^{(m)})_{m=0}^{\infty}, (\overline{A_1}^{(n-1)}W^{(n-1)})_{n=1}^{-N-1},$$
$$(\overline{A_2}^{(n-1)}W^{(n-1)})_{n=-N}^{M}, (\overline{A_0}^{(n-1)}W^{(n-1)})_{n=M+1}^{\infty}\Big)$$

$$\Big((-V^{(m)}\overline{T_0}^{(m)})_{m=0}^{\infty}, (T_0^{(n-1)}V^{(n-1)})_{n=1}^{-N-1},$$
$$(T^{(n-1)}V^{(n-1)})_{n=-N}^{M}, (T_1^{(n-1)}V^{(n-1)})_{n=M+1}^{\infty}\Big)$$

In the formulas above there is no second term when $N = -1$.

Case 1(c) : $M \leq 0$, $N \geq 0$

$$\Big((-\overline{A_1}^{(m)}W^{(m)})_{m=0}^{-M-1}, (-\overline{A}^{(m)}W^{(m)})_{m=-M}^{N},$$
$$(-\overline{A_0}^{(m)}W^{(m)})_{m=N+1}^{\infty}, (W^{(n-1)}A_1^{(n-1)})_{n=1}^{\infty}\Big)$$

$$\Big((-W^{(m)}A_0{}^{(m)})_{m=0}^{-M-1}, (-W^{(m)}A_2{}^{(m)})_{m=-M}^{N},$$
$$(-W^{(m)}A_1{}^{(m)})_{m=N+1}^{\infty}, (\overline{A_0}^{(n-1)}W^{(n-1)})_{n=1}^{\infty}\Big)$$

$$\Big((-V^{(m)}\overline{T_1}^{(m)})_{m=0}^{-M-1}, (-V^{(m)}\overline{T}^{(m)})_{m=-M}^{N},$$
$$(-V^{(m)}\overline{T_0}^{(m)})_{m=N+1}^{\infty}, (T_1^{(n-1)}V^{(n-1)})_{n=1}^{\infty}\Big)$$

When $M = 0$ in the above formulas we simply omit the first term.

<u>Case 2</u> : $M + N < 0$

Case 2(a) : $N < 0$, $M \leq 0$

$$\Big((-\overline{A_2}^{(m)}W^{(m)})_{m=0}^{-M-1}, (-\overline{A_0}^{(m)}W^{(m)})_{m=-M}^{\infty},$$
$$(W^{(n-1)}A_2^{(n-1)})_{n=1}^{-N-1}, (W^{(n-1)}A_1^{(n-1)})_{n=-N}^{\infty}\Big)$$

$$\Big((-W^{(m)}A^{(m)})_{m=0}^{-M-1}, (-W^{(m)}A_1{}^{(m)})_{m=-M}^{\infty},$$
$$(\overline{A}^{(n-1)}W^{(n-1)})_{n=1}^{-N-1}, (\overline{A_0}^{(n-1)}W^{(n-1)})_{n=-N}^{\infty}\Big)$$
$$\Big((-V^{(m)}\overline{T_2}^{(m)})_{m=0}^{-M-1}, (-V^{(m)}\overline{T_0}^{(m)})_{m=-M}^{\infty},$$
$$(T_2^{(n-1)}V^{(n-1)})_{n=1}^{-N-1}, (T_1^{(n-1)}V^{(n-1)})_{n=-N}^{\infty}\Big)$$

In the formulas above there is no first term when $M = 0$.

Case 2(b) : $N < 0,\ M > 0$

$$\Big((-\overline{A_0}^{(m)}W^{(m)})_{m=0}^{\infty}, (W^{(n-1)}A_0^{(n-1)})_{n=1}^{M},$$
$$(W^{(n-1)}A_2^{(n-1)})_{n=M+1}^{-N-1}, (W^{(n-1)}A_1^{(n-1)})_{n=-N}^{\infty}\Big)$$
$$\Big((-W^{(m)}A_1{}^{(m)})_{m=0}^{\infty}, (\overline{A_1}^{(n-1)}W^{(n-1)})_{n=1}^{M},$$
$$(\overline{A}^{(n-1)}W^{(n-1)})_{n=M+1}^{-N-1}, (\overline{A_0}^{(n-1)}W^{(n-1)})_{n=-N}^{\infty}\Big)$$
$$\Big((-\overline{V^{(m)}T_0}^{(m)})_{m=0}^{\infty}, (T_0^{(n-1)}V^{(n-1)})_{n=1}^{M},$$
$$(T_2^{(n-1)}V^{(n-1)})_{n=M+1}^{-N-1}, (T_1^{(n-1)}V^{(n-1)})_{n=-N}^{\infty}\Big)$$

Case 2(c) : $N \geq 0,\ M < 0$

$$\Big((-\overline{A_1}^{(m)}W^{(m)})_{m=0}^{N-1}, (-\overline{A_2}^{(m)}W^{(m)})_{m=N}^{-M-1},$$
$$(-\overline{A_0}^{(m)}W^{(m)})_{m=-M}^{\infty}, (W^{(n-1)}A_1^{(n-1)})_{n=1}^{\infty}\Big)$$
$$\Big((-W^{(m)}A_0{}^{(m)})_{m=0}^{N-1}, (-W^{(m)}A^{(m)})_{m=N}^{-M-1},$$
$$(-W^{(m)}A_1{}^{(m)})_{m=-M}^{\infty}, (\overline{A_0}^{(n-1)}W^{(n-1)})_{n=1}^{\infty}\Big)$$
$$\Big((-V^{(m)}\overline{T_1}^{(m)})_{m=0}^{N-1}, (-V^{(m)}\overline{T_2}^{(m)})_{m=N}^{-M-1},$$
$$(-V^{(m)}\overline{T_0}^{(m)})_{m=-M}^{\infty}, (T_1^{(n-1)}V^{(n-1)})_{n=1}^{\infty}\Big)$$

In the formulas above there is again no first term when $N = 0$.

From these representations and from Proposition 3.3.11, one gets the following

$$\dim\, \mathrm{Ker}(D_{M,N}) = \begin{cases} M+N+1 & \text{for } M+N \geq 0 \\ 0 & \text{for } M+N < 0, \end{cases}$$

and

$$\dim\, \mathrm{Ker}({D_{M,N}}^*) = \begin{cases} 0 & \text{for } M+N \geq 0 \\ -(M+N+1) & \text{for } M+N < 0. \end{cases}$$

Thus index calculation follows. Using the analysis done in section 3.3, one gets the following two relations

$$Q_{M,N}D_{M,N} = \begin{cases} I - \mathrm{Proj}_{\mathrm{Ker}D_{M,N}} & \text{for } M+N \geq 0 \\ I & \text{for } M+N < 0, \end{cases}$$

and

$$D_{M,N}Q_{M,N} = \begin{cases} I & \text{for } M+N \geq 0 \\ I - \mathrm{Proj}_{\mathrm{Ker}{D_{M,N}}^*} & \text{for } M+N < 0. \end{cases}$$

The relation $Ker(Q_{M,N}) = Coker(D_{M,N})$ follows from the same property of the parametrix of each component of $Q_{M,N}$.

The proof that $Q_{M,N}$ is compact is the same as in the unilateral case. ■

4. DIRAC TYPE OPERATORS ON THE CLASSICAL AND QUAUNTUM PUNCTURED DISK

4.1 Classical Dirac operator on the punctured disk

In this section the analysis of Atiyah, Patodi, Singer is revisited in the simple case of semi-infinite cylinder $\mathbb{R}_+ \times S^1$, or equivalently a punctured disk. Using complex coordinates of the latter, the construction of a parametrix of a version of the d-bar operator is made and one proves norm estimates on its components by using different techniques than those in [2].

Let $\mathbb{D}^* = \{z \in \mathbb{C} \,:\, 0 < |z| \leq 1\}$ be the punctured disk. Consider the following Dirac type operator on $\mathbb{D}^*$:

$$D = -2\bar{z}\frac{\partial}{\partial \bar{z}}.$$

In polar coordinates $z = re^{i\varphi}$ the operator D has the following representation:

$$D = -r\frac{\partial}{\partial r} + \frac{1}{i}\frac{\partial}{\partial \varphi} = -r\frac{\partial}{\partial r} + B$$

where $B = \frac{1}{i}\frac{\partial}{\partial \varphi}$ is the boundary operator.

Studying D, subject to the APS boundary condition would like to be done, on the Hilbert space $L^2(\mathbb{D}^*, d\mu)$ with measure $\mu(z)$ given by the following formula:

$$d\mu(z) = \frac{1}{2i|z|^2} dz \wedge d\bar{z}. \tag{4.1}$$

Recall the APS condition. Define $P_{\geq 0}$ to be the spectral projection of B in $L^2(S^1)$ onto the non-negative part of the spectrum of B. Equivalently, $P_{\geq 0}$ is the orthogonal projection onto $\mathrm{span}\{e^{in\varphi}\}_{n\geq 0}$. Then one says that D satisfies the APS boundary condition when its domain consists of those functions $f(z) = f(r, \varphi)$ on $\mathbb{D}^*$ which have only negative frequencies at the boundary, see [2] and [5] for more details. More precisely:

$$\mathrm{dom}(D) = \left\{ f \in L^2(\mathbb{D}^*, d\mu) \ : \ \ Df \in L^2(\mathbb{D}^*, d\mu), \ \ P_{\geq 0} f(1,\cdot) = 0 \right\}. \tag{4.2}$$

Notice that, by the change of variable, $t = -\ln r$, the Dirac operator, D on $L^2(\mathbb{D}^*, d\mu)$, is equivalent to the operator, $\frac{\partial}{\partial t} + \frac{1}{i}\frac{\partial}{\partial \varphi}$ on $L^2(\mathbb{R}_+ \times S^1)$, since one has:

$$d\varphi \wedge dt = \frac{1}{2i|z|^2} dz \wedge d\overline{z}.$$

This matches the APS setup.

One proceeds the in the same way as in [2] by considering the spectral decomposition of the boundary operator B, which in our case amounts to Fourier decomposition:

$$f(z) = \sum_{n\in\mathbb{Z}} f_n(r) e^{-in\varphi}. \tag{4.3}$$

This yields the following decomposition of the Hilbert space $L^2(\mathbb{D}^*, d\mu)$:

$$L^2(\mathbb{D}^*, d\mu) = \bigoplus_{n\in\mathbb{Z}} \left(L^2((0,1], \frac{dr}{r}) \otimes \left[e^{-in\varphi}\right] \right). \cong \bigoplus_{n\in\mathbb{Z}} L^2 \left((0,1], \frac{dr}{r} \right) \tag{4.4}$$

Now one considers the decomposition of D and its inverse. The theorem below is a special case of Proposition 2.5 of [2] but a proof is supplied that generalizes to the noncommutative setup. Define $\overline{A}^{(n)} f(r) := -rf'(r) - nf(r)$ on the maximal domain in $L^2((0,1], \frac{dr}{r})$, and let $\overline{A_0}^{(n)}$ be the operator $\overline{A}^{(n)}$ but with domain $\{f(r) \in \mathrm{dom}(\overline{A}^{(n)}) : f(1) = 0\}$. One has:

Theorem 4.1.1 *Let D be the Dirac operator defined above on the domain (4.2). With respect to the decomposition 4.4 one has*

$$D \cong \bigoplus_{n>0} \overline{A}^{(n)} \oplus \bigoplus_{n\leq 0} \overline{A_0}^{(n)}.$$

Moreover, there exists an operator Q such that $DQ = I = QD$, and

$$Q = \bigoplus_{n\in\mathbb{Z}} Q^{(n)} = Q^{(0)} + \tilde{Q}$$

where $\tilde{Q}$ is bounded.

Proof Staring with a function $g(z) \in L^2(\mathbb{D}^*, d\mu)$ we want to solve the following equation

$$Df(z) = g(z)$$

with $f(z)$ satisfying the APS boundary condition. The Fourier decomposition (4.3) yields

$$\sum_{n\in\mathbb{Z}} \left(-rf_n'(r) - nf_n(r)\right) e^{-in\varphi} = \sum_{n\in\mathbb{Z}} g_n(r)e^{-in\varphi}.$$

Therefore we must solve the differential equation $-rf_n'(r) - nf_n(r) = g_n(r)$ where additionally $f_n(1) = 0$ for $n \le 0$. This, and the requirement that f is square integrable, assures that there is a unique solution given by the following formula:

$$f_n(r) = Q^{(n)}g_n(r) = \begin{cases} -\displaystyle\int_0^r \left(\frac{\rho}{r}\right)^n g_n(\rho)\frac{d\rho}{\rho} & n > 0 \\ \displaystyle\int_r^1 \left(\frac{\rho}{r}\right)^n g_n(\rho)\frac{d\rho}{\rho} & n \le 0 \end{cases}.$$

This gives the formula for the parametrix: $Q = \oplus_{n\in\mathbb{Z}} Q^{(n)}$. Showing $QD = DQ = I$ is a simple computation and is omitted.

The goal is to prove that $\tilde{Q} = \oplus_{n\neq 0} Q^{(n)}$ is bounded. One has

$$\|\tilde{Q}\| \le \sup_{n\in\mathbb{Z}\setminus\{0\}} \left\|Q^{(n)}\right\|.$$

In what follows it is shown that the $Q^{(n)}$ are uniformly bounded, in fact of order $O(\frac{1}{|n|})$. The main tool is the following inequality, see [13].

Lemma 4.1.2 *(Schur-Young Inequality) Let $T : L^2(Y) \longrightarrow L^2(X)$ be an integral operator:*

$$Tf(x) = \int K(x,y)f(y)dy.$$

Then one has

$$\|T\|^2 \leq \left(\underset{x\in X}{sup} \int_Y |K(x,y)|dy \right) \left(\underset{y\in Y}{sup} \int_X |K(x,y)|dx \right).$$

For negative n one can rewrite $Q^{(n)}$ as

$$Q^{(n)} g_n(r) = \int_0^1 K(r,\rho) g_n(\rho) \frac{d\rho}{\rho}$$

with integral kernel $K(r,\rho) = \chi(r/\rho)\,(r/\rho)^{|n|}$. Here the characteristic function $\chi(t) = 1$ for $t \leq 1$ and is zero otherwise. Next we estimate:

$$\sup_r \int_r^1 \frac{r^{|n|}}{\rho^{|n|+1}} d\rho = \sup_r \frac{1}{|n|}(1 - r^{|n|}) \leq \frac{1}{|n|}.$$

Similarly one has:

$$\sup_\rho \int_0^\rho \frac{r^{|n|-1}}{\rho^{|n|}} dr = \sup_\rho \frac{1}{|n|} \cdot \frac{1}{\rho^{|n|}} \cdot \rho^{|n|} = \frac{1}{|n|}.$$

Thus one has by the Schur-Young inequality that $\|Q^{(n)}\| \leq \frac{1}{|n|}$. A similar computation for positive n gives $\|Q^{(n)}\| \leq \frac{1}{|n|}$ for all $n \neq 0$. Hence one has that $\tilde{Q}$ is bounded. ■

Since in this chapter going beyond the analysis on the semi-infinite cylinder is not attempted, the $n = 0$ term will simply be ignored. In [2] this was not an issue as $Q^{(0)}$ is continuous when mapping into an appropriate local Sobolev space.

4.2 Dirac operators on the quantum punctured disk

In this section the non-commutative punctured disk and the quantum analog of the Dirac operator of the previous section is constructed. In particular, a non-commutative Fourier decomposition of that operator is discussed. It's also worth mentioning that a version of a quantum punctured disk was previously considered in [16].

One starts with defining several auxiliary objects needed for the current construction. Let $\ell^2(\mathbb{Z})$ be the Hilbert space of square summable bilateral sequences, and let $\{e_k\}_{k\in\mathbb{Z}}$ be its canonical basis. The following two operators are needed: let U be the shift operator given by:

$$Ue_k = e_{k+1}$$

and let K be the label operator defined by the following formula:

$$Ke_k = ke_k.$$

By the functional calculus, if $f : \mathbb{Z} \to \mathbb{C}$, then $f(K)$ is a diagonal operator and satisfies the relation $f(K)e_k = f(k)e_k$.

Next assume a sequence $\{w(k)\}_{k\in\mathbb{Z}}$ of real numbers are given with the following properties:

$$\begin{aligned} &1.\ w(k) < w(k+1) \\ &2.\ \lim_{k\to\infty} w(k) =: w_+ \text{ exists} \\ &3.\ \lim_{k\to-\infty} w(k) = 0 \\ &4.\ \sup_k \frac{w(k)}{w(k-1)} < \infty. \end{aligned} \tag{4.5}$$

In particular one has $w(k) > 0$.

The function $w : \mathbb{Z} \to \mathbb{C}$ gives a diagonal operator $w(K)$ as above. From this the weighted shift operator $U_w := Uw(K)$ is defined which plays the role of a noncommutative complex coordinate on the punctured disk.

Clearly:

$$\begin{aligned} U_w e_k &= w(k)e_{k+1} \\ U_w^* e_k &= w(k-1)e_{k-1}. \end{aligned}$$

Consider the commutator $S := [U_w^*, U_w]$, for which one has $Se_k = (w^2(k) - w^2(k-1))e_k$. If one lets $S(k) := w^2(k) - w^2(k-1)$, then one can write $S = S(K)$. Notice that S is a trace class operator and a simple computation gives $\mathrm{tr}(S) = w_+^2$.

The quantum punctured disk $C^*(U_w)$ is defined to be the C^*-algebra generated by U_w. General theory, see [9], gives the following short exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(U_w) \xrightarrow{\sigma} C(S^1) \longrightarrow 0$$

where $\mathcal{K}$ is the ideal of compact operators and σ is the noncommutative "restriction to the boundary" map.

Let $b \in C^*(U_w)$ and consider the densely defined weight on $C^*(U_w)$ by

$$\tau(b) = \mathrm{tr}\left(S\left(U_w^* U_w\right)^{-1} b\right)$$

(compare with 4.1). This weight will be used to define the Hilbert space $\mathcal{H}$ on which the Dirac operator will live. This is done by the GNS construction for the algebra $C^*(U_w)$ with respect to τ. In other words $\mathcal{H}$ is obtained as a Hilbert space completion

$$\mathcal{H} = \overline{(C^*(U_w), \langle \cdot, \cdot \rangle_\tau = \| \cdot \|_w^2)}$$

where $\|b\|_w^2 = \tau(bb^*)$.

Now it is time to define the operator that is studied in this chapter, the quantum analog of the operator of the previous section. Define D by the following formula:

$$Db = -S^{-1} U_w^* \left[b, U_w\right]. \tag{4.6}$$

Let, as before, $P_{\geq 0}$ be the orthogonal L^2 projection onto $\mathrm{span}\{e^{in\varphi}\}_{n \geq 0}$. The APS boundary conditions on D amount to the following choice of the domain:

$$\mathrm{dom}(D) = \left\{b \in \mathcal{H} \;:\; \|Db\|_w^2 < \infty, \quad P_{\geq 0}\sigma(b) = 0\right\}. \tag{4.7}$$

There are certain subtleties in this definition which are clarified in the statement of Proposition 4.2.2 at the end of this section.

The next proposition describes a (partial) Fourier series decomposition of the Hilbert space $\mathcal{H}$. Define

$$a(k) := \frac{w(k)^2}{S(k)},$$

and let

$$\ell_a^2(\mathbb{Z}) = \{\{g(k)\}_{k\in\mathbb{Z}} \ : \ \|g\|_a^2 = \sum_{k\in\mathbb{Z}} a(k)^{-1}|g(k)|^2 < \infty\}.$$

Now one is ready for the Fourier decomposition of $\mathcal{H}$ which is just like (4.4).

Proposition 4.2.1 *Let $\mathcal{H}$ be the Hilbert space defined above. Then the formula*

$$b = \sum_{n\in\mathbb{Z}} g_n(K)\,(U^*)^n$$

defines an isomorphism of Hilbert spaces

$$\bigoplus_{n\in\mathbb{Z}} \ell_a^2(\mathbb{Z}) \cong \mathcal{H}. \tag{4.8}$$

Proof The proof is identical to the one in [17]. In particular we have

$$\|b\|_w^2 = \sum_{n\in\mathbb{Z}} \operatorname{tr}\left(S(K)w^{-2}(K)|g_n(K)|^2\right).$$

■

The main reason for considering the Fourier decomposition is that it (again partially) diagonalizes the operator D. This is the subject of the next lemma. Before stating some more notation is needed. Consider the ratios:

$$c^{(n)}(k) := \frac{w(k+n)}{w(k)}$$

and notice that since $\{w(k)\}$ is an increasing sequence we have:

$$\begin{aligned} c^{(n)}(k) &= 1 \qquad \text{for } n = 0 \\ c^{(n)}(k) &> 1 \qquad \text{for } n > 0 \\ c^{(n)}(k) &< 1 \qquad \text{for } n < 0. \end{aligned}$$

The coefficients $c^{(n)}$ are needed to define the following operators in $\ell^2_a(\mathbb{Z})$. The first is:

$$\overline{A}^{(n)}g(k) = a(k)(g(k) - c^{(n)}(k)g(k+1))$$

with domain

$$\mathrm{dom}(\overline{A}^{(n)}) = \left\{ g \in \ell^2_a(\mathbb{Z}) \ : \ \|\overline{A}^{(n)}g\|_a < \infty \right\}.$$

Additionally consider the operator $\overline{A_0}^{(n)}$ which is the operator $\overline{A}^{(n)}$ but with domain

$$\mathrm{dom}(\overline{A_0}^{(n)}) = \{g \in \mathrm{dom}(\overline{A}^{(n)}) : g_\infty := \lim_{k\to\infty} g(k) = 0\}.$$

The last definition makes sense since by the analysis of [17] the limit $\lim_{k\to\infty} g(k)$ exists for $g \in \mathrm{dom}(\overline{A})$. One has the following proposition, which is a quantum analog of the first part of Theorem 4.1.1.

Proposition 4.2.2 *With respect to the decomposition (4.8) one has:*

$$D \cong \bigoplus_{n>0} \overline{A}^{(n)} \oplus \bigoplus_{n\leq 0} \overline{A_0}^{(n)}.$$

Equivalently:

$$Db = \sum_{n>0} \overline{A}^{(n)} g_n(K)(U^*)^n + \sum_{n\leq 0} \overline{A_0}^{(n)} g_n(K)(U^*)^n$$

where

$$b = \sum_{n\in\mathbb{Z}} g_n(K)\,(U^*)^n\,.$$

Proof The proof is a direct calculation identical to the one in [17]. ■

4.3 Construction of the parametrix

In this section we construct and analyze in detail the inverse (= a parametrix) Q for the operator D. The construction is fairly similar to the one done in section 4 in [17], however the norm estimates are quite different. Somewhat surprisingly the norm estimates below hold for any choice of sequence of weights $\{w(k)\}$ satisfying (4.5).

We start with a lemma containing estimates of sums through integrals. Recall that the sequence $\{w(k)\}$ is increasing with limits at $\pm\infty$ equal, correspondingly, to w^+ and 0.

Lemma 4.3.1 *If $f(t)$ is a decreasing continuous function on $(0,(w^+)^2)$ then*

$$\sum_{l<k} f(w(k)^2)S(k) = \sum_{l<k} f(w(k)^2)(w(k)^2 - w(k-1)^2) \leq \int_{w(l)^2}^{w_+^2} f(t)dt \tag{4.9}$$

$$\sum_{k\leq l} f(w(k)^2)S(k) \leq \int_0^{w(l)^2} f(t)dt \tag{4.10}$$

$$\sum_{k\in\mathbb{Z}} f(w(k-1)^2)S(k) \geq \int_0^{w_+^2} f(t)dt. \tag{4.11}$$

The proof of the statements of the lemma follows from a straightforward comparison of the Riemann sums of the left hand side with the integrals on the right hand side.

The presentation in this section is as follows. First we discuss the kernels of the $\overline{A}^{(n)}$ operators for the three cases $n = 0$, $n > 0$, $n < 0$. Secondly one constructs the parametrices for all three cases. Thirdly the norm estimates of the parametrices are discussed, and finally a summary the analysis in the main result of this chapter is made.

Below it is shown that the operator D has no kernel by analyzing the terms in the decomposition of Proposition 4.2.2.

Proposition 4.3.2 *The operators $\overline{A}^{(n)}$ for $n \geq 0$ and $\overline{A_0}^{(n)}$ for $n < 0$ have no kernel.*

Proof We start with $n = 0$. Here $c^{(n)}(k) = 1$ and it is clear that the only solution of that $\overline{A}^{(0)}R^{(0)} = 0$ is, up to a constant, $R^{(0)} = 1$. But one has

$$\|R^{(0)}\|_a^2 = \sum_{k\in\mathbb{Z}} \frac{1}{a(k)} = \sum_{k\in\mathbb{Z}} \frac{S(k)}{w(k)^2} \geq \text{const} \sum_{k\in\mathbb{Z}} \frac{S(k)}{w(k-1)^2} = \infty$$

where we used condition 4 of (4.5) as well as (4.11) for $f(t) = 1/t$. Therefore $R^{(0)}(K) \notin \ell_a^2(\mathbb{Z})$ and hence $\overline{A}^{(0)}$ has no kernel.

Next we discuss the kernel of $\overline{A}^{(n)}$ when $n > 0$. It is not too hard to see that any element of the kernel has to be proportional to

$$R^{(n)}(k) := \prod_{l=k}^{\infty} c^{(n)}(l) = \frac{(w_+)^n}{w(k)w(k+1)\cdots w(k+n-1)}.$$

The norm calculation gives

$$\|R^{(n)}\|_a^2 = \sum_{k\in\mathbb{Z}} \frac{1}{a(k)} |R^{(n)}(k)|^2 = \sum_{k\in\mathbb{Z}} \frac{1}{a(k)} \prod_{l=k}^{\infty} |c^{(n)}(l)|^2.$$

Since $|c^{(n)}(l)| > 1$ and $\sum_{k\in\mathbb{Z}} \frac{1}{a(k)} = \infty$, the sum above diverges and hence $R^{(n)}(K) \notin \ell_a^2(\mathbb{Z})$.

Finally we discuss the kernel of the operator $\overline{A_0}^{(n)}$ when $n < 0$. Yet again the kernel is formally one dimensional and spanned by

$$R^{(n)}(k) = \prod_{l=k}^{\infty} c^{(n)}(l) = \frac{w(k+n)w(k+n-1)\cdots w(k-1)}{(w_+)^{-n}}.$$

While one can easily show that $R^{(n)} \in \ell_a^2(\mathbb{Z})$, one however has $\lim_{k\to\infty} R^{(n)}(k) = 1 \neq 0$, so this means $R^{(n)} \notin \operatorname{dom}\left(\overline{A_0}^{(n)}\right)$. Thus the result follows. ■

The second portion of the discussion is the construction of the parametrices for all three cases. Since there are no kernels (and cokernels) involved we simply compute the inverses of operators $\overline{A}^{(n)}$. Thus, given $g(k)$, one needs to solve the equation $\overline{A}^{(n)} f(k) = g(k)$ where additionally $\lim_{k\to\infty} f(k) = 0$ for $n \leq 0$ is needed. This is done in a similar manner to the methods in [6], [17]. In the case when $n > 0$ one arrives at the following formula:

$$f(k) = -\sum_{l<k} \frac{R^{(n)}(k)}{R^{(n)}(l)a(l)} g(l) = -\sum_{l<k} \frac{w(l)\cdots w(l+n-1)}{w(k)\cdots w(k+n-1)} \cdot \frac{S(l)}{w(l)^2} g(l).$$

Similarly in the case $n \le 0$ one has:

$$f(k) = \sum_{k \le l} \frac{R^{(n)}(k)}{R^{(n)}(l)} \cdot \frac{g(l)}{a(l)} = \sum_{k \le l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)} \cdot \frac{S(l)}{w(l)^2} g(l).$$

The right hand sides of the above equation give the parametrices $Q^{(n)}$ for all three cases. Thus one has the following:

$$\begin{aligned} Q^{(n)}g(k) &= -\sum_{l<k} \frac{S(l)}{w(l)^2} g(l) && \text{for } n = 0 \\ Q^{(n)}g(k) &= -\sum_{l<k} \frac{w(l)\cdots w(l+n-1)}{w(k)\cdots w(k+n-1)} \cdot \frac{S(l)}{w(l)^2} g(l) && \text{for } n > 0 \\ Q^{(n)}g(k) &= \sum_{k \le l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)} \cdot \frac{S(l)}{w(l)^2} g(l) && \text{for } n < 0. \end{aligned} \tag{4.12}$$

A summary of the above analysis is made in the following proposition.

Proposition 4.3.3 *Let $Q^{(n)}$ be defined by the formulas above, then we have* $\overline{A}^{(n)} Q^{(n)} = I$, $Q^{(n)}\overline{A}^{(n)} = I$ *for* $n > 0$ *and* $\overline{A_0}^{(n)} Q^{(n)} = I$, $Q^{(n)}\overline{A_0}^{(n)} = I$ *for* $n \le 0$.

The next question in hand is the boundedness for the parametrices in the cases $n > 0$ and $n < 0$. The difficulty comes for $k \to \infty$: while the ratios of weights are always less than 1, the series $\sum_{k\in\mathbb{Z}} \frac{S(k)}{w(k)^2}$ is not summable and we cannot replicate the estimates of [6] and [17]. In fact the integral operators $Q^{(n)}$ are not Hilbert-Schmidt. The trick is to estimate most but not all weight ratios by one. The remaining sums, containing potentially divergent terms, are estimated by integrals using Lemma 4.3.1. We have the following result.

Proposition 4.3.4 *The operators $Q^{(n)}$ defined above are bounded operators in $\ell^2_a(\mathbb{Z})$ when $n \neq 0$.*

Proof First consider the case that $n > 0$. Applying the Schur-Young inequality and the inequalities (4.9), and (4.10) one has

$$
\begin{aligned}
&\left\|Q^{(n)}\right\|_a^2 \le \\
&\le \sup_k \left(\sum_{l<k} \frac{w(l)\cdots w(l+n-1)}{w(k)\cdots w(k+n-1)} \cdot \frac{S(l)}{w(l)^2} \right) \\
&\qquad \times \sup_l \left(\sum_{l<k} \frac{w(l)\cdots w(l+n-1)}{w(k)\cdots w(k+n-1)} \cdot \frac{S(k)}{w(k)^2} \right) \\
&\le \sup_k \left(\frac{1}{w(k)} \sum_{l<k} \frac{S(l)}{w(l)} \right) \sup_l \left(w(l) \sum_{l<k} \frac{S(k)}{w(k)^3} \right) \\
&\le \sup_k \left(\frac{1}{w(k-1)} \sum_{l \le k-1} \frac{S(l)}{w(l)} \right) \sup_l \left(w(l) \int_{w(l)^2}^{w_+^2} t^{-\frac{3}{2}} dt \right) \\
&\le \sup_k \left(\frac{1}{w(k-1)} \int_0^{w(k-1)^2} t^{-\frac{1}{2}} dt \right) \cdot 2 \sup_l \left(1 - \frac{w(l)}{w_+} \right) \le 2 \cdot 2 = 4.
\end{aligned}
$$

Thus $Q^{(n)}$ is bounded for $n > 0$. Next consider the case $n < 0$. Here one has quite similar estimates:

$$
\begin{aligned}
&\left\|Q^{(n)}\right\|_a^2 \le \\
&\le \sup_k \left(\sum_{k \le l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)} \cdot \frac{S(l)}{w(l)^2} \right) \\
&\qquad \times \sup_l \left(\sum_{k \le l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)} \cdot \frac{S(k)}{w(k)^2} \right) \\
&\le \sup_k \left(w(k-1) \sum_{k \le l} \frac{S(l)}{w(l)^2 w(l-1)} \right) \sup_l \left(\frac{1}{w(l-1)} \sum_{k \le l} \frac{S(k)}{w(k)} \right) \\
&\le \left(\sup_l \frac{w(l)}{w(l-1)} \right) \sup_k \left(w(k-1) \sum_{k-1<l} \frac{S(l)}{w(l)^3} \right) \sup_l \left(\frac{1}{w(l-1)} \sum_{k \le l} \frac{S(k)}{w(k)} \right) \\
&\le \left(\sup_l \frac{w(l)}{w(l-1)} \right) \sup_k \left(w(k-1) \int_{w(k-1)^2}^{w_+^2} t^{-\frac{3}{2}} dt \right) \sup_l \left(\frac{1}{w(l-1)} \int_0^{w(l)^2} t^{-\frac{1}{2}} dt \right) \\
&\le 4 \left(\sup_l \frac{w(l)}{w(l-1)} \right)^2 < \infty.
\end{aligned}
$$

Thus $Q^{(n)}$ is bounded for $n < 0$ and this completes the proof. ■

Finally one puts together the previous information about the parametrix Q of the Dirac operator D defined in section 3. The main result of this chapter is now stated.

Theorem 4.3.5 *Let D be the operator (4.6) with domain (4.7). Then there exists an operator Q such that $QD = DQ = I$. Moreover, with respect to the decomposition (4.8) one has*

$$Q = \bigoplus_{n\in\mathbb{Z}} Q^{(n)} = Q^{(0)} + \tilde{Q} \tag{4.13}$$

where the operators $Q^{(n)}$ are given by (4.12) and $\tilde{Q}$ is bounded.

Proof By Proposition 4.2.2 one can decompose D as $\bigoplus_{n>0} \overline{A}^{(n)} \oplus \bigoplus_{n\leq 0} \overline{A_0}^{(n)}$ which in turn gives the decomposition (4.13) of Q. One has that

$$\|\tilde{Q}\|_w = \sup_{n\neq 0} \left\|Q^{(n)}\right\|_a .$$

Then from Proposition 4.3.4, one has the following inequalities

$$\|Q\|_w^2 \leq 4\left(\sup_l \frac{w(l)}{w(l-1)}\right)^2 < \infty$$

where the last inequality follows from the assumptions in (4.5). To see that one has $DQ = QD = I$ we use the decompositions of Q and D and Proposition 4.3.3. This completes the proof. ■

4.4 The balanced quantum Dirac operators

In this section a version of the constructions of the previous sections that is more like the theory of [17] and chapter 3 is studied. The main objects: the Hilbert space and the Dirac operator are called balanced since in their definitions the left multiplication is not preferred over the right multiplication.

Since the results for the balanced Dirac operators are completely analogous to the "unbalanced" case and the proofs require only trivial modification, the main steps of the construction are only stated. The only significant difference between the two cases are the estimates on the components of the parametrix.

In order to avoid unnecessary complications old notation is recycled. As before the starting point is the choice of a sequence of weights $\{w(k)\}_{k\in\mathbb{Z}}$ satisfying (4.5). The Hilbert space $\mathcal{H}$ is the space of power series:

$$b=\sum_{n\in\mathbb{Z}} g_n(K)\,(U^*)^n$$

but this time with a different, balanced norm:

$$\begin{aligned}\|b\|_w^2 &= \mathrm{tr}\left(S^{1/2}w(K)^{-1}bb^*w(K)^{-1}S^{1/2}\right) = \\ &= \sum_{n\in\mathbb{Z}} \mathrm{tr}\left(\sqrt{S(K)S(K+n)}w^{-1}(K)w^{-1}(K+n)|g_n(K)|^2\right).\end{aligned}$$

The balanced Dirac operator is

$$Db=-S^{-1/2}U^*\,[b,U_w]\,w(K)S^{-1/2}$$

with the domain:

$$\mathrm{dom}(D)=\left\{b\in\mathcal{H}\ :\ \|Db\|_w^2<\infty,\quad P_{\geq 0}\sigma_{\mathrm{circ}}(b)=0\right\}.$$

As before the Dirac operator splits into Fourier components. To describe them the coefficients of the previous sections must be modified. Actually, the coefficients

$$c^{(n)}(k):=\frac{w(k+n)}{w(k)}$$

stay the same, but we need to change:

$$a^{(n)}(k):=\frac{w(k)w(k+n)}{\sqrt{S(k)S(k+n)}}.$$

Those are used for the following previously defined operators in $\ell_a^2(\mathbb{Z})$. The first operator is:

$$\overline{A}^{(n)}g(k)=a^{(n)}(k)(g(k)-c^{(n)}(k)g(k+1))$$

with domain

$$\mathrm{dom}(\overline{A}^{(n)})=\left\{g\in\ell_a^2(\mathbb{Z})\ :\ \|\overline{A}^{(n)}g\|_{\ell_a^2(\mathbb{Z})}<\infty\right\},$$

and the second operator $\overline{A_0}^{(n)}$ is the operator $\overline{A}^{(n)}$ but with domain

$$\mathrm{dom}(\overline{A_0}^{(n)}) = \{g \in \mathrm{dom}(\overline{A}^{(n)}) : g_\infty := \lim_{k\to\infty} g(k) = 0\}.$$

With that notation, the Proposition 4.2.2 remains true. In particular one has:

$$D \cong \bigoplus_{n>0} \overline{A}^{(n)} \oplus \bigoplus_{n\le 0} \overline{A_0}^{(n)}.$$

The problem of inverting the operator D is tackled as in the previous section. The components of the inverse are given by formulas like (4.12) with the only modification coming from the different $a^{(n)}$ coefficients. One ends up with the following expressions for the parametrices:

$$Q^{(n)}g(k) = -\sum_{l<k} \frac{\sqrt{S(l)S(l+n)}}{w(l)w(l+n)} g(l) \qquad \text{for } n = 0$$

$$Q^{(n)}g(k) = -\sum_{l<k} \frac{w(l)\cdots w(l+n-1)}{w(k)\cdots w(k+n-1)} \cdot \frac{\sqrt{S(l)S(l+n)}}{w(l)w(l+n)} g(l) \qquad \text{for } n > 0$$

$$Q^{(n)}g(k) = \sum_{k\le l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)} \cdot \frac{\sqrt{S(l)S(l+n)}}{w(l)w(l+n)} g(l) \qquad \text{for } n < 0.$$

One can verify directly that for the operator $Q = \bigoplus_{n\in\mathbb{Z}} Q^{(n)}$ then $QD = DQ = I$. The following is the main result of this section.

Proposition 4.4.1 *The operators $Q^{(n)}$ defined above are bounded operators in $\ell^2_a(\mathbb{Z})$ when $n \neq 0$.*

Proof The Schur-Young inequality is used and one follows the steps of the proof of the Proposition 4.3.4, with some modifications. Details for $n < 0$ are only shown, the other case is completely analogous.

There are two sums that we need to estimate. The first sum is:

$$\Sigma_1^n(k) := \sum_{k\le l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)} \cdot \frac{\sqrt{S(l)S(l+n)}}{w(l)w(l+n)}.$$

Using Cachy-Schwarz inequality we estimate:

$$\begin{aligned}
\Sigma_1^n(k) \leq& \\
\leq& \left(\sum_{k\leq l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)}\cdot\frac{S(l)}{w(l)^2}\right)^{1/2} \\
&\times\left(\sum_{k\leq l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)}\cdot\frac{S(l+n)}{w(l+n)^2}\right)^{1/2} \\
\leq& \left(w(k-1)\sum_{k\leq l}\frac{S(l)}{w(l)^2 w(l-1)}\right)^{1/2}\left(w(k+n)\sum_{k\leq l}\frac{S(l+n)}{w(l+n)^3}\right)^{1/2}.
\end{aligned}$$

Since the weights in the denominator are bigger than the corresponding weights in the numerator, their ratios were estimated by one. The first term on the rights hand side of the above was already estimated in the proof of Proposition 4.3.4. The second term is essentially the same as the first:

$$\sup_k \left(w(k+n)\sum_{k\leq l}\frac{S(l+n)}{w(l+n)^3}\right) = \sup_k\left(w(k)\sum_{k\leq l}\frac{S(l)}{w(l)^3}\right).$$

It follows that $\Sigma_1^n(k)$ is bounded uniformly in n.

The second sum in the Schur-Young inequality is

$$\Sigma_2^n(l) := \sum_{k\leq l}\frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)}\cdot\frac{\sqrt{S(k)S(k+n)}}{w(k)w(k+n)}$$

and it is bounded in the same fashion as the first sum:

$$\begin{aligned}
\Sigma_2^n(l) \leq& \\
\leq& \left(\sum_{k\leq l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)}\cdot\frac{S(k)}{w(k)^2}\right)^{1/2} \\
&\times\left(\sum_{k\leq l} \frac{w(k+n)\cdots w(k-1)}{w(l+n)\cdots w(l-1)}\cdot\frac{S(k+n)}{w(k+n)^2}\right)^{1/2} \\
\leq& \left(\frac{1}{w(l-1)}\sum_{k\leq l}\frac{S(k)}{w(k)}\right)^{1/2}\left(\frac{1}{w(l+n)}\sum_{k\leq l}\frac{S(k+n)}{w(k+n)}\right)^{1/2}.
\end{aligned}$$

Again the first term above was already estimated in the proof of Proposition 4.3.4, and the second term is essentially the same as the first. It follows that $\Sigma_1^n(k)$ is uniformly bounded. Repeating the same steps for $n > 0$ gives the boundedness of Q for the balanced Dirac operator. ■

5. CLASSICAL LIMIT OF THE D-BAR OPERATOR ON QUANTUM DOMAINS

5.1 Continuous Fields of Hilbert Spaces

In this section a review of some aspects of the theory of continuous fields of Hilbert spaces is made. The main reference here is Dixmier's book [11].

Definition: A *continuous field of Hilbert spaces* is a triple, denoted $(\Omega, \mathcal{H}, \Gamma)$, where Ω is a locally compact topological space, $\mathcal{H} = \{H(\omega) \ : \ \omega \in \Omega\}$ is a family of Hilbert spaces, and Γ is a linear subspace of $\prod_{\omega\in\Omega} H(\omega)$, such that the following conditions hold:

1. for every $\omega \in \Omega$, the set of $x(\omega)$, $x \in \Gamma$, is dense in $H(\omega)$,
2. for every $x \in \Gamma$, the function $\omega \mapsto \|x(\omega)\|$ is continuous,
3. let $x \in \prod_{\omega\in\Omega} H(\omega)$; if for every $\omega_0 \in \Omega$ and every $\varepsilon > 0$, there exists $x' \in \Gamma$ such that $\|x(\omega) - x'(\omega)\| \leq \varepsilon$ for every ω in some neighborhood (depending on ε) of ω_0, then $x \in \Gamma$.

The point of this definition is to describe a continuous arrangement of a family of different Hilbert spaces. If they are all the same, then the space Γ of continuous functions on Ω with values in that Hilbert space clearly satisfies all the conditions. Below the following terminology will be used.

We say that a section $x \in \prod_{\omega\in\Omega} H(\omega)$ is approximable by Γ at ω_0 if for every $\varepsilon > 0$, there exists an $x' \in \Gamma$ and a neighborhood of ω_0 such that $\|x(\omega) - x'(\omega)\| \leq \varepsilon$ for every ω in that neighborhood. In this terminology condition 3 of the above definition says that if a section is approximable by Γ at every $\omega \in \Omega$, then $x \in \Gamma$.

The above definition is a little cumbersome to work with, namely, trying to describe Γ in full detail is usually very difficult since the third condition isn't easy to verify. The following proposition, proved in [11], makes it easier to construct continuous fields.

Proposition 5.1.1 *Let Ω be a locally compact topological space, and let $\mathcal{H} = \{H(\omega) : \omega \in \Omega\}$ be a family of Hilbert spaces. If Λ is a linear subspace of $\prod_{\omega\in\Omega} H(\omega)$ such that*

1. *for every $\omega \in \Omega$, the set of $x(\omega)$, $x \in \Lambda$, is dense in $H(\omega)$,*

2. *for every $x \in \Lambda$, the function $\omega \mapsto \|x(\omega)\|$ is continuous,*

then Λ extends uniquely to $\Gamma \subset \prod_{\omega\in\Omega} H(w)$ such that $(\Omega, \mathcal{H}, \Gamma)$ is a continuous field of Hilbert spaces.

Here one says that if a linear subspace Λ of $\prod_{\omega\in\Omega} H(\omega)$ satisfies the two conditions above then Λ generates the continuous field of Hilbert spaces $(\Omega, \mathcal{H}, \Gamma)$. In fact, Γ is simply constructed as a local completion of Λ, i.e. Γ consists of all those sections $x \in \prod_{\omega\in\Omega} H(\omega)$ which are approximable by Λ at every $\omega \in \Omega$.

Next one considers morphisms of continuous fields of Hilbert spaces. For this one has the following definition.

Definition: Let $(\Omega, \mathcal{H}, \Gamma)$ be a continuous field of Hilbert spaces and let $T(\omega) : H(\omega) \to H(\omega)$ be a collection of operators acting on the Hilbert spaces $H(\omega)$. Define $T = \prod_{\omega\in\Omega} T(\omega) : \prod_{\omega\in\Omega} H(\omega) \to \prod_{\omega\in\Omega} H(\omega)$. We say that $\{T(\omega)\}$ is a continuous family of bounded operators in $(\Omega, \mathcal{H}, \Gamma)$ if

1. $T(\omega)$ is bounded for each ω,

2. $\sup\limits_{\omega\in\Omega} \|T(\omega)\| < \infty$,

3. T maps Γ into Γ.

The proposition below contains an alternative description of the third condition above, so it is more manageable.

Proposition 5.1.2 *With the notation of the above definition, the following three conditions are equivalent:*

1. *T maps Γ into Γ,*

2. *T maps Λ into Γ,*

3. *for every $x \in \Lambda$ and for every $\omega \in \Omega$, $T(\omega)x(\omega)$ is approximable by Λ at ω.*

Proof The items above are arranged from stronger to weaker. The proof that condition (2) is equivalent to condition (3) is a simple consequence of the way that Γ is obtained from Λ described in the paragraph following Proposition 5.1.1. Condition (2) implies condition (1) because $\sup_{\omega\in\Omega} \|T(\omega)\| < \infty$ and so, if $x(\omega)$ and $y(\omega)$ are locally close to each other, so are $T(\omega)x(\omega)$ and $T(\omega)y(\omega)$. ■

5.2 D-bar operators on non-commutative domains

In this section another review of a variety of constructions needed to formulate and prove the results of this chapter is made. Those constructions include the definitions of the quantum disk, the quantum annulus, Hilbert spaces of L^2 "functions" on those quantum spaces, and d-bar operators that were discussed in [17] and chapter 3. Other items discussed in this section are APS boundary conditions, inverses of d-bar operators subject to APS conditions, conditions on weights, and a construction of the generating subspace Λ of the continuous field of Hilbert spaces. The main results are stated at the end of this section.

In the following formulas one lets $\mathbb{S}$ be either $\mathbb{N}$ or $\mathbb{Z}$. Let $t \in (0, 1)$ be a parameter. Let $\{e_k\}$, $k \in \mathbb{S}$ be the canonical basis for $\ell^2(\mathbb{S})$. Given a t-dependent, bounded sequence of numbers $\{w_t(k)\}$, called weights, the weighted shift U_{w_t} is an operator in $\ell^2(\mathbb{S})$ defined by: $U_{w_t}e_k = w_t(k)e_{k+1}$. The usual shift operator U satisfies $Ue_k = e_{k+1}$.

If $\mathbb{S} = \mathbb{N}$ then the shift U_{w_t} is called a unilateral shift and it will be used to define a quantum disk. If $\mathbb{S} = \mathbb{Z}$ then the shift U_{w_t} is called a bilateral shift and it will used it to define a quantum annulus (also called a quantum cylinder). Chapter 3 contained similar quantum analogs however there, they were t-independent. For the choice of weights (2.1) the shifts U_{w_t} are the quantum complex coordinates z_t described in the summary of this chapter.

The following condition on the one-parameter family of weights $w_t(k)$ is required.

Condition 1. The weights $w_t(k)$ form a positive, bounded, strictly increasing sequence in k such that the limits $w_\pm := \lim_{k\to\pm\infty} w_t(k)$ exist, are positive, and independent of t.

Consider the following commutator $S_t = U_{w_t}^* U_{w_t} - U_{w_t} U_{w_t}^*$. It is a diagonal operator $S_t e_k = S_t(k)\, e_k$, where

$$S_t(k) := w_t(k)^2 - w_t(k-1)^2.$$

Moreover S_t is a trace class operator with easily computable trace:

$$\mathrm{tr}(S_t) = \sum_{k\in\mathbb{S}} S_t(k) = (w_+)^2 - (w_-)^2 \tag{5.1}$$

in the bilateral case, and $\mathrm{tr}(S_t) = (w_+)^2$ in the unilateral case. Additionally S_t is invertible with unbounded inverse.

More conditions on the $w_t(k)$'s and the $S_t(k)$'s are assumed. Those conditions were simply extracted from the proofs in the next section to make the estimates work. They are possibly not optimal, but they cover the motivating example described in the summary of this chapter, see chapter 2.

Condition 2. The function $t \mapsto w_t(k)$ is continuous for every k, and for every $\varepsilon > 0$, $w_t(k)$ converges to $w_\pm$ as $k \to \pm\infty$ uniformly on the interval $t \geq \varepsilon$.

Condition 3. If $h_1(t) := \sup_{k\in\mathbb{S}} S_t(k)$ then $h_1(t) \to 0$ as $t \to 0^+$.

Condition 4. The supremum $h_2(t) := \sup\limits_{k\in\mathbb{S}} \left|1 - \frac{S_t(k+1)}{S_t(k)}\right|$ exists, and is a bounded function of t, and $h_2(t) \to 0$ as $t \to 0^+$.

Condition 5. The supremum $h_3(k) := \sup\limits_{t\in[0,1)} \left|1 - \frac{w_t(k-1)}{w_t(k)}\right|$ exists for every k, and $h_3(k) \to 0$ as $k \to \pm\infty$.

Notice that the last condition implies that

$$w_t(k) \leq \text{const}\, w_t(k-1) \tag{5.2}$$

where the const above does not depend on t and k. This observation will be used in the proofs in the next section.

Before moving on, one verifies that the weight sequence (2.1) in the example in the summary of this chapter satisfies all of the conditions. First we compute:

$$S_t(k) = \frac{t}{(1+kt)(1+(k+1)t)}.$$

Conditions 1 and 2 are all easily seen to be true with $w_+ = 1$. For *conditions* 3, 4, and 5 simple computations give $h_1(t) = t/(1+t)$, $h_2(t) = 2t/(1+2t) = O(t)$, and $h_3(k) = (k+1+\sqrt{k^2+k})^{-1} = O(1/k)$, and so, by inspection, these weights meet all the required conditions. Examples of bilateral shifts satisfying the above conditions are:

$$w_t^2(k) = \alpha + \beta\frac{tk}{1+t|k|}.$$

For this example $h_1(t) = \beta t/(1+t)$, $h_2(t) = O(t)$, $h_3(k) = O(1/k)$, $w_+^2 = \alpha+\beta$, and $w_-^2 = \alpha - \beta$. Another similar example is $w_t^2(k) = \alpha + \beta\tan^{-1}(tk)$.

Next we proceed to the definition of the continuous field of Hilbert spaces over the interval $I = [0,1)$. Let $C^*(U_{w_t})$ be the C^*-algebra generated by U_{w_t}. Then, in the unilateral case, the algebra $C^*(U_{w_t})$ is called the non-commutative disk. There is a canonical map:

$$C^*(U_{w_t}) \xrightarrow{r} C(S^1)$$

called the restriction to the boundary map.

In the bilateral case the algebra $C^*(U_{w_t})$ is called the non-commutative cylinder, and one also has restriction to the boundary maps:

$$C^*(U_{w_t}) \stackrel{r=r_+\oplus r_-}{\longrightarrow} C(S^1) \oplus C(S^1).$$

Even though a similar definiton for the quantum disk and annulus was made in chapter 3, the repetition of this definition is made to distinguish the dependence of the parameter t in this chapter for the use of the theory of continuous fields of Hilbert spaces.

One then defines the Hilbert space $\mathcal{H}_t$, for $t > 0$, to be the completion of $C^*(U_{w_t})$ with respect to the inner product given by:

$$\|a\|_t^2 = \mathrm{tr}\left(S_t^{1/2} a S_t^{1/2} a^*\right).$$

For $t = 0$ one sets $\mathcal{H}_0 = L^2(\mathbb{D}_{w_+})$ in the unilateral/disk case and $\mathcal{H}_0 = L^2(\mathbb{A}_{w_-,w_+})$ in the bilateral/annulus case where $\mathbb{D}_{w_+} := \{z \in \mathbb{C} : |z| \le w_+\}$ is the disk of radius w_+, and $\mathbb{A}_{w_-,w_+} := \{z \in \mathbb{C} : w_- \le |z| \le w_+\}$ is the annulus with inner radius w_- and outer radius w_+. In what follows the norm subscript will usually be skipped as it will be clear from other terms subscript which Hilbert space norm or operator norm is used. Also notice that setting $w_- = 0$ reduces most annulus formulas below to the disk case.

It was proved in chapter 3 Lemma 5.1, see also [6], that if $a \in \mathcal{H}_t$ then it can be written as

$$a = \sum_{n\ge 0} U^n \alpha_n(K) + \sum_{n \ge 1} \beta_n(K)(U^*)^n,$$

where $\alpha_n(K)$ and $\beta_N(K)$ are diagonal operators in $\ell^2(\mathbb{S})$ given by

$$\alpha_n(K)e_k = \alpha_n(k)e_k,$$

and similar for $\beta_n(K)$, for some sequences $\{\alpha_n(k)\}$ and $\{\beta_n(k)\}$. Additionally one has the following formula for the norm:

$$\begin{aligned} \|a\|^2 = &\sum_{n=0}^{\infty}\sum_{k\in\mathbb{S}} S_t(k+n)^{1/2}S_t(k)^{1/2}\left|\alpha_n(k)\right|^2 \\ &+\sum_{n=1}^{\infty}\sum_{k\in\mathbb{S}} S_t(k+n)^{1/2}S_t(k)^{1/2}\left|\beta_n(k)\right|^2 < \infty. \end{aligned} \tag{5.3}$$

Consequently we will identify elements of $\mathcal{H}_t$ with sequences $\{x(k)\}_{k\in\mathbb{S}}$, where

$$x(k) = \sum_{n\geq 0} U^n\alpha_n(k) + \sum_{n\geq 1}\beta_n(k)(U^*)^n$$

such that equation (5.3) is true.

We now proceed to the construction of a continuous field of Hilbert spaces. By the remark after Proposition 5.1.1 we need to specify a generating linear space $\Lambda \subset \prod_{t\in I}\mathcal{H}_t$. We define it to consists of all those $x = \{x_t \;:\; t\in I\}$ such that there exists $N > 0$, (depending on x), and such that for every $n \leq N$ there are functions $f_n, g_n \in C([(w_-)^2, (w_+)^2])$, such that for $t > 0$:

$$x_t(k) = \sum_{n\leq N} U^n f_n\left(w_t(k)^2\right) + \sum_{n\leq N} g_n\left(w_t(k)^2\right)(U^*)^n\,, \tag{5.4}$$

and for $t = 0$:

$$x_0(r,\varphi) = \sum_{n\leq N} f_n(r^2)e^{in\varphi} + \sum_{n\leq N} g_n(r^2)e^{-in\varphi}. \tag{5.5}$$

This definition is motivated by the fact that formally $x_t \to x_0$ as $t \to 0$. Moreover by polar decomposition, $U_{w_t} = w_t(K)U$ and $U_{w_t}^* = U^*w_t(K)$; so they both belong to Λ. It follows by Lemma 5.3.7 that polynomials in U_{w_t} and $U_{w_t}^*$ belong to Γ in other words they are, according to our definition, continuous sections. Now we proceed to the definitions of the quantum d-bar operators. The operator D_t in $\mathcal{H}_t$ is given by the following expression:

$$D_t a = S_t^{-1/2}\left[a, U_{w_t}\right]S_t^{-1/2}$$

for $t > 0$, and for $t = 0$, $D_0 = \partial/\partial\overline{z}$. Of course one needs to specify the domain of D_t since it is an unbounded operator. For reasons indicated in the summary of this chapter, one considers the operators subject to the APS boundary conditions in this chapter. Let $P^{\pm}$ be the spectral projections in $L^2(S^1)$ of the boundary operators $\pm\frac{1}{i}\frac{\partial}{\partial\varphi}$ onto the interval $(-\infty, 0]$. The domain of D_t is then defined to be:

$$\mathrm{dom}(D_t) = \{a \in \mathcal{H}_t \ : \ \|D_t a\| < \infty,\ r(a) \in \mathrm{Ran}\ P^+\}$$

for the disk. For the annulus one sets:

$$\mathrm{dom}(D_t) = \{a \in \mathcal{H}_t \ : \ \|D_t a\| < \infty,\ r_+(a) \in \mathrm{Ran}\ P^+,\ r_-(a) \in \mathrm{Ran}\ P^-\}.$$

Here the maps r, $r_{\pm}$ are the restriction to the boundary maps, that by the results of [17] and chapter 3, continue to make sense for those $a \in \mathcal{H}_t$ for which $\|D_t a\| < \infty$.

If $t = 0$ the domain of D_0 consists of all those first Sobolev class functions f on the disk or the annulus for which the APS condition holds i.e. either $r(f) \in \mathrm{Ran}\ P^+$ or $r_+(f) \in \mathrm{Ran}\ P^+,\ r_-(f) \in \mathrm{Ran}\ P^-$, depending on the case. Here, by slight notational abuse, the symbols r, $r_{\pm}$ are the classical restriction to the boundary maps.

It was verified in chapter 3 that the above defined operators D_t are invertible, with bounded, and even compact inverses Q_t. Using chapter 3 one can immediately write down the formulas for Q_t. If $x \in \Lambda$ one has the following for $t > 0$:

$$\begin{aligned} &Q_t x_t(k) = \\ &-\sum_{n=0}^{N} U^n \left(\sum_{i\geq k} \frac{w_t(k+1)\cdots w_t(k+n)}{w_t(i+1)\cdots w_t(i+n)} \cdot \frac{S_t(i)^{1/2} S_t(i+n+1)^{1/2}}{w_t(k+n)} f_{n+1}(w_t(i)^2)\right) \\ &+\sum_{n=1}^{N} \left(\sum_{i\leq k} \frac{w_t(i)\cdots w_t(i+n-1)}{w_t(k)\cdots w_t(k+n-1)} \cdot \frac{S_t(i)^{1/2} S_t(i+n-1)^{1/2}}{w_t(i+n-1)} g_{n-1}(w_t(i)^2)\right) (U^*)^n . \end{aligned}$$

For the disk the second sum is from 0 to k, while for the annulus it is from $-\infty$ to k.

For $t = 0$ one has

$$D_0x_0 = \sum_{n=0}^{N} \frac{e^{i(n+1)\varphi}}{2}\left(2rf_n'(r^2) - \frac{n}{r}f_n(r^2)\right) + \sum_{n=1}^{N}\left(2rg_n'(r^2) + \frac{n}{r}g_n(r^2)\right)\frac{e^{-i(n-1)\varphi}}{2}.$$

for both the disk and annulus. From this one can compute the inverse Q_0 of D_0. A straightforward calculation gives the following result:

$$Q_0x_0 = -\sum_{n=0}^{N} e^{in\varphi}\int_{r^2}^{(w_+)^2} f_{n+1}(\rho^2)\frac{r^{n-1}}{\rho^n}d(\rho^2) + \sum_{n=1}^{N} e^{-in\varphi}\int_{(w_-)^2}^{r^2} g_{n-1}(\rho^2)\frac{\rho^{n-1}}{r^n}d(\rho^2),$$

for the annulus, and the same formula with w_- replaced by 0 for the disk.

The main results of this chapter are now ready to be stated. They are summarized in the following two theorems:

Theorem 5.2.1 *Given $I = [0,1)$, let $\mathcal{H} = \{\mathcal{H}_t \ : \ t \in I\}$ be the family of Hilbert spaces defined above and let Λ be the linear subspace of $\prod_{t\in I}\mathcal{H}_t$ defined by (5.4) and (5.5). Also let the conditions on $w_t(k)$ and $S_t(k)$ hold. Then Λ generates a continuous field of Hilbert spaces denoted below by $(I, \mathcal{H}, \Gamma)$.*

Theorem 5.2.2 *Let $Q_t : \mathcal{H}_t \to \mathcal{H}_t$ be the collection of operators for $t \in [0,1)$ defined above. Then $\{Q_t\}$ is a continuous family of bounded operators in the continuous field $(I, \mathcal{H}, \Gamma)$.*

One finishes this section by shortly indicating that the above results are also valid for families of d-bar operators studied in [6]. Let us quickly review the differences. The Hilbert space $\mathcal{H}_t$ studied in [6] is the completion of $C^*(U_{w_t})$ with respect to the following inner product:

$$\|a\|_t^2 = \mathrm{tr}(S_t a a^*).$$

The quantum d-bar operator D_t of [6], acting in $\mathcal{H}_t$, is given by the following formula:

$$D_t a = S_t^{-1}[a, U_{w_t}].$$

It turns out that Theorems 5.2.1 and 5.2.2 are also true for the above spaces and operators. In fact the proofs are even easier in this case and *Condition* 4, designed to handle expressions like $S_t(k+n)^{1/2}S_t(k)^{1/2}$ is not even needed.

The next section will contain all the analysis needed to prove the two theorems.

5.3 Continuity and the classical limit

The two theorems from the above section will be proven by a series of steps that verify the assumptions in the definitions of the continuous field of Hilbert spaces and the continuous family of bounded operators. The annulus case will be the one that is mainly concentrated on since the disk case is in some respects simpler. Most of the formulas for the annulus are true also in the disk case with a modification: replacing w_- by zero. The summation index in the annulus case extends to $-\infty$ and in couple of places the corresponding sums need to be estimated. This is not the issue in the disk case where the summation starts at zero. However the major difficulty in the disk case are the w_t terms in the denominator in the formula for the parametrix since they go to zero as t goes to zero. In the end the proofs that are described below work in both cases, but much shorter arguments are possible in the annulus case.

One first verifies that Λ generates a continuous field of Hilbert spaces. To this end one needs to check two things: the density in $\mathcal{H}_t$ of $x(t)$, $x \in \Lambda$, and the continuity of the norm. The density is immediate, since, for example, the canonical basis elements of $\mathcal{H}_t$, see the proof of Lemma 3.4.1 in chapter 3, come from Λ.

The verification of the continuity of the norm is done in two steps: continuity at $t = 0$, and at $t > 0$. If $x \in \Lambda$, i.e. x is given by formulas (5.4) and (5.5) then, for $t > 0$, the norm of x_t in $\mathcal{H}_t$ is

$$\begin{aligned} \|x_t\|^2 = &\sum_{n=0}^{N}\sum_{k\in\mathbb{S}} S_t(k+n)^{1/2}S_t(k)^{1/2}\left|f_n\left(w_t(k)^2\right)\right|^2 \\ &+ \sum_{n=1}^{N}\sum_{k\in\mathbb{S}} S_t(k+n)^{1/2}S_t(k)^{1/2}\left|g_n\left(w_t(k)^2\right)\right|^2, \end{aligned} \tag{5.6}$$

while for $t = 0$ the norm of x_0 is

$$\|x_0\|^2 = \sum_{n=0}^{N} \int_{(w_-)^2}^{(w_+)^2} \left|f_n\left(r^2\right)\right|^2 d\left(r^2\right) + \sum_{n=1}^{N} \int_{(w_-)^2}^{(w_+)^2} \left|g_n\left(r^2\right)\right|^2 d\left(r^2\right). \tag{5.7}$$

The next lemma is needed to handle the product of S terms with different arguments.

Lemma 5.3.1 *For $n \geq 1$ one has*

$$\sup_{k \in \mathbb{S}} \left| \frac{S_t(k+n)}{S_t(k)} - 1 \right| \leq (2 + h_2(t))^{n-1} h_2(t)$$

where $h_2(t)$ is the function defined in Condition 4.

Proof The proof is by induction. For $n = 1$ one gets *Condition* 4. The inductive step is

$$\begin{aligned} \left| \frac{S_t(k+n+1)}{S_t(k)} - 1 \right| &= \left| \frac{S_t(k+n+1)}{S_t(k+n)} \left(\frac{S_t(k+n)}{S_t(k)} - 1 \right) + \frac{S_t(k+n+1)}{S_t(k+n)} - 1 \right| \\ &\leq (1 + h_2(t))\, (2 + h_2(t))^{n-1} h_2(t) + h_2(t) \\ &\leq (2 + h_2(t))^n h_2(t) \end{aligned}$$

and the lemma is proved. ■

Now we are ready to discuss the continuity of norms (5.6) and (5.7) as $t \to 0^+$.

Proposition 5.3.2 *If x_t is in Λ then*

$$\lim_{t \to 0^+} \|x_t\| = \|x_0\|$$

Proof Without loss of generality one can assume that $x_t(k) = U^n f_n\left(w_t(k)^2\right)$ and $x_0(r, \varphi) = f_n(r^2) e^{in\varphi}$, as the proof is identical for the g terms, and the elements of Λ are finite sums of such x's. One has

$$\left|\|x_t\|^2 - \|x_0\|^2\right| = \left|\sum_{k\in\mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} \left|f_n(w_t(k)^2)\right|^2 - \int_{(w_-)^2}^{(w_+)^2} \left|f_n(r^2)\right|^2 d(r^2)\right|$$
$$\leq \left|\sum_{k\in\mathbb{S}} S_t(k) \left|f_n(w_t(k)^2)\right|^2 - \int_{(w_-)^2}^{(w_+)^2} \left|f_n(r^2)\right|^2 d(r^2)\right| +$$
$$+ \left|\sum_{k\in\mathbb{S}} \left(S_t(k+n)^{1/2} S_t(k)^{1/2} - S_t(k)\right) \left|f_n(w_t(k)^2)\right|^2\right|.$$

Since f_n is continuous and hence bounded, One can estimate:

$$\left|\|x_t\|^2 - \|x_0\|^2\right| \leq \left|\sum_{k\in\mathbb{S}} S_t(k) \left|f_n(w_t(k)^2)\right|^2 - \int_{(w_-)^2}^{(w_+)^2} \left|f_n(r^2)\right|^2 d(r^2)\right| +$$
$$+ \text{const} \left|\sum_{k\in\mathbb{S}} S_t(k) \left[\left(\frac{S_t(k+n)}{S_t(k)}\right)^{1/2} - 1\right]\right|.$$

Using $S_t(k) = w_t(k)^2 - w_t(k-1)^2$ and *Condition* 3, one sees that the first term inside of the absolute value is a difference of a Riemann sum and the integral to which it converges as $t \to 0^+$. Hence this term is zero in the limit. As for the second term, since by (5.1), $\sum_{k\in\mathbb{S}} S_t(k) = (w_+)^2 - (w_-)^2 = \text{const}$, Lemma 5.3.1 shows that it also goes to zero, because, by *Condition* 4, $h_2(t) \to 0$ as $t \to 0^+$. ■

The first theorem can now be proved.

Proof (of Theorem 5.2.1) One has already verified that Λ satisfies some of the properties of Proposition 5.1.1. What remains is the proof of the continuity of the norm for $t > 0$. Notice that by *Condition* 2 all the terms in formula (5.6) are continuous in t, $t > 0$. Thus one needs to show that the series (5.6) converges uniformly in t (away from $t = 0$). Assuming again that $x_t(k) = U^n f_n\left(w_t(k)^2\right)$, and using the boundedness of f_n one has:

$$\left|\|x_t\|^2 - \sum_{k=L+1}^{M-1} S_t(k+n)^{1/2} S_t(k)^{1/2} \left|f_n(w_t(k)^2)\right|^2\right| \leq$$
$$\leq \text{const} \sum_{k\geq M} S_t(k+n)^{1/2} S_t(k)^{1/2} + \text{const} \sum_{k\leq L} S_t(k+n)^{1/2} S_t(k)^{1/2}.$$

The Cauchy-Schwarz inequality is used to estimate the first term:

$$\begin{aligned}\sum_{k\geq M} S_t(k+n)^{1/2}S_t(k)^{1/2} &\leq \left(\sum_{k\geq M} S_t(k+n)\right)^{1/2}\left(\sum_{k\geq M} S_t(k)\right)^{1/2} \\ &\leq \sum_{k\geq M}^{\infty} S_t(k) = w_+^2 - w_t^2(M).\end{aligned} \tag{5.8}$$

The second term is only present in the annulus case and can be estimated in an analogous way.

By *Condition* 2 again, the difference $w_+^2 - w_t^2(M)$ is small for large M, uniformly in t on the intervals $t \geq \varepsilon > 0$, and so, for $t > 0$, $\|x_t\|$ is locally, the uniform limit of continuous functions and hence continuous. Therefore Λ generates a continuous field of Hilbert spaces $(I, \mathcal{H}, \Gamma)$. ■

The next concern is with the parametrices $Q_t(k)$. To verify that they form a continuous family of bounded operators in $(I, \mathcal{H}, \Gamma)$ one must check that they are uniformly bounded and that Q maps Γ into itself. One starts with the former assertion.

Proposition 5.3.3 *The norm of Q_t is uniformly bounded in t.*

Proof First one writes $Q_t x_t(k)$ in a more compact form:

$$Q_t x_t(k) = -\sum_{n=0}^{N} U^n T_t^{(1,n)} f_{n+1}(w_t(k)^2) + \sum_{n=1}^{N} T_t^{(2,n)} g_{n-1}(w_t(k)^2)\,(U^*)^n$$

where

$$\begin{aligned}T_t^{(1,n)} f(k) &= \sum_{i\geq k} \frac{w_t(k+1)\cdots w_t(k+n)}{w_t(i+1)\cdots w_t(i+n)} \cdot \frac{S_t(i)^{1/2}S_t(i+n+1)^{1/2}}{w_t(k+n)} f(i) \\ T_t^{(2,n)} g(k) &= \sum_{i\leq k} \frac{w_t(i)\cdots w_t(i+n-1)}{w_t(k)\cdots w_t(k+n-1)} \cdot \frac{S_t(i)^{1/2}S_t(i+n-1)^{1/2}}{w_t(i+n-1)} g(i).\end{aligned}$$

Here the operators $T_t^{(1,n)}$ and $T_t^{(2,n)}$ are integral operators between weighted l^2 spaces, namely: $T_t^{(1,n)} : l^2_{n+1} \mapsto l^2_n$ and $T_t^{(2,n)} : l^2_{n-1} \mapsto l^2_n$ where

$$l_n^2 := \{f : \sum_{k\in\mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} |f(k)|^2 < \infty\}.$$

See the weighted ℓ^2-space defined in Lemma 3.4.1 in chapter 3 for similarties. There the space was independent of the parameter t.

The main technique used to estimate the norms will be the Schur-Young inequality which was used in chapter 4, though it is stated again for convienence: if $T : L^2(Y) \longrightarrow L^2(X)$ is an integral operator $Tf(x) = \int K(x,y)f(y)dy$, then one has

$$\|T\|^2 \leq \left(\sup_{x\in X} \int_Y |K(x,y)|dy\right) \left(\sup_{y\in Y} \int_X |K(x,y)|dx\right).$$

The details can be found in [13].

The following two integral estimates will also be used, with t independent right hand sides:

$$\sum_{i<k} \frac{S_t(k)}{w_t(k)} \leq \int_{w_t(i)^2}^{(w_+)^2} \frac{dx}{\sqrt{x}} = 2(w_+ - w_t(i)) \leq 2(w_+ - w_-), \tag{5.9}$$

$$\sum_{k\leq i} \frac{S_t(k)}{w_t(k)} \leq \int_{(w_-)^2}^{w_t(i)^2} \frac{dx}{\sqrt{x}} = 2(w_t(i) - w_-) \leq 2(w_+ - w_-). \tag{5.10}$$

Such estimates were described and used in [18] and chapter 4 and are simply obtained by estimating the area under the graph of $x^{-1/2}$, like in the integral test for series.

First one estimates the norm of $T_t^{(1,n)}$. Repeatedly using the monotonicity of $w_t(i)$ and the Cauchy-Schwarz inequality, one has, like in chapter 4:

$$\begin{aligned}
\|T_t^{(1,n)}\|^2 &\leq \left(\sup_{k\in\mathbb{S}} \sum_{i\geq k} \frac{S_t(i)^{1/2} S_t(i+n+1)^{1/2}}{w_t(i+n)}\right) \left(\sup_{i\in\mathbb{S}} \sum_{k\leq i} \frac{S_t(k)^{1/2} S_t(k+n)^{1/2}}{w_t(i+n)}\right) \\
&\leq \left[\sup_{k\in\mathbb{S}} \left(\sum_{i\geq k} \frac{S_t(i)}{w_t(i)}\right) \left(\sum_{i\geq k} \frac{S_t(i+n+1)}{w_t(i+n)}\right)\right]^{1/2} \\
&\qquad \times \left[\sup_{i\in\mathbb{S}} \left(\sum_{k\leq i} \frac{S_t(k)}{w_t(k)}\right) \left(\sum_{k\leq i} \frac{S_t(k+n)}{w_t(k+n)}\right)\right]^{1/2}.
\end{aligned} \tag{5.11}$$

Using inequalities (5.2), (5.9) and (5.10) one sees that the norm of $T_t^{(1,n)}$ is bounded uniformly in n and t. The estimate on $T_t^{(2,n)}$ is essentially the same. Therefore one has

$$\|Q_t\| \leq \sup_{n\in\mathbb{N}} \|T_t^{(1,n)}\| + \sup_{n\in\mathbb{N}} \|T_t^{(2,n)}\| \leq \text{const}$$

and this completes the proof. ■

Next one needs to prove that Q maps Γ into itself. This requires checking condition (3) of Proposition 5.1.2. Thus one needs to show that, given $x \in \Lambda$, Qx is approximable by Λ at every $t \in I$. The hardest part is to show that this is true around $t = 0$, which will be done now.

Let $x \in \Lambda$ be given by formulas (5.4), (5.5), and define

$$\widetilde{g_n}(r^2) := \int_{(w_-)^2}^{r^2} g_{n-1}(\rho^2)\frac{\rho^{n-1}}{r^n} d(\rho^2),$$

and similarly

$$\widetilde{f_n}(r^2) := \int_{r^2}^{(w_+)^2} f_{n+1}(\rho^2)\frac{r^{n-1}}{\rho^n} d(\rho^2),$$

and set

$$y_t(k) := \sum_{n\leq N} U^n \widetilde{f_n}\left(w_t(k)^2\right) + \sum_{n\leq N} \widetilde{g_n}\left(w_t(k)^2\right)(U^*)^n,$$

and for $t = 0$:

$$y_0(r,\varphi) := \sum_{n\leq N} \widetilde{f_n}(r^2)e^{in\varphi} + \sum_{n\leq N} \widetilde{g_n}(r^2)e^{-in\varphi}.$$

Notice that one has $y \in \Lambda$ since clearly $\widetilde{f_n}(r^2)$, $\widetilde{g_n}(r^2)$ are in $C([(w_-)^2, (w_+)^2])$, and also one has obviously, $Q_0x_0 = y_0$ which was the motivating property of the above construction of y. It will be shown that $x \in \Lambda$ is approximable by $y \in \Lambda$ at $t = 0$. This is stronger than proving that x is approximable by Λ at $t = 0$.

Proposition 5.3.4 *With the above notation the following is true:*

$$\lim_{t\to 0^+} \|Q_t x_t - y_t\| = 0.$$

Proof The details for a single g_n term in the finite sum will be shown. The first thing to do is to obtain a pointwise estimate. Adding and subtracting one gets:

$$\left|\sum_{i\leq k}\frac{w_t(i)\cdots w_t(i+n-1)}{w_t(k)\cdots w_t(k+n-1)}\frac{S_t(i)^{1/2}S_t(i+n-1)^{1/2}}{w_t(i+n-1)}g_{n-1}\left(w_t(i)^2\right)-\widetilde{g_n}\left(w_t(k)^2\right)\right|$$
$$\leq\sum_{i\leq k}\left|\frac{w_t(i)\cdots w_t(i+n-2)}{w_t(k)\cdots w_t(k+n-1)}-\frac{w_t(i)^{n-1}}{w_t(k)^n}\right|S_t(i)\left|g_{n-1}\left(w_t(i)^2\right)\right|+$$
$$+\sum_{i\leq k}\frac{w_t(i)\cdots w_t(i+n-2)}{w_t(k)\cdots w_t(k+n-1)}\left|S_t(i)^{1/2}S_t(i+n-1)^{1/2}-S_t(i)\right|\left|g_{n-1}\left(w_t(i)^2\right)\right|+$$
$$+\left|\sum_{i\leq k}\frac{w_t(i)^{n-1}}{w_t(k)^n}g_{n-1}\left(w_t(i)^2\right)S_t(i)-\int_{(w_-)^2}^{w_t(k)^2}\frac{\rho^{n-1}}{w_t(k)^n}g_{n-1}(\rho^2)\,d(\rho^2)\right|:=I+II+III.$$

Let us discuss the structure of the above terms. The expression inside the absolute value in term I unfortunately in general does not go to zero as t goes to zero. To go around it one shows that the expression is small for large k which then lets us use the smallness of $S_t(i)$ to get the desired limit. This term is the trickiest to handle. Term II is the most straightforward to estimate along the lines of the proof of Proposition 5.3.2. Finally expression III is a difference between an integral and its Riemann sum, but because of the small denominator it has to be estimated carefully.

To handle term I one needs the following observation.

Lemma 5.3.5 *With the above notation one has:*

$$\left|1-\frac{w_t(k)^{n-1}}{w_t(k+1)\cdots w_t(k+n-1)}\right|\leq\sum_{j=0}^{n-1}jh_3(k+n-j)$$

where $h_3(k)$ is the sequence of Condition 5.

Proof To prove the statement one writes

$$\frac{w_t(k)^{n-1}}{w_t(k+1)\cdots w_t(k+n-1)}$$
$$=\frac{w_t(k)}{w_t(k+1)}\frac{w_t(k)w_t(k+1)}{w_t(k+1)w_t(k+2)}\frac{w_t(k)\cdots w_t(k+n-2)}{w_t(k+1)\cdots w_t(k+n-1)}$$

and uses an elementary inequality:

$$|1-x_1\cdots x_n|\leq|1-x_1|+\ldots+|1-x_n|$$

if $|x_k| \leq 1$. ■

One concentrates on the expression inside the absolute value in term I:

$$\begin{aligned}
J &:= \left| \frac{w_t(i)\cdots w_t(i+n-2)}{w_t(k)\cdots w_t(k+n-1)} - \frac{w_t(i)^{n-1}}{w_t(k)^n} \right| \\
&\leq \left| \frac{w_t(i)\cdots w_t(i+n-2)}{w_t(k)\cdots w_t(k+n-1)} - \frac{w_t(i)^{n-1}}{w_t(k)\cdots w_t(k+n-1)} \right| + \\
&+ \left| \frac{w_t(i)^{n-1}}{w_t(k)\cdots w_t(k+n-1)} - \frac{w_t(i)^{n-1}}{w_t(k)^n} \right|
\end{aligned}$$

Factoring one gets:

$$\begin{aligned}
J &\leq \frac{1}{w_t(k+n-1)} \left| 1 - \frac{w_t(i)^{n-2}}{w_t(i+1)\cdots w_t(i+n-2)} \right| + \\
&+ \frac{1}{w_t(k)} \left| 1 - \frac{w_t(k)^{n-1}}{w_t(k+1)\cdots w_t(k+n-1)} \right| .
\end{aligned}$$

Using lemma 5.3.5 yields:

$$\begin{aligned}
J &\leq \frac{1}{w_t(k+n-1)} \sum_{j=0}^{n-2} j h_3(i+n-1-j) + \frac{1}{w_t(i)} \sum_{j=0}^{n-1} j h_3(k+n-j) \\
&=: \frac{1}{w_t(k+n-1)} h_4(i) + \frac{1}{w_t(i)} h_5(k).
\end{aligned}$$

The functions $h_4(k)$ and $h_5(k)$ above are t independent and go to zero as $k \to \pm\infty$. Consequently:

$$\begin{aligned}
I(k) &\leq \text{const} \frac{1}{w_t(k+n-1)} \sum_{i \leq k} S_t(i) h_4(i) + \text{const}\, h_5(k) \sum_{i \leq k} \frac{S_t(i)}{w_t(i)} \\
&\leq \text{const} \frac{1}{w_t(k+n-1)} \sum_{i \leq k} S_t(i) h_4(i) + \text{const}\, h_5(k) =: I_1 + I_2.
\end{aligned}$$

To handle both the I_1 and I_2 term, one uses the following lemma. This is the tricky part of the argument.

Lemma 5.3.6 *If $h(k) \to 0$ as $k \to \pm\infty$ then*

$$\lim_{t \to 0^+} \sum_{k \in \mathbb{S}} S_t(k) h(k) = 0$$

Proof One splits the sum:

$$\sum_{k\in\mathbb{S}} S_t(k)h(k) = \sum_{|k|\leq N} S_t(k)h(k) + \sum_{|k|>N} S_t(k)h(k)$$
$$\leq \text{ const} \sum_{|k|\leq N} S_t(k) + \text{const} \sup_{|k|>N} h(k)$$

and first chooses N such that $\sup_{|k|>N} h(k) \leq \varepsilon/2$ and then chooses $\delta > 0$ such that $\sum_{|k|\leq N} S_t(k) \leq \varepsilon/2$ for all $t \leq \delta$. The last inequality is possible because of *Condition 3.* ■

As a corollary one also has:

$$\lim_{t\to 0^+} \sum_{k\in\mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} h(k) = 0, \tag{5.12}$$

obtained by estimating like in (5.8):

$$\sum_{k\in\mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} h(k) \leq$$
$$\leq \left(\sum_{k\in\mathbb{S}} S_t(k+n)\right)^{1/2} \left(\sum_{k\in\mathbb{S}} S_t(k)h(k)^2\right)^{1/2} \leq \text{const} \left(\sum_{k\in\mathbb{S}} S_t(k)h(k)^2\right)^{1/2}.$$

One now proceeds to show that I_1 and I_2 are small for small t. This is more straightforward with the I_2 term. Namely one has $\|I_2\|^2 \leq \text{ const} \sum_{k\in\mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} h_5^2(k)$ which by (5.12) goes to zero as t goes to zero.

To estimate I_1 notice first that

$$I_1(k) \leq \text{const} \sum_{i\leq k} \frac{S_t(i)}{w_t(i)} h_4(i) \leq \text{const} \sum_{i\leq k} \frac{S_t(i)}{w_t(i)} \leq \text{const}$$

by (5.10). Consequently one has:

$$\|I_1\|^2 = \sum_{k\in\mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} I_1^2(k) \leq \text{const} \sum_{k\in\mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} I_1(k)$$
$$\leq \text{const} \sum_{i,k\in\mathbb{S}} \frac{S_t(k+n)^{1/2} S_t(k)^{1/2}}{w_t(k+n)} S_t(i) h_4(i) \leq \text{const} \sum_{i\in\mathbb{S}} S_t(i) h_4(i).$$

The sum over k above is estimated as in (5.11), and one can use Lemma 5.3.6 again to conclude that $\|I_1\|^2$ goes to zero as t goes to zero.

Estimating the term II comes next. This is done analogously to the way the second term in Proposition 5.3.2 was treated. Using the boundedness of g_{n-1}, the definition of $h_2(t)$, and (5.10), one has:

$$\begin{aligned} II(k) &\leq \sum_{i \leq k} \frac{\left|S_t(i+n-1)^{1/2} S_t(i)^{1/2} - S_t(i)\right|}{w_t(i+n-1)} \left|g_{n-1}(w_t(i)^2)\right| \\ &\leq \text{const} \sum_{i \leq k} \frac{S_t(i)}{w_t(i)} \left| \left(\frac{S_t(i+n-1)}{S_t(i)} \right)^{1/2} - 1 \right| \\ &\leq \text{const}\, h_2(t). \end{aligned}$$

Consequently $\|II\|^2 \leq \text{const}\, h_2^2(t)$ which goes to zero by *Condition* 4.

Finally $III(k)$ is estimated. It is clear that this expression is small for small t and a fixed k, as a difference between an integral and its Riemann sum. However this is not enough in the disk case when $w_t(k)^n$ in the denominator is small for small t. To overcome this difficulty one first replaces g_{n-1} by its step function approximation and then deal directly with the remaining integral of ρ^{n-1}.

With this strategy in mind one estimates:

$$\begin{aligned} III(k) &= \left| \sum_{i \leq k} \frac{w_t(i)^{n-1}}{w_t(k)^n} g_{n-1}\left(w_t(i)^2\right) S_t(i) - \int_{(w_-)^2}^{w_t(k)^2} \frac{\rho^{n-1}}{w_t(k)^n} g_{n-1}(\rho^2)\, d(\rho^2) \right| \\ &\leq \left| \sum_{i \leq k} \left(\frac{w_t(i)^{n-1}}{w_t(k)^n} g_{n-1}\left(w_t(i)^2\right) S_t(i) - \int_{w_t(i-1)^2}^{w_t(i)^2} \frac{\rho^{n-1}}{w_t(k)^n} g_{n-1}\left(w_t(i)^2\right)\, d(\rho^2) \right) \right| + \\ &+ \left| \sum_{i \leq k} \int_{w_t(i-1)^2}^{w_t(i)^2} \frac{\rho^{n-1}}{w_t(k)^n} \left(g_{n-1}\left(w_t(i)^2\right) - g_{n-1}(\rho^2)\right)\, d(\rho^2) \right| =: III_1(k) + III_2(k). \end{aligned}$$

Since continuous functions on a closed interval are uniformly continuous, the function

$$h_5(t) := \sup_{i \in \mathbb{S}} \sup_{\rho^2 \in [(w_t(i-1)^2, (w_t(i))^2]} \left| g_{n-1}\left(w_t(i)^2\right) - g_{n-1}\left(\rho^2\right) \right|$$

goes to zero as $t \to 0^+$. Consequently, using the definition of $h_5(t)$, term III_2 can be estimated as follows:

$$III_2(k) \leq h_5(t) \int_{(w_-)^2}^{w_t(k)^2} \frac{\rho^{n-1}}{w_t(k)^n}\, d(\rho^2) \leq h_5(t) w_t(k) \int_0^1 u^{n-1}\, d(u^2) \leq \mathrm{const}\, h_5(t).$$

This means that $\|III_2\|$ goes to zero as $t \to 0^+$.

When estimating III_1 one first eliminates g_{n-1} using its boundedness:

$$\begin{aligned} III_1(k) &= \left| \sum_{i\leq k} \int_{w_t(i-1)^2}^{w_t(i)^2} \left(\frac{w_t(i)^{n-1}}{w_t(k)^n} g_{n-1}\left(w_t(i)^2\right) - \frac{\rho^{n-1}}{w_t(k)^n} g_{n-1}\left(w_t(i)^2\right) \right) d(\rho^2) \right| \\ &\leq \mathrm{const} \sum_{i\leq k} \int_{w_t(i-1)^2}^{w_t(i)^2} \left(\frac{w_t(i)^{n-1}}{w_t(k)^n} - \frac{\rho^{n-1}}{w_t(k)^n} \right) d(\rho^2). \end{aligned}$$

What is left is the difference between the integral of ρ^{n-1} and its upper sum which is handled like in the error estimate of the integral test for series. This is summarized in the following sequence of inequalities.

$$\begin{aligned} III_1(k) &\leq \mathrm{const} \sum_{i\leq k} \left(\frac{w_t(i)^{n-1}}{w_t(k)^n} - \frac{w_t(i-1)^{n-1}}{w_t(k)^n} \right) S_t(i) \\ &\leq \mathrm{const} \left(\sum_{i\leq k} \frac{w_t(i)^{n-1}}{w_t(k)^n} S_t(i) - \sum_{i\leq k-1} \frac{w_t(i)^{n-1}}{w_t(k)^n} S_t(i+1) \right) \\ &\leq \mathrm{const} \sum_{i\leq k-1} \frac{w_t(i)^{n-1}}{w_t(k)^n} S_t(i) \left(1 - \frac{S_t(i+1)}{S_t(i)} \right) + \mathrm{const}\, \frac{S_t(k)}{w_t(k)}. \end{aligned}$$

Notice that

$$\frac{S_t(k)^2}{w_t(k)^2} = S_t(k)\, \frac{w_t(k)^2 - w_t(k-1)^2}{w_t(k)^2} \leq S_t(k) \leq h_1(t).$$

Hence, using the monotonicity of $w_t(i)$ one has

$$III_1(k) \leq \mathrm{const}\, h_2(t) \sum_{i\leq k-1} \frac{w_t(i)^n}{w_t(k)^n} \frac{S_t(i)}{w_t(i)} + \mathrm{const} \sqrt{h_1(t)} \leq \mathrm{const} \left(h_2(t) + \sqrt{h_1(t)} \right),$$

and again $\|III_1\|$ goes to zero as $t \to 0^+$. The proof of the proposition is complete. ■

To proceed further one needs a better understanding of Γ, the space of continuous sections of our continuous field. One has the following useful result.

Lemma 5.3.7 *For $t > 0$ consider the following function*

$$x_t(k) = \sum_{n \le N} U^n F_n(t,k) + \sum_{n \le N} G_n(t,k)(U^*)^n$$

such that the functions $t \mapsto F_n(t,k)$ and $t \mapsto G_n(t,k)$ are continuous for every k, and such that $|F_n(t,k)|$ and $|G_n(t,k)|$ are bounded (in both variables), then x_t is approximable by Λ at every $t > 0$.

Proof Without a loss of generality one can assume that $x_t(k) = U^n F_n(t,k)$ as the proof is identical for the G terms, and it will extend to finite sums of such x's. Given $t_0 \in I$ and $\varepsilon > 0$, let $y \in \Lambda$ be such that for $t > 0$

$$y_t(k) := U^n f_n(w_t(k)^2),$$

where one chooses $f_n \in C([(w_-)^2, (w_+)^2])$ such that $\|F_n(t_0, \cdot) - f_n\left(w_{t_0}(\cdot)^2\right)\| \le \varepsilon/2$. This is always possible since the space of sequences of the form $k \to f_n(w_{t_0}(k)^2)$, where $f_n \in C([(w_-)^2, (w_+)^2])$, is a dense subspace in the Hilbert space l^2_n.

The goal is to show that

$$\|x_t - y_t\| \le \varepsilon$$

for all t sufficiently close to t_0. By the construction of f_n this is true at $t = t_0$. It will proven that $t \to \|x_t - y_t\|$ is continuous for $t > 0$ which will imply the above inequality. But the inequality means that x is approximable by Λ at $t = t_0$, which is exactly what we want to achieve.

The proof that $t \to \|x_t - y_t\|$ is continuous is analogous to the last part of the proof of Theorem 5.2.1, that the norm is continuous for elements of Λ and $t > 0$. Indeed, by the continuity assumptions, $\|x_t - y_t\|^2$ is an infinite sum of continuous functions:

$$\|x_t - y_t\|^2 = \sum_{k \in \mathbb{S}} S_t(k+n)^{1/2} S_t(k)^{1/2} \left|F_n(t,k) - f_n(w_t(k)^2)\right|^2.$$

The series converges uniformly around t_0 because, by the boundedness assumptions, one can estimate the remainder as follows:

$$\sum_{k\geq M} S_t(k+n)^{1/2} S_t(k)^{1/2} \left|F_n(t,k) - f_n(w_t(k)^2)\right|^2 \leq \text{const} \sum_{k\geq M} S_t(k+n)^{1/2} S_t(k)^{1/2}.$$

For large M this is small by (5.8). In the annulus case there is also a remainder at $-\infty$ which also goes to zero by an analogous estimate. As a consequence $t \to \|x_t - y_t\|$ is indeed continuous for $t > 0$ and the lemma is proved. ■

All the tools to finish the proof the second theorem are now available.

Proof (of Theorem 5.2.2) What remains is to show that $Q_t x_t$ is approximable by Λ for $t > 0$ since Propositions 5.3.3 and 5.3.4 establish the other properties of $\{Q_t\}$ needed to conclude that they form a continuous family of bounded operators in $(I, \mathcal{H}, \Gamma)$.

To prove that $Q_t x_t$ is approximable by Λ for $t > 0$ one uses Lemma 5.3.7 with

$$\begin{aligned}
F_n(t,k) &= \sum_{i\geq k} \mathcal{F}_n(t,i) \\
&:= \sum_{i\geq k} \frac{w_t(k+1)\cdots w_t(k+n)}{w_t(i+1)\cdots w_t(i+n)} \cdot \frac{S_t(i)^{1/2} S_t(i+n+1)^{1/2}}{w_t(k+n)} f_{n+1}(i) \\
G_n(t,k) &= \sum_{i\leq k} \mathcal{G}_n(t,i) \\
&:= \sum_{i\leq k} \frac{w_t(i)\cdots w_t(i+n-1)}{w_t(k)\cdots w_t(k+n-1)} \cdot \frac{S_t(i)^{1/2} S_t(i+n-1)^{1/2}}{w_t(i+n-1)} g_{n-1}(i).
\end{aligned}$$

Thus one needs to show that $F_n(t,k)$ and $G_n(t,k)$ are continuous and bounded functions of t, for $t > 0$. This will be done for the $F_n(t,k)$ term only as the argument is analogous for the $G_n(t,k)$ term. In fact, in the disk case the $G_n(t,k)$ is only a finite sum, so the continuity for $t > 0$ follows immediately from *Condition* 2.

Each $\mathcal{F}_n(t,i)$ is continuous on the intervals $t \geq \varepsilon > 0$ by *Condition* 2, so one must show that for each k, the series defining $F_n(t,k)$ converges uniformly in t. To estimate the tail end of the series one uses 5.2, 5.8, and the boundedness of $f_{n+1}(i)$ to get

$$\left|\sum_{i=M}^{\infty} \frac{w_t(k+1)\cdots w_t(k+n)}{w_t(i+1)\cdots w_t(i+n)} \cdot \frac{S_t(i)^{1/2} S_t(i+n+1)^{1/2}}{w_t(k+n)} f_{n+1}(i)\right| \leq$$

$$\leq \frac{\text{const}}{w_t(k+n)} \sum_{i=M}^{\infty} S_t(i)^{1/2} S_t(i+n+1)^{1/2} \leq \frac{\text{const}}{w_t(k+n)} \left(w_+^2 - w_t^2(M)\right),$$

which goes to zero uniformly on the intervals $t \geq \varepsilon > 0$ as M goes to infinity by *Condition* 2. Hence $F_n(t,k)$ is a uniform limit of continuous functions and consequently it is continuous for $t > 0$ and for each k.

Next the goal is to show that $F_n(t,k)$ and $G_n(t,k)$ are bounded. Indeed one has:

$$|F_n(t,k)| \leq \text{const} \sum_{i \geq k} \frac{S_t(i)^{1/2} S_t(i+n+1)^{1/2}}{w_t(i+n)} \leq$$

$$\leq \text{const} \left(\sum_{i \geq k} \frac{S_t(i)}{w_t(i)}\right)^{1/2} \left(\sum_{i \geq k} \frac{S_t(i+n+1)}{w_t(i+n)}\right)^{1/2} \leq \text{const},$$

where the inequalities (5.2), (5.9) and (5.10) are used. Similar argument works also for estimating $|G_n(t,k)|$. Thus the assumptions of Lemma 5.3.7 are satisfied and $Q_t x_t$ is approximable by Λ at every t. Hence the collection $\{Q_t\}$ is a continuous family of bounded operators. This finishes the proof. ∎

6. DIRAC TYPE OPERATORS ON THE SOLID TORUS WITH GLOBAL BOUNDARY CONDITIONS

6.1 The Dirac Operator

As always this chapter begins with the necessary notation. Let

$$\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| \leq 1\}$$

be the unit disk and let $S^1 = \left\{e^{i\theta} \in \mathbb{C} \ : \ 0 \leq \theta \leq 2\pi\right\}$ be the unit circle. Also let $\mathbb{T}^2$ be the two dimensional torus and let ST^2 be the solid torus: $ST^2 = \mathbb{D} \times S^1 \subset \mathbb{C} \times S^1$. The boundary of ST^2 is $\mathbb{T}^2$. The operators that we are studying will be acting in the Hilbert space $\mathcal{H} = L^2(ST^2, \mathbb{C}^2) \cong L^2(ST^2) \otimes \mathbb{C}^2$, i.e. the space of square-integrable complex vector-valued functions on the solid torus. The inner product of $F, G \in \mathcal{H}$ will be denoted as $\langle F, G\rangle$.

One proceeds to the definitions of the main object that is studied in this chapter. One considers the following formally self-adjoint Dirac operator D defined on $\mathcal{H}$ by

$$D = \begin{pmatrix} \frac{1}{i}\frac{\partial}{\partial\theta} & 2\frac{\partial}{\partial\bar{z}} \\ -2\frac{\partial}{\partial z} & -\frac{1}{i}\frac{\partial}{\partial\theta} \end{pmatrix}. \tag{6.1}$$

Notice that D can act on either $L^2(\mathbb{D} \times S^1) \otimes \mathbb{C}^2$ or $L^2(\mathbb{C} \times S^1) \otimes \mathbb{C}^2$.

The domain of D is defined to be:

$$\mathrm{dom}(D) = \left\{F \in H^1(ST^2) \otimes \mathbb{C}^2 \ : \ \exists F^{ext} \in H^1_{loc}\left((\mathbb{C} \times S^1) \setminus ST^2\right) \otimes \mathbb{C}^2\right\} \tag{6.2}$$

such that (1), (2), and (3) hold:

1. $F^{ext}|_{\mathbb{T}^2} = F|_{\mathbb{T}^2}$,

2. $DF^{ext} = 0$,

3. $F^{ext} \in L^2\left((\mathbb{C} \times S^1) \setminus ST^2\right) \otimes \mathbb{C}^2$.

Here H^1 is the first Sobolev space.

The first task is to study the kernel of D. For a function $F \in L^2(ST^2) \otimes \mathbb{C}^2$, by using the polar decomposition $z = re^{i\varphi}$, one has the Fourier decomposition as follows:

$$F = \sum_{m,n\in\mathbb{Z}} \begin{pmatrix} f_{m,n}(r) \\ g_{m,n}(r) \end{pmatrix} e^{in\varphi + im\theta}. \tag{6.3}$$

The norm of F can then be expressed as:

$$\|F\|^2 = \langle F, F\rangle = \sum_{m,n\in\mathbb{Z}} \int_0^1 \left(|f_{m,n}|^2 + |g_{m,n}|^2\right) r\, dr.$$

The next idea is to solve the equation $DF = 0$ without any kind of conditions imposed. This is done in the following proposition.

Proposition 6.1.1 *Let D be the operator defined by (6.1) and acting in the Hilbert space $L^2\left(ST^2 \setminus (\{0\} \times S^1)\right) \otimes \mathbb{C}^2$. Then the kernel of D consists of those $F \in L^2(ST^2 \setminus (\{0\} \times S^1)) \otimes \mathbb{C}^2$ for which the coefficients of (6.3) satisfy the following relations: for $m \neq 0$ and any n*

$$f_{m,n+1}(r) = \frac{m}{|m|}(-A_{m,n} I_{n+1}(|m|r) + B_{m,n} K_{n+1}(|m|r)) \tag{6.4}$$

and

$$g_{m,n}(r) = A_{m,n} I_n(|m|r) + B_{m,n} K_n(|m|r), \tag{6.5}$$

while if $m = 0$ and any n then $f_{0,n}(r) = A_{0,n} r^{-n}$ and $g_{0,n}(r) = B_{0,n} r^n$. Here $A_{m,n}, B_{m,n}$ are constants and I_n, K_n are the modified Bessel functions of the first and second kind respectively.

Proof To solve the equation $DF = 0$, it is enough to solve the equation

$$\sum_{m,n\in\mathbb{Z}} \begin{pmatrix} m & e^{i\varphi}\left(\frac{\partial}{\partial r} - \frac{n}{r}\right) \\ e^{-i\varphi}\left(-\frac{\partial}{\partial r} - \frac{n}{r}\right) & -m \end{pmatrix} \begin{pmatrix} f_{m,n}(r) \\ g_{m,n}(r) \end{pmatrix} e^{in\varphi + im\theta} = 0$$

by the Fourier decomposition (6.3). There are two cases to consider. First for $m \neq 0$ and any n, letting $t = |m|r$, $\tilde{f}_{m,n}(t) = f_{m,n}(t/|m|)$, and $\tilde{g}_{m,n}(t) = g_{m,n}(t/|m|)$ one arrives at the following system of differential equations:

$$\begin{cases} \tilde{f}'_{m,n+1}(t) + \dfrac{n+1}{t}\tilde{f}_{m,n+1}(t) + \dfrac{m}{|m|}\tilde{g}_{m,n}(t) = 0 \\ \tilde{g}'_{m,n}(t) - \dfrac{n}{t}\tilde{g}_{m,n}(t) + \dfrac{m}{|m|}\tilde{f}_{m,n+1}(t) = 0. \end{cases}$$

Substituting the second equation into the first yields:

$$\tilde{g}''_{m,n}(t) + \frac{\tilde{g}'_{m,n}(t)}{t} - \left(1 + \frac{n^2}{t^2}\right)\tilde{g}_{m,n}(t) = 0,$$

which is equation (6.21). This implies that $\tilde{g}_{m,n}(t)$ is a linear combination of the two modified Bessel functions, i.e. $\tilde{g}_{m,n}(t) = A_{m,n}I_n(t) + B_{m,n}K_n(t)$. Then, using the second equation from the above system and using the relations (6.22), one obtains the desired result.

If $m = 0$ then the above system of differential equations reduces to:

$$\begin{cases} f'_{0,n}(r) + \dfrac{n}{r}f_{0,n}(r) = 0 \\ g'_{0,n}(r) - \dfrac{n}{r}g_{0,n}(r) = 0. \end{cases}$$

This system is uncoupled and the solutions are easily seen to be $f_{0,n}(r) = A_{0,n}r^{-n}$ and $g_{0,n}(r) = B_{0,n}r^n$. Thus this completes the proof. ■

In the following proposition the domain condition of D is rephrased in terms of the Fourier coefficients to explicitly write down the boundary condition.

Proposition 6.1.2 *Suppose that F is in the domain of D and has Fourier decomposition given by (6.3). Then*

$$|m|K_{n+1}(|m|)g_{m,n}(1) - mK_n(|m|)f_{m,n+1}(1) = 0 \tag{6.6}$$

if $m \neq 0$ and $f_{0,n}(1) = 0$ for $n \leq 0$, and $g_{0,n}(1) = 0$ for $n \geq 0$.

Proof The goal is to find a function F^{ext} for $r \geq 1$ such that $DF^{ext} = 0$ and $F^{ext}|_{r=1} = F|_{r=1}$. First solving the system $DF^{ext} = 0$ for F^{ext} yields

$$F^{ext} = \sum_{m,n\in\mathbb{Z}} \begin{pmatrix} f^{ext}_{m,n}(r) \\ g^{ext}_{m,n}(r) \end{pmatrix} e^{in\varphi+im\theta}$$

with $f^{ext}_{m,n+1}(r)$ and $g^{ext}_{m,n}(r)$ given by (6.4) and (6.5) for $m \neq 0$. Additionally $f^{ext}_{0,n}(r) = A_{0,n}r^{-n}$ and $g^{ext}_{0,n}(r) = B_{0,n}r^n$. First consider $m \neq 0$. In order for the solutions to agree on the boundary of the disk, the coefficients $A_{m,n}$ and $B_{m,n}$ must solve the following system of equations

$$\begin{pmatrix} I_n(|m|) & K_n(|m|) \\ -\frac{m}{|m|}I_{n+1}(|m|) & \frac{m}{|m|}K_{n+1}(|m|) \end{pmatrix} \begin{pmatrix} A_{m,n} \\ B_{m,n} \end{pmatrix} = \begin{pmatrix} g_{m,n}(1) \\ f_{m,n+1}(1) \end{pmatrix}.$$

The solution is:

$$A_{m,n} = |m|K_{n+1}(|m|)g_{m,n}(1) - mK_n(|m|)f_{m,n+1}(1)$$

and

$$B_{m,n} = |m|I_{n+1}(|m|)g_{m,n}(1) + mI_n(|m|)f_{m,n+1}(1)$$

If $m = 0$ one gets $f_{0,n}(1) = A_{0,n}$ and $g_{0,n}(1) = B_{0,n}$.

Our boundary condition requires that F^{ext} is square integrable on the complement of ST^2 in $\mathbb{C} \times S^1$. Because of the asymptotic properties of the modified Bessel functions, see (6.26), this forces $A_{m,n} = 0$ for $m \neq 0$. If $m = 0$ the integrability of powers of r force $A_{0,n} = 0$ for $n \leq 0$, and $B_{0,n} = 0$ for $n \geq 0$. The statement follows from the above formulas for $A_{m,n}$, $A_{0,n}$, and $B_{0,n}$. ■

Combining the above two propositions we obtain the following corollary.

Corollary 6.1.3 *Let D be the operator defined by (6.1) subject to the boundary conditions (6.2). Then its kernel is trivial.*

Proof Let $F \in L^2(ST^2 \setminus (\{0\} \times S^1)) \otimes \mathbb{C}^2$ be a solution of $DF = 0$, as in Proposition 6.1.1. Its extension to a solution on $\mathbb{C} \times S^1 \setminus (\{0\} \times S^1)$ is clearly given by the same formula. Since the powers of r are either not square integrable at zero or at infinity, it is clear that $A_{0,n} = 0$ and $B_{0,n} = 0$.

If $m \neq 0$ then in order for F to be regular at zero we must have $B_{m,n} = 0$ by the asymptotic expansion of $K_n(t)$ near zero, see (6.25). Then Proposition 6.1.2 implies that $A_{m,n} = 0$, hence the kernel of D is trivial. ∎

It turns out that the boundary condition is self-adjoint as demonstrated in the next proposition.

Proposition 6.1.4 *The operator D defined by (6.1) subject to the boundary conditions (6.2) is self-adjoint.*

Proof It is clear that D is formally self-adjoint. From the standard elliptic theory [5] the domain of D and its adjoint consists of (vector-valued) functions of the first Sobolev class. Thus the only thing that one needs to check are the boundary conditions of the adjoint. To this end one inspects the boundary integral in Green's formula. Let F, G be H^1 functions on the solid torus. Using Proposition 2.2 from [17] and the Fourier decompositions:

$$F = \begin{pmatrix} f \\ g \end{pmatrix} = \sum_{m,n\in\mathbb{Z}} \begin{pmatrix} f_{m,n}(r) \\ g_{m,n}(r) \end{pmatrix} e^{in\varphi+im\theta}$$

$$G = \begin{pmatrix} p \\ q \end{pmatrix} = \sum_{m,n\in\mathbb{Z}} \begin{pmatrix} p_{m,n}(r) \\ q_{m,n}(r) \end{pmatrix} e^{in\varphi+im\theta},$$

one obtains:

$$\begin{aligned}\langle DG, F\rangle - \langle G, DF\rangle &= 2\left(\langle p, \partial g/\partial\overline{z}\rangle - \langle q, \partial f/\partial z\rangle - \langle \partial q/\partial\overline{z}, f\rangle + \langle \partial p/\partial z, g\rangle\right) \\ &= 2\sum_{m,n\in\mathbb{Z}} \left(\overline{p_{m,n+1}(1)}g_{m,n}(1) - \overline{q_{m,n}(1)}f_{m,n+1}(1)\right).\end{aligned}$$

Now suppose that F is in the domain of D, so it satisfies the conditions of Proposition 6.1.2. For G to be in the domain of the adjoint of D one needs the above expression to be equal to zero. This gives:

$$\sum_{m\in\mathbb{Z}\setminus\{0\},n\in\mathbb{Z}}\left(\frac{m}{|m|}\frac{K_n(|m|)}{K_{n+1}(|m|)}\overline{p_{m,n+1}(1)}-\overline{q_{m,n}(1)}\right)f_{m,n+1}(1)+$$
$$+\sum_{n<0}\overline{p_{0,n+1}(1)}g_{0,n}(1)-\sum_{n\geq 0}\overline{q_{0,n}(1)}f_{0,n+1}(1) \ = 0.$$

For $m\neq 0$ the above equation will equal zero for arbitrary F only if

$$|m|K_{n+1}(|m|)q_{m,n}(1)-mK_n(|m|)p_{m,n+1}(1)=0.$$

Additionally one must have $p_{0,n}(1)=0$ for $n\leq 0$, and $q_{0,n}(1)=0$ for $n\geq 0$. All together those requirements are exactly the same as the original boundary condition. Hence D and D^* have the same domain and the proof is complete. ■

The next goal is to construct the inverse of D. This is done by explicit solving of a non-homogeneous system of differential equations for the Fourier components and adjusting the integration constants to get the regularity at $r=0$ and so that the boundary condition is satisfied.

Proposition 6.1.5 *Let D be the operator defined by (6.1) subject to the boundary conditions (6.2). Then the operator Q given by the formula (6.9) below is the inverse to D, in other words $QD=DQ=I$.*

Proof To compute the inverse of D one solves the equation $DF=G$, which will reduce to solving a non-homogeneous second order ordinary differential equation. The idea is to use the Fourier decomposition (6.3). One first considers the case $m\neq 0$. Letting $t=|m|r$, $\tilde{f}_{m,n}(t)=f_{m,n}(t/|m|)$, and similarly for other functions, $DF=G$ becomes the system of differential equations:

$$\begin{cases}\tilde{g}'_{m,n}(t)-\dfrac{n}{t}\tilde{g}_{m,n}(t)+\dfrac{m}{|m|}\tilde{f}_{m,n+1}(t)=\dfrac{\tilde{p}_{m,n+1}(t)}{|m|}\\ -\tilde{f}'_{m,n+1}(t)-\dfrac{n+1}{t}\tilde{f}_{m,n+1}(t)-\dfrac{m}{|m|}\tilde{g}_{m,n}(t)=\dfrac{\tilde{q}_{m,n}(t)}{|m|}.\end{cases} \tag{6.7}$$

Next by substituting the first equation into the second one arrives at the following second order differential equation:

$$\begin{aligned}&\tilde{g}''_{m,n}(t) + \frac{1}{t}\tilde{g}'_{m,n}(t) - \left(1 + \frac{n^2}{t^2}\right)\tilde{g}_{m,n}(t) = \\ &= \frac{1}{m}\tilde{q}_{m,n}(t) + \frac{\tilde{p}'_{m,n+1}(t)}{|m|} + \frac{n+1}{t}\cdot\frac{\tilde{p}_{m,n+1}(t)}{|m|} =: h_{m,n}(t),\end{aligned} \tag{6.8}$$

where the right hand side of the above equation was denoted by $h_{m,n}(t)$. Notice that equation (6.8) is the non-homogeneous version of the Bessel differential equation (6.21). General theory of ordinary differential equations tells us that the general solution to equation (6.8) is

$$\tilde{g}_{m,n}(t) = c_1^{(m,n)}(t)I_n(t) + c_2^{(m,n)}(t)K_n(t)$$

where $c_1^{(m,n)}(t), c_2^{(m,n)}(t)$ solve the following system

$$\begin{pmatrix} I_n(t) & K_n(t) \\ I'_n(t) & K'_n(t)\end{pmatrix}\begin{pmatrix} c_1^{(m,n)}(t) \\ c_2^{(m,n)}(t)\end{pmatrix}' = \begin{pmatrix} 0 \\ h_{m,n}(t)\end{pmatrix}.$$

The solution of this system is

$$\begin{aligned} c_1^{(m,n)}(t) &= A_{m,n} + \int_{|m|}^{t} sK_n(s)h_{m,n}(s)ds \\ c_2^{(m,n)}(t) &= B_{m,n} - \int_0^t sI_n(s)h_{m,n}(s)ds,\end{aligned}$$

where $A_{m,n}$ and $B_{m,n}$ are constants. The boundary condition (6.6) and the regularity at $t = 0$ imply that we must have $c_1^{(m,n)}(t)$ equal to zero on the boundary, in other words where $t = |m|$. The boundary condition and regularity also imply that $c_2^{(m,n)}(t)$ goes to zero as $t \to 0$. After performing the integration by parts we obtain that $A_{m,n} = K_n(|m|)p_{m,n+1}(|m|)$ and $B_{m,n} = 0$, and

$$\begin{aligned} c_1^{(m,n)}(t) &= \frac{1}{m}\int_{|m|}^{t} sK_n(s)\tilde{q}_{m,n}(s)ds + \frac{1}{|m|}\int_{|m|}^{t} sK_{n+1}(s)\tilde{p}_{m,n+1}(s)ds \\ c_2^{(m,n)}(t) &= -\frac{1}{m}\int_0^t sI_n(s)\tilde{q}_{m,n}(s)ds + \frac{1}{|m|}\int_0^t sI_{n+1}(s)\tilde{p}_{m,n+1}(s)ds.\end{aligned}$$

Next one uses the first equation of (6.7) to solve for the general $\tilde{f}_{m,n+1}(t)$ term to get:

$$\frac{m}{|m|}\tilde{f}_{m,n+1}(t) = \frac{\tilde{p}_{m,n+1}(t)}{|m|} + \frac{n}{t}\tilde{g}_{m,n}(t) - \tilde{g}'_{m,n}(t).$$

A straightforward calculation using the relations between the derivatives and indices of the modified Bessel functions (6.22) and the formula for the Wronskian of the modified Bessel functions (6.24) yields:

$$\tilde{f}_{m,n+1}(t) = \frac{m}{|m|}\left(-c_1^{(m,n)}(t)I_{n+1}(t) + c_2^{(m,n)}(t)K_{n+1}(t)\right).$$

Next consider the case when $m = 0$. The system of differential equations reduces to the following

$$\begin{cases} g'_{0,n}(r) - \frac{n}{r}g_{0,n}(r) = p_{0,n+1}(r) \\ f'_{0,n+1}(r) + \frac{n+1}{r}f_{0,n+1}(r) = -q_{0,n}(r) \end{cases}$$

This system is an uncoupled system and can be solved using an integration factor in each equation. Therefore the formula for the parametrix to D is

$$QG := \sum_{m\in\mathbb{Z}\setminus\{0\},n\in\mathbb{Z}} \begin{pmatrix} f_{m,n}(r) \\ g_{m,n}(r) \end{pmatrix} e^{in\varphi+im\theta} + \sum_{n\in\mathbb{Z}} \begin{pmatrix} f_{0,n}(r) \\ g_{0,n}(r) \end{pmatrix} e^{in\varphi} \tag{6.9}$$

where for $m \neq 0$:

$$\begin{aligned} f_{m,n+1}(r) &= |m|I_{n+1}(|m|r)\int_r^1 K_n(|m|\rho)q_{m,n}(\rho)\rho d\rho \\ &+ mI_{n+1}(|m|r)\int_r^1 K_{n+1}(|m|\rho)p_{m,n+1}(\rho)\rho d\rho - |m|K_{n+1}(|m|r)\int_0^r I_n(|m|\rho)q_{m,n}(\rho)\rho d\rho \\ &+ mK_{n+1}(|m|r)\int_0^r I_{n+1}(|m|\rho)p_{m,n+1}(\rho)\rho d\rho \end{aligned}$$

$$\begin{aligned} g_{m,n}(r) &= -mI_n(|m|r)\int_r^1 K_n(|m|\rho)q_{m,n}(\rho)\rho d\rho \\ &- |m|I_n(|m|r)\int_r^1 K_{n+1}(|m|\rho)p_{m,n+1}(\rho)\rho d\rho - mK_n(|m|r)\int_0^r I_n(|m|\rho)q_{m,n}(\rho)\rho d\rho \\ &+ |m|K_n(|m|r)\int_0^r I_{n+1}(|m|\rho)p_{m,n+1}(\rho)\rho d\rho \end{aligned}$$

and

$$f_{0,n+1}(r) = \begin{cases} -\displaystyle\int_0^r \frac{\rho^n}{r^{n+1}} q_{0,n}(\rho)\rho d\rho & n \geq 0 \\ \displaystyle\int_r^1 \frac{\rho^n}{r^{n+1}} q_{0,n}(\rho)\rho d\rho & n < 0, \end{cases}$$

and

$$g_{0,n}(r) = \begin{cases} -\displaystyle\int_r^1 \frac{r^n}{\rho^{n+1}} p_{0,n+1}(\rho)\rho d\rho & n \geq 0 \\ \displaystyle\int_0^r \frac{r^n}{\rho^{n+1}} p_{0,n+1}(\rho)\rho d\rho & n < 0. \end{cases}$$

It is is now a routine exercise to verify that $DQ = QD = I$. Thus this completes the proof. ■

6.2 The Parametrix

Now that the parametrix has been constructed the next goal is to show that it is a compact operator. This is the main result of the chapter.

Theorem 6.2.1 *The Dirac operator D, defined by (6.1) and subject to the boundary conditions (6.2) has a bounded inverse. Moreover that inverse is a compact operator.*

Proof Consider the following integral operators in $L^2([0,1], rdr)$ for $i, j = 0, 1$ and $m \neq 0$:

$$R_{ij}^{(m,n)} f(r) := |m| \int_r^1 I_{n+i}(|m|r) K_{n+j}(|m|\rho) f(\rho)\rho d\rho,$$

$$S_{ij}^{(m,n)} f(r) := |m| \int_0^r K_{n+i}(|m|r) I_{n+j}(|m|\rho) f(\rho)\rho d\rho,$$

and for $n \geq 0$:

$$T_1^{(0,n)} f(r) := \int_r^1 \frac{r^n}{\rho^{n+1}} f(\rho)\,\rho d\rho,$$

$$T_2^{(0,n)} f(r) := \int_0^r \frac{\rho^n}{r^{n+1}} f(\rho)\,\rho d\rho.$$

Then one can rewrite formula (6.9) for Q in the following way:

$$\begin{aligned} f_{m,n+1} &= R_{10}^{(m,n)} q_{m,n} + \frac{|m|}{m} R_{11}^{(m,n)} p_{m,n+1} - S_{10}^{(m,n)} q_{m,n} + \frac{|m|}{m} S_{11}^{(m,n)} p_{m,n+1} \\ g_{m,n} &= -\frac{|m|}{m} R_{00}^{(m,n)} q_{m,n} - R_{01}^{(m,n)} p_{m,n+1} - \frac{|m|}{m} S_{00}^{(m,n)} q_{m,n} + S_{11}^{(m,n)} p_{m,n+1} \end{aligned} \tag{6.10}$$

and

$$f_{0,n+1} = \begin{cases} -T_2^{(0,n)} q_{0,n} & n \geq 0 \\ T_1^{(0,-n-1)} q_{0,n} & n < 0, \end{cases}$$

and

$$g_{0,n} = \begin{cases} -T_1^{(0,n)} p_{0,n+1} & n \geq 0 \\ T_2^{(0,-n-1)} p_{0,n+1} & n < 0. \end{cases}$$

It will be shown that all ten integral operators above are Hilbert-Schmidt by estimating the Hilbert-Schmidt norms. It turns out that the HS norm of each integral operator goes to zero as $|m| + |n|$ goes to infinity. This implies that Q is a compact operator as the norm limit of compact operators, since it is (up to a shift in the n index) a direct sum of compact operators with decreasing norms.

To show that $T_2^{(0,n)}$ and $T_1^{(0,n)}$ are Hilbert-Schmidt one simply computes:

$$\|T_1^{(0,n)}\|_2^2 = \|T_2^{(0,n)}\|_2^2 = \int_0^1 \int_0^r \left(\frac{\rho}{r}\right)^{2n+1} d\rho dr = \frac{1}{4(n+1)}. \tag{6.11}$$

For the other operators one has:

$$\|R_{ij}^{(m,n)}\|_2^2 = |m|^2 \int_0^1 \int_r^1 I_{n+i}^2(|m|r) K_{n+j}^2(|m|\rho)\, r\rho d\rho dr,$$

and

$$\|S_{ij}^{(m,n)}\|_2^2 = |m|^2 \int_0^1 \int_0^r K_{n+i}^2(|m|r) I_{n+j}^2(|m|\rho)\, r\rho d\rho dr.$$

Clearly one has $R_{11}^{(m,n)} = R_{00}^{(m,n+1)}$ and $S_{11}^{(m,n)} = S_{00}^{(m,n+1)}$ and additionally, using the inequality (6.27), one can conclude that:

$$\begin{aligned} \|R_{10}^{(m,n)}\|_2^2 \leq \|R_{00}^{(m,n)}\|_2^2 \leq \|R_{01}^{(m,n)}\|_2^2 \\ \|S_{01}^{(m,n)}\|_2^2 \leq \|S_{00}^{(m,n)}\|_2^2 \leq \|S_{10}^{(m,n)}\|_2^2. \end{aligned}$$

Consequently, one only has to estimate the Hilbert-Schmidt norm for $R_{01}^{(m,n)}$ and $S_{10}^{(m,n)}$. Indeed one has:

$$\|R_{01}^{(m,n)}\|_2^2 = \frac{1}{|m|^2}\int_0^{|m|}\int_t^{|m|} K_{n+1}^2(s)I_n^2(t)stdsdt = \frac{1}{|m|^2}\int_0^{|m|}\int_0^{s} K_{n+1}^2(s)I_n^2(t)stdtds,$$

where we've changed to new variables $t = |m|r$, $s = |m|\rho$, and used Fubini's Theorem. The next idea is to estimate the above expression in two ways to show that it goes to zero when $|m| + |n|$ increases.

From (6.31) one has $I_n^2(t) \le \frac{t}{n} I_n(t)I_n'(t)$ which by integration yields:

$$\int_0^s I_n^2(t)dt \le \frac{s}{2n} I_n^2(s). \tag{6.12}$$

Next, using $t \le s$, (6.12), and (6.30), one gets for $n \neq 0$:

$$\|R_{01}^{(m,n)}\|_2^2 \le \frac{1}{2|n||m|^2}\int_0^{|m|} s^3 K_{n+1}^2(s)I_n^2(s)ds \le \frac{1}{2|n||m|^2}\int_0^{|m|} sds = \frac{1}{4|n|}.$$

On the other hand from (6.32) one has $I_n^2(t) \le 2I_n(t)I_n'(t)$, yielding:

$$\int_0^s I_n^2(t)dt \le I_n^2(s). \tag{6.13}$$

So using inequalities (6.13) and (6.30) again, one gets for $n \neq 0$:

$$\|R_{01}^{(m,n)}\|_2^2 \le \frac{1}{|m|^2}\int_0^{|m|} s^2 K_{n+1}^2(s)I_n^2(s)ds \le \frac{1}{|m|^2}\int_0^{|m|} ds = \frac{1}{|m|}.$$

Finally, if $n = 0$, one notices that the recurrence relations (6.22) imply:

$$tI_0^2(t) = (tI_1(t)I_0(t))' - tI_1^2(t) \le (tI_1(t)I_0(t))'.$$

Hence one obtains an integral estimate:

$$\int_0^s I_0^2(t)tdt \le sI_1(s)I_0(s) \le sI_0^2(s),$$

which will be used to estimate the norm above as follows:

$$\|R_{01}^{(m,0)}\|_2^2 \le \frac{1}{|m|^2}\int_0^{|m|} s^2 K_1^2(s)I_0^2(s)ds \le \frac{1}{|m|}.$$

For the norm of $S_{10}^{(m,n)}$ observe that, after a change of variables, one has:

$$\|S_{10}^{(m,n)}\|_2^2 = \frac{1}{|m|^2}\int_0^{|m|}\int_0^t I_n^2(s)K_{n+1}^2(t)stdsdt = \|R_{01}^{(m,n)}\|_2^2.$$

This shows that all of the operators are indeed Hilbert-Schmidt operators. Moreover one has the estimates:

$$\|R_{ij}^{(m,n)}\|_2^2 \leq \frac{\text{const}}{\sqrt{1+m^2+n^2}}, \tag{6.14}$$

and

$$\|S_{ij}^{(m,n)}\|_2^2 \leq \frac{\text{const}}{\sqrt{1+m^2+n^2}}. \tag{6.15}$$

It follows by the remarks at the beginning of the proof that Q is compact. Thus the proof of the theorem is complete.

■

Theorem 6.2.2 *The operator Q, defined by (6.9), is a p-th Schatten-class operator for all $p > 3$.*

Proof Notice that the p-th Schatten norm of Q can be estimated as follows:

$$\|Q\|_p^p \leq \text{const}\sum_{m,n,i,j}\left(\|R_{ij}^{(m,n)}\|_p^p + \|S_{ij}^{(m,n)}\|_p^p\right) + \sum_{n,i}\|T_i^{(0,n)}\|_p^p. \tag{6.16}$$

This is because Q is (essentially) a direct sum of two by two matrices with entries made up of the ten integral operators we studied above, see (6.10).

To bound $\|R_{ij}^{(m,n)}\|_p^p$ and the other norms we use the following interpolation estimate for the p-th Schatten norm: if a is a Hilbert-Schmidt operator and $p \geq 2$ then

$$\|a\|_p^p \leq \|a\|_2^2\,\|a\|^{p-2}. \tag{6.17}$$

The estimate easily follows from the definition of the p-th Schatten norm. We have already obtained estimates on the Hilbert-Schmidt norms of $R_{ij}^{(m,n)}$ and the other operators in (6.14), (6.15), and (6.11), so by the above interpolation we need estimates on the operator norms. The main tool used to establish such estimates for the operator norms of integral operators is the Schur-Young inequality, see [13]. This Lemma has been stated in previous chapters, however it is stated again for convenience. It is stated in the lemma below.

Lemma 6.2.3 *(Schur-Young Inequality) Let $\mathcal{K} : L^2(Y) \longrightarrow L^2(X)$ be an integral operator:*

$$\mathcal{K}f(x) = \int K(x,y)f(y)dy$$

Then one has:

$$\|\mathcal{K}\|^2 \leq \left(\sup_{x \in X} \int_Y |K(x,y)|dy \right) \left(\sup_{y \in Y} \int_X |K(x,y)|dx \right).$$

The kernels of the integral operators are products of modified Bessel functions, and the difficulty here is to estimate the integrals of such products. The main technical step in those estimates is summarized in the following lemma.

Lemma 6.2.4 *Consider the following expressions for $m \neq 0$:*

$$\mathcal{I}_1^{(m,n)} = \frac{1}{|m|} \sup_{0 \leq s \leq |m|} \int_0^s K_{n+1}(s)I_n(t)tdt,$$

$$\mathcal{I}_2^{(m,n)} = \frac{1}{|m|} \sup_{0 \leq t \leq |m|} \int_t^{|m|} K_{n+1}(s)I_n(t)sds.$$

There is a constant such that for $i = 1, 2$:

$$\mathcal{I}_i^{(m,n)} \leq \frac{const}{\sqrt{1 + m^2 + n^2}}.$$

The proof of this lemma will be postponed until the main line of the argument is finished. Now one turns to estimating $\|R_{ij}^{(m,n)}\|$ for $m \neq 0$. Using Lemma 6.2.3 one has:

$$\|R_{ij}^{(m,n)}\|^2$$
$$\leq m^2 \left(\sup_{0 \leq r \leq 1} \int_r^1 K_{n+i}(|m|\rho)I_{n+j}(|m|r)\rho d\rho \right) \left(\sup_{0 \leq \rho \leq 1} \int_0^\rho K_{n+i}(|m|\rho)I_{n+j}(|m|r)rdr \right)$$

Changing variables in both integrals one gets:

$$\|R_{ij}^{(m,n)}\|^2 \\ \leq \left(\frac{1}{|m|}\sup_{0\leq t\leq |m|}\int_t^{|m|} K_{n+i}(s)I_{n+j}(t)sds\right)\left(\frac{1}{|m|}\sup_{0\leq s\leq |m|}\int_0^{s} K_{n+i}(s)I_{n+j}(t)tdt\right).$$

By the monotonicity (6.27) the right hand side is biggest when $i = 1$ and $j = 0$, and so

$$\|R_{ij}^{(m,n)}\|^2 \leq \mathcal{I}_1^{(m,n)} \cdot \mathcal{I}_2^{(m,n)}.$$

It follows from Lemma 6.2.4 that:

$$\|R_{ij}^{(m,n)}\| \leq \frac{\text{const}}{\sqrt{1+m^2+n^2}}. \tag{6.18}$$

Attention is now turned to estimating $\|S_{ij}^{(m,n)}\|$. By Lemma 6.2.3 one has:

$$\|S_{ij}^{(m,n)}\|^2 \\ \leq m^2\left(\sup_{0\leq r\leq 1}\int_0^r I_{n+i}(|m|\rho)K_{n+j}(|m|r)\rho d\rho\right)\left(\sup_{0\leq \rho\leq 1}\int_\rho^1 I_{n+i}(|m|\rho)K_{n+j}(|m|r)rdr\right)$$

Clearly the expression on the right hand side of the above inequality is the same as the expression in the estimate of $\|R_{ij}^{(m,n)}\|^2$. It follows that

$$\|S_{ij}^{(m,n)}\| \leq \frac{\text{const}}{\sqrt{1+m^2+n^2}}.$$

Consider now the case $m = 0$. Lemma 6.2.3 is used once again to compute:

$$\|T_2^{(0,n)}\|^2 \leq \left(\sup_{0\leq r\leq 1}\int_0^r \left(\frac{\rho}{r}\right)^{n+1} d\rho\right)\left(\sup_{0\leq \rho\leq 1}\int_\rho^1 \left(\frac{\rho}{r}\right)^{n} dr\right) \leq \frac{\text{const}}{1+n^2}$$

and similarly

$$\|T_1^{(0,n)}\|^2 \leq \left(\sup_{0\leq r\leq 1}\int_r^1 \left(\frac{r}{\rho}\right)^{n} d\rho\right)\left(\sup_{0\leq \rho\leq 1}\int_0^\rho \left(\frac{r}{\rho}\right)^{n+1} dr\right) \leq \frac{\text{const}}{1+n^2}.$$

Either way one has for $i = 1, 2$:

$$\|T_i^{(0,n)}\| \leq \frac{\text{const}}{\sqrt{1+n^2}}.$$

Combining (6.14), (6.18), and using (6.17) one gets:

$$\|R_{ij}^{(m,n)}\|_p^p \leq \frac{\text{const}}{(1+m^2+n^2)^{\frac{p-1}{2}}} \ ,$$

and exactly the same estimates for $\|S_{ij}^{(m,n)}\|_p^p$ and $\|T_i^{(0,n)}\|_p^p$. Consequently, by (6.16) one gets:

$$\|Q\|_p^p \leq \sum_{m,n} \frac{\text{const}}{(1+m^2+n^2)^{\frac{p-1}{2}}} \ ,$$

where the series is summable when $p > 3$. This concludes the proof of the theorem.

■

The proof of Lemma 6.2.4 is now stated.

Proof (of Lemma 6.2.4) Since both $K_n(z)$ and $I_n(z)$ are symmetric for positive and negative n, see (6.20), one will only need to consider the case when $n \geq 0$.

Using (6.32) and integrating by parts one has:

$$\int_0^s I_n(t)tdt \leq 2\int_0^s I'_{n+1}(t)tdt = 2sI_{n+1}(s) - 2\int_0^s I_{n+1}(t)dt \leq 2sI_{n+1}(s).$$

Consequently we get:

$$\mathcal{I}_1^{(m,n)} = \frac{1}{|m|}\sup_{0\leq s\leq|m|}\int_0^s K_{n+1}(s)I_n(t)tdt \leq \frac{1}{|m|}\sup_{0\leq s\leq|m|} 2sK_{n+1}(s)I_{n+1}(s).$$

Now one bounds $\mathcal{I}_1^{(m,n)}$ in two different ways. First observe:

$$\mathcal{I}_1^{(m,n)} \leq 2\sup_{0\leq s\leq|m|} K_{n+1}(s)I_{n+1}(s) \leq \frac{2}{n+1},$$

by (6.29). On the other hand one has:

$$\mathcal{I}_1^{(m,n)} \leq \frac{2}{|m|}\sup_{0\leq s\leq\infty} sK_{n+1}(s)I_{n+1}(s) \leq \frac{2}{|m|},$$

by inequality (6.30). It follows that $\mathcal{I}_1^{(m,n)} \leq \text{const}/\sqrt{1+m^2+n^2}$.

One estimates $\mathcal{I}_2^{(m,n)}$ in the same fashion, however the process is somewhat more complicated. Using (6.34) and integrating by parts one gets:

$$\int_t^{|m|} sK_{n+1}(s)ds \leq -2\int_t^{|m|} sK'_n(s)ds \leq 2tK_n(t) + \int_t^{|m|} K_n(s)ds.$$

Using (6.34) again yields:

$$\int_t^{|m|} sK_{n+1}(s)ds \leq 2tK_n(t) + 4K_n(t).$$

It follows that:

$$\mathcal{I}_2^{(m,n)} = \frac{1}{|m|} \sup_{0\leq t\leq |m|} \int_t^{|m|} K_{n+1}(s)I_n(t)sds \leq \frac{1}{|m|} \sup_{0\leq t\leq |m|} \left(2tK_n(t)I_n(t) + 4K_n(t)I_n(t)\right).$$

If $n > 0$ one estimates the above expression in two ways using (6.30) and (6.29). First one has:

$$\mathcal{I}_2^{(m,n)} \leq \frac{1}{|m|}(2|m|+4)\frac{1}{2n}.$$

Secondly:

$$\mathcal{I}_2^{(m,n)} \leq \frac{1}{|m|}\left(2 + \frac{4}{2n}\right).$$

If $n = 0$ one has:

$$\mathcal{I}_2^{(m,n)} \leq \frac{1}{|m|} \sup_{0\leq t<\infty} I_0(t) \int_t^{\infty} K_1(s)sds,$$

and one needs to show that the function $I_0(t)\int_t^\infty K_1(s)sds$ is bounded. It follows from the asymptotic behavior (6.25) and (6.26) that the limit of $I_0(t)\int_t^\infty K_1(s)sds$ at $t = 0$ is $\int_0^\infty K_1(s)sds < \infty$. On the other hand using L'Hospital's rule one gets:

$$\lim_{t\to\infty} I_0(t)\int_t^\infty K_1(s)sds = \lim_{t\to\infty} \frac{I_0^2(t)K_1(t)t}{I_0'(t)} = \lim_{t\to\infty} \frac{I_0^2(t)K_1(t)t}{I_1(t)} = \frac{1}{2},$$

by (6.26) again. Thus, in a similar fashion to $\mathcal{I}_1^{(m,n)}$, one has

$$\mathcal{I}_2^{(m,n)} \leq \mathrm{const}/\sqrt{1+m^2+n^2}.$$

Therefore the proof of the lemma is complete. ■

In conclusion a somewhat more complicated proof of Lemma 6.2.4 is possible without the use of the non-elementary inequality (6.28). Estimating along the lines of the Hilbert-Schmidt norm bound in the proof of Theorem 6.2.1 it is enough to employ the inequalities (6.31), (6.32), (6.34), and (6.33) instead of monotonicity of $K_n(t)I_n(t)$.

6.3 The modified Bessel functions

This section contains all of the relevant information on the modified Bessel functions. The references that are used are [1], [3], [24], and [33]. A short argument will be given for those results that are not from any of these references.

6.3.1 Basic properties

The main reference of this subsection is [1].

The modified Bessel functions of integer order n can be defined by the following expressions:

$$I_n(t) = \frac{1}{\pi} \int_0^{\pi} e^{t \cos \alpha} \cos(n\alpha)\ d\alpha$$

and

$$K_n(t) = \int_0^{\infty} e^{-t \cosh \alpha} \cosh(n\alpha)\ d\alpha \tag{6.19}$$

where in both formulas t is a positive real number.

Both functions are symmetric in n:

$$I_n(t) = I_{-n}(t) \text{ and } K_n(t) = K_{-n}(t). \tag{6.20}$$

Consequently, without the loss of generality, it will be assumed that n is a non-negative integer.

One has the following power series representation for $I_n(t)$:

$$I_n(t) = \sum_{k=0}^{\infty} \frac{(t/2)^{n+2k}}{k!(n+k)!}$$

It follows that both modified Bessel functions are positive.

They are two independent solutions of the second-order differential equation:

$$\frac{d^2x}{dt^2} + \frac{1}{t}\frac{dx}{dt} - \left(1 + \frac{n^2}{t^2}\right) x(t) = 0 \tag{6.21}$$

which is called the modified Bessel equation.

They satisfy the recurrence relations with derivatives:

$$I_n'(t) = I_{n+1}(t) + \frac{n}{t} I_n(t) \text{ and } K_n'(t) = -K_{n+1}(t) + \frac{n}{t} K_n(t), \tag{6.22}$$

as well as:

$$I_n'(t) = I_{n-1}(t) - \frac{n}{t} I_n(t) \text{ and } K_n'(t) = -K_{n-1}(t) - \frac{n}{t} K_n(t), \tag{6.23}$$

The Wronskian of the two functions is:

$$W(K_n(t), I_n(t)) = \det \begin{pmatrix} K_n(t) & I_n(t) \\ K_n'(t) & I_n'(t) \end{pmatrix} = I_n(t)K_{n+1}(t) + I_{n+1}(t)K_n(t) = 1/t. \tag{6.24}$$

They have the following expansions near zero for $n \geq 0$:

$$I_n(t) \sim \frac{1}{\Gamma(n+1)} \left(\frac{t}{2}\right)^n \text{ and } K_n(t) \sim \begin{cases} -\ln\left(\frac{t}{2}\right) - \gamma & \text{if } n = 0 \\ \frac{\Gamma(n)}{2}\left(\frac{2}{t}\right)^n & \text{if } n > 0 \end{cases} \tag{6.25}$$

where γ is the Euler-Mascheroni constant. The expansions at infinity are:

$$I_n(t) \sim \frac{e^t}{\sqrt{2\pi t}} \text{ and } K_n(t) \sim e^{-t}\sqrt{\frac{\pi}{2t}}. \tag{6.26}$$

In the following subsections lesser known results about the modified Bessel functions are stated.

6.3.2 Monotonicity

The modified Bessel functions have simple monotonicity properties in the argument t: $I_n'(t) > 0$ and $K_n'(t) \leq 0$ on $(0, \infty)$, which says that $I_n(t)$ is increasing and $K_n(t)$ is decreasing. The first inequality follows from (6.22). The second inequality follows immediately from the integral representation (6.19).

Additionally there are the following monotonicity properties in the order n:

$$I_{n+1}(t) \leq I_n(t) \text{ and } K_n(t) \leq K_{n+1}(t). \tag{6.27}$$

The first inequality was proven in [33]. It also follows from Turan - type inequality [3]:

$$I_{n-1}(t)I_{n+1}(t) - I_n^2(t) \leq 0.$$

For the second inequality one estimates

$$K_{n+1}(t) = \int_0^\infty e^{-t\cosh\alpha}\left(\cosh(n\alpha)\cosh\alpha + \sinh(n\alpha)\sinh\alpha\right) d\alpha \geq$$
$$\geq \int_0^\infty e^{-t\cosh\alpha}\cosh(n\alpha)\cosh\alpha d\alpha \geq \int_0^\infty e^{-t\cosh\alpha}\cosh(n\alpha)d\alpha = K_n(t).$$

One also has monotonicity of the product:

$$(K_n(t)I_n(t))' \leq 0 \tag{6.28}$$

i.e. $K_n(t)I_n(t)$ is a decreasing function of t, see [24].

6.3.3 Product estimates

For $n \geq 1$ one has:

$$\lim_{t\to 0^+} K_n(t)I_n(t) = \frac{1}{2n}.$$

This is a simple consequence of the asymptotics of $I_n(t)$ and $K_n(t)$ as $t \to 0$, see (6.25). Since $I_n(t)K_n(t)$ is decreasing on $(0,\infty)$, we have:

$$K_n(t)I_n(t) \leq \frac{1}{2n}. \tag{6.29}$$

Additionally one has:

$$tK_n(t)I_n(t) \leq tK_{n+1}(t)I_n(t) \leq 1. \tag{6.30}$$

The inequality follows from (6.27) and from the Wronskian formula (6.24) since both terms on the left-hand side of that equation are positive.

6.3.4 Derivative estimates

The proofs in this paper use the following two inequalities with derivatives of the modified Bessel functions of the first kind. They are, for $n > 0$:

$$I_n(t) \leq \frac{t}{n} I_n'(t), \tag{6.31}$$

and

$$I_{n-1} \leq I_n(t) \leq 2\, I_n'(t). \tag{6.32}$$

To prove them notice that from (6.22) one gets $I_n'(t) - \frac{n}{t} I_n(t) > 0$, which gives (6.31). Secondly, (6.22) and (6.23) give:

$$2\, I_n'(t) = I_{n+1}(t) + I_{n-1}(t) \geq I_{n-1}(t) \geq I_n(t),$$

which is (6.32).

Both inequalities above are also a direct consequence of the following stronger result of [24]:

$$\frac{t I_n'(t)}{I_n(t)} > \sqrt{t^2 \frac{n}{n+1} + n^2}$$

For the modified Bessel functions of the second kind one has the following useful result from [24]:

$$\frac{s K_n'(s)}{K_n(s)} \leq -\sqrt{s^2 + n^2}.$$

An analog of (6.31) and obtainable in the same way from (6.23) is:

$$K_n(s) \leq -\frac{s}{n} K_n'(s). \tag{6.33}$$

However for the applications of this chapter one only needs the following estimate: combining (6.22) and (6.23) gives:

$$-2\, K_n'(t) = K_{n+1}(t) + K_{n-1}(t) \geq K_{n+1}(t) \geq K_n(t). \tag{6.34}$$

7. DIRAC TYPE OPERATORS ON THE QUANTUM SOLID TORUS WITH GLOBAL BOUNDARY CONDITIONS

7.1 Non-commutative Torus and the quantum Dirac operator

In this section we define the main objects of this chapter: the quantum solid torus, the Hilbert spaces of L^2 "functions", and of course the Dirac type operators that we will be studying. Let $\{e_{k,l}\}$ be the canonical basis in $\ell^2(\mathbb{Z}_{\geq 0} \times \mathbb{Z})$. Define the following two operators: $Ue_{k,l} = e_{k+1,l}$ and $Ve_{k,l} = e^{2k\pi i\theta}e_{k,l+1}$ with $k \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}$. The two label operators K and L where $Ke_{k,l} = ke_{k,l}$ and $Le_{k,l} = le_{k,l}$ will also be needed for later computations. Returning to the two operators U and V, notice that V is a unitary and that $U^*U = 1$. Also notice that one has the commutation relation

$$VU = e^{2\pi i\theta}UV \ . \tag{7.1}$$

Let $C^*(U,V)$ be the C^*-algebra generated by U and V. This and relation (7.1) gives the noncommutative (quantum) solid torus. Recall that T^2_θ is the standard 2 dimensional quantum torus. In other words, if u and v are unitaries such that $vu = e^{2\pi i\theta}uv$, then $T^2_\theta := C^*(u,v)$. It will be seen later on, that the analysis through out this paper is not dependent on θ. Let $\mathcal{K}$ be the ideal of compact operators. One would like to have a short exact sequence like in [17] for example.

Proposition 7.1.1 *One has the following short exact sequence:*

$$0 \longrightarrow \mathcal{K} \otimes C(S^1) \longrightarrow C^*(U,V) \longrightarrow T^2_\theta \longrightarrow 0.$$

Proof Let $\mathcal{T}$ be the Toeplitz algebra and let $U'e_{k,l} = e^{-2\pi il\theta}e_{k+1,l}$ and $U_1e_{k,l} = e^{-2\pi ikl\theta}e_{k,l}$. Then general theory tells us that $\mathcal{T} \cong C^*(U')$. One would like to show that $C^*(U', V') \cong \mathcal{T} \otimes_\theta \mathbb{Z}$ where $V'e_{k,l} = e_{k,l+1}$. In order to do this, one needs to write down the automorphism and the action on the generators for U' and $\mathbb{Z}$. For $n \in \mathbb{Z}$ one chooses the automorphism $\varphi_n(U) = e^{2\pi i n\theta}U$. Recall that the Hilbert space is $\ell^2(\mathbb{Z} \times \mathbb{N})$ and so for $f(l,k) \in \ell^2(\mathbb{Z} \times \mathbb{N})$, one has the following actions on the generators $1 \in \mathbb{Z}$ and $U' \in \mathcal{T}$:

$$V'f(l,k) = \psi(1)f(l,k) = f(l-1,k) \quad \text{and} \quad \psi(U')f(l,k) = e^{-2\pi i l\theta}f(l,k+1).$$

Knowing this it follows from general theory that $C^*(U', V') \cong \mathcal{T} \otimes_\theta \mathbb{Z}$. Also note since U_1 is a unitary operator and $U = U_1^{-1}U'U_1$ and $V = U_1^{-1}VU_1$, one has $C^*(U, V) \cong C^*(U', V')$. In order to get the exact sequence, one needs to show two things, first for the ideal $\mathcal{I}$, which will be defined in a moment, one has $\mathcal{I} \cong \mathcal{K} \otimes C(S^1)$ and second $(\mathcal{T} \otimes_\theta \mathbb{Z})/\mathcal{I} \cong T_\theta^2$. Let $\mathcal{I}$ be the subring of $\mathcal{T} \otimes_\theta \mathbb{Z}$ such that $I - U'(U')^*$ is a projection. It is clear that this a subring, but it needs to be verified that $\mathcal{I}$ really is an ideal. However it will follow immediately once one sees how the automorphism acts on involutions and products. Indeed one has

$$(\varphi_n(U'))^* = e^{-2\pi i n\theta}(U')^* = \varphi_{-n}((U')^*)$$
$$(\varphi_n(fg) = \varphi_{n_f}(f)\varphi_{n_g}(g) = e^{2\pi i(n_f+n_g)\theta}fg\ .$$

Since in the quotient ring, $I - (U')(U')^*$ gets mapped to zero, ones sees that $U' \mapsto \tilde{U}'$ where $\tilde{U}'$ is a unitary. Since U' and V' generate $\mathcal{T} \otimes_\theta \mathbb{Z}$ there respective quotient classes $\tilde{U}'$ and $\tilde{V}'$ will generate the quotient ring. Moreover since V' is already a unitary and U' and V' satisfy the commuatation relation (7.1), $\tilde{U}'$ and $\tilde{V}'$ will satisfy the same relation. Since T_θ^2 has a universal representation equivalent to this it follows that $(\mathcal{T} \otimes_\theta \mathbb{Z})/\mathcal{I} \cong T_\theta^2$. To show the other isomorphism, let $a \in \mathcal{T}$ be a compact operator. Then $aV^n \in \mathcal{I}$ and consider the following map $aV^n \mapsto \left(ae^{2\pi i n K\theta}\right) \otimes (V')^n$. First one has $\left(ae^{2\pi i n K\theta}\right) \otimes (V')^n \in \mathcal{K} \otimes C(S^1)$. This map clearly preserves multiplication making it a homomorphism, moreover it isn't too difficult to see that this map is indeed an isomorphism. Therefore one has $\mathcal{I} \cong \mathcal{K} \otimes C(S^1)$. Thus the result follows. ■

The next goal is to define the quantum Dirac operator that will be studied, however before that one needs a few more items. The next item is that must be defined is the Hilbert space in which the Dirac operator will be acting on, however in order to do this, a few more things must be defined. For $n \geq 0$, let $a^{(n)}(k)$ be a sequence of numbers such that

$$\sum_{k=0}^{\infty} \frac{1}{a^{(n)}(k)} < \infty$$

and the above sum goes to zero as $n \to \infty$. For $f \in C^*(U, V)$, define the formal series:

$$f_{\text{series}} = \sum_{n \geq 0, m \in \mathbb{Z}} V^m U^n f^+_{m,n}(k) + \sum_{n \geq 1, m \in \mathbb{Z}} f^-_{m,n}(k) V^m (U^*)^n$$

where

$$f^+_{m,n}(k) = \langle e_{k,0}, (U^*)^n V^{-m} f e_{k,0} \rangle \quad \text{and} \quad f^-_{m,n}(k) = \langle e_{k,0}, f U^n V^{-m} e_{k,0} \rangle.$$

With the above series define a norm by

$$\|f_{\text{series}}\|^2 = \sum_{k=0}^{\infty} \sum_{n \geq 0, m \in \mathbb{Z}} \frac{1}{a^{(n)}(k)} |f^+_{m,n}(k)|^2 + \sum_{k=0}^{\infty} \sum_{n \geq 1, m \in \mathbb{Z}} \frac{1}{a^{(n)}(k)} |f^-_{m,n}(k)|^2$$

Let $\mathcal{H}_0$ be the Hilbert space where its elements are the above formal series, f_{series} such that $\|f_{\text{series}}\|$ is finite. The following proposition can be deduced from the above.

Proposition 7.1.2 *If $f \in C^*(U, V)$, then the series f_{series} converges to f, in the L^2 sense, in $\mathcal{H}_0$ and moreover $C^*(U, V)$ is dense in $\mathcal{H}_0$.*

Proof Since the norm on f_{series} is finite, the L^2-convergence will follow from showing a norm equivanlence calculation. We only show the computation for the "positive" vector as it is the same computation for the "negative" vector. One has

$$\begin{aligned}\|f_{\text{series}}\|^2 &= \sum_{n\geq 0, m\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{1}{a^{(n)}(k)} \overline{f^+_{m,n}(k)} f^+_{m,n}(k) \\ &= \sum_{n\geq 0, m\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{1}{a^{(n)}(k)} \left(V^m U^n f^+_{m,n}(k)\right)^* V^m U^n f^+_{m,n}(k) \\ &= \sum_{n\geq 0, m\in\mathbb{Z}} \sum_{k=0}^{\infty} \frac{1}{a^{(n)}(k)} \left(\langle e_{k,0}, f e_{k,0}\rangle\right)^* \langle e_{k,0}, f e_{k,0}\rangle = \|f^* f\| = \|f\|^2 \ .\end{aligned}$$

Hence the norms are equivalent. The density argument is along the same lines as the proof done in Lemma 5.1 in [17]. In fact choose the function

$$\delta_i(k) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} .$$

Then it is clear one has $V^m U^n \delta_i(k)$, $\delta_i(k) V^m (U^*)^n \in C^*(U, V)$. However since $a^{(n)}(k)$ is inversely summable this implies that $V^m U^n \delta_i(k)$ and $\delta_i(k) V^m (U^*)^n$ have finite norm in $\mathcal{H}_0$, hence they are elements in $\mathcal{H}_0$. But this implies that $C^*(U, V)$ is dense in $\mathcal{H}_0$ since all elements of $C^*(U, V)$ can be written as finite sums of $V^m U^n \delta_i(k)$ and $\delta_i(k) V^m (U^*)^n$. This completes the proof. ■

The main reason for the reciprocals of the $a^{(n)}(k)$ being in the norm is to approximate a Riemann sum. Recall that for a partition on $[0, 1]$ with points x_k, one has

$$\int_0^1 f(x)dx = \sum_{k=0}^{\infty} f(x_k)(x_k - x_{k-1}) = \sum_{k=0}^{\infty} f(x_k)\Delta x \ .$$

The idea is to think of the reciprocals of $a^{(n)}(k)$ to be the Δx. Now one can define the Hilbert space, $\mathcal{H}$, that the Dirac type operator will be acting on. One defines $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$. Now that the Hilbert space has been defined we can finally begin to define the Dirac type operator. Let $c_1^{(n)}(k)$ and $c_2^{(n)}(k)$ be sequences of numbers such that $c_1^{(n)}(k), c_2^{(n)}(k) \leq 1$ and $\prod_k (c_1^{(n)}(k))^{-1}, \prod_k (c_2^{(n)}(k))^{-1}$ exist and are finite and there exists a constant, κ, which does not depend on k or n such that $1/\kappa \leq 1/c_1^{(n)}(k) \leq \kappa$.

Let $\ell^2_{a^{(n)}}(\mathbb{N}) = \{h : \sum_k \frac{1}{a^{(n)}(k)}|h(k)|^2 < \infty\}$. Define the following Jacobi type difference operators

$$\begin{aligned} B^{(n)}h(k) &= a^{(n)}(k)(h(k) - c_2^{(n)}(k-1)h(k-1)) : \ell^2_{a^{(n+1)}}(\mathbb{N}) \to \ell^2_{a^{(n)}}(\mathbb{N}) \\ \overline{B}^{(n)}h(k) &= a^{(n+1)}(k)(h(k) - c_1^{(n)}(k)h(k+1)) : \ell^2_{a^{(n)}}(\mathbb{N}) \to \ell^2_{a^{(n+1)}}(\mathbb{N}) \end{aligned} \tag{7.2}$$

where $\mathrm{dom}(B) = \{h \in \ell^2_{a^{(n+1)}}(\mathbb{N}) \ : \ \|Bh\|_{\ell^2_{a^{(n)}}(\mathbb{N})} < \infty\}$ and the domain of $\overline{B}$ is similar. Now the definition of the quantum Dirac operator that will be studied can be stated. Let

$$\begin{aligned} \delta_0(f) &= -\sum_{m\in\mathbb{Z},n\geq 0} V^m U^{n+1}\overline{B}^{(n)} f^+_{m,n}(K) + \sum_{m\in\mathbb{Z},n\geq 1} B^{(n-1)} f^-_{m,n}(K) V^m (U^*)^{n-1} \\ \delta_2(f) &= -\sum_{m\in\mathbb{Z},n\geq 1} V^m U^{n-1} B^{(n-1)} f^+_{m,n}(K) + \sum_{m\in\mathbb{Z},n\geq 0} \overline{B}^{(n)} f^-_{m,n}(K) V^m (U^*)^{n+1} \end{aligned}$$

and $\delta_1 = [L, \ \cdot\ ]$. It is easy to see that

$$\delta_1(f) = \sum_{m\in\mathbb{Z},n\geq 0} mV^m U^n f^+_{m,n}(K) + \sum_{m\in\mathbb{Z},n\geq 1} m f^-_{m,n}(K) V^m (U^*)^n$$

Define the quantum Dirac type operator, D to be

$$D = \begin{pmatrix} \delta_1 & \delta_0 \\ \delta_2 & -\delta_1 \end{pmatrix}. \tag{7.3}$$

As with any unbounded operator one must define what the domain of D is, for now we take the maximal domain of D that is

$$\mathrm{dom}(D) = \{F \in \mathcal{H} \ : \ \|DF\| < \infty\} \ . \tag{7.4}$$

In the commutative case there was a nice Fourier decomposition for the Dirac operator. In this case there is a Fourier decomposition for the Dirac type operator however it is not very pratical. Instead two propositions will be stated that relate D to a finite difference operator with matrix coefficients.

What is to come is a lot of notation but this is for convenience and it will make the chapter easier to read. Let $a_1(k+1) = a^{(n)}(k+1)$ and $a_2(k) = a^{(n+1)}(k)$. The products

$$J_1(n) = \prod_{k=0}^{\infty} c_1^{(n)}(k) \quad \text{and} \quad J_2(n) = \prod_{k=0}^{\infty} c_2^{(n)}(k)$$

are well defined since $c_1^{(n)}(k) \leq 1$ and $c_2^{(n)}(k) \leq 1$ for all k and n. Also by the definition of $c_1^{(n)}(k)$ and $c_2^{(n)}(k)$, the products $1/J_1(n)$ and $1/J_2(n)$ exist and are finite. Also define the following two sums

$$s_1(n) = \sum_{k=0}^{\infty} \frac{1}{a_1(k)} \quad \text{and} \quad s_2(n) = \sum_{k=0}^{\infty} \frac{1}{a_2(k)}$$

Notice that both $s_1(n)$ and $s_2(n)$ go to zero as $n \to \infty$ by definition of $a_1(k)$ and $a_2(k)$. Define the following Jacobi type difference operator with matrix valued coefficients

$$\mathcal{A}^{(m,n)} \begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = A^{(m,n)}(k+1) \left[\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} - C^{(m,n)}(k) \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \right] \tag{7.5}$$

where

$$A^{(m,n)}(k+1) = \begin{pmatrix} a_2(k)c_1^{(n)}(k) & 0 \\ m & a_1(k+1) \end{pmatrix} \tag{7.6}$$

and

$$C^{(m,n)}(k) = \begin{pmatrix} \frac{1}{c_1^{(n)}(k)} & \frac{-m}{a_2(k)c_1^{(n)}(k)} \\ \frac{-m}{a_1(k+1)c_1^{(n)}(k)} & c_2^{(n)}(k) + \frac{m^2}{a_1(k+1)a_2(k)c_1^{(n)}(k)} \end{pmatrix}. \tag{7.7}$$

Notice that $\det A^{(m,n)}(k+1) = a_2(k)a_1(k+1)c_1^{(n)}(k) \neq 0$ for any k and n which means that the inverse of $A^{(m,n)}(k+1)$ exists for any k and n. This will be needed to compute the parametrix later. The domain must also be stated for $\mathcal{A}^{(m,n)}$. One has the following

$$\text{dom } \mathcal{A}^{(m,n)} = \left\{ h \in \ell^2_{a^{(n)}}(\mathbb{N} \times \mathbb{N}) \ : \|\mathcal{A}^{(m,n)} h\|_{a^{(n)}} < \infty \right\}. \tag{7.8}$$

The next two propositions will show that studying D and finding it's parametrix will boil down to studying $\mathcal{A}^{(m,n)}$ and its respective parametrix. In fact one will only need to study the "positive" vector as the "negative" one will produce the same result. Because of this, the "+" sign will be dropped for simplicity.

Proposition 7.1.3 *For $F \in \mathcal{H}$ where $F = (f, g)^t$ and $f, g \in \mathcal{H}_0$, solving $DF = 0$ is equivalent to solving the following equation:*

$$\mathcal{A}^{(m,n)} \begin{pmatrix} g_{m,n}(k+1) \\ f_{m,n+1}(k+1) \end{pmatrix} = 0$$

Proof Using the definition of D and shifting the first sum one gets

$$DF = \sum_{n \geq 0, m \in \mathbb{Z}} V^m \begin{pmatrix} U^{n+1}(m f_{m,n+1}(k) - \overline{B}^{(n)} g_{m,n}(k)) \\ U^n(-B^{(n)} f_{m,n+1}(k) - m g_{m,n}(k)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This is equivalent to solving the following system of equations

$$\begin{cases} m f_{m,n+1}(k) - a_2(k)(g_{m,n}(k) - c_1^{(n)}(k) g_{m,n}(k+1)) = 0 \\ a_1(k+1)(f_{m,n+1}(k+1) - c_2^{(n)}(k) f_{m,n+1}(k)) + m g_{m,n}(k) = 0 \end{cases}$$

where the defintions of $a_1(k+1)$ and $a_2(k)$ have been used and a shift in k in the second equation has been made. Then rewriting the above using linear algebra produces

$$\begin{pmatrix} a_2(k) c_1^{(n)}(k) & 0 \\ m & a_1(k+1) \end{pmatrix} \begin{pmatrix} g_{m,n}(k+1) \\ f_{m,n+1}(k+1) \end{pmatrix} - \begin{pmatrix} a_2(k) & -m \\ 0 & a_1(k+1) c_2^{(n)}(k) \end{pmatrix} \begin{pmatrix} g_{m,n}(k) \\ f_{m,n+1}(k) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The first matrix is $A^{(m,n)}(k+1)$, therefore factoring it out of the left side of the equation will produce the desired result. This completes the proof. ■

The next lemma will show the relation between solving the equation $DF = G$ and solving a finite difference equation.

Proposition 7.1.4 *For $F, G \in \mathcal{H}$ where $F = (f, g)^t$, $G = (p, q)^t$, and $f, g, p, q \in \mathcal{H}_0$, solving $DF = G$ is equivalent to solving the following equation:*

$$\mathcal{A}^{(m,n)} \begin{pmatrix} g_{m,n}(k+1) \\ f_{m,n+1}(k+1) \end{pmatrix} = \begin{pmatrix} p_{m,n+1}(k) \\ -q_{m,n}(k+1) \end{pmatrix}$$

Proof The proof follows the exact same lines as Proposition 7.1.3, therefore it will be omitted. ■

It follows from Propositions 7.1.3 and 7.1.4 that one only needs to study the properties of $\mathcal{A}^{(m,n)}$ to compute the kernel of D, compute the parametrix, Q, to D and show the compactness of Q. Stating the main result will close out this section and it will be proven throughout the remainder of this paper. One has the following theorem.

Theorem 7.1.5 *The quantum Dirac operator D, defined by (7.3) subject to the boundary condition to be defined in equation (7.11), is an invertible operator whose inverse Q, is a compact operator.*

It should be noted that what is to follow is a very general analysis of finite difference operators with matrix coeffients, with a structure of $\mathcal{A}^{(m,n)}$ and the coeffients have a structure like $A^{(m,n)}(k+1)$ and $C^{(m,n)}(k)$. The analysis carried out will work for generic $a_1(k)$, $a_2(k)$, $c_1^{(n)}(k)$, and $c_2^{(n)}(k)$ that satisfy the conditions that were defined above.

One example that can be used is when $a^{(n)}(k) = S^{-1/2}(k)S^{-1/2}(k+n)$ where $S(k) = w^2(k) - w^2(k-1)$. Also $c^{(n)}(k) = w(k)/w(k+n+1)$, $c_1^{(n)}(k) = c^{(n-1)}(k)$ and $c_2^{(n)}(k) = c^{(n)}(k)$. Then $\delta_0 = S^{-1/2}(k)[\ \cdot\ , U_W]S^{-1/2}(k)$ and $\delta_2 = \delta_0^*$ where $U_W = UW(k)$.

7.2 Kernel of the Quantum Dirac operator

Before one solves for the kernel it will be important to know some of the properties of the $C^{(m,n)}(k)$ matrix. First notice that $\det C^{(m,n)}(k) = c_2^{(n)}(k)/c_1^{(n)}(k)$. For later calculations one needs the infinite product of $C^{(m,n)}(k)$ to exist. We now take a moment to prove that the product really does exist.

Proposition 7.2.1 *The product*

$$C^{(m,n)} := \prod_{k=0}^{\infty} C^{(m,n)}(k)$$

exists.

Proof The goal is to write $C^{(m,n)}(k)$ as $id + B(k)$, since by [35], if the series $\sum_k \|B(k)\|$ converges and $\det C^{(m,n)}(k) \neq 0$, then the infinite product of $C^{(m,n)}(k)$ converges. First one already has $\det C^{(m,n)}(k) \neq 0$ for all k, m, n. Seocnd notice that there exists positive numbers $b(k)$ and $b'(k)$ such that $1/c_1^{(n)}(k) = 1 + b(k)$, $c_2^{(n)}(k) = 1 - b'(k)$, $\sum_k b(k) < \infty$, and $\sum_k b'(k) < \infty$ since $\prod_k (c_1^{(n)}(k))^{-1}$ and $\prod_k c_2^{(n)}(k)$ exist. Therefore one has

$$C^{(m,n)}(k) = id + \begin{pmatrix} b(k) & \frac{-m}{a_2(k)c_1^{(n)}(k)} \\ \frac{-m}{a_1(k+1)c_1^{(n)}(k)} & -b'(k) + \frac{m^2}{a_1(k+1)a_2(k)c_1^{(n)}(k)} \end{pmatrix} := id + B(k) \ .$$

Next one computes the matrix norm of $B(k)$ to get

$$\|B(k)\|$$
$$= \max\left\{ b(k) + \frac{|m|}{a_1(k+1)c_1^{(n)}(k)}, b'(k) + \frac{|m|}{a_2(k)c_1^{(n)}(k)} + \frac{m^2}{a_1(k+1)a_2(k)c_1^{(n)}(k)} \right\} .$$

By the definition of $a_1(k)$ and $a_2(k)$ either term in the above maximum has a finite sum for all $m \neq 0$, therefore $\sum_k \|B(k)\| < \infty$. Therefore one can deduce that $\prod_k C^{(m,n)}(k)$ exists. Thus the proof is complete. ∎

Next one can compute the determinant for $C^{(m,n)}$. Indeed one has

$$\det C^{(m,n)} = \lim_{k\to\infty} \prod_{i=0}^{k} \det C^{(m,n)}(i) = \lim_{k\to\infty} \prod_{i=0}^{k} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)} = \frac{\prod_{i=0}^{\infty} c_2^{(n)}(i)}{\prod_{i=0}^{\infty} c_1^{(n)}(i)} = \frac{J_2(n)}{J_1(n)}$$

The next proposition will show the structure that arises from the infinite product of the $C^{(m,n)}(k)$ matrices.

Proposition 7.2.2 *The infinite product $C^{(m,n)}$ has the following structure:*

$$C^{(m,n)} = \begin{pmatrix} \prod_{i=0}^{\infty} \frac{1}{c_1^{(n)}(i)} + F_0(m^2) & -mF_1(m^2) \\ -mF_2(m^2) & \prod_{i=0}^{\infty} c_2^{(n)}(i) + F_3(m^2) \end{pmatrix}$$

where the $F_j(m^2)$ are power series in m^2 with positive coefficients that grow faster than any polynomial, for $j = 0, 1, 2, 3$ and

$$F_3(m^2) = \left(\prod_{i=0}^{\infty} \frac{1}{c_1^{(n)}(i)} + F_0(m^2) \right)^{-1} \left(m^2 F_1(m^2) F_2(m^2) - F_0(m^2) \prod_{i=0}^{\infty} c_2^{(n)}(i) \right).$$

Proof First we show by induction that for each k the product $\prod_{i=0}^{k} C^{(m,n)}(i)$ is of the form

$$\prod_{i=0}^{k} C^{(m,n)}(i)$$
$$= \begin{pmatrix} \prod_{i=0}^{k} \frac{1}{c_1^{(n)}(i)} + \sum_{i=0}^{k} u_0(n,k)(m^2)^i & -m \sum_{i=0}^{k} u_1(n,k)(m^2)^i \\ -m \sum_{i=0}^{k} u_2(n,k)(m^2)^i & \prod_{i=0}^{k} c_2^{(n)}(i) + \sum_{i=0}^{k} u_3(n,k)(m^2)^i \end{pmatrix}$$

where the above sums are polynomials in m^2 with positive coefficients. The case $k = 0$ is trivial. Assume the claim is true for k. We only show the computation for the first entry in the matrix as the rest is similar. Indeed the upper corner is equal to

$$\prod_{i=0}^{k+1} \frac{1}{c_1^{(n)}(i)} + \frac{1}{c_1^{(n)}(k+1)} \sum_{i=0}^{k} u_0(n,k)(m^2)^i + \frac{m^2}{a_2(k) c_1^{(n)}(k)} \sum_{i=0}^{k} u_2(n,k)(m^2)^i$$
$$= \prod_{i=0}^{k+1} \frac{1}{c_1^{(n)}(i)} + \sum_{i=0}^{k+1} u_0(n,k+1)(m^2)^i .$$

It is clear now that there is a recurrence relation for the coefficients which are still positive and that one has a polynomial in m^2 for the $k+1$ term. Therefore by induction the claim follows. Next one needs to show as $k \to \infty$ the polynomials converge to power series. In order to show this, one will use the Weierstrass Analytic Convergence Theorem (WACT). Again we will only show one of the entries as the rest are similar. Let $F_0^k(m^2) = \sum_{i=0}^k u_0(n,k)(m^2)^i$. It is clear that $F_0^k(m^2)$ is analytic for each open disk of radius R since it is just a polynomial. The goal is to show that $F_0^k(m^2)$ converges to a function $F_0(m^2)$ uniformly, then the WACT will imply that $F_0(m^2)$ is analytic and hence the power series representation. However by the Weierstrass M-test one has

$$\sum_{i=0}^{\infty} \sup_{0 \le m \le R} u_0(n)(m^2)^i \le R^2 \sum_{i=0}^{\infty} u_0(n) \le R^2 \cdot const \sum_{i=0}^{\infty} \frac{1}{a_1(i)}$$

where the last inequality is true because the coefficients, $u_0(n)$ are comprised of the products of $c_1^{(n)}(i)^{-1}$, $a_1(i)^{-1}$, and $a_2(i)^{-1}$. Moreover the constant *const* comes from just taking the supremum of all the numbers after factoring out $a_1(i)^{-1}$. From the conditions on $a_1(i)$ it is clear that the last sum is finite and hence $F_0^k(m^2)$ converges uniformly. Thus the result follows. ∎

The next goal is to solve the equation $DF = 0$, however again using the decomposition given in Proposition 7.1.3 one only needs to solve the equation $\mathcal{A}^{(m,n)}(x(k+1), y(k+1))^t = 0$ where the dependence on m and n have been suppressed. Solving the equation will give information about the kernel of D and tell us if in fact D is invertible or not.

Proposition 7.2.3 *Let $\mathcal{A}^{(m,n)}$ be the operator given by equation (7.5), then*

$$Ker\ \mathcal{A}^{(m,n)} = \left\{ \left(\prod_{i=0}^{k} C^{(m,n)}(i) \right) \alpha \right\}$$

for some vector α.

Proof As stated above the only equation that needs to be solved is

$$\mathcal{A}^{(m,n)} \begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which by equation (7.5) becomes the following equation

$$A^{(m,n)}(k+1) \left[\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} - C^{(m,n)}(k) \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above equation is just like a finite difference operator in [17] except the above has matrix-valued coeffiecents. Thus using the results from [17] and the fact that $A^{(m,n)}(k+1)$ is invertible, one sees the above difference equation has solutions of the form

$$\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} = \left(\prod_{i=0}^{k} C^{(m,n)}(i) \right) \alpha$$

for some arbitrary vector α. Hence the result follows finishing the proof. ∎

In chapter 6, one had special solutions that solved the system of differential equations that arose from the Dirac operator. They were called the modified Bessel functions, with $I_n(|m|r)$ denoting the solution that grew with large $|m|$ and was zero at $|m| = 0$, and $K_n(|m|r)$ which decayed with large $|m|$, more importantly, it was square integrable for large $|m|$ and it was infinite for $|m| = 0$. We would like to have in spirit a similar setup for the special solutions in the kernel of $\mathcal{A}^{(m,n)}$ in the non-commutative case. We therefore define the $I_{m,n}(k)$ and $K_{m,n}(k)$ special solutions to be

$$I_{m,n}(k) = \left(\prod_{i=0}^{k-1} C^{(m,n)}(i) \right) I_{m,n}(0) = \left(\prod_{i=0}^{k-1} C^{(m,n)}(i) \right) \begin{pmatrix} I^{(1)}_{m,n}(0) \\ I^{(2)}_{m,n}(0) \end{pmatrix} \tag{7.9}$$

and

$$K_{m,n}(k) = \left(\prod_{i=0}^{k-1} C^{(m,n)}(i) \right) K_{m,n}(0) = \left(\prod_{i=0}^{k-1} C^{(m,n)}(i) \right) \begin{pmatrix} K^{(1)}_{m,n}(0) \\ K^{(2)}_{m,n}(0) \end{pmatrix} \tag{7.10}$$

where one requires $\mathcal{A}^{(m,n)} I_{m,n}(0) = 0$ for regularity at $k = 0$. When $k = 0$ and $m \neq 0$, the regularity condition implies that $B^{(n)} I^{(2)}_{m,n}(0) + m I^{(1)}_{m,n}(0) = 0$ which implies that

$$I^{(1)}_{m,n}(0) = -\frac{a_1(0)}{m} I^{(2)}_{m,n}(0) \ .$$

However if $m = 0$ one gets $B^{(n)} f_{0,n+1}(0) = 0$ which implies that $f_{0,n+1}(0) = 0$. We summarize in the following proposition:

Proposition 7.2.4 *If $m \neq 0$ the following initial condition is true:*

$$I^{(1)}_{m,n}(0) = -1, \quad \textit{and} \quad I^{(2)}_{m,n}(0) = \frac{m}{a_1(0)} \ .$$

If $m = 0$ one has the following initial condition $f_{0,n+1}(0) = 0$.

It also follows from the existence of $C^{(m,n)}$ that $K_{m,n}(\infty)$ and $I_{m,n}(\infty)$ exist. Next one can finally define the boundary condition. First one writes a function F in a Fourier decomposition on the boundary of the solid torus. Indeed for $F \in L^2(T^2_\theta) \otimes \mathbb{C}^2$ one has

$$F = \sum_{m,n \in \mathbb{Z}} V^m U^n \begin{pmatrix} F^{(1)}_{m,n}(\infty) \\ F^{(2)}_{m,n}(\infty) \end{pmatrix} \ .$$

Let $\mathcal{V} \subset L^2(T^2_\theta) \otimes \mathbb{C}^2$ be a subspace such that for $F \in \mathcal{V}$ the following is true:

1.) if $m > 0$, then $\dfrac{F^{(1)}_{m,n}(\infty)}{F^{(2)}_{m,n}(\infty)} > 0$

2.) if $m < 0$, then $\dfrac{F^{(1)}_{m,n}(\infty)}{F^{(2)}_{m,n}(\infty)} < 0$

3.) if $m = 0$, then $F^{(1)}_{0,n}(\infty) = 0$, for $n \geq 0$, and $F^{(2)}_{0,n}(\infty) = 0$, for $n < 0$

4.) $\dfrac{F^{(1)}_{m,n}(\infty)}{|m| F^{(2)}_{m,n}(\infty)} \to 0$ as $|m| \to \infty$.

The boundary condition on D will be stated in a dense domain of D; it is

$$\mathrm{dom}(D) = \{F \in \mathcal{H} \ : \ \|DF\| < \infty, F|_{L^2(T^2_\theta) \otimes \mathbb{C}^2} \in \mathcal{V}\}. \tag{7.11}$$

The next proposition reformats the boundary condition in to a more applicable form.

Proposition 7.2.5 *The boundary conditions defined in equation (7.11) are equivalent to the following, if $m \neq 0$, then*

$$\frac{K^{(1)}_{m,n}(\infty)}{|m|K^{(2)}_{m,n}(\infty)} \to 0 \quad as \quad |m| \to \infty \ .$$

Moreover if $m > 0$ then one has $K^{(1)}_{m,n}(\infty) > 0$ and $K^{(2)}_{m,n}(\infty) > 0$ and if $m < 0$ one has $K^{(1)}_{m,n}(\infty) < 0$ and $K^{(2)}_{m,n}(\infty) > 0$. If $m = 0$ then the boundary condition is equivalent to $g_{0,n}(\infty) = 0$.

Notice in the above since the "negative" terms are not present the other condition for $m = 0$ is not present. One thing that needs to be discussed is the linear independence of the solutions $I_{m,n}(k)$ and $K_{m,n}(k)$. It will turn out that these solutions are independent.

Proposition 7.2.6 *For $m \neq 0$, the solutions $I_{m,n}(k)$ and $K_{m,n}(k)$ are linear independent.*

Proof First the case $m > 0$. Recall that one has

$$I_{m,n}(k) = \left(\prod_{i=0}^{k-1} C^{(m,n)}(i)\right) \begin{pmatrix} -1 \\ \frac{m}{a_1(0)} \end{pmatrix}$$

$$\text{and} \quad K_{m,n}(k) = \left(\prod_{i=0}^{k-1} C^{(m,n)}(i)\right) \begin{pmatrix} K^{(1)}_{m,n}(0) \\ K^{(2)}_{m,n}(0) \end{pmatrix} .$$

Using the formula for $I_{m,n}(k)$ and the proof of Proposition 7.2.2 one write out the components to get

$$I^{(1)}_{m,n}(k) = -\left(\prod_{i=0}^{k-1} \frac{1}{c^{(n)}_1(i)} + F^{k-1}_0(m^2) + \frac{m^2 F^{k-1}_1(m^2)}{a_1(0)}\right),$$

$$I^{(2)}_{m,n}(k) = mF^{k-1}_2(m^2) + \frac{m}{a_1(0)}\left(\prod_{i=0}^{k-1} c^{(n)}_2(i) + F^{k-1}_3(m^2)\right)$$

where $F_j^{k-1}(m^2)$, is the $(k-1)^{\text{th}}$ partial sum of the infinite series $F_j(m^2)$ for $j = 0, 1, 2, 3$. Since each series is positive for $m > 0$, in fact the series are positive for all $m \neq 0$, we see that $I^{(1)}_{m,n}(k)$ is negative and $I^{(2)}_{m,n}(k)$ is positive. Next for the $K_{m,n}(k)$ an alternate, yet equivalent, way to write the solution is

$$K_{m,n}(k) = \left(\prod_{i=k}^{\infty} C^{(m,n)}(i)\right)^{-1} K_{m,n}(\infty).$$

Recall that the components of $K_{m,n}(\infty)$ are both positive, this is the required boundary condition for $m > 0$, and the matrix $C^{(m,n)}(i)^{-1}$ has all positive entries since $m > 0$. Therefore mutliplying by a matrix with positive enties over and over again to a vector with positive components results in a vector with positive components. Hence $K^{(1)}_{m,n}(k)$ and $K^{(2)}_{m,n}(k)$ are positive. Therefore since one of the components of $I_{m,n}(k)$ is negative and both components of $K_{m,n}(k)$ are positive, it is impossible for them to be linear dependent. For the case $m < 0$ one does the same process above with slight adjustments to take into account that $m < 0$. In particular the matrix $C^{(m,n)}(i)^{-1}$ does not have all positive entries anymore and so the $K_{m,n}(k)$ solutions must be written out in a fashion similar to the $I_{m,n}(k)$ solutions. This will result in both components of $I_{m,n}(k)$ being negative and one of the components of $K_{m,n}(k)$ being positive again showing they can not be linear dependent. Thus the proof is complete. ■

Proposition 7.2.7 *The operator $\mathcal{A}^{(m,n)}$ subject to the equivalent boundary conditions in Proposition 7.2.5 has trivial kernel.*

Proof If $m \neq 0$ notice that one can write any element $x_{m,n}(k) \in \operatorname{Ker} \mathcal{A}^{(m,n)}$ as

$$x_{m,n}(k) = c_1^{(m,n)} I_{m,n}(k) + c_2^{(m,n)} K_{m,n}(k)$$

since by Proposition 7.2.6 they are linear independent. However if $k = 0$ then one gets for some constant, $c_3^{(m,n)}$ in k,

$$c_3^{(m,n)} I_{m,n}(0) = x_{m,n}(0) = c_1^{(m,n)} I_{m,n}(0) + c_2^{(m,n)} K_{m,n}(0)$$

which implies that $K_{m,n}(0)$ is a scalar multiple of $I_{m,n}(0)$, but this is impossible by the linear independence of the solutions, hence $c_2^{(m,n)} = 0$. Next if $k = \infty$ then one gets for some constant $c_4^{(m,n)}$ in k,

$$c_4^{(m,n)} K_{m,n}(\infty) = x_{m,n}(\infty) = c_1^{(m,n)} I_{m,n}(\infty) + c_2^{(m,n)} K_{m,n}(\infty)$$

which implies that $K_{m,n}(\infty)$ is a scalar multiple of $I_{m,n}(\infty)$, but again this is impossible by the linear independence of the solutions, hence $c_1^{(m,n)} = 0$. Therefore if $m \neq 0$ the kernel is trivial. Next is the case if $m = 0$ one can write the solution in the form

$$x_{0,n}(k) = \begin{pmatrix} 0 \\ \left(\prod_{i=k}^{\infty} \frac{1}{c_2^{(n)}(i)}\right) g_{0,n}(\infty) \end{pmatrix}.$$

However by Proposition 7.2.5 one has $g_{0,n}(\infty) = 0$, thus the kernel is trivial for $m = 0$. Therefore the proof is finished. ■

The kernel of D with the prescibed boundary conditions has now been eliminated. Since the kernel is trivial we now stand a chance in building an inverse to D and showing that it is compact. Constructing the parametrix will be the main discussion of the next section.

7.3 Parametrix to the quantum Dirac type operator

Now it time to discuss the non-homogeneuous equation $DF = G$ which leads to the parametrix of the quantum D. Once again by Proposition 7.1.4, this equation reduces to solving $\mathcal{A}^{(m,n)}(x, y)^t = (p, q)^t$ where again the dependence on m and n have been suppressed. Upon solving this system the special solutions, (7.9) and (7.10), will appear and more analysis will be discussed for them. This analysis is necessary for showing that the parametrix is compact. Before one can compute the parametrix we need to make a choice of a perpendicular vector. In other words, for a vector $x = (x_1, x_2)^t$ we take $x^{\perp} = (x_2, -x_1)^t$.

Proposition 7.3.1 *Let $\mathcal{A}^{(m,n)}$ be the finite differnce operator defined by equation (7.5), then $\mathcal{A}^{(m,n)}$ subject to the boundary conditions given by Propositions 7.2.4 and 7.2.5, is an invertible operator with inverse $Q^{(m,n)}$ given by (7.14) below.*

Proof One wishes to solve the equation $\mathcal{A}^{(m,n)}(x,y)^t = (p,q)^t$. Since the kernel of the quantum D with the boundary conditions applied to it is trivial and hence the kernel of $\mathcal{A}^{(m,n)}$ with these conditions will be trivial, there is a chance for the parametrix to exist and in fact it does. The goal is to solve the equation

$$\mathcal{A}^{(m,n)} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix},$$

which becomes the following difference equation with matrix coefficients

$$A^{(m,n)}(k+1) \left[\begin{pmatrix} x(k+1) \\ y(k+1) \end{pmatrix} - C^{(m,n)}(k) \begin{pmatrix} x(k) \\ y(k) \end{pmatrix} \right] = \begin{pmatrix} p(k) \\ -q(k+1) \end{pmatrix}$$

with $A^{(m,n)}(k+1)$ and $C^{(m,n)}(k)$ are the same as in formulas (7.6) and (7.7) respectively. Relabling $h(k) = (x(k), y(k))^t$ and $r_{m,n}(k+1) = (p_{m,n+1}(k), -q_{m,n}(k+1))^t$, the system becomes $A^{(m,n)}(k+1)(h(k+1) - C^{(m,n)}(k)h(k)) = r_{m,n}(k+1)$. Here the dependence on m and n have been reintroduced for tracking purposes. We will solve this for the case $m = 0$ and $m \neq 0$. These cases will be done separate since they will be solved in two different ways which will be needed for later analysis. In both formulas, for convenience, one sets the product $\prod_{j=k}^{k-1} C^{(m,n)}(j) = 1$ for any m and n. For the case $m = 0$, one solves recursively to get

$$Q^{(0,n)} r_{0,n}(k) = \sum_{i=0}^{k} \left(\prod_{j=i}^{k-1} C^{(0,n)}(j) \right) \left(A^{(0,n)}(i) \right)^{-1} r_{0,n}(i)$$

This form will be needed for estimating purposes on the parametrix $Q^{(0,n)}$ when $m = 0$. For the case $m \neq 0$ we solve the equation by using variation of constants to get

$$Q^{(m,n)}r_{m,n}(k) = \prod_{i=0}^{k-1} C^{(m,n)}(i) \sum_{i=0}^{k} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j) \right)^{-1} \left(A^{(m,n)}(i) \right)^{-1} r_{m,n}(i) + \alpha I_{m,n}(k)$$

for some parameter α. This form will be needed to figure out the parameter α so that one can write the parametrix in terms of our special solutions $I_{m,n}(k)$ and $K_{m,n}(k)$ and estimating the parametrix when $m \neq 0$. To see the full details on these to solution methods see [17]. To apply the boundary conditions in equation (7.11), one needs to know that $Q^{(m,n)}r_{m,n}(\infty)$ is well defined. Looking at the above formula for $Q^{(m,n)}r_{m,n}(k)$, using Proposition 7.2.1 and the summability of $A^{(m,n)}(k)^{-1}$ shows that the limit as $k \to \infty$ exists. Therefore applying the boundary condition, equation (7.11), one has

$$Q^{(m,n)}r_{m,n}(\infty) = C^{(m,n)} \sum_{i=0}^{\infty} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j) \right)^{-1} \left(A^{(m,n)}(i) \right)^{-1} r_{m,n}(i) + \alpha I_{m,n}(\infty)$$
$$= \beta K_{m,n}(\infty)$$

for the same α and some other constant β. Since one has formulas for $K_{m,n}(k)$ and $I_{m,n}(k)$, one has $K_{m,n}(\infty) = C^{(m,n)}K_{m,n}(0)$ and $I_{m,n}(\infty) = C^{(m,n)}I_{m,n}(0)$.

The goal is to solve for α, so one considers $K_{m,n}(0)^{\perp}$ and taking the inner product to both sides of the above equation with $K_{m,n}(0)^{\perp}$ then multiplying by $(C^{(m,n)})^{-1}$ one gets

$$\left\langle \sum_{i=0}^{\infty} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j) \right)^{-1} \left(A^{(m,n)}(i) \right)^{-1} r_{m,n}(i), K_{m,n}(0)^{\perp} \right\rangle + \alpha \left\langle I_{m,n}(0), K_{m,n}(0)^{\perp} \right\rangle = 0$$

which can now be solved for α to get

$$\alpha = \frac{-1}{\langle I_{m,n}(0), K_{m,n}(0)^{\perp} \rangle} \left\langle \sum_{i=0}^{\infty} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j) \right)^{-1} \left(A^{(m,n)}(i) \right)^{-1} r_{m,n}(i), K_{m,n}(0)^{\perp} \right\rangle .$$

This part of the goal is to write $Q^{(m,n)}$ as a linear combination of the special solutions. Assuming $I_{m,n}(0)$ and $K_{m,n}(0)^{\perp}$ are linear independent, which can be done since assume otherwise, then there exists a non-zero constant c such that $I_{m,n}(0) = cK_{m,n}(0)^{\perp}$ but this would say $\langle I_{m,n}(0), K_{m,n}(0)\rangle = 0$, but a simple calculation shows this is not possible; then by the projection theorem one has $x = x_1 I_{m,n}(0) + x_2 K_{m,n}(0)$ where

$$x_1 = \frac{\left\langle x, K_{m,n}(0)^{\perp}\right\rangle}{\langle I_{m,n}(0), K_{m,n}(0)^{\perp}\rangle} \qquad \text{and} \qquad x_2 = \frac{\left\langle x, I_{m,n}(0)^{\perp}\right\rangle}{\langle K_{m,n}(0), I_{m,n}(0)^{\perp}\rangle},$$

and

$$x = \sum_{i=0}^{k} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j)\right)^{-1} \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i).$$

Therefore using the formula for α, the above formula, and the formulas for $I_{m,n}(k)$ and $K_{m,n}(k)$, equations (7.9) and (7.10) respectively, one gets

$$\begin{aligned} Q^{(m,n)} r_{m,n}(k) &= \prod_{i=0}^{k-1} C^{(m,n)}(i) \\ &\times \left[\frac{-\left\langle \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j)\right)^{-1} \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i), K_{m,n}(0)^{\perp}\right\rangle}{\langle I_{m,n}(0), K_{m,n}(0)^{\perp}\rangle} I_{m,n}(0) \right. \\ &\left. + \frac{\left\langle \sum_{i=0}^{k} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j)\right)^{-1} \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i), I_{m,n}(0)^{\perp}\right\rangle}{\langle K_{m,n}(0), I_{m,n}(0)^{\perp}\rangle} K_{m,n}(0) \right] \\ &= c_1^{(m,n)}(k) I_{m,n}(k) + c_2^{(m,n)}(k) K_{m,n}(k) \end{aligned}$$

where

$$c_1^{(m,n)}(k) = \frac{-\left\langle \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j)\right)^{-1} \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i), K_{m,n}(0)^{\perp}\right\rangle}{\langle I_{m,n}(0), K_{m,n}(0)^{\perp}\rangle} \tag{7.12}$$

and

$$c_2^{(m,n)}(k) = \frac{\left\langle \sum_{i=0}^{k} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j)\right)^{-1} \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i), I_{m,n}(0)^{\perp}\right\rangle}{\langle K_{m,n}(0), I_{m,n}(0)^{\perp}\rangle}. \tag{7.13}$$

Thus one has the nice formula for $Q^{(m,n)}$, namely

$$
\begin{aligned}
Q^{(m,n)} r_{m,n}(k) &= c_1^{(m,n)}(k) I_{m,n}(k) + c_2^{(m,n)}(k) K_{m,n}(k) \quad & m \neq 0 \\
Q^{(m,n)} r_{0,n}(k) &= \sum_{i=0}^{k} \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{a_1(i)} \prod_{j=i}^{k-1} c_2^{(n)}(j) \end{pmatrix} r_{0,n}(i) \quad & m = 0
\end{aligned}
\tag{7.14}
$$

where the coefficients $c_1^{(m,n)}(k)$ and $c_2^{(m,n)}(k)$ are given by (7.12) and (7.13) respectively. All that remains to show is $\mathcal{A}^{(m,n)} Q^{(m,n)} = Q^{(m,n)} \mathcal{A}^{(m,n)} = id_{m,n}$. The case $m \neq 0$ amounts to showing that the formula for $Q^{(m,n)}$ is a solution to the original difference equation, thus this will be omitted. The case $m = 0$ amounts to showing that the formula for $Q^{(0,n)}$ is a solution to the original equation for $m = 0$, but this is easy since both (7.6) and (7.7) are diagonal in this case. Therefore the proof is complete. ■

The value $\langle K_{m,n}(0), I_{m,n}(0)^{\perp} \rangle$ will appear quite frequently throughout the rest of this paper, therefore it will be labeled. Denote this initial value as follows

$$
\tau := \langle K_{m,n}(0), I_{m,n}(0)^{\perp} \rangle \ . \tag{7.15}
$$

Since the special soutions solve a finite difference equation they will satisfy some type of recurrence relation. Analyzing the recurrence relations the special solutions satisfy is the next goal as these will be vital in proving the compactness of the parametrix $Q^{(m,n)}$. We first start with some more notation since the relations will be similar for each special solution. Let $H_{m,n}(k)$ be either $I_{m,n}(k)$ or $K_{m,n}(k)$ and $H_{m,n}(k) = (H_{m,n}^{(1)}(k), H_{m,n}^{(2)}(k))^t$ like at the end of the previous section, then by the formulas of these solutions it is easy to see that one has

$$
H_{m,n}(k+1) = C^{(m,n)}(k) H_{m,n}(k). \tag{7.16}
$$

This tells us some recurrence relations between the components of either $I_{m,n}(k)$ or $K_{m,n}(k)$. Indeed one has the following

$$\begin{aligned}
H^{(1)}_{m,n}(k+1) - \frac{1}{c^{(n)}_1(k)} H^{(1)}_{m,n}(k) &= -\frac{m}{a_2(k)c^{(n)}_2(k)} H^{(2)}_{m,n}(k) \\
H^{(2)}_{m,n}(k+1) - c^{(n)}_2(k) H^{(2)}_{m,n}(k) &= -\frac{m}{a_1(k+1)c^{(n)}_1(k)} H^{(1)}_{m,n}(k) \\
&\quad + \frac{m^2}{a_2(k)a_1(k+1)c^{(n)}_1(k)} H^{(2)}_{m,n}(k).
\end{aligned} \tag{7.17}$$

Then using the relation

$$H_{m,n}(k) = (C^{(m,n)}(k))^{-1} H_{m,n}(k+1),$$

one can produce two more equations to get

$$\begin{aligned}
H^{(2)}_{m,n}(k) - \frac{1}{c^{(n)}_2(k)} H^{(2)}_{m,n}(k+1) &= \frac{m}{a_1(k+1)c^{(n)}_2(k)} H^{(1)}_{m,n}(k+1) \\
H^{(1)}_{m,n}(k) - c^{(n)}_1(k) H^{(1)}_{m,n}(k+1) &= \frac{m}{a_2(k)c^{(n)}_2(k)} H^{(2)}_{m,n}(k+1) \\
&\quad + \frac{m^2}{a_2(k)a_1(k+1)c^{(n)}_2(k)} H^{(1)}_{m,n}(k+1).
\end{aligned} \tag{7.18}$$

The next lemma is an algebraic lemma that mainly uses tools from linear algebra to write the parametrix into a form more suitable to estimate the Hilbert-Schmidt norm. Computing the Hilbert-Schmidt norm is necessary to show that parametrix, Q, is compact since it will turn out that Q is essentially a direct sum of Hilbert-Schmidt operators with norms going to zero as $|m|, n \to \infty$. Before the lemma is stated a few integral operators will be defined and the lemma will show that $Q^{(m,n)}$ will be a sum of these integral operators.

For $\alpha, \beta = 1, 2$, $m \neq 0$, and a sequence of numbers $\{r(k)\}$, define the following integral operators:

$$\begin{aligned}
X^{\alpha\beta}_{m,n} r(k) &= I^{(\alpha)}_{m,n}(k) \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-\beta} \frac{c^{(n)}_1(j)}{c^{(n)}_2(j)} \right) \frac{K^{(\beta)}_{m,n}(i-\beta+1)}{a_\beta(i-\beta+1)} r(i) \\
Y^{\alpha\beta}_{m,n} r(k) &= K^{(\alpha)}_{m,n}(k) \sum_{i=0}^{k} \left(\prod_{j=0}^{i-\beta} \frac{c^{(n)}_1(j)}{c^{(n)}_2(j)} \right) \frac{I^{(\beta)}_{m,n}(i-\beta+1)}{a_\beta(i-\beta+1)} r(i)
\end{aligned} \tag{7.19}$$

where $X^{\alpha\beta}_{m,n}, Y^{\alpha\beta}_{m,n} : \ell^2_{a_\beta}(\mathbb{N}) \to \ell^2_{a_\alpha}(\mathbb{N})$. Also for $\alpha, \beta = 1, 2$ and $m = 0$ define another integral operator by

$$Z_{0,n} r(k) = \sum_{i=0}^{k} \left(\prod_{j=i}^{k-1} c_2^{(n)}(j) \right) \frac{r(i)}{a_1(i)} \tag{7.20}$$

where $Z_{0,n} : \ell^2_{a_1}(\mathbb{N}) \to \ell^2_{a_2}(\mathbb{N})$.

Lemma 7.3.2 *The parametrix, $Q^{(m,n)}$ for the operator $\mathcal{A}^{(m,n)}$ from above for $m \neq 0$ has the following equivalent formula:*

$$Q^{(m,n)} \begin{pmatrix} r^{(1)}_{m,n}(k) \\ r^{(2)}_{m,n}(k) \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} p^{(1)}_{m,n}(k) \\ p^{(2)}_{m,n}(k) \end{pmatrix}$$

where

$$\begin{aligned} p^{(1)}_{m,n}(k) &= X^{12}_{m,n} r^{(1)}_{m,n}(k) + Y^{12}_{m,n} r^{(1)}_{m,n}(k) + X^{11}_{m,n} r^{(2)}_{m,n}(k) + Y^{11}_{m,n} r^{(2)}_{m,n}(k) \\ p^{(2)}_{m,n}(k) &= X^{22}_{m,n} r^{(1)}_{m,n}(k) + Y^{22}_{m,n} r^{(1)}_{m,n}(k) + X^{21}_{m,n} r^{(2)}_{m,n}(k) + Y^{21}_{m,n} r^{(2)}_{m,n}(k) \ . \end{aligned}$$

Moreover when $m = 0$ the parametrix has the following formula:

$$Q^{(0,n)} \begin{pmatrix} r^{(1)}_{0,n}(k) \\ r^{(2)}_{0,n}(k) \end{pmatrix} = \begin{pmatrix} 0 \\ Z_{0,n} r^{(2)}_{0,n}(i) \end{pmatrix} \ .$$

Proof One starts with the case $m \neq 0$. One first needs to establish several little facts in linear algebra that we will use for the main argument. The first fact is for vectors. If one has vectors $x = (x_1, x_2)^t$ and $y = (y_1, y_2)^t$, then $\langle x^\perp, y\rangle = -\langle x, y^\perp\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean inner product on $\mathbb{R}^2$. This fact is easily seen to be true by just writing out the formula.

The next fact is for a 2×1 vector and a 2×2 matrix. Again the fact follows by just doing the computation so it will be omitted. If R is a 2×2 matrix and $x = (x_1, x_2)^t$, then one has $(Rx)^\perp = (\det\ R) \left(R^t\right)^{-1} x^\perp$. The next item is to write out a formula for the perpendiculars to the special solutions. Recall our special solutions were given by equations (7.9) and (7.10), then indeed one has using the second fact and the formula for the determinant of the $C^{(m,n)}(k)$ matrix

$$
\begin{aligned}
I_{m,n}(k)^{\perp} &= \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)} \left[\left(\prod_{i=0}^{k-1} C^{(m,n)}(i) \right)^{-1} \right]^t I_{m,n}(0)^{\perp} \\
K_{m,n}(k)^{\perp} &= \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)} \left[\left(\prod_{i=0}^{k-1} C^{(m,n)}(i) \right)^{-1} \right]^t K_{m,n}(0)^{\perp}.
\end{aligned} \tag{7.21}
$$

Now one can begin showing the result of this lemma. Recall one has

$$
\begin{aligned}
Q^{(m,n)} r_n(k) &= c_1^{(m,n)}(k) I_{m,n}(k) + c_2^{(m,n)}(k) K_{m,n}(k) \\
&= (I_{m,n}(k), K_{m,n}(k)) \begin{pmatrix} c_1^{(m,n)}(k) \\ c_2^{(m,n)}(k) \end{pmatrix} \\
&= \begin{pmatrix} I_{m,n}^{(1)}(k) & K_{m,n}^{(1)}(k) \\ I_{m,n}^{(2)}(k) & K_{m,n}^{(2)}(k) \end{pmatrix} \begin{pmatrix} c_1^{(m,n)}(k) \\ c_2^{(m,n)}(k) \end{pmatrix}.
\end{aligned}
$$

Next one works with the vector of the cees. Only the argument for the $c_1^{(m,n)}(k)$ will be shown as the other is completely anagolous. The goal is to transform the vector of cees, the coefficient vector, into a matrix times the vector $r_{m,n}(i)$ and this is done by manipulating the inner product. Recall that one has

$$
\begin{aligned}
c_1^{(m,n)}(k) &= \frac{-\left\langle \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-1} C^{(m,n)}(j)\right)^{-1} \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i), K_{m,n}(0)^{\perp} \right\rangle}{\langle I_{m,n}(0), K_{m,n}(0)^{\perp}\rangle} \\
&= \frac{1}{\tau} \sum_{i=k+1}^{\infty} \left\langle r_{m,n}(i), \left[\left(A^{(m,n)}(i)\right)^{-1}\right]^t \left[\left(\prod_{j=0}^{i-1} C^{(m,n)}(j)\right)^{-1}\right]^t K_{m,n}(0)^{\perp} \right\rangle \\
&= \frac{1}{\tau} \sum_{i=k+1}^{\infty} \left\langle r_{m,n}(i), \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} \left[\left(A^{(m,n)}(i)\right)^{-1}\right]^t K_{m,n}(i)^{\perp} \right\rangle \\
&= \frac{1}{\tau} \sum_{i=k+1}^{\infty} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} \left\langle \left[\left(A^{(m,n)}(i)\right)^{-1}\right]^t \begin{pmatrix} K_{m,n}^{(2)}(i) \\ -K_{m,n}^{(1)}(i) \end{pmatrix}, r_{m,n}(i) \right\rangle \\
&= \frac{1}{\tau} \sum_{i=k+1}^{\infty} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} \left(\left[\left(A^{(m,n)}(i)\right)^{-1}\right]^t \begin{pmatrix} K_{m,n}^{(2)}(i) \\ -K_{m,n}^{(1)}(i) \end{pmatrix} \right)^t r_{m,n}(i) \\
&= \frac{1}{\tau} \sum_{i=k+1}^{\infty} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} (K_{m,n}^{(2)}(i), -K_{m,n}^{(1)}(i)) \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i).
\end{aligned}
$$

Similarly one also has

$$
c_2^{(m,n)}(k) = \frac{1}{\tau} \sum_{i=0}^{k} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} (I_{m,n}^{(2)}(i), -I_{m,n}^{(1)}(i)) \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i).
$$

These are to be understood as a row of a matrix times a column vector. Realizing this one gets

$$
Q^{(m,n)} r_n(k) = \frac{1}{\tau} \begin{pmatrix} I_{m,n}^{(1)}(k) & K_{m,n}^{(1)}(k) \\ I_{m,n}^{(2)}(k) & K_{m,n}^{(2)}(k) \end{pmatrix} \times
$$
$$
\begin{pmatrix} \sum_{i=k+1}^{\infty} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} K_{m,n}^{(2)}(i) & -\sum_{i=k+1}^{\infty} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} K_{m,n}^{(1)}(i) \\ \sum_{i=0}^{k} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} I_{m,n}^{(2)}(i) & -\sum_{i=0}^{k} \prod_{j=0}^{i-1} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)} I_{m,n}^{(1)}(i) \end{pmatrix} \left(A^{(m,n)}(i)\right)^{-1} r_{m,n}(i).
$$

Next we focus on multiplying the second matrix with $\left(A^{(m,n)}(i)\right)^{-1}$. The second column is a straight forward calculation, however the first column is a bit more complex. Both entries require the use of the recurrence relations, (7.17) and (7.18).

Only the first entry will be shown, i.e. the top left corner entry of the matrix, since both are similar. Indeed the top left entry is equal to

$$\begin{aligned}
&\sum_{i=k+1}^{\infty}\prod_{j=0}^{i-1}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\left(\frac{K_{m,n}^{(2)}(i)}{a_2(i-1)c_1^{(n)}(i-1)}+\frac{mK_{m,n}^{(1)}(i)}{a_2(i-1)a_1(i)c_1^{(n)}(i-1)}\right)\\
&=\frac{1}{m}\sum_{i=k+1}^{\infty}\prod_{j=0}^{i-2}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\left(\frac{mK_{m,n}^{(2)}(i)}{a_2(i-1)c_2^{(n)}(i-1)}+\frac{m^2K_{m,n}^{(1)}(i)}{a_2(i-1)a_1(i)c_2^{(n)}(i-1)}K_{m,n}^{(1)}(i)\right)\\
&=-\frac{1}{m}\sum_{i=k+1}^{\infty}\prod_{j=0}^{i-2}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\left(c_1^{(n)}(i-1)K_{m,n}^{(1)}(i)-K_{m,n}^{(1)}(i-1)\right)
\end{aligned}$$

where the last equality comes from recurrence relation (7.18). Then by using recurrence relation (7.17), the above is equal to

$$\frac{1}{m}\sum_{i=k+1}^{\infty}\prod_{j=0}^{i-2}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\frac{m}{a_2(i-1)}K_{m,n}^{(2)}(i-1)=\sum_{i=k+1}^{\infty}\prod_{j=0}^{i-2}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\frac{1}{a_2(i-1)}K_{m,n}^{(2)}(i-1).$$

Then after computing the other entries one gets

$$Q^{(m,n)}r_n(k)=\frac{1}{\tau}\begin{pmatrix} I_{m,n}^{(1)}(k) & K_{m,n}^{(1)}(k)\\ I_{m,n}^{(2)}(k) & K_{m,n}^{(2)}(k)\end{pmatrix}\times$$
$$\begin{pmatrix} \sum_{i=k+1}^{\infty}\prod_{j=0}^{i-2}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\frac{1}{a_2(i-1)}K_{m,n}^{(2)}(i-1) & \sum_{i=k+1}^{\infty}\prod_{j=0}^{i-1}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\frac{1}{a_1(i)}K_{m,n}^{(1)}(i)\\ \sum_{i=0}^{k}\prod_{j=0}^{i-2}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\frac{1}{a_2(i-1)}I_{m,n}^{(2)}(i-1) & \sum_{i=0}^{k}\prod_{j=0}^{i-1}\frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\frac{1}{a_1(i)}I_{m,n}^{(1)}(i)\end{pmatrix} r_{m,n}(i).$$

Multipling out the two matrices and applying it to the vector $r_{m,n}(i)$, and using the definitions of the integral operators, equation (7.19) will show the desired result. The case $m = 0$ immediately follows from equations (7.14) and (7.20). Thus the proof is complete. ■

The next section will contain all the estimates of $Q^{(m,n)}$ and show that it is indeed a Hilbert-Schmidt operator with decresing norms. At the very end, as stated in the introduction, the section will contain the proof of the main theorem of this chapter.

7.4 Analysis of the Parametrix

This section will contain a series of lemmas and propositions to get the final estimates on $Q^{(m,n)}$ and thus the ability to prove the main result of this chapter. Other than the boundary condition, there is a distinction between positive m and negative m. From this point on the two cases will be considered as different things happen. In fact only the case $m > 0$ will really be addressed with minor details about the case $m < 0$, since the computations are virtually identical with some minor adjustments. First and for most we set

$$I^{(1)}_{m,n}(k) = \begin{cases} -I^{(1)}_{m,n}(k) & m > 0 \\ -I^{(1)}_{m,n}(k) & m < 0 \end{cases} \quad \text{and} \quad I^{(2)}_{m,n}(k) = \begin{cases} I^{(2)}_{m,n}(k) & m > 0 \\ -I^{(2)}_{m,n}(k) & m < 0 \end{cases}.$$

We also set

$$K^{(1)}_{m,n}(k) = \begin{cases} K^{(1)}_{m,n}(k) & m > 0 \\ -K^{(1)}_{m,n}(k) & m < 0 \end{cases} \quad \text{and} \quad K^{(2)}_{m,n}(k) = \begin{cases} K^{(2)}_{m,n}(k) & m > 0 \\ K^{(2)}_{m,n}(k) & m < 0 \end{cases}.$$

Each solution will have their own recurrence relations that stem from the generic ones, see equations (7.17) and (7.18). For $m > 0$ the $I_{m,n}(k)$ solutions have the following relations:

$$\begin{aligned} I^{(1)}_{m,n}(k+1) - \frac{1}{c^{(n)}_1(k)} I^{(1)}_{m,n}(k) &= \frac{m}{a_2(k)c^{(n)}_1(k)} I^{(2)}_{m,n}(k) \\ I^{(2)}_{m,n}(k+1) - c^{(n)}_2(k) I^{(2)}_{m,n}(k) &= \frac{m}{a_1(k+1)c^{(n)}_1(k)} I^{(1)}_{m,n}(k) \\ &\quad + \frac{m^2}{a_1(k+1)a_2(k)c^{(n)}_1(k)} I^{(2)}_{m,n}(k) \\ \frac{1}{c^{(n)}_2(k)} I^{(2)}_{m,n}(k+1) - I^{(2)}_{m,n}(k) &= \frac{m}{a_1(k+1)c^{(n)}_2(k)} I^{(1)}_{m,n}(k+1) \\ c_1(k) I^{(1)}_{m,n}(k+1) - I^{(1)}_{m,n}(k) &= \frac{-m}{a_2(k)c^{(n)}_2(k)} I^{(2)}_{m,n}(k+1) \\ &\quad + \frac{m^2}{a_1(k+1)a_2(k)c^{(n)}_2(k)} I^{(1)}_{m,n}(k+1). \end{aligned} \tag{7.22}$$

Also for $m > 0$ the $K_{m,n}(k)$ solutions have the following relations:

$$
\begin{aligned}
\frac{1}{c_1^{(n)}(k)}K_{m,n}^{(1)}(k) - K_{m,n}^{(1)}(k+1) &= \frac{m}{a_2(k)c_1^{(n)}(k)}K_{m,n}^{(2)}(k) \\
c_2^{(n)}(k)K_{m,n}^{(2)}(k) - K_{m,n}^{(2)}(k+1) &= \frac{m}{a_1(k+1)c_1^{(n)}(k)}K_{m,n}^{(1)}(k) \\
&\quad - \frac{m^2}{a_1(k+1)a_2(k)c_1^{(n)}(k)}K_{m,n}^{(2)}(k) \\
K_{m,n}^{(2)}(k) - \frac{1}{c_2^{(n)}(k)}K_{m,n}^{(2)}(k+1) &= \frac{m}{a_1(k+1)c_2^{(n)}(k)}K_{m,n}^{(1)}(k+1) \\
K_{m,n}^{(1)}(k) - c_1^{(n)}(k)K_{m,n}^{(1)}(k+1) &= \frac{m}{a_2(k)c_2^{(n)}(k)}K_{m,n}^{(2)}(k+1) \\
&\quad + \frac{m^2}{a_1(k+1)a_2(k)c_2^{(n)}(k)}K_{m,n}^{(1)}(k+1).
\end{aligned}
\tag{7.23}
$$

As stated before one would like to compute the Hilbert-Schmidt norm; also it would be useful to compute it a similar way done in [20]. However there, the authors established some positivity, increasing, and decreasing facts about the special solutions. Here one would like to establish the same kind of facts, except now we're interested in the components of the special solutions since the special soluions are vector valued. These facts will be based on if m is positive or negative. We first assume m is positive, then we will see that it is completely analogous if m is negative and to keep brevity, we will omit most of those proofs. The upcoming lemma will discuss the positivity of the components to the special solutions.

Lemma 7.4.1 *If $m \neq 0$, then $I_{m,n}^{(1)}(k)$, $I_{m,n}^{(2)}(k)$, $K_{m,n}^{(1)}(k)$, and $K_{m,n}^{(2)}(k)$ are all positive.*

Proof Using the proof of Proposition 7.2.6 and the above definitions will show the desired result. Thus the proof is complete. ■

Now it is time to establish whether or not the components to the special solutions are either increasing or decreasing. We are interested in this since in the classical case the authors used this knowledge about the special solutions to calculate esitimates for those solutions that played a role in estimating the parametrix to the classical Dirac operator. It is expected that these types of estimates will be used here as well to estimate the parametrix to the quantum Dirac operator. Using the previous lemma and the help of the recurrence relations for the components, this can be tackled.

Lemma 7.4.2 *If $m \neq 0$, then one has the following inequalities for all k:*

$$I^{(1)}_{m,n}(k) < I^{(1)}_{m,n}(k+1) \ , \qquad I^{(2)}_{m,n}(k) < \frac{1}{c^{(n)}_2(k)} I^{(2)}_{m,n}(k+1) \ ,$$
$$K^{(1)}_{m,n}(k+1) < \frac{1}{c^{(n)}_1(k)} K^{(1)}_{m,n}(k) \ , \quad K^{(2)}_{m,n}(k+1) < K^{(2)}_{m,n}(k) \ .$$

Proof We first start with the $K_{m,n}(k)$ solution. Using recurrence relation (7.23), the fact that $m > 0$ and the previous lemma one has

$$0 < \frac{m}{a_1(k+1)} K^{(1)}_{m,n}(k+1) = c^{(n)}_2(k) K^{(2)}_{m,n}(k) - K^{(2)}_{m,n}(k+1).$$

Using the above and the fact that $c^{(n)}_2(k) \leq 1$ one has

$$K^{(2)}_{m,n}(k+1) < c^{(n)}_2(k) K^{(2)}_{m,n}(k) \leq K^{(2)}_{m,n}(k)$$

which implies desired inequality. Also using recurrence relation (7.23), the fact that $m > 0$ and the previous lemma one has

$$0 < \frac{m}{a_2(k) c^{(n)}_1(k)} K^{(2)}_{m,n}(k) = \frac{1}{c^{(n)}_1(k)} K^{(1)}_{m,n}(k) - K^{(1)}_{m,n}(k+1)$$

implying the other inequality. The proofs for the $I_{m,n}(k)$ are similar, except one uses the recurrence relation (7.22). If $m < 0$, then one uses again the same recurrence relations that are tailored to the specific $I_{m,n}(k)$ and $K_{m,n}(k)$ solutions respectively. Therefore the desired result has been shown. ■

Then next lemma establishes estimates on the components of the special solutions in a similar fashion to the estimates of the modified Bessel functions in terms of their indices shown in [20]. Let

$$\varepsilon(m,n) = \sum_{k=0}^{\infty} \frac{a_2(k)}{m^2 + a_1(k) a_2(k)} \ .$$

Lemma 7.4.3 *For $m \neq 0$ the following inequalities hold:*

$$I^{(2)}_{m,n}(k) \leq |m|\varepsilon(m,n) I^{(1)}_{m,n}(k+1)$$
$$K^{(1)}_{m,n}(k+1) \leq |m| \left(\varepsilon(m,n) + \frac{K^{(1)}_{m,n}(\infty)}{|m| K^{(2)}_{m,n}(\infty)} \right) K^{(2)}_{m,n}(k) \ .$$

Moreover $\varepsilon(m,n) \to 0$ as $|m|, n \to \infty$.

Proof First one considers the case $m > 0$. Using the recurrence relations (7.22) one has

$$\begin{aligned}
\frac{I^{(2)}_{m,n}(k)}{I^{(1)}_{m,n}(k+1)} &= \frac{c^{(n)}_1(k)}{\frac{m}{a_2(k)} + \frac{1}{\frac{m}{a_1(k)} + c^{(n)}_2(k-1) \frac{I^{(2)}_{m,n}(k-1)}{I^{(1)}_{m,n}(k)}}} \\
&= \frac{c^{(n)}_1(k) \left(\frac{m}{a_1(k)} + c^{(n)}_2(k-1) \frac{I^{(2)}_{m,n}(k-1)}{I^{(1)}_{m,n}(k)} \right)}{1 + \frac{m^2}{a_1(k)a_2(k)} + c^{(n)}_2(k-1) \frac{m}{a_2(k)} \frac{I^{(2)}_{m,n}(k-1)}{I^{(1)}_{m,n}(k)}} \\
&\leq c^{(n)}_1(k) \left(\frac{\frac{m}{a_1(k)}}{1 + \frac{m^2}{a_1(k)a_2(k)}} + c^{(n)}_2(k-1) \frac{I^{(2)}_{m,n}(k-1)}{I^{(2)}_{m,n}(k)} \right) \\
&\leq \frac{\frac{m}{a_1(k)}}{1 + \frac{m^2}{a_1(k)a_2(k)}} + \frac{I^{(2)}_{m,n}(k-1)}{I^{(2)}_{m,n}(k)}
\end{aligned}$$

since one has $c^{(n)}_1(k) \leq 1$ and $c^{(n)}_2(k) \leq 1$ for all k and n. Rearranging the terms it follows that

$$\frac{1}{m} \left(\frac{I^{(2)}_{m,n}(k)}{I^{(1)}_{m,n}(k+1)} - \frac{I^{(2)}_{m,n}(k-1)}{I^{(1)}_{m,n}(k)} \right) \leq \frac{\frac{1}{a_1(k)}}{1 + \frac{m^2}{a_1(k)a_2(k)}} \ .$$

Suming both sides and telescoping the left side one gets

$$\frac{1}{m} \left(\frac{I^{(2)}_{m,n}(k)}{I^{(1)}_{m,n}(k+1)} \right) \leq \sum_{k=0}^{\infty} \frac{\frac{1}{a_1(k)}}{1 + \frac{m^2}{a_1(k)a_2(k)}} = \varepsilon(m,n).$$

From this the first inequality follows. To obtain the second one, a similar argument will work except one will use the recurrence relation (7.23) which is tailored to the $K_{m,n}(k)$ solution and the telescoping sum will produce the ratio at $k = \infty$. The next step is to show $\varepsilon(m,n)$ goes to zero as m and n go to infinity. First notice that one has immediately that

$$\varepsilon(m,n) \leq \sum_{k=0}^{\infty} \frac{1}{a_1(k)}$$

where the sum on the right goes to zero as $n \to \infty$ from the condition on $a_1(k)$ and hence $\varepsilon(m,n) \to 0$ as $n \to \infty$. Now for any $\eta > 0$ pick $N > 0$ such that one has

$$\sum_{k>N} \frac{1}{a_1(k)} \leq \frac{\eta}{2} \ ,$$

and pick $M > 0$ such that

$$\sum_{k \leq N} \frac{\frac{1}{a_1(k)}}{1 + \frac{m^2}{a_1(k)a_2(k)}} \leq \frac{\eta}{2}$$

for $m > M$. It now follows that $\varepsilon(m,n) \to 0$ as $m \to \infty$. To do the case $m < 0$ case, one follows the same type of analysis done above. Thus the desired result follows. ■

The following result will produce integral estimates on the components to special solution $K_{m,n}(k)$ through their respective opposite component, i.e. an integral estimate of $K^{(1)}_{m,n}(k)$ through $K^{(2)}_{m,n}(k)$.

Lemma 7.4.4 *The following summation estimates are true for $m \neq 0$:*

$$\sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-2} c_1^{(n)}(j) \right) \frac{K^{(2)}_{m,n}(i-1)}{a_2(i-1)} \leq \frac{1}{|m|} \prod_{j=0}^{k-1} c_1^{(n)}(j) K^{(1)}_{m,n}(k)$$
$$\sum_{i=k+1}^{\infty} \frac{K^{(1)}_{m,n}(i)}{a_1(i)} \leq \frac{1}{|m|} K^{(2)}_{m,n}(k) \ .$$

Proof First is the case $m > 0$. The second inequality falls directly from the recurrence relation for the $K_{m,n}(k)$ special solution, i.e. equation (7.23) and realizing one has a telescoping sum. The first will come from equation (7.23) but the difference side is not quite telescoping and needs a little work. Using equation (7.23) and multiplying both sides of the equation by a product of $c_1^{(n)}(j)$'s one has

$$\sum_{i=k+1}^{\infty}\left(\prod_{j=0}^{i-2}c_1^{(n)}(j)\right)\frac{K_{m,n}^{(2)}(i-1)}{a_2(i-1)}$$
$$=\frac{1}{m}\sum_{i=k+1}^{\infty}\left(\prod_{j=0}^{i-2}c_1^{(n)}(j)K_{m,n}^{(1)}(i-1)-\prod_{j=0}^{i-1}c_1^{(n)}(j)K_{m,n}^{(1)}(i)\right)$$
$$\leq\frac{1}{m}\prod_{j=0}^{k-1}c_1^{(n)}(j)K_{m,n}^{(1)}(k)$$

where the last inequality is true because the difference above it is now a telescoping sum. As always, the case $m < 0$ is similar. This completes the proof. ■

Throughout the proof of the compactness of the parametrix in chapter 6, one needed to know how the product of the two special solutions behaved, in other words they needed to know if the product was increasing or decreasing. It will turn out that we need to know similar information for the product of the components of the special solutions in the present case. This brings us to the next lemma.

Lemma 7.4.5 *Let $n \geq 0$. If $m \neq 0$ then*

$$K_{m,n}^{(1)}(k)I_{m,n}^{(2)}(k)\leq\tau\prod_{i=0}^{k-1}\frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}\ \textit{and}\ K_{m,n}^{(2)}(k)I_{m,n}^{(1)}(k)\leq\tau\prod_{i=0}^{k-1}\frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}$$
$$I_{m,n}^{(1)}(k+1)K_{m,n}^{(2)}(k)\leq\frac{\tau}{c_1(k)}\prod_{i=0}^{k-1}\frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}\ \textit{and}\ I_{m,n}^{(2)}(k)K_{m,n}^{(1)}(k+1)\leq\frac{\tau}{c_1^{(n)}(k)}\prod_{i=0}^{k-1}\frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}\ .$$

Proof There are two cases to consider: $m > 0$ and $m < 0$. We start with the case $m > 0$. Using the matrix valued recurrance relation (7.16) and properties of the inner product one gets

$$\langle K_{m,n}(k+1),I_{m,n}(k+1)^{\perp}\rangle$$
$$=\prod_{i=0}^{k}\frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}\left\langle\prod_{i=0}^{k}C^{(m,n)}(i)K_{m,n}(0),\prod_{i=0}^{k}\left(C^{(m,n)}(i)^{-1}\right)^{t}I_{m,n}(0)^{\perp}\right\rangle$$
$$=\prod_{i=0}^{k}\frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}\langle K_{m,n}(0),I_{m,n}(0)^{\perp}\rangle=\prod_{i=0}^{k}\frac{c_2(i)}{c_1(i)}\tau.$$

Writing out the above inner product gives

$$\begin{aligned}\langle K_{m,n}(k+1), I_{m,n}(k+1)^{\perp}\rangle &= K^{(1)}_{m,n}(k+1)I^{(2)}_{m,n}(k+1) + K^{(2)}_{m,n}(k+1)[-I^{(1)}_{m,n}(k+1)]\\ &= K^{(1)}_{m,n}(k+1)I^{(2)}_{m,n}(k+1) + K^{(2)}_{m,n}(k+1)I^{(1)}_{m,n}(k+1)\end{aligned}$$

when $m > 0$. Then using this and the above equality implies that

$$K^{(1)}_{m,n}(k)I^{(2)}_{m,n}(k) \leq \tau \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)} \quad \text{and} \quad K^{(2)}_{m,n}(k)I^{(1)}_{m,n}(k) \leq \tau \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}.$$

The case when $m < 0$ is similar. Writing out the same inner product formula as above gives

$$\begin{aligned}\langle K_{m,n}(k+1), I_{m,n}(k+1)^{\perp}\rangle &= K^{(1)}_{m,n}(k+1)I^{(2)}_{m,n}(k+1) + K^{(2)}_{m,n}(k+1)[-I^{(1)}_{m,n}(k+1)]\\ &= K^{(1)}_{m,n}(k+1)I^{(2)}_{m,n}(k+1) + K^{(2)}_{m,n}(k+1)I^{(1)}_{m,n}(k+1)\end{aligned}$$

when $m < 0$. Then using this and the equality at the beginning of the proof gives

$$K^{(1)}_{m,n}(k)I^{(2)}_{m,n}(k) \leq \tau \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)} \quad \text{and} \quad K^{(2)}_{m,n}(k)I^{(1)}_{m,n}(k) \leq \tau \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)}.$$

To get the second set of inequalities one uses the above equation and using recurrence relations (7.22) and (7.23) for $m > 0$ to get

$$\begin{aligned}\tau \prod_{i=0}^{k-1} \frac{c_2(i)}{c_1^{(n)}(i)} &= I^{(1)}_{m,n}(k)K^{(2)}_{m,n}(k) + I^{(2)}_{m,n}(k)K^{(1)}_{m,n}(k)\\ &= \left(c_1^{(n)}(k)I^{(1)}_{m,n}(k+1) - \frac{m}{a_2(k)}I^{(2)}_{m,n}(k)\right)K^{(2)}_{m,n}(k)\\ &\qquad + I^{(2)}_{m,n}(k)\left(c_1^{(n)}(k)K^{(1)}_{m,n}(k+1) + \frac{m}{a_2(k)}K^{(2)}_{m,n}(k)\right)\\ &= c_1^{(n)}(k)\left(I^{(1)}_{m,n}(k+1)K^{(2)}_{m,n}(k) + I^{(2)}_{m,n}(k)K^{(1)}_{m,n}(k+1)\right) .\end{aligned}$$

From this last equality it follows that

$$I^{(1)}_{m,n}(k+1)K^{(2)}_{m,n}(k) \leq \frac{\tau}{c_1^{(n)}(k)} \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)} \text{ and } I^{(2)}_{m,n}(k)K^{(1)}_{m,n}(k+1) \leq \frac{\tau}{c_1^{(n)}(k)} \prod_{i=0}^{k-1} \frac{c_2^{(n)}(i)}{c_1^{(n)}(i)} .$$

If $m < 0$ one does the same trick using the recurrence relations keeping track that $m < 0$ now. Thus the result follows. ■

The next item to discuss is the computations of the Hilbert-Schmidt norms of the integral operators $X^{\alpha\beta}_{m,n}$, $Y^{\alpha\beta}_{m,n}$, $Z^{12}_{0,n}$ and $Z^{21}_{0,n}$. Computing these are necessary to estimate the Hilbert-Schmidt norm of $Q^{(m,n)}$.

Proposition 7.4.6 *If $m \neq 0$, then the integral operators $X^{\alpha\beta}_{m,n}$ and $Y^{\alpha\beta}_{m,n}$ defined in equation (7.19) are Hilbert-Schmidt operators for $\alpha, \beta = 1, 2$. Moreover if $m = 0$ the integral operators $Z^{12}_{0,n}$ and $Z^{21}_{0,n}$ defined in equation (7.20) are Hilbert-Schmidt operators.*

Proof We start with the case $m \neq 0$. Using the definition of $X^{\alpha\beta}_{m,n}$ and $Y^{\alpha\beta}_{m,n}$, it's easy to see that for $\alpha, \beta = 1, 2$ one has

$$
\begin{aligned}
\|X^{\alpha\beta}_{m,n}\|^2_{HS} &= \sum_{k=0}^{\infty} \frac{\left(I^{(\alpha)}_{m,n}(k)\right)^2}{a_\alpha(k)} \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-\beta} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\right)^2 \cdot \frac{\left(K^{(\beta)}_{m,n}(i-\beta+1)\right)^2}{a_\beta(i-\beta+1)} \\
\|Y^{\alpha\beta}_{m,n}\|^2_{HS} &= \sum_{k=0}^{\infty} \frac{\left(K^{(\alpha)}_{m,n}(k)\right)^2}{a_\alpha(k)} \sum_{i=0}^{k} \left(\prod_{j=0}^{i-\beta} \frac{c_1^{(n)}(j)}{c_2^{(n)}(j)}\right)^2 \cdot \frac{\left(I^{(\beta)}_{m,n}(i-\beta+1)\right)^2}{a_\beta(i-\beta+1)} \ .
\end{aligned}
\tag{7.24}
$$

There are eight sums to estimate however it can be reduced to four since by Fubini's Theorem one has

$$
\begin{aligned}
\|X^{11}_{m,n}\|^2_{HS} = \|Y^{11}_{m,n}\|^2_{HS}, &\quad \|X^{22}_{m,n}\|^2_{HS} = \|Y^{22}_{m,n}\|^2_{HS} \\
\|X^{12}_{m,n}\|^2_{HS} = \|Y^{21}_{m,n}\|^2_{HS}, &\quad \|X^{21}_{m,n}\|^2_{HS} = \|Y^{12}_{m,n}\|^2_{HS}.
\end{aligned}
$$

As always there are two cases, $m > 0$ and $m < 0$, though the details for the case $m > 0$ will be shown only as the $m < 0$ is completely analogous like always. First one considers $\|X^{11}_{m,n}\|^2_{HS}$. Using Lemmas 7.4.2, 7.4.4, 7.4.5 and the fact that one has

$$\|X^{11}_{m,n}\|^2_{HS} = \sum_{k=0}^{\infty} \frac{\left(I^{(1)}_{m,n}(k)\right)^2}{a_1(k)} \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-1} \frac{c^{(n)}_1(j)}{c^{(n)}_2(j)}\right)^2 \cdot \frac{\left(K^{(1)}_{m,n}(i)\right)^2}{a_1(i)}$$
$$\leq \frac{J_1^2(n)}{J_2^2(n)} \kappa \sum_{k=0}^{\infty} \frac{\left(I^{(1)}_{m,n}(k)\right)^2 K^{(1)}_{m,n}(k+1)}{a_1(k)} \sum_{i=k+1}^{\infty} \frac{K^{(1)}_{m,n}(i)}{a_1(i)}$$
$$\leq \frac{\tau\kappa}{m} \cdot \frac{J_1(n)}{J_2(n)} \sum_{k=0}^{\infty} \frac{I^{(1)}_{m,n}(k) K^{(1)}_{m,n}(k+1)}{a_1(k)}.$$

The above is then equal to

$$\frac{\tau\kappa J_1(n)}{m J_2(n)} \sum_{k=0}^{\infty} \frac{I^{(1)}_{m,n}(k) K^{(1)}_{m,n}(k+1)}{a_1(k)} \cdot \frac{K^{(2)}_{m,n}(k)}{K^{(2)}_{m,n}(k)}$$
$$\leq \frac{J_2(n)}{J_1(n)} \frac{\tau^2 \kappa J_1(n)}{J_2(n)} \sum_{k=0}^{\infty} \left(\varepsilon(m,n) + \frac{K^{(1)}_{m,n}(\infty)}{|m| K^{(2)}_{m,n}(\infty)}\right) \frac{1}{a_1(k)}$$
$$= \tau^2 \kappa \left(\varepsilon(m,n) + \frac{K^{(1)}_{m,n}(\infty)}{|m| K^{(2)}_{m,n}(\infty)}\right) s_1(n)$$

by using Lemmas 7.4.3, 7.4.5, and the fact that $1/c^{(n)}_1(k)$ is less than κ. Therefore one has

$$\|X^{11}_{m,n}\|^2_{HS} \leq \tau^2 \kappa \left(\varepsilon(m,n) + \frac{K^{(1)}_{m,n}(\infty)}{|m| K^{(2)}_{m,n}(\infty)}\right) s_1(n) \ .$$

Next one estimates $\|X^{22}_{m,n}\|^2_{HS}$. Using Lemmas 7.4.2, 7.4.4, and 7.4.5 respectively, one gets

$$\|X^{22}_{m,n}\|^2_{HS} = \sum_{k=0}^{\infty} \frac{\left(I^{(2)}_{m,n}(k)\right)^2}{a_2(k)} \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-2} \frac{c^{(n)}_1(j)}{c^{(n)}_2(j)}\right)^2 \frac{\left(K^{(2)}_{m,n}(i-1)\right)^2}{a_2(i-1)}$$
$$\leq \frac{J_1(n)}{J_2^2(n)} \sum_{k=0}^{\infty} \frac{\left(I^{(2)}_{m,n}(k)\right)^2 K^{(2)}_{m,n}(k)}{a_2(k)} \sum_{i=k+1}^{\infty} \left(\prod_{j=0}^{i-2} c^{(n)}_1(j)\right) \frac{K^{(2)}_{m,n}(i-1)}{a_2(i-1)}$$
$$\leq \frac{J_1^2(n)}{J_2^2(n)} \frac{\tau}{m} \cdot \frac{J_2(n)}{J_1(n)} \sum_{k=0}^{\infty} \frac{I^{(2)}_{m,n}(k) K^{(2)}_{m,n}(k)}{a_2(k)} \ .$$

The above is then equal to

$$\frac{\tau}{m} \cdot \frac{J_1(n)}{J_2(n)} \sum_{k=0}^{\infty} \frac{I^{(2)}_{m,n}(k) K^{(2)}_{m,n}(k)}{a_2(k)} \cdot \frac{I^{(1)}_{m,n}(k+1)}{I^{(1)}_{m,n}(k+1)} \leq \tau^2 \kappa \sum_{k=0}^{\infty} \frac{\varepsilon(m,n)}{a_2(k)} = \tau^2 \kappa \varepsilon(m,n) s_2(n)$$

by using Lemmas 7.4.3 and 7.4.5 and the fact that $1/c_1^{(n)}(k)$ is less than κ. Thus one has

$$\|X^{22}_{m,n}\|^2_{HS} \leq \tau^2 \kappa \varepsilon(m,n) s_2(n) \ .$$

Very similar arguments, using the same lemmas as above, will show that

$$\begin{aligned} \|X^{12}_{m,n}\|^2_{HS} &\leq \tau^2 \kappa \left(\varepsilon(m,n) + \frac{K^{(1)}_{m,n}(\infty)}{|m| K^{(2)}_{m,n}(\infty)} \right) s_1(n) \\ \|X^{21}_{m,n}\|^2_{HS} &\leq \tau^2 \kappa \varepsilon(m,n) s_2(n) \ . \end{aligned} \tag{7.25}$$

Therefore this shows that for $m \neq 0$ the Hilbert-Schmidt norm is finite for all of the operators. Next one considers the case $m = 0$. Then one has

$$\|Z_{0,n}\|^2_{HS} = \sum_{k=0}^{\infty} \frac{1}{a_2(k)} \sum_{i=0}^{k} \left(\prod_{j=i}^{k-1} c_2^{(n)}(j) \right)^2 \frac{1}{a_1(i)} \leq s_1(n) s_2(n)$$

since again $c_2(k) \leq 1$. This shows that for $m = 0$ the Hilbert-Schmidt norms is finite for both operators, thus the proof is finished. ■

It is now time to show that $Q^{(m,n)}$ is a Hilbert-Schmidt operator for all $m \in \mathbb{Z}$ and $n \geq 0$.

Theorem 7.4.7 *The parametrix $Q^{(m,n)}$ for $m \neq 0$ and $n \geq 0$ is a Hilbert-Schmidt operator and the Hilbert-Schmidt norm of $Q^{(m,n)}$ goes to zero as $|m|, n \to \infty$. Moreover for $m = 0$ the parametrix is a Hilbert-Schmidt operator and the Hilbert-Schmidt norm of $Q^{(0,n)}$ goes to zero as $n \to \infty$.*

Proof One starts with the case $m = 0$. It follows from Lemma 7.3.2 and Proposition 7.4.6 that one has

$$\|Q^{(0,n)}\|_{HS} \leq \|Z_{0,n}\|_{HS} \leq \sqrt{s_1(n)s_2(n)}$$

which is finite, showing that $Q^{(0,n)}$ is a Hilbert-Schmidt operator. Since the sums on the right side in the above inequality go to zero as $n \to \infty$, by the conditions on $a_1(k)$ and $a_2(k)$ it follows that $\|Q^{(0,n)}\|_{HS}$ goes to zero as $n \to \infty$. Next one considers the case $m \neq 0$. It also follows from Lemma 7.3.2, Proposition 7.4.6, the triangle inequality and combining like terms that one has

$$\begin{aligned}\|Q^{(m,n)}\|_{HS} &\leq \frac{1}{\tau} \sum_{\alpha,\beta=1,2} \left(\|X^{\alpha\beta}_{m,n}\|_{HS} + \|Y^{\alpha\beta}_{m,n}\|_{HS}\right) \\ &\leq 4\sqrt{\kappa}\left(\sqrt{\varepsilon(m,n)}\left(\sqrt{s_1(n)} + \sqrt{s_2(n)}\right) + \sqrt{s_1(n)}\sqrt{\frac{K^{(1)}_{m,n}(\infty)}{|m|K^{(2)}_{m,n}(\infty)}}\right)\end{aligned}$$

which is finite. This means that $Q^{(m,n)}$ is a Hilbert-Schmidt operator. To see that the right hand side of the above inequality goes to zero as $|m|, n \to \infty$ notice that from Lemma 7.4.3 implies that $\varepsilon(m,n) \to 0$ as $|m|, n \to \infty$ so that the first term in the sum goes to zero as $|m|, n \to \infty$. Now the boundary condition given in Proposition 7.2.5 implies that

$$\frac{K^{(1)}_{m,n}(\infty)}{|m|K^{(2)}_{m,n}(\infty)} \to 0 \quad \text{as} \quad |m| \to \infty.$$

Using this and the fact that $s_1(n)$ goes to zero as $n \to \infty$ it follows that

$$\sqrt{s_1(n)}\sqrt{\frac{K^{(1)}_{m,n}(\infty)}{|m|K^{(2)}_{m,n}(\infty)}} \to 0 \quad \text{as} \quad |m|, n \to \infty.$$

Therefore $\|Q^{(m,n)}\|_{HS}$ goes to zero as $|m|, n \to \infty$. The proof is now complete. ■

We can now close out this chapter by proving the main theorem of this chapter, that is, the Dirac operator defined by equation (7.3) subject to the boundary condition () has a compact inverse.

Proof (Proof of Theorem 7.1.5) It follows from Proposition 7.1.4 that parametrix Q, to the Dirac operator D is essentially a direct sum of $Q^{(m,n)}$. Theorem 7.4.7 shows that $Q^{(m,n)}$ is a Hilbert-Schmidt operator for all $m \in \mathbb{Z}$ and $n \geq 0$. Morever the same theorem showed that the Hilbert-Schmidt norms of $Q^{(m,n)}$ go to zero as $|m|, n \to \infty$. This means since Q is essentially a direct sum of those operators, one must have Q to be a compact operator. Since $\mathcal{A}^{(m,n)}Q^{(m,n)} = Q^{(m,n)}\mathcal{A}^{(m,n)} = id_{m,n}$ by Proposition 7.3.1, it again follows from Proposition 7.1.4 that $DQ = QD = id$. Thus the proof is complete. ■

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