PERIODIC ORBITS OF PIECEWISE MONOTONE MAPS

A Dissertation

Submitted to the Faculty

of

Purdue University

by

David Cosper

In Partial Fulfillment of the

Requirements for the Degree

of

Doctor of Philosophy

May 2018

Purdue University

Indianapolis, Indiana

ACKNOWLEDGMENTS

I would like to thank my advisor Michał Misiurewicz for sticking with me all these years.

TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

Cosper, David Ph.D., Purdue University, May 2018. Periodic Orbits of Piecewise Monotone Maps. Major Professor: Michał Misiurewicz.

Much is known about periodic orbits in dynamical systems of continuous interval maps. Of note is the theorem of Sharkovsky. In 1964 he proved that, for a continuous map f on $\mathbb{R}$, the existence of periodic orbits of certain periods force the existence of periodic orbits of certain other periods. Unfortunately there is currently no analogue of this theorem for maps of $\mathbb{R}$ which are not continuous. Here we consider discontinuous interval maps of a particular variety, namely piecewise monotone interval maps. We observe how the presence of a given periodic orbit forces other periodic orbits, as well as the direct analogue of Sharkovsky's theorem in special families of piecewise monotone maps. We conclude by investigating the entropy of piecewise linear maps. Among particular one parameter families of piecewise linear maps, entropy remains constant even as the parameter varies. We provide a simple geometric explanation of this phenomenon known as entropy locking.

1. INTRODUCTION

The subject of continuous interval dynamics is very rich, with results dating back many decades. As a result, most open problems pertaining to continuous interval dynamics are either difficult or highly technical. Thus, this leads us to a natural next step in the study of interval dynamics: piecewise continuous.

One of the most basic objects of study in dynamical systems is periodic points. A point x will be called *periodic* under the map f if there exists natural number n such that $f^n(x) = x$, and the smallest integer n satisfying this property will be called the *period* of x. In 1964, Oleksandr Sharkovsky showed that the presence of points of a given period can force the existence of periodic points of certain other periods [1]. This rather well-known result is referred to as the Sharkovsky Theorem.

Theorem 1.0.1 (Sharkovsky) *Let f be a continuous interval map. Consider the ordering:*

$$\begin{array}{llllllll}
3 & >_S & 5 & >_S & 7\ldots & & \\
2\cdot 3 & >_S & 2\cdot 5 & >_S & 2\cdot 7\ldots & & \\
\ldots & & & & & & \\
2^n\cdot 3 & >_S & 2^n\cdot 5 & >_S & 2^n\cdot 7\ldots & & \\
\ldots & >_S & 2^2 & >_S & 2^1 & >_S & 1
\end{array}$$

If f has a periodic point of period p, then f has a periodic point of period q for every $p >_S q$ in the above ordering.

In Section 2.1, we give several preliminary results for kneading theory. We then use these tools in Section 2.2 to outline a proof of the Sharkovsky Theorem and give a geometric interpretation of the theorem.

The Sharkovsky Theorem leads us to a very simple question: for piecewise continuous interval maps, what are the possible sets of periods of periodic orbits? This

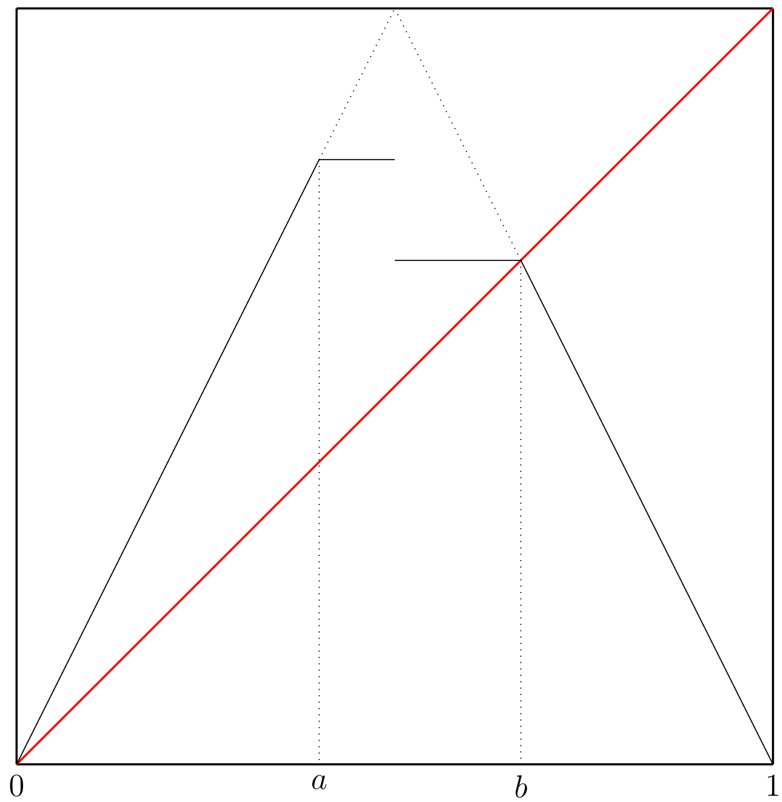


Fig. 1.1. An example of a two-sided truncated tent map.

question is not completely unstudied. For instance in [2], this question is studied in the case of Lorenz-like maps. However, in full generality this question is widely untouched. Here we shall restrict our study to maps f with one increasing piece and one decreasing piece of monotonicity. In Section 2.3, we show using kneading theory that it suffices instead to study the family of two-sided truncated tent maps (see Figure 1.1).

In Chapter 3, we use the parameter space $\mathcal{TS}$ associated to the family of truncated tent maps to give a simple visual understanding of which periodic points appear under a map $T_{a,b}$. For each possible periodic orbit Q, there exists a parameter, called a peak, which splits the parameter space into two subsets. Parameters in one subset are guaranteed to correspond to maps which have a periodic orbit Q, while parameters in the other are guaranteed to correspond to maps which do not have Q as a periodic orbit. In Section 3.1, we use these ideas to investigate special families in $\mathcal{TS}$ and compare these to the order from Sharkovsky's Theorem.

In Section 3.2, we describe extremal points in $\mathcal{TS}$, which are peaks corresponding to truncated tent maps which have only one periodic point. These points give insight into understanding how far we can generalize results similar to Sharkovsky's Theorem. Specifically, these Extremal points will tell us that given a function f with a periodic point of period p, we cannot guarantee the existence of a periodic point of any other period without some additional information. Section 3.3 will be devoted to understanding the structure of the set of peaks in $\mathcal{TS}$. These will have, for the most part, a very intuitive geometric description.

In Chapter 4 we investigate a special collection of piecewise monotone maps. These maps will have formula

$$T_{\lambda,\mu,b}(x) = \begin{cases} 1 + \lambda x + b & \text{if} \quad x \leq 0, \\ 1 - \mu x & \text{if} \quad x \geq 0, \end{cases} \tag{1.1}$$

For a function $T_{\lambda,\mu,b}$ which has a jump discontinuity at a point 0, there are two potential values of the function at 0. Denote these values by $T_{\lambda,\mu,b}(0_-)$ and $T_{\lambda,\mu,b}(0_+)$. We say that $T_{\lambda,\mu,b}$ has *matching* if there exists integers m and k such that $f^k(c_-) = f^m(c_+)$. In [3], Botella-Soler, Oteo, Ros and Glendinning observed numerically for special values of λ and μ that the topological entropy and Lyapunov exponent remain constant in an interval of b values close to 0. In [4], Bruin, Carminati, Marmi and Profeti explained this phenomenon using matching. In Chapter 4 we present a surprisingly simple proof of existence of these entropy plateaus of topological entropy. Our proof primarily relies on Euclidean geometry and matching.

2. PRELIMINARIES

In this chapter we discuss basic, well-understood topics that will be used throughout this dissertation. In Section 2.1 we present some main results in the kneading theory of unimodal maps. We then extend these results to include discontinuous maps which are "unimodal", i.e., increasing on one continuous piece and decreasing on another.

Section 2.2 is devoted to the theorem of Sharkvosky, a well known result pertaining to periodic orbits of continuous interval maps. An outline of the proof shall be given, as well as an explanation of the theorem's relevance to our results.

Finally, Section 2.3 details results pertaining to Markov graphs. Given a map $f : J \to J$ and a partition $\mathcal{U} = \{U_0, \ldots, U_{n-1}\}$ of J, the Markov graph is a directed graph which describes how elements of $\mathcal{U}$ cover each other under iteration by f. In particular, we can may find the Markov graphs of partitions which are prescribed by periodic orbits. We shall details this technique and describe how it may used to determine other periodic orbits of f.

2.1 Kneading Theory

Kneading Theory is a tool for studying interval dynamics that was first introduced in a 1977 preprint by Milnor and Thurston, [5]. The use of symbolics to study interval maps, which was the starting point of their work, appeared earlier, for instance [6] and [7]. The central idea of kneading theory is that, given an interval map $f : I \to I$, we can assign to every point $x \in I$ a code, called the itinerary of x, which describes the point's behavior under f. Of particular interest are the codes assigned to critical values, since the graph of f can fold only around these points (hence the term "kneading"). These ideas are central mechanisms in the results of this dissertation, and hence we will now state some standard results of the subject. We then extend these ideas to the case where the map is piecewise monotone. We shall use notation similar to that of [8].

We shall say that a mapping of the interval $J = [\alpha, \beta]$ into itself is *unimodal* if the following conditions are satisfied.

1. f is continuous,
2. There exists $c \in J$ so that f is increasing on $[\alpha, c]$ and decreasing on $[c, \beta]$,
3. $f^2(c) \leq f^3(c)$,

For $x \in J$ we define its *itinerary* $\underline{I}_f(x)$ under the mapping f to be the sequence $I_0(x)I_1(x)I_2(x)\ldots$, where

$$I_0(x) = \begin{cases} R & \text{if } \; x > c, \\ C & \text{if } \; x = c, \\ L & \text{if } \; x < c, \end{cases} \tag{2.1}$$

and $I_j(x) = I_0(f^j(x))$. We adopt the convention that the itinerary terminates if $I_j(x) = C$ for some j. We will call a sequence $\underline{A}$ of Rs, Ls, and Cs *admissible* if it is either an infinite sequence of Rs and Ls, or a finite (possibly empty) sequence of Rs and Ls followed by a C. Note that all itineraries are admissible. For finite sequences $\underline{A}$, we will write $|\underline{A}|$ to denote their length. The *kneading sequence* of the map f will refer to $\underline{I}_f(f(c))$, i.e., the itinerary of the critical value. If $\underline{A} = A_0A_1A_2\ldots$, we define the shift operation σ by $\sigma(\underline{A}) = A_1A_2A_3\ldots$. If $\underline{A} = C$, then σ will be undefined. Note that if $x \neq c$, then $\underline{I}_f(f(x)) = \sigma(\underline{I}_f(x))$.

Remark 2.1.1 *Often in this dissertation there will be maps f such that for some interval (a, b), $c \in (a, b)$ and $f((a, b)) = f(c)$. We call this a* turning interval. *However, in such cases we shall treat the interval (a, b) as a point and proceed as usual.*

We will now define the *parity lexicographical ordering* of admissible sequences. We shall first say that $L < C < R$. Let $\underline{A} \neq \underline{B}$ be two admissible sequences. Let i be the first index for which $A_i \neq B_i$. If $|\underline{A}| = l < \infty$ and $l < |\underline{B}|$, then $A_{l-1} = C$ and $B_{l-1} \neq C$. Therefore, such an i will always exist. We will say that $\underline{A} < \underline{B}$ if either

1. There are an even number of Rs in $A_0A_1\ldots A_{i-1} = B_0B_1\ldots B_{i-1}$ and $A_i < B_i$.

2. There are an odd number Rs in $A_0A_1\ldots A_{i-1} = B_0B_1\ldots B_{i-1}$ and $A_i > B_i$.

We shall also define a finite sequence $\underline{A}$ to be *positive* if it has an even number of Rs and *negative* if it has an odd number of Rs. A sequence $\underline{A}$ is *maximal* if $\sigma^n(\underline{A}) \leq \underline{A}$, for every $n \geq 0$.

Proposition 2.1.2 *Suppose that f is a unimodal map.*

1. *If $\underline{I}_f(x) < \underline{I}_f(y)$, then $x < y$.*

2. *If $x < y$, then $\underline{I}_f(x) \leq \underline{I}_f(y)$.*

In particular, if f is a piecewise expanding map, then the inequality in part 2 is strict.

Let $\underline{K}(f)$ denote the kneading sequence of a unimodal map f. We shall say that an admissible sequence $\underline{A}$ is *dominated* by $\underline{K}(f)$ if one of the following conditions is satisfied for all $n \geq 0$:

- $\sigma^n(\underline{A}) < \underline{K}(f)$ if $\underline{K}(f)$ is infinite,

- $\sigma^n(\underline{A}) < (\underline{D}L)^\infty$ if $\underline{K}(f) = \underline{D}C$ and $\underline{D}$ is positive,

- $\sigma^n(\underline{A}) < (\underline{D}R)^\infty$ if $\underline{K}(f) = \underline{D}C$ and $\underline{D}$ is negative.

Note that for a unimodal map f every itinerary is dominated by the kneading sequence of f.

Proposition 2.1.3 *If $\underline{K}(f)$ dominates the admissible sequence $\underline{A}$, then the sets*

$$L_{\underline{A}} = \{x \in (\alpha, \beta) \,|\, \underline{I}_f(x) < \underline{A}\}$$

and

$$R_{\underline{A}} = \{x \in (\alpha, \beta) \,|\, \underline{I}_f(x) > \underline{A}\}$$

are open.

Proposition 2.1.4 *If $\underline{K}(f)$ dominates the admissible sequence $\underline{A} \geq \underline{I}_f(f^2(c))$, then there exists $x \in J = [\alpha, \beta]$ such that $\underline{I}_f(x) = \underline{A}$.*

We would now like to extend some of these results to the case where $f : J \to J$ is a piecewise monotone map, with $J = [\alpha, \beta]$. More precisely, we want to consider all maps f satisfying the following properties:

1. There exists $c \in J$ so that f is increasing and continuous on $[\alpha, c)$ and decreasing and continuous on $(c, \beta]$.

2. If $\kappa = \sup\{\lim_{x \to c^-} f(x), \lim_{x \to c^+} f(x)\}$, then $f(\kappa) \leq f^2(\kappa)$.

We will denote the set of all such maps by $\mathcal{F}$. It is important to note that we shall define itineraries for elements of $\mathcal{F}$ in the same manner as the unimodal case. This means we will use the parity lexicographical ordering in this case as well.

Proposition 2.1.5 *Suppose that $f \in \mathcal{F}$.*

- *If $\underline{I}_f(x) < \underline{I}_f(y)$, then $x < y$.*

- *If $x < y$, then $\underline{I}_f(x) \leq \underline{I}_f(y)$.*

In particular, if f is an expanding map, then the inequality in part 2 is strict.

The kneading sequence of a unimodal map is the itinerary of the critical value. However, elements of $\mathcal{F}$ can be discontinuous at the critical point c. Moreover, if we fix the value of the function at c, we will lose information about f. To accommodate for this, we will consider itineraries of both possible "critical values". More precisely, let $f(c_-) = \lim_{x \to c^-} f(x)$ and $f(c_+) = \lim_{x \to c^+} f(x)$. We will define the *left kneading sequence* of f to be $\underline{I}_f(f(c_-))$ and the *right kneading sequence* of f to be $\underline{I}_f(f(c_+))$. Note that in the continuous case the left and right kneading sequences coincide (at the kneading sequence).

We now direct our attention to finding an analogue of domination for elements of $\mathcal{F}$. To simplify matters, we will consider only cases where the left and right kneading sequences are infinite. Let $f \in \mathcal{F}$ have left and right kneading sequences $\underline{K}_-$ and $\underline{K}_+$, respectively. Assume additionally that $|\underline{K}_-|$ and $|\underline{K}_+|$ are infinite. We will say that the kneading sequences $\underline{K}_-$ and $\underline{K}_+$ *dominate* admissible sequence $\underline{A}$ if both of the following are satisfied:

- $\sigma^n(\underline{A}) < \underline{K}_-$ whenever $A_{n-1} = L$,
- $\sigma^n(\underline{A}) < \underline{K}_+$ whenever $A_{n-1} = R$.

It is important to note that every itinerary is dominated by the left and right kneading sequences.

Proposition 2.1.6 *Let* $\kappa = \sup\{\lim_{x \to c^-} f(x), \lim_{x \to c^+} f(x)\}$*. If* $\underline{K}_-(f)$ *and* $\underline{K}_+(f)$ *dominate the admissible sequence* $\underline{A} \geq \underline{I}_f(f(\kappa))$*, then there exists* $x \in J$ *so that* $\underline{I}_f(x) = \underline{A}$*.*

Proof Consider the two sets

$$U_L = \{x \,|\, \underline{I}_f(x) < \underline{A}\}$$

and

$$U_R = \{x \,|\, \underline{I}_f(x) > \underline{A}\}.$$

We begin by showing that U_L and U_R are both open sets. Let $y \in U_R$ and denote it's itinerary by $\underline{I}_f(y) = B_0B_1B_2\ldots$. Since $\underline{I}_f(y) > \underline{A}$, there exists n such that $A_n \neq B_n$ and $A_i = B_i$ for $i < n$. There are now two cases: when $B_n = C$ and when $B_n \neq C$. Suppose that $B_n \neq C$. Then there exists $\epsilon > 0$ sufficiently small so that if $x \in (y - \epsilon, y + \epsilon)$, then $\underline{I}_f(x) = B_0B_1\ldots B_{n-1}B_n\ldots$. Since $\underline{I}_f(y) > \underline{A}$ and the n is the first index on which the two sequences disagree, $\underline{I}_f(x) > \underline{A}$.

Now suppose that $\underline{I}_f(y) = \underline{B}C$, with $|\underline{B}| = n$. Thus we have two cases: $\underline{B}$ is either positive or negative. Since the proof is analogous in both cases, we will assume that $\underline{B}$ is positive. This implies $\underline{A} = \underline{B}L\underline{A}^*$. Consider the sequence $\underline{A}^*$. Since the preceding letter is L, $\underline{A}^* < \underline{K}_-(f)$. If $\underline{A}^* = C_0C_1C_2\ldots$ and $\underline{K}_-(f) = K_0K_1K_2\ldots$, there must exists m so that $C_m \neq K_m$ and $C_i = K_i$ for $i < m$. By continuity there exists $\epsilon > 0$ sufficiently small so that if $x \in (y-\epsilon, y+\epsilon)$, then either $\underline{I}_f(x) = \underline{B}R\ldots$ or $\underline{I}_f(x) = \underline{B}LK_0\ldots K_m\ldots$. Either case implies $\underline{I}_f(x) > \underline{A}$. Therefore we have proven that U_R is open. The proof showing U_L is open is similar.

Since U_L and U_R are open, then

$$U_L^c = \{x \,|\, \underline{I}_f(x) \geq \underline{A}\}$$

and

$$U_R^c = \{x \,|\, \underline{I}_f(x) \leq \underline{A}\}$$

are closed. Moreover, $U_L^c \cup U_R^c = J$, and J is a connected set. Therefore $U_L^c \cap U_R^c \neq \emptyset$ and hence there exists $x \in J$ such that $\underline{I}_f(x) = \underline{A}$. ∎

In this dissertation, we aren't so much interested in the map f as we are the set of periodic orbits of f. We choose to express these ideas in terms of kneading sequences and itineraries. Therefore, if $\mathcal{I}(f)$ denotes the set of itineraries of f, we would like to find a family of simple maps that are somewhat representative of the family $\mathcal{F}$. Thus we turn our attention to the family of two-sided truncated tent maps. Recall that the tent map $T : [0,1] \to [0,1]$ is defined by

$$T(x) = \begin{cases} 2x & \text{if} \quad x \leq 1/2, \\ 2 - 2x & \text{if} \quad 1/2 \leq x \leq 1. \end{cases} \tag{2.2}$$

Proposition 2.1.7 *Every admissible sequence $\underline{A} \geq L^\infty$ is contained in $\mathcal{I}(T)$.*

Proof The tent map T has kneading sequence $\underline{K}(T) = RL^\infty$ and $\underline{I}_T(0) = L^\infty$. The result then follows from Proposition 2.1.4. ∎

A truncated tent map $T_{a,b}$ is defined by

$$T_{a,b}(x) = \begin{cases} 2x & \text{if} \quad 0 \le x \le a, \\ 2a & \text{if} \quad a \le x \le \frac{1}{2}, \\ 2-2b & \text{if} \quad \frac{1}{2} \le x \le b, \\ 2-2x & \text{if} \quad b \le x \le 1. \end{cases} \tag{2.3}$$

We will denote by $\mathcal{TS}$ the set of all parameters $(\!(a,b)\!)$, with $a \in [0, \frac{1}{2}]$ and $b \in [\frac{1}{2}, 1]$. Here we use notation $(\!(a,b)\!)$ for the parameter to avoid confusion with interval (a,b). Notice that we assign two values at $x = \frac{1}{2}$. Fortunately, this will not cause any contradictions in our results (in fact, we lose some information by not considering both possible images of $\frac{1}{2}$).

The family $\mathcal{TS}$ is much more convenient to work with for many reasons. The most immediate reason is that a map $T_{a,b}$ is simple to formulate both algebraically and geometrically. This family will also help draw parallels between unimodal maps and elements of $\mathcal{F}$. However, the most important property of $\mathcal{TS}$ is that if gives a "good representation" of $\mathcal{F}$. More concisely, if $f \in \mathcal{F}$, then there must exist $(\!(a,b)\!) \in \mathcal{TS}$ so that f and $T_{a,b}$ have the same collection of itineraries on their respective cores, which we will define momentarily.

Proposition 2.1.8 *If $\underline{A}$ is a maximal sequence, there exists $a \in [0, 1/2]$ such that $\underline{K}(T_{a,1-a}) = \underline{A}$, i.e., there exists continuous truncated tent map with kneading sequence $\underline{A}$.*

Proof Let T be the full tent map. By Proposition 2.1.7, there exists $\alpha \in [0,1]$ with $\underline{I}(\alpha) = \underline{A}$. Since the left and right laps of T are full laps, there must exist $a \in [0, 1/2]$ and $b \in [1/2, 1]$ such that $T(a) = T(b) = \alpha$. The graph of T is symmetric about $1/2$, which implies $b = 1 - a$. Therefore it can be seen that the map $T_{a,1-a}$ is a continuous truncated tent map. Since $\underline{A}$ is maximal, $\underline{K}(T_{a,1-a}) = \underline{A}$. ∎

We will say a sequence $\underline{A} = A_0 A_1 A_2 \ldots$ is *left maximal* if $\underline{A}$ is admissible and $\sigma^n(\underline{A}) \le \underline{A}$ whenever $A_{n-1} = L$ and $\underline{A}$ is *right maximal* if $\underline{A}$ is admissible and

$\sigma^n(\underline{A}) \leq \underline{A}$ whenever $A_{n-1} = R$. If $\underline{A}$ is both left and right maximal, we say simply that $\underline{A}$ is *maximal*. Note that left kneading sequences are left maximal and right kneading sequences are right maximal. If $\underline{A} = A_0A_1A_2\ldots$ and $\underline{B} = B_0B_1B_2\ldots$ are admissible sequences and left and right maximal, respectively, we will say that $\underline{A}$ and $\underline{B}$ are *comaximal* if $\sigma^n(\underline{B}) \leq \underline{A}$ whenever $B_{n-1} = L$ and $\sigma^n(\underline{A}) \leq \underline{B}$ whenever $A_{n-1} = R$, $n \geq 1$. Note that left and right kneading sequences are comaximal.

Proposition 2.1.9 *Let left maximal sequence $\underline{A}$ and right maximal sequence $\underline{B}$ be comaximal. There exists $((a,b)) \in \mathcal{TS}$ such that $K_-(T_{a,b}) = \underline{A}$ and $K_+(T_{a,b}) = \underline{B}$.*

Proof If $\underline{A} = \underline{B}$, then $\underline{A}$ is maximal and by Proposition 2.1.8 there exists continuous function $T_{a,1-a}$ with kneading sequence $\underline{A}$. Suppose then, without loss of generality, that $\underline{A} < \underline{B}$. Since $\underline{A}$ and $\underline{B}$ are comaximal, then $\sigma^n(\underline{B}) \leq \underline{A} < \underline{B}$ when $B_{n-1} = L$. Thus $\underline{B}$ is also left maximal, and hence maximal. Thus by Proposition 2.1.8 there exists $b \in [1/2, 1]$ such that $\underline{K}(T_{1-b,b}) = \underline{B}$. Since $\underline{B}$ dominates $\underline{A}$, there exists $x \in [0,1]$ such that $\underline{I}_{T_{a-b,b}}(x) = \underline{A}$. Let $a \in T^{-1}_{(1/2,1/2)}(x) \cap [0, 1/2]$. Then the function $T_{a,b}$ has left kneading sequence $\underline{A}$ and right kneading sequence $\underline{B}$. ■

Given $f \in \mathcal{F}$, the core of f is the minimal bounded invariant interval under f. Recall that $\kappa = \sup\{f(c_+), f(c_-)\}$. Then the *core* of f, if it exists, will be the interval $[f(\kappa), \kappa]$. Note that the core exists if and only if $f^2(\kappa) \geq f(\kappa)$.

Proposition 2.1.10 *For every $f \in \mathcal{F}$ there exists $((a,b)) \in \mathcal{TS}$ such that $\underline{K}_-(T_{a,b}) = \underline{K}_-(f)$ and $\underline{K}_+(T_{a,b}) = \underline{K}_+(f)$. In particular, $\mathcal{I}(f|_{J_1}) = \mathcal{I}(T_{a,b}|_{J_2})$, where J_1 and J_2 are the cores of f and $T_{a,b}$, respectively.*

Proof Since $\underline{K}_-(f)$ and $\underline{K}_+(f)$ are left and right maximal, resp., and comaximal, there exists, by Proposition 2.1.9, $((a,b)) \in \mathcal{TS}$ such that $\underline{K}_-(T_{a,b}) = \underline{K}_-(f)$ and $\underline{K}_+(T_{a,b}) = \underline{K}_+(f)$. Since f and $T_{a,b}$ have the same kneading sequences, it follows from Proposition 2.1.6 that $\mathcal{I}(f|_{J_1}) = \mathcal{I}(T_{a,b}|_{J_2})$. ■

2.2 Sharkovsky's Theorem

Sharkovsky's Theorem is a classic result of interval dynamics. The theorem let's us deduce the existence of periodic orbits of certain periods given the existence of periodic orbits of another period. There are many proofs of Sharkovsky's Theorem using various techniques. Here we shall use kneading theory to outline a basic proof. More details of this proof can be found in [8]. Additionally, we are interested only in unimodal maps, and therefore we will state the theorem only for unimodal maps. However, let it be known that this theorem holds true for all continuous maps on $\mathbb{R}$.

Theorem 2.2.1 (Sharkovsky) *Let f be a unimodal map. Consider the ordering:*

$$\begin{array}{lllllll}
3 & >_S & 5 & >_S & 7\dots & & \\
2\cdot 3 & >_S & 2\cdot 5 & >_S & 2\cdot 7\dots & & \\
\dots & & & & & & \\
2^n\cdot 3 & >_S & 2^n\cdot 5 & >_S & 2^n\cdot 7\dots & & \\
\dots & >_S & 2^2 & >_S & 2^1 & >_S & 1
\end{array}$$

If f has a periodic point of period p, then f has a periodic point of period q for every $p >_S q$ in the above ordering.

Let $\underline{A}$ be a finite nonempty sequence of Ls and Rs and let $\underline{B}$ be admissible. We define $\underline{A} * \underline{B}$ as follows:

- If $\underline{B}$ is infinite and $\underline{A}$ is positive, then $\underline{A} * \underline{B} = \underline{A}B_0\underline{A}B_1\underline{A}B_2\dots$
- If $\underline{B} = B_0B_1\dots B_{n-1}C$ is finite and $\underline{A}$ is positive, then $\underline{A} * \underline{B} = \underline{A}B_0\underline{A}B_1\underline{A}\dots\underline{A}B_{n-1}\underline{A}C$.
- If $\underline{B}$ is infinite and $\underline{A}$ is negative, then $\underline{A} * \underline{B} = \underline{A}\check{B}_0\underline{A}\check{B}_1\underline{A}\check{B}_2\dots$
- If $\underline{B} = B_0B_1\dots B_{n-1}C$ is finite and $\underline{A}$ is negative, then $\underline{A} * \underline{B} = \underline{A}\check{B}_0\underline{A}\check{B}_1\underline{A}\dots\underline{A}\check{B}_{n-1}\underline{A}C$.

where $\check{L} = R$, $\check{R} = L$, and $\check{C} = C$. We call the operation $*$ the $*$-*product* of $\underline{A}$ and $\underline{B}$.

Let us now observe the order of periodic sequences of Rs and Ls with respect to the parity lexicographical ordering. If period $q \geq 2$ if fixed, there are only finitely many sequences of period q. Let $(\underline{B_1})^\infty, (\underline{B_2})^\infty, \ldots, (\underline{B_k})^\infty$ be the maximal sequences of period q. We define the *min-max* of period q, denoted by $\underline{P}_q$, to be the minimal element of $\{\underline{B_1}, \underline{B_2}, \ldots, \underline{B_k}\}$. Note that this set will always be non-empty since for any $q \geq 2$, there is at least one sequence of period q, and thus we can shift this sequence until it is maximal. The following lemma precisely describes the min-max for a period q.

Lemma 2.2.2 *The min-max $\underline{P}_q$ are of the following form:*

1. *If $q \geq 3$ is odd, $\underline{P}_q = RLR^{i-2}$.*

2. *If $q = 2^n \cdot k$, where $k \geq 3$ is odd, then $\underline{P}_q = R^{*n} * (RLR^{k-2})$.*

3. *If $q = 2^n$, $n > 0$, then $\underline{P}_q = R^{*n} * R$, and $\underline{P}_1 = L$.*

where $R^{*n} * \underline{A} = R * R * \cdots * R * \underline{A}$.

The min-max $\underline{P}_q$ represents, in essence, the smallest periodic orbit of period q in the parity lexicographical ordering. All other sequences of period q are larger than $\underline{P}_q$ or have a shift which is larger. Recall that for a map f, the kneading sequence $\underline{K}(f)$ must dominate every itinerary of f. Therefore if $\underline{K}(f) = (\underline{P}_q)^\infty$, then f has only one periodic orbit of period q: the orbit $(\underline{P}_q)^\infty$. So the kneading sequence $(\underline{P}_q)^\infty$ is the smallest kneading sequence which will allow a periodic orbit of period q. Once again consider the ordering in Theorem 2.2.1.

Theorem 2.2.3 *Let s, t be two integers. If $s <_S t$, then $\underline{P}_s < \underline{P}_t$.*

Using Proposition 2.1.6 and Theorem 2.2.3, we arrive at an immediate proof for Sharkovsky's Theorem. If f is a map with periodic orbit of period p, then there exists a point x_0 in this orbit such that $\underline{I}_f(x_0)$ is a maximal periodic sequence of period p. Since $\underline{K}(f)$ dominates $\underline{I}_f(x_0)$, then $\underline{K}(f)$ will also dominate $(\underline{P}_p)^\infty$. Thus

f has a periodic point with itinerary $(\underline{P}_p)^\infty$. Moreover, by Theorem 2.2.3 $\underline{K}(f)$ will also dominate $(\underline{P}_q)^\infty$ for all $q <_S p$, and therefore f will have periodic orbits of all periods $q <_S p$. However, the reader may find this result unsatisfying, since the proof relies entirely on kneading theory. Fortunately, there is a very intuitive geometric interpretation which we shall now detail.

Let us again consider the full tent map $T : [0,1] \to [0,1]$. Since $\underline{K}(T) = RL^\infty$, it follows from Proposition 2.1.6 that every admissible sequence occurs as an itinerary for some point under T. This includes all periodic sequences, which occur as itineraries of periodic points. Consider all periodic orbits of some fixed period q, listed as $\{Q_1, \ldots, Q_k\}$. Each Q_i has a maximum element α_k. Let us suppose that these maximal elements are ordered $\alpha_1 < \alpha_2 < \cdots < \alpha_k$. A continuous truncated tent map $T_{a,1-a}$, $a \in [0, \frac{1}{2}]$, can be thought of as a full tent map which has had particular orbits removed, specifically those orbits which eventually land in the interval $(a, 1-a)$. In other words, if $G = \bigcup_{i=0}^{\infty} T^{-i}((a,b))$, then

$$T|_{I \setminus G} = T_{a,1-a}|_{I \setminus G}. \tag{2.4}$$

It is immediate from this observation that $\mathcal{I}(T_{a_1,1-a_1}) \subset \mathcal{I}(T_{a_2,1-a_2})$, for every $a_1 < a_2$. Thus we have the following Proposition.

Remark 2.2.4 *If $\underline{I}_T(x)$ is maximal, then $\underline{I}_T(x) = \underline{I}_{T_{a,1-a}}(x)$ for all $a \in [0, \frac{1}{2}]$ such that $x \notin (a, 1-a)$.*

Proposition 2.2.5 *Suppose that $a \in [0, \frac{1}{2}]$ so that $\underline{I}_T(T(a))$ is maximal. Then α_i is a q-periodic point under $T_{a,1-a}$ if and only if $T(a) \geq \alpha_i$.*

Proof Suppose that α_i is a q-periodic point under $T_{a,1-a}$. Then there are two cases. If $\underline{I}_{T_{a,1-a}}(\alpha_i)$ is dominated by $\underline{K}(T_{a,1-a})$, then $\underline{I}_{T_{a,1-a}}(\alpha_i) < \underline{K}(T_{a,1-a}) = \underline{I}_{T_{a,1-a}}(T(a))$. By Proposition 2.1.5, $\alpha_i < T(a)$. If $\underline{I}_{T_{a,1-a}}(\alpha_i) = \underline{K}(T_{a,1-a})$, then it must be that $T(a) = \alpha_i$.

Now suppose that $\alpha_i \leq T(a)$. Because $\underline{I}_T(T(a))$ is maximal, then $\alpha_i = T(a)$ implies that α_i is q-periodic under $T_{a,1-a}$. If $\alpha_i < T(a)$, then it follows from Proposition 2.1.5 that $\underline{I}_T(T(a)) = \underline{K}(T_{a,1-a})$ dominates $\underline{I}_T(\alpha_i)$. By Proposition 2.1.6

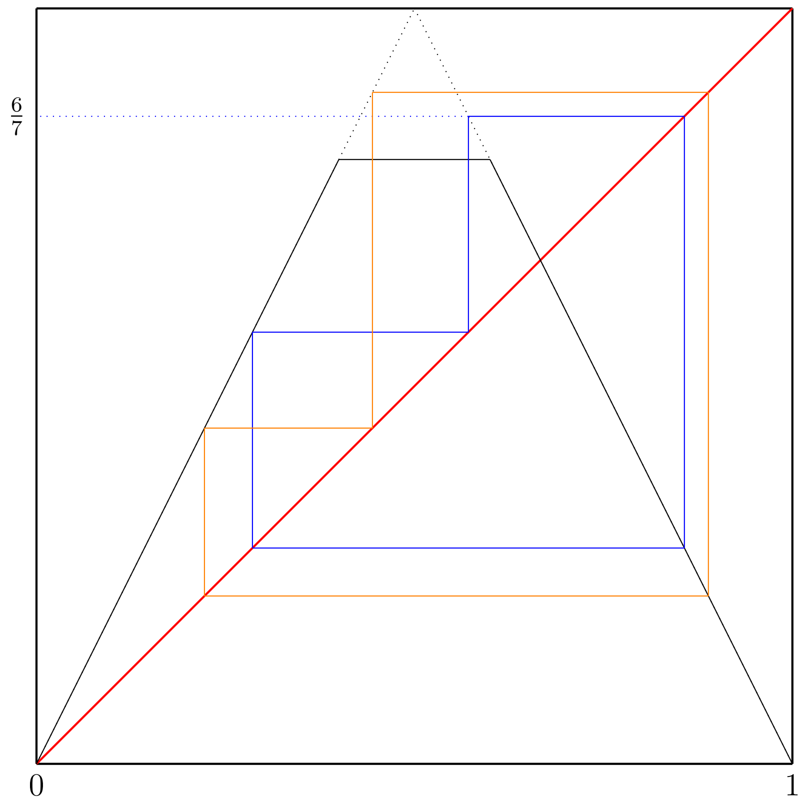


Fig. 2.1. This truncated tent map cannot have the min-max periodic orbit $(RLR)^\infty$, and hence does not have a period 3 periodic point.

there exists point x with itinerary $\underline{I}_T(\alpha_i)$ under the function $T_{a,1-a}$. Additionally, we can see that $T^i(x) \notin (a, 1-a)$ for all i. Since T is piecewise expanding, then $\underline{I}_{T_{a,1-a}}(x) = \underline{I}_T(\alpha_i)$ implies $x = \alpha_i$. ■

Using the above notation, $\underline{I}_T(\alpha_1) = \underline{P}_q^\infty$, and therefore we can see that geometrically $\underline{P}_q^\infty$ represents the lowest truncation which admits a period q orbit (see Figure 2.1 for period 3 orbits). Recall that for every admissible sequence $\underline{A}$ there exists $x \in [0,1]$ such that $\underline{I}_T(x) = \underline{A}$. Moreover, since T is piecewise expanding, then $\underline{I}_T(x) = \underline{I}_T(y)$ if and only if $x = y$. Therefore for every min-max $\underline{P}_q$, there exists x_q such that $\underline{I}_T(x_q) = (\underline{P}_q)^\infty$. Using Theorem 2.2.3, we see that $x_s < x_t$ if and only if

$s <_S t$. Hence the Sharkovsky Theorem can be understood to be the order in which the sets $\mathcal{I}(T_{a,1-a})$ lose periods as a decreases from $\frac{1}{2}$ to 0.

2.3 Periodic Orbits of Interval Maps

Suppose we have a function $f : [\alpha, \beta] \to [\alpha, \beta]$. Let $\mathcal{U} = \{U_0, U_1, \dots, U_{n-1}\}$ be a partition of interval $[\alpha, \beta]$ with endpoints $x_0 = \alpha, x_1, \dots, x_n = \beta$. A partition $\mathcal{U}$ is called a *Markov* partition if $f(x_i) = x_j$ for every i, i.e., if the boundary points of the partition elements form an invariant set. The Markov graph associated to a partition $\mathcal{U}$ is the directed graph whose vertices are U_i and where the directed edges are the pairs (U_i, U_j) such that $U_j \subset f(U_i)$, with the arrow beginning at U_i and terminating at U_j.

Markov graphs are generally used as a method to allow us to utilize combinatorial or symbolic techniques to study interval maps. For the contents of this dissertation, Markov graphs will give us a simple method of generating periodic orbits of a map via the use of loops in the graph. Of interest to us will be the Markov graphs associated to periodic points, particularly ones associated to min-max orbits. The proof of the following Proposition is straightforward.

Proposition 2.3.1 *Let $f \in \mathcal{F}$ and Q be a periodic orbit of f with points $q_0 < q_1 < \cdots < q_n$. Then the partition $\{[q_0, q_1], [q_1, q_2], \dots, [q_{n-1}, q_n]\}$ forms a Markov partition of $\langle Q \rangle$, the convex hull of Q.*

Lemma 2.3.2 *Let $f \in \mathcal{F}$ with partition $\mathcal{U} = \{U_0, \dots, U_{n-1}\}$. Suppose that the discontinuity c is in the partition element U_c and that G is the Markov graph associated to $\mathcal{U}$. Then for every path in G which does not pass through U_c, there exists x with trajectory passing through that path.*

Proof Let $V_0 \to V_1 \to V_2 \to \dots$ be a path in G. Recall that $V_i \to V_{i+1}$ if and only if $V_{i+1} \subset f(V_i)$. Since f is continuous on V_0, there exists a closed interval $J_1 \subset V_0$ such that $f(J_1) = V_2$. Suppose, by induction, that there exist closed intervals

$J_n \subset \cdots \subset J_1 \subset V_0$ such that $f^i(J_i) = V_i$ for every $1 \le i \le n$. Now $f(V_n) \supset V_{n+1}$, so there is a closed interval $V_n^* \subset V_n$ such that $f(V_n^*) = V_{n+1}$. But, as $f^n(J_n) = V_n \supset V_n^*$ and f^n is continuous on J_n, there exists closed interval $J_{n+1} \subset J_n$ with $f^n(J_{n+1}) = V_n^*$, meaning also that $f^{n+1}(J_{n+1}) = V_{n+1}$. Since f is invariant on a compact space, the intersection of nested, closed intervals $\bigcap_{n=1}^{\infty} J_n$ will be nonempty. A point x in this intersection will have the desired properties. ■

By the preceding Lemma, we can use the Markov graph generated by a particular periodic orbit Q to assess what other periodic orbits are forced to exists by Q. A classic example is that "Period 3 implies everything". We know, by the Sharkovsky Theorem, that if a continuous interval map has a period 3 point, then it must have periodic points of all other periods. Observing the Markov graph for the min-max period 3 orbit results in the same conclusion (see figure).

Due to their periodic structure, periodic orbits appear as the result of loops in the Markov graph. Two loops,L_1 and L_2, in the Markov graph are said to be *linked loops* if for any $v_1 \in L_1$ and $v_2 \in L_2$, there exists a path between v_1 and v_2 which lies entirely in $L_1 \cup L_2$.

For the remainder of this section, all of the results shall be for a unimodal function f. From now on, unless otherwise stated, we would like to consider only Markov partitions generated by periodic orbits. We would also like to restrict our functions to the convex hulls of these periodic orbits. We shall now simply refer to these graphs as the *Markov graph of a periodic orbit*. We shall identify the Markov graphs of particular min-max orbits of unimodal maps that will be used in this dissertation. We begin with the case when Q_n is periodic of odd period n.

Lemma 2.3.3 *Let Q_n be the periodic orbit of odd period n associated to the min-max P_n. The Markov graph associated to Q_n will contain the following paths:*

- $I_1 \to I_1$
- $I_1 \to I_2 \to \cdots \to I_{n-1}$

- $I_{n-1} \to I_k$*, for every odd* $K < n$*.*

The $*$-product plays a strong role in the formulation of min-maxes. By extension, it will also play a role in understanding how the Markov graphs associated to various min-maxes are related to each other. The following Lemma will be used in Section 3.2.

Lemma 2.3.4 *Let* Q_n *be the periodic orbit associated to* $\underline{P}_{3\cdot 2^n}$*. The Markov graph associated to* Q_n *has linked loops of lengths* 2^n *and* 2^{n+1} *and isolated loops of lengths* 2^k *for* $0 \leq k < n$*.*

Proof First we discuss the linked loops. For $n = 0$, this is immediate. Suppose that $Q_{3\cdot 2^{n-1}}$ has linked loops of lengths 2^{n-1} and 2^n. Consider the Markov graph of $Q_{3\cdot 2^n}$. Recall that $\underline{P}_{3\cdot 2^n} = R * \underline{P}_{3\cdot 2^{n-1}}$. This means that over half of the letters in $\underline{P}_{3\cdot 2^n}$ are Rs. If Q_L represents the smaller $3 \cdot 2^{n-1}$ elements of Q_n and Q_R represents the larger $3 \cdot 2^{n-1}$ elements of Q_n, set $J_L = \langle Q_L \rangle$ and $J_R = \langle Q_R \rangle$. Note that the turning point c is in J_L.

Consider now $f^2|_{J_L}$. Then Q_L is a periodic orbit of period $3 \cdot 2^{n-1}$ under f^2. Moreover, this orbit is exactly the orbit $Q_{3\cdot 2^{n-1}}$ with the orientation reversed, as can be seen from the itineraries. Thus the Markov graph of Q_L under f^2 will have linked loops of lengths 2^{n-1} and 2^n, denoted L_{n-1} and L_n, respectively.

Let V_1 and V_2 be vertices in L_n, with $V_1 \to V_2$. This means that $V_2 \subset f^2(V_1)$. Since periodic orbits form Markov Partitions, there must exists interval $U_1 \subset J_R$ such that $f(V_1) = U_1$ and $f(U_2) \supset V_2$. If we now consider the Markov graph of Q_n, this means that the graph contains a loop λ_{n+1} where every other vertex of λ_{n+1} is an element of L_n. A similar argument can be made using L_{n-1} to create loop λ_n. Moreover, since L_n and L_{n-1} are linked, the loops λ_{n+1} and λ_n must be linked.

Now we consider loops of lengths 2^k for $k < n$. Since $\underline{P}_{3\cdot 2^n}$ is a min-max, then it cannot dominate any other period $3 \cdot 2^n$ orbit. In particular, this means that the Markov graph of Q_n can only generate one orbit of length $3 \cdot 2^n$: the one generated by passing through the two linked loops. However, if the graph had other linked loops of lengths 2^k for $k < n$, then we could generate other period $3 \cdot 2^n$ orbits, which would

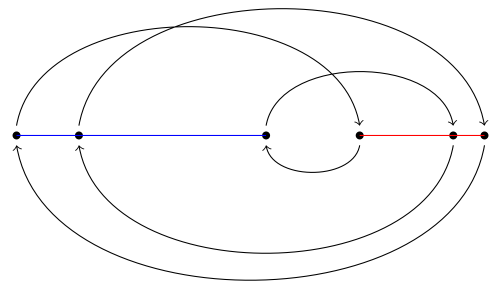


Fig. 2.2. The period 6 orbit with itinerary $[R * (RLR)]^\infty$. Notice that the first return map on the blue interval is f^2 and that the three points form a periodic orbit under f^2.

lead to a contradiction. However, it follows from Sharkovsky's Theorem that Q_n must force period 2^k orbits for all k. Thus the Markov graph of Q_n must contain isolated loops of lengths 2^k for $0 \leq k < n$. ∎

For ease, we will often relax our language when speaking about periodic orbits associated to min-maxes. We will simply refer to the associated Markov graph as the Markov graph of $\underline{P}_q$.

3. PERIODIC ORBITS OF PIECEWISE MONOTONE MAPS

In Section 2.2 we discussed the Sharkovsky order for continuous maps. However a discontinuous analogue of this order does not yet exist. In other words, given a discontinuous map g on $\mathbb{R}$ with a periodic point of period q, it is not known what other periods of periodic points must occur for g. In full generality, this problem can be difficult to study. Therefore, we shall restrict our discussion to piecewise monotone maps; in particular, we shall study maps with two laps, one increasing on the left and the other decreasing on the right. In Section 2.1, we denoted the family of these "unimodal" maps by $\mathcal{F}$. By Proposition 2.1.10, for every $f \in \mathcal{F}$ there is a parameter $(\!(a, b)\!) \in \mathcal{TS}$ such that $T_{a,b}$ and f exactly the same set of itineraries on their cores. We can additionally show that $T_{a,b}$ and f have the exact same set periodic points everywhere, with one potential exception being a fixed point. Therefore, to study periodic points of maps from $\mathcal{F}$, it suffices to study the two parameter family $\mathcal{TS}$.

In Section 2.2 we stated that the Sharkovsky order may be interpreted as the order in which periods disappear as the tent map is continuously truncated. This inspires a natural two parameter analogue for two-sided truncated tent maps. if we allow the left and right truncations to move independently, how does this affect the "thresholds" for periodic orbits? Let us consider the period 3 orbits in Figure 2.1. We shall now allow the left and right truncations to move independently. For a continuous truncated tent map, there could be no other periodic orbits of period 3 once the orbit with itinerary $(RLR)^\infty$ was removed. However, Figure 3.1 shows such a map $T_{a,b}$ with periodic orbit of itinerary $(RLL)^\infty$, but no orbit with itinerary $(RLR)^\infty$. Thus, we can see that the geometric notion offered by the min-max in the continuous case does not generalize to the discontinuous case. Furthermore, using similar ideas we can generate a function

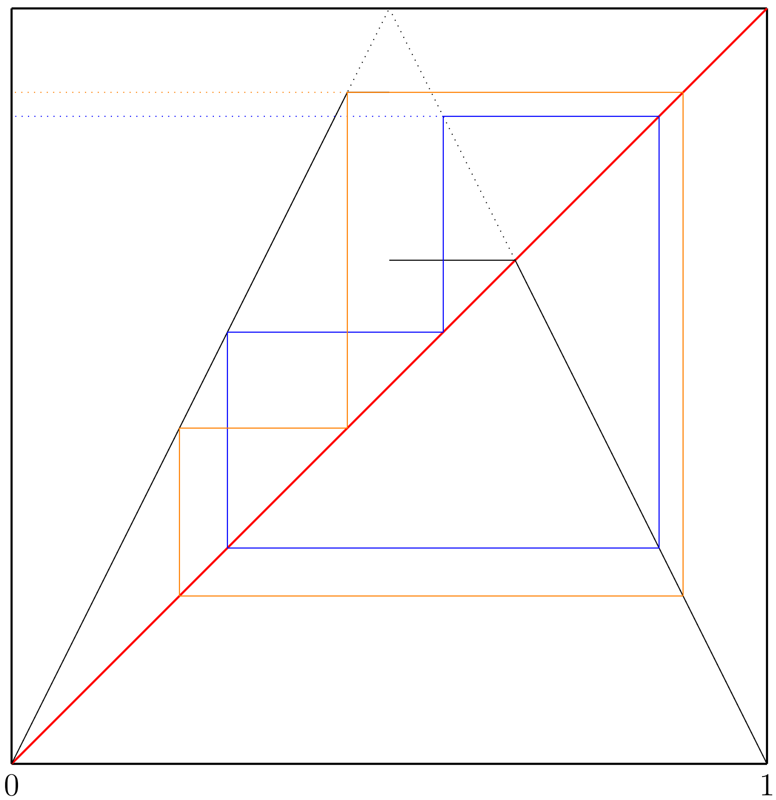


Fig. 3.1. This truncated tent map cannot have the orbit $(RLR)^\infty$. However, it does have the orbit $(RLL)^\infty$.

which has a period 2 point, but not period 3 point (see Figure 3.2). Hence we can also see that the Sharkovsky order also fails to generalize to the discontinuous case.

Let Q be a periodic orbit under the full tent map T, $x_L = \max\{x \in Q \,|\, x < \frac{1}{2}\}$, and $x_R = \min\{x \in Q \,|\, x > \frac{1}{2}\}$. We now formulate the 2-parameter analogue of Proposition 2.2.5. The proof follows easily by using equation 2.4.

Lemma 3.0.1 *Suppose that* $(\!(a, b)\!) \in \mathcal{TS}$ *so that* $\underline{I}_T(T(a))$ *is left maximal and* $\underline{I}_T(T(b))$ *is right maximal. Let* Q *be a periodic orbit such parameter* $(\!(x_L, x_R)\!)$ *defined as before. Then* Q *is a periodic orbit of* $T_{a,b}$ *if and only if* $x_L \leq a$ *and* $x_R \geq b$.

The parameter $(\!(x_L, x_R)\!)$ functions as a threshold on the parameters for which Q exists as an orbit of $T_{a,b}$. The parameter $(\!(x_L, x_R)\!)$ shall be called the *peak* of the

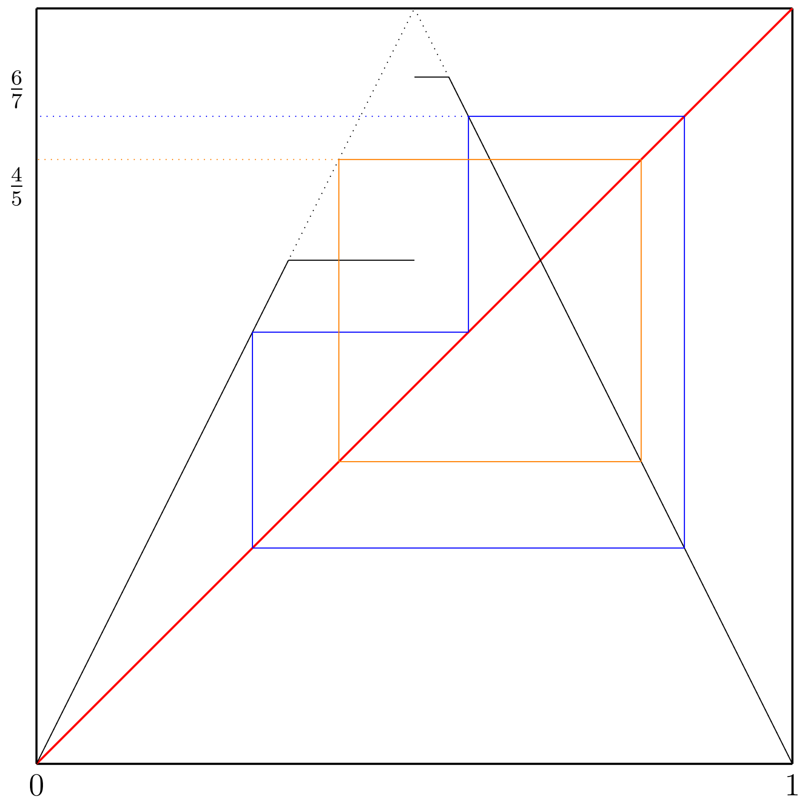


Fig. 3.2. This truncated tent map cannot have the orbit $(RL)^\infty$. However, it does have the orbit $(RLR)^\infty$.

periodic orbit Q. We shall denote the set of all peaks by $\mathcal{P}$. Since $\underline{I}_T(T(x_L))$ and $\underline{I}_T(T(x_R))$ are left and right maximal, respectively, Lemma 3.0.1 gives us an intuitive geometric interpretation of the relationship between peaks.

Lemma 3.0.2 *Let Q_1 and Q_2 be periodic orbits under T with peaks $(\!(x_1, y_1)\!)$ and $(\!(x_2, y_2)\!)$, respectively. Then T_{x_2,y_2} has periodic orbit Q_1 if and only if $x_1 \le x_2$ and $y_1 \ge y_2$.*

Proof This is immediate from Lemma 3.0.1.

Figure 3.3 details this geometric interpretation. We are also able to generate an image of the set $\overline{\mathcal{P}}$. With this geometric intuition, we would now like to begin

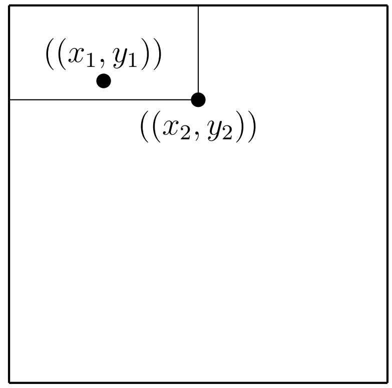


Fig. 3.3. Proposition 3.0.2.

attacking this period ordering problem. In Section 3.1, we analyze special vertical and horizontal families of $\mathcal{TS}$. These families will demonstrate how the "forcing" orders of periods can vastly differ from the continuous case.

In Section 3.2, we identify peaks which force no other periodic orbits, known as extremal points. Section 3.3 will be devoted entirely to characterizing the structure of the set $\overline{\mathcal{P}}$.

■

3.1 Horizontal/Vertical Families and Order Type

We will be investigating how a periodic orbit of period n forces a periodic orbit of period m. We shall write $n \underset{per}{\Longrightarrow} m$ if the existence of a period n point forces the existence of a period m point. In Section 3.2 we shall see that, in full generality, given a periodic orbit of period n we cannot guarantee the existence of another periodic point. Here we will study the relation $\underset{per}{\Longrightarrow}$ for particular vertical and horizontal families in $\mathcal{TS}$ and observe the differences from the relation $\underset{per}{\Longrightarrow}$ in the continuous case, i.e., the Sharkovsky order.

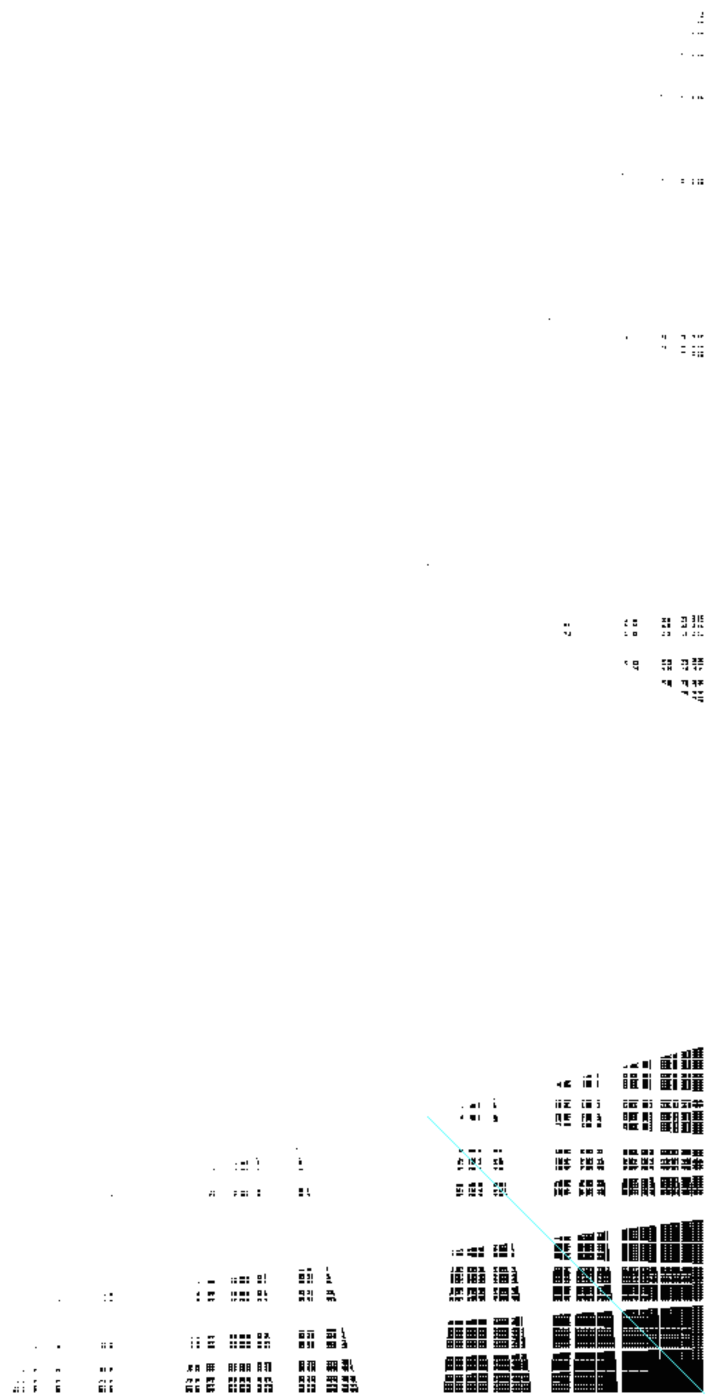


Fig. 3.4. This is a picture featuring all peaks up to period 20. Note that the reverse diagonal corresponds to continuous functions.

Proposition 3.1.1 *The relation* $\underset{per}{\Longrightarrow}$ *forms a linear ordering on vertical and horizontal families in* $\mathcal{TS}$.

Proof Let H be a horizontal family in $\mathcal{TS}$, i.e., $H = \{((a,b))|b = b_0\}$. Fix a period q. Then for every periodic orbit Q of period q, there is a peak $((\alpha_Q, \beta_Q))$ associated to it. Let

$$P_{H,q} = \{((\alpha_Q, \beta_Q))|\beta_Q \geq b_0\}.$$

Since there are only finitely many periodic orbits of period q, there must be a value a_q which is minimal among the α_Q which occur in $P_{H,q}$. Clearly, if $q_1 \neq q_2$, then $a_{q_1} \neq a_{q_2}$. Therefore, the set of a_q, $q = 1, \ldots, \infty$, has some linear order in $[0, \frac{1}{2}] \times b_0$. By Lemma 3.0.2, the order on the indices of a_q must coincide with the order induced by $\underset{per}{\Longrightarrow}$ on H. An analogous proof works for vertical families. ■

We will begin by studying the one-dimensional families $H_n = \{((a,b))|b = \frac{2^n-1}{2^n}\}$ in $\mathcal{TS}$, $n \geq 2$. It will be necessary for us to keep track of the number of Rs in an itinerary. If $\underline{A}$ denotes some finite sequence of Rs and Ls, define $\gamma(\underline{A})$ to be the number of Rs in that sequence. Define the substitution

$$s_k(A) = \begin{cases} L & \text{if} \quad A = L \\ L^k R & \text{if} \quad A = R, \end{cases} \tag{3.1}$$

and define $S_k(\underline{A}) = s_k(A_0)s_k(A_1)s_k(A_2)\ldots$, $k \geq 1$. Clearly $S_{k+1}(\underline{A}) = S_1(S_k(\underline{A}))$.

Lemma 3.1.2 *If* $\underline{A} < \underline{B}$, *then* $S_k(\underline{A}) < S_k(\underline{B})$.

Proof Let $\underline{A} = \underline{E}a_m a_{m+1}\ldots$ and $\underline{B} = \underline{E}b_m b_{m+1}\ldots$, where $\underline{E}$ is a finite sequence of Ls and Rs and $a_m \neq b_m$. Then we must consider two cases: $a_m = R$ and $a_m = L$. First note that the parity of $S_k(\underline{E})$ is the same as that of $\underline{E}$. The substitution yields:

$$\begin{gathered} S_k(\underline{A}) = S_k(\underline{E})s_k(a_m)s_k(a_{m+1})\ldots \\ \textit{and} \\ S_k(\underline{B}) = S_k(\underline{E})s_k(b_m)s_k(b_{m+1})\ldots. \end{gathered} \tag{3.2}$$

If $a_m = L$, then the parity of $\underline{E}$ and $S_k(\underline{E})$ is even and thus $b_m = R$. So $s_k(a_m) = L$ and $s_k(b_m) = L^k R$. Thus (3.2) becomes

$$\begin{aligned} S_k(\underline{A}) &= S_k(\underline{E})L\dots \\ &\mathit{and} \\ S_k(\underline{B}) &= S_k(\underline{E})L^k R s_k(b_{m+1})\dots. \end{aligned} \tag{3.3}$$

There are two possibilities: either $\underline{A} = \underline{E}L^{k+1}\dots$ or $\underline{A} = \underline{E}LR\dots.$ In either case, the equations in 3.3 become

$$\begin{aligned} S_k(\underline{A}) &= S_k(\underline{E})L^{k+1}\dots \\ &\mathit{and} \\ S_k(\underline{B}) &= S_k(\underline{E})L^k R s_n(b_{m+1})\dots. \end{aligned}$$

Since the L does not affect the parity, $S_k(\underline{A}) < S_k(\underline{B})$. The case of this proof when $a_m = R$ is very similar and will be omitted. ■

Lemma 3.1.3 *Itineraries of points via $T_{\frac{1}{2},\frac{2^n-1}{2^n}}(x)$, $n \geq 2$, do not contain consecutive Rs.*

Proof An itinerary $\underline{I}_{T_{\frac{1}{2},\frac{2^n-1}{2^n}}}(x)$ contains two consecutive Rs if and only if there exists some $m \geq 0$ such that $T^m_{\frac{1}{2},\frac{2^n-1}{2^n}}(x) > \frac{1}{2}$ and $T^{m+1}_{\frac{1}{2},\frac{2^n-1}{2^n}}(x) > \frac{1}{2}$. However, this can only happen if $T^m_{\frac{1}{2},\frac{2^n-1}{2^n}}(x) \in (\frac{1}{2}, \frac{3}{4})$. By definition, $T_{\frac{1}{2},\frac{2^n-1}{2^n}}$ sends elements in $(\frac{1}{2}, \frac{3}{4})$ to $\frac{1}{2^{n-1}}$. Thus if $T^m_{\frac{1}{2},\frac{2^n-1}{2^n}}(x) \in (\frac{1}{2}, \frac{3}{4})$, then the $(m+1)$ term of $\underline{I}_{T_{\frac{1}{2},\frac{2^n-1}{2^n}}}(x)$ must be L (or C if $n = 2$). ■

By finding the induced map of $T_{\frac{1}{2},\frac{2^n-1}{2^n}}$ on the interval $[0, \frac{1}{2}]$, we arrive at a simple inductive relationship for the itineraries of $T_{\frac{1}{2},\frac{2^n-1}{2^n}}$.

Proposition 3.1.4 *Let $\mathcal{I}_L(T_{\frac{1}{2},\frac{2^n-1}{2^n}})$ be the set of itineraries under $T_{\frac{1}{2},\frac{2^n-1}{2^n}}$ which begin with an L. Then $\mathcal{I}_L(T_{\frac{1}{2},\frac{2^n-1}{2^n}}) = S_{n-1}(\mathcal{I}(T))$, where T is the full tent map.*

Proof We argue by induction. Let $n = 2$ and begin by finding the first return map of $T_{\frac{1}{2},\frac{3}{4}}$ on the interval $[0, \frac{1}{2}]$.

Denote this first return map by ρ_2. Then ρ_2 can be written as follows (see Figure 3.5):

$$\rho_2(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{2} & \text{if} \quad \frac{1}{4} \leq x \leq \frac{3}{8}, \\ 2 - 4x & \text{if} \quad \frac{3}{8} \leq x \leq \frac{1}{2}. \end{cases} \tag{3.4}$$

Note that the set of itineraries for $\rho_2(x)$ will be $\mathcal{I}(T)$. Moreover, if $\underline{A}$ is an itinerary of a point under the map ρ_2, then this corresponds to the itinerary $S_1(\underline{A})$ under $T_{\frac{1}{2},\frac{3}{4}}$. This can be seen as a point x in $[\frac{3}{8}, \frac{1}{2}]$ is to the left of $\frac{1}{2}$ and is mapped exactly once to the right of $\frac{1}{2}$ before returning to $[0, \frac{1}{2}]$. Thus an R under the action of ρ_2 corresponds to a block LR under the action of $T_{\frac{1}{2},\frac{3}{4}}$.

Now suppose that $\mathcal{I}(T_{\frac{1}{2},\frac{2^n-1}{2^n}}) = S_{n-1}(\mathcal{I}(T))$ and consider $\mathcal{I}(T_{\frac{1}{2},\frac{2^{n+1}-1}{2^{n+1}}})$. Denote by ρ_n the first return map of $T_{\frac{1}{2},\frac{2^n-1}{2^n}}$ on the interval $[0, \frac{1}{2}]$. Then ρ_n can be written as follows:

$$\rho_n(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x \leq \frac{1}{4}, \\ \frac{1}{2^n} & \text{if} \quad \frac{1}{4} \leq x \leq \frac{2^{n+1}-1}{2^{n+2}}, \\ 2 - 4x & \text{if} \quad \frac{2^{n+1}-1}{2^{n+2}} \leq x \leq \frac{1}{2}. \end{cases} \tag{3.5}$$

Notice that the set of itineraries for ρ_n will be $\mathcal{I}(T_{\frac{1}{2},\frac{2^n-1}{2^n}}) = S_{n-1}(\mathcal{I}(T))$. Therefore, if $\underline{A}$ is an itinerary of a point under the map ρ_n, then this corresponds to the itinerary $S_1(\underline{A})$ under $T_{\frac{1}{2},\frac{2^n-1}{2^n}}$. Thus

$$\mathcal{I}(T_{\frac{1}{2},\frac{2^{n+1}-1}{2^{n+1}}}) = S_1(\mathcal{I}(T_{\frac{1}{2},\frac{2^n-1}{2^n}})) = S_1(S_{n-1}(\mathcal{I}(T)) = S_n(\mathcal{I}(T)).$$

■

Using Lemma 3.1.2, we can see that if $\underline{A}$ dominates $\underline{B}$, then $S_k(\underline{A})$ dominates $S_k(\underline{B})$. However, it would be a mistake to conclude that min-maxes are preserved under S_k. Unfortunately, S_k radically changes the period of a periodic itinerary based on the number or Rs. As a result, the order induced by $\underset{per}{\Longrightarrow}$ is not the same as the

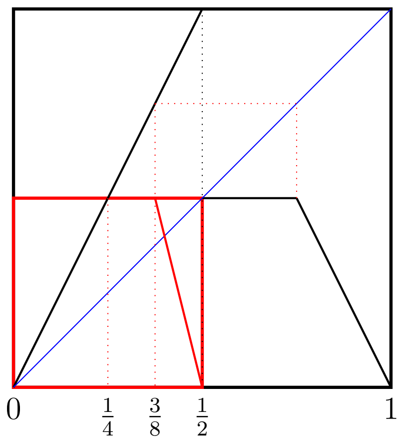


Fig. 3.5. Graph of $T_{\frac{1}{2},\frac{3}{4}}$ with first return graph on interval $[0,\frac{1}{2}]$.

Sharkovsky order. Using 3.0.2 and 3.1.1, we can understand the order induced by $\underset{per}{\Longrightarrow}$ to be the order in which periods disappear as we truncate the maps $T_{a,\frac{2^n-1}{2^n}}$. Since in 3.1.4 we consider the induced map on the interval $[0,\frac{1}{2}]$, a truncated map $T_{a,\frac{2^n-1}{2^n}}$ will have a truncated first return map, which we will denote $\rho_{n,a}$. Here we shall restrict our arguments to the case $n = 2$.

Remark 3.1.5 *The following arguments will follow inductively for general $n \geq 2$. However, we use $n = 2$ for simplicity.*

Lemma 3.1.6 *The induced map of $T_{a,\frac{3}{4}}$ on the interval $[0,\frac{1}{2}]$ will be*

$$\rho_{2,a} = \begin{cases} 2x & \textit{if } \ 0 \leq x \leq a, \\ \frac{1}{2} & \textit{if } \ a \leq x \leq \frac{3}{8}, \\ 2-4x & \textit{if } \ \frac{3}{8} \leq x \leq a, \\ 2-4a & \textit{if } \ a \leq x \leq \frac{1}{2}. \end{cases} \tag{3.6}$$

Moreover, as $a \to 0$, $\rho_{2,a}$ loses periods of periodic orbits in the Sharkovsky order.

Proof Finding the equation of the first return map $\rho_{2,a}$ is a simple calculation. To see that $\rho_{2,a}$ loses periods in the Sharkovsky order, simply recall that as we truncate the full tent map (and ρ_1 is essentially a full tent map) we lose periods in the Sharkovsky order. Moreover, note that a truncated tent map $T_{a,1-a}$ will have the same set of itineraries as a map

$$T^* = \begin{cases} 2x & \text{if } \; 0 \le x \le \frac{1}{2}, \\ 2-2x & \text{if } \; \frac{1}{2} \le x \le 2a, \\ 2-4a & \text{if } \; 2a \le x \le 1. \end{cases} \tag{3.7}$$

Therefore, the truncation of ρ_2 from the right (which is what happens for $\rho_{2,a}$) yields the same result as if it is truncated from the top. Hence as $a \to 0$, $\rho_{2,a}$ loses periods of periodic orbits in the Sharkovsky order. ■

Consider the Markov graphs of periodic points under $T_{\frac{1}{2},\frac{3}{4}}$. Since $T_{\frac{1}{2},\frac{3}{4}}$ is not unimodal, we cannot expect the results of Section 2.3 to hold here. However, note that the first return map ρ_2 is unimodal. Thus, we may question how the substitution S_1 relates the Markov graphs of $T_{\frac{1}{2},\frac{3}{4}}$ and ρ_2.

Lemma 3.1.7 *Let Q be a periodic orbit under $T_{\frac{1}{2},\frac{3}{4}}$ and Q_ρ be the corresponding periodic orbit under the induced map ρ_1. Let G be the Markov graph of Q under $T_{\frac{1}{2},\frac{3}{4}}$ and G_ρ the Markov graph of Q_ρ under ρ_1. Then for every path λ_ρ in G_ρ which does not pass through the turning point c, there exists path λ in G with the properties:*

1. *If $V_1 \to V_2$ is in λ_ρ and V_1 has interior entirely left of $c = \frac{1}{4}$, then λ contains the segment $V_1 \to V_2$.*
2. *If $V_1 \to V_2$ is in λ_ρ and V_2 has interior entirely right of $c = \frac{1}{4}$, then λ contains the segment $V_1 \to V_1^* \to V_2$, where V_1 has interior in $[0, \frac{1}{2}]$ and V_1^* has interior in $[\frac{1}{2}, 1]$.*

Despite S_k changing periods, some min-maxes are, in a certain sense, preserved under S_k. The min-max $(\underline{P}_{2^n})^\infty$ dominates only sequences of the form $(\underline{P}_{2^m})^\infty$, with $m < n$. We observe how the period changes under S_1.

Lemma 3.1.8 *Let* $r_n = \gamma(\underline{P}_{2^n})$. *Then* $r_0 = 1$ *and*

$$r_n = \begin{cases} 2r_{n-1} + 1 & \textit{if } \; n \textit{ is even,} \\ 2r_{n-1} - 1 & \textit{if } \; n \textit{ is odd.} \end{cases}$$

Proof We argue inductively. For $n = 0$, we have that $P_1 = R^\infty$, and hence $r_0 = 1$. Now let n be even and suppose that $r_{n-1} = 2r_{n-2} - 1$. Since $\underline{P}_{2^n} = R * \underline{P}_{2^{n-1}}$, it follows from the definition of $*$-product that

$$\gamma(\underline{P}_{2^n}) = 2^{n-1} + [2^{n-1} - r_{n-1}] = 2^n - r_{n-1}. \tag{3.8}$$

Using our inductive hypothesis for r_{n-1}, equation 3.8 becomes

$$\gamma(\underline{P}_{2^n}) = 2^n - [2r_{n-2} - 1] = 2[2^{n-1} - r_{n-2}] + 1 = 2r_{n-1} + 1.$$

A symmetric proof will work when n is odd. Hence we get the result

$$\gamma(P_{2^n}) = r_n = \begin{cases} 2r_{n-1} + 1 & \text{if } \; n \text{ is even,} \\ 2r_{n-1} - 1 & \text{if } \; n \text{ is odd.} \end{cases}$$

. ■

Proposition 3.1.9 *The periods of* $T_{\frac{1}{2},\frac{3}{4}}$ *which force finitely many periods are those periods* A_n*, where* $A_0 = 2$ *and*

$$A_n = \begin{cases} 2A_{n-1} + 1 & \textit{if } \; n \textit{ is even,} \\ 2A_{n-1} - 1 & \textit{if } \; n \textit{ is odd.} \end{cases}$$

In particular, the linear order of the relation $\underset{per}{\Longrightarrow}$ *associated with the family* H_2 *contains the sequence*

$$\ldots \underset{per}{\Longrightarrow} A_n \underset{per}{\Longrightarrow} \ldots \underset{per}{\Longrightarrow} 7 \underset{per}{\Longrightarrow} 3 \underset{per}{\Longrightarrow} 2. \tag{3.9}$$

Proof Let ρ_2 be as in Proposition 3.1.4. Then for every itinerary $\underline{A}$, there is a point $x \in [0, \frac{1}{2}]$ with $I_{\rho_2}(x) = \underline{A}$. In particular, there exists points x_n with $\underline{I}_{\rho_2}(x_n) = (\underline{P}_{2^n})^\infty$, for every n. Additionally, itinerary $(\underline{P}_{2^n})^\infty$ only dominates periodic orbits of the form $(\underline{P}_{2^k})^\infty$, $k \leq n$. Therefore, by Lemma 3.1.2, every sequence $\sigma \circ S_1(\underline{P}_{2^n})^\infty$ only dominates periodic orbits of the form $(\sigma \circ S_1(\underline{P}_{2^k}))^\infty$, $k \leq n$. Hence the periods of $T_{\frac{1}{2},\frac{3}{4}}$ which force only finitely many other periods will be $A_n = |S_1(\underline{P}_{2^n})|$. By Lemma 3.1.8

$$\gamma(\underline{P}_{2^n}) = r_n = \begin{cases} 2r_{n-1} + 1 & \text{if } \ n \text{ is even,} \\ 2r_{n-1} - 1 & \text{if } \ n \text{ is odd.} \end{cases}$$

Hence we find that

$$A_n = |S_1(\underline{P}_{2^n})| = 2^n + r_n = \begin{cases} 2^n + 2r_{n-1} + 1 = 2A_{n-1} + 1 & \text{if } \ n \text{ is even,} \\ 2^n + 2r_{n-1} - 1 = 2A_{n-1} - 1 & \text{if } \ n \text{ is odd.} \end{cases}$$

Therefore the forcing ordering of $T_{\frac{1}{2},\frac{3}{4}}$ contains the sequence (3.9). ■

Remark 3.1.10 *Note that* $A_n + A_{n+1} = 5 \cdot 2^n$.

Two ordered sets X and Y with orders $<_x$ and $<_y$ are said to have the same *order type* if there exists a bijection f between X and Y such that $a_1 <_x b_1$ implies $f(a_1) <_y f(b_1)$ and $a_2 <_y b_2$ implies $f^{-1}(a_2) <_x f^{-1}(b_2)$. As a simple example, the natural numbers, with standard order, have the same order type as the positive even integers. However, the natural numbers have a different order type then, say, the Sharkovsky order.

The line $b = 1 - a$ in $\mathcal{TS}$ represents the parameters corresponding to continuous truncated tent maps. The relation $\underset{per}{\Longrightarrow}$ on this family is exactly the order given by $>_S$. A priori, it is possible that the order type of relation $\underset{per}{\Longrightarrow}$ for all horizontal and vertical families is the same as that of $>_S$. However, this turns out not to be the case. We show this by finding the order type of family H_2.

Proposition 3.1.11 $|S_1(\underline{P}_{3\cdot 2^n})| = 5 \cdot 2^n$, *where* $n \geq 0$.

Proof Since $s_1(R) = LR$ and $s_1(L) = L$, then $|S_1(\underline{P}_{3\cdot 2^n})| = |\underline{P}_{3\cdot 2^n}| + \gamma(\underline{P}_{3\cdot 2^n})$. It is known that $\underline{P}_{3\cdot 2^n} = R^{*n} * (RLR) = R * R * \cdots * R * (RLR)$. Using this, we wish to show that $\gamma(\underline{P}_{3\cdot 2^n}) = 2^{n+1}$. We proceed by induction. The $k = 0$ case is trivial. Therefore, assume that $\gamma(\underline{P}_{3\cdot 2^k}) = 2^{k+1}$. Then $\underline{P}_{3\cdot 2^{k+1}} = R * \underline{P}_{3\cdot 2^k}$. Hence $\gamma(\underline{P}_{3\cdot 2^{k+1}}) = 3 \cdot 2^k + (3 \cdot 2^k - 2^{k+1}) = (3-1) \cdot 2^{k+1} = 2^{k+2}$. Finally, we may conclude that $|\underline{P}_{3\cdot 2^n}| + \gamma(\underline{P}_{3\cdot 2^n}) = 3 \cdot 2^n + 2^{n+1} = 5 \cdot 2^n$. ∎

Since the relation $\underset{per}{\Longrightarrow}$ for H_2 is linearly ordered , then for every q there exists a_q such that $T_{a,\frac{3}{4}}$ has a periodic point of period q if and only if $a \geq a_q$. In other words, a_q is acting as a lower threshold on the parameters where a period q point exists for $T_{a,\frac{3}{4}}$. Let $(\underline{D}_q)^\infty = \underline{I}_T(T(a_q))$. This $\underline{D}_q$ acts as a "min-max" of period q in H_2.

Let us again consider $T_{\frac{1}{2},\frac{3}{4}}$ with induced map $\rho_2 : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$. Earlier, we determined that $S_1(I_{\rho_2}(x)) = I_{T_{\frac{1}{2},\frac{3}{4}}}(x)$. Now since $\underset{per}{\Longrightarrow}$ is linearly ordered in H_2, we can obtain this order by considering increasingly shorter truncations of $T_{a,\frac{3}{4}}$. In other words, the order induced by $\underset{per}{\Longrightarrow}$ is exactly the order periods disappear under the maps $T_{a,\frac{3}{4}}$ as a approaches 0. Here we would like to point out that the truncated tent map $T_{a,\frac{3}{4}}$ has a corresponding truncated induced map $\rho_{2,a}$ on the interval $[0, \frac{1}{2}]$. The map $\rho_{2,a}$ corresponds exactly to the continuous truncated tent map.

Proposition 3.1.12 $\underline{D}_{5\cdot 2^n} = \sigma(S_1(\underline{P}_{3\cdot 2^n}))$.

Proof We consider the Markov graph of $\sigma(S_1(\underline{P}_{3\cdot 2^n}))$, call it G_σ. If G is the Markov graph of $\underline{P}_{3\cdot 2^n}$, then by 2.3.4 G has linked loops of lengths 2^n and 2^{n+1}. Additionally, these loops follow orbit patterns of $\underline{P}_{2^n}$ and $\underline{P}_{2^{n+1}}$ and G has no other linked loops. Therefore, by Lemma 3.1.7, G_σ must contain linked loops of lengths A_n and A_{n+1}. By Remark 3.1.9, G_σ generates only one periodic orbit of length $5 \cdot 2^n$. This orbit must be $\sigma(S_1(\underline{P}_{3\cdot 2^n}))$. Since $\sigma(S_1(\underline{P}_{3\cdot 2^n}))$ forces only one periodic orbit of length $5 \cdot 2^n$, then it must be that $\sigma(S_1(\underline{P}_{3\cdot 2^n})) = \underline{D}_{5\cdot 2^n}$.

∎

Proposition 3.1.13 *A periodic point of period* $5 \cdot 2^n$ *will force periodic orbits of all but finitely many periods* k *under the map* $T_{\frac{1}{2},\frac{3}{4}}$.

Proof Let $\underline{D}_{5\cdot 2^n}$ be the min-max of period $5 \cdot 2^n$. This min-max will have a Markov graph with linked loops of length A_n and A_{n+1}. Since $(A_n, A_{n+1}) = 1$ for all n, then the equation

$$p \cdot A_n + q \cdot A_{n+1} = k,$$

with $p, q \geq 1$, will have a solution for sufficiently large k. Therefore a periodic point of period $5 \cdot 2^n$ will force all but finitely many periods. ■

Consider the periodic orbit $S_1((RLR)^\infty) = (RLLRL)^\infty = \underline{D}_5$. Then the Markov graph of this periodic orbit under $T_{a_5, \frac{3}{4}}$ must contain the subgraph

$$\begin{tikzcd} I_1 \arrow[r] & I_2 \arrow[r, shift left] & I_3 \arrow[l, shift left] \arrow[ll, bend left] \end{tikzcd} \tag{3.10}$$

Using the subgraph in 3.10 and Lemma 2.3.2, the Markov graph associated to $(RLLRL)^\infty$ generates periodic points of all periods. Therefore, $5 \underset{per}{\Longrightarrow} q$ for every q. By Proposition 3.1.13, the relation $\underset{per}{\Longrightarrow}$ in H_2 has the sequence

$$5 \underset{per}{\Longrightarrow} \ldots \underset{per}{\Longrightarrow} 5 \cdot 2 \underset{per}{\Longrightarrow} \ldots \underset{per}{\Longrightarrow} 5 \cdot \underset{per}{\Longrightarrow} \ldots . \tag{3.11}$$

Compare this with Proposition 3.1.9 and we arrive at the following:

Theorem 3.1.14 *The relation* $\underset{per}{\Longrightarrow}$ *in* H_2 *has order*

$$5 \underset{per}{\Longrightarrow} \ldots \underset{per}{\Longrightarrow} 5 \cdot 2 \underset{per}{\Longrightarrow} \ldots \underset{per}{\Longrightarrow} 5 \cdot 2^n \underset{per}{\Longrightarrow} \ldots | \tag{3.12}$$

$$\ldots \underset{per}{\Longrightarrow} A_n \underset{per}{\Longrightarrow} \ldots \underset{per}{\Longrightarrow} 3 \underset{per}{\Longrightarrow} 2, \tag{3.13}$$

where $5 \cdot 2^n$ *and* $5 \cdot 2^{n+1}$ *have finitely many integers between them.*

Corollary 3.1.15 *The ordering of periods induced by relation* $\underset{per}{\Longrightarrow}$ *in a one parameter subfamily of* $\mathcal{TS}$ *need not have the same order type as the Sharkovsky order.*

The order type observed in the family $((a, \frac{3}{4}))$ is not unique to this family. Consider the vertical family $((\frac{3}{8}, b))$. We claim that the order type of the period forcing order of this family is the same as that of the family $((a, \frac{3}{4}))$. In fact, we can explicitly calculate the period forcing order for the family $((\frac{3}{8}, b))$.

Lemma 3.1.16 *Let $\underline{I}_{T_{\frac{3}{8},b}}(x)$ be the itinerary of a point under $T_{\frac{3}{8},b}$. Then $\underline{I}_{T_{\frac{3}{8},b}}(x)$ cannot contain the finite subsequence LRL.*

Proof Under the tent map, an itinerary $\underline{I}_T(x)$ contains the sequence LRL if and only if there exists n such that $T^n(x) \in (\frac{3}{8}, \frac{1}{2})$. However, $T_{\frac{3}{8},b}$ sends elements of the interval $[\frac{3}{8}, \frac{1}{2}]$ to $\frac{3}{4}$. ■

Lemma 3.1.17 *Štefan sequences $(RLR^k)^\infty$ are contained in $\mathcal{I}(T_{\frac{3}{8},\frac{1}{2}})$.*

Proof Štefan sequences do not contain the subsequence LRL, which gives us the conclusion. ■

Lemma 3.1.18 *Markov Graphs of Štefan orbits of period $n+1$ under $T_{\frac{3}{8},\frac{1}{2}}$,must contain linked loops of lengths n and 1. In particular, the linear order on periods induced by $\underset{per}{\Longrightarrow}$ in the family $(\!(\frac{3}{8}, b)\!)$ will contain the sequence*

$$3 \underset{per}{\Longrightarrow} 5 \underset{per}{\Longrightarrow} 7 \underset{per}{\Longrightarrow} 9 \underset{per}{\Longrightarrow} \dots \tag{3.14}$$

Proof From Lemma 2.3.3, it is known that Markov graphs of Štefan orbits have linked loops of length 1 and $q-1$ in the continuous case. Thus we need only show that the truncation associated to parameter $(\!(\frac{3}{8}, \frac{1}{2})\!)$ does not annihilate this structure.

Consider then the intervals I_k from Lemma 2.3.3. Štefan orbits have periodic itinerary LR^n. If Q is the periodic orbit associated to LR^n, then the elements of Q divide $\langle Q \rangle$ into n intervals. Listed from left to right, these intervals are

$$I_n, I_{n-2}, I_{n-4}, \dots I_2, I_1, I_3, \dots I_{n-3}, I_{n-5}$$

Note that if x_L is the left most point of Q, then $x_L < \frac{3}{8}$, and therefore the map $T_{\frac{3}{8},\frac{1}{2}}$ has periodic point with itinerary $(LR^n)^\infty$ for all n. Moreover, $\frac{3}{8} \in I_n$ and so we must show that $T(\frac{3}{8}) = \frac{3}{4}$ is not contained in I_1 so that $T(I_n)$ covers I_1. However, this can be checked using itineraries. Note that $\underline{I}_T(\frac{3}{4}) = RC$. If y_R is the right endpoint of I_1, then $\underline{I}_T(y_R) = RR\dots$, and thus $y_R < \frac{3}{4}$. Hence the Markov graph of LR^n under $T_{\frac{3}{8},\frac{1}{2}}$ contains the loop of length n and 1 ■

Lemma 3.1.19 *The period forcing order for the family* $(\!(\frac{3}{8}, b)\!)$ *contains the sequence*

$$\ldots \underset{per}{\Longrightarrow} 12 \underset{per}{\Longrightarrow} 8 \underset{per}{\Longrightarrow} 6 \underset{per}{\Longrightarrow} 4. \tag{3.15}$$

Proof For a period $2(k+1)$ the Markov graph associated to the periodic point with itinerary $(RLR^{2k})^\infty$ will have non-connected loops of length $2 \cdot m$ for every $1 < m \leq k$. This gives the conclusion. Therefore, period $2(k+1)$ can only force periods $2 \cdot m$. ■

Theorem 3.1.20 *The order on periods induced by* $\underset{per}{\Longrightarrow}$ *for the family* $(\!(\frac{3}{8}, b)\!)$ *is the sequence*

$$3 \underset{per}{\Longrightarrow} 5 \underset{per}{\Longrightarrow} 7 \underset{per}{\Longrightarrow} \ldots | \ldots \underset{per}{\Longrightarrow} 8 \underset{per}{\Longrightarrow} 6 \underset{per}{\Longrightarrow} 4 \tag{3.16}$$

So far we have seen two ordinal types for one-parameter families. However, vertical and horizontal families have demonstrated only one order type. Thus, we end this section with a conjecture.

Conjecuture 3.1.21 *The order type on periods induced by relation* $\underset{per}{\Longrightarrow}$ *is the same for all vertical and horizontal families in* $\mathcal{TS}$.

3.2 Extremal Points

Let $(\!(a, b)\!)$ be a peak. We say $(\!(a, b)\!)$ is extremal if the only periodic orbits of $T_{a,b}$ are $orb(a)$ and the fixed orbit 0. Now we would like to address the following question: can the set $\mathcal{E}$ of extremal points of $\mathcal{TS}$ be fully characterized? These points loosely reflect the boundary of the regions in $\mathcal{TS}$ which force non-fixed periodic orbits. We shall begin by considering the following three sets of peaks:

- $\mathcal{E}(RL^n)$, the set of peaks corresponding to periodic orbits of the form $(RL^n)^\infty$,
- $\mathcal{E}(LR^{n+1})$, the set of peaks corresponding to periodic orbits of the form $(LR^{n+1})^\infty$,
- $\mathcal{E}(R^2L^n)$, the set of peaks corresponding to periodic orbits of the form $(R^2L^n)^\infty$.

For all families, we consider $n \geq 1$.

Proposition 3.2.1 *Let* $\langle\!\langle x_n, y_n \rangle\!\rangle \in \mathcal{E}(RL^n)$, $n \geq 1$. *Then* $x_{k+1} \geq x_k$, $y_{k+1} \geq y_k$, *and* $\langle\!\langle x_n, y_n \rangle\!\rangle$ *is extremal.*

Proof Let $a_0 < a_1 < \cdots < a_n$ be the points of the periodic orbit $(RL^n)^\infty$. Using the itinerary, we can see that the temporal ordering is identical to the spatial ordering, that is, $T(a_k) = a_{k+1}$ for $k < n$ and $T(a_n) = a_0$. If $J_k = [a_{k-1}, a_k]$, $k = 1, \ldots n$, then we can write the Markov graph associated to this periodic orbit as

$$J_1 \longrightarrow J_2 \longrightarrow \quad \bullet \quad \bullet \quad \bullet \quad \longrightarrow J_n$$

Now if $\langle\!\langle x_n, y_n \rangle\!\rangle$ is the point of $\mathcal{TS}$ corresponding to $(RL^n)^\infty$, then the truncation of map T_{x_n, y_n} will be on the interval J_n. Thus the only periodic orbits T_{x_n, y_n} can have are $(RL^n)^\infty$ and a fixed point. Hence $\langle\!\langle x_n, y_n \rangle\!\rangle$ is an extremal point.

The periodic point of the tent map with itinerary $(L^nR)^\infty$ is a solution of the equation $2 - 2^{n+1}x = x$. So $x = \frac{2}{2^{n+1}+1}$ is the smallest point in the periodic orbit. Therefore, the largest point of the orbit which is less than $\frac{1}{2}$ will be $x_n = \frac{2^n}{2^{n+1}+1}$ and the only point larger than $\frac{1}{2}$ is $y_n = \frac{2^{n+1}}{2^{n+1}+1}$. We can see that x_n and y_n are both increasing with n. ■

We have established that elements of $\mathcal{E}(RL^n)$ are extremal. However, it is possible that some extremal points have been overlooked in the "nooks" between extremal points. Using Lemma 3.0.1 gives us a very simple technique for determining if there are any extremal points between elements of $\mathcal{E}(RL^n)$.

Proposition 3.2.2 *Let* $\langle\!\langle x_n, y_n \rangle\!\rangle$ *and* $\langle\!\langle x_{n+1}, y_{n+1} \rangle\!\rangle$ *correspond to* RL^n *and* RL^{n+1}, *respectively. If* $\langle\!\langle x, y \rangle\!\rangle$ *is a parameter such that* $x < x_n$ *and* $y > y_{n+1}$, *then* $\langle\!\langle x, y \rangle\!\rangle$ *is not a peak.*

Proof Denote by $a_0 < a_1 < \cdots < a_n$ the orbit for L^nR and by $b_0 < b_1 < \cdots < b_n < b_{n+1}$ the orbit of $L^{n+1}R$. It can be seen that $L^{k+1}R < L^kR$ for all k. Therefore, we get that $b_0 < a_0 < b_1 < a_1 < \ldots b_n < a_n < b_{n+1}$. Let $J_k = [b_k, a_k]$ for $k = 0, \ldots, n$

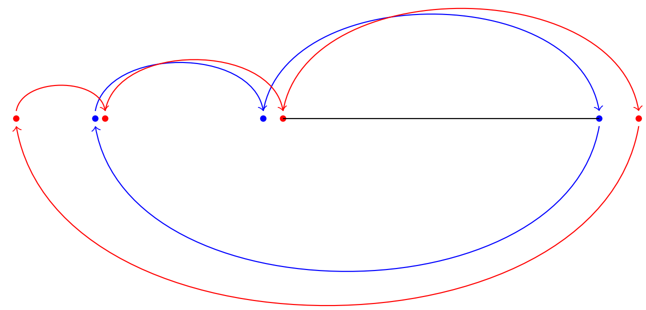


Fig. 3.6. In Proposition 3.2.2 we create the Markov graph associated to two adjacent elements of $\mathcal{E}(RL^n)$. The shaded interval represents the truncated interval.

and $I_m = [a_m, b_m + 1]$ for $m = 0, \ldots, n$. The map T_{x_n,y_n} is truncated on the interval J_n, and thus J_n collapses under T_{x_n,y_n}. The Markov graph of T_{x_n,y_n} will be

$$I_1 \to I_2 \to \cdots \to I_n \to J_1 \to J_2 \to \cdots \to J_n.$$

Therefore, the function T_{x_n,y_n} will have only periodic orbits for $(RL^n)^\infty$, $(RL^{n+1})^\infty$, and a fixed point. By Lemma 3.0.1, any $(\!(x, y)\!)$ such that $x < x_n$ and $y < y_n + 1$ can only have a fixed point. Since a condition for extremality is forcing a non-fixed orbit, then no such $(\!(x, y)\!)$ is extremal. ∎

With the previous Proposition, we have now characterized all extremal points above the line $b = 1 - a$ in $\mathcal{TS}$.

Proposition 3.2.3 *The set $\mathcal{E}(RL^n)$ contains the only extremal points above the line $b = 1 - a$ in $\mathcal{TS}$.*

Proof First note that if $(\!(x, y)\!)$ is an extremal point and $a \geq x$ and $b \leq y$, then $(\!(a, b)\!)$ is not extremal. Therefore Propositions 3.2.1 and 3.2.2 characterize all extremal points in $[\frac{2}{5}, \frac{1}{2}] \times [\frac{4}{5}, 1]$. Now the negative diagonal contains all parameters corresponding to continuous truncated tent maps $T_{a,1-a}$, $a \in [0, \frac{1}{2}]$. As a decreases,

$\mathcal{I}(T_{a,1-a})$ loses periodic orbits in the order prescribed by Sharkovsky's Theorem. Note that $(\!(a, 1-a)\!)$ is not extremal so long as $a \geq \frac{2}{5}$. However, $T_{a,1-a}$ has no periodic orbit (other than the fixed point 0) for $a < \frac{2}{5}$. Therefore, there can be no peak, and hence not extremal point, in the set $[0, \frac{2}{5}] \times [\frac{3}{5}, 1]$. In particular, there is no extremal point above $b = 1 - a$ outside of $\mathcal{E}(RL^n)$. ■

Proposition 3.2.4 *Let* $(\!(x_n, y_n)\!) \in \mathcal{E}(LR^{n+1})$, $n \geq 1$. *Then* $(\!(x_n, y_n)\!)$ *is extremal.*

Proof Consider the orbit with itinerary $(LR^{n+1})^\infty$ There are two cases to consider: when n is even and when n is odd. Suppose that n is even. Let $a_1, a_2, \ldots, a_{n+2}$ be the temporal ordering of the orbit (that is $T(a_i) = a_{i+1}$). If a_1 is the smallest point of the orbit, then this temporal ordering corresponds to the spatial ordering

$$a_1 < a_{n+1} < a_{n-1} < \cdots < a_3 < a_2 < a_4 < \cdots < a_n < a_{n+2}$$

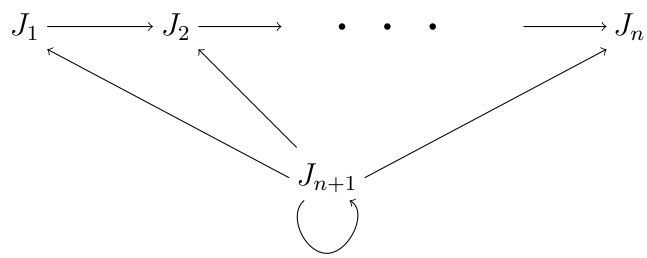
where the fixed point $\frac{2}{3}$ is in the interval $[a_3, a_2]$. Let $J_1 = [a_1, a_{n+1}]$, $J_2 = [a_2, a_3]$, and $J_m = [a_{m-1}, a_{m+1}]$, $m \leq n+1$ odd, and $J_m = [a_{m+1}, a_{m-1}]$, $m \leq n+1$ even. Then the Markov graph of this map will be

$$\begin{tikzcd} J_1 \arrow[r] & J_2 \arrow[r] \arrow[loop below] & \cdots \arrow[r] & J_n \end{tikzcd}$$

If T_{x_n,y_n} is the truncated tent map for the parameter $(\!(x_n, y_n)\!)$ corresponding to $(RLR^n)^\infty$, n even, then the truncation is on the interval J_1. Hence it follows from the Markov graph that the only periodic orbits of T_{x_n,y_n} are the fixed point and RLR^n. The proof for n odd is the same, save for the fact that the spatial ordering would have $a_2 < a_3$ and the fixed point would be on the interval $[a_2, a_3]$, so we relabel our J_m accordingly. ■

Proposition 3.2.5 *Let* $(\!(x, y,)\!) \in \mathcal{E}(R^2L^n)$. *Then* $(\!(x, y)\!)$ *is extremal.*

Proof Same as with elements of $\mathcal{E}(RL^n)$, here the spatial ordering of a periodic orbit $(R^2L^n)^\infty$ is the same as the temporal ordering. Thus the orbit will be $a_1 < a_2 < \cdots < a_{n+1} < a_{n+2}$, and $T(a_i) = a_{i+1}$ for $i < n+2$ and $T(a_{n+2}) = a_1$. Let $J_i = [a_i, a_{i+1}]$, $i = 0, \ldots, n+1$. Then the Markov graph will be



If T_{x_n,y_n} is the truncated tent map for the peak $(\!(x_n, y_n)\!)$ corresponding to $(R^2L^n)^\infty$, then the truncation for T_{x_n,y_n} occurs on the interval J_n. Thus we can see from the Markov graph that the only periodic orbits of T_{x_n,y_n} will be the fixed point and R^2L^n.

■

Proposition 3.2.6 *Elements of $\mathcal{E}(LR^{n+1})$ converge to the point $(\!(\frac{1}{3}, \frac{7}{12})\!)$. Additionally, if n is even (odd), then $x_n > \frac{1}{3}$ ($x_n < \frac{1}{3}$) and $y_n > \frac{7}{12}$ ($y_n < \frac{7}{12}$).*

Proof A tedious calculation shows that the peak corresponding to $(LR^{n+1})^\infty$ is given by $(\!(x_n, y_n)\!)$ where

$$x_n = \frac{(-1)^n + 2^{n+1}}{2^{n+1} \cdot 3},$$

and

$$y_n = \frac{2^{n+3} \cdot 3 - 2^{n+2} \cdot 3 + (-1)^{n+2} + 2^{n+1}}{2^{n+3} \cdot 3}.$$

Taking the limit as $n \to \infty$ clearly yields $x_n \to \frac{1}{3}$ and $y_n \to \frac{7}{12}$.

If $(\!(x_n, y_n)\!) \in \mathcal{E}(LR^{n+1})$, then $\underline{I}_T(x_n) = (LR^{n+1})^\infty$ and $\underline{I}_T(y_n) = (RRLR^{n-1})^\infty$. Then we have

$$\underline{I}_T(x_n) = (LR^{n+1})^\infty > LR^\infty = \underline{I}_T(\frac{1}{3})$$

and

$$\underline{I}_T(y_n) = (RRLR^{n-1})^\infty > RRLR^\infty = \underline{I}_T(\frac{7}{12})$$

if n is even. The inequalities are reversed if n is odd. ■

Corollary 3.2.7 *Let $(\!(x_{2k}, y_{2k})\!)$ be elements of $\mathcal{E}(LR^{n+1})$ corresponding to orbits $(LR^{2k-1})^\infty$ and $(\!(x_{2k+1}, y_{2k+1})\!)$ be elements corresponding to orbits $(LR^{2k})^\infty$. Then:*

1. *$x_{2k} > x_{2(k+1)}$*

2. $y_{2k} > y_{2(k+1)}$

3. $x_{2k+1} < x_{2(k+1)+1}$

4. $y_{2k+1} < y_{2(k+1)+1}$

Proof We observe the itineraries. In case 1, $2k+1$ is odd, so we get

$$\underline{I}_T(x_{2k}) = LR^{2k+1}LR\cdots > LR^{2k+1}RR\cdots = \underline{I}_T(x_{2(k+1)}),$$

which implies $x_{2k} > x_{2(k+1)}$. In case 2, $2k-1$ is odd, so we get

$$\underline{I}_T(y_{2k}) = RRLR^{2k-1}LR\cdots > RRLR^{2k-1}RR\cdots = \underline{I}_T(y_{2(k+1)}),$$

which implies $y_{2k} > y_{2(k+1)}$. In case 3, $2k+2$ is even, so

$$\underline{I}_T(x_{2k+1}) = LR^{2k+2}LR\cdots < LR^{2k+2}RR\cdots = \underline{I}_T(x_{2k+3}),$$

which implies $x_{2k+1} < x_{2k+3}$. In case 4, $2k$ is even, so

$$\underline{I}_T(y_{2k+1}) = RRLR^{2k}RRL\cdots < RRLR^{2k}RRR\cdots = \underline{I}_T(y_{2k+3}),$$

which implies $y_{2k+1} < y_{2k+3}$. ■

Proposition 3.2.8 *Let $(\!(x_{2k}, y_{2k})\!) \in \mathcal{E}(LR^{n+1})$ corresponding to $(LR^{2k-1})^\infty$. If $(\!(x, y)\!) \in \mathcal{TS}$ such that $x < x_{2(k+1)}$ and $y < y_{2k}$, then $(\!(x, y)\!)$ is not extremal.*

Proof We will show that the parameter $(\!(x_{2(k+1)}, y_{2k})\!)$ does not generate any periodic orbits other than LR^{2k-1} and LR^{2k+1}. Let $n = 2k$ and denote by u_i the spatial ordering of the periodic orbit LR^{n-1} and w_j the spatial ordering of the orbit LR^{n+1}. So w_1 has itinerary $(LR^{n+1})^\infty$, and therefore $T(w_1)$ has itinerary $(R^{n+1}L)^\infty$. Observe that $\sigma^\alpha(R^{n+1}L) < (R^{n+1}L)$ if α is odd and $\sigma^\alpha(R^{n+1}L) > (R^{n+1}L)$ if α is even, thus w_{k+2} corresponds to itinerary $(R^{n+1}L)^\infty$ (and similarly u_{k+1} has itinerary $(R^{n-1}L)^\infty$). Additionally observe that

$$w_{k+2} < u_{k+1} < w_{k+3}. \tag{3.17}$$

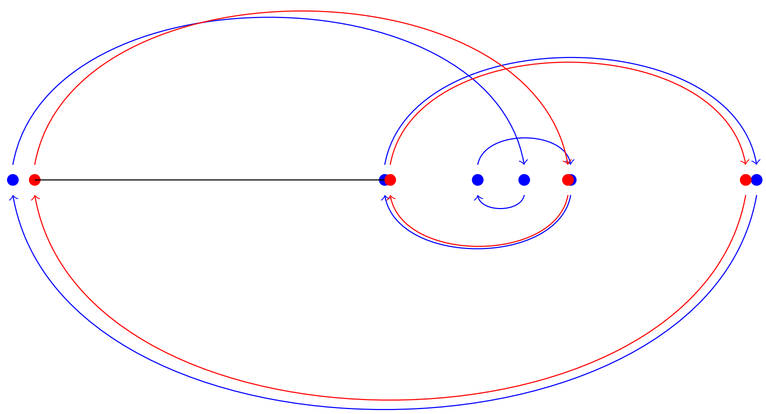


Fig. 3.7. This pictures describes how two adjacent elements of $\mathcal{E}(LR^{n+1})$ are arranged. The shaded region is where the truncation occurs.

Now let $a_1 = u_{k+1}$ and $a_i = T(a_{i-1})$ and $b_1 = w_{k+2}$ and $b_j = T(b_{j-1})$. Then (3.17) can be written as

$$b_1 < a_1 < b_3. \tag{3.18}$$

Now define intervals $J_i =< a_i, b_i >$ and $K_i =< b_{i+2}, a_i >$, where $i = 1, \ldots, n$ and $< \cdot, \cdot >$ ignores orientation. Then the Markov graph can be written as

$$K_1 \to K_n \to J_1 \to J_2 \to \ldots J_n =< a_n, b_n > .$$

However for parameter $(x_{2(k+1)}, y_{2k})$, the truncated interval is exactly $[a_n, b_n]$. Thus the Markov graph does not generate any periodic orbits and the proposition follows. ■

Proposition 3.2.9 *Let* $((x_n, y_n))$ *and* $((x_{n+1}, y_{n+1}))$ *be elements of* $\mathcal{E}(R^2L^n)$. *Then the set* $[0, x_n) \times (y_{n+1}, 1]$ *contains a peak, and hence an extremal point.*

Proof The points $((x_n, y_n))$ and $((x_{n+1}, y_{n+1}))$ correspond to periodic orbits R^2L^n and R^2L^{n+1}, respectively. Let $k = n + 2$ Let $a_1 < \cdots < a_{k+1}$ be the points of the orbit R^2L^{n+1} and $b_1 < \cdots < b_k$ be the points of the orbit R^2L^n. Note that $T(a_i) = a_{i+1 \mod k+1}$ and $T(b_j) = b_{j+1 \mod k}$. Using the itineraries, we get the ordering

$$a_1 < a_2 < b_1 < a_3 < b_2 < \cdots < a_k < b_{k-1} < b_k < a_{k+1}.$$

Let $J_i = [a_i, b_{i-1}]$, $i = 2, \ldots, k$, and $K_j = [b_j, a_{j+2}]$, $j = 1, \ldots, k-2$. There are also two special intervals $A = [a_1, a_2]$ and $B = [b_k, a_{k+1}]$. Then the Markov graph is shown in Figure 3.8.

Since there are linked loops in the Markov graph, the map $T_{x_n, y_{n+1}}$ has more periodic points, and hence the set $[0, x_n) \times (y_{n+1}, 1]$ contains a peak. ■

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & J_2 \arrow[r] & J_3 \arrow[r] & \cdots \arrow[r] & J_k \arrow[r] & B \arrow[lllll, bend right] \arrow[llll, bend right] \\
 & K_1 \arrow[r] & K_2 \arrow[r] & \cdots \arrow[r] & K_{k-2} &
\end{tikzcd}$$

Fig. 3.8. The Markov graph of $T_{x_n, y_{n+1}}$, where $((x_n, y_n))$ and $((x_{n+1}, y_{n+1}))$ are in $\mathcal{E}(R^2L^n)$.

Proposition 3.2.10 *Let $((x_n, y_n))$ and $((x_{n+1}, y_{n+1}))$ be elements of $\mathcal{E}(LR^{n+1})$. Then the set $((x_n, y_{n+1}))$ contains a peak, and hence an extremal point.*

Propositions 3.2.9 and 3.2.10 show that there are other extremal points living in the "nooks" between certain elements of $\mathcal{E}(R^2L^n)$ and $\mathcal{E}(LR^{n+1})$. As of this moment, we do not understand extremal points outside the sets $\mathcal{E}(RL^n)$, $\mathcal{E}(LR^{n+1})$, and $\mathcal{E}(R^2L^n)$. However, we can get a grasp on how many peaks are in these nooks. To look further, we require a a number theoretic result. The proofs of Theorem 3.2.11 and Corollaries 3.2.12 and 3.2.13 are due to Henryk Iwaniec.

Theorem 3.2.11 *Let k, p, q be positive integers with $(p, q) = 1$. Let $N(x)$ be the number of integers m, n with $(m, n) = 1$ and $0 < m \leq x$ such that*

$$pm + qn = k. \tag{3.19}$$

Then we have

$$|N(x) - \frac{\phi(k)}{k}\frac{x}{q}| \leq 2^{\omega(k)}, \tag{3.20}$$

where $\phi(k)$ is the Euler function and $\omega(k)$ denotes the number of distinct prime factors of k.

Proof We relax the condition $(m, n) = 1$ by using he formula

$$\sum_{d|l} \mu(d) = \begin{cases} 1 & \text{if} \quad l = 1 \\ 0 & \text{otherwise} \end{cases}$$

where $\mu(d)$ is the Möbius function. We apply this formula for $l = (m, n)$, so $d|(m, n)$ implies $d|m$ and $d|n$. Hence

$$N(x) = \sum_{0<m\leq x} \sum_{\substack{(m,n)=1 \\ pm+qn=k}} 1 = \sum_{d|k} \mu(d) \sum_{0<m\leq \frac{x}{2}} \sum_{pm+qn=\frac{k}{d}} 1$$

Note that

$$\sum_{d|(m,n)} \mu(d) = \begin{cases} 1 & \text{if} \quad (m, n) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore $N(x)$ becomes

$$N(x) = \sum_{0<m\leq x} \sum_{\substack{(m,n)=1 \\ pm+qn=k}} \sum_{d|(m,n)} \mu(d) = \sum_{0<m\leq x} \sum_{pm+qn=k} \sum_{d|(m,n)} \mu(d).$$

$$= \sum_{d|k} \mu(d) \sum_{0<m_1\leq \frac{x}{d}} \sum_{pm_1+qn_1=\frac{k}{d}} 1 = \sum_{d|k} \mu(d) \sum_{0<m\leq \frac{x}{d}} \sum_{pm+qn=\frac{k}{d}} 1$$

The last double sum is just a single sum over $0 < m \leq \frac{x}{d}$ in an arithmetic progression $m \equiv a \mod q$ with some a, specifically $a = \overline{p}k/d$ where $\overline{p}$ denotes the multiplicative inverse of p modulo q. Observe that the number of integers in an interval of length x is equal to $x+\delta$ with $|\delta| \leq 1$. Hence the number of integers in an interval of lengths y which are $\equiv a \mod q$ is $y/q+\delta$ with $|\delta| \leq 1$. By these observations

$$N(x) = \sum_{d|k} \mu(d)(\frac{x}{dq} + \delta(d))$$

with $|\delta(d)| \leq 1$. This completes the proof. ■

Corollary 3.2.12 *For $x > q2^{\omega(k)}k/\phi(k)$ we have $N(x) > 0$.*

Corollary 3.2.13 *Fix $(p,q) = 1$. If k is sufficiently large, specifically if*

$$\phi(k)2^{-\omega(k)} > pq, \tag{3.21}$$

then there exist m, n such that $pm + qn = k$ where $(m,n) = 1$.

Proof Apply Corollary 3.2.12 with $x = \epsilon + q2^{\omega(k)}k/\phi(k)$ so we get integers m, n with $(m,n) = 1$, $0 < m \leq x$ which satisfy 3.19. We have $m > 0$ and

$$pm \leq px = \epsilon p + pq2^{\omega(k)}k/\phi(k) < \epsilon p + k$$

Hence $pm < k$ so $n > 0$, or $pm = k$ so $n = 0$. In the last case $m = 1$, $k = p$ and the condition 3.21 implies $p > pq$ which doesn't happen.

■

Consider again the point $((x_n, y_{n+1}))$, where $((x_n, y_n))$ and $((x_{n+1}, y_{n+1}))$ are elements of $\mathcal{E}(R^2L^n)$. By 3.2.9, $[0, x_n) \times (y_{n+1}, 1]$ contains a peak. To see all the peaks in this set, we can use a technique similar to the one used in Section 3.1. Specifically, we will find the map induced by $T_{x_n, y_{n+1}}$ on a special subinterval of $[0,1]$.

Lemma 3.2.14 *Take J_{k-1} to be as it is defined in Proposition 3.2.9. Then the map induced by $T_{x_n, y_{n+1}}$ on the interval J_{k-1} is Lorenz-like.*

Proof Following the Markov graph in Figure 3.8, we can see that J_{k-1} maps onto B in an order reversing fashion in two steps. The set B then maps onto $A \cup J_2$ in an order reversing fashion. Thus J_{k-1} maps onto $A \cup J_2$ in an order preserving fashion in three steps. Note that A and J_2 are adjacent intervals.

Now the interval A can be split into A_L and A_R, where A_L in an order preserving fashion onto J_2 and A_R maps onto K_1. Thus, following the Markov graph, we can see that the induced map ρ on J_{k-1} will be have two full, order preserving laps. The left lap will have slope 2^{k+1}, while the right lap will have slope 2^k. Moreover, the middle of J_{k-1} will contain a preimage of K_1, which eventually maps to the gap (x_n, y_{n+1}) and is annihilated. ■

Since the induced map ρ is Lorenz-like and has two full laps, then the set of periodic points for ρ must be the same as those for the angle doubling map z^2, namely twist periodic orbits. Let Q be a periodic orbit with points $q_1 < q_2 < q_3 \cdots < q_m$. This orbit is a *twist* periodic orbit under the map f if $f(q_i) = q_{i+j \mod m}$, for some fixed j where $(j, m) = 1$. For Lorenz like maps this also means that j elements of the orbit have to be to the left and $m - j$ orbits have to be to the right.

Since ρ is an induced map, the periodic points of ρ can be extended to a periodic orbit of $T_{x_n, y_{n+1}}$. Thus if Q is a twist periodic orbit of ρ with rotation $\frac{j}{m}$, then this can be extended to a periodic orbit Q^* which is of period $j \cdot k + (m - j) \cdot (k + 1)$. Therefore, we get the following.

Theorem 3.2.15 *The set* $[0, x_n) \times (y_{n+1}, 1]$ *contains peaks corresponding to periodic points of periods* $m_1 \cdot k + m_2 \cdot (k + 1)$*, where* $(m_1, m_2) = 1$*. In particular, there are peaks corresponding to all but finitely many periods.*

Proof The statement about peaks is explained prior to the theorem. Using Corollary 3.2.13, it follows that for sufficiently high period p there exists m_1 and m_2 such that $m_1 \cdot k + m_2 \cdot (k + 1) = p$. In particular, we can find m_1 and m_2 for all but finitely many periods.

■

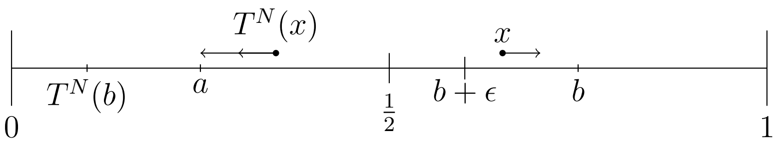


Fig. 3.9. As x moves towards b, $T^N(x)$ moves toward $T^N(b)$ and hence toward a.

3.3 Characterizing $\overline{\mathcal{P}}$

Let T denote the full tent map and $\mathcal{M}$ the set of parameters $((a, b))$ such that $orb(a) \cap (a, b) = \emptyset = orb(b) \cap (a, b)$. Let $\mathcal{D}$ denote the set of $((a, b)) \in \mathcal{M}$ such that either $a \in orb(b)$ or $b \in orb(a)$ under T. For $((a, b)) \in \mathcal{M}$ we will say that *a has space* if $a \notin \overline{orb(b) \setminus \{a\}}$. Similarly we will say that *b has space* if $b \notin \overline{orb(a) \setminus \{b\}}$. This section is devoted to characterizing the closure of $\mathcal{P}$, the set of peaks. We will prove that $\overline{\mathcal{P}} = \mathcal{M} \setminus \mathcal{O}$ where

$$\mathcal{O} = \{((a, b)) \in \mathcal{M} \setminus \mathcal{P} \,|\, a \text{ or } b \text{ negative periodic, or } a = 0\}.$$

We begin by proving that $\mathcal{M} \setminus \mathcal{O} \subset \overline{D}$.

Lemma 3.3.1 *Suppose $((a, b)) \in \mathcal{M}$. If $b \neq 1/2$ is not periodic and $a \notin orb(b)$, then for every $\epsilon > 0$ there exists $b' \in (b - \epsilon, b)$ such that $((a, b')) \in \mathcal{M}$ and either $a \in orb(b')$ or b' is an positive periodic point and has space. If $a \neq 1/2$ is not periodic and $b \notin orb(a)$, then for every $\epsilon > 0$ there exists $a' \in (a, a + \epsilon)$ such that $((a', b)) \in \mathcal{M}$ and either $b \in orb(a')$ or a' is an positive periodic point and has space.*

Proof We begin with the proof of the first statement. Without loss of generality, let $0 < \epsilon < b - \frac{1}{2}$. We let

$$A_x = \{n \,|\, T^n(x) \in [a, x]\},$$

and consider the set $A = \bigcup_{x \in (b-\epsilon, b)} A_x$. To see that A is not empty, we note that every number $\frac{k}{2^n}$ is eventually mapped to $\frac{1}{2}$. Since the diadic rationals are dense, there exists k and n with $\frac{k}{2^{n+1}} \in (b-\epsilon, b)$; this implies $n \in A$. Let $N = \min A$ and now consider the set $B = \{x \in (b-\epsilon, b) \mid f^N(x) \in [a, x]\}$. We claim that $(\!(a, b')\!) \in \mathcal{D}$, where $b' = \sup B$.

Note that since $(\!(a, b)\!) \in \mathcal{M}$, then $T^n(b) < a$ whenever $T^n(b) < 1/2$ and $T^n(b) > b$ whenever $T^n(b) > 1/2$ for any n. Additionally, since N is minimal, then $1/2 \notin T^n((b-\epsilon, b))$ for all $0 < n < N$. Consequently T^N is monotone on $(b-\epsilon, b)$.

Suppose then that $T^N(b) < a$. Since T^N is monotone on $(b-\epsilon, b)$ and $T^N(x) \in [a, x]$ for some $x \in (b-\epsilon, b)$, then T^N must be monotonically decreasing on $(b-\epsilon, b)$ (see Figure 1). Thus $b' = \sup B = T^{-N}(a)$. By the same token, if $T^N(b) > b$, then T^N is monotonically increasing on $(b-\epsilon, b)$ (see Figure 2). Because $T^N(x) \to T^N(b)$ as $x \to b$ and $T^N(b) > b$, it follows that there exists $b' \in (b-\epsilon, b)$ such that $T^N(b') = b'$. Moreover, if $b' < x < b$, then $b' < T^N(x) < T^N(b)$. Since T^N is expanding, $T^N(x) > x$ for all $x \in (b', b)$, and thus $b' = \sup B$.

As a result, $T^N(b') = a$ if $T^N(b) < a$, and $T^N(b') = b'$ if $T^N(b) > b$. In addition, it was shown that T^N was increasing (order preserving) on $(b-\epsilon, b)$ if $T^N(b) > b$. It then follows that b' must be an positive periodic point in this case. In both cases $b' < b$, so $orb(a) \cap (a, b') = \emptyset$. Moreover $T^i(b') \notin [a, b']$ for all $i < N$, by minimality of N. It is then clear that in both cases that $orb(b) \cap (a, b') = \emptyset$, and thus $(\!(a, b')\!) \in \mathcal{M}$. Additionally, b' must have space since $b' < b$ and $\omega(a) \subset [0, 1] \setminus (a, b)$. In particular, if b' is periodic, then it also has space.

The proof of the second statement is analogous. We need only exchange the roles of a and b, replace $(b-\epsilon, b)$ with $(a, a+\epsilon)$, and consider $\inf B$ instead of the supremum.
■

Lemma 3.3.2 *Suppose $(\!(a, b)\!) \in \mathcal{M}$. If b is an positive periodic orbit, has space and $b \notin orb(a)$, then for every $\epsilon > 0$ there exists $b' \in (b, b+\epsilon)$ such that $(\!(a, b')\!) \in \mathcal{M}$ and $a \in orb(b')$. If a is an positive periodic orbit, has space and $a \notin orb(b)$, then for every $\epsilon > 0$ there exists $a' \in (a-\epsilon, a)$ such that $(\!(a', b)\!) \in \mathcal{M}$ and $b \in orb(a')$.*

Proof We begin with the proof of the first statement. Let b be an positive periodic point of period p. Then there must be at least two elements of $orb(b)$ which are greater

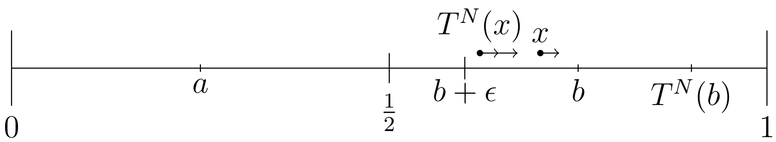


Fig. 3.10. As x moves towards b, $T^N(x)$ will "catchup" to x from the left.

than 1/2. Moreover, since $(a, b) \in \mathcal{M}$, b is the smallest element of $orb(b)$ which is greater than 1/2. Let b^* be the smallest element of $orb(b)$ greater than b. Since b and b^* are different points, then $I(b)$ and $I(b^*)$ are distinct itineraries. Let M be the smallest integer such that $I_M(b) \neq I_M(b^*)$. Then $T^M(b)$ and $T^M(b^*)$ are on opposite sides of 1/2. Moreover, T^M is continuous on interval $I_b = [b, b^*]$, so $T^M(I_b)$ is also a closed interval. Since $orb(b) \subset [0,1] \setminus (a, b)$, then it is clear that $(a, b) \subset T^M(I_b)$. Thus there exists $a^* \in I_b$ such that $T^i(a^*) \notin (a, b)$ for $0 \leq i < M$ and $T^M(a^*) = a$.

Since b has space, then there exists a δ ball around b which contains no element of $orb(a)$. Therefore we may assume that $0 < \epsilon < \delta$. Now let

$$A_x = \{n \,|\, T^n(x) \in \{a, a^*\}\},$$

and consider $A = \bigcup_{x \in (b, b+\epsilon)} A_x$. The existence of a^* implies that $A \neq \emptyset$. Now let $N = \min A$ and $b' = \min\{x \in (b, b+\epsilon) \,|\, T^N(x) \in \{a, a^*\}\}$. We claim that $(\!(a, b')\!) \in \mathcal{D}$.

Suppose first that $T^N(b') = a$. We wish to show that $(\!(a, b')\!) \in \mathcal{M}$. Since $T^N(b') = a$ and ϵ was chosen so that $(b, b+\epsilon) \cap \omega(a) = \emptyset$, then $T^i(b') \notin (a, b')$ for all $i \geq N$. Since N is minimal, then T^i is monotone on $(b, b+\epsilon)$ for $i \leq N$. Moreover if $i = kp \leq N$, then $T^i(b') > b'$. Therefore if $T^i(b') \in (a, b')$ for $i < N$, then either $a^* \in T^i((b, b+\epsilon))$ or $a \in T^i((b, b+\epsilon))$. By the intermediate value theorem there is an element c of $(b, b+\epsilon)$ with $c < b'$ and $T^i(c) \in \{a, a^*\}$, a contradiction. Hence $T^i(b') \notin (a, b')$ for all $i < N$ and $(\!(a, b')\!) \in \mathcal{M}$. The argument for $T^N(b') = a^*$ is identical. Additionally, it follows from the definition of b' that $a \in orb(b')$.

Now we consider the second statement. If a is an positive periodic orbit with at least two elements less than $1/2$, then the proof of the second statement is analogous to the first. Simply exchange the roles of a and b, replace $(b, b+\epsilon)$ with $(a-\epsilon, a)$, and consider $a' = \max\{x \in (b, b+\epsilon) \,|\, T^N(x) \in \{b, b^*\}\}$. We will now consider the case when the orbit of a has one element less than $1/2$.

If the orbit of a has only one element less than $1/2$, then the itinerary of the orbit must be $(LR^{p-1})^\infty$, where $p-1$ is even. In other words, the orbit of a is Štefan. Now fix $\epsilon > 0$ so that $orb(b) \cap (a-\epsilon, a) = \emptyset$. Let

$$A_x = \{n \,|\, T^n(x) \in [x, b]\}$$

and consider $A = \bigcup_{x \in (a-\epsilon, a)} A_x$. Since T is a piecewise expanding map, then A is non-empty. Now let $N = \min A$ and $a' = \sup\{x \in (a-\epsilon, a) \,|\, T^N(x) \in [x, b]\}$. We claim that $(\!(a', b)\!) \in \mathcal{D}$.

First note that since $T^i(x) \notin (a, b)$ for all $i < N$ and $x \in (a-\epsilon, a)$, then T^N is monotone on $(a-\epsilon, a)$. Since a is an positive periodic point, $T^N(x) < x$ for all $x \in (a-\epsilon, a)$ when $p|N$. Since A is non-empty, it must be that $p \nmid N$. Therefore $T^N(a) > 1/2$ and hence $a' \in T^{-N}(b)$. Since $(\!(a, b)\!) \in \mathcal{M}$ and ϵ was chosen so that $orb(b) \cap (a-\epsilon, a) = \emptyset$, then $T^i(b) \notin (a', b)$ for all i and $T^i(a') \notin (a', b)$ for all $i \geq N$. Suppose then that $T^i(a') \in (a', b)$, for some $i < N$. If m is the smallest integer such that $T^m(a') \in (a', b)$, then T^m is monotone on $(a-\epsilon, a)$. Therefore, by the same argument as before, $p \nmid m$ and there must exist some $x_0 \in (a-\epsilon, a)$ with $T^m(x_0) = b$. This is a contradiction, so it must be that $(\!(a', b)\!) \in \mathcal{M}$. ■

Proposition 3.3.3 *$\mathcal{M} \setminus \mathcal{O} \subset \overline{\mathcal{D}}$.*

Proof Let $(\!(a, b)\!) \in \mathcal{M} \setminus \mathcal{O}$ and suppose that $a, b \neq 1/2$. If $(\!(a, b)\!) \in \mathcal{D}$, then there is nothing to prove. Suppose then that $(\!(a, b)\!) \notin \mathcal{D}$. If neither a nor b is periodic, then $(\!(a, b)\!)$ satisfies the hypothesis of Lemma 3.3.1, and therefore $(\!(a, b)\!) \in \overline{\mathcal{D}}$ by Lemmas 3.3.1 and 3.3.2. If either a or b is periodic and has space, then $(\!(a, b)\!) \in \overline{\mathcal{D}}$ by Lemma 3.3.2. Suppose then that, say, a is periodic and does not have space. Since the orbit of a is finite, b must have space. Thus, if b is not periodic, then $((a,b)) \in \overline{\mathcal{D}}$ by Lemmas 3.3.1 and 3.3.2. Moreover, if b is periodic, then $((a,b))$ satisfies the hypothesis of Lemma 3.3.2 and hence $((a,b)) \in \overline{\mathcal{D}}$. The proof is the same if we initially assume b is periodic.

Now consider the case where $a = 1/2$ and $b \neq 1/2$. Since the orbit of a is finite, then b has space. Hence if b is periodic, then $((a,b)) \in \overline{\mathcal{D}}$ by Lemma 3.3.2 and if b is not periodic then $((a,b)) \in \overline{\mathcal{D}}$ by Lemmas 3.3.1 and 3.3.2. The situation the same when $b = 1/2$ and $a \neq 1/2$, so the proof will be similar.

Finally consider the case where both a and b are $1/2$. Let $b' = \frac{2^{n-1}+1}{2^n}$ and note that $T(b') > b'$ and $T^i(b') < 1/2$ for $2 \leq i \leq n-2$. Since $T^{n-1}(b') = 1/2$, then $a \in orb(b')$ and thus $((a,b')) \in \mathcal{D}$. We may then conclude that $((1/2,1/2)) \in \overline{\mathcal{D}}$. ■

Proposition 3.3.3 tells us that for any $((a,b)) \in \mathcal{M} \backslash \mathcal{O}$, either $((a,b))$ is a dominating pair or $((a,b))$ may be approximated by a dominating pair. We will now show that $\mathcal{D} \subset \overline{\mathcal{P}}$.

Lemma 3.3.4 *Suppose $((a,b)) \in \mathcal{M}$. If $a,b \neq 1/2$, with $f^m(a) = b$, and b is not periodic, then for every $\epsilon > 0$ there exists $a' \in (a-\epsilon, a+\epsilon)$ so that $((a', T^m(a'))) \in \mathcal{M}$ and either $((a', T^m(a'))) \in \mathcal{P}$ or $T^m(a')$ is periodic and has space. If $b,a \neq 1/2$, with $T^m(b) = a$ and a is not periodic, then for every $\epsilon > 0$ there exists $b' \in (b-\epsilon, b+\epsilon)$ so that $((T^m(b'), b')) \in \mathcal{M}$ and either $((T^m(b'), b')) \in \mathcal{P}$ or $T^m(b')$ is periodic and has space.*

Proof We prove the first statement. The proof of the second statement follows in the same manner, with the orientation of intervals reversed. Fix $\epsilon > 0$ so that $1/2 \notin T^i((a-\epsilon, a+\epsilon))$ for $0 \leq i \leq m$. We would like to consider only $x \in (a-\epsilon, a+\epsilon)$ such that $T^m(x) < T^m(a)$. The map T^m is monotone on $(a-\epsilon, a+\epsilon)$, and therefore we suppose, without loss of generality, that T^m is order reversing on $(a-\epsilon, a+\epsilon)$. This means for $x \in (a-\epsilon, a+\epsilon)$, $T^m(x) < T^m(a)$ if and only if $x > a$. We now consider the set

$$A = \{n > m \mid \exists_{x \in (a, a+\epsilon)} T^n(x) \in \{x, T^m(x)\}\}.$$

To see that A is non-empty, recall that $T^n((a, a+\epsilon)) = [0,1]$ for n sufficiently large. Now let $N = \inf A$ and $a' = \inf\{x | x \in (a, a+\epsilon), T^N(x) \in \{x, T^m(x)\}\}$. Then either $T^N(a') = a'$ or $T^N(a') = T^m(a')$. Suppose that $T^N(a') = a'$ and let $b' = T^m(a')$. Then $(\!(a', b')\!) \in \mathcal{M}$ if and only if $T^i(a') \notin (a', b')$ for all i. By construction $T^i(a') \notin (a', b')$ for $i \leq m$. If $T^i(a') \in (a', b')$ for some $n < i < N$, then there must exists $x \in (a, a+\epsilon)$ such that $x < a'$ with $T^i(x) \in \{x, T^m(x)\}$, which contradicts minimality. Thus $(\!(a', b')\!)$ is in $\mathcal{M}$ in this case.

If $T^N(a') = b'$, then b' is periodic. Again $(\!(a', b')\!) \in \mathcal{M}$ by similar arguments. Moreover, b' has space since T^m was orientation preserving. ■

Lemma 3.3.5 *Let $(\!(a, b)\!) \in \mathcal{M}$. If $a \neq 1/2$, $T^m(a) = b$ and b is an positive periodic orbit, then for every $\epsilon > 0$ there exists $a' \in (a-\epsilon, a+\epsilon)$ so that $(\!(a', T^m(a'))\!) \in \mathcal{P}$. If $b \neq 1/2$, $T^m(b) = a$ and a is an positive periodic orbit, then for every $\epsilon > 0$ there exists $b' \in (b-\epsilon, b+\epsilon)$ so that $(\!(T^m(b'), b')\!) \in \mathcal{P}$.*

Proof We begin with the proof of the first statement. Fix $\epsilon > 0$ so that $1/2 \notin T^i((a-\epsilon, a+\epsilon))$ and so that no element of $orb(b)$ is contained in $T^i((a-\epsilon, a+\epsilon))$ for all $0 \leq i \leq m$. Let b^* be smallest element of $orb(b)$ which is larger than b and let $J_b = [b, b^*]$. Now if b dominates a, then there is nothing to prove. So we assume $a \notin orb(b)$. This means that if $T^i(J_b)$ contains $1/2$ for some i, then $T^i(J_b)$ must also contain $(a-\epsilon, a+\epsilon)$ as well. Thus, there exists p so that $(a-\epsilon, a+\epsilon) \subset T^p(J_b)$ and $(a-\epsilon, a+\epsilon) \not\subset T^i(J_b)$ for all $i < p$. In particular, for every $x \in (a-\epsilon, a+\epsilon)$ there is a unique $x_b \in J_b$ so that $T^p(x_b) = x$. We would like to consider only $x \in (a-\epsilon, a+\epsilon)$ so that $T^m(x) > T^m(a)$. Since $1/2 \notin T^i((a-\epsilon, a+\epsilon))$ for all $i \leq m$, then T^m is monotone on $(a-\epsilon, a+\epsilon)$. Suppose, without loss of generality, that T^m is order preserving on $(a-\epsilon, a+\epsilon)$. Then for $x \in (a-\epsilon, a+\epsilon)$, $T^m(x) > T^m(a)$ if and only if $x \in (a, a+\epsilon)$. We would like to consider only $T^m((a-\epsilon, a+\epsilon) \cap J_b$. We set

$$A = \{n > m \,|\, \exists_{x \in (a, a+\epsilon)} T^n(x) \in \{x, x_b\}\}.$$

To see that A is non-empty, observe that $T^n((a, a+\epsilon)) = [0,1]$ for n sufficiently large. Now let $N = \min A$ and $a' = \min\{x \in (a, a+\epsilon) | T^N(x) \in \{x, x_b\}\}$. Note that if

$T^N(x) = x_b$, then x is periodic of period $N + p$. We wish to show that $(\!(a', b')\!) \in \mathcal{M}$, where $b' = T^m(x')$. Since a' and b' are in the same periodic orbit for both cases, we need only check that $T^i(a') \notin (a', b')$ for all i. Now since N is minimal, T^N is monotone on $(a, a + \epsilon)$. Then suppose by way of contradiction that there is some i with $T^i(a') \in (a', b')$. Since a' is periodic, then $i < N$. Moreover ϵ was chosen so that $T^i((a, a + \epsilon)) \cap (a - \epsilon, b + \epsilon) = \emptyset$, and so $i > m$. However this means that T^i is monotone on $(a, a + \epsilon)$. Since $T^i(a') \in (a', b')$ and $T^i(a) \notin (a, b)$, then by the IVT there exists some x with $T^i(x) \in \{x, x_b\}$. But this is a contradiction, as N is the minimal such integer. Thus $(\!(a', b')\!) \in \mathcal{M}$.

The proof of the second statement is analogous to that of the first if the orbit of a has at least two elements less than 1/2. Suppose then that the orbit of a has only one element less than 1/2. Choose ϵ so that the orbit of a is not in $T^i((b - \epsilon, b + \epsilon))$ for $0 < i < m$ and so that $1/2 \notin T^i((b - \epsilon, b + \epsilon))$ for $0 < i \leq m$. This means that T^m is monotone on $(b - \epsilon, b + \epsilon)$. Without loss of generality, suppose that T^m is orientation reversing. We consider $(b, b + \epsilon)$ so that $T^m((b, b + \epsilon)) \subset [0, a]$.

Now for every $x \in (b, b + \epsilon)$ there exists a unique $x_a \in [0, a)$ so that $T(x_a) = x$. Let

$$A_x = \{n > m \,|\, T^n(x) \in x, x_a\}$$

and consider the set $A = \bigcup_{x \in (b, b+\epsilon)} A_x$. Let $N = \min A$ and $b' = \inf\{x \in (b, b + \epsilon) | T^N(x) \in \{x, x_a\}\}$. We claim that $(\!(a', b')\!)$ is a peak, where $a' = T^m(b')$. By construction, $b'\ inorb(a')$ and $a' \in orb(b')$, so we now show $(\!(a', b')\!) \in \mathcal{M}$.

By the assumption on ϵ, $T^i(b') \notin (a', b')$ for $i \leq m$. Now assume by way of contradiction that $(\!(a', b')\!)$ is not in $\mathcal{M}$ and $i > m$ is the smallest integer such that $T^i(b') \in (a', b')$. Then T^i must be monotone on $(b, b + \epsilon)$. Since $T^i((b, b + \epsilon))$ must contain an element of the orbit of a, it follows from the IVT that there exists $x \in (b, b + \epsilon)$ such that $T^i(x) \in x, x_a$, which contradicts the definition of N. Thus $(\!(a', b')\!)$ must be in $\mathcal{M}$, and thus a peak. ■

Before we draw any conclusions, we would now like to direct the attention of the reader to the structure of $\mathcal{M}$. Recall that $(\!(a, b)\!) \in [0, 1/2] \times [1/2, 1]$ and note that $((0, b)) \in \mathcal{M}$ if and only if $b = 1/2$ or $b = 1$. However, if $0 < a < 1/4$, then $a < T(a) < 1/2$ and thus $((a, b))$ is not in $\mathcal{M}$. We conclude that $((0, 1))$ and $((0, 1/2))$ are isolated in $\mathcal{M}$ and therefore cannot be approximated by peaks. Now, excluding these two exceptional cases, Proposition 3.3.3 and Lemmas 3.3.4 and 3.3.5 can be used to show that any $((a, b)) \in \mathcal{M} \setminus \mathcal{P}$ where neither a nor b is an negative periodic, can be approximated by peaks, so long as neither a nor b equals $1/2$. The case when either $a = 1/2$ or $b = 1/2$ is a special case which we will address first.

Lemma 3.3.6 *If $((a, b)) \in \mathcal{M}$ with either $a = 1/2$ or $b = 1/2$, then $((a, b)) \in \overline{\mathcal{P}}$.*

Proof Assume $((a, b)) \neq ((1/2, 1/2))$. First suppose that $a = 1/2$ and $b \in orb(a)$. Then it must be that $b = 1$. The pair $((1/2, 1))$ can be approximated by peaks $((\frac{2^n}{2^{n+1}+1}, \frac{2^{n+1}}{2^{n+1}+1}))$.

Suppose then that $a = 1/2$ and $b \notin orb(a)$. Then by Proposition 3.3.3, $((1/2, b)) \in \overline{\mathcal{D}}$. If $((1/2, b)) \in \mathcal{D}$, this means $1/2 \in orb(b)$ with $b \neq 1/2$. It then follows from Lemmas 3.3.4 3.3.5 that $((1/2, b)) \in \overline{\mathcal{P}}$. Now consider the case when $((1/2, b)) \notin \mathcal{D}$. Fix $\epsilon > 0$ so that $1/2 \notin (b - \epsilon, b + \epsilon)$. By Lemmas 3.3.1 and 3.3.2 that there exists $((1/2, b'))$ so that $((1/2, b')) \in \mathcal{D}$ and $b' \in (b - \epsilon, b)$. By the choice of ϵ, we know that $b' \neq 1/2$. Therefore by Lemmas 3.3.4 and 3.3.5, $((1/2, b')) \in \overline{\mathcal{P}}$ and hence $((1/2, b)) \in \overline{\mathcal{P}}$.

A similar proof will also work for when $b = 1/2$. If both a and b are $1/2$, then we may let $b' = \frac{2^{n-1}+1}{2^n}$ to get $a \in orb(b')$ and proceed as before. ∎

Theorem 3.3.7 *$\mathcal{M} \setminus \mathcal{O} \subset \overline{\mathcal{P}}$.*

Proof Let $((a, b)) \in \mathcal{M} \setminus \mathcal{O}$. The case for when a or b is $1/2$ is covered in Lemma 3.3.6. Therefore we now assume that $a, b \neq 1/2$ and fix $\epsilon > 0$ sufficiently small so that $|a - 1/2| > \epsilon$ and $|b - 1/2| > \epsilon$. By Proposition 3.3.3 there exists$((\alpha, \beta)) \in \mathcal{D}$ such that $|b - \beta| < \frac{\epsilon}{2}$ and $|a - \alpha| < \frac{\epsilon}{2}$. It then follows from Lemmas 3.3.4 and 3.3.5 that there exists peak $((a', b'))$ with $|\alpha - a'| < \frac{\epsilon}{2}$ and $|\beta - a'| < \frac{\epsilon}{2}$. It then follows from the triangle inequality that $|a - a'| < \epsilon$ and $|b - b'| < \epsilon$. ∎

Theorem 3.3.8 $\overline{\mathcal{P}} = \mathcal{M} \setminus \mathcal{O}$.

Proof First note that since $\mathcal{M}^C$ is an open set, then $\mathcal{M}$ is closed. Therefore $\overline{\mathcal{P}} \subset \mathcal{M}$ since all peaks are contained in $\mathcal{M}$. Also note that we have already shown pairs $(\!(0,1)\!)$ and $(\!(0,1/2)\!)$ are isolated in $\mathcal{M}$, and thus are not in the closure of peaks. Now consider $(\!(a,b)\!) \in \mathcal{O}$. Suppose that, for a very small ϵ, there exists $(\!(a',b')\!) \in \mathcal{P}$ such that $|a - a'| < \epsilon$ and $|b - b'| < \epsilon$. Without loss of generality suppose that a is an negative periodic orbit with period p. Then either $T^p(a')$ or $T^{2p}(a') \in (a', b')$, but this means that $(\!(a',b')\!) \notin \mathcal{M}$, a contradiction. Therefore it must be that $\mathcal{O} \cap \overline{\mathcal{P}} = \emptyset$ and thus $\overline{\mathcal{P}} \subset \mathcal{M} \setminus \mathcal{O}$. Now if $(\!(a,b)\!) \in \mathcal{M} \setminus \mathcal{O}$, then by Theorem 3.3.7 $(\!(a,b)\!) \in \overline{\mathcal{P}}$. Therefore $\overline{\mathcal{P}} = \mathcal{M} \setminus \mathcal{O}$. ■

4. MATCHING

This chapter is devoted to studying a particular subfamily of $\mathcal{F}$, specifically elements of $\mathcal{F}$ which are piecewise affine. The contents of this chapter can be found in its entirety in [9]. Using an affine conjugacy, we can bring such maps each to a form where the discontinuity occurs at 0 and the right limit at 0 of the value is 1. The formula will then be

$$T_{\lambda,\mu,b}(x) = \begin{cases} 1 + \lambda x + b & \text{if} \quad x \le 0, \\ 1 - \mu x & \text{if} \quad x \ge 0, \end{cases} \tag{4.1}$$

where $\lambda, \mu > 0$; see Figure 4.1. Note that if λ and μ are understood to be fixed, we will simply write T_b for $T_{\lambda,\mu,b}$.

Define $y_b = \max\{T_b(0_-), T_b(0_+)\}$ and $x_b = T_b(y_b)$. We want to consider our map on a compact interval instead of the whole real line. The natural candidate for this interval is $[x_b, y_b]$. If this interval is invariant for T_b, then it is the smallest invariant interval. If this interval is not invariant, then the trajectory of x_b escapes to $-\infty$, and there is no invariant interval. The necessary and sufficient condition for this interval to be invariant is $T_b(x_b) \in [x_b, y_b]$. Since always $T_b(x_b) < y_b$, our condition becomes

$$T_b(x_b) \ge x_b. \tag{4.2}$$

While we could translate (4.2) to inequalities in λ, μ and b, we would never use them in that form.

We also want the map to be (eventually) piecewise expanding, so we assume that

$$\lambda \ge 1 \quad \text{and} \quad \mu > 1. \tag{4.3}$$

However, if in both (4.2) and (4.3) we have equalities, then the map on the left lap is the identity. This is a highly degenerate case, so we will assume that

$$\text{if} \quad T_b(x_b) = x_b \quad \text{then} \quad \lambda > 1. \tag{4.4}$$

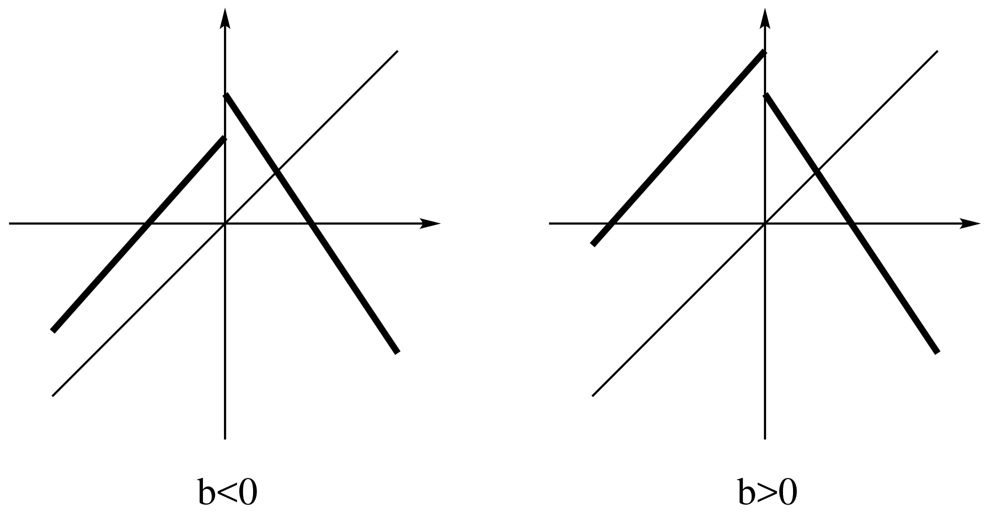


Fig. 4.1. The maps $T_{\lambda,\mu,b}$.

Throughout most of the paper we will consider maps $T_b = T_{\lambda,\mu,b}$ satisfying (4.2), (4.3) and (4.4). We will denote the family of those maps by $\mathcal{T}$.

The map T_b has a fixed point

$$z = \frac{1}{1+\mu}$$

on the right lap. Note that its position does not depend on b, so we do not need a subscript b here.

For a piecewise continuous piecewise monotone map f (with the finite number of laps), the usual definition of its topological entropy is

$$h_{\mathrm{top}}(f) = \lim_{n\to\infty} \frac{1}{n} \log c_n, \tag{4.5}$$

where c_n is the number of laps of f^n. In [10] it is shown that this agrees with the standard Bowen's definition of topological entropy.

In 2013, V. Botella-Soler, J. A. Oteo, J. Ros and P. Glendinning [3] observed numerically that for certain values of λ and μ both Lyapunov exponent and topological entropy of $T_{\lambda,\mu,b}$ remain constant as b varies in some interval of values close to 0, although the kneading sequence varies. They proved it for $\lambda = 1$ and μ equal to 2 or the golden ratio. Their proofs rely on the algebraic properties of the slopes.

In 2014, H. Bruin, C. Carminati, S. Marmi and A. Profeti [4] connected the phenomenon of plateaus of the exponent and topological entropy with *matching with index zero*, which was defined as the existence of $k > 0$ such that $T_b^k(0_-) = T_b^k(0_+)$, and the derivatives from the left and right also match. In their talk they sketched the proof for $\lambda = 1$ and $\mu = 2$ and noted that the plateaus occur for $\lambda = 1$ and μ a quadratic Pisot number.

In this paper we prove existence of plateaus of the topological entropy (we call this phenomenon *entropy locking*) for all λ, μ satisfying (4.3) and (4.4). We do not use algebraic properties of the slopes. Our proof is simple and its central part is based on the ideas from the Euclidean geometry of the plane. Such connection is expected if the system itself is defined in geometric terms, but here it comes as a surprise.

Proposition 4.0.1 *For a map $T_b \in \mathcal{T}$, $I(x) < I(y)$ if and only if $x < y$.*

This Proposition is almost identical to Proposition 2.1.2. The only detail that is different, is that we can use the strict inequalities on both sides of the equivalence. This follows from the fact that our maps have iterates that are piecewise expanding, so different points have different itineraries. Let us state it as a lemma.

Lemma 4.0.2 *For a map $T_b \in \mathcal{T}$, there is n such that T_b^n is expanding on each lap.*

Proof If $\lambda > 1$, then T_b itself is expanding on each lap. Assume that $\lambda = 1$. Then, by (4.2) and (4.4), $T_b(x_b) - x_b > 0$, and for each $x \in [x_b, 0)$ we have $T_b(x) - x = T_b(x_b) - x_b$. This means that at least one of the points $T_b^i(x)$, $0 \le i \le n$, belongs to the right lap of T_b, provided $n > |x_b|/(T_b(x_b) - x_b)$. Therefore, for such n the map T_b^n is expanding with the constant at least μ on each lap. ∎

4.1 Matching

We are interested in the conditions under which $T_b^k(0_-)$ and $T_b^k(0_+)$ coincide for some k. We start with a simple geometric lemma.

Lemma 4.1.1 *Let f be a map conjugated to $T_{\lambda,\mu,0} \in \mathcal{T}$ via an orientation preserving affine map. Let c be the turning point of f and let $x < c < y$. Then $f(x) = f(y)$ if and only if*

$$\frac{x-c}{c-y} = \frac{\mu}{\lambda}. \tag{4.6}$$

Proof Assume that (4.6) is satisfied. Then

$$f(x) - f(c) = \lambda(x-c) = \mu(c-y) = f(y) - f(c),$$

and therefore $f(x) = f(y)$.

Now assume that $f(x) = f(y)$. Then

$$\lambda(x-c) = f(x) - f(c) = f(y) - f(c) = \mu(c-y),$$

and (4.6) follows. ■

Now we can prove the main result of this section. In the proof we will be using the notation $\langle x, y\rangle$ for $[x, y]$ if $x < y$ and $[y, x]$ if $y < x$.

Theorem 4.1.2 *Let $T_b = T_{\lambda,\mu,b} \in \mathcal{T}$, and let $\underline{A}$ be a finite (possibly empty) sequence of symbols R and L. Set $n = |RL\underline{A}C|$. Assume that $K_-(T_b) = RL\underline{A}R\ldots$ and $K_+(T_b) = RL\underline{A}L\ldots$. Then $K(T_0) = RL\underline{A}C$ if and only if $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$.*

Proof We use the ideas from the Euclidean geometry. We consider the graph of T_b, then draw some additional lines, identify similar figures and use proportions.

Thus, consider the graph of T_b. It consists of two branches. From the assumptions on the kneading sequences it follows that $b \neq 0$. If $b < 0$, then the left branch ends lower than the right branch; if $b > 0$ then the right branch ends lower than the right

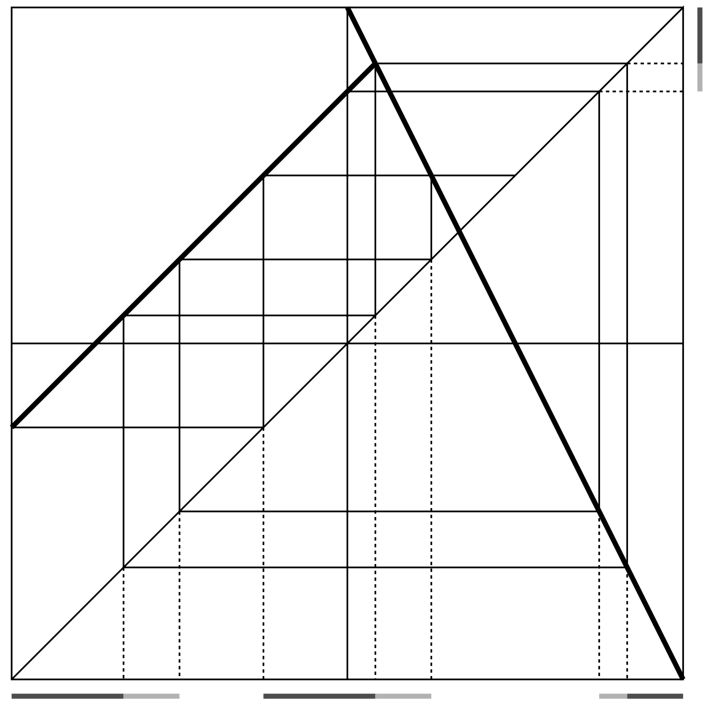


Fig. 4.2. The proof of Theorem 4.1.2. The proportion of the lengths of the light gray and dark gray intervals stays λ/μ.

one. Extend the lower branch until it crosses the higher one (see Figure 4.2). This happens at the point $(c, T_b(c))$, where $1 + \lambda c + b = 1 - \mu c$, so

$$c = \frac{-b}{\mu + \lambda}. \tag{4.7}$$

Now we define a continuous map f of $[x_b, y_b]$ to itself by

$$f(x) = \begin{cases} 1 + \lambda x + b & \text{if} \quad x \le c, \\ 1 - \mu x & \text{if} \quad x \ge c. \end{cases}$$

We claim that $f^i(c) \notin \langle 0, c \rangle$ for $i = 1, 2, \ldots, n-1$. Indeed, suppose that $f^i(c) \in \langle 0, c \rangle$ for some $i \in [1, n-1]$ and $f^k(c) \notin \langle 0, c \rangle$ for all $k \in [1, i-1]$. Then $f^k(c) = T_b^k(c)$ for $k \in [1, i]$. Set $U = \langle T_b(0_-), T_b(0_+) \rangle$, and note that $T_b(c) \in U$.

Since both $K_-(T_b)$ and $K_+(T_b)$ begin with RL, the interval U lies to the right of the fixed point z, while 0 and c are to the left of z. Therefore $i \geq 2$.

We have

$$T_b(0_+) - T_b(c) = -\mu\left(0 - \frac{-b}{\mu+\lambda}\right) = \frac{-\mu b}{\mu+\lambda} \tag{4.8}$$

and

$$T_b(c) - T_b(0_-) = \lambda\left(\frac{-b}{\mu+\lambda} - 0\right) = \frac{-\lambda b}{\mu+\lambda}. \tag{4.9}$$

Since $K_+(T_b)$ and $K_-(T_b)$ agree on the first $n-1$ places, then $0 \notin T_b^k(U)$ for $k \leq i$. Therefore, T_b^i is affine on U. Thus, we get

$$|T_b^i(0_+) - T_b^i(c)| \geq \frac{\mu^2|b|}{\mu+\lambda} > |c|$$

and

$$|T_b^i(c) - T_b^i(0_-)| \geq \frac{\lambda\mu|b|}{\mu+\lambda} > |c|,$$

where we get the final inequality because $\mu > 1$ and $\lambda \geq 1$. Thus, $K_+(T_b)$ and $K_-(T_b)$ disagree on the $i-1$st index, which is a contradiction. This proves that $f^i(c) \notin \langle 0, c\rangle$ for $i = 1, 2, \ldots, n-1$.

It follows from the assumption on kneading sequences that $n \geq 3$, and therefore $f^2(c) \notin \langle 0, c\rangle$, so $f^2(c) < c$. This implies that the interval $[f^2(c), f(c)]$ is invariant under f. Since f has the same slopes as T_0, then $f|_{[f^2(c),f(c)]}$ is conjugate to T_0 via an orientation preserving affine map. Moreover, it follows from our claim that $f^i(c) = T_b^i(c)$ for all $i \in [0, n]$. Because of the assumptions on kneading sequences, none of the intervals $T_b^i(U)$, $0 < i < n-1$, contains 0. Therefore the map T_b^{n-1} is affine on U.

From this and from the formulas (4.8) and (4.9) it follows that

$$\frac{T_b^n(0_+) - T_b^n(c)}{T_b^n(c) - T_b^n(0_-)} = \frac{\mu}{\lambda}. \tag{4.10}$$

Assume that $K(T_0) = RL\underline{A}C$. Then $f^n(c) = c$, so $T_b^n(c) = c$. We have $|T_b^n(0_-) - c| > |c|$ and $|T_b^n(0_+) - c| > |c|$, and thus, $T_b^n(0_-)$ and $T_b^n(0_+)$ are not contained in the interval $\langle 0, c\rangle$. Hence, $T_b^{n+1}(0_+) = f(T_b^n(0_+))$ and $T_b^{n+1}(0_-) = f(T_b^n(0_-))$. The assumption $x < c < y$ of Lemma 4.1.1 for $x = T_b^n(0_+)$ and $y = T_b^n(0_-)$ is satisfied by the assumption on the kneading sequences of T_b. Thus, from (3.17) and Lemma 4.1.1 we get $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$.

Assume now that $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$. Then

$$T_b^{n+1}(0_-) \leq \min(T_b(0_-), T_b(0_+)) < T_b(c),$$

so again $T_b^n(0_-)$ and $T_b^n(0_+)$ are not contained in the interval $\langle 0, c\rangle$. By (3.17) and Lemma 4.1.1 we get $f^n(c) = c$. Since for $i < n$ the point $f^i(c)$ is between $T_b^i(0_-)$ and $T_b^i(0_+)$ and both $K_-(T_b)$ and $K_+(T_b)$ begin with $RL\underline{A}$, and moreover, $f^i(c) \notin \langle 0, c\rangle$ for $i = 1, 2, \ldots, n-1$, we see that the kneading sequence of f also begins with $RL\underline{A}$. Since $f^n(c) = c$, the next symbol is C. The maps T_0 and f are conjugate, so their have the same kneading sequences. Therefore, $K(T_0) = RL\underline{A}C$. ■

Remark 4.1.3 *Suppose that the assumptions of Theorem 4.1.2 are satisfied. If* $b < 0$ *then* $T_b(0_-) < T_b(0_+)$*, so* $K_-(T_b) < K_+(T_b)$*. This implies that* $\underline{A}$ *is even. Similarly, if* $b > 0$ *then* $\underline{A}$ *is odd.*

We can prove a kind of converse to the above remark.

Proposition 4.1.4 *Fix parameters* $\lambda \geq 1$*,* $\mu > 1$*, such that* $K(T_0) = RL\underline{A}C$*. Then if if* $\underline{A}$ *is even (respectively, odd), there exists* $\varepsilon > 0$ *such that if* $b \in (-\varepsilon, 0)$ *(respectively,* $b \in (0, \varepsilon)$*) then the assumptions of Theorem 4.1.2 are satisfied, and thus,* $T_b^{n+1}(0_-) = T_b^{n+1}(0_+)$*.*

Proof If $|b|$ is sufficiently small, then both $K_-(T_b)$ and $K_+(T_b)$ begin with $RL\underline{A}$. Thus, we have to show that the next symbol is R for $K_-(T_b)$ and L for $K_+(T_b)$. By making the construction from the proof of Theorem 4.1.2, we see that $T^n(0_-)$ and $T^n(0_+)$ are on the opposite sides of c. Moreover, both $|T^n(0_-) - c|$ and $|T^n(0_+) - c|$ are larger than $|c|$, so the nth terms of $K_-(T_b)$ and $K_+(T_b)$ are distinct. Taking into account the order in the set of itineraries (as in Remark 4.1.3), we get the assertion of the proposition. ■

4.2 Topological entropy

Entropy locking refers to intervals of the parameter b where topological entropy of T_b remains constant. It turns out that the intervals of parameter b satisfying Theorem 4.1.2 are intervals with entropy locking.

We need some estimates of the topological entropy for piecewise continuous piecewise monotone interval maps (when using this term, we always assume that the number of pieces is finite). They are known, but they are difficult to find in the literature. Since the proofs are simple, we provide them here.

For a piecewise continuous piecewise monotone interval map f we will say that α is an *anti-Lipschitz constant* if for every x, y from the same lap we have $|f(x) - f(y)| \geq \alpha|x-y|$. In particular, a map with an anti-Lipschitz constant larger than 1 is piecewise expanding.

An *s-horseshoe* for f is an interval J and a partition $D = \{J_1, \ldots, J_s\}$ of J into s subintervals such that $J \subset f(J_i)$ and f is continuous and monotone on each J_i. The following theorem was proved in [10].

Theorem 4.2.1 *If f is a piecewise continuous piecewise monotone interval map, then for every $\varepsilon > 0$ there exist n and s, such that f^n has an s-horseshoe and $(1/n) \log s > h_{\text{top}}(f) - \varepsilon$.*

Now we can prove the promised estimates.

Theorem 4.2.2 *If f is a piecewise continuous piecewise monotone interval map with an anti-Lipschitz constant α and a Lipschitz constant β, then $\log \alpha \leq h_{\text{top}}(f) \leq \log \beta$.*

Proof We use formula (4.5). If the interval on which f is acting has length γ, then the length of each lap of f^n is not larger than γ/α^n. Therefore $c_n \geq \alpha^n$, and thus, $h_{\text{top}}(f) \geq \log \alpha$.

Take $\varepsilon > 0$. By Theorem 4.2.1, there exist n and s, such that f^n has an s-horseshoe and $(1/n) \log s > h_{\text{top}}(f) - \varepsilon$. Let an interval J and a partition $D = \{J_1, \ldots, J_s\}$ be this horseshoe. Then the length of each J_i is at least the length of J divided by β^n.

Therefore, $s \le \beta^n$, and hence, $\log\beta > h_{\mathrm{top}}(f) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we get $h_{\mathrm{top}}(f) \le \log\beta$. ■

From this theorem and Lemma 4.0.2, we get immediately the following corollary.

Corollary 4.2.3 *All maps from $\mathcal{T}$ have strictly positive topological entropy.*

Any map $T_{\lambda,\lambda,b} \in \mathcal{T}$ has both anti-Lipschitz and Lipschitz constants equal to λ. Therefore we get immediately another corollary to Theorem 4.2.2.

Corollary 4.2.4 *If $T_{\lambda,\lambda,b} \in \mathcal{T}$, then its topological entropy is $\log\lambda$.*

Now we are ready to prove the main result of this section. We will refer to piecewise continuous piecewise affine interval maps with the absolute value of the derivative constant, as *maps of constant slope*. In $\mathcal{T}$, these are maps of the form $T_{\lambda,\lambda,b}$.

We will be using often a certain long assumption, so it makes sense to give it a short name.

Definition 4.2.5 *We will say that T_b satisfies the* kneading assumption *if $T_b = T_{\lambda,\mu,b} \in \mathcal{T}$ and there exists a finite (possibly empty) sequence $\underline{A}$ of symbols R and L, such that $K(T_0) = RL\underline{A}C$, $K_-(T_b) = RL\underline{A}R\dots$, and $K_+(T_b) = RL\underline{A}L\dots$.*

Theorem 4.2.6 *Assume that T_b satisfies the kneading assumption and is topologically conjugate to a map of constant slope. Then $h_{\mathrm{top}}(T_b) = h_{\mathrm{top}}(T_0)$.*

Proof By the assumption, $T_{\lambda,\mu,b}$ is conjugate to $T_{\alpha,\alpha,d}$ for some α and d. By Corollary 4.2.4,

$$\log\alpha = h_{\mathrm{top}}(T_{\alpha,\alpha,d}) = h_{\mathrm{top}}(T_{\lambda,\mu,b}). \tag{4.11}$$

Set $n = |RL\underline{A}C|$. From Theorem 4.1.2 it follows that $T^{n+1}_{\lambda,\mu,b}(0_+) = T^{n+1}_{\lambda,\mu,b}(0_-)$. Hence, $T^{n+1}_{\alpha,\alpha,d}(0_+) = T^{n+1}_{\alpha,\alpha,d}(0_-)$. Since the kneading sequences are preserved by a conjugacy, the left and right kneading sequences of $T_{\alpha,\alpha,d}$ are

$K_-(T_{\alpha,\alpha,d}) = RL\underline{A}R\ldots$ and $K_+(T_{\alpha,\alpha,d}) = RL\underline{A}L\ldots$, respectively. Thus, we can use Theorem 4.1.2 again, and we get $K(T_{\alpha,\alpha,0}) = RL\underline{A}C$. For unimodal maps the topological entropy is determined by the kneading sequence, and therefore

$$h_{\text{top}}(T_{\lambda,\mu,0}) = h_{\text{top}}(T_{\alpha,\alpha,0}). \tag{4.12}$$

By Corollary 4.2.4,

$$h_{\text{top}}(T_{\alpha,\alpha,0}) = \log \alpha. \tag{4.13}$$

From (4.11), (4.12) and (4.13) we get $h_{\text{top}}(T_{\lambda,\mu,0}) = h_{\text{top}}(T_{\lambda,\mu,b})$. ∎

4.3 Transitivity

While Theorem 4.2.6 is quite strong, it contains an assumption that may be not easy to verify in concrete situations. Namely, we assume that T_b is topologically conjugate to a map of constant slope. In this section we will try to replace this assumption by weaker ones, which are easier to verify.

The first idea is to assume that T_b is topologically transitive. The following theorem can be found for instance in [11].

Theorem 4.3.1 *If f is a piecewise continuous piecewise monotone topologically transitive interval map with topological entropy $\log \beta > 0$, then it is topologically conjugate to a map of constant slope β.*

In view of this theorem and Corollary 4.2.3, we get the following corollary to Theorem 4.2.6.

Corollary 4.3.2 *Assume that T_b satisfies the kneading assumption and is topologically transitive. Then $h_{\text{top}}(T_b) = h_{\text{top}}(T_0)$.*

We will further improve this corollary, by replacing the assumption that T_b is topologically transitive by another assumption, which is maybe a little weaker, but easier to check. This assumption will be

$$T_{\lambda,\mu,0}(x_0) < z. \tag{4.14}$$

It can be easily written as an inequality on parameters

$$\lambda + \mu < \lambda\mu^2. \tag{4.15}$$

It is known that it is equivalent to $T_{\lambda,\mu,0}$ being totally transitive; however, we will not use this fact. We will say that $T_b = T_{\lambda,\mu,b}$ satisfies (4.14) if $T_0 = T_{\lambda,\mu,0}$ satisfies it.

Definition 4.3.3 *The set $\mathcal{T}_{\mathrm{KAT}}$ is the set of all maps T_b satisfying both the kneading assumption and* (4.14)*.*

Lemma 4.3.4 *Assume that $T_b \in \mathcal{T}_{\mathrm{KAT}}$. Then*

$$T_b(1-\mu) \le 1. \tag{4.16}$$

Proof If $b \le 0$, then $y_b = 1$, so (4.16) holds. Assume that $b > 0$. If $T_b(1-\mu) > 1$, then $K_+(T_b) = RLRL\dots$. By the kneading assumption, $K_-(T_b) = RLR\dots$. We have $T_b(x_b) = 1 + \lambda(1-\mu-\mu b) + b$ and $T_0(x_0) = 1 + \lambda(1-\mu)$. Since $b < \lambda\mu b$, we get $T_b(x_b) < T_0(x_0)$. By this and (4.14), $T_b(x_b) < z$, so the next term in $K_-(T_b)$ is R. Thus, by the kneading assumption, $K(T_0) = RLRC$. Then $1 - \mu(1 + \lambda - \lambda\mu) = T_0^4(0) = 0$, so $\lambda = 1/\mu < 1$, a contradiction. Thus, (4.16) holds. ∎

Lemma 4.3.5 *Assume that $T_b \in \mathcal{T}_{\mathrm{KAT}}$. Let U be an interval containing z. Then*

$$\bigcup_{i=0}^{\infty} T_b^i(U) = [x_b, y_b].$$

Proof Suppose first that $b \le 0$. Then $[x_b, y_b] = [1-\mu, 1]$. Since the interval U contains z, then all sets $T_b^i(U)$ must contain z as well. Moreover, $\mu > 1$, so the length of $T_b^i(U)$ is expanding exponentially with i until we reach an m such that $T_b^m(U)$ contains $[z, 1]$. Therefore $T_b([z,1]) = [1-\mu, z] \subset T_b^{m+1}(U)$. Hence, $T_b^m(U) \cup T_b^{m+1}(U) = [x_b, y_b]$.

Now assume that $b > 0$. By Lemma 4.3.4, (4.16) holds. As in the case $b \le 0$, we get $T_b^m(U) \cup T_b^{m+1}(U) \supset [1-\mu, 1]$ for some m. Since $T_b(1-\mu) \le 1$, the interval $T_b([1-\mu, 0])$ contains $[1, y_b]$. Since $T_b([1, y_b]) = [x_b, 1-\mu]$, we get $T_b^m(U) \cup T_b^{m+1}(U) \cup T_b^{m+2}(U) \cup T_b^{m+3}(U) = [x_b, y_b]$. ∎

Theorem 4.3.6 *Assume that $T_b \in \mathcal{T}_{\mathrm{KAT}}$. Then T_b is topologically transitive.*

Proof Let U be an open subinterval of $[x_b, y_b]$. We will show that $V = \bigcup_{i=0}^{\infty} T_b^i(U)$ is dense in $[x_b, y_b]$. Since $\mu > 1$, the length of $T_b^n(U)$ increases exponentially with n. Thus, there exists k such that $0 \in T_b^k(U)$. Therefore, V contains an interval containing 0. Let W be the largest such interval contained in V. We can write $W = W_L \cup W_R$, where $W_L = \{x \in W : x \leq 0\}$ and $W_R = \{x \in W : x \geq 0\}$. Since V is invariant and $\mu > 1$, then, by the same reason as for U, it must happen that 0 belongs to the interior of $T_b^m(W_L)$ and $T_b^n(W_R)$ for some positive integers m and n. If m and n are minimal such integers, then $T_b^m(W_L)$ and $T_b^n(W_R)$ are intervals, and therefore they are contained in W.

Suppose that V is not dense. We claim that then $m \geq 2$ and $n \geq 2$. In view of Lemma 4.3.5, in order to prove the claim, it is enough to show that if m or n is 1, then $z \in W$.

Assume first that $b < 0$. If $m = 1$, then $T_b(W_L) \subset W$ and in particular $T_b(0_-) \in W$. Therefore the interval $[0, T_b(0_-)]$ is contained in W. We claim that $T_b(0_-) \geq z$. Indeed, if $T_b(0_-) < z$, then $K_-(T_b)$ starts with RR, which is impossible by the kneading assumption, and this proves the claim. Therefore, $z \in [0, T_b(0_-)]$. If $n = 1$, then $T_b(W_R) \subset W$, and in particular $T_b(0_+) = 1 \in W$. Thus, $z \in [0, 1] \subset W$.

Now assume that $b > 0$. If $m = 1$, then $T_b(W_L) \subset W$ and in particular $T_b(0_-) = 1 + b \in W$. Thus, $z \in [0, 1 + b] \subset W$. If n=1, then $T_b(W_R) \subset W$ and it follows that $z \in [0, 1] \subset W$. This completes the proof of the claim.

By our choice of m and n, T_b^m is affine on W_L and T_b^n is affine on W_R. Additionally, since $T_b(0_-) > 0$, $T_b(0_+) > 0$ and $m, n \geq 2$, we have $I(x) = LR\ldots$ for every $x \in W_L$ and $I(x) = RR\ldots$ for every $x \in W_R$. In such a way, we get lower bounds on the lengths of $T_b^m(W_L)$ and $T_b^n(W_R)$:

$$\begin{aligned} \lambda\mu|W_L| &\leq |T_b^m(W_L)|, \\ \mu^2|W_R| &\leq |T_b^n(W_R)|. \end{aligned} \tag{4.17}$$

We also know that $T_b^m(W_L) \subset W$ and $T_b^n(W_R) \subset W$, so from (4.17) we get

$$\begin{aligned} \lambda\mu|W_L| &\le |W|, \\ \mu^2|W_R| &\le |W|. \end{aligned} \tag{4.18}$$

We add the first inequality in (4.18) multiplied by by μ to the second one multiplied by λ, and taking into account that $|W_L| + |W_R| = |W|$, we get

$$\lambda\mu^2|W| \le (\lambda + \mu)|W|,$$

which contradicts (4.15) (which, as we noticed, is equivalent to (4.14)). This completes the proof. ■

Now from Corollary 4.3.2 and Theorem 4.3.5 we get an improved corollary.

Corollary 4.3.7 *Assume that $T_b \in \mathcal{T}_{\mathrm{KAT}}$. Then $h_{\mathrm{top}}(T_0) = h_{\mathrm{top}}(T_b)$.*

4.4 Beyond transitivity

Theorem 4.3.6 gives sufficient conditions for transitivity of $T_b = T_{\lambda,\mu,b}$. The assumption of this theorem is that $T_b \in \mathcal{T}_{\mathrm{KAT}}$, that is, that T_b satisfies the kneading assumption (Definition 4.2.5) and satisfies (4.14). Two simple examples will show that both assumptions are essential.

First, we establish a necessary condition for transitivity.

Lemma 4.4.1 *Suppose $T_b \in \mathcal{T}$. If $z \notin T_b([x_b, 0])$, then T_b is not transitive.*

Proof Let ε be sufficiently small so that $(z-\varepsilon, z+\varepsilon) \cap T_b([x_b, 0]) = \emptyset$. Since $\mu > 1$, z is repelling, and therefore $T_b^{-1}((z-\varepsilon, z+\varepsilon)) \subset (z-\varepsilon, z+\varepsilon)$. Hence, if V is an open interval such that $(z-\varepsilon, z+\varepsilon) \cap V = \emptyset$, then $T_b^n(V) \cap (z-\varepsilon, z+\varepsilon) = \emptyset$ for all n. Thus, T_b is not transitive. ■

Example 4.4.2 *Set $\lambda = 1$ and find μ such that the kneading sequence of $T_0 = T_{\lambda,\mu,0}$ is $RLRRRC$. Elementary computations show that μ is the real solution of the equation*

$\mu^3 - \mu^2 - 1 = 0$ ($\mu \approx 1.46557$). We can deduce from the kneading sequence that $T_0(x_0) > z$, so T_0 does not satisfy (4.14)*. Moreover, $T_b(x_b) > z$ for sufficiently small b. It follows from Proposition 4.1.4 that T_b satisfies the kneading assumption for sufficiently small $b > 0$. Hence, for $b > 0$ sufficiently small, T_b satisfies the kneading assumption, but not* (4.14)*, and is not transitive.*

Example 4.4.3 *Set $\lambda = 1$ and $\mu = 2$. Then $K(T_0) = RLC$ and $T_0(x_0) < z$. Therefore, T_b satisfies* (4.14) *for any b. However, for $b = -\frac{3}{4}$ we have $T_b(0_-) < z$, so by Lemma 4.4.1, T_b is not transitive. In particular, it cannot satisfy the kneading assumption.*

We will show that also the topological entropies of T_0 and T_b are different. Both maps are Markov. For T_0, the Markov partition consists of two intervals, and the topological entropy is the logarithm of the positive solution of the equation $x^2 - x - 1 = 0$, that is, the logarithm of the golden ratio $\phi = \frac{1+\sqrt{5}}{2} \approx 1.618$.

For the map T_b, we have $T_b^6(0_+) = 0$ and $T_b^3(0_-) = 0$. A Markov partition $\mathcal{P}$ of $[x_b, y_b] = [-1, 1]$ is given by the orbits of 0_+ and 0_- and consists of 7 intervals. One can find easily its entropy using the rome method (see [12] or [13]). It is equal to the logarithm of the positive solution of the equation $x^6 - x^3 - x^2 - x - 1 = 0$, that is, approximately $\log 1.3803$. Hence, $h_{\mathrm{top}}(T_0) \neq h_{\mathrm{top}}(T_b)$ for $b = -3/4$. A reader who does not believe in approximate values can check that

$$x^6 - x^3 - x^2 - x - 1 = (x^4 + x^3 + 2x^2 + 2x + 3)(x^2 - x - 1) + (4x + 2),$$

so $\phi^6 - \phi^3 - \phi^2 - \phi - 1 = 4\phi + 2 > 0$.

Remember that the reason we started to consider transitivity of T_b was that we do not know any other simple way of verifying that T_b is conjugate to a map of constant slope. However, the maps $T_0 \in \mathcal{T}$ are known to be conjugate to maps of constant slope (this basically follows from [14] and [8], although it is not stated explicitly there). Thus, we can state the following conjecture.

Conjecuture 4.4.4 *Every $T_b \in \mathcal{T}$ is topologically conjugate to a map of constant slope.*

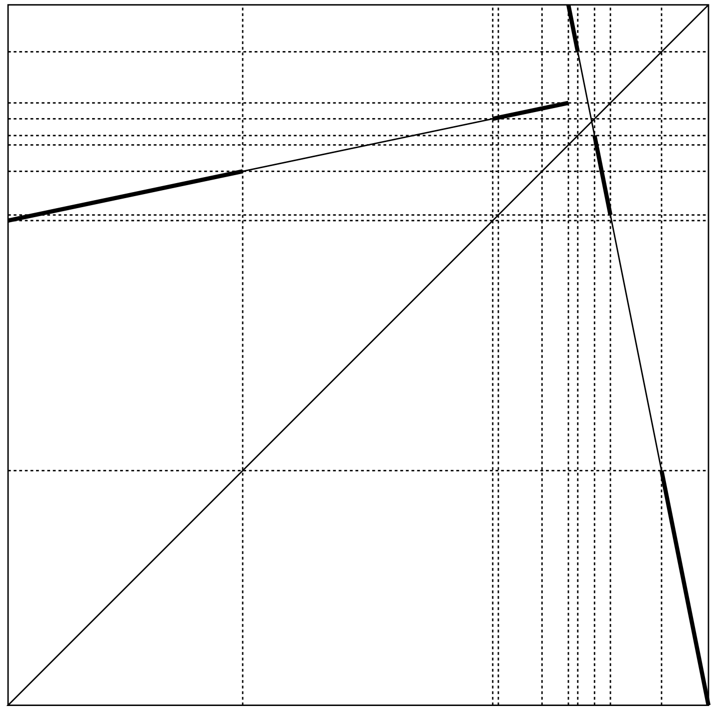


Fig. 4.3. The map from Example 4.4.5.

If this conjecture is true, then by Theorem 4.2.6 every map $T_{\lambda,\mu,b} \in \mathcal{T}$ satisfying the kneading assumption would have the same topological entropy as $T_{\lambda,\mu,0}$.

To illustrate the problems which one may encounter when trying to prove this conjecture, we present another example. In it $\lambda < 1$, so the map does not belong to $\mathcal{T}$, and we believe that such example does not exist in $\mathcal{T}$. However, we do not know any compelling reasons for that.

Example 4.4.5 *Set $\lambda = 0.21$, $\mu = 5$. For $b = 0$ the orbit of the turning point is the Štefan periodic orbit of period 5 (and thus, the entropy of T_0 is the logarithm of the positive zero of the polynomial $x^5 - 2x^3 - 1$, approximately $\log 1.5129$). However, for $b = -0.7$ the map T_b is not transitive (see Figure 4.3).*

The union J of the intervals marked by thick lines in Figure 4.3 on the graph is invariant. It consists of the intervals $J_1 = [-4, -2.325]$, $J_2 = [-0.54, 0]$, $J_3 = [0, 0.067]$, $J_4 = [0.1866, 0.3]$, $J_5 = [0.665, 1]$. *We have*

$$T_b(J_1) \subset J_2, \quad T_b(J_2) = J_4, \quad T_b(J_3) = J_5, \quad T_b(J_4) \subset J_2 \cup J_3, \quad T_b(J_5) = J_1. \tag{4.19}$$

The entropy restricted to J is not larger than the entropy given by the Markov graph that we obtain by replacing in (4.19) *inclusions by equalities. That is, it is not larger than the logarithm of the positive zero of the polynomial $x^5 - x^3 - 1$, approximately* $\log 1.2365$.

The complement of J (call it G) has three components,

$$G_1 = (-2.325, -0.54), \quad G_2 = (0.067, 0.1866), \quad G_3 = (0.3, 0.665).$$

We have

$$T_b(G_1) \supset G_2, \quad T_b(G_2) \supset G_2 \cup G_3, \quad T_b(G_3) \supset G_1 \tag{4.20}$$

and the images of G_i do not intersect any other components of G than stated in (4.20). *Therefore G contains an invariant Cantor set C and the entropy of T_b restricted to C is equal to the logarithm of the positive zero of the polynomial $x^3 - x^2 - 1$, approximately* $\log 1.4656$.

Thus, the semiconjugacy with the map of constant slope and the same entropy maps any component of the complement of C (including the components of J) to a point. In particular, the factor map has different kneading sequences than the original one.

Observe that while T_b is not piecewise expanding, one can check using (4.19) *and* (4.20) *that T_b^n is piecewise expanding for $n \geq 195$.*

LIST OF REFERENCES

LIST OF REFERENCES

[1] O. M. Sharkovsky, "Co-existence of cycles of a continuous mapping of the line into itself," *Ukrain. Mat. Ž.*, vol. 16, pp. 61–71, 1964.

[2] L. Alsedà, J. Llibre, M. Misiurewicz, and C. Tresser, "Periods and entropy for Lorenz-like maps," *Ann. Inst. Fourier (Grenoble)*, vol. 39, no. 4, pp. 929–952, 1989.

[3] V. Botella-Soler, J. A. Oteo, J. Ros, and P. Glendinning, "Lyapunov exponent and topological entropy plateaus in piecewise linear maps," *J. Phys. A*, vol. 46, no. 12, pp. 125101, 26, 2013.

[4] S. M. Henk Bruin, Carlo Carminati and A. Profeti, "Matching in a family of piecewise affine interval maps," 2016.

[5] J. Milnor and W. Thurston, "On iterated maps of the interval," in *Dynamical systems (College Park, MD, 1986–87)*, vol. 1342 of *Lecture Notes in Math.*, pp. 465–563, Springer, Berlin, 1988.

[6] N. Metropolis, M. L. Stein, and P. R. Stein, "On finite limit sets for transformations on the unit interval," *J. Combinatorial Theory Ser. A*, vol. 15, pp. 25–44, 1973.

[7] W. Parry, "Symbolic dynamics and transformations of the unit interval," *Trans. Amer. Math. Soc.*, vol. 122, pp. 368–378, 1966.

[8] P. Collet and J.-P. Eckmann, *Iterated maps on the interval as dynamical systems*. Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009. Reprint of the 1980 edition.

[9] D. Cosper and M. Misiurewicz, "Entropy locking," *Fundamenta Mathematicae*, vol. 241, 2018.

[10] M. Misiurewicz and K. Ziemian, "Horseshoes and entropy for piecewise continuous piecewise monotone maps," in *From phase transitions to chaos*, pp. 489–500, World Sci. Publ., River Edge, NJ, 1992.

[11] L. Alsedà and M. Misiurewicz, "Semiconjugacy to a map of a constant slope," *Discrete Contin. Dyn. Syst. Ser. B*, vol. 20, no. 10, pp. 3403–3413, 2015.

[12] L. Block, J. Guckenheimer, M. Misiurewicz, and L. S. Young, "Periodic points and topological entropy of one-dimensional maps," in *Global theory of dynamical systems (Proc. Internat. Conf., Northwestern Univ., Evanston, Ill., 1979)*, vol. 819 of *Lecture Notes in Math.*, pp. 18–34, Springer, Berlin, 1980.

[13] L. Alsedà, J. Llibre, and M. Misiurewicz, *Combinatorial dynamics and Entropy in Dimension One*, vol. 5 of *Advanced Series in Nonlinear Dynamics*. World Scientific Publishing Co., Inc., River Edge, NJ, second ed., 2000.

[14] M. Misiurewicz and E. Visinescu, "Kneading sequences of skew tent maps," *Ann. Inst. H. Poincaré Probab. Statist.*, vol. 27, no. 1, pp. 125–140, 1991.

VITA

VITA

A son of the south who will seek to use his skills in service of his country.