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RESTRICTIONS TO INVARIANT SUBSPACES OF COMPOSITION OPERATORS ON THE HARDY SPACE OF THE DISK

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ABSTRACT

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The main result of this paper is found in Chapter 3, specifically Theorem 3.3.4. This theorem shows that if a linear fractional map of the unit disk into itself has rational coefficients and fixed points zero and one, then the composition operator on the Hardy space of the disk based on that map does not act the same on its invariant subspaces $z^k H^2$. Preliminaries on composition operators, Toeplitz operators, weighted composition operators, and the Hardy space are given in Chapter 1. Chapter 1 also contains some theorems on reproducing kernel functions that will aid our calculations in the main result. Preliminaries on linear fractional maps and known results for isometric and power compact composition operators are given in Chapter 2. Chapter 4 provides some further results about general composition operators for which zero is a fixed point of the symbol.

1. INTRODUCTION

1.1 Motivation

The topic of invariant subspaces is one that occurs naturally in linear algebra and operator theory. In the case of composition operators, some work has been done in this area. For examples, see [1], [2], [3] and [4]. To reduce this workload, it is of interest to see if any set of obvious subspaces reduces to a single situation - that is, to see if the operator behaves the same way on each subspace in the set. More formally, the question in mind is this: for two subspaces K, L of a Hilbert space H, is $C_\varphi|_K$ unitarily equivalent to $C_\varphi|_L$? In the Hardy Space $H^2(\mathbb{D})$, when $\varphi(0) = 0$, the subspaces $z^k H^2 = \{z^k f : f \in H^2\}$ are all invariant for C_φ. We show that for certain linear fractional symbols, the operator does *not* act the same on these subspaces.

1.2 Composition Operators on the Hardy Space H^2

This dissertation will discuss operators only on the Hilbert space $H^2(\mathbb{D})$, which we will shorten to H^2. The vectors in this space are analytic functions from the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ into the complex plane $\mathbb{C}$. For an analytic function f to be in the space H^2, it is further required that

$$\|f\|^2 = \sup_{0<r<1} \int_{\partial\mathbb{D}} |f(re^{i\theta})|^2 \, d\theta < \infty$$

where $d\theta$ is normalized arc-length on the boundary of the disk. If f is represented by the power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, an equivalent definition for $\|f\|$ is given by

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2$$

If we also let $g(z) = \sum_{n=0}^{\infty} b_n z^n$, then the inner product on this space is given by

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n}$$

One class of functions of particular interest is the set of reproducing kernel functions, denoted by K_α for $\alpha \in \mathbb{D}$. These are defined by

$$K_\alpha(z) = \frac{1}{1 - \overline{\alpha} z} \text{ for } z \in \mathbb{D}$$

These kernel functions satisfy $f(\alpha) = \langle f, K_\alpha \rangle$ and

$$\|K_\alpha\|^2 = \frac{1}{1 - |\alpha|^2}$$

K_α is a function in H^2 but also defines the linear functional $\langle f, K_\alpha \rangle = f(\alpha)$. In particular, we are interested in the linear functional given by the kernel for evaluation at 0, which is also the function that is identically 1. We will call this operator O:

$$Of = \langle f, K_0 \rangle = \langle f, 1 \rangle = f(0)$$

The results in this paper are for composition operators on H^2, which we may now define.

Definition 1.2.1 *For φ an analytic self-map of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, the composition operator C_φ is defined for each point $f \in H^2$ by*

$$(C_\varphi f)(z) = f(\varphi(z))$$

for each $z \in \mathbb{D}$.

We immediately see the relationship between reproducing kernel functions and composition operators: a bounded operator A is a composition operator if and only if the set of reproducing kernel functions is invariant under A^* [5, Theorem 1.4].

Every composition operator on H^2 is bounded. The norm of such operators is not generally known, but we are particuarly interested in symbols φ where $\varphi(0) = 0$, and in this case $\|C_\varphi\| = 1$ [5, Corollary 3.3]. Furthermore, if $\varphi(0) = 0$, then we see that the subspaces of H^2 given by

$$z^k H^2 = \{z^k f : f \in H^2\}$$

are invariant for C_φ: let $\psi(z) = \frac{\varphi(z)}{z}$ which is also in H^2. Then

$$(C_\varphi)(z^k f) = \varphi^k (f \circ \varphi) = z^k(\psi^k(f \circ \varphi))$$

which is an element of $z^k H^2$.

1.3 Toeplitz Operators and Weighted Composition Operators

Definition 1.3.1 *Let $g \in L^\infty(\partial\mathbb{D})$. Then the Toeplitz operator T_g on H^2 is defined by*

$$T_g f = P f g$$

for $f \in H^2$, where P is the orthogonal projection of L^2 onto H^2.

Although our operators will be on the space H^2, we are also interested in the space

$$H^\infty(\mathbb{D}) = \{f \text{ analytic on } \mathbb{D} : \|f\|_\infty = \sup_{|z|<1} |f(z)| < \infty\}$$

Although the product of two H^2 functions need not be in H^2, the product of an H^∞ function and an H^2 function is in H^2. In particular, this allows for a simpler definition of Toeplitz operators in this case:

Definition 1.3.2 *For $g \in H^\infty(\mathbb{D})$, the analytic Toeplitz operator T_g on H^2 is defined by*

$$(T_g f)(z) = g(z) f(z)$$

for each function $f \in H^2$.

Note that for linear fractional maps φ fixing 0, $\psi(z) = \frac{\varphi(z)}{z}$ is an H^∞ function so that we may use this simpler definition to define T_ψ.

Another Toeplitz operator of interest is the operator T_z. Due to the nature of the inner product on H^2, T_{z^k} is a unitary operator from H^2 onto $z^k H^2$. Then $T^*_{z^k}$ is necessarily a unitary operator from $z^k H^2$ to H^2. Note that $T^*_z = PT_{\overline{z}}$, but since the norm on H^2 is defined as an integral along the boundary of the disk, we have $\overline{z} = \frac{1}{z}$, so we can write

$$T^*_z f = \frac{f(z) - f(0)}{z}$$

In particular, this allows us to write $T_z T^*_z$ as $I - O$, a fact which we shall use often. In addition, it is clear that for any k, $T^*_{z^k} T_{z^k} = I$.

We also want to define a weighted composition operator on H^2:

Definition 1.3.3 *Let $g \in H^\infty$ and φ be an analytic self-map of the disk. Then the weighted composition operator $W_{g,\varphi}$ is defined by*

$$(W_{g,\varphi} f)(z) = T_g C_\varphi = g(z)(f \circ \varphi)(z)$$

Weighted composition operators naturally arise because of the way that Toeplitz operators and composition operators intertwine:

$$C_\varphi T_g = T_{g\circ\varphi} C_\varphi$$

Now, since T_{z^k} is a unitary operator from H^2 to $z^k H^2$, we can relate the restriction of composition operators to these subspaces with weighted composition operators on H^2.

Theorem 1.3.4 *If $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic H^∞ function and $\varphi(0) = 0$, then $C_\varphi | z^k H^2 \cong T_{\psi^k} C_\varphi$, where $\psi(z) = \frac{\varphi(z)}{z}$ and $T_{\psi^k} C_\varphi$ is an operator on H^2.*

Proof Using the unitary operator T_{z^k}, we have

$$\begin{aligned} T_{z^k}^* C_\varphi|_{z^k H^2} T_{z^k} &= T_{z^k}^* C_\varphi T_{z^k} \\ &= T_{z^k}^* T_{\varphi^k} C_\varphi \\ &= T_{z^k}^* T_{z^k} T_{\psi^k} C_\varphi \\ &= I T_{\psi^k} C_\varphi \\ &= T_{\psi^k} C_\varphi \end{aligned} \tag{1.1}$$

In Equation (1.1), we may drop the reference to $z^k H^2$ because C_φ is necessarily acting on $z^k H^2$ due to the operator T_{z^k}. ■

We would also like to use the information known about self-adjoint weighted composition operators and semigroups.

Definition 1.3.5 *A* strongly continuous one-parameter semigroup *on H^2 is a map $A : \mathbb{R}_+ \to L(H^2)$ such that*

1. $A_0 = I$
2. *For all* $t, s \geq 0, A_{s+t} = A_s A_t$
3. *For all* $f \in H^2, \|A_t f - f\| \to 0$ *as* $t \downarrow 0$.

In particular, we see that if two operators are in the same semigroup, one is necessarily a power (or root) of the other. Since *positive* operators have unique positive roots and powers, in the case of a semigroup of positive operators, those unique roots and powers are found within the semigroup.

1.4 Theorems for Reproducing Kernel Functions

In this section we will provide some useful relations between T_z, T_z^*, and reproducing kernel functions. First, note that the space H^2 also contains reproducing kernel functions for derivatives, and these are given by

$$K_\alpha^{(n)} = \frac{z^n}{(1-\overline{\alpha} z)^{n+1}}$$

so that $\langle f, K_\alpha^{(n)}\rangle = f^{(n)}(\alpha)$. Also important is the fact that the set

$$\left\{K_{\alpha_i}^{(j)} : 1 \leq i \leq m, 0 \leq j \leq n\right\}$$

for finite values m, n is a linearly independent set.

Theorem 1.4.1 *$T_z^* K_\alpha^{(n)} = K_\alpha^{(n-1)} + \overline{\alpha} K_\alpha^{(n)}$ when $n > 0$, and $T_z^* K_\alpha = \overline{\alpha} K_\alpha$.*

Proof Note that $K_\alpha^{(n)} = \frac{z^n}{(1-\overline{\alpha}z)^{n+1}}$. Then $T_z^* K_\alpha^{(n)} = T_z^* \frac{z^n}{(1-\overline{\alpha}z)^{n+1}} = \frac{z^{n-1}}{(1-\overline{\alpha}z)^{n+1}} = \frac{z^{n-1}}{(1-\overline{\alpha}z)^{n}} + \overline{\alpha}\frac{z^n}{(1-\overline{\alpha}z)^{n+1}} = K_\alpha^{(n-1)} + \overline{\alpha} K_\alpha^{(n)}$.

When $n = 0$, we have $T_z^* K_\alpha = \frac{1}{z}(\frac{1}{1-\overline{\alpha}z} - 1) = \frac{1}{z}(\frac{1}{1-\overline{\alpha}z} - \frac{1-\overline{\alpha}z}{1-\overline{\alpha}z}) = \frac{1}{z}(\frac{\overline{\alpha}z}{1-\overline{\alpha}z}) = \frac{\overline{\alpha}}{1-\overline{\alpha}z} = \overline{\alpha} K_\alpha$. ■

Note that by iteratively applying T_z^*, this lemma also gives another corollary.

Corollary 1.4.2 *$T_{z^j}^* K_\alpha^{(n)}$ is a linear combination of $\{K_\alpha^{(m)} : m \leq n\}$.*

In addition, we can also say something now about T_z applied to reproducing kernel functions.

Corollary 1.4.3 *$T_z K_\alpha^{(n)}$ is a linear combination of $\{K_\alpha^{(m)} : m \leq n\} \cup \{1\}$.*

Proof Note first that $T_z^* f = \frac{f-f(0)}{z}$. By Thoerem 1.4.1,

$$\begin{aligned} T_z^* K_\alpha &= \bar{\alpha} K_\alpha \\ \frac{K_\alpha - 1}{z} &= \bar{\alpha} K_\alpha \\ K_\alpha - 1 &= \bar{\alpha} z K_\alpha \\ \frac{1}{\bar{\alpha}}(K_\alpha - 1) &= z K_\alpha \end{aligned}$$

So zK_α is a linear combination of K_α and 1. Note that for $n \geq 1$, $K_\alpha^{(n)}(0) = 0$.

Then

$$
\begin{aligned}
T_z^* K_\alpha^{(n)} &= K_\alpha^{(n-1)} + \bar{\alpha} K_\alpha^{(n)} \\
\frac{K_\alpha^{(n)}}{z} &= K_\alpha^{(n-1)} + \bar{\alpha} K_\alpha^{(n)} \\
K_\alpha^{(n)} &= z K_\alpha^{(n-1)} + \bar{\alpha} z K_\alpha^{(n)} \\
\frac{1}{\bar{\alpha}} K_\alpha^{(n)} - \frac{1}{\bar{\alpha}} z K_\alpha^{(n-1)} &= z K_\alpha^{(n)} \qquad (1.2)
\end{aligned}
$$

Using Equation (1.2) iteratively, we see that

$$
\begin{aligned}
z K_\alpha^{(n)} &= \frac{1}{\bar{\alpha}} K_\alpha^{(n)} - \frac{1}{\bar{\alpha}} z K_\alpha^{(n-1)} \\
&= \frac{1}{\bar{\alpha}} K_\alpha^{(n)} - \frac{1}{\bar{\alpha}} \left(\frac{1}{\bar{\alpha}} K_\alpha^{(n-1)} - \frac{1}{\bar{\alpha}} z K_\alpha^{(n-2)} \right) \\
&\vdots \\
&= \sum_{i=0}^{n} \frac{(-1)^{n-i}}{\bar{\alpha}^{n+1-i}} K_\alpha^{(i)} - \frac{(-1)^{n+1}}{\bar{\alpha}^{n+1}} 1
\end{aligned}
$$

So $zK_\alpha^{(n)}$ is a linear combination of $\{K_\alpha{}^{(m)} : m \leq n\} \cup \{1\}$ as desired. ■

Applying this result iteratively leads to another corollary:

Corollary 1.4.4 $T_{z^j} K_\alpha^{(n)}$ *is a linear combination of*

$$
\left\{ K_\alpha^{(m)} : m \leq n \right\} \cup \left\{ z^i : 0 \leq i \leq j \right\}
$$

2. SPECIFIC CASES

Our main result in Chapter 3 denies the unitary equivalence among restrictions of C_φ to the invariant subspaces z^kH^2 when φ is a linear fractional map of the disk into itself with rational coefficients. In this chapter, we will provide the known results for (power) compact and isometric composition operators, and offer some preliminaries on composition operators with linear fractional symbols.

2.1 C_φ Is (Power) Compact

This case is somewhat trivial due to a theorem by Hammond [6]:

Theorem 2.1.1 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map with $\varphi(0) = 0$. Suppose, for some integer $n \geq 1$, $\|C_\varphi|_{z^nH^2}\|$ is strictly greater than $\|C_\varphi\|_e$. Then*

$$\|C_\varphi|_{z^nH^2}\| < \|C_\varphi|_{z^{n-1}H^2}\|$$

Thus we can dismiss any (power) compact operator:

Corollary 2.1.2 *$\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic map with $\varphi(0) = 0$. If C_φ is (power) compact, then $C_\varphi|_{z^jH^2} \ncong C_\varphi|_{z^kH^2}$ for $j \neq k$.*

Proof Since the essential norm is 0 for compact operators, and the restriction of a compact operator to an invariant subspace is compact, Theorem 2.1.1 will hold for all restrictions of a compact operator C_φ to the subspaces z^kH^2. Since the norms are all different, we see that the restrictions cannot be unitarily equivalent. Furthermore, if $(C_\varphi)^n$ is compact for some power n, we see that $(C_\varphi)^n = C_{\varphi^n}$ and φ^n is an analytic function from the disk into itself that also fixes 0. Furthermore, $(C_\varphi|_{z^jH^2})^n = C_{\varphi^n}|_{z^jH^2}$. If $C_\varphi|_{z^jH^2} \cong C_\varphi|_{z^kH^2}$, then $(C_\varphi|_{z^jH^2})^n \cong (C_\varphi|_{z^kH^2})^n$, which in turn implies that $C_{\varphi^n}|_{z^jH^2} \cong C_{\varphi^n}|_{z^kH^2}$. However, since C_{φ^n} is a compact composition operator, this is a contradiction. ■

2.2 C_φ Is Isometric

In [7], Schwartz proved that a composition operator C_φ on H^2 is an isometry if and only if φ is an inner function fixing 0. In stark contrast to the compact case, we find here that C_φ acts much the same on its invariant subspaces, due to a thoerem of Jones [3]:

Theorem 2.2.1 *Let φ be an inner function that fixes a point in $\mathbb{D}$ and suppose that I, J are invariant subspaces for C_φ. Then $C_\varphi|_{IH^2}$ is similar to $C_\varphi|_{JH^2}$. Furthermore, if $\varphi(0) = 0$, then these restrictions are unitarily equivalent.*

This shows that if φ is inner and fixes 0, not only are the restrictions of C_φ to z^kH^2 unitarily equivalent, they are also unitarily equivalent to the restriction of C_φ to any of its other invariant subspaces.

2.3 C_φ Is Linear Fractional

The question of whether the restrictions of C_φ to z^kH^2 are unitarily equivalent is a difficult question for a general φ fixing 0. However, due to the progress already made on composition operators with linear fractional symbols, we are able to give some definitive answers in this case. To get to those answers, we require some preliminaries.

Our most important tool is Cowen's formula for the adjoint of C_φ:

Theorem 2.3.1 *(Cowen's adjoint formula.) Let $\varphi(z) = (az+b)(cz+d)^{-1}$ be a linear fractional transformation mapping $\mathbb{D}$ into itself. Then $\sigma(z) = (\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})^{-1}$ maps $\mathbb{D}$ into itself, $g(z) = (-\overline{b}z + \overline{d})^{-1}$ and $h(z) = cz + d$ are in H^∞, and*

$$C_\varphi^* = T_g C_\sigma T_h^*$$

Proof The proof is found in [5, Theorem 9.2]. ■

In addition, in contrast to other types of composition operators, we have a simple method of determining compactness.

Theorem 2.3.2 *If $\varphi : \mathbb{D} \to \mathbb{D}$ is linear fractional, then $||\varphi||_\infty < 1$ if and only if C_φ is compact.*

Proof Since linear fractional maps have finite angular derivatives everywhere that the map itself is finite, this follows from [5, Corollary 3.14]. ■

It follows from this statement that when φ is linear fractional, C_φ is not power compact if and only if φ has a fixed point on the unit circle. By the previous section, we are not interested in power compact operators, so we will assume our symbol φ has fixed points $0, w$ where $|w| = 1$. By the following theorem, we will be able to assume $w = 1$.

Theorem 2.3.3 *Let $\psi : \mathbb{D} \to \mathbb{D}$ be an analytic map with $\psi(0) = 0$ and $\psi(e^{i\theta}) = e^{i\theta}$ for some $\theta \in [0, 2\pi)$. Then $C_\varphi = C_{e^{i\theta}z} C_\psi C_{e^{-i\theta}z}$ is a composition operator with fixed points $0, 1$ and is unitarily equivalent to C_ψ.*

Proof First, we will show that $C_{e^{i\theta}z}$ is unitary. By Theorem 2.3.1, $C^*_{e^{i\theta}z} = C_{e^{-i\theta}z}$. Now we can see that

$$\begin{aligned} C_{e^{i\theta}z} C_{e^{-i\theta}z} &= C_{e^{-i\theta}z} C_{e^{i\theta}z} \\ &= C_{e^{i\theta}e^{-i\theta}z} \\ &= C_z = I \end{aligned}$$

so $C_{e^{i\theta}z}$ is unitary. Now if $\varphi = e^{-i\theta}\psi(e^{i\theta}z)$, then

$$C_{e^{i\theta}z} C_\psi C_{e^{-i\theta}z} = C_{e^{-i\theta}\psi(e^{i\theta}z)} = C_\varphi$$

Now we check the fixed points of φ:

$$\varphi(1) = e^{-i\theta}\psi((e^{i\theta})(1)) = e^{-i\theta}\psi(e^{i\theta}) = e^{-i\theta}(e^{i\theta}) = 1$$

$$\varphi(0) = e^{-i\theta}\psi((e^{i\theta})(0)) = e^{-i\theta}\psi(0) = (e^{-i\theta})(0) = 0$$

Thus $C_\varphi \cong C_\psi$ and φ has fixed points $0, 1$. ■

Now we need only consider linear fractional symbols φ that fix 0 and 1, so we may write $\varphi(z) = \frac{az}{1-cz}$, noting that $0 < |a|, |c| < 1$ and $a + c = 1$.

Of course, this situation is only of concern if we have suspicion that these restrictions may be unitarily equivalent. Hamnmond [6] showed that for this class of operators, $\|C_\varphi|_{z^k H^2}\| = \|C_\varphi\|_e$ for any positive integer k. In addition, each restriction has no eigenvalues [8], although every point of the essential spectrum (which is not lost under restriction) corresponds to an eigenvalue of infinite multiplicity for the adjoint [5, Lemma 7.25]. The adjoints of these restrictions maintain all of these eigenvalues; we shall prove this fact in Chapter 4. Despite all this evidence, we will show in the next chapter that at least when the constant a is rational, these restrictions are not unitarily equivalent.

3. UNITARY EQUIVALENCE OF THE OPERATORS $T_{\psi^k}C_\varphi$ ON H^2

In the previous chapter, we reduced the case of linear fractional maps to the situation where 0 and 1 are the only fixed points of the symbol φ. Furthermore, we have shown in our introductory remarks that $C_\varphi|_{z^kH^2} \cong T_{\psi^k}C_\varphi$ for $\psi(z) = \frac{\varphi(z)}{z}$. In this chapter, we will discuss $T_{\psi^k}C_\varphi$ as its own operator on H^2, where $\varphi(z) = \frac{az}{1-cz}$. We will show that when a is rational, the operators $T_{\psi^j}C_\varphi$ and $T_{\psi^k}C_\varphi$ are not unitarily equivalent for positive integers $j \neq k$. Since our result is only for rational a, from this point we will assume that a and c are real. The following work will then give us our desired result about restrictions of C_φ.

3.1 The Operator $T_{\psi^k}C_\varphi$ on H^2

Theorem 3.1.1 *Let $\varphi : D \to D$ be defined by $\varphi(z) = \frac{az}{1-cz}$ with $a, c > 0$ and $a+c = 1$. Let $\psi(z) = \frac{\varphi(z)}{z} = \frac{a}{1-cz}$. Then $(T_{\psi^k}C_\varphi)^* = aT^*_{z^{k-1}}C_\sigma T_{z^{k-1}}$, where $\sigma(z) = az + c$.*

Proof Note that by Theorem 2.3.1, $(T_\psi C_\varphi)^* = aC_\sigma$. Additionally, $T_{\psi^k}C_\varphi = T^*_{z^{k-1}}T_\psi C_\varphi T_{z^{k-1}}$ as shown in Theorem 1.3.4. Then

$$(T_{\psi^k}C_\varphi)^* = (T^*_{z^{k-1}}T_\psi C_\varphi T_{z^{k-1}})^* = T^*_{z^{k-1}}(T_\psi C_\varphi)^* T_{z^{k-1}} = aT^*_{z^{k-1}}C_\sigma T_{z^{k-1}}$$

as desired. ∎

Theorem 3.1.2 *If $A = T_\psi C_\varphi$ with ψ, φ as in Theorem 3.1.1, then AA^*, A^*A are both self-adjoint weighted composition operators. Furthermore, $(a^{a-1}A^*A)^{\frac{1}{a}} = AA^*$.*

Proof First, note that $A^* = aC_\sigma$ as in Theorem 2.3.1. Then $AA^* = aT_\psi C_\varphi C_\sigma = T_{a\psi}C_{\sigma\circ\varphi}$ and $A^*A = aC_\sigma T_\psi C_\varphi = T_{a\psi\circ\sigma}C_{\varphi\circ\sigma}$, so both are weighted composition operators. In addition, both A^*A and AA^* are self-adjoint, and also positive, by construction.

In [9], Cowen and Ko proved that a self-adjoint weighted composition operator $T_\beta C_\alpha$ belongs to a strongly continuous one-parameter semigroup if $\alpha'(0) = (1 - |\alpha(0)|)^2$, and that in this case, the index t for the semigroup is given by $\alpha'(0) = (1+t)^{-2}$. In our first case, $AA^* = T_{a\psi}C_{\sigma\circ\varphi}$, we have

$$(\sigma \circ \varphi)(0) = \sigma(0) = c$$

$$(\sigma \circ \varphi)'(0) = \sigma'(\varphi(0))\varphi'(0) = \sigma'(0)\varphi'(0) = a^2$$

Note that $(1-c)^2 = a^2$ as desired. The semigroup index is then given by $t_{AA^*} = \frac{1-a}{a}$. For our second case, $A^*A = T_{a\psi\circ\sigma}C_{\varphi\circ\sigma}$, we have

$$(\varphi \circ \sigma)(0) = \varphi(c) = \frac{ac}{1-c^2} = \frac{a}{(1+c)(1-c)} = \frac{ac}{(1+c)(a)} = \frac{c}{1+c}$$

$$(\varphi \circ \sigma)'(0) = \varphi'(c)\sigma'(0) = \left(\frac{a}{(1-c^2)^2}\right)(a) = \frac{a^2}{(1+c)^2(1-c)^2} = \frac{1}{(1+c)^2}$$

Note that

$$\left(1 - \frac{c}{1+c}\right)^2 = \left(\frac{1+c}{1+c} - \frac{c}{1+c}\right)^2 = \left(\frac{1+c-c}{1+c}\right)^2 = \frac{1}{(1+c)^2}$$

as desired. The semigroup index is given by $t_{A^*A} = c = 1-a$.

Now, since both AA^* and A^*A are positive operators, their positive roots and powers are unique, and by virtue of being in the same semigroup, one is a power of the other. The power is given by $(t_{AA^*})/(t_{A^*A}) = \frac{1-a}{a}/(1-a) = \frac{1}{a}$. Because [9] does not take into consideration our factor of a present in both of our weights, we also require the factor of a^{a-1}, so that in combination with the a already present in the weight of A^*A, we have $(aa^{a-1})^{\frac{1}{a}} = (a^a)^{\frac{1}{a}} = a$, which is the factor present in AA^*.

■

3.2 An Illustrative Example

In this section, let φ specifically represent the function $\varphi(z) = \frac{1}{2}z/(1-\frac{1}{2}z)$.

Theorem 3.2.1 *Let* $\varphi(z) = \frac{1}{2}z/(1-\frac{1}{2}z)$*, so that* $\psi(z) = \frac{1}{2}/(1-\frac{1}{2}z)$ *and* $\sigma(z) = \frac{1}{2}z + \frac{1}{2}$*. Additionally, let* $A_n = T_{\psi^{n+1}}C_\varphi$ *and* $A = A_0$*. Then* $A_n^* = \frac{1}{2}T_{z^n}^* C_\sigma T_{z^n}$ *and* $(\sqrt{2}A^*A)^2 = AA^*$*.*

Proof This is a restatement of 3.1.1 and 3.1.2 with the value $a = \frac{1}{2}$. ■

Theorem 3.2.2 *Using the definitions in Theorem 3.2.1, let* S_n *be defined by*

$$S_n = (\sqrt{2}A_n^*A_n)^2 - A_nA_n^*$$

Then S_n *is an operator with range equal to the linear span of*

$$\left\{K_{\frac{1}{2}}^{(m)}, K_{\frac{1}{3}}^{(m)}, m \leq n-1\right\}$$

Proof We will prove this inductively. Note that S_{n+1} can be written as

$$S_{n+1} = T_z^*[S_n]T_z - T_z^*[2A_n^*A_nOA_n^*A_n]T_z + T_z^*[A_nOA_n^*]T_z$$

where $O(f) = \langle f, K_0\rangle = f(0)$. We will show first that the range of S_1 is the linear span of $\{K_{\frac{1}{2}}, K_{\frac{1}{3}}\}$ and then show the general inductive case.

Since clearly $S_0 = 0$, we have

$$S_1 = T_z^*[A_0OA_0^*]T_z - T_z^*[2A_0^*A_0OA_0^*A_0]T_z$$

For the first term, we have

$$\begin{aligned}
T_z^*[A_0OA_0^*]T_zf &= \frac{1}{2}T_z^*T_\psi C_\varphi OC_\sigma T_zf \\
&= \frac{1}{2}T_z^*T_\psi C_\varphi O\sigma f(\sigma) \\
&= \frac{1}{2}T_z^*T_\psi C_\varphi \frac{1}{2}f(\frac{1}{2}) \\
&= \frac{1}{2}T_z^*\frac{1}{2}f(\frac{1}{2})\psi(z) \\
&= \frac{1}{8}f(\frac{1}{2})\psi(z) \\
&= \frac{1}{16}f(\frac{1}{2})K_{\frac{1}{2}}
\end{aligned}$$

For the second term, we have $A_0^*A_0 = \frac{1}{2}T_\zeta C_\tau$, where $\tau(z) = \frac{z+1}{3-z}$ and $\zeta(z) = \frac{2}{3-z}$. So

$$\begin{aligned}
T_z^*[2A_0^*A_0OA_0^*A_0]T_z &= \frac{1}{2}T_z^*T_\zeta C_\tau OT_\zeta C_\tau T_zf \\
&= \frac{1}{2}T_z^*T_\zeta C_\tau O\zeta\tau f(\tau) \\
&= \frac{1}{2}T_z^*T_\zeta C_\tau \frac{2}{9}f(\frac{1}{3}) \\
&= \frac{1}{2}T_z^*\zeta\frac{2}{9}f(\frac{1}{3}) \\
&= \frac{1}{27}f(\frac{1}{3})\zeta \\
&= \frac{2}{81}f(\frac{1}{3})K_{\frac{1}{3}}
\end{aligned}$$

Thus, we have $S_1f = \frac{1}{16}f(\frac{1}{2})K_{\frac{1}{2}} - \frac{2}{81}f(\frac{1}{3})K_{\frac{1}{3}}$. Since we are free to choose functions with any desired values at $\frac{1}{2}$ and $\frac{1}{3}$, we see that the range is the linear span of $K_{\frac{1}{2}}$ and $K_{\frac{1}{3}}$ and is 2-dimensional.

Now, assume that the range of S_n is the linear span of $\{K_{\frac{1}{2}}^{(m)}, K_{\frac{1}{3}}^{(m)}, m \leq n-1\}$. For now, consider the operator

$$U_{n+1} = [S_n] - 2A_n^*A_nOA_n^*A_n + A_nOA_n^*$$

and note that $T_z^* U_{n+1} T_z = S_{n+1}$. We will first show the range of U_{n+1} is

$$\left\{ K_{\frac{1}{2}}^{(m)}, K_{\frac{1}{3}}^{(m)}, m \leq n \right\}$$

Since the rank of S_n is $2n$, i.e. finite, we can write S_n in the generic form

$$S_n f = \sum_{i=0}^{n-1} \alpha_i \langle f, u_i \rangle K_{\frac{1}{2}}^{(i)} + \beta_i \langle f, v_i \rangle K_{\frac{1}{3}}^{(i)}$$

for an appropriate orthonormal basis $\{u_0, v_0, ...\}$. Now consider the second term of U_{n+1}:

$$2A_n^* A_n O A_n^* A_n f = T_{z^n}^* T_\zeta C_\tau T_{z^n} O \frac{1}{2} T_{z^n}^* T_\zeta C_\tau T_{z^n} f$$

Here we denote the resultant value of $O\frac{1}{2}T_{z^n}^* T_\zeta C_\tau T_{z^n} f$ by λ_f. Continuing, we have

$$T_{z^n}^* T_\zeta C_\tau T_{z^n} \lambda_f = \lambda_f T_{z^n}^* T_\zeta \tau^n = 2\lambda_f T_{z^n}^* \frac{(z+1)^n}{(3-z)^{n+1}}$$

Note that $\frac{(z+1)^n}{(3-z)^{n+1}}$ is a linear combination of $\{K_{\frac{1}{3}}^{(i)}, i = 0, ..., n\}$. By Corollary 1.4.2, we can rewrite the previous expression:

$$2\lambda_f T_{z^n}^* \frac{(z+1)^n}{(3-z)^{n+1}} = \sum_{i=0}^{n} \lambda_f \gamma_i K_{\frac{1}{3}}^{(i)}$$

for some constants γ_i.

Continuing to the next term of U_{n+1}, we have

$$A_n O A_n^* = T_{\psi^{n+1}} C_\varphi O \frac{1}{2} T_{z^n}^* C_\sigma T_{z^n} f$$

Denote the value of $O\frac{1}{2}T_{z^n}^* C_\sigma T_{z^n} f$ by ω_f. Then we have

$$T_{\psi^{n+1}} C_\varphi O \frac{1}{2} T_{z^n}^* C_\sigma T_{z^n} f = \omega_f \psi^{n+1}$$

Note that $\psi^{n+1} = \frac{1}{(2-z)^{n+1}}$ is a linear combination of $\{K_{\frac{1}{2}}{}^{(i)}, i = 0, ..., n\}$. So now we have

$$\omega_f \psi^{n+1} = \sum_{i=0}^{n} \omega_f \delta_i K_{\frac{1}{2}}^{(i)}$$

for some constants δ_i.

Putting this all together, we have

$$U_{n+1}f = \sum_{i=0}^{n-1} \alpha_i \langle f, u_i\rangle K_{\frac{1}{2}}^{(i)} + \beta_i \langle f, v_i\rangle K_{\frac{1}{3}}^{(i)} + \sum_{i=0}^{n} \lambda_f \gamma_i K_{\frac{1}{3}}^{(i)} + \sum_{i=0}^{n} \omega_f \delta_i K_{\frac{1}{2}}^{(i)}$$

Defining α_n, β_n as 0, we can write that expression as

$$\sum_{i=0}^{n} (\alpha_i \langle f, u_i\rangle + \omega_f \delta_i) K_{\frac{1}{2}}^{(i)} + (\beta_i \langle f, v_i\rangle + \lambda_f \gamma_i) K_{\frac{1}{3}}^{(i)}$$

Note that the coefficients of the kernels are determined entirely by the value of the various derivatives of the function f at $\frac{1}{2}$ and $\frac{1}{3}$, and thus any value is possible for λ_f and ω_f, we see that U_{n+1} has range $\{K_{\frac{1}{2}}^{(m)}, K_{\frac{1}{3}}^{(m)}, m \leq n\}$. Furthermore, since any function f in zH^2 can also have any combination of possible values for those derivatives, $U_{n+1}T_z$ has the same range. By Theorem 1.4.1, $T_z^* U_{n+1} T_z = S_{n+1}$ has range

$$\{K_{\frac{1}{2}}^{(m)}, K_{\frac{1}{3}}^{(m)}, m \leq n\}$$

as well. This completes the proof.

■

Corollary 3.2.3 *Let $\varphi(z) = \frac{1}{2}z/(1 - \frac{1}{2}z)$. Then $C_\varphi|_{z^j H^2}$ is not unitarily equivalent to $C_\varphi|_{z^k H^2}$ for any two positive integers $j \neq k$.*

Proof By Theorem 1.3.4, two such operators would be unitarily equivalent to A_{j-1} and A_{k-1} respectively, as defined in Theorem 3.2.1. If we have $A_{j-1} \cong A_{k-1}$, then we have $S_{j-1} \cong S_{k-1}$ as defined in Theorem 3.2.2. However, S_{j-1} and S_{k-1} do not have the same rank and therefore cannot be unitarily equivalent, leading to a contradiction.

■

3.3 The Rational Case

For this section, we will use the definitions given in Theorem 3.1.1, but we will now assume also that a is rational, so that $a = \frac{p}{q}$ for integers p, q. We will use p and q only to indicate powers of our operators; we will continue to use the constants a and c for simplicity of notation.

Theorem 3.3.1 *Let $A = T_\psi C_\varphi$ with ψ, φ as in Theorem 3.1.1. Let a be rational, so that $a = \frac{p}{q}$ for integers p, q. Then $(a^{a-1}A^*A)^q = (AA^*)^p$.*

Proof From Theorem 3.1.2, we have $(a^{a-1}A^*A)^{\frac{1}{a}} = AA^*$. Once the exponent $\frac{1}{a}$ is rewritten as $\frac{q}{p}$, raising both sides to the power of p gives the result. ■

The advantage of Theorem 3.3.1 is that we now may look at integer powers of operators in the general rational case, which makes our task much simpler. As we have seen in the case when $a = \frac{1}{2}$, we now show more generally that the rank of our operator S_n is different for each integer value of n. Before we approach our main result, we state two lemmas to simplify the following computations.

Lemma 3.3.2 *Let ψ, φ, σ be as in Theorem 3.1.1, $\zeta = \psi \circ \sigma$, $\tau = \varphi \circ \sigma$, and $G = T_{z^n}^* T_\zeta C_\tau T_{z^n}$ for a specific integer n. Then $G^\ell(1)$ is a linear combination of $\{K_{w_r}^{(j)} : 1 \le r \le \ell, 0 \le j \le n\}$, where $w_r = \frac{rc}{1+rc}$ for integers r.*

Proof First, note that we have an explicit formula for ζ and τ:

$$\zeta(z) = (\psi \circ \sigma)(z) = \frac{1}{1 + c - cz}$$
$$\tau(z) = (\varphi \circ \sigma)(z) = \frac{c + (1-c)z}{1 + c - cz}$$

Consider the vectors $K_{w_r}^{(n)}$ where $w_r = \frac{rc}{1+rc}$ for real numbers r. Then

$$T_\zeta C_\tau K_{w_r}^{(n)} = \left(\frac{1}{1 + c - w_r c}\right)^{n+1} \left(\frac{(c + (1-c)z)^n}{(1 - \frac{(r+1)c}{1+(r+1)c} z)^{n+1}}\right)$$

This is a linear combination of the vectors $K_{w_{r+1}}^{(j)}$ for $j \le n$.

Also note that $\zeta\tau^n$ is a linear combination of $K_{w_1}^{(j)}$ for $j \le n$.

Before continuing, note that we write constants $\alpha_{r,j}$ so that they depend on the point in the disk w_r and the jth derivative for the reproducing kernel in question.

When the constant would change but r and j do not, we will again write $\alpha_{r,j}$. From here, we see that

$$
\begin{aligned}
(T^*_{z^n}T_\zeta C_\tau T_{z^n})^\ell(1) &= (T^*_{z^n}T_\zeta C_\tau T_{z^n})^{\ell-1}T^*_{z^n}(\zeta\tau^n) \\
&= (T^*_{z^n}T_\zeta C_\tau T_{z^n})^{\ell-1}T^*_{z^n}\left(\sum_{j=0}^{n}\alpha_{1,j}K^{(j)}_{w_1}\right) \\
&= (T^*_{z^n}T_\zeta C_\tau T_{z^n})^{\ell-2}T^*_{z^n}T_\zeta C_\tau T_{z^n}\left(\sum_{j=0}^{n}\alpha_{1,j}K^{(j)}_{w_1}\right) \qquad (3.1) \\
&= (T^*_{z^n}T_\zeta C_\tau T_{z^n})^{\ell-2}T^*_{z^n}T_\zeta C_\tau\left(\sum_{j=0}^{n}\alpha_{1,j}K^{(j)}_{w_1}+\sum_{j=0}^{n}\alpha_j z^j\right) \qquad (3.2) \\
&= (T^*_{z^n}T_\zeta C_\tau T_{z^n})^{\ell-2}T^*_{z^n}\left(\sum_{j=0}^{n}\alpha_{2,j}K^{(j)}_{w_2}+\sum_{j=0}^{n}\alpha_{1,j}K^{(j)}_{w_1}\right) \\
&\vdots \\
&= \sum_{r=1}^{\ell}\sum_{j=0}^{n}\alpha_{r,j}K^{(j)}_{w_r}
\end{aligned}
$$

Note that Equation (3.1) is a result of Corollary 1.4.2, and Equation (3.2) is a result of Corollary 1.4.4. From here, we see that $G^\ell(1)$ is a linear combination of $\{K^{(j)}_{w_r} : 1 \le r \le \ell, 0 \le j \le n\}$ as desired. ■

This lemma gives rise to another:

Lemma 3.3.3 *Let ψ, φ, σ be as in Theorem 3.1.1, $\zeta = \psi \circ \sigma$, $\tau = \varphi \circ \sigma$, and $G = T^*_{z^n}T_\zeta C_\tau T_{z^n}$ for a specific integer n. Then $T_{\psi^{n+1}}C_\varphi G^\ell(1)$ is a linear combination of $\{K^{(j)}_{b_s} : 0 \le s \le \ell, 0 \le j \le n\}$, where $b_s = \frac{c(1+sc)}{1+sc^2}$.*

Proof By the previous lemma, $G^\ell(1)$ is a linear combination of

$$\{K^{(j)}_{w_s} : 1 \le s \le \ell, 0 \le j \le n\}$$

where $w_s = \frac{sc}{1+sc}$. Letting $b_s = \frac{c(1+sc)}{1+sc^2}$, note that

$$T_\psi C_\varphi K^{(j)}_{w_s} = \frac{a^{j+1}}{(1-w_s a)^{j+1}}\frac{z^j}{(1-b_s z)^{j+1}} = \frac{a^{j+1}}{(1-w_s a)^{j+1}}K^{(j)}_{b_s}$$

Note also that $\psi\varphi^n$ is a scalar multiple of $K_{b_0}^{(n)}$. Again, as in the previous lemma, we write constants $\alpha_{s,j}$ so that they depend on the point in the disk b_s and the jth derivative for the reproducing kernel in question. When the constant would change but s and j do not, we will again write $\alpha_{s,j}$. Then we have

$$\begin{aligned}
T_{\psi^{n+1}}C_\varphi \sum_{s=1}^{\ell}\sum_{j=0}^{n} \alpha_{s,j}K_{w_r}^{(j)} &= T_{z^n}^* T_\psi C_\varphi T_{z^n} \sum_{s=1}^{\ell}\sum_{j=0}^{n} \alpha_{s,j}K_{w_s}^{(j)} \\
&= T_{z^n}^* T_\psi C_\varphi \left(\sum_{s=1}^{\ell}\sum_{j=0}^{n} \alpha_{s,j}K_{w_s}^{(j)} + \sum_{j=0}^{n} z^j \right) \qquad (3.3) \\
&= T_{z^n}^* \left(\sum_{s=1}^{\ell}\sum_{j=0}^{n} \alpha_{s,j}K_{b_s}^{(j)} + \sum_{j=0}^{n} \psi\varphi^j \right) \\
&= T_{z^n}^* \sum_{s=0}^{\ell}\sum_{j=0}^{n} \alpha_{s,j}K_{b_s}^{(j)} \\
&= \sum_{s=0}^{\ell}\sum_{j=0}^{n} \alpha_{s,j}K_{b_s}^{(j)} \qquad (3.4)
\end{aligned}$$

Note that Equation (3.3) is due to Corollary 1.4.4, and Equation (3.4) is due to Corollary 1.4.2. Thus $T_{\psi^{n+1}}C_\varphi G^\ell(1)$ is a linear combination of

$$\{K_{b_s}^{(j)} : 0 \le s \le \ell, 0 \le j \le n\}$$

as desired. ∎

Now we are prepared for the main result:

Theorem 3.3.4 *Let $A_n = T_{\psi^{n+1}}C_\varphi$ with ψ, φ as in Theorem 3.1.1. Let a be rational, so that $a = \frac{p}{q}$ for integers p, q. Then for positive integers n, the operator*

$$S_n = (a^{a-1}A_n^*A_n)^q - (A_nA_n^*)^p$$

has range equal to the linear span of

$$\{K_{\frac{rc}{1+rc}}^{(j)}, K_{\frac{(sc+1)c}{1+sc^2}}^{(k)} : 1 \le r \le q-1, 0 \le s \le p-1, 0 \le j,k \le n-1\}$$

and the operator S_n has rank $n(p+q-1)$.

Proof We will show this inductively. Note that $S_0 = 0$ by Theorem 3.3.1. This case will be sufficient for induction.

Now, assume that the range of S_n is the linear span of

$$\{K^{(j)}_{\frac{rc}{1+rc}}, K^{(k)}_{\frac{(sc+1)c}{1+sc^2}} : 1 \leq r \leq q-1, 0 \leq s \leq p-1, 0 \leq j,k \leq n-1\}$$

so that the rank of S_n is $n(p+q-1)$. Now we have

$$\frac{1}{a^p}S_{n+1} = (T^*_{z^{n+1}}C_\sigma T_\psi C_\varphi T_{z^{n+1}})^q - (T^*_{z^{n+1}}T_\psi C_\varphi T_{z^{n+1}}T^*_{z^{n+1}}C_\sigma T_{z^{n+1}})^p$$

We have moved the constant to the left side; however, the range of $\frac{1}{a^p}S_n$ is the same as S_n, so for the rest of the proof we will ignore this constant. Now consider the first term. Letting $\zeta = \psi \circ \sigma$ and $\tau = \varphi \circ \sigma$, we can rewrite it as

$$(T^*_{z^{n+1}}C_\sigma T_\psi C_\varphi T_{z^{n+1}})^q = T^*_z(T^*_{z^n}T_\zeta C_\tau T_{z^n}(I-O))^{q-1}T^*_{z^n}T_\zeta C_\tau T_{z^n}T_z$$

Letting $G = T^*_{z^n}T_\zeta C_\tau T_{z^n}$ as in Lemma 3.3.2, we can again rewrite the first term as

$$T^*_z(G - GO)^{q-1}GT_z$$

where $O(f) = \langle f, K_0\rangle = f(0)$. Moving to the second term, we will use G to rewrite it as well:

$$\begin{aligned}
&(T^*_{z^{n+1}}T_\psi C_\varphi T_{z^{n+1}}T^*_{z^{n+1}}C_\sigma T_{z^{n+1}})^p \\
&= T^*_{z^{n+1}}T_\psi C_\varphi T_{z^{n+1}}(T^*_{z^{n+1}}C_\sigma T_\psi C_\varphi T_{z^{n+1}})^{p-1}T^*_{z^{n+1}}C_\sigma T_{z^{n+1}} \\
&= T^*_z(T^*_{z^n}T_\psi C_\varphi T_{z^n}(I-O)(T^*_{z^n}C_\sigma T_\psi C_\varphi T_{z^n}(I-O))^{p-1}T^*_{z^n}C_\sigma T_{z^n})T_z \\
&= T^*_z(T^*_{z^n}T_\psi C_\varphi T_{z^n}(I-O)(G-GO)^{p-1}T^*_{z^n}C_\sigma T_{z^n})T_z \\
&= T^*_z(T_{\psi^{n+1}}C_\varphi(I-O)(G-GO)^{p-1}T^*_{z^n}C_\sigma T_{z^n})T_z \\
&= T^*_z(T_{\psi^{n+1}}C_\varphi(G-GO)^{p-1}T^*_{z^n}C_\sigma T_{z^n} - T_{\psi^{n+1}}C_\varphi O(G-GO)^{p-1}T^*_{z^n}C_\sigma T_{z^n})T_z
\end{aligned}$$

Now, let $S_{n+1} = T_z^* U T_z$. Then from our work above, we can write

$$U = (G - GO)^{q-1} G - T_{\psi^{n+1}} C_\varphi (G - GO)^{p-1} T_{z^n}^* C_\sigma T_{z^n} \\ + T_{\psi^{n+1}} C_\varphi O (G - GO)^{p-1} T_{z^n}^* C_\sigma T_{z^n}$$

We would like to expand the first term. Note that the range of O is the constants, so for the sake of computing the range we need only know what operators are to the left of O. If we expand the first term of U from the left and stop when we reach a factor of O, then the terms given are scalar multiples of $G^\ell O$ for $\ell \leq q-1$, along with G^q. Expanding the second term in the same way, we see that the terms are scalar multiples of $T_\psi^{n+1} C_\varphi G^m O$ for $m \leq p-1$, along with $T_{\psi^{n+1}} C_\varphi G^{p-1} T_{z^n}^* C_\sigma T_{z^n}$. Note that

$$G^q - T_{\psi^{n+1}} C_\varphi G^{p-1} T_{z^n}^* C_\sigma T_{z^n} = S_n$$

whose range is already known. Also note that the third term involves O as well. Therefore, letting $f \in zH^2$, we can write

$$Uf = S_n f + \sum_{\ell=1}^{q-1} G^\ell \alpha_{f,\ell} + \sum_{m=0}^{p-1} T_{\psi^{n+1}} C_\varphi G^m \beta_{f,m}$$

where $\alpha_{f,\ell}$ and $\beta_{f,m}$ are constants depending on the input function f and the operators in front of $G^\ell O$ and $T_{\psi^{n+1}} C_\varphi G^m O$. Even though f is in zH^2 so that $f(0) = 0$, these constants are generally non-zero, because zH^2 is not invariant for the operators in front of each instance of O. Furthermore, any combination of constants is possible by appropriate choice of f.

By Lemmas 3.3.2 and 3.3.3, we know the effect of G^ℓ and $T_{\psi^{n+1}} C_\varphi G^m$ on constants. Therefore, the range of the second and third terms of U on vectors in zH^2 is the linear span of vectors

$$\{K^{(j)}_{\frac{rc}{1+rc}}, K^{(k)}_{\frac{(sc+1)c}{1+sc^2}} : 1 \leq r \leq q-1, 0 \leq s \leq p-1, 0 \leq j, k \leq n\}$$

Note that the range of S_n is a subset of this, and the range of S_n on zH^2 is therefore also a subset. So the range of U on vectors in zH^2 is therefore the linear span of

$$\{K^{(j)}_{\frac{rc}{1+rc}}, K^{(k)}_{\frac{(sc+1)c}{1+sc^2}} : 1 \leq r \leq q-1, 0 \leq s \leq p-1, 0 \leq j,k \leq n\}$$

Recall that $S_{n+1} = T_z^* U T_z$. The range of U on zH^2, which we have found, is the same as the range of UT_z on H^2. By Theorem 1.4.1, T_z^* has no effect on this range, since none of these kernels are for evaluation at 0. So the range of S_{n+1} is

$$\{K^{(j)}_{\frac{rc}{1+rc}}, K^{(k)}_{\frac{(sc+1)c}{1+sc^2}} : 1 \leq r \leq q-1, 0 \leq s \leq p-1, 0 \leq j,k \leq n\}$$

and the rank is $(n+1)(p+q-1)$ as desired. ■

Corollary 3.3.5 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be defined by $\varphi(z) = \frac{az}{1-cz}$ with $a, c > 0$ and $a+c = 1$. Furthermore, let a be a rational number so that $a = \frac{p}{q}$ for integers p, q. Then for any nonnegative integers $j \neq k$, $C_\varphi|_{z^j H^2}$ is not unitarily equivalent to $C_\varphi|_{z^k H^2}$.*

Proof By Theorem 1.3.4, it is equivalent to state that $A_j = T_{\psi^j} C_\varphi$ and $A_k = T_{\psi^k} C_\varphi$ are not unitarily equivalent. If $A_j \cong A_k$, then $S_j \cong S_k$ where S_n is as defined in Theorem 3.3.4. However, that theorem shows that S_j and S_k do not have the same rank, and thus cannot be unitarily equivalent to each other. Therefore $A_j \ncong A_k$, so $C_\varphi|_{z^j H^2} \ncong C_\varphi|_{z^k H^2}$. ■

4. FURTHER RESULTS

4.1 Compact Perturbations

Our result here depends on a lemma found in [10]:

Lemma 4.1.1 *Suppose $b \in C(\partial U)$ and $b(1) = 0$. Suppose further that the function $\theta \to b(e^{i\theta})$ is differentiable at $\theta = 0$. Then for every non-automorphic, linear fractional φ that maps the disk into itself with fixed point 1, the operator $T_b C_\varphi$ is compact on H^2.*

This result allows us to connect the weighted composition operators discussed in Chapter 2.

Theorem 4.1.2 *Let $\varphi(z) = \frac{az}{1-cz}$ for some real constants a, c with $a + c = 1$. Let $\psi(z) = \frac{\varphi(z)}{z} = \frac{a}{1-cz}$. Then $T_{\psi^k} C_\varphi - T_{\psi^{k-1}} C_\varphi$ is compact.*

Proof First, rewrite the difference as one operator:

$$\begin{aligned} T_{\psi^k} C_\varphi - T_{\psi^{k-1}} C_\varphi &= (T_{\psi^k} - T_{\psi^{k-1}}) C_\varphi \\ &= T_{(\psi^k - \psi^{k-1})} C_\varphi \\ &= T_{\psi^{k-1}} T_{(\psi - 1)} C_\varphi \end{aligned}$$

Now consider the first weight $\psi - 1$:

$$(\psi - 1)(z) = \frac{a}{1 - cz} - 1 = \frac{a - 1 + cz}{1 - cz} = \frac{c(1 - z)}{1 - cz} = ((1 - z)\frac{c}{a})\psi$$

Let $b(z) = \frac{c}{a}(1 - z)$. Then we can rewrite

$$T_{\psi^{k-1}} T_{(\psi-1)} C_\varphi = T_{\psi^k} T_b C_\varphi$$

Now b clearly satisfies the hypotheses of Lemma 4.1.1, so T_bC_φ is compact. Since compact operators are an ideal, $T_{\psi^k}T_bC_\varphi = T_{\psi^k}C_\varphi - T_{\psi^{k-1}}C_\varphi$ is compact.

■

Now we can say what this means for restrictions of composition operators:

Theorem 4.1.3 *Let $\varphi(z) = \frac{az}{1-cz}$ for some real constants a, c with $a + c = 1$. Then for $j \neq k$, $C_\varphi|_{z^kH^2}$ is unitarily equivalent to a compact perturbation of $C_\varphi|_{z^jH^2}$.*

Proof First, note that by Theorem 4.1.2, $T_{\psi^k}C_\varphi - T_{\psi^{k-1}}C_\varphi = K$ for some compact K. This means that

$$\begin{aligned}
T_{\psi^k}C_\varphi - T_{\psi^{k-1}}C_\varphi &= K \\
T^*_{z^k}C_\varphi|_{z^kH^2}T_{z^k} - T^*_{z^{k-1}}C_\varphi|_{z^{k-1}H^2}T_{z^{k-1}} &= K \\
T^*_{z^k}C_\varphi|_{z^kH^2}T_{z^k} - T^*_{z^{k-1}}C_\varphi|_{z^{k-1}H^2}T_{z^{k-1}} - K &= 0 \\
T^*_{z^{k-1}}(T^*_zC_\varphi|_{z^kH^2}T_z - C_\varphi|_{z^{k-1}H^2} - T_{z^{k-1}}KT^*_{z^{k-1}})T_{z^{k-1}} &= 0
\end{aligned}$$

The operator inside the parentheses, $B = T^*_zC_\varphi|_{z^kH^2}T_z - C_\varphi|_{z^{k-1}H^2} - T_{z^{k-1}}KT^*_{z^{k-1}}$, is operating on $z^{k-1}H^2$, so that the entire left side is an operator on H^2. Because B is $z^{k-1}H^2$-invariant, this equation is only true if

$$T^*_zC_\varphi|_{z^kH^2}T_z - C_\varphi|_{z^{k-1}H^2} - T_{z^{k-1}}KT^*_{z^{k-1}} = 0$$

So now we see that

$$T^*_zC_\varphi|_{z^kH^2}T_z = C_\varphi|_{z^{k-1}H^2} + T_{z^{k-1}}KT^*_{z^{k-1}}$$

Since compact operators are an ideal, $L_{k-1} = T_{z^{k-1}}KT^*_{z^{k-1}}$ is compact and we see that for any positive integer k, $C_\varphi|_{z^kH^2}$ is unitarily equivalent to a compact perturbation of $C_\varphi|_{z^{k-1}H^2}$.

Now, for positive integers $j \neq k$, assume $j < k$.

Then we know that

$$\begin{aligned}
C_\varphi|_{z^jH^2} &= T_z^* C_\varphi|_{z^{j+1}H^2} T_z - L_j \\
&= T_z^*(T_z^* C_\varphi|_{z^{j+2}H^2} T_z - L_{j+1}) T_z - L_j \\
&\vdots \\
&= T_{z^{k-j}}^* C_\varphi|_{z^kH^2} T_{z^{k-j}} - \sum_{i=0}^{k-1} T_{z^i}^* L_{j+i} T_{z^i}
\end{aligned}$$

Note that the summation taken as one operator, i.e. $L = \sum_{i=0}^{k-1} T_{z^i}^* L_{j+i} T_{z^i}$, is compact. We then see that

$$T_{z^{k-j}}^* C_\varphi|_{z^kH^2} T_{z^{k-j}} = C_\varphi|_{z^jH^2} + L$$

Thus $C_\varphi|_{z^kH^2}$ is unitarily equivalent to a compact perturbation of $C_\varphi|_{z^jH^2}$.

■

This result nicely complements our strategies in Chapter 3. For two restrictions to be unitarily equivalent, the compact difference in Theorem 4.1.2 would need to be 0; but we have seen that for two certain unitary-invariant expressions in the operator and its adjoint, the compact difference is growing in size with each restriction.

4.2 Spectra

In [8], Neophytou gives the following theorem:

Theorem 4.2.1 *Let* $\varphi(z) = \frac{sz}{1-(1-s)z}$ *for* $0 < s < 1$. *Then the point-spectrum of* C_φ^* *on* H^2 *is the set*

$$\{\lambda : 0 < |\lambda| < \sqrt{s}\} \cup \{1\}$$

In addition, the eigenspaces for eigenvalues other than 1 are shown to be infinite-dimensional. The eigenvalue 1 has a one-dimensional eigenspace, coming from the fact that $C_\varphi 1 = 1 \circ \varphi = 1$. We are able to show when C_φ^* is compressed to $z^k H^2$, the point spectrum is the same otherwise.

Theorem 4.2.2 *Let $\varphi(z) = \frac{sz}{1-(1-s)z}$ for $0 < s < 1$. Then the point-spectrum of C_φ^* compressed to $z^k H^2$ is the set*

$$\{\lambda : 0 < |\lambda| < \sqrt{s}\}$$

and every eigenspace is infinite-dimensional.

Proof First, we will show that nothing is added to the point-spectrum under compression. Since $z^k H^2$ is invariant for C_φ, C_φ^* has a block matrix of the form

$$\begin{pmatrix} A^* & B^* \\ 0 & C^* \end{pmatrix}$$

where $H^2 = (z^k H^2)^\perp \bigoplus z^k H^2$.

An eigenvector for C^* and eigenvalue λ can be extended to an eigenvector $[g, f]$ on H^2 if the equation $A^*g + B^*f = \lambda g$, i.e. $(A^* - \lambda I)g = B^*f$, is solvable. We need only show that $A^* - \lambda I$ is onto, so that B^*f is in its range.

Note that $A^* : (z^k H^2)^\perp \to (z^k H^2)^\perp$ is a finite rank operator since it operates on a finite-dimensional space. Then we need only to make sure that $A^* - \lambda I$ does not have 0 as an eigenvalue, but we know that the eigenvalues of A^* are $\{s^i : 0 \leq i \leq k-1\}$ [5, Proposition 7.32]. If λ is one of these values, A^* is not full rank, but these values are already in the spectrum of C_φ^*. Otherwise, $A^* - \lambda I$ has full rank, so B^*f is in its range. This means that any eigenvector of C^* is an eigenvector for C_φ^*, so nothing is added to the point-spectrum.

Since each eigenspace is infinite dimensional, and the projection of an eigenvector to $z^k H^2$ is either 0 or another eigenvector, each eigenvalue of C_φ^* is still an eigenvalue for C^* of infinite multiplicity.

Thus, $\sigma_p(C_\varphi^*|_{z^k H^2}) = \sigma_p(C_\varphi^*)$ as desired. ∎

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