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MOVING WINDOW UNIT ROOT TEST: LOCATING REAL ESTATE PRICE BUBBLES IN SEOUL APARTMENT MARKET

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SINGAPORE MANAGEMENT UNIVERSITY

2007

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SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE IN ECONOMICS

SINGAPORE MANAGEMENT UNIVERSITY

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SHI SHU PING

Abstract

Bubbles are characterized by rapid expansion followed by a contraction. Evans (1991) shows that stationarity tests suggested by Hamilton and Whiteman (1985) and Diba and Grossman (1988) are incapable of detecting periodically collapsing bubbles. Phillips, Wu, and Yu (2006) advanced the forward recursive unit root test which improves the power significantly in the presence of periodically collapsing bubbles. In this paper, we consider rolling window unit root test with a pre-selected optimum window. A combining use of conventional unit root test and forward recursive unit root test is suggested from the results of power comparison. Furthermore, we apply those three methods to test the existence of bubbles in Seoul apartment market and to locate the bubble period if they were present.

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I Introduction

The problem of assessing the contribution of market fundamentals and rational bubbles has always been of a high concern. Hamilton and Whiteman (1985) and Diba and Grossman (1988) recommend the strategy of stationarity test for asset prices and observable market fundamentals. The stationarity test does not preclude the influence of unobserved market fundamentals. The basic rationale of this strategy is the following. If the first level difference of the asset price and the dividend are stationary, and the level of asset price and dividend are cointegrated, the asset price in the market is dominated by the market fundamentals. At the presence of speculative bubble, asset price has an explosive tendency and stationarity can not be attained even after taking multiple difference. Due to the unobservable variables in market fundamentals, the existence of bubbles can not be concluded by the evidence of non-stationarity. However, the reverse inference is possible.

Evans (1991) shows that the stationarity tests suggested by Diba and Grossman (1988) are incapable of detecting periodically collapsing bubbles. Phillips, Wu and Yu (2006) prove that the forward recursive unit root test can improve the power of ADF test significantly at the present of periodically collapsing phenomenon of bubbles. They argue that this characteristic can be detected by adding increments gradually to a subset of the sample period of ADF test. Based on the recursive procedure, supremum test is proposed in order to conclude the existence of bubbles. The authors also provided the asymptotic and finite sample properties of the sup test in their paper. Furthermore, the forward recursive procedure is recommended for its ability to

locating the bubble period.

In this paper, we consider rolling window ADF test with a pre-selected window size. Based on power comparison results, a combination of forward recursive ADF test and conventional ADF test is suggested. Moreover, we apply those three methods to the Seoul apartment market to test the existence of bubbles in this market.

II Review of the Real Estate Price Bubbles Literature

Bubbles may exist in any assets whose fundamental value is hard to assess. This kind of uncertainty is the source of speculation, hence the source of the bubbles. According to Stiglitz (1990), "if the reason that the price is high today is only because investors believe that the selling price will be high tomorrow – when 'fundamental' factors do not seem to justify such a price – then a bubble exists." (For further discussion, please refer to Appendix C)

Summary measures commonly used to assess housing market conditions are the affordability ratio and price-to-rent ratio. If the affordability ratio rises above its long-term average, this could be an indication of those prices are overvalued. The price-to-rent ratio is interpreted as the cost of owing versus renting a house. When house prices are high relative to rents, potential buyers will prefer to rent, which in turn should exert a downward pressure on house prices. Hence, a continuous upward price-to-rent ratio also suggests the existence of speculative bubbles. However, the non-stable long-run relationship between house price and disposal income as well as the relationship

between house price and rent is the main critique of these measurements.

Both linear and nonlinear econometric methods are widely used to explore house market conditions. The standard approach in the house market is that analysts run a regression of one-period net return of house investment, or house price, on a constant and a group of explanatory variables known at the beginning of the period.

Chung and Kim (2004) relate house price to income and bond yield by regressing a simple linear model. Income and bond yield represents "normal" demand and "speculative demand" respectively.

Hu et al. (2006) use a nonparametric method to test the relationship between housing price and the housing price growth rate in the Chinese housing market. They argue that the significant relation between these two variables implies the existence of bubbles.

The existence of bubbles are confirmed when house price deviate from its market fundamentals significantly. The most widely accepted market fundamental model is called the General Arbitrage-free model suggested by Poterba (1984, 1991) and Topel and Rosen (1988), which is a combination of non-arbitrage condition and rational expectation assumption in housing market. Based on this market fundamental model, a number of studies concerning bubble testing have been carried out.

Scott (1990) and Brooks et al. (2001) employ variance bound tests, suggested by Shiller (1981) and LeRoy and Porter (1981), to test the rationality of real estate share prices. The study of Scott focuses on price indices of 13 REITS. Brooks et al. exam the price of U.K. property stocks.

Lim (2003) examines housing price in Korea by applying two bubble test methods based on the General Arbitrage-free model. One is a modified volatility test (MRS test) suggested by Mankiw et al. (1985), the other one combines the unit root test suggested by Diba and Grossman (1998b) with the cointegration test by Campell and Shiller (1987). The results of these two methods are surprisingly different. The MRS test rejects the null hypothesis of market efficiency, which implies the existence of an irrational bubble. The unit root and cointegration tests, however, shows no evidence of the existence of bubbles.

Qin (2005) employs the Markov switching ADF approach advanced by Hall et al. (1999) to examine Seoul housing prices. They distinguish the expanding phase from the collapsing phase of a bubble by allowing for different parameters in a two-state Markov chain model. Their study suggests that positive bubbles possibly exists between January 1986 to June 1991, and between July, 1998 to June, 2003, while a negative bubble may have occurred between July, 1991 and June, 1998.

Qin (2006) estimates two models to determine the accountability of apartment price volatility in Seoul. A Kalman filter is used to capture the misspecification error within the General Arbitrage-free and rational expectation framework. It is suggested that both a misspecification error and a bubble proxy are responsible for the shoot up apartment price, and that a bubble explains the main part of the price movement.

III Housing Market Fundamental

1. Basic Model

The basic equations of the General Arbitrage-free model, proposed by Poterba (1984, 1991) and Topel and Rosen (1988), consists of a non-arbitrage condition and a rational expectation assumption. Another implicit assumption is that all individuals in the market are risk neutral.

The non-arbitrage condition states that excluding the depreciation maintenance expense and the tax cost of housing, the rent revenue plus the expected added return should equal the average return of alternative assets. Alternatively, it can be interpreted that the rental price of a house is its amortized price including allowance for interest rate, depreciation and capital gains. Mathematically,

$$R_t + g_e P_t = (r_t + \delta) P_t \tag{1}$$

$$g_e = \left(P^e_{t+1,t} - P_t\right) / P_t \tag{2}$$

where R_t is real rental income; P_t is the current real asset value of the housing unit; r_t is the real return available on alternative assets; δ is the summation of depreciation maintenance expense and property tax cost as a share of house value; $P^e_{t+1,t}$ is the expected housing price in period $t+1$ based on the information which is available in period t; and g_e is the expected housing price appreciation rate.

The second assumption in the General Arbitrage-free model is about rational expectation: people determine their expectation of housing price based on the present

information in the housing market. That is

$$P_{t+1}^{e} = E\left(P_{t+1}|I_t\right) \tag{3}$$

Upon combing equations (1)-(3) and solving for P_t, the house price determinant equation (4) indicates that housing price equals the present value of the summation rent income and the property price which can be sold at the end of the ownership.

$$P_t = \frac{R_t + E\left(P_{t+1}|I_t\right)}{1+\delta+r_t} \tag{4}$$

To simplify the analysis, we normalize the permanent component of housing price P^* and rent R^*; i.e. we let $P^* = R^* = 1$, $p_t = \ln P_t - \ln P^*$, $r_t = \ln R_t - \ln R^*$ and $\rho = (1+\delta+r_t)^{-1}$. Hence, $p_t = \ln P_t$ and $r_t = \ln R_t$. It is readily shown that

$$p_t \approx (2\rho - 1) + \rho\left(r_t + E_t p_{t+1}\right) \tag{5}$$

Following the recursive method, it can be showed that

$$p_t = \frac{2\rho-1}{1-\rho} + \rho r_t + \sum_{i=2}^{\infty} \rho^i E_t r_{t+i-1} + \lim_{i\to\infty} \rho^i E_t \rho_{t+i} \tag{6}$$

When the transversality condition

$$\lim_{i\to\infty} \rho^i E_t \rho_{t+i} = 0$$

holds,

$$p_t^f \equiv \frac{2\rho-1}{1-\rho} + \rho r_t + \sum_{i=2}^{\infty} \rho^i E_t r_{t+i-1} \tag{7}$$

The logarithm of housing fundamental price then equals to the present value of expected logarithm of real rental revenue in the future. While, if the transversality condition fails,

$$p_t = p_t^f + b_t \tag{8}$$

where

$$b_t = \lim_{i \to \infty} \rho^i E_t \rho_{t+i} \tag{9}$$

At the presence of rational speculative bubbles in the housing market, housing price will experience a long-sustained rapidly growth period and the scale of the bubbles will be time-dependent. By the definition of b_t, it can be shown that this component may be modeled by an $AR(1)$ process. That is,

$$b_t = \rho^{-1} b_{t-1} + \varepsilon_t \tag{10}$$

where

$$E_{t-1}\varepsilon_t = 0$$

Since $\rho^{-1} = 1 + r_t + \delta$, where r_t is the real return available on alternative assets, the coefficient of the first order autoregressive AR process is greater than one in magnitude, implying an explosive tendency.

Therefore, the additional component b_t in housing price is one way to capture the explosive tendency of rational speculative bubbles in the housing market.

2. Bubble Cycle Model

Rational speculative bubbles can take the form of time-dependent explosive AR (1) process. However, Blanchard (1979) and Blanchard and Watson (1982) argued, explosive trend is not the whole story of bubbles, and periodically collapsing is its another important characteristic. Resembling to business cycles, which are characterized by expansion and contraction, they are called bubble cycles.

As Evans(1991) criticizes, such kind of periodically collapsing, however, makes the stationary and cointegration test result not so convincing, since the spiral trend may show stationary properties when applying conventional Augmented-Dickey-Fuller test. Evans (1991) put forward a model to describe financial bubble cycles.

$$b_{t+1} = \rho^{-1} b_t u_{t+1}, \qquad \text{if } b_t < \alpha \tag{11}$$

$$b_{t+1} = \left[\varsigma + (\pi\rho)^{-1} \theta_{t+1} (b_t - \rho\varsigma)\right] u_{t+1}, \text{ if } b_t >= \alpha \tag{12}$$

where $0 < \rho < 1$ and $u_t = \exp(y_t - \tau^2/2)$ with $y_t \sim N(0, \tau^2)$. ζ is the remaining size after bubble collapsing. θ_t measures the probability of bubble collapse. It follows a Bernoulli process which takes 1 if the bubbles survive at period t; otherwise, it takes 0.

The bubble cycle model has the property that $Eb_{t+1} = \rho^{-1} b_t$. When the size of a bubble is smaller than α, the bubble survive anyway; while, when it exceed the critical point, it collapses with certain probability, which is captured by θ_t. That is θ_t equals to zero when Bubbles collapse.

IV Empirical Strategy

However, Evans (1991) did not propose any econometric method to test the existence of bubble cycles. Phillips, Wu, and Yu (2006) worked on this issue by advancing a forward recursive test procedure. Their method implemented a right-side Augmented-Dickey-Fuller test and a sup test, which we apply in this paper.

The forward recursive test procedure is implemented as follows: Firstly, we applied

Augmented-Dickey-Fuller (ADF) test with the null hypothesis of unit roots against explosive alternatives for each time series x_t . It is specified as the following:

$$x_t = u + \delta x_{t-1} + \sum_{i=1}^{k} \beta_i \triangle x_{t-i} + \varepsilon_t,\ \varepsilon_t \sim N\left(0, \delta^2\right) \tag{13}$$

k is the number of lags in the test. Significance test are used to determine the lag order as Ng and Perron (1995) did. The null hypothesis is $H_0 : \delta = 1$ and the explosive alternative hypothesis is $H_1 : \delta > 1$.

In the forward recursive test procedure, the above test is implemented repeatedly. Starting from a small sample size, we increase one observation in each subsequence regression until it covers the whole sample period. Suppose f_0 is the fraction of total observations which is included in the first regression. It suggests that the sample size in first regression is $n_1 = [f_0 n]$, where $[.]$ signifies the integer part of its argument. The subsequence regression sample size is $n_i = n_1 + i - 1$,where $i = 1, 2, \ldots, n - n_1$.

Denote each ADF statistic as ADF_i, the sup forward ADF statistic is defined as

$$\sup ADF_i = \max\left(ADF_i\right) \tag{14}$$

ADF in this paper refers to conventional Augmented-Dickey-Fuller statistic which is implemented with all observations. That is

$$ADF = ADF_{n-n_1}.$$

Another moving window procedure frequently considered is rolling window. Distinguished from the forward recursive test procedure, the key step in rolling window procedure is to select an optimum window for the ADF regression.

Suppose m_0 is the minimum window for rolling regression. The regression coefficient is represented as $ADF_{m_0,i}$, where $i = 1, \ldots, n - m_0$. We choose the rolling window in the sequence $\{m_0, m_0 + 1, \ldots, m_0 + m\}$, where $m_0 + m$ is the maximum window of regression. The corresponding regression statistics are obtained in following procedure:

$$\begin{array}{ll} ADF_{m_0,i} & \text{where } i = 1, \ldots, n - m_0 \\ ADF_{m_0+1,i} & \text{where } i = 1, \ldots, n - m_0 - 1 \\ \ldots & \ldots \\ ADF_{m_0+m,i} & \text{where } i = 1, \ldots, n - m_0 - m \end{array}$$

Based on above regressions, we compute the distance between optimum window and minimum window $\hat{\jmath}$ in sense to capture the most explosive tendency. That is

$$\hat{\jmath} = \arg\max_{j} \left\{ \sup_{m_o+j} ADF : j = 1, \ldots, m \right\}$$

where

$$\sup ADF_{m_0+j} = \max_{i} \left\{ ADF_{m_0+j,i} : i = 1, \ldots, n - m_0 - j \right\}$$

Therefore, the optimum rolling window is $m_0 + \hat{\jmath}$ and $\sup ADF_{m_0+\hat{\jmath}}$ is the corresponding ADF statistic.

Furthermore, from equation (7), we can get that

$$(1 - \rho)\, p_t - \rho r_t = 2\rho - 1 + \sum_{i=2}^{\infty} \rho^i \triangle E_t r_{t+i} \tag{15}$$

It suggests that the linear combination of the logarithm of housing price and rent (LHS) should have the same time series properties as the first difference of logarithm rent (RHS).

Therefore, we have following statements:

Remark 1 *If p_t and r_t are all stationary, we have no evidence to prove that there is bubble in the market.*

Remark 2 *If $\triangle r_t \sim I(0)$, p_t, r_t are cointegrated and the cointegration vector is $[1-\rho, \rho]$, we have no evidence to prove that there is bubble in the market.*

Remark 3 *If either of the above statement is violated, there may be bubbles in housing market.*

V Power Comparison

Combining the periodically collapse bubble model equations (11) and (12) with the equation (16), which is derived from equation (6) by assuming that rental r_t follows unit root process (equation (17)), we simulate the series of housing price. For the convenience, we write down all the equations here.

$$b_{t+1} = \rho^{-1} b_t u_{t+1}, \qquad \text{if } b_t < \alpha \tag{11}$$

$$b_{t+1} = \left[\varsigma + (\pi\rho)^{-1} \theta_{t+1} (b_t - \rho\varsigma)\right] u_{t+1}, \text{ if } b_t < \alpha \tag{12}$$

$$p_f = \frac{1-g}{g} + \frac{1-g}{(1+g)\, g^2} u + \frac{r_t}{g}, \tag{16}$$

$$r_t = c + r_{t-1} + \varepsilon_t, \varepsilon_t \sim N\left(0, \sigma^2\right) \tag{17}$$

where $u_t = \exp\left(y_t - \tau^2/2\right); y_t \sim N\left(0, \tau^2\right); 1 + g = \rho^{-1}$.

To facilitate comparisons, we set parameters as following, : $g = 0.05, b_0 = 0.5, \alpha = 1, \varsigma = b_0, \tau = 0.0025, u = 0.373, \sigma = 0.1574, r_0 = 1.3$ and sample size equals 200,which is the same as Evans (1991) and Phillips, Wu, and Yu (2006). Table 2 illustrates power of each approach under this setting at 5% size. We can see the improvement of forward recursive method and rolling window method. Meanwhile, forward recursive method performs best among those three methods.

In order to check the robustness of the power, the growth rate of bubble g takes the value of 0.01, 0.03 and 0.05; Meanwhile, we choose the value of the initial size of bubble b_0 equals to 0.00, 0.05, 0.10, 0.15, 0.20 and 0.50; the probability of bubble collapse π , which is the probability of $\theta_t = 0$ in the Bernoulli process, takes the value of 0.1, 0.25, 0.9, 0.95, 0.99; and the standard error of bubble varies from 0.01 to 0.09.

Conclusions for the power comparison are following: For conventional ADF test, the initial size of bubble, the growth rate of bubble and the volatility of bubble do not affect its testing power.

Meanwhile, as long as the size of bubbles, the growth rate of bubble or the probability of collapsing is small, the forward recursive approach and rolling window approach do not have obvious improvement in testing the existence of bubbles in term of power. When the growth rate of bubble is 0.01 and the initial bubble size equals to zero (Table 3), the power of forward ADF test and rolling ADF test are worse than conventional ADF test due to smaller sample size. The same result happens to the case when the probability of bubble collapsing π equals to 0.1 and 0.25 (Table 4).

However, when the initial size of bubble does not equal to zero and it grows faster

$(g > 0.01)$, the relative power of forward ADF begins to increase comparing with conventional ADF test. While for rolling window ADF test, the advantage stands out only when bubble size is large enough and growth faster as we can see that its power is greater than conventional ADF test when $g = 0.5$ and $b_0 \geq 0.15$ in Table 3.

Furthermore, higher probability of bubble collapsing will enforce the advantage of forward recursive ADF test and rolling window ADF test. As we can see in Table 4, when the probability of bubble collapsing reaches 0.9, the advantage of forward recursive ADF test and rolling window ADF test are obvious given initial bubble size are large enough. And forward recursive ADF test performs better than rolling window ADF test.

In addition, given the initial size of bubble, the probability of collapsing and the growth rate of bubble, power of forward ADF test and rolling window ADF test are immune to the volatility of power (Table 5).

To sum up, forward recursive unit root test is proved with high power comparing with conventional unit root test when the growth rate of bubbles, probability of bubble collapsing and size of bubbles are large enough. Rolling window unit root test shows higher power than conventional unit root test when bubble size are large as well, however, forward recursive unit root test is still the best among those three methods.

Hence, we recommend a combination of conventional unit root test and forward recursive unit root test for practical usage.

Remark 4 *Suppose bubbles are detected by both conventional unit root test and forward recursive method, the advantage of using forward recursive method is that it*

can locate the exact bubble period.

Remark 5 *If bubbles are spied by forward recursive method but not by conventional unit root, it suggests that there exist bubbles with high periodically collapsing probability, fast growth rate and large initial bubble size.*

Remark 6 *If bubbles are alarmed with conventional unit root test but not with forward recursive unit root test, the only possibility is that the initial size of bubbles and the growth rate of bubbles are still very small, and the possibility of bubble collapsing is also very low.*

Remark 7 *If none of all three methods detected the existence of bubbles, our non-bubble conclusion is more powerful than using conventional unit root test by reducing the possibility of misidentification caused by the periodically collapsing characteristic of bubbles.*

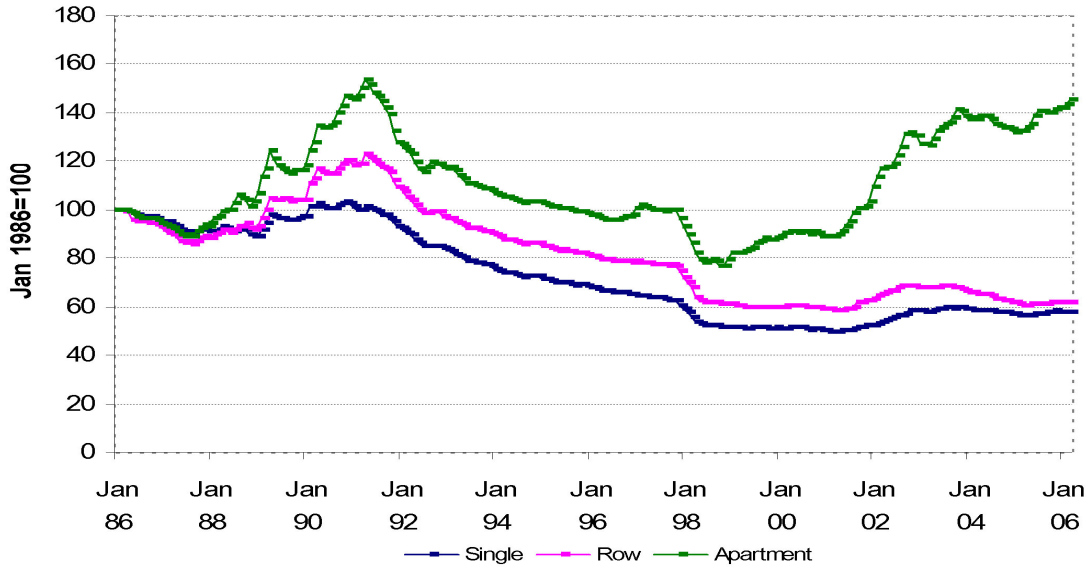
VI Seoul Apartment Market

The apartment price of Seoul has been volatile in the last twenty years. Starting from August 1987, the Seoul house price increased sharply and peaked at April 1991. During this boom, the real apartment house price index enlarged 66.6%, while the row and single real house price indices only increased 38.5% and 8.52% respectively. After April 1991, there was a seven-year long continued decrease of the housing price. By April 1998, the apartment, row, single house price indices were only 60.03%,

66.71% and 66.50% of their previous respective peak value. Since then, the apartment price became almost flat for two months and successively experienced another rising trend again. (Figure 1)

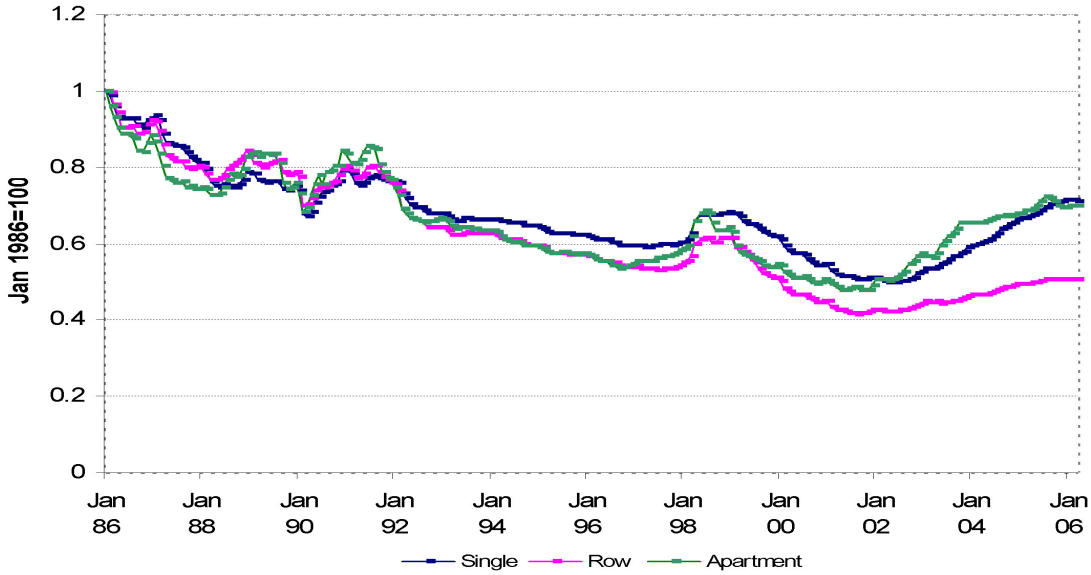
Meanwhile, the price-rent ratio roared up as the housing price went up, especially the apartment price-rent ratio (Figure 2). Due to the spiral housing price, the existence of bubble in Seoul apartment market during this period has been the subject of a good deal of recent discussion. In the following section, we would apply the strategy suggested above to test the existence of bubbles in this market.

Figure 1: Seoul Real Housing Price Indices



Data Resources: Seoul Housing Price Index (Sep 2003=100,Monthly): CEIC; Consumer Price Index (Monthly,all cities): IFS; Data Period: From Jan1986 to Mar 2006.

Figure 2: Seoul Housing Price-rent Ratio



Data Resources: Seoul Housing Price Index (Sep 2003=100,Monthly): CEIC; Consumer Price Index (Monthly,all cities): IFS; Data Period: From Jan1986 to Mar 2006.

VII Empirical Results

Figure 3 illustrates the values of $\sup ADF$ under different rolling windows. l refers to proportion of the rolling window we chosen $m_0 + j$ to the whole sample period n, that is $l = \frac{m_0+j}{n}$. It is considered from 0.15 to 0.4, which covers from 3 years' rolling window to over 8 years'. Optimum rolling window refers to the corresponding window of maximum $\sup ADF$. As it is showed in the graph, the peak of $\sup ADF$ curve of Seoul apartment price and apartment rental are reached at 0.22 and 0.18 separatively. It implies that our optimum rolling window sizes $m_0 + \hat{j}$ are the integer part of $0.22 \times n$ and $0.18 \times n$, which equal to 53 and 43. Thus, their regression

windows are $[i, i+53]$ and $[i, i+43]$, where $i = 0, 1, \ldots n-53$ and $i = 0, 1, \ldots n-43$ respectively.

For forward recursive method, the fraction f_0 we chosen equals to 0.3. Thus, the sample window for first regression is from January 1986 to January 1992. Then, we add one month each time in successive regressions until March 2006.

Table 1 reports the results of conventional ADF test, forward recursive ADF test and rolling window ADF test. $\sup ADF_i$ and $\sup ADF_{m_0+\hat{\jmath}}$ are obtained from forward recursive ADF test and rolling window ADF test separatively. Both Seoul apartment price index and rental index series, ranged from January 1986 to March 2006, are tested here. Critical values for each method in this table are obtained from 5000 times' Monte Carlo simulation. Optimum rolling window is re-selected at each simulation for rolling window approach.

Several conclusions can be drawn from Table 1: Firstly, Seoul apartment rental is tested stationary under all three different statistics. The results can be seen even clear from Figure 4 and Figure 5. The ADF statistics for apartment rental in Seoul are all under the 5% critical values. It indicates that Seoul apartment rental r_t is a stationary series during January 1986 to March 2006.

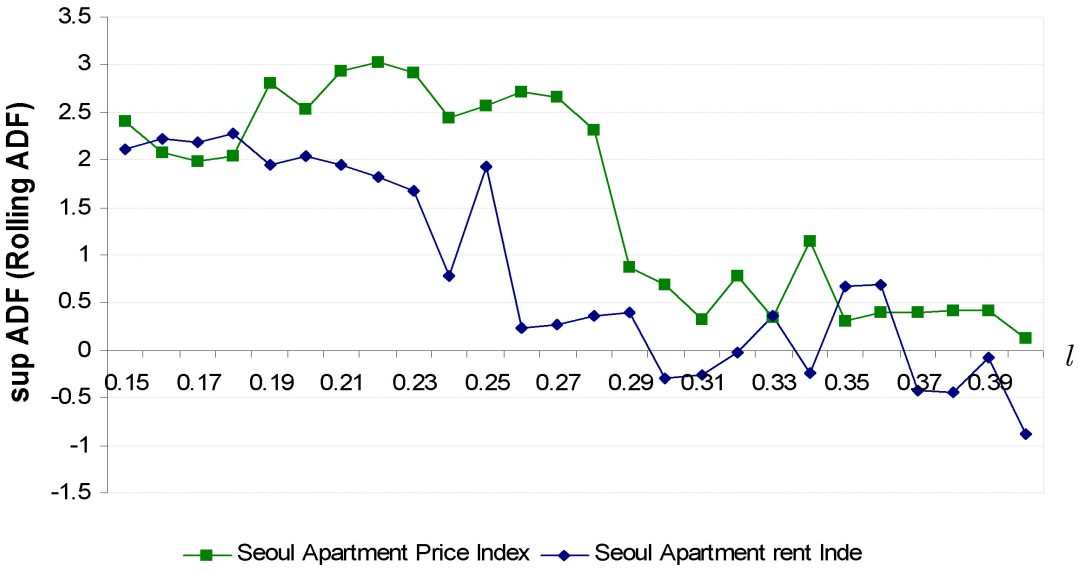
Secondly, we do not detect any nonstationary factor in Seoul apartment price index either using all three methods. From Figure 3, we can see that all the forward recursive ADF statistics including the last point, which is the value of conventional ADF test, fall below the 10% critical value line.

There are two peaks in Figure 5, which locate at windows {May 1998: October

2002} and {May 1999: October 2003}. It suggests that Seoul apartment price in these two periods grow faster than other periods'. However, these two points on $ADF_{m_0+\hat{\jmath}}$ curve are still under the10% critical value line. We can not reject null hypothesis.

According to Lemma 1 and corollary 3, we can infer that non bubble presents at Seoul apartment market during this period. In other words, apartment price in Seoul is driven by market fundamental.

Figure 3: Optimum Rolling Windows Selection



Note: l is the proportion of sub-window size to the whole sample size, ranged from 0.15 to 0.4. The corresponding expression using window size $[m_0, m_0 + m]$ is from 3 years' monthly data to over 8 years'.

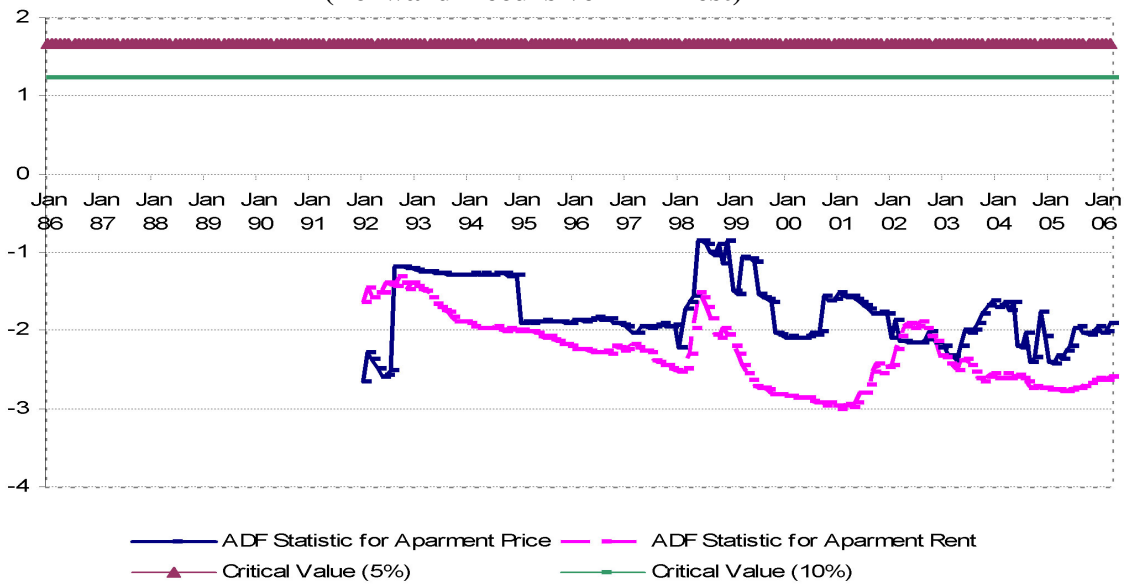
Table 1: Forward ADF Test and Rolling ADF Test

	Conventional ADF Test	Forward ADF Test ($\sup ADF_f$)	Rolling ADF Test ($\sup ADF_{m_0+\hat{j}}$)
HPI	-1.7512	-0.84810370	3.0257829
Rent	-3.0196	-1.3180561	1.8295795
Critical values for explosive alternative			
1%	0.84522811	2.4486125	10.897909
5%	0.22023759	1.6679288	7.0055413
10%	-0.38080786	1.2364278	5.9140605
Critical values for stationary alternative			
10%	-2.6559274	-1.2379664	2.1469900
5%	-3.0142079	-1.5424851	1.7781104
1%	-3.6347292	-2.0523479	1.1028712

Note: Critical values are all based on 5000 times Monte Carlo simulation. For rolling ADF test, the optimal rolling window is reselected at each simulation.

Figure 4: ADF Statistic for Seoul Apartment Price and Rent

(Forward Recursive ADF Test)



Note: The fraction of first regression sample size to the whole sample f_0 equals to 0.3 for both real apartment price index and real apartment rental index.The straight lines in the graph represent 5% and 10% finite sample critical values respectively.

Figure 5: ADF Statistic for Seoul Apartment Price and Rent

(Rolling Window ADF Test)

Note: The optimal windows for real apartment price index and real apartment rental index from January 1986 to March 2006 are $\hat{j} = [l^* \times n] = 53$ and 43. The straight lines in the graph are 5% and 10% finite sample critical values.

VIII Conclusions

In this paper, we analysis the time series properties of housing price and rental based on General Arbitrage-free model, which is resemble to stock price and dividend in financial market. As it is discussed in the literature, stationarity of both series or cointegration of housing price and rental is the implicit requirement for fundamental market.

It is commonly believed that housing price should be explosive at the presence of bubbles, which formalize the idea of identifying the existence of bubbles by performing stationary and cointegration tests. However, its periodically collapsing characteristic put forward higher requirements to conventional stationarity test, which refers to unit root test in most cases.

Based on Evans (1991)'s periodically collapsing bubble model, we compare conventional unit root test with forward recursive unit root test proposed by Phillips, Wu, and Yu (2006), and rolling window unit root test, which is considered in this paper. By setting different parameters to bubble cycle models, more abundant results are obtained from power comparison. We prove that conventional unit root test performs better than the other two when bubble sizes are small or probability of collapsing is lower; while, forward recursive method and rolling window method outperform conventional unit root test when bubble sizes are large and probability of bubble collapsing is high or bubble growth rate is fast. In addition, forward recursive method always does a better job than rolling window approach. Therefore, a combining usage of conventional unit root test and forward recursive unit root test is suggested.

All three methods show that real apartment price index and real rental index in Seoul are stationary from January 1986 to March 2006. Hence, our conclusion is that apartment price in Seoul market during this period is driven by market fundamental. Furthermore, as we argued above, this conclusion is stronger than conclusions from applying only conventional unit root test.

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A Appendix:Tables

Table 2: Power Comparison

Approaches	ADF	Forward ADF Statistic	Rolling ADF Statistic
Power	0.1170	0.2136	0.1664

NOTE: Power is defined as the probability of rejecting the null hypothesis (unit root)with size 5%. Parameters in this comparison are set as following:$g = 0.05$, $b_0 = 0.5$, $\alpha = 1$, $\varsigma = b_0$, $\tau = 0.0025$, $u = 0.373$, $\sigma = 0.1574$, $r_0 = 1.3$ and sample size equals 200.

Table 3: Power Comparison

(Bubble Growth Rate and Initial Size of Bubble)

Power	g=0.01			g=0.03			g=0.05		
$\tau = 0.0025$ π =0.9	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic
b0=0.00	0.1354	0.0928	0.0548	0.1242	0.0952	0.0612	0.1252	0.0952	0.0588
b0=0.05	0.1314	0.0942	0.0546	0.1176	0.1414	0.0642	0.1160	0.1414	0.1008
b0=0.10	0.1240	0.0938	0.0566	0.1250	0.1596	0.0618	0.1196	0.1596	0.1078
b0=0.15	0.1268	0.0868	0.0504	0.1276	0.1712	0.0700	0.1112	0.1712	0.1114
b0=0.20	0.1304	0.0922	0.0612	0.1204	0.1790	0.0662	0.1096	0.1790	0.1246
b0=0.50	0.1364	0.0908	0.0584	0.1262	0.2136	0.0710	0.1170	0.2136	0.1664

NOTE: PPower is defined as the probability of rejecting the null hypothesis (unit root)with size 5%. Number of replication is 5000 .

Table 4: Power Comparison

(Bubble Growth Rate and Probability of Collapsing)

Power	π =0.1			π =0.25			π =0.9		
b0=0.5 $\tau = 0.0025$	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic
g=0.05	0.1256	0.0948	0.0554	0.1070	0.0570	0.0768	0.1170	0.2216	0.1584

NOTE: Power is defined as the probability of rejecting the null hypothesis (unit root)with size 5%. Number of replication is 5000 .

Table 4(cont.): Power Comparison

(Bubble Growth Rate and Probability of Collapsing)

Power	π =0.95			π =0.99		
b0=0.5 σ_b =0.037	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic
g=0.05	0.1190	0.3376	0.2298	0.4124	0.7316	0.5826

NOTE: Power is defined as the probability of rejecting the null hypothesis (unit root)with size 5%. Number of replication is 5000 .

Table 5: Power Comparison

(Volatility of Bubble)

Power	τ =0.01			τ =0.025			τ =0.05		
(g=0.05 π =0.9)	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic
b0=0.5	0.1054	0.2100	0.1530	0.1084	0.2110	0.1574	0.1040	0.2142	0.1574

NOTE: PPower is defined as the probability of rejecting the null hypothesis (unit root)with size 5%. Number of replication is 5000 .

Table 5(cont.): Power Comparison

(Volatility of Bubble)

Power	τ =0.07			τ =0.09		
(g=0.05 π =0.9)	ADF	Forward ADF Statistic	Rolling ADF Statistic	ADF	Forward ADF Statistic	Rolling ADF Statistic
b0=0.5	0.1018	0.2122	0.1628	0.1040	0.2246	0.1530

NOTE: Power is defined as the probability of rejecting the null hypothesis (unit root)with size 5%. Number of replication is 5000 .

B Technical Appendix

1. **The General Arbitrage-free Model**

Step One: Log-linearism:

From equation (4) , $p_t = \ln P_t - \ln P^*$, $r_t = \ln R_t - \ln R^*$.

Let $P^* = R^* = 1$ and $\rho = (1 + \delta + r_t)^{-1}$.

Thus, $p_t = \ln P_t$ and $r_t = \ln R_t$.

$$e^{p_t} = \rho \left[e^{r_t} + E\left(e^{p_{t+1}} | I_t \right) \right]$$

Since

$$e^{p_t} = \sum_{n=0}^{\infty} \frac{p_t^n}{n!} = 1 + p_t + \frac{p_t}{2} + \ldots$$

$$1 + p_t \approx \rho \left[1 + r_t + E_t \left(1 + p_{t+1} | I_t \right) \right]$$

$$p_t \approx (2\rho - 1) + \rho \left[r_t + E\left(p_{t+1} | I_t \right) \right]$$

Step Two:

$$\begin{aligned}
p_t &\approx (2\rho - 1) + \rho \left[r_t + E\left(p_{t+1} | I_t \right) \right] \\
&= (2\rho - 1) + \rho \left\{ r_t + E_t \left[(2\rho - 1) + \rho \left(r_{t+1} + E_{t+1} p_{t+2} \right) \right] \right\} \\
&= (2\rho - 1) + \rho (2\rho - 1) + \rho r_t + \rho^2 E\left(r_{t+1} | I_t \right) + \rho^2 E\left(p_{t+2} | I_t \right) \\
&= (2\rho - 1) \left(1 + \rho + \rho^2 + \ldots \right) + \rho r_t + \sum_{i=2}^{\infty} \rho^i E_t r_{t+i-1} + \lim_{i \to \infty} \rho^i E_t p_{t+i}
\end{aligned}$$

Since $|\rho| < 1$,

$$p_t \approx \frac{2\rho - 1}{1 - \rho} + \rho r_t + \sum_{i=2}^{\infty} \rho^i E\left(r_{t+i-1} | I_t \right) + \lim_{i \to \infty} \rho^i E_t p_{t+i}$$

Step Three: Suppose $p_t = p_t^f + b_t$, then,

$$p_t \approx \frac{2\rho - 1}{1 - \rho} + \rho r_t + \sum_{i=2}^{\infty} \rho^i E\left(r_{t+i-1} | I_t\right) + b_t$$

where

$$b_t = \lim_{i \to \infty} \rho^i E_t p_{t+i}$$

$$\begin{aligned}
E_t b_{t+1} &= \lim_{i \to \infty} \rho^i E_t p_{t+i+1} \\
&\approx \lim_{i \to \infty} \rho^i \left(\frac{1}{\rho} E_{t+1} p_{t+i} - E_{t+1} r_{t+i} - \frac{2\rho - 1}{\rho} \right) \\
&= \lim_{i \to \infty} \rho^{i-1} E_{t+1} p_{t+i} - \lim_{i \to \infty} E_{t+1} r_{t+i} - \lim_{i \to \infty} \rho^{i-1} \frac{2\rho - 1}{\rho} \\
&\triangleq \frac{1}{\rho} b_t
\end{aligned}$$

Thus b_t may be modeled by $b_t = \frac{1}{\rho} b_{t-1} + \varepsilon_t$ where $E_{t-1}\varepsilon_t = 0$. Since $\rho^{-1} = 1 + \delta + r_t > 1$, the process is explosive.

Step Four: Cointegration Test

If there is no bubble in the market $b_t = 0$; $p_t = p_t^f$

$$
\begin{aligned}
(1-\rho)\, p_t &= 2\rho - 1 + \rho\,(1-\rho)\, r_t + (1-\rho) \sum_{i=2}^{\infty} \rho^i E_t r_{t+i-1} \\
&= 2\rho - 1 + \rho\,(1-\rho)\, r_t + \sum_{i=2}^{\infty} \rho^i E_t r_{t+i-1} - \sum_{i=2}^{\infty} \rho^{i+1} E_t r_{t+i-1} \\
&= 2\rho - 1 + \rho\,(1-\rho)\, r_t + \sum_{i=2}^{\infty} \rho^i E_t r_{t+i-1} - \sum_{i=3}^{\infty} \rho^i E_t r_{t+i-2} \\
&= 2\rho - 1 + \rho\,(1-\rho)\, r_t + \rho^2 E_t r_{t+1} + \sum_{i=3}^{\infty} \rho^i \left(E_t r_{t+i-1} - E_t r_{t+i-2}\right) \\
&= 2\rho - 1 + \rho\,(1-\rho)\, r_t + \rho^2 E_t r_{t+1} + \sum_{i=3}^{\infty} \rho^i \triangle E_t r_{t+i-1} \\
&= 2\rho - 1 + \rho\,(1-\rho)\, r_t + \rho^2 E_t r_{t+1} - \rho^{2i} \triangle E_t r_{t+1} + \sum_{i=3}^{\infty} \rho^i \triangle E_t r_{t+i-1} \\
&= 2\rho - 1 + \rho r_t - \rho^2 r_t + \rho^2 E_t r_{t+1} - \rho^2 E_t r_{t+1} + \rho^2 r_t + \sum_{i=3}^{\infty} \rho^i \triangle E_t r_{t+i} \\
&= 2\rho - 1 + \rho r + \sum_{i=2}^{\infty} \rho^i \triangle E_t r_{t+i}
\end{aligned}
$$

Thus,

$$
(1-\rho)\, p_t - \rho r_t = 2\rho - 1 + \sum_{i=2}^{\infty} \rho^i \triangle E_t r_{t+i}
$$

2. **Market Fundamental with $AR(1)$ Rental Process:**

Since $p_t^f = \frac{2\rho-1}{1-\rho} + \rho r_t + \sum_{i=2}^{\infty} \rho^i E_t r_{t+i}$

We assume $r_t = u + r_{t-1} + \varepsilon_t$, that is

$$
E_t r_{t+j} = ju + r_t.
$$

Therefore,

$$
\begin{aligned}
p_t^f &= \frac{2\rho-1}{1-\rho} + \rho r_t + \lim_{j\to\infty} \left[\rho^2 (2u + r_t) + \rho^3 (3u + r_t) + \ldots + \rho^j (ju + r_t)\right] \\
&= \frac{2\rho-1}{1-\rho} + \rho r_t + \lim_{j\to\infty} \left[u \left(2\rho^2 + 3\rho^3 + \ldots + j\rho^j\right) + r_t \left(\rho^2 + \rho^3 + \ldots + \rho^j\right)\right]
\end{aligned}
$$

Let $S = \lim_{j\to\infty}\left(2\rho^2 + 3\rho^3 + \ldots + j\rho^j\right),$

$$
\begin{aligned}
S\rho &= \lim_{j\to\infty}\left(2\rho^3 + 3\rho^4 + \ldots + j\rho^{j+1}\right) \\
S\left(1-\rho\right) &= \lim_{j\to\infty}\left(\rho^2 + \rho^2 + \rho^3 + \ldots + \rho^j - j\rho^{j+1}\right) \\
S\left(1-\rho\right) &= \rho^2 + \frac{\rho^2}{1-\rho} \\
S &= \frac{\rho^2}{1-\rho} + \frac{\rho^2}{\left(1-\rho\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
p_t^f &= \frac{2\rho-1}{1-\rho} + \rho r_t + \lim_{j\to\infty}\left[u\left(2\rho^2 + 3\rho^3 + \ldots + j\rho^j\right) + r_t\left(\rho^2 + \rho^3 + \ldots + \rho^j\right)\right] \\
&= \frac{2\rho-1}{1-\rho} + \rho r_t + \left[\frac{\rho^2}{1-\rho} + \frac{\rho^2}{\left(1-\rho\right)^2}\right]u + r_t\frac{\rho^2}{1-\rho} \\
&= \frac{2\rho-1}{1-\rho} + \frac{\rho r_t}{1-\rho} + \frac{\rho^2\left(2-\rho\right)}{\left(1-\rho\right)^2}u
\end{aligned}
$$

C Appendix: Non-bubble conditions

Various formal definition of rational bubbles exist in the literature. Stiglitz (1990) gives the basic intuition of a bubble, which is people's high expectation of future price. Dixit and Pindyck(1994) analysis the characteristic of bubbles theoretically. Here we try to formalize Stiglitz's intuition via a well defined model.

1. Assumptions

We assume that there are three status in both housing market and financial market: market with bubbles, fundamental market and market after bubbles collapsed.

There is a possibility of switching among those three status. Suppose market begin at t_0 with high price and the probability of bubbles does not burst at time t_0 is

$$p_1(t) = e^{-b(t-t_0)}$$

That is $p(t+\tau)/p(t) = e^{-b\tau}$, which is independent of t. In other words, If the price is high at some time, the probability that it will still be high τ time later is $e^{-b\tau}$ regardless of how long the high price has lasted. It implies that there is no need to keep track of how long price has been high.

An equivalent way to describe the process of bubble burst: it occurs with probability b per unit time, which means that the hazard rate for bubble burst is b. The probability that a high house price ends in next dt unit of time approaches bdt as dt approaches 0.

$$p_1'(t) = -bp_1(t)$$

Bubbles are characterized by rapid expansion followed by a contraction. When market starts from market fundamental state, the possibility of bubble emerging is s per unit time. Alternatively, the probability of successfully maintaining at market fundamental state at each period is

$$p_2(t) = e^{-s(t-t_0)}$$

Meanwhile, There is no reason for fundamental state to collapse. Therefore, the probability for bubble emerging is

$$1 - e^{-s(t-t_0)}$$

In general, bubbles will damage the economy greatly and a great contraction will follow after bubble collapsing. Taking housing market as an example, in the presence of bubbles, the demand of housing are lifted up significantly due to speculative investment. Accordingly, without certain guidance of government or a completely mature market, exceed supply will be provided by the market and such kind of hidden abnormal demand will not show up until bubble burst. The exceed supply will drive the asset price below market fundamental when bubble collapsed. The larger the bubble size is , the greater the depression will be.

At bubble collapsing state, regardless of government activating policies or market auto-adjustment function, the possibility of market recovery always exists. We assume that the probability of market recovery, from bubble collapsing state to market fundamental state, is α per unit time, which can be interpreted in the same way of bubble collapsing probability. Furthermore, there is no jump within these states. It

implies that collapsed market need to be recovered to fundamental state before bubbles re-emerging.

The transitivity probability can be summarized in the following table:

Status	B	F	C
B	$(1-b)$	–	b
F	s	$(1-s)$	–
C	–	a	$(1-a)$

where B represents the state of in the presence of bubbles, F is market fundamental state, C is the state after bubbles collapsing.

2. Dynamic Programming

Under the given probability, investors decide to invest or not, which is the determinant factor of bubble emergence. To choose investor's best strategy under certain probability, we adopt dynamic program by looking at a brief interval of time and using V_i to summarize what occurs after the end of the interval, which is the continuous value of asset.

Suppose asset price at bubbles state at time t_0 is p_B, p_f is the price for market fundamental state and price after bubble collapsing is p_c. The time interval of length $\triangle t$ goes to zero; V_i denotes the expected value of being in state i at $t + \triangle t$; r is the expected average revenue rate of assets.

The value of asset at time 0 is evaluated based on its expected future revenue. According to our analysis above, bubble asset will retain its value at period t with $e^{-b\triangle t}$ probability, and with $1 - e^{-b\triangle t}$ probability, investors suffer a lost from bubble

collapsing. That is:

$$V_B(0) = \int_0^{\triangle t} e^{-bt}.e^{-rt}.p_B.dt + e^{-r\triangle t}.E\left[V_B(\triangle t)\right] \tag{1}$$
$$= \int_0^{\triangle t} e^{-bt}.e^{-rt}.p_B.dt + e^{-r\triangle t}\left[e^{-b\triangle t}.V_B(\triangle t) + \left(1 - e^{-b\triangle t}\right).V_c(\triangle t)\right]$$

For the asset with market fundamental value, its expected revenue depends on the probability of bubble re-emerging $e^{-s\triangle t}$:

$$V_f(0) = \int_0^{\triangle t} e^{-rt}.\left(1 - e^{-st}\right).p_f.dt + e^{-r\triangle t}.E\left[V_f(\triangle t)\right] \tag{2}$$
$$= \int_0^{\triangle t} e^{-rt}.\left(1 - e^{-st}\right).p_f.dt + e^{-r\triangle t}\left[e^{-s\triangle t}.V_B(\triangle t) + \left(1 - e^{-s\triangle t}\right).V_f(\triangle t)\right]$$

Market recovery is the hope of people who holds asset at a state of bubble collapsing. At each period, we assume that the probability of market recovery is $e^{-a\triangle t}$. Therefore, the expected revenue is a weighting average of market fundamental asset value and bubble collapsing asset value.

$$V_c(0) = \int_0^{\triangle t} e^{-rt}.\left(1 - e^{at}\right).p_c.dt + e^{-r\triangle t}.E\left[V_c(\triangle t)\right] \tag{3}$$
$$= \int_0^{\triangle t} e^{-rt}.\left(1 - e^{at}\right).p_c.dt + e^{-r\triangle t}.\left[e^{-a\triangle t}.V_f(\triangle t) + \left(1 - e^{-a\triangle t}\right).V_c(\triangle t)\right]$$

Solutions for those three equations are as following (Appendix II):

$$r.V_B = p_B + b\left(V_c - V_B\right) \tag{4}$$

$$r.V_f = p_f + s\left(V_B - V_f\right) \tag{5}$$

$$r.V_c = p_c + a\left(V_f - V_c\right) \tag{6}$$

3. Non-bubble condition

Investment decision is made based on the expected value of assets. Speculative investment only occur when the expected value of bubble asset is higher than market fundamental state. In other word, if the expected value of bubble asset is lower than the asset at market fundamental state, bubbles will die out. Therefore, non-bubble condition is:

$$V_B \leq V_f \tag{7}$$

The threshold condition is the relationship of asset prices when the expected revenue of bubble asset equals to the expected value of fundamental asset. That is when the value of bubble asset equals to the value of market fundamental asset

$$V_B = V_f$$

Combining equation (4) and (5),

$$p_B + b\left(V_c - V_B\right) = p_f + s\left(V_B - V_f\right)$$
$$V_B - V_c = \frac{1}{b}\left(p_B - p_f\right)$$

It implies that the loss of investing in bubble asset depends on the size of bubbles in term of asset price $p_B - p_f$ and the probability of bubble collapsing b.

From (4) and (6),

$$r\left(V_B - V_c\right) = p_B - p_c - (a+b)\left(V_B - V_c\right)$$
$$V_B - V_c = \frac{1}{a+b+r}\left(p_B - p_c\right)$$

This equation emphasizes the importance of price level after bubble collapsing and the probability of market recovery to possible loss of investing in bubble asset.

Combine above two equations, we can get

$$\frac{1}{b}\left(p_B - p_f\right) = \frac{1}{a+b+r}\left(p_B - p_c\right)$$
$$p_B = p_f + \frac{b}{a+r}\left(p_f - p_c\right)$$

Thus, the non-bubble condition can be re-written as

$$\frac{p_B - p_f}{p_f - p_c} \leq \frac{b}{a+r} \tag{8}$$

As we can see, the numerator is the gain from holding a bubble asset currently if the bubble does not collapse; while, the denominator is the loss when the bubble burst. It suggests that when the gain-loss ratio is smaller than a certain critical value, which is $\frac{b}{a+r}$ in the above equation, the bubble can not longer exist. The critical value increases when b become larger; while, a and r are inversely related to the critical value.

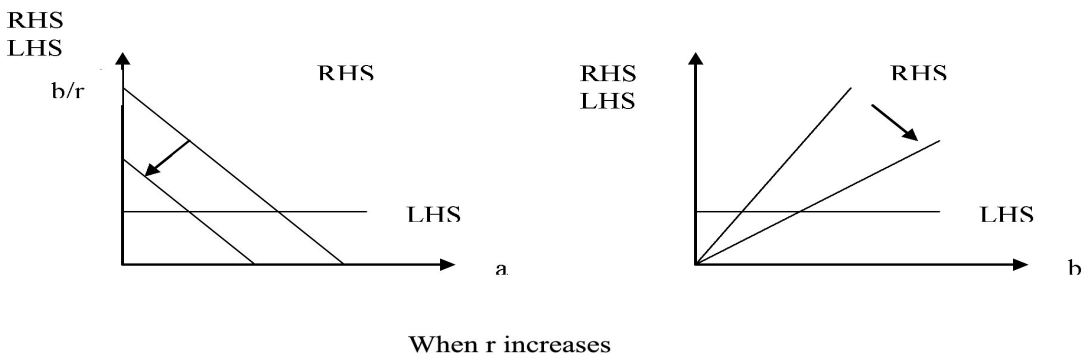
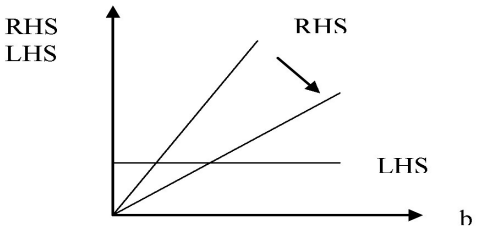
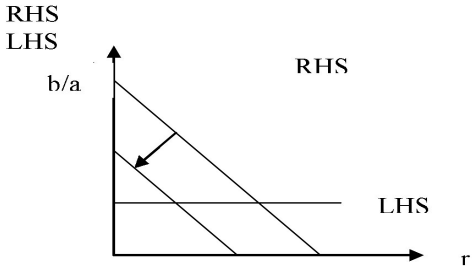


Figure 1(a) **Figure 1(b)**

From Figure 1, RHS line shifts down in both graphs when r increases. Therefore, range where LHS (gain-lost ratio) is smaller than RHS (critical value) become smaller after the shifting. It implies that bubbles are more likely to exist in the market. Intu-

itively, people are more tend to venture when the expected average revenue is higher.

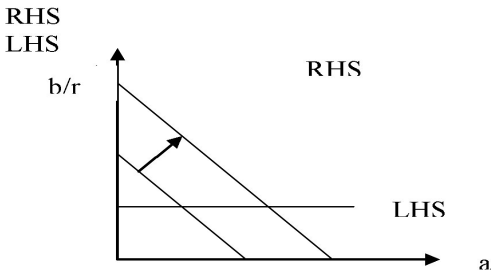


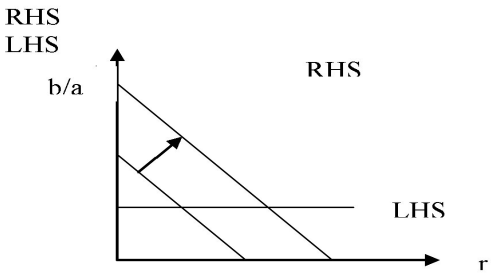


When a increases

Figure 2(a) **Figure 2(b)**

Figure 2 shows the shift-up effect of increasing a in b-graph and r-graph. It suggests that bubbles are more likely to present when the probability of market recovery a increases. When people are confident on either government's recovery policy or market's auto-adjustment function, they are more tempt to invest on bubble asset.





When b increases

Figure 3(a) **Figure 3(b)**

The possibility of bubble collapsing is the resistent force for speculative investment. Fixed market recovery probability and the expected revenue, people will hesitate to invest with higher risk, which is presented by higher expected possibility of bubble collapsing (Figure 3).

In addition, the probability of emerging of bubbles s is uncorrelated with the probability of bubble existence.

To sum up, people's expectation is the determinant factor of bubbles. Higher expected market revenue, and over-confident over the market will cause the emerging of bubbles. Meanwhile, it can also be a possible channel for economic governors to control bubbles.