8-2017

On random assignment problems

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PHD DISSERTATION

On Random Assignment Problems

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July 20, 2017

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Submitted to School of Economics in partial fulfillment of
the requirements for the Degree of Doctor of Philosophy in Economics

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Abstract

This dissertation studies the standard random assignment problem (Bogomolnaia and Moulin (2001)) and investigates the scope of designing a desirable random assignment rule. Specifically, I ask the following two questions:

1. Is there a reasonably restricted domain of preferences on which there exists an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule?

2. Moreover, if the answer is in the affirmative, what is that rule?

As a starting point, attention is restricted to the connected domains (Monjardet (2009)). It is shown that if a connected domain admits a desirable random assignment rule, it is structured in a specific way: a tier structure is fixed such that each tier contains at most two objects and every admissible preference respects this tier structure. A domain structured in this way is called a *restricted tier domain*. In addition, on such a domain, the probabilistic serial (PS) rule is characterized by either *sd-efficiency* and *sd-envy-freeness* or *sd-strategy-proofness*, *sd-efficiency*, and *equal-treatment-of-equals*.

Since these domains are too restricted, it becomes an important question whether we can find some unconnected domains on which desirable rules exist. To facilitate such an investigation, the adjacency notion in Monjardet (2009) is weakened to block-adjacency, which refers to a flip between two adjacent blocks. Hence block-connectedness can be defined accordingly. Block-connected domains include connected domains as well as many unconnected ones. A sufficient condition called "path-nestedness" is proposed for the equivalence between *sd-strategy-proofness* and the *local sd-strategy-proofness* on a block-connected domain, called the *block-adjacent sd-strategy-proofness*.

Next, a class of domains, *sequentially dichotomous domains*, is proposed. A partition of the object set is called a direct refinement of another partition if from the latter to the former, exactly one block breaks into two and all the other blocks are inherited. Then a sequence of partitions is called a partition-path if it starts from the coarsest partition, ends at the finest partition, and along the sequence every partition is a direct refinement of its previous one. Hence a partition-path plots a way of differentiating objects by dichotomous

divisions. Fix a partition-path, the corresponding *sequentially dichotomous domain* is the collection of preferences that respect all the partitions along the partition-path.

Every such domain satisfies path-nestedness and hence the PS rule is shown to be *sd-strategy-proof* by verifying *block-adjacent sd-strategy-proofness*. In addition, every such domain is maximal for the PS rule to be *sd-strategy-proof*. Hence *sequentially dichotomous domains* significantly expand the scope of designing a desirable rule beyond what is indicated by the *restricted tier domains*.

The last part of this dissertation investigates realistic preference restrictions, which are modeled as follows. Each object can be evaluated according to a large set of characteristics. The planner chooses a subset of these characteristics and a ranking of them. Then she describes each object as a list according to the chosen characteristics and their ranking. Being informed of such a description, each agent's preference that is assumed to be lexicographically separable with respect to the ranking proposed by the planner. Hence a description induces a collection of admissible preferences.

It is shown that, under two technical assumptions, whenever a description induces a preference domain which admits an *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule, it is a binary tree, i.e., for each feasible combination of the top-t characteristic values, the following-up characteristic takes two feasible values. In addition, whenever a description is a binary tree, the PS rule is *sd-strategy-proof* on the induced preference domain. In order to show *sd-strategy-proofness* of the PS rule on the domain induced by a binary tree, the domain is shown to be contained by a *sequentially dichotomous domain* and then the result stating the *sd-strategy-proofness* of the PS rule on *sequentially dichotomous domains* is invoked.

Contents

Acknowledgment

I want to express my deepest gratitude to Prof. Shurojit Chatterji, who led me to the career in research and provided excellent guidance throughout my PhD years. Also I'd like to thank Huaxia Zeng who was both a good friend and a good tutor. In addition, I want to thank Prof. William Thomson, Prof. Takashi Kunimoto, and Prof. Jingyi Xue. They all helped a lot in my research. Especially, I was really impressed by their kindness and eagerness to help.

I thank sincerely Sijia and my family, who supported me throughout these years. In addition, I thank Prof. Anthony Tay as the director of my PhD program and two administrative staff, Thor Qiu Ling and Amelia Tan. Last but not the least, I want to thank my friends especially Jiang Liang.

1 Overview

We consider the problem of allocating several indivisible objects to a group of agents, each of whom consumes at most one object. Classical examples include assigning college seats to applicants (Gale and Shapley (1962)), houses to residents (Shapley and Scarf (1974)), and jobs to workers (Hylland and Zeckhauser (1979)).

From a design point of view, a primary target is to allocate the objects efficiently, i.e., no reallocation among the agents strictly improves some agents' welfare without hurting any others. To achieve efficiency is not very difficult if the planner, who has the authority to implement allocations, knows every agent's true preference. However, this is usually not true, i.e., agents' preferences on objects is private information. To cope with this problem, the planner is supposed to design an allocation rule which provides incentives for the agents to truthfully reveal their preferences. This requirement on an allocation rule is summarized by strategy-proofness, which says that, in a revelation game associated to the rule, truth-telling is a weakly dominant strategy for each agent. A strategy-proof rule is attractive not only in that it makes evaluating whether an allocation is efficient possible, but also that reporting the true preference is readily optimal for each agent without worrying about other agents' strategic behavior. In addition, if a rule is not strategy-proof, i.e., a profitable manipulation is possible for some agent, agents will try to find such a manipulation which is usually costly. So taking the potential cost of manipulation into account, strategy-proofness can be interpreted also as a way of achieving efficiency.

The literature has introduced various classes of efficient and strategy-proof rules, e.g., serial dictatorship rules (Svensson (1999)), hierarchical exchange rules (Pápai (2000)), restricted endowment inheritance rules (Ehlers et al. (2002)) and trading cycles rules (Pycia and Ünver (2017)).

However, none of these rules satisfies any fairness requirement.[1] For instance, two

[1]Fairness in many realistic assignment problems is a property even more concerned than efficiency. This is because of the fact that the objects to be allocated are collectively owned, for example, public houses. Also fairness is concern maybe because of legal issues. For example, Abdulkadiroglu and Sönmez (2003) noted that in USA, when the parents think the allocation of school seats are ”unfair”, they can sue the local government.

agents reporting the same strict preference always receive distinct objects and hence are never treated equally. Consequently, one of them must envy the other. Instead of allocating deterministic objects, the literature has resorted to random assignment rules that assign to each agent a lottery on objects to restore ex ante fairness.[2] Thus, agents representing the same preference may receive the same lottery, and the random assignment rule satisfies a classic fairness axiom: *equal treatment of equals*.

Since ordinal preferences on deterministic objects are collected to establish the random assignment, one needs to extend agents' preferences over deterministic objects to assess lotteries. A standard practice is to adopt the stochastic dominance extension.[3] A lottery is viewed at least as good as another if the former (first-order) stochastically dominates the latter according to the ordinal preference over objects. Equivalently, under the von-Neumann-Morgenstern hypothesis, a lottery (first-order) stochastically dominates another one if and only if it delivers an expected utility weakly higher than the expected utility delivered by the other lottery for every cardinal utility representing her ordinal preference on objects. By adopting the stochastic dominance extension, ex ante efficiency and strategy-proofness are defined and referred to as *sd-efficiency* and *sd-strategy-proofness*.[4] Beyond *equal treatment of equals*, ex ante fairness, in random rules can be strengthened by *sd-envy-freeness* which requires that each agent always prefers her own lottery to any others.

There are essentially two random assignment rules in the literature: the Random Serial Dictatorship (or RSD) rule (Abdulkadiroğlu and Sönmez (1998)) and the Probabilistic Serial (or PS) rule (Bogomolnaia and Moulin (2001)). In deterministic assignment models, serial dictatorship rules are known to be (ex-post) strategy-proof and efficient (Svensson (1999)). As a uniform randomization among all serial dictatorship rules, the RSD rule treats equals equally and inherits ex ante incentive property, i.e., *sd-strategy-proofness* from ex-post strategy-proofness of serial dictatorship rules. However, the RSD rule fails

[2]Another strand of the literature resorts to the possibility of side payment, for example Miyagawa (2001). However, side payment is not allowed in many realistic applications either morally or legally.

[3]For other preference extensions, please refer to Cho (2012) and Aziz et al. (2014).

[4]Henceforth, we add prefix "sd-" to emphasize that the corresponding axiom is established with respect to the stochastic dominance extension.

sd-efficiency, for which Abdulkadiroğlu and Sönmez (2003) and Kesten (2009) provide excellent explanations.

The PS rule is initially introduced by Crès and Moulin (2001) to deal with the scheduling problem and later introduced to the standard random assignment problem by Bogomolnaia and Moulin (2001). The PS rule is fundamentally different from the RSD rule as it specifies directly a random assignment for each preference profile, rather than using a mixture of some deterministic assignments to determine the random assignment. The PS rule treats the objects as infinitely divisible and agents consume the objects as time flows. When time starts, each agent consumes her favorite object at the uniform speed, until some object(s) are exhausted. Then agents reformulate their preferences by removing the exhausted object(s), and resume consuming their favorite object in the remaining ones at the uniform speed. This procedure proceeds until all the objects are exhausted. Finally, the share of an object consumed by an agent is interpreted as the probability she receives this object. The axiomatic performance of the PS rule is very different from the RSD rule. It is *sd-efficient* and *sd-envy-free* since at each point in time each agent is consuming her favorite available object. However, the major drawback of the PS rule is that it is manipulable, i.e., not *sd-strategy-proof*. This happens because the consumption procedure is sensitive to unilateral deviations, which will be elaborated by Example 1 in chapter 3.

From the above discussion of the RSD rule and the PS rule, there seems to be a fundamental conflict between *sd-strategy-proofness* and *sd-efficiency*. Such a conflict is formally established in the following impossibility result.

Proposition 1 (Bogomolnaia and Moulin (2001)) *There exists no sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule on the universal domain.*[5]

Recently, this impossibility has been established on some restricted preference domains, e.g., single-peaked domains and single-dipped domains by Kasajima (2013), Chang and

[5]The universal domain is referred to as the collection of all strict preferences. Throughout this paper, we assume that the preference is strict. This impossibility is established for the situations where the number of agents is the same as the number of objects, which is also a standard assumption we make. In addition, this number is supposed to be at least four since when there are less than four objects and agents the RSD is also sd-efficient.

Chun (2016) and Altuntaş (2016).

These results raise two natural questions and also the central questions addressed in this dissertation:

1. Is there a reasonably restricted domain of preferences on which there exists an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule?

2. Moreover, if the answer is in the affirmative, what is that rule?

We start our research by introducing a class of restricted domains of preferences: *restricted tier domains*. We show that, on a *restricted tier domain*, a rule is *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* (or *sd-efficient* and *sd-envy-free*) if and only if it is the PS rule. More importantly, we prove that a *restricted tier structure* is necessary for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* (or *equal-treatment-of-equals*) rule, provided that the domain satisfies a mild richness condition: connectedness (Monjardet (2009)). These results are collected in Chapter 3, titled as random assignments on preference domains with a tier structure.[6]

To construct a *restricted tier domain*, objects are first partitioned into several tiers each of which contains one or two objects; and all preferences are required to respect a common ranking of these tiers while the relative rankings of objects within a tier may vary arbitrarily. Such a common ranking of 1-or-2-object tiers is referred to as a restricted tier structure. As an example, consider a skyscraper with two apartments on each floor. A restricted tier structure can be generated according to floors (for instance, from the top down to the bottom), i.e., all agents prefer higher apartments to lower ones. Between two apartments on the same floor, however, the preferences may be arbitrary across agents. For another example, consider a road from the downtown to the suburb along which houses of similar quality are located on both sides. A restricted tier structure can be generated according to the distance away from the downtown.

Theorem 1 shows that a rule on a *restricted tier domain* is *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* if and only if it is the PS rule. Recall that the PS rule is manipulable on the universal domain since the lotteries prescribed by the PS

[6] All results in Chapter 3 are co-authored with Huaxia Zeng.

rule are sensitive to unilateral deviations. Intuitively, the restricted tier structure embedded in a *restricted tier domain* reduces such sensitivity, and therefore restores appropriate incentive properties on the PS rule.[7] At every preference profile of a *restricted tier domain*, according to the PS rule, all agents first equally share each tier of object(s), and moreover, within a tier with two objects, say a and b, each agent in the (weak) majority group (e.g., the group of agents with cardinality $l \geqslant \frac{n}{2}$ who prefer a to b, provided that n is the total number of agents) consumes $\frac{1}{l}$ of her preferred object a and obtains $\frac{2}{n} - \frac{1}{l}$ of her less preferred object b, while each agent in the minority group (i.e., the complementary group of the (weak) majority group) merely consumes $\frac{2}{n}$ of her preferred object b. Consequently, any individual preference misrepresentation does not affect the manipulator's share on each tier, and cannot increase the consumption of her sincerely preferred object in each 2-object tier. Therefore, we restore *sd-strategy-proofness* of the PS rule on a *restricted tier domain*. Moreover, in the verification of this characterization, we find that *sd-envy-freeness* is endogenized in an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule, and essentially *sd-efficiency* and *sd-envy-freeness* pin down all random assignments induced to the PS rule. Therefore, the PS rule is also uniquely characterized by *sd-efficiency* and *sd-envy-freeness* on a *restricted tier domain*. This result is presented as the Corollary 1 of the Theorem 1.

As the restricted tier structure helps to restore *sd-strategy-proofness* of the PS rule, it provides one particular sufficient condition for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule. More importantly, we characterize *restricted tier domains* for the existence of such an admissible rule. We restrict attention to the class of connected domains which have been widely studied in the voting literature (e.g., Monjardet (2009), Sato (2013), Chatterji et al. (2013) and Chatterji et al. (2016)), and recently have been adopted for characterizing random assignment rules in Cho (2012) and Cho (2016a). A pair of preferences is said to be adjacent if they are identical up to a switch of two consecutively ranked objects. Given a domain, a undirected graph is constructed such that the vertex set is the preferences in the domain and

[7] Kojima and Manea (2010) restrict such sensitivity by increasing the copies of objects, and hence restore *sd-strategy-proofness* of the PS rule.

an edge is drawn between two adjacent preferences. Correspondingly, a domain is said to be connected if this graph is connected. Theorem 2 proves that if a connected domain admits an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* rule, it must be a *restricted tier domain*. This axiomatically justifies the necessity of the restricted tier structure, and clearly specifies a boundary between the impossibility and possibility for designing desirable strategy-proof random assignment rules. Furthermore, when we weaken the fairness axiom from *sd-envy-freeness* to *equal treatment of equals*, the domain characterized is, surprisingly, not enlarged at all (see Theorem 3).

In addition to the above-mentioned domain and rule characterizations, we extend our preference restriction, i.e., restricted tier structures, to a generalized model where each agent has an outside option, and the number of agents may differ from the number of objects. This domain strictly nests the one investigated by Bogomolnaia and Moulin (2002), where a scheduling problem is addressed. Imagine that there is a public service center which can serve one agent in each slot, each agent wants to be served in a earlier slot. In addition, each agent has a deadline above which receiving serve slot is perceived to be of no value. The major difficulty is that these individual deadlines are private information, hence the planner needs a schedule rule that incentivizes the agents to report their true deadlines and hence design an efficient service plan. Treating service slots as objects and each agent has an outside option, each agent has a common preference on slots, i.e., earlier is better, and the only difference among agents is how do they rank their outside options. On such a restricted domain, Bogomolnaia and Moulin (2002) characterize the PS rule as the unique one that satisfies either *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals* or *sd-efficiency* and *sd-envy-freeness*. Such characterizations strongly suggest the use of the PS rule in scheduling problems.

On a *restricted tier domain* with outside options, called an *augmented restricted tier domain*, we also establish these two characterizations of the PS rule. This generalization suggests the use of the PS rule in a scheduling problem where there might be slight perturbation of rankings of service slots. For example, each day has two service slots, one in the morning and the other in the afternoon, and every agent agrees on that being served in an earlier day is better, however some agents prefer being served in the morning while some afternoon.

The results in Chapter 3 may be interpreted as both negative and positive. On the one hand, in some realistic situations, for example the house allocation in a skyscraper or along a road, the restricted tier structure seems to be an appropriate assumption. Then our characterization of the PS rule supports its application in these situations. On the other hand, a *restricted tier domain* is restrictive, and does not give much freedom for agents to spell their preferences. More importantly, we identify the *restricted tier domain* as a boundary for the compatibility of these canonical axioms under connectedness.

In other words, in order to find domains restricted in a more reasonable way, we must go beyond connected ones. However, this is not an easy task because of mainly two reasons. The first is that when we go beyond connected domains, we do not have any domain structure such as the adjacency notion for connected domains. The second is that, verifying *sd-strategy-proofness* is usually difficult, which is even more difficult on unconnected domains. Hence it helps a lot if we have a domain structure which facilitates the analysis on unconnected domains and simplifies the verification of the *sd-strategy-proofness*.

To do this, we weaken the adjacency notion of Monjardet (2009) so that two preferences are block-adjacent if they are different only in a flip of two adjacent blocks.[8] Particularly, the relative rankings within two blocks flipped remain unchanged. Accordingly, we say a domain is block-connected if between two arbitrary preferences we can arrange a sequence of admissible preferences such that every two contiguous preferences are block-adjacent. Hence, the class of block-connected domains contains the connected domains as well as many unconnected domains.

On a block-connected domain, we propose a sufficient condition between *sd-strategy-proofness* and a weaker notion, called *block-adjacent sd-strategy-proofness*, which requires that reporting the true preference always delivers a lottery that stochastically dominates the lottery induced by any preference that is block-adjacent to the sincere one. Hence, given a block-connected domain, an arbitrarily fixed random assignment rule is *sd-strategy-proof* as long as it is *block-adjacent sd-atrategy-proof*.

[8] A block in this dissertation refers to a subset of objects.

Sato (2013) defines local strategy-proofness on a connected domain which requires that reporting the true preference always delivers an alternative that is as least as good as the alternative induced by any preference that is adjacent to the sincere one. In addition, he shows that on a connected domain, if strategy-proofness is equivalent to local strategy-proofness, then the domain satisfies a structural condition called non-restoration. Afterwards, Cho (2016a) considers incentives for random mechanisms and verifies the non-restoration as a sufficient condition for the equivalence between *local sd-strategy-proofness* and *sd-strategy-proofness*.

The sufficient condition we propose for the equivalence between *block-adjacent sd-strategy-proofness* and *sd-strategy-proofness* on block-connected domains is called path-nestedness. Pick an arbitrary pair of preferences, it is possible to arrange a sequence between them such that each contiguous pair is block-adjacent and satisfying the following structural condition. Let A_1, A_2 be two blocks flipped between two contiguous preferences and A_3, A_4 be two blocks flipped between two contiguous preferences ranked latter along the sequence, path-nestedness requires either $A_3 \cup A_4 \subset A_1$ or $A_3 \cup A_4 \subset A_2$ or $(A_3 \cup A_4) \cap (A_1 \cup A_2) = \emptyset$.

The equivalence result in chapter 4 and it is supposed to be useful either when the interesting domain is block-connected but not connected or when the interesting domain is connected but violates non-restoration. We illustrate the latter with Example 5. As to the former, we use path-nestedness to show that on a quite flexible class of preference domains, each of which is block-connected and quite large in size, the PS rule is *sd-strategy-proof*. These domains are called *sequentially dichotomous domains* and presented in chapter 5.

Recall that, in the literature there are essentially two random assignment rules: random priority rule (Abdulkadiroğlu and Sönmez (1998)) and the probabilistic serial rule (Bogomolnaia and Moulin (2001)). The random priority rule is the uniform randomization on serial dictatorship rules. It is *sd-strategy-proof* and treating equals equally, but not *sd-efficient*. Quite a few papers are devoted to understanding why the random priority rule is *sd-inefficient* and under what conditions it becomes *sd-efficient*, including Abdulkadiroğlu and Sönmez (2003), Kesten (2009), Manea (2008), and Manea (2009).

The second rule, the probabilistic serial rule (PS), is *sd-efficient* and treating equals equally, but not *sd-strategy-proof*. The issue of why the PS rule is not *sd-strategy-proof* and under what conditions it becomes *sd-strategy-proof* is largely neglected. The only paper in this line, as far as my knowledge, is Kojima and Manea (2010). However they are essentially dealing with the so-called ”large assignment problems”, that is each object has sufficiently many copies. For the baseline model discussed in the beginning, we still don't know much. To fill this gap, we propose, in chapter 5, a class of preference domains and shows that on such domains the PS rule is *sd-strategy-proof*. We call them *sequentially dichotomous domains*. In addition, each of these domains is shown to be maximal for the PS rule to be *sd-strategy-proof*, i.e., not a single preference can be added into a *sequentially dichotomous domain* while the *sd-strategy-proofness* of the PS rule is preserved.

The results associated with *sequentially dichotomous domains* contribute to the literature from another point of view. The literature of random assignment comes with many impossibilities. It starts from Bogomolnaia and Moulin (2001) who proved that there is no acceptable rule on the universal domain. This impossibility was then strengthened to the single-peaked domain by Kasajima (2013), and then to the single-peaked domain when all preferences have a common peak by Chang and Chun (2016). In light of these impossibilities, one might be pessimistic about the prospect of finding a reasonably restricted preference domain on which there is an acceptable rule. In addition, normatively speaking, the characterization result with respect to the *restricted tier domains* should be treated largely as an impossibility. In this line of the literature, the results in chapter 5 are probably the first possibility results.

To define a *sequentially dichotomous domain*, we need some preliminary definitions. We say a partition on the object set is a direct refinement of another if from the latter to the former there is exactly one block that breaks into two smaller blocks and all the other blocks are inherited. Then we say a sequence of partitions is a path if it starts from the coarsest partition, ends at the finest partition, and along the sequence each partition is a direct refinement of the previous one. In other words, a path plots a way to differentiate objects by *sequentially dichotomous divisions*. Given a partition and a preference, we say this preference respects this partition if, for every pair of blocks in this partition, every object in one block is better than every object in the other. Then a collection of preferences is said to be a *sequentially dichotomous domain* if we can find a path such that a preference is included if and only if it respects every partition along the path.

It is worth noting that a *sequentially dichotomous domain* satisfies minimal richness, i.e., every object is found the top of some admissible preference. In addition, the size of a *sequentially dichotomous domain* is quite large. Given the number of objects being n, a *sequentially dichotomous domain* contains 2^{n-1} preferences, exactly the same size as the single-peaked domain.

Surprisingly, *sequentially dichotomous domains* have shown up in the literature on ”Condorcet domain.” A preference domain is a Condorcet domain if majority rule does not generate Condorcet cycles. Classical papers in this literature include Black (1948), Black et al. (1958), Abello (1981), Fishburn (1997), Fishburn (2002), and so on. An excellent survey is by Monjardet (2009).

It turns out that each *sequentially dichotomous domain* is a maximal Condorcet domain. In addition, the size of a *sequentially dichotomous domain* is the largest in a class of maximal Condorcet domains. This class is called the symmetric domains, requiring that whenever a preference is admissible, its total reversal is also admissible.

The structure of *sequentially dichotomous domains* has already been found in Danilov and Koshevoy (2013), who describe such a structure in a much more abstract manner from the view point of operations research.

As discussed in chapter 3 and 5, on some domains it is impossible to design a desired random assignment rule, examples include Bogomolnaia and Moulin (2001), Kasajima (2013), Chang and Chun (2016), and the results presented in chapter 3, while it is sometimes possible to find a rule satisfying desired properties, an example is the PS rule on *sequentially dichotomous domains*.

In theory, we simply assume what preferences are admissible and a larger domain is preferred. It is interesting to ask, in reality, how should we model preference restrictions, if any, and what is its implication on the scope of designing acceptable rules. We address this issue in the final chapter, i.e. chapter 6.

Facing these questions, the first observation is that, in reality, the way people figure out their preferences on the objects is fundamentally affected by the way these objects are described. Since in reality, when people are required to submit their preferences on the objects, the information they have is usually a description of the objects. Take house allocation as an example, people are usually required to express their preferences on houses before they can really live in the houses and consume the housing service. Rather, the information they have for them to figure out their preferences is usually a description of the houses, which is usually provided by the authority who organize the allocation. Take the working task allocation as another example, it is impossible that workers express their preferences on tasks after they have experienced every task. Rather, the information they have for them to figure out their preferences is usually a description of the tasks, provided by the manager or their team leader.

Hence let's take a closer look on the descriptions of objects in reality and the way descriptions affect people's formulation of their preferences on the objects. A frequently seen description in reality is in the following format:

Table 1: A typical description of objects in reality

House / Characteristic	#1	#2	#3	#4	#5	#6
1. Type (no. of rooms)	2-Room	2-Room	3-Room	3-Room	4-Room	5-Room
2. floor area (approximately in m^2)	30	35	60	65	90	110
3. no. of bedrooms	1	2	3	3	4	3
4. no. of bathrooms	1	2	2	2	3	2

- Source: Website of Housing Development Board of Singapore. See Flat Type.

That is, each object is described as a combination of various characteristic values. Notice that the characteristics presented are deliberately chosen from a much larger set of characteristics. Essentially a house can be evaluated from much more dimensions, for example how long does it take to the nearest subway station, which floor the house is on, how many public primary schools in the community, etc. In addition, even the characteristics chosen here can be expressed in different ways. For example, the floor area may be expressed as a series of binary choices: is the area larger than 30, 40, 50, and so on.

The reason we need to take such a close look on the descriptions of objects is as we mentioned, different descriptions induce people to formulate their preferences in different ways. Observing that objects are described as combinations of various characteristic values, we make the central behavioral assumption in this paper that *people figure out their preferences on objects that are lexicographically separable with respect to the given description.* As illustrated by the above table, a typical resident tends to compare a pair of houses first by their type and then floor area if they are of the same type and then number of bedrooms if they have the same values of both type and area, and so on.

Here are two arguments that support this behavioral assumption. First, in reality sometimes people are required to submit their marginal preferences immediately when they are informed of the characteristic values in a sequential way. For example, a frequently observed practice is as follows. First they choose a specific type on a webpage and then they are redirected to another webpage where they choose a floor area from a collection of admissible choices which depend on which type they have chosen. After they have chosen a floor area, they are led to a third page to choose a number of bedrooms and so on until the end of the characteristic list.

Second, the set of feasible combinations of characteristic values is usually very sparse relative to the whole Cartesian product. For example, in Table 1, the whole Cartesian product has $4 \times 6 \times 4 \times 3 = 288$ elements. However, the feasible set has only 6 combinations. In other words, the characteristics are heavily interdependent. Such interdependence makes our behavioral assumption not that restrictive as it appears. For example, according to Table 1, knowing that a house with more rooms is always having a larger area, there is not much difference in whether a person compares two houses according to first the number of rooms and then area or first the area and then the number of rooms.

According to the behavioral assumption, by choosing a specific description of the objects, i.e., a subset of characteristics and a ranking of them, the planner actually imposes a preference restriction. In chapter 6, we model the situation as follows. The set of available objects is a subset of the Cartesian product of a finite set of characteristics. The planner chooses a specific description, i.e., a subset of the characteristics and a ranking of them. Facing the description provided by the planner, the agents report lexicographically separable preferences. Then the planner assign a lottery to each agent according to a prescribed rule.

Within this setting, we investigate how choices of descriptions affect the scope for designing an acceptable rule. Specifically we ask the following three questions one by one.

1. Given an arbitrary object set, does each possible description induces a preference domain on which an acceptable rule exists?

2. If the answer to the above question is negative, what characterizes a good description so that an acceptable rule exists on the induced domain?

3. Given a good description, what allocation rule should we use?

Let us examine the questions with a specific object set illustrated in Table 2. That is, each object can be evaluated according to each of three characteristics, $c = \{1, 2\}$ $c' = \{a, b, c\}$ and $c'' = \{x, y\}$.

Table 2: An object set

Characteristic \ Object	o_1	o_2	o_3	o_4	o_5
c	1	1	1	2	2
c'	a	b	c	a	b
c''	x	y	y	x	y

The answer to the first question is obviously negative. Consider the description such that c and c' are chosen and c is ranked above c', illustrated by Table 3.

Table 3: A bad description

Characteristic \ Object	o_1	o_2	o_3	o_4	o_5
1: c	1	1	1	2	2
2: c'	a	b	c	a	b

Then according to the behavioral assumption, the following three preferences are admissible.

$$\begin{array}{ccccccccc} o_1 & \succ & o_3 & \succ & o_2 & \succ & o_4 & \succ & o_5 \\ o_1 & \succ' & o_2 & \succ' & o_3 & \succ' & o_4 & \succ' & o_5 \\ o_2 & \succ'' & o_1 & \succ'' & o_3 & \succ'' & o_4 & \succ'' & o_5 \end{array}$$

This domain exhibits the elevating structure and hence implies nonexistence of an acceptable rule. (See definition 2 and Theorem 3.)

From the description illustrated by Table 3, it's evident that a necessary condition for a description to induce a domain admitting an acceptable rule is that the last characteristic can not take more than two feasible values, conditional on the previous characteristics.

A simple way to meet this necessary condition is to reverse the ranking of c and c', as illustrated by Table 4.

Table 4: Another bad description

Characteristic \ Object	o_1	o_4	o_2	o_5	o_3
1: c'	a	a	b	b	c
2: c	1	2	1	2	1

Now the induced domain will not exhibit the structure illustrated by $\succ$, $\succ'$, and $\succ''$ above. Does this domain admits an acceptable rule?

To answer the question, we strengthen the impossibility with respect to the elevating property in a manner so that, if n satisfies two technical assumptions (Assumptions 1 and 2 in Appendix D.1), then the three preferences forming the structure below imply nonexistence of an acceptable rule.

$$\begin{array}{ccccccccc} E & \succ & B & \succ & D & \succ & C & \succ & F \\ E & \succ' & B & \succ' & C & \succ' & D & \succ' & F \\ E & \succ'' & C & \succ'' & B & \succ'' & D & \succ'' & F \end{array}$$

Specifically, let B, C, D be three nonempty blocks of objects and $\succ, \succ', \succ''$ three admissible preferences such that (1) B, C, D take consecutive positions in three preferences, (2) B and C take the first two positions in two preferences, and (3) in the third preference, B

takes the first position and C takes the third position. E is the common upper contour set that can be empty. Notice that the sizes of three blocks are arbitrary and the ranking of objects within each block is allowed to be arbitrary across preferences.[9]

According to this impossibility, we know that the description illustrated by Table 4, i.e., c and c' are chosen and c' is ranked above c, admits no acceptable rule since the induced domain includes three preferences exhibiting the structure that leads to impossibility, as illustrated below.

$$\begin{array}{ccccc}
\{o_1, o_4\} & \succ & \{o_3\} & \succ & \{o_2, o_5\} \\
\{o_1, o_4\} & \succ & \{o_2, o_5\} & \succ & \{o_3\} \\
\{o_2, o_5\} & \succ & \{o_1, o_4\} & \succ & \{o_3\}
\end{array}$$

Hence, according to this impossibility, we have a stronger necessary condition on descriptions that induce domains admitting an acceptable rule. The condition is that, for each feasible combination of values of the top-t ranked characteristics, the following-up characteristic can take at most two feasible values. Whenever this condition is violated, the induced domain include three preferences that exhibit the block elevating structure that leads to impossibility.

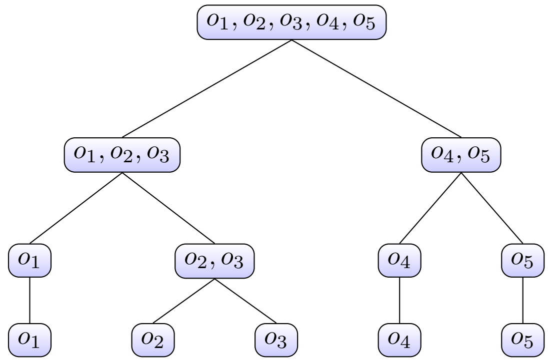
For the object set illustrated in Table 2, a description that satisfies the necessary condition is as follows: all three characteristics are chosen and c ranked the first, c'' the second, and c' the last. This description is illustrated by Table 5 below. It's evident that the necessary condition is satisfied: the first characteristic c takes two values 1 and 2; conditional on c's value the second characteristic c'' takes two values x and y; conditional on a combination of the first two characteristics, the last characteristic c' takes either one value or two values.

[9] We provide the proof subject to two technical assumptions in Appendix D.1. Although we can not prove these two assumptions analytically, we conjecture them to be true. In addition, we provide Matlab code to verify them given specific n and we have verified the assumptions with these codes for all the cases where $n \leqslant 1000$.

Table 5: A good description

Characteristic \ Object	o_1	o_2	o_3	o_4	o_5
1: c	1	1	1	2	2
2: c''	x	y	y	x	y
3: c'	a	b	c	a	b

We call a description satisfying this necessary condition a binary tree. The reason we name it in this way can be seen easily from the following representation of the above table. Particularly, according to the first characteristic c, objects are divided into two subsets, i.e., $\{o_1, o_2, o_3\}$ and $\{o_4, o_5\}$. Then according to the second characteristic, $\{o_1, o_2, o_3\}$ breaks into $\{o_1\}$ and $\{o_2, o_3\}$ and $\{o_4, o_5\}$ breaks into $\{o_4\}$ and o_5. Finally, according to the last characteristic, $\{o_2, o_3\}$ breaks into $\{o_2\}$ and $\{o_3\}$.



However since the impossibility justifies the binary tree as only a necessary condition, we still don't know whether there is an acceptable rule on the domain induced by the description in Table 5. To justify the condition as also a sufficient condition, we verify that the PS rule is sd-strategy-proof on the domain induced by a binary tree. Hence a description induces a preference domain on which there is an acceptable rule if and only if it is a binary tree, subject to two technical assumptions. To show sd-strategy-proofness of the PS rule on the domain induced by a binary tree, we utilize Theorem 6, which shows that the PS rule is sd-strategy-proof on any *sequentially dichotomous domain*. It then suffices to show that as long as the description is a binary tree, the induced domain is covered by a *sequentially dichotomous domain*.

2 Model

This section presents the general model, which can be seen as the common model setting for all chapters. Settings specific to each chapter will be presented chapter-wisely.

Let $A \equiv \{a, b, \dots\}$ be a finite set of objects and $I \equiv \{1, 2, \dots, n\}$, $n \geqslant 4$, a finite set of agents. For most of this dissertation (except for the extension part of Chapter 3), we assume $|A| = |I| = n$. Each agent i is equipped with a complete, transitive and antisymmetric binary relation P_i over A, i.e., a linear order. Let $\mathbb{P}$ denote the set consisting of *all* strict preferences over A. The set of admissible preferences is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the *preference domain*. Thus, $\mathbb{P}$ is referred to as the universal domain. Given $P_i \in \mathbb{D}$ and $a \in A$, let $r_k(P_i)$, $k = 1, \dots, n$, denote the kth ranked object according to P_i, and $B(P_i, a) = \{x \in A | x \;\; P_i \;\; a\}$ denote the (strict) upper contour set of a in P_i. A *preference profile* $P \equiv (P_1, \dots, P_n) \equiv (P_i, P_{-i}) \in \mathbb{D}^n$ is an n-tuple of admissible preferences.

Let $\Delta(A)$ denote the set of lotteries, or probability distributions, over A. Given $\lambda \in \Delta(A)$, λ_a denotes the probability assigned to object a. A (random) **assignment** is a bi-stochastic matrix $L \equiv [L_{ia}]_{i \in I, a \in A}$, namely a non-negative square matrix whose elements in each row and each column sum to unity, i.e., $L_{ia} \geqslant 0$ for all $i \in I$ and $a \in A$, $\sum_{a \in A} L_{ia} = 1$ for all $i \in I$, and $\sum_{i \in I} L_{ia} = 1$ for all $a \in A$. Evidently, in a bi-stochastic matrix L, each row is a lottery, i.e., $L_i \in \Delta(A)$ for all $i \in I$. Let $\mathcal{L}$ denote the set of all bi-stochastic matrices. Agents assess lotteries according to (first-order) stochastic dominance. Given $P_i \in \mathbb{D}$ and lotteries $\lambda, \lambda' \in \Delta(A)$, λ *stochastically dominates* λ' according to P_i, denoted $\lambda \;\; P_i^{sd} \;\; \lambda'$, if $\sum_{l=1}^{k} \lambda_{r_l(P_i)} \geqslant \sum_{l=1}^{k} \lambda'_{r_l(P_i)}$ for all $1 \leqslant k \leqslant n$. Analogously, given $P \in \mathbb{D}^n$, we say an assignment L stochastically dominates L' according to P, denoted $L \;\; P^{sd} \;\; L'$, if $L_i \;\; P_i^{sd} \;\; L'_i$ for all $i \in I$.

A *rule* is a mapping $\varphi : \mathbb{D}^n \to \mathcal{L}$. Given $P \in \mathbb{D}^n$, $\varphi_{ia}(P)$ denotes the probability of agent i receiving object a, and thus $\varphi_i(P)$ denotes the lottery assigned to agent i.

Given $P \in \mathbb{D}^n$, an assignment L is *sd-efficient* if it is not stochastically dominated by any another assignment L', i.e., $[L' \;\; P^{sd} \;\; L] \Rightarrow [L' = L]$. Accordingly, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-efficient** if the assignment $\varphi(P)$ is *sd-efficient* for all $P \in \mathbb{D}^n$.

Next, a rule is *sd-strategy-proof* if for every agent, her lottery under truthtelling always stochastically dominates her lottery induced by any misrepresentation, according to her true preference. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-strategy-proof** if for all $i \in I$, $P_i, P'_i \in \mathbb{D}$, and $P_{-i} \in \mathbb{D}^{n-1}$, $\varphi_i(P_i, P_{-i})\ \ P_i^{sd}\ \ \varphi_i(P'_i, P_{-i})$.

Last, we require that every agent weakly prefer her own lottery to any other's. Given $P \in \mathbb{D}^n$, an assignment L is *sd-envy-free* if $L_i\ \ P_i^{sd}\ \ L_j$ for all $i, j \in I$. Accordingly, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-envy-free** if $\varphi(P)$ is *sd-envy-free* for all $P \in \mathbb{D}^n$. As a weaker notion of fairness, we say that an assignment $L \in \mathcal{L}$ satisfies *equal treatment of equals* if for all $i, j \in I$, $[P_i = P_j] \Rightarrow [L_i = L_j]$. Similarly, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ satisfies **equal treatment of equals** if $\varphi(P)$ satisfies *equal treatment of equals* for all $P \in \mathbb{D}^n$. Evidently, *sd-envy-freeness* implies *equal treatment of equals*.

3 Random Assignments on Preference Domains with a Tier Structure

In this chapter we answer the central questions by proposing a class of connected domains: *restricted tier domains*. As a simpler starting point, we ask, in stead of the original question, what preference restriction restores the *sd-strategy-proofness* of the PS rule? As long as we find such a preference restriction, it can serve as a candidate answer to the original question. We start our investigation by presenting a heuristic example.

Example 1 Let $A = \{a, b, c, d\}$. Let $P \equiv (P_{-4}, P_4)$ and $P' \equiv (P_{-4}, P'_4)$ be two preference profiles below which specify an possible manipulation of agent 4. In the PS rule, the corresponding consumption procedures are depicted in Figure 1.

$$P = \begin{pmatrix} P_1 : a \succ c \succ b \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4 : b \succ a \succ c \succ d \end{pmatrix} \qquad P' = \begin{pmatrix} P_1 : a \succ c \succ b \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P'_4 : a \succ b \succ c \succ d \end{pmatrix}$$

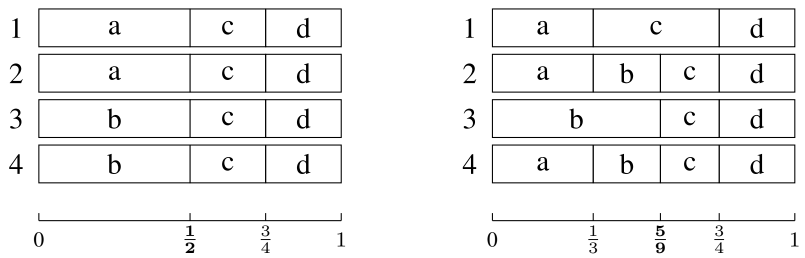


Figure 1: Consumption procedures under P and P' in the PS rule

Observe that $\varphi_{4a}(P) + \varphi_{4b}(P) = \frac{1}{2} < \frac{5}{9} = \varphi_{4a}(P') + \varphi_{4b}(P')$. Thus, agent 4 can profitably manipulate at profile P via P'_4. This indicates that the PS rule is vulnerable to small manipulations like P_4 and P'_4 which differ on the relative rankings of exactly one pair of objects. Note that in profile P, each of a and b is most preferred by two agents. Therefore, as shown in Figure 1, objects a and b are exhausted simultaneously at time $\frac{1}{2}$, and all agents turn to objects in $\{c, d\}$ at the same time. However, in profile P',

a is top ranked in the preferences of agents 1, 2 and 4, and therefore is exhausted in a shorter time: $\frac{1}{3}$. This indicates that agent 3, who prefers b the most, only consumes $\frac{1}{3}$ of b while all others exhaust a. Furthermore, since c is the second best in agent 1's preference while a and b occupy the top two positions in all others' preferences, after time $\frac{1}{3}$, agent 1 starts to consume c while agents 2, 3 and 4 are going to equally share the rest of object b. Consequently, agent 4 obtains $\frac{2}{9}$ of b, and therefore has $\frac{5}{9}$ of a and b combined which is more desirable than that under profile P.

This manipulation is made possible by the following two facts. First, according to the PS rule, agents are myopic and greedy: every agent consumes her favorite object among what are not exhausted at every time point. Second, more specifically, agent 1's preference differs to others in both profiles P and P' in the sense that c is ranked in between a and b in P_1 while all others rank both a and b above c.

Observe that in preferences P_1, P_2 and P_3, object b occupies three distinct ranking positions, and more specifically, is elevated successively from the third position in P_1 to the second in P_2, and to the top in P_3. Now, we impose an additional restriction on all agents' preferences to avoid such 3-position elevating phenomenon: both a and b occupy the top two positions, and c and d obtain the other two positions. Thus, preference P_1 is no longer admissible, and more importantly, all preferences preserve a common tier structure: both a and b are ranked above c and d. Accordingly, let $\bar{P} \equiv (\bar{P}_1, P_2, P_3, P_4)$ and $\bar{P}' \equiv (\bar{P}_1, P_2, P_3, P'_4)$ where for instance, $\bar{P}_1 = P_2$. The consumption procedure at $\bar{P}$ (specified in Figure 2 below) remains identical to that in Figure 1, while the consumption procedure at $\bar{P}'$ becomes significantly simpler than that at profile P', and is depicted in Figure 2 below.

$$\bar{P} = \begin{pmatrix} \bar{P}_1 : a \succ b \succ c \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4 : b \succ a \succ c \succ d \end{pmatrix} \qquad \bar{P}' = \begin{pmatrix} \bar{P}_1 : a \succ b \succ c \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P'_4 : a \succ b \succ c \succ d \end{pmatrix}$$

Now, the manipulation of agent 4 at $\bar{P}$ via P'_4 is non-profitable, i.e., the lottery assigned to agent 4 at $\bar{P}$ first-order stochastically dominates that at $\bar{P}'$ according to her sincere preference P_4. First, it is evident that agent 4 obtains identical shares of objects c and d across profiles $\bar{P}$ and $\bar{P}'$. Second, more importantly, due to the common tier structure,

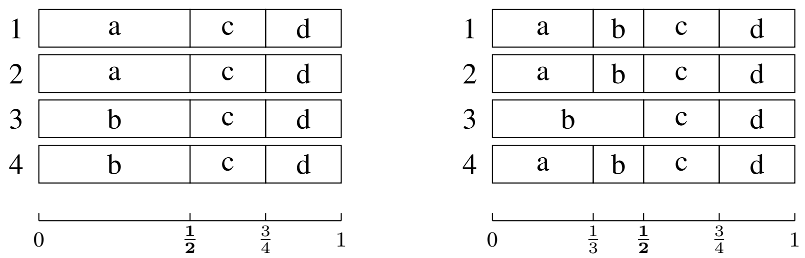


Figure 2: Consumption procedures under $\bar{P}$ and $\bar{P}'$ in the PS rule

the combined share of a and b assigned to agent 4 at $\bar{P}'$ is fixed to $\frac{1}{2}$ which is identical to that at profile $\bar{P}$. Last, the switch of a and b in $\bar{P}_1$ and $\bar{P}_1'$ makes agent 4 worse off as she consume less of b at $\bar{P}'$ than that at $\bar{P}$, i.e., agent 4 gets $\frac{1}{6}$ of b at $\bar{P}'$, and $\frac{1}{2}$ at $\bar{P}$. $\square$

3.1 Restricted tier domains: A possibility result

Now, we formally establish the preference restriction introduced in Example 1: all object are partitioned into tiers; each tier consists of one or two objects; and all admissible preferences respect a common ranking of tiers.

Let $\mathcal{P} \equiv (A_k)_{k=1}^T$ denote a **tier structure**, i.e., (i) tier $A_k \subseteq A$ is not empty, $k = 1, \dots, T$, (ii) $A_k \cap A_{k'} = \emptyset$ for all $k \neq k'$, (iii) $\cup_{k=1}^T A_k = A$. According to an arbitrary tier structure, we have a tier domain where the relative rankings over tiers in every preference are identical. Moreover, we impose an additional restriction: every tier contains *at most* two objects, and then construct a *restricted tier domain*.

Definition 1 *A domain $\mathbb{D}$ is a **restricted tier domain** if there exists a restricted tier structure $\mathcal{P} \equiv (A_k)_{k=1}^T$ such that*

1. *For all $1 \leqslant k \leqslant T$, $|A_k| \leqslant 2$;*

2. *Given $P_i \in \mathbb{D}$ and $a, b \in A$, $[a \in A_k, b \in A_{k'}$ and $k < k'] \Rightarrow [a \;\; P_i \;\; b]$.*

Let $\mathbb{D}(\mathcal{P})$ denote the restricted tier domain containing all admissible preferences.

Remark 1 Given a tier structure where some tier contains more than two objects, let $\mathbb{D}$ be the tier domain containing *all* admissible preferences. Then, there are three preferences analogous to P_1, P_2 and P_3 in Example 1, and consequently, by a similar argument in Example 1, the PS rule fails *sd-strategy-proofness*.

Remark 2 In an auction model, Bikhchandani et al. (2006) study a particular class of tiered domains, named "order-based domains" where all (quasi linear) cardinal preferences they examine induce an identical ordinal preference on objects at each payment level. More recently, tiered domains are examined in two-sided matching (Akahoshi (2014) and Kandori et al. (2010)), school choice (Kesten (2010) and Kesten and Kurino (2013)), and spectrum license auctions (Serizawa and Zhou (2016)).

Remark 3 Let $\mathcal{P} \equiv (A_k)_{k=1}^{T}$ be a tier structure with $|A_k| = 2$ for all $1 \leqslant k \leqslant T$. The cardinality of the *restricted tier domain* $\mathbb{D}(\mathcal{P})$ is 2^T.

On a *restricted tier domain*, we can escape from the impossibility in Proposition 1 by restoring *sd-strategy-proofness* of the PS rule. Moreover, Theorem 1 below shows that the PS rule is the unique one on a *restricted tier domain* satisfying *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals*.

Theorem 1 *On a restricted tier domain, a rule is sd-strategy-proof, sd-efficient and equal-treatment-of-equals if and only if it is the PS rule.*

Proof: Given $\mathcal{P} \equiv (A_k)_{k=1}^{T}$, let $\mathbb{D} \subseteq \mathbb{D}(\mathcal{P})$ be a *restricted tier domain*.

Due to the restricted tier structure embedded in $\mathbb{D}$, at each preference profile, we can clearly specify the random assignment induced by the PS rule as shown in Fact 1 below.

Fact 1 *Given $P \in \mathbb{D}^n$, let L be the random assignment induced by the PS rule. Then, the following two conditions: for each $1 \leqslant k \leqslant T$,*

1. $L_{iA_k} \equiv \sum_{x \in A_k} L_{ix} = \frac{|A_k|}{n}$ *for all* $i \in I$.
2. *Assume* $A_k = \{a, b\}$. *Let* $I_k \equiv \{i \in I | a \;\; P_i \;\; b\}$ *and* $l \equiv |I_k|$.

   (i) *If* $\frac{n}{2} \leqslant l \leqslant n$, *then*

   - $L_{ia} = \frac{1}{l}$ *and* $L_{ib} = \frac{2}{n} - \frac{1}{l}$ *for all* $i \in I_k$;

– $L_{ja} = 0$ *and* $L_{jb} = \frac{2}{n}$ *for all* $j \in I \backslash I_k$.

(ii) *If* $0 \leqslant l < \frac{n}{2}$, *then*

– $L_{ia} = \frac{2}{n}$ *and* $L_{ib} = 0$ *for all* $i \in I_k$;

– $L_{ja} = \frac{2}{n} - \frac{1}{n-l}$ *and* $L_{jb} = \frac{1}{n-l}$ *for all* $j \in I \backslash I_k$.

The verification of Fact 1 is routine, and we hence omit it. We first intuitively explain two conditions in Fact 1. In the assignment L , all agents first equally share every tier. Next, in a particular tier with two objects, say $A_k = \{a, b\}$, the set of agents who prefer a to b, i.e., $I_k \equiv \{i \in I | a \ \ P_i \ \ b\}$, is either a (weak) majority, i.e., $\frac{n}{2} \leqslant |I_k| \leqslant n$, or a (strict) minority, i.e., $0 \leqslant |I_k| < \frac{n}{2}$. If I_k is a (weak) majority, then all agents in I_k share a equally and exclusively, and hence each receives the share $\frac{1}{|I_k|}$ of a; $I \backslash I_k$ only consume b, and each of them receives the share $\frac{2}{n}$ of b. Moreover, all agents in I_k split what remains of b, and hence each obtains the share $\frac{1-(n-|I_k|)\times \frac{2}{n}}{|I_k|} = \frac{2}{n} - \frac{1}{|I_k|}$ of b. If I_k is a (strict) minority, then $I \backslash I_k$ is a (strict) majority, i.e., $\frac{n}{2} < |I \backslash I_k| \leqslant n$, and objects a and b are shared in an opposite symmetric way.

It is evident that the PS rule is always *sd-efficient* and *equal-treatment-of-equals*. We verify that the PS rule is *sd-strategy-proof* on $\mathbb{D}$. Given $i \in I$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, let L and L' be two random assignments induced by the PS rule at profile $P \equiv (P_i, P_{-i})$ and $P' \equiv (P_i', P_{-i})$ respectively. We show $L_i \ P_i^{sd} \ L_i'$.

According to condition 1 above, we know that for every $1 \leqslant k \leqslant T$, $\sum_{t=1}^{k} L_{iA_t} = \sum_{t=1}^{k} L'_{iA_t}$. Therefore, to complete the verification, it suffices to show that given $1 \leqslant k \leqslant T$, assuming $A_k = \{a, b\}$ and $a \ P_i \ b$, we have $L_{ia} \geqslant L'_{ia}$. If $a \ P_i' \ b$, condition 2 above implies $L_{ia} = L'_{ia}$. Next, assume $b \ P_i' \ a$. Let l be the number of agents who prefer a to b at P, i.e., $l \equiv |\{j \in I | a \ \ P_j \ \ b\}|$. Thus, $1 \leqslant l \leqslant n$ and the number of agents who prefer a to b at P' must be $l - 1$. If $\frac{n}{2} < l \leqslant n$, condition 2(i) implies $L_{ia} = \frac{1}{l} > 0 = L'_{ia}$. If $1 \leqslant l \leqslant \frac{n}{2}$, condition 2(i) (if $l = \frac{n}{2}$) or condition 2(ii) (if $1 \leqslant l < \frac{n}{2}$) implies $L_{ia} = \frac{2}{n}$. Moreover, since $L'_{ia} \leqslant \frac{2}{n}$ by condition 1, we have $L_{ia} \geqslant L'_{ia}$. Therefore, $L_{ia} \geqslant L'_{ia}$ as required and hence $L_i \ P_i^{sd} \ L_i'$. In conclusion, the PS rule is *sd-strategy-proof* on domain $\mathbb{D}$. This completes the verification of the sufficiency part of Theorem 1.

Henceforth, we prove the necessity part of Theorem 1. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ a rule which satisfies all three axioms. Fix $P \equiv (P_1, \dots, P_n) \in \mathbb{D}^n$ and $L \equiv \varphi(P)$ for the verifications below. Specifically, we show that L satisfies conditions 1 and 2 of Fact 1.

Lemma 1 *For all $k \in \{1, \dots, T\}$ and $i \in I$, $L_{iA_k} = \frac{|A_k|}{n}$.*

The proof of Lemma 1 is in Appendix A.1.

Thus, random assignment L satisfies condition 1 of Fact 1.

Lemma 2 *Given $1 \leqslant k \leqslant T$, assume $A_k = \{a, b\}$ and let $I_k = \{i \in I | a \quad P_i \quad b\}$. The following statements hold.*

(i) *For all $i, j \in I_k$, $L_{ia} = L_{ja}$.*

(ii) *For all $i \in I_k$ and $j \in I \backslash I_k$, $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$.*

The proof of Lemma 2 is in Appendix A.2.

Lemma 3 *Random assignment L satisfies sd-envy-freeness.*

Proof: Given $a \in A$, assume $a \in A_k$. Given $i \in I$, assume $a = r_l(P_i)$. If $A_k = \{a\}$, or $|A_k| = 2$ and $a = \min(P_i, A_k)$, then Lemma 1 implies $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{k} \frac{|A_k|}{n} = \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$. If $|A_k| = 2$ and $a = \max(P_i, A_k)$, then Lemmas 1 and 2 imply $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{k-1} \frac{|A_k|}{n} + L_{ia} \geqslant \sum_{t=1}^{k-1} \frac{|A_k|}{n} + L_{ja} = \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$. Therefore, $L_i \quad P_i^{sd} \quad L_j$ for all $j \neq i$. Thus, φ satisfies *sd-envy-freeness*.

Now, given $1 \leqslant k \leqslant T$, assume $A_k = \{a, b\}$, and let $I_k \equiv \{i \in I | a \quad P_i \quad b\}$ and $l \equiv |I_k|$. By *sd-envy-freeness*, we first know that for each pair $i, j \in I_k$, or each pair $i, j \in I \backslash I_k$, $L_{ia} = L_{ja}$ and $L_{ib} = L_{jb}$. Next, by *sd-efficiency* and feasibility, we know that

(i) If $\frac{n}{2} \leqslant l \leqslant n$, then

- $L_{ia} = \frac{1}{l}$ and $L_{ib} = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$;
- $L_{ja} = 0$ and $L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$.

(ii) If $0 \leqslant l < \frac{n}{2}$, then

- $L_{ia} = \frac{2}{n}$ and $L_{ib} = 0$ for all $i \in I_k$;

- $L_{ja} = \frac{2}{n} - \frac{1}{n-l}$ and $L_{jb} = \frac{1}{n-l}$ for all $j \in I \backslash I_k$.

Thus, random assignment L satisfies condition 2 of Fact 1. Therefore, L is induced by the PS rule. This completes the verification of the necessity part of Theorem 1. ■

According to the verification of Theorem 1, on a *restricted tier domain*, we also characterize the PS rule under *sd-efficiency* and *sd-envy-freeness*.

Corollary 1 *On a restricted tier domain, a rule is sd-efficient and sd-envy-free if and only if it is the PS rule.*

Proof: The sufficiency part holds evidently. We focus on the necessity part. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be an *sd-efficient* and *sd-envy-free* rule. Fixing $P \equiv (P_1, \ldots, P_n) \in \mathbb{D}^n$, let $L \equiv \varphi(P)$. Fix $1 \leqslant k \leqslant T$. First, *sd-envy-freeness* implies that $L_{iA_k} = L_{jA_k}$ for all $i, j \in I$. Hence, feasibility implies that $L_{iA_k} = \frac{|A_k|}{n}$ for all $i \in I$. Thus, L satisfies condition 1 of Fact 1. Furthermore, in the proof of the necessity part of Theorem 1, note that the verification of Lemma 3 only relies on the application of *sd-efficiency* and *sd-envy-freeness*. Hence, L must also satisfy condition 2 of Fact 1. Therefore, φ is the PS rule. ■

Remark 4 Since every preference profile on a *restricted tier domain* has rich support on a partition (Heo (2014a)) and is recursively decomposable in the sense of Cho (2016b), by invoking either Theorem 1 in Heo (2014a) or Theorem 3 in Cho (2016b), we can also establish Corollary 1. We use the following examples to illustrate. Let $A \equiv \{a, b, c, d\}$, $\mathcal{P} \equiv (A_1, A_2)$ where $A_1 \equiv \{a, b\}$ and $A_2 \equiv \{c, d\}$), and $I \equiv \{1, 2, 3, 4\}$. Consider two preference profiles on the *restricted tier domain* $\mathbb{D}(\mathcal{P})$ specified below.

$$P = \begin{pmatrix} P_1 : a \succ b \succ c \succ d \\ P_2 : a \succ b \succ c \succ d \\ P_3 : b \succ a \succ c \succ d \\ P_4 : b \succ a \succ c \succ d \end{pmatrix} \qquad \bar{P} = \begin{pmatrix} \bar{P}_1 : a \succ b \succ c \succ d \\ \bar{P}_2 : a \succ b \succ c \succ d \\ \bar{P}_3 : b \succ a \succ d \succ c \\ \bar{P}_4 : a \succ b \succ d \succ c \end{pmatrix}$$

According to profile P, in the tier A_2, since every agent prefers c to d, we refine the tier structure $\mathcal{P}$ to $\mathcal{P}' \equiv (A_1, A_2^1, A_2^2) \equiv (\{a, b\}, \{c\}, \{d\})$. Thus, profile P has rich support on partition $\mathcal{P}'$, and hence the PS rule is the unique one satisfying *sd-efficiency* and *sd-envy-freeness*.

According to profile $\bar{P}$, we first partition the objects into $A_1 \equiv \{a, b\}$ and $A_2 \equiv \{c, d\}$ and construct a type 1 decomposition in Cho (2016b). Then, *sd-efficiency* and *sd-envy-freeness* requires that each agent should consume $\frac{1}{2}$ of $\{a, b\}$ and $\frac{1}{2}$ of $\{c, d\}$. Next, we partition $A_2 \equiv \{c, d\}$ into $\{c\}$ and $\{d\}$ and agents into $\{1, 2\}$ and $\{3, 4\}$. Thus, we construct a type 2 decomposition in Cho (2016b). Then, by *sd-efficiency* and *sd-envy-freeness*, agents 1 and 2 both consume $\frac{1}{2}$ of c; and agents 3 and 4 both consume $\frac{1}{2}$ of d. Last, we "partition" $\{a, b\}$ into $\{a$, $\frac{1}{2}$ of $b\}$, and $\{\frac{1}{2}$ of $b\}$, and partition agents into $\{1, 2, 4\}$ and $\{3\}$. In this way, we construct a type 3 decomposition in Cho (2016b). Then, *sd-efficiency* and *sd-envy-freeness* implies that agents $1, 2$ and 4 share a equally and each obtains $\frac{1}{6}$ of b, while agent 3 receives $\frac{1}{2}$ of b.

3.1.1 Relation to the literature

In the literature, there are two main strands on the axiomatic characterizations of the PS rule. The first strand focuses on identifying axioms that characterize the PS rule on the universal domain.[10] For instance, Bogomolnaia and Heo (2012) proposes the axiom *bounded invariance*, and characterizes the PS rule along with *sd-efficiency* and *sd-envy-freeness*.[11] Recently, Bogomolnaia (2015) adopts different preference extension approach for lotteries: *lexicographic preference extension* to establish a weaker incentive notion, and then shows that the PS rule is unique for *sd-efficiency*, *sd-envy-freeness* and strategy-

[10]We discuss here only the characterizations in the model where ordinal preferences are strict; each agent receives exactly one object; and each object has one unit. There are also interesting characterizations of the PS rule in other environments, e.g., Heo and Yılmaz (2015) add indifferences in preferences; Heo (2014b) allows each agents to consume more than one object; Liu and Pycia (2011a) increase the copies of each object to infinity; while both infinitely many copies and multiple-unit consumption are allowed in Liu and Pycia (2011b).

[11]Bounded invariance requires that whenever an agent's unilateral deviation does not involve her top k ranked objects, the allocation of each of these k objects remains unchanged. Hashimoto et al. (2014) weaken bounded invariance and characterize the PS rule accordingly.

proof on lexicographic preference extension.[12] Alternatively, Hashimoto et al. (2014) neglect the incentive issue in random assignment rules, and characterize the PS rule with a new axiom *ordinal fairness* which in fact strengthens *sd-efficiency* and *sd-envy-freeness* combined.[13]

In the second strand, restrictions are imposed on either preference domains or preference profiles, and the PS rule is characterized by canonical axioms, i.e., the axioms studied in our paper. Bogomolnaia and Moulin (2002) introduce an assignment model with preference restrictions which can be described by the following realistic application. Consider a public service center which is able to serve only one agent in each time slot. All agents want to be served earlier and differ only on their deadlines of services beyond which the services are perceived of no value. The deadlines are private information of agents, and the planner wants to truthfully elicit them and then schedule an efficient and fair service plan. Then, they characterize the PS rule by either *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals* or *sd-efficiency* and *sd-envy-freeness*. As we mentioned above, the assignment model studied in Bogomolnaia and Moulin (2002) is nested in our generalized model with outside options, and the same characterization results are established. Alternatively, various restrictions are introduced on preference profiles, and the PS assignments are characterized via *sd-efficiency* and *sd-envy-freeness*, e.g., the *full support* requirement in Liu and Pycia (2011a), *rich support on a partition* in Heo (2014a) and *rich preferences* in Cho (2016b).[14]

[12] Fix a preference and two distinct lotteries over objects. We rearrange each lottery according to the preference from the worst object up to the best object. One lottery is evaluated better than the other according to the lexicographic preference extension, if we can find one object which has strictly higher probability in the former lottery than that in the latter one while for any less preferred object, the probabilities in both lotteries are identical. The lexicographic preference extension induce a linear order over all lotteries while stochastic-dominance preference extension only produces a partial order over all lotteries.

[13] Ordinal fairness requires that whenever an agent is assigned an object a with strictly positive probability, the probability of this agent receiving an object better than a is no greater than the probability of any other agent getting an object better than a.

[14] The *full support* requires that in a preference profile, each preference in the universal domain is adopted by some agent. In a preference profile which is *rich support on a partition*, we first observe that all agents preference share a common ranking on a partition of objects where some block of the partition may contain

Our paper lies in the same vein of the second strand. We focus on the incentive property of rules on restricted preference domains (provided that each agent's domain is assumed to be identical), and canonically characterize the PS rule on *restricted tier domains*. More importantly, we axiomatically justify the necessity of our domain restriction for the existence of *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rules. Our characterization result implies that the PS assignments are unique for *sd-efficiency* and *sd-envy-freeness* on profiles of restricted tier preferences.

In the verification of our domain characterization theorems (Theorems 2 and 3), we introduce an important notion called *the elevating property* which is a sufficient domain condition for the incompatibility of *sd-strategy-proofness*, *sd-efficiency* and *sd-envy-freeness* or *equal treatment of equals*. To the best of our knowledge, the *elevating property* covers all existing literature related to the study of impossibility on the existence of *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rules (e.g., Bogomolnaia and Moulin (2001), Kasajima (2013), Chang and Chun (2016) and Altuntaş (2016)). More importantly, in contrast to this literature which proposes some domain conditions and establishes negative results, we formulate the elevating property in a greater more sense so that the avoidance of the *elevating property* becomes a critical and informative condition which then is adopted to characterize *restricted tier domains*.

3.2 Necessity: A characterization of restricted tier domains

We have proposed a class of restricted domains, *restricted tier domains*, which is sufficient for the admission of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* (or *equal-*

more than 2 objects. Moreover, in each block of the partition, the full support requirement holds. In a *rich preference* profile, for each agent j and each object a, we can find an agent k (either $k = j$ or $k \neq j$) who prefers a the most and moreover, after partitioning all objects into two blocks according to agent j's preference: objects better than or identical to a and objects worse than a, we note that agent k also prefers the first block to the second one. Cho (2016b) studies economies with random assignments, and shows that if an economy is able to be decomposed into several feasible sub-economies via his *recursive decomposability* condition, then the PS assignment is the unique one satisfying *sd-efficiency* and *sd-envy-freeness* in the economy. For more detailed relation of our paper to Heo (2014a) and Cho (2016b), please refer to Remark 4.

treatment-of-equals) rule, specifically the PS rule. Despite of the significant restriction and small cardinality of *restricted tier domains* (recall Remark 3), we show in this section that a restricted tier structure is necessary for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* (or *equal-treatment-of-equals*) rule, provided a mild richness condition.

We first introduce the richness condition: *connectedness* (Monjardet (2009)). Two preferences $P_i, P_i' \in \mathbb{D}$ are **adjacent**, denoted $P_i \sim^A P_i'$, if there exist $x, y \in A$ such that

(i) $x = r_k(P_i) = r_{k+1}(P_i')$ and $y = r_{k+1}(P_i) = r_k(P_i')$ for some $1 \leqslant k \leqslant n-1$; and

(ii) $r_l(P_i) = r_l(P_i')$ for all $l \neq k$.

Accordingly, a domain $\mathbb{D}$ is **connected** if for every pair of distinct preferences $P_i, P_i' \in \mathbb{D}$, there exists a sequence of consecutively adjacent preferences (in other words, a *path*) $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i', i.e., $P_i^1 = P_i$, $P_i^t = P_i'$ and $P_i^k \sim^A P_i^{k+1}$, $k = 1, \ldots, t-1$. Intuitively, connectedness implies that the difference of two preferences in the domain can be reconciled via a sequence of local switchings. When a domain is interpreted as a collection of opinions in a society (Puppe (2016)), connectedness implies that the society's opinions are sufficiently dispersed.

Remark 5 The notion of connectedness is introduced in Monjardet (2009) for the study of maximal Condorcet domain. Recently, it has been identified by Sato (2013) as a necessary condition for the equivalence of local and global strategy-proofness in deterministic voting. Note that many well studied domains are connected, including the universal domain (Gibbard (1973) and Satterthwaite (1975)), the single-peaked domain (Moulin (1980) and Demange (1982)), the single-dipped domain (Barberà et al. (2012)), and maximal single-crossing domains (Saporiti (2009) and Carroll (2012)).

We now present the domain characterization result.

Theorem 2 *If a connected domain admits an sd-strategy-proof, sd-efficient and sd-envy-free rule, it is a restricted tier domain.*

Proof: First, note that if $\mathbb{D}$ contains exactly two preferences, by connectedness, it is evident that it is a *restricted tier domain*. Henceforth, we assume that $\mathbb{D}$ contains at least three preferences. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* rule. To prove Theorem 2, we first introduce an important terminology, the *elevating property*.

Definition 2 *A domain satisfies the* ***elevating property*** *if there exist three preferences* $\bar{P}_i, P_i, \hat{P}_i$*, three objects* a, b, c *and a ranking position* $1 \leqslant k \leqslant n-2$ *such that the following three conditions are satisfied.*

1. $a = r_k(\bar{P}_i) = r_k(P_i) = r_{k+1}(\hat{P}_i)$.

2. $b = r_{k+2}(\bar{P}_i) = r_{k+1}(P_i) = r_k(\hat{P}_i)$.

3. $c = r_{k+1}(\bar{P}_i) = r_{k+2}(P_i) = r_{k+2}(\hat{P}_i)$.

4. $B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$.[15]

We use Table 6 below to illustrate the *elevating property*:

$$\begin{array}{lcccccccc}
\text{Ranking:} & & & k & & k+1 & & k+2 & \\
\bar{P}_i: & \underbrace{\cdots\cdots\cdots}_{B(\bar{P}_i,a)} & \succ & a & \succ & c & \succ & b & \succ \cdots\cdots\cdots \\
 & \shortparallel & & & & & & & \\
P_i: & \overbrace{\underbrace{\cdots\cdots\cdots}_{B(P_i,a)}}^{B(P_i,a)} & \succ & a & \succ & b & \succ & c & \succ \cdots\cdots\cdots \\
 & \shortparallel & & & & & & & \\
\hat{P}_i: & \overbrace{\cdots\cdots\cdots}^{B(\hat{P}_i,b)} & \succ & b & \succ & a & \succ & c & \succ \cdots\cdots\cdots
\end{array}$$

Table 6: The elevating property

Recall preferences P_1, P_2 and P_3 in Example 1. Note that objects a, b and c cluster in 3 ranking positions of these preferences; three corresponding upper contour sets are empty (and hence identical); and moreover, object b is elevated from the third ranking position in P_1 to the second in P_2, and then is successively elevated to the top of P_3. (This

[15]Note that within the upper contour set, the relative rankings of objects in three preferences are arbitrary.

is a problem. Either remove this part or change the preference in Example 1.) Many well known voting domains satisfy the *elevating property*.[16] In a contrary, since each object takes at most two positions in *all* preferences of a *restricted tier domain*, it is evident that *restricted tier domains* always violate the elevating property. Lemma 4 below shows that domain $\mathbb{D}$ must violate the elevating property since it is the key for the incompatibility of *sd-strategy-proofness*, *sd-efficiency* and *sd-envy-freeness*.

Lemma 4 *Domain $\mathbb{D}$ violates the elevating property.*

Proof: Suppose that $\mathbb{D}$ satisfy the *elevating property*. Specifically, assume that $\mathbb{D}$ contains three preferences in Table 6. Let $B \equiv B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$ for notational convenience. Thus, $|B| = k - 1$. In the detailed verification below, we consider four particular profiles:

(i) P, where every agent presents preference P_i in Table 6,

(ii) $(\bar{P}_1, P_{-1})$, where agent 1 deviates at P via $\bar{P}_i$ in Table 6,

(iii) $(\hat{P}_2, P_{-2})$, where agent 2 deviates at P via $\hat{P}_i$ in Table 6, and

(iv) $(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})$, where agent 2 deviates at $(\bar{P}_1, P_{-1})$ via $\hat{P}_i$ in Table 6.

First, at all four profiles, *sd-envy-freeness* and feasibility imply that the cumulative probability placed on subset B for each agent is fixed to $\frac{k-1}{n}$ which is identical to that given by the PS rule. Next, at all these four preference profiles, we only focus on the probabilities assigned to objects a and b. We first show that at profiles P, $(\bar{P}_1, P_{-1})$ and $(\hat{P}_2, P_{-2})$, these probabilities induced by φ are the same as those induced by the PS rule. Last, we show that, under $(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})$, *sd-strategy-proofness* implies that agent 2's probability of receiving a is the same as that given by the PS rule, while *sd-efficiency* requires that

[16] See for instance, the universal domain (Gibbard (1973)), the single-peaked domain (Moulin (1980) and Demange (1982)), the single-dipped domain (Barberà et al. (2012)), all the maximal single-crossing domains (Saporiti (2009)), the multi-dimensional single-peaked domain (Barberà et al. (1993)) and the separable domain (Le Breton and Sen (1999)), some linked domains (Aswal et al. (2003)) and some circular domains (Sato (2010)).

the probability of agent 2 getting b is higher than that given by the PS rule. Consequently, every agent other than 1 and 2 envies agent 2.

By *sd-envy-freeness* and feasibility, it is evident that $\sum_{x\in B}\varphi_{ix}(P)=\sum_{x\in B}\varphi_{ix}(\bar{P}_1,P_{-1})=\sum_{x\in B}\varphi_{ix}(\hat{P}_2,P_{-2})=\sum_{x\in B}\varphi_{ix}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})=\frac{k-1}{n}$ for all $i\in I$.

Now, we start with profile P. By *sd-envy-freeness*, $\varphi_{ix}(P)=\frac{1}{n}$ for all $i\in I$ and $x\in A$. Next, we consider profile $(\bar{P}_1,P_{-1})$.

Claim 1: The following two statements hold:

(i) $\varphi_{ia}(\bar{P}_1,P_{-1})=\frac{1}{n}$ for all $i\in I$ by *sd-envy-freeness*.

(ii) $\varphi_{1b}(\bar{P}_1,P_{-1})=0$ by *sd-efficiency*, and $\varphi_{ib}(\bar{P}_1,P_{-1})=\frac{1}{n-1}$ for all $i\neq 1$ by *sd-envy-freeness*.

Next, we consider profile $(\hat{P}_2,P_{-2})$.

Claim 2: The follow two statements hold:

(i) $\varphi_{2a}(\hat{P}_2,P_{-2})=0$ by *sd-efficiency*, and $\varphi_{2b}(\hat{P}_2,P_{-2})=\frac{2}{n}$ by *sd-strategy-proofness* according to $\varphi_2(P)$.

(ii) $\varphi_{ia}(\hat{P}_2,P_{-2})=\frac{1}{n-1}$ for all $i\neq 2$ and $\varphi_{ib}(\hat{P}_2,P_{-2})=\frac{2}{n}-\frac{1}{n-1}$ for all $i\neq 2$ by *sd-envy-freeness* and Claim 2(i).

Last, we consider profile $(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})$.

Claim 3: The following two statements hold:

(i) $\varphi_{1a}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})=\frac{1}{n-1}$ by *sd-strategy-proofness* according to $\varphi_1(\hat{P}_2,P_{-2})$ and Claim 2, and $\varphi_{1b}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})=0$ by *sd-efficiency*.

(ii) $\varphi_{2a}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})=0$ by *sd-efficiency*, and $\varphi_{2b}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})=\frac{1}{n}+\frac{1}{n-1}$ by *sd-strategy-proofness* according to $\varphi_2(\bar{P}_1,P_{-1})$ and Claim 1.

Now, by feasibility and Claim 3 (i) and (ii), we know that for all $i\notin\{1,2\}$,

$$\begin{aligned}\varphi_{ia}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}}) &= \frac{1}{n-2}\Big[1-\sum_{j\in\{1,2\}}\varphi_{ja}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})\Big]=\frac{1}{n-1},\\ \varphi_{ib}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}}) &= \frac{1}{n-2}\Big[1-\sum_{j\in\{1,2\}}\varphi_{jb}(\bar{P}_1,\hat{P}_2,P_{-\{1,2\}})\Big]=\frac{1}{n-2}(1-\frac{1}{n}-\frac{1}{n-1}).\end{aligned}$$

Consequently, between agent 2 and any agent $i \notin \{1, 2\}$, we have $\sum_{x \in B \cup \{a,b\}} \varphi_{2x}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}}) = \frac{k-1}{n} + 0 + (\frac{1}{n} + \frac{1}{n-1}) > \frac{k-1}{n} + \frac{1}{n-1} + \frac{1}{n-2}(1 - \frac{1}{n} - \frac{1}{n-1}) = \sum_{x \in B \cup \{a,b\}} \varphi_{ix}(\bar{P}_1, \hat{P}_2, P_{-\{1,2\}})$. This contradicts *sd-envy-freeness* and we hence completes the verification of Lemma 4.

■

Henceforth, we will use the information of violating the *elevating property* in Lemma 4 to characterize the restricted tier structure in domain $\mathbb{D}$.

Lemma 5 *For every path $\{P_i^k\}_{k=1}^t$, if there exists $1 \leqslant l \leqslant n-1$ such that $r_l(P_i^1) = a$, $r_{l+1}(P_i^1) = b$ and $r_l(P_i^k) = b$ for all $k = 2, \ldots, t$, then $r_{l+1}(P_i^k) = a$ for all $k = 2, \ldots, t$.*

Proof: Given a path $\{P_i^k\}_{k=1}^t$, consider $l = n-1$. Thus, $r_{n-1}(P_i^1) = a$, $r_n(P_i^1) = b$, and $r_{n-1}(P_i^k) = b$ for all $k = 2, \ldots, t$. Suppose $r_n(P_i^2) \equiv c \neq a$. Thus, $cP_i^1 b$ (recall that b is the bottom ranked object) and $bP_i^2 c$. Therefore, the local switching pair in P_i^1 and P_i^2 is b and c. Consequently, it must be the case that $r_{n-1}(P_i^1) = c \neq a$. Contradiction! Therefore, $r_n(P_i^2) = a$. Next, consider P_i^3, and suppose $r_n(P_i^3) \equiv c \neq a$. Thus, $cP_i^2 a$ (recall that a is the bottom ranked object) and $aP_i^3 c$. Therefore, the local switching pair in P_i^2 and P_i^3 is c and a. Consequently, it must be the case that $r_{n-1}(P_i^2) = c \neq b$. Contradiction! Therefore, $r_n(P_i^3) = a$. Applying the same argument along the path, we can show that $r_n(P_i^k) = a$ for all $k = 4, \ldots, t$. We next adopt an induction argument.

Induction Hypothesis: Given $1 \leqslant l \leqslant n-1$, for every path $\{P_i^k\}_{k=1}^t$, if there exists $l < l' \leqslant n-1$ such that $r_{l'}(P_i^1) = a$, $r_{l'+1}(P_i^1) = b$, and $r_{l'}(P_i^k) = b$ for all $k = 2, \ldots, t$, then $r_{l'+1}(P_i^k) = a$ for all $k = 2, \ldots, t$.

Now, given a path $\{P_i^k\}_{k=1}^t$, assume that $r_l(P_i^1) = a$, $r_{l+1}(P_i^1) = b$; and $r_l(P_i^k) = b$ for all $k = 2, \ldots, t$. We will show that $r_{l+1}(P_i^k) = a$ for all $k = 2, \ldots, t$.

Since $P_i^1 \sim^A P_i^2$, it is evident that $r_{l+1}(P_i^2) = a$. Suppose that there exists $2 \leqslant \bar{k} \leqslant t$ such that $r_{l+1}(P_i^{\bar{k}}) \neq a$. Assume $r_{l+1}(P_i^{\bar{k}}) = c$. Evidently, $\bar{k} > 2$ and $c \notin \{a, b\}$. Moreover, we can assume $r_{l+1}(P_i^k) = a$ for all $2 \leqslant k \leqslant \bar{k} - 1$. Since $P_i^{\bar{k}-1} \sim^A P_i^{\bar{k}}$, $r_l(P_i^{\bar{k}-1}) = r_l(P_i^{\bar{k}}) = b$, and $r_{l+1}(P_i^{\bar{k}-1}) = a \neq c = r_{l+1}(P_i^{\bar{k}})$, it must be the case that $r_{l+2}(P_i^{\bar{k}-1}) = c$ and $r_{l+2}(P_i^{\bar{k}}) = a$. Now, consider the path $\{P_i^{\bar{k}}, P_i^{\bar{k}-1}, \ldots, P_i^2\}$. Since $r_{l+1}(P_i^{\bar{k}}) = c$, $r_{l+2}(P_i^{\bar{k}}) = a$ and $r_{l+1}(P_i^k) = a$ for all $k = \bar{k} - 1, \ldots, 2$, induction hypothesis implies $r_{l+2}(P_i^k) = c$ for all $k = \bar{k} - 1, \dots, 2$. Furthermore, since $P_i^1 \sim^A P_i^2$, $r_{l+2}(P_i^1) = r_{l+2}(P_i^2) = c$. Along the sub-path $\{P_i^k\}_{k=1}^{\bar{k}}$, since a, b and c take positions l, $l+1$ and $l+2$ in every preference, it is easy to verify that the sets of top $l-1$ ranked objects are identical for all preferences. Thus, $B(b, P_i^{\bar{k}}) = B(b, P_i^{\bar{k}-1}) = B(a, P_i^1)$. Consequently, preferences $P_i^{\bar{k}}, P_i^{\bar{k}-1}$ and P_i^1 indicates that domain $\mathbb{D}$ satisfies the *elevating property* (see the table below). Contradiction to Lemma 4!

$$
\begin{array}{lccccccccc}
\text{Ranking:} & & & l & & l+1 & & l+2 & & \\
P_i^{\bar{k}}: & \underbrace{\cdots\cdots\cdots}_{B(P_i^{\bar{k}},b)} & \succ & b & \succ & c & \succ & \color{red}{a} & \succ & \cdots\cdots\cdots \\
 & \shortparallel & & & & & & & & \\
P_i^{\bar{k}-1}: & \overbrace{\underbrace{\cdots\cdots\cdots}_{B(P_i^{\bar{k}-1},b)}}^{B(P_i^{\bar{k}-1},b)} & \succ & b & \succ & \color{red}{a} & \succ & c & \succ & \cdots\cdots\cdots \\
 & \shortparallel & & & & & & & & \\
P_i^1: & \overbrace{\cdots\cdots\cdots}^{B(P_i^1,a)} & \succ & \color{red}{a} & \succ & b & \succ & c & \succ & \cdots\cdots\cdots
\end{array}
$$

Therefore, $r_{l+1}(P_i^k) = a$ for all $k = 2, \dots, t$. This completes the verification of induction hypothesis and hence the lemma. ■

Lemma 6 *For every path $\{P_i^k\}_{k=1}^t$, if there exists $1 \leqslant l \leqslant n-1$ such that $r_l(P_i^1) = a$, $r_{l+1}(P_i^1) = b$, and $r_{l+1}(P_i^k) = a$ for all $k = 2, \dots, t$, then $r_l(P_i^k) = b$ for all $k = 2, \dots, t$.*

Proof: The verification of this lemma is symmetric to Lemma 5. The induction argument in the proof of Lemma 5 starts from the bottom (i.e., $l = n-1$) and proceeds successively up to the top (i.e., $l = 1$). To verify this lemma, an analogous induction argument can be adopted from the top (i.e., $l = 1$) down to the bottom (i.e., $l = n-1$). ■

Lemma 7 *Given $P_i, P_i' \in \mathbb{D}$, assume $P_i \sim^A P_i'$, $a = r_l(P_i) = r_{l+1}(P_i')$ and $b = r_{l+1}(P_i) = r_l(P_i')$. In every preference, objects a and b occupy positions l and $l+1$, i.e., $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$ for all $P_j \in \mathbb{D}$.*

Proof: It is evident that $\{r_l(P_i), r_{l+1}(P_i)\} = \{a, b\}$ and $\{r_l(P_i'), r_{l+1}(P_i')\} = \{a, b\}$. Next, fix an arbitrary $P_j \in \mathbb{D} \backslash \{P_i, P_i'\}$, and we show $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$. Since

$\mathbb{D}$ is connected and $P_i \sim^A P_i'$, it is true that there exists a path $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ such that $\{P_i^1, P_i^2\} = \{P_i, P_i'\}$ and $P_i^t = P_j$. We assume $P_i^1 = P_i$ and $P_i^2 = P_i'$. The verification of the situation $P_i^1 = P_i'$ and $P_i^2 = P_i$ is symmetric and we hence omit it.

If $r_l(P_i^k) = b$ for all $k = 3, \ldots, t$, then Lemma 5 implies $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$. Next, we assume that there exists $3 \leqslant k \leqslant t$ such that $r_l(P_i^k) \neq b$. We highlight the subset $\{k_j\}_{j=1}^{\nu} \subseteq \{3, \ldots, t\}$ such that $r_l(P_i^{k_j}) \neq r_l(P_i^{k_j - 1})$, $j = 1, \ldots, \nu$. Since there exists $3 \leqslant k \leqslant t$ such that $r_l(P_i^k) \neq b$, the set of preferences $\{P_i^{k_j}\}_{j=1}^{\nu}$ is not empty. Moreover, we can separate the path $\{P_i^k\}_{k=1}^t$ into $\nu + 1$ parts according to the l-th ranked object in each preference, i.e., $r_l(P_i^2) = \cdots = r_l(P_i^{k_1 - 1})$, $r_l(P_i^{k_1}) = \cdots = r_l(P_i^{k_2 - 1})$, ... , $r_l(P_i^{k_{\nu-1}}) = \cdots = r_l(P_i^{k_\nu - 1})$, and $r_l(P_i^{k_\nu}) = \cdots = r_l(P_i^t)$ (see the table below).

$P_i^2, \ldots, P_i^{k_1-1}$	;	$P_i^{k_1}, \ldots, P_i^{k_2-1}$	;	$P_i^{k_2}, \ldots\ldots, P_i^{k_{\nu-1}-1}$;	$P_i^{k_{\nu-1}}, \ldots, P_i^{k_\nu - 1}$	;	$P_i^{k_\nu}, \ldots, P_i^t$
the same lth ranked alternative		the same lth ranked alternative			the same lth ranked alternative		the same lth ranked alternative

Evidently, $r_l(P_i^2) = \cdots = r_l(P_i^{k_1-1}) = b$. Then, we apply Lemma 5 on the sub-path $\{P_i^1, P_i^2, \ldots P_i^{k_1-1}\}$ and obtain $r_{l+1}(P_i^k) = a$ for all $k = 2, \ldots, k_1 - 1$.

Claim 1: $r_l(P_i^{k_1}) = a$.

Evidently, $r_l(P_i^{k_1}) \neq r_l(P_i^{k_1-1}) = b$. Suppose $r_l(P_i^{\bar{k}}) = c \neq a$. Thus, $c \notin \{a, b\}$. Since $P_i^{k_1-1} \sim^A P_i^{k_1}$ and $r_l(P_i^{k_1-1}) = b \neq c = r_l(P_i^{k_1})$, it must be the case that $r_{l+1}(P_i^{k_1}) = a$. Furthermore, since $r_{l+1}(P_i^{k_1-1}) = r_{l+1}(P_i^{k_1}) = a$, it is true that $r_{l-1}(P_i^{k_1}) = b$ and $r_{l-1}(P_i^{k_1-1}) = c$. Now, we can apply Lemma 6 on the sub-path $\{P_i^{k_1}, P_i^{k_1-1}, \ldots, P_i^2\}$ and obtain $r_{l-1}(P_i^k) = c$ for all $k = k_1 - 1, \ldots, 2$. Moreover, since $P_i^1 \sim^A P_i^2$, $r_{l-1}(P_i^1) = r_{l-1}(P_i^2) = c$. Furthermore, it is easy to verify that the set of top $l - 2$ ranked objects in each preference of the sub-path $\{P_i^k\}_{k=1}^{k_1}$ is identical. Thus, $B(c, P_i^1) = B(c, P_i^{k_1-1}) = B(b, P_i^{k_1})$. Consequently, preferences $P_i^1, P_i^{k_1-1}$ and $P_i^{k_1}$ indicates that domain $\mathbb{D}$ satisfies the *elevating property* (see the table below). Contradiction to Lemma 4!

$$
\begin{array}{rcccccccc}
\text{Ranking:} & & & l-1 & & l & & l+1 & \\
P_i^1: & \underbrace{\cdots\cdots\cdots}_{B(P_i^1,c)} & \succ & c & \succ & a & \succ & b & \succ \cdots\cdots\cdots \\
 & \parallel & & & & & & & \\
P_i^{k_1-1}: & \overbrace{\underbrace{\cdots\cdots\cdots}_{B(P_i^{k_1-1},c)}}^{B(P_i^{k_1-1},c)} & \succ & c & \succ & b & \succ & a & \succ \cdots\cdots\cdots \\
 & \parallel & & & & & & & \\
P_i^{k_1}: & \overbrace{\cdots\cdots\cdots}^{B(P_i^{k_1},b)} & \succ & b & \succ & c & \succ & a & \succ \cdots\cdots\cdots
\end{array}
$$

This completes the verification of the claim.

Now, we know $r_l(P_i^{k_1}) = \cdots = r_l(P_i^{k_2-1}) = a$. Applying Lemma 5 on $\{P_i^{k_1-1}, P_i^{k_1}, \ldots, P_i^{k_2-1}\}$, we have $r_{l+1}(P_i^{k_1}) = \cdots = r_{l+1}(P_i^{k_2-1}) = b$. Along the path $\{P_i^k\}_{k=1}^t$, repeatedly applying the symmetric argument above, we finally have $\{r_l(P_j), r_{l+1}(P_j)\} = \{a, b\}$. ■

Now, we are ready to reveal the *restricted tier structure* in domain $\mathbb{D}$. If there exists $a \in A$ such that $r_1(P_i) = a$ for all $P_i \in \mathbb{D}$, let $A_1 = \{a\}$. If there exist $P_i, P_i' \in \mathbb{D}$ such that $r_1(P_i) \equiv a \neq b \equiv r_1(P_i')$, connectedness implies that there must exist $\bar{P}_i, \bar{P}_i' \in \mathbb{D}$ such that $\bar{P}_i \sim^A \bar{P}_i'$, $r_1(\bar{P}_i) = a$ and $r_1(\bar{P}_i') = b$. Thus, $r_2(\bar{P}_i) = b$ and $r_2(\bar{P}_i') = a$. Then, Lemma 7 implies $\{r_1(P_i), r_2(P_i)\} = \{a, b\}$ for all $P_i \in \mathbb{D}$. Then, let $A_1 = \{a, b\}$.

Assume $|A_1| = l$ (either $l = 1$ or $l = 2$). If there exists $x \in A$ such that $r_{l+1}(P_i) = x$ for all $P_i \in \mathbb{D}$, let $A_2 = \{x\}$. Next, assume that there exist $P_i, P_i' \in \mathbb{D}$ such that $r_{l+1}(P_i) \equiv x \neq y \equiv r_{l+1}(P_i')$. Since the set of top l ranked objects in every preference is identical, connectedness implies that there must exist $\bar{P}_i, \bar{P}_i' \in \mathbb{D}$ such that $\bar{P}_i \sim^A \bar{P}_i'$, $r_{l+1}(\bar{P}_i) = x$ and $r_{l+1}(\bar{P}_i') = y$. Thus, $r_{l+2}(\bar{P}_i) = y$ and $r_{l+2}(\bar{P}_i') = x$. Then, Lemma 7 implies $\{r_{l+1}(P_i), r_{l+2}(P_i)\} = \{x, y\}$ for all $P_i \in \mathbb{D}$. Then, let $A_2 = \{x, y\}$.

Applying the symmetric argument repeatedly, since A is finite, we can generate tiers $A_1, A_2, \ldots, A_T$ such that (i) $A_k \cap A_{k'} = \emptyset$ for all $1 \leqslant k < k' \leqslant T$ and $\cup_{k=1}^T A_k = A$, (ii) $1 \leqslant |A_k| \leqslant 2$ for all $1 \leqslant k \leqslant T$, and (iii) for all $1 \leqslant k < k' \leqslant T$, $[a \in A_k$ and $b \in A_{k'}] \Rightarrow [aP_ib$ for all $P_i \in \mathbb{D}]$. In conclusion, domain $\mathbb{D}$ is a *restricted tier domain*. ■

We now weaken *sd-envy-freeness* to *equal treatment of equals*, and investigate the connected domains which admit an *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule. Surprisingly, such weakening does not expand the characterized domains, i.e., they are still *restricted tier domains*.

Theorem 3 *If a connected domain admits an sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule, it is a restricted tier domain.*

Lemmas 5 - 7 remain valid for the proof of Theorem 3. However, the proof of Lemma 4 becomes significantly complicated as we weaken *sd-envy-freeness* to *equal treatment of equals*. Therefore, we relegate the proof of Theorem 3 to the Appendix A.3.

As *restricted tier domains* are characterized in Theorems 2 and 3 under different fairness axioms, it suggests that the source of restriction power that pins down the *restricted tier domains* arises mainly from the resolution of the conflict between *sd-strategy-proofness* and *sd-efficiency* under the *elevating property*. Since the weakening of fairness axiom in Theorem 3 does not expand the characterized domain in Theorem 2 and more importantly the proof of Theorem 2 is significantly simpler and conveys the central logic of the proof of Theorem 3, we believe that Theorem 2 is of special interest and present it in the first place.

Our proof of Theorem 3 uses the proof strategy introduced by Chang and Chun (2016) in their impossibility which says that there is no *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule on a domain that includes three particular preferences such that one object takes the last three ranking positions respectively and all the other objects are identically ranked in these three preferences. Their preference structure is actually a special case of our *elevating property*! In the Appendix, we establish the impossibility of the existence of an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule on a domain satisfying the *elevating property* which hence generalizes the impossibility theorem in Chang and Chun (2016). We believe that our generalization is significant since it allows first the elevated object to take arbitrary three consecutive positions; second the other objects to be arbitrarily ranked as long as the truncation sets up to the elevating positions in three preferences are the same; and more importantly, our impossibility result under the *elevating property* appears to be informative and is repeatedly referred to for establishing Theorem 3. Last, our proof slightly improves theirs in logical conciseness and fluency (see for instance, footnote **22**).

We conclude this section by emphasizing insightful light shed by our domain characterization results on the direction of identifying a unified necessary and sufficient condition for the existence of an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule. When we encounter with a preference domain which fails connectedness but admits an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* or *equal-treatment-of-equals* rule, we first partition the domain into several connected subdomains. Thus, Theorem 2 or 3 implies that each subdomain must be a *restricted tier domain*. Therefore, to completely revealing the domain structure, one needs to resolve this problem: *what are the relations among the restricted tier structures of these subdomains*? For instance, more specifically, if two *restricted tier subdomains* share an identical set of tiers, how is this set of tiers systematically organized in two distinct restricted tier structures?

3.3 A generalized model with outside options

In this section, we extend our model to situations in which the number of agents may differ from the number of objects, and each agent has an outside option. In the generalized model, the characterizations of the PS rule in Theorem 1 still hold. This extension can be viewed as a strengthening of Bogomolnaia and Moulin (2002) since their domain is strictly nested in the class of domains investigated in this section.

Let $m \equiv |A|$ and $n \equiv |I|$. Moreover, there is an object $\varnothing$ with at least n copies. Object $\varnothing$ can be interpreted as an individual outside option for each agent. Each agent i has a strict preference order P_i over $A \cup \{\varnothing\}$. An object $a \in A$ is *acceptable* if $a \; P_i \; \varnothing$. Let $\mathcal{A}(P_i)$ denote the set of acceptable objects in P_i.

Since the number of agents may differ from the number of objects, it may be that an object is not fully shared by all agents. Accordingly, the definition of an assignment $[L_{ia}]_{i \in I, a \in A \cup \{\varnothing\}}$ is modified in such a way that (i) $L_{ia} \geqslant 0$ for all $i \in I$ and $a \in A \cup \{\varnothing\}$, (ii) $\sum_{a \in A \cup \{\varnothing\}} L_{ia} = 1$ for all $i \in I$, and (iii) $0 \leqslant \sum_{i \in I} L_{ia} \leqslant 1$ for all $a \in A$.

All original axioms of *sd-strategy-proofness*, *sd-efficiency*, *sd-envy-freeness*, and *equal* *treatment of equals* apply without any modification. Also, the definition of the PS rule remains unchanged. Evidently, the PS rule remains *sd-efficient* and *sd-envy-free*.

However, we need to modify the definition of a *restricted tier domain* of preferences. Notably, we require restricted tier structure only on the acceptable objects.

Definition 3 *A domain $\mathbb{D}$ is an* ***augmented restricted tier domain*** *if there exists a restricted tier structure $\mathcal{P} \equiv (A_k)_{k=1}^{T}$ (over A, not $A \cup \{\varnothing\}$) such that*

1. *For all $1 \leqslant k \leqslant T$, $|A_k| \leqslant 2$;*

2. *Given $P_i \in \mathbb{D}$, $\mathcal{A}(P_i) = \cup_{k=1}^{t} A_k$ for some $0 \leqslant t \leqslant T$;*

3. *Given $P_i \in \mathbb{D}$ and $a, b \in A$, $[a \in A_k, b \in A_{k'}, a, b \in \mathcal{A}(P_i)$ and $k < k'] \Rightarrow [a \;\; P_i \;\; b]$.*

Example 2 Let $|A| = 5$ and $\mathcal{P} \equiv (A_1, A_2, A_3)$ where $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3\}$, and $A_3 = \{a_4, a_5\}$. Then $\mathbb{D} = \{P_1, P_2, P_3, P_4\}$ is an *augmented restricted tier domain* associated to $\mathcal{P}$.

$$
\begin{array}{lccccccccccc}
P_1\colon & \varnothing & \succ & \cdots & & & & & & & & \\
P_2\colon & a_1 & \succ & a_2 & \succ & \varnothing & \succ & \cdots & & & & \\
P_3\colon & a_2 & \succ & a_1 & \succ & a_3 & \succ & \varnothing & \succ & \cdots & & \\
P_4\colon & a_1 & \succ & a_2 & \succ & a_3 & \succ & a_5 & \succ & a_4 & \succ & \varnothing
\end{array}
$$

In words, an agent with P_1 perceives every object as unacceptable, and all unacceptable objects are ranked arbitrarily. An agent with P_2 perceives only A_1 as acceptable and she prefers a_1 to a_2. An agent with P_3 perceives A_1 and A_2 as acceptable and A_3 as unacceptable. In addition, she prefers all objects in A_1 to all objects in A_2 according to the tier structure. Last, in P_4, all objects are acceptable, and ranked according to the tier structure restriction. $\square$

Analogous to Theorems 1, Theorem 4 below characterizes the PS rule on *augmented restricted tier domains*.

Theorem 4 *On an augmented restricted tier domain, a rule is sd-strategy-proof, sd-efficient and equal-treatment-of-equals if and only if it is the PS rule.*

Proof: Given $\mathcal{P} \equiv (A_k)_{k=1}^T$, let $\mathbb{D} \subseteq \mathbb{D}(\mathcal{P})$ be a *augmented restricted tier domain.*

Given $P \in \mathbb{D}^n$ and $1 \leqslant k \leqslant T$, let $N_k \equiv \{i \in I | A_k \subseteq \mathcal{A}(P_i)\}$ denote the set of agents whose acceptable set includes tier A_k, and $n_k \equiv |N_k|$. Given $1 \leqslant k \leqslant T$, if $n_k > 0$ (equivalently, tier A_k is acceptable for some agent), it is true that $n_{k'} > 0$ for all $1 \leqslant k' \leqslant k-1$ (equivalently, each tier $A_{k'}$, $1 \leqslant k' < k-1$, is also acceptable for some agent). Therefore, given $1 \leqslant k \leqslant T$ with $n_k > 0$, we can define $r_{k'} = \sum_{t=1}^{k'-1} \frac{|A_t|}{n_t}$ for all $1 \leqslant k' \leqslant k+1$. Note that it is either $r_k \leqslant 1$ or $r_k \geqslant 1$.[17]

Due to the augmented restricted tier structure embedded in $\mathbb{D}$, at each preference profile, we can clearly specify the random assignment induced by the PS rule as shown in Fact 2 below.

Fact 2 *Given a profile $P \in \mathbb{D}^n$, let L be the random assignment induced by the PS rule. Then, the following five conditions hold.*

1. *Given $i \in I$, assume $\mathcal{A}(P_i) = \cup_{t=1}^k A_t$. Then, $L_{i\varnothing} = \max(0, 1 - r_{k+1})$ and $L_{ia} =$ for all $a \notin \mathcal{A}(P_i)$.*

Given $1 \leqslant k \leqslant T$, assume $A_k = \{a\}$ or $A_k = \{a, b\}$, and $n_k > 0$. If $A_k = \{a, b\}$, let $I_k = \{i \in I | a\ P_i\ b\}$ and $|I_k \cap N_k| = l$.

2. *If $r_k \geqslant 1$, then $L_{iA_k} = 0$ (equivalently, $L_{ia} = 0$ for all $a \in A_k$) for all $i \in N_k$.*

3. *If $r_k < 1$ and $A_k = \{a\}$, then $L_{ia} = \min(\frac{1}{n_k}, 1 - r_k)$ for all $i \in N_k$.*

4. *If $r_k < r_{k+1} < 1$ and $A_k = \{a, b\}$, we have*

   - $[\frac{n_k}{2} < l \leqslant n_k] \Rightarrow \begin{cases} L_{ia} = \frac{1}{l} \text{ and } L_{ib} = \frac{2}{n_k} - \frac{1}{l} & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = 0 \text{ and } L_{ib} = \frac{2}{n_k} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

   - $[0 \leqslant l \leqslant \frac{n_k}{2}] \Rightarrow \begin{cases} L_{ia} = \frac{2}{n_k} \text{ and } L_{ib} = 0 & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = \frac{2}{n_k} - \frac{1}{n_k - l} \text{ and } L_{ib} = \frac{1}{n_k - l} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

[17] For instance, in profile P, if all tiers are acceptable for all agents, then $r_k \leqslant 1$ for all $1 \leqslant k \leqslant T+1$. If one agent accepts all tiers and all others do not accept any tier, then $r_k \geqslant 1$ for all $1 \leqslant k \leqslant T+1$. In particular, recall the consumption procedure at profile P in the PS rule, and note that if $0 \leqslant r_k < 1$, then r_k is identical to the time at which all tiers $A_1, \ldots, A_{k-1}$ are exhausted, and all agents in N_k are about to consume A_k.

5. *If* $r_k < 1 \leqslant r_{k+1}$ *and* $A_k = \{a, b\}$*, we have*

- $[\frac{n_k}{2} < l \leqslant n_k] \Rightarrow \begin{cases} L_{ia} = \min(\frac{1}{l}, 1 - r_k) \text{ and } L_{ib} = \max(1 - r_k - \frac{1}{l}, 0) & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = 0 \text{ and } L_{ib} = 1 - r_k & \text{for all } i \in N_k \backslash I_k. \end{cases}$

- $[0 \leqslant l \leqslant \frac{n_k}{2}] \Rightarrow \begin{cases} L_{ia} = 1 - r_k \text{ and } L_{ib} = 0 & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = \max(1 - r_k - \frac{1}{n_k - l}, 0) \text{ and } L_{ib} = \min(\frac{1}{n_k - l}, 1 - r_k) & \text{for all } i \in N_k \backslash I_k. \end{cases}$

The verification of Fact 2 is routine, and we hence omit it.

It is evident that the PS rule satisfies *sd-efficiency* and *equal treatment of equals*. We first show that the PS rule is *sd-strategy-proof* on $\mathbb{D}$.

Fix $i \in I$, $P \in \mathbb{D}^n$ and $P'_i \in \mathbb{D}$. Assume $\mathcal{A}(P_i) = \cup_{t=1}^{k} A_t$ and $\mathcal{A}(P'_i) = \cup_{t=1}^{k'} A_t$. Let L and L' be two random assignments induced by the PS rule at profiles P and (P'_i, P_{-i}) respectively. We show $L_i \;\; P_i^{sd} \;\; L'_i$.

According Fact 2, we know $L_{iA_t} = L'_{iA_t}$ for all $1 \leqslant t \leqslant \min(k, k')$. Moreover, given $1 \leqslant t \leqslant \min(k, k')$, assume $A_t = \{a, b\}$ and $a \; P_i \; b$. By a similar argument in verifying *sd-strategy-proofness* of the PS rule in Theorem 1, we have $L_{ia} \geqslant L'_{ia}$.

Let $\underline{l} \equiv \sum_{t=1}^{\min(k,k')} |A_t|$ and $\bar{l} \equiv \sum_{t=1}^{\max(k,k')} |A_t|$. Given $x \in A \cup \{\varnothing\}$, assume $x = r_l(P_i)$. If $1 \leqslant l \leqslant \underline{l}$, then $\sum_{t=1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$. If $l > \underline{l}$ and $k \leqslant k'$, it is evident that $\sum_{t=1}^{l} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$ by condition 1 of Fact 2.

Last, assume $l > \underline{l}$ and $k > k'$. Observe that $L'_{iz} = 0$ for all $z \in \cup_{t=k'+1}^{k} A_t$ by condition 1 of Fact 2. Therefore, if $\underline{l} < l \leqslant \bar{l}$, we have $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{\underline{l}} L_{ir_t(P_i)} + \sum_{t=\underline{l}+1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{\underline{l}} L'_{ir_t(P_i)} + \sum_{t=\underline{l}+1}^{l} L'_{ir_t(P_i)} = \sum_{t=1}^{l} L'_{ir_t(P_i)}$. Furthermore, if $\bar{l} < l \leqslant |A| + 1$, it is evident that $\sum_{t=1}^{l} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$ by condition 1 of Fact 2.

Therefore, $\sum_{t=1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{l} L'_{ir_t(P_i)}$ for all $1 \leqslant l \leqslant |A| + 1$. Hence, $L_i \;\; P_i^{sd} \;\; L'_i$ as required. In conclusion, the PS rule is *sd-strategy-proof* on domain $\mathbb{D}$.

Henceforth, we prove that on domain $\mathbb{D}$, the PS rule is the unique one satisfying *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals*. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ a rule which satisfies all three axioms.

Fixing $P \equiv (P_1, \ldots, P_n) \in \mathbb{D}^n$, let $L \equiv \varphi(P)$. We first show that L satisfies *sd-envy-freeness*, and then show that L satisfies all five conditions of Fact 2. Given $1 \leqslant k \leqslant T$, recall $N_k \equiv \{i \in I | A_k \subseteq \mathcal{A}(P_i)\}$ and $n_k \equiv |N_k|$. Moreover, let $k^* \equiv \max\big\{k \in$

$\{1,\ldots,T\}|L_{iA_k} > 0$ for some $i \in I\big\}$ be the *maximum* index in $\{1,\ldots,T\}$ such that some agent consumes strictly positive proportion of A_{k^*}. Consequently, $n_{k^*} > 0$. Hence, $n_k > 0$ for all $1 \leqslant k \leqslant k^*$, and $r_k = \sum_{t=1}^{k-1} \frac{|A_t|}{n_k}$, $1 \leqslant k \leqslant k^*+1$, is well-defined.

First, taking each tier A_k as one combined object and applying Theorem 5.1 in Bogomolnaia and Moulin (2002), we have the following three statements.

(i) Given $i \in I$, assume $\mathcal{A}(P_i) = \cup_{t=1}^{k} A_t$. Then, we have

- $L_{i\varnothing} = \max(0, 1 - r_{k+1})$ and $L_{ia} = 0$ for all $a \notin \mathcal{A}(P_i)$.
- $\sum_{t=1}^{k'} L_{iA_t} \geqslant \sum_{t=1}^{k'} L_{jA_t}$ for all $0 \leqslant k' \leqslant \min(k, k^*)$ and $j \neq i$.

(ii) Given $1 \leqslant k < k^*$, $L_{iA_k} = \frac{|A_k|}{n_k}$ for all $i \in N_k$.

(iii) $L_{iA_{k^*}} = 1 - r_{k^*} \leqslant \frac{|A_{k^*}|}{n_{k^*}}$ for all $i \in N_{k^*}$.

According to the first part of statement (i) above, condition 1 of Fact 2 is satisfied in L.

Lemma 8 *Given $1 \leqslant k \leqslant k^*$, assume $A_k = \{a, b\}$ and let $I_k = \{i \in I | a\ P_i\ b\}$. The following two statements hold.*

(1) *For all $i, j \in N_k \cap I_k$, $L_{ia} = L_{ja}$.*

(2) *For all $i \in N_k \cap I_k$ and $j \in N_k \backslash I_k$, $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$.*

Proof: The verification of this lemma follows from a modification of the proof of Lemma 2. Specifically, fix all preferences in profile P whose acceptable set do not include A_k, and apply all proofs of Lemma 2 with respect to the remaining preferences in P with the following modifications:

- Change I in the proof of Lemma 2 to N_k.
- Change I_k in the proof of Lemma 2 to $N_k \cap I_k$.
- Change $I \backslash I_k$ in the proof of Lemma 2 to $N_k \backslash I_k$.
- Change n in the proof of Lemma 2 to n_k.

- If $k < k^*$, change $\frac{2}{n}$, $\frac{n}{2}$ and $\frac{1}{n}$ in the proof of Lemma 2 to $\frac{2}{n_k}$, $\frac{n_k}{2}$ and $\frac{1}{n_k}$ respectively. Moreover, whenever Lemma 1 is referred to in the proof of Lemma 2, change it to statement (ii) above.

- If $k = k^*$, change $\frac{2}{n}$, $\frac{n}{2}$ and $\frac{1}{n}$ in the proof of Lemma 2 to $1-r_k$, $\frac{1}{1-r_k}$ and $\min(\frac{1}{n_k}, 1-r_k)$ respectively. Moreover, whenever Lemma 1 is referred to in the proof of Lemma 2, change it to statement (iii) above.

■

Lemma 9 *Random assignment L satisfies sd-envy-freeness.*

Proof: Fix $i \in I$ and assume $\mathcal{A}(P_i) = \cup_{t=1}^{k} A_t$. Thus, $\varnothing = r_{l^*}(P_i)$ where $l \equiv \sum_{t=1}^{k} |A_t| + 1$. Evidently, *sd-efficiency* implies $\sum_{t=1}^{l^*} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l^*} L_{jr_t(P_i)}$ for all $j \neq i$.

Next, given $a \in A$, assume $a \in A_s$ and $a = r_l(P_i)$. We consider three cases.

Case 1: $s > \min(k, k^*)$.

Statements (i) - (iii) above imply $\sum_{t=1}^{l} L_{ir_t(P_i)} = 1 \geqslant \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$.

Case 2: $s \leqslant \min(k, k^*)$ and moreover, either $A_s = \{a\}$, or $|A_s| = 2$ and $a = \min(P_i, A_s)$.

The second part of statement (i) above implies $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{s} L_{iA_s} \geqslant \sum_{t=1}^{s} L_{jA_s} = \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $j \neq i$.

Case 3: $s \leqslant \min(k, k^*)$ and moreover, $|A_s| = 2$ and $a = \max(P_i, A_s)$.

Let $j \neq i$. If A_s is not included in $\mathcal{A}(P_j)$, the first part of statement (i) above implies $L_{ja} = 0$. If A_s is included in $\mathcal{A}(P_j)$, Lemma 8 implies $L_{ia} \geqslant L_{ja}$. Therefore, $L_{ia} \geqslant L_{ja}$. Furthermore, since $\sum_{t=1}^{s-1} L_{iA_t} \geqslant \sum_{t=1}^{s-1} L_{jA_t}$ by the first part of statement (i) above, we have $\sum_{t=1}^{l} L_{ir_t(P_i)} = \sum_{t=1}^{s-1} L_{iA_t} + L_{ia} \geqslant \sum_{t=1}^{s-1} L_{jA_t} + L_{ja} = \sum_{t=1}^{l} L_{jr_t(P_i)}$.

In conclusion, $\sum_{t=1}^{l} L_{ir_t(P_i)} \geqslant \sum_{t=1}^{l} L_{jr_t(P_i)}$ for all $1 \leqslant l \leqslant |A| + 1$ and $j \neq i$. Therefore, L satisfies *sd-envy-freeness*. ■

Last, we use the following 5 claims to show that conditions 2 - 5 of Fact 2 are satisfied in L.

Claim 1: $r_{k^*} < 1 \leqslant r_{k^*+1}$.

According to the definition of k^*, there exist $i \in N_{k^*}$ such that $L_{iA_{k^*}} > 0$. Fix such an agent i. By statement (ii) above, we know $r_{k^*} = \sum_{t=1}^{k^*-1} \frac{|A_t|}{n_t} = \sum_{t=1}^{k^*-1} L_{iA_t} < 1$. Moreover, by statement (iii), we have $r_{k^*+1} = \sum_{t=1}^{k^*} \frac{|A_t|}{n_t} = r_{k^*} + \frac{|A_{k^*}|}{n_{k^*}} \geqslant 1$. This completes the verification of the claim.

Given $1 \leqslant k \leqslant T$, assume $A_k = \{a\}$ or $A_k = \{a, b\}$, and $n_k > 0$. If $A_k = \{a, b\}$, let $I_k = \{i \in I | a\ P_i\ b\}$ and $|I_k \cap N_k| = l$.

Claim 2: Condition 2 of Fact 2 is satisfied in L.

According to the hypothesis of condition 2, since $r_k \geqslant 1$, Claim 1 implies $k \geqslant k^* + 1$. Then, the definition of k^* implies $L_{iA_k} = 0$ for all $i \in I$ (hence for all $i \in N_k$). This completes the verification of the claim.

Claim 3: Condition 3 of Fact 2 is satisfied in L.

According to the hypothesis of condition 3, since $r_k < 1$, Claim 1 implies $k \leqslant k^*$. Fix $i \in N_k$. If $k < k^*$, then statement (ii) above implies $L_{ia} = \frac{|A_k|}{n_k} = \frac{1}{n_k}$. Since $k + 1 \leqslant k^*$ and $r_{k^*} < 1$, it must be the case that $r_k + \frac{|A_k|}{n_k} = r_k + \frac{1}{n_k} = r_{k+1} \leqslant r_{k^*} < 1$. Thus, $\frac{1}{n_k} < 1 - r_k$, and hence $L_{ia} = \min(\frac{1}{n_k}, 1 - r_k)$. If $k = k^*$, statement (iii) above implies $L_{ia} = 1 - r_{k^*} = \min(\frac{1}{n_{k^*}}, 1 - r_{k^*})$. In conclusion, $L_{ia} = \min(\frac{1}{n_k}, 1 - r_k)$ for all $i \in N_k$. This completes the verification of the claim.

Claim 4: Condition 4 of Fact 2 is satisfied in L.

According to the hypothesis of condition 4, since $r_{k+1} < 1$, Claim 1 implies $k + 1 \leqslant k^*$, and hence $k < k^*$. Therefore, $L_{ia} + L_{ib} = L_{iA_k} = \frac{|A_k|}{n_k} = \frac{2}{n_k}$ for all $i \in N_k$ by statement (ii) above. Recall $I_k = \{i \in I | a\ \ P_i\ \ b\}$ and $|N_k \cap I_k| = l$. Then, *sd-efficiency* and *sd-envy-freeness* (recall Lemma 9) imply

- $[\frac{n_k}{2} < l \leqslant n_k] \Rightarrow \begin{cases} L_{ia} = \frac{1}{l} \text{ and } L_{ib} = \frac{2}{n_k} - \frac{1}{l} & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = 0 \text{ and } L_{ib} = \frac{2}{n_k} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

- $[0 \leqslant l \leqslant \frac{n_k}{2}] \Rightarrow \begin{cases} L_{ia} = \frac{2}{n_k} \text{ and } L_{ib} = 0 & \text{for all } i \in N_k \cap I_k; \\ L_{ia} = \frac{2}{n_k} - \frac{1}{n_k - l} \text{ and } L_{ib} = \frac{1}{n_k - l} & \text{for all } i \in N_k \backslash I_k. \end{cases}$

Claim 5: Condition 5 of Fact 2 is satisfied in L.

According to the hypothesis of condition 5, since $r_k < 1 \leqslant r_{k+1}$, Claim 1 implies $k = k^*$. Thus, by statement (iii), $L_{ia} + L_{ib} = L_{iA_{k^*}} = 1 - r_{k^*} \leqslant \frac{|A_{k^*}|}{n_{k^*}} = \frac{2}{n_{k^*}}$ for all $i \in N_{k^*}$. Recall $I_{k^*} \equiv \{i \in I | a \ \ P_i \ \ b\}$ and $l \equiv |N_{k^*} \cap I_{k^*}|$. We know either $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$ or $0 \leqslant l \leqslant \frac{n_{k^*}}{2}$.

First, assume $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$. Subsequently, two cases are separately considered.

Case 1: $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$ **and** $\frac{1}{l} \leqslant 1 - r_{k^*}$.

If $l = n_{k^*}$, then *sd-envy-freeness* implies $L_{ia} = \frac{1}{l}$, and hence $L_{ib} = 1 - r_{k^*} - \frac{1}{l}$ for all $i \in N_k \cap I_k$. If $\frac{n_{k^*}}{2} < l < n_{k^*}$, *sd-efficiency* first implies $L_{ia} = 0$, and hence $L_{ib} = 1 - r_{k^*}$ for all $i \in N_k \backslash I_k$. Consequently, *sd-envy-freeness* implies $L_{ia} = \frac{1}{l}$ for all $i \in N_k \cap I_k$. Hence, $L_{ib} = 1 - r_{k^*} - \frac{1}{l}$ for all $i \in N_k \cap I_k$.

Case 2: $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$ **and** $\frac{1}{l} > 1 - r_{k^*}$.

If $l = n_{k^*}$, *sd-envy-freeness* implies $L_{ia} = L_{ja}$ for all $i, j \in N_k \cap I_k$. Moreover, since $\frac{1}{l} > 1 - r_{k^*}$, it is true that $L_{ia} = 1 - r_{k^*}$, and hence $L_{ib} = 0$ for all $i \in N_k \cap I_k$. If $\frac{n_{k^*}}{2} < l < n_{k^*}$, *sd-efficiency* first implies $L_{ia} = 0$, and hence $L_{ib} = 1 - r_{k^*}$ for all $i \in N_k \backslash I_k$. Next, *sd-envy-freeness* implies $L_{ia} = L_{ja}$ for all $i, j \in N_k \cap I_k$. Since $\frac{1}{l} > 1 - r_{k^*}$, it is true that $L_{ia} = 1 - r_{k^*}$ for all $i \in N_k \cap I_k$. Hence, $L_{ib} = 0$ for all $i \in N_k \cap I_k$.

In conclusion, if $\frac{n_{k^*}}{2} < l \leqslant n_{k^*}$,

$$L_{ia} = \min(\frac{1}{l}, 1 - r_{k^*}) \text{ and } L_{ib} = \max(1 - r_{k^*} - \frac{1}{l}, 0) \qquad \text{for all } i \in N_k \cap I_k;$$

$$L_{ia} = 0 \text{ and } L_{ib} = 1 - r_{k^*} \qquad \text{for all } i \in N_k \backslash I_k.$$

By a symmetric argument, if $0 \leqslant l \leqslant \frac{n_{k^*}}{2}$,

$$L_{ia} = 1 - r_{k^*} \text{ and } L_{ib} = 0 \qquad \text{for all } i \in N_k \cap I_k;$$

$$L_{ia} = \max(1 - r_{k^*} - \frac{1}{n_{k^*} - l}, 0) \text{ and } L_{ib} = \min(\frac{1}{n_{k^*} - l}, 1 - r_{k^*}) \qquad \text{for all } i \in N_k \backslash I_k.$$

Thus, all five conditions of Fact 2 are verified. Therefore, φ is the PS rule. This completes the verification of Theorem 4. ■

Analogous to Corollary 1, the verification of Theorem 4 implies that the PS rule is the unique one satisfying *sd-efficiency* and *sd-envy-freeness* on an *augmented restricted tier domain*.

Corollary 2 *Let $\mathbb{D}$ be an augmented restricted tier domain. A rule is sd-efficient and sd-envy-free if and only if it is the PS rule.*

Proof: The sufficiency part hold evidently. We focus on the necessity part. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be an *sd-efficient* and *sd-envy-free* rule. Fixing $P \equiv (P_1, \ldots, P_n) \in \mathbb{D}^n$, let $L \equiv \varphi(P)$. First, according to Theorem 4.1 in Bogomolnaia and Moulin (2002), we have statements (i) - (iii) in the proof of the necessity part of Theorem 4. Furthermore, in the proof of the necessity part of Theorem 4, note that the verification of Claims 1 - 5 only relies on the application of statements (i) - (iii) and the axioms of *sd-efficiency* and *sd-envy-freeness*. Therefore, we assert that φ is the PS rule. ■

3.4 Final remarks

In this chapter, we have shown that if a connected domain admits an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* (or *sd-envy-free*) rule, the domain is a *restricted tier domain*, and this rule must be the PS rule.

Our results may be interpreted as both negative and positive. On the one hand, a *restricted tier domain* is restrictive, and does not give much freedom for agents to spell their preferences. On the other hand, in some realistic situations, for example the house allocation in a skyscraper or along a road, the restricted tier structure seems to be an appropriate assumption. Then our characterization of the PS rule supports its application in these situations.

More importantly, we identify the *restricted tier domain* as a boundary for the compatibility of these canonical axioms, within connected domains. Since connectedness is a mild and economically reasonable domain richness assumption and the axioms we impose are both canonical and normatively desirable, our characterizations suggest that a restricted tier structure must be embedded and the PS rule should be used to determine all random assignments.

For further research, it would be interesting to investigate the analogous characterization problem for more general class of domains beyond connectedness. Another interesting problem is related to the domain characterization under different preference extension approaches (e.g., Cho (2012) and Aziz et al. (2014)) other than the stochastic-dominance extension.

4 The Equivalence between Local and Global SD-Strategy-Proofness on Block-Connected Domains

In this chapter we introduce a new adjacency notion: block-adjacency. Two preferences are block-adjacent if they are different only in a flip between two adjacent blocks. Accordingly, a undirected graph can be constructed for a given domain such that the vertex set is the domain itself and an edge links two block-adjacent preferences. In addition we call a domain block-connected if its corresponding undirected graph is connected.

The first section in this chapter formalizes the above notions and the second section proposes a condition on a block-connected domain which implies the equivalence between local and global incentive compatibility.

4.1 Block-connected domains

We start from the formal definition of block-adjacency.

Definition 4 *Two preferences $P_0, \tilde{P}_0 \in \mathbb{D}$ are **block-adjacent** if there are two nonempty and disjoint subsets $A_1, A_2 \subset A$ such that*

1. *$a\ \tilde{P}_0\ b$ if and only if $b\ P_0\ a$, for all $a \in A_1$ and $b \in A_2$*
2. *$a\ \tilde{P}_0\ b$ if and only if $a\ P_0\ b$, for all $a, b \in A$ either $a \notin A_1$ or $b \notin A_2$*
3. *$\nexists\ a \in A_1\ b \in A_2\ c \in A\backslash\{A_1, A_2\}$ such that $a\ P_0\ c\ P_0\ b$ or $b\ P_0\ c\ P_0\ a$.*

For any two preferences, it's either that they are not block-adjacent or that they are block-adjacent with respect to a unique pair of nonempty and disjoint subsets $A_1, A_2 \subset A$. We denote for two block-adjacent preferences, $P_0, \tilde{P}_0$, the corresponding pair of object subsets as $FB_1(P_0, \tilde{P}_0)$ and $FB_2(P_0, \tilde{P}_0)$, where FB represents "flipped block." In addition, we refer to the change from a preference to a block-adjacent one as a **block-adjacent reversal**.

Given the notion of block-adjacency, we define block-connected domains. A domain is block-connected if between any two admissible preferences a sequence of admissible preferences can be arranged such that any two contiguous preferences along the path are block-adjacent. Formally:

Definition 5 *A preference domain* $\mathbb{D}$ *is* ***block-connected*** *if for any two admissible preferences* $P_0, \tilde{P}_0 \in \mathbb{D}$*, there is a sequence of admissible preferences* $P_1, \cdots, P_M \in \mathbb{D}$ *such that (i)* $P_1 = P_0$*, (ii)* $P_M = \tilde{P}_0$*, and (iii) for each* $m = 1, \cdots, M-1$*,* P_m *and* P_{m+1} *are block-adjacent.*

For a pair of preferences $P_0, \tilde{P}_0$ admissible to a block-connected domain, we call a sequence in the above definition a **path from** $\mathbf{P_0}$ **to** $\tilde{\mathbf{P}}_\mathbf{0}$. Particularly, when we say a path $P_1, \cdots, P_M$ is from P_0 to $\tilde{P}_0$, P_0 is the start and $\tilde{P}_0$ is the end, i.e., $P_1 = P_0$ and $P_M = \tilde{P}_0$.

On a block-connected domain, a mechanism is called *block-adjacent sd-strategy-proof* if reporting true preference always leads to a lottery that stochastically dominates the lottery delivered by reporting a preference that is block-adjacent to the sincere one.

Definition 6 *A rule defined on a block-connected domain* $\varphi : \mathbb{D}^n \to \mathcal{L}$ *satisfies* ***block-adjacent sd-strategy-proofness*** *if for all* $i \in I$*,* $P_i, \tilde{P}_i \in \mathbb{D}$*, and* $P_{-i} \in \mathbb{D}^{n-1}$*, such that* $\tilde{P}_i$ *and* P_i *are block-adjacent,* $\varphi(P_i, P_{-i}) \;\; P_i^{sd} \;\; \varphi(\tilde{P}_i, P_{-i})$.

The incentive compatibility defined above is a local notion. Since block-adjacency is weaker than the adjacency notion in Cho (2016a), *block-adjacent sd-strategy-proofness* is stronger than the ”local sd-strategy-proofness” defined by Cho (2016a). Example 5 illustrates this point.

4.2 The equivalence between local and global incentive compatibilities

In this section, we introduce a condition on a block-connected domain that is sufficient to guarantee the equivalence between *block-adjacent sd-strategy-proofness* and *sd-srategy-proofness*. However, some preliminary definitions are needed.

Given two pairs of nonempty and pairwise disjoint subsets of objects, we say one pair is nested in or disjoint with the other pair if either the union of the former pair is a subset of either one of the latter pair or the unions of these two pairs are disjoint; formally

Definition 7 *Let A_1, A_2 and A_3, A_4 be two pairs of nonempty and pair-wisely disjoint subsets of objects, i.e., $\emptyset \neq A_1, A_2, A_3, A_4 \subset A$, $A_1 \cap A_2 = \emptyset$, and $A_3 \cap A_4 = \emptyset$. We say A_1, A_2 are* ***nested in or disjoint with*** *A_3, A_4, denoted $\{A_1, A_2\} \sqsubseteq \{A_3, A_4\}$, if either $(A_1 \cup A_2) \cap (A_3 \cup A_4) = \emptyset$ or $A_1 \cup A_2 \subset A_3$ or $A_1 \cup A_2 \subset A_4$.*

Recall that a path is a sequence of preferences, along which adjacent preferences are different only in a flip between two adjacent blocks. We now define a special class of paths which guarantees the equivalence between *block-adjacent sd-strategy-proofness* and *sd-strategy-proofness*. Particularly, we call a path nested if along the path, the blocks flipped latter are nested in or disjoint with the blocks flipped earlier.

Definition 8 *A path $P_1, \cdots, P_M$ is* ***nested*** *if for all $1 \leqslant m' < m \leqslant M-1$*
$\{FB_1(P_m, P_{m+1}), FB_2(P_m, P_{m+1})\} \sqsubseteq \{FB_1(P_{m'}, P_{m'+1}), FB_2(P_{m'}, P_{m'+1})\}$.

A domain is called path-nested if between any two preferences, there is a nested path connecting them; formally:

Definition 9 *A domain $\mathbb{D}$ is* ***path-nested*** *if for all distinct $P_0, \tilde{P}_0 \in \mathbb{D}$, there is a nested path $P_1, \cdots, P_M \in \mathbb{D}$ such that $P_1 = P_0$ and $P_M = \tilde{P}_0$.*

By definition, a path-nested domain needs first to be block-connected. The following are two weakly connected domains, one is path-nested and the other is not.

Example 3 *Figure 3 depicts a block-connected domain which is path-nested. Boldface sequences of letters denote the preferences, for example,* ***abcd*** *refers to the preference $a \succ b \succ c \succ d$. A dotted line denotes a block-adjacency relation and the two blocks connected with a hyphen denote the associated flipped pair of blocks.*

A nested path from ***cadb*** *to* ***bdac*** *is highlighted by red and thick lines: from* ***cadb*** *to* ***bcad*** *$\{a, c, d\}$ is flipped with $\{b\}$; then from* ***bcad*** *to* ***bdca*** *$\{a, c\}$ is flipped with $\{d\}$; finally from* ***bdca*** *to* ***bdac*** *$\{a\}$ is flipped with $\{c\}$. It can be checked easily that from an arbitrary preference to another, there is a nested path.*

Example 4 *Figure 4 depicts a block-connected domain that is not path-nested. From* ***abcd*** *to* ***cadb****, there are two paths, one through* ***cdab*** *and the other through* ***acdb****. It is evident that neither one of them is nested.*

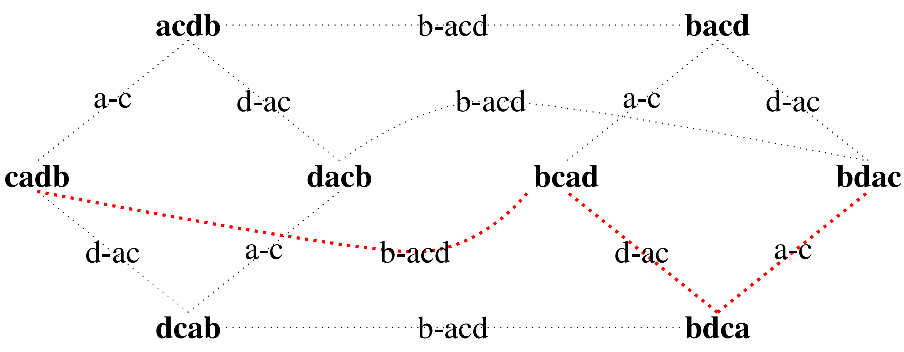


Figure 3: A Path-Nested Domain.

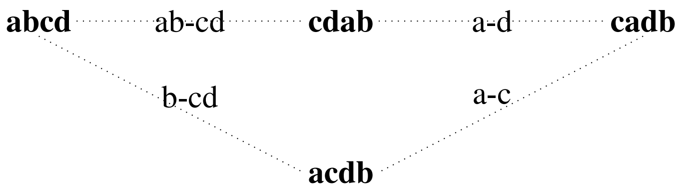


Figure 4: A Block-Connected Domain that is not Path-Nested

Theorem 5 shows that path-nestedness is sufficient to guarantee the equivalence between *block-adjacent sd-strategy-proofness* and *sd-strategy-proofness*.

Theorem 5 *Block-adjacent sd-strategy-proofness is equivalent to sd-strategy-proofness on a path-nested domain.*

The following is a sketch of the proof. Consider the path-nested domain depicted in Figure 3. Let the unilateral deviation be from $acdb$ to $bdca$. Consider the nested path connecting them as follows.

$$
\begin{array}{llllllllll}
P_1: \; a \;\; c \;\; d \;\; b & & & & & & & \\
\downarrow (b-acd) & L_1\, P_1^{sd}\, L_2 & + & L_2\, P_1^{sd}\, L_3 & + & L_3\, P_1^{sd}\, L_4 & \Rightarrow L_1\, P_1^{sd}\, L_4 \\
P_2: \; b \;\; a \;\; c \;\; d & & & \Uparrow & & \Uparrow & \\
\downarrow (ac-d) & & & L_2\, P_2^{sd}\, L_3 & & L_3\, P_2^{sd}\, L_4 & \\
P_3: \; b \;\; d \;\; a \;\; c & & & & & \Uparrow & \\
\downarrow (a-c) & & & & & L_3\, P_3^{sd}\, L_4 & \\
P_4: \; b \;\; d \;\; c \;\; a & & & & & &
\end{array}
$$

The lotteries L_1, L_2, L_3, L_4 denote the deviating agent's lottery when she reports respectively P_1, P_2, P_3, P_4. To verify *sd-strategy-proofness*, it suffices to establish $L_1 \; P_1^{sd} \; L_4$. By *block-adjacent sd-strategy-proofness*, $L_1 \; P_1^{sd} \; L_2$, $L_2 \; P_2^{sd} \; L_3$, and $L_3 \; P_3^{sd} \; L_4$. According to $L_2 \; P_2^{sd} \; L_3$ and the fact that b is not involved in this local deviation, $L_{2b} = L_{3b}$. Then since the ranking of a, c, d is the same in P_1 and P_2, $L_2 \; P_2^{sd} \; L_3$ implies $L_2 \; P_1^{sd} \; L_3$. Similarly $L_3 \; P_3^{sd} \; L_4$ implies $L_3 \; P_2^{sd} \; L_4$, which then implies $L_3 \; P_1^{sd} \; L_4$. Finally the transitivity of P_1^{sd} establishes $L_1 \; P_1^{sd} \; L_4$.

The proof follows the above logic. It is in Appendix B.1.

4.2.1 Necessity of path-nestedness

In this subsection we discuss the necessity of path-nestedness. We show first that whenever *block-adjacent sd-strategy-proofness* is equivalent to *sd-strategy-proofness*, the domain is block-connected.

Proposition 2 *If block-adjacent sd-strategy-proofness is equivalent to sd-strategy-proofness, then the domain is block-connected.*

Proof: Let $\mathbb{D}$ be a domain which is not block-connected.[18] Let $P_0 \in \mathbb{D}$ and $\overline{\mathbb{D}} \subset \mathbb{D}$ be the block-connected sub-domain that contains P_0. By definition, $\overline{\mathbb{D}}$ exists but may be singleton. Since $\mathbb{D}$ is not block-adjacent, $\mathbb{D} \backslash \overline{\mathbb{D}} \neq \emptyset$. Let $x = r_1(P_0)$ and $y \in A \backslash \{x\}$. Define

$$\varphi(P) = \begin{cases} \delta_y, & \text{if } P_1 \in \overline{\mathbb{D}} \\ \delta_x, & \text{otherwise} \end{cases}$$

where δ_y denotes the degenerated lottery that gives full probability to y. It is evident that φ is *block-adjacent sd-strategy-proof* but not *sd-strategy-proof.* ■

With the domain illustrated in Figure 4, we show that path-nestedness is not necessary to the equivalence between *block-adjacent sd-strategy-proofness* and *sd-strategy-proofness*.

Lemma 10 *There is a block-connected but not path-nested domain on which block-adjacent sd-strategy-proofness is equivalent to sd-strategy-proofness.*

[18] This proof is inspired by the proof of Proposition 3.1 in Sato (2013).

Proof: Consider the domain illustrated in Figure 4, i.e., $\mathbb{D} \equiv \{abcd, cdab, cadb, acdb\}$. It is easy to see that this domain is block-connected but not path-nested. We will show that on this domain, a *block-adjacent sd-strategy-proof* rule is *sd-strategy-proof*.

	a	**b**	**c**	**d**
abcd	$l_a + \delta_1 + \delta_2$	$l_b + \delta_3$	$l_c - \delta_2 - \delta_4$	$l_d - \delta_1 - (\delta_3 - \delta_4)$
cdab	l_a	l_b	l_c	l_d
cadb	$l_a + \delta_1$	l_b	l_c	$l_d - \delta_1$
acdb	$l_a + \delta_1 + \delta_2$	l_b	$l_c - \delta_2$	$l_d - \delta_1$

Consider an arbitrary fixed preference profile and an arbitrary agent who unilaterally deviates. We need only to specify the lottery for the unilateral deviator, hence we denote his lottery as $\varphi(P_i)$ where P_i is this agent's reported preference. Without loss of generality, let $(l_a, l_b, l_c, l_d) \equiv \varphi(cdab)$. The *block-adjacent sd-strategy-proofness* implies non-manipulability between $cdab$ and $cadb$, which is equivalent to the existence of $\delta_1 \in [0, l_d]$ such that $\varphi(cadb) = (l_a + \delta_1, l_b, l_c, l_d - \delta_1)$. Similarly, the block-adjacent *sd-strategy-proofness* implies non-manipulability between $cadb$ and $acdb$, which is equivalent to the existence of $\delta_2 \in [0, l_c]$ such that $\varphi(cadb) = (l_a + \delta_1 + \delta_2, l_b, l_c - \delta_2, l_d - \delta_1)$. Last, the *block-adjacent sd-strategy-proofness* implies non-manipulability between $acdb$ and $abcd$, which is equivalent to the existence of $\delta_4 \in [0, l_c - \delta_2]$ and δ_3 such that $\delta_3 - \delta_4 \in [0, l_d - \delta_1]$ such that $\varphi(abcd) = (l_a + \delta_1 + \delta_2, l_b + \delta_3, l_c - \delta_2 - \delta_4, l_d - \delta_1 - (\delta_3 - \delta_4))$.

To check *sd-strategy-proofness*, we need only to check non-manipulability between $abcd$ and $cadb$ and between $cdab$ and $acdb$. This is easily verified for any $\delta_1, \delta_2, \delta_3, \delta_4$ given above. ■

4.3 Final remarks

Theorem 5 is useful either when the domain of interest is block-connected but not connected, or when it is connected but violates non-restoration, the sufficient condition on a connected domain to guarantee the equivalence between local and global sd-strategy-proofness in Cho (2016a). As to the former, we use Theorem 5 to show that on a quite flexible class of preference domains, each of which is block-connected and quite large in domain size, the PS rule is *sd-strategy-proof*. As to the latter, the following is an example.

Example 5 Consider a domain that consisting of the following five preferences.[19] Among these preferences, the adjacency relations are illustrated by solid lines. It is easy to see that this domain is connected but it does not satisfy non-restoration. Specifically, from $xyvwz$ to $xyzvw$, there is only one path which involves two flips between x and y. Hence we can not invoke the result in Cho (2016a) to simplify the verification of *sd-strategy-proofness*.

However, we start from *block-adjacent sd-strategy-proofness* rather than *sd-strategy-proofness*, we can draw two more links, illustrated by dotted lines. It is easy to check path-nestedness of this domain. Then invoking Theorem 5, we have *sd-strategy-proofness* on this domain whenever *block-adjacent sd-strategy-proofness* is satisfied.

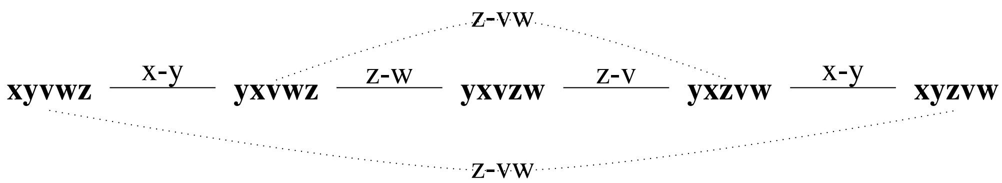


Figure 5: A Connected Domain that Violates non-Restoration

Another remark is that, although we focus on random assignment rules, the equivalence result can be applied to all stochastic-dominance ordinal mechanisms, i.e., centralized mechanisms where agents report ordinal preferences, receive lotteries, and the lotteries are evaluated according to the stochastic-dominance extensions of ordinal preferences.

[19]The preferences in this example come from Table 1 in Sato (2013).

5 Strategy-Proofness of the Probabilistic Serial Rule on Sequentially Dichotomous Domains

This chapter proposes a class of domains, *sequentially dichotomous domains*. It is shown that the PS rule is *sd-strategy-proof* on any such domain. In addition, any such domain is shown to be maximal for the PS rule to be *sd-strategy-proof*. The results in this chapter significantly enlarge the scope of designing a satisfactory random assignment rule.

The organization of this chapter is as follows. The first section presents the domains. Section 2 proves *sd-strategy-proofness* of the PS rule. Section 3 presents the maximality result. The last section presents some final remarks.

5.1 Sequentially dichotomous domains

This section defines the *sequentially dichotomous domains*. However, before we present the domains, several preliminary definitions are needed.

A **partition** of the object set A is a set of nonempty subsets of A such that every object a in A is in exactly one of these subsets, i.e., $\mathbf{A} \equiv \{A_k \subset A : A_k \neq \emptyset\}$ such that $\bigcup_{A_k \in \mathbf{A}} A_k = A$ and $A_k \cap A_l = \emptyset$ for all distinct $A_k, A_l \in \mathbf{A}$. A typical element of a partition is called a **block** and denoted $A_k \in \mathbf{A}$. We denote the collection of all partitions of A by $\mathcal{A}$ and define a binary relation, called direct refinement, on $\mathcal{A}$. A partition is called a direct refinement of another if there is exactly one block of the latter partition broken into two smaller blocks in the former partition and all the other blocks are inherited; formally:

Definition 10 *A partition $\mathbf{A}'$ is a* ***direct refinement*** *of another partition $\mathbf{A}$, if there are blocks $A_k \in \mathbf{A}$ and $A'_i, A'_j \in \mathbf{A}'$ such that $A_k = \mathbf{A}\backslash\mathbf{A}'$ and $\{A'_i, A'_j\} = \mathbf{A}'\backslash\mathbf{A}$.*

We define a **partition-path** as a sequence $(\mathbf{A}_t)_{t=1}^{T} \subset \mathcal{A}$ such that $\mathbf{A}_1 = \{A\}$, $\mathbf{A}_T = \{\{a\} : a \in A\}$, and $\mathbf{A}_{t+1}$ is a direct refinement of $\mathbf{A}_t$ for every $t = 1, \cdots, T-1$. Let $\mathbf{A}'$ be a direct refinement of $\mathbf{A}$, it is evident that $|\mathbf{A}'| = |\mathbf{A}| + 1$ and hence $T = n$ for any partition-path. Henceforth we denote a partition-path as $(\mathbf{A}_t)_{t=1}^{n}$. A partition-path plots a sequence from the coarsest partition to the finest partition by sequentially breaking one block into two. For each $t \in \{1, \cdots, n-1\}$, let $A_{t*} \equiv \mathbf{A}_t \backslash \mathbf{A}_{t+1}$ be the block in $\mathbf{A}_t$ that breaks into two smaller blocks. For each $t = 2, \cdots, n$, let $\{A_{t1}, A_{t2}\} \equiv \mathbf{A}_t \backslash \mathbf{A}_{t-1}$ be the two blocks whose union is a block in $\mathbf{A}_{t-1}$. Hence from $\mathbf{A}_1$ to $\mathbf{A}_2$, A_{1*} breaks into A_{21} and A_{22}; from $\mathbf{A}_2$ to $\mathbf{A}_3$, A_{2*} breaks into A_{31} and A_{32}, etc. Two partition-paths are plotted in Figure 6, one with darkened arrows and the other with darkened and dotted arrows.

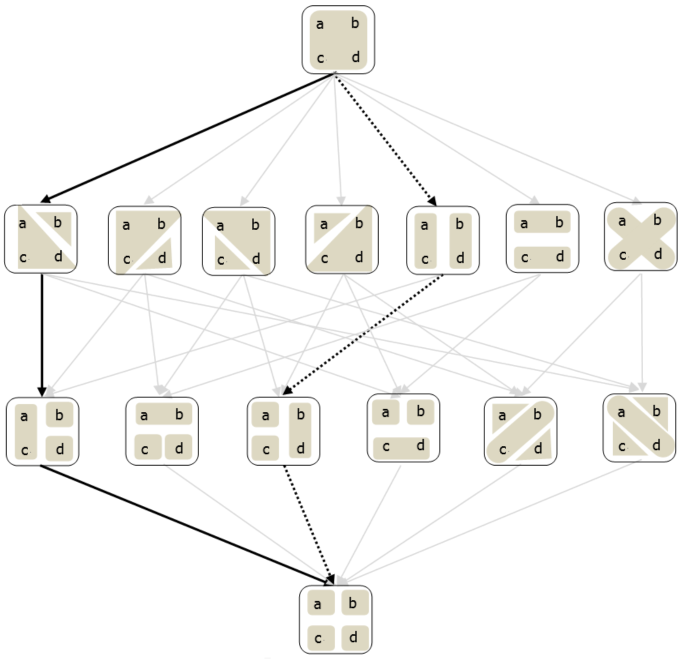


Figure 6: Direct refinement relation when $A = \{a, b, c, d\}$. A shade covering several objects means a block containing these objects and each square containing objects and shade(s) is a partition. An arrow pointing from one partition to another means the latter is a direct refinement of the former.

Given a strict preference $P_0 \in \mathbb{P}$ and a block A_k, we say P_0 **clusters objects in** A_k if all the objects in A_k are ranked next to each other in P_0, i.e., for each $x \in A \backslash A_k$, either $[a \ P_0 \ x$ for all $a \in A_k]$ or $[x \ P_0 \ a$ for all $a \in A_k]$. We say a preference $P_0 \in \mathbb{P}$ **respects a partition A** if, for each $A_k \in \mathbf{A}$, P_0 clusters objects in A_k. Given a partition $\mathbf{A}$, the collection of *all* preferences that respect this partition is called the domain that respects $\mathbf{A}$. It is denoted $\mathbb{D}_{\mathbf{A}}$, i.e., $\mathbb{D}_{\mathbf{A}} \equiv \{P_0 \in \mathbb{P} | P_0 \text{ respects } \mathbf{A}\}$. Finally given a partition-path $(\mathbf{A}_t)_{t=1}^n$, we say a preference **respects the partition-path** if it respects every partition along the partition-path. A *sequentially dichotomous domain* is hence defined as the collection of preferences that respect a common partition-path.

Definition 11 *A preference domain $\mathbb{D} \subset \mathbb{P}$ is a* ***sequentially dichotomous domain*** *if there is a partition-path $(\mathbf{A}_t)_{t=1}^n$ such that $P_0 \in \mathbb{D}$ if and only if P_0 respects $\mathbf{A}_t$ for all $t = 1, \cdots, n$, i.e., $\mathbb{D} = \bigcap_{t=1}^n \mathbb{D}_{\mathbf{A}_t}$.*

Notice that the definition above imposes a richness condition: every preference respecting the partitions along the partition-path is included. This facilitates our analysis as we focus on verifying *sd-strategy-proofness* and the fact that if a rule is *sd-strategy-proof* on a domain, it is *sd-strategy-proof* on every sub-domain. Here are two examples of *sequentially dichotomous domains*.

Example 6 Let $A = \{a, b, c, d\}$ and consider the partition-path plotted in Figure 6 with darkened arrows, i.e., $(\mathbf{A}_t)_{t=1}^4$ where $\mathbf{A}_1 = \{\{a, b, c, d\}\}$, $\mathbf{A}_2 = \{\{a, c, d\}, \{b\}\}$, $\mathbf{A}_3 = \{\{a, c\}, \{d\}, \{b\}\}$, and $\mathbf{A}_4 = \{\{a\}, \{c\}, \{d\}, \{b\}\}$. Let $\mathbb{D}$ be the corresponding *sequentially dichotomous domain*.

It is evident that the collection of preferences respecting $\mathbf{A}_1$ is the universal domain, i.e., $\mathbb{D}_{\mathbf{A}_1} = \mathbb{P}$. However, not every preference respecting $\mathbf{A}_1$ is in $\mathbb{D}$ since it needs to respect in addition $\mathbf{A}_2$. To respect $\mathbf{A}_2$ is equivalent to ranking b either at the top or at the bottom. However not every preference ranking b at the top (or at the bottom) is admissible since it is required to respect in addition $\mathbf{A}_3$. For a preference in $\bigcap_{t=1}^2 \mathbb{D}_{\mathbf{A}_t}$ to respect $\mathbf{A}_3$, it suffices that a and c be ranked next to each other. Last, it is evident that every preference in $\mathbb{P}$ respects $\mathbf{A}_4$ trivially. Hence the domain $\mathbb{D} \equiv \{P_1, \cdots, P_8\}$ is a *sequentially dichotomous domain*.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a	c	d	d	b	b	b	b
c	a	a	c	a	c	d	d
d	d	c	a	c	a	a	c
b	b	b	b	d	d	c	a

Similarly, let us consider the partition-path plotted in Figure 6 with darkened and dotted arrows, i.e., $(\mathbf{A}'_t)_{t=1}^4$ where $\mathbf{A}'_1 = \{\{a, b, c, d\}\}$, $\mathbf{A}'_2 = \{\{a, c\}, \{b, d\}\}$, $\mathbf{A}'_3 = \{\{a\}, \{c\}, \{b, d\}\}$, and $\mathbf{A}'_4 = \{\{a\}, \{c\}, \{b\}, \{d\}\}$.

From similar procedure above, we can find the *sequentially dichotomous domain* with respect to $(\mathbf{A}'_t)_{t=1}^4$ as such that includes preferences P'_1 to P'_8 below.

P'_1	P'_2	P'_3	P'_4	P'_5	P'_6	P'_7	P'_8
a	c	a	c	b	b	d	d
c	a	c	a	d	d	b	b
b	b	d	d	a	c	a	c
d	d	b	b	c	a	c	a

This section ends with several remarks on the *sequentially dichotomous domains*.

Remark 6 A *sequentially dichotomous domain* $\mathbb{D}$ satisfies minimal richness, i.e., for each $a \in A$, there is a preference $P_0 \in \mathbb{D}$ such that $r_1(P_0) = a$. This is already illustrated by the domains in Example 6.

Remark 7 Given n be the number of objects, a *sequentially dichotomous domain* has 2^{n-1} preferences, exactly the same size as the single-peaked domain. As in Example 6, when $n = 4$, a *sequentially dichotomous domain* includes 8 preferences.

Remark 8 Two different partition-paths may lead to the same *sequentially dichotomous domain*. Consider the domain containing P'_1 to P'_8 in the above example. This domain can also be seen as respect to the partition-path $(\mathbf{A}''_t)_{t=1}^4$, where $\mathbf{A}''_3 \equiv \{\{a, c\}, \{b\}, \{d\}\}$ and $\mathbf{A}''_t \equiv \mathbf{A}'_t$ for $t = 1, 2, 4$.

However such nonuniqueness is not a problem for our analysis since we always start from a given domain and as long as this domain can be structured as a *sequentially dichotomous domain* according to some partition-path, our analysis applies.

Remark 9 A *sequentially dichotomous domain* can be expressed as a union of several *restricted tier domains*. And given a *sequentially dichotomous domain* and a partition-path with respect to which the domain is defined, the involving ordered partitions can be recovered. For example, the domain in Example 6 defined with respect to $(\mathbf{A}_t)_{t=1}^4$ is the union of four *restricted tier domains* with the ordered partitions being, respectively,

$$\begin{aligned}
&\mathcal{P}^1: \quad A_1^1 = \{a, c\}, \quad A_2^1 = \{d\}, \qquad A_3^1 = \{b\} \\
&\mathcal{P}^2: \quad A_1^2 = \{d\}, \qquad A_2^2 = \{a, c\}, \quad A_3^2 = \{b\} \\
&\mathcal{P}^3: \quad A_1^3 = \{b\}, \qquad A_2^3 = \{a, c\}, \quad A_3^3 = \{d\} \\
&\mathcal{P}^4: \quad A_1^4 = \{b\}, \qquad A_2^4 = \{d\}, \qquad A_3^4 = \{a, c\}
\end{aligned}$$

In addition, the domain in Example 6 defined with respect to $(\mathbf{A}'_t)_{t=1}^4$ is the union of two *restricted tier domains* with the ordered partitions being, respectively, $\mathcal{P}^1 : A_1^1 = \{a, c\}, A_2^1 = \{b, d\}$ and $\mathcal{P}^2 : A_1^2 = \{b, d\}, A_2^2 = \{a, c\}$.

Remark 10 A *sequentially dichotomous domain* can be understood as a lexicographic (non-separable) preference domain. Imagine that the object set is the Cartesian product of T characteristics, each of which takes two values. We illustrate this with the following example. Suppose the objects in consideration are four houses which are labeled according to their relative positions, particularly, house 1 on northeast, house 2 on northwest, house 3 on southeast, and house 4 on southwest. An agent first expresses her taste between south and north. Suppose she prefers houses on the north, then houses 1 and 2 are supposed to be better than houses 3 and 4. Conditional on north, she expresses her taste between east and west. And conditional on south, she expresses her taste between east and west. Suppose she prefers east conditional on north and west conditional on south, her preference on houses is $1 \succ 2 \succ 4 \succ 3$.

Remark 11 A *sequentially dichotomous domain* is a Condorcet domain (Monjardet (2009)), on which the majority rule applies without a cycle. This is a very interesting feature of this domain. In the current paper, we focus on the *sd-strategy-proofness* of the PS rule. This interesting feature of *sequentially dichotomous domains* also guarantees *sd-strategy-proofness* of some random voting rules, for example the maximal lotteries (Fishburn (1984) and Brandl et al. (2016)).

5.2 Strategy-proof probabilistic serial rule

This section shows that the PS rule is *sd-strategy-proof* on a *sequentially dichotomous domain*. We show first, by Lemma 11, that any *sequentially dichotomous domain* is block-connected. Second, Lemma 12 implies *block-adjacent sd-strategy-proofness* of the PS rule on any *sequentially dichotomous domain*. Then, by invoking Theorem 5, we have what is desired.

Lemma 11 *Any sequentially dichotomous domain is path-nested.*

This lemma can be verified easily according to the definitions of the *sequentially dichotomous domain* and path-nestedness. Hence the proof is omitted.

Due to Theorem 5, to show *sd-strategy-proofness* of the PS rule on a *sequentially dichotomous domain*, it suffices to show *block-adjacent sd-strategy-proofness*. Actually, what we can show is slightly stronger:

Lemma 12 *Let $P \in \mathbb{P}^n$, $\tilde{P}_1 \in \mathbb{P}$, if there are two nonempty subsets of objects $A_1, A_2 \subset A$ such that*

1. *for all $i \in I$, P_i clusters objects in A_1, A_2, and $A_1 \cup A_2$ respectively;*
2. *$\tilde{P}_1$ is block-adjacent to P_1 with the reversal between A_1 and A_2 such that $a\ P_1\ b$ and $b\ \tilde{P}_1\ a$ for $a \in A_1$ and $b \in A_2$.*

then:

1. *$PS_{1,a}(P) \geqslant PS_{1,a}(\tilde{P}_1, P_{-1})$ for all $a \in A_1$,*
2. *$PS_{1,b}(P) \leqslant PS_{1,b}(\tilde{P}_1, P_{-1})$ for all $b \in A_2$, and*
3. *$PS_{1,x}(P) = PS_{1,x}(\tilde{P}_1, P_{-1})$ for all $x \in A \backslash A_1 \cup A_2$.*

The proof of Lemma 12 is in Appendix C.1.

Lemma 12 says that if one agent performs a block-adjacent reversal with respect to two blocks $A_1, A_2 \subset A$ and it is known that in every other's preference these two objects are adjacent, then for every object that has moved downward in the deviator's preference, the probability that the deviator gets this object is non-increasing. It is evident that this lemma implies *block-adjacent sd-strategy-proofness* on a *sequentially dichotomous domain*.

The statement of Lemma 12 is stronger than that the PS rule satisfies *block-adjacent sd-strategy-proofness* on a *sequentially dichotomous domain* in two respects. First, the lemma does not require the preferences to be taken from a *sequentially dichotomous domain*. Rather, the requirement is that, in every preference, objects in A_1, A_2, and $A_1 \cup A_2$ be respectively clustered. Second, *block-adjacent sd-strategy-proofness* does not require the probability of every object that moves down the deviator's preference to be non-increasing. Rather, *block-adjacent sd-strategy-proofness* requires only the probability of every upper contour set to be non-increasing. For example, consider a block-adjacent reversal from $abcd$ to $cabd$. *Block-adjacent sd-strategy-proofness* requires that the probability of getting a, b combined be non-increasing. Particularly, it is allowed that the probability of getting b increases as long as the decrease in a's probability exceeds the increase in b's probability.

We are now ready to present the theorem.

Theorem 6 *The PS rule is sd-strategy-proof on a sequentially dichotomous domain.*

The theorem follows directly from Lemma 11, Lemma 12, and Theorem 5.

Remark 12 As the other important random assignment rule, the random serial dictatorship, is known to be *sd-inefficient* on the universal domain. However, what kind of preference restriction guarantees the *sd-efficiency* for it is deserved effort. Some preliminary results are in Appendix E.

5.3 Maximality

From Theorem 6, a *sequentially dichotomous domain* guarantees the *sd-strategy-proofness* of the PS rule. The next interesting question is can we expand a *sequentially dichotomous domain* while preserving the *sd-strategy-proofness* of the PS rule? The answer is negative, as indicated by Theorem 7. The theorem shows that given an arbitrary *sequentially dichotomous domain*, whenever an additional preference is admissible, the PS rule becomes manipulable.

Theorem 7 *A sequentially dichotomous domain is maximal for the probabilistic serial rule to be sd-strategy-proof.*

The proof of Theorem 7 is in Appendix C.2.

In the proof, we fix an arbitrary *sequentially dichotomous domain* $\mathbb{D}$ and an arbitrary preference out of it $\tilde{P}_0 \in \mathbb{P}\backslash\mathbb{D}$. We first compare the fixed preference with the partition-path according to which the *sequentially dichotomous domain* is defined. Then according to such comparison, we identify two preferences within the *sequentially dichotomous domain* $P_0, \bar{P}_0 \in \mathbb{D}$ and then construct two preference profiles consisting of only $\tilde{P}_0, P_0, \bar{P}_0$. In these two preference profiles, one agent unilaterally deviates. Finally we calculate the relevant probabilities specified by the PS rule and show that this deviation is profitable.

5.4 Final remarks

In this chapter, we introduced a class of preference domains, *sequentially dichotomous domains*. We first show that the PS rule is *sd-strategy-proof* on any such domain. Then we show that each such domain is maximal for the PS rule to be *sd-strategy-proof*.

A remaining interesting question is whether the class of *sequentially dichotomous domains* is uniquely maximal. In other words, if we know already that the PS rule is *sd-strategy-proof* on a given domain, can we structure it as a sub-domain of a *sequentially dichotomous domain*? Unfortunately, the next example proves this to be false.

Example 7 Consider a domain $\mathbb{D}$ consisting of only two preferences as follows

$$P_0 : a \succ b \succ c \succ d$$

$$\bar{P}_0 : c \succ a \succ d \succ b$$

It is easy to check that the PS rule is *sd-strategy-proof* on $\mathbb{D}$. However $\mathbb{D}$ can never be structured as a *sequentially dichotomous domain*. The key insight is that we can not find a dichotomous partition that both P_0 and $\bar{P}_0$ respect it.

Interestingly the pattern indicated by $\mathbb{D}$ is an important structure for some computer science studies, for example Rossin and Bouvel (2006). In their language, a permutation, i.e., a linear order like $\bar{P}_0$ given P_0 already present, is "separable" if we can find a partition-path that both P_0 and $\bar{P}_0$ respect it. It seems that separability is a convenient and fundamental structure for a computer either to generate permutations fast or compare two sets of permutations fast.

For CS studies, it is perfectly justified that the pattern of $\mathbb{D}$ is excluded artificially due to computational targets. However, there is no economically reasonable argument, in my understanding, to exclude such a pattern artificially. Hence, although it might be possible that by excluding such pattern, we can establish the uniqueness of the *sequentially dichotomous domain* for the PS rule to be *sd-strategy-proof*, this exercise is not of much economic interest.

However, it is still interesting that we find some economically reasonable richness condition, under which the uniqueness of the *sequentially dichotomous domain* can be established. This is left for future studies.

Another remark is a conjecture: any *sequentially dichotomous domain* is maximal for the existence of a random assignment rule satisfying *sd-strategy-proofness*, *sd-efficiency*, and *equal-treatment-of-equals*. However, even from the results presented in this chapter, verifying this conjecture is still a very difficult question.

6 How to Describe Objects?

This chapter models the preference restrictions as an implication of specific choices of object descriptions. Specifically, each object can be evaluated according to a large number of characteristics. The planner chooses a subset and a ranking of the chosen ones. Then she describes each object to the agents as a sequence of values according to the ranking of chosen characteristics. After that each agent reports a lexicographically separable preference according to the informed description. So a specific description induces a collection of admissible preferences.

The question we ask is what are the descriptions which induce the good preference domains, in the sense that there is an *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule. The answer provided is a characterization of these descriptions: binary trees, i.e., conditional on each feasible combinations of the top-t characteristics, the next characteristic takes two feasible values. This characterization is under two technical assumptions. (See Theorem 8.)

In addition, the domain induced by a binary tree is shown to be covered by a *sequentially dichotomous domain*. Hence the PS rule is *sd-strategy-proof* on such a domain. This result in addition to the other nice properties suggests the use of the PS rule.

The next section formally defines the notions mentioned above. Section 2 presents the results and the last section concludes.

6.1 Descriptions of objects and induced preference domains

Let $\mathcal{C}$ denote the collection of characteristics, according to each of which an object can be evaluated. Hence the object set is a subset of a Cartesian product $A \subset \prod_{c \in \mathcal{C}} A_c$ where A_c is the collection of all possible values of characteristic c. For each object $a \in A$ and each characteristic $c \in \mathcal{C}$, we denote object's value of characteristic c by a_c. Without loss of generality, we assume no unused characteristic value, i.e., for each $c \in \mathcal{C}$ and $v \in A_c$ there is an object $a \in A$ such that $a_c = v$.

Given $A \subset \prod_{c \in \mathcal{C}} A_c$, the planner chooses a subset of the characteristics and specifies a ranking of these chosen characteristics. We call the pair of the chosen subset and the ranking of characteristics a description of the objects. Formally it is defined as below.

Definition 12 *A **description** of the object set $A \subset \prod_{c \in \mathcal{C}} A_c$ is a pair (C, σ) where*

- *$C \subset \mathcal{C}$ is a subset of characteristics such that for each pair of distinct objects $x, y \in A$ there is a characteristic $c \in C$ that gives $x_c \neq y_c$, and*
- *σ is an one-to-one mapping $\sigma : C \to \{1, \cdots, |C|\}$.*

The condition imposed on the chosen set of characteristics is requiring that the chosen characteristics be sufficiently informative so that an agent can differentiate each object from the others according to the description given by the planner.

Given a description (C, σ), an $t \in \{1, \cdots, |C|\}$, and an object $a \in A$, let $a_t^\sigma \equiv a_{\sigma^{-1}(t)}$ be a's value of the characteristic which is t-th ranked according to σ and let $A_t^\sigma \equiv A_{\sigma^{-1}(t)}$ be the admissible value set of the t-th ranked characteristic. In addition, for any $t \in \{2, \cdots, |C|\}$ and any combination of values of the top-$(t-1)$ ranked characteristics $(v_1, \cdots, v_{t-1}) \in \prod_{\tau=1}^{t-1} A_\tau^\sigma$, let $A_t^\sigma|(v_1, \cdots, v_{t-1}) \equiv \{v_t \in A_t^\sigma | \exists a \in A \text{ s.t. } (a_1^\sigma, \cdots, a_{t-1}^\sigma, a_t^\sigma) = (v_1, \cdots, v_{t-1}, v_t)\}$. For notational convenience, we write $A_1^\sigma | v_0 \equiv A_1^\sigma$.

After choosing a description (C, σ), the planner disclose the information of objects to the agents. Facing the chosen description of objects, an agent compares each pair of objects lexicographically according to the characteristics in C and the ranking specified by σ. Formally the preference and the collection of these preferences are defined as below.

Definition 13 *Given a description (C, σ) of $A \subset \prod_{c \in C} A_c$, a preference $P_0 \in \mathbb{P}$ is **lexicographically separable with respect to** (C, σ) if there is a strict preference P_0^c on each A_c such that $x \ P_0 \ y$ if and only if there exists $t \in \{1, \cdots, |C|\}$ s.t. $\left(x_1^\sigma, \cdots, x_{t-1}^\sigma\right) = \left(y_1^\sigma, \cdots, y_{t-1}^\sigma\right)$ and $x_t^\sigma \ P_0^{\sigma^{-1}(t)} \ y_t^\sigma$.*

*In addition, let **the domain induced by the description** (C, σ) be the collection of* all *lexicographically separable preferences with respect to (C, σ) and denoted as $\mathbb{D}_{(C,\sigma)}$.*

The preferences on A_c's that spell a lexicographically separable preference are called marginal preferences. We now present the example discussed in the introduction in the language just defined.

Example 8 *Consider the object set illustrated by Table 2. Let* $\mathcal{C} \equiv \{c, c', c''\}$*,* $A_c \equiv \{1, 2\}$*,* $A_{c'} \equiv \{a, b, c\}$*, and* $A_{c''} \equiv \{x, y\}$*. Now the object set in Table 2 can be expressed as a subset of* $\prod_{c \in \mathcal{C}} A_c$*.*

Consider the description illustrated in Table 4. The characteristic subset chosen is $C \equiv \{c, c'\}$ *and the ranking is* $\sigma(c') = 1$ *and* $\sigma(c) = 2$*. The domain induced by description* (C, σ) *is such one that includes* P_1 *to* P_{24} *and is described as follows.*

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_3	o_3	o_3	o_3
o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_1	o_4	o_1	o_4
o_3	o_3	o_3	o_3	o_2	o_2	o_5	o_5	o_4	o_1	o_4	o_1
o_2	o_2	o_5	o_5	o_5	o_5	o_2	o_2	o_2	o_2	o_5	o_5
o_5	o_5	o_2	o_2	o_3	o_3	o_3	o_3	o_5	o_5	o_2	o_2

P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}	P_{23}	P_{24}
o_3	o_3	o_3	o_3	o_2	o_2	o_5	o_5	o_2	o_2	o_5	o_5
o_2	o_2	o_5	o_5	o_5	o_5	o_2	o_2	o_5	o_5	o_2	o_2
o_5	o_5	o_2	o_2	o_3	o_3	o_3	o_3	o_1	o_4	o_1	o_4
o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_4	o_1	o_4	o_1
o_4	o_1	o_4	o_1	o_4	o_1	o_4	o_1	o_3	o_3	o_3	o_3

Consider another description, illustrated in Table 5. Now the description is $(\bar{C}, \bar{\sigma})$ *where* $\bar{C} \equiv \{c, c', c''\}$ *and* $\bar{\sigma}$ *is such that* $\bar{\sigma}(c) = 1$*,* $\bar{\sigma}(c') = 3$*, and* $\bar{\sigma}(c'') = 2$*. Then the domain induced by this description is such that includes all preferences* $\bar{P}_1$ *to* $\bar{P}_{16}$ *below.*

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$	$\bar{P}_6$	$\bar{P}_7$	$\bar{P}_8$	$\bar{P}_9$	$\bar{P}_{10}$	$\bar{P}_{11}$	$\bar{P}_{12}$	$\bar{P}_{13}$	$\bar{P}_{14}$	$\bar{P}_{15}$	$\bar{P}_{16}$
o_1	o_1	o_1	o_1	o_2	o_3	o_2	o_3	o_4	o_5	o_4	o_5	o_4	o_5	o_4	o_5
o_2	o_3	o_2	o_3	o_3	o_2	o_3	o_2	o_5	o_4	o_5	o_4	o_5	o_4	o_5	o_4
o_3	o_2	o_3	o_2	o_1	o_1	o_1	o_1	o_1	o_1	o_1	o_1	o_2	o_2	o_3	o_3
o_4	o_4	o_5	o_5	o_4	o_4	o_5	o_5	o_2	o_2	o_3	o_3	o_3	o_3	o_2	o_2
o_5	o_5	o_4	o_4	o_5	o_5	o_4	o_4	o_3	o_3	o_2	o_2	o_1	o_1	o_1	o_1

6.2 Results

This section presents two results. First, whenever the problem size satisfies the two technical assumptions 1 and 2, the objects should be described as a binary tree since this is the only way the induced domain admits an acceptable rule.

Theorem 8 *Let (C,σ) be a description and $\mathbb{D}_{(C,\sigma)}$ the corresponding induced domain. In addition, let n satisfy the Assumptions 1 and 2. If there is an sd-strategy-proof sd-efficient and equal-treatment-of-equals rule defined on $\mathbb{D}_{(C,\sigma)}$, then $\left|A_t^{\sigma}|(v_1,\cdots,v_{t-1})\right| \leqslant 2$ for all $t \in \{1,\cdots,|C|\}$ and $(v_1,\cdots,v_{t-1}) \in \prod_{\tau=1}^{t-1} A_{\tau}^{\sigma}$, i.e., the description is a binary tree.*

In order to prove Theorem 8, we show an impossibility result, which states that whenever a domain exhibits the "block elevating" property and two technical assumptions are satisfied, there is no possibility of finding an acceptable rule. We first formally define the block elevating property and then the impossibility.

A domain $\mathbb{D}$ satisfies the **block elevating property** if there are three admissible preferences and three nonempty blocks such that the block (can be empty) ranked above these three blocks in all three preferences is the same, three blocks are ranked next to each other in all three preferences, one block is ranked last among the three blocks in two of the three preferences and the second among the three in the third preference; formally:

Definition 14 *A domain $\mathbb{D}$ satisfies the **block elevating property** if there are three preferences $\bar{P}_0, P_0, \hat{P}_0 \in \mathbb{D}$, three nonempty blocks $B, C, D \subset A$ and two blocks $E, F \subset A$ which can be empty such that $B \cup C \cup D \cup E \cup F = A$ and three preferences are as follows, where $B\ P_0\ C$ means $b\ P_0\ c$ for all $b \in B$ and $c \in C$.*

$$\begin{array}{ccccccccc}
E & \bar{P}_0 & B & \bar{P}_0 & D & \bar{P}_0 & C & \bar{P}_0 & F \\
E & P_0 & B & P_0 & C & P_0 & D & P_0 & F \\
E & \hat{P}_0 & C & \hat{P}_0 & B & \hat{P}_0 & D & \hat{P}_0 & F
\end{array}$$

Table 7: The Block Elevating Property

The block elevating property is a generalization of the elevating property in Table 6, which requires all three blocks to be singletons. The impossibility with respect to the block elevating property is as follows.

Proposition 3 *Let $\mathbb{D}$ be a domain satisfying block elevating property. If n satisfies Assumptions 1 and 2, then $\mathbb{D}$ admits no sd-strategy-proof, sd-efficient, and equal-treatment-of-equals rule.*

The proof of Proposition 3 is in Appendix D.3. The proof is by contradiction, i.e., suppose $\mathbb{D}$ satisfies block elevating property and admits an acceptable rule, then we specify a series of preference profiles consisting of only three preferences illustrated in Table 7. We characterize the random assignments of these profiles according to the three axioms. Finally a contradiction is identified.

The identification of the contradiction relies on two assumptions, each of which compares zero with an expression of a floor function[20], conditional on that the real number is strictly larger than the integer identified by the floor function for this real number. I am unable to verify the comparisons analytically. However, I provide two Matlab codes to verify, for a specific n, whether these two assumptions hold, and I have verified them to be true for all n no larger than 1000. As to 1000, it is not that when n goes beyond it the impossibility fails but that my laptop is not that powerful.

Proof: [Proof of Theorem 8] In stead of showing it directly, we show its contrapositive statement. Let (C, σ) be a description and $\mathbb{D}_{(C,\sigma)}$ its induced domain, in addition let $t^* \in \{1, \cdots, |C|\}$ and $(v_1, \cdots, v_{t^*-1}) \in \prod_{\tau=1}^{t^*-1} A^\sigma_\tau$ be such that $\left| A^\sigma_{t^*} | (v_1, \cdots, v_{t^*-1}) \right| \geqslant 3$, we show that $\mathbb{D}_{(C,\sigma)}$ satisfies the block elevating property.

Pick any three values $u_{t^*}, u'_{t^*}, u''_{t^*} \in A^\sigma_{t^*} | (v_1, \cdots, v_{t^*-1})$ and let $B \equiv \{b \in A | (b^\sigma_1, \cdots, b^\sigma_{t^*-1}) = (v_1, \cdots, v_{t^*-1})$ and $b^\sigma_{t^*} = u_{t^*}\}$, $C \equiv \{c \in A | (c^\sigma_1, \cdots, c^\sigma_{t^*-1}) = (v_1, \cdots, v_{t^*-1})$ and $c^\sigma_{t^*} = u'_{t^*}\}$, and $D \equiv \{d \in A | (d^\sigma_1, \cdots, d^\sigma_{t^*-1}) = (v_1, \cdots, v_{t^*-1})$ and $d^\sigma_{t^*} = u''_{t^*}\}$. Consider the marginal preferences $(\bar{P}^c_0)_{c \in C}$, $(P^c_0)_{c \in C}$, and $(\hat{P}^c_0)_{c \in C}$ such that

$$
\begin{aligned}
& r_1(\bar{P}_0^{\sigma^{-1}(\tau)}) = r_1(P_0^{\sigma^{-1}(\tau)}) = r_1(\hat{P}_0^{\sigma^{-1}(\tau)}) = v_\tau && \text{for all } \tau \leqslant t^* - 1 \\
& u_{t^*} \ \bar{P}_0^{\sigma^{-1}(t^*)} \ u''_{t^*} \ \bar{P}_0^{\sigma^{-1}(t^*)} \ u'_{t^*} \ \bar{P}_0^{\sigma^{-1}(t^*)} \ v_{t^*} && \text{for all } v_{t^*} \in A^\sigma_{t^*} \backslash \{u_{t^*}, u'_{t^*}, u''_{t^*}\} \\
& u_{t^*} \ P_0^{\sigma^{-1}(t^*)} \ u'_{t^*} \ P_0^{\sigma^{-1}(t^*)} \ u''_{t^*} \ P_0^{\sigma^{-1}(t^*)} \ v_{t^*} && \text{for all } v_{t^*} \in A^\sigma_{t^*} \backslash \{u_{t^*}, u'_{t^*}, u''_{t^*}\} \\
& u'_{t^*} \ \hat{P}_0^{\sigma^{-1}(t^*)} \ u_{t^*} \ \hat{P}_0^{\sigma^{-1}(t^*)} \ u''_{t^*} \ \hat{P}_0^{\sigma^{-1}(t^*)} \ v_{t^*} && \text{for all } v_{t^*} \in A^\sigma_{t^*} \backslash \{u_{t^*}, u'_{t^*}, u''_{t^*}\}
\end{aligned}
$$

It's evident that these marginal preferences give the following preferences $\bar{P}_0, P_0, \hat{P}_0 \in \mathbb{D}_{(C,\sigma)}$, which completes the proof.

[20] A floor function identifies for a real number the largest integer no larger than the real number itself.

$$
\begin{array}{ccccccc}
B & \bar{P}_0 & D & \bar{P}_0 & C & \bar{P}_0 & A\backslash(B \cup C \cup D) \\
B & P_0 & C & P_0 & D & P_0 & A\backslash(B \cup C \cup D) \\
C & \hat{P}_0 & B & \hat{P}_0 & D & \hat{P}_0 & A\backslash(B \cup C \cup D)
\end{array}
$$

■

Given Theorem 8, an interesting question arises: what rule can we use on a domain induced by a binary tree? The following result shows that the PS rule is *sd-strategy-proof* on the domain induced by a binary tree.

Theorem 9 *Let (C, σ) be a description and $\mathbb{D}_{(C,\sigma)}$ the corresponding induced domain. If (C, σ) is a binary tree, then the PS rule is sd-strategy-proof on $\mathbb{D}_{(C,\sigma)}$.*

The proof of Theorem 9 is in Appendix D.4.

We prove the theorem by showing that when the description is a binary tree, the induced domain is a sub-domain of a sequentially dichotomous domain. Then Theorem 6 implies the desired conclusion.

6.3 Final remarks

The result in this chapter suggests that, if the planner believes that agents report preferences lexicographically separable according to the ranking of the chosen characteristics, she should describe the objects as a binary tree, i.e., given any feasible values of the top-t ranked characteristics, the following up characteristic can take at most two feasible values. In addition, due to the fact that the PS rule is *sd-strategy-proof* on the domain induced by a binary tree, she should use the PS rule to allocate the objects after agents report their preferences.[21]

However since I assume explicitly that the problem size n satisfies two technical assumptions, before following the above suggestions, the planner needs to check these two assumptions. Although I can not prove them to be true analytically, I conjecture that they are true. For applications, a planner can use the Matlab code I provide to check whether

[21]In addition to *sd-strategy-proofness*, PS rule is favored also in the sense that it satisfies *sd-efficiency* and *sd-envy-freeness*, a fairness axiom stronger than *equal treatment of equals*.

these two assumptions hold. In addition, if the problem size is smaller than 1000, I have already checked them to be true so the suggestion above can be adopted directly.

7 Conclusion and Further Research

This dissertation studies the scope of designing a satisfactory random assignment rule and provides some answers. Specifically, whenever the preference domain is connected, it is nearly impossible to find a satisfactory rule. However, there is a very large class of unconnected domains, i.e., *sequentially dichotomous domains*, on which possibility holds.

Besides the answers, I think this dissertation raised several following-up questions. Some of them are listed for further research. First, under what reasonable richness conditions, can the *sequentially dichotomous domain* be verified as the unique maximal domains for the PS rule being *sd-strategy-proof*? Second, I conjecture that any *sequentially dichotomous domain* is maximal for the existence of an *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule. The last conjecture is with respect to the characterization of the binary trees as the unique ones that induce good domains. Specifically, it would be good if the two technical assumptions imposed can be verified analytically.

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A Appendix to Chapter 3

A.1 Proof of Lemma 1

Let $\bar{P} \in \mathbb{D}^n$ be such that $\bar{P}_i = \bar{P}_j$ for all $i, j \in I$. Then *equal treatment of equals* implies $\varphi_{ia}(\bar{P}) = \frac{1}{n}$ for all $i \in I$ and all $a \in A$. Hence, $\varphi_{iA_k}(\bar{P}) = \frac{|A_k|}{n}$ for all $i \in I$ and all $k \in \{1, \ldots, T\}$. According to P and $\bar{P}$, we can separate all agents into two groups: $\hat{I} = \{i \in I | P_i \neq \bar{P}_i\}$ and $I \backslash \hat{I} = \{i \in I | P_i = \bar{P}_i\}$. Given $S \subseteq \hat{I}$, let $\bar{P}^S \in \mathbb{D}^n$ be such that $\bar{P}^S_i = P_i$ for all $i \in S$ and $\bar{P}^S_i = \bar{P}_i$ for all $i \notin S$. Thus, $\bar{P}^S = (P_S, \bar{P}_{-S})$. Evidently, $\bar{P}^\emptyset = \bar{P}$ and $\bar{P}^{\hat{I}} = P$.

Now, given $i \in \hat{I}$, *sd-strategy-proofness* implies $\sum_{t=1}^k \varphi_{iA_t}(\bar{P}^{\{i\}}) = \sum_{t=1}^k \varphi_{iA_t}(\bar{P}^\emptyset)$ for all $k \in \{1, \ldots, T\}$, which in turn implies $\varphi_{iA_k}(\bar{P}^{\{i\}}) = \frac{|A_k|}{n}$ for all $k \in \{1, \ldots, T\}$. Furthermore, *equal treatment of equals* implies $\varphi_{jA_k}(\bar{P}^{\{i\}}) = \frac{|A_k|}{n}$ for all $j \neq i$ and all $k \in \{1, \ldots, T\}$. Therefore, given $i \in \hat{I}$, $\varphi_{jA_k}(\bar{P}^{\{i\}}) = \frac{|A_k|}{n}$ for all $j \in I$ and all $k \in \{1, \ldots, T\}$. We continue with an induction argument on $S \subseteq \hat{I}$.

Induction Hypothesis: Given $1 < l \leqslant |\hat{I}|$, for all $S \subseteq \hat{I}$ with $1 \leqslant |S| \leqslant l - 1$, we have $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \in I$ and $k \in \{1, \ldots, T\}$.

Let $S \subseteq \hat{I}$ with $|S| = l$. We will show $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \in I$ and all $k \in \{1, \ldots, T\}$. Given $j \in S$, by *sd-strategy-proofness* and induction hypothesis, $\sum_{t=1}^k \varphi_{jA_t}(\bar{P}^S) =$
$\sum_{t=1}^k \varphi_{jA_t}(\bar{P}^{S \backslash \{j\}}) = \sum_{t=1}^k \frac{|A_t|}{n}$ for all $k \in \{1, \ldots, T\}$, which in turns implies $\varphi_{jA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $k \in \{1, \ldots, T\}$. Furthermore, *equal treatment of equals* implies $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \notin S$ and all $k \in \{1, \ldots, T\}$. Therefore, $\varphi_{iA_k}(\bar{P}^S) = \frac{|A_k|}{n}$ for all $i \in I$ and all $k \in \{1, \ldots, T\}$. This completes the verification of induction hypothesis. Therefore, $L_{iA_k} = \frac{|A_k|}{n}$ for all $i \in I$ and $k \in \{1, \ldots, T\}$.

A.2 Proof of Lemma 2

Assume $|I_k| = l$. If $l = 0$, statements (i) and (ii) are satisfied vacuously. Henceforth, assume $1 \leqslant l \leqslant n$. We consider three cases.

Case 1: $l = 1$.

Statement (i) is satisfied vacuously. Assume $I_k = \{i\}$. By *sd-efficiency*, either $L_{ib} = 0$, or $L_{ja} = 0$ for all $j \neq i$. Suppose $L_{ib} > 0$. Then, $L_{ja} = 0$ for all $j \neq i$. Consequently, $L_{ia} = 1$ and $L_{ia} + L_{ib} > 1$. Contradiction! Therefore, $L_{ib} = 0$. Then, Lemma 1 implies $L_{ia} = \frac{2}{n}$. Moreover, since $L_{ja} + L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$, it is evident that $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$ for all $i \in I_k$ and all $j \in I \backslash I_k$. This completes the verification of statement (ii) in Case 1.

Case 2: $l = n$.

Statement (ii) is satisfied vacuously. We focus on statement (i). Let $\bar{P} \in \mathbb{D}^n$ be such that $\bar{P}_i = \bar{P}_j$ for all $i, j \in I$ and $a\ \bar{P}_i\ b$ for all $i \in I$. Thus, according to P and $\bar{P}$, we can separate all agents into two groups: $\hat{I} = \{i \in I | P_i \neq \bar{P}_i\}$ and $I \backslash \hat{I} = \{i \in I | P_i = \bar{P}_i\}$. Given $S \subseteq \hat{I}$, let $\bar{P}^S \in \mathbb{D}^n$ be such that $\bar{P}_i^S = \bar{P}_i$ for all $i \in \hat{I} \backslash S$ and $\bar{P}_i^S = P_i$ for all $i \notin \hat{I} \backslash S$. Thus, $\bar{P}^{\emptyset} = \bar{P}$ and $\bar{P}^{\hat{I}} = P$. First, *equal treatment of equals* implies $\varphi_{ia}(\bar{P}) = \frac{1}{n}$ for all $i \in I = I_k$. Next, we provide an induction argument on S.

Induction Hypothesis: Given $0 < s \leqslant |\hat{I}|$, for all $S \subseteq \hat{I}$ with $0 \leqslant |S| < s$ and all $i \in I$, we have $\varphi_{ia}(\bar{P}^S) = \frac{1}{n}$.

Given $S \subseteq \hat{I}$ with $|S| = s$, we show $\varphi_{ia}(\bar{P}^S) = \frac{1}{n}$ for all $i \in I$. Given $i \in S$, *sd-strategy-proofness* and induction hypothesis imply $\varphi_{ia}(\bar{P}^S) = \varphi_{ia}(P_i, \bar{P}_{-i}^S) = \varphi_{ia}(\bar{P}_i, \bar{P}_{-i}^S) = \varphi_{ia}(\bar{P}^{S \backslash \{i\}}) = \frac{1}{n}$. Furthermore, in $\bar{P}^S$, for all $j \in I \backslash S$, *equal treatment of equals* implies $\varphi_{ja}(\bar{P}^S) = \frac{1 - \sum_{i \in S} \varphi_{ia}(\bar{P}^S)}{|I \backslash S|} = \frac{1 - s \times \frac{1}{n}}{n - s} = \frac{1}{n}$. Therefore, $\varphi_{ia}(\bar{P}^S) = \frac{1}{n}$ for all $i \in I$. This completes the verification of induction hypothesis. Therefore, $L_{ia} = L_{ja}$ for all $i, j \in I_k = I$. This completes the verification of statement (i) in Case 2.

Case 3: $1 < l < n$.

First, *sd-efficiency* implies either $L_{ib} = 0$ for all $i \in I_k$, or $L_{ja} = 0$ for all $j \in I \backslash I_k$. If $L_{ib} = 0$ for all $i \in I_k$, then Lemma 1 implies $L_{ia} = \frac{2}{n}$ for all $i \in I_k$. Thus, $L_{ia} = L_{ja}$ for all $i, j \in I_k$. Moreover, since $L_{ja} + L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$, it is evident that $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$ for all $i \in I_k$ and $j \in I \backslash I_k$.

Next, assume $L_{ja} = 0$ for all $j \in I \backslash I_k$, and $L_{ib} > 0$ for some $i \in I_k$. By Lemma 1, $L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$. Moreover, since $L_{ia} + L_{ib} = \frac{2}{n}$ for all $i \in I_k$, it is evident that $L_{ia} \geqslant L_{ja}$ and $L_{ib} \leqslant L_{jb}$ for all $i \in I_k$ and $j \in I \backslash I_k$. Hence, statement (ii) is verified.

Last, we verify statement (i). We first claim $l > \frac{n}{2}$. Suppose not, i.e., $l \leqslant \frac{n}{2}$ and hence $|I \backslash I_k| = n - l \geqslant \frac{n}{2}$. Since $L_{ja} = 0$ for all $j \in I \backslash I_k$, Lemma 1 implies $L_{jb} = \frac{2}{n}$ for all $j \in I \backslash I_k$. Consequently, $\sum_{i \in I} L_{ib} = \sum_{i \in I_k} L_{ib} + \sum_{j \in I \backslash I_k} L_{jb} = \sum_{i \in I_k} L_{ib} + (n-l)\frac{2}{n} > 1$. Contradiction! Therefore, $l > \frac{n}{2}$.

Let $\bar{P} \equiv (\bar{P}_{I_k}, P_{-I_k}) \in \mathbb{D}^n$ be such that $\bar{P}_i = \bar{P}_j$ for all $i, j \in I_k$, and $a \;\; \bar{P}_i \;\; b$ for all $i \in I_k$. We divide I_k into two groups: $\hat{I} = \{i \in I_k | P_i \neq \bar{P}_i\}$ and $I_k \backslash \hat{I} = \{i \in I_k | P_i = \bar{P}_i\}$. Given $S \subseteq \hat{I}$, let $\bar{P}^S \in \mathbb{D}^n$ be such that $\bar{P}_i^S = \bar{P}_i$ for all $i \in \hat{I} \backslash S$, and $\bar{P}_i^S = P_i$ for all $i \notin \hat{I} \backslash S$. Evidently, $\bar{P}^{\emptyset} = \bar{P}$ and $\bar{P}^{\hat{I}} = P$.

Since $l > \frac{n}{2}$, *sd-efficiency* implies $\varphi_{ja}(\bar{P}^{\emptyset}) = 0$ for all $j \in I \backslash I_k$, and hence $\sum_{i \in I_k} \varphi_{ia}(\bar{P}^{\emptyset}) = 1$. Moreover, since *equal treatment of equals* implies $\varphi_{ia}(\bar{P}^{\emptyset}) = \varphi_{ja}(\bar{P}^{\emptyset})$ for all $i, j \in I_k$, it is true that $\varphi_{ia}(\bar{P}^{\emptyset}) = \frac{1}{l}$ for all $i \in I_k$. Next, we provide an induction argument on S.

Induction Hypothesis: Given $0 < s \leqslant |\hat{I}|$, for all $S \subseteq \hat{I}$ with $0 \leqslant |S| < s$, $\varphi_{ia}(\bar{P}^S) = \frac{1}{l}$ for all $i \in I_k$.

Let $S \subseteq I_k$ with $|S| = s$. We show $\varphi_{ia}(\bar{P}^S) = \frac{1}{l}$ for all $i \in I_k$. Since $l > \frac{n}{2}$, *sd-efficiency* implies $\varphi_{ja}(\bar{P}^S) = 0$ for all $j \in I \backslash I_k$. Thus, $\sum_{i \in I_k} \varphi_{ia}(\bar{P}^S) = 1$. Given $i \in S$, *sd-strategy-proofness* and induction hypothesis imply $\varphi_{ia}(\bar{P}^S) = \varphi_{ia}(P_i, \bar{P}^S_{-i}) = \varphi_{ia}(\bar{P}_i, \bar{P}^S_{-i}) = \varphi_{ia}(\bar{P}^{S \backslash \{i\}}) = \frac{1}{l}$. Furthermore, in $\bar{P}^S$, for all $j \in I_k \backslash S$, *equal treatment of equals* implies $\varphi_{ja}(\bar{P}^S) = \frac{1 - \sum_{i \in S} \varphi_{ia}(\bar{P}^S)}{l-s} = \frac{1 - s \times \frac{1}{l}}{l-s} = \frac{1}{l}$. Therefore, $\varphi_{ia}(\bar{P}^S) = \frac{1}{l}$ for all $i \in I_k$. This completes the verification of induction hypothesis. Therefore, $L_{ia} = \frac{1}{l}$ for all $i \in I_k$, and hence, $L_{ia} = L_{ja}$ for all $i, j \in I_k$. This completes the verification of statement (i) in Case 3, and hence the lemma.

A.3 Proof of Theorem 3

To prove Theorem 3, it suffices to show that if domain $\mathbb{D}$ satisfies the *elevating property*, there exists no *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule.

Suppose that $\mathbb{D}$ satisfy the *elevating property*, e.g., domain $\mathbb{D}$ contains preferences $\bar{P}_i, P_i, \hat{P}_i$ in Table 6. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be a rule satisfying *sd-strategy-proofness*, *sd-efficiency*, and *equal treatment of equals*. Let $\bar{n} \equiv \frac{n}{2}$ if n is even, and $\bar{n} \equiv \frac{n-1}{2}$ if n is odd. We search for a contradiction. We first provide the sketch of proofs.

We consider the following four groups of preference profiles: **Profile Groups I - IV**. In particular, for the case of odd number of agents, we consider two additional groups of preference profiles: **Profile Groups V** and **VI**. See Table 8 below. Note that every preference profile in these groups consists of only preferences of $\bar{P}_i$, P_i and $\hat{P}_i$.

Profile Group I: n **is either even or odd**	**Profile Group II:** n **is either even or odd**
$P^{1,0} = (P_1, P_2, \ldots, P_n)$	
$P^{1,1} = (\hat{P}_1, P_2, \ldots, P_n)$	$P^{2,1} = (P_1, P_2, \ldots, P_{n-1}, \bar{P}_n)$
$\vdots$	$P^{2,2} = (\hat{P}_1, P_2, \ldots, P_{n-1}, \bar{P}_n)$
$\vdots$	$\vdots$
$P^{1,m} = (\hat{P}_1, \ldots, \hat{P}_m, P_{m+1}, \ldots, P_n)$	$P^{2,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots, P_{n-1}, \bar{P}_n)$
$\vdots$	$\vdots$
$P^{1,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}}, P_{\bar{n}+1}, \ldots, P_n)$	$P^{2,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, \mathbf{P_{\bar{n}}}, P_{\bar{n}+1}, \ldots, P_{n-1}, \bar{P}_n)$
	$P^{2,\bar{n}+1} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, \hat{P}_{\bar{n}}, P_{\bar{n}+1}, \ldots, P_{n-1}, \bar{P}_n)$
Profile Group III: n **is either even or odd**	**Profile Group IV:** n **is either even or odd**
$P^{3,0} = (\hat{P}_1, \ldots, \hat{P}_{n-1}, \hat{P}_n)$	
$P^{3,1} = (\hat{P}_1, \ldots, \hat{P}_{n-1}, P_n)$	$P^{4,1} = (\hat{P}_1, \ldots, \hat{P}_{n-2}, \hat{P}_{n-1}, \bar{P}_n)$
$\vdots$	$P^{4,2} = (\hat{P}_1, \ldots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n)$
$\vdots$	$\vdots$
$P^{3,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_n)$	$P^{4,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_{n-1}, \bar{P}_n)$
$\vdots$	$\vdots$
$P^{3,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{n-\bar{n}}, P_{n-\bar{n}+1}, \ldots, P_n)$	$P^{4,\bar{n}} = (\hat{P}_1, \ldots, \mathbf{\hat{P}_{n-\bar{n}}}, P_{n-\bar{n}+1}, \ldots, P_{n-1}, \bar{P}_n)$
Profile Group V: n **is odd**	**Profile Group VI:** n **is odd**
$P^{5,1} = (P_1, P_2, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$	$P^{6,1} = (\hat{P}_1, \ldots, \hat{P}_{n-3}, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n)$
$P^{5,2} = (\hat{P}_1, P_2, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$	$P^{6,2} = (\hat{P}_1, \ldots, \hat{P}_{n-3}, \hat{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$
$\vdots$	$P^{6,3} = (\hat{P}_1, \ldots, \hat{P}_{n-3}, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$
$\vdots$	$\vdots$
$P^{5,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$	$P^{6,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$
$\vdots$	$\vdots$
$P^{5,\bar{n}} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, P_{\bar{n}}, P_{\bar{n}+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$	$P^{6,\bar{n}} = (\hat{P}_1, \ldots, \mathbf{\hat{P}_{n-\bar{n}}}, P_{n-\bar{n}+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$
$P^{5,\bar{n}+1} = (\hat{P}_1, \ldots, \hat{P}_{\bar{n}-1}, \hat{P}_{\bar{n}}, \mathbf{P_{\bar{n}+1}}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$	

Table 8: Preference Profile Groups

We first show that for every preference profile of each profile group and every agent, the sum of probabilities over objects a, b and c equals to $\frac{3}{n}$ (see Lemma 13). Then, in the rest of verification, we only focus on the random assignments of objects a, b and c.

At every preference profile in profile groups I - IV, we fully characterize the random assignment of objects a, b and c(see Claims 1 - 5). Then, we realize that when n is an even number, the probability of assigning object c to agent $\bar{n}$ under profile $P^{2,\bar{n}}$ is distinct from that under profile $P^{4,\bar{n}}$. This formulates a contradiction against *sd-strategy-proofness* since from $P^{2,\bar{n}}$ to $P^{4,\bar{n}}$, agent $\bar{n}$ unilaterally deviates from P_i to $\hat{P}_i$, and object c shares the same upper contour set in both P_i and $\hat{P}_i$.

When n is an odd number, in addition to profile groups I - IV, we consider profile groups V and VI. At every preference profile in both profile groups V and VI, we focus on characterizing probabilities of assigning object c to every agent (see Claims 6 - 8). Eventually, we observe that the probability of assigning object c to agent $\bar{n}+1$ under profile $P^{5,\bar{n}+1}$ is distinct from that under profile $P^{6,\bar{n}}$. This formulates a similar contradiction against *sd-strategy-proofness*.

Lemma 13 *For every profile P in profile groups I - VI, $\varphi_{ia}(P) + \varphi_{ib}(P) + \varphi_{ic}(P) = \frac{3}{n}$.*

Proof: The verification of this lemma is routine. In each profile group, repeatedly applying *sd-strategy-proofness* and *equal treatment of equals*, we have the result. Due to the tediousness, we omit the detailed proof. ■

Now, we consider profile groups I - IV. According to Lemma 13, we only focus on the random assignments over a, b and c in each preference profile.

Claim 1 *In profile group I, for each $m = 0, 1, \ldots, \bar{n}$, the random assignment $\varphi(P^{1,m})$ over a, b and c is specified below*

	a	b	c
1	0	$\frac{2}{n}$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	0	$\frac{2}{n}$	$\frac{1}{n}$
$m+1$	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n}$

Proof: For $m = 0$, *equal treatment of equals* implies $\varphi_{ix}(P^{1,0}) = \frac{1}{n}$ for all $i \in I$ and $x \in \{a, b, c\}$.

Next, we show $\varphi_{ic}(P^{1,m}) = \frac{1}{n}$ for all $i \in I$ and $m = 1, \ldots, \bar{n}$. We specify an induction hypothesis: given $1 \leqslant m \leqslant \bar{n}$, for all $0 \leqslant l < m$, $\varphi_{ic}(P^{1,l}) = \frac{1}{n}$ for all $i \in I$. We will show $\varphi_{ic}(P^{1,m}) = \frac{1}{n}$ for all $i \in I$. Notice that profiles $P^{1,m-1}$ and $P^{1,m}$ are different only in agent m's preference, i.e., $P_m^{m-1} = P_i$ and $P_m^m = \hat{P}_i$ in Table 6. Then *sd-strategy-proofness* and induction hypothesis imply $\varphi_{mc}(P^{1,m}) = \varphi_{mc}(P^{1,m-1}) = \frac{1}{n}$. Moreover, *equal treatment of equals* and feasibility imply $\varphi_{ic}(P^{1,m}) = \frac{1}{n}$ for all $i \in I$.

Last, for $m = 1, \ldots, \bar{n}$, note that $|\{1, \ldots, m\}| \leqslant \frac{n}{2}$ and all agents in $\{m+1, \ldots, n\}$ prefer a to b in profile $P^{1,m}$. Consequently, by *sd-efficiency*, feasibility and Lemma 13, we have $\varphi_{ia}(P^{1,m}) = 0$ for all $i = 1, \ldots, m$.

Finally, by Lemma 13, feasibility and *equal treatment of equals*, we have the claim. ■

Claim 2 *In profile group II, the random assignment $\varphi(P^{2,1})$ over a, b and c is specified below*

	a	b	c
1	$\frac{1}{n}$	$\frac{1}{n-1}$	$\frac{n-2}{n(n-1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\frac{1}{n}$	$\frac{1}{n-1}$	$\frac{n-2}{n(n-1)}$
n	$\frac{1}{n}$	0	$\frac{2}{n}$

Proof: The verification is routine and we hence omit it. ■

Claim 3 *In profile group II, for each $m = 2, \ldots, \bar{n}$ (if n is even), and for each $m = 2, \ldots, \bar{n}, \bar{n}+1$ (if n is odd), the random assignment $\varphi(P^{2,m})$ over a, b and c is specified* *below*

$$
\begin{array}{cccc}
 & a & b & c \\
1 & 0 & \frac{3}{n}-\alpha(m) & \alpha(m) \\
\vdots & \vdots & \vdots & \vdots \\
m-1 & 0 & \frac{3}{n}-\alpha(m) & \alpha(m) \\
m & \frac{1}{n-(m-1)} & \frac{1-(m-1)[\frac{3}{n}-\alpha(m)]}{n-m} & \frac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\alpha(m)}{n-m} \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & \frac{1}{n-(m-1)} & \frac{1-(m-1)[\frac{3}{n}-\alpha(m)]}{n-m} & \frac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\alpha(m)}{n-m} \\
n & \frac{1}{n-(m-1)} & 0 & \frac{3}{n}-\frac{1}{n-(m-1)}
\end{array}
$$

where $\alpha(m) = \frac{n^2-(m+1)n+(3m-4)}{n(n-1)[n-(m-1)]}$

Proof: The verification of this claim consists of 4 steps. The first 3 steps are valid for $m = 2, \ldots, \bar{n}, \bar{n}+1$ no matter n is even or odd. The last steps is verified under the cases of even and odd number of agents separately.

Step 1, we show $\varphi_{na}(P^{2,m}) = \frac{1}{n-(m-1)}$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$. Notice that $P^{2,m}$ and $P^{1,m-1}$ are different merely in agent n's preferences, i.e., $P^{2,m}_n = \bar{P}_i$ and $P^{1,m-1}_n = P_i$ in Table 6. Then, *sd-strategy-proofness* implies $P^{2,m}_{na} = P^{1,m-1}_{na} = \frac{1}{n-(m-1)}$. This completes the verification of step 1.

Step 2, we show $\varphi_{nb}(P^{2,m}) = 0$ and $\varphi_{nc}(P^{2,m}) = \frac{3}{n}-\frac{1}{n-(m-1)}$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$. Given $m \in \{1, \ldots, \bar{n}, \bar{n}+1\}$, since all agents other than n prefer b to c, *sd-efficiency* and feasibility imply $\varphi_{nb}(P^{2,m}) = 0$. Then, by Lemma 13 and Step 1, we have $\varphi_{nc}(P^{2,m}) = \frac{3}{n} - \frac{1}{n-(m-1)}$. This completes the verification of step 2.

Step 3, we show $\varphi_{ic}(P^{2,m}) = \alpha(m)$ for all $i = 1, \ldots, m-1$ and $m = 2, \ldots, \bar{n}, \bar{n}+1$. By *equal treatment of equals*, it suffices to show $\varphi_{m-1,c}(P^{2,m}) = \alpha(m)$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$.

Notice that for all $m = 2, \ldots, \bar{n}, \bar{n}+1$, profiles $P^{2,m}$ and $P^{2,m-1}$ are different merely in agent $m-1$'s preferences, i.e., $P^{2,m-1}_{m-1} = P_i$ and $P^{2,m}_{m-1} = \hat{P}_i$ in Table 6.

Now, we prove Step 3 by an induction argument on $m = 2, \ldots, \bar{n}, \bar{n}+1$.

Initial statement: for $m = 2$, by *sd-strategy-proofness* and Claim 2, we have

$$\varphi_{1,c}(P^{2,2}) = \varphi_{1,c}(P^{2,1}) = \frac{n-2}{n(n-1)} = \frac{n^2-(2+1)n+(3\times 2-4)}{n(n-1)[n-(2-1)]} = \alpha(2).$$

Induction Hypothesis: Given $2 \leqslant m \leqslant \bar{n}$, for all $2 \leqslant l < m+1$, we have $\varphi_{l-1,c}(P^{2,l}) = \alpha(l)$.

We show $\varphi_{m,c}(P^{2,m+1}) = \alpha(m+1)$ by the following elaboration.

$$\begin{aligned}
\varphi_{m,c}(P^{2,m+1}) &= \varphi_{m,c}(P^{2,m}) && \text{by } \textit{sd-strategy-proofness} \\
&= \tfrac{1-\varphi_{nc}(P^{2,m})-\sum_{i=1}^{m-1}\varphi_{ic}(P^{2,m})}{n-m} && \text{by } \textit{equal treatment of equals} \text{ and feasibility} \\
&= \tfrac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\varphi_{m-1,c}(P^{2,m})}{n-m} && \text{by Step 2 and } \textit{equal treatment of equals} \\
&= \tfrac{1-[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\alpha(m)}{n-m} && \text{by induction hypothesis} \\
&= \alpha(m+1) && \text{by simplifying the expression}
\end{aligned}$$

This completes the verification of induction hypothesis and hence step 3.

Step 4, we show $\varphi_{ia}(P^{2,m}) = 0$ for all $i = 1, \ldots, m-1$ and $m = 2, \ldots, \bar{n}$ (if n is even) or $m = 2, \ldots, \bar{n}, \bar{n}+1$ (if n is odd).[22] Given $m \in \{2, \ldots, \bar{n}\}$ (if n is even), or $m \in \{2, \ldots, \bar{n}, \bar{n}+1\}$ (if n is odd), suppose that $\varphi_{ia}(P^{2,m}) = \beta > 0$ for some $i = 1, \ldots, m-1$. Thus, *sd-efficiency* implies that $\varphi_{jb}(P^{2,m}) = 0$ for all $j = m, \ldots, n-1$. Consequently, since $\varphi_{nb}(P^{2,m}) = 0$ by Step 2, we know $\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) = 1$.

Evidently, *equal treatment of equals* implies $\varphi_{ia}(P^{2,m}) = \beta$ for all $i = 1, \ldots, m-1$. Thus, by Lemma 13 and Step 3, we have $\varphi_{ib}(P^{2,m}) = \frac{3}{n} - \alpha(m) - \beta$ for all $i = 1, \ldots, m-1$. Therefore, *equal treatment of equals* implies

$$\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) = (m-1)\times\left[\frac{3}{n} - \alpha(m) - \beta\right] < (m-1)\times\left[\frac{3}{n} - \frac{n^2-(m+1)n+(3m-4)}{n(n-1)[n-(m-1)]}\right].$$

To induce the contradiction $\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) < 1$, we show $(m-1)\times\left[\frac{3}{n} - \frac{n^2-(m+1)n+(3m-4)}{n(n-1)[n-(m-1)]}\right] \leqslant 1$. Equivalently, we show $-2m^2n + 3mn^2 + m - n^3 - 2n^2 + 2nm - 1 \leqslant 0$.

Consider the function $f(\theta) = -2\theta^2 n + 3\theta n^2 + \theta - n^3 - 2n^2 + 2n\theta - 1$, $\theta \in \mathbb{R}$. We know $f'(\theta) = -4n\theta + 3n^2 + 1 + 2n$ and $f''(\theta) = -4n < 0$ for all $\theta \in \mathbb{R}$. It is evident

[22] In Chang and Chun (2016), the verification related to this step is simply an application of *sd-efficiency*. However, due to the complexity of $\alpha(m)$, mere *sd-efficiency* is not enough for the verification.

that $f'(\theta)$ is a strictly decreasing function on $\mathbb{R}$. Now, we consider the case n is even and the case n is odd separately.

Case 1: n is even. Thus, $\bar{n} = \frac{n}{2}$. Since $f'(\frac{n}{2}) = (n+1)^2 > 0$, it must be the case that $f'(\theta) > 0$ for all $2 \leqslant \theta \leqslant \frac{n}{2}$. Therefore, f is a strictly increasing function on $2 \leqslant \theta \leqslant \frac{n}{2}$. Next, since $f(\frac{n}{2}) = -(n-\frac{1}{4})^2 - \frac{15}{16} < 0$, we have $f(\theta) < 0$ for all $2 \leqslant \theta \leqslant \frac{n}{2}$.

Case 2: n is odd. Thus, $\bar{n} = \frac{n-1}{2}$. Since $f'(\frac{n+1}{2}) = n^2 + 1 > 0$, it must be the case that $f'(\theta) > 0$ for all $2 \leqslant \theta \leqslant \frac{n+1}{2}$. Therefore, f is a strictly increasing function on $2 \leqslant \theta \leqslant \frac{n+1}{2}$. Since $f(\frac{n+1}{2}) = -\frac{1}{2}(n-1)^2 < 0$, we have $f(\theta) < 0$ for all $2 \leqslant \theta \leqslant \frac{n+1}{2}$.

In conclusion, no matter n is even or odd, we have $-2m^2n + 3mn^2 + m - n^3 - 2n^2 + 2nm - 1 = f(m) < 0$, and hence, $\sum_{i=1}^{m-1} \varphi_{ib}(P^{2,m}) < 1$. Contradiction! This completes the verification of step 4.

Finally, Lemma 13, feasibility and *equal treatment of equals* give the rest of characterizations in the claim. ■

Claim 4 *In profile group III, for each $m = 0, 1, \ldots, \bar{n}$, the random assignment $\varphi(P^{3,m})$ over a, b and c is specified below*

	a	b	c
1	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-m$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$	$\frac{1}{n}$
$n-m+1$	$\frac{2}{n}$	0	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	$\frac{2}{n}$	0	$\frac{1}{n}$

Proof: The verification follows from a similar argument in the proof of Claim 1. ■

Claim 5 *In profile group IV, for each $m = 1, \ldots, \bar{n}$, the random assignment $\varphi(P^{4,m})$*

over a, b and c is specified below

	a	b	c
1	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-m$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$	$\frac{1}{n}$
$n-m+1$	$\frac{2}{n}$	0	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\frac{2}{n}$	0	$\frac{1}{n}$
n	$\frac{2}{n}$	0	$\frac{1}{n}$

Proof: The verification of the claim consists of 4 steps.

Step 1, we show $\varphi_{n,a}(P^{4,m}) = \frac{2}{n}$ for all $m = 1, \ldots, \bar{n}$. Notice that, for all $m = 1, \ldots, \bar{n}$, $P^{4,m}$ and $P^{3,m}$ are different merely in agent n's preferences, i.e., $P_n^{4,m} = \bar{P}_i$ and $P_n^{3,m} = P_i$ in Table 6. By *sd-strategy-proofness*, we have $\varphi_{n,a}(P^{4,m}) = \varphi_{n,a}(P^{3,m}) = \frac{2}{n}$. This completes the verification of step 1.

Step 2, we show $\varphi_{n,b}(P^{4,m}) = 0$ and $\varphi_{n,c}(P^{4,m}) = \frac{1}{n}$ for all $m = 1, \ldots, \bar{n}$. The verification of this step follows from the same verification of Step 2 in the proof of Claim 3.

Step 3, we show $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i \in I$ and $m = 1, \ldots, \bar{n}$. First, since $\varphi_{n,c}(P^{4,1}) = \frac{1}{n}$ by Step 2, feasibility and *equal treatment of equals* imply $\varphi_{ic}(P^{4,1}) = \frac{1}{n}$ for all $i = 1, \ldots, n-1$. Therefore, $\varphi_{ic}(P^{4,1}) = \frac{1}{n}$ for all $i \in I$.

Next, we specify an induction hypothesis: given $2 \leqslant m \leqslant \bar{n}$, for all $1 \leqslant l < m$, $\varphi_{ic}(P^{4,l}) = \frac{1}{n}$ for all $i \in I$. We will show $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i \in I$. Notice that, $P^{4,m}$ and $P^{4,m-1}$ are different merely in agent $(n-m+1)$'s preferences, i.e., $P^{4,m}_{n-m+1} = P_i$ and $P^{4,m-1}_{n-m+1} = \hat{P}_i$ in Table 6. Then *sd-strategy-proofness* and induction hypothesis imply $\varphi_{n-m+1,c}(P^{4,m}) = \varphi_{n-m+1,c}(P^{4,m-1}) = \frac{1}{n}$. Then, by *equal treatment of equals*, we know $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i = n-m+1, \ldots, n-1$. Moreover, by Step 2 and feasibility, we have $\varphi_{jc}(P^{4,m}) = \frac{1}{n}$ for all $j = 1, \ldots, n-m$. Therefore, $\varphi_{ic}(P^{4,m}) = \frac{1}{n}$ for all $i \in I$. This completes the verification of induction hypothesis and hence step 3.

Step 4, we show $\varphi_{i,b}(P^{4,m}) = 0$ for all $i = n-m+1, \ldots, n-1$ and $m = 2, \ldots, \bar{n}$. Given $m \in \{2, \ldots, \bar{n}\}$, suppose $\varphi_{i,b}(P^{4,m}) = \alpha > 0$ for some $i \in \{n-m+1, \ldots, n-1\}$. Then *equal treatment of equals* and Step 4 imply $\varphi_{i,b}(P^{4,m}) = \alpha$ and $\varphi_{i,a}(P^{4,m}) = \frac{2}{n} - \alpha$ for all $i = n-m+1, \ldots, n-1$. Moreover, by *sd-efficiency*, it must be the case that $\varphi_{i,a}(P^{4,m}) = 0$ for all $i = 1, \ldots, n-m$. Then, Lemma 13 and Step 3 imply $\varphi_{i,b}(P^{4,m}) = \frac{2}{n}$ for all $i = 1, \ldots, n-m$ Thus, the feasibility of a implies $\alpha = \frac{1}{m-1}\frac{2}{n}(m - \frac{n}{2}) \leqslant 0$ since $m \leqslant \bar{n} \leqslant \frac{n}{2}$. Contradiction!

Finally, Lemma 13, feasibility and *equal treatment of equals* give the rest of characterizations in the claim. ■

Now we have the contradiction for the case of even number of agents. Let n be even. Notice that $P^{2,\bar{n}}$ and $P^{4,\bar{n}}$ are different merely in agent $\bar{n}$'s preference, i.e., $P^{2,\bar{n}}_{\bar{n}} = P_i$ and $P^{4,\bar{n}}_{\bar{n}} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 6. Then *sd-strategy-proofness* requires $\varphi_{\bar{n},c}(P^{2,\bar{n}}) = \varphi_{\bar{n},c}(P^{4,\bar{n}})$. Thus, we have

$$\frac{1 - [\frac{3}{n} - \frac{1}{n-(\frac{n}{2}-1)}] - (\frac{n}{2}-1)\frac{n^2-(\frac{n}{2}+1)n+(3\times\frac{n}{2}-4)}{n(n-1)[n-(\frac{n}{2}-1)]}}{n - \frac{n}{2}} = \frac{1}{n} \Leftrightarrow n^2 - n - 2 = n^2 - n. \text{ Contradiction!}$$

When n is odd, profiles $P^{2,\bar{n}+1}$ and $P^{4,\bar{n}}$ are different merely in agent $(\bar{n}+1)$'s preferences, i.e., $P^{2,\bar{n}+1}_{\bar{n}+1} = P_i$ and $P^{4,\bar{n}}_{\bar{n}+1} = \hat{P}_i$ in Table 6. However, we cannot induce a contradiction similar to that above since we can verify that $\varphi_{\bar{n}+1,c}(P^{2,\bar{n}+1}) = \frac{1}{n} = \varphi_{\bar{n}+1,c}(P^{4,\bar{n}})$. Henceforth, we assume that n is an odd number. Hence, $n \geqslant 5$. We proceed the verification on profile groups V and VI.

Claim 6 *In profile group V, the random assignment $\varphi(P^{5,1})$ over a, b and c is specified below*

	a	b	c
1	$\frac{1}{n}$	$\frac{1}{n-2}$	$\frac{2}{n} - \frac{1}{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$\frac{1}{n}$	$\frac{1}{n-2}$	$\frac{2}{n} - \frac{1}{n-2}$
$n-1$	$\frac{1}{n}$	0	$\frac{2}{n}$
n	$\frac{1}{n}$	0	$\frac{2}{n}$

Proof: The verification is routine and we hence omit it. ■

Claim 7 *In profile group V, for each $m = 2, \ldots, \bar{n}, \bar{n}+1$, the random assignment $\varphi(P^{5,m})$ over a, b and c is specified below*[23]

$$\begin{array}{ccccc}
 & a & b & c \\
1 & - & - & \gamma(m) \\
\vdots & \vdots & \vdots & \vdots \\
m-1 & - & - & \gamma(m) \\
m & - & - & \frac{1-2\times(\frac{3}{n}-\frac{1}{n-(m-1)})-(m-1)\gamma(m)}{n-(m+1)} \\
\vdots & \vdots & \vdots & \vdots \\
n-2 & - & - & \frac{1-2\times(\frac{3}{n}-\frac{1}{n-(m-1)})-(m-1)\gamma(m)}{n-(m+1)} \\
n-1 & \frac{1}{n-(m-1)} & 0 & \frac{3}{n}-\frac{1}{n-(m-1)} \\
n & \frac{1}{n-(m-1)} & 0 & \frac{3}{n}-\frac{1}{n-(m-1)}
\end{array}$$

where $\gamma(m) = \frac{n^4-2(m+2)n^3+(m^2+11m-5)n^2-(7m^2+m-8)n+(6m^2-6m-4)}{n(n-1)(n-2)(n-(m-1))(n-m)}$.

Proof: The verification of this claim consists of 3 steps.

Step 1, we show $\varphi_{ia}(P^{5,m}) = \frac{1}{n-(m-1)}$ for $i = n-1, n$ and all $m = 2, \ldots, \bar{n}, \bar{n}+1$. Notice that $P^{5,m}$ and $P^{2,m}$ are different merely in agent $(n-1)$'s preferences, i.e., $P^{5,m}_{n-1} = \bar{P}_i$ and $P^{2,m}_{n-1} = P_i$ in Table 6. Then *sd-strategy-proofness* implies $\varphi_{n-1,a}(P^{5,m}) = \varphi_{n-1,a}(P^{2,m}) = \frac{1}{n-(m-1)}$. This completes the verification of step 1.

Step 2, we show $\varphi_{ib}(P^{5,m}) = 0$ and $\varphi_{ic}(P^{5,m}) = \frac{3}{n} - \frac{1}{n-(m-1)}$ for $i = n-1, n$ and all $m = 2, \ldots, \bar{n}, \bar{n}+1$. The verification simply follows from an application of *sd-efficiency*, *equal treatment of equals*, feasibility and Lemma 13. Therefore, we omit the details.

Step 3, we show $\varphi_{ic}(P^{5,m}) = \gamma(m)$ for all $i = 1, \ldots, m-1$ and $m = 2, \ldots, \bar{n}, \bar{n}+1$. By *equal treatment of equals*, it suffices to show $\varphi_{m-1,c}(P^{5,m}) = \gamma(m)$ for all $m = 2, \ldots, \bar{n}, \bar{n}+1$.

[23] A dash "–" in the random assignment matrix represents that the probability of assigning one object to a corresponding agent is *not* specified.

First, notice that for all $m = 2, \ldots, \bar{n}, \bar{n}+1$, profiles $P^{5,m}$ and $P^{5,m-1}$ are different merely in agent $m-1$'s preferences, i.e., $P^{5,m-1}_{m-1} = P_i$ and $P^{5,m}_{m-1} = \hat{P}_i$ in Table 6.

Now, we prove Step 3 by an induction argument on $m = 2, \ldots, \bar{n}, \bar{n}+1$.

Initial statement: for $m = 2$, by *sd-strategy-proofness* and Claim 6, we have

$$\begin{aligned}
\varphi_{1,c}(P^{5,2}) &= \varphi_{1,c}(P^{5,1}) \\
&= \frac{2}{n} - \frac{1}{n-2} \\
&= \frac{n^4 - 2\times(2+2)n^3 + (2^2 + 11\times 2 - 5)n^2 - (7\times 2^2 + 2 - 8)n + (6\times 2^2 - 6\times 2 - 4)}{n(n-1)(n-2)(n-(2-1))(n-(2+1))} \\
&= \gamma(2).
\end{aligned}$$

Induction Hypothesis: Given $2 \leqslant m \leqslant \bar{n}$, for all $2 \leqslant l < m+1$, we have $\varphi_{l-1,c}(P^{5,l}) = \gamma(l)$.

We show $\varphi_{m,c}(P^{5,m+1}) = \gamma(m+1)$ by the following elaboration.

$$\begin{aligned}
\varphi_{m,c}(P^{5,m+1}) &= \varphi_{m,c}(P^{5,m}) && \text{by } \textit{sd-strategy-proofness} \\
&= \tfrac{1-\varphi_{n-1,c}(P^{5,m})-\varphi_{nc}(P^{5,m})-\sum_{i=1}^{m-1}\varphi_{ic}(P^{5,m})}{n-(m+1)} && \text{by } \textit{equal treatment of equals} \text{ and feasibility} \\
&= \tfrac{1-2\times[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\varphi_{m-1,c}(P^{5,m})}{n-(m+1)} && \text{by Step 2 and } \textit{equal treatment of equals} \\
&= \tfrac{1-2\times[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\gamma(m)}{n-(m+1)} && \text{by induction hypothesis} \\
&= \gamma(m+1) && \text{by simplifying the expression}
\end{aligned}$$

This completes the verification of induction hypothesis and hence step 3.

Finally, by feasibility and *equal treatment of equals*, we have $\varphi_{ic}(P^{5,m}) = \frac{1-2\times[\frac{3}{n}-\frac{1}{n-(m-1)}]-(m-1)\gamma(m)}{n-(m+1)}$ for all $i = m, \ldots, n-2$ and $m = 2, \ldots, \bar{n}, \bar{n}+1$. This completes the verification of the claim. ■

Claim 8 *In profile group VI, for each $m = 2, \ldots, \bar{n}$, the random assignment $\varphi(P^{6,m})$*

over a, b and c is specified below

	a	b	c
1	$-$	$-$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-m$	$-$	$-$	$\frac{1}{n}$
$n-m+1$	$-$	$-$	$\frac{1}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	$\frac{1}{n}$
$n-1$	$\frac{2}{n}$	0	$\frac{1}{n}$
n	$\frac{2}{n}$	0	$\frac{1}{n}$

Proof: The verification of this claim consists of 3 steps.

Step 1, we show $\varphi_{ia}(P^{6,m}) = \frac{2}{n}$ for $i = n-1, n$ and all $m = 2, \ldots, \bar{n}$. For each $m = 2, \ldots, \bar{n}$, notice that $P^{6,m}$ and $P^{4,m}$ are different merely in agent $(n-1)$'s preferences, i.e., $P^{6,m}_{n-1} = P_i$ and $P^{4,m}_{n-1} = \bar{P}_i$ in Table 6. Then, *sd-strategy-proofness* implies $\varphi_{n-1,a}(P^{6,m}) = \varphi_{n-1,a}(P^{4,m}) = \frac{2}{n}$. Then, *equal treatment of equals* implies $\varphi_{na}(P^{6,m}) = \frac{2}{n}$. This completes the verification of step 1.

Step 2, we show $\varphi_{ib}(P^{6,m}) = 0$ and $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for $i = n-1, n$ and all $m = 2, \ldots, \bar{n}$. The verification simply follows from an application of *sd-efficiency*, *equal treatment of equals*, feasibility and Lemma 13. Therefore, we omit the details.

Step 3, we show $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for all $i = 1, \ldots, n-2$ and $m = 3, \ldots, \bar{n}$. First, in $P^{6,2}$, according to Step 2, by feasibility and *equal treatment of equals*, we have $\varphi_{ic}(P^{6,2}) = \frac{1}{n}$ for all $i = 1, \ldots, n-2$.

Next, notice that $P^{6,3}$ and $P^{6,2}$ are different merely in agent $n-2$'s preferences, i.e., $P^{6,3}_{n-2} = P_i$ and $P^{6,2}_{n-2} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 6. Then, *sd-strategy-proofness* implies $\varphi_{n-2,c}(P^{6,3}) = \varphi_{n-2,c}(P^{6,2}) = \frac{1}{n}$. Last, by feasibility, *equal treatment of equals* and Step 2, we know $\varphi_{ic}(P^{6,3}) = \frac{1}{n}$ for all $i = 1, \ldots, n-3$. Next, we provide an induction argument.

Induction Hypothesis: Given $4 \leqslant m \leqslant \bar{n}$, for all $3 \leqslant l < m$, $\varphi_{ic}(P^{6,l}) = \frac{1}{n}$ for all $i = 1, \ldots, n-2$.

We will show $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for all $i = 1, \dots, n-2$. Notice that $P^{6,m}$ and $P^{6,m-1}$ are different merely in agent $n-m+1$'s preference, i.e., $P^{6,m}_{n-m+1} = P_i$ and $P^{6,m-1}_{n-m+1} = \hat{P}_i$ in Table 6. Then, *sd-strategy-proofness* and induction hypothesis imply $\varphi_{n-m+1,c}(P^{6,m}) = \varphi_{n-m+1,c}(P^{6,m-1}) = \frac{1}{n}$. Furthermore, *equal treatment of equals* implies $\varphi_{i,c}(P^{6,m}) = \frac{1}{n}$ for all $i = n-m+1, \dots, n-2$. Last, by feasibility, *equal treatment of equals* and Step 2, we have $\varphi_{ic}(P^{6,m}) = \frac{1}{n}$ for all $i = 1, \dots, n-m$. This completes the verification of induction hypothesis, hence Step 3 and the claim. ■

Now we have the contradiction for the case of odd number of agents. Now, $\bar{n} = \frac{n-1}{2}$. Notice that $P^{5,\bar{n}+1}$ and $P^{6,\bar{n}}$ are different only in agent $(\bar{n}+1)$'s preference, i.e., $P^{5,\bar{n}+1}_{\bar{n}+1} = P_i$ and $P^{6,\bar{n}}_{\bar{n}+1} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 6. Then *sd-strategy-proofness* requires $\varphi_{\bar{n}+1,c}(P^{5,\bar{n}+1}) = \varphi_{\bar{n}+1,c}(P^{6,\bar{n}})$. Thus, we have

$$\frac{1 - 2 \times \left[\frac{3}{n} - \frac{1}{n-[(\bar{n}+1)-1]}\right] - [(\bar{n}+1)-1]\gamma(\bar{n}+1)}{n - [(\bar{n}+1)+1]} = \frac{1}{n} \Leftrightarrow \frac{n^3 - 6n^2 + 11n - 2}{n(n^3 - 6n^2 + 11n - 6)} = \frac{1}{n}. \text{ Contradiction!}$$

In conclusion, a domain satisfying the *elevating property* admits no *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule. Therefore, the connected domain $\mathbb{D}$ in Theorem 3 must violate the *elevating property*. Last, applying Lemmas 5 - 7, we show that domain $\mathbb{D}$ is a *restricted tier domain*. This completes the verification of Theorem 3.

B Appendix to Chapter 4

B.1 Proof of Theorem 5

The necessity part is evident by definition. We prove the sufficiency part.

Let $\mathbb{D}$ be a path-nested domain and $\varphi : \mathbb{D}^n \to \mathcal{L}$ a *block-adjacent sd-strategy-proof* rule. Let $P \in \mathbb{D}^n$ and $P'_i \in \mathbb{D}\backslash\{P_i\}$. In addition, let $L \equiv \varphi(P)$ and $L' \equiv \varphi(P'_i, P_{-i})$.

Fix a nested path from P_i to P'_i and denote it as $(P_m)_{m=1}^M$ where $P_1 = P_i$, $P_M = P'_i$, and for all $1 \leqslant m' < m \leqslant M-1$, either

1. $FB_1(P_m, P_{m+1}) \cup FB_2(P_m, P_{m+1}) \subset FB_1(P_{m'}, P_{m'+1})$, or
2. $FB_1(P_m, P_{m+1}) \cup FB_2(P_m, P_{m+1}) \subset FB_2(P_{m'}, P_{m'+1})$, or
3. $[FB_1(P_m, P_{m+1}) \cup FB_2(P_m, P_{m+1})] \cap [FB_1(P_{m'}, P_{m'+1}) \cup FB_2(P_{m'}, P_{m'+1})] = \emptyset$.

Let $L^m \equiv \varphi(P_m, P_{-i})$ for all $m = 1, \cdots, M$. To prove the theorem, it suffices to show $L_i^1 \; P_1^{sd} \; L_i^M$. In the following, we show $L_i^m \; P_1^{sd} \; L_i^{m+1}$ for each $m \in \{1, \cdots, M-1\}$. Then the transitivity of P_1^{sd} implies what is desired.

Fix $m \in \{2, \cdots, M-1\}$, we show $L_i^m \; P_1^{sd} \; L_i^{m+1}$ by the following induction.

Initial statement: $L_i^m \; P_m^{sd} \; L_i^{m+1}$ by *block-adjacent sd-strategy-proofness.*

Induction statement: $L_i^m \; P_\alpha^{sd} \; L_i^{m+1}$ implies $L_i^m \; P_{\alpha-1}^{sd} \; L_i^{m+1}$ for each $2 \leqslant \alpha \leqslant m$.

Proof: Either one of the following three cases happens.

Case 1: $FB_1(P_\alpha, P_{\alpha+1}) \cup FB_2(P_\alpha, P_{\alpha+1}) \subset FB_1(P_{\alpha-1}, P_\alpha)$.

We illustrate the situation as follows.

$$P_{\alpha-1}: \quad \cdots\cdots \succ \underbrace{\cdots \succ FB_1(P_\alpha, P_{\alpha+1}) \succ FB_2(P_\alpha, P_{\alpha+1}) \succ \cdots}_{FB_1(P_{\alpha-1}, P_\alpha)} \succ FB_2(P_{\alpha-1}, P_\alpha) \succ \cdots\cdots$$

$$P_\alpha: \quad \cdots\cdots \succ FB_2(P_{\alpha-1}, P_\alpha) \succ \underbrace{\cdots \succ FB_1(P_\alpha, P_{\alpha+1}) \succ FB_2(P_\alpha, P_{\alpha+1}) \succ \cdots}_{FB_1(P_{\alpha-1}, P_\alpha)} \succ \cdots\cdots$$

By $L_i^m \; P_\alpha^{sd} \; L_i^{m+1}$, $L_{ia}^m = L_{ia}^{m+1}$ for all $a \in \backslash [FB_1(P_\alpha, P_{\alpha+1}) \cup FB_2(P_\alpha, P_{\alpha+1})]$, and $\sum_{a \in B_k(P_\alpha, FB_1(P_\alpha, P_{\alpha+1}) \cup FB_2(P_\alpha, P_{\alpha+1}))} L_{ia}^m \geqslant \sum_{a \in B_k(P_\alpha, FB_1(P_\alpha, P_{\alpha+1}) \cup FB_2(P_\alpha, P_{\alpha+1}))} L_{ia}^{m+1}$ for all $k = 1, \cdots, |FB_1(P_\alpha, P_{\alpha+1}) \cup FB_2(P_\alpha, P_{\alpha+1})|$.[24] Then it is easy to check $L_i^m \; P_{\alpha-1}^{sd} \; L_i^{m+1}$.

The same logic applies to both the remaining two cases. However, for a better understanding, these two cases are illustrated as follows.

Case 2: $FB_1(P_\alpha, P_{\alpha+1}) \cup FB_2(P_\alpha, P_{\alpha+1}) \subset FB_2(P_{\alpha-1}, P_\alpha)$.

$$P_{\alpha-1}: \quad \cdots\cdots \succ FB_1(P_{\alpha-1}, P_\alpha) \succ \underbrace{\cdots \succ FB_1(P_\alpha, P_{\alpha+1}) \succ FB_2(P_\alpha, P_{\alpha+1}) \succ \cdots}_{FB_2(P_{\alpha-1}, P_\alpha)} \succ \cdots\cdots$$

$$P_\alpha: \quad \cdots\cdots \succ \underbrace{\cdots \succ FB_1(P_\alpha, P_{\alpha+1}) \succ FB_2(P_\alpha, P_{\alpha+1}) \succ \cdots}_{FB_2(P_{\alpha-1}, P_\alpha)} \succ FB_1(P_{\alpha-1}, P_\alpha) \succ \cdots\cdots$$

Case 3:

$[FB_1(P_\alpha, P_{\alpha+1}) \cup FB_2(P_\alpha, P_{\alpha+1})] \cap [FB_1(P_{\alpha-1}, P_\alpha) \cup FB_2(P_{\alpha-1}, P_\alpha)] = \emptyset$.

$$P_{\alpha-1}: \quad \cdots \succ FB_1(P_\alpha, P_{\alpha+1}) \succ FB_2(P_\alpha, P_{\alpha+1}) \succ \cdots \succ FB_1(P_{\alpha-1}, P_\alpha) \succ FB_2(P_{\alpha-1}, P_\alpha) \succ \cdots$$

$$P_\alpha: \quad \cdots \succ FB_1(P_\alpha, P_{\alpha+1}) \succ FB_2(P_\alpha, P_{\alpha+1}) \succ \cdots \succ FB_2(P_{\alpha-1}, P_\alpha) \succ FB_1(P_{\alpha-1}, P_\alpha) \succ \cdots$$

By verifying the induction statement, we prove the theorem.

[24]Note that for arbitrary $P_0 \in \mathbb{P}$ and subset of objects $\bar{A} \subset A$, $B_k(P_0, \bar{A})$ denotes the collection of the top ranked k objects in $\bar{A}$ according to P_0.

C Appendix to Chapter 5

C.1 Proof of Lemma 12

For illustration purpose, we start from the definition of the PS rule. The PS rule treats the objects as if they are divisible and specifies random assignment for a given preference profile as follows. Starting from time 0, all agents consume their most preferred object at unit speed until some objects reach exhaustion. Then agents reformulate their preferences by removing the objects exhausted and then resume consuming their most preferred objects in the available set, until some objects reach exhaustion. This procedure is repeated until all objects reach exhaustion. The ending time of this procedure is 1, since we have n agents consuming n objects with unit speed. Finally, the share of an object consumed by an agent is interpreted as the probability of this agent obtaining this object. Formally the PS rule is defined as follows. We borrow the notation from Kojima and Manea (2010).

Definition 15 *Fix a preference profile $P \in \mathbb{D}^n$, $PS(P)$ is the random assignment $[L_{ia}]_{i\in I, a\in A} \in \mathcal{L}$ calculated as follows. For any $a \in A' \subset A$, let $N(a, A') = \{i \in I | a \ P_i \ b, \forall b \in A' \backslash \{a\}\}$ be the set of agents whose most preferred object in A' is a. Let $A^0 = A$, $t^0 = 0$, and $L^0_{ia} = 0$ for any $i \in I$ and $a \in A$. For any $v \geqslant 1$, given $t^0, A^0, [L^0_{ia}]_{i\in I,a\in A}, \cdots, t^{v-1}, A^{v-1}, [L^{v-1}_{ia}]_{i\in I,a\in A}$, define*

$$
\begin{aligned}
t^v &= \min_{a\in A^{v-1}} \max \left\{ t \in [0,1] | \sum_{i\in I} L^{v-1}_{ia} + |N(a, A^{v-1})|(t - t^{v-1}) \leqslant 1 \right\} \\
A^v &= A^{v-1} \backslash \left\{ a \in A^{v-1} | \sum_{i\in I} L^{v-1}_{ia} + |N(a, A^{v-1})|(t^v - t^{v-1}) = 1 \right\} \\
L^v_{ia} &= \begin{cases} L^{v-1}_{ia} + t^v - t^{v-1} & \text{if } i \in N(a, A^{v-1}) \\ L^{v-1}_{ia} & \text{otherwise.} \end{cases}
\end{aligned}
\tag{1}
$$

Since A is a finite set, there exists $\bar{v}$ such that $A^{\bar{v}} = \emptyset$. Let $PS(P) = L^{\bar{v}}$.

Fix a preference profile P, we call the sequence generated by applying the PS rule to P, $(t^v, A^v, L^v)^{\bar{v}}_{v=0}$, the corresponding **consumption procedure**. Evidently, for each $v \in \{0, \cdots, \bar{v}\}$, A^v is the collection of objects which are still available at time t^v. In other words, if $a \in A^{v-1} \backslash A^v$, a is available at t^{v-1} and reaches exhaustion at t^v, i.e., t^v is exactly the time when a reaches exhaustion. For each $a \in A$, let $t_a \equiv t^v$ where $a \in A^{v-1} \backslash A^v$ denote the time when a reaches exhaustion.

Recall that a *sequentially dichotomous domain* is an intersection of domains, each of which respects a partition. Then for a better understanding of the consumption procedure specified by the PS rule when the preferences are from a *sequentially dichotomous domain*, we investigate the consumption procedure subject to a given partition, denoted as $\left(t^v\big|_{\mathbf{A}}, A^v\big|_{\mathbf{A}}, L^v\big|_{\mathbf{A}}\right)$.

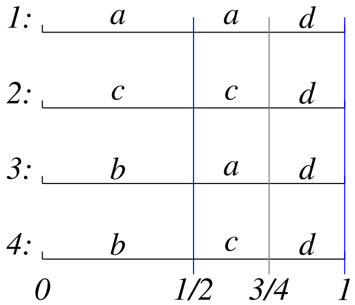
Given a preference profile $P \in \mathbb{D}^n_{\mathbf{A}}$, every agent's preference clusters every block in $\mathbf{A}$. Let $\{a, b\} \in \mathbf{A}$ be a block, then for every agent either a is ranked just above b or b is ranked just above a. Hence if a reaches exhaustion before b, then agents who prefer a to b start to consume b immediately when a reaches exhaustion and all the others keep consuming b until it reaches exhaustion. If instead b reaches exhaustion before a, then agents who prefer b to a start to consume a immediately when b reaches exhaustion and all the others keep consuming a until it reaches exhaustion. So, if we focus on only blocks rather than objects, we can simply ignore the time when a reaches exhaustion if it reaches exhaustion before b and ignore the time when b reaches exhaustion if it reaches exhaustion before a. In other words, what we care about is only the time when the whole block reaches exhaustion. Formally **the consumption procedure subject to $\mathbf{A}$**, $\left(t^v\big|_{\mathbf{A}}, A^v\big|_{\mathbf{A}}, L^v\big|_{\mathbf{A}}\right)$, is defined as follows.

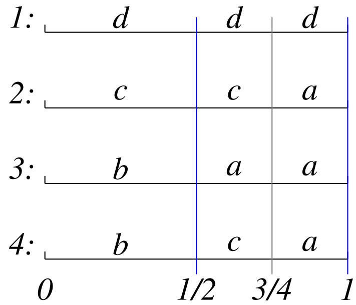
Let $V\big|_{\mathbf{A}} \equiv \{v \in \{0, \cdots, \bar{v}\} | \exists A_k \in \mathbf{A} \text{ s.t. } t^v = \max_{a \in A_k} t_a\}$ be the collection of time points when a block reaches exhaustion.

- $t^v\big|_{\mathbf{A}}$ is the subsequence of $(t^v)_{v=0}^{\bar{v}}$ involving elements in $V\big|_{\mathbf{A}}$. We record only the time points when a block in $\mathbf{A}$ reaches exhaustion.
- $A^v\big|_{\mathbf{A}}$ is the subsequence of $(A^v)_{v=0}^{\bar{v}}$ involving elements in $V\big|_{\mathbf{A}}$.
- $L^v\big|_{\mathbf{A}}$ is a matrix $[L^v_{iA_k}]_{i \in I, A_k \in \mathbf{A}}$ defined only for elements in $V\big|_{\mathbf{A}}$, where $L^v_{iA_k} \equiv \sum_{a \in A_k} L^v_{ia}$.

When we focus on the consumption procedure subject to a partition, an important observation is that the consumption procedure subject to $\mathbf{A}$ should not change when the preference profile is changing in a way that the "ranking" of the blocks remain the same. That is, the change involves only permutations within blocks won't change the consumption procedure subject to the partition. Here is an example.

Example 9 *Let $P \equiv (P_1, P_2, P_5, P_6)$ and $\bar{P} \equiv (P_3, P_2, P_5, P_6)$ where the preferences are from Example 6. The consumption procedures of two profiles are illustrated as follows.*





It is evident that all involved preferences respect $\mathbf{A}_2 = \{\{a, c, d\}, \{b\}\}$. In addition the consumption procedure subject to $\mathbf{A}_2 = \{\{a, c, d\}, \{b\}\}$ is not changing: In time interval $(0, 1/2]$ agents 1 and 2 consume objects in block $\{a, c, d\}$ and agents 3 and 4 consume $\{b\}$. Then in time interval $(1/2, 1]$ all agent consume objects in $\{a, c, d\}$.

We formalize the observation illustrated in Example 9. Given a partition $\mathbf{A}$, a preference respecting it, $P_0 \in \mathbb{D}_{\mathbf{A}}$, induces a (strict) preference on $\mathbf{A}$ in a natural way: a block is said to be ranked above another block if every object in the former is ranked above every object in the latter. Given a partition $\mathbf{A}$, $\mathbb{P}(\mathbf{A})$ denotes the collection of all strict preferences, i.e., linear orders, on $\mathbf{A}$.

Definition 16 *For fixed partition $\mathbf{A}$, **a preference on the blocks** $P_0^{\mathbf{A}} \in \mathbb{P}(\mathbf{A})$ **is induced by a preference on objects** $P_0 \in \mathbb{D}_{\mathbf{A}}$ if, for each pair of blocks $A_k, A_l \in \mathbf{A}$, $A_k \ P_0^{\mathbf{A}} \ A_l$ if and only if $a \ P_0 \ b$ for all $a \in A_k$ and $b \in A_l$.*

It is easy to see that a preference $P_0 \in \mathbb{D}_{\mathbf{A}}$ induces a *unique* preference $P_0^{\mathbf{A}}$ on $\mathbf{A}$. However the converse is not true: two different preferences $P_0, P_0' \in \mathbb{D}_{\mathbf{A}}$ may induce the same preference $P_0^{\mathbf{A}}$ on $\mathbf{A}$. For example, preferences P_1 and P_3 in Example 6 induce the same preference on $\mathbf{A}_2$: $\{a, c, d\} \succ \{b\}$.

Now we formalize the observation in Example 9 as the following lemma.

Lemma 14 *For $P, \bar{P} \in \mathbb{D}^n_{\mathbf{A}}$ such that $P^{\mathbf{A}}_i = \bar{P}^{\mathbf{A}}_i$ for all $i \in I$, the consumption procedures subject to* **A** *for two preference profiles are the same, which implies $\sum_{a\in A_k} PS_{ia}(P) = \sum_{a\in A_k} PS_{ia}(\bar{P})$ for all $i \in I$ and $A_k \in \mathbf{A}$.*

We are now ready to prove Lemma 12.

Let $L \equiv PS(P)$ and $\tilde{L} \equiv PS(\tilde{P}_1, P_{-1})$. Let $(t^v, A^v, [L^v_{ia}]_{i\in I, a\in A})^{\bar{v}}_{v=0}$ and $\left(\tilde{t}^v, \tilde{A}^v, [\tilde{L}^v_{ia}]_{i\in I, a\in A}\right)^{\bar{\bar{v}}}_{v=0}$ be respectively consumption procedures for P and $(\tilde{P}_1, P_{-1})$. In addition, let $B \equiv \{a \in A | aP_1x, \forall x \in A_1 \cup A_2\}$ and $C \equiv A \backslash (A \cup A_1 \cup A_2)$ be respectively the upper and lower contour sets of objects in $A_1 \cup A_2$ according to P_1. Evidently, these two sets are the same for $\tilde{P}_1$.

Before proceeding to the proof, we clarify some notations.

- For each $a \in A$, let $t_a \equiv t^v$ s.t. $a \in A^{v-1}\backslash A^v$, i.e., t_a is the time when a reaches exhaustion in the consumption procedure of applying the PS rule to P. For each $a \in A$, let $\tilde{t}_a \equiv \tilde{t}^v$ s.t. $a \in \tilde{A}^{v-1}\backslash \tilde{A}^v$, i.e., $\tilde{t}_a$ is the time when a reaches exhaustion in the consumption procedure of applying the PS rule to $(\tilde{P}_1, P_{-1})$. Similarly, for each $\bar{A} \subset A$, let $t_{\bar{A}} \equiv \max\{t_a : a \in \bar{A}\}$, i.e., $t_{\bar{A}}$ is the time when all the objects in $\bar{A}$ reach exhaustion in the consumption procedure of applying the PS rule to P. For each $\bar{A} \subset A$, let $\tilde{t}_{\bar{A}} \equiv \max\{\tilde{t}_a : a \in \bar{A}\}$, i.e., $\tilde{t}_{\bar{A}}$ is the time when all the objects in $\bar{A}$ reach exhaustion in the consumption procedure of applying the PS rule to $(\tilde{P}_1, P_{-1})$.

- For each $a \in A$ and $v \in \{0, 1, \cdots, \bar{v}\}$, let $S_a(t^v) \equiv 1 - \sum_{i\in I} L^v_{ia}$, i.e., $S_a(t^v)$ is the remaining share of a at time t^v in the consumption procedure of applying the PS rule to P. For each $a \in A$ and $v \in \{0, 1, \cdots, \bar{\bar{v}}\}$, let $\tilde{S}_a(\tilde{t}^v) \equiv 1 - \sum_{i\in I} \tilde{L}^v_{ia}$, i.e., $\tilde{S}_a(\tilde{t}^v)$ is the remaining share of a at time $\tilde{t}^v$ in the consumption procedure of applying the PS rule to $(\tilde{P}_1, P_{-1})$.

- For each $a \in A$ and each $i \in I$, let $t_i(a) \equiv \max\{t_x : xP_ia\}$ be the time when all the objects in the upper contour set of a in P_i reach exhaustion when the PS rule is applied to P. Hence $t_i(a)$ is also the time when agent i starts to consume a whenever there is still some of a available, i.e., $S_a(t_i(a)) > 0$. In addition, for a block $\bar{A}$, let $t_i(\bar{A}) = \min\{t_i(a) : a \in \bar{A}\}$ be the time when the upper contour set of

$\bar{A}$ reaches exhaustion before deviation. Similarly, for each $a \in A$ and each $i \in I$, let $\tilde{t}_i(a) \equiv \max\{\tilde{t}_x : x\tilde{P}_i a\}$ be the time when all the objects in the upper contour set of a in $\tilde{P}_i$ reach exhaustion when the PS rule is applied to $(\tilde{P}_1, P_{-1})$. In addition, for a block $\bar{A}$, let $\tilde{t}_i(\bar{A}) = \min\{\tilde{t}_i(a) : a \in \bar{A}\}$ be the time when the upper contour set of $\bar{A}$ reaches exhaustion after deviation.

We prove the lemma in three steps.

Step 1: $L_{1a} = \tilde{L}_{1a}$ for all $a \in B \cup C$.

Consider a partition $\mathbf{A} \equiv \{A_1 \cup A_2, \{a\} : a \in A\backslash (A_1 \cup A_2)\}$. Then it is evident that the profiles of preferences on $\mathbf{A}$ induced by P and $(\tilde{P}_1, P_{-1})$ are the same. Hence by Lemma 14, we have what we want. In addition, by Lemma 14, we have the following claims which are used subsequently.

Claim 9 *For each $i \in I$, $t_i(A_1 \cup A_2) = \tilde{t}_i(A_1 \cup A_2)$, i.e., each agent starts to consume $A_1 \cup A_2$, if any, at the same time before and after agent 1's deviation.*

Claim 10 *$t_{A_1 \cup A_2} = \tilde{t}_{A_1 \cup A_2}$, i.e., each agent stops consuming $A_1 \cup A_2$, if any, at the same time before and after the deviation.*

Claim 11 *$t_B = \tilde{t}_B$, i.e., B reaches exhaustion at the same time before and after the deviation. In addition, $S_a(t_B) = \tilde{S}_a(t_B)$ for each $a \in A_1 \cup A_2$, that is when B reaches exhaustion the remaining share of each object in $A_1 \cup A_2$ is not changing because of agent 1's deviation.*

Step 2: $L_{1a} \geqslant \tilde{L}_{1a}$ for all $a \in A_1$.

If $\sum_{a \in A_1} S_a(t_B) = 0$, i.e., when B reaches exhaustion and agent 1 is about to consume objects in A_1, A_1 already reached exhaustion, then by Claim 11 $L_{1a} = \tilde{L}_{1a} = 0$ for all $a \in A_1$ and hence the result holds trivially.

If $\sum_{a \in A_2} S_a(t_B) = 0$, i.e., when B reaches exhaustion the objects in A_2 already reached exhaustion, then it is evident that the consumption procedures of applying the PS rule to P and $(\tilde{P}_1, P_{-1})$ are the same. Which then implies $L_{1a} = \tilde{L}_{1a}$ for all $a \in A_1$ and hence the result holds trivially.

In addition, if $\sum_{a \in A_1} \tilde{L}_{1a} = 0$, the result holds trivially.

Hence we show Step 2 when $\sum_{a\in A_1} S_a(t_B) > 0$, $\sum_{a\in A_2} S_a(t_B) > 0$, and $\sum_{a\in A_1} \tilde{L}_{1a} > 0$.

Notice that before deviation t_B is the time when agent 1 starts to consume A_1. After deviation, when B reaches exhaustion, agent 1 starts to consume A_2 and will turn to A_1 when A_2 reaches exhaustion. Hence $\tilde{t}_{B\cup A_2} > \tilde{t}_B = t_B$ is the time when agent 1 starts to consume A_1 after deviation. That is agent 1 starts to consume A_1 latter after the deviation.

Then to show Step 2, it suffices to show the following two statements.

1. $S_a(t_B) \geqslant \tilde{S}_a(\tilde{t}_{B\cup A_2})$ for each $a \in A_1$, i.e., when agent 1 is about to consume objects in A_1, she finds that the remaining share of each object in A_1 is less;

2. For each $i \in I\backslash\{1\}$ with $\sum_{a\in A_1} L_{ia} > 0$, agent i starts consuming A_1 after agent 1's deviation at a time no latter than the time when she started to consume A_1 before deviation, i.e., all the agents who compete with agent 1 in consuming A_1 will still do so after the deviation.

Recall that $\tilde{t}_{B\cup A_2} > t_B$, then to show statement 1, it suffices to show for each agent $i \in I\backslash\{1\}$ who started to consume A_1 before agent 1's deviation at a time before t_B will start consuming A_1 after agent 1's deviation at a time no latter than that time. Notice that this new statement is implied by the statement 2. Hence what we need to show is just statement 2.

To show this, recall first Claim 9 which says that each agent starts to consume $A_1 \cup A_2$ at the same time before and after agent 1's deviation. Then pick any $i \in I\backslash\{1\}$ with $\sum_{a\in A_1} L_{ia} > 0$, if $a \ P_i \ b$ for all $a \in A_1$ and $b \in A_2$, we know that she starts to consume A_1 at the same time before and after agent 1's deviation. Suppose instead, $b \ P_i \ a$ for all $a \in A_1$ and $b \in A_2$, since we know this agent starts to consume A_1 immediately when all the objects in A_2 reach exhaustion, what we need is just $t_{B\cup A_2} > \tilde{t}_{B\cup A_2}$. This can be seen from the consumption procedures for P and $(\tilde{P}_1, P_{-1})$ subject to the partition $\{A_1, A_2, \{a \in A : a \notin A_1 \cup A_2\}\}$. Hence we have shown that for each $i \in I\backslash\{1\}$ with $\sum_{a\in A_1} L_{ia} > 0$, if she prefers A_1 to A_2, she starts to consume A_1 at the same time before and after the deviation; if she prefers A_2 to A_1, she starts to consume A_1 after the deviation at a time earlier than before deviation, which proves Step 2.

Step 3: $L_{1a} \leqslant \tilde{L}_{1a}$ for all $a \in A_2$.

By exchanging the roles of P_1 and $\tilde{P}_1$, Step 3 is implied by Step 2.

C.2 Proof of Theorem 7

Let $\mathbb{D}$ be a *sequentially dichotomous domain* and $(\mathbf{A}_t)_{t=1}^n$ the corresponding path. Let $\tilde{P}_0$ be an arbitrary preference out of $\mathbb{D}$, we show that the PS rule defined on the combination of $\mathbb{D}$ and $\tilde{P}_0$, i.e., $PS : \left(\mathbb{D} \cup \{\tilde{P}_0\}\right)^n \to \mathcal{L}$, is manipulable.

Let $\underline{t} \equiv \min\{t \in \{1, \cdots, n\} : \tilde{P}_0 \text{ does not observe } \mathbf{A}_t\}$. Since $\tilde{P}_0 \in \mathbb{P} \backslash \mathbb{D}$, $\underline{t}$ is well-defined. In addition, since $\tilde{P}_0$ observes $\mathbf{A}_1$ trivially, $\underline{t} \geqslant 2$ and $\tilde{P}_0$ observes $\mathbf{A}_{\underline{t}-1}$. Recall that from $\mathbf{A}_{\underline{t}-1}$ to $\mathbf{A}_{\underline{t}}$, $A_{(\underline{t}-1)*}$ breaks into $A_{\underline{t}1}$ and $A_{\underline{t}2}$. Let $a \equiv r_1(\tilde{P}_0, A_{(\underline{t}-1)*})$ be the most preferred objects according to $\tilde{P}_0$ in $A_{(\underline{t}-1)*}$. Without loss of generality, let $a \in A_{\underline{t}1}$. In addition, let $c \equiv r_1(\tilde{P}_0, A_{\underline{t}2})$ be the most preferred objects according to $\tilde{P}_0$ in $A_{\underline{t}2}$. Since $\tilde{P}_0$ does not observe $\mathbf{A}_{\underline{t}}$, there must be $b \in A_{\underline{t}1} \backslash \{a\}$ ranked below c. Let $b \equiv r_1(\tilde{P}_0, \{x \in A_{\underline{t}1} : c \; \tilde{P}_0 \; x\})$ be the most preferred object in $A_{\underline{t}1}$ that is ranked below c. In addition, let $C \equiv \{x \in A_{\underline{t}1} \backslash \{a\} : x \; \tilde{P}_0 \; c\}$. Notice that C may be empty. Hence $\tilde{P}_0$ can be illustrated as follows.

$$\tilde{P}_0 : \quad \cdots\cdots \succ \overbrace{\underbrace{a \succ \cdots C \cdots}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \succ \cdots\cdots$$

There must be an unique $\bar{t} \geqslant \underline{t}$ such that $a, b \in A_{(\bar{t}-1)*}$, $a \in A_{\bar{t}1}$, and $b \in A_{\bar{t}2}$, i.e., from $\mathbf{A}_{\bar{t}-1}$ to $\mathbf{A}_{\bar{t}}$, a and b are split into two separate blocks. Respectively, let $B \equiv A_{\bar{t}1} \backslash \{a\}$ and $D \equiv A_{\bar{t}2} \backslash \{b\}$. In the following, we identify two preferences $P_0, \bar{P}_0 \in \mathbb{D}$. Then we construct two preference profiles consisting of only $\tilde{P}_0$, P_0 and $\bar{P}_0$ and show that there is a profitable manipulation. To do this, we need to consider four cases.

Case 1: $C = \emptyset$ or $r_1(\tilde{P}_0, C) \notin B \cup D$.

Since $\tilde{P}_0$ observes $\mathbf{A}_{\underline{t}-1}$, $\tilde{P}_0^{\mathbf{A}_{\underline{t}-1}}$ is well-defined. Let $P_0^{\mathbf{A}_{\underline{t}-1}} = \bar{P}_0^{\mathbf{A}_{\underline{t}-1}} = \tilde{P}_0^{\mathbf{A}_{\underline{t}-1}}$. Second, let $A_{(\bar{t}-1)*}$ be the first ranked block in $\mathbf{A}_{\bar{t}-1}$ according to both P_0 and $\bar{P}_0$. Third, let $A_{\bar{t}1} P_0^{\mathbf{A}_{\bar{t}}} A_{\bar{t}2}$ and $A_{\bar{t}2} \bar{P}_0^{\mathbf{A}_{\bar{t}}} A_{\bar{t}1}$. Fourth, let respectively a the first ranked object in $A_{\bar{t}1}$ and b the first ranked object in $A_{\bar{t}2}$ according to both P_0 and $\bar{P}_0$. Last, let P_0 and $\bar{P}_0$ have the same ranking of the objects contained in the same block. It's evident that $P_0, \bar{P}_0 \in \mathbb{D}$. Hence, P_0 and $\bar{P}_0$ are illustrated below.

$$P_0: \quad \cdots\cdots \succ \underbrace{\overbrace{a \succ \cdots B \cdots}^{=A_{\bar{t}1}} \succ \overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c \cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

$$\bar{P}_0: \quad \cdots\cdots \succ \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a \succ \cdots B \cdots}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c \cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

Let $P \equiv (\tilde{P}_1, P_2, P_3, \cdots, P_n)$ and $P' \equiv (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$. We calculate the probabilities specified by the PS rule as follows.

$$\begin{array}{lcccc} PS(P): & a & B & b & D \\ 1: & \frac{1}{n} & 0 & 0 & 0 \\ 2\cdots n: & \frac{1}{n} & \frac{|B|}{n-1} & \frac{1}{n-1} & \frac{|D|}{n-1} \end{array}$$

For P, all agents equally share a. After that agents 2 to n consume $B \cup \{b\} \cup D$ while agent 1 consumes c if C is empty and $r_1(\tilde{P}_0, C)$ if not. By *sd-efficiency*, $PS(P)_{1x} = 0$ for all $x \in B \cup \{b\} \cup D$. Then *equal-treatment-of-equals* implies the other entries.

$$\begin{array}{lcccc} PS(P'): & a & B & b & D \\ 1: & \frac{1}{n-1} & 0 & 0 & 0 \\ 2: & 0 & 0 & \frac{1}{n-1} + \frac{|B|}{n-2} + \frac{1-\frac{1}{n-1}-\frac{|B|}{n-2}}{n-1} & \frac{|D|}{n-1} \\ 3\cdots n: & \frac{1}{n-1} & \frac{|B|}{n-2} & \frac{1-\frac{1}{n-1}-\frac{|B|}{n-2}}{n-1} & \frac{|D|}{n-1} \end{array}$$

For P', agents other than 2 equally share a. By *sd-efficiency*, $PS(P)_{1x} = 0$ for all $x \in B \cup \{b\} \cup D$. During the period when agents other than 2 consume a, agent 2 consumes b. When a is exhausted, agents 3 to n start to consume B if B is nonempty and b is b is empty; agent 1 starts to consume c if C is empty and $r_1(\tilde{P}_0, C)$ if not; and agent 2 still consume b. It's evident that B will be exhausted before b or $r_1(\tilde{P}_0, C)$. After that agents 3 to n join agent 2 in consuming b, so they equally share the remaining share of b, i.e., $1 - \frac{1}{n-1} - \frac{|B|}{n-2}$. Last, agents 2 to n equally share D.

Now we have a contradiction against *sd-strategy-proofness*:

$$
\begin{aligned}
\textit{sd-strategy-proofness} \Rightarrow & \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P) = \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P') \\
& \Rightarrow \frac{1}{n} + \frac{|B|}{n-1} + \frac{1}{n-1} + \frac{|D|}{n-1} = \frac{1}{n-1} + \frac{|B|}{n-2} + \frac{1-\frac{1}{n-1}-\frac{|B|}{n-2}}{n-1} + \frac{|D|}{n-1} \\
& \Rightarrow n = 2 : \text{contradiction.}
\end{aligned}
$$

Case 2: $r_1(\tilde{P}_0, C) \in B$.

In order to calculate the random assignments specified by the PS rule, we need to consider two sub-cases. Given $r_1(\tilde{P}_0, C) \in B$, there is an upper contour set in C contained in B. Let B_1 be the largest such upper contour set, i.e., $B_1 \equiv \max_{k\leqslant|C|}\{U_k(\tilde{P}_0, C) \subset B\}$[25]. Hence either $B_1 = C$ or $r_1(\tilde{P}_0, C\backslash B_1) \notin B$. We consider the sub-cases: (i) $B_1 = C$ or $r_1(\tilde{P}_0, C\backslash B_1) \notin D$, or (ii) $r_1(\tilde{P}_0, C\backslash B_1) \in D$.

Sub-case 2.1: $B_1 = C$ or $r_1(\tilde{P}_0, C\backslash B_1) \notin D$.

For this sub-case, we use the same preferences $\tilde{P}_0$, P_0, and $\bar{P}_0$ as in Case 1 and the same profiles $P \equiv (\tilde{P}_1, P_2, P_3, \cdots, P_n)$ and $P' \equiv (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$.

Given B_1, there must be a largest lower contour set in B according to P_0 such that does not contain any object in B_1, i.e., let $\bar{B}_1 \equiv \min_{k\leqslant|B|}\{U_k(P_0, B) : (B\backslash U_k(P_0, B)) \cap B_1 = \emptyset\}$. For reader to understand the consumption procedure better, preferences are illustrated below with B_1 and $\bar{B}_1$ explicitly located.

$$
\begin{aligned}
\tilde{P}_0 : &\quad \cdots\cdots \succ \overbrace{\underbrace{a \succ B_1 \succ C\backslash B_1}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \succ \cdots\cdots \\
P_0 : &\quad \cdots\cdots \succ \underbrace{\overbrace{a \succ \bar{B}_1 \succ B\backslash\bar{B}_1}^{=A_{\bar{t}1}} \succ \overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots \\
\bar{P}_0 : &\quad \cdots\cdots \succ \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a \succ \bar{B}_1 \succ B\backslash\bar{B}_1}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots
\end{aligned}
$$

[25] $U_k(\tilde{P}_0, C)$ denotes the collection of the most preferred k objects in C according to $\tilde{P}_0$, i.e., $U_k(\tilde{P}_0, C) \equiv \{r_1(\tilde{P}_0, C), \cdots, r_k(\tilde{P}_0, C)\}$

We calculate the probabilities specified by the PS rule as follows.

$$\begin{array}{rcccc} PS(P): & a & B & b & D \\ 1: & \frac{1}{n} & \frac{|\bar{B}_1|}{n} & 0 & 0 \\ 2\cdots n: & \frac{1}{n} & \frac{|\bar{B}_1|}{n}+\frac{|B\backslash\bar{B}_1|}{n-1} & \frac{1}{n-1} & \frac{|D|}{n-1} \end{array}$$

For P, all agents equally share a. By construction, after a is exhausted agent 1 consumes B_1 and all the others consume $\bar{B}_1$. Notice that the time when agent 1 stops consuming B_1 is exactly the time when the other agents stop consuming $\bar{B}_1$. Hence all the agents start consuming $\bar{B}_1$ at the same time and stop consuming $\bar{B}_1$ also at the same time. Then each agent consumes exactly $\frac{|\bar{B}_1|}{n}$ share of $\bar{B}_1$. After that agent 1 consumes c if $B_1 = C$ and $C\backslash B_1$ if not while all the others consume $B\backslash\bar{B}_1$ and then b and D. Notice that *sd-efficiency* implies that agent 1 consumes no share of b or D.

$$\begin{array}{rcccc} PS(P'): & a & B & b & D \\ 1: & \frac{1}{n-1} & \frac{|\bar{B}_1|}{n-1} & 0 & 0 \\ 2: & 0 & 0 & \frac{1}{n-1}+\frac{|\bar{B}_1|}{n-1}+\frac{|B\backslash\bar{B}_1|}{n-2}+\frac{1-\frac{1}{n-1}-\frac{|\bar{B}_1|}{n-1}-\frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} & \frac{|D|}{n-1} \\ 3\cdots n: & \frac{1}{n-1} & \frac{|\bar{B}_1|}{n-1}+\frac{|B\backslash\bar{B}_1|}{n-2} & \frac{1-\frac{1}{n-1}-\frac{|\bar{B}_1|}{n-1}-\frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} & \frac{|D|}{n-1} \end{array}$$

For P', all agents other than 2 equally share a and during they do this agent 2 consumes b. After that agents other than 2 equally share $\bar{B}_1$ and agent 2 still consumes b. Then agent 1 starts to consume c if $C = B_1$ and $C\backslash B_1$ if not and agents 3 to n equally share $B\backslash\bar{B}_1$. During this time period agent 2 is still consuming b. After that, all agents other than 1 equally share the remaining share of b, and D after b is exhausted. The zero entries in the above table are implied by *sd-efficiency*.

Now we have a contradiction against *sd-strategy-proofness*:

$$\textit{sd-strategy-proofness} \Rightarrow \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P) = \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P')$$

$$\Rightarrow \frac{1}{n}+\frac{|\bar{B}_1|}{n}+\frac{|B\backslash\bar{B}_1|}{n-1}+\frac{1}{n-1}+\frac{|D|}{n-1}$$

$$=\frac{1}{n-1}+\frac{|\bar{B}_1|}{n-1}+\frac{|B\backslash\bar{B}_1|}{n-2}+\frac{1-\frac{1}{n-1}-\frac{|\bar{B}_1|}{n-1}-\frac{|B\backslash\bar{B}_1|}{n-2}}{n-1}+\frac{|D|}{n-1}$$

$\Rightarrow n = 2$: contradiction.

Sub-case 2.2: $r_1(\tilde{P}_0, C\backslash B_1) \in D$.

Given $r_1(\tilde{P}_0, C\backslash B_1) \in D$, let D_1 be the largest upper contour set in $C\backslash B_1$ contained in D, i.e., $D_1 \equiv \max_{k\leqslant|C\backslash B_1|}\{U_k(\tilde{P}_0, C\backslash B_1) \subset D\}$. Similarly, let $\bar{D}_1 \equiv \min_{k\leqslant|D|}\{U_k(P_0, D) : (D\backslash U_k(P_0, D)) \cap D_1 = \emptyset\}$. For this sub-case, we use the same preferences and profiles as in the previous sub-case except a small change in P_0: D is ranked above b.

For reader to understand the consumption procedure better, preferences are illustrated below with B_1, $\bar{B}_1$, D_1 and $\bar{D}_1$ explicitly located.

$$\tilde{P}_0 : \quad \cdots\cdots \succ \overbrace{\underbrace{a \succ B_1 \succ D_1 \succ C\backslash(B_1 \cup D_1)}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \succ \cdots\cdots$$

$$P_0 : \quad \cdots\cdots \succ \underbrace{\overbrace{a \succ \bar{B}_1 \succ B\backslash\bar{B}_1}^{=A_{\bar{t}1}} \succ \overbrace{\bar{D}_1 \succ D\backslash\bar{D}_1 \succ b}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

$$\bar{P}_0 : \quad \cdots\cdots \succ \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a \succ \bar{B}_1 \succ B\backslash\bar{B}_1}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots$$

We calculate the probabilities specified by the PS rule as follows.

$$PS(P): \quad \begin{array}{rcccc}
 & a & B & b & D \\
1: & \frac{1}{n} & \frac{|\bar{B}_1|}{n} & 0 & \frac{|B\backslash\bar{B}_1|}{n-1} + \frac{|\bar{D}_1| - \frac{|B\backslash\bar{B}_1|}{n-1}}{n} \\
2\cdots n: & \frac{1}{n} & \frac{|\bar{B}_1|}{n} + \frac{|B\backslash\bar{B}_1|}{n-1} & \frac{1}{n-1} & \frac{|\bar{D}_1| - \frac{|B\backslash\bar{B}_1|}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1}
\end{array}$$

For P, all agents equally share a and $\bar{B}_1$. After that, agent 1 consumes D_1 and all the others consumes $B\backslash\bar{B}_1$. It's evident that $B\backslash\bar{B}_1$ is exhausted faster and after this agents 2 to n join agent 1 in consuming $\bar{D}_1$. When $\bar{D}_1$ is exhausted, agent 1 starts consume c if $C\backslash(B_1 \cup D_1) = \emptyset$ and $C\backslash(B_1 \cup D_1)$ is otherwise. Agents 2 to n equally share $D\backslash\bar{D}_1$.

$$PS(P'): \quad \begin{array}{rcccc}
 & a & B & b & D \\
1: & \frac{1}{n-1} & \frac{|\bar{B}_1|}{n-1} & 0 & \frac{|B\backslash\bar{B}_1|}{n-2} + \frac{|\bar{D}_1| - \frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} \\
2: & 0 & 0 & \alpha + \frac{1-\alpha}{n-1} & 0 \\
3\cdots n: & \frac{1}{n-1} & \frac{|\bar{B}_1|}{n-1} + \frac{|B\backslash\bar{B}_1|}{n-2} & \frac{1-\alpha}{n-1} & \frac{|\bar{D}_1| - \frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} + \frac{|D\backslash\bar{D}_1|}{n-2}
\end{array}$$

where $\alpha = \frac{1}{n-1} + \frac{|\bar{B}_1|}{n-1} + \frac{|B\backslash\bar{B}_1|}{n-2} + \frac{|\bar{D}_1| - \frac{|B\backslash\bar{B}_1|}{n-2}}{n-1} + \frac{|D\backslash\bar{D}_1|}{n-2}$.

For P', agents other than 2 equally share a. When they consume a, agent 2 consumes b. After a is exhausted, agents other than 2 equally share $\bar{B}_1$ and agent 2 still consumes b. After $\bar{B}_1$ is exhausted, agent 1 consumes D_1, agents 3 to n equally share $B\backslash\bar{B}_1$, and agent 2 still consumes b. It's evident that $B\backslash\bar{B}_1$ is exhausted faster. After that agents other than 2 equally share the remaining share of $\bar{D}_1$ and agent 2 still consumes b. Then agent 1 consumes c if $C\backslash(B_1 \cup D_1) = \emptyset$ and $C\backslash(B_1 \cup D_1)$ if otherwise, agents 3 to n equally share $D\backslash\bar{D}_1$, and agent 2 still consumes b. After that agents other than 1 consume the remaining share of b.

Now we have a contradiction against *sd-strategy-proofness*:

$$
\begin{aligned}
\textit{sd-strategy-proofness} \Rightarrow \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P) &= \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P') \\
\Rightarrow & \frac{1}{n} + \frac{|\bar{B}_1|}{n} + \frac{|B\backslash\bar{B}_1|}{n-1} + \frac{1}{n-1} + \frac{|\bar{D}_1| - \frac{|B\backslash\bar{B}_1|}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1} \\
&= \alpha + \frac{1-\alpha}{n-1} \\
\Rightarrow & n > |B| + |D| = \frac{3(n-1)}{n-2}|D\backslash\bar{D}_1| + n(n-2) > n : \text{contradiction.}
\end{aligned}
$$

Case 3: $r_1(\tilde{P}_0, C) \in D$ and $B \neq \emptyset$.

Given $r_1(\tilde{P}_0, C) \in D$, let D_1 be the largest upper contour set in C contained in D, i.e., $D_1 \equiv \max_{k\leqslant|C|}\{U_k(\tilde{P}_0, C) \subset D\}$. Similarly, let $\bar{D}_1 \equiv \min_{k\leqslant|D|}\{U_k(P_0, D) : (D\backslash U_k(P_0, D)) \cap D_1 = \emptyset\}$. For this sub-case, we use the same preferences and profiles as in Case 1.

For reader to understand the consumption procedure better, preferences are illustrated below with D_1 and $\bar{D}_1$ explicitly located.

$$\tilde{P}_0: \quad \cdots\cdots \succ \quad \overbrace{\underbrace{a \succ D_1 \succ C\backslash D_1}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \quad \succ \cdots\cdots$$

$$P_0: \quad \cdots\cdots \succ \quad \underbrace{\overbrace{a \succ \cdots B \cdots}^{=A_{\bar{t}1}} \succ \overbrace{b \succ \bar{D}_1 \succ D\backslash \bar{D}_1}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c \cdots\cdots}_{=A_{\underline{t}2}} \quad \succ \cdots\cdots$$

$$\bar{P}_0: \quad \cdots\cdots \succ \quad \underbrace{\overbrace{b \succ \bar{D}_1 \succ D\backslash \bar{D}_1}^{=A_{\bar{t}2}} \succ \overbrace{a \succ \cdots B \cdots}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c \cdots\cdots}_{=A_{\underline{t}2}} \quad \succ \cdots\cdots$$

$P = (\tilde{P}_0, P_2, P_3, \cdots, P_n)$ and $P' = (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$

$PS(P):$

	a	B	b	D
$1:$	$\frac{1}{n}$	0	0	$\frac{\lvert B\rvert}{n-1} + \frac{1}{n-1} + \frac{\lvert\bar{D}_1\rvert - \frac{\lvert B\rvert}{n-1} - \frac{1}{n-1}}{n}$
$2\cdots n:$	$\frac{1}{n}$	$\frac{\lvert B\rvert}{n-1}$	$\frac{1}{n-1}$	$\frac{\lvert\bar{D}_1\rvert - \frac{\lvert B\rvert}{n-1} - \frac{1}{n-1}}{n} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-1}$

For P, all agents equally share a. Then agent 1 consumes D_1 and all the others equally share B and b. It's evident that $B \cup \{b\}$ is exhausted faster. After that all agents share the remaining of $\bar{D}_1$. Then agent 1 consumes c if $C\backslash D_1 = \emptyset$ and $C\backslash D_1$ if otherwise and all the other agents share $D\backslash\bar{D}_1$.

$PS(P'):$

	a	B	b	D
$1:$	$\frac{1}{n-1}$	0	0	$\frac{\lvert B\rvert}{n-1} + \frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-1}}{n-1} + \frac{\lvert\bar{D}_1\rvert - \frac{\lvert B\rvert}{n-1} - \frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-1}}{n-1}}{n}$
$2:$	0	0	$\frac{1}{n-1} + \frac{\lvert B\rvert}{n-1} + \frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-1}}{n-1}$	$\frac{\lvert\bar{D}_1\rvert - \frac{\lvert B\rvert}{n-1} - \frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-1}}{n-1}}{n} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-1}$
$3\cdots n:$	$\frac{1}{n-1}$	$\frac{\lvert B\rvert}{n-1}$	$\frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-1}}{n-1}$	$\frac{\lvert\bar{D}_1\rvert - \frac{\lvert B\rvert}{n-1} - \frac{1-\frac{1}{n-1}-\frac{\lvert B\rvert}{n-1}}{n-1}}{n} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-1}$

For P', all agents except 2 equally share a and during they do this agent 2 consumes b. After a is exhausted, agent 1 consumes D_1, agent 2 still consumes b, and all the others consume B. It's evident that B is exhausted faster. Then agents 3 to n join agent 2 in consuming the remaining share of b and agent 1 still consumes D_1. After b is exhausted, all agents equally share the remaining share of $\bar{D}_1$. Then agent 1 consumes c if $C\backslash D_1 = \emptyset$ and $C\backslash D_1$ if otherwise and all the other agents share $D\backslash\bar{D}_1$.

Now we have a contradiction against *sd-strategy-proofness*:

$$
\begin{aligned}
\textit{sd-strategy-proofness} \Rightarrow & \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P) = \sum_{x\in\{a,b\}\cup B\cup D} PS_{2x}(P') \\
&\Rightarrow \frac{1}{n} + \frac{|B|}{n-1} + \frac{1}{n-1} + \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1} \\
&\quad = \frac{1}{n-1} + \frac{|B|}{n-1} + \frac{1 - \frac{1}{n-1} - \frac{|B|}{n-1}}{n-1} + \frac{|\bar{D}_1| - \frac{|B|}{n-1} - \frac{1-\frac{1}{n-1}-\frac{|B|}{n-1}}{n-1}}{n} + \frac{|D\backslash\bar{D}_1|}{n-1} \\
&\Rightarrow (1-n)|B| = 0 : \text{contradiction.}
\end{aligned}
$$

Case 4: $r_1(\tilde{P}_0, C) \in D$ and $B = \emptyset$.

In this case $A_{\bar{t}1} = \{a\}$. We use the same preferences and profiles as in Case 3 except a small change in P_0: D is ranked above b.

$$
\begin{aligned}
\tilde{P}_0 :&\quad \cdots\cdots \succ \overbrace{\underbrace{a \succ D_1 \succ C\backslash D_1}_{\subset A_{\underline{t}1}} \succ \underbrace{c \succ \cdots}_{\subset A_{\underline{t}2}} \succ \underbrace{b \succ \cdots}_{\subset A_{\underline{t}1}} \succ \cdots\cdots}^{A_{(\underline{t}-1)*}} \succ \cdots\cdots \\
P_0 :&\quad \cdots\cdots \succ \underbrace{\overbrace{a}^{=A_{\bar{t}1}} \succ \overbrace{\bar{D}_1 \succ D\backslash\bar{D}_1 \succ b}^{=A_{\bar{t}2}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots \\
\bar{P}_0 :&\quad \cdots\cdots \succ \underbrace{\overbrace{b \succ \cdots D \cdots}^{=A_{\bar{t}2}} \succ \overbrace{a}^{=A_{\bar{t}1}} \succ \cdots}_{=A_{\underline{t}1}} \succ \underbrace{c\cdots\cdots}_{=A_{\underline{t}2}} \succ \cdots\cdots
\end{aligned}
$$

$P = (\tilde{P}_0, P_2, P_3, \cdots, P_n)$ and $P' = (\tilde{P}_1, \bar{P}_2, P_3, \cdots, P_n)$

$PS(P)$:		a	D	b
	1 :	$\frac{1}{n}$	$\frac{\lvert\bar{D}_1\rvert}{n}$	0
	$2\cdots n$:	$\frac{1}{n}$	$\frac{\lvert\bar{D}_1\rvert}{n} + \frac{\lvert D\backslash\bar{D}_1\rvert}{n-1}$	$\frac{1}{n-1}$

For P, all agents equally share a. Then they equally share $\bar{D}_1$. After that agent 1 consumes c if $C\backslash D_1 = \emptyset$ and $C\backslash D_1$ if otherwise and all the others equally share $D\backslash\bar{D}_1$ and then b.

$PS(P')$:

	a	D	b
$1:$	$\frac{1}{n-1}$	$\frac{\lvert\bar{D}_1\rvert}{n-1}$	0
$2:$	0	0	$\frac{\lvert\bar{D}_1\rvert}{n-1}+\frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}+\frac{1-\frac{\lvert\bar{D}_1\rvert}{n-1}-\frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}}{n-1}$
$3\cdots n:$	$\frac{1}{n-1}$	$\frac{\lvert\bar{D}_1\rvert}{n-1}+\frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}$	$\frac{1-\frac{\lvert\bar{D}_1\rvert}{n-1}-\frac{\lvert D\backslash\bar{D}_1\rvert}{n-2}}{n-1}$

For P', all agents other than 2 equally share a and during they do this agent 2 consumes b. After a being exhausted, agents other than 2 equally share $\bar{D}_1$ and agent 2 still consumes b. It's evident that $\bar{D}_1$ is exhausted faster. After that agent 1 consumes c if $C\backslash D_1\emptyset$ and $C\backslash D_1$ if otherwise, agent 3 to n equally share $D\backslash\bar{D}_1$, and agent 2 still consumes b. It's evident that $D\backslash\bar{D}_1$ is exhausted faster. After that, agents 3 to n join agent 2 in consuming b so they equally share the remaining share of b.

Now we have a contradiction against *sd-strategy-proofness*:

$$\begin{aligned}
\textit{sd-strategy-proofness} \Rightarrow & \sum_{x\in\{a,b\}\cup D} PS_{2x}(P) = \sum_{x\in\{a,b\}\cup D} PS_{2x}(P') \\
\Rightarrow & \frac{1}{n}+\frac{|\bar{D}_1|}{n}+\frac{|D\backslash\bar{D}_1|}{n-1}+\frac{1}{n-1} \\
& =\frac{|\bar{D}_1|}{n-1}+\frac{|D\backslash\bar{D}_1|}{n-2}+\frac{1-\frac{|\bar{D}_1|}{n-1}-\frac{|D\backslash\bar{D}_1|}{n-2}}{n-1} \\
\Rightarrow & (n-1)^2=-|\bar{D}_1| : \text{contradiction.}
\end{aligned}$$

D Appendix to Chapter 6

D.1 Two Technical Assumptions

Assumption 1

$$f(m_1, m_2) \equiv -\, n(m_1 + m_2)\left(\left[\frac{m_2 n}{m_1 + m_2}\right]_-\right)^2$$
$$+ \left[(n^2 - n + 1)m_1 + (2n^2 - n)m_2\right]\left(\left[\frac{m_2 n}{m_1 + m_2}\right]_-\right) - n^2(n-1)m_2 \leqslant 0$$

*for all positive integers m_1, m_2 such that $m_1 + m_2 < n$ and $\frac{m_2 n}{m_1+m_2}$ is **not an integer**, where $[x]_-$ denotes for a real number the largest integer which is no greater than x.*

Assumption 2

$$g(m_1, m_2, m_3) \equiv \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(\bar{n}_5 - 1)}) - (\bar{n}_5 - 1) \times \gamma(\bar{n}_5)}{n - (\bar{n}_5 + 1)} \neq \frac{m_3}{n}$$

*for all positive integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 \leqslant n$ and $\frac{m_2 n}{m_1+m_2}$ is **not an integer**.*

where $\bar{n}_5 = \left[\frac{m_2 n}{m_1 + m_2}\right]_- + 1,$

$$\gamma(k) = \frac{\Gamma_1(k)m_1 + \Gamma_2(k)m_2 + \Gamma_3(k)m_3}{n[n^4 - 2(k+1)n^3 + (k^2 + 5k - 1)n^2 - (3k^2 + k - 2)n + 2(k^2 - k)]}$$

and $\Gamma_1(k) = 2(k-2)n^2 - 2(k^2 - k - 2)n + 2(k^2 - k - 2)$

$$\Gamma_2(k) = -2n^3 + 4kn^2 - 2(k^2 + k - 1)n + 2(k^2 - k)$$

$$\Gamma_3(k) = n^4 - 2(k+1)n^3 + (k^2 + 5k - 1)n^2 - (3k^2 + k - 2)n + 2(k^2 - k)$$

D.2 Matlab Codes for Checking The Assumptions Given a Fixed n.

- Check $f(m_1, m_2) \leqslant 0$ for all positive integers m_1, m_2 such that $m_1 + m_2 < n$ and $\frac{m_2 n}{m_1+m_2}$ is **not an integer**.

```
clear;clc
syms n m1 m2 x
a=sym('-n*(m1+m2)');
b=sym('(n^2-n+1)*m1+(2*n^2-n)*m2');
c=sym('-n^2*(n-1)*m2');
eqn = a*x^2+b*x+c == 0;
solx = solve(eqn,x); %solve equation ax^2+b+c=0.

n=1000; % Fix an n

flag=1;
for m1=1:n-2
for m2=1:(n-m1-1)
fprintf('n=%d m1=%d m2=%d',[n m1 m2])
fprintf('\n')
x=floor(m2*n/(m1+m2));
if x<m2*n/(m1+m2)
if x<max(eval(solx)) && x>min(eval(solx))
flag=0;
fprintf('n=%d m1=%d m2=%d x_1^* x_2^* x',[n m1 m2 min(eval(solx))...,
max(eval(solx)) floor(m2*n/(m1+m2)) m2*n/(m1+m2)])
fprintf('\n')
break
end
end
end
if flag==0
break
end
end
```

- Check $g(m_1, m_2, m_3) \neq \frac{m_3}{n}$ for all positive integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 \leqslant n$ and $\frac{m_2 n}{m_1+m_2}$ is **not an integer**.

```
n=1000; % Fix an n
for m1=1:n-2
```

```
for m2=1:n-m1-1
for m3=1:n-m1-m2
fprintf('n=%d m1=%d m2=%d m3=%d',[n m1 m2 m3])
fprintf('\n')
if floor(m2*n/(m1+m2))<m2*n/(m1+m2)
nbar5=floor(m2*n/(m1+m2))+1;
A1=2*(nbar5 -2)*n^2 -2*(nbar5^2-nbar5 -2)*n+2*(nbar5^2-nbar5 -2);
A2=-2*n^3+4*nbar5*n^2 -2*(nbar5^2+nbar5 -1)*n+2*(nbar5^2-nbar5);
A3=n^4 -2*(nbar5+1)*n^3+(nbar5^2+5*nbar5 -1)*n^2...,
-(3*nbar5^2+nbar5 -2)*n+2*(nbar5^2-nbar5);
gamma=(A1*m1+A2*m2+A3*m3)/(n^(n^4  -2*(nbar5+1)^n^3...,
+(nbar5^2+5*nbar5 -1)*n^2 -(3*nbar5^2+nbar5 -2)*n  +2*(nbar5^2-nbar5)));
m=m1+m2+m3;
f=(m3-2*(m/n-m1/(n-(nbar5 -1)))-(nbar5 -1)*gamma)/(n-(nbar5+1));
if abs(f - m3/n) < eps
fprintf('n=%d m1=%d m2=%d m3=%d nbar5=%f gamma=%f f=%f'...,
,[n m1 m2 m3 nbar5 f])
flag=0;
break
end
end
end
if flag==0
break
end
end
if flag==0
break
end
end
```

D.3 Proof of Proposition 3

Let $E, B, C, D, F \subset A$ with $m_1 \equiv |B| \geqslant 1$, $m_2 \equiv |C| \geqslant 1$, $m_3 \equiv |D| \geqslant 1$. Let $m \equiv m_1 + m_2 + m_3$. In addition, given a real number x, $[x]_{-}$ denotes the largest integer which is smaller or equal to x. Finally given a random assignment L and a subset of objects $B \subset A$, we denote $L_{i,B} = \sum_{x \in B} L_{i,x}$.

Let $\mathbb{D} \equiv \{\bar{P}_i, P_i, \hat{P}_i\}$ where the preferences are from Table 7. To prove the theorem, it suffices to prove $\mathbb{D}$ admits no good rule. Suppose not, and let $\varphi : \mathbb{D}^n \longrightarrow \mathcal{L}$ be a good rule.

Lemma 15 *For any $P \in \mathbb{D}^n$, $\varphi_{i,B}(P) + \varphi_{i,C}(P) + \varphi_{i,D}(P) = \frac{m}{n}$ for all $i \in I$.*

This lemma can be proved by applying repeatedly *equal treatment of equals* and *sd-strategy-proofness*. The proof is standard and hence omitted.

Notice that, since $\frac{m_1 n}{m_1+m_2} + \frac{m_2 n}{m_1+m_2} = n$, it's either both $\frac{m_1 n}{m_1+m_2}$ and $\frac{m_2 n}{m_1+m_2}$ are integers or neither one of them is an integer. I'll show two contradictions, one for each case. When $\frac{m_2 n}{m_1+m_2}$ is an integer, the contradiction is identified. While Assumptions 1 and 2 are needed to identify the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is not an integer.

In the following ,I'll construct six groups of profiles and characterize the random assignments of B, C, D for each of these profiles. The contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is an integer can be found using profile groups I to IV. To find the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is not an integer, we need in addition profile groups V and VI.

Firstly I list all profiles.

Profile group I:

$$
\begin{aligned}
P^{1,0} &= (P_1, \cdots, P_n) \\
P^{1,1} &= (\hat{P}_1, P_2, \cdots, P_n) \\
&\vdots \\
P^{1,k} &= (\hat{P}_1, \cdots, \hat{P}_k, P_{k+1}, \cdots, P_n) \\
&\vdots \\
P^{1,\bar{n}_1} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_1}, P_{\bar{n}_1+1}, \cdots, P_n)
\end{aligned}
$$

where $\bar{n}_1 = \frac{m_2 n}{m_1+m_2}$ when $\frac{m_2 n}{m_1+m_2}$ is an integer and $\bar{n}_1 = \left[\frac{m_2 n}{m_1+m_2}\right]_-$ otherwise.

Profile group II:

$$
\begin{aligned}
P^{2,1} &= (P_1, \cdots, P_{n-1}, \bar{P}_n) \\
P^{2,2} &= (\hat{P}_1, P_2 \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{2,k} &= (\hat{P}_1, \cdots, \hat{P}_{k-1}, P_k, \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{2,\bar{n}_2} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_2-1}, P_{\bar{n}_2}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}
$$

where $\bar{n}_2 = \frac{m_2 n}{m_1+m_2}$ when $\frac{m_2 n}{m_1+m_2}$ is an integer and $\bar{n}_2 = \left[\frac{m_2 n}{m_1+m_2}\right]_- + 1$ otherwise.

Profile group III:

$$
\begin{aligned}
P^{3,0} &= (\hat{P}_1, \cdots, \hat{P}_n) \\
P^{3,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-1}, P_n) \\
&\vdots \\
P^{3,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_n) \\
&\vdots \\
P^{3,\bar{n}_3} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_3}, P_{n-\bar{n}_3+1}, \cdots, P_n)
\end{aligned}
$$

where $\bar{n}_3 = \frac{m_1 n}{m_1+m_2}$ when $\frac{m_1 n}{m_1+m_2}$ is an integer and $\bar{n}_3 = \left[\frac{m_1 n}{m_1+m_2}\right]_-$ otherwise.

Profile group IV:

$$
\begin{aligned}
P^{4,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-1}, \bar{P}_n) \\
P^{4,2} &= (\hat{P}_1, \cdots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{4,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{4,\bar{n}_4} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_4}, P_{n-\bar{n}_4+1}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}
$$

where $\bar{n}_4 = \frac{m_1 n}{m_1+m_2}$ when $\frac{m_1 n}{m_1+m_2}$ is an integer and $\bar{n}_4 = \left[\frac{m_1 n}{m_1+m_2}\right]_-$ otherwise.

Profile group V:

$$
\begin{array}{rcl}
P^{5,1} & = & (P_1, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
P^{5,2} & = & (\hat{P}_1, P_2 \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
 & \vdots & \\
P^{5,k} & = & (\hat{P}_1, \cdots, \hat{P}_{k-1}, P_k, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
 & \vdots & \\
P^{5,\bar{n}_5} & = & (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_5-1}, P_{\bar{n}_5}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)
\end{array}
$$

where $\bar{n}_5 = \left[\frac{m_2 n}{m_1+m_2}\right]_- + 1$ and $\frac{m_2 n}{m_1+m_2}$ is not an integer.

Profile group VI:

$$
\begin{array}{rcl}
P^{6,1} & = & (\hat{P}_1, \cdots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n) \\
P^{6,2} & = & (\hat{P}_1, \cdots, \hat{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
P^{6,3} & = & (\hat{P}_1, \cdots, \hat{P}_{n-3}, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
 & \vdots & \\
P^{6,k} & = & (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
 & \vdots & \\
P^{6,\bar{n}_6} & = & (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_6}, P_{n-\bar{n}_6+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)
\end{array}
$$

where $\bar{n}_6 = \left[\frac{m_1 n}{m_1+m_2}\right]_-$ and $\frac{m_2 n}{m_1+m_2}$ is not an integer.

Now we characterize the random assignments for the preference profiles through a series of claims.

Claim 12 *For each preference profile $P^{1,k}$, $\varphi(P^{1,k})$ specifies probabilities on B, C, and D as follows*

$$
\begin{array}{cccc}
 & B & C & D \\
1 & 0 & \frac{m_1+m_2}{n} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
k & 0 & \frac{m_1+m_2}{n} & \frac{m_3}{n} \\
k+1 & \frac{m_1}{n-k} & \frac{m_1+m_2}{n} - \frac{m_1}{n-k} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n & \frac{m_1}{n-k} & \frac{m_1+m_2}{n} - \frac{m_1}{n-k} & \frac{m_3}{n}
\end{array}
$$

Proof: Verification of the claim consists of three steps.

Step 1: We show $\varphi_{i,D}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$ and all $k = 0, 1, \cdots, \bar{n}_1$.

First, by *equal treatment of equals*, $\varphi_{i,D}(P^{1,0}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n$. Second, we show for all $k = 1, \cdots, \bar{n}_1$ if $\varphi_{i,D}(P^{1,k-1}) = \frac{m_3}{n}$ for all $i \in I$, then $\varphi_{i,D}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$. Notice that $P^{1,k}$ and $P^{1,k-1}$ are different only in agent k's preference, i.e., $P^{1,k}_k = \hat{P}_i$ and $P^{1,k-1}_k = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{k,D}(P^{1,k}) = \varphi_{k,D}(P^{1,k-1}) = \frac{m_3}{n}$. Hence by feasibility and *equal treatment of equals*, $\varphi_{i,D}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$.

Step 2: We show $\varphi_{i,B}(P^{1,k}) = 0$ for all $i = 1, \cdots, k$ and all $k = 0, 1, \cdots, \bar{n}_1$. Fix an k and suppose without loss of generality $\varphi_{1,B}(P^{1,k}) = \beta > 0$. Then *sd-efficiency* implies $\varphi_{i,C}(P^{1,k}) = 0$ for all $i = k+1, \cdots, n$ and *equal treatment of equals* implies $\varphi_{i,C}(P^{1,k}) = \frac{m_2}{k}$ for all $i = 1, \cdots, k$.

$$\begin{aligned}\varphi_{1,B}(P^{1,k}) + \varphi_{1,C}(P^{1,k}) + \varphi_{1,D}(P^{1,k}) &= \beta + \frac{m_2}{k} + \frac{m_3}{n} \\ &> \frac{m_2}{k} + \frac{m_3}{n} \\ &\geqslant \frac{m}{n}\end{aligned}$$

where the last inequality comes from $k \leqslant \bar{n}_1 \leqslant \frac{m_2 n}{m_1+m_2}$: a contradiction against Lemma 15.

Step 3: Lemma 15 and *equal treatment of equals* imply all other entries. ■

Claim 13 *For each preference profile $P^{2,k}$, $\varphi(P^{2,k})$ specifies probabilities on B, C, and D as follows*

	B	C	D
1	0	$\frac{m}{n} - \alpha(k)$	$\alpha(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	0	$\frac{m}{n} - \alpha(k)$	$\alpha(k)$
k	$\frac{m_1}{n-(k-1)}$	$\frac{m_2-(k-1)\times(\frac{m}{n}-\alpha(k))}{n-k}$	$\frac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\alpha(k)}{n-k}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$\frac{m_1}{n-(k-1)}$	$\frac{m_2-(k-1)\times(\frac{m}{n}-\alpha(k))}{n-k}$	$\frac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\alpha(k)}{n-k}$
n	$\frac{m_1}{n-(k-1)}$	0	$\frac{m}{n} - \frac{m_1}{n-(k-1)}$

where $\alpha(k) = \frac{(k-2)m_1-(n-(k-1))m_2+(n-1)(n-(k-1))m_3}{n(n-1)(n-(k-1))}$.

Proof: Verification of the claim consists of six steps.

Step 1: We show $\varphi(P^{2,1})$ specifies probabilities on B, C, and D as follows

$$\begin{array}{cccc} & B & C & D \\ 1 & \frac{m_1}{n} & \frac{m_2}{n-1} & \frac{m_2+m_3}{n} - \frac{m_2}{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & \frac{m_1}{n} & \frac{m_2}{n-1} & \frac{m_2+m_3}{n} - \frac{m_2}{n-1} \\ n & \frac{m_1}{n} & 0 & \frac{m_2+m_3}{n} \end{array}$$

First notice that $P^{2,1}$ and $P^{1,0}$ are different only in agent n's preference, i.e., $P^{2,1}_n = \bar{P}_1$ and $P^{1,0} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n,B}(P^{2,1}) = \varphi_{n,B}(P^{1,0}) = \frac{m_1}{n}$. Hence feasibility and *equal treatment of equals* imply $\varphi_{i,B}(P^{2,1}) = \frac{m_1}{n}$ for all $i \in I$.

Second $\varphi_{n,C}(P^{2,1}) = 0$. Suppose not, then *sd-efficiency* implies $\varphi_{i,D}(P^{2,1}) = 0$ for all $i = 1, \cdots, n-1$. Hence feasibility implies $\varphi_{n,D}(P^{2,1}) = m_3 \geqslant 1$: a contradiction against Lemma 15.

Last, feasibility and *equal treatment of equals* imply all other entries.

Step 2: We show $\varphi_{n,B}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ for all $k = 2, \cdots, \bar{n}_2$. Fix an k. Notice that $P^{2,k}$ and $P^{1,k-1}$ are different only in agent n's preference, i.e., $P^{2,k}_n = \bar{P}_i$ and $P^{2,k}_n = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n,B}(P^{2,k}) = \varphi_{n,B}(P^{1,k-1}) = \frac{m_1}{n-(k-1)}$.

Step 3: We show $\varphi_{n,C}(P^{2,k}) = 0$ for all $k = 2, \cdots, \bar{n}_2$. Fix an k and suppose $\varphi_{n,C}(P^{2,k}) > 0$. Then *sd-efficiency* implies $\varphi_{i,D}P^{2,k} = 0$ for all $i = 1, \cdots, n-1$ and hence $\varphi_{n,D}P^{2,k} = m_3$: a contradiction against Lemma 15.

Step 4: We show $\varphi_{i,D}(P^{2,k}) = \alpha(k)$ for all $i = 1, \cdots, k-1$ and all $k = 2, \cdots, \bar{n}_2$.

First we show $\varphi_{1,D}(P^{2,2}) = \alpha(2)$. Notice that $P^{2,2}$ and $P^{2,1}$ are different only in agent 1's preference, i.e., $P^{2,2}_1 = \hat{P}_i$ and $P^{2,1}_1 = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{1,D}(P^{2,2}) = \varphi_{1,D}(P^{2,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-1}$.

$$\begin{aligned}\alpha(2) &= \frac{(2-2)m_1 - (n-(2-1))m_2 + (n-1)(n-(2-1))m_3}{n(n-1)(n-(2-1))} \\ &= \frac{(n-1)m_3 - m_2}{n(n-1)} \\ &= \frac{m_2+m_3}{n} - \frac{m_2}{n-1}.\end{aligned}$$

Second, we show an induction: If $\varphi_{i,D}(P^{2,k}) = \alpha(k)$ for all $i = 1, \cdots, k-1$ and an $k \in \{2, \cdots, \bar{n}_2 - 1\}$, then $\varphi_{i,D}(P^{2,k+1}) = \alpha(k+1)$ for all $i = 1, \cdots, k$. Notice that $P^{2,k+1}$ and $P^{2,k}$ are different only in agent k's preference, i.e., $P^{2,k+1}_k = \hat{P}_i$ and $P^{2,k}_k = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{k,D}(P^{2,k+1}) = \varphi_{k,D}(P^{2,k})$. Hence for all $i = 1, \cdots, k$

$\varphi_{i,D}(P^{2,k+1})$	$= \varphi_{k,D}(P^{2,k+1})$	by *equal treatment of equals*
	$= \varphi_{k,D}(P^{2,k})$	by *sd-strategy-proofness*
	$= \frac{m_3 - (\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\varphi_{k-1,D}(P^{2,k})}{n-k}$	by feasibility and *equal treatment of equals*
	$= \frac{m_3 - (\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\alpha(k)}{n-k}$	by induction hypothesis
	$= \alpha(k+1)$	by simplifying expression.

Step 5: We show $\varphi_{i,B}(P^{2,k}) = 0$ for all $i = 1, \cdots, k-1$ and all $k = 2, \cdots, \bar{n}_2$. Fix an k. Suppose without loss of generality $\varphi_{1,B}(P^{2,k}) = \beta > 0$. Then *equal treatment of equals* implies $\varphi_{i,B}(P^{2,k}) = \beta$ for all $i = 1, \cdots, k-1$. Hence Lemma 15 and Step 4 imply $\varphi_{i,C}(P^{2,k}) = \frac{m}{n} - \alpha(k) - \beta$ for all $i = 1, \cdots, k-1$ and *sd-efficiency* implies $\varphi_{i,C}(P^{2,k}) = 0$ for all $i = k, \cdots, n-1$.

Now we show $(k-1)\times(\frac{m}{n}-\alpha(k)-\beta)<m_2$: a contradiction against feasibility.

$$
\begin{aligned}
&(k-1)\times(\frac{m}{n}-\alpha(k)-\beta)<m_2\\
\Leftarrow&(k-1)\times(\frac{m}{n}-\alpha(k))\leqslant m_2\\
\Leftrightarrow&(k-1)\times\left[\frac{m}{n}-\frac{(k-2)m_1-(n-(k-1))m_2+(n-1)(n-(k-1))m_3}{n(n-1)(n-(k-1))}\right]-m_2\leqslant 0\\
\Leftrightarrow&(k-1)\times[(n-1)(n-(k-1))(m_1+m_2)-(k-2)m_1+(n-(k-1))m_2]\\
&-n(n-1)(n-(k-1))m_2\leqslant 0\\
\Leftrightarrow&-n(m_1+m_2)(k-1)^2+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right](k-1)-n^2(n-1)m_2\leqslant 0
\end{aligned}
$$

Let $f(\theta)=-n(m_1+m_2)(\theta-1)^2+[(n^2-n+1)m_1+(2n^2-n)m_2](\theta-1)-n^2(n-1)m_2$. To verify the Step, it suffices to show $f(\theta)\leqslant 0$ for all $k=2,\cdots,\bar{n}_2$.

From the functional form of $f(\theta)$, we have first-order derivative and the second order derivative as follows

$$
\begin{aligned}
f'(\theta)&=-2n(m_1+m_2)(\theta-1)+(n^2-n+1)m_1+(2n^2-n)m_2\\
f''(\theta)&=-2n(m_1+m_2)
\end{aligned}
$$

When $\frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}$ is an integer, $\bar{\mathbf{n}}_\mathbf{2}=\frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}$.

$$
\begin{aligned}
f(\bar{n}_2)=&-n(m_1+m_2)\left(\frac{m_2 n}{m_1+m_2}-1\right)^2\\
&+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right]\left(\frac{m_2 n}{m_1+m_2}-1\right)-n^2(n-1)m_2\\
=&\frac{1}{m_1+m_2}\{-n[(n-1)m_2-m_1]^2\\
&+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right][(n-1)m_2-m_1]-n^2(n-1)m_2(m_1+m_2)\}\\
=&\frac{1}{m_1+m_2}\left[-(n^2+1)m_1^2-\left((n-\frac{1}{2})^2+\frac{3}{4}\right)m_1m_2\right]<0.
\end{aligned}
$$

$$
\begin{aligned}
f'(\bar{n}_2) =& -2n(m_1+m_2)(\frac{m_2 n}{m_1+m_2}-1)+(n^2-n+1)m_1+(2n^2-n)m_2 \\
=& \frac{1}{m_1+m_2}[-2n(m_1+m_2)((n-1)m_2-m_1)+(n^2-n+1)m_1(m_1+m_2) \\
&+(2n^2-n)m_2(m_1+m_2)] \\
=& \frac{1}{m_1+m_2}[(n^2+n+1)m_1^2+nm_2^2+(n^2+2n+1)m_1m_2]>0
\end{aligned}
$$

By $f''(\theta)<0$ and $f'(\bar{n}_2)>0$, $f'(\theta)>0$ for all $\theta \leqslant \bar{n}_2$, that is $f(\theta)$ is increasing through 2 to $\bar{n}_2$. Then $f(\bar{n}_2)<0$ implies $f(\theta)<0$ for all $\theta \leqslant \bar{n}_2$, which is what we want.

When $\frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}$ is not an integer, $\bar{\mathbf{n}}_\mathbf{2} = \left[\frac{\mathbf{m_2 n}}{\mathbf{m_1+m_2}}\right]_- + \mathbf{1}$.

$$
\begin{aligned}
f(\bar{n}_2) =& -n(m_1+m_2)\left(\left[\frac{m_2 n}{m_1+m_2}\right]_-\right)^2 \\
&+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right]\left(\left[\frac{m_2 n}{m_1+m_2}\right]_-\right)-n^2(n-1)m_2 \leqslant 0
\end{aligned}
$$

where the last inequality comes from Assumption 1 in Appendix D.1.

$$
\begin{aligned}
f'(\bar{n}_2) =& -2n(m_1+m_2)(\frac{m_2 n}{m_1+m_2}-\delta)+(n^2-n+1)m_1+(2n^2-n)m_2 \\
=& \frac{1}{m_1+m_2}[-2n(m_1+m_2)((n-\delta)m_2-\delta m_1)+(n^2-n+1)m_1(m_1+m_2) \\
&+(2n^2-n)m_2(m_1+m_2)] \\
=& \frac{1}{m_1+m_2}[m_1n(n-1)+m_2n(m_1(n-2)-m_2)+m_1m_2+m_1^2 \\
&+2\delta(m_1^2n+m_2^2n+2m_1m_2n)]>0
\end{aligned}
$$

where the last inequality comes from $m_2 \leqslant (n-2)$ and $m_1 \geqslant 1$.

Step 6: Lemma 15 and *equal treatment of equals* imply all other entries. ■

Claim 14 *For each preference profile $P^{3,k}$, $\varphi(P^{3,k})$ specifies probabilities on B, C, and D as follows*

$$
\begin{array}{cccc}
 & B & C & D \\
1 & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-k & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
n-k+1 & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n}
\end{array}
$$

Proof: This claim can be verified by the similar arguments that verify Claim 12. ■

Claim 15 *For each preference profile $P^{4,k}$, $\varphi(P^{4,k})$ specifies probabilities on B, C, and D as follows*

$$
\begin{array}{cccc}
 & B & C & D \\
1 & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-k & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
n-k+1 & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n} \\
n & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n}
\end{array}
$$

Proof: Verification of the claim consists of five steps.

Step 1: We show $\varphi_{n,B}(P^{4,k}) = \frac{m_1+m_2}{n}$ for all $k = 1, \cdots, \bar{n}_4$. Fix an k. Notice that $P^{4,k}$ and $P^{3,k}$ are different only in agent n's preference, i.e., $P^{4,k}_n = \bar{P}_i$ and $P^{3,k}_n = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n,B}(P^{4,k}) = \varphi_{n,B}(P^{3,k}) = \frac{m_1+m_2}{n}$.

Step 2: We show $\varphi_{n,C}(P^{4,k}) = 0$ and $\varphi_{n,D}(P^{4,k}) = \frac{m_3}{n}$ for all $k = 1, \cdots, \bar{n}_4$. Fix an k. Suppose $\varphi_{n,C}(P^{4,k}) > 0$, then *sd-efficiency* implies $\varphi_{i,D}(P^{4,k}) = 0$ for all $i = 1, \cdots, n-1$ and hence $\varphi_{n,D}(P^{4,k}) = m_3$: a contradiction against Lemma 15. Given $\varphi_{n,C}(P^{4,k}) = 0$, Lemma 15 implies $\varphi_{n,D}(P^{4,k}) = \frac{m_3}{n}$.

Step 3: We show $\varphi_{i,D}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$ and all $k = 1, \cdots, \bar{n}_4$. First *equal treatment of equals* and Step 2 imply $\varphi_{i,D}(P^{4,1}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$. Second we prove an induction: For any $k = 2, \cdots, \bar{n}_4$, if $\varphi_{i,D}(P^{4,k-1}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$, then $\varphi_{i,D}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$. Notice that $P^{4,k-1}$ and $P^{4,k}$ are different only in agent $(n-k+1)$'s preference, i.e., $P^{4,k-1}_{n-k+1} = \hat{P}_i$ and $P^{4,k}_{n-k+1} = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-k+1,D}(P^{4,k}) = \varphi_{n-k+1,D}(P^{4,k-1}) = \frac{m_3}{n}$. Hence feasibility and *equal treatment of equals* imply $\varphi_{i,D}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$.

Step 4: We show $\varphi_{i,C}(P^{4,k}) = 0$ for all $i = n-k+1, \cdots, n-1$ and all $k = 2, \cdots, \bar{n}_4$. Fix an k and suppose without loss of generality $\varphi_{n-1,C}(P^{4,k}) = \beta > 0$. By *equal treatment of equals*, $\varphi_{i,C}(P^{4,k}) = \beta$ for all $i = n-k+1, \cdots, n-1$. Then Lemma 15 implies $\varphi_{i,B}(P^{4,k}) = \frac{m_1+m_2}{n} - \beta$ for all $i = n-k+1, \cdots, n-1$ and *sd-efficiency* implies $\varphi_{i,B}(P^{4,k}) = 0$ for all $i = 1, \cdots, n-k$. Then we have a contradiction against feasibility

$$
\begin{aligned}
m_1 &= (n-k) \times 0 + (k-1) \times (\frac{m_1+m_2}{n} - \beta) + \frac{m_1+m_2}{n} \\
&< k \times \frac{m_1+m_2}{n} \leqslant m_1
\end{aligned}
$$

where the last inequality comes from $k \leqslant \bar{n}_4 \leqslant \frac{m_1 n}{m_1+m_2}$.

Step 5: Lemma 15 and *equal treatment of equals* imply all other entries. ■

Now we have the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is an integer.

$$
\begin{aligned}
P^{2,\bar{n}_2} =& (\hat{P}_1, \cdots, \hat{P}_{\frac{m_2 n}{m_1+m_2}-1}, P_{\frac{m_2 n}{m_1+m_2}}, \cdots, P_{n-1}, \bar{P}_n) \\
P^{4,\bar{n}_4} =& (\hat{P}_1, \cdots, \hat{P}_{n-\frac{m_1 n}{m_1+m_2}}, P_{n-\frac{m_1 n}{m_1+m_2}+1}, \cdots, P_{n-1}, \bar{P}_n) \\
=& (\hat{P}_1, \cdots, \hat{P}_{\frac{m_2 n}{m_1+m_2}}, P_{\frac{m_2 n}{m_1+m_2}+1}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}
$$

Hence $P^{2,\bar{n}_2}$ and $P^{4,\bar{n}_4}$ are different only in agent $\frac{m_2 n}{m_1+m_2}$'s preference, i.e., $P^{2,\bar{n}_2}_{\frac{m_2 n}{m_1+m_2}} = P_i$ and $P^{2,\bar{n}_2}_{\frac{m_2 n}{m_1+m_2}} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 15. Then *sd-strategy-proofness* implies $\varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{2,\bar{n}_2}) = \varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{4,\bar{n}_4})$. Now we have the contradiction as the following elaboration:

$$\begin{aligned}
&\varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{2,\bar{n}_2}) = \varphi_{\frac{m_2 n}{m_1+m_2},D}(P^{4,\bar{n}_4})\\
\Leftrightarrow &\frac{m_3 - (\frac{m}{n} - \frac{m_1}{n-(\bar{n}_2-1)}) - (\bar{n}_2 - 1) \times \alpha(\bar{n}_2)}{n - \bar{n}_2} = \frac{m_3}{n}\\
\Leftrightarrow &- m_1 n(m_1 + m_2)((n+1)m_1 + m_2) = 0\text{: contradiction!}
\end{aligned}$$

To find the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is not an integer, we characterize the assignment of D for the profiles in groups V and VI.

Let k^* be such that $k^* - 1 = n - \frac{m_1}{\frac{m}{n} - \frac{m_3}{2}}$, which is equivalent to $\frac{m}{n} - \frac{m_1}{n-(k^*-1)} - \frac{m_3}{2} = 0$. I first present two types of assignments and later I will show that both assignments are possible for profiles in group V by Claim 16, 17, and 18.

Assignment 1:

$$\begin{array}{cccc}
 & B & C & D\\
1 & - & - & \gamma(k)\\
\vdots & \vdots & \vdots & \vdots\\
k-1 & - & - & \gamma(k)\\
k & - & - & \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\gamma(k)}{n-(k+1)}\\
\vdots & \vdots & \vdots & \vdots\\
n-2 & - & - & \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\gamma(k)}{n-(k+1)}\\
n-1 & \frac{m_1}{n-(k-1)} & 0 & \frac{m}{n} - \frac{m_1}{n-(k-1)}\\
n & \frac{m_1}{n-(k-1)} & 0 & \frac{m}{n} - \frac{m_1}{n-(k-1)}
\end{array}$$

$$\begin{aligned}
\text{where}\quad &\gamma(k) = \frac{\Gamma_1(k)m_1 + \Gamma_2(k)m_2 + \Gamma_3(k)m_3}{n[n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)]}\\
\text{and}\quad &\Gamma_1(k) = 2(k-2)n^2 - 2(k^2-k-2)n + 2(k^2-k-2)\\
&\Gamma_2(k) = -2n^3 + 4kn^2 - 2(k^2+k-1)n + 2(k^2-k)\\
&\Gamma_3(k) = n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)
\end{aligned}$$

Assignment 2:

	B	C	D
1	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$-$	$-$	0
k	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	0
$n-1$	$\frac{m_1}{n-(k-1)}$	$\frac{m}{n} - \frac{m_1}{n-(k-1)} - \frac{m_3}{2}$	$\frac{m_3}{2}$
n	$\frac{m_1}{n-(k-1)}$	$\frac{m}{n} - \frac{m_1}{n-(k-1)} - \frac{m_3}{2}$	$\frac{m_3}{2}$

Claim 16 *If* $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$, $\varphi(P^{5,k})$ *specifies probabilities on* B, C, *and* D *as assignment 1 for all* $k = 1, \cdots, \bar{n}_5$.

Proof: Verification of the claim consists of four steps.

Step 1: We show, if $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$, $\varphi(P^{5,1})$ specifies probabilities on B, C, and D as follows

	B	C	D
1	$\frac{m_1}{n}$	$\frac{m_2}{n-2}$	$\frac{m_2+m_3}{n} - \frac{m_2}{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$\frac{m_1}{n}$	$\frac{m_2}{n-2}$	$\frac{m_2+m_3}{n} - \frac{m_2}{n-2}$
$n-1$	$\frac{m_1}{n}$	0	$\frac{m_2+m_3}{n}$
n	$\frac{m_1}{n}$	0	$\frac{m_2+m_3}{n}$

First notice that $P^{5,1}$ and $P^{2,1}$ are different only in agent $(n-1)$'s preference, i.e., $P^{5,1}_{n-1} = \bar{P}_i$ and $P^{2,1}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{5,1}) = \varphi_{n-1,B}(P^{2,1}) = \frac{m_1}{n}$ and hence feasibility and *equal treatment of equals* imply $\varphi_{i,B}(P^{5,1}) = \frac{m_1}{n}$ for all $i \in I$.

Second we show $\varphi_{n-1,C}(P^{5,1}) = \varphi_{n,C}(P^{5,1}) = 0$. Suppose not, let $\beta \equiv \varphi_{n-1,C}(P^{5,1}) = \varphi_{n,C}(P^{5,1}) > 0$, then *sd-efficiency* implies $\varphi_{i,D}(P^{5,1}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1,D}(P^{5,1}) = \varphi_{n,D}(P^{5,1}) = \frac{m_3}{2}$. Then Lemma 15 requires $\frac{m_1+m_2+m_3}{n} =$

$\frac{m_1}{n} + \beta + \frac{m_3}{2}$. Then $\beta > 0$ implies $\frac{m_1+m_2+m_3}{n} - \frac{m_1}{n} - \frac{m_3}{2} > 0$ which is equivalent to $\frac{m_3}{m_2} < \frac{2}{n-2}$: contradiction!

All the other entries are implied by Lemma 15 and *equal treatment of equals*.

Step 2: We show $\varphi_{n-1,D}(P^{2,k}) = \varphi_{n,D}(P^{2,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)}$ for all $k = 1, \cdots, \bar{n}_5$.

Fix an k. First notice that $P^{5,k}$ and $P^{2,k}$ are different only in agent $(n-1)$'s preference, i.e., $P^{5,k}_{n-1} = \bar{P}_i$ and $P^{2,k}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{5,k}) = \varphi_{n-1,B}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ and hence *equal treatment of equals* implies $\varphi_{n,B}(P^{5,k}) = \varphi_{n-1,B}(P^{5,k}) = \frac{m_1}{n-(k-1)}$.

Second we show $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = 0$. Suppose not, let $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = \beta > 0$, *sd-efficiency* implies $\varphi_{i,D}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1,D}P^{5,k} = \varphi_{n,D}P^{5,k} = \frac{m_3}{2}$. Then we have a contradiction:

$$\varphi_{n,B}(P^{5,k}) + \varphi_{n,C}(P^{5,k}) + \varphi_{n,D}(P^{5,k}) = \varphi_{n,B}(P^{5,1}) + \varphi_{n,C}(P^{5,1}) + \varphi_{n,D}(P^{5,1})$$

$$\Leftrightarrow \frac{m_1}{n-(k-1)} + \beta + \frac{m_3}{2} = \frac{m_1}{n} + 0 + \frac{m_2+m_3}{n}\text{: contradiction!}$$

where the contradiction comes from $\frac{m_1}{n-(k-1)} \geqslant \frac{m_1}{n}$, $\beta > 0$, and that $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$ implies $\frac{m_3}{2} \geqslant \frac{m_2+m_3}{n}$.

Lastly, Lemma 15 implies what we want.

Step 3: We show $\varphi_{1,D}(P^{5,2}) = \gamma(2)$. Notice that $P^{5,2}$ and $P^{5,1}$ are different only in agent 1's preference, i.e., $P^{5,2}_1 = \hat{P}_i$ and $P^{5,2}_1 = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{1,D}(P^{5,2}) = \varphi_{1,D}(P^{5,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-2} = \frac{-2m_2}{n(n-2)} + \frac{m_3}{n}$. Notice that $B(2) = 0$, $C(2) = -2n^3+8n^2-10n+4$, and $D(2) = n^4-6n^3+13n^2-12n+4$. Then

$$\gamma(2) = \frac{0m_1 + (-2n^3+8n^2-10n+4)(k)m_2 + (n^4-6n^3+13n^2-12n+4)(k)m_3}{n[n^4-6n^3+13n^2-12n+4]}$$
$$= \frac{-2m_2}{n(n-2)} + \frac{m_3}{n}.$$

Step 4: We show an induction: For any $2 \leqslant k < \bar{n}_5$, if $\varphi_{i,D}(P^{5,k}) = \gamma(k)$ for all $i = 1, \cdots, k-1$, then $\varphi_{i,D}(P^{5,k+1}) = \gamma(k+1)$ for all $i = 1, \cdots, k$. By *equal treatment of equals*, it suffices to show $\varphi_{k,D}(P^{5,k+1}) = \gamma(k+1)$. Notice that $P^{5,k+1}$ and $P^{5,k}$ are different only in agent k's preference, i.e., $P^{5,k+1}_k = \hat{P}_i$ and $P^{5,k}_k = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then

$$\begin{aligned}
\varphi_{k,D}(P^{5,k+1}) &= \varphi_{k,D}(P^{5,k}) && \text{by } \textit{sd-strategy-proofness} \\
&= \frac{m_3 - 2\times\varphi_{n-1,D}(P^{5,k}) - (k-1)\times\varphi_{k-1,D}(P^{5,k})}{n-(k+1)} && \text{by feasibility and } \textit{equal treatment of equals} \\
&= \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\gamma(k)}{n-(k+1)} && \text{by Step 2 and hypothesis assumption} \\
&= \gamma(k+1) && \text{by simplifying the expression}
\end{aligned}$$

■

Claim 17 *If* $\frac{m_3}{m_2} < \frac{2}{n-2}$ *and* $\bar{n}_5 < k^*$, $\varphi(P^{5,k})$ *specifies probabilities on* B, C, *and* D *as assignment 2 for each* $k = 1, \cdots, \bar{n}_5$.

Proof: Verification of the claim consists of four steps.

Step 1: We show, if $\frac{m_3}{m_2} < \frac{2}{n-2}$, $\varphi(P^{5,1})$ specifies probabilities on B, C, and D as follows

	B	C	D
1	$\frac{m_1}{n}$	$\frac{m_2+m_3}{n}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$\frac{m_1}{n}$	$\frac{m_2+m_3}{n}$	0
$n-1$	$\frac{m_1}{n}$	$\frac{m_2-(n-2)\times\frac{m_2+m_3}{n}}{2}$	$\frac{m_3}{2}$
n	$\frac{m_1}{n}$	$\frac{m_2-(n-2)\times\frac{m_2+m_3}{n}}{2}$	$\frac{m_3}{2}$

First by the same argument showing the Step 1 in Claim 16, $\varphi_{i,B}(P^{5,1}) = \frac{m_1}{n}$.

Second we show $\varphi_{n-1,C}(P^{5,1}) = \varphi_{n-1,C}(P^{5,1}) > 0$. Suppose not, $\varphi(P^{5,1})$ is specified as by the Step 1 in Claim 16. Then $\varphi_{1,D}(P^{5,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-2} \geqslant 0$: contradicting against $\frac{m_3}{m_2} < \frac{2}{n-2}$.

Lastly, *sd-efficiency* implies $\varphi_{i,D}(P^{5,1}) = 0$ for all $i = 1, \cdots, n-2$. All the other entries are implied by Lemma 15 and *equal treatment of equals*.

Step 2: We show $\varphi_{n-1,B}(P^{5,k}) = \varphi_{n,B}(P^{5,k}) = \frac{m_1}{n-(k-1)}$ for all $k = 1, \cdots, \bar{n}_5$. Fix an k. Notice that $P^{5,k}$ and $P^{2,k}$ are different only in agent $(n-1)$'s preference, i.e., $P^{5,k}_{n-1} = \bar{P}_i$ and $P^{2,k}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{5,k}) = \varphi_{n-1,B}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ and hence *equal treatment of equals* implies $\varphi_{n,B}(P^{5,k}) = \varphi_{n-1,B}(P^{5,k}) = \frac{m_1}{n-(k-1)}$.

Step 3: For any $k < k^*$, if $\varphi_{i,D}(P^{5,k-1}) = 0$ for all $i = 1, \cdots, n-2$, then $\varphi_{i,D}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$. By *sd-efficiency*, it suffices to show $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) > 0$. Suppose not. First, by Step 2 and Lemma 15, $\varphi_{n-1,D}(P^{5,k}) = \varphi_{n,D}(P^{5,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)}$. Second, notice that $P^{5,k}$ and $P^{5,k-1}$ are different only in agent k's preference, i.e., $P_k^{5,k} = \hat{P}_i$ and $P_k^{5,k-1} = P_i$ where $\hat{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* and *equal treatment of equals* imply $\varphi_{i,D}(P^{5,k}) = \varphi_{k,D}(P^{5,k-1}) = 0$ for all $i = 1, \cdots, k$. Last, feasibility and *equal treatment of equals* imply $\varphi_{i,D}(P^{5,k}) = \frac{m_3 - 2\left(\frac{m}{n} - \frac{m_1}{n-(k-1)}\right)}{n-(k+1)}$. Then by $k < k^*$, we have a contradiction: $\varphi_{i,D}(P^{5,k}) < \frac{m_3 - 2\left(\frac{m}{n} - \frac{m_1}{n-(k^*-1)}\right)}{n-(k+1)} = 0$.

■

Claim 18 *If* $\frac{m_3}{m_2} < \frac{2}{n-2}$ *and* $\bar{n}_5 \geqslant k^*$, $\varphi(P^{5,k})$ *specifies probabilities on* B, C, *and* D *as assignment 2 for each* $k = 1, \cdots, k^*$ *and as assignment 1 for each* $k = k^* + 1, \cdots, \bar{n}_5$.

Proof: Verification of the claim consists of two steps.

By Claim 17, $\varphi(P^{5,k})$ specifies probabilities on B, C, and D as assignment 2 for each $k = 1, \cdots, k^*$.

Step 1: $\varphi(P^{5,k^*})$ specifies probabilities on B, C, and D as follows.

	B	C	D
1	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k^*-1	$-$	$-$	0
k^*	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	0
$n-1$	$\frac{m_1}{n-(k^*-1)}$	0	$\frac{m_3}{2}$
n	$\frac{m_1}{n-(k^*-1)}$	0	$\frac{m_3}{2}$

Step 2: For any $k > k^*$, if $\varphi_{n-1,C}(P^{5,k-1}) = \varphi_{n,C}(P^{5,k-1}) = 0$, then $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = 0$. Suppose not, then $\varphi_{i,D}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$ and

hence $\varphi_{n-1,D}(P^{5,k}) = \varphi_{n,D}(P^{5,k}) = \frac{m_3}{2}$. By Step 2 and Lemma 15, $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)} - \frac{m_3}{2}$. Then by $k > k^*$, we have a contradiction: $\varphi_{n-1,C}(P^{5,k}) = \varphi_{n,C}(P^{5,k}) < \frac{m}{n} - \frac{m_1}{n-(k^*-1)} - \frac{m_3}{2} = 0$.

■

Claim 19 *For each preference profile $P^{6,k}$, $\varphi(P^{6,k})$ specifies probabilities on B, C, and D as follows*

	B	C	D
1	$-$	$-$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-k$	$-$	$-$	$\frac{m_3}{n}$
$n-k+1$	$-$	$-$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	$\frac{m_3}{n}$
$n-1$	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$
n	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$

Proof: Verification of the claim consists of three steps.

Step 1: We show $\varphi_{n-1,B}(P^{6,k}) = \varphi_{n,B}(P^{6,k}) = \frac{m_1+m_2}{n}$ for all $k = 2, \cdots, \bar{n}_6$. Fix an k. Notice that $P^{6,k}$ and $P^{4,k}$ are different only in agent $(n-1)$'s preference, i.e., $P^{6,k}_{n-1} = \bar{P}_i$ and $P^{4,k}_{n-1} = P_i$ where $\bar{P}_i$ and P_i are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-1,B}(P^{6,k}) = \varphi_{n-1,B}(P^{4,k}) = \frac{m_1+m_2}{n}$. Hence *equal treatment of equals* implies $\varphi_{n-1,B}(P^{6,k}) = \varphi_{n,B}(P^{6,k}) = \frac{m_1+m_2}{n}$.

Step 2: We show $\varphi_{n-1,C}(P^{6,k}) = \varphi_{n,C}(P^{6,k}) = 0$ and $\varphi_{n-1,D}(P^{6,k}) = \varphi_{n,D}(P^{6,k}) = \frac{m_3}{n}$ for all $k = 2, \cdots, \bar{n}_6$. Fix an k. By Lemma 15, it suffices to show $\varphi_{n-1,C}(P^{6,k}) = \varphi_{n,C}(P^{6,k}) = 0$. Suppose not, then *sd-efficiency* implies $\varphi_{i,D}(P^{6,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence feasibility and *equal treatment of equals* imply $\varphi_{n-1,D}(P^{6,k}) = \varphi_{n,D}(P^{6,k}) = \frac{m_3}{2}$. Then $\frac{m_1+m_2}{n} + 0 + \frac{m_3}{2} > \frac{m}{n}$: contradiction against Lemma 15.

Step 3: We show $\varphi_{i,D}(P^{6,k}) = \frac{m_3}{n}$ for all $i = 1 \in I$ and all $k = 3, \cdots, \bar{n}_6$.

We first show $\varphi_{i,D}(P^{6,3}) = \frac{m_3}{n}$ for all $i = 1 \in I$. Notice that, by Step 2 and *equal treatment of equals*, $\varphi_{n-2,D}(P^{6,2}) = \frac{m_3}{n}$. Notice also that $P^{6,3}$ and $P^{6,2}$ are different only in agent $(n-2)$'s preference, i.e., $P^{6,3}_{n-2} = P_i$ and $P^{6,2}_{n-2} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-2,D}(P^{6,3}) = \varphi_{n-2,D}(P^{6,2}) = \frac{m_3}{n}$. Then Step 2 and *equal treatment of equals* imply what we want.

Now we show an induction: for any $3 \leqslant k < \bar{n}_6$, if $\varphi_{i,D}(P^{6,k}) = \frac{m_3}{n}$ for all $i = 1 \in I$, then $\varphi_{i,D}(P^{6,k+1}) = \frac{m_3}{n}$ for all $i = 1 \in I$. Notice that $P^{6,k+1}$ and $P^{6,k}$ are different only in agent $(n-k)$'s preference, i.e., $P^{6,k+1}_{n-k} = P_i$ and $P^{6,k}_{n-k} = \hat{P}_i$ where P_i and $\hat{P}_i$ are from Table 7. Then *sd-strategy-proofness* implies $\varphi_{n-k,D}(P^{6,k+1}) = \varphi_{n-k,D}(P^{6,k}) = \frac{m_3}{n}$. Hence Step 2 and *equal treatment of equals* imply what we want.

■

Now we have the contradiction to prove the theorem for the case where $\frac{m_2 n}{m_1+m_2}$ is not an integer.

$$P^{5,\bar{n}_5} = (\hat{P}_1, \cdots, \hat{P}_{\left[\frac{m_2 n}{m_1+m_2}\right]_-}, P_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$$

$$P^{6,\bar{n}_6} = (\hat{P}_1, \cdots, \hat{P}_{n-\left[\frac{m_1 n}{m_1+m_2}\right]_-}, P_{n-\left[\frac{m_1 n}{m_1+m_2}\right]_-+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$$

Notice that $\left[\frac{m_2 n}{m_1+m_2}\right]_- = \left[n - \frac{m_1 n}{m_1+m_2}\right]_- = \left(n - \left[\frac{m_1 n}{m_1+m_2}\right]_-\right) - 1$. Then $P^{5,\bar{n}_5}$ and $P^{6,\bar{n}_6}$ are different only in agent $\left(\left[\frac{m_2 n}{m_1+m_2}\right]_- + 1\right)$'s preference, i.e., $P^{5,\bar{n}_5}_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1} = P_i$ and $P^{6,\bar{n}_6}_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1} = \hat{P}_i$. Hence *sd-strategy-proofness* implies $\varphi_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1,D}(P^{5,\bar{n}_5}) = \varphi_{\left[\frac{m_2 n}{m_1+m_2}\right]_-+1,D}(P^{6,\bar{n}_6})$.

If $\varphi(P^{5,\bar{n}_5})$ is in the form of Assignment 2, the contradiction is evident: $0 \neq \frac{m_3}{n}$.

If $\varphi(P^{5,\bar{n}_5})$ is in the form of Assignment 1, the contradiction is verified by Assumption 2 in Appendix D.1.

$$\frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(\bar{n}_5-1)}) - (\bar{n}_5 - 1) \times \gamma(\bar{n}_5)}{n - (\bar{n}_5 + 1)} \neq \frac{m_3}{n}$$

D.4 Proof of Theorem 8

Fix a description (C, σ) that satisfies the condition in Statement **??**, we show that the induced preference domain $\mathbb{D}_{(C,\sigma)}$ is covered by a sequentially dichotomous domain.

For each $t^* \in \{2, \cdots, |C|\}$, let

$$\mathbf{A}^{\sigma}_{t^*} \equiv \left\{ B \subset A \middle| \exists (v_1, \cdots, v_{t^*-1}) \in \prod_{\tau=1}^{t^*-1} A^{\sigma}_{\tau} \text{ s.t. } b \in B \text{ whenever } (b^{\sigma}_1, \cdots, b^{\sigma}_{t^*-1}) = (v_1, \cdots, v_{t^*-1}) \right\},$$

i.e., objects are grouped according to their values of the top-(t^*-1) ranked characteristics.

Now we construct a sequence of partitions $(\mathbf{A}_t)_{t=1}^n$ using $(\mathbf{A}_{t^*})_{t^*=2}^{|C|}$ as the backbones.

- $\mathbf{A}_1 \equiv \{A\}$
- $\mathbf{A}_{|\mathbf{A}_{t^*}|} \equiv \mathbf{A}_{t^*}$, for each $t^* \in \{2, \cdots, |C|\}$
- for each $t^* \in \{2, \cdots, |C| - 1\}$, the partitions $\mathbf{A}_t$ with $|\mathbf{A}_{t^*}| < t < |\mathbf{A}_{t^*+1}|$ are defined as follows
  - pick any $B \in \mathbf{A}_{t^*} \backslash \mathbf{A}_{t^*+1}$, due to statement **??**, there are two blocks $C, D \in \mathbf{A}_{t^*+1} \backslash \mathbf{A}_{t^*}$ such that $B = C \cup D$
  - label blocks in $\mathbf{A}_{t^*} \backslash \mathbf{A}_{t^*+1}$ as $\left(B^1, \cdots, B^{|\mathbf{A}_{t^*+1}|-|\mathbf{A}_{t^*}|}\right)$ and blocks in $\mathbf{A}_{t^*+1} \backslash \mathbf{A}_{t^*}$ as $\left(C^1, D^1, \cdots, C^{|\mathbf{A}_{t^*+1}|-|\mathbf{A}_{t^*}|}, D^{|\mathbf{A}_{t^*+1}|-|\mathbf{A}_{t^*}|}\right)$ such that $B^m = C^m \cup D^m$ for all $m = 1, \cdots, |\mathbf{A}_{t^*+1}| - |\mathbf{A}_{t^*}|$
  - define $\mathbf{A}_{|\mathbf{A}_{t^*}|+m} \equiv (\mathbf{A}_{t^*} \backslash \{B^m\}) \bigcup \{C^m, D^m\}$ for each $m = 1, \cdots, |\mathbf{A}_{t^*+1}| - |\mathbf{A}_{t^*}|$.

The following two claims prove that the induced domain is covered by a sequentially dichotomous domain and hence Theorem 6 implies what we want.

Claim 20 *The sequence $(\mathbf{A}_t)_{t=1}^n$ is a path.*

By definition, $\mathbf{A}_1 = \{A\}$ and $\mathbf{A}_n = \{\{a\} : a \in A\}$, it suffices to show for each $t \in \{1, \cdots, n-1\}$, $\mathbf{A}_{t+1}$ is a direct refinement of $\mathbf{A}_t$, i.e., there is exactly one block $A_k \in \mathbf{A}_t$ and two blocks $A_i, A_j \in \mathbf{A}_{t+1}$ such that $A_k = A_i \cup A_j$ and for each $A_l \in \mathbf{A}_t \backslash \{A_k\}$ there is $A_i \in \mathbf{A}_{t+1}$ such that $A_l = A_i$. This is obvious from the construction of $(\mathbf{A}_t)_{t=1}^n$.

Claim 21 *Every preference $P_0 \in \mathbb{D}_{(C,\sigma)}$ observes every partition in the path $(\mathbf{A}_t)_{t=1}^n$.*

This is evident from the definition of lexicographically separable preferences and the path $(\mathbf{A}_t)_{t=1}^n$.

E Two Necessary Preference Restrictions for the Random Priority Rule to be sd-Efficient

The major virtue of random priority rule (RP henceforth) in theory is sd-strategy-proofness. In realistic applications, it has additional desired properties. The first one is that it is easy to implement. Since RP is a convex combination of deterministic allocations induced by serial dictatorships, to implement it, the authority simply randomly draws an order of agents and then implement the corresponding serial dictatorship. The second is its transparency, which comes directly from the way it is implemented: once the order of agents is settled down, agents just line up and each takes one object when it's her turn to do so. The third is that it is easily adjustable to context-born requirements. For example, when we allocate offices to professors, a commonly adopted practice is that senior professors should have higher priorities. To observe this requirement, it needs only to exclude some orders from the pool of admissible orders.

However, RP has its major drawback: it is not ex ante sd-efficient (See Bogomolnaia and Moulin (2001)), which causes strong opposition. After all, the major justification of pursuing sd-strategy-proofness is that it makes it easily tractable to pursuing other desired axioms, a major one of which is efficiency. So if it's sure that RP is not sd-efficient, the support of sd-strategy-proofness and hence RP itself is severely weakened.

However, RP being ex ante inefficient is built on the universal domain, which should not be the case in realistic applications. In stead of allowing all possible preferences, a reasonable assumption should be that agents preferences are intercorrelated in some way, as addressed by Che et al. (2015). For example, within a particular school district, when we observe that there are several parents who rank school 1 as the best and school 2 as the worse, it seems impossible to have another parent who ranks school 2 over school 1.

Then a natural question is: is it possible that in reality the RP is actually sd-efficient?

To answer this question, we need to investigate the preference restrictions which induce inefficiency for RP.

E.1 Definitions and Notations

Let A denote the object set and I denote the agent set. Typical elements in A are denoted a, b, x, y, and typical element in I are denoted i, j, l. We assume $|A| = |I| = n \geqslant 4$[26]. A preference of agent on objects is a linear order on A and typically denoted as P_i. By $B(a, P_i)$ we mean the upper contour set of a in P_i, i.e., $B(a, P_i) = \{x \in A : xP_ia\}$. A preference profile is a list of preferences, one for each agent. Given a preference profile P, we refer to P_i as the preference of agent i. A random assignment, or a bi-stochastic matrix, is denoted as L. In addition, L_{ia} denotes the probability that agent i gets object a.

We focus on RP rule, which is a convex combination of serial dictatorships. By a prior, we mean an order of agents, i.e., a one-to-one bijection $\sigma : I \to \{1, \cdots, n\}$. And $\sigma(i) = m$ means according to σ agent i is ranked as the m-th in I. Hence $\sigma(i) < \sigma(j)$ means according to σ agent i is ranked before j. Given a prior σ and a preference profile P, we denote $Pr_i(\sigma, P) = a$ when the serial dictatorship defined by σ gives agent i object a.

Definition 17 *Given a pair of preferences, P_i and $\bar{P}_i$, we say they are adjacent if there is an index $1 \leqslant k < n$ and two objects $x, y \in A$ such that $r_k(P_i) = r_{k+1}(\bar{P}_i) = x$, $r_{k+1}(P_i) = r_k(\bar{P}_i) = y$, and $B(x.P_i) = B(y, \bar{P}_i)$.*

$$
\begin{array}{lcccc}
\textit{ranking:} & \cdots & k & k+1 & \cdots \\
 & & & & \\
P_i : & \underbrace{\cdots\cdots\cdots}_{B(x,P_i)} & x & y & \cdots \\
 & \| & & & \\
\bar{P}_i : & \overbrace{\cdots\cdots\cdots}^{B(y,\bar{P}_i)} & y & x & \cdots
\end{array}
\tag{2}
$$

When P_i and $\bar{P}_i$ are adjacent through a reversal between x and y, we denote it by $P_i \sim^{\{xy\}} \bar{P}_i$.

Lemma 16 *Let P be an arbitrary profile and $L = RP(P)$, for any $i \in I$ and any $a \in A$, $L_{ia} > 0$ if and only if there is a prior σ such that $Pr_i(\sigma, P) = a$,*

[26]When $n \leqslant 3$, RP is sd-efficient.

Given a preference profile P and a random assignment L, we define a relation on A as $a\tau(P,L)b \Leftrightarrow \exists i \in I : aP_ib, L_{ib} > 0$.

Lemma 17 (Lemma 3 in Bogomolnaia and Moulin (2001)) *The random assignment L is sd-efficient at profile P if and only if $\tau(P,L)$ is acyclic.*

Proof: Suppose $\tau(P,L)$ is a cycle, let

$$a_1 \underset{i_1}{\tau(P,L)} a_2 \underset{i_2}{\tau(P,L)} a_3 \cdots a_K \underset{i_K}{\tau(P,L)} a_1$$

where i_k is the agent such that $a_k P_{i_k} a_{k+1}$ and $L_{i_k a_{k+1}} > 0$.

By Lemma 16, there exists this cycle if and only if for each $k = 1, \cdots, K$ there is a σ_k such that $Pr_{i_k}(\sigma_k, P) = a_{k+1}$ (let $a_{K+1} = a_1$). For each $k = 1, \cdots, K$, define another (infeasible) matching μ_k such that $\mu_k(i_k) = a_k$ and $\mu_k(j) = Pr_j(\sigma_k, P)$ for all $j \neq i_k$. Then it's easy to see that $\{\mu_1, \cdots, \mu_K\}$ dominates $\{Pr(\sigma_1, P), \cdots, Pr(\sigma_K, P)\}$, which is equivalent to saying that L is sd-inefficient at P, by Abdulkadiroğlu and Sönmez (2003). ■

E.2 Conditions

Condition 1 *A domain $\mathscr{D}$ satisfies **condition 1** if for any $P_i, \bar{P}_i \in \mathscr{D}$ and any $x, y \in A$ such that xP_iy, $y\bar{P}_ix$ and $r_k(P_i) = y$, $|B(x,\bar{P}_i) \backslash B(x,P_i)| \geqslant n-k$.*

$$\begin{array}{lccccccc}
\bar{P}_i: & \underbrace{\cdots \quad y \quad \cdots}_{B(x,\bar{P}_i)} & x & \cdots & \cdots & \cdots \\
P_i: & \overbrace{\cdots\cdots\cdots\cdots}^{B(x,P_i)} & x & \cdots & y & \cdots \\
 & & & & \vdots & \\
\text{ranking:} & & & & k &
\end{array} \tag{3}$$

Condition 2 *A domain $\mathscr{D}$ satisfies **condition 2** if there exists no adjacent pair $P_i, \bar{P}_i \in \mathscr{D}$.*

E.3 Results

The following two results showing that both Condition 1 and 2 are necessary for RP to be sd-efficient. In other words, whenever either one of these two conditions is violated, the RP rule is for sure inefficient.

Proposition 4 *The RP is sd-efficient on a domain $\mathscr{D}$ only if $\mathscr{D}$ satisfies condition 1.*

Proof: We prove the contrapositive statement. Let $P_i, \bar{P}_i \in \mathscr{D}$ be such that xP_iy, $y\bar{P}_ix$, and $|B(x,\bar{P}_i)\backslash B(x,P_i)| \leqslant n-k-1$, where $r_k(P_i) = y$. Let $P \in \mathscr{D}^I$ be a profile such that agents $i_1, \cdots, i_k$ report P_i and agents $i_{k+1}, \cdots, i_{k+(n-k)}$ report $\bar{P}_i$. In addition, denote $L \equiv RP(P)$. It sufices to show L is sd-inefficient at P.

Consider a prior $\sigma_1 : i_1, \cdots, i_k, i_{k+1}, \cdots, i_{k+(n-k)}$. According to σ_1, agent i_k takes y, which gives $L_{i_k,y} > 0$. Consider in addition another prior $\sigma_2 : i_1, \cdots, i_{|B(x,P_i)|-1}, i_{k+1}, \cdots, i_{k+(n-k)}, \cdots$. According to σ_2, agents $i_1, \cdots, i_{|B(x,P_i)|-1}$ take $B(x,P_i)\backslash\{x\}$. After that, agents $i_{k+1}, \cdots, i_{k+(n-k)}$ take objects in $B(x,\bar{P}_i)\backslash B(x,P_i)$, if there are some remaining. Notice that it takes at most $n-k-1$ agents to take away $B(x,\bar{P}_i)\backslash B(x,P_i)$ and there are $n-k$ agents waiting to do so. Hence there must be an agent in $i_{k+1}, \cdots, i_{k+(n-k)}$ who takes x. That agent is actually $i_{k+|B(x,\bar{P}_i)\backslash B(x,P_i)|+1}$, which gives $L_{i_{k+|B(x,\bar{P}_i)\backslash B(x,P_i)|+1},x} > 0$. Combining the above two observations, we know L is sd-inefficient at P. ■

Proposition 5 *The RP is sd-effficient on a domain only if $\mathscr{D}$ satisfies condition 2.*

Proof: We prove the contrapositive statement. Let $P_i, \bar{P}_i \in \mathscr{D}$ and two objects $x, y \in A$ such that $x = r_k(P_i) = r_{k+1}(\bar{P}_i)$, $y = r_{k+1}(P_i) = r_k(\bar{P}_i)$ for some $k = 1, \cdots, n-1$, and $B(x,P_i) = B(y,\bar{P}_i)$. To prove the statement, it suffices to show that there exists a profile P which consists of only these two preferences and $L \equiv RP(P)$ is sd-inefficient at P.

When n is even, let P be a profile, according to which agents $i_1, \cdots, i_{n/2}$ report P_i and the remaining agents, $i_{n/2+1}, \cdots, i_n$, report $\bar{P}_i$.

If $k \leqslant n/2$, consider two order of agents σ_1 and σ_2 depicted in (4).

According to σ_1, agents $i_1, \cdots, i_{k-1}$ take the first $k-1$ ranked objects. And after this, agents $i_{n/2+1}$ and $i_{n/2+2}$ take y and x respectively. Hence we have $L_{i_{n/2+2},x} > 0$.

According to σ_2, agents $i_{n/2+1}, \cdots, i_{n/2+(k-1)}$ take the first $k-1$ ranked objects. And after this, agents i_1 and i_2 take x and y respectively. Hence we have $L_{i_2,y} > 0$. Combining these two observations, we know L is sd-inefficient at P.

$$\begin{array}{cccccccc} \text{order:} & 1 & \cdots & k-1 & k & k+1 & \cdots \\ \hline \sigma_1: & i_1 & \cdots & i_{k-1} & i_{n/2+1} & i_{n/2+2} & \cdots \\ \sigma_2: & i_{n/2+1} & \cdots & i_{n/2+(k-1)} & i_1 & i_2 & \cdots \end{array} \tag{4}$$

If $k \geqslant n/2+1$, consider two orders of agents σ_3 and σ_4, where $\sigma_3(l) = i_l$ for all $l = 1, \cdots, n$, $\sigma_4(l) = i_{n/2+l}$ and $\sigma_4(n/2+l) = i_l$ for $l = 1, \cdots, n/2$. According to σ_3, agent i_{k+1} takes x, which gives $L_{i_{k+1},x} > 0$. According to σ_4, agent $i_{k-n/2+1}$ takes y, which gives $L_{i_{k-n/2+1},y} > 0$. Combining these two observations, we know L is sd-inefficient at P.

The case where n is odd can be verified similarly, we omit it here. ■

Remark 13 Almost all domains in the voting literature violate the Condition 2: the universal domain (Gibbard (1973)), linked domains (Aswal et al. (2003)), circular domains (Sato (2010)), the single-peaked domain (Moulin (1980), Demange (1982)), the single-dipped domain (Barberà et al. (2012)), maximal single-crossing domains (Saporiti (2009)), generalized single-peaked domains (Nehring and Puppe (2007)), the multi-dimensional single-peaked domain (Barberà et al. (1993)) and the separable domain (Barberà et al. (1991)). Therefore, our impossibility result also prevails on these domains.

Unfortunately, these two conditions combined is **not sufficient** to guarantee efficiency, as indicated by the following example.

Example 10 A domain which satisfies both conditions but gives a profile for which the random priority is inefficient. Consider the following four preference. It's easy to verify that the domain consisting of these preferences satisfies both conditions.

$$\begin{array}{lllll} P_1: c & a & b & d \\ P_2: c & b & d & a \\ P_3: d & a & c & b \\ P_4: d & b & c & a \end{array}$$

Let $P = (P_1, P_2, P_3, P_4)$.

Consider prior $\sigma : 2 < 4 < 3 < 1$, we have $Pr(\sigma, P) = (b, c, a, d)$. Consider prior $\sigma' : 1 < 3 < 4 < 2$, we have $Pr(\sigma', P) = (c, a, d, b)$. Then let $L = RP(P)$, we have $L_{1b} > 0$ and $L_{2a} > 0$: L is inefficient at P.